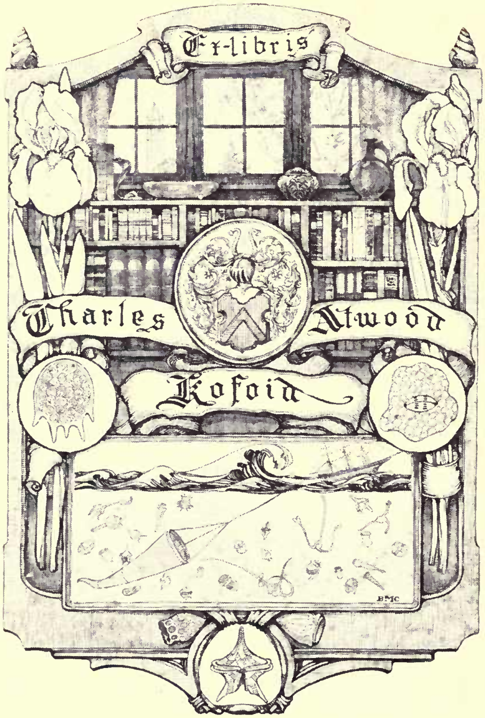
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AN ACCOUNT OF THE ASTRONOMICAL DISCOVERIES OF KEPLER.

J. Craig

AN

ACCOUNT

OF THE

ASTRONOMICAL DISCOVERIES

OF

KEPLER:

INCLUDING

AN HISTORICAL REVIEW

OF

THE SYSTEMS

WHICH HAD SUCCESSIVELY PREVAILED BEFORE HIS TIME.

BY

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FELLOW OF THE ROYAL SOCIETY OF EDINBURGH.

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TO

THE RIGHT HONOURABLE

THE EARL OF LAUDERDALE,

THE

FOLLOWING ACCOUNT

OF THE

ASTRONOMICAL DISCOVERIES

OF

KEPLER,

IS GRATEFULLY INSCRIBED,

BY HIS LORDSHIP'S

MOST OBEDIENT AND DEVOTED SERVANT,

R. SMALL.

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CHAP.

AN ACCOUNT

OF THE

ASTRONOMICAL DISCOVERIES

OF

KEPLER, &c.

AS the discoveries of Kepler have contributed more than all other causes to raise the science of astronomy to its present state of improvement, they not only deserve full and particular explication, but also all the circumstances which led to them, and even the mistakes committed in their prosecution, become interesting objects of curiosity. It is a just observation of his, that we not only pardon Columbus and the Portuguese navigators, for relating their errors; the former in the discovery of America, and the latter in the circumnavigation of Africa; but should be deprived of much instruction and satisfaction if those errors were omitted. My principal intention, therefore, in the present publication, is to give a more full and particular account of Kepler's discoveries, than any to be found in the usual systems, or the general histories of Astronomy; and to extract the account from his own investigations. These are chiefly contained in his Commentary on the Motions of Mars; and I have often regretted that a work, containing such invaluable discoveries, should not be more generally and distinctly known. This work claims attention for another reason, that it exhibited, even prior to the publication of Bacon's *Novum Organum*, a more perfect example, than perhaps ever was given, of legitimate connection between theory and experiment; of experiments suggested by theory, and of theory submitted without prejudice to the test and decision of experiments. But, in order to form a just estimate of those discoveries, nay, perhaps, a distinct conception of the investigations by which they were produced, it seemed absolutely necessary to prefix an account of the more ancient astronomical theories, and of the principal phenomena which they were contrived and supposed to explain.

CHAP. I.

Of the principal Motions and Inequalities of the Celestial Bodies.

1. In the contemplation of the heavens, the attention of mankind must have been first attracted by the splendor and configurations of the celestial bodies, and the constant diurnal revolution in which they seem in general to partake. They all appear to be placed in the vast concave surface of the ethereal expanse, at equal distances from the spectator's eye, and indeed from the earth itself; for no change of the spectator's situation, upon the surface of the earth, is found to produce any general and sensible change on their relative positions. It was concluded, therefore, that this expanse was a sphere, of which the earth is the centre; and of such

such immense magnitude, that the earth, in comparison with it, was to be considered only as a single point: and, by the constant rotation of this expanse upon its axis, the whole phenomena of the diurnal revolution, called the first motion, were explained. First motion, or diurnal revolution.

2. But in attending to the circumstances of the diurnal revolution, it was discovered that seven of the celestial bodies, to wit, the Sun, the Moon, Mercury, Venus, Mars, Jupiter, and Saturn, though constantly carried round the earth in the course of the diurnal revolution, neither described their circles with the same accuracy as the rest, nor in the same precise portion of time. On marking the points of the horizon, where on any particular day they rose and set, they were always found after some interval to rise and set at different points, more to the north, or to the south: and, in the same manner, the points where they crossed the meridian were also observed to vary, though within certain limits; and their transits over it to be made at unequal distances, both from the zenith and from either pole. Those diurnal circles were seldom great circles of the sphere: they were rather a perpetual spiral than a series of circles, and resembled a thread regularly wound about a cylinder, and continually passing and re-passing between its extremities. One of the most early distinctions, made among the celestial bodies, was actually taken from the different velocities with which they performed their diurnal revolution. The stars, afterwards called fixed, were considered as the swiftest of all: the Sun, Mercury, Venus, Mars, Jupiter, and Saturn, were conceived to move with somewhat less rapidity: and the Moon was reckoned

Peculiar motions of some celestial bodies.

reckoned the slowest of all, as taking the longest time to complete her diurnal circle.

3. Though the fixed stars, therefore, as they were never seen to change their positions, either with respect to one another, or with respect to the poles of the equator and the horizon, appeared to have no motion except the first and simple one produced by the diurnal revolution of the starry sphere, it was evident that the sun, the moon, and the planets, were subjected to peculiar motions of their own. To explain these, the ancients supposed every one of those bodies to be placed in a great circle of a separate and peculiar sphere, having different poles from those of the diurnal sphere, and revolving on a different axis: and, as each of these great circles intersected the equator, and declined from it to the north and south, it necessarily followed, that those seven bodies declined also from the equator, and were generally found in some of the smaller circles of the diurnal sphere; and though partaking of the diurnal revolution, yet on account of their own proper motions, or rather the proper motions of the spheres in which they were fixed, they did not complete their diurnal circles in the same precise time which the diurnal revolution required. In consequence of thus appropriating to each planet a sphere of its own, and to each sphere a motion peculiar to itself, new notions concerning the velocities of the celestial bodies were adopted, the reverse of what had formerly prevailed. The moon, which before was regarded as the slowest of all, was now reckoned by far the most rapid; the sun and the planets were supposed to perform their peculiar revolutions more slowly; and the other celestial bodies, which had been esteemed the swiftest, were now considered

Ancient explication by means of peculiar spheres.

dered as absolutely fixed; since they were found to be destitute of every motion except the diurnal revolution.

4. This separation of the peculiar motions of the sun, moon, and planets from the general diurnal revolution, was an important step in the progress of astronomy; for it banished the perplexing spirals in which those bodies seemed to move, and subjected their motions to more accurate examination. But the difficulties which this examination presented were as great, and the irregularities as inexplicable, as when the peculiar motions were confounded with the general diurnal revolution: and the natural difficulty was increased by the false maxims connected with the very low state of physical knowledge. It was established among the ancients as a sacred and inviolable principle, that all the celestial motions must be uniform and circular: and the great object to which they applied themselves in all their theories, was to delineate the circles, and compositions of circles, by which, motions, apparently irregular, might be resolved into such as were circular and uniform. Though therefore the distinguishing between the motion peculiar to the sphere of each planet, and the diurnal revolution common to the whole heavens, rendered the ancient spirals unnecessary, and seemed at first happily to account for the observed appearances; yet, on more close examination, the irregularities which presented themselves, could not be explained by a supposition so simple, and other contrivances were required to represent them.

Insufficiency of the explication.

5. The irregularities of the motion of the sun were indeed less striking than those of the moon and planets, but they were equally real. For;

though the equinoctial points, that is, the points where the ecliptic, or the sun's peculiar great circle intersects the plane of the equator, were diametrically opposite, the sun employs 187 days in passing from the vernal to the autumnal equinox; whereas, in describing the opposite semi-circle of the ecliptic, from the autumnal equinox to the vernal, he employs no more than 178: and, in like manner, though the solstitial points are in the plane of a circle perpendicular to that which joins the equinoctial points, the interval between the vernal equinox and the summer solstice, consists of 94½ days, whereas the interval between that solstice and the autumnal equinox, consists of only 92½ days. The sun, therefore, in the two first of these intervals, must proceed in his orbit more slowly than in the two last. Nor is it only by observations at the distance of several months, but also from the observations of almost every day, that the inequality of the solar motions may be rendered evident. By the returns of the sun to the same equinoctial point, and especially by the comparison of two such returns as remote as possible from each other, it has been found that the time, in which he performs his annual revolution of 360°, consists of 365 d. 5 h. 49′ 8″; and consequently that his motion for one day ought to be through an arch of 59′ 8″. If his motion, therefore, were perfectly uniform, all that would be necessary to find his place in the ecliptic, or his longitude from Aries, on any given day, would be to multiply the number of days elapsed, from his passing that equinoctial point, by 59′ 8″. But, if we compare the mean or uniform longitudes thus deduced, with the longitudes obtained by daily and actual observation, we shall find the sun's place in the ecliptic sometimes more advanced by nearly two degrees than it

Inequality of the solar motions.

it ought to be, on the supposition of uniform motion, and at other times left as far behind. In fact, the sun's actual rate of motion is never found equal to the mean rate, except in two nearly opposite points of his orbit; whereas in two other opposite points, distinguished by the name of the apsides, at the distance of 90° from these, and in a line passing through the centre of the earth, it is in the one slower than the mean rate, and in the other swifter; and in both the retardation and acceleration are at their maximum.

6. It was also discovered by Hipparchus, from a comparison of his own observations with the more ancient ones of Timochares, that the intersections of the ecliptic with the equator were not fixed, but variable points; and that they had a motion westwards, in an opposite direction to the solar motion: for it appeared by this comparison, that the sun returned sooner to either of the equinoctial points than to any particular star in the ecliptic; and, by similar comparisons of ancient observations with modern ones, it has been ascertained, that if a fixed star had coincided with one of the equinoctial points at the sun's setting out from it, he will, at his next return to the same equinox, fall short of the star 50″. 33. This retrograde motion of the equinoctial points is called the *precession of the equinoxes*; and it was found to produce variations in the longitudes, the right ascensions, and the declinations of all the fixed stars. It has also been discovered, by a like comparison of ancient and modern observations, that the solar apsides are not fixed points in the ecliptic, but moveable in a contrary direction to the motion of the equinoxes; for the points, of both the greatest and the least differences between the mean and

Precession of the equinoxes, and motion of the solar apsides.

true places of the sun, are found to have advanced more than a whole sign in longitude beyond their positions, in the days of Hipparchus and Ptolemy: and, according to the most accurate determinations, the arch through which they move annually, consists of about 65″. 5. Thus, three kinds of solar revolutions came to be distinguished; the tropical, to the equinoctial or solstitial points, consisting of 365 d. 5 h. 49′. 8″, which brings back the seasons in their order, and is properly called the year; the sidereal, to any particular star or point of the ecliptic, consisting of 365 d. 6 h. 9′ 34″; and the anomalistical, to either of the apsides, bringing back all the solar equations in their order, and consisting of 365 d. 6 h. 15′ 20″.

7. As all the phenomena of the moon were more striking to common observers than those of the sun, her irregularities were also more various and perplexing. In order to discover their laws and progress, it was necessary, in the same manner as with respect to the sun, to determine her mean rate of motion; that, by a comparison with this as a standard, her variations from it might be ascertained. But the determination was a matter of the greatest difficulty. It is probable that the attention of astronomers was first turned to her synodical revolutions, or those which she makes with respect to the sun, and which bring back all her various phases or changes of appearance. The determination, however, of the mean duration of these was so difficult, that for a long time little more seems to have been known concerning it, than that it consisted of about 29½ days; and when Meton, an Athenian, in the year 430 before the Christian æra, discovered, or more probably learned from oriental astronomers, that the

moon

moon performed 235 synodical revolutions in the time of 19 tropical revolutions of the sun; the discovery, though imperfect, was considered as so important, that it was publicly fixed up in letters of gold in the principal cities of Greece; and the period, or cycle of 19 years, has thence been distinguished by the name of the golden number. Golden number.

In this synodical revolution the moon moves through more than 360° of longitude in the zodiac; for, during the continuance of it, the sun advances about 29° in the ecliptic, and therefore her motion in longitude must be made through 389 degrees, before she can return to her conjunction with him. Her tropical revolution, therefore, with respect to any particular point of the zodiac, and her synodical revolution, are to each other nearly as 360 to 389; but, by means of Meton's cycle, or perhaps some other still less perfect, the mean time of both, and consequently the mean motion of the moon in longitude, for any particular portion of time, was in some manner, however imperfectly, obtained.

8. This determination, however, of the moon's mean motion in longitude, though imperfect, was sufficient to demonstrate, that her real motion was unequal: for it was found that, if her mean and true places had at any time coincided, in about 7 days after they generally differed 5 or 6 degrees; in about 14 days they again coincided; in 21 days they again differed 5 or 6 degrees in a contrary direction; and, in 27 days, and something more than an half, they again coincided as at first. It appeared therefore, that there were two opposite points of the lunar orbit, afterwards called apsides as in the solar orbit, in which the differences between the moon's mean and real places disappeared; and

Inequality of the moon's motion in longitude.

and two others, at equal distances from either of these, in which they came to their greatest amount; so that, in the one semicircle, the velocity was increasing from its least to its mean, and then to its greatest rate; and, in the other, decreasing in a reversed order, though in the same proportion. To ascertain the period of those inequalities, the ancients employed lunar eclipses, the observations of them being the only ones in which the place of the moon was not deranged by parallax: and, if in any two eclipses the motions of the moon had been found equal and of the same kind, either increasing in velocity, or decreasing, they considered this as a proof, that in both she occupied the same point of her orbit, or was in the same degree of anomaly. Consequently, they were enabled, by their records of her retardations and accelerations, to compute how many repetitions of those inequalities had been made in the interval, or how many revolutions the moon had made with respect to the apsides of her orbit: and their conclusions were confirmed, if two other eclipses could be found, whose interval was the same, where the moon's motions were also equal, but in the opposite simicircle of anomaly, with the velocity decreasing, as in the former it had been increasing. Hipparchus accordingly found two such pairs of eclipses, at the interval of 126,007 days, and 1 hour, during which he computed the moon to have performed 4267 synodical revolutions, and 4612 tropical revolutions, wanting 7° 30′; while the returns of the inequality now mentioned, that is, the revolutions in the orbit were 4573. It became evident, therefore, that the returns of this inequality had neither any connection with the synodical revolution, (because in 4267 lunations, the moon had returned to the same degree of anomaly

maly 4573 times); nor with the tropical revolution, (because during those 4573 revolutions of anomaly, she had been nearly 4612 times in the same degree of longitude). The lunar apsides, therefore, were variable, and not fixed points in the zodiac, and appeared to advance, during every lunation, about 3 degrees in longitude, so as in 9 years to complete the circuit of the heavens: and if the greatest difference between the mean and true places of the moon had happened, that is, if the greatest equation of 5 or 6 degrees had been applied, in one lunation, at any particular point of longitude, it was not applicable in the next lunation at the same point, but at one about 3° more advanced. By this investigation also of Hipparchus, a new cycle for calculating the lunar motions was discovered, namely, that in 251 lunations the moon performs 269 revolutions of anomaly: for the ratio of 4267 to 4573, is the same with that of 251 to 269.

9. It was likewise discovered, chiefly by the observation of lunar eclipses, that, as the ecliptic intersects the equator, so the lunar orbit intersects the ecliptic in two opposite points, called nodes; and that these opposite points are, like the apsides, variable, though in a contrary direction, in their longitude. For it was only in the nodes that the moon was seen to be without latitude; in all the other points of her orbit she was found to be either on the north, or on the south of the ecliptic; till, at the distance of 90° from the nodes, her latitude, or deviation from the ecliptic, amounted to 5 degrees: and, since in lunar eclipses she must always be very near one or other of the nodes, it appeared that she was not involved in the shadow of the earth, at the same constant

Motion of the moon's nodes.

constant points of the zodiac. The position of the nodes, therefore, was variable; and their motion became also evident from her occultations of stars situated in the ecliptic. When an occultation of such stars, as for example, of Regulus, happens, the moon is certainly in one of the nodes: but, as in something less than 5 years, she is seen to pass 5 degrees to the north or the south of Regulus; and as this is the greatest latitude at which she is known to arrive, she must have departed from the node to the distance of 90 degrees. To determine the rate and direction of the motion of the nodes, Hipparchus compared all the lunar eclipses which had been observed from the times of the Chaldean astronomers to his own, and found that, in 5458 lunations, the moon had been in the same node 5923 times. By the same comparison another cycle was discovered of great importance in the calculation of lunar eclipses: for, in 223 lunations, that is, in 18 years and 10 days, the moon had made 242 revolutions to the same node, 241 and 10° 40′ more to the same degree of longitude, and 239 to the same degree of anomaly: so that while the anomalistical revolution was longer than the tropical, and consequently the apsides were continually advancing in the ecliptic, the revolution in latitude, or to the nodes, was shorter; and the motion of these points therefore retrograde.

10. Modern astronomers, possessing many advantages above the ancients, for determining the mean times of the various lunar revolutions, have accordingly applied themselves to this subject with the greatest care. D. Cassini, for example, compared a lunar eclipse, recorded by Ptolemy to have been observed at Babylon 720 years before the Christian æra, with an eclipse observed by himself in

in the year of that æra 1717. The Chaldean eclipse, he finds, must have happened at Paris on the 19th of March, at 6 h. 48′, O. S; and his own was observed there, on the 9th of September, at 6 h. 2′. The moon's longitude in the former was 5 s. 21° 27′, and in the latter 11 s. 27° 34′; and the interval consists of 890,288 days, wanting 46 minutes. In this interval he calculated, that the moon had performed 32,585 revolutions to the same point of longitude, and 6 s. 6° 7′ more; and consequently, dividing the interval by the number of revolutions, that the mean time of every one was 27 d. 7 h. 43′ 5″. But, as there might be some uncertainty about the number of the revolutions, and the omission of one in so long a period would not produce a difference of above one minute in the length of each, he also made comparisons between less distant eclipses, and by these established his conclusion. The moon's mean diurnal motion, therefore, in longitude, according to D. Cassini, is 13° 10′ 35″; and multiplying this by 365¼, he found her annual motion; and this product again by 100, her secular motion; that is, the remainders of both, after subtracting the complete revolutions. By still later investigations of the same kind, the tropical revolution, at the beginning of the eighteenth century, is found to be performed in 27 d. 7 h. 43′ 4″.648; and consequently the mean diurnal motion in longitude, to be 13° 10′ 35″.02847, and the mean secular motion 10 s. 7° 53′ 35″.

Mean tropical revolution of the moon.

From the time of a tropical revolution thus ascertained, that of a sidereal one is easily inferred; for, in 27 d. 7 h. 43′ 4″.648, the equinoctial points move in precession through 4″ of longitude; and as the moon's mean motion in 1 hour is through 32′ 56″.458, she will employ 7″ of time in moving through

Sidereal revolution.

through 4″. The time, therefore, of a mean sidereal revolution of the moon, in the 18th century, is 27 d. 7 h. 43′ 11″.648.

From the same investigation, the time will also easily follow of a mean synodical revolution. For the times of the mean tropical revolutions of the sun and moon are given, and the arches they describe are inversely as the times. Let t be the time of a tropical revolution of the moon, and T that of the sun; and let the arch described by the sun, in the time of a synodical revolution, be called x, and consequently that described by the moon in the same time $360° + x$. Since, therefore, $T : t :: 360 + x : x$, we shall have $T - t : t :: 360 : x$; and this arch therefore will be found: and, since the time of the moon's tropical revolution, or of describing 360°, is given, the whole time of describing $360° + x$ will be found to consist of 29 d. 12 h. 44′ 2″.8291.

Synodical revolution.

The time, again, of an anomalistical revolution is, in the same manner, determined by comparisons between the places occupied by the apsides, in ages distant from one another. By observing the points where the moon's mean and true places coincide, the longitude of the apsis is given; for it is, in these circumstances, the same with the longitude of the moon: and, from the nature of the lunar motion about the time of the observation, it may always be distinguished, whether the apsis under consideration be the apogee, or the perigee. Or the place of either apsis may be more accurately found, by observing the points of longitude where the greatest equations of contrary kinds, or, in general, where any two equal equations of contrary kinds take place, because the apsides are situated precisely in the middle between them. To these means of discovering the position of the apsides, the

Anomalistical revolution.

the moderns add observations of the lunar diameter; for, as its apparent magnitude varies between the two extremes of 29′ 30″, and 33′ 30″, the moon is evidently in the apogee in the former case, and in the perigee in the latter. Accordingly, by the comparison of such places, the apsides were found to perform their revolution with respect to either of the equinoctial points, in 3231 d. 8 h. 24′ 57″.6; and with respect to any particular star, in 3232 d. 11 h. 14′ 31″: so that their mean diurnal motion towards the former is 6′ 41″.069815, and towards the latter 6′ 40″.931992; and the time of the anomalistical revolution is 27 d. 13 h. 18′ 34″.022.

Revolution of the nodes.

Finally, the places of the nodes were determined by observations of the moon's latitude, especially in eclipses; and, by like comparisons between them, they are found to perform their retrograde revolution, with respect to the equinoctial points, in 6798 d. 4 h. 52′ 52″; so that the time of the moon's revolution, with respect to the nodes, consists of 27 d. 5 h. 5′ 49″.1709.

Thus, no less than five kinds of lunar revolutions came to be distinguished, the tropical, which brings back the moon to the same longitude; the sidereal, which brings her back to the longitude of some particular star; the synodical, in which she goes through all her varieties of phasis, and returns to her conjunction with the sun; the anomalistical, in which she returns to the apogee of her orbit, and during which, the inequality above mentioned performs its course; and her revolution to either of the nodes, which brings back her whole varieties of latitude.

11. The greatest amount of the inequality hitherto spoken of, and called the equation of the orbit, the first inequality, and the *inæqualitas soluta*,

luta, was fixed by Hipparchus and Ptolemy, at 5° 1′: and, while the attention of astronomers was principally turned to lunar eclipses, no other had been distinguished, at least accurately ascertained. But when *armillæ* were employed to observe the moon in other situations, and especially when the observations were made in the meridian, in order to avoid the errors in longitude produced by parallax, a second inequality was discovered, which was connected, not with the anomalistical, but with the synodical revolution of the moon; disappearing in conjunctions and oppositions, and coming to its greatest amount in quadratures. What was most perplexing about this second inequality was, that it did not return in every quadrature; but though, in some, it amounted to 2° 39′, and increased the first inequality to 7° 40′, in other quadratures it totally disappeared. This extraordinary increase of the first inequality, in certain quadratures, had been established by Hipparchus; and Ptolemy happily succeeded in determining its laws and progress: and, by these celebrated astronomers, the mean amount of both was fixed with singular precision, at 6° 20′. Modern astronomers reckon it at 6° 18′ 32″. The name of the evection was given to this second inequality; and as the former being unconnected with the synodical revolution was called the *inæqualitas soluta*, so this, being connected with it, was termed the *inæqualitas alligata*.

Second inequality of the moon, or the evection.

12. As the first inequality came to be discovered in consequence of determining the general mean motion of the moon in longitude (10); and, after the application of the first equation, the difference which still remained between her true and calculated places, produced the discovery of the second;

so

so the difference, which Tycho Brahé, in consequence of the extensive range and superior accuracy of his observations, continued to find between those places, discovered to him a third lunar inequality. This was, like the second, connected with the synodical revolution, and he termed it the variation. He found that the moon, in passing from the quadratures to the syzigies, was always more accelerated than the first and second equations represented, and, from the syzigies to the quadratures, more retarded; and that this variation came to its greatest amount in the octants, that is at the diſtance of 45° from either syzigy. The greateſt amount of this variation, now fixed at 37′ 4″, he reckoned to be 40′ 30″.

Third inequality, or the variation.

13. The same illustrious astronomer discovered a fourth inequality of the lunar motion in longitude. It was not connected, like the former, with any of the lunar revolutions, but depended upon the annual, or more properly, upon the anomalistical revolution of the sun: and it took place at any time of the lunation without distinction, and in any point of the moon's anomaly. When the sun was in his apogee, the motion of the moon was found to be quicker, than what was represented after the application of all the equations now mentioned; and ſlower when the sun was in his perigee. Another equation therefore upon this account was necessary, and it was named the annual equation. Its greateſt amount, which T. Brahé reckoned at 9′ 56″, is now fixed at 11′ 49″.

Fourth inequality, or annual equation.

14. T. Brahé also discovered, by means of simple observation, that the inclination of the lunar orbit to the ecliptic was variable, and that the motion of the nodes was not equable. For he found that

the

Inequalities of latitude. the greatest latitudes of the moon, in conjunctions and oppositions, did not exceed 4° 58′ 30″; whereas, in quadratures, they increased to 5° 17′ 30″: and that, though there was no sensible difference, in conjunctions and oppositions, between the mean and true places of the node, the difference, in other cases, rose to 1° 46′; and produced, in the neighbourhood of the node, a difference of 12′ between the observed and the calculated latitudes. (A)

15. The peculiar motions and inequalities of the planets were not less perplexing than those of the moon. They all, like the sun, seem to describe orbits round the earth, and the general direction of their peculiar motions is from west to east, among the stars of the zodiac. But this general direction is affected by several remarkable interruptions. It is not with an uniform progress, but after a variety of successive doublings and vicissitudes, that they complete their periodical revolutions through the heavens. After continuing

Motions of the planets, to advance for a considerable time in their course, according to the order of the signs, they gradually slacken their pace and become entirely stationary: then, after remaining for some time stationary, they begin to move in the opposite direction, from east to west: and they do not return to their progressive course, till, having described their retrograde arches, they become stationary a second time. These interruptions and vicissitudes were long objects of the greatest admiration, not only in themselves, but likewise on account of their apparent dependence upon the sun. In certain configurations of the sun with the several planets and the earth, every planet is seen to be arrested in its course, as by some superior influence to that which usually conducted it, and then necessitated to return

turn backwards through some part of its former track. The motions also of all the planets are slower, in describing their retrograde arches, and their apparent magnitudes greater, than when their course is progressive: but the description of the retrograde arches is attended with this remarkable difference, that in the middle of them Mercury and Venus are always found in conjunction with the sun, and Mars, Jupiter, and Saturn always in opposition. In the middle again of their progressive arches, all the five are in conjunction: and, in the orbits of Venus and Mercury, this is called their superior conjunction.

16. It is in the interval between two successive oppositions of Mars, Jupiter, and Saturn to the sun, or of their conjunctions with him; and between two successive conjunctions, of the same kind, of the sun with Mercury and Venus; that all the vicissitudes now mentioned take place: that is, they return and perform their course in every synodical revolution. The interval, for example, between two successive superior conjunctions of Mercury, consists of about 115 days; during 93 of which the planet's motion is progressive, and during the remaining 21 or 22 it is retrograde: for, from the superior conjunction, this planet proceeds particularly of Mercury, for about 46 days in the same direction with the sun, though with greater velocity, to its greatest elongation or digression from him, which varies from 17° 36′ to 28° 30′; and then after becoming for about half a day stationary, it begins its retrograde motion and continues retrograde, till it has not only reached its inferior conjunction with the sun, but till it has attained a second elongation, in a contrary direction to the former, and is now left behind the sun. After this second elongation,

 which

which is very seldom equal to the first, the planet becomes for about half a day stationary ; and then begins a second time to proceed in the same direction with the sun, till, after about 46 days more, it returns to its superior conjunction. In this manner, with a continual repetition of such variations, is Mercury always seen from the earth, to make his progress through the zodiac : and as he never departs farther from the sun than between 17° 36′ and 28° 30′, the intervals of his mean returns to the same longitude must consist each of a solar year. The progress of the planet Venus is attended with similar irregularities. The synodical revolution of this planet is performed in nearly 584 days. During 542 of these her motion is progressive, and quicker than that of the sun: for, from an elongation, varying between 44° 57′, and of Venus, 47° 48′, to the westward of the sun, she not only overtakes him in the superior conjunction, but also gets beyond the place he occupies, and attains to a like elongation to the eastward: in the firſt of which circumstances she is called the morning star, and in the last, the evening star. She then becomes stationary for about 36 hours, when her retrograde motion begins ; and it continues till, after passing the inferior conjunction, she becomes again stationary at her greatest western elongation. As this planet is likewise retained as an attendant upon the sun, it is obvious that her mean returns to the same point of longitude must be also made in the time of a solar revolution ; and that her revolution through the zodiac must be performed sooner than the return of all the vicissitudes now described.

17. A synodical revolution of the planet Mars is performed in about 780 days, and during 707 of

these

these his motion is progressive, and in general more rapid than that of the sun. From an elongation of about 134° to the weftward of the sun, he not only comes up with the sun in the conjunction, but passes beyond the conjunction, and attains to a similar eastern elongation. In this configuration he is, as it were, arrested in his course, becomes for two days stationary, and then seems to be drawn backward to the opposition. This retrograde motion continues for about 73 days, and through an arch from 10 to 18 degrees; till after the opposition the planet is found to the westward of the sun, and before the renewal of its progressive course besomes stationary a second time. The period necessary for the return of all these vicissitudes is longer than the period of the planet's revolution in the zodiac: for this is performed in 686 days. The apparent motions and irregularities of Jupiter and Saturn are similar to those of Mars, excepting only that the completion of their vicissitudes takes up less time than their revolution in the zodiac. The mean synodical revolution of the former employs 398 days, in 120 of which the planet's motion is retrograde, through an arch of about 10 degrees; his stations continue about 4 days each, and they happen when his elongations from the sun arise to 115°. The synodical revolution again of Saturn employs 378 days, in 138 of which his motion is retrograde, through an arch of 6 or 7 degrees, and each of his stations continues about 8 days. To these it may be added, that the synodical revolutions of the planet discovered by Herschel are performed in about 370 days, that its retrograde motion continues about 142 days, through an arch of 4°, and that each of its stations continues about 11 days.

and of the fuperior planets.

 18. But

18. But it was not enough to know that all the vicissitudes now described return and complete their course, in the time of a synodical revolution; for it was equally necessary to obtain the motion of the planets in the zodiac separate from all those synodical derangements: and, that this might be accomplished, the ancients endeavoured to compare their motions with the sun's mean progress on the ecliptic, as a standard which was not affected by any inequality. In making the comparison with respect to Mars, Jupiter, and Saturn, called the superior planets, they turned their attention chiefly to their oppositions; as being not only the points in which they were moſt observable, but in which also they were thought to occupy the same places in the zodiac, as if they had not been subjected to any synodical vicissitudes. By a comparison of such oppositions it was discovered that these vicissitudes were not the only irregularities to which the superior planets were subjected; but that their progressive motions in the zodiac were also, like those of the sun and moon, inequable in themselves. The mean returns of those planets to their oppositions were made, as has been said, in the following intervals, of Mars in 779 d. 22 h. 28′ 26″, of Jupiter in 398 d. 21 h. 15′ 44″.6, and of Saturn in 378 d. 2 h. 8′ 7″.8. But the real oppositions to the mean place of the sun seldom, or never, returned in these intervals, and very seldom after any other equal intervals: and the points where they took place were very seldom at equal distances in the zodiac. The oppositions, for example, of Jupiter, as they return almoſt every year, and this planet performs its revolution round the zodiac in nearly 12 years, ought to happen at the distance of about one sign from one another. But when Jupiter

is

is in the neighbourhood of Libra, they happen at a less distance; and, in the neighbourhood of Aries, at a greater. His motion therefore, abstracting from all consideration of his occasional accelerations and retrogradations, is evidently subject to another inequality, and of a different kind: and his progress in one part of his orbit is slower than in another. This, although probably the latest in being discovered, was called the first inequality, and, as it had no dependence upon the synodical revolution, the *inæqualitas soluta*. The other, more striking one, whose returns always keep pace with the synodical revolution, was called the second inequality, and the *inæqualitas alligata*.

First and second inequalities of the planets.

19. In order then to discover the course and law of the first inequality, in any planetary orbit, it was necessary to find the mean motion of the planet, or the time which it employs in performing its revolution in the zodiac: for, in the same manner as in the cases of the sun and moon, the comparison of the real place, observed at any time when it was divested of the second inequality, with the calculated mean place, would give the correspondent first inequality, or ſhew the points where it disappeared. The moſt effectual method for this purpose was, in the orbits of the ſuperior planets, to employ the same oppositions by which this inequality was at first discovered: only taking care to choose the most distant ones which might be found. This was necessary upon two accounts; first, of the errors which might be supposed to have been committed in observation; and secondly, of the still more important errors which might arise from the observations having been made in different, or opposite, degrees of anomaly. For these errors, even the last of them, unless where

the inequality was very great, by being thus distributed among a very great number of revolutions, would disappear. But a previously necessary step was to approximate to the time of a revolution, by a comparison of less distant oppositions: for, till this was nearly found, it could not be known how many revolutions had been performed in the interval.

D. Cassini's procedure, in determining the mean motions of Saturn, will serve as an example. First, to approximate to the time of a revolution, he compared together the three oppositions, of the 16th of September 1701, at 2 h. 0′; of the 10th of September 1730, at 12 h. 27′; and of the 23d of September 1731, at 15 h. 31′. The observed longitude of the planet, in the first of these oppositions, was 11 s. 23° 21′ 16″; in the second, 11 s. 17° 53′ 57″; and in the third, 0s 0° 30′ 50″. The interval between the first and second consists of 29 years, of which seven are bissextile, wanting 5 d. 13 h. 33′; and between the second and third, of 378 d. 3 h. 24′: and the difference of the two first longitudes is 5° 27′ 19″, and of the two last 12° 36′ 53″. Therefore to find the time in which Saturn will describe the 5° 27′ 19″ necessary to complete his circle; that is, to bring him to the same point of longitude which he occupied at the first opposition; he had this analogy, 12° 36′ 53″: 5° 27′ 19″: : 378 d. 3 h. 24′: 163 d. 12 h. 41′. By adding this quantity therefore to the first interval, he had, for a first approximation to the time of a tropical revolution of Saturn, 29 y. 164 d. 23 h. 8′: whence the mean annual motion of the planet is found to be 12° 13′ 23″.8′; and the mean diurnal motion 2′ 0″.466.

Mean motion of Saturn,

Next, to determine the time of a revolution more accurately, he employed an ancient observation

tion made at Babylon, on the 24th of the month Tybi, in the 519th year of the æra of Nabonassar, and which time he found to be the 1st of March, in the 228th year before the christian æra, according to the Julian calendar. In this observation, Saturn, when in opposition to the sun, was in conjunction with the star marked γ in the south shoulder of Virgo; and Cassini computes that this conjunction took place at 6 h. on the meridian of Babylon, or at 3 h. on the meridian of Paris. Hence he concludes, by taking into the account the precession of the equinoxes, that, in the 228th year before the christian æra, Saturn was in opposition on the 2d of March O. S. at 1 h. on the meridian of Paris, in 5 s. 8° 25′ of longitude. With this ancient opposition he compares that of February 15, 1714, at 8 h. 15′, in 5 s. 7° 56′ 46″ of longitude; and that of February 28, 1716, at 16 h. 55′ in 5 s. 21° 3′ 14″. The interval between the first and second oppositions consists of 1943 y. 105 d. 7 h. 15′; and in the second the planet wanted 28′ 14″ to complete its circle. The interval again between the two last is 378 d. 8 h. 40′, and the difference of their longitudes is 13° 6′ 28″. Therefore 13° 6′ 28″ : 28′ 14″ : : 378 d. 8 h. 40′ : 13 d. 14 h.; and consequently 13 d. 14 h. added to the first interval, gives 1943 y. 118 d. 21 h. 15′ for the period which brings Saturn to the point of longitude which he occupied in the first opposition. Dividing this period by 29 y. 164 d. 23 h. 8′, the approximate time before found of a tropical revolution, it appeared that the planet had performed 66 such revolutions in 1943 y. 118 d. 21 h. 15′: so that the mean time of each was 29 y. 162 d. 4 h. 27′, and the planet's mean annual motion in longitude 12° 13′ 35′.233.

The same astronomer endeavoured, by a like procedure,

cedure, to determine the mean motion of Jupiter.

of Jupiter, To approximate to this he first employed the modern oppositions of 1699, 1710, 1711, which gave for the time of a revolution 11 y. 313 d. 16 h. 34′; secondly, those of 1672, 1731, 1732, which gave 11 y. 315 d. 7 h. 13′; and thirdly, those of 1620, and 1703, which gave 11 y. 315 d. 11 h. 25′: a mean of which is 11 y. 314 d. 19 h. 47′. Then, for a more accurate determination, he compared Ptolemy's oppositions, first of the year 133 with those of 1698 and 1699; secondly, of the year 136 with that of 1713; and thirdly that of 137 with the opposition of 1714. The results for the time of a tropical revolution of Jupiter were 11 y. 315 d. 11 h. 25′; 11 y. 315 d. 17 h. 6′; and 11 y. 315 d. 16 h. 32′. A mean of these is 11 y. 315 d. 14 h. 36′=4430 d. 14 h. 36′: and consequently the mean annual motion of Jupiter in longitude is 30° 20′ 31″.83, and his mean diurnal motion 4′ 59″.

By similar researches, first from modern observations, and secondly from a comparison of these with the ancient oppositions of Ptolemy, in the years 130, 135, and 139, he found the time

and of Mars. of a tropical revolution of Mars to consist of 686 d. 22 h. 18′; his mean annual motion in longitude to be 6 s. 11° 17′ 9″.5; and his mean diurnal motion to be 31′ 26″: and care was taken, in all these comparisons, to associate the oppositions, where it was found, by a previous knowledge of the theories of those planets, that the planet was nearly in the same degree of anomaly: and special regard was had to such as were at a considerable distance from the apsides.

Sidereal revolutions. 20. The times of the mean sidereal revolutions of the planets are, on account of the precession of the

the equinoxes, longer than those of the tropical revolutions. According to the determinations of D. Cassini and Halley, they are, for Saturn, 29 y. 174 d. 6 h. 36′ 26″=10759 d. 6 h. 36′ 26″; for Jupiter, 4332 d. 12 h. 20′ 25″; and for Mars, 686 d. 23 h. 27′ 30″.

21. When, by researches of this kind, the ancients had determined the mean motions of those three planets, they were enabled to compare together the mean with the observed places, and consequently to find their differences. By following the course of such differences through all parts of the zodiac, they perceived that, in the same manner as in the orbits of the sun and moon, there were two opposite points, or *apsides*, of every planetary orbit, in which the differences entirely disappeared, and two other opposite points, in the middle between the former, where they came to their greatest amount, though in different directions: for, in the one semicircle, the mean place always preceded the true, and in the opposite semicircle, it was always left as far behind. The motions therefore of the superior planets, as viewed from the earth, were evidently unequal, independently of their annual vicissitudes; and equations were necessary in their orbits as well as in those of the sun and moon. These were called the equations of the centre; and, in the orbit of Saturn, they were found to arise to 6° 30′; in that of Jupiter to 5° 30′; and of Mars to 10° 39′. Equations of the centre.

22. The motions of Venus and Mercury, called the inferior planets, differed in two essential circumstances from those of the superior. For, as has been seen, the general progress of the superior planets in their orbits, and the times of their revolutions,

lutions, had no dependence upon the revolutions of the sun; and it was only their stations, accelerations, and retrogradations, that were connected with his positions, and completed their course in the times of his synodical revolutions. But it was, on the contrary, the apparent revolutions of Venus and Mercury in their orbits which were connected with the sun, for they were performed in the precise time of the sun's tropical revolution; and their stations and other vicissitudes of second inequality had no dependence on any of his periods. The mean motions therefore of Venus and Mercury, or their mean progress in their orbits, were the same with the mean motion of the sun: and the determination of it did not require from the ancients such laborious researches as have been now exemplified. But, judging their real progress to be unequal, they found as great difficulty in determining it, as in determining the real progress of the superior planets. For, as the unequal distances of the times and points of the oppositions shewed the digressions of the superior planets from the point opposite to the sun to be unequal, so the visible unequal digressions of Venus and Mercury from the sun himself, were supposed to evidence like inequalities in the progress of these inferior planets in their orbits: and, the prejudices of the ancients in favour of the perfect regularity of the celestial motions, not permitting them to believe that these apparent unequal digressions were really unequal, no hypothesis was left to account for their differences, except what was founded on the difference of the distances from the earth at which the digressions were viewed. Ascribing therefore to the inferior planets inequalities to which they were not subject, they supposed the orbits of Venus and Mercury to have, each of them, an apogee, in which the

First inequalities of Venus and Mercury.

arches

arches of digression, subtending the smallest angle at the earth, would appear smallest; a perigee, in which they appeared greatest; and two other points in the middle between these apsides, where the apparent magnitude of those arches would be a mean between both extremes. The first inequality therefore of each inferior planet followed as a necessary consequence; for its apparent velocity, independent of all secondary vicissitudes, would be always inversely at its distance: and, as in the progress of the planet from its apogee to its return, all its first inequalities completed their course, so the variations of the second inequalities, or greatest digressions, depended upon the degree of anomaly which the planet occupied at the time of their taking place. But though the variations of the greatest digressions of Venus seemed to agree with these suppositions, they were not sufficient to explain the variations of the digressions of Mercury: for though his greatest digression subtended the smallest angle at the apogee, the perigee was not the point of the orbit where it subtended the greatest angle; and the angle which it subtended was found, or at least believed, to be the greatest, at two points on either side of the perigee, and 60° distant from it.

23. As the stations and other vicissitudes of second inequality of the inferior planets had no connection with any of the solar periods, it was a business of equal importance, to determine their laws and progress, as to determine the motions in the orbits. The inferior planets, as has been said, are never in opposition; their conjunctions, till telescopes were invented, were all invisible; their stations were not instantaneous; and, from their nearness, especially that of Mercury, to the sun, all

all the observations of them were made near the horizon, and were therefore precarious and doubtful. The only method, therefore, which remained to the ancients, was to find two observations of either of those planets, as distant as possible from each other, in which the planet was both in the same degree of longitude in the zodiac, and at the same distance in the same direction from the mean place of the sun: for its returns to the same point of longitude shewed that it had performed round the earth a certain number of revolutions in the zodiac, and its returns to the same distance from the sun shewed that it had also completed a certain number of revolutions of second inequality: and the number of both being supposed to have been, either observed during the interval, or known by other means, the times in which each was performed were consequently given. Accordingly, it was found in this manner by Hipparchus, that in 8 years, wanting 2 d. 6 h.83, there had been five returns of all the vicissitudes of the second inequality of Venus; so that the mean time of each consisted of 583 d. 22 h.; and that in 46 y. 1 d. 6 h. 15′, the second inequalities of Mercury had returned 145 times, so that every one of his mean synodical revolutions employed about 115 days.

Second inequality.

25. The ancients also found that the same equtions, between the mean and true places of the planets, did not continue to be applicable in the same points of longitude, and that therefore their apsides were not fixed points in the zodiac, but moveable, like those of the sun and moon, according to the order of the signs. But they considered the motions of all as equal, and the same in quantity with the precession of the equinoxes.

Motion of the planetary apsides.

25. As

25. As all the great circles, in which the planets performed their peculiar motions, were early known to be distinct from the equator, so it was also soon discovered that neither did they coincide with the solar orbit, or ecliptic; but that they were all inclined to it in different angles, and intersected it in opposite points, called, as in the lunar orbit, nodes. It was only at these nodes that any planet was ever seen in the ecliptic: and at all the other points of their orbits they were observed to have latitude, north or south, according to the passage of the planet through the ascending or descending node. The latitudes were also seen to be extremely variable, yet not so as to exceed the limits of the zodiac. These intersections of the planetary orbits with the ecliptic, afforded another method of determining the times of their revolutions, and consequently of finding their mean motions. For, if a planet has been observed in two successive passages through the same node, the interval, if the node has no motion, is evidently the time of its tropical revolution: and, if the node has any motion, it must be added to the motion of the planet, or subtracted from it.

Nodes of the orbits.

CHAP.

CHAP. II.

Of the more Ancient Theories and Planetary Systems, and especially of the Ptolemaic System.

26. THE purpose intended in all systems was to give a just representation of the different phenomena of the celestial bodies, and especially to explain the causes and laws of those inequlities in their motions, which observation from time to time discovered. When these motions were so perplexed, and apparently irregular, as has been seen in the foregoing account, it is evident that the explication of them was an attempt of the greatest difficulty; and that, instead of its being surprising that the ancients were unsuccessful in it, the ingenuity which was capable of reducing them to any settled rules whatever, is entitled to our higheſt praise. The natural difficulty of accounting for so many complicated appearances was also increased by physical prejudices; and by this in particular, that all the celestial motions, notwithstanding their apparent irregularity, were perfectly circular and uniform. A circular motion was considered as not only the most perfect of all motions, and therefore the only one suited to the nature of such divine and perfect beings as the stars were conceived to be; but also as the only motion which they could perform without extrinsic violence. The great object therefore of the ancients in all their systems was, as has been formerly mentioned, to delineate the circles, and compositions of circles, by means of which all the apparently unequal motions of the celestial bodies might

might be resolved into such as were regular and uniform.

27. The most celebrated of the ancient systems appears to have been invented by Apollonius, not long after Aristotle; and, though it takes its name and received many improvements from Ptolemy, to have been carried to its perfection chiefly by Hipparchus. In this system the earth is supposed to be immoveably fixed in the centre of the starry sphere, and round the earth all the motions, both diurnal and periodical, of the celestial bodies are represented to be performed. The order, in which those bodies, or rather the spheres in which they move, are arranged in it, is the following. The lowest of all the spheres, or the least distant from the earth, was that of the moon: and this place was justly assigned to it, both on account of the occultations by the moon of all the celestial bodies which she encounters in her progress through the zodiac, and because she is more than any other deranged by the diurnal parallax. The sphere of Mercury was placed next to the moon. Then followed those of Venus, the Sun, Mars, Jupiter, and Saturn, in the order now enumerated; and beyond Saturn the sphere of the fixed stars was placed, at a distance so immense, that no change of the observer's situation upon the earth made any change on the relative positions of the bodies situated in it, and to its semidiameter the semidiameter of the earth had no sensible proportion. This sphere was called the eighth, or the *primum mobile*; it uniformly revolved round the earth, from east to west, in 24 sidereal, or in 23 h. 56′ 4″.098 mean solar hours; and, in its revolution, all the inferior orbits were supposed to partake, with degrees of velocity modified by their respective periodical re-

Ptolemaic system.

Eighth sphere or primum mobile.

 volutions

volutions in directions nearly opposite. But this system was not implicitly received by all the ancients, nor does it appear upon what principles all its parts were formed. Many objections were made, especially to the arrangement of Venus and Mercury: and Géber of Seville, in particular, an Arabian astronomer, remarks that in this part Ptolemy had directly contradicted his own doctrines and observations: for, though he supposed the diurnal parallaxes of Venus and Mercury to be insensible, and the sun to have a parallax of 2′ 51″, he notwithstanding placed them below the sun; and so far below him, that the parallax of Venus ought sometimes to arise to 3′, and of Mercury to no less than 7′. Géber also adduces, as an argument against the arrangement of Ptolemy, a circumstance in which that arrangement was not really in fault, and which the modern invention of telescopes would have turned into a presumption in its favour. For he demonstrates, from the latitudes of Venus and Mercury, that these two planets ought to be frequently seen passing over the solar disk; and as no such phenomenon had ever been observed, he infers that there was strong reason to suspect the theory to be false. It was probably on account of these and similar difficulties, that another system was formed, which received its denomination from Plato, and seems never to have been destitute of illustrious adherents. In it the sun was placed next to the moon, after him Venus, and next Mercury; in other respects it agreed with the Ptolemaic. Much regard was also at all times paid to the ancient Egyptian system; in which Venus and Mercury were supposed to be *satellites*, or attendants upon the sun, and to revolve round him continually, while he performed his annual revolution round the earth. Among the disciples of this

Platonic and Egyptian systems.

system

system were Vitruvius, Martianus Capella, Macrobius, Bede, &c. if not Plato himself.

28. Though some philosophers, especially Anaxagoras and Democritus, taught that the celestial bodies moved in spaces void of all resistance, their doctrine, as it assigned no physical causes of the celestial motions, was never prevalent among astronomers. The current opinion was, that those bodies were fixed in transparent solid spheres, to which motions were communicated, from some original power or cause, by mechanical operations: and, it being likewise an inviolable maxim that all the celestial motions were perfectly regular and circular, these two principles seem to have given rise to the first and most ancient method of accounting for the observed inequalities, and to which the name of the concentric theory was given. For when the same sphere, or the same great circle of it, in which any moveable celestial body was placed, could not admit of all the motions peculiar to that body, another subordinate one, like a smaller wheel moving upon the circumference of a greater, with different and even contrary motions, was supposed to be adapted to it: and the body was conceived to be placed in the circumference of this subordinate sphere or circle, and to be carried round by its motion on its axis. A subordinate sphere or circle, which thus moved in the circumference of a large one, was called an epicycle; and the larger one, on which it moved, was called its deferent. By means of an epicycle, one motion apparently irregular, was resolved into two that were circular and uniform: and when the observed motion was so irregular and complicated, as not to be represented by one epicycle, there was no other expedient but to super-add more. In short, the

Rise of the concentric theory.

whole celestial physics of the ancients hinged on two principles, supposed to be unquestionable; *first*, that all the celestial motions were perfectly circular; and *secondly*, that they were really uniform, when referred to their proper centres: and when an ancient astronomer had investigated the circles and epicycles which seemed best to agree with his observations, he rested satisfied that he had divined the true system of nature.

Concentric solar theory.

29. The representation made by this concentric theory of the solar inequalities in longitude, was as follows. Let C (fig. 1.) be the centre both of the earth and of the circle FBD; and let HGK be a smaller circle, called an epicycle, whose cente B moves uniformly in the circumference FBD, from west to east, or *in consequentia*, while the sun moves also uniformly, and with the same velocity in the circumference of the epicycle, *in antecedentia* in the upper part, but *in consequentia* in the lower. If the point G of the epicycle, called its apogee, as being most distant from the earth, be supposed, at the beginning of the anomalistical revolution, to be placed in the point A of CF produced; and if, when it comes to G, the arch GH be taken similar to FB, the point H will be the place of the sun when the centre of the epicycle has moved from F to B. If then in CF, to which BH is parallel, we take CE=BH, and on E as a centre with the distance EA=CF describe the circle AHP, the sun would be seen from E to move in this circle equably; for the angle AEH is equal to the angle FCB: but seen from C, the centre of the earth, he will appear to move in it inequably; for the angle ACH, in the first semicircle of anomaly, that is, in the passage of the sun from A to P, is always less than AEH or FCB; and his true place

place H will be less advanced in longitude than his mean place B. When, again, the centre of the epicycle, or the mean place of the sun, having described a semicircle, shall have come to D, the sun having also described a semicircle of the epicycle, will be found in P, the perigee of the orbit AHP, and his mean and true places B and H will be seen from C to coincide, as they did in A, the apogee. But in the sun's passage from P to A, that is in the second semicircle of anomaly, his true place H, as seen from C, will be always more advanced in longitude than his mean place B: for, in this semicircle, the angle PCH is always greater than PEH, or DCB. The angle EHC, or BCH, which is the difference between the mean and true places of the sun, is called the equation of the orbit: as being that quantity which added to the true motion ACH of the planet in its orbit AHP in the first semicircle of anomaly, and subtracted from it in the opposite semicircle, will render it equal to the mean motion AEH, or FCB: and it is evident that this equation, or difference, will be greatest in N or M, where the centre B of the epicycle is 90° distant from either apsis. Any lines drawn from E, the centre of the orbit AHP, to the true place of the sun in H, and from C the centre of the earth and of the deferent FBD to the mean place of the sun in B, are equally called the line of the mean motion of the sun; because these lines are always parallel, and mark the same point in the zodiac: and any line drawn from C to the true place of the sun is the line of the sun's true motion.

Apogee and perigee.

Equation of the orbit.

Deferent.

30. It was thus that the ancients originally proceeded in their representation of the solar inequalities, and the representation seemed to be suffi-

ciently justified by observation: at least, till the days of T. Brahé, no observations had been made with sufficient accuracy to subject it to suspicion. Their success also, while no lunar inequality except the simple anomalistical one was discovered, was equal in the application of the same concentric theory to the motions of the moon: and having, in two cases, thus successfully, by means of one subordinate sphere or epicycle, reconciled apparent inequality of motion with real uniformity, it was natural to suppose that other inequalities, though more various and complicated, might be explained in a similar manner, and required only the addition of other epicycles. The same method of procedure therefore was continued, and every new inequality which observation discovered, was accounted for by a new sphere or epicycle producing it, till the whole number employed in the system amounted to 34. Aristotle, on narrower examination, found these insufficient, and added to them 22: but ftill they were deemed insufficient; and the number was at last increased to 72. But though it was not till long after the days of Aristotle, that the theory was carried to such a degree of extravagance, the multiplication of epicycles rendered it, even in his time, almost as intricate and complex as the appearances which it was intended to explain. Some examples of this kind will occur on the revival of it by Copernicus and T. Brahé: and when Hipparchus and Ptolemy introduced eccentric orbits, and by means of them somewhat diminished the multiplicity of the spheres employed by their predecessors, they were thought to do a signal service to astronomy.

31. The manner in which Hipparchus explained the solar inequalities, in his eccentric theory, was to

to this purpose. Let O (fig. 2.) be the centre of the earth and of the starry sphere. Let BCDE be the ecliptic, or great circle in the *primum mobile* in which the sun seems to perform his annual revolutions; and, in the same plane, but on a different centre Z, let the circle ALP be described. This is the circle, or orbit, in which the sun is supposed actually to move, and to describe round its centre equal arches or angles in equal times: or rather, he is supposed to be carried round by the equable motion of the circle itself; which, because its centre is not occupied by the earth, is called an excentric circle. It is evident, in this representation, that if the earth were placed in Z, a spectator on it would perceive the sun, since he is supposed to move uniformly in the excentric, to move also uniformly in the ecliptic. But the earth is placed in O, at the distance OZ from the centre of the excentric; and therefore, when his motions are referred to the ecliptic by a spectator in O, they must appear unequal. When, for example, he departs from A the apogee of the excentric, and comes to K, he would be seen from Z in the point R of the ecliptic; but from O, the centre of the earth, he is seen in C, a point less advanced in longitude. On the contrary, when he departs from P, the perigee of the excentric, and comes to N, his place in the ecliptic, as seen from Z, is the point V; but seen from O it is the point F, more advanced in longitude than V. Any line, as ZK, drawn from the centre of the excentic to the sun, or any parallel to it drawn from O, is called the line of mean motion, and determines the mean anomaly AZK; and any line, as OK, drawn from the sun to the centre of the earth, is called the line of true motion, and determines the true anomaly AOK; and the angle OKZ, which is the difference

Excentric solar theory.

 between

between the mean and true anomalies, is the equation of the orbit. In the apogee and perigee this equation vanishes, in the same manner as in the concentric theory, because there the lines of mean and true motion coincide; and at the points L, L, where a perpendicular to the line of apsides, drawn through O, meets the excentric, it comes to its greatest amount. Thus, by the single supposition that the solar orbit was excentric to the earth, Hipparchus supplied the place of the epicycle added to the concentric: nor is it difficult to perceive that the representations, given by both theories of the solar inequalities, were in their effects precisely the same.

32. In both these theories, it is evident that the inequalities of the sun were considered as purely optical: and what was principally required was to find the point O, in the line AP of the apsides, in which the earth must be situated, in order to give to the solar motions the just inequality which observation required; and to determine the longitude A of the solar apogee, that is the point B of the ecliptic to which it is referred from O. Without finding the just excentricity OZ, the calculated differences, or equations, between the sun's mean and true places, would not correspond with the observed differences: and without discovering the position of the apogee, the calculated equations, however accurate, would not be applied in their proper places. In these investigations the procedure of Hipparchus was as follows.

Let B (fig. 3.) be the place of the sun at the vernal equinox, BD an arch of the excentric equal to his mean motion for 94½ days, that is, from the vernal equinox to the summer solstice (5.), DF an arch equal to his mean motion for 92½ days, or from

from that solstice to the autumnal equinox: let the chord BF be drawn, and from D another chord DEG perpendicular to BF. The point E of the intersection of these chords is evidently the point where the earth must be situated: for it is the only point from which B, D, F, G, can appear at the distance of 90° from one another, and as they appear actually in the heavens.

It was required therefore to determine the excentricity EC, or the distance of the point E from C, the centre of the solar orbit. Since the arch BDF of the mean motion, and which the ancients supposed to be the only real motion of the sun, from B the vernal to F the autumnal equinox, is given by means of the annual revolution, and consists of 184° 20′, its half BH will consist of 92° 10′. Through H draw the diameter HC; and, through C the centre of the orbit, the diameter CK perpendicular to HC, and meeting DE in L; and join CE. If from BD, the mean motion for $94\frac{1}{2}$ days, and = 93° 9′, we subtract BH=92° 10′, the remainder DH will be found=59′: and if from BH we subtract the quadrant KH, the remainder BK will be =2° 10′. Therefore CL and EL, the sines of the arches DH, BK, will be given in parts of CD=10,000; the former to wit =172, and the latter =378. Therefore the excentricity EC, being the hypothenuse of the right-angled triangle ELC, will be found in the same parts: for $EC^2=CL^2+EL^2$, and therefore EC=415. Solar excentricity.

By producing EC to meet the excentric in A and P, we shall have the line AP of the apsides: and the longitude of A the apogee will be found from the same triangle ELC; for CE:EL::R: sin. ECL=KCA=BEA=65° 30′. Since B therefore represents the point of Aries, the place A of the apogee fell short of the solstitial point D, 24° 30′, in Longitude of the solar apogee.

in the days of Hipparchus. It is now between 9° and 10° more advanced.

33. When the excentricity and the longitude of the apsides were thus determined, the equations of the orbit, or the differences between the mean and true places of the sun, were obtained by a very simple trigonometrical computation. For in the excentric theory, (fig. 2) when this equation comes to its greatest magnitude, in the points L, L, or M, M, where perpendiculars to the line of apsides drawn through O or Z meet the excentric, the excentricity OZ becomes the tangent of the angle ZMO to the radius of the orbit ZM: and at any other point K, with the excentricity OZ, the radius ZK, and the angle KZO the supplement of the mean anomaly AZK, the equation ZKO will be found by the analogy $KZ+ZO:KZ-ZO::\tan.\frac{1}{2}AZK:\tan.\frac{1}{2}(ZOK-ZKO)$. The calculation will be precisely the same in the concentric theory if EC be taken in CA, (fig 1.) equal to the radius of the epicycle, that is to the excentricity OZ, (fig. 2.) and will produce the same results: and the excentric theory will be found to differ from the concentric only in simplicity. It is probably needless to observe that the parallels OL and ZM will mark the same point in the zodiac.

To calculate the solar equations.

34. As the first inequality of the moon (8) was the only one which for a long time was known to the ancients, it was, like the solar inequality, represented by either of the theories now described; and Hipparchus appears to have used both promiscuously. But as the greatest equations of the moon did not take place in the same fixed points of the zodiac, but were found to advance about 3° eastward in every tropical revolution, this peculiarity

rity was accounted for, in the excentric theory, by an orbit AGPH (fig. 4.) not fixed and unmoveable like the solar orbit; but moveable upon the point T where the centre of the earth is placed, and describing with its centre C, round that point, a small circle Cc. If therefore, in any one revolution, that orbit had been in the position AGPH, with the greatest equations EK and FL, it would in the next tropical revolution come into the position a g p h, and the greatest equations would be the arches e k and f l, equal indeed to the former, but more advanced in longitude. The effect thus represented was produced by the different durations of the tropical and anomalistical revolutions: for as the latter are about five hours longer than the former, (10) it must happen that, when the moon returns to the longitude of the point B, she has not yet returned to the apogee of her orbit; nor will she return to it, till after advancing about 3° farther in the zodiac. Excentric theory of the moon's first inequalities.

35. In the concentric theory of the moon there was no difference from the solar concentric theory, except that in the latter the revolutions of the sun in the circumference of the epicycle, and of the centre of the epicycle in the deferent, were made in times precisely equal; whereas in the former they were made in unequal times; the moon's revolution in the epicycle requiring 27 d. 13 h. &c. and the revolution of the centre of the epicycle requiring only 27 d. 7 h. &c. Consequently when the centre of the epicycle, after a tropical revolution, returns to its first situation in A, (fig. 5.) the moon has not yet returned to E, the apogee of the epicyle; nor will return to it, till the centre has advanced in longitude 3° beyond A. The independency therefore of the anomalistical revolution upon Concentric theory of the same inequalities.

upon the tropical, and the advanced position of the greatest equations in the zodiac, became as evident on this supposition, as on the former.

36. What was principally required in these theories was to ascertain, in the first, the ratio of the excentricity, and in the second, that of the semidiameter of the epicycle, to the semidiameter of the orbit; and to investigate the moon's equations. The procedure in the concentric theory was to this purpose. Ptolemy had observed three lunar eclipses; the first in the year 133 of the christian æra, when the moon was seen in 7 s. 13° 15′; the second in 134, in 0 s. 25° 10′; and the third in 136, in 5 s. 14° 5′: and Copernicus found that, in time reckoned on the meridian of Cracow, the first had happened on the 6th of May, at 11 h. 15′; and that the interval between the first and second consisted of 531 d. 23 h. 29′; and, between the second and third, of 502 d. 5 h. 30′. The apparent motion, therefore, of the moon in longitude, in passing from 7 s. 13° 15′ to 0 s. 25° 10′, after neglecting the complete revolutions, was 161° 55′; and, in passing from 0 s. 25° 10′ to 5 s. 14° 5′, it was 138° 55′. But the mean motion of the moon, that is, of the centre of the epicycle, known by the time of her tropical revolutions, was in the first interval, 169° 37′, and in the second, 137° 34′: and her anomalistical motion in the epicycle, being the difference between the mean motion of the moon and that of her apogee, was also, in the first interval 110° 21′; and, in the second, 81° 36′. Let then ABC (fig. 6.) be the lunar epicycle, and E its centre revolving *in consequentia* round T, the centre of the earth: and let it be supposed that the first eclipse was observed in the point A of the epicycle, a place somewhat more advanced in longitude than the

To find, in the concentric theory, the semid. of the epicycle, the mean place E or D. Since the first eclipse was observed in A, it is evident, that, if the moon's apparent motion in longitude had been, in the first interval, equal to her mean motion, the second eclipse would have also happened at the point A; the angle ATE continuing in that case invariable. But since the moon's apparent motion, in the first interval, is 7° 42′ less than the mean, the second eclipse will not happen in the point A, but in a point B, less advanced in longitude by 7° 42′: and the anomalistical arch of 110° 21′, which she also described in this interval, she must have described *in antecedentia*, and therefore in the most distant part of her epicycle, and its position must have been such as to subtend, at the centre of the earth, the angle ATB of 7° 42′. In like manner, since the apparent motion of the moon in longitude is, in the second interval, greater by 1° 21′ than the mean motion, the third eclipse will not happen at the same point B with the second, but at a point C more advanced in longitude than B by 1° 21′: and the moon, when describing in the same interval the anomalistical arch BC of 81° 36′, must have been descending to the lower part of the epicycle; and even moving in that lower part; because the angle BTC of 1° 21′ is described *in consequentia*.

Since then the arch AB is towards the apogee of the epicycle, the line TA, if it be not a tangent to the epicycle, will meet it again in some point F towards the perigee. Join BF, CF, BC.

In the triangle BFT all the angles are given: for AFB standing on the arch AB=110° 21′, and consequently =55° 10′ 30″, is the supplement of BFT, and BTF=7° 42′; and therefore the side FT, in parts of the semid. of the circumscribing circle, will be found =73698, and BF=13399.

The

The angles also of the triangle CFT are given; for CTF=BTF—BTC=6° 41′; and AFC the supplement of CFT stands on the arch ABC=191° 57′, and is therefore =95° 58′ 30″. Therefore FT, in parts of the semid. of a circle described about CFT=99980, and CF = 11060; and therefore, in parts of BF found formerly, CF = 8151. But in the triangle BFC, the angle BFC, standing on the arch BC = 81° 36′, is equal to 40° 48′, and therefore with the sides BF = 13399 and CF = 8151, the remaining side BC will be found = 8980. But BC being the chord of 81° 36′, will be = 130684, in parts of the semid. of the epicycle = 100000; and therefore, in like parts, FT = 1072684, and CF=118637; and consequently the arch CKF of which it is the chord = 72° 46′ 10″. Consequently, FHA = CFA—CKF = 95° 16′ 50″, and its chord FA = 147786; and therefore TFA = TF + FA = 1220470. Suppose E the centre of the epicycle found, and the line TED of apsides drawn: and then, since AT. TF = DT. TK, and DT. TK + KE² =TE², this square will be given; and therefore also TE = 1148558, in parts of KE = 100000. Therefore in parts of TE = 100000, we shall have KE = 8706. By the same method of investigation, Copernicus found, from three lunar eclipses observed by himself, that TE: KE:: 100000:8604.

and the equations. The semidiameter of the epicycle being determined, the equations were thus investigated.—From the centre E draw EG perpendicular to AF, and meeting the epicycle in H. Since TE, TF, and FA are given, in parts of KE = 100000, GF will be found, in the same parts = 73893, and GT = 1146577. Therefore, in the right-angled triangle TGE, the angle TEG, which is measured by the arch KH, will be found

found = 86° 38′ 30″; and consequently its supplement HED, or the arch HD = 93° 21′ 30″: and since HA = ½FA = 47° 38′ 25″, the arch AD of the moon's distance from the apogee, in the first eclipse, will be = HD—HA = 45° 43′; BD the anomaly, in the second eclipse, will be = BA—AD = 64° 38′; and CBD the anomaly, in the third eclipse, will be = CB + BD = 146° 14′. Since also TEG = 86° 38′ 30″, and TGE a right angle, the angle GTE or ATD, the equation at the first eclipse, will be = 3° 22′; at the second eclipse the equation will be BTD = ATB —ATD = 4° 20′; and at the third eclipse the equation will be CTD = BTD—BTC = 2° 59′; and in the same manner the equations in all other degrees of anomaly might be derived.

37. In the excentric theory the procedure was to this purpose. Let D (fig. 7.) be the excentric place of the earth in the circle ABC, in the circumference of which the moon performs her anomalistical revolution; and A, B, C, the three eclipses observed from D. Let E be the centre of the circle; join AD, BD, CD, AE, BE, CE; produce any of the lines AD, drawn from the moon to the earth, to meet the circle again in G; and join GB, GC. Since all the lunar motions are supposed to be uniform round the centre E, and their periods given, the anomalistical angles AEB, BEC will be given, for they are equal to the differences between the mean motion of the moon, and the motion of the apogee, in the intervals of the eclipses. As also the apparent motion of the moon in these intervals is known by her observed longitude in each eclipse, the angles ADB, BDC are likewise given; for they are equal to the dif- ferences

To find the same things in the excentric theory.

ferences between the motion of the apogee and the moon's apparent motion. Thus

Moon's mean mot. 1st interval	169° 37′;
Motion of the apogee	59 16 ;
Therefore AEB	110 21 ;
Moon's apparent motion	161° 55′
Motion of the apogee	59 16
and ADB	102 39.
Moon's mean mot. 2d interval	137° 34′;
Motion of the apogee	55 58 ;
Therefore BEC	81 36
Moon's apparent motion	138° 55′
Motion of the apogee	55 58
and BDC	82 57

Then 1st, in the triangle DBG, with the given angles BDG the supplement of ADB, and BGD the half of AEB, and BD assumed of any definite length, suppose = 200000, DG may be found. 2dly, In the triangle DCG, with DG and the given angles DGC = $\frac{1}{2}$AEC, and CDG the supple. of ADC, the side DC may be found. 3dly, In the triangle BDC, with the sides BD, DC, and the included angle BDC we may find DBC and BC. 4thly, In the isosceles triangle BEC with BC and BEC we may find BE. 5thly, In the triangle EBD, with the sides BD, BE, and the included angle EBD = EBC—DBC, we may find the excentricity DE in parts of BD assumed, and the angle BDE, which is the distance of the moon from the apogee in the second eclipse. 6thly, Since

Since BE is found in parts of BD assumed, if we make BE = 100000, the excentricity ED will be found in the same parts, and the result will be the same as in the concentric theory.

38. Though both these theories were equivalent in their effects, and therefore equally fitted to represent the first lunar inequality, yet when the evection, or second inequality, came to be considered, they were insufficient to represent both combined; and Ptolemy found it necessary to invent another much more complicated. In this the moon was still supposed to perform her anomalistical revolution in the circumference of an epicycle, but the deferent on which the centre of that epicyle moved was no longer concentrical, but excentrical, to the earth; and the epicycle revolved on this deferent, in such a manner, as in all mean conjunctions and oppositions to be always in its apogee, and, in all quadratures, in its perigee. This effect was produced by two supposed contrary motions performed round the centre of the earth; to wit, of the epicycle *in consequentia*, and of the apsides together with the centre of the deferent *in antecedentia*; and the rates of motion were such, that the angular distances on both sides of the mean place of the sun were always equal. The representation of it was thus effected.

Ptolemy's theory of the evection.

Let E. (fig. 8.) be the centre of the earth, O the centre of the deferent AFP supposed to revolve round E, and to describe with its apogee A the circle ADCB in the course of one lunation; and let this circle be divided into four quadrants by the perpendicular diameters AC, BD. If at the time of the mean conjunction in the line EA the apogee of the deferent be conceived to coincide with the centre of the epicycle, and, if, for the sake of simplicity, we suppose the sun to continue in this line during the whole lunation, the consequenses will be these: C will be the point of the full moon or opposition, B and D the points of quadrature, and the apogee A moving *in antecedentia* from the line EA will against the first quadrature come to D, having described the arch AD, or angle AED of 90°, while the centre of the epicycle GH, by a like motion *in consequentia*, will describe the equal angle AEK. Though therefore their angular distance from the sun in EA is only 90°, they will be diametrically opposite to each other; that is, the centre of the epicycle will be in the perigee of the deferent, and the epicycle, which represents the lunar inequality, being now at the least distance from the earth, will be seen under the greatest possible angle, to wit, LEK or NEB much greater than MEA; and the equation NEB will now rise to 7° 40′, whereas MEA, the greatest equation which can take place at the conjunction, is only 5° 1′. Thus the evection or second lunar inequality was by this theory represented to be merely an optical amplification of the first, and wholly produced by the approach of the lunar epicycle, especially at the quadratures, to the earth: and the theory also explained the perplexing circumstance of its not coming to its full amount in every quadrature; for, if the moon should then happen to be in the apogee or the perigee of her epicycle, as the first inequality could not take place in such quadrature, so neither could the second: or if the moon should be then found in some point Q of the epicyle not far distant from either of its apsides, the inequality would indeed take place, but in a much less degree. It is true that the sun does not continue in the line EA during the whole lunation, and consequently neither

neither will the points B and D be the precise points of quadrature, nor will the angles AED, AEK, made by the line of apsides and the line of the moon's mean motion, with the line of the sun's mean motion, be equal, for the sun advances *in consequentia* at the mean daily rate of 59′ 8″: in order therefore to render these angles always equal, the daily motion ascribed to the centre of the epicycle was 13° 11′, and to the lunar apogee only 11° 2′, and the equality of the angles AED, AEK, was in all cases thus preserved.

39. In this representation, however, the principle of uniform motion in perfect circles was not in all respects inviolably maintained. For, if by means of the supposed contrary revolutions of the epicycle and the apogee of the deferent, the arch AR of the zodiac intercepted, (fig. 9.) at the first octant for example, between the lines EKR and EA, or the angle AER which is subtended at the centre of the earth, was = 90°, the arch AK of the deferent which the centre of the epicycle really described, or the angle AOK at the centre of the orbit which AK subtended, was of necessity greater than 90°; and it evidently followed that the motion of the centre of the epicycle, though apparently equal in the zodiac, was really unequal in its own proper orbit. Objection to this theory.

40. The merits of the ancient astronomers, in discovering and separating between two inequalities, whose returns were so irregular, and their connection so intricate, were certainly very great: and when Ptolemy formed a theory which might subject them to any settled rules, he was justly entitled to the credit he obtained. But, when the excentricity necessary for representing the second in-

equality came to be afterwards investigated; and the distance, at which it placed the moon in quadrature from the earth, came to be compared with the distances given by her diurnal parallax, an objection altogether unanswerable was presented to his theory. For, let RS (fig. 9, n°. 2) be the excentric orbit, in the pofition which Ptolemy assigned to it in the quadratures, M its centre, and T the centre of the earth; and let $RTk = 5° 1'$, be the angle under which the semi-diameter of the epicycle would appear from T, if its centre were in R, and $STh = 7° 42'$, the angle under which it appears when the centre is in S. It is evident, that the distances RT and ST, are inversely as the tangents of these angles; for, taking in TR the line $Ty = TS$, and drawing yz parallel to Rk, and meeting Tk in z, we have $Ty : TR :: yz : Rk$; that is $TS : TR ::$ tan. yTz : ($Rk = Sh =$) tan. STh; or $TS : TR ::$ tan. RTk : tan. STh. Therefore $\frac{1}{2}$ (tan. STh + tan. RTk) : $\frac{1}{2}$ (tan. STh — tan. RTk) :: $\frac{1}{2}$ (TR + TS) : $\frac{1}{2}$ (TR — TS); that is, 1114935 : 237117 :: 1000000 : 213164 = TM. Hence TR, the distance of the lunar apogee from the centre of the earth, will be 1213164. The distance, according to Ptolemy and the other ancients, is found, by their observations of parallax, to be equivalent to 60 semi-diameters of the earth; and consequently, by subtracting from it a 60th part, we shall have 1192945 for the distance of the apogee of the excentric from the earth's surface. The semi-diameter of the epicycle, found in like parts by the analogy R : tan. RTh :: TR : Rk, will be = 106304; and this added to 1192945, will give 1299449 for the distance of the moon from the earth's surface, when the centre of the epicycle is in the apogee of the excentric, and the moon at the same time in the apogee of the epicycle. But when

Another more decifive.

when this centre comes in the quadratures to the perigee of the excentric, and the moon happens at this time to be also in the perigee of the epicycle, her distance from the surface of the earth will be little more than the half of this quantity. For, if to double the excentricity TM, we add the semi-diameter of the epicycle, and subtract the sum = 532832 from 1192945, the remainder, which is the moon's least distance from the earth's surface, will not exceed 660113. The apparent diameter, therefore, and the diurnal parallaxes of the moon ought, as Copernicus justly observed, to be found, in some quadratures, nearly twice as great as in some oppositions. But by all the ancient observations, the difference was found so inconsiderable as hardly to deserve attention.

41. Ptolemy accounted more happily for the latitudes of the moon, ascribing them to their proper cause, the intersection of the plane of the lunar orbit in the plane of the ecliptic; but, as he supposed the inclination of these planes invariable, it is probable that many of the inequalities in latitude had escaped his notice. The inclination, he agreed with Hipparchus in fixing at 5°; and, as the latitudes did not return regularly with any of the other lunar revolutions, this variation of them was also rightly ascribed to the revolution of the nodes round the ecliptic in a retrograde direction. This revolution of the nodes was supposed to be performed in 18 y. 223 d. 12 h. 6′; after which period the varieties of latitude returned regularly in their order.

Latitudes of the moon.

42. The inequalities of the planets were so various and intricate, that the explications of them were for a long time extremely imperfect, and so

 partial,

partial, that no Grecian astronomer before Ptolemy had supposed it practicable to give a compleat theory of all. In the more ancient times the explications of them appear to have been made by orbits concentric to the earth, and charged with epicycles: but, as Ptolemy had found no method of representing the second inequalities, except by means of epicycles, so, to avoid the perplexity occasioned by the multiplication of them, he gave the preference to an excentric orbit for the representation of the first; and, by the superior simplicity of the representation, the authority of the excentric theory was for many centuries established. With respect to those first inequalities, at least of the superior planets, it appears to have been originally supposed, that they might be sufficiently accounted for by the more simple solar hypothesis (31). For, if the planet, in consequence of its second inequalities, be represented as moving in the circumference *abc* (fig. 2) of an epicycle, the centre A, L, P, Q, of this epicycle, will represent the places which it would occupy, if it were divested of all second inequality: and it was thought a sufficient explication of the first inequality, to suppose that this centre moved equally round Z, the centre of the orbit, and consequently inequably round O, the centre of the earth. But when, according to Kepler's conjectures on this subject, they endeavoured to account for the inequality of the latitudes in opposition, and of the digressions of the planets, especially their greatest digressions, from the point opposite to the sun, by variations of their distances from O, it appeared that the point Z, round which the planet moved equably, could not be the same with the centre of the orbit; for, both the latitudes, and also the angles *a*O*c*, *d*O*f*, which the epicycle subtended at the centre of the earth, were

were found to be greater at the apogee, and less at the perigee than the limits of the excentric A L P Q permitted; and that, consequently the centre of the orbit occupied a place nearer than Z to the centre of the earth. It was therefore a matter of much greater difficulty to form an hypothesis for the motions of the planets, than for those of the sun: for, if their first inequalities required one determined excentricity, or distance between the centre of uniform motion and the centre of the earth, the variations of their latitudes, and of their second inequalities, showed that this was not the excentricity of the orbit in which the epicycle moved, and that this orbit evidently required another. In what proportion the distance ZO between the centre of uniform motion and the centre of the earth, ought to be divided by the centre of the orbit, appears to have been for a long time a matter of much uncertainty. But Ptolemy tells us that, on applying himself to investigate the measure of the approach of the centre of the epicycle, within the circle ALPO at the apogee, and its consequent withdrawing from it beyond the perigee, he found, by multiplied observations, that the centre of the orbit lay precisely in the middle, between Z the centre of uniform motion, and O the centre of the earth. This is the famous principle, known by the name of the bisection of the excentricity: and, as Ptolemy gives no account of the means by which it was discovered, nor of the observations from which it was inferred, his assuming it has justly excited the wonder of all astronomers. The greater part believed him to have assumed it merely from conjecture, and not to have derived it, as Kepler more generously supposed, from any observations; and there seems to be some reason for thinking,

Theory of the superior planets.

Bisection of the excentricity.

that it came to him by tradition, from the more ancient astronomy of the east.

43. In consequence of adopting this principle of the bisection, Ptolemy's explication of the first inequalities of Mars, Jupiter, and Saturn, was to this effect. Let ABC (fig. 10.) be the orbit of any superior planet, and D its centre: let T be the earth at the distance DT from D the centre of the orbit, ADP the line of apsides, and B the centre of the epicycle in the circumference of which the planet is supposed to move. This centre B, does not, like the sun, move equably round D, the centre of ABC, called the deferent of the epicycle; but round the centre E of another circle FGL, called the equant; and the point E is so taken between D and the apogee A, that its distance ED from D, the centre of the deferent, may be = TD. The first inequality therefore of the planet was supposed to be produced by the earth's place in T, at the distance TE from the centre of the equant, called the excentricity of the equant: for, on this account the lines TB of the true motion of the centre of the epicycle, and EB, or its parallel TM, of the mean motion, could never coincide, except at the apsides A and P; and every where else they would make an angle with each other, as EBT, or its equal BTM, subtractive from the mean place of the centre in the first six signs of anomaly, in order to find the true, and additive in the six last. This angle is called the equation of the centre, to wit, of the epicycle; and it is manifest that it must be much greater, than if the centre had moved equably in the circumference ABC, and its only excentricity had been DT, called the excentricity of the orbit.

Deferent & equant.

Excentricity of the equant, and the orbit.

Equation of the centre.

44. To

44. To explain again the second inequality, or the change of a superiour planet's motion from direct to retrograde, and from retrograde to direct, and their stations between these two opposite changes, the ancients supposed that, while the centre B of the epicycle moved inequably round the earth in T, though equably round the point E, the planet itself performed various revolutions in the circumference of the epicycle; not in a similar direction indeed to those of the sun and moon in their epicycles, but in a contrary direction, to wit, *in consequentia*, or from K to H and O in the upper part of it, and *in antecedentia*, in the lower. These motions also were supposed to be made uniformly round the centre of the epicycle; and, as in the middle, between the two opposite directions of the motions, there are two small arches of the epicycle sensibly coincident with the tangents TO, TQ drawn to it from the earth, they supposed, that in describing these the planet would seem to be stationary. Hence therefore arises another equation, the difference, to wit, between the place of the centre of the epicycle, and the place of the planet in its circumference, as referred to the zodiac from T the centre of the earth. For example, supposing the planet in O, this equation to the angle OTB, called the equation of the argument, and of the orbit: and as, when the centre of the epicycle was supposed to move in the circumference of the equant, the calculated amount of this angle was always less towards the apogee, and greater towards the perigee than the observed amount, this was one of the reasons by which Kepler conceived Ptolemy to be induced, when he placed the centre of the circle in which the epicycle actually moved nearer than the centre of the equant to the earth. The circumstance, concerning the revolutions of the

Equation of the argument, or orbit.

the superior planets in their epicycles, which was most surprizing to the ancients, was their close connection with the motion of the sun, and their being always made in the precise time of his synodical revolutions to the same planets; so that, if the mean place of the sun was found in B, the planet in conjunction with the sun, or in the line TB, was always in the true apogee K of its epicycle, and, producing TB to V, the planet in opposition to the mean place of the sun in V, was always in the true perigee R of its epicycle; and in all other situations of the planet, as at Z, the line BZ of the planet's true motion in the epicycle, was always parallel to TX, the line of the sun's mean motion round the earth. The point H of the epicycle, found by drawing through its centre B a line EB, from the centre of the equant, is called the mean apogee; and it is on the distance of the planet, as HBZ, from this mean apogee, that the amount of the equation ZTB partly depends; and this distance HBZ is part of the argument of that equation. In forming this part of the argument, all that is necessary is to subtract the mean motion of the centre B of the epicycle, from the mean motion of the sun; for we have seen, that HBZ is always equal to MTX, or BE*a*: and it is manifest that, the more slowly the centre B revolves in the deferent ABC, the sun's returns to the planet, and consequently the revolutions of all its second inequalities will be the more frequent.

45. To find the excentricity TE, and the place of the apogee, which corresponded to the first inequality of any superior planet, its observed oppositions were employed, because in these it was not affected by any second inequality: and Ptolemy supposed, for a first approximation, that the centre of

of the deferent coincided with the centre of the equant. In this approximation, therefore, his procedure was the same as in deducing from lunar eclipses the same elements in the excentric theory of the moon (37). Only here, the motion of the apsides of the planets being considered as the same with that of the fixed stars in precession, there was no necessity, in a short period, for distinguishing between the anomalistical and the tropical revolutions. But as the deferent did not coincide with the equant, this approximation was by no means sufficient. For, let ABC (fig. 11), be the equant described on the centre D, and MNO the planet's apparent path in the zodiac described on E, the centre of the earth: let ED, the excentricity of the equant, be bisected, as Ptolemy directed, in F, and on this point as a centre, with the semidiameter FG = DA, or EM, describe the deferent, or orbit GKL. The places of the centre of the epicycle, or of the planet itself in opposition, at the times represented by A, B, C, will be marked in this orbit by lines drawn to those points from D the centre of the equant, and cutting the orbit in G, K, L: and, if lines be drawn from G, K, L to E, the centre of the earth, and meeting the ecliptic in P, Q, R, they will represent the longitudes in which the three oppositions were observed. But these known arches PQ, QR, of the ecliptic, are not subtended by the known arches AB, BC, of the equant, but the arches subtended by AB, BC, are the unknown ones MN, NO; and, till the differences PM, QN, OR, between these and PQ, QR, be found, the excentricity and the position of the line HS of apsides cannot be accurately determined. The method therefore of Ptolemy, to obtain those arches, proceeded thus: first, in any one of the triangles, as B D E, which have for their common

To find the excentricity.

common base the excentricity DE of the equant; and for their sides lines drawn to given points, as B, of the same circle, from its centre D, and from E the centre of the earth, he determined, with BD the semidiameter = 100000, and with DE the excentricity, and the included angle BDE, to both which he had approximated, the angle DBE.

and longitude of the apogee.

Secondly, in the triangle KDF, with the same angle at D, and the sides DF = ½DE, and FK = BD, he determined the side DK. Lastly, in the triangle KDE, with the same angle at D and the including sides, he determined the angle EKD. By these means he found the angle BEK = EKD — EBD, and consequently the arch QN; and in the same manner the other arches OR and PM. Having added these to, or subtracted them from, the given arches PQ, QR, according as their positions required, he proceeded anew, and with the arches MN, NO, instead of PQ, QR, deduced another excentricity DE, and another place of the apogee (37), that with these he might determine three other small arches, by which MN, NO, might be corrected: and he repeated the same approximations, till at last the corrections of MN, NO became insensible. By investigations of this kind the excentricity of the equant of Saturn was fixed at 11388, of Jupiter at 9167, and of Mars at 20000, in parts of which the semi-diameter of each orbit contained 100000.

46. The situation of the centre of uniform motion, in a point different from the centre of the orbit, rendered two operations necessary in calculating the equations of the centre, that is, the angles TBE, or BTM; (fig. 10), and they could not, as in the solar theory, be obtained by a single operation. First, since the longitude of A, the apogee,

apogee, is found, and the mean motion of the centre of the epicycle given, (18), the angle AEB of mean anomaly is also given: and therefore, in the triangle DBE, with DB the semi-diameter of the deferent, and DE $= \frac{1}{2}$TE the excentricity of the equant, the angle DBE may be found. Next, in the triangle DBT, with the sides DB, DT, and the included angle BDT $=$ BED $\pm$ DBE, the angle TBD may be found; and, if required, the radius vector, or distance TB between the centres of the epicycle and the earth. The two angles TBD, BDE, make up the whole equation of the centre, or of the first inequality; and the first of these was by the later disciples of Ptolemy, called the optical equation, as arising merely from the situation of the observer: the other was called the physical equation, and, according to this theory, though perhaps without the knowledge of its author, was a real variation of the planet's velocity, and an exception from the supposed universal law of uniform motion.

To calculate the equations of the centre.

47. In order to calculate the second inequalities, it was necessary to determine the semi-diameter of the epicycle, that is, its ratio to the semi-diameter of the orbit. This was obtained by observing the planet out of opposition, and comparing its place with the calculated place of the centre of the epicycle; that is, the place which it was found to occupy when the equation TBE was applied. If the planet was observed in Z, the difference would be the angle BTZ. But the angle HBZ, the distance of the planet from the mean apogee H of the epicycle is given, it being always equal to HE*a*, or MTX, the distance of the mean place of the sun from the mean place of the planet; (43), and adding to this, or subtracting from it, HBK $=$ TBE,

Semi-diameters of the epicycle.

TBE, the result will be KBZ, the planet's distance from the true apogee. Therefore, with TBZ, the suppl. of KBZ, the side TB, and the angle BTZ, it was easy to determine, in the triangle BTZ, the side BZ. In this manner, supposing the semi-diameter of each orbit = 100000, the semi-diametar of the epicycle of Saturn was found to contain 10833 of such parts, that of Jupiter 19166, and of Mars 65833. But, if the Ptolemaic theory thus determined the ratios of the epicycles to their orbits, it contained no principle whereby to determine the ratios of the orbits to one another.

To calculate the equation of the orbit.

48. When the semi-diameter of the epicycle was determined, it was easy to calculate, for any given time, the equation of the orbit; or the derangement made by the second inequality on the planet's place in the zodiac. The argument of this equation, called also the equation of the argument, is the distance KBZ, from the true apogee of the epicycle, now described; and with this, and the sides BZ, BT of the triangle BTZ, the angle, or equation, BTZ may be always found. The greatest equation of this kind will evidently take place, when a line drawn from the centre of the earth to the planet is a tangent to the epicycle. But the equations of the orbit do not continue equal, in all degrees of anomaly, even when the planet is found in the same point of the epicycle; for, near the apogee of the excentric, its semi-diameter must appear under a less angle than towards the perigee; and, in all such cases, an operation, equivalent to a new determination of it, was necessary. As the planet also, towards the apogee, described its epicycle in less time than towards the perigee, another deviation from the law of uniform motion was evidently committed.

According

According to this theory, the greatest equations of the centre do not take place, as in the solar orbit, at the points C, N, where a perpendicular to the line of apsides, drawn through the centre of the earth, meets the orbit; but in two other points *c*, *n*, found by bisecting TD in Y, and drawing through Y a line parallel to CN. The points *c*, *n*, are those also in which the epicycle is at its mean distances from the earth; for $cT = cD$.

49. Though the ancients employed epicycles to explain the whole second inequalities of the planets, it did not follow, that either their stations or their retrogradations would be always visible; and no such vicissitudes had ever been observed, notwithstanding their epicycles, in the motions of the sun and moon. Apollonius therefore demonstrated that these phenomena depended on a certain ratio, between the velocity of the centre of the epicycle, and the velocity of the planet in its circumference; and, that if this were not less than the ratio of the semi-diameter of the epicycle to its distance from the earth, the planet never could appear retrograde.

His demonstration depends upon the following lemma:

Let DGE (fig. 13) be a triangle whose base DE is so divided in C, that DC is not less than the adjacent side DG, the ratio of DC to CE will be greater than that of the angle DEG to the angle EDG.

Let DC be equal to DG, join GC, to which draw from D the parallel DA, to meet EG produced; and, from G to DA, the line GB parallel to DC. Thus DC will be = GB, and GC = DB; and a circle described on G with the semi-diameter GB, will pass through D, because DC = GB = DG.

The

The ratio therefore of the triangle ABG to the triangle DGB, will be greater than that of the sector FGB to the triangle DGB; because the sector FGB is only a part of the triangle ABG. But the ratio of the sector FBG to the triangle DGB, is greater than that of the sector FBG to the sector DGB; and therefore the ratio of the triangle AGB to the triangle DGB, is much greater than that of the sector FBG to the sector DGB. But the triangle AGB is to the triangle DGB :: AB : BD :: AG : GE :: DC : CE, and the sector FGB is to the sector DGB, as the angle FGB to the angle DGB. Therefore the ratio of DC to CE is much greater than that of FGB to DGB, that is of the angle DEG to the angle EDG.

If DC were greater than DG, the truth of the lemma would be still more evident.

The proposition then, that, if the ratio of the velocity of the centre D, to the velocity of the planet in G, be not less than that of DC to CE, (fig. 12), the planet never can appear retrograde, was thus demonstrated.

When a planet will appear retrograde.

If this should be supposed possible, it must be in some arch GC, near the perigee of the epicycle, where the motion is actually retrograde. Join DG, which will be equal to DC.

Since the base DE, of the triangle DGE, is so divided in C, that DC is not less than the adjacent side DG, the ratio of the angle DEG, to the angle EDG, will be less than that of DC to CE. But the ratio of the velocity of the centre D, to the velocity of the planet in G, was supposed not to be less than that of DC to CE; and therefore must be much greater than that of DEG to EDG. But the velocity of the planet in G, is as the angle CDG, or EDG; and therefore the velocity of the centre D, must be as an angle LED, greater than GED. The

The truth of the proposition is therefore manifest. If D were at rest while the planet described the angle GDC, the planet would, no doubt, appear to be retrograde, because the arch GC subtends the visual angle GED; but, since D is not at rest, but has described in the same time the angle DEL, greater than DEG *in consequentia*, the planet must also have appeared to move *in consequentia*, through an arch PL, or to have described an angle PEL = DEL — DEG.

When a planet will appear retrograde.

It is equally manifest that, if the ratio of the velocity of D to the velocity of the planet in G were less than that of DC to CE, the planet would appear to be retrograde in describing an arch GC near the perigee; for then DEL would be less than DEG, and DEL — DEG would be negative, that is, PEL would lie on the other side of PE.

Finally, if the ratios were equal, the planet must appear for some time to be stationary; because then DEL — DEG = 0. The stations therefore of a planet will happen when any small retrograde arch, which it describes of the epicycle, and the contemporary direct arch described by the centre of the epicycle, shall subtend the same visual angle; or when the velocity of the centre shall be to the velocity of the planet, as the distance of the planet from the centre to the distance of the planet from the earth.

The reason is hence also evident why the sun and moon, though supposed to move in epicycles, are never seen to be retrograde, or so much as stationary, even while describing the retrograde arches of those epicycles. For the velocity of the sun, and the centre of his epicycle are precisely equal, and of the moon and centre of her epicycle nearly so: whereas the distances are, in the former case, as 24 to 1, and in the latter nearly as 10 to 1.

If the line ED be produced to meet the epicycle again in A, and other lines be also drawn from E, on both sides of EA, to meet the epicycle, as in G, B, and in H, K; and if the intercepted parts BG, KH, be bisected in P, M, by perpendiculars from D, the ratio of PG to GE, or of MH to HE, will be less than that of DC to CE: and, as the lines diverge from DE, this ratio will continually diminish, till at last, when LE, NE, become tangents to the epicycle, it entirely vanishes. The planet therefore may be found in a point G, or H, of its epicycle such, that the velocity of D shall be to the velocity of G, as PG to GE, or as MH to HE: so that its first station will be in G, and its second in H.

The distance GC, or HC, of either of these points from C the perigee of the epicycle, and consequently the length of the whole retrograde arch GH, was thus investigated. The ratio of PG to GE, being the same with that of the velocity of D to the velocity of G, is given, and therefore that of BG to GE, and of BE to GE. But the ratio of DC, in parts of the semi-diameter of the planet's orbit $=$ 100000, to CE is given; and consequently that of AE to CE; and, since AE.CE $=$ BE.GE, the rectangle BE.GE, will be also given in parts of CE. If therefore we make BE : GE :: BE.GE : GE^2, the side GE of this square will also be found in parts of CE; and consequently BE, BG and PG. Therefore, in the right angled triangle PDG, with PG and DG, we may find the angle PDG, and the side DP; and, in the right-angled triangle EPD, with DP and PE, we may find EDP. If, therefore, we subtract PDG from EDP, the remainder GDC will be measured by the required arch GC, the distance of the planet at the first point of station from the perigee C. In the same

same manner HC may be found; and the retrograde arch GH is = GC + HC.

50. The varying latitudes of the superior planets were accounted for by supposing the planes of their orbits were, not only like that of the moon, inclined to the ecliptic; but that the planes of the epicycles were also inclined to the orbits. Let AB (fig. 14), be the plane of the ecliptic, and CD that of any superior planet's orbit, intersecting each other in a line passing through the centre of the earth, or rather of the ecliptic, and called, as in the lunar orbit, the line of nodes. The inclination of these planes was supposed to be invariable, and stated at 2° 30′ for the orbit of Saturn, at 1° 30′ for Jupiter, and for Mars at 1°. On this single supposition there could be no varieties of latitude, except those produced by its change from north to south, and from south to north, according to the passage of the centre of the epicycle, from the one side of the plane of the ecliptic to the other, and arising from 0° 0′ in the nodes, to the greatest amount in the limits; and those occasioned by the position of the earth in a point different from the centre of the orbit. But if the plane EF, or GH, of the epicycle, shall so cross the plane of the orbit, as to be parallel to AB the plane of the ecliptic, the planet, when viewed from the earth, in E, or H, the apogee of the epicycle, will have a less distance from AB; and, when viewed in the perigee F, or G, a greater distance from AB, than if the plane of the epicycle had co-incided with the plane of the orbit: and, as the revolutions of the planets in the epicycles, had no connection with the revolutions of the epicycles in the orbits, the varieties of the observed latitudes would evidently be very great. But these suppositions were found

Latitudes of the superior planets.

found insufficient, and there was added to them a libration of the plane of the epicycle on a diameter perpendicular to the line of apsides. The inclination therefore of the epicycle was conceived to be variable; and the variation, in Saturn's orbit, increased to 4° 30′; in that of Jupiter, to 2° 30′; and in that of Mars, to 2° 15′. It does not however appear, that this variation of inelination was admitted by Ptolemy himself; and Kepler was of opinion that, on the contrary, the varieties of latitude were partly the cause, which suggested to him the bisection of the excentricity; and that this was the principle from which he endeavoured to deduce them, and not from any supposed oscillation of the epicycle.

51. Though the motions of the inferior planets differed from those of the superior in this essential circumstance, that their mean revolutions in their orbits were alone connected with the solar revolution, and their revolutions in their epicycles totally unconnected with it, the Ptolemaic theory, at least of Venus, did not essentially differ from that of the superior planets. Let T (fig. 15), be the centre of the earth, or of the ecliptic ABCD; let E be the centre of the solar orbit GHKL, and O the centre of the orbit of Venus MNP. Join TO producing it to meet the orbit in M and Q, and in OM, taking OF = OT, the point F will be the centre of the equant of Venus, and about which, and not about O, her mean motion, that is, the revolution of the centre of her epicycle was performed; and as this was always equal to the mean motion of the sun, it followed that the lines of both were always parallel, as FM to EL, FN to ER, FQ to ES, &c. As F is thus the centre of the

Theory of Venus.

the uniform motion of Venus, and T of her apparent motion, the first inequalities were represented by drawing from the centre of her epicycle, in N or P for example, lines to T and F; and the equations of this centre were the angles TNF, TPF. As Ptolemy supposed that the excentricity TF of the equant of Venus had the same ratio to its semi-diameter, which the solar excentricity had to the semi-diameter of the solar orbit, the equations were the same as those of the sun; but, as he also supposed that D was the longitude of the apogee of Venus, and A that of the solar apogee, the points differed where the same equations were applied; for example, the greatest equation of Venus is at the point N or P, but of the sun in H or V. But some of Ptolemy's disciples, particularly the framers of the Alphonsine tables, supposed that the line of the apsides of Venus coincided with the line AET of the solar apsides, and consequently that her orbit was in the position XYZ; and others, that her orbit, and that of the sun, were in all respects co-incident; and that, as in the solar theory, the centres of the orbit and the equant were the same.

First inequality.

52. The second inequalities of the inferior planets were likewise represented in the same manner as those of the superior planets, by means of an epicycle. But, though the planet was always in conjunction with the sun in the apogee of its epicycle, those conjunctions, as the periods of the revolutions in the epicycle and the orbit were different, happened in all points of the orbit indifferently; and, neither had the stations any connection with the revolution of the centre of the epicycle, that is, of the sun, or with its position. As also the greatest digressions of Venus never exceeded

Second inequality.

exceeded 48°, and those of Mercury did not amount to 28°, it was evident that the limits of their epicycles did not permit them to be in opposition to the sun, like the superior planets, when they came to the perigees of their epicycles; and that there also they would again be found in conjunction with him. But both kinds of conjunction were alike invisible to the ancients; and, their observations of the inferior plants being therefore limited to their greatest digressions, they endeavoured to account for the variations of these by the excentricities of their orbits; which made the semi-diameters of the epicycles appear, at different points of anomaly, under different angles. Yet the digressions of Mercury were found to be so extremely various as not to admit of this explication; and it was found necessary to have recourse also to variations of the excentricity, and oscillations of the apogee; and by these the theory of this planet was rendered much more perplexed and intricate than that of Venus.

Theory of Mercury.

53. In the theory of Mercury the excentricity of the equant was not, as in the orbits of the superior planets, double of the excentricity of the orbit; but, on the contrary, the centre of the equant bisected the mean distance between the centres of the orbit and the earth. Let TA (fig. 16), be the mean excentricity of the orbit, let B be the point in which it is bisected by the centre of the equant CDEFG, and let BC, BD, BE, &c. be the line of the mean motion of the centre of the epicycle, always parallel, as in the case of Venus, to TC the line of the mean motion of the sun, and performing its revolution *in consequentia*, from C to D, E, &c. On the centre A, with the semi-diam. AB = BT, describe the small circle KLMBNO: and, to represent

present the supposed variations of excentricity and oscillations of the apogee, let the centre of the orbit be conceived to revolve in a direction contrary to that of BC, though with the same velocity, in the circumference of this small circle, from K, to L, M, B, &c. The orbit, in these different positions of its centre, is marked by the dotted lines of the figure; and without a complicated scheme of this kind, it was thought that the whole varieties of the greatest digressions of Mercury, between the extremes of 28° and 16°, could not be justly represented. For, if the line PQ of the apsides should remain immoveable, as the first inequality would disappear at these apsides, and become greatest at points equally distant from them in the zodiac, so, with respect to the second inequality, the greatest digressions of the planet from the sun would take place in the perigee, because there the epicycle approaching nearest to the earth, would be seen under the greatest angle; the least digressions would, on the contrary, happen in the apogee; and the mean digressions about the points of the zodiac, or of the orbit equally distant from them. They would not, indeed, invariably return, like the first inequalities, with the return of the centre of the epicycle to these particular points of the orbit; because the periods of the revolutions in the epicycle and in the orbit are different; and, where a digression from the sun had taken place, there might, in a following revolution, be a conjunction with him. But, if any digressions did take place, they would observe these constant rules, and the apparent magnitude of the epicycle would always be the same in the same degree of anomaly reckoned from the immoveable line PQ. But, as the greatest digressions observed no such constant rules, it became necessary to suppose that this line PQ of the apsides was moveable upon the point T, and in such a manner, that when the line BC of mean motion had revolved through two signs, and had come to the situation BD, the centre of the excentric should also move through two signs in the opposite direction, in the circumference of the small circle, from K to L; and carry along with it the whole excentric orbit, and consequently the line of apsides PTQ into a new situation RTS. By these means the centre of the epicycle was brought to a point of the line BD, much nearer both to the earth and to the perigee, than if the situation of the orbit had been immoveable: for, on this supposition, it would have been found in Y, with the anomaly PKY, whereas, on the supposition of a moveable orbit, it is found in Z, with a much greater anomaly RLZ. In like manner, when the line of mean motion had proceeded through two signs more to BE, and the centre of the excentric through as many, *in antecedentia* to M, the line of apsides would come into the position VTX, and the centre of the epicycle to the point *a*, very near the perigee X; and consequently, the apparent magnitude of the semi-diameter of the epicycle, that is, the greatest digression from the sun, would at this time be very great. Indeed, in no other point could the digressions appear greater; for, as the line TP of the apsides only oscillates, and does not revolve, about the point T, it would happen that, when the line of mean motion proceeded farther to the situation BF in the perigee of the equant, the line of apsides would be in the position CTF, having returned from TV to TP. But, as the orbit now coincided with the equant CDEF, the centre of the epicycle would be found in F, the common perigee of both, and at a greater distance from the

the earth than when it was in *a*; for TF is evidently greater than T*a*. The appearances in the other half of the revolution are perfectly similar. Two points *a* and *b* were thus assigned, in every revolution of the centre of the epicycle, where the greatest digressions of Mercury came to their *maximum*, and each of them at the distance of 60° from the perigee of the equant; whereas, in the immoveable orbit they could not have come to their *maximum*, except at the perigee. By this theory also, the first inequalities were represented as more variable than in an immoveable orbit, and the path, in which the planet actually moved, as an oval, and not a circle. (B).

54. It was impossible to determine the excentricities of the inferior planets, and the longitudes of their apogees, in the same manner as those of the superior planets, because in all their conjunctions they were invisible; and as their greatest digressions were extremely variable, and of necessity, especially in the case of Mercury, observed too near the horizon, not to be greatly affected by derangements from refraction, the conclusions from them with respect to these elements, were neither made so early, nor with such certainty.

The greatest digression of an inferior planet is the difference between the place of the planet, in the point of its greatest elongation from the sun, and the mean place of the sun determined by calculation; for the lines of the mean motion of the sun and of the planet, being always either parallel or coincident, (50) terminate in the same points of the zodiac. This difference is the angle LTM, or FTO, (fig. 18), or the arch of the zodiac which measures it: and, as it is the angle which is subtended by the semi-diameter of the epicycle, it follows, Longitude of the apogee of an inferior planet.

follows, that the greater the distance of the epicycle from the eye in T, the less will be the angle which it subtends. If then, among a number of such observed angles, two equal ones, as LTM, FTO, shall be found in opposite directions from the sun; that is, an evening digression LTM, equal to the morning digression FTO; it is evident, that the line of apsides ATH will, according to the ancient principles, bisect the angle MTO, formed at the centre of the earth by the lines of the true motion of the planet, or the angle DTB, formed by the lines of the mean motion of the sun. If also two other greatest digressions KTP, GTQ, of contrary kinds, shall be found, differing in magnitude from the former, but equal between themselves, it will be known whether the apsis A, whose longitude was determined, were the apogee or the perigee; for, if the two last digressions were the greatest, it must have been the apogee. Nor is it an objection to this procedure, that Ptolemy introduced into his theory oscillations of the line of apsides, for what is required is, its longitude in its mean position.

55. When the longitude of the apogee was determined, the excentricity CT of the orbit was found in this manner. Let the greatest digressions ETV, and HTY, be observed, when the first, being in the apogee, is a *minimum*, and the last, at the perigee, a *maximum*. Or, though the position of the apsides has not been determined, if these greatest digressions were certainly of the least and greatest kinds, the mean place of the sun was of necessity, at the time of the former, in the apogee, and, of the latter, in the perigee: and, in the theory of Venus, these two points are in the same straight line; and, in that of Mercury, supposed in

in it. Now, since the epicycles, though appearing in the heavens, and represented in the figure, as unequal, were notwithstanding, according to the theory, really equal; their distances ET and HT will be reciprocally proportional to the tangents of the angles ETS, HTX; (40) that is, TH : TE :: tan. ETS : tan. HTX. These distances therefore will be given, in parts of the semi-diameter of the orbit = 100000; and consequently, in the same parts, the excentricity TC = $\frac{1}{2}$ (ET − HT). The semi-diameter EV, or HY, also of the epicycle is given in the same parts; for R : sin. HTY :: TH : HY. By these means Ptolemy determined that the mean excentricity of the orbit of Mercury was 9498, and the semi-diameter of his epicycle 35738; and the excentricity of the orbit of Venus 2339, and the semi-diameter of her epicyle 72094; in parts of which the semi-diameter of each orbit contained 100000.

Excentricity of the orbit.

Semidiam. of the epicycle.

56. But though these were the excentricities of the orbits, and their centres, as C, the points from which the epicycles, in the apsides, would appear of the same magnitude, it remained to enquire, whether an epicycle would continue to appear at C of the same magnitude in all the other points of the orbit; that is, if equants were not also necessary to the inferior planets, and at what distances from the centre of the earth the centres of these equants should be placed: and we find, that though Ptolemy gives no account of his procedure in this enquiry in the case of the superior planets, he thus describes it with respect to Mercury and Venus. Let AB, (fig. 19), be the line of the apsides of Venus, T the centre of the earth, E the centre of the equant, and let AED, the angular distance of the sun, or centre of the epicycle, from

from A the apogee of Venus, be, for example, = 90°. In this situation of the centre of the epicycle, let DTG be an observed greatest evening digression of the planet, and at some other time, when the sun occupies the same point of the equant, let DTF be an observed greatest morning digression. These digressions, though unequal when viewed from T, will, notwithstanding, be equal when viewed from E, and a mean of them will be the angle DTF. Therefore, the other acute angle FDT, in the right-angled triangle DTF, is given; and, since the side DF, the semi-diameter of the epicycle, is given in parts of the semi-diameter ED of the equant = 100000, (54) the secant TD of the angle FDT to the radius DF will be given in the same parts. Therefore, in the right-angled triangle DET, with TD and ED, the required excentricity TE will readily be found. In this manner Ptolemy discovered that the excentricity of the equant of Venus, as in the case of the superior planets, was double the excentricity of the orbit; whereas that of the equant of Mercury was only one-half of the excentricity of the orbit.

Excentricity of the equant.

57. When the excentricities of the orbit and equant, and the longitude of the apogee were determined, the place of an inferiour planet, at any given time in the zodiac, was to be found, in the same manner as of the superior planets, by calculating the equations of the centre and of the orbit. (45. 47). But as the excentricity of Venus was inconsiderable, and her orbit and equant seemed almost to coincide, and the centre of the equant of Mercury bisected the excentricity of the orbit, instead of the double operation for the equation of the centre, it was calculated by one operation, as

Equations of the centre.

as in the solar orbit. (33). Yet, when great precision was studied, it was previously necessary, in the orbit of Mercury, and even of Venus, to investigate the variations of the excentricity. Thus, let L (fig. 16) be the place of the centre of the deferent, at the distance, suppose of 60° from K, and join AL. Then, in the triangle TAL, the varied excentricity TL was investigated from TA and AL, and the angle TAL given. Or, according to a more approved hypothesis, which considered the line of apsides as immoveable, and the motion of the centre of the deferent as only imaginary in the small circle KLM, while its real motion was a libration in the diameter KB, the varied excentricity was found by drawing to KB from L the perpendicular L*m*; and was = TA + A*m*, and, universally, = TA ± cos. KAL.

The equation also of the orbit was calculated in the same manner as for the superior planets, (47), excepting that the argument of this equation was not the difference between the mean motion of the sun and the mean motion of the planet in its orbit, but between the former and the mean motion of the planet in its epicycle; and that it was previously necessary to investigate the variations of the apparent magnitude of the semi-diameter of the epicycle. For this last purpose the procedure was as follows. Let AB, (fig. 20), be the semi-diameter of the epicycle of Venus at its mean distance from the earth, and = 71890, in parts of the semi-diameter of the orbit. and let BC = 2500 be the difference between its greatest and least apparent semi-diameters, of which the first takes place in the perigee, and the second in the apogee. To find then the semi-diameter in any other point of the orbit, describe on B, with the distance BC, the circle CED; in which let CE be equal to the given anomaly.

maly. The required variation of the semi-diameter AB will be BF, the cosine of CE; subtractive from AB near the apogee, that is, in the first and fourth quadrants of anomaly, and additive to AB in the second and third.

Latitudes of the inferior planets.

58. The various latitudes of the inferior planets were explained in this manner. Let ABC (fig. 21) be the plane of an orbit, BD the line of apsides, B the apogee, and AC the intersection of this plane with the ecliptic. It was supposed, first, that the plane of the orbit oscillated on the line AC; coinciding with the ecliptic, when the centre of the epicycle was in A or C; receding from it, on the departure of this centre from A, till it came to B the apogee; and again approaching to the ecliptic, in the progress of the epicycle from B to C: and this osciliatory motion of the plane of the orbit was called the deviation. Secondly, it was supposed, that the plane of the epicycle was inclined to the plane of the orbit, intersecting it in the line EF; and that it oscillated on this line according to a law the contrary of the deviation; the inclination disappearing at the apsides, and coming in the nodes to its greatest amount. Finally, it was supposed that another oscillation of the epicycle, called the reflexion, took place upon a line GH, perpendicular to EF, and observed the law of the deviation. The nodes were placed at the distance of 90° from the apsides, and both were supposed to move according to the apparent motion given by the precession of the equinoxes to the fixed stars. Though the latitudes of the moon and the planets required that reductions should be made from their peculiar orbits to the ecliptic, Ptolemy does not appear to have perceived their necessity; nor, till the later times, were they attended to by his followers.

Deviation, inclination and reflection.

59. Notwithstanding the labour employed, and the ingenuity displayed in the formation of this system, its imperfections were so great and numerous, that it was impossible it should always continue to maintain its credit. Though the original introduction of excentric orbits by Hipparchus banished some of the ancient epicycles, and promised greater simplicity than had been attained in the ancient concentric theory, the more extensive application, which Ptolemy and his followers made of the excentric theory, rendered it at last almost as perplexed and intricate as the theory which it had displaced. Its suppositions were often dissimilar, and even inconsistent with one another; and instead of answering to the title of a system, it was an assemblage of parts connected by no general principle of union, and between which there subsisted no known mutual relations: for, though the ratio of an epicycle to its peculiar orbit might be determined, those of the orbits to one another were entirely arbitrary. In many respects it was destitute of the authority even of sense, to which it principally appealed; and, notwithstanding the pretensions of reducing the celestial phenomena to the rules of geometrical calculation, it was not uncommon to find differences of hours and days, and even of months, between the calculation and the fact; nay, it frequently happened, that the predictions, made according to its rules, entirely failed. But the objection, which seems to have struck at the credit of the Ptolemaic theory more than all its inaccuracy in representing the phenomena, was its contradiction to the supposed inviolable law of circular and uniform motion, which it was the principal object of all systems to establish and confirm. Not only did its oscillations and librations produce perpetual deviations, both from the plane and

Imperfections of the Ptolemaic system.

and the circumference of the circle; but also the uniformity aimed at by the equant itself was purely imaginary; for it took place in an orbit where the celestial body was hardly ever found; and, by the introduction of it, a real inequality of velocity was acknowledged in the orbit, which the body actually described. The position also of the centre of the equant was regulated by no general law: for, in the theories of Venus and the superior planets, its distance from the centre of the earth was bisected by the centre of the orbit; in the theory of Mercury, on the contrary, the mean excentricity of the orbit was bisected by it; in the lunar theory, it continually varied its position; and, in the solar theory, it coincided with the centre of the orbit. Finally, by the complicated motions ascribed to some of the circles, especially in the theories of the moon and Mercury, those bodies were brought, in some parts of their orbits, so near to the earth, that their position was wholly inconsistent with all observations of diurnal parallax.

CHAP.

CHAP. III.

Of the Copernican System.

60. THOUGH the imperfections of the Ptolemaic system were not immediately perceived, especially during the confusion which attended the decline and destruction of the Roman empire, their effects did not fail, in process of time, to become fully evident. In the ninth century, on the revival of science in the east, under the encouragement of the caliphs, surnamed Abassides, Ptolemy's astronomical tables were found to deviate so widely from the actual situations of the celestial bodies, as to be no longer useful in calculations: and it became necessary for the Saracen astronomers at Bagdat to form tables entirely new. The Saracens carried their astronomical knowledge with them into Spain; and, in the thirteenth century again, the new tables were found unfit to represent the celestial motions; and, to supply their place, the tables, called Alphonsine, were constructed, by the direction of Alphonso the 10th, king of Castile. The errors even of the Alphonsine tables became, in the fifteenth century, equally sensible with the former. But though it was hence evident, that the revolutions of the celestial bodies were not precisely what the excentric circles and epicycles of the ancients represented, such was the veneration of astronomers for Hipparchus and Ptolemy, from whom they derived all that was valuable in their science; and, indeed, so great was the merit of subjecting motions, so intricate as the apparent celestial re-

volutions, to any settled rules; that no suspicions concerning the principles of the theory appear to have been entertained. All that was proposed even by eminent astronomers, such as Purbach and Muller, surnamed Regiomontanus, was only to correct the theory by more accurate observations: and none of the amendments, which they introduced, was inconsistent with its principles. It was the celebrated Copernicus, a native of Thorn in Polish Prussia, who first called in question the principles themselves, and to whom the exclusive honour belongs, of substituting for the Ptolemaic a new and more beautiful system, representing the celestial motions with much more simplicity, and, after its principles were fully understood, with incomparably greater accuracy.

Copernicus.

61. It does not appear that Copernicus originally meditated such a total revolt from the authority of Ptolemy, as that to which, in the course of forming his theory, he was eventually led. Though equally sensible with others of the deviation of the Ptolemaic tables from the actual state of the heavens, the chief cause of his dissatisfaction with Ptolemy's theory related to his explications of the first planetary inequalities, and was his departure in these explications from the principle of uniform motion in perfect circles, which all astronomers considered as sacred and inviolable. When Hipparchus introduced an excentric orbit into the solar theory, no trespass against this principle was committed; because the sun was supposed to move uniformly round its centre: but when Ptolemy extended the application of excentrics to the planetary orbits, and supposed every epicycle to move uniformly round a point, not in the centre of the deferent, the principle was undeniably abandoned:

for

for the uniformity attained was in a foreign orbit, called an equant; and there was a real and evident inequality in its own. It was this part of the theory of Ptolemy which, Copernicus tells us, he chiefly disapproved; and, as he found the explication of the first inequality, by means of an excentric with an equant, irreconcileable with his favourite principle, he seems to have had at first no higher purpose in view, than to substitute an explication of a different kind, and more consonant to that principle. His original design.

62. The theory which he proposed to substitute, for explaining the first inequalities in a manner more consistent with the principle of uniformity, was the ancient concentric one: and it was in restoring this from its neglected state, that the distinguishing and essential part of his system, in which he ventured to depart, not only from the authority of Ptolemy, but from all the established opinions of mankind, seems first to have presented itself to his thoughts. When both inequalities were represented by means of concentric circles with epicycles, the necessary multiplicity of epicycles confounded the imagination; and the chief recommendation of the excentric theory, and even the original cause of framing it, was the banishing several of the epicycles by which the imagination had been perplexed. The same desire therefore of simplicity, which led Ptolemy to substitute an excentric orbit, in the explication of the first inequality, for the concentric with its epicycle, seems to have had equal influence, in suggesting to Copernicus a like substitution for the epicycles used in the explication of the second. He discovered in the annual orbit of the sun, or earth, an universal epicycle, which explained the second inequalities Probable motives.

 more

more advantageously than all the various separate ones with which the planetary orbits had been encumbered: and in fact the solar orbit had been already employed in this explication, at least in some degree; for while the ancients formed the argument of the equation of the second inequality, by taking the difference between the places of the sun and the planet (47), they were actually converting the sun's orbit into an epicycle. These seem to have been the motives by which Copernicus was led to conceive the bold design of attributing motion to the earth; and, by the application which he made of her annual orbit, he found the simplicity which he sought; not like Ptolemy, at the expence of the sacred principle of uniformity, but in some sense perfectly consistent with it. This attachment indeed to the doctrines of uniform circular motion, which made him reject the excentric of Ptolemy, was merely a prejudice connected with the imperfect state of physical knowledge; for the motions of the planets are in reality neither circular, nor uniform: but, in the present instance, it produced the happiest and most important effects, and proved the introduction of all that is true and valuable in astronomy.

63. When the design was thus conceived of ascribing motion to the earth, and displacing her from the centre of the planetary system, Copernicus found that, bold as it was, it was not destitute of support from many powerful arguments, and even from several striking astronomical phenomena. In particular, he found that an acknowledgment of its propriety was made, however undesignedly, in the whole theory of the epicycles, which, by the annual orbit of the earth, he proposed to abolish. The ratio of the semi-diameter of

of the epicycle of Mars, to the semi-diameter of his orbit, exceeded that of 6 to 10; and, in the theory of Venus it was still greater; for it exceeded that of 7 to 10; and therefore, when the distance of the former planet from the earth varied from 4 to 16, and of the latter from 3 to 17, it was undeniably absurd to consider the earth as the centre of their motions: and the absurdity was the same, though not so evident, of supposing the earth to be the centre of the motions of the other planets. Copernicus found also that the Ptolemaic arrangement of the inferior planets had not always been generally received; for Plato, and his followers, placed them beyond the sun; and he saw that the reasons for inverting this order, and including their orbits between those of the sun and moon, were unsatisfactory and inconclusive. The distance of the sun from the earth, Ptolemy reckoned at 1160 of the earth's semi-diameters, and that of the moon in apogee, at 64. The interjacent space between these two orbits he filled up, first, with the epicycle of Mercury of 177 in diameter, and next with that of Venus of 910; thus bringing Mercury's epicycle almost in contact with the lunar orbit, and the epicycle of Venus almost in contact with the solar orbit: and the principal reason which he assigns for crowding them in these positions, is the improbability of supposing a space so vast to have been left wholly empty; forgetting that, by this very arrangement, he had allotted, to the single epicycle of Venus, a space more than four times, or according to a more probable deduction, more than six times as extensive in breadth, as the space allotted to the earth, the moon, and Mercury, all together. The other reason alleged for this arrangement, viz. the propriety of the sun's holding the middle station, between

Confirmations in his purpose.

the planets whose digressions permitted them to come into opposition with him, and those whose digressions were more limited, was both false and frivolous: for the moon's digressions were unlimited, though her orbit was arranged on the same side of the sun with those of Venus and Mercury. No reason also appeared from this arrangement, why the digressions of Venus and Mercury should be limited, why their revolutions should be so intimately connected with those of the sun, or why any one of all the planets, the superior ones not excepted, should be placed nearer to the earth than any other. On these accounts Copernicus could not fail to consider the theory of the inferior planets, attributed to the ancient Egyptians, and held by several Latin astronomers, and particularly by Martianus Capella in the fifth century, as much more worthy of attention than the Ptolemaic, and as giving a much more consistent explication of their phenomena. In this, the sun was the centre about which Venus and Mercury performed their revolutions: and, as the earth was not included within their orbits, it was impossible that they should be seen from the earth, to make any greater digressions from the sun than the limits of these orbits would allow: and the reason was also manifest, both of the relative positions assigned them with respect to the sun, and of the intimate connection of their apparent annual circuit round the earth with the apparent solar revolution. This theory Copernicus applied to the superior planets, and found the application attended with like success; for, though their oppositions shewed that the earth was included within their orbits, their near approaches to the earth in their oppositions, and the vast distances to which they removed in their conjunctions, made it impossible that the

Egyptian system.

earth

earth could be the centre of their motions. This variation of distance was especially remarkable in the planet Mars; who, in oppositions, appears equal in size to Jupiter, but towards his conjunctions no larger than a star of the second or third magnitude; and afforded an unquestionable proof that none of the superior orbits approaches so near the earth. It was from this extension of the Egyptian theory concerning the inferior planets to the superior, and making the sun the centre of all the planetary orbits, that the transition seems to have been more immediately made to the doctrine of the motion, or revolution, of the earth, like any other planet, round the sun. For, as the variations of distance shewed the earth to be nearest to the orbits of Mars and Venus, and as she was evidently within the former, and without the latter, no reason appeared why she should not partake of the revolutions round the sun, in which so many other bodies, and some of them thought to be of greater magnitude, on both sides of her, were supposed to be involved: nay, on the contrary, strong probability appeared that she did partake in those revolutions; for, on this principle, all the varieties of the distances of the planets, and all the circumstances of their second inequalities were at once and easily explained: and they could not be explained otherwise, without the improbable supposition of the annual revolution of the whole planetary orbits, round her centre. The only celestial body, which could not be subjected to the general law of describing an orbit round the sun as a centre, was the moon: for, though her oppositions, like those of Mars, proved that the earth was included within her orbit, her parallaxes shewed her distances to be much less than the least distance of Venus; and the variations of her distance

Extension of it.

Transition from it to his own system.

were so inconsiderable as not to require, or even to admit, of any other centre of her motions than the earth. In the interjacent space, therefore, between the orbits of Mars and Venus, and where the former system made the sun to move, Copernicus placed the orbit of the moon with the earth in its centre; and supposed both together, like some great planet, to revolve round the sun, in the precise time of an apparent solar revolution. The sun was now considered as the only immoveable body in the system: for Copernicus was not influenced by the objection of his apparent progress through the zodiac; being convinced, from innumerable examples, how unavoidably we ascribe to surrounding objects all the real motions of which we are not sensible: and, by the immoveable position of the sun in the centre, and the continual revolutions of the earth and planets round it, at different distances, and in different times, not only were the second inequalities explained in general, without the embarrassment of epicycles, but the causes also of the different times of the revolutions in these imaginary epicycles, and of their different magnitudes, became fully evident. In particular, the cause became evident, why both the direct and retrograde arches of Jupiter were greater, and required longer time than those of Saturn, and less than those of Mars; and the like arches of Venus greater than those of Mercury: and why those vicissitudes returned more frequently in Saturn than in Jupiter and Mars; and more frequently in Mercury than in Mars and Venus. It is true that no reciprocations of this kind had been observed in any of the fixed stars: but this Copernicus boldly, though justly ascribed to their immense distance, in comparison with which the diameter of the whole terrestrial orbit, though bearing a sensible ratio to the

the distance even of the remotest planet, entirely vanished. When he had thus ascribed a periodical revolution to the earth, the transition was more easy to the doctrine of her diurnal rotation, by which his theory was completed; for, if it was improbable that the sun, carrying along with him the whole planetary orbits, should revolve annually round the earth; it was much more improbable that all these, together with the immense sphere of the fixed stars, should revolve round her every 24 hours, with a rapidity incomparably greater, and almost indeed inconceivable.

64. There were also some ancient authorities which seemed to encourage Copernicus in the opinions he was thus led to conceive, or at least tended to introduce them to the world with less appearance of absolute innovation. He first found, as he tells us, in Cicero's writings, a tradition, transmitted by Theophrastus, of the opinion of Nicetas of Syracuse, which made the sun, the moon, and the whole starry heavens immoveable, and ascribed their constant apparent diurnal revolutions to the sole rotation of the earth on an axis. Next, he found in Plutarch, not only a similar tradition, that the same doctrine of the diurnal rotation of the earth was asserted by Heraclides of Pontus, and Zephantus the Pythagorean; but also, what he thought had a reference to her annual revolution, and that this was said by Philolaus of Crotona, another disciple of Pythagoras, to be performed about the central fire, or sun. We have also seen the favourable opinion which he entertained, and the more extensive use which he made, of what is called the ancient Egyptian system; where Venus and Mercury are considered as *satellites* to the sun: and this opinion could not fail to be

Ancient authorities.

be confirmed, when he saw that Ptolemy, though not in words, had in effect adopted it, by making the mean place of the sun the centre of both their epicycles; and that the framers of the Alphonsine tables had, at least with respect to Venus, expressly adopted it, by considering the solar orbit not only as her equant, but even as the deferent of her epicycle. But the reasons which were decisive with Copernicus, and far outweighed all authorities, in the formation of his system, were certainly the satisfactory explication which it gave of all the circumstances of the second inequalities, and the symmetry and proportion, which he calls admirable, of all its parts. It was not a mere assemblage, like the Ptolemaic, of unconnected parts in arbitrary positions: but, as the ratio of his general epicycle to every particular orbit was given, the ratios were also given of the orbits to one another; and the position of every one was determinate, and not arbitrarily assumed.

Copernican system.

65. In the Copernican system, the sun is placed in the centre of the universe, and Copernicus expresses a peculiar satisfaction at contemplating him in this situation, the most commodious for diffusing light and heat to the whole celestial bodies: for he supposed that the fixed stars, equally with the planets, derived their splendour from him. The planets perform round him their periodical revolutions, in the following order determined by the ratio of every orbit to that of the earth; Mercury, Venus, the Earth attended by her satellite the moon, Mars, Jupiter, and Saturn; and after these may now be added the planet discovered by Herschel. The earth, instead of continuing to be the centre of the motions of the sun and planets, is degraded to become herself a planet, and is the centre

centre only of the motions of the moon. It has been since discovered, that Jupiter has four moons, and Saturn five, which they carry along with them, as our moon is carried along by the earth, in accomplishing their revolutions about the sun. To the five moons, *or satellites*, of Saturn, Herschel has also discovered that two more ought to be added, and that his own planet is accompanied by two. Beyond these, and at an immense distance, is placed the sphere of the fixed stars; and its diurnal revolution, together with that of all the moveable celestial bodies referred to it, is considered in this system merely as apparent, and produced by the diurnal rotation of the terrestrial globe.

66. Notwithstanding the simplicity and symmetry of this system, and all the advantages by which it was recommended, Copernicus was so much aware of the objections that would be made to it, and the prejudices which it would have to encounter, that he was deterred from publishing it to the world, and forbore, for thirty years, to communicate it, except to some confidential friends. Many of the astronomical phenomena, by which it is supported, were then undiscovered: the rotation, for example, of other celestial bodies on their axes, had not been observed: none of the changes had been seen, in the phases especially of Venus and Mercury, which this system rendered necessary: the principle of gravitation and its important consequences were almost wholly unknown: and, till the aberration of the fixed stars was discovered, it seemed altogether incredible, that the translation of the earth, in her annual revolution, from one extremity of her immense orbit to another, should produce no change on

Long hesitation of Copernicus to publish it.

on the apparent magnitudes, or the relative positions, of the fixed stars. It was not, therefore, till near the close of life, nor even then without the importunities of his friends, particularly of Schomberg cardinal of Capua, and Gisius bishop of Culm, that his consent to the publication was obtained : and, when the first edition of his work was completed, under the inspection of the eminent George Joachim Rheticus, at Nurenberg, on the 24th of May, 1543, the illustrious author, a few days after receiving a copy, died in his 72d year at Frawenberg.

67. The explication which the Copernican system gave of the second inequalities of the inferior planets, is as follows. Let ABC (fig. 22), represent part of the zodiac, DEF the orbit of the earth, GHK the orbit of Venus, and S the sun, supposed to be in the centre of both : and let Venus be in the superior part of her orbit, or that which is beyond the sun ; and from T the earth, let the tangents TG, TK, be drawn to the orbit of Venus, and produced to meet the zodiac in A and C. In the portion GHK of this orbit, which is intercepted by these tangents, and most distant from the earth, the planet will be referred, by the observer in T, to the portion ABC of the zodiac ; and will appear to move in it from west to east, according to the order of the signs, and in the same direction in which she moves in her orbit from G to H, and K. But, supposing the earth to continue at rest in T, and Venus to proceed through the inferior part of her orbit KLG, she will still be referred from T to the same portion ABC of the zodiac, only she will appear to move in it in the contrary direction, from east to west, or from C through B to A. The real direction of her

her motion has, notwithstanding, suffered no alteration, and she has continued to describe her orbit round the sun in S without any interruption, and always according to the order of the signs: and it is the situation only of the earth, in a point without the orbit, which renders to a spectator in the earth an appearance of alteration unavoidable, and makes her seem to change her course from direct to retrograde. It is true that, while Venus performs these motions, the earth does not continue immoveable in T, but proceeds in her orbit also according to the order of the signs. But, as she proceeds more slowly than either Venus or Mercury, no difference will be produced, upon either the direct or the retrograde arches of those planets, except in some circumstances relating to their magnitudes, the times in which they are described, and the portions of the zodiac to which they are referred. From the contemporary motion also of the earth, it becomes evident why Venus and Mercury seem to move, in the superior parts of their orbits, with greater rapidity than in the inferior: for, in the former case, the real motion of the earth and the real motion of the planet being in opposite directions, the planet must seem to move with the sum of both velocities; and, in the latter, with the difference only of both velocities, because the direction of the real motions is the same. The phenomenon also of the double conjunction of the inferior planets, and their never coming to an opposition with the sun, and for which no reason appeared in the system of Ptolemy, was in that of Copernicus a necessary consequence.

Explication of the second inequality of the inferior planets,

68. The explication of the second inequalities of the superior planets is drawn from principles entirely similar. Let S (fig. 23), be the sun, ABC the

the orbit of the earth, DEF the orbit of the planet, GH a portion of the zodiac, E the planet supposed at rest; and let the earth be supposed to move, according to the order of the signs, in that part of her orbit which is between the planet and the sun. Here it is equally obvious, as in the former caſe, that during the progress of the earth from A to B, where the observer finds the planet in opposition, and from thence to C; that is, while the earth describes the arch ABC comprehended between two tangents drawn to her orbit from E, the planet must appear retrograde. For, when the earth is in A, the planet's place in the zodiac will be G, marked by the line AEG, drawn from the centre of the earth through E; when the earth advances to I, the place of the planet in the zodiac will be K; and it will appear in M, when the earth proceeds to L; and in H, when the earth arrives at C. But in that part of the earth's orbit which is most remote from the planet, or in the whole progress of the earth from C to N, and thence to the conjunction in P, and from P through O, till her return to A, the motions of the planet will appear direct: for NM is more advanced in the zodiac than CH, and OK than NM, and AG more advanced than all the three. It is true here also, that, during this progress of the earth through all the signs, the planet has not continued immoveable, but has advanced through some portion of its own orbit: but, as its progress is much slower than that of the earth, the only difference produced upon the appearances will relate, as before, to the extent of the retrograde and direct arches, the duration of their description, and the points of longitude in the zodiac by which they are terminated. The reason is also equally evident, why a superior planet, in describing its retrograde arch, will

and of the superior.

will seem to move with the difference only, and, in describing its direct arch, with the sum, of both velocities.

69. Between the direct and retrograde motions of the planet, there must be a time when the one changes into the other; and when, of consequence, the planet must appear stationary, and seem to partake of no motion except the diurnal revolution. If the earth continued at rest in the point T of her orbit, (fig. 24), the planet Venus, for example, would always appear stationary when she was found in any of the two tangents, as TK, which might be drawn from T to the planetary orbit: only the change of her motion from direct to retrograde would not be instantaneous, because there is a small portion of the orbit, towards K, which is to sense confounded with the tangent. But as the earth does not continue fixed, but moves in her orbit from T to D, Venus, though found in the tangent TK, will not immediately appear stationary, but seem to continue the course she held before she arrived at K, because the tangent itself alters its direction. In a short time, however, the visual ray MN, drawn from the earth to Venus, will cease to be a tangent to the orbit, and will meet it in another point O: yet, as long as the contemporary arches, TM and KN, described by the earth and the planet, shall be so situated with respect to each other, as to render the lines MN and TK, which include them, parallel, the planet will appear stationary, and be referred to the same point in the starry heavens. For the distance between all parallel straight lines, which can be drawn from the earth, in any of all her different positions, bears no sensible ratio to the immense distance of the celestial sphere. It is obvious that the duration of the stations

Stations of the planets.

stations will be longer, than if the earth had continued immoveable. (C.)

70. It was in this simple and convincing manner that Copernicus accounted for the second inequalities of the planets, by substituting the orbit of the earth for the three epicycles of the superior planets, and the two deferents of the inferior. By this alone the causes of all the circumstances and varieties of these inequalities were, in the most satisfactory manner, exhibited to view: whereas the ancient system shewed no cause, either of the phenomenon itself, or of the magnitude of the epicycle by which it was represented, or of the ratio between the number of the revolutions in the epicycle and the number of the revolutions in the real or supposed orbit.

Method of determining them.

In determining the times and points in which the planets became stationary, Copernicus, notwithstanding his new explication of their second inequalities, continued to employ the ancient method suggested by Apollonius (49), in which the orbits are considered as circular and concentrical to the sun: and it is evident that this method was alike applicable to his system as to the Ptolemaic. Indeed, the orbits are still considered as circular and concentrical to the sun in the greatest part of the modern solutions of this problem; and solutions, on such conditions, are supposed to be sufficiently exact for any useful purpose to which they can be applied. (D.)

71. An equal advantage of the system of Copernicus was, that it determined the mutual ratios of the distances of the planets, from the common centre of their motions; and consequently, it was no longer an aggregate of parts disproportioned and

and arbitrarily assumed. When a celestial body is referred to one point of the starry sphere from the centre of the earth, and to another from any station on its surface; that is, when it is subject to a diurnal parallax; it was known that this parallax was an indication of its distance from the centre of the earth being less than the semi-diameter of that sphere; and presented the means of comparing the distance with the semi-diameter of the earth. But the comparison was in general inaccurate, because the ratio of the semi-diameter of the earth to the distances of all the celestial bodies, except the moon, was so inconsiderable, that it was incapable of being ascertained by the ancient observations. But, by the annual motion of the earth, two stations were found whose distance was not less than the semi-diameter, or even the diameter of the whole terrestrial orbit; and there was no planet in the system whose place in the starry sphere did not appear to be changed, by viewing it from stations so remote. Let ʘ (fig. 27), be the sun, or rather the centre of the terrestrial orbit BAT; let T, M, J, S, be the Earth, Mars, Jupiter, and Saturn, in their several orbits; and let CDEF be a portion of the starry sphere. The differences of the places of these planets, called their annual parallaxes, as seen from T the centre of the earth, and from ʘ the centre of the sun, are the arches CF, CE, CD, of the zodiac; or, since the parallels TC, ʘG, when produced, mark the same point of the zodiac, the arches GF, GE, GD, which measure the angles GʘF, GʘE, GʘD, or their equals ʘMT, ʘJT, ʘST, under which the semi-diameter of the terrestrial orbit is viewed from every planet. It is indeed supposed, that one of the stations is in the centre of the orbit; but, from the theory of the first inequalities formed on op-

Annual parallax,

positions, the place of the planet, as viewed from the centre of the orbit, may at any time be calculated: nor, whether this centre be occupied by the earth, or by the sun, will there be any difference on the result of the calculation. The difference between the place of the planet, observed from the earth, and its place thus calculated for the time of the observation, is the required parallax: and, with the difference between either of these and the place of the sun, the ratio between the semi-diameters of the planetary and terrestrial orbits was given.

72. Copernicus, for example, determined in this manner the ratio between the distances of Saturn and the earth, from the sun. He observed the longitude of Saturn, on the 24th of February, 1514, at 17 h. in the line TSC, and found it = 209°. Then, either by his own theory of the first inequalities, or by that of Ptolemy (46), (for, except as to the supposed positions of the earth and sun, these theories chiefly differed in form,) he calculated for the same instant the longitude as seen from the sun, in the line ʘSD, or which would have been seen from the earth, in the point L of her orbit, and found it to be 203° 16′; that is, the heliocentric longitude was 5° 44′ less than the geocentric. The two visual rays, therefore, or lines TC and ʘD, proceeding from the earth and sun, intersected each other at the centre of Saturn, in an angle CSD, or TSʘ, of 5° 44′: and this angle, which in the Ptolemaic system was called the equation of the orbit, or of the argument, is the parallax. Having also the place of the sun in K, to which the place of the earth, as seen from K or ʘ, is always opposite, the difference between this and the heliocentric longitude of Saturn, gave him the

Angles of parallax, commutation, and elongation.

the angle S⊖T, called the angle of commutation; and which, in the present case, was = 67° 35′. Consequently, he had also, in the triangle S⊖T, the elongation, or angle ST⊖, = 106° 43′; and therefore the ratio of the distance T⊖, of the earth from the sun, to the distance S⊖, of Saturn from the sun, was given: for sin. TS⊖ : sin. ST⊖ :: ⊖T : ⊖S; and this ratio, supposing ⊖T = 1, is that of 1 to 9.6.

The procedure with respect to every other planet was the same: for the parallaxes ⊖JT, ⊖MT of Jupiter and Mars, as well as those of the inferior planets, are in like manner the differences between the observed, or geocentric, longitudes TJ, TM, and the calculated, or heliocentric, longitudes ⊖J, ⊖M, obtained from the theory of the first inequalities; and the commutations J⊖T, M⊖T, the differences between the heliocentric longitudes of the planet and the earth, as well as the elongations JT⊖, MT⊖, the differences between the geocentric longitudes of the planet and the sun, are given. On these principles, the distances of the planets in their order from the sun were found to be, to one another, as the numbers 4, 7, 10, 15, 52, 95; and though the investigation was not made with all the advantages afforded by the Copernican system, because the true form of the orbits was not yet discovered, and no consideration was had of their situation in different planes, it became the foundation of one of Kepler's most celebrated laws, that the squares of the periodical times of the planets are to one another as the cubes of their distances from the sun. The procedure indeed is not materially different from the procedure of Ptolemy, in finding the ratio between the semi-diameter of any single orbit and the semi-diameter of its epicycle. For the paral-

To find the ratios of the orbits to one another.

lax ⊙ST is the same, and found in the same manner, with the equation of the orbit S⊙H; and the commutation S⊙T, which is its argument, is the same with ⊙SH, the supplement of DSH, the planet's mean motion from the true apogee D of the epicycle; which was the argument of the equation of the orbit (48), and, to the astonishment of the ancients, found always equal to D⊙K, the sun's mean motion from the same apogee. But from Ptolemy's procedure, nothing more could be found than the ratio between a single orbit and its particular epicycle: whereas, in the Copernican system, the orbit of the earth being made the general epicycle, the mutual ratios of the orbits easily followed, and the reasons became manifest of the order in which they were arranged. It is obvious that all the circumstances of the second inequalities depend upon the annual parallaxes. The less the parallax, when the angles of commutation are equal, the less will always be the extent of the direct and retrograde arches of the planet, and the greater their number, during the course of every revolution in their orbits: and, as the distance of every planet from the sun is variable, upon account of the excentricity of their orbits, the magnitude of these arches will not be constant, nor the times of describing them the same.

73. Another no less signal advantage of the Copernican system was, that it led to a simple and satisfactory explication of the latitudes of the planets, and pointed out the reason of their being found very different, even in the same distances from the nodes. Ptolemy, to explain the latitudes of the superior planets, supposed a double inclination, first of the planes of the orbits to the plane of the ecliptic; and, secondly, of the epicycles to the

the orbits; and, for the inferior planets, besides the same double inclination, no less than three oscillatory motions. But, in the system of Copernicus, there was no necessity for such improbable suppositions. For, let ABCD (fig. 28), be the orbit of the earth, in the plane of the ecliptic EFG, and let KFL be the orbit of the planet Mars, for example, inclined to the ecliptic in the angle KFE, and intersecting it in the line of nodes FH, passing through the centre S of ABCD; and, through the same centre, draw ESG perpendicular to FH, in the plane of the ecliptic; and also KSL likewise perpendicular to FH in the plane of the orbit. A perpendicular KE drawn from the point K of the orbit to ES, or from L to SG, will mark its greatest elevation above, or depression below, the plane of the ecliptic: and the angle under which this perpendicular is seen from S will be the planet's heliocentric latitude, (because S is also supposed to be the centre of the sun,) and will be equal to the angle KFE of the inclination: and the only variations to which this heliocentric latitude is subject, arise from its gradual increase, as the planet moves from the ascending node H, where it is = 0, to the limit K, where it comes to its greatest amount; and from its gradual decrease, in the descent of the planet to the opposite node F. But the angle KTE is the geocentric latitude, and this, according to the Copernican system, is variable, not only on account of the variations of the heliocentric latitudes, between the limits and the nodes; but much more on account of the variations of distance between the planet and earth, produced by the revolution of the latter in her orbit. Even at the same point of the orbit N, the same heliocentric latitude MN is seen from T, under the angle MTN, and from A under a dif-

Heliocentric and geocentric latitudes.

ferent angle MAN; and these angles would be still more various at B, C, V, &c.

74. To be enabled at any time to calculate the latitudes, whether heliocentric or geocentric, it was necessary to determine the inclinations of the planetary orbits to the ecliptic: and Copernicus, for this purpose, employed the following method. He determined, by some theory of the first inequalities, the excentricity of the planet's orbit, and the place of its aphelion; whence, as he supposed every orbit to be circular, the distance of the planet from the sun, in the centre of the ecliptic, could be easily calculated for any given point of anomaly; and having also the ratio between this distance and the semi-diameter of the terrestrial orbit (72), together with the observed latitudes; all the conditions necessary for the solution were given. But his success in this part of his undertaking did not correspond to the justness of his principles; for his observations of latitude were chiefly taken from Ptolemy, and his deference for this ancient astronomer prevented him from perceiving that they were not always to be depended upon. Ptolemy had, for example, transmitted an observation of Mars, in opposition at his perihelion, in which his latitude, observed from the earth, was 6° 50′ south; and, to deduce hence the inclination of the orbit, Copernicus thus proceeded.

To determine the inclination of the orbit.

Let A (fig. 29), be the centre of the ecliptic, that is, the mean place of the sun, B the place of the earth, C Mars in opposition, and CBD the observed latitude. Then, since AC the distance of Mars from the centre of the ecliptic was given, by the theory of the first inequalities, in parts of the semi-diameter of the planet's own orbit; and, consequently, from the determined ratio of the orbits, was

was found = 1370 in parts of AB = 1000, and the angle CBD was given; he had this analogy for the angle ACB, AC : AB :: sin. CBD : sin. ACB = 4° 59′; and therefore the inclination CAB = CBD − ACB was found to be = 1° 51′, and, as is now ascertained, very nearly accurate. Copernicus also found this result to be in some degree confirmed by another observation transmitted by Ptolemy of the planets north latitude: for producing the line of apsides CA to F, where FA is = 1670 in parts of AE = 1000, and, with these lines and the angle EAF of inclination now determined, calculating the angle EFA, he found it = 2° 40′; whence the geocentric latitude, when the planet, in aphelion, is in opposition to the earth in E, ought to be 4° 31′ = FEG: now Ptolemy tells us, that he observed this to be = 4° 20′. But, by another observation of Ptolemy's, near the conjunction in the perihelion, from which he deduced that the south latitude CED, if observed in the conjunction, would be no more than 5′, it was found, by resolving the triangle CAE, that the inclination CAD was no greater than 9′; and by a like procedure, founded on Ptolemy's observations, the results for the inclinations of Saturn and Jupiter, deduced from oppositions in the limits, were 2° 44′, and 1° 42′; but, from conjunctions in the limits, only 2° 16′, and 1° 18′. These results Copernicus considered as sufficient evidences that the inclinations of the orbits to the plane of the ecliptic were variable; and, as no variation could be ascribed to the plane of the ecliptic, he supposed the planes of the orbits to oscillate backwards and forwards, each on its line of nodes; and the oscillation of the orbit of Mars was supposed to be through an arch, or angle, of 1° 41′, of Jupiter through 24′, and of Saturn through 28′;

and in such a manner, that the inclination always increased in proportion to the increase of anomaly. Nor was his deference for Ptolemy the only cause of concealing from him the full value and important consequences of his principles; but the two unhappy preconceived opinions, that every orbit was perfectly circular, and that the planes of the orbits intersected the plane of the ecliptic in lines passing through its centre, contributed equally to the same effect: for by these means all the distances, as AF and AE, whether of the planet or of the earth from the sun, which entered into his investigation, were false. It is evident, that the system of Copernicus afforded an equally simple and satisfactory explication of the various latitudes of the inferior planets: but the same causes operated here also, and prevented the just use and application of its principles. His theory for the latitudes of Venus and Mercury, was even more complicated than for the latitudes of the superior planets; and, except in name and form, was very little different from that of Ptolemy. Nor was he more successful in his deductions of the longitudes and motions of the nodes; for, in the orbits of the inferior planets, he considered the line of nodes as coincident with the line of apsides; and in the orbit of Mars it was perpendicular to the line of apsides; and, in all orbits, he gave no different motion, either to their apsides or to their nodes, except the apparent motion of the fixed stars, on account of the regression of the equinoxes.

75. Copernicus seems, like Ptolemy, to have thought the inclinations of the superior planets so inconsiderable, that reductions from the planes of their orbits to the plane of the ecliptic might be neglected as unnecessary; and, accordingly, in his investigation

investigation of the distance of Saturn from the sun, (72), no reduction was employed; but he saw the necessity of them in the orbits of the moon and the inferior planets, for that otherwise their calculated and observed places would not in general coincide. His methods, with respect to Venus and Mercury, were of necessity equally complicated with his theory of their latitudes; and, as they were not required by his system, need not be repeated; but, with respect to the moon, they were of the same simple kind which are now employed. Let ABC (fig. 30), be the orbit of the earth, and *a*BP the orbit of the moon, or of any planet, intersecting the plane of the earth's orbit, that is, of the ecliptic, in the line of nodes BD. If a line PC be drawn from the planet in P perpendicular to the plane of the ecliptic, it will represent the latitude of the planet, geocentric if seen from the earth, and heliocentric if seen from the sun in S, supposed to be the centre of the ecliptic; for it is the sine of the angle PSC, or of the arch of a circle of latitude by which PSC is measured; and the inclination of the orbits will be the angle CBP. The arch BP of the orbit, computed *in consequentia* from the ascending node B, is called the argument of latitude; because, when the angle CBP of inclination is given, it is the condition, or circumstance, on which the magnitude of the angle PSC of heliocentric latitude depends. When the inclination CBP, and this argument of latitude BP are given, the arch BC, or the distance of the planet from the node reckoned upon the ecliptic, may be found by this analogy R : cos. CBP :: tan. BP : tan. BC; and the difference between BP and BC is the required reduction, subtractive from BP in the first and third quadrants

Reduction to the ecliptic.

Argument of latitude.

quadrants of anomaly, and additive to it, in the second and fourth. (E.)

Vicissitudes of the seasons.

76. The vicissitudes of the seasons are generally explained, in the Copernican system, by supposing that, while the inclination of the axis of the equator to the plane of the ecliptic is the same as in the Ptolemaic system, its positions in the annual translation of the earth through all the signs of the ecliptic continue sensibly parallel to one another; by which means all the circles parallel to the equator, are daily intersected in variable proportions, by the great circle on the earth's surface which separates light and darkness; and several of them are generally found wholly within, or wholly without its limits. But the effects produced by this invariable direction of the axis of the equator, were explained by Copernicus himself in a less simple manner; viz. by ascribing to the earth an annual rotation upon the axis of the ecliptic; or more strictly speaking, upon an axis of its own perpendicular to the ecliptic. For he supposed the annual revolution of the earth round the sun to resemble the menstrual revolution of the moon round the earth, or that of a ball fixed at one extremity of a lever, the other extremity of which turns round upon a pivot; and seems to have considered this as the only method in which a revolution about a centre could be performed. Consequently it became necessary for him, in explaining the vicissitudes of the seasons, to ascribe also to the earth the annual rotation now spoken of, and performed in a contrary direction to the annual revolution: and otherwise, if we abstract from the diurnal rotation, only one hemisphere of the earth would have been presented to the sun, and

and it would have been presented to him invariably; while the opposite hemisphere was involved in perpetual darkness.

77. But while the merits of Copernicus were so great, in these essential and distinguishing parts of his system, which related to the second inequalities of the planets, all their first inequalities, and all the inequalities in general of the sun and moon, remained in their former obscure and imperfect state; and all the explications of them, which he attempted to give, differed from those of Ptolemy chiefly in outward form. A minute detail of his whole doctrines on these subjects would be unnecessary, and it will be sufficient to produce some of the most remarkable.

78. Copernicus was dissatisfied with Ptolemy's lunar theory, chiefly on two accounts; the first arising merely from the ancient prejudice in favour of uniform motion in perfect circles, with which the explication of the second lunar inequality was inconsistent, (39); but the other unanswerable and decisive, that the variations of the moon's distance, which this explication rendered necessary, were irreconcileable with all observations, both of her diurnal parallax, and her apparent diameter, (40). His own lunar theory, therefore, was to this purpose. Let A (fig. 35), be the centre of the earth, and of the lunar deferent BCD; let C be the centre of the epicycle EFGH, carried about in the circumference of the deferent; let E be the centre of a smaller epicycle KLM, carried about in the circumference of the greater, and let AC be the line of the mean motion of the sun, or earth. Copernicus supposed, that C the centre of the greater epicycle moved *in consequentia* from C to D, with a velocity

Lunar theory of Copernicus.

velocity equal to that of the moon's mean motion in the zodiac; from which, subtracting the apparent mean motion of the sun, the remainder CD was the mean motion of the moon from the sun; so that her return to the line AC was made in the time of a synodical revolution. But the centre E of the smaller epicycle was supposed to be carried from E, the apogee of the greater, towards F, G, H, with a motion equal to that of the moon in mean anomaly; while the moon was carried in the circumference of the smaller, from K its perigee, towards L, M, with a velocity double of her rate of mean motion from the sun. By these means he supposed that the first and second lunar inequalities would be justly represented: for, supposing both to disappear at an opposition to the sun in C, and that then the centre C of the greater epicycle, the centre E of the smaller, and K the centre of the moon, were in the same right line AC, it would happen that when the centre of the greater epicycle came to the quadrature in D, and the centre of the smaller one to F, the moon herself having a motion double of CD would be found in M, having described the whole semicircle KLM; and, joining MA, the equation, or angle MAD, would be the greatest possible, and amount to 7° 40′. But in other situations of the centre E of the smaller epicycle at the opposition, and which would necessarily take place, on account of the difference between the synodical and anomalistical revolutions, the place both of it and of the moon at the quadrature might be less remote from D, than E and M in the present figure, and the equation would be liable to all possible varieties, from 7° 40′ to 0° 0′. To represent the first inequality separately from the second, the smaller epicycle was supposed to describe with its perigee K the circle

circle KN; to this a tangent AN was drawn from A, and the angle KAN, which never exceeded 5°, was the first inequality; while CAO, formed by a tangent drawn from A to the circle MO described by the apogee of the smaller epicycle, represented both inequalities, and amounted to 7° 40′. It will however be understood, that though both are represented at the opposition in C, they could not meet together and be accumulated, except at the quadratures in B and D. Hence it was easy to find the semi-diameters of the epicycles, at least their ratio to the semi-diameter of the lunar orbit; for, in the right angled triangles CNA, COA, CN is the sine of 5° and CO the sine of 7° 40′ to the same radius CA = 100000, and consequently CN = 8716, and CO = 13341. Their difference NO is = 4625; so that FN, the semi-diameter of the smaller epicycle, is = 2312, and CF, the semi-diameter of its deferent, = 11028. This was the principal part of the lunar theory of Copernicus, nearly as complicated as that of Ptolemy, and liable also, like his, to various objections. One in particular was made by Kepler, that it rendered the distance MA of the moon, in quadrature from the earth, sometimes greater than the semi-diameter of the orbit, though all observations concurred to make it less.

79. Nor was Copernicus more successful in his explications of the first inequalities of the planets. In ascertaining the mean motions and revolutions of the superior planets, his procedure was the same with the ancient one; for, whether the centre of the system were occupied by the sun or by the earth, the place of a planet in opposition viewed from either was the same. But, as Copernicus considered Ptolemy's representation of the first

first inequalities, by an excentric with an equant, to be a trespass against the law of uniform circular motion, he returned, as was mentioned, to the more ancient concentric theory with its epicycles, which Ptolemy had abandoned; and one of his explications was to this purpose. Let A (fig. 36), be the sun, or centre of the ecliptic, BCD the orbit of a superior planet, and BD the line of apsides. On B, with a semi-diameter BF equal to $\frac{3}{4}$ of Ptolemy's excentricity of the equant, describe the greater epicycle FGHK; and on F, its apogee, with the semi-diameter FN equal to the remaining $\frac{1}{4}$, describe the smaller epicycle LMNO. Copernicus supposed, nearly as in his lunar theory, that B the centre of the greater epicycle, was carried round the orbit *in consequentia*, from B, to C, D; that F, the centre of the smaller, was, in the same precise time, carried round the circumference FGHK, from F to G, H, K; and that the planet moved with a double velocity in the circumference LMNO, from L to M, N, O; and it was also supposed that, when the centre F was in the apogee of the greater epicycle, the place of the planet was in L, the perigee of the smaller. By these means, when the centre B advanced to C 90° distant, the centre of the smaller epicycle was found in G; but the planet in N, having described the whole semi-circle LMN; and the equation NAC, which at 90° of anomaly is subtractive from the mean longitude AC, would come to its greatest amount. When again the centre B, or C, had described another quadrant, and was brought to D, and the centre of the smaller epicycle, having also described another quadrant, was found in H the perigee of the greater; the planet had moved through its other semicircle NOL to L the apogee of the smaller epicycle, and therefore the equation would disappear.

Theory of the first inequalities of the superior planets.

disappear. This theory, as has been said, hardly differs from that of Ptolemy, except in outward form. No new excentricity is introduced, and the mutual ratio of the epicycles is equivalent to the ancient bisection; for, if from L the perigee of the smaller epicyle we take in LA the part LP = AB, the part AP will be precisely equal to half the excentricity of Ptolemy's equant; and if on P, with the semi-diameter PL, we describe a circle, this will be the planet's path in the heavens according to the excentric theory, and nearly its path according to the present theory of Copernicus. Indeed it was the object of Copernicus rather to confirm Ptolemy's explication of the first inequalities, than to depart from it; and to remove from it the supposed absurdity of assigning to the planets a motion which was uniform only in a fictitious orbit. But his explication was much more complicated than that of Ptolemy; and, though the motions in all his three circles were uniform, the real path of the planet was no longer circular, but extended beyond the circle in all points except the apsides; for example, towards 90° of anomaly, the line PN, drawn from P the centre of the Ptolemaic orbit, to the planet, is the secant of the angle LPN. The uniformity attained was therefore merely imaginary.

80. Another explication given by Copernicus of the first inequalities of the superior planets was less complicated, and in form more resembling the Ptolemaic. It was by means of an excentric orbit FCDE (fig. 37), the distance AB of whose centre, from A the centre of the ecliptic, was equal to the semi-diameter of the former greater epicycle; and he only retained the smaller epicycle, in the circumference of which the planet was supposed to move,

move, with half only of the former velocity, but in the same direction. Two uniform equal motions were, by this theory, substituted for the single unequable motion of Ptolemy; the one of the centre of the epicycle round the centre of the orbit, and the other of the planet round the centre of the epicycle: but it is evident that this theory was in all respects equivalent to the former. In neither of them, did Copernicus so much attempt to investigate the excentricity and place of the aphelion, as to examine the conclusions of Ptolemy respecting these elements; and to make trial how far the equations deduced from them agreed with observation. In this examination his procedure was to this purpose. Having three mean oppositions, that is, three observations of a planet when the centre of the epicycle was in the points D, E, F, and the planet in the points G, H, K, he either adopted Ptolemy's determinations of the excentricity AB, and the longitude ABC of the aphelion; or he approximated to them in the manner of Ptolemy, by supposing CDEF to be the orbit described by the planet, and not by the centre of the epicycle (45). With these elements, thus rudely found, he had first, in the triangle FBA, the excentricity AB in parts of FB = 100000, and the angle FBA, known by the motion of the centre of the epicycle from C, the supposed place of the aphelion; and could therefore find the base FA, and the angles AFB, FAB; and secondly, in the triangle FAK, he had the angle KFN = FBC, and therefore KFA = KFN + AFB, and consequently, with the sides KF, FA, the angle FAK; which, subtracted from FAB, gave the distance KAC of the planet from the aphelion; and afforded one part of the evidence by which he was to judge how far his conclusion, or assumption, for the

Another theory.

the place of the aphelion erred from the truth. In the same manner, by means of the triangles DBA, DAG, he calculated the distance GAC of the planet from the supposed aphelion, at the time of the opposition in D; and, by means of the triangles EBA, EAH, the distance HAC of the planet from the aphelion, at the time of the opposition in E. After these operations, by adding together KAC, and GAC, and subtracting GAC from HAC, he obtained the angles KAG, GAH; and, these being found equal to the observed differences of longitude in Ptolemy's three oppositions, he concluded that, as far as the observations could be depended upon, Ptolemy's determinations of the excentricity and place of the aphelion were just. But, on applying the same elements to some oppositions observed by himself, he found that the calculated differences of longitude did not agree so perfectly with his observed differences; and he was therefore obliged to make some alterations of the Ptolemaic suppositions: these alterations were especially found necessary in the theory of Mars; where, instead of fixing AB at 1500, in parts of BC = 10000, that is, at $\frac{3}{4}$ of Ptolemy's excentricity of the equant, he fixed it at 1460, while the semi-diameter of the epicycle continued at 500; so that, in his calculations for this planet, he did not precisely adopt the principle of the bisection.

Excentricity and place of the aphelion.

In the former theory, the manner of the calculation would have been different. Let E (fig. 36) be the centre of the greater epicycle in any one of the oppositions, and F its apogee: and let Q be the centre of the smaller epicycle. Then, on account of the equal motion of both centres, FEQ = BAE; and, on account of the double velocity of R the planet, EQR will be either = 2FEQ, or the suppl. of 2FEQ to 180°; and therefore,

The same in the first theory.

 adopting

adopting Ptolemy's elements, varied to suit the concentric form, the points Q and R are given. Join AQ, AR, RQ. Then, in the triangle AEQ, with the sides AE, EQ, and the included angle AEQ, the angles EQA, EAQ, and the base AQ may be found: and, in the same manner, in the triangle ARQ, with AQ, QR, and AQR = EQR − EQA, the angle QAR. Consequently the whole angle EAR is given; and, subtracted from EAB, gives the required distance RAB, from the aphelion B.

81. The Copernican system produced an alteration in the manner of considering the mean motions of the inferior planets; for it represented their only motions in longitude to be performed in what the ancients called their epicycles. The mean time therefore of a revolution in the ancient orbit was not now required, it being the same with that of a solar revolution, but only the time of a revolution in the epicycle. But though the manner of considering the motions of the inferior planets was simplified, the investigation of them continued to be attended with the same difficulties as in the ancient system; and, till telescopes were invented, the greatest digressions continued to be the only means of determining the times of their revolutions. According to the principles of the Copernican system, a right line CE, (fig. 38), drawn to the planet from the earth in C, at the time of a greatest digression, is a tangent to the orbit: and, as it is also perpendicular to a right line BE drawn to the planet from the sun in B, the parallax BEC must at this time be = 3 signs, or 90°. If then the planet's motion were circular and uniform, and the sun in the centre of the circular orbit; and if also the instant of the greatest digression could

Mean motions of the inferior planets.

could be exactly observed, the planet seen from C, in two successive greatest digressions of the same kind, and in the same point of geocentric longitude, would be precisely in the same point of its heliocentric orbit; and the interval would be the precise time of its revolution round the sun. Even, if the observations of two such digressions had been made in distant ages, from the same point of the earth's orbit, the time of any one revolution might be ascertained, by the knowledge of the whole number of revolutions performed in the interval. But none of these required conditions takes place: for neither is the planet's motion uniform, nor its orbit a circle whose centre is occupied by the sun; nor, since the tangent and the orbit are, for some time about the greatest digression, sensibly coincident, can the instant of any one be observed with precision. Yet, by employing the observations of distant ages, and thus distributing the errors arising from those causes among a great number of particular revolutions, the method has been applied with considerable success to the motions of Venus, and the time of her heliocentric revolution determined with considerable accuracy. Cassini's procedure in this investigation was as follows. He compared an observation of a greatest digression, made by Ptolemy in the year 136, and which he found must have taken place at Paris on the 25th of December, at 4 h., with another made by Julius Byrgius, in 1594, and which also took place at Paris on the 17th of December, at 4 h. 30′. The geocentric longitude of the planet, in the first of these observations, was 1 s. 20° 13′ 45″, and in the second 1 s. 23° 1′ 36′, differing 2° 47′ 51″. As in both observations the right lines drawn from the earth and the sun to the planet, are supposed to be perpendicular to each other, the same must have been the difference between their heliocentric longitudes: and from a ruder determination of the motions of Venus, deduced from less distant observations, Cassini found that the time, in the year 1594, when the heliocentric longitude was the same as on the 25th day of December 136, at 4 h. must have been, December the 15th, at 10 h. 36′. In this interval he knew from his ruder determination, that Venus had made 2370 revolutions in her orbit: and, therefore, dividing the interval by 2370, he found that the time of each consisted of 224 d. 16 h. 39′ 4″; and consequently, that the mean annual motion of the planet in longitude was 19 s. 14° 47′ 45″, that is, 7 s. 14° 47′ 45″ more than a compleat revolution; and the mean diurnal motion 1° 36′ 8″. But the application of this method to Mercury was, on account of the greater excentricity and the difficulty of observing this planet, attended with less success; indeed hardly with any, till Kepler, by remarking that the time of a greatest digression must be also the time when a telescope shews the disk of the planet accurately dicholomized, and, happily distinguishing such instants, attained to a determination of the time of a revolution much more precise than he had presumed to hope. The more accurate method of modern astronomers of determining those mean motions, is by observations of conjunctions, and especially of the transits of the inferior planets over the solar disk. By these methods the time of a tropical revolution of Venus is now fixed at 224 d. 16 h. 40′ 30″.6, and of Mercury, at 87 d. 23 h. 14′ 34″.4.

82. In determining the place of the aphelion for an inferior planet, Copernicus did nothing more than adopt the methods used by Ptolemy to the forms

forms of his own system. Let A be the sun in the centre of the terrestrial orbit CD, B the centre of the orbit of Venus, and AB its line of apsides produced to meet the earth's orbit in C and D. From the same reasonings which Ptolemy employed, Copernicus also supposed that, if two greatest digressions ECB, GCB, in opposite directions, were observed from C to be precisely equal, the line of apsides must be BC bisecting the intercepted arch GE of the orbit, and passing through the centre of the earth in C; and if from the opposite point D of the earth's orbit two other greatest digressions BDF, BDH should also be observed precisely equal, but different from the former, the line of apsides would again bisect the intercepted arch FH, and pass through the centre of the earth in D. If therefore the latter digressions were greater than the former, the point of bisection L was the planet's aphelion, where it was most distant from the sun, and nearest to the earth; and if they were less, the point K of bisection must have been the perihelion. If, again, the opposite equal digressions should be of their mean magnitude, the earth must have been in a line perpendicular to the line of apsides; and in both cases the longitude of the apsides was given by means of the longitude of the earth. In this investigation Copernicus employed not only the principles, but the observations of Ptolemy, and concluded as he had done, that the longitude of the aphelion of Venus was 1 s. 25°, and of Mercury 6 s. 10°.

Aphelia of the inferior planets.

Hence the excentricity of the orbit easily followed; for, if the digressions towards the opposite apsides were of the greatest and least kinds, he drew from these apsides to the orbit the tangents CE, DF perpendicular to the semi-diameters BE, BF, and consequently the lines BC, BD were the se-

Excentricities of the orbits.

cants of the angles BCE, BDF to the radius BE. Half their sum is AC, the semi-diameter of the orbit of the earth: and assuming this = 100000, BE, the semi-diameter of the orbit of Venus was = 7193, and AB, its excentricity, = 208; and the semi-diameter of the orbit of Mercury = 3773, and its excentricity = 948.

83. Copernicus also employed the ancient method of investigating the position of the centre of uniform motion in an inferior orbit; and of determining if it was different from the centre of the orbit, which he had now found. Let AB (fig. 39), be the line of the apsides of Mercury, F the centre of his orbit, and E the earth in a line EG perpendicular to AB; though no more than in Ptolemy's investigation was this absolutely necessary, it being sufficient if the angle AGE was given: it is required to find the point G, in which the centre of the planet's orbit ought to be situated, in order to make the digressions on both sides of it appear equal to a spectator at E; that is, the distance GF of this point from F the real centre of the orbit, and if it differed from the excentricity AB found by the former investigation.

Here Copernicus employed two of Ptolemy's greatest digressions, the one CEG = 20° 15′, and the other DEG = 26° 15′. Their sum gave the whole angle CED = 46° 30′, and bisecting this by the line EF, which meets the line of apsides in the real centre F, the angle CEF was found to be = 23° 15′. Therefore GEF = CEF − CEG = 3°, and consequently GF the tangent of this angle to the radius EG = 100000, was found = 524. This was little more than the half of 948, the excentricity of the orbit: so that in Ptolemy's language this excentricity was bisected by the centre

and of the point of uniform motion.

centre of the equant; and, contrary to what happened in the case of the superior planets, the orbit at the apsides extended beyond the equant. On the contrary, the distance of the centre of the uniform motion of Venus from the centre of her orbit was found, by a similar procedure, to be = 416 parts of EG; that is, to be double of 208, the excentricity of the orbit: and consequently the same bisection took place in her theory, which had been introduced into the theories of the superior planets.

84. The substitutions which Copernicus made for the Ptolemaic equant, in the theories of the inferior planets, were different from what he employed in those of the superior. Instead of supposing the inferior planet to move in an epicycle carried round in the circumference of an excentric orbit, (80), he supposed the centre of the orbit to revolve in what he called an hypocycle. Let A, (fig. 40), be the centre of the terrestrial orbit BCDE, let the diameter BAD coincide with the line of the apsides of Venus, in its mean situation, and let the diameter CAE be perpendicular to BAD. Let also AF = 216 be the mean excentricity of the orbit of Venus, and AG = 416, what Ptolemy called the excentricity of her equant. If FG be bisected in H, and on H, with the semidiameter HF, a circle FLGK be described, this will be the hypocycle in which the centre of the orbit is supposed to revolve. The law of the revolution is, that when the earth is in B, the longitude of the aphelion of Venus, the centre of the orbit is F the perihelion of the hypocycle; when Hypocycle. the earth has revolved to C, the centre of the orbit having a double velocity is found in G, the aphelion of the hypocycle, and has described the whole

 semi-

semi-circle FLG; and when the earth has proceeded to D, the centre is again found in F, having described the other semi-circle GKF. Thus the physical part of the first inequality, which Ptolemy made to arise from the distance between the centres of the orbit and equant, was explained as effectually in the case of the inferior planets by means of this hypocycle, as in the case of the superior by means of the epicycle LGO (fig. 37); for, whether the planet shall describe such an epicycle in the circumference of its orbit, or the centre of the orbit shall describe the hypocycle FLGK, the appearances to a spectator from the earth will evidently be the same: and thus also the second inequalities were much more diversified than if this hypocycle had not been employed; for, when the earth was in D, the longitude of the planet's perihelion, the greatest digression was MDA of the least kind; when the earth was in C, 90° from D or B, it was NCA much greater; and the greatest of all the digressions OBA took place when the earth was in B the longitude of the planet's aphelion, that is, of its perigee.

Calculations.

The calculations were much the same with those formerly exemplified. Let the earth be in the given point Q, and the position of P the centre of the planet's orbit will be also given, for FHP = 2DAQ. Join QA, QP, AP; and draw RS parallel to QA. 1. In the triangle AHP, with the given sides AH, HP, and the angle AHP, the excentricity AP, and the angle PAH, are given. 2. In the triangle PAQ, with the sides AP, AQ, and the angle QAP = QAH — PAH, the side QP, and the angle PQA = QPR are given: and QPR is in Ptolemy's language the equation of the centre, and the point R the mean perigee of the epicycle (47). 3. The arch RT, or angle RPT, is is the motion of the planet from the mean perigee of its orbit, or epicycle, and consequently given: and, if with this and the angle QPR, we form the angle QPT, we may thence, having the sides PQ, PT, find the equation of the orbit PQT.

85. The same principle, with some variations, was employed in the orbit of Mercury. In this AE, (fig. 41) = 948 is the excentricity of the orbit, and AF what Ptolemy called the excentricity of the equant, EF is bisected in G, and on this centre, with the semi-diameter GE = 262, the hypocycle is described. In this the centre of the orbit was supposed to move *in consequentia*, with a velocity double that of the earth, and in such a manner that, when the earth is in B the longitude of the aphelion, the centre of the orbit is in E, and when the earth has described the quadrant BC, the centre of the orbit is found in F. But this representation was not sufficient to display all the irregularities of Mercury; and therefore Copernicus added an epicycle NM nearly equal to EF, and supposed the planet, not indeed to move in its circumferences, but to librate in its diameter, according to this law, that when the earth was in B, the planet should be in M, the perihelion of the epicycle; when the earth had described the quadrant BC, the planet should be in N the aphelion; and that it should return to M, against the time that the earth was diametrically opposite to B. By these means the orbit was not only displaced, according to the different positions of the earth, but even its magnitude was varied: for, when the earth was in the line of apsides, as at B, the semi-diameter of the planet's path was EM less than EH, and its digressions especially of the earth were at the point opposite to B, the least possible. When

Theory of Mercury.

When again the earth was in C, the semi-diameter of the planet's path was FN, greater than FH = EH, and therefore the digression NCA greater than the former, both on this account, and also because the earth is at a less distance from the orbit. By this representation Copernicus also explained the phenomenon which, according to Ptolemy, was so singular in the orbit of Mercury; that the greatest of all the digressions should take place, not when the earth is in B the longitude of the planet's aphelion, nor yet at C where the semi-diameter FN of the planet's path is greatest; but when the earth is in the point K, or L, 60° removed from B. For, as the motion of the centre of the orbit is double to that of the earth, its place at that time in the hypocycle will be O, found by taking the angle EGO = 2BAK; and the planet's place in the diameter of the epicycle will be P, found by taking MHR = EGO, and drawing from R to MN the perpendicular RP; so that the digression becomes PKA, greater than NCA, though FN be greater than OP, and even greater than MBA at the planet's perigee.

86. To avoid the prolixity produced by this double epicycle, Copernicus afterwards employed a different representation. Let A, (fig. 42), be the centre of the circle BCD, divided into quadrants by the diameters BD, CC, and on the same centre with a semi-diameter double of GF in the former figure, describe the hypocycle GQF, meeting BD in F and G. Also on F, with the semi-diameter FH = AB, describe the circle HEK, and on G the equal circle MNO. Suppose that these two last circles move *in consequentia* round the centre A of BCD, with a velocity equal to the difference between the velocity of Mercury and the earth; that

Another theory.

that Mercury moves from H *in consequentia*, with a velocity equal to that of the earth ; and that the centre of HEK is, in the time of an annual revolution, first carried from F upwards to G, and thence back again to F. By these means, when the centre is in F, and Mercury in H, the planet, since its orbit revolves on the centre A, will appear as if it described the smaller circle HLO, and its digressions will therefore be of the least kind ; and they will also be of the least kind, when the planet is in O, and the centre of the orbit in G. But when the planet comes to C, 90° from H or B, and the centre of the orbit HEK to A, it will appear to describe the circle BCD, and its digressions will be the greatest possible. The planet however, though at the distance of 90° from H, is not therefore 90° distant from the apsides of its orbit, because these apsides revolve also ; and the difference between their velocity and that of the planet, will make the greatest digressions take place, as the observations were believed to require, when the planet is at the distance of 60° from the aphelion.

87. The principal cause which engaged Copernicus in all this waste of labour and ingenuity, was the unhappy prejudices so often mentioned in favour of uniform and circular motion, joined to his excessive deference for Ptolemy. This deference was indeed the sole cause of that most vexatious part of his labour, which regarded the motions of Mercury : for his situation on the low and foggy banks of the Vistula, rendered it impossible for him to make observations himself on that excentric planet ; and he had not presumed, like Kepler, to suspect that Ptolemy, instead of deducing his theory from observations, had sometimes corrected the observations to suit the theory. When he introduced

introduced therefore such a complication of circles, and perplexed compositions of their motions, to account for the singular phenomenon of Mercury's greatest digressions taking place at 60° on each side of his least distance from the earth, it was because he had no opportunity of discovering that the phenomenon was imaginary; and the observations, from which it was inferred, either inaccurately made, or unfaithfully transmitted.

Copernican system unfavourably received.

88. The system of Copernicus was not received, on its appearance, with any degree of that approbation which it deserved, and which it now universally obtains. Its cold reception, indeed, fully justified the hesitation and tardiness of its author, to communicate it to the world. Yet, his want of success in explaining the latitudes and first inequalities of the planets in longitude, and the intricacy of his theories on these subjects, were not the principal causes of rejecting his opinions. On the contrary, those were the parts of his labours which, on their first publication, were chiefly valued: and his theory of Mercury, especially, notwithstanding its being encumbered with more epicycles than his explication of the second inequalities had banished, excited the admiration of many eminent astronomers. But his system was chiefly opposed, on account of all in it that was valuable and distinguishing: and the substitution of the diurnal and annual motions of the earth, for the apparent diurnal revolution of the heavens, and the annual motion of the sun, was such a violent contradiction, both of the philosophical principles of the age, and the immediate evidence of sense, that all its advantages were undervalued, and proved insufficient to procure to it general credit. The conception of Copernicus, which represented the

the distance of the fixed stars from the sun to be so immense, that in comparison with it, the whole diameter of the terrestrial orbit shrunk into an imperceptible point, was too great to be adopted suddenly by men accustomed to refer all magnitudes to the earth, and to consider the earth as the principal object in the universe. Instead of being reckoned an answer to the objection against the annual revolution of the earth, that her axis was not found directed to different stars, it was rather considered as the subterfuge of one who had invented, and therefore tried to vindicate, an absurdity: and, when in answer to another equally powerful objection, that no varieties of phase were seen in the planets, especially in Venus and Mercury, Copernicus could only express his hopes that such varieties would be discovered in future times, his reply, though it now raises admiration, could not in his own times make the least impression on those who opposed his system. The earth was universally supposed to be so immense and ponderous as to be incapable of any kind of motion: and the diurnal rotation, in particular, was thought to be decisively confuted by the consideration of centrifugal force; which would throw off all bodies, animate and inanimate, from its surface. These objections, and many others of no force in themselves, but in that age deemed irresistible, by reason of the low state of human knowledge, prevented the Copernican system from being generally considered in any other light than as a mere hypothesis, and were the principal causes of the celebrity for some time maintained by the system of T. Brahé.

CHAP.

CHAP. IV.

Of the System of Tycho Brahé.

89. THIS celebrated astronomer was a Dane, of a noble family in Schonen, a province now subject to Sweden. His strong and early propensity for astronomical studies, the opposition given him by his family in indulging it, his perseverance in these studies so as to devote to them his whole time and fortune, and the truly royal encouragement given him by Frederick, the first king of this name in Denmark, are very generally known. This enlightened prince conferred on him the island of Wien, or Huen, at the entrance of the Baltic, as a proper and undisturbed retreat for his observations; built for him a castle distinguished by the name of Uraniburg, erected an observatory, and contributed to the expence of his astronomical instruments and assistants. In this sequestered place he employed himself for fifteen years, from 1582 to 1598, in those assiduous observations of the celestial bodies, which have been the principal foundations both of his own fame, and of the whole of modern astronomy. But, after the death of his patron, his labours suffered the most mortifying interruption; and the persecutions he experienced, not only from envy, but from bigotry, to which science, in all its branches, is a constant object of jealousy, obliged him to abandon his native country. He found, however, a new patron in the Emperor of Germany, Rhodolph the second, and an asylum at Prague under his protection: and, assisted by Rhodolph's liberality, he resumed his studies

studies with his former ardour. But a final period was soon after put to them, by his death, in the year 1601, of an acute distemper, at the age of 55.

90. It was impossible for T. Brahé, with his abilities and advantages, not to perceive the incomparable superiority of the Copernican system to the Ptolemaic, both in the arrangement of the planets, and in the simple and satisfactory explication which it gave of their second inequalities. What Copernicus assumed from more doubtful evidence, and perhaps partly from theory, T. Brahé found to be ascertained by indubitable observations, that Venus and Mercury were not always less distant than the sun from the centre of the earth, as in the system of Ptolemy, but that they were frequently much farther distant; and that Venus sometimes approached much nearer to the earth than Mercury. The places therefore assigned to them by Ptolemy could not possiby be just; nor, however it might be contended that the earth was the centre of the solar revolutions, could it with any propriety be reckoned the centre of the revolutions of the inferior planets: and the representation of these, given in the ancient Egyptian system, was found to be unquestionable, that as they constantly attended the sun, and frequently crossed his apparent orbit, the sun was the centre about which their motions were performed. But T. Brahé's observations also demonstrated, that the planet Mars sometimes approached nearer than the sun to the earth, and was found within the solar orbit, and sometimes removed to a much greater distance; and induced him to conclude, that the sun, and not the earth, was the centre also of the motions of Mars: and, since the distances of Jupiter and Saturn from the earth were likewise extremely variable, he concluded

System of T. Brahé.

cluded in general, that the sun was the centre of the whole planetary system. His approbation however of the principles of Copernicus went no farther. The motion of the earth in any sense, either annual or diurnal, he conceived to be impossible; and, as the principal objections against it had never received any satisfactory confutation, he considered them as unanswerable. He applied himself therefore to form another hypothesis, which might be more consistent with celestial observations than the Ptolemaic, and less contradictory to first appearances and established principles than the Copernican. Accordingly, in this hypothesis, the earth continued, as before, immoveable in the centre of the universe: and, notwithstanding that his observations on comets had demolished the whole ancient fabrick of solid spheres, he thought it less repugnant to reason, that the immense sphere of the stars, and together with it those of the moon, the sun, and the five planets, though connected by no conceivable principle, should daily revolve round the earth, than that the earth should be subjected to a daily rotation upon an axis. But, while the sun, the moon, and the planets, were thus carried round the immoveable earth, by the diurnal revolution of the starry sphere, from east to west, he explained the periodical revolutions, which are made in a contrary direction from west to east, by supposing the earth to be the centre of the motions only of the moon and sun: while the latter, as his observations required, was the centre of the revolutions of the five planets; with which, as *satellites* at certain assignable distances, he was always accompanied in revolving round the earth. The invention indeed of this system is not solely to be attributed to T. Brahé, for it had originally occurred to Copernicus, before his prejudices against

against the motion of the earth were fully conquered: nor was it adopted in its full extent by all T. Brahé's followers; for Longomontanus, his assistant and disciple, did not admit the doctrine of diurnal revolution of the heavens.

91. T. Brahé's explication of the second inequalities of the inferior planets was perfectly the same with that given in the supposed system of the ancient Egyptians; and, as to its effects, the same with that given by Copernicus. For, whether the earth revolve round the sun, or the sun round the earth, carrying along with him those orbits of which he is the centre, the appearances, to a spectator upon the earth, of the motions performed in them will be perfectly the same. The explication indeed, given in this system of the second inequalities of the inferior planets, did not differ in effect from the theory of Ptolemy. For, though Ptolemy did not suppose them to describe their obits, or, in his language, their epicycles, about the sun as their centre, but round centres nearer to the earth; these centres were always in the line of the mean motion of the sun; and the ratio between their distances and that of the sun, from the earth, did not enter into his calculations.

His explication of the second inequalities;

92. As it may be more difficult to conceive the manner in which the second inequalities of the superior planets are produced, in this system, than that of those of the inferior, the following is a description of it. Let T (fig. 43) be the earth, ABCD the solar orbit, HK the orbit of Jupiter, and let Jupiter in H be in conjunction with the sun in A. While the sun describes his annual revolution *in consequentia*, Jupiter also moves *in consequentia*

 through

through nearly 30 degrees of his orbit; and therefore his next conjunction with the sun will not be in the point E of the zodiac, but in a point G 30° more to the east. In the fourth part therefore of the interval between the conjunctions, the sun, having described somewhat more than a fourth part of his annual circle, will be found in B, and the situation of the line HK in Jupiter's orbit, will be LM. The planet therefore having advanced about 7° 30′ in that orbit, will be found in N; and, by a line TNO, drawn from the earth, will be referred to the point O of the zodiac. When again the sun has proceeded from B to C, the situation of HK will be PQ; Jupiter, having departed about 15° from P, will be found in R the point opposite to the sun; and by a line TRF, drawn from the earth, will be referred to F. Further, when Jupiter has described 7° 30′ more of his orbit, and the sun has come to D, the situation of HK will be SV; and the planet, being found in X, will be referred by the line TXY to Y. Finally, at the next conjunction, the orbit will be restored to its first position; and the sun, having proceeded to *a*, and Jupiter having advanced about 30° from H, he will be found in Z, and referred from the earth, by a line TZG to G. Thus, in his progress in the zodiac, from E to G, his motion will first appear to be direct, between E and O; then retrograde, between O and Y; then a second time direct, between Y and G; and about O and Y, he will seem to be stationary. While Jupiter therefore seems to perform these motions in the zodiac, his real path, according to this system, is the looped curve HNRXZ; and, it is evident that, the greater the distance of the planet from the earth, and the slower its motion, if a superior planet; but the quicker if inferior; the greater will be the number of

particularly in the superior orbits.

of such reduplications; in the course of its own periodical revolution, in the former case; and in the course of the solar revolution, in the latter: and, if the distance of the mean place of the same planet from the earth, and its velocity, should be variable, so will likewise be the extent of its retrograde arches in the zodiac, and the time of their description. By this representation, the intricacy and perplexity of T. Brahé's system, in comparison with the Copernican, may clearly appear: for, in the latter, nothing more is necessary to bring back the whole reciprocations of a planet, than the difference between its velocity and the velocity of the earth; whereas, in the former, the whole orbit must be displaced, and the difference, between the velocities of the planet and the sun, is not sufficient.

93. This system had the same advantages with the Copernican, in explaining the various latitudes of the planets, and in determining their various distances from the sun; that is, the ratios of the semi-diameters of their orbits to the semi-diameter of the orbit of the earth. But the imperfections of these determinations continued to be very great: and neither did T. Brahé, more than Copernicus, avail himself of all the advantages afforded by his hypothesis.

Imperfect methods of reduction.

The great cause of the imperfection of T. Brahé's investigations on these subjects, was the perplexity of his system, which prevented him from accurately distinguishing between the real and apparent orbit of a planet. The orbit of the earth, according to the Copernican system, is NH (fig. 44), and of a superior planet MK; and, if their planes intersect each other in a line MG, passing through G the place of the sun, the place of the node in the zodiac

 will

will be A; and the orbit of the earth will be referred from G to the great circle AEB, and of the planet to the great circle AFD. But if K be the place of the planet in opposition, and H that of the earth, the planet will be referred from H by the line HKC, to the point C of the zodiac, CHE, or its sine CFE will be its apparent latitude, and AC its apparent orbit. Now T. Brahé, by placing the earth in the centre of the system, was led to consider CFE as the real latitude: and, having compared the observed latitudes in opposition, concluded the greatest of them to be the latitude in the limits, and consequently the measure of the inclination CAE, which angle he supposed to continue constant. With any other observation, therefore, of latitude in opposition, he calculated in the right-angled triangle CAE, from the angle CAE and the side CE given, the sides AE and AC, and the difference between them was his required reduction between the ecliptic and the orbit. But, as Kepler first and justly remarked, the heliocentric place of the planet in the zodiac is in the line GKF drawn from G through the planet; and, since GKF and HKC are in the plane of the same circle CFE of latitude, and the angle CHE of observed latitude in oppositions is always greater than that of heliocentric latitude, the line GKF will meet the arch CFE in a point F, between C and E; and the reductions from AC to AE will be all too great. Neither is the circle AC the same circle with that which is described on the same centre H to pass through the node and the planet in either limit, but another of the smaller circles of the sphere: and, if with this reduction applied to AE, the observed distance from the node, so as to make it = AC, and the line AC thus determined, the angle CAE should be calculated, it would, contrary

contrary to the supposition, be found a variable angle. All the calculations therefore of latitude or distance founded on these suppositions were unavoidably deranged, and it was impossible for T. Brahé and his disciples to reject the Copernican or Ptolemaic oscillations of the orbit. One phenomenon especially rendered them necessary, namely, that the difference of the observed latitudes in the opposite limits was very great: for, in the northern limit they did not exceed 4° 34′, whereas in the southern limit they arose to 6° 26′; and, without supposing the orbit to oscillate, it must have been believed to be infracted in the line where its plane intersects the plane of the ecliptic.

94. In explaining the first inequalities of the planets, T. Brahé adopted the concentrical orbit of Copernicus with its two epicycles, and for the same reasons; namely, that by Ptolemy's introduction of an equant, a real, and not merely an optical, inequality took place in his excentric orbit, (79); and that the distances of the planet, given by this excentric theory, from the centre of the ecliptic were often inconsistent with the observations of parallax. But the magnitudes of his epicycles were not precisely the same with those of the epicycles of Copernicus: that is, if his scheme had been reduced to the Ptolemaic form, the centre of his orbit would not have bisected the excentricity of the equant, but have divided it in a different ratio. First inequalities.

95. The principal merit of T. Brahé, and in which he far excelled all the preceding astronomers of whom we have any knowledge, was that of a zealous, indefatigable, and most ingenious, observer of the heavens; and indeed, in the time Great merit of T. Brahé.

in which he lived, this was the chief and most important distinction which an astronomer could attain. He properly considered observations as the only foundation of a just astronomy; and finding astronomy as it then stood, in a great measure destitute of this foundation, he extended them to the greatest part of the celestial phenomena. Though he had to contrive and form the greatest part of his instruments, he determined, without any assistance from the pendulum, and by the laborious method of distances, the positions of no less than 777 fixed stars; the parallaxes, refractions, diameters, and whole peculiarities, of the sun, moon, planets, and even the comets which then appeared, were subjected to his examination; and, by the uncommon magnitude of his instruments, and the ingenuity of their construction, he not only attained to an accuracy before unknown, but also made several perfectly new discoveries, still allowed to be of the most delicate and subtle kind, and most apt to elude observation.

96. Before we can properly understand his explications of the lunar inequalities which he discovered, a short account is necessary of his lunar theory.

T. Brahé's lunar theory.

Though, for the same reasons with Copernicus, he rejected the Ptolemaic hypothesis, he did not imitate him in representing the first lunar inequality by a single epicycle; but he divided its semidiameter, which consisted of 8696 parts of the semi-diameter of the orbit = 100000, into two portions; one BF (fig. 45) = 5797.3, and the other FM = 2898.6: and, having on A the centre of the earth described the lunar orbit BCD, he described on B, with the semi-diameter BF, the epicycle FGH; and on F its apogee, with the semi-

semi-diameter FM, the epicycle KLM. He supposed the centre of the greater of these to move in the circumference of the orbit, *in consequentia*, or from B to C, and the centre of the smaller in the circumference of the greater, *in antecedentia*, or from F to G, H; and both to complete their circuits in 27 d. 13 h. 18′ 35″, the time of an anomalistical revolution. But he supposed the moon to move in the circumference of the smaller epicycle, *in consequentia*, with a velocity double of the former; and in such a manner that, if at the conjunction the line of apsides had happened to coincide with the line of syzigy, her centre should be in K; but towards the quadrature, when the centre of the greater epicycle had described the quadrant BC, and the centre of the smaller the quadrant FG, the moon's centre should be found in M, having described the semi-circle KLM. The first inequality was therefore represented by the angle CAM; and when, as in the present case, the conjunction happened at the apsis, the inequality arose to its greatest amount at C the quadrature. In other cases, when the moon at the conjunction was not found in K, so neither at the quadrature would she be found in M; and the equation would be less, or even wholly disappear.

First inequality.

To represent the evection, or second inequality, he took in AB (fig. 46), the line AN = 2174 of the parts now mentioned, and with the semi-diameter AN described on N the hypocycle AOP; in the circumference of which the centre of the orbit was supposed to perform a revolution *in consequentia*, in half the time of the synodical revolution, and in such a manner, that in the conjunction it should be found in A, and consequently at the quadrature in P, having described the semicircle AOP. The orbit therefore would be displaced;

Second inequality, or the evection.

placed; and, if the conjunctions had happened near its apsides, both equations would be accumulated at the quadrature: and the whole equation would not, as before, be the angle HPM of nearly 5°, but a greater angle CAM, amounting to 7° 48″.

Third inequality, or the variation.

97. With these preparations, the variation, or third inequality, and which T. Brahé first discovered, was thus explained. He supposed another small circle to be described, at the conjunction or opposition, on B the centre of the greater epicycle, (fig. 47), with a semi-diameter BQ = 1163, and that its diameter QR continued always perpendicular to a line BA, or SO, drawn from its centre to the centre of the orbit. In this diameter he supposed the centre of the greater epicycle, carrying with it the smaller one ML, to librate; and that the libration, from S or B to R, from R to Q, and from Q back to S, was also performed in half the time of the synodical revolution. In the conjunction, therefore, the centre of the epicycle FG coincided with S, the centre of QR; but, against the time of the first octant, when the centre of the orbit was in O, at the distance of 90° from A, the centre of FG was not found in S, but at the extremity R of the diameter QR: and the mean place of the moon had advanced beyond the point S, where it would have been, if not affected by this libration, and required an equation ROS of 40′ 30″. But, against the first quadrature, the centre of the epicycle FG would librate back to the centre S of QR, and there would be no such variation.

Fourth, or annual equation.

98. As to the annual equation, which depends upon the anomaly of the sun, and in virtue of which

which the moon moves with greatest velocity when the sun is in the apogee of his orbit, no representation of it was given. It was sufficient to have discovered the existence of this inequality, and the law which it observed; and T. Brahé applied its equation, converted into time, as a correction of the time for which he had made any calculation of the moon's longitude. To correct the time, or to correct the longitude corresponding to it, are operations perfectly equivalent; and in this practice T. Brahé was followed both by Kepler and Horrox.

99. The last discovery of T. Brahé, relating to the variable inclination of the lunar orbit, and the variable motion of the nodes, he explained by ascribing to each of the poles of the orbit a revolution, in a small circle, round the point which it would have always occupied, if the inclination had never varied from its mean amount. According to his observations, the inclination, in conjunctions and oppositions, was $4^{\circ}\ 58'\ 30''$; and it rose to $5^{\circ}\ 17'\ 30''$ in quadratures. Since the mean of these is $5^{\circ}\ 8'$, he laid off from A the pole of the ecliptic (fig. 48), in the great circle AD, an arch $AB = 5^{\circ}\ 8'$, and supposed B to be the mean place of the pole of the lunar orbit; so that, if the inclination had been constant, it would have been always measured by AB; and the point E, at the distance of 90° both from A and B, would have been the constant position of the node. To represent therefore the variations of both, he described on the centre B, with the semi-diameter $BM = 9'\ 30''$, the circle MPLN; and supposed the pole of the lunar orbit to revolve in this circle, with double the velocity of the moon in her synodical revolution; and in such a manner that, if NP be perpen-

Variable inclination of the orbit.

perpendicular to AB or ML, the pole at every syzigy was found in M, at every quadrature in L, at the first and third octants in P, and at the second and fourth in N. According to this representation, the least inclination AM at the syzigies was found, by subtracting MB = 9′ 30″ from AB = 5° 8′, to be no greater than 4° 58′ 30″; and the greatest inclination AL at the quadratures, by adding LB to AB, to arise to 5° 17′ 30″. At the second and fourth octants, it was AN, measuring the angle AGN, formed by perpendiculars from each pole to the arches EC and HK of the orbits of the earth and moon, and equal to EGH their inclination; and, by resolving the right angled spherical triangle ABN, found to be = 5° 8′ 9″.

Variable motion of the nodes.

In other positions of the pole of the lunar orbit, as at Q, when the moon's distance from the quadrature is = 22° 30′, and consequently the angle QBL = 45°, the inclination is measured by the arch AQ; and, by resolving the triangle ABQ, it is found to be = 5° 14′ 18″. The unequal motion of the nodes was a consequence of these suppositions, and each of them appeared to librate continually to and from the point E, through an arch double of EG; for, if the distance AB of the two poles were invariable, the node would always be in the point E, where perpendiculars from them to both orbits intersect each other; and this would also be its position, when the lunar pole was in L or M: but at an octant, as when this pole is in N, and the lunar orbit in the position HGK, such perpendiculars will intersect each other in the point G of the ecliptic; and the variation of the longitude of the node will be EG, measuring the angle EAG = DAO, that is BAN. This equation of the node BAN, is to be found by resolving the triangle ABN, where, with the former data BAN, its greatest

greatest amount will come out = 1° 46′ 9″; and in other cases, as when the pole is in Q, and it is then = 1° 13′ 28″.4, by resolving the triangle ABQ.

These variations of the inclination and places of the nodes must obviously produce correspondent variations of the moon's latitudes, and they will not continue to be the same even in the same degrees of longitude: and variations must also follow on the reductions to the ecliptic. The methods however of calculating these need not be explained. Variations of latitude.

It is here impossible to forbear remarking, both the accuracy of T. Brahé's observations on a subject so delicate, and the ingenuity of his explications. The only correction made on his conclusions is, that the mean inclination is now stated at 5° 8′ 52″, instead of 5° 8′, and consequently BN the semi-diameter of the circle in which the lunar pole revolves at 8′ 49″: and, though Sir I. Newton assigned the cause of the phenomenon, no alteration was made, either by him or others, on the manner of deducing the various latitudes, till T. Mayer, in his lunar theory, found a method of calculating them by a single operation.

100. Great however as the ingenuity and merits of T. Brahé appear to be, we can hardly fail to perceive the inferiority of his system to the Copernican, in perspicuity, simplicity, and symmetry of parts. The periodical motions which he ascribed to the planets were double, and performed round two different centres; for, considering the orbits of the planets as epicycles, a planet had first to move in the circumference of its epicycle round the sun, and then the sun, carrying along with him all the planetary epicycles, had to move round the centre of the ecliptic: and, if we also take into consideration T. Brahé's system preferred to the Copernican.

consideration the diurnal revolution, the system of T. Brahé will be found more complex than that of Ptolemy, who gave one common centre to all the revolutions, both periodical and diurnal. But, notwithstanding the perplexed nature of this system, and that it gave no account why the sun, moon, and planets, should obey the diurnal revolution of a sphere in which they did not move; or why the planets should obey the annual revolution of the solar sphere, from which the distance of some was immense; or why the earth, though placed between the spheres of Mars and Venus, should alone resist the influence, whatever it might be, which carried these bodies and so many others on each side of it round the sun; such was the difficulty of conceiving and admitting the motion of the earth, that this intricate and incoherent system was preferred to the simple and beautiful system of Copernicus, by all the vulgar; and for a long time rivalled, and even surpassed it, in reputation, among the learned.

101. The first steps which tended to abate the general prejudice against the Copernican system, appear to have been taken by its author's friends and disciples, Reinholdus and Rheticus, men of eminence, and joint professors of the mathematical sciences at Wittemberg. The astronomical tables, called the Prutenic, which Reinholdus calculated on the principles of Copernicus, were found to correspond more accurately with the state of the heavens than any before in use; and many astronomers were thereby induced to think favourably of the system on which they were founded. But a circumstance of greater importance towards its reception, was the spirit of philosophical speculation, and particularly the ardour in astronomical researches,

Progress of the latter to higher reputation.

researches, which the Copernican system contributed to excite. The important doctrine, for example, of the composition of motion, taught by the celebrated and unfortunate Gallileo, and founded on reason and experiment, decisively confuted the grand objection against the rotation of the earth, drawn from the motion of projectiles, and especially of falling bodies on its surface; and this decisive confutation, of an objection thought to be so powerful, weakened the influence of others, and produced suspicions of their being alike founded on ignorance and mistake.

102. But what principally contributed to remove the general prejudice against the Copernican system, was the invention of telescopes, and the application of them, especially by Gallileo, to the observation of celestial objects. His discoveries, by means of the telescopes, were so many and striking as to attract the attention even of the vulgar, and to render familiar and acceptable those parts of the theory, which originally appeared most improbable and paradoxical. By his observations, for example, of the moon, and similar, though later observations made by others on the planets, these bodies appeared to be of the same nature with the earth, and some of them far to exceed the earth in magnitude: and when such bodies were seen to perform revolutions through the heavens, the annual revolution of the earth ceased to be thought such a violent and improbable supposition, as it had appeared, while the evidences presented of their magnitude were of a less familiar kind. Of the same tendency were his discoveries of the *satellites* of Jupiter, and the later discoveries, when telescopes were more improved, of like *satellites* to Saturn.

Causes of it.

Telescopic discoveries especially by Gallileo.

When

When Copernicus had displaced the earth from the immoveable station which Ptolemy supposed her to occupy, and made the sun the common centre, both of her orbit and of the planetary revolutions, he had been obliged to except the moon from this general rule, and to acknowledge that no other centre than the earth could be given to the lunar orbit. This therefore seemed to be an incoherent and contradictory part of his hypothesis, and he appeared here to depart from its boasted simplicity, and to return to the complex motion in epicycles, which it was one of the principal merits of his system to destroy. But, after those examples, the revolution of the moon, in an epicycle round the earth, could no longer be viewed as a singular and solitary fact; and the attendance of the moon upon the earth in her annual orbit, in the same manner as so many various satellites of other planets were seen to do, could no longer be deemed contradictory to the common course of nature. Of the same kind also, though less decisive in its tendency, for the argument applied equally to the system of T. Brahè, was the discovery made by Gallileo and others, of the phases of Mars, Mercury, and Venus. It was objected to the Copernican system by the disciples of Ptolemy, that, on the supposition of its truth, these planets ought to be seen from the earth under varieties of form, similar to those of the moon: and we have seen the acknowledgement of Copernicus, that such variations of form ought undoubtedly to take place; and his prediction, that they would become evident in future times. His prediction was fulfilled by the telescope; the phases of Venus were recognized almost immediately after the date of that invention, and the discovery

discovery of those of Mercury and Mars soon followed it; and if like variations of form are not also observable in Jupiter and Saturn, the principles of the system sufficiently explain the reason. The telescope also furnished strong probabilities in favour of the diurnal rotation of the earth; for it discovered the rotation of the sun about an axis, and, besides those of some other planets, the extremely rapid rotation of Jupiter. On account of this, while the whole heavens appear from the earth to revolve in 24 hours, they must appear from Jupiter to revolve in 9 hours 56′: and the reasons for concluding both revolutions to be merely apparent, are the same. By these discoveries, the original prejudices against the system of Copernicus were in a great degree removed, and it rose in the esteem of astronomers, not only above the system of Ptolemy, but above the more coherent one of T. Brahé. Even the diurnal rotation of the heavens became more questionable than the authority of sense had represented it: and, though no annual parallax of the fixed stars had been discovered, the irregular path, which the annual revolution of the planetary orbits represented the planets as describing through the heavens, seemed to be a more improbable and violent supposition, than that the diameter of the terrestrial orbit should be considered as evanescent, when compared with the immense distance of the fixed stars. But perhaps no cause finally contributed so much to promote the credit of the Copernican system, as the authority and discoveries of Kepler; and by him such laws were demonstrated to be established, among the celestial bodies of which it is composed, as were wholly inconsistent with every other.

But

But the value of Kepler's discoveries rises far beyond that of being adminicles to any system: and it was chiefly to exhibit that value in a just and convincing light, that the foregoing detail, of systems and of the previous state of astronomy, has been given.

CHAP.

CHAP. V.

Of the Preparations to Kepler's Discoveries, and of his Original Intentions.

103. KEPLER was born in the year 1571, in the dutchy of Wirtemberg, of a family ranked among the *noblesse*, or gentry; but, in consequence of military service, the only business thought reputable by the *noblesse* of many countries, reduced to indigence. By the favour of his prince he was educated at Tubingen, a seat of learning then so eminent, that masters of science were frequently selected from among its scholars, to supply the universities of other countries. Though he prosecuted his studies with success, and was a disciple of Mæstlinus, an astronomer of eminence, and of the Copernican school, he informs us, that he had no peculiar predilection for astronomy. His passion was rather for studies more flattering to the ambition of a youthful mind; and when his prince selected him, in 1591, to fill the vacant astronomical chair at Gratz, in Stiria, it was purely from deference to his authority, and the persuasions of Mæstlinus, who had high expectations from his talents, that he reluctantly accepted of the office. He appears to have thought it unsuitable to his pretensions; and the state of astronomy was besides so low, uncertain, and in many respects visionary, that he had no hope of attaining to eminence in it. But what he undertook with reluctance, and as a temporary provision conferred on a dependant by his prince, soon

Occasion of Kepler's application to astronomy.

engaged his ardour, and engrossed almost his whole attention.

His first astronomical publications.

104. The first fruits of his application to astronomical studies appeared in his *Mysterium Cosmographicum*, published about two years after his settlement in Gratz. Though he had adopted the Copernican arrangement of the planets, the principles of connection subsisting between its parts appeared to him to be in a great degree unknown. No causes especially were assigned for the different positions of the centre of uniform motion in the different orbits, nor for the irregular distances of the planets from the sun; and no proportion was discovered between these distances and the times of the planetary revolutions. He considered the arrangement as deficient and unsatisfactory, till all those causes should be discovered: and in the work now mentioned he supposed that he had discovered them, at least he traced out several similar analogies, in the Pythagorean, or Platonic doctrines concerning numbers, in the proportions of the regular solids in geometry, and in the divisions of the musical scale: and these analogies seemed to assign the reason why the primary planets should be only six in number. Hasty and juvenile as this production was, it displayed so many marks of genius, and such indefatigable patience in the toil of calculation, that, on presenting it to T. Brahé, it procured him the esteem of this illustrious astronomer, and even excited his anxiety for the proper direction of talents so uncommon. Accordingly, not contented with exhorting Kepler to prefer the road of observation to the more uncertain one of theory, T. Brahé added a generous and unsolicited invitation to live with him at Uraniburg; where his whole observations would be open to Kepler's perusal,

perusal, and those advantages found for making others, which his situation at Gratz denied. The opinion of other astronomers, concerning this production, was no less favourable: and, were there no other evidence of his just and general conceptions, the remark, which afterward led to such important consequences, that the cause of the equant, whatever it might be, ought to operate universally, is a sufficient attestation of them; and a proof that, even in this early period of his studies, the possibility at least of deducing the equations of a planet, from the relation between its different degrees of velocity and distances from the sun, had presented itself to his thoughts.

105. Notwithstanding the ardent desire of Kepler to be admitted to the perusal of T. Brahé's observations, it was probable that the distance of the place, and the difficulty of the journey, to one in Kepler's situation, would have for ever prevented the gratification of it. But the persecutions now arose which drove T. Brahé from his native country, and from which he at last found a refuge in Prague, the capital of Bohemia. Thither, accordingly, Kepler repaired in the year 1600; and, that he might be under no necessity of returning to Gratz, he obtained, by T. Brahé's interest with the Emperor, his patron, the appointment of imperial mathematician. On his arrival at Prague, another circumstance occurred, equally important to his success and fame, and which he piously ascribes to the kind direction of Providence. T. Brahé, with his assistant Longomontanus, was employed about his theory of Mars; no doubt, induced by the favourable opportunity of verifying it, by observations of this planet in its approaching opposition in the sign of Leo. Kepler's attention

His introduction to T. Brahé.

was therefore, of course, directed to the same planet; where the great degree of excentricity renders its inequalities peculiarly remarkable, and leads with proportional advantage to the discovery of their laws; and it would have been directed to observations much less instructive, had those astronomers been engaged about any other planet.

106. The former planetary theories were in general unlike, and founded upon different principles. The ancients, as Ptolemy and his followers, considered every planet separately, and supposed that their several motions and inequalities arose from causes peculiar to every orbit, and with which the other orbits had no connection. The moderns, again, considered the orbits as connected by a common principle; and, remarking all that was similar in their motions, endeavoured to derive it from a common cause. But they disagreed about this principle; for the cause of the second inequalities of the planets, according to Copernicus, was the annual revolution of the earth; while T. Brahé and his disciples ascribed them to the attendance of the planets upon the sun, in his annual revolution. But, notwithstanding these and other less important differences, the effects of all the theories, in representing the motions of the planets, and on the calculations of their places, were very nearly equal, and Kepler found them almost equally erroneous: for the longitudes of Mars, calculated for August 1598, and August 1608, even from the Prutenic tables of Reinholdus, fell short of the observed longitudes, the first nearly 4°, and the last no less than nearly 5°; and errors of this magnitude were not peculiar to the Prutenic tables, for similar and even greater errors were found in all the others. These errors, so demonstrative of the

imperfect

imperfect state of the science to which they adhered, Kepler chiefly imputed to the practice universally followed by astronomers, of observing the mean oppositions only of a planet, and forming their theories for the first inequalities upon these. It was not at all surprizing that this practice should have been introduced; for the points of the zodiac, which a planet occupies in oppositions, are not, as in lunar eclipses, distinguished by any sensible marks, but must be determined by calculation: and to determine when the mean place of the sun was opposite to a planet, required only the knowledge of the mean solar motions; whereas, to determine the same thing of the true place of the sun, required an accurate solar theory; of which, in the ruder ages of their science, astronomers were destitute. As the authority of the ancients who had introduced this practice, continued to be followed, so Copernicus, for example, when he observed a planet about the time of an expected opposition, calculated the mean place of the sun known by his mean motion; and if this differed precisely six signs, or 180°, from the observed place of the planet, the instant of the observation was the required instant of the opposition: but, if the difference was not 180° precisely, he endeavoured, from a comparison of the motions of the planet viewed for several nights successively, to find out the instant when this precise difference of 180° took place. (G.) Thus he supposed the eye to be placed in the centre A of the earth's orbit, round which the mean motions of the sun or earth were believed to be performed, (fig. 49); and the observations of the planet to be made in the points J, H, K, of the planetary orbit, marked by lines passing from A through the earth in O, M, N, and not from the centre of the sun in B. Consequently,

 when

when he had obtained as many longitudes of the planet in such points, and their correspondent times, as he thought necessary, and proceeded with these to investigate the elements of his theory, the elements investigated were, in Ptolemy's language, the distance AE of the centre of the planet's equant from the point A, and the longitude marked by the line AEF, which he reckoned the line of apsides, in the zodiac. Now this was the practice against which Kepler chiefly objected, and to which he chiefly ascribed the imperfection of all the former theories: and, without immediately deciding on their respective merits, the principal improvement, which he originally had in view, was to introduce the practice of employing apparent oppositions, which supposed the observations to be made from the centre of the sun in B, and to form his projected theory on these. Consequently, in his theory, the line of apsides would be BEC, drawn from the sun through the centre of the equant, and not AEF from the centre of the ecliptic: and the elements to be investigated for the first inequalities, would be the longitude marked in the zodiac by this line BEC, and the excentricity BE; or the distance of the centre of uniform motion from the sun, and not AE its distance from the centre of the ecliptic, or earth's orbit OMN. This was Kepler's principal, at least his original, design; and the immediate reason of his anxious desire, to obtain a perusal of T. Brahé's observations, was to ascertain the propriety of the alteration. (H.)

Kepler's original designs.

107. The necessity of this substitution was partly suggested to Kepler by the principles implied in the Copernican system; and, with the strong propensity by which he was distinguished, of enquiring

quiring into the physical causes, especially of astronomical *phenomena*, the design of introducing it was what he could hardly fail to entertain. After T. Brahé's observations on comets had exploded the doctrine of solid spheres, it seemed to the disciples of Copernicus more likely than ever, that the force which controuled and gave motion to the planets proceeded from the sun: and Kepler tells us, that he could conceive no reason, why, in particular, they should be retarded at the points of their orbits most distant from the sun, and accelerated in the parts least distant, except that this force became more languid, in proportion to the increase of the distance; and grew more vigorous in proportion as the distance was diminished. But, if the line of apsides should pass, not through B the sun, but through the centre A of the earth's orbit, the planet's motion will not be slowest in C the point of greatest distance from the sun, and quickest at the point D of least distance, but at the points F and G of greatest and least distance from the centre A, where no force could be conceived to reside. Or, if to evade this argument, the acceleration and retardation of a planet should be ascribed to some innate affection or propensity of the planet, or of the intelligence or power by which it might be supposed to be animated; it was equally inconceivable how this intelligence should be induced to overlook the sun, an object of so great magnitude and splendour, and with respect to which it might regulate its variations of velocity according to geometrical principles; and in preference to the sun to fix its attention on an imaginary point, distinguished by no mark, and placed at no greater distance from the sun than two or three solar semi-diameters: especially, when Copernicus had acknowledged this point to be also

Reasons inducing him.

 variable

variable in position, and had taught that the excentricity BA of the earth's orbit was changed, though the position of the sun in B remained the same.

But there was also a practical reason which urged Kepler to his projected innovation, namely, his perceiving that the ancient practice in a great degree defeated the design for which oppositions were observed, and involved the planets more or less in those second inequalities, which it was the sole purpose in observing their oppositions to avoid. Indeed, in the rare case of an observation of this kind, in the line ABH of the solar apsides, the lines AH and BH of mean and true oppositions were coincident; and the planet, considered with respect to either, equally divested of every degree of second inequality. But in the observations on either side of ABH, and especially at the distance of 90° from it, the apparent opposition will happen in the line BQP, for example, and at the time marked by the line AQ; and the mean opposition will not happen till afterwards, when the planet is found in the line AN, and at the time which this line marks. Now, at this time, when the earth has proceeded to N, the planet will not be seen to have advanced in its orbit; a line drawn to it from the earth in N, will not even be parallel to PQ, and make it seem stationary in P; but, as its motion is slower than that of the earth, that motion will seem to be retrograde; and the planet's geocentric place will be *p*, a point less advanced than P; or, which is the same, in *k* less advanced than K; that is, it will be involved in the second inequality, and, instead of being seen in the same point from the centres of the earth and sun, lines drawn from the earth to the sun and the planet will make with each other the angle BN*k*. In the

the opposite semi-circle, again, of the earth's anomaly, the mean oppositions will be more early than the apparent, and the planet will be more advanced in longitude: and these differences may arise, in the case of Mars, to 5° 19′; of Jupiter, to 30′; and of Saturn, to 15′: that is, to intervals of many days. Corrections, therefore, on all the mean oppositions, except those observed in the line of the solar apsides, seemed to Kepler to be absolutely necessary: on those, to wit, observed in the first semi-circle of the earth's anomaly, by additions to the planet's longitude and subtractions from the time; and, on those observed in the opposite semi-circle, by additions to the time and subtractions from the longitude: and he believed, that without such corrections it was vain to expect more perfection from any theory, however sound all the other principles might be which should be employed in forming it. The consequent difference, therefore, made on the ancient theories, would be, that the planet's excentricity should be measured from the centre of the sun in B, and that the line of apsides should be translated from its former position AEF. to a new position BEC; and the greatest accelerations would take place, as they ought, at the planet's least distances, and the greatest retardations at its greatest distances from the sun in B, and not from the imaginary point A. By these means, not only would the equations for the first inequality be varied, both in magnitude and position; but much greater variations, since the centre of the orbit was also brought from L to T, might be expected upon the equations of second inequality and the calculated latitudes.

In short, Kepler proposed to demonstrate, that the planets moved in orbits, whose planes intersected

sected one another, in lines passing through the centre of the sun, and not through the centre of the ecliptic; and this was the great innovation which he originally intended, and was chiefly desirous to introduce into astronomy. In fact, excepting only the system of Copernicus, it was an improvement more important, and of greater consequence, to simplify the science, than any which had been introduced in all the preceding ages; and his successful and decisive establishment of its truth and propriety, may be justly ranked among his greatest discoveries; and equally deserves our attention with those which have been more generally celebrated.

108. On Kepler's coming to Prague to live with T. Brahé, he found greater necessity for his projected innovations than he had formerly believed. T. Brahé and Longomontanus had made a catalogue of all the mean oppositions of Mars observed from the year 1580, and formed a theory representing them, as they said, so nearly, that its errors in longitude never exceeded two minutes. In this theory, the place of the apogee of Mars, for the year 1585, was in 4 s. $23^{\circ}\ 45'$; the whole excentricity was 20160, or nearly the same with that of Ptolemy's equant; and of this they had allotted 16380 for the semi-diameter of the greater epicycle. Consequently, to express their theory in Ptolemy's language, they did not bisect the excentricity of the equant by the centre of the orbit, but divided it in the ratio of 12600 to 7560. Satisfied with the accuracy of these elements, they had calculated the equations of the centre, that is, of the first inequality, for every degree of anomaly; and, making an addition of $1'\ 40''$ to the mean motion of the Prutenic tables, had formed a new

Confirmations of them.

new table of the mean motions of the planet, of its apogee, and of its nodes, for 400 years. But the ancient difficulties about the second inequalities, or the equations of the orbit, and the latitudes even in opposition, remained unconquered; and not only abated their confidence in their theory, but put a stop to their farther procedure in completing it. Now these were the very imperfections to which Kepler had supposed that the principles on which they formed their theory would lead. For, by employing mean oppositions, and making EA the excentricity of their equant, they had displaced the planetary orbit, describing it on the centre L, a point of the line EA: and the error produced on the equations of second inequality could not fail, in some configurations of the earth with the planet and the sun, by thus displacing the orbit from its proper centre T, to be considerable. For example, if the earth should be in the point I, and the planet falsely represented in S, while its real place was in R, the error of the equation of the orbit would be the angle RIS. If also the planes of the orbits should intersect each other in a line passing through the sun in B, their supposition, of this intersection taking place in the centre A of the ecliptic, could not fail to produce correspondent errors on the calculated latitudes.

109. But, convincing as these reasons appeared to Kepler, and at last, as we are informed, to T. Brahé himself, the first proposal of the alteration gave rise to much debate between those astronomers. The reasons certainly were not obvious why T. Brahé's hypothesis should not be supposed as perfect as the intended hypothesis, which was to rest on apparent oppositions; and it seemed to be impossible that a theory, which was affirmed

to

to represent the mean oppositions so exactly, in all points of the orbit without distinction, should be false. This debate, and the desire of giving satisfaction to T. Brahé about the subject of it, was the occasion of engaging Kepler in all the laborious and unpleasant investigations contained in the first part of his commentary; in which he shews, by demonstrations adapted, not only to the three different astronomical systems, but even to both the kinds of theory, concentric as well as excentric, that though a false position were given to the orbit, the longitudes of a planet might be so represented, by a proper position of the centre of the equant, as never to err, in oppositions, above 5 minutes from those given by observation; but, that by the same false position of the orbit, and though no other error of theory should be committed, the second inequalities and the latitudes would be much more considerably, and indeed very greatly, deranged: and he demonstrates, in particular, that when the planet is in 9 s. 8° 33′ 6″ of mean longitude, and the earth in 9 s. 16° 0′ 16″, the error produced on the calculated longitude, or the angle RIS, would arise to 1° 3′ 32″. These investigations, therefore, however unpleasant and laborious, served to confirm Kepler in his purpose; and that part of them, which related to the similarity of the excentric and concentric theories in their effects, served also another important purpose; for it demonstrated, that the labour and ingenuity employed in the formation of those theories, and in the substitution of the one for the other, had been in a great measure expended in vain, and were of little importance to the real improvement of astronomy. (I.)

Purpose of the first part of Kepler's commentary.

110. After

110. After thus examining the principles of the ancient theories, and pointing out their defects, his next step was to examine the structure of T. Brahé's theory in particular, in order to discover, if in all parts of it the principles had been preserved, and if its success in the representation even of the first inequalities, were as compleat as had been supposed. By this examination, errors, sometimes very considerable, were discovered in those observations of mean opposition, from which its elements had been deduced. In mean oppositions, the computed mean place of the sun, and the observed place of the planet, should be precisely six signs distant. But in none of all the ten contained in T. Brahé's table, from 1580 to 1600, and employed in the formation of his theory, did this essential condition accurately take place; and in the greater part the mean place of the sun was at a considerable distance from the point opposite to the planet. For in that table the oppositions, and their circumstances, stood as follows: Structure of T. Brahé's theory.

Times

		Times of opp.			Obs. long. in eclip.				M. long. ☉			
		d.	h.	′	s.	°	′	″	s.	°	′	″
1580.	Nov.	17	9	40	2	6	46	0	8	6	48	32
1582.	Dec.	28	12	16	3	16	46	10	9	16	50	58
1585.	Jan.	31	19	35	4	21	10	26	10	21	10	13
1587.	Mar.	7	17	22	5	25	10	20	11	25	5	57
1589.	Apr.	15	13	34	7	3	58	10	1	3	53	32
1591.	June	8	16	25	8	26	42	0	2	26	45	24
1593.	Aug.	24	2	13	11	12	43	45	5	12	34	36
1595.	Oct.	29	21	22	1	17	56	15	7	17	56	17
1597.	Dec.	13	13	35	3	2	28	0	9	2	28	51
1600.	Jan.	19	9	40	4	8	18	0	10	8	18	43

Difference.			Latitudes.				M. long. ♂				Long. Apogee ♂			
′	″		°	′	″		s.	°	′	″	s.	°	′	″
2	22	—	1	40	0	N	1	27	29	46	3	25	21	40
4	48	—	4	6	0	N	3	11	34	56			22	17
0	13	+	4	32	10	N	4	22	37	46				55
4	23	+	3	38	12	N	6	3	27	46			23	32
4	38	+	1	6	45	N	7	16	53	7			24	10
13	24	—	3	39	0	S	9	7	47	30				48
9	9	+	6	3	0	S	11	10	53	50			25	26
0	2	+	0	5	15	N	1	8	26	47			27	35
0	51	—	3	33	0	N	2	24	55	47			29	5
0	43	—	4	30	50	N	4	6	46	16			30	6

Precession.			Reduction.			Obs. long. orbit.		Calculated.	
°	′	″	′	″		′	″	′	″
27	58	50	4	10	+	50	10	50	40
28	0	38	5	20	+	51	30	51	26
	2	25	0	36	—	9	50	9	41
	4	10	5	10	—	5	10	4	50
	5	55	3	35	—	54	35	54	33
	7	47	10	20	+	42	0	40	23
	9	40	8	45	—	35	0	34	36
	11	27	0	12	+	56	5	57	14
	13	20	6	0	+	34	0	32	20
	15	5	0	45	+	18	45	19	57

The

The error in some of these oppositions appears from this table to have been very great. Particularly, on the 8th of June 1591, the sun was advanced no less than 13′ 24″ beyond the point opposite to the planet. No restraint, therefore, was laid on Kepler, in the introduction of his theory, from the accuracy observed in the structure of the former; for this error was about three times greater than any difference which the proposed transposition of the orbit could produce.

111. These errors arose from the inaccurate conceptions entertained by T. Brahé and his assistant, concerning oppositions; their supposing, to wit, that they took place, not when the sun was distant 180° from the point of the ecliptic in which the planet was observed, but from the point of the ecliptic in which it would be found after applying the reduction. It was with design, therefore, that they observed the planet in the points marked in the table; because they thought them the only points, where the application of the proper reductions would produce the effect they wished. But, even if this principle had been just, their rules of reduction were not constant. The nodes of the orbit of Mars were in 1 s. 17°, and 7 s. 17°, and consequently the limits in 4 s. 17°, and 10 s. 17°; and, though therefore the reductions ought to have been greatest at 45° from either, yet at the observation of 1582, of the planet in 3 s. 16° 46′ 10″, that is, at the distance of 30° 17′ 50″ from the limit, their reduction was 4′ 48″; and at the observation of 1597, in 3 s. 2° 28′ 0″, at the greater distance of 44° 32′ from the limit, it was only 51″. With the inclination which they gave to the planet's orbit, the just reductions were those which are marked in a subsequent part of the table: and

Cause of these errors.

and upon the application of them, the places of the planet and the sun were not found precisely opposite, but stood as follows :—

Long. ♂ reduced to the orbit.	M. long. ☉.	Differences.
50′ 10″	48′ 32″	1′ 38″ +
51 30	50 58	1 28 +
9 50	10 13	0 23 —
5 10	5 57	0 47 —
54 35	53 32	1 3 +
42 0	45 24	3 24 —
35 0	34 36	0 24 +
56 5	56 17	0 12 —
34 0	28 51	5 9 +
18 45	18 43	0 2 +

No errors of any consequence were found to have been committed about the planet's mean longitude; nor, when the theory was formed, and the true longitudes calculated according to its principles, did the calculated places differ, any where, above 2′ from the places given by observation, except at the opposition of 1591. In this the heliocentric longitude, which, as deduced from the observation, was 8s 20° 42′ 0″, came out, by T. Brahé's calculation, from his theory, 8 s. 26° 40′ 23″, as in the table But, by a more accurate calculation made by Kepler, on the same principles, it was found to be 8 s. 26° 34′ 43″; from which, subtracting 10′ 30″ for the reduction to the ecliptic, it becomes 8s. 26° 24′ 13″; and, therefore, as the sun's mean longitude was at this time 2s. 26° 45′ 24″, the error of the theory, instead of 2′, arises to no less than 21′ 11″.

112. While the planet's apparent orbit was confounded, by T. Brahé and his disciples, in the same

same manner as by the ancients, with its real orbit; they justly thought that, if they should continue, like the ancients, to reckon the opposition from the point of the ecliptic where the planet was observed, it would not be perfectly divested of the second inequality: for, the supposed inclination being far greater than the truth, the planet in the point E of observation (fig. 44), was brought too near the node. But by taking from the node A an arch AB of the ecliptic equal to AC, that is, by reckoning the opposition from the point B, in which the planet was found after the application of the reduction, Kepler remarks, that it is as truly involved in the second inequality, as if no such correction had been made, and the ancient practice had been continued. For, let AFD be the real or heliocentric orbit; which, as the heliocentric latitudes in opposition are less than the geocentric, must intersect the circle of latitude CE, in a point F between C and E; and let CB be joined, cutting AF in D. The arch AD therefore is shorter than either of the sides AC, AB, of the isosceles triangle ABC, and the point B more remote from A than the point D, which the planet, according to this practice, would occupy in its real orbit; that is, the sun, when opposite to B, has passed the point of his opposition to the planet in D. Demonstration of them.

But, though the observations were made erroneous, by this inaccuracy in computing the oppositions, they were rendered much more erroneous by the excessive reductions, even if they had been regularly applied, which the mistake of the apparent for the real orbit produced. T. Brahé's reduction was BE, for example, the difference between AE the observed distance from the node, and AC the distance in the supposed orbit calculated from CE and CAE given (93). But AC is an arch only of the apparent orbit; and, unless the planet at an opposition be actually in the node, must always be greater than AF the distance in the real orbit. The inclination EAF of the real orbit, as will afterwards appear, does not much exceed 1° 50′, and consequently the reduction from it, even when AE is = 45°, will not exceed 1′: but T. Brahé considered CAE as the inclination of the orbit; and, as this angle was valued at 4° 58′ in the northern semi-circle, and at 6° 26′ in the southern, the reductions rose to 8′, and even to 10′. Though therefore the error committed, if AC had been the real orbit, by making AB = AC, and counting B the point of opposition, would have been inconsiderable, the effects of it, when AC is an arch only of the apparent orbit, on the observations, may sometimes arise to 9′.

On these accounts it was evident to Kepler, that a theory founded on such erroneous principles, could not be depended upon. But he still continued his examination, extending it to the manner in which T. Brahé had deduced, from those principles, the times and points of his oppositions. Here, however, no considerable errors were found; and the greatest, which took place at the opposition of 1597, appeared to arise from inaccurate observations unavoidably produced by his persecutions. (J.)

CHAP.

CHAP. VI.

Of Kepler's Theory, founded on apparent Oppositions, and of its total Failure.

113. THE principal reason which induced Kepler to substitute apparent for mean oppositions, in the formation of his planetary theories, was, that a planet could not be found perfectly divested of its second inequalities, except in the points of apparent opposition. But since the point of the orbit, which a planet occupies in an apparent opposition, cannot be accurately determined without just and legitimate reductions from the ecliptic, a necessary previous step to the introduction of his theory, was to investigate the inclination of the orbit on which the reductions depend. The determination again, not only of the inclination, but also of the place of the nodes, depended on the diurnal parallax; and it was not yet known with certainty how far any observation might be affected by it.

114. The diurnal parallax of Mars was a subject to which T. Brahé had paid great attention; and especially about the opposition of the planet in Cancer, in the year 1582, he had made a great number of observations for the express purpose of discovering it. By these he supposed it to be ascertained, that towards oppositions this parallax was remarkably greater than that of the sun; and concluded that, from 1′ 9″, its amount at the point of the orbit most distant from the earth, it arose to no less than 8′ 35″, at the point least distant.

 But,

But, on examining the observations, it was found that they were not made with all the ingenuity which a subject so delicate required; for the fixed stars, with which he had compared the planet, were generally at a great distance from it, and also near the ecliptic: and, therefore, the star with which the comparison was made in the morning was commonly set, or so near the horizon as to be deranged by refraction, before the evening observation; and it became necessary to supply its place with another. Kepler also found reason to suppose, that in ascribing so great a parallax to Mars, T. Brahé had been misled by the calculations of his assistants, who, in making them, had entirely mistaken the intention of their master. He had, at different times of the same days, taken various distances of the planet from various zodiacal stars; and had committed to his assistants the care of calculating, without regard to any theory, by a comparison of those distances, what parallax they would produce. But, to Kepler's surprize, they had applied themselves to calculate what place the planet would occupy, according to the forms and proportions of the concentric theory of Copernicus; and from the difference between this and any of the observed places, endeavoured to deduce the horizontal parallax. It became necessary, therefore, that Kepler should have recourse to the original observations of 1582, as much as if they had not been subjected to any previous examination.

Inaccuracy in the investigations of the parallax of Mars.

115. T. Brahé had endeavoured to investigate the horizontal parallax of Mars from the sum of two parallaxes in longitude, found by observations of the planet made on each side of the nonagesimal; that is, by the excess of the apparent motion in

in longitude above the real motion: and this real motion he carefully ascertained by the observations of successive days. The most remarkable results which Kepler found from the examination of this procedure, were the following:

1. From one pair of such observations made on the 26th of December 1582, at an interval of 10h. 46′, the sum of the parallaxes in longitude was -	1′ 10″.0
2. From a pair made on the 16th of January 1583, at an interval of 9h. 30′	0′ 22″.5
3. From a pair on the 17th, at an interval of 9h. 40′, it was -	1′ 6″.6
4. From another pair on the 17th, at an interval of 9h. 18′, - -	2′ 19″.2
5. From a pair on the 18th, at an interval of 7 h. 53′, it was - -	1′ 46″.2

The only conclusion which he thought himself entitled to draw from these results, so far below the common opinions of the age, was, that the determination of the horizontal parallax of Mars was a matter of the greatest delicacy, and almost an unconquerable difficulty: and this conclusion was confirmed by other observations in different years, all of which were found equally indecisive.

Parallax of Mars uncertain.

In the examination of the foregoing observations, Kepler employed also the converse method; and, supposing the planet's horizontal parallax, on the 17th of January, 1583, for example, to be 4′, he calculated the sum which this supposition would give for the parallaxes of longitude; in the above-mentioned interval of 9h. 40′, beginning at 5h. 20′ and ending at 15h, the times of T. Brahé's observations. But, instead of 1′ 6″.6 given by the observations, the result was 4′ 8″.

Nor were the conclusions more decisive which he attempted to draw from some observations of his own. The method of deducing the horizontal parallax from motions in longitude having proved so unsuccessful, he attempted to employ observations of latitude. These he made in February, 1604, at a time peculiarly favourable, when Mars was stationary, and when, at the times of observation, a circle of latitude passing through the planet, nearly coincided with a vertical: and the instruments used were an iron sextant of $2\frac{1}{2}$ feet radius, and a brass quadrant of $3\frac{1}{2}$. But his observations, after all the attention he could bestow, appeared to him inferior to those of T. Brahé: and on the whole he was obliged to rest in this general conclusion, that the horizontal parallax of Mars certainly did not exceed 4′, even in his nearest approaches to the earth, and that, very probably, it was a great deal less. (K.)

116. When Kepler had thus satisfied himself that no considerable errors could take place in regard to the observations on which he was to found his theory, by adopting the common opinions concerning the parallax of Mars; his next object was to investigate the longitudes of the nodes. T. Brahé's procedure in this business was founded on a false principle; for it supposed AC (fig. 44), the path of the planet seen from the earth in H, to be its real orbit, and that the angle CAE was constant: and, therefore, unless the latitude EC, from which, and the angle CAE, he calculated the distance AE, was very small, his conclusion for the place of the node A was not to be depended upon. But Kepler's procedure consisted in observing the planet when actually in the node: and consequently, when the observed longitude was cleared of

Investigation of the longitudes of the nodes.

of refraction and parallax, the true longitude of the node was found.

Four observations were accordingly found, in T. Brahé's register, of the planet in the ascending, and two in the descending node. By the first four it was represented to have been in the former, on the 7th of March 1590, the 23d of January 1592, the 10th of December 1593, and the 28th of October 1595: and by the two last, in the latter, on the 9th of May 1589, and the 29th of December 1594.

The longitudes of the nodes were therefore found by calculating, for these times, the longitudes of the planet. This was done by T. Brahé's theory. For example; the mean longitude of Mars, on the 29th of December 1594, was 7s. 27° 14′ 30″; and on the 28th of October 1595, it was 1s. 5° 31′ 0″; and the equations corresponding to these were 11° 30′ 30″ subtractive, and 11° 17′ 0″ additive. The heliocentric longitude therefore of the descending node was 7s. 15° 44′, and of the ascending node 1s. 16° 48′. The same must have also been their longitude very nearly in the other observations: for, though their position is not absolutely invariable, the variation in the course of a few years is almost insensible. (L.)

117. After determining, at least nearly the longitude of the nodes, the remaining and most important business was, to investigate the inclination of the orbit to the ecliptic. For this purpose Kepler employed three methods. One, which is applicable to all the planets except Mercury, requires observations of latitude made in the limits, when the distance of the planet from the earth and the sun are equal: and it supposes that the mutual ratio of the orbits and the heliocentric distance of

Inclination of the orbit.

the planet may be nearly, at least, determined by some of the former theories. Thus, since the node G (fig. 52), is in 1s. 17°, and the other F in 7s. 17°, one of the limits C will be in 4s. 17°, and the other in 10s. 17°. On C then, with the distance CB, describe a circle ABH, cutting the terrestrial orbit in A and H; so that, by joining AB, AC, there will be formed, upon the base AB, the isosceles triangle CAB. When the planet is in C, in 4s. 17°, its heliocentric distance CB, calculated from T. Brahé's theory, is 1666.6 in parts of AB = 1000: and it is required to find, in this case, what the relative positions of the sun, the planet, and the earth, will be, when AC becomes equal to BC. For this purpose, draw from C the line CD, bisecting AB in D. Then, since AC = 1666.6, and AD = 500, the angle CAD will be found = 72° 23′, and thence ACB = 2ACD = 34° 54′. Therefore, since Mars, when seen from the sun in 4s. 17° is in the line BC, and ACB is = 34s. 54′, he will, at the time when AC is = BC, be seen from the earth in 5s. 21° 54′: and, since CAB = 72° 23′, the sun will at the same time be in 8s. 4° 27′. Or, if the earth should be in the point H of her orbit, the planet would be seen from the earth in 3s. 12° 6′, and the sun in 0s. 28° 33′.

First method to determine it.

When C represents the opposite limit, the configuration will be somewhat different. For, then CB, according also to T. Brahé's theory, is = 1375: and CAB will therefore be found = 68° 41′, and ACB = 42° 28′. Consequently, in this limit, the planet will be seen from the earth in 9s. 4° 22′, and the sun in 6s. 25° 51′, when AC becomes equal to BC; or, in 11s. 29° 28′, while the sun is in 2s. 6° 9′.

As the invariable inclination of the planetary

orbits

orbits to the ecliptic, which was established by this investigation, may justly be ranked among Kepler's great discoveries, the methods of investigation merit a particular detail.

Five observations were found, in T. Brahé's register, of the planet near to 5s. 21° 24′, while the sun was also nearly in 8s. 4° 27′. On the 9th of November 1588, at 18h. 30′, the planet was seen in 5s. 25° 31′, with 1° 36′ 45″ of north latitude, while the sun was in 7s. 28°. Since, therefore, it was neither precisely in the limit, nor the sun precisely in 8s. 4° 27′, the angle CAB will be only 62° 29′, instead of 72° 23′, and, AC being greater than BC, the observed latitude must be less than the inclination.

On the 4th of the following December, Mars was observed in 6s. 9° 19′ 24″, with 1° 53′ 50″ of north latitude, and the sun was in 8s. 23°. The elongation CAB was therefore 73° 40′ 36″; and, consequently AC being less than BC, the observed latitude was greater than the inclination. But the planet was more than 18° beyond the limit, and the inclination at the point of heliocentric longitude which it then occupied, was not the greatest inclination. On balancing the effects of those opposite causes, the greatest inclination was stated at 1° 50′.

On the 21st of October 1586, at 18h. Mars was seen in 5s. 0° 7′, with 1° 36′.6 of latitude, and the sun was in 7s. 8°; so that the elongation was only 67° 33′, and the planet had not yet reached the limit. On both these accounts the latitude must have been considerably less than the inclination.

On the 1st of November following, at 16h. 40′, Mars was seen in 5s. 5° 52′, with 1° 47′ of latitude, and the longitude of the sun was 7s. 19° 40′.

Consequently

Consequently the elongation was 73° 48′, and the planet 16° short of the limit. Kepler considers this observation as very accurate, and in the limit it would have given a latitude = 1° 50′.

On the 30th of the same month, at 19h. 30′, Mars appeared in 5s. 20° 4′, that is, almost in the limit, with 2° 13′ 30″ of north latitude. But, the sun's longitude being 8° 18″, the elongation was 87° 56′ instead of 72° 23′. The distance AC therefore, was much less than BC, and the observed latitude considerably greater than the inclination.

Two observations were also found of the planet, near the same limit, from the earth towards the point H.

On the 9th of March 1596, at 8 h. the planet's observed longitude was 2s. 15° 14′, with 1° 49′ 50″ of north latitude, and the sun's longitude 0s. 0° 0′. The elongation therefore was too great, but the planet was 26° short of the limit; and, balancing the effects, Kepler states the inclination at 1° 50′.

On the 22d of April 1583, at 9h. 45′, the planet's geocentric longitude was 4s. 1° 17′, with 1° 50′ 40″ of latitude, and the sun's longitude 1 s. 11°. The elongation therefore was too great by about 8°; but the planet had passed 19° beyond the limit, and the inclination in the limit is again stated at 1° 50′.

No more than two observations of the planet were found in the opposite limit, and in both the earth was on the same side of it.

The first was on the 5th of September 1589, at 7 h. 15′, when the planet's geocentric longitude was 8s. 16° 45′ 40″, with 1° 41′ 40″ of south latitude, and the sun's longitude 6s. 2°. The elongation therefore was 74° 45′ 40″ instead of 68° 41′, and the planet fell 17° 36′ short of the limit.

limit. Balancing the opposite effects of these causes, the inclination at this limit also appears to be 1° 50′.

The other was on the 1st of the following November, at 6 h. 10′; when the observed longitude was 9s. 20° 50′, with 1° 36′ of latitude, and the sun's longitude 7 s 19°. The elongation therefore was only 61° 50′, and the planet also 16° beyond the limit. These causes combined, must render the latitude less than the inclination, which again appears to be about 1° 50′.

In the same investigation Kepler employed another method, equally of his own invention, and which required, neither any pre-conceived opinion concerning the ratio of the orbits, nor the aid of calculation to distinguish the observations proper for the purpose. It depends on this principle, that when two planes ABCD, and ABEF (fig. 53), intersect each other, two perpendiculars CB, EB, drawn in both planes to the point B of the line of intersection AB, will contain an angle CBE equal to the angle DAF contained by two other perpendiculars drawn in the same planes to any other point A of the line AB, and also equal to the inclination of the planes. If, therefore, there be any observations of a planet, when both the sun and the earth are in the line of its nodes, and the geocentric place of the planet 90° from the nodes, the observed latitude will at once give the inclination.

Second method.

Though observations with such conditions are rare, four were found in T. Brahé's collection, in circumstances not much different. One was, that of April 22d, 1583, already employed. The sun was in 1s. 11°, that is, 5 or 6 degrees below the node, and consequently the earth as much above the opposite one. But, though this tended to in-

crease

crease the apparent latitude, the elongation, which was only 80° 17′, instead of 90°, tended rather more to diminish it; and Kepler concluded, that it might be considered as very nearly equal to the inclination.

Again, on the 12th of November 1584, at 13h. 20′, the sun was in 8 s. 1°, that is, 14° or 15° below the line of nodes, and Mars was seen in 4s. 23° 14′, with 2° 12′ 24″ of north latitude. The planet therefore had passed the limit, and was not in the point of greatest inclination. But the elongation was 97° 46′, instead of 90°, and the latitude, on account of the nearness of the earth, much more increased than by the other cause diminished; and on balancing both, the inclination appeared to be 1° 50′.

On the 26th of April 1585, at 9h. 42′, Mars was observed in 4s. 21° 6′, with 1° 49′ 45″ of latitude, while the sun was in 1s. 16°, that is, almost in the node. The elongation was too great, being 95° 6′: but the planet was beyond the limit, and therefore the latitude somewhat less than the inclination.

Also, on the 16th of October 1591, at 6h. 30′, Mars, towards the opposite limit, was observed in 10s. 27° 20′, with 2° 10′ 30″ of south latitude decreasing, while the sun was in 7s. 2° 30′, that is, 15° above the node. The observed latitude therefore, as the earth was too near the limit, was greater than the inclination. The latitude had decreased 28′, from the 2d to the 16th of October; and, if it had continued to decrease at the same rate for the next fourteen days, about which time the sun would have occupied the node, it would have been reduced to 1° 45′. But, on the withdrawing of the earth from the planet, the rate of decrease varies, and the conclusion of 1° 50′ for the inclination is confirmed.

This method of determining the inclination may be made general, for any geocentric position of the planet; for any two lines BG, BK, drawn in both planes to any point B of the line of nodes, will include an angle GBK, equal to the angle HAL, included between two other lines AH, AL, respectively parallel to BG, BK, and also drawn in both planes to any other point A of the line of nodes. Extension of it.

Thus, in the observation of April 26, 1585, when the sun's longitude was 1s. 16°, the planet was seen from B in 4s. 21° 6′, with a latitude GBK = 1° 49′ 45″. This therefore is equal to the heliocentric latitude HAL of the planet, in 4s. 21° 6′ of heliocentric longitude; and as this point is distant 4° 6′ from the limit, or 85° 54′ from the node, the inclination, or heliocentric latitude in the limit, will be found by this analogy, sin. 85° 54′ : sin. 1° 49′ 45″ :: sin. 90° : sin. 1° 50′.

To confirm this conclusion from so many observations, Kepler also employed the method originally used by Copernicus, though it supposes a pre-conceived opinion concerning the ratio of the orbits. Let A (fig. 29), be the sun, B the earth, C the planet Mars in his orbit, and D the point of the ecliptic to which he is referred by a circle of latitude. Let CBD, the observed latitude, at the opposition, for example, of the 24th of August 1593, be 6° 3′ south; and let it be previously known, that in this point of the orbit AB : : AC :: 1000 : 1389. Then, since AC : AB :: sin. ABC : sin. ACB, this angle will be found = 4° 21′ 6″, so that, subtracting it from CBD, the heliocentric latitude CAD will be found = 1° 41′ 54″: and, since the longitude of Mars in this opposition was 11s. 12° 30′, and consequently his distance Third method.

distance from the node = 64° 30′, the analogy, sin. 64° 30′ : sin. 90° : sin. 1° 41′ 54″ : sin. 1° 50′ 20″, will give this last angle for the required inclination.

118. It was by such careful and multiplied investigations, and from so many different observations, in the most various configurations of the planet with the sun and the earth, at 60° or 70° of elongation, as in the first method, at both quadratures, reckoned from the nodes as in the second, and at any opposition, taken at pleasure, as in the third, that Kepler established this most important conclusion, and which, though fundamental to every just planetary theory, had never been before established, that the inclination of the orbits is invariable and constant; and that, in the orbit of Mars in particular, instead of amounting to 4° 33′ in the northern limit, and 6° 26′ in the southern, as T. Brahé had supposed, it was very little more than 1° 50′. By this discovery the oscillations, which, in consequence of the variety of observed latitudes, Ptolemy had ascribed to the epicycle, and Copernicus, on the contrary, to the orbit of the planet, and T. Brahé's equally improbable infraction of the orbit, were demonstrated to have no existence. Ptolemy appears to have been led to his improbable doctrine, in the same manner with T. Brahé, by the intricacy and perplexity of his hypothesis, which prevented him from perceiving, that the whole varieties of latitude might have been justly represented by the single supposition of the parallelism of the plane of the epicycle to the plane of the ecliptic: and besides, his observations of latitude were not sufficiently numerous, nor did he pay equal attention to

Important discovery, that the inclination was invariable.

to all; but, misled by this theory, frequently rejected those which disagreed remarkably with the greater number. Copernicus again was misled by his reverence for Ptolemy, and his not forming a just estimation of the value * of his own theory, nor placing in it sufficient confidence. Though he beheld, as we are told, with great satisfaction, its explication of the increase of the latitudes as the earth approached the planets, and of their decrease on the withdrawing of the earth; when he found that increase and decrease not entirely consonant to Ptolemy's account of his observations, he introduced oscillations similar to Ptolemy's; and, as it was impossible that the ecliptic should be subjected to them, he was obliged to transfer them to the planetary orbits. This doctrine, however, of an inclination which varied, not according to the situation of the planet in the oscillating orbit, but of the earth in an orbit which did not oscillate, was what Kepler tells us he always thought incredible, even before he had the good fortune of perusing T. Brahé's observations; and his opinion was not more judiciously pre-conceived, than we have seen it to be decisively confirmed, by experiments most numerous and most fairly conducted.

* Divitiarum suarum ignarus.

Determination of the oppositions.

119. After having thus determined the longitude of the nodes, and the inclination of the orbit to the ecliptic, he next proceeded to ascertain the points of the ecliptic where the apparent oppositions on which he was to form his theory took place; and he was now enabled to reduce them justly and legitimately to the points of the orbit which the planet really occupied. His procedure in deducing, from observations near the opposition,

tion, the time and point of the ecliptic at which the opposition actually happened, was the same in principle with those since employed by the modern astronomers (G); and the same laborious attention and accuracy were displayed in it which distinguished all his investigations. The oppositions which he thus determined were twelve in number, consisting of the ten employed by T. Brahé, and two observed by himself; and he had the advantage of verifying some of them by means of the observations made by David Fabricius, in East Friesland. The mean longitudes of Mars were calculated from T. Brahé's tables, and the whole catalogue is as follows. (M.)

	Times Merid. Uraniburg.					Long. ♂ in the orbit.				Latitudes.				M. longit. ♂			
			d.	h.	′	s.	°	′	″	°	′	″		s.	°	′	″
1.	1580.	Nov.	18	1	31	2	6	28	35	1	40	0	N	1	25	49	31
2.	1582.	Dec.	28	3	58	3	16	55	30	4	6	0	N	3	9	24	55
3.	1585.	Jan.	30	19	24	4	21	36	10	4	32	10	N	4	20	8	19
4.	1587.	Mar.	6	7	23	5	25	43	0	3	41	0	N	6	0	47	40
5.	1589.	Apr.	14	6	23	7	4	23	0	1	12	45	N	7	14	18	26
6.	1591.	June	8	7	43	8	26	43	0	4	0	0	S	9	5	43	55
7.	1593.	Aug.	25	17	27	11	12	16	0	6	2	0	S	11	9	55	4
8.	1595.	Oct.	31	0	39	1	17	31	40	0	8	0	N	1	7	14	9
9.	1597.	Dec.	13	15	54	3	2	28	0	3	33	0	N	2	23	11	56
10.	1600.	Jan.	18	14	2	4	8	38	0	4	30	50	N	4	4	35	50
11.	1602.	Feb.	20	14	13	5	12	27	0	4	10	0	N	5	14	59	37
12.	1604.	Mar.	28	16	23	6	18	37	10	2	26	0	N	6	27	0	12

120. In order to understand distinctly the procedure of Kepler in the formation of his theory, it seems necessary to recall to our recollection the principles or foundations of the ancient theories. Ptolemy, in particular, supposed, 1. That the motions of every planet, however unequal in appearance, were notwithstanding performed in circles.

The

Ptolemy's fundamental principles.

The inequalities of Mars, for example, were very great; for the arch, intercepted between the opposition in 3s. 2° 28′, and that in 8s. 26° 43′, is only 5s. 24° 15′, and less than a semi-circle; but the corresponding mean longitudes are 2s. 23° 11′ 56″ and 9s. 5° 43′ 55″; and the difference between them is 6s. 12° 31′ 59″ greater than a semi-circle. The planet therefore employs more than half the time of a compleat revolution to describe an arch of 5s. 24° 15′, which is less than a semi-circle; and of consequence it will describe the supplemental arch of 6s. 35° 45′, which is greater than a semi-circle, in less than half the time of a compleat revolution. The arches, however, described by the planet, seem to be circular; for its velocity regularly increases to its *maximum* in one particular point of the zodiac, and declines to its least rate at the opposite point; and the transitions from the greatest velocity to the least, and from the least to the greatest, are gradual: whereas, if it moved in straight lines, making angles with each other, as in a pentagon, for example, according to a supposition once made by Kepler, the transitions would be sudden, and take place in more points of the zodiac than two. 2. That though the motions of every planet seemed to be performed in circles, their inequalities, abstracting even entirely from those of the second kind, demonstrated that the eye, or the earth could not possibly occupy the centre of any one of them. As, therefore, there were two methods of reconciling apparent inequality of motion with real uniformity, either by a composition of two circles, one of which was concentrical to the earth, or by a single circle excentrical to it, Ptolemy preferred the latter method; partly for its simplicity, and partly because he knew of no way to represent the second

second inequalities, except by epicycles, and was unwilling unnecessarily to multiply them. But, 3. Ptolemy also says, that he found, by the comparison of numerous observations, the centre of his epicycle to approach in the apogee of the planet much nearer to the earth, and to depart in the perigee much farther from it than the limits of the excentric, by which he represented the first inequalities, could permit; and that, on applying himself to find the measure of this approach of the epicycle within the apogee of his excentric, or rather equant, and of its departure beyond the perigee, he discovered that the centre of the circle in which the epicycle revolved, precisely bisected the distance between the centres of the equant and of the earth. On these three principles Ptolemy founded his whole theory of the superior planets; and Copernicus, though in a different form, religiously followed Ptolemy. The principle last mentioned, was the celebrated bisection of the excentricity, which he assumed without giving any account of the means by which it was discovered: and his adopting it, in this arbitrary manner, justly excited the wonder of all astronomers, who ascribed it to no better cause than mere conjecture.

121. Ptolemy, in the formation of his theory for the three superior planets, had no necessity for employing more than three observed oppositions: for these, with this assumed principle of the bisection, were sufficient to determine all its elements; the excentricity, the longitude of the apogee, and the correction which might be necessary of the mean motion of the planet. But, as Kepler had always thought that the bisection of the excentricity was a principle unwarrantably

 assumed,

assumed, and had considered it as one of the principal causes of the errors and imperfections of the Ptolemaic theory, he resolved, not to assume, but to investigate the ratio in which the excentricity of the equant was to be divided: and, on his repairing, in the year 1600, to T. Brahé, he found, to his great satisfaction, that this astronomer had also departed from Ptolemy's authority, and had divided the excentricity in a different ratio. Even Copernicus, who, except in the distinguishing parts of his system, had a greater deference for Ptolemy's authority than any other astronomer, was evidently found to account it in this instance doubtful; and he also had substituted for the bisection, in the orbit of Mars, a different ratio of division. Kepler therefore only retained the two first of Ptolemy's principles, that the planet's orbit was circular, and that the circle in which it moved uniformly, was excentrical to the earth, or, in the language of Copernicus, to the sun. But his rejection of the bisection rendered his procedure in forming his theory much more difficult; and he found that, instead of three oppositions, which, in conjunction with this principle, had been sufficient, the ratio in which the excentricity was to be divided, could not be determined from less than four. The procedure also was the more unpleasant, that no direct or geometrical method of investigation could be found; and it became necessary to have recourse to the indirect and tedious methods of false position.

T. Brahé rejects the Ptolemaic bisection.

122. On the centre B (fig. 54), describe the circle DEFG, and let the diameter HK represent the line of apsides, supposed for a few years to be immoveable. In this line let the eye be placed at A, and in the same line, though in the opposite

site direction from B, let C be the centre of the equant, about which the planet describes angles proportional to the times. Let D, E, F, G, be four oppositions, in which the places of the planet, as seen from A, are in the lines AD, AE, AF, AG. As the four angles at A, included between these lines, are given by observation, it is required to determine such angles FCH, FAH, of mean and true anomaly, that the points D, E, F, G, may be found in the circumference of a circle, whose centre B shall be found in some point of the line AC, and to find its distance from A, and C.

These angles FCH, FAH, are first to be assumed: and to assume them, is, to suppose given two of the things that are to be investigated; to wit, the longitude of the aphelion H, and the planet's mean longitude, or the position of the line FC. But the truth or falsehood of these assumptions will be discovered by deducing their consequences.

Since the position of AH, and the mean longitude or position of CF, &c. are given, the angles at A and C formed with the line ACH, by the lines drawn from these points to the points D, E, F, G, will be given: and, since AC is the common base of the four triangles CFA, CEA, CDA, CGA, if it be denoted by any known number, the four lines AF, AE, AD, AG, will be also given in parts of AC. Since also in the other four triangles FAE, EAD, DAG, GAF, the sides with the included angles are given, the angles AFE, AFG, ADE, ADG, will be given; that is, the two angles EFG, EDG. But these, by the supposition, are angles at the circumference of a circle, and the opposite angles of the quadrilateral figure DEFG inscribed in it: therefore their sum should

 be

be equal to two right angles: and, if it be either greater or less, one of the assumptions for the longitudes of AH and CF, and perhaps both, are false.

Formation of Kepler's theory.

Retaining, therefore, the same mean longitudes, let the longitudes of the aphelion H be changed; by which means, though the angles FAH, FCH, of anomaly are varied, the former equations AFC, AEC, &c. will continue: and let the operation be repeated, till the sum of the angles at the circumference be found a second time. If the difference between it and two right angles be greater than before, the change upon the position of ACH has been made for the worse; and, if its longitude by the change was increased, it must be now diminished; or, if diminished, it must be now increased. But, if the result comes nearer than the former to the sum of two right angles, it shews that you are in the way of coming at the truth: and a comparison of the present error with the former, will assist you in judging how much the longitude of AH should be again diminished, or increased. The result, however, upon the angles at D and F, will not be in the simple ratio of the change made on the position of AH: but the operation must be, probably many times, repeated, before the sum of these angles be found accurately, or even but nearly, equal to two right angles.

When you have at last obtained a sum of the two angles EFG, EDG, equal, or at least very nearly equal, to the sum of two right angles, and thus found that D, E, F, G, are points in the circumference of the same circle, it is next to be enquired, if B the centre of it lies in the line AC. For otherwise the motion will not be slowest at the greatest distance from the earth, or the sun, in A; as Ptolemy supposed that his observations required,

required, and Kepler, that also physical causes rendered necessary. In prosecuting this enquiry, from the sides AE, AG of the triangle AEG, which were before found, and the given angle GAE = GAD + DAE, the side GE, and the angle AGE, are first to be determined Next, in the isosceles triangle BEG, where the angle GBE is given, being double of the angle GFE at the circumference, and consequently the angle at the base BGE, the semi-diameter BG is to be determined in parts of AC. Finally, in the triangle AGB, with the given sides AG, BG, and the included angle AGB = BGE — AGE, the angle BAG is to be determined: and, if this shall differ from CAG, or HAG, before assumed, the centre B does not fall in the line AC; and the suppositions for the longitudes of FC and ACH must, one, or perhaps both, be false.

But no new supposition can be made for the position of AH, unless one be also made for FC, and the other mean longitudes, because we have already investigated the only longitude of AH, which, with the assumed mean longitude, will permit the four points D, E, F, G, to be in a circle. In varying therefore the position of AH, we must also vary the position of FC, and the other mean longitudes; and then the position of AH may be changed five or six times, if necessary, and the first operation as often repeated, till, with the new assumptions for FC, EC, and the rest, the sum of the angles at F and D be again found equal to two right angles. When this shall be accomplished, the second part of the process, namely, to find the angle BAG, is to be repeated: and, when this angle BAG is compared with the new assumption for CAG, it will be discovered, whether you are approaching to your purpose, or departing from it: and you will have directions in what manner other assumptions are to be made, with which all the former operations are to be repeated, till at last BAG, and CAG, come out equal, while D, E, F, G, are in the circumference of the circle described on the centre B.

When this is at last accomplished, if the semidiameter BG of this circle be denoted by any number of known parts, suppose 100000, BA its excentricity, and CA the excentricity of the equant, will easily be found in the same parts; and the suppositions for the place of the aphelion, and the mean longitude of the planet, at any one of the oppositions, made immediately before the last repetition of the operations, will be established. (N.)

123. This then was the method in which Kepler investigated the elements of his first theory of Mars, which he afterward distinguished by the name of the vicarious theory; and, when the detail of it is so fatiguing, he appeals to his readers if he, who had gone over all its steps no less than seventy times, was not justly entitled to their sympathy; and if it was at all surprizing that four years had been consumed in the formation of it. If we consider that logarithms were not then invented, the justice of his appeal will be readily admitted.

Vicarious theory.

The elements of the theory deduced from this painful investigation were, that, in the opposition of 1587, on the 6th of March, at 7 h. 23′, the longitude of the aphelion of Mars was 4 s. 28° 48′ 55″; that the planet's mean longitude was 6 s. 0° 51′ 35″, that is, 3′ 55″ more advanced, than according to T. Brahé's tables; that the excentricity of the equant was 18564, in parts of the semi-diameter of the orbit = 100000, and that the excentricity

excentricity of the orbit was 11332. The excentricity therefore of the equant was not bisected, as in the Ptolemaic theory, but divided by the centre of the orbit in the ratio of 11332 to 7232: and, according to the forms of Copernicus and T. Brahé, 14988 was allotted for the semi-diameter of the greater, and 3628 for the semi-diameter of the smaller epicycle. The improvement therefore of astronomy, which Kepler originally meditated, was now completed. In it he agreed with preceding astronomers, in supposing the orbit to be circular, and that there was a fixed point within it, about which the planet described angles proportional to the times: but he differed from them with respect to its relative position to the eye; for drawing his line of apsides through the centre of the sun, he rendered the real motions slowest, as the physical causes seemed to require, at the points most distant from the sun; and avoided the absurdity of assigning such points to the greatest aud least velocities as seemed inconsistent with their apparent cause.

124. It only remained that this vicarious theory should be verified, by the exactness with which it represented the heliocentric longitudes, in all the observed oppositions. But, that the verification might be legitimate, it was previously necessary to determine the motion of the aphelion. This determination, however, could not be precisely accurate; for it depended entirely on the observations transmitted by Ptolemy, and these were not placed entirely beyond suspicion.

Let A, (fig. 49), be the centre of the terrestrial orbit MN; let E be the centre of the equant of Mars, and B the centre of the sun. According to Ptolemy's accounts, the line AB of the solar apsides

apsides was in 2s. 5° 30′, and AEF the line of the apsides of Mars, in 3° 25′ 30″; so that the angle BAE was = 50°. According also to Ptolemy, the solar excentricity AB was = 4153, in parts of the semi-diameter AM = 100000; and AE, the excentricity of the equant of Mars, in the same parts = 30380, so that by resolving the triangle AEB, the angle ABE comes out = 123° 27′. The longitude therefore of BEC, the new line of apsides, according to Kepler's theory, was 4s. 2° 3′ in the days of Ptolemy: and Kepler was disposed to consider it as no small presumption in favour of this conclusion, that BE the excentricity of the equant, given by the solution of this triangle, is, in parts of the semi-diameter TC = 100000, very nearly the same with what had been determined by his own theory.

Motion of the aphelion of Mars.

As Ptolemy's account of the precession was irreconcileable with every other, it was impossible for Kepler to deduce from it the longitude of the apogee in 1587, and he was obliged to compare the place of it with that of a fixed star. By an observation of Ptolemy, the longitude of Cor leonis, in the year 140, was 4s. 2° 30′; and, as Kepler found the apogee then in 4s. 2° 3′, its distance from that star, *in antecedentia*, must have been 27′. But T. Brahé found the same star, in 1587, to be in 4s. 24° 5′; and, according to the vicarious theory, the apogee was also in 4s. 28° 49′, and distant from Cor leonis 4° 44′, *in consequentia*. Its whole motion, therefore, with respect to this star, in 1447 years, that is, from 140 to 1587, was 5° 11′; and consequently its annual motion nearly 13″, or in 30 years 6′ 29″. If we add to this the motion of the fixed stars in precession, and which, according to T. Brahé, amounts to 25′ 30″ in 30 years, the motion of the aphelion of Mars, with respect

respect to the equinoctial points, will, in 30 years, be 31′ 59″ *in consequentia*, and its annual motion = 1′ 4″. Nor does this determination differ much from the most accurate determination of later times; and in particular, in De la Lunde's tables, it is reckoned at 1′ 7″.

As Kepler purposed also to verify his theory by means of the latitudes of the planet, so he endeavoured on this occasion to determine the motion of the nodes. He collected from Ptolemy's accounts, that the distance of the northern limit from Cor leonis was, about the year 140, 3° 29′, or 3° 30′, *in antecedentia*. But, in 1587, its distance from the same star, also *in antecedentia*, was 7° 45′. The difference is 4° 15′, and this therefore was the retrograde motion of the limit, or of the nodes, from Cor leonis in 1447 years. The annual motion, therefore, of the nodes, with respect to the fixed stars, is 10″.56, and for 30 years 5′ 17″, *in antecedentia*; and, subtracting this from 25′ 30″, the precession for 30 years, the remainder 20′ 13″, shews that the motion of the nodes, with respect to the equinoctial points, is *in consequentia*. De la Lande makes it 19′ 54″.

125. After thus determining the motion of the aphelion, Kepler proceeded to the verification of his theory, by calculating from it the longitudes of the planet in all the observed oppositions, and comparing them with the observed longitudes. He sets before us every step of his twelve calculations; but an example of one of them will be here sufficient. Suppose that it is required to calculate the planet's longitude at the opposition of 1602.

In

Verification of the theory, by longit. in opposition.

	s.	°	′	″
In 1587, the longitude of the aphelion now found was -	4	28	48	55
Its motion, *in consequentia*, for nearly 15 years, is - -			15	56
Longitude of the aphelion, at the opposition of 1602 - -	4	29	4	51
Mean longitude of the planet, from T. Brahé's tables + 3′ 55″	5	15	3	32
Mean anomaly, or angle FCH (fig. 54) - -		15	58	41

Since BC is = 7232 in parts of BF = 100000, we have, in the triangle BCF, sin. BFC = BC. sin. FCH = 7232.37528 = 1991. Therefore the angle BFC, or the physical equation is = $1^\circ\ 8'\ 26''$.

Since also FBH = FCH — BFC = $14^\circ\ 50'\ 15''$, and BA = 11332, we have, in the triangle AFB, tan. $\frac{1}{2}$ (BAF — AFB) = tan. $\frac{1}{2}$FBH $\left(\frac{BF - BA}{BF + BA}\right)$ = 79643.13021 = 10370. Whence $\frac{1}{2}$(BAF — AFB) = $5^\circ\ 55'\ 14''$, and the optical equation BFA = $1^\circ\ 29'\ 53''.5$. The whole equation therefore is $2^\circ\ 38'\ 19''.5$; and the true

	s.	°	′	″
anomaly FAH = - -	0	13	20	21.5
Adding to this the longitude of the aphelion - -	4	29	4	51
The calculated long. of Mars, at the opposition of 1603, is -	5	12	25	12.5
The observed longitude was -	5	12	27	—
Difference, though one of the greatest, no more than -			1	47.5

The verification therefore of the theory, as far as depended on longitudes in opposition, was compleat. As it was superior to all that preceded it, in its principle, in the accuracy of its structure, and in the legitimate manner of determining all the

the conditions by which it could be affected; so likewise it was superior in the exactness with which it represented all the longitudes in opposition. Its errors never exceeded 2′; that is, they never exceeded the supposed apparent magnitude of the planet: and if we consider how far it surpassed in accuracy the theory of T. Brahé, it will not be surprizing that Kepler indulged the most sanguine expectations from it.

126. Contrary however to all expectation and probability, and notwithstanding the present verification by oppositions, this theory, on which so much time and labour had been expended, was found to be false, and soon received the most decisive confutation. The principle especially of dividing the excentricity in a different ratio from bisection, and which Kepler had supposed to be of the greatest importance, was found altogether inadmissible, not merely in his own theory, but even in the ancient ones: and it seemed probable to him, that, when Ptolemy originally introduced the bisection, when Copernicus was so scrupulous in departing from it, and when T. Brahé hesitated and left his theory unfinished, they had been induced by the reasons which convinced himself of the impropriety of rejecting it. Refutation of it.

The first refutation of the *vicarious* theory arose from the latitudes in opposition, and especially from those observed near to the apsides and the limits. Let DE (fig. 55), be a line in the plane of the orbit of Mars, and HL a line in the plane of the ecliptic, both passing through A the centre of the sun. Let D be a point towards the northern limit, and E towards the southern: and, let AD, AH, be in the plane of the same circle of latitude, as also AE, AL. Let the earth, at the opposition First by latitudes in opposition.

of

of 1585, be in B, and at the opposition of 1593 in C. Since the planet's longitude, in the former, was 4s. 21°, and, in the latter, 10s. 12°; and, since the longitude of the earth's aphelion is 9s. 5°, it is evident that the distance AC of the earth in 10s. 12° from the sun, must be greater than AB, its distance in 4s. 21°. On T. Brahé's supposition, that the excentricity of the earth's orbit is 3584, in parts of the semi-diameter = 100000, AC will be = 101400, and AB = 97500. If the point B were precisely in the limit, the angle BAD of heliocentric latitude would be = 1° 50′ (118), but, as it is 4 or 5 degrees from the limit, BAD at this point will be only 1° 49′ 30″. But the observed latitude HBD, was 4° 32′ 10″, and therefore BDA = 2° 42′ 40″; and consequently $AD = AB \frac{\sin. HBD}{\sin. BDA} = 163000$. Or, if according to the theories founded on mean oppositions, A were the centre, not of the sun, but of the earth's orbit, and consequently AB = 100000; AD = $AB \frac{\sin. HBD}{\sin. BDA}$, would be found = 167000.

In the same manner the angle CAE of heliocentric latitude, at the opposition of 1593, is found to be 1° 39′: for the points C and E are 64° distant from the node, and sin. 90° : sin. 1° 50′ :: sin. 64° : sin. 1° 39′. But the geocentric latitude ECL was 6° 3′, and consequently AEC = 4° 24′. Therefore $AE = AC \frac{\sin. ECL}{\sin. AEC} = 139300$. Or, if AC were = 100000, then AE = 137380.

If the excentricity of the earth's orbit should not amount to 3584, but, as Kepler suspected, to little more than one-half of it, the results for AD and AE would be different; greater, to wit, than the first, and less than the second.

From

From these lengths of AD and AE, the former in 4s. 21°, and the latter in 10s. 12° of longitude, their lengths at the apsides of Mars in 4s. 28°, and 10s. 28°, may at least nearly be deduced; for AD, at the distance of 8 degrees from the aphelion, would in the aphelion be 150 parts longer; and AE, at the distance of 16 degrees from the perihelion, would in the perihelion be 300 parts shorter. When, therefore, DA and AE coincide with the line of apsides,

AD will certainly be	between	163150 &	167350;
and AC -	between	139000 &	137080:
so that DE -	between	302150 &	304430;
and DK = $\frac{1}{2}$DE	between	151075 &	152215;
and the excentricity AK = AD — DK, certainly	between	12075 &	15135.
Or, in parts of the semi-diameter of the planet's orbit = 100000, -	between	8000 &	9943.

But, by the theory now investigated, AK was found to be = 11332. Some of the principles assumed in forming it must therefore be false. One of these was, that the orbit of the planet is a perfect circle; and the other, that there is a fixed point in the line of apsides, about which the planet describes equal angles in equal times. Since then the observations were accurate, one of these principles, or perhaps both, must be false. The same conclusion holds equally in the theories formed on mean oppositions: for the excentricity, instead of arising to 12352, as T. Brahé had determined, is by these observations of latitude reduced to 9943.

127. It was on account of such observations of latitude, that Kepler supposed Ptolemy to have introduced

Probable origin of the bisection.

introduced the doctrine of the bisection of the excentricity. For, as the excentricity of the equant in Kepler's theory is 18564, the half of this, to wit, 9282, differs only 310 parts from 8972, the mean between 8000 and 9943, the two extreme results from the present observations of latitude: and if, conversely, we apply the principle of bisection to the calculation of the longitudes in opposition, the differences from the observed longitudes will be found, at least in 90° of distance from the apsides, to be very inconsiderable. Even in the octants, they were not of such consequence as to attract the attention of Ptolemy, whose methods of observation appear to have been very imperfect, in comparison with those of later times.

For example, if at the opposition of 1593, where the planet's mean anomaly was 6s. 11° 3′ 16″, we substitute in the analogy BD : BC :: sin. DCK : sin. BDC for the physical equation, (fig. 54), 9282 for 7232, the amount of BC, according to the vicarious theory, this equation will become 1° 1′ 12″; and by substituting also 9282 for 11332 in the analogy for the optical equation, it will become 1° 13′ 26″. CDA, therefore, the sum of both, will be 2° 14′ 38″; and, consequently, the longitude = 11s. 12° 13′ 37″, differing only 3′ 5″ in defect from that given by the vicarious theory.

If the bisection be in like manner employed, at the opposition of 1582, when the planet's distance from the aphelion was about 41°, the calculated longitude will become 3s. 17° 4′ 45″; greater by 7′ 41″ than according to the vicarious theory, and by 9′ 15″ than according to the observation.

When Ptolemy therefore was induced, as Kepler supposed, by the latitudes in opposition, to adopt the principle of the bisection, it was impossible that these errors produced by it upon the longitudes

tudes should divert him from his purpose: for, as he did not pretend to greater accuracy of observation than within 10′, and those errors never exceeded 8′ or 9′, and in many parts of the orbit were much more inconsiderable, they were all within the limits of his acknowledged errors of observation. But the advantage which Kepler enjoyed, of perusing such observations as those made by T. Brahé, no longer permitted him to ascribe errors of theory to inaccuracy of observation: and he was perfectly convinced by his present deductions, that, if the orbit were a perfect circle, there was no fixed point within it, about which the planet could describe equal angles in equal times. From these deductions it undeniably appeared that, if at 90° and 270° of anomaly the bisection might with propriety be introduced, yet, near the apsides, the distance BC of the centres of the orbit and equant was greater, and, near the octants, less, than 9282, the half of 18564. If the orbit, therefore, was to be accounted circular, and its excentricity BA, as the latitudes required = 9282, no method seemed to be left for extinguishing the differences between the observed and calculated longitudes, except the libration of the centre of the equant; and this was a supposition, both improbable, and apparently incapable of being derived from any natural cause. These eight minutes therefore of difference, produced by the bisection, between the observed and calculated longitudes, Kepler informs us, were the sole cause of his farther researches in astronomy, and of the total reformation of it, which he eventually introduced. If, like Ptolemy, he could have neglected these 8′, and ascribed them to errors of observation, he would have rested content with the principle of

Cause of all the future discoveries made by Kepler.

the bisection, and accounted it a sufficient correction of his former theory.

Second refutation,

128. The second refutation of the vicarious theory, notwithstanding the accuracy with which it represented the longitudes in opposition, was derived from longitudes observed at points not in opposition, when the planet was near the apsides.

For example, on the 5th of March, 1600, at 12h. Mars was observed in 3s. 29° 12′ 30″, with 3° 23′ of north latitude. His mean longitude was 4s. 29° 14′ 58″, and the longitude of his aphelion 4s. 29° 2′ 45″. The mean anomaly therefore was 12′ 13″, and, by the vicarious theory, the equation was 2′ subtractive. The true heliocentric longitude therefore was 4s. 29° 12′ 58″; while at the same time the sun's longitude was 11s. 25° 45′ 51″.

Let A (fig. 56), be the sun, B Mars, and C the earth. Since CB is in 3s. 29° 12′ 30″, and AB in 4s. 29° 12′ 58″, the angle CBA of annual parallax will be = 30° 0′ 28″; and, since CA is in 11s. 25° 45′ 51″, the angle BCA of elongation will be = 123° 26′ 39″. Since also CA the distance of the earth from the sun, in this part of the orbit, calculated for T. Brahé's excentricity = 3584, is 99302, we shall have $AB = CA \frac{\text{sin. } BCA}{\text{sin. } CBA}$ = 165680. Or, by the theory of mean oppositions, where CA is the semi-diameter of the earth's orbit and = 100000, AB will be = 166846.

by longitudes out of opposition.

Again, on the 30th of July, 1593, at 13h. 45′, Mars was observed in 11s. 17° 39′ 30″ of longitude, with 6° 6′ 15″ of south latitude. His mean longitude was 10s. 26° 16′ 38″; and, as the aphelion was in 4s. 28° 55′ 43″, the mean anomaly was 5s. 27° 20′ 55″, and the equation 32′ subtractive.

tractive. The true heliocentric longitude therefore was 10s. 25° 44′ 38″, and the sun's longitude was 4s. 17° 3′.

Let then D be Mars, and E the earth. The parallax EDA will now be 21° 44′ 52″, and the elongation AED = 149° 23′ 30″: and since, according to the theory of apparent oppositions, AE is the distance of the earth from the centre of the sun, and in the present case = 102689, we shall have $AD = AE \frac{\text{sin. AED}}{\text{sin. EDA}} = 140080$. Or, by the theory of mean oppositions, where AE is = 100000, $AD = AE \frac{\text{sin. AED}}{\text{sin. EDA}} = 136409$.

As the distance of AB from the aphelion was only 12′ 11″, any correction for its length in the aphelion would be insensible; but, as AD was 3° 11′ 7″ distant from the perihelion, its length in the perihelion would be less by 15, and consequently = 140065, or 136394.

The heliocentric latitude FAB, at the point B of the orbit, is = 1° 48′; and, therefore AF in the plane of the orbit 82 parts longer than AB; and the heliocentric latitude GAD is = 1° 36′, and therefore AG 72 parts longer than AD.

Therefore AF is between	165762	and	166928
AG between	140137	and	136466
Consequently FG -	305899		303394
and HG = ½FG -	152949.5		151697
and the excentricity AH = AF – HF =	12812.5		15231;
and in parts of HF or HG = 100000, it is between	8377	and	10106.

A mean of these is 9241.5, differing very little from 9282, found by bisecting 18564, the whole excentricity of the vicarious theory.

Since, therefore, the excentricity deduced from latitudes, and from longitudes out of opposition, differed so remarkably from that given by the longitudes in opposition, and investigated in forming the vicarious theory, one at least of the principles of that theory must be false. If the orbit of the planet were a perfect circle, it undeniably appeared, there could be no fixed point within it, about which the planet moved uniformly; or, if the centre of uniform motion were a fixed point in the line of apsides, the orbit of the planet could not be a perfect circle.

CHAP.

CHAP. VII.

Of Kepler's Solar Theory, that is, his Theory of the Second Inequalities.

129. AS Kepler's labours had been attended with so little success, while, in imitation of former astronomers, he began his researches from the first inequalities of the planets, he resolved to reverse this procedure, and to begin from their second inequalities; that is, he resolved to investigate the whole circumstances of the annual motion of the earth, by which these second inequalities were produced. The theory, which he had formed from observations of longitude in opposition, had been directly contradicted by observations equally authentic of other kinds; and the principles of it partly subverted, and partly rendered doubtful; and he expected by this alteration of procedure, to determine which of those doubtful principles was to be retained, or whether they ought not both to be rejected. Among the reasons which led him to this alteration of procedure, the investigations immediately preceding were the chief; for, as in these, the distances of the earth from the sun were necessary conditions, it was evident that, if the excentricity of the terrestrial orbit were not justly determined, all the distances would be false, and consequently all the investigations founded on them. The knowledge, therefore, of all the circumstances of the second inequality, that is, an accurate and authentic solar theory, was considered by Kepler as the key to all

Cause of beginning his researches from the solar theory.

the mysteries of astronomy; and in fact, by his use of it, the greatest part of them has been happily disclosed.

Suspicion of the necessity of an equant to the earth.

130. One of the propensities, by which Kepler was remarkably distinguished from other men, was, an unconquerable desire of discovering the causes of natural phenomena, and of tracing them up to general analogies and laws. Accordingly, when, in his most early and juvenile production, he endeavoured to assign a cause for the planetary equants, or second epicycles; that is, a reason why the accelerations and retardations of the planets were not wholly optical, but partly real; it occurred to him that, if this cause were just, its operation ought to be general, and to produce similar effects on all the moveable bodies in the system. But as no equant, or second epicycle, had been found necessary to the earth, or sun, an objection to his conclusions arose from this circumstance, and his speculations were left unfinished. From that period, however, he entertained a suspicion, that, notwithstanding the contrary opinion of all preceding astronomers, the earth, or sun, required an equant, in the same manner with every one of the other planets; and, if the necessity of it should be demonstrated, that is, if the centre of the earth's uniform motion should be found different from the centre of the orbit; it evidently followed that, not only would the excentricity, and the whole *radii vectores* in that orbit be different from those employed by T. Brahé and his predecessors, but the excentricities, and indeed the whole elements of the planetary orbits, would also differ from the ancient determinations. This suspicion was strongly confirmed by his correspondence with T. Brahé; who wrote

wrote to him, while he was yet in Stiria, that the orbit ascribed by Copernicus to the earth, did not always appear to be of the same magnitude, but sensibly varied its ratio to the three superior orbits; and especially, that in some particular situations of Mars, the angles, which its semi-diameter subtended at that planet, sometimes differed by no less than 1° 45′. Expressions, which Kepler reckoned of the same import, occurred also in the letters which T. Brahé published; and particularly where he mentions, that some inequality, which seemed to arise from variations of the solar excentricity, affected the longitudes of this planet in particular situations.

131. On receiving this intelligence, Kepler immediately perceived that the error, which he had always suspected in the solar theory, of confounding together the orbit and the equant, gave a sufficient explication of it; and, that there was no necessity for having recourse to the improbable supposition of a variation of the excentricity, or of the magnitude of the semi-diameter. Let DE (fig. 57), be the line of the earth's apsides, A the sun, and B the centre of the orbit. When the planet is in T, a point of DE produced, and observed from the earth in the opposite quadratures Q and R, it was evident, that whatever were the position of the centre C of uniform motion in DE, and whether it coincided with B or not, the angles of parallax QTC, RTC, under which the supposed semi-diameter of the orbit appeared from T, would be always equal; because they were the same with the angles QTB, RTB. But, if the planet were in F, a point of the line FC, perpendicular to DE in the centre of uniform motion C, and observed from the earth in the opposite apsides D and E,

the angles DFC, EFC, of parallax could not be equal, unless C should coincide with B; and they who believed these points to coincide, and therefore considered DC and EC as the semi-diameters of the orbit, would of necessity be led to conclude it variable in magnitude. Accordingly, when Kepler came to examine T. Brahé's observations, the variations of parallax were found to be precisely what he had supposed. None took place, in the first configuration, when the planet in T was observed from the earth in Q and R; nor indeed from any other opposite points as X, Y, of any line perpendicular to the line of apsides: But, in the second configuration, when the planet in F was observed from the earth at D and E, or from L and K, making with FC the equal angles FCL, FCK, the variations of parallax were considerable; and, as this was the law which they constantly observed, he concluded it to be impossible that the centres of the equant and orbit could coincide.

First proof of this necessity.

No pairs of observations indeed of Mars in quadrature, from the earth precisely in the apsides, were to be found in T. Brahé's whole collection; but two of a less perfect kind were found from the earth at L and K, whose distance LCD, KCE, from the apsides were nearly equal; and where, if Kepler's suspicions were just, the parallaxes would also be affected, as he supposed, though, no doubt, in a less degree. On the 30th of May, 1585, at 5h. when the heliocentric longitude of Mars was 6s. 13° 28′ 16″, and the earth at K in 8s. 17° 51′ 46″, the planet was observed in 5s. 6° 37′; and, on the 20th of January, 1591, at 0h., when the heliocentric longitude was also 6s. 13° 28′ 16″, and the earth at L, in 4s. 9° 4′ 46″, the planet's geocentric longitude was 7s. 21° 34′. Now, at these

these times, the angles of commutation LCF, KCF, being the differences between the heliocentric longitudes of the planet and the earth, were equal, for each was = 64° 23′ 20″: but the parallaxes or differences between the heliocentric and geocentric longitudes of the planet were unequal; for KFC was only = 36° 51′, whereas LFC was = 38° 6′; that is, greater than KFC by no less than 1° 15′. The consequence was, that CL must be greater than CK, and that the point C, about which the equal angles LCF, KCF, were described in equal times, was not the centre of the orbit, but a point in the line of apsides more distant from the sun. On applying himself to investigate the distance CB of this point from the centre of the orbit, he found it to be 1837, in parts of BD = 100000; that is, only 45 of such parts greater than 1792, the half of 3584, which was the whole excentricity of T. Brahé. As it was therefore only in the circumstances here described, that the differences now mentioned took place, the centre of the earth's orbit, in the same manner with those of the orbits of the superior planets, appeared to bisect the excentricity of the equant; and the coincidence of the orbit and equant could no longer be admitted. (O.)

132. As Kepler thus found means to account for the inequalities of parallax, without having recourse to the improbable supposition of variations in the magnitude of the terrestrial orbit, and merely by displacing the point of uniform motion from its centre; his next experiment, to ascertain the propriety of the new position given to that point, was of a less timid kind; and consisted, not like the former, of a few observations subject to rare conditions, but of any number of observations

taken

taken promiscuously, and subject only to this single condition, that the planet in all should occupy the same point of heliocentric longitude.

Accordingly, the observations of Mars, employed in this experiment, were those of March 5th, 1590, at 7h. 10′; January 21st, 1592, at 6h. 41′; December 8th, 1593, at 6h. 12′; and October 26th, 1595, at 5h. 44′. At the first of these times, the true heliocentric longitude of the planet, calculated by T. Brahé's theory, was 1s. 15° 53′ 45″, and increased, at the others, by the proper additions for the precession.

Second more decisive proof.

Let C (fig. 58), be the centre of the earth's uniform motion, and A the sun, in the line of apsides AC; let D be the place assumed for the aphelion, and to be ascertained by the result; and let the earth, at the time of the first observation, be in E, with 5s. 22° 58′ 46″ of mean longitude; at the time of the second, in F with 4s. 10° 5′ 57″; of the third, in G with 2s. 27° 13′ 12″; and of the fourth, in H with 1s. 14° 20′ 25″. It is required to determine the lines CE, CF, CG, CH; for, if they all come out equal, Kepler observes, that his opinions concerning the position of the centre of the equant must, notwithstanding the former experiment, be false; but, if unequal, his cause is decisively gained.

In 1590, the planet, at the point M of the line CM, was in 1s. 15° 53′ 45″ of heliocentric longitude, and the geocentric longitude observed from E was 0s. 25° 6′. Consequently, the parallax EMC = 20° 47′ 45″, and, from the commutation ECM = 127° 5′ 1″, it appears that the elongation CEM was = 32° 7′ 14″. Therefore,

CE =

$$CE = CM. \frac{\sin. EMC}{\sin. CEM} = 66774; \text{ \& in like manner}$$

$$CF = CM. \frac{\sin. FMC}{\sin. CFM} = 67467;$$

$$CG = CM. \frac{\sin. GMC}{\sin. CGM} = 67794;$$

$$CH = CM. \frac{\sin. HMC}{\sin. CHM} = 67478.$$

All these are in parts of CM = 100000; and it therefore clearly appears, that when the earth was in G, with 2s. 27° 13′ 12″ of longitude, and consequently near the perihelion P in 3s. 5° 30′, the distance CG was the greatest; CE, at 77° from the perihelion, was the least; and CF, and CH, nearly equal, and of an intermediate length between both extremes. CH, indeed, as being more remote from P, ought to be less than CF; but the error may be justly ascribed to the minuteness of the angles at C and M, in the triangle CHM.

The consequence was what Kepler had always believed. The point C, about which the planet describes the equal angles ECF, FCG, GCH, in equal times, (for each of them is determined by the time of the revolution of Mars), is not the centre of the earth's orbit EFGH. The centre of this orbit is a point B nearer to A the sun; and, on investigating the distance BC of the point C from the centre of the orbit, by means of the distances CE, CF, CG, and the given angles EAF, FAG, he found it to be = 1023, in parts of CM = 100000; or = 1530, in parts of BE = 100000; that is, somewhat less than the half of T. Brahé's excentricity. It clearly appeared, therefore, that, even in this ruder form of investigation, where both the mean motions and the equations were taken from T. Brahé's tables, constructed on the ancient principles, an equant was necessary for the earth

earth as much as for the planets; and that her motions are accelerated in approaching the sun, and retarded in withdrawing from him; not in appearance only, but in reality. (P.)

Third proof. 133. The next step of Kepler's procedure was, to investigate what might be the distance AB in the opposite direction, between the centre of the orbit and the sun; for, if this also should be but one-half of T. Brahé's excentricity, the evidence was compleat, that the equant and orbit were circles wholly different. In this investigation the procedure was more accurate than in the former; for, though the same observations were employed, the reductions of them to the proper times were made with greater care; the heliocentric longitude of Mars was calculated from the principles of the vicarious theory; and true longitudes of the earth substituted for mean longitudes. The distances of the earth from the sun, calculated on these conditions, were as follows.

	Long. of the earth.		Distances from the sun.
E in	5s. 24° 0′ 25″	(fig. 59)	AE = 67467
F in	4 10 17 8	-	AF = 66632
G in	2 25 53 24	-	AG = 66429
H in	1 11 41 34	-	AH = 67220; all

in parts of AM = 100000. From these distances it appeared, even without the aid of calculation, that the centre of the orbit very nearly bisected the excentricity of the equant. For, according to T. Brahé's tables, the longitude of P, the earth's perihelion, was 3s. 5° 30′; and, consequently, the distance AG, at 9° from this *apsis*, between the earth and the sun, could not be much greater than the perihelion distance; and AE, at 79° from it, could not be much less than the mean distance.

Therefore

Therefore their difference 1038, in parts of AM = 100000, or 1539, in parts of BE = 100000, though less than the just excentricity, shews that this cannot exceed 1800, the half of the excentricity deduced from the solar equations. On more particular investigation, this excentricity of 1800 was established beyond suspicion; and, in further confirmation of it, Kepler reversed the procedure; and with this excentricity, and the distance AM of Mars from the sun = 147700, that is, nearly equal to 147443, found by the direct method (128) and on the supposition that the planet between the apsides retired within the circle, calculated the geocentric longitudes, and found them hardly to differ from those given by observation. (Q.)

134. As a doctrine so important required various and multiplied evidence, and as the proof of it now given had employed longitudes deduced by the vicarious theory, which, as well as the mean longitudes to which they were supposed to correspond, might be suspected of inaccuracy; so, in the next proof which he produced, nothing is supposed to be ascertained concerning the planet, except the time of its periodical revolution. A supposition is, indeed, made for its heliocentric longitude; but this is merely for the purpose of establishing or correcting it by the methods of false position. It is obvious that the accuracy of the parallaxes EMA, FMA, &c. (fig. 60), and, consequently, of the distances EA, FA, &c. of the earth from the sun, depends on the justness of the position given to the line AM. By assuming it therefore, and calculating accordingly the lines EA, FA, &c., Kepler proceeded to deduce, from these and the given angles at A, which they included, the angles EGF, and EHF. If these came

Fourth proof.

came out equal, as they ought, since the earth's orbit was supposed to be circular, the position assumed for AM was just; and, if not, new suppositions for it were to be made.

When this, after repeated trials, was at last effected, and the just heliocentric longitude of Mars, at the time of his first observation, found to be 6s. 5° 19′ 42″, he proceeded, in his usual manner, to investigate the excentricity AB, and the longitude of the earth's aphelion. The result for the first was 1653, and for the latter 9s. 10° 19′; and he considers it as evident, that, with a more just place of the aphelion, the excentricity would rise to 1800.

It also appeared that AM, the distance of Mars from the sun, in the point which he now occupied of his orbit, was = 162851, in parts of BE = 100000.

Here likewise, reversing the procedure, and diminishing the heliocentric longitude of Mars 1′ 30″, and supposing, as before, the orbit of the earth to retire within the circle, he calculated, with the excentricity 1800, and the place of the aphelion in 9s. 5° 30′, the geocentric longitudes. These were found to be

Calculated	4s. 26° 55′ 0″;	5s. 8° 11′ 40″;
Observed	4 26 54 30;	5 8 12 0;
Calculated	7s. 8° 49′ 0″;	7s 9° 44′ 20″;
Observed	7 8 48 15;	7 9 47 10: and

he also tells us that, on calculating, with the same suppositions, the geocentric longitude of Mars, for the 9th of February, 1604, at the time when the planet was observed by himself in the meridian in 6s. 26° 18′ 48″, he found it to be 6s. 26° 17′ 30″.

With these suppositions also the distance AM was,

was, in this last investigation, found to be somewhat less than 163100. (R.)

135. The last evidence which Kepler produced, that an equant which should not coincide with the orbit was necessary to the earth, consisted, in investigating with any number of longitudes of the earth, and their distances from the sun, at the times when Mars was in the same point of his orbit, and with as many geocentric longitudes of the planet, whether any different combinations of these data would give the same results for the heliocentric longitude and distance of the planet from the sun; when the earth's distances were calculated for a circular orbit, whose excentricity was = 1800.

When the periodical time of Mars is known, it is evident that, though his heliocentric longitudes were unknown, we may, by reckoning from any particular instant as an epoch, find any number of others, in all of which his longitude must be the same. At five such times observations of the geocentric longitude were made from the earth in E, F, G, H, K, and the apparent longitudes of the sun were given. The object, therefore, of the investigation was, to find what results any pairs of these observations would give for the length and position of AM. Fifth proof.

With the data at G and F, the triangle GAF was resolved to find its base GF, and the angles at the base AGF, AFG; and by subtracting these from the given angles of elongation AGM, AFM, the angles at the base of the triangle MFG were found, and consequently its side FM. With this therefore, and FA, and the given angle AFM, in the triangle AFM, the side AM and the angle FAM were both found; so that the length of AM was

was by this investigation = 166208, and, by subtracting FAM = 21° 26′ 32″ from the longitude of the point F, its position was in 5s. 8° 14′ 32″.

A like investigation from E and H gave 166179 for the length of AM, and 5s. 8° 13′ 8″ for its position; and on employing the observation in K, with AM = 166208, the position found was 5s. 8° 15′ 4″.

Nothing therefore could appear with fuller evidence, than that the distances AE, AF, AG, AH, AK, of the earth from the sun, calculated for a circular orbit, whose excentricity was 1800, were at least nearly just; and that, therefore, this was the true excentricity of the terrestrial orbit. Wherefore T. Brahé's excentricity of 3600, the results for the length and position of AM, investigated in the same manner, would have been wholly irreconcileable. (S.)

Other evidences.

136. These demonstrations of the necessity of an equant to the earth, and of the bisection of its excentricity by the centre of the orbit, received farther, and indeed intuitive, confirmation from the variations of the sun's apparent diameter; which were not above half of what would have been produced by an excentricity of 3600. When the distances also of the earth from the sun were calculated on the principle of bisection, and the equations deduced, it was found that the differences between them and the ancient equations, calculated on the supposition of the coincidence of the equant and orbit, were altogether inconsiderable. Even at the octants, where the greatest differences might have been expected, the new equations were 1° 27′ 24″, at 45° of anomaly, and 1° 27′ 28″, at 135°; the ancient equations, at both these points, being 1° 27′ 31″. (T.)

137. But

137. But in all this attention and labour, bestowed by Kepler upon the solar theory, his object was not merely to render the representations of the solar motions more correct, nor even to furnish him with the means of perfecting the planetary theories, but it was also to introduce a method of deriving equations, from a more just and natural principle than the imaginary equant, which all former astronomers had employed. This principle was the relation between the velocity of a planet in any point of its orbit, and the distance of that point from the sun; or, in other words, the acceleration of the planet, in approaching the sun, and its retardation in retiring from him. Such a variation of velocity had been, at least virtually, acknowledged by Ptolemy in assigning equants to the planets, and by Copernicus and T. Brahé, in ascribing to the superior planets epicycles, and to the inferior hypocycles; but especially by Ptolemy in his doctrine of the bisection. Kepler's design of applying, to the important purpose of deducing the equations of the planets, a principle thus universally acknowledged to regulate their motions, appears to have been conceived from the most early period of his astronomical studies: but as no equant was then thought necessary to the earth, and the principle had not been demonstrated to be general in its operation, his speculations concerning it were interrupted. But, now that T. Brahé's observations had enabled him to demonstrate in the most convincing manner, that its operation affected the earth as well as the other planets, and that the retardation of every planet in retiring from the sun, and its acceleration in approaching him, was the general law of the whole system, he was encouraged to resume a design so happily conceived,

New method of equations.

ceived, and so constantly entertained; and to endeavour to carry it into execution.

138. Though the consequences of endeavouring to derive the equations of a planet, from the relation between its velocity and its distance from the sun, were as important as the ingenuity of the conception was admirable, the evidences, drawn from fact and experiment, on which the law was founded, were more satisfactory, than the geometrical demonstration which he attempted. This demonstration was not general, for it is applicable only to circular orbits, and even to the very small arches only of those which are next to the apsides; nor are the principles rigorously exact, for it depends upon a compensation or balancing of errors by one another; and though the proportion, between the times of describing these small arches and their distances from the sun, be strictly true, it is deduced only in an approximate manner. But, as every step which Kepler took towards his invaluable discovery, that the planets in revolving round the sun describe areas proportional to the times, must be deemed of the greatest importance, the omission of any one would be unpardonable.

The proposition to be demonstrated is, that on the Ptolemaic hypothesis of an equant, whose excentricity is bisected by the centre of the orbit, the times of describing very small arches, at the aphelion and perihelion, are proportional to the distances of these apsides from the sun; and the demonstration is to this purpose.

Let A (fig. 62), be the sun, B the centre of the orbit, C the centre of the equant; let AC be bisected by the centre B, and on B and C let the equal circles DFEG and HLKM be described. Let FD be a very small portion of the orbit at the aphelion,

aphelion, and draw CFL to meet the equant in L; and, in the same manner, let EG be a very small portion of the orbit at the perihelion, and draw CMG to meet the equant in M: so that, according to the received principles of Ptolemy, the arch HL of the equant will measure the time of describing DF; and the arch KM, the time of describing EG.

1. Because HL, DF, do not sensibly differ from straight lines, and the two triangles CHL, CDF, may be considered as right-lined similar triangles, we shall have CH : CD :: HL : DF. But, on account of the bisection of AC in B, the lines CD, CH (= BD), and AD are arithmetically proportional; and an arithmetical mean between two terms, whose difference is inconsiderable in comparison with either term, is but insensibly greater than a geometrical mean. Consequently, CH : CD :: AD : BD; only the former ratio is insensibly greater than the latter. Therefore HL : DF :: AD : BD (= BE).

2. It follows also, from like reasons, that CE : CK :: EG : KM. But CE, CK (= BE), and AE are, in the same manner, arithmetically proportional; and, therefore CE : CK :: BE : AE; only the former ratio is insensibly less than the latter: and therefore EG : KM :: BE : AE.

3. Since then the ratio of AD to BD is but insensibly less than that of BE to AE, because AD, BD, or BE, and AE are also, in the same manner, arithmetically proportional; and it has been proved, that HL : DF is but insensibly greater than AD : BD the least; and, that EG : KM, is but insensibly less than BD : AE the greatest; Kepler supposes it to follow, that the ratio of HL to DF is accurately the same with that of EG to KM. That is, HL : DF :: EG : KM.

 4. If,

4. If, therefore, the two small arches DF and EG be taken equal, either of them will be a mean proportional between HL and KM. Therefore HL : KM :: ($HL^2 : DF^2$ ::) $AD^2 : BD^2$; or HL : KM :: ($EG^2 : KM^2$ ::) $BE^2 : AE^2$; of which ratios $AD^2 : BD^2$ is the least, and $BE^2 : AE^2$ the greatest.

5. But, as has been seen, AD : BD :: BE (= BD) : AE; and, consequently AD : AE :: $AD^2 : BD^2$ :: $BE^2 : AE^2$. Therefore HL : KM :: AD : AE. That is, the time of describing a very small arch DF, at the aphelion, is to the time of describing the equal arch EG, at the perihelion, as the aphelion distance AD to the perihelion distance AE; and, in general, Kepler concluded, though from reasonings less geometrical, that, in all other points of the orbit, as N, O, P, Q, marked by lines drawn from A, the time of describing any small arch, at N for example, is to the time of describing an equal arch, at any other point P, as AN to AP.

Of this demonstration the three last steps seem to be wholly unnecessary. For, since HL : DF :: AD : BD; and EG : KM :: (BE =) BD : AE; it follows, when DF is taken = EG, that HL : KM :: AD : AE; and, indeed, it is remarked by Kepler himself, that this conclusion does not follow from the two proportions HL : DF :: EG : KM, and AD : BD :: BD : AE, except by means of a compensation of errors; the ratio of HL to DF being as much less than the ratio of EG to KM, as the ratio of AD to BD is less than that of BD to AE.

An easier, and less exceptionable, demonstration, for the case of small arches at the apsides, will be as follows.

Since the two triangles CDF, CHL, may be considered as right-lined and similar, CD : CH :: DF

DF : HL. Consequently, CD . HL = CH . DF: or, because CD = AE, and CH = BD, we have AE . HL = BD . DF.

Since also the two triangles CEG, CKM, may be considered as right-lined and similar, CE : CK :: EG : KM. Consequently, CE . KM = CK . EG; or, because CE = AD, and CK = BD, we have AD . KM = BD . EG.

If, then, the two small arches DF and EG be taken equal, BD . DF will be = BD . EG; and consequently AE . HL = AD . KM. Therefore HL : KM :: AD : AE.

From this proposition Kepler appears to have derived another, which he employs afterwards (148). It is this, that the diurnal motions, at the apsides, are in the inverse duplicate proportion of the distances from the sun. The derivation may be made in this manner.

It has been seen that AE . HL = BD . DF, and BD . EG = AD . KM; and therefore BD : HL :: AE : DF, and BD : KM :: AD : EG.

Suppose, then, that the very small arches DF and EG are described in equal times, and the arches HL and KM, which are the measures of these times, will be equal. Consequently, BD : HL :: BD : KM; and, therefore AE : DF :: AD : EG; and, hence AE . EG = AD . DF.

But, since ADF and AEG may be considered as right-angled plane triangles, it is manifest, that DF = AD. tan. DAF, and EG = AE. tan. EAG; and therefore, by substituting these values, AE^2. tan. EAG = AD^2. tan. DAF. Therefore AE^2 : AD^2 :: tan. DAF : tan. EAG.

But, since the angles DAF and EAG are very small, the ratio of their tangents will not sensibly differ from the ratio of the angles themselves; and therefore AE^2 : AD^2 :: DAF : EAG.

 It

It may justly excite surprize, that Kepler does not appear to have been fully sensible of the important uses which he might have made of this proposition, in many of his future disquisitions.

139. Since, then, it appeared, that the times employed by any planet, in describing very small and equal arches of its orbit, were always proportional to the distances of these arches from the sun; or, in other words, that the greater the distance of the planet, the more feebly and slowly is it always impelled forwards, and the more forcibly and rapidly the less the distance; it was impossible that Kepler should refrain from a variety of speculations, concerning the power or virtue which produced such effects, and the source from which it flowed. The whole of these it would be unnecessary to repeat, and they are only of importance, as they assisted or retarded his progress in his future investigations.

In these speculations, one of the two things contained in this alternative seemed evident, that the force, which produced the planetary revolutions, must reside, either in the sun, or in the planet; and, as no planet, without some animal activity, could be supposed to change its place; nor, even though endowed with animation, without some mechanical means or bodily organs, by which the translation might be made; and since, after T. Brahé's observations on comets had exploded the ancient transparent solid spheres, no probability appeared of its possessing such means or organs; nothing seemed to be left, except to ascribe these revolutions to some force proceeding from the sun. In confirmation of this opinion, the sun was also seen to be the source of light; and as light proceeded from him in straight lines, and in all directions,

tions, and illuminated all bodies, not equally, but according to various circumstances by which it was modified, a probability seemed to arise, that the present force, in like manner, proceeded from the sun; not producing equal velocities in every planet, nor even uniform velocity in the same planet, but modified also by various circumstances; such as the distance, the resistance of the medium, and the supposed disposition or proneness of the planet to rest, and which he termed its *vis inertiæ*. But the example, which Kepler thought most applicable to his subject, was that of a magnet, to which W. Gilbert, in England, had compared the earth, in her relation to the moon; and by supposing the sun to possess a similar virtue, and to have a constant rotation upon the axis of the ecliptic, a cause seemed to be presented from which the periodical motions of the planets might be derived. For, as a magnet seems not to attract iron in all its parts, but in those which are at equal distances from its poles, gives only to the iron a direction parallel to its own fibres; so Kepler conceived, that neither does the sun attract planets placed in the zodiac, but, sending forth magnetical rays or fibres, and whirling continually upon his axis, impels forwards, with these extended fibres, the planets from west to east in the direction of his own rotation. According to this conception, the planets in their revolution were not active, but merely passive: nor was it an objection to the theory, but rather a confirmation of it, that the planets were not all carried with the same velocity; for, as the rotation of the sun, with his magnetical fibres extending in the direction of the zodiac, was the common cause of their revolutions, and the impulse of those fibres could not communicate equal velocity to every planet, but only in

Force moving the planets in their orbits.

the inverse ratio of the resistance compounded with the distance from the source of power; like the impulse of a feeble lever, which is the more easily bent, the greater that the distance is of its extremity from the *fulcrum*; so those planets, whose mass was greatest, and which from their greater distance were impelled by more feeble and dispersed rays, would be impelled with least celerity; and none of them, not even Mercury, though impelled most forcibly, would attain in its revolution the same celerity with the rotation of the sun. So far, indeed, did Kepler carry his speculations on this part of the subject, as to attempt to deduce the time of a solar rotation from the period of a planet's revolution; and, having observed that the ratio of the semi-diameter of the sun to the semi-diameter of the orbit of Mercury, was nearly the same with that of the semi diameter of the earth to the semi-diameter of the lunar orbit, he supposed that the ratio of the revolutions to the rotations might also be the same: and he thence deduced this consequence, that the time of a solar rotation consisted of nearly three days.

140. But, though the impulse of magnetical fibres, extended from the sun, and carried round by his continual rotation, was considered as the cause of the periodical revolutions of the planets, no explication could be derived from it of the variations of their velocities and distances from the sun; because the rotation was supposed to be performed with perfect equability. It was therefore necessary to suppose also, that the planets were endowed with some intrinsic force, and peculiar energy of their own, by which they could vary their distances from the sun, and consequently their velocities: but, while the prejudice in favour of

of circular orbits continued, it was hardly possible to form any probable conceptions, either about the nature of this force, or the manner of its operation.

Cause of their variations of velocity.

By supposing the centre H, (fig. 63), of the epicycle GD, to revolve in the circle KH, and in the direction KH, round A its centre, which is also the centre of the sun, and the planet to revolve in the circumference of this epicycle in the contrary direction from G to D, a representation may be made of an excentric orbit perfectly circular. For, as was formerly explained (29), if the arch KH of the circle described by the centre of the epicycle should be always similar to the arch GD of the epicycle described by the planet, the path of the planet through the heavens would be the excentric circle CDP, whose excentricity AB is equal to HD.

But, whatever might be the nature of the intrinsic force of the planet, no means could be conceived, by which this effect should be produced, which did not involve great absurdities. For, 1, it was necessary to suppose that the planet had some propensity, or was somehow excited, to endeavour to extricate itself from the force of the impelling ray, or fibre, AG; and to ascribe to it a power of moving in a contrary direction, from G to D: and of this no planet, even though endowed with animation and intelligence, seemed, without bodily and mechanical organs, to be capable. 2. If this should be supposed possible, and the arches GD and KH should be so described as to be always similar; and therefore AH, the ray from the sun, always parallel to BD, the line drawn from the planet to the centre of the orbit; the consequence would be, that AH would not be carried round A with the invariable and constant velocity

velocity of the sun's rotation, but with the inconstant velocity of the line BD. For it is to be observed, that the epicycle GD represents only the optical inequality of the planet, the real inequality must be represented by the variable motion of BD; and therefore AH would not only be variable in its motion, though the rotation of the sun be invariable; but variable according to the distance AD of the point D from the sun, a point entirely foreign and unconnected with it.

Other suppositions were found to involve equal absurdities; and the last which Kepler made, before his prejudice in favour of circular orbits was removed; a prejudice which he feelingly laments, as involving him in many vexatious and fruitless labours, and robbing him of much precious * time; was, that the planet did not at all move in the circumference of the epicycle, but only librated backwards and forwards in its diameter GHP, (No. 2.): but, though it did not move in the circumference, the libration GO, or PO, was determined as if it had moved in it in the manner now described. But besides that no explication could be found of the cause which rendered the motion of H variable, it was evident, that though astronomers might calculate the distance AD, by means of the imaginary arch GD of a supposed epicycle, no rule, or mark, could be conceived by which the planet, even if endowed with intelligence and memory, might be enabled to preserve its distance AD, such as the limits of the excentric required; except the variations which it might perceive and recollect on the apparent solar diameter.

* Nocens temporis fur.

141. Unsatisfactory, however, as all the attempts of Kepler were, to account for the causes by which a planet is determined, and the means by which it

is

is enabled to describe a circular orbit excentric to the sun, since it was undeniable, that the times in which it described very small and equal arches of it, were always as its distances from the sun, he resolved to carry into execution his long conceived design of applying this principle to the deduction of equations. As his procedure for this purpose was the original of the important and celebrated discovery, that all bodies revolving round the sun describe areas proportional to the times, it deserves a full and particular explication.

Formation of a new method of equations.

His first attempt, for deriving the equations of a planet from this natural principle, was to divide the excentric orbit into 360 degrees, or equal parts, as if these had been the least possible; and, supposing that the distance, during the description of any one of these, suffered no variation, he proceeded to calculate for the orbit of the sun or earth, with which alone he was at present concerned, the distance of every particular degree or part. (T) He next summed all these distances, and thence had this analogy, as the sum of the 360 distances of the earth from the sun is to the time of the annual revolution, that is, to 365 d. 6 h., so is the sum of any given number of distances, reckoning them in their order *in consequentia* from the aphelion, to the time in which the degrees correspondent to this given number of distances are described. As the periodical time 365 d. 6 h. did not consist of an integral number of days, so, to facilitate his calculation, he designed it, in the usual manner of astronomers, by the integral number of 360° of mean anomaly; and then his analogy was thus expressed, as 36000000 (supposing the sum of the 360 distances to be equal to the sum of 360 semi-diameters) is to 360°, so is 305396, the sum of the distances from A of the 3 degrees, for

for example, of the excentric DH (fig. 61), which are next to the aphelion, to 3° 1′ 7″, the time, or mean anomaly, corresponding to these 3 degrees of the excentric orbit. The difference 3′ 7″ is the physical equation CHB, produced by the real retardation of the earth in this most distant part of the excentric; and adding to it the optical equation BHA, which may be easily calculated, especially when the distance HA is given, or, in orbits so little excentric as that of the earth, may be taken equal to the physical part, the result for the whole equation CHA will be 6′ 14″.

First attempt.

142. But, as this method was tedious and mechanical, and the physical equation could not be obtained for any particular degree of the excentric, without previously calculating and summing the distances of all the preceding degrees of anomaly, it was necessary to think of other methods. Accordingly, it occurred to Kepler, that the whole plane of the excentric might be considered as composed of the sum of the distances from A of all the minute equal parts into which its circumference might be divided, and which are here supposed to be 360; and he recollected that Archimedes, in attempting the quadrature of the circle, had divided it, by lines drawn from the centre, into an indefinite number of triangles. In his next attempt therefore, besides dividing the circumference of the excentric into 360 equal parts, he divided its plane or area into 360 parts, by drawing lines not only from the centre, in imitation of Archimedes, but also from the excentric point A, to all the points of equal division in the circumference.

Second attempt.

Let ABC (fig. 64), be the line of apsides, A the sun, B the centre of the orbit, and let the semicircle CFK be divided into any indefinite number of of equal parts, in the points D, E, F, &c. and to all these points let straight lines from A and B be drawn. It is evident that the sectors CBD, DBE, EBF, &c. are all equal; and that the areas CBD, CBE, CBF, &c. will represent the excentric anomalies as accurately, as the arches CD, CE, CF, &c. or the angles at the centre which these arches of the excentric measure.

But as the area of the semi-circle CFK may be considered as composed of the whole sum of the lines drawn from B to the indefinite number of points D, E, F, &c. in its circumference; and this sum may be also considered as equivalent to the sum of the indefinite number of sectors, into which it is by these lines divided; so the same things may be observed concerning the lines which are drawn from A. They may likewise be considered as composing the area of the semi-circle CFK, and as equivalent to the sum of the indefinite number of small areas CAD, DAE, EAF, &c. into which the semi-circle is divided by lines from A. If, therefore, the areas CAD, CAE, CAF, &c. could be exactly calculated, Kepler supposed that they would be equivalent to the required sum of the distances from A of the points of equal division in the arches CD, CE, CF, &c. and consequently represent the times in which these arches were described; and that the areas CAD, CAE, CAF, &c. would, equally with the actual sums of their distances, become the mean anomalies to the excentric anomalies CD, CE, CF, &c. or to the sectors CBD, CBE, CBF, &c. by which they are represented. That is, he concluded this to be a just analogy, "As the semi-circular area CFK is "to half the periodical time, or 180°, so are the "particular areas CAD, CAE, CAF, &c. to the "times in which the arches CD, CE, CF, &c. "described,

"described, or to their correspondent mean anomalies."

Since, then, the sector CBD is the excentric anomaly of the earth in the point D, and the area CAD the mean anomaly; their difference, the area of the triangle DAB, will be the physical equation at this point: and, as the angle ADB is the optical equation, both parts will be obtained by the resolution of the same triangle.

143. The area of the triangle ABD, or the physical equation, is easily determined. When the earth is in F, and the excentric anomaly CBF = 90, this equation becomes the area of the triangle AFB; which is $= \frac{1}{2}$FB.AB $= \frac{100000.1800}{2} =$ 90,000,000. This area will be reduced to parts of mean anomaly by this analogy, as the whole area of the circle, that is, 31,415,926,536, is to 360°, so is 90,000,000 to 3713″ = 1° 1′ 53″. The mean anomaly therefore, when the earth is in this point of the excentric, will be = 91° 1′ 53″; and if the optical equation be calculated by means of its tangent AB, it will be found = 1° 1′ 52″. The whole equation therefore corresponding to 91° 1′ 53″ of mean anomaly is 2° 1′ 45″, and the correspondent true anomaly, that is, the angle CAF is = 88° 58′ 8″. Or, by merely doubling the physical part, the whole equation will be 2° 3′ 46″, and the true anomaly = 88° 58′ 7″.

Calculation of the equations by its principles.

When the area AFB is found and expressed in parts of mean anomaly, the physical equation, at any other point D of the excentric, will be determined by means of the well known proposition, that all triangles which have the same base are as their altitudes. Through D draw to BE, for example, the line DM parallel to ABC, the line of apsides;

apsides; and from D, E, M, the lines DN, EO, MP, perpendicular to it, and join MA. The triangles DAB, MAB, are equal, and the triangles EAB, MAB, are to each other as their altitudes EO, MP, or DN. That is, EAB : DAB :: sin. EBC : sin. DBC.

Let the earth therefore be in E, and the excentric anomaly EBC $= 45^{\circ}$. The physical equation, or area BAE, will be found by the analogy, sin. FBC : sin. EBC :: triang. FBA : triang. EBA; that is, sin. 90° : sin. 45° :: $3713''$: $2625''$ $=$ EBA $= 43' \, 45''$. The mean anomaly, therefore, or the area CAE $=$ CBE $+$ BAE $= 45^{\circ} \, 43' \, 45''$; and calculating the optical equation, or the angle BEA, by means of the formula, tan. $\frac{1}{2}$(BAE — BEA) $=$ tan. $\frac{1}{2}$CBE $\left(\frac{EB - AB}{EB + AB}\right)$, it will be found $= 43' \, 14''$. The whole equation therefore in $45^{\circ} \, 43' \, 45''$ of mean anomaly, is $1^{\circ} \, 26' \, 59''$, and the true anomaly $= 44^{\circ} \, 16' \, 46''$.

This then was Kepler's new method for calculating the equations of a planet in an excentric circular orbit; in which he substituted, for the planet's supposed uniform progress in an imaginary equant, a principle which took place in nature, that the times of describing any given arches of that orbit, are as the sums of the distances of their indefinitely small divisions from the sun; or, as the areas bounded by these arches, and included between two lines drawn to the sun from the extremities of each: and this was the original draught of the celebrated law by which the planets are found to be regulated, that in performing their revolutions they describe about the sun areas proportional to the times. It was afterwards found that this law was universal, and that it not only regulated the planets and the other celestial bodies which

which revolve about the sun, but also the moon in her revolutions about the earth, and all other secondary planets in their revolutions about their primary ones. As the equations for the solar orbit, calculated on this principle, suited the phenomena with equal, or even greater accuracy, than those derived from the ancient principles, the law, as far as this orbit was concerned, seemed to be sufficiently established.

Imperfection of this method.

144. But, notwithstanding the justness of the principle from which this method of equations was derived, and the accuracy with which these corresponded with the *phenomena*; Kepler was well aware that his reasonings in forming it had not been unexceptionable. Besides that the planetary orbits, as he afterwards discovered, were not precisely circular, he saw that the planes, or areas, which he employed, were not accurate measures of those distances, or sums of distances, which he had found to be equivalent to the times or mean anomalies. For, when the area of a circle is divided into 360 equal parts, by 180 diameters, since each of these diameters is denoted by 200,000, the sum of the whole must be 36,000,000; but, when lines, as AF, AQ; AE, AR; AD, AS; are drawn from the excentric point A to the extremities of these diameters, they will form with all of them, except the diameter which passes through A, such triangles as AFQ, AER, ADS; and, as the sum of any two sides of a triangle is greater than the third, the whole sum of the lines drawn from A must be greater than the sum of the 180 diameters passing through B; that is, the sum of the distances from A is greater than the area of the circle employed as their measure; and, consequently, any particular sum of the distances, as of

of those belonging to the arches CD, CE, CF, &c. of the excentric, must be greater than the area CDA, CEA, CFA, &c. In fact, the areas employed in this method, instead of being equivalent to the sums of the distances from A, were equivalent only to the sums of the distances from such points as T and V, where perpendiculars from A meet the various diameters. But, what Kepler counted very remarkable (miraculi loco), it afterwards appeared, when the truth came to be discovered, that the errors committed by this procedure exactly balanced one another: for if, on the one hand, the area CAD, for example, was too small to measure the distances from A of all the points of equal division in the circular arch CD; that is, to represent the time employed by the earth in describing this circular arch; those distances, on the other hand, were uniformly greater than the real distances of the earth from the sun; the real orbit not being a circle, but an ellipse. (U.)

145. By the foregoing investigations, however, the solar theory, that is, the theory of the annual motion of the earth, which Copernicus justly considered as the cause of the second inequalities of the planets, was carried to a degree of perfection never before attained: and Kepler, having demonstrated that the bisection of the excentricity took place in the earth's orbit, in the same manner as the latitudes and longitudes, except those at opposition, required it in the orbit of Mars; having made the corrections, which this principle rendered necessary, on the former erroneous distances between the earth and sun; having found that the inequalities of the motion of the earth were to be

ascribed to the same common cause which produced the first inequalities of the planets; and having, from the consideration of this cause, derived a new and accurate method of investigating the earth's equations—he was enabled to proceed to the theory of the planets, with greater advantage than any of his predecessors; and to view their motions more perfectly separated from every mixture with the solar motions, and unaffected by the errors of the solar theory.

CHAP. VIII.

Of the Theory of Mars resumed, and the Application to this Planet of the physical Method of Equations; together with its important Consequences.

146. IN Kepler's former attempt to investigate a theory for Mars, he had endeavoured to deduce all its elements from observed oppositions, after the manner of the ancients; and the only difference was, that he employed real instead of mean oppositions. But though the equations, calculated by the theory then investigated, gave the planet's longitudes with considerable accuracy, the excentricity and distances of the planet from the sun were totally irreconcileable with the variations, both of latitude and longitude, produced by the annual motion of the earth. (126.128.) That these distances, therefore, might be determined with greater certainty, he had reversed the ancient order of procedure, and had begun his researches with investigating the elements of the solar theory; that is, the whole peculiarities of the annual motion of the earth: for, while these were unknown, or assumed erroneously, it was impossible that the theory of Mars, which was in a great degree dependent on them, could be accurate. As this was now therefore accomplished, it was to be expected that no farther difficulties about the theory of this planet should remain; and certainly, had its orbit been a precise circle, the expectation would have been fully gratified. For, when the distances of the earth from the sun were correctly given, the

 distance

distance of Mars from the sun was also given, by means of the angles of parallax and elongation (72); and, in a circular orbit excentric to the sun, from any three such distances of a planet given, and the angles at the sun given which are included by lines drawn from the planet, the magnitude and position of the orbit with respect to the point occupied by the sun, that is, the excentricity and longitude of the aphelion, may be always readily determined. (P.)

Deduction of the elements of Mars, with the corrected distances of the earth from the sun.

147. Accordingly, in the investigation of these elements, three distances of Mars carefully deduced from his parallaxes and elongation, and the earth's distances calculated by the improved solar theory (T), were now employed. (fig. 66.)

	Helioc. long. Mars.		Dist. of Mars from the sun.
1st.	1s. 14° 16′ 52″	-	147750 = AE.
2d.	5 8 19 4	-	166225 = AF.
3d.	6 5 24 21	-	163100 = AG.

Let A be the sun, and from A let the lines AE, AF, AG, be drawn of the given lengths, and so as to make with each other the given angles EAF = 114° 2′ 12″, and FAG = 25° 5′ 17″. It is required to determine the magnitude and position of the circle passing through E, F, G.

In the triangle EAF, with AE, AF, and the included angle, the angle AEF will be found = 35° 10′ 17″; and from the like data, in the triangle EAG, the angle AEG will be found = 20° 26′ 13″. Therefore GEF = 14° 44′ 4″, and its double GBF = 29° 28′ 8″; and its chord GF = 50868, in parts of the semi-diameter BG = 100000.

Again, in the triangle GAF, the angle AGF will, from like data, be found = 78° 44′ 1″, and GF = 77187, in parts of AG = 163100. Therefore

fore AG will be = 107486, in parts of BG = 100000; and, since GBF = 29° 28′ 8″, we shall have BGA = 3° 28′ 5″ (= AGF — BGF.)

Therefore, in the triangle BGA, with the sides AG, BG, and the angle BGA, the angle GAB will be found = 38° 15′ 45″; and, since the position of AG is in 6s. 5° 24′ 21″, that of ABC will be in 4s. 27° 8′ 36″; and the excentricity AB will be also found = 9768, in parts of BG = 100000, and BG = 151740, in parts of AG = 163100.

But how false these conclusions are, will appear by repeating the same operation with a combination of any other three distances of Mars, or even from one other distance combined with any two of the present three; for the results both for the excentricity and the longitude of the aphelion would be found entirely different. As it was thus found impossible to determine these elements from three given longitudes, and three distances of the planet in those points of longitude from the sun, though deduced from observation with the greatest care; and as these data would have been quite sufficient, if the orbit had been circular; strong cause of suspicion was given that it was not circular, and that the ancient principle on this subject was less sacred and inviolable than had been supposed.

148. A different course was therefore to be pursued, and Kepler, laying aside all preconceived opinions concerning the form of the orbit, now resolved to investigate the aphelion and perihelion distances of the planet from the sun, purely by means of observations.

As it was therefore previously known, by the vicarious theory, that the aphelion of Mars was, very nearly at least, in 4s. 28° 53′, he endeavoured to collect from T. Brahé's register all the

 observations

observations of the planet made near to this point of heliocentric longitude: and from one observation in 1585, three about the end of 1586, two in 1588, one in 1590, and two in 1600, he found, that when the planet was in 4 s. 29° 18′ 36″ of heliocentric longitude, its geocentric longitudes, the heliocentric longitudes of the earth, and the distances of the earth from the sun, calculated according to his improved solar theory, were these (fig. 67.)

		Times.			Geoc. long. Mars.			
		d.	h.	′	s.	°	′	″
1585.	Feb.	17	10	0	4	15	12	30
1587.	Jan.	5	9	31	6	2	8	30
1588.	Nov.	22	9	11½	6	2	35	40
1590.	Oct.	10	8	35	5	20	13	30
1600.	Mar.	6	6	17	3	29	18	30

Long. of the Earth.				Distances Earth.
s.	°	′	″	
5	9	22	37	99170 = AD.
3	25	21	16	98300 = AE.
2	10	55	8	98355 = AF.
0	26	58	46	99300 = AG.
5	26	31	36	99667 = AH.

Let, then, A be the sun, B the centre of the orbit of Mars, C the aphelion, in or near to which the planet is seen from the earth in D, E, F, G, H. The procedure of Kepler to determine AC, was first to assume it = 166700; and with this, and the given angles of elongation CDA, CEA, CFA, CGA, CHA, and the given distances of the earth from the sun, to calculate the parallaxes DCA, ECA, FCA, GCA, HCA: for, if these concurred to place the planet in the same point of heliocentric longitude, the assumption for AC was undoubtedly

doubtedly just; and if not, new suppositions for it were to be made, and the calculations of parallax repeated, till this effect should at last be produced. At present it happened that no new suppositions for AC were necessary; since the results for the planet's heliocentric longitude, after corrections for the precession, were, from the observation of

New and more successful method.

1585 at D	1587 at E	1588 at F
4s. 29° 18′ 19″	4s. 29° 19′ 21″	4s. 29° 20′ 40″

1590 at G	1600 at H
4s. 29° 20′ 30″	4s. 29° 28′ 44″.

By the vicarious theory they should have been
4s. 29° 17′ 0″ 4s. 29° 18′ 36″ 4s. 29° 20′ 12″
4s. 29° 21′ 48″ 4s. 29° 29′ 51″.

It follows, therefore, that the assumption for AC was just; and after the reduction to the orbit was added, the heliocentric latitude at this point being 1° 48′, it became 166780.

The observations of the planet towards the perihelion P were less numerous: but from one of 1589, one of 1591, and two of 1593, it appeared that the necessary conditions of the investigation were,

	Times.			Geoc. long. Mars.
1589.	Nov.	1d.	16h. 10′	9s. 20° 59′ 15″
1591.	Sept.	19d.	5h. 42′	9s. 14° 20′ 30″
1593.	Aug.	6d.	5h. 14′	11s. 16° 56′ 0″

Long. of the Earth.	Distances.
1s. 19° 13′ 56″	98730 = AK.
0s. 5° 47′ 5″	99946 = AL.
10s. 23° 26′ 13″	101183 = AM.

 The

The assumption for AP was less fortunate than for AC, and it required repeated trials before it was ascertained, that the only supposition which would give nearly the same results for the heliocentric longitude, was 138,430, in the plane of the ecliptic, or 138,500, in the plane of the orbit. These results, after corrections for the precession, were,

1589 at K	1591 at L	1593 at M
10s. 29° 54′ 53″	10s. 29° 56′ 30″	10s. 29° 58′ 6″

By the vicarious theory they should have been

10s. 29° 52′ 56″	10s. 29° 54′ 31″	10s. 29° 56′ 7″

From these results for the planet's heliocentric longitude, the longitude of the apsides was thus investigated.

The heliocentric longitude of Mars, near the perihelion, on the 1st of November, at 6 h. 10′, was 10s. 29′ 54′ 53″; and near the aphelion, on the 22d of November, 1588, at 9h. 11′ 30″, it was 4s. 29° 20′ 12″. The interval was 343d. 21 h. 52′ 30″, and the time of half a revolution of the planet is 343d. 11 h. 46′. The interval therefore was 10h. 6′ 30″ greater than the time of half a revolution. The arch described in this interval was 180° 34′ 41″, that is, 180° 33′ 53″, after subtracting 48″ for the precession. If, therefore, this excess of 33′ 53″ above 180° had been described, after the planet's being in the perihelion, the longitude of the aphelion would have been precisely 4 s. 29° 20′ 12″.

But it was demonstrated (138), that the diurnal motions of a planet at the apsides, are very nearly in the inverse duplicate ratio of the distances. That is, the diurnal arch described at the aphelion, is to the diurnal arch described at the perihelion as as AP^2 to AC^2. Since, then, the mean diurnal motion 31′ 27″ is given, the diurnal motion at the aphelion will be 26′ 13″, and at the perihelion 38′ 2″.

Therefore, since Mars, if he were supposed to depart from the precise point of the aphelion, would, at the end of half his periodical time, be found precisely in the perihelion, it follows that, if we begin to reckon this time 24 hours after his leaving the aphelion, the arch described during the course of it, will begin at a point more advanced in longitude than the aphelion by 26′ 13″, and terminate at a point more advanced than the perihelion by 38′ 2″; and that it will consist of 11′ 49″ more than 180 degrees.

But the present interval is longer than half the periodical time by 10h. 6′ 30″; and consequently, whatever may have been the point at which the arch described in it began, it must upon this account exceed 180°. If, therefore, we suppose, that half the excess of the interval, that is, 5h. 3′ 15″, elapsed before the planet reached the aphelion, and half after it passed the perihelion; since the motion in 5h. 3′ 15″ is 5′ 16″ about the aphelion, and 8′ 1″ towards the perihelion, the point where the motion begins must be carried forward 5′ 16″ beyond the place which the planet occupied at the beginning of the interval, and will be found in 4s. 29° 25′ 28″; and the point where it ends must be brought back 8′ 1″: That is, the intercepted arch must be diminished 13′ 17″; and, as it amounted to 180° 33′ 53″ in half the periodical time, and 10h. 6′ 30″ more; so now, when this excess of time is taken away, it will consist of 180° 20′ 36″.

Since then it appeared, that when the arch described in half the periodical time, consisted of

180°

Longitude of the aphelion.

180° 11′ 49″, this time must have begun to be reckoned 24 hours after the planet had passed the aphelion; it follows, that when the arch consists, as at present, of 180° 20′ 36″, the time of describing it must begin to be reckoned 41 hours and 54 minutes after the planet was in the aphelion. As therefore the planet, when near the aphelion, describes an arch of 45′ 42″ in 41 h. 54′, the aphelion must be brought back 45′ 42″ from 4 s. 29° 25′ 28″, the point whence the motion began to be reckoned, and will fall in 4 s. 28° 39′ 46″. By the vicarious theory the longitude of the aphelion, in November 1588, was 4 s. 28° 50′ 44″, differing from the present determination no more than 10′ 58″.

Correction of mean longitude.

By thus altering the longitude of the aphelion, in 1588, from 4 s. 28° 50′ 44″ to 4 s. 28° 39′ 46″, the planet's mean longitude at that epoch also must be altered. For, if at the time when Mars was supposed to be in the aphelion, without any equation, he was in fact 10′ 58″ beyond it, an addition of 4′ must be made to his true place, in order to find the mean. When the planet, therefore, was in 4 s. 28° 50′ 44″ of true longitude, the mean longitude was 4 s. 28° 54′ 44″.

Excentricity of the orbit.

To determine the excentricity of the orbit, corrections of the distances now found would be necessary, if the planet were in any considerable degree remote from the apsides; but at present such corrections would be insensible. Since then

The aphelion dist. AC is found to be =	166780,
and the perihelion distance AP =	138500,
a mean of them, that is, the semi-diameter BC will be =	152640,
and the excentricity AB = BC — AP =	14140,
or in parts of BC = 100000, AB will be =	9264.

But

But the excentricity of the equant given by the vicarious theory, was 18564; and its half 9282, differs only 18 from the present result. The bisection therefore takes place in this orbit, in the same manner as in the orbit of the earth; and, if it were really circular, the equations might be derived from the same natural principles.

To confirm this conclusion for the length and position of AC, they were calculated by the method of art. 135, from the observations at D and F, and AC was found = 166725, and its longitude in 1587 was 4s. 29° 19′ 49″, differing only 28″ from the result by the present investigation. Nor did the combination of any one of these, with the observation at G, produce any sensible difference.

149. As the planet Mars thus appeared to be governed by the same law which regulated the motions of the earth or sun, and the bisection taking place in this orbit demonstrated, that the times of describing any arches of it, were as the sums of their distances from the sun (138), it was to be expected, that the equations might be derived from this principle with the same accuracy as in the solar orbit. It was, indeed, vain to make the attempt, with the uncertain and contradictory elements investigated by the common methods (147); but, now that these elements had been determined with so much accuracy by another method, no reason appeared why Kepler should dread any disappointment.

Accordingly he proceeded to the calculation of the equations, and in the same manner as in the solar theory considered the circular areas CAD, DAE, (fig. 64), included between two lines drawn to the sun in A, from the extremities of the arches CD, DE, &c. as equivalent to the sum of the

distances

distances from the sun of all the minute equal parts into which those arches were divided, and consequently to the times in which they were described. Nor was it possible that the equations of Mars thus calculated, should have been less accurate, if the orbit had been truly circular, than those which he had formerly deduced from the same principle for the sun.

Calculation of the equations of Mars, by the new method.

The excentricity AB, being now found to be 9264, is, in 90° of excentric anomaly CBF, the tangent of the optical equation AFB to the radius FB = 100000, and this optical equation is therefore = $5^{\circ}\ 17'\ 34''$; and the physical equation is the area AFB = $\frac{1}{2}$AB . FB = 463,200,000; to which the whole area of the circle bears the same ratio that 360° bears to $5^{\circ}\ 18'\ 28''$. The mean anomaly therefore is $95^{\circ}\ 18'\ 28''$, and subtracting from it the whole equation = $10^{\circ}\ 36'\ 2''$, the true anomaly, or angle CAF, will be = $84^{\circ}\ 42'\ 26''$. But by the vicarious theory, which is sufficiently accurate as to longitudes, the true anomaly ought to be $84^{\circ}\ 42'\ 2''$; so that an error is in this case committed of $24''$ in excess.

Its failure

When the excentric anomaly is 45°, or 135°, the error will be more considerable. For, since R : sin. CBE :: ar. ABF : ar. ABE, this last area will be found = $3^{\circ}\ 45'\ 12''$; so that the mean anomalies will be $48^{\circ}\ 45'\ 12''$, and $138^{\circ}\ 45'\ 12''$; and, by resolving the triangles ABE, ABG, the optical equation at the first octant will be found = $3^{\circ}\ 31'\ 6''$, and at the second $4^{\circ}\ 0'\ 35''$. The true anomalies therefore will be CAE = $41^{\circ}\ 28'\ 54''$, and CAG = $130^{\circ}\ 59'\ 25''$. By the vicarious theory they are $41^{\circ}\ 20'\ 33''$, and $131^{\circ}\ 7'\ 26''$; so that there is an error in the first of $8'\ 21''$ in excess, and in the second of $8'\ 1''$ in defect; and the planet is represented as moving too rapidly about the

the apsides, and too slowly towards the mean distances.

But, notwithstanding these errors, Kepler was not immediately induced to believe that the ancient doctrine of precisely circular orbits was false, and merely founded in prejudice; but was rather inclined to suspect the falsehood of his new method of equations. For, in calculating the equations in the ancient manner, by means of an equant, on the principle of the bisection, the errors committed, (which were indeed as great, but of a contrary kind), were considered as evidences of the falseness of Ptolemy's principles, and the present errors might equally be considered as evidences of the falseness of his own.

It first occurred to him, that his errors might be ascribed to the substitution of areas for the sums of the distances, and his supposing that these areas were adapted equally with the distances to represent the times. But, on examination, this was found to be impossible. For, after calculating with the greatest care the excess of the sum of the whole distances in the semi-circle above its area; that is, the excess of the sum of the distances from the excentric point A, above the sum of the distances from the centre B; it amounted to no more than 7000 parts of the semi-circular area which contained 18000000; and if this area be equivalent to 180°, the sum of the excesses must be equivalent to little more than 4′ 30″; and these 4′ 30″ are not accumulated at any particular point of the excentric, but diffused over the whole semicircle. Besides the effect of the substitution of areas for the sums of distances, would chiefly be to represent them as too short in the intermediate parts of the excentric; as at the point F 90° from the apsides, where BF = 100000, would be substituted

not owing to its own imperfections.

stituted for AF, the secant of the greatest optical equation BFA, and = 100432. The times therefore would, in consequence of this imperfection, be rendered too short in the intermediate parts, and the planet's motions represented as too rapid: whereas, the effects of the present errors are to make the motions too rapid about the apsides, and too slow in the intermediate parts.

It was more evidently impossible, that his errors could arise from his rejection of the double epicycle of Copernicus and T. Brahé, (fig. 36,) which represents the planet's path as an oval running out beyond the circle at 90° of excentric anomaly 246 parts of 100000; and the effect of the method of areas, substituted for the distances in this theory, would have been to increase the rapidity much more. (V.)

Suspicion that the orbit was not circular.

150. But though Kepler was inclined to ascribe the errors of his equations to the imperfection of the method of areas from which they were deduced, the considerations now mentioned rendered this no longer possible, for it was undeniable that this method would have led to errors directly contrary. No conceivable cause therefore remained to which they could be imputed, except what he had begun to suspect, the form which he and all preceding astronomers had falsely ascribed to the planetary orbits; and, on examination, convincing evidence was found of the justice of his suspicions; and that the false opinion of orbits precisely circular, notwithstanding all the authority on which it had been established, was the real and only cause of all his past vexatious labours and disappointment.

His first proof of the falseness of this opinion, was the circumstance which first gave rise to his suspicions,

suspicions, that when, on the supposition of its truth, the longitude of the aphelion, the excentricity of the orbit, and its ratio to the solar orbit, were calculated, though by the most legitimate deduction, and from the most unquestionable and accurate distances, they were totally irreconcileable with the determination of the same elements drawn from direct and immediate observation; and that the results for them, given by any one combination of distances, were generally irreconcileable with those given by any other. Thus the distances AE (fig. 66), in longitude 1 s. 14° 16′ 42″; AF in longitude 5 s. 8° 19′ 4″, and AG in 6 s. 5° 24′ 21″, had been carefully deduced from accurate observations; (133, 134, 135.) but when they were supposed to be the distances of the planet in the points now mentioned of a circular orbit, the aphelion distance AC, calculated from them, came out = 166562, the perihelion distance AD = 136918, the mean distance BD = 151740, and the excentricity AB = 14822 (147). Whereas, by another investigation (148), from multiplied and equally accurate observations, and independent of all theory about the form of the orbit, they were AC = 166780, AD = 138500, BD = 152640, and AB = 14140. Proof of the suspicion being just.

A proof still more evident, that the orbit could not be circular, and which even gave some indications of its real form, arose from a comparison of the distances AE, AF, AG, deduced from observation, with the results for them calculated in an orbit precisely circular. The method of calculation was the same as in the solar theory; for, in October 1590, the longitude of the aphelion was 4 s. 28° 41′ 40″, and from the heliocentric longitudes of the planet in E, F, G, the angles CAE, CAF, CAG, were therefore given; and with BE

=

= 152640, and AB = 14140, the distances required for the circular orbit were found, by the resolution of the triangles AFB, AGB, AEB, to be

AF = 166605, AG = 163883, AE = 148539;
whereas by actual observation they were
AF = 166225, AG = 163100, AE = 147750;
shorter than the former by

350,	783,	789;

so that the path of the planet, in moving between the apsides, clearly appears to come within the circle. At F in 5s. 8°, where the planet is within 10° of the aphelion, the deficiency is least: but at E, where the distance from the aphelion is 104°, and at G, where it is 37°, the deficiency is much greater; and indeed too great, especially at G, no doubt from some inaccuracy, especially of observation. The conclusion therefore was undeniable, that the orbit of the planet is not a circle, but a curve, which coincides with the circle at the apsides, and then retires more and more within it till, at 90° and 270° of excentric anomaly, it comes to its greatest deviation from the circle.

Another argument to the same purpose arose from the errors committed in attempting to deduce the equations for an excentric circular orbit, by the method of areas (149). In that method the area of the circle, though somewhat less than the sum of the distances of the indefinite number of equal parts of its circumference from the excentric point which the sun occupied, was supposed to be equivalent to the whole time of the planet's revolution; and any particular area of it, included by an arch of the circle, and two lines drawn to the same excentric point from its extremities, to be equivalent also to the time in which the planet described

described that particular arch. But if the real path of the planet should not be a circle, but a curve retiring within the circle at all points except the apsides, the errors committed, by employing circular areas to measure the times in which the arches of this curve are described, would be unavoidable. For, every particular area, bounded in the manner now mentioned, would, every where, except about the apsides, be greater than the sum of the distances of the correspondent arch; and especially towards the points in 90° or 270° of excentric anomaly; and therefore it would represent the time as too long, and the motion of the planet while it describes the arch, as too slow. But no such errors can be committed by employing the areas of the curve which the planet really describes; for, as its area is less than that of the circle, and yet measures the same periodical time, the particular portions of this time will, on account of its form, be more properly distributed. The particular areas of the curve, nearly coinciding at the apsides with the circular areas, will there represent a longer time; and, at the distance of 90° from the apsides, retiring more within the circle, will there represent a shorter time. The errors therefore committed by the circular areas will be remedied, and the planet's motions rendered, as they ought to be, slower at the apsides, and quicker in the intermediate parts, than by the former calculations.

It was not however by this argument that Kepler was first led to believe that the orbit was not circular; nor was it from any hope of escaping the errors produced by employing circular areas, that he was induced to forsake the ancient opinion. On the contrary, so far was he from imagining this to be an evidence of its falsity, that, after a long time spent in vain and perplexing attempts to reconcile with a circular orbit the equations given by circular areas, he abandoned altogether his method of areas, and gave up every hope of being able to apply the natural principle on which it was founded to any real use. Nor did he resume this method till he was convinced by other arguments, that the circular form ascribed to the orbit was the great cause of his failure; and especially till he had found the distances in a circular orbit to be totally inconsistent with those deduced from observation.

151. But after Kepler was thus convinced that the ancient circular form ascribed to the orbit was merely a prejudice, a theory which he precipitately formed about the cause which produced its deviation from the circle, involved him in equal difficulties, and occasioned equal vexation and loss of time. Some account was given (140) of his attempts to assign a natural cause of the excentricity of a planet's orbit, and to explain the means by which it was enabled to describe a perfect circle round the sun, though not situated in its centre. One of these consisted in supposing the planet to be possessed of some intrinsic force, by which it endeavoured to extricate itself from the influence of the solar fibres, and to describe an epicycle in a contrary direction; and the supposed law of the description was, that the line DH, (fig. 63), drawn from the planet to the centre of the epicycle, should be equal to the excentricity AB, and always parallel to the line of apsides ABC; and from thence it followed, that the line DB, drawn from the planet to the centre of the orbit, would be also parallel to AH, a line drawn from the sun to the centre of the epicycle. But though the orbit DC thus described, was evidently a perfect circle, the explication

explication was rejected; for, no representation having been made of the physical equation, it was necessary that the motion of AH should be variable, even while the distance of H from the sun continued invariable; and consequently the planet's motion in the epicycle would be also variable, and regulated by the laws of the extraneous force impelling AH, which it had no conceivable means, even if it were endowed with intelligence, to know. This theory, however, Kepler now, in some degree, resumed; and overlooking the first difficulty about the inequable motion of the centre H of the epicycle, that is, of the line AH, supposed the planet to move always equably in the circumference of the epicycle, at the rate of the mean velocity of H. By these means he expected that the errors of his equations would be remedied, and the planet brought within the excentric as far as the observations required. For, when it revolved in the epicycle from G towards D, L, always at the mean rate of H; and the line AH again in passing, for example, from the aphelion C to G, in the first quadrant of anomaly, moved slower than its mean rate; the planet's place, when AC coincided with AG, would not be in the line HD parallel to ABC, but in a line HL inclined to it, and entering within the circle CD; and, as the angles GHL and CBD were no longer equal, the arches GL and CD would be no longer similar. Consequently, the distance AL of the planet from the sun, when the centre of the epicycle was in H, would be less than AD, which would have been its distance if GL had been similar to CD; the greater distance AD would take place in a position of the centre of the epicycle nearer to the aphelion C, and in a less degree of anomaly; and the physical equations, consisting of an accumulation

Precipitate theory concerning the oval orbit.

tion of distances, would become more accurate; for this accumulation would consist both of greater and of more numerous distances towards the aphelion, and there measure longer time; and of more dispersed and shorter distances towards 90° from the aphelion, and there measure shorter time, and represent a more rapid motion than formerly. In this theory, therefore, the cause which brought the planet within the circle in all the parts between the apsides, was, that in the first semicircle of anomaly, the intrinsic virtue of the planet anticipated the solar virtue impelling it forwards; and made it attain to the distances correspondent to given arches of the excentric, before the solar virtue had impelled the centre of the epicycle to describe these arches; and that, in the second semicircle, the latter virtue anticipated the former, and impelled the centre of the epicycle to describe the arches, before the intrinsic virtue had made the planet attain to their proper distances. Accordingly Kepler tells us, that confiding in this theory, which seemed to promise effects so consonant to observation, and supposing that he could not fail to find a method of reducing it to calculation, he began again to entertain the most sanguine hopes of conquering all the difficulties which the motions of Mars had so long presented; and was induced to reject the more just hypothesis, to which he had once given the preference; that the planet did not at all revolve in the epicycle, but only librated upwards and downwards in its diameter. Some account of the vexatious investigations, in which this precipitancy engaged him, will be deemed not unworthy of attention.

152. His first fruitless labour, in the prosecution of his theory, was to describe the curve in which

which it supposed the planet to revolve. For this purpose he transformed the concentric KHH, with its epicycle GDL, into an excentric CDP, whose excentricity AB was equal to the semi-diameter DH of the epicycle; and it was well known, and had indeed been formerly demonstrated, that these two forms were perfectly equivalent.

According to this representation, it is evident that, if the planet were always impelled forward by equal degrees of the solar virtue, it would move as equably in this excentric, as it is supposed to do by its own intrinsic virtue in the epicycle; and would not only describe round its centre the equal arches CD, DE, EF, &c. (fig. 68), or angles, or even sectors, CBD, DBE, EBF, &c. in equal times; but also its distances DA, EA, FA, from the sun, at the end of those times, would be the same, both as to magnitude and position, as they would have been after describing, with the same uniformity, similar arches, or angles, or sectors, of the epicycle.

But, since it is only the planetary virtue which acts equably, and its effects, in the passage of the planet from the aphelion C to the perihelion K, anticipate the effects of the solar virtue, the planet in the time CBD, for example, will attain to the distance DA before the solar virtue has impelled it forwards through the arch CD; and since, in passing from the perihelion K the effects of the planetary virtue are anticipated by the solar, the planet will, on the contrary, be impelled forwards through an arch $Kh = CD$, before it attains to the distance hA; and, when it is actually found in the lines DA, hA, its distances from the sun will be less than the distances of the points D, h. At all the points, therefore, D, E, F, &c. of this excentric, the path of the planet will retire within it, and its distances, at the end of the equal times CD, DE, EF, &c., though the same in magnitude with the lines DA, EA, FA, &c. will not be the same in position.

Attempts to describe the oval;

As it is therefore only the magnitude of those lines, as DA, which is determined by the supposed equal action of the planetary virtue, which carries the planet through the arch CD in the time represented by the sector CBD; and the position of DA depends upon the inequable action of the solar virtue, which requires the time represented by the area CAD to carry the planet through the arch CD; so Kepler first thought of determining its position by this analogy, as the area CAD is to the sector CBD, so is the arch CD described by the solar virtue in the time CAD, to the arch CL, which will be described by the same virtue in the time CBD; that is, since arches of a circle are as their correspondent sectors, by the analogy CAD : CBD :: CBD : CBL. Thus the sector CBD was considered as a mean proportional between the areas CAD and CBL; and, by converting the area CAD into circular parts, and dividing the angle CBD by the line BL, in the ratio of the number of the circular parts in CAD, to the number of circular parts in CBD, he imagined that the arch CL, or the position of the planet when its distance LA was = DA, would be found. But the imperfections of this attempt were many. For, 1. the area CAD is not precisely equal to the sum of the distances from A (149), and therefore does not denote the time of describing CD with perfect accuracy: 2. Though any single distance from A be to the indefinitely small arch corresponding to it, inversely, as the distance from B to its correspondent indefinitely small arch, it does not follow that this will continue to be the ratio of ac-

cumulations

cumulations of these distances and arches; for if, for example, BD were = 10, and the minute arch corresponding to it = $10'$, while AD was = 12, the arch which is inversely to the given arch $10'$ as AD to BD, would be found by the analogy $12 : 10 :: 10' : 8'\frac{1}{3}$, and if AD were = 11, the analogy would be $11 : 10 :: 10' : 9'\frac{1}{11}$; but $12 + 11 = 23$, is not to 20 as $20'$ to $8'\frac{1}{3} + 9'\frac{1}{11} = 17'\frac{14}{33}$, for $23 : 20 :: 20 : 17\frac{9}{23}$. 3dly. Though CBD were an accurate mean proportional between CAD and CBL, the angle CBL could not be determined geometrically, because there was no geometrical method of dividing the angle CBD in a given ratio. 4thly. Though the angle CBL were accurately determined, and consequently the circular arch CL, the arch of the curve which the planet really describes will not thereby be determined; and therefore the position of LA = DA will remain unknown.

A second attempt, therefore, was to this purpose. Since the sectors CBD, CBE, &c. represent the times in which the planet, moving equably round B, would attain to the distances DA, EA; and the areas CA λ, CA *m*, of the real orbit which the planet, moving unequably by the impulse of the solar virtue, describes before it attains to the distances λ A, *m* A, respectively equal to DA, EA, also represent the same times; let the lines LA, MA, be so drawn from the given point A, as to cut off from the plane of the excentric the areas CAL, CAM, respectively equal to CBD, CBE; or, in other words, so as to render the space *f* DL, which is cut off from CBD, equal to *f* AB, the space added to it: and on A as a centre with the distances DA, EA, describe arches cutting the lines LA, MA, in the points λ, *m*. These will nearly be the points of the orbit to which the planet will come in the times CBD, CBE, and where it attains to the required distances. But the defects in point of geometrical accuracy were similar to those of the former method. The areas were not precisely equal to the sums of the distances. There was no known method of dividing a semicircular area in a given ratio by a line drawn from a given excentric point in the diameter: and it was uncertain if the differences of the circular areas CAL, CAM, from the areas CA λ, CA *m*, of the orbit, were every where proportional.

As no geometrical method, therefore, could be found of describing the curve in which this theory supposed the planet to move, Kepler endeavoured to supply the want of it by the assistance of his vicarious theory.

Let A (fig. 69), be the sun, ABE the line of apsides, AD the excentricity of the equant = 18564, and AC = 11332 the excentricity of the orbit LGM. In this theory the angle LDG will represent the mean anomaly, and the angle LAG the true, when the planet is in the point G. But the distance AG will be false, because AC, the excentricity of the orbit, is false, and its centre C does not bisect the line AD. Let AD, therefore, be bisected in B, and on B with the semi-diameter BE = CL, describe the excentric EHF, in which the planet would, by its own intrinsic virtue, move equably round B, and make the angle EBH = LDG: so that, after the time EBH or LDG, it will attain to its just distance AH. But the position of the planet is not in the line AH, but in some point of the line AG; because the vicarious theory is, with respect to longitudes, sufficiently exact. If, therefore, on the centre A, with the semi-diameter AH, an arch, HK, be described meeting AG in K, the line KA will be the required

required distance of the planet from the sun, and both its magnitude and position will be sufficiently accurate.

These were all the attempts that could be made for describing the curve required by the present theory: and Kepler observes, that in all of them it is an oval, and not an ellipse; and that the segment, or lunula, CFKP (fig. 68), cut off by it from the plane of the excentric, is broadest towards the perihelion.

153. As it was impossible, even after attaining to the description of this oval, to deduce its equations by the method of areas, without attaining also to its quadrature, so the quadrature of it was another labour in which Kepler found himself, in consequence of his precipitancy, engaged. But this labour was equally fruitless with the former; and the whole result of it was to find, that if the oval should be supposed sensibly equal to an ellipse of the same excentricity, and described upon the same greater axis, the lunula cut off by it from the area of the semicircle would be but insensibly greater than half the area of a circle, whose diameter is equal to the excentricity AB: or, supposing the oval to pass through the point X, that $BT.TX = AB^2$. Consequently, since BT : AB :: AB : TX, the extreme breadth of the lunula, and AB = 9264, we shall have TX = 858: and since also BT : TX :: $BT^2 : AB^2$; that is, as the area of the circle described upon BT to the area of the circle described upon AB; this last area will be found = 269500000, and subtracting it from 31415900000, the area of the circle described on BT, the remainder will be the area of the oval, and = 31146400000. (W.) and to find its quadrature.

But, though the accurate quadrature of the oval should

should be supposed to be thus attained, it was still impossible to deduce its equations by the method of areas, unless means should be also found of dividing it in any given ratio by lines drawn from A. When the orbit was considered as a perfect circle, and the planet in any given point F of excentric anomaly FBC, it was easy, with the given excentricity AB, to determine the area FAB, which, added to the given sector FBC, makes up the area FAC, or represents the time in which the planet would describe the arch CF: and it was also easy with the same data to determine the true anomaly, or angle CAF. But because the planet, in descending from C, retires within the circle, it was necessary to determine not only the angle CAF, but also the portion of the oval area intercepted between the lines CA and FA, or at least the portion of the lunula intercepted between them, that by subtracting it from CAF, we may find the oval area. This, however, was found impracticable by any geometrical methods.

154. Finding thus no assistance from geometry, it was necessary for him to have recourse to other methods. In one of these, he was freed from the necessity of attempting the particular quadrature of the oval; for, by continuing to suppose it sensibly coincident with a perfect ellipse, he was enabled to derive its areas from the known ratio between it and a circle described on its greater axis as a diameter. For let CPK be an ellipse described on CK, the diameter of the circle CFK as its greater axis; let A be the focus of it, which is occupied by the sun; let BP be the conjugate axis meeting the circle in F, and *o m n* an ordinate to the greater axis; and let *n* A, *m* A, be joined; it was known that the areas CPK and CFK, or

C *m* A

C *m* A and C *n* A, were always to each other in the constant ratio of BP to BF; and if therefore the quadrature of the circle was supposed to be attained, that of the ellipse was attained also.

In this method, therefore, the circular area C *n* A was found by the usual analogy; as the whole periodical time, or 360°, is to the whole area of the excentric, so is the time given since the passage of the planet from the aphelion to the area C *n* A. But to find the angle CA *n* of this area at the sun, it was necessary to employ the methods of false position; and to assume the excentric anomaly CB *n* such, that when the triangle *n* AB was added to the sector CB *n*, their sum should be precisely equal to the area C *n* A. When this was effectuated, it was easy to find the true anomaly, or angle CA *m*: for, with the angle AB *n*, and the sides *n* B,BA, the angle CA *n* is given; and the tangents of the angles CA *n* and CA *m* to the radius *o* A, are to each other in the constant ratio of BF to BP.

For example, let the mean anomaly be = 95° 18′ 28″. The equivalent area CAF will be found by this analogy, 360° : 31415926536 :: 95° 18′ 28″ : 8317172671 = CAF. Then to find the angle CAF, let CBF be assumed = 90°, so that the sector CBF will be = ¼ of the circular area, that is = 7853981634, and BF = 100000. Therefore the area of the triangle FAB = ½ AB.BF = 4632 × 100000 = 463200000; and this added to the sector CBF, gives, for the area CAF, the sum 8317181634, insensibly greater than the former. It appears, therefore, that the assumption for CBF was nearly accurate; and since its sine BF is = 100000, and PF the breadth of the lunula at this point = 858, the semi-axis minor BP of the ellipse will be = 99142. Wherefore, since in

An attempt to deduce the equations, by means of the relation between the ellipse and the circle.

in the triangle PAB, we have PB : BA :: R : tan. APB, this angle will be found $= 5^\circ\ 20'\ 18''$, and consequently the true anomaly CAP $= 84^\circ\ 39'\ 42''$. By the vicarious theory it is $= 84^\circ\ 42'\ 2''$.

Again, let the mean anomaly be $= 48^\circ\ 45'\ 12''$; and since the whole circular area may be represented by 360°, as well as by its superficial amount, the area of the triangle FAB, represented in the same manner, will be $= 19108''$. To find then the angle CA n, let CB n be assumed $= 45^\circ$, and consequently its sine $n\,o = 70711$. The area, therefore, of the triangle n AB, which is to FAB as 70711 to 100000 will be $= 19108'' \times 70711 = 13512'' = 3^\circ\ 45'\ 12''$. This, added to the sector CB $n = 45^\circ$, will give the area CA $n = 48^\circ\ 45'\ 12''$, that is, equal to the given mean anomaly, and shews that the assumption for CB n was just. But BF : BP :: on : $om = 70104$: and A $o =$ cos. CB $n +$ AB $= 70711 + 9264 = 79975$. Therefore, since A o : om :: R : tan. m A o, this angle, that is the true anom. CA m, will be found $= 41^\circ\ 14'\ 9''$. The vicarious theory gives it $= 41^\circ\ 20'\ 33''$.

A comparison of the effects of the different theories is as follows:

Mean

	°	′	″	°	′	″	°	′	″
Mean anomalies - - - -	48	45	12	95	18	28	138	45	12.
True anomaly, supposing the orbit and equant to coincide - - - -	41	40	14	84	40	44	130	40	46.
True anomaly, on the principle of the bisection and doubling the optical equation - -	40	45	52	84	37	48	131	45	0.
True anomaly, on the same principle, but with both parts of the equation calculated, after Ptolemy - - - -	41	15	31	84	41	22	131	15	31.
True anomaly, by the vicarious theory - -	41	20	33	84	42	2	131	7	26.
True anomaly, by the method of areas calculated for a circular orbit - - -	41	28	54	84	42	36	130	59	25.
True anomaly, by the method of areas in the present elliptical orbit - - - -	41	14	9	84	39	42	131	14	5.
True anomaly, by a mean of the circular and elliptical areas - - - - -	41	21	33	84	41	9	131	6	45.

From

From this comparison it appears, that, if the vicarious theory be considered as the standard, the calculation by the elliptical areas of the present method approaches nearer to the truth than the calculation by circular areas: and, what may perhaps seem surprizing, that with a very small increase of excentricity, it would be nearly equivalent to the Ptolemaic method. As this, therefore, was found to be incorrect and imperfect, because it made the motions too slow about the apsides, and too quick about the mean distances, it followed that the present method was equally imperfect; and that it produced errors of a contrary kind to those produced by areas in a circular orbit: though it was yet uncertain whether the errors arose from the theory itself, or from the imperfect manner in which its principles had been expressed in the calculations. It likewise appeared, that a mean of the results, from the circular and elliptical areas, approached very nearly to the truth, that is, to the results from the vicarious theory: and, if this circumstance did not first suggest to Kepler the idea of another ellipse, which should every where bisect the space cut off from the circular area by the present ellipse, it certainly at least confirmed him in his intention of introducing it.

155. As the calculations hitherto made, on the principles of the oval theory, had been in many respects deficient as to geometrical accuracy, so that it might be certainly determined whether the errors produced arose from the theory, or from the methods of calculation, Kepler next applied himself to deduce the equations immediately from the distances in the oval, and to employ the sums of these in place of the elliptical areas. The calculation and summation of these distances was indeed a most

a most laborious business; but the errors, whatever they might be, which arose from the areas not being precise measures of the corresponding distances; from the ratio of any sum of distances to the sum of the corresponding arches not continuing the same with the ratio of any particular distance to its particular arch; from the impossibility of obtaining a geometrical quadrature of the areas; and from the differences between circular and oval sectors; would, by means of this labour, be avoided (152).

In calculating the distances, he employed the excentric circle CHK (fig. 70), in which the planet, moving equably, as the theory supposed it to do in the epicycle, would describe equal arches CD, or angles CBD, about the centre B in equal times: and dividing the circumference into 360 equal parts, as being the least possible, he calculated for the excentricity AB = 9165, the distance of every one of these 360 divisions from the sun in A: and, lest any error should arise from any variation of distance in the time of describing a single division, he took a mean of the two distances by which every particular division was terminated. Thus the distances, as DA, of the planet from the sun, at all the particular points of equal division in the excentric, or at all the particular times marked by these points, were accurately determined as to length; and they were also the distances of the planet, at the same particular times, in the various points of its oval orbit from the sun: but their positions were not just; for the planet does not in fact describe equal angles, CBD, in equal times about the centre B, but, in consequence of the solar force acting inversely as the distance, describes in equal times unequal angles about this centre. Those distances, however, were the real distances

An attempt by actual summation of the distances.

distances of the planet, and their sum, therefore, was equivalent to the time of describing the whole oval, and found to be = 36075562. Since, then, the arches described by the planet are inversely as the distances, to find the length of the oval arch corresponding to every distance, Kepler had this analogy; as the whole oval circumference is to the whole sum of distances, so is the distance of every particular arch to the arch itself. Thus supposing CD to be an arch of the excentric = 1°, the distance of the planet from the sun at the end of the time CD will be DA; and as the whole length of the oval circumference is to 36075562, so is DA to the oval arch CE: and if, on the centre A, with the distance DA, an arch DE were described, and on the centre C, with the distance CE considered as a straight line, another arch were described cutting the former in E, this would be nearly the point of the orbit in which the planet would be found after the time CBD. In the very same manner the length of the next oval arch, advancing from the point E, may be also found; and the position of its distance with respect to the distance AE.

But to determine the position of AE in a more accurate manner, that is the angle CAE of true anomaly, it was necessary to find the angle CBE, at the centre of the orbit; and this is not measured by the known oval arch CE, but by the unknown arch CG of the excentric, greater than CE. But, since CG and CE, when viewed from B, appear to be equal, let them, for a first approximation, be supposed actually equal; and also, as if the very small arch GF were insensible, let each of them be supposed = CF. On these suppositions, the angle FBC, measured by CF = CE, is given: and therefore, with its supplement FBA, and the sides

sides FB, BA, the base AF of the triangle ABF may be found; and therefore EF = AF — AE. But EF is almost accurately equal to EG, the quantity by which the oval arch CE falls within the excentric at its extremity E. By bisecting, therefore, the line EG, the quantity will be also found by which the whole arch CE would fall within the excentric, if the approach to B of all its parts were uniform. The visible magnitude, therefore, of CE, viewed from B, will be given, and the angle CBE, which was assumed less than the truth, and = CBF, will be corrected. With this corrected angle, therefore, and the sides AE, AB, the true anomaly CAE was finally determined.

By this method of procedure, no equation, except for the first degree of anomaly, could be separately calculated, and all the rest, to the 180th degree, required the equation immediately preceding to be found. But this was not its only defect; for the foundation of the whole was wanting; that is, the length of the whole oval circumference was unknown: nor could Kepler find any means to determine it, except the indirect methods of false position, in which it was assumed, and the assumption verified by adding together, after the 180th operation. all its particular arches CE, or rather angles CAE, in the order in which they had been calculated: for if the sum came out greater or less than 180°, the assumption was evidently false. In fact, it was not till after many most laborious and obstinate calculations of this kind, that the whole length of the semi-oval circumference was determined to be = 179° 14′ 15″; that is, 45′ 45″ shorter than the circumference CHK.

In this method, laborious as it was, Kepler calculated the equations for every point of the excentric, that is, according to the present representa-

 tion,

tion, in which the motion was supposed to be equable round the centre B, for every degree of mean anomaly, no less than three several times. First, with an excentricity = 9165, and consequently too small; and with the length of the semi-oval assumed too great: for it was supposed to exceed 180°, and consequently the sum of the angles CAE came out also greater than 180°. Secondly, with the same excentricity, and the semi-oval assumed = 179° 14′ 15″: and as the equations in the first quadrant, and especially at 90° of mean anomaly, came out too small, it appeared that the excentricity was too small. Thirdly, with the same supposition for the length of the semi-oval, and the excentricity = 9230: and, after the calculation of 180 true anomalies, the results were—

Mean anom.	True anom.	By the vicarious theory.	Differences.
°	° ′ ″	° ′ ″	′ ″
45	38 2 24	38 4 54	2 30 —;
90	79 26 49	79 27 41	0 52 —.
135	126 56 25	126 52 0	4 25 +.

The planet, therefore, still moved too slowly about the apsides, and too rapidly about the mean distances: and any increase of excentricity would have increased the errors. But, as the equations approached nearer to those of the vicarious theory than any given by the former calculations, Kepler considered them as confirmations of his present hypothesis; and began to flatter himself again with the expectation, that, the more exactly its principles were expressed, the effects would always approach so much nearer to the truth. (X.)

156. This method, however, was so vexatious, and its foundations so unsettled, that it was impossible

possible for Kepler to continue satisfied with it. The argumentation employed in it was of that vicious kind, which is called *reasoning in a circle:* for the length of the oval which the planet was supposed to describe, could not be ascertained without determining the deviation of all its points from the excentric, and the measures of this deviation could not be determined without ascertaining the length of the oval circumference. It seemed therefore to follow that, either a more legitimate method of calculation should be found, or that the theory should be abandoned, as involving an absurdity unexampled in any other law of nature. There was also reason to suspect, that even in this method, the theory had not been justly represented. The oval which the planet, in consequence of the solar force, and its own intrinsic force, was supposed to describe, had been divided into unequal portions, each of which was inversely as the time, or distance from the sun. But, according to the theory, it is only the solar force which acts in the inverse ratio of the distances; and the planet's intrinsic force is invariable in every distance. There seemed, therefore, to be some reason to doubt, whether the portions of the oval orbit, which is the joint production of both forces, were accurately measured. Accordingly Kepler, laying wholly aside the oval orbit produced by the composition of two forces, and all considerations of its quadrature, resolved to return to the original principles of his theory, and to deduce his equations immediately from these.

Let A (fig. 63), be the centre of the sun, and from it, with the distance AK, describe the circle KH, in which the centre of the epicycle is supposed to revolve, and with the distance AC another circle CG, in which its aphelion revolves.

According to the principles of the theory, while the centre of the epicycle passes from K to H, or its aphelion from C to G, the planet moves equably in its circumference, from G to L, describing angles, GHL, proportional to the times; and the line AH or AG moves unequably, describing angles, KAH, which are inversely as the distances AL of the planet from the sun. It is therefore required to find these distances AL, and angles KAH, and thence the true anomalies CAL.

Deduction of the equations in the original form of the theory.

Since the planet moves equably in the epicycle, the mean anomalies may be represented by the angles GHL, or the arches GL of its circumference, divided into 360 degrees, as its smallest equal divisions; and, therefore, from those angles, with the semi-diameter AH = 100000, and the semi-diameter HL of the epicycle = 9264, the distances AL, and angles HAL, corresponding to every complete degree of mean anomaly, may easily be found. When these whole distances are calculated, their sum will be nearly equal to 36075562, as in the former method, and the 360th part of this sum is 100210. Every particular angle KAH, therefore, will be found by this analogy, as every particular distance AL, in its order, is to the mean distance 100210, so is 1° or 60′, to the arch KH, or angle KAH, corresponding to the given distance AL. Thus all the different angles KAH will be found, which are described by the centre of the epicycle; and if, beginning with the first, we add to it the second, and to their sum a third, and a fourth, till the whole 180 are summed, we shall have a table of all the mean anomalies, as GHL, accompanied with all their corresponding distances AL, and angles HAL and KAH; and, therefore, by subtracting from every angle KAH, its correspondent angle HAL, we shall also have the

the true anomaly KAL. Thus, if GHL were = 45°, this table would shew that the angle KAH corresponding to it was = 41° 27′ 0″, and HAL = 3° 30′ 17″; so that KAL = KAH — HAL = 37° 56′ 43″. The effects of this method in the following mean anomalies were

Mean anom.	True anom.	Vicarious theory.	Differences.
45°	37° 56′ 43″	38° 4′ 54″	8′ 11″ —.
90	79 26 35	79 27 0	0 25 —.
120	110 28 8	110 18 30	9 38 +.
150	144 16 49	144 8 0	8 49 +.

It clearly appeared, therefore, that though the principles of the theory were now fully introduced and fairly expressed in the calculation, the errors became greater than ever; and the planet rendered still slower about the apsides, and quicker than before, about the mean distances. The failure therefore of the calculation, by the method of art. 154, was not so much owing to its imperfections in point of geometrical accuracy, or to misrepresentation of the theory, as to the falsity of its principles. (Y.)

157. But a more decisive evidence of the imperfection, or indeed the falsity, of the theory, arose from an accurate investigation of the planet's actual distances from the sun, as given by observation in all the various degrees of anomaly, and a comparison of them with the distances calculated for the present oval orbit. The investigation of the distances obtained by observation was made for both semicircles of anomaly, and for the corresponding points of both: and as by a like investigation full evidence was given that the orbit was not circular (150), equal evidence was now given that neither did the planet so move in

epicycle, as to describe the oval orbit supposed to be produced, by the combination of its own uniform motion with the unequable motion arising from the solar impulse. As the consequence of this investigation was so important, some examples of it are justly entitled to our attention.

Distances given by observation.

To find the distance, for example, in the 87th degree of mean anomaly, he employed two observations; one of the 6th of May 1589, at 11h. 20′, when the planet's observed longitude was = 6s. 27° 7′ 20″, the sun's apparent longitude = 1s. 25° 48′ 40″, and his distance from the earth = 101365. In this observation, also, the planet's mean heliocentric longitude was 7s. 26° 0′ 36″, and therefore its true heliocentric longitude 7s. 15° 32′ 13″, by the vicarious theory. But, as this theory did not, in 1589, represent the longitude in the opposition of the preceding April, with greater certainty than within 2′ 12″ (125), he added to this observation another, made on the 27th of December 1594, at 19h. 15′, when the mean heliocentric longitude was 7s. 26° 13′ 39″, the observed longitude 8s. 8° 46′ 30″, the sun's longitude 9s. 16° 47′ 10″, and his distance from the earth 98232.

Let A (fig. 72), be the sun, B the place of the earth, at the time of the first observation in 7s. 25° 48′ 40″, and D its place at the second in 3s. 16° 47′ 10″; and though the planet is not in both observations at the same precise point of heliocentric longitude, let C represent its position in both. Since, then, in the first observation, the elongation ABC is = 151° 18′ 40″, and AB = 101365, if we shall assume AC = 154200, the parallax ACB will be found = 18° 23′ 43″; and consequently AC in 7s. 15° 31′ 3″ of heliocentric longitude. But, if in this first observation AC be assumed

assumed = 154200, it must in the second be assumed 32 parts shorter; for in this part of the orbit a difference of one degree in longitude varies the distance 240 parts, and the present difference is 13′, or, after subtracting the precession, 8′. Therefore, if with AC = 154168, AD = 98232, and the elongation ADC = 38° 0′ 40″, we calculate the parallax ACD, it will be found = 23° 6′ 21″; and, consequently, AC in 7s. 15° 40′ 9″ of heliocentric longitude; that is, its position at the second observation, will differ 9′ 6″ from its position in the first. But the difference ought to be somewhat greater; for the difference of the mean longitudes is 13′ 3″, including the precession, and therefore the true longitude ought to differ 12′ 37″; and consequently AC, in the second observation, ought to have been in 7s. 15° 43′ 40″. This Kepler found would be effected by assuming AC = 154400 in the first observation, and = 154368 in the second; and these therefore he concluded to be their lengths, the first in 87° 9′ 24″ of mean anomaly, and the second in 87° 16′ 30″. No reduction of these distances to the plane of the orbit was thought necessary, because the latitude in the first observation was only 6′ 40″ north, and in the second 3′ 31″ also north. Corrections for refraction had been applied both to the observed longitudes and latitudes, but the diurnal parallaxes were considered as insensible.

For the distance AF, towards the correspondent point of the opposite semi-circle, he employed an observation of the 17th of December 1595, at 7h. 6′, when the planet was seen from E the earth to be in 1s. 11° 31′ 27″, with 1° 40′ 44″ of north latitude. The mean heliocentric longitude was 2s. 2° 4′ 22″, and the aphelion being in 4s. 28° 58′ 10″, and consequently the mean anomaly

 =

= 273° 6′ 12″; the true heliocentric longitude, by the vicarious theory, was 2s. 12° 32′ 22″. The sun's longitude was 9s. 5° 39′ 3″, and his distance AE = 98200. The parallax AFE therefore was = 31° 0′ 55″, and the elongation AEF = 125° 52′ 24″; and from these angles, and the distance AE, the planet's distance AF in the plane of the ecliptic came out = 154432. From this 60 parts are to be subtracted, because the planet was 15′ 36″ nearer the aphelion than in the observation of 1589; but, on the contrary, 15.5 parts are to be added as the reduction to the plane of the orbit, because the latitude was considerable. The distance therefore of the planet from the sun, when the complement of the mean anomaly is 87° 9′ 24″ was found to be = 154387, differing only 13 from AC = 154400. Such also was Kepler's attention, that though this observation might have been safely confided in, because the heliocentric longitudes of Mars, given by the vicarious theory for the year 1595, were very nearly accurate (125), he chose to employ another, made on the 29th of October, 1597, at 17 h. from which the distance AF in the same point of the orbit was found to be = 154272, or more probably, supposing an error of 3′ to have been committed in the observed longitude, = 154387, that is, precisely equal to the former. This error of observation may be supposed to be very probable, for T. Brahé, after his expulsion from Denmark, had not yet obtained a fixed place of residence.

In the same manner, at 70° 55′ 0″ of mean anomaly, the distance of the planet from the sun was by one observation found to be 158090, and by another 158111; and in the correspondent point of the opposite semi-circle, to wit, at 289° 5′ 0″ of mean anomaly, it was 158217.

At

At 43° 23′ 31″ of mean anomaly, it was found, by a single observation, to be, 163100; and at 316° 36′ 29″, it was by one observation = 163051, and by another = 162996.

At 11° 87′ 0″, it was by one observation = 166180, and by another, = 166208; and at 348° 23′ 0″, it was by one observation = 166230.

In the same manner also, towards the perihelion, it was found to be = 147820, or = 147700, in 113° of mean anomaly, and = 147443 or 147750 in 247°.

Finally, in 162° and 198° of mean anomaly, it was between 138954 and 139000. (Z.)

A comparative view of these observed distances, and the distances calculated for the same degrees of anomaly, in an oval orbit, where the mean distance was 152350, and the excentricity = 14115, is as follows:

Mean anom. ° ′ ″	Observed dist.	Compl. mean anom. ° ′ ″	Observed dist.	Calculated.
11 37 0	166194	11 37 0	166230	166179
43 23 31	163100	43 23 31	163028	162895
70 55 0	158100	70 55 0	158217	157530
87 9 24	154387	87 9 24	154400	153697
113 0 0	147760	113 0 0	147591	146928
162 0 0	138954	162 0 0	139000	138991

It seemed, therefore, to be clearly established, that the orbit of the planet, in all points except the apsides, went beyond the oval assigned to it by the present

present theory; and that, towards the mean distances, the difference exceeded 600 parts.

The planes of the orbits pass through the centre of the sun.

158. Before the distances of Mars from the sun, found by this laborious investigation, were employed by Kepler for his present purpose of shewing that the planet's real orbit extended beyond the oval, in all points except the apsides, he applies it to demonstrate the propriety of the innovation which he had originally intended to introduce into astronomy (106). He had conceived, that the plane of every planetary orbit, and the line of its apsides, ought to pass through the centre of the sun, and not through the centre of the ecliptic: and he was persuaded that, till this substitution should be adopted, it was vain to expect any important improvement of the science. The arguments hitherto adduced in support of this opinion were imperfect, and not fully conclusive; but the equality of the planet's distances from the sun, which he had now so laboriously established, in so many corresponding points of both semi-circles of anomaly, supplied him with a strict demonstration of it, and enables us to rank it among his greatest and most important discoveries.

Let A (fig. 73,*) be the centre of the sun, AG the line of the apsides of Mars, DGE his excentric described on the centre C, G its aphelion in 4s. 29°, AC the excentricity = 14140 in parts of which the semi-diameter of the earth's orbit contains 100000, F the centre of the equant, GFE the mean anomaly when the planet is in E, equal to GFD, the complement of the mean anomaly when the planet is in D. Suppose each of those angles = 87° 9′ 24″; so that, if the lines AE, AD, be drawn, each of them, as has been now investi-

* In fig. 73, FE FD ought to be joined.

gated,

gated, will be = 154400. Let also B be the centre of the earth's orbit, and H of her equant; OABN her line of apsides, where the aphelion N is in 9s 5° 30′, and AB the excentricity = 1800. Join also BE, BD. Since N is in 9s 5° 30′, and E in 7s 15° 30′ (157), the angle EAB will be = 50°, and AEB being a very small angle, the angle EBA will be obtuse; and consequently EA longer than EB. In like manner, since D is in 2s. 12° 30′ (157), and O the earth's perihelion in 3s. 5° 30′, the angle OAD will be = 23°, and its supplement DAB = 157°, and consequently BD longer than AD, or AE. Therefore BD is much longer than BE; that is, the lines drawn from the centre of the earth's orbit to any two correspondent points of the opposite semi-circles are unequal. It is even in general manifest, that there are no points within the orbit, except those which lie in the line GA passing through the centre of the sun, from which equal straight lines can possibly be drawn to any two such correspondent points.

It is true that by joining BC, and producing it to L, we may suppose this to be a new line of apsides, and it may be said, that since D is less remote than E from its aphelion L, it is not to be wondered at that BD should be longer than BE. But whatever lines of this kind may be drawn, it has been proved by the present investigations (157), that the lines AE, AD, must always continue equal. Now it was before proved (see note on Article 106), that no new line of apsides, drawn through the centre of the orbit described on C, could, equally with AG, suit the observations in opposition: and if, to suit these, a third line HKF of apsides should be drawn through H and F, the centres of uniform motion, and the centre of the orbit should be thereby displaced from C to K, the

the orbit would also be displaced, and come into the situation MPQ; and consequently the lines AE, AD, terminating in it, could no longer continue equal, as the observations out of opposition have been now found to require.

It is also to be considered, that the orbit of Mars was proved (150) to retire within the excentric in all points between the apsides; and it has now been proved (157), that any two points equally remote from either apsis fall equally within it. The orbit, therefore, must be described on GFCAR as its greater axis, and not on MFKHP: and if the distances from the point H should be calculated, they would be found altogether irregular, and incapable of being bounded by any known curve.

To complete the evidence of a fact so important, Kepler afterwards added two other proofs equally decisive. 1. That the places of the nodes were found to lie precisely in a line passing thro' the centre of the sun; whereas, if they had been situated in a line passing through the centre of the ecliptic, the longitude of the ascending node of Mars must have been $1^{\circ}\ 1'\ 33''$ less, and that of the descending node $1^{\circ}\ 1'\ 33''$ greater than those given by observation: and 2, That the inclination of the orbits could not otherwise be invariable, as had been before demonstrated (117): for had their intersection taken place in the centre of the ecliptic, the heliocentric latitude at the northern limit must have been two minutes less, and at the southern two minutes greater than it was actually found.

It was thus undeniably ascertained, that the motion of every planet is in one invariable plane passing through the centre of the sun: and the discovery was of greater importance to the improvement

provement and simplification of astronomy than any which had ever before been introduced, except, perhaps, the Copernican system alone.

159. The undoubted conclusion from the investigations of Article 157 was, that the real path of the planet went beyond the oval which it was erroneously supposed to describe: but Kepler justly thought that a conclusion so important ought not to rest on any single method of proof, and he therefore employed another equally decisive. It was drawn from observations of the planet before and after its oppositions, when the distances of the earth from the line of syzigy were equal.

Observed distances investigated by another method.

Let A (fig. 74) be the sun, B the earth before the opposition, and ABD the angle of elongation; and C the earth after the opposition, and ACD the angle of elongation; and let AD be the line of syzigy. The planet therefore, at the first observation, will be seen in some point G of the line BD; and at the second, in some point as H of the line CD; and from the given interval of the observations, the angle GAH, described by the planet, will be found with sufficient accuracy by means of the vicarious theory. The distances also of the planet from the sun, in the degrees of anomaly given by the vicarious theory, will be also found by the usual methods of calculation, so as not to err widely from the truth; at least, if the interval be not too great, and especially, if the opposition has happened, either near the apsides, or at the points 90° remote from them, the differences of the distances will be nearly found; nor will the form of the orbit, whether circular or oval, make any considerable alteration on them.

If,

If, therefore, with these distances AG, AH, we resolve the triangles ABG, ACH, in which the angles of elongation are given by observation, and the sides AB, AC, by the solar theory, in order to find the angles of commutation BAG, CAH, and thence the angle GAH; this angle, if AG, AH, be too long, will come out less than the amount of it given by the interval of the observations; and, if AG, AH, be too short, it will come out greater; that is, if for AG, AH, we have used AE, AF, the angle will be EAF instead of GAH; and, if we have used AK, AL, it will be KAL. We shall also discover, by resolving the triangles ABG, ACH, with the given angles BGA, CHA, of parallax, instead of the sides AG, AH, whether any error has been committed by the vicarious theory, as to the positions of AG, AH; for, if AG has been erroneously advanced to AE, and AH to AL, we shall find for AG a distance AE too long, and for AH a distance AL too short.

Accordingly, the observations before and after the opposition on the 28th of December, 1582, were these—

	Nov. 23 16 0 (d. h. ′)	Dec. 26 8 30 (d. h. ′)	Dec. 30 8 10 (d. h. ′)	Jan. 26 6 15 (d. h. ′)
	s. ° ′ ″	s. ° ′ ″	s. ° ′ ″	s. ° ′ ″
Geoc. long. ♂	3 26 38 30	3 17 40 30	3 16 0 30	3 8 20 30
Geoc. latit. -	2 49 10 N.	4 7 0 N.	4 8 0 N.	2 52 12 N.
Mean anomaly	2 7 28 13	1 19 30 10	1 17 51 35	1 4 8 15
Heliocentric long.	3 0 43 34	3 16 7 10	3 17 57 32	4 0 9 40
Reduced -	3 0 42 42	3 16 6 23	3 17 56 45	4 0 9 30
Longitude Sun	8 11 40 40	9 15 4 12	9 19 8 31	10 16 33 20
Distance Sun	98345	98226	98258	98624
Distance Mars	AG = 158920	AE = 163082	AF = 158842	AH = 164116
In the Orbit -	158960	163147	158907	164196

The difference of the two distances AE, AF, in the second and third observations, is = 4240, and AF, though nearest to the aphelion, is the shortest. From the previous calculation this difference ought to have been only 336. Taking therefore a mean of both, we shall have AE = $\frac{322054 - 336}{2}$ = 160859; and AF = $\frac{322054 + 336}{2}$ = 161195; and with these resolving the triangles ABE, ACF, we shall find AE in 3s. 16° 5′, and AF in 3s. 17° 55′, each less advanced by about 1′ 30″ than according to the vicarious theory. They seem therefore to be nearly the just distances; though, on account of the smallness of the angles at E and F, they are not to be depended upon.

Greater confidence may be placed on the results from the first and fourth observations. The difference between AG and AH is = 5236. But by the previous calculation it ought to be about 5570. Therefore AG = $\frac{321356 - 5570}{2}$ = 158790; and AH = $\frac{321356 + 5570}{2}$ = 164363; and with these resolving the triangles ABG, ACH, AG will be found in 3s 0° 41′, and AH in 4s. 0° 8′ 30″; each less advanced also by about 1′ than according to the vicarious theory. The results therefore for AG and AH seem to be almost accurate, especially as, by the observations of four successive days about the opposition, the planet's longitude in it given by the vicarious theory appeared to be 1′ 30″ too great; and they confirm the results from the two other observations, because the angles at G and H are so considerable, that an error of 1′ in observing could not produce an error of above 50 parts on either distance.

Kepler

Kepler investigated, in the same manner, no less than 28 distances, in different points of the orbit; and for the verification of them, he employed them conversely in deducing the geocentric longitudes; and the differences found between the longitude thus calculated, and the longitude observed, were in general very inconsiderable. Only when the planet was in Cancer, they fell 4 minutes short of the observed longitudes, and in the opposite part of the orbit as much exceeded them. But it was impossible that those errors should have proceeded from false distances; for the effects produced by this cause could not have been the same on both sides of an opposition, but would have been contrary; and they were more probably supposed to arise from an error in the position of the aphelion. (Z.)

160. By these two different methods were the distances of Mars from the sun investigated in no less than forty different points of the orbit: but before they should be compared with the distances given by the oval theory, Kepler thought it necessary to employ them in a new investigation of the ratio of the orbits. This he had formerly attempted by observations near the apsides; but he now proposed to add the testimony of some of the present distances.

New investigation of the ratios of the orbits.

It appeared from five observations, that when Mars was in 11° 37′ of mean anomaly, or more accurately in 11° 52′, according to a correction of 15′ found necessary on the vicarious theory, his distance from the sun in the plane of the ecliptic was 166180, or 166208 (157); and therefore, as this point was 23° distant from the northern limit, the distance in the plane of the orbit must be 166250, or 166278. It also appeared, that when the complement of the mean anomaly was 10° 9′ 41″, or more correctly, 9° 54′ 41″, the distance was 166311, or subtracting 1′ 30″ from the longitude given by the vicarious theory, 166208; and, consequently, by an addition of 1° 57′ 19″ to the complement of mean anomaly, so as to make it 11° 52′, the distance will be diminished, and become = 166113, or in the plane of the orbit = 166193. It was likewise found, that when the complement of the mean anomaly was 8° 2′ 21″, or more accurately 7° 47′ 51″, the distance in the plane of the ecliptic was 166396; and therefore at a point 4° 4′ 9″ more remote from the aphelion, it will be = 166204, and in the plane of the orbit 166284. A mean of the results then, when the complement of mean anomaly was 11° 52′, is 166238; and a mean of the results in art. 135, probably still more accurate, is 166260.

Since, then, at 11° 52′ of mean anomaly, or of its complement, the distance is 166260, it is evident that, whatever be the form of the orbit, the excess of the aphelion distance above 166260, cannot be greater than 250; and, if the orbit be circular, it will be less. The aphelion distance therefore, by the present investigation, is 166510. In the investigation of art. 148, it was 166780, but from less accurate observations.

The perihelion distance again was found in art. 148, to be = 138500. But, in art. 157, it was found that the distance, when the complement of mean anomaly was 160° 45′ 30′, or more correctly, 160° 30′ 30″, came out = 138984 in the plane of the ecliptic, or 139000. Suppose it then = 139000, when the planet is in 11s. 21° of heliocentric longitude; and consequently, since this point is 35° beyond the limit, the distance in the plane of the orbit will be = 139019. The excess

of

of this above the perihelion distance cannot be greater than 876, and if the orbit were circular it would be less. The perihelion distance therefore will be = 138173, less than 138500 by 327.

Since, then, the aphelion distance now found is = - - 166510,
and the perihelion distance = - 138173,
half their sum, or the semi-diameter of the orbit, will be = - - 152342;
and the excentricity = - 14169,
or, in parts of the semi-diameter of the earth's orbit = - - 100000,
the excentricity will be = - 9301.

But, as an excentricity so great did not agree with the other observations; and as Kepler had found, by the greatest variety of trials, that 9265, the excentricity of art. 148, was the most probable, and gave the most accurate physical equations, he adapted this to the present aphelion distance, in order to discover another perihelion distance, by means of the analogy 100000 + 9265 : 100000 — 9265 :: 166510 : 138274; whence the semi-diameter of the orbit came out = 152400. But, as too much confidence might thus seem to be placed on the aphelion distance = 166510, and too little on the perihelion distance = 138173, the conclusions which he finally adopted were

Aphelion dist.	Perihelion dist.	Semi-d. of the orbit.	Excentricity.
166465	138234	152350	14115;

whereas before they were

166780	138500	152640	14140;

and it was on the elements now investigated, that the comparison of art. 157 was made; and by which it was so clearly established, that the oval retired as far within the real orbit as the circle went beyond it.

Decisive confutation of the oval theory.

161. Kepler appears to have made a much more minute comparison than that now mentioned; and to have submitted to the labour of deducing the distances for his oval orbit from the present elements, in all the 40 points of mean anomaly which he had employed; and finding, that all of them which lay between the apsides, fell generally as much short of the observed distances, as the calculated circular distances had exceeded them; the evidence was rendered undeniable and compleat, that neither is the orbit of the planet a circle, nor does it retire within the circle so far as the oval of his present theory required, but passes nearly in the middle between them.

Nor was he satisfied with a comparison thus laborious and minute; for he also conversely calculated what geocentric longitudes would be given by the distances of his oval theory; and found that in all the seven oppositions employed in his second method of investigation (159), these longitudes, as must have been expected from distances erring in defect, were always too far advanced before the opposition, and too little advanced after it. This especially was evident at the oppositions of 1589 and 1591 in the descending, and of 1582 and 1595 in the ascending, semi-circle of anomaly; in all which the distance from the apsides was considerable; and as the oval in these points retired too far within the orbit, and the distances which it gave erred sometimes 660 parts in defect, the errors produced on the geocentric longitudes rose sometimes to 20′. Kepler informs us also, that this was observed by D. Fabricius, in East Friesland, as well as by himself; and so nearly was he anticipated by this astronomer in the discovery of the real form of the orbit, that, on communicating to him the oval theory, his answer was, that all its distances

distances towards the intermediate parts were too short.

An evidence to the same purpose was also given by the errors of the equations of art. 156, though calculated according to the original and strictest principles of the theory. For they concurred in representing the motions of the planet as by far too rapid in the intermediate parts of the orbit, and consequently shewed the times or distances in those parts to be shortened a great deal too much. The conclusion, therefore, was, that the oval theory, in endeavouring to avoid the errors of the circle in excess, commits others equally conspicuous in defect; and the physical causes which Kepler supposed to produce it, to use his own language, fly off in smoke.

Accidental discovery of the true law of the distances.

162. When it thus appeared that the breadth of the lunula, cut off by the real orbit of Mars from the excentric, was but half of that cut off by the oval, and that, even at 90° from the apsides, where it was greatest, it did not exceed 660 parts of a semi-diameter = 152350; Kepler tells us, that the vexation which he suffered was very great; not merely from the mortification of finding a theory confuted on which he had founded the most sanguine expectations, but principally, from anxious and fruitless speculations concerning the causes of his disappointment. But while his mind was thus occupied, accident, as he candidly informs us, was more friendly to him than all the exertions of his ingenuity. The 660 parts of a semi-diameter = 152350 were equivalent to 432 parts of a semi-diameter = 100000, that is, nearly to 429 the half of 858, which he had found to be in the same parts, the extreme breadth of the lunula cut off in the oval theory; and, happening

to turn his attention to the greatest optical equation of Mars, which is between 5° 18′ and 5° 19′, he perceived that 429 was also the excess of the secant of 5° 18′ above the radius 100000. By the accidental discovery of this coincidence, he tells us he was suddenly roused as from sleep, and a new light seemed to dart in upon him. His reasonings on the discovery were to this purpose. It is at the points F and Q (fig. 64), of the excentric, which are equally remote from either apsis, that the greatest optical equation AFB takes place; and these are also the points where the orbit retires farthest within the circle, and the extreme breadth of the lunula, instead of amounting to 858, arises only to 429. If, therefore, the semi-diameter BF or BQ, each = 100000, should be employed instead of the secants AF and AQ, each of which is = 100429, as the distances of the planet in these points F and Q of excentric anomaly from the sun, they would be the very distances required by the observations; and, extending this reasoning to all the other divisions of the excentric, he concluded in general, that if, at the opposite points of it, E and R, D and S, &c., he should substitute, for AE, AR, and AD, AS, &c. the lines VE, VR, TD, TS, &c., determined by perpendiculars AV, AT, &c. drawn from A to the diameters ER, DS, &c.; the distances would be every where of the just amount required by the observations, and would neither err in excess, as those given by the circular theory, nor as those given by the oval theory in defect.

But though the propriety of substituting BF for AF, at 90° of excentric anomaly, was evident, it remained to be proved, whether the like substitution of VE for AE, and of TD for AD, &c., was equally admissible in the other points of the orbit.

orbit. Accordingly, to all the true anomalies of art. 157, which from the longitude of the aphelion and the heliocentric longitude of the planet were given, he calculated the excentric anomalies CBD, CBE, and the optical equations ADB, AEB. Then, with the distances AD, AE, or AS, AR, given in the circular theory, he calculated, by the analogy sec. ADB : R :: AD : DT, all the lines TD, VE, or TS, VR, and found their coincidence with the observed distance in these points almost perfect; at least, the differences were altogether insignificant. The following is a comparative view of both.

In the descending semi-circle.

Observed distances.	Calculated.	Differences.
166194	166228	34 +.
163022	163160	138 +.
158101	158074	27 —.
154400	154338	62 —.
147760	147918	158 +.
139000	139093	93 +.

In the ascending semi-circle.

Observed distances.	Calculated.	Differences.
166348	166228	120 —.
163100	163160	60 +.
158217	158074	143 —.
154278	154338	60 +.
147871	147918	47 +.
138984	139093	109 +.

Where two conclusions for the observed distance had been made, a mean of them was taken; and it is evident, that as the differences are so irregular, and at the same time so inconsiderable,

 they

they are principally to be ascribed to errors of observation.

It is evident that the substitution now described was the same in effect, as if, in the original form of the theory (fig. 63, No. 1), AM were substituted for AD; or if, in the separate epicycle, (fig. 63, No. 2), AO were substituted for AM or AN, and AS for AQ or AR; and, that by it the planet is not at all supposed to move in the circumference GNR of its epicycle, but only to librate upwards and downwards in its diameter; and it therefore became unnecessary to institute a like comparison with the distances given by the second method of investigation (159), because in it the librations of the planet, or the variations of its distance from the sun, had been the only subjects of consideration.

Diametral and circumferential distances.

It appeared therefore, from the most numerous observations made in all points of the orbit promiscuously, that the lines TD, TS, or VE, VR, which Kepler called *diametral* distances, were the exact measures of the real distances of the planet from the sun, in those points of the orbit which it occupied, when, according to the circular theory, it would have been found in the points D, S, or E, R, with the distances AD, AS, or AE, AR; which he called circumferential; as supposing the planet to revolve in the circumference of its epicycle, and not to librate in its diameter. (Aa.)

163. It was now to be expected, that when Kepler had discovered a method of calculating the just distances of the planet in all points of its orbit from the sun, that no farther difficulty would occur in deducing from them just equations. But, notwithstanding this important discovery, he was not immediately successsful; and, by a false application

cation of his distances, though in themselves just and accurate, his fears of a new disappointment were again revived. Accordingly, in one of those lively sallies of fancy, by which he was as much distinguished as by sagacity of perception, he tells us that, though the last and most secret retreat of the truth was discovered, the reluctant inhabitant did not immediately deliver herself up to his possession, and applies to her the verses of Virgil:—

> Malo me Galatea petit, lasciva puella;
> Et fugit ad salius, et se cupit ante videri.

The cause of this new alarm he thus describes.

On the points A, B, (fig. 77), let the equal circles HKS, GDT, be described, and let AB be the excentricity of the orbit. Let the equal arches HK, GD, represent the same excentric anomaly, and join DK, which will be parallel and equal to AB. On K describe, with the distance DK, the epicycle LD, and draw AK producing it, if necessary, to meet the epicycle again in L; by which means LD will be an arch of the epicycle similar to GD. Join BD, and from D draw to AL, AG, the perpendiculars DE, DC. It is evident, from what has been now proved, that the accurate distance of the planet from the sun, at the time when, if it moved in the circle ADT, it would be found in the point D, is the line AE; and it is required to know what time it has employed in attaining this excentric anomaly GBD. Now, because GC, the versed sine of GD, that is, after reduction to the dimensions of the epicycle, the libration LE, subtracted from AL = AG, gives the accurate length of AE, Kepler tells us, he supposed that the position of its extremity E was to be sought for in the semi diameter BD of the excentric, and not in the perpendicular DC; and to find it, he described on A as a centre, with the distance

distance AE, the arch EZF, meeting BD in Z. According to this supposition AZ would have been the accurate distance of the planet, in position as well as magnitude, when GBD was its excentric anomaly; and the angle GAZ would have been the true anomaly; though, as was afterward discovered, the true anomaly corresponding to the time, or area DAG, ought to have been the angle GAF, determined by the line AF, drawn to A from the point F, where the arch EZF meets the perpendicular DC.

False position of the diametral distances.

The mistake was first discovered by experiment. On calculating the mean anomaly, or circular area DAG, corresponding to the excentric anomaly GBD, he found that, whether the calculation was made by taking the sum of the diametral distances from A of all the points of equal division in GD, or by adding the triangle DAB to the sector GBD, the substitution of the angle GAZ for GAF, always rendered the true anomaly greater by 5′ 30″, about the first and fourth octants, than the result from the vicarious theory; and, about the second and third octants, nearly as much less. As the equations therefore, even when deduced in this manner from librations, differed from the truth, he ascribed his errors to the librations; and, notwithstanding the just distances which they were found to give, he for some time abandoned them. The ellipse therefore seemed now to be his only resource for applying to any use, the natural principle of the motions being inversely as the times; and he supposed that, in having recourse to the ellipse, he was returning to a theory perfectly different from his theory of librations.

His argumentation in returning to the ellipse, was to this purpose: "The circle gave true anomalies erring in excess, and the oval substituted

for

for it, and which had been considered in his calculations as coincident with an ellipse, gave them erring in defect; and, since the errors were of contrary kinds and equal, another ellipse must therefore be the only curve which can annihilate them. But, if the path of the planet be an ellipse, it must follow, that the point Z cannot be substituted for F, because this would render the plane of the orbit too broad towards the aphelion, and transform the ellipse into an oval. For, let K, *k*, be two positions of the centre of the epicycle equally remote from the opposite apsides. Since the angles EDB, *ed*B, are right, and the sines DE, *de*, of the excentric anomalies equal, it is evident that the arch EZ of less curvature will cut off a smaller portion DZ of the semi-diameter DB, than that which the arch *ez* of greater curvature cuts off from the semi-diameter *d*B; and therefore the curve which passes through the points Z, *z*, will retire more within the circle GMT at *d*, than at D. As it is, therefore, only an ellipse which can bisect the lunulæ cut off from the plane of the excentric by the ellipse which was substituted for the oval, and consequently remove the errors both of the circular and of the oval theories; and the orbit produced in the manner now described, is an oval, and not an ellipse, the errors will not be removed by it, and the equations will continue to be false."

As Kepler, therefore, had no solicitude about the equations which his new ellipse would yield, because it was every where to bisect the former lunulæ; so neither was he very solicitous, lest its distances should differ from those given by observation, because the best observations of the age left them to a considerable degree uncertain. But when no reason could be discovered, making it necessary

necessary for a planet to abandon the librations which produced distances so accurate; and to prefer, as the equations testified, an elliptical orbit; and when his utmost attention and study for this purpose, were for a long time entirely unavailing, he tells us, that his perplexity was extreme, and almost approaching to derangement of mind. This perplexity, however, was at last found to be so groundless, as to appear ridiculous: and the librations in the diameter of the epicycle were discovered to be so far from inconsistent with an elliptical orbit, that they actually produced it, and suggested a method by which it might be described; and the equations deduced from its areas were confirmed, both by the vicarious theory and the most accurate observations.

The librations produce an ellipse.

164. The identity of an elliptical orbit, and the curve described by a planet in consequence of its librations, in the diameter of an epicycle, may be thus demonstrated.

Let A and B be two points in the line GT, and on them as centres let the equal circles HKS and GDT be described. From the extremities of the diameters GT, HS, let the equal and similarly situated arches GD, HK, be taken, and let DK be joined. On the centre K, with the distance KD, describe the epicycle LD, and through A and K draw the line AK, producing it, if necessary, to meet the epicycle again in L. From D draw to GT and AK, the perpendiculars DC, DE; and on A, with the distance AE, describe the circle EF, meeting DC in F. The point F will be in the circumference of an ellipse, whose transverse axis is GT, the diameter of the circle GDT, and its excentricity AB the distance of the centres of the circles HKS and GDT.

Through

Through B, the centre of GDT, draw the semidiameter BM perpendicular to GT, and cutting the circle HKS in N; and join BD, AD, AF, AN.

Because A is the centre of the circle EF, the lines AE, AF, are equal; and, therefore $DA^2 - AE^2 = DA^2 - AF^2$. But $DA^2 = DC^2 + AC^2$ and $AF^2 = FC^2 + AC^2$; and therefore

$$DA^2 - AE^2 = DC^2 - FC^2.$$

But the triangles DKE, DBC, are similar; for, since GD = HK, the lines DK and GB are parallel; and the angles at C and E are right. Therefore $DC^2 : DB^2 :: DE^2 : DK^2$; that is, since DB = AK, and therefore DK = AB, and since also DB = BM, $DC^2 : BM^2 :: DE^2 : AB^2$. But

$$DE^2 = DA^2 - AE^2 = DC^2 - FC^2; \text{ and}$$

$AB^2 = AN^2 - BN^2 = BM^2 - BN^2$. Therefore

$$DC^2 : BM^2 :: DC^2 - FC^2 : BM^2 - BN^2 : \text{Or}$$

$$DC^2 : FC^2 :: BM^2 : BN^2; \text{ and}$$

$$DC : FC :: BM : BN.$$

But since AN = BM, the line BN will be the semi-conjugate axis of the ellipse, whose semi-transverse axis is GB = BM, and its excentricity AB = KD: and, since DC : FC :: BM : BN, and DC perpendicular to the axis GBT, the point F will be a point in the circumference of the same ellipse, and FC an ordinate to the axis GBT.

Cor. If GT be also the transverse axis of the ellipse GKT (fig. 78), the distance AF of the point F, where the ellipse is cut by the perpendicular DC, drawn to GT from any point D of equal division in the circumference GMT, will be equal to the diametral distance XD of that point D; and the sum of the distances of such points F from A, the focus of the ellipse, whether in the whole elliptical circumference, or in the particular arch GF, will be equal to the correspondent sum of the diametral distances, either in the whole circular

cular circumference GMT, or the particular arch GD.

For, if the diameter D*d* pass through the two opposite points of division D, *d*, and there be drawn through A the line E*e* parallel to D*d*, and from D and *d*, the perpendiculars DE, *de* to E*e*, it is evident that AF = AE = DX.

Equations in the ellipse.

165. As the curve described by a planet, in consequence of its librations, was thus demonstrated to be an ellipse, there was no necessity for attempting its particular quadrature, in order to deduce the just equations; because they followed from the known relation between the ellipse and the circle: and his procedure was founded on the following reasoning.

1. If the circumference GMT be divided into any number of indefinitely small equal parts, the circular area GMT will be a just measure of the diametral distances of all the points of division, from any excentric point A in the diameter GT. For, let D, *d*, be two opposite points of the division, join D*d*, and draw to it from A the perpendicular AX. The lines XD, X*d*, are the diametral distances of the points D, *d*, from A, and the sum of the lines XD, X*d*, is equal to the sum of the semi-diameters BD, B*d*. The whole sum, therefore, of all the diametral distances will be equal to the sum of as many semi-diameters; and, as the area GMT is a just measure of the whole sum of the semi-diameters, so will it also be of the whole sum of diametral distances.

2. Supposing the same division of the circular circumference GMT, let GT be the diameter which passes through the excentric point A, and from A let the line AD be drawn to any point D of the division cutting off from the semi-circle the arch

arch GD of excentric anomaly, and from its area, the area GDA: this area GDA will also be a like measure of the sum of diametral distances in the arch GD. For the area GDA consists of two parts, the sector GBD, and the triangular area ABD; of which the former increases in the ratio of the number of semi-diameters drawn to the points of division in GD, and the latter in the ratio of the sines of excentric anomaly terminating in these points, and multiplied into $\frac{AB}{2}$. But the sum of the diametral distances consists also of two parts; of which the first is the actual sum of the semi-diameters, and the second the sum of as many cosines BX, or BC multiplied in AB, of excentric anomaly as there are sines: and, since this sum of cosines increases in the ratio of the sines, it will increase equally with the triangular area ABD. As therefore the whole area GMT is a just measure of the whole sum of the diametral distances, the particular circular area GDA is a like measure of the sum of diametral distances corresponding to the points of equal division in GD. (Bb.)

3. If the excentric point A be one of the foci of the ellipse GKT, described on the diameter GT as its greater axis, and perpendiculars as DC be drawn from all the points of equal division in the circumference GMT, to the axis GT, cutting the ellipse in as many points F, the whole sum of the lines AF, which may be drawn from A to the points of unequal division in GKT, is equal to the whole sum of diametral distances of the points of division in GMT; and the particular sum of the lines AF drawn to the points of division in GF, is equal to the particular sum of the diametral distances in GD. The whole area therefore GMT, being

being a just measure of the sum of diametral distances, will also be a just measure of the sum of elliptical distances; and the particular circular area GDA, will be a like measure of the particular sum of elliptical distances in GF.

But the whole circular area GMT is to the whole elliptical area GKT, as the particular circular area GDA is to the particular elliptical area GFA. The determination, therefore, of this area GFA, by means of the relation between the circle and the ellipse, will be equivalent to the summation of the elliptical distances in GF, or the diametral distances in GD; and the angle GAF, in which this area terminates at the sun in A, will be the true anomaly corresponding to the mean anomaly, which is represented either by the elliptical area GFA, or by the circular area GDA; for these areas are always to each other in the same constant ratio. (Cc.)

Termination of the anomalistical arch.

166. The anomalistical arch GF, which the planet describes in the time represented by the area GFA, or GDA, was, according to this reasoning, made to terminate in the ordinate DFC. But Kepler was so prepossessed with the opinion, that it ought to terminate, not in the point F of this ordinate, but in a point of the semi-diameter BD, as for example Z, where the arch EF described on A, with the distance AE or AF, meets that semi-diameter, that he was doubtful about the truth of his reasoning; for the ellipse was divided by such ordinates into unequal arches, and the effect produced was to determine the times of describing these; whereas the end which seemed to be proposed, was to estimate the times of describing equal arches. It was therefore by trial and experience, that his conviction of the truth of his

his reasoning was principally established; and, indeed, the fact was ascertained before the reasonings were employed.

His principal experiment was to this purpose. He calculated the whole diametral distances for every degree of excentric anomaly, and adding them together in their order, not only found the whole sum, which was = 36000000; but also the particular sum corresponding to the degrees of the same anomaly in any given arch GD. Then, drawing from the given point D the ordinate DC, cutting the ellipse in F, he joined AD, AF; and supposing the whole sum 36000000, to be equivalent to the whole elliptical area, and the particular sum corresponding to the degrees in GD to be equivalent to the particular elliptical area GFA, he had this analogy, as 36000000 is to 314159265 36, the whole area of the circle, so is the sum corresponding to the divisions of GD, or the equivalent area GFA to the circular area GDA. The result for the area GDA left no doubt about the justice of the supposition; for, when calculated by this analogy, it came out perfectly the same as if he had calculated it by finding the amount of the triangular area ABD, and adding this area to the sector GBD. (Dd.)

He found also that, by supposing the anomalistical arch to terminate in the semi-diameter BD, and not in the ordinate DC, the same error ensued which had formerly made him abandon the theory of librations. For, an area GZA, and not GFA, became the mean anomaly corresponding to the excentric arch GD; and the angle GAZ, the true anomaly: and this angle, in the same manner as formerly, was found to exceed the truth in the first and fourth quadrants, and to fall short of it in the second and third. But the angle GAF was always found to be the same with the true anomaly of the vicarious theory; that is, it entirely agreed with the observations.

167. In this manner it was that Kepler discovered and established the celebrated law, by which all the planets are regulated in describing their various orbits; namely, that every such orbit is an ellipse, in one of the foci of which the sun is situated; and that, in revolving round this common focus, the planet describes areas, as GFA, of its ellipse proportional to the times in which it moves through the elliptical arches, as GF, by which those areas are subtended. It only remained therefore, to explain or investigate the methods of deducing the equations of a planet from the principles now established.

From the excentric anomaly, to find the mean.

The first problem proposed for that purpose was, from the given excentricity and excentric anomaly, to find the correspondent mean anomaly.

The excentric anomaly is the angle or sector GBD, or the arch GD which measures them: or it is the elliptical arch GF, because this arch is divided by ordinates to the axis into the same number of parts with GD. The required mean anomaly, again is the circular area GDA, or the elliptical area GFA, because each of them has a constant and determined ratio to the sum of the diametral or elliptical distances, that is, to the time; and therefore will justly measure it.

The circular area, GDA, consists of two parts. The first is the sector GBD; which, because the arch GD is given, will be also given in parts of which the whole area of the circle contains 360. For example, if the arch GD, or the angle GBD, were = 10°, the sector, or area, GBD, would be also = 10°.

The

The other part of the area GDA, is the triangular space ABD. To find this in the most expeditious manner, Kepler previously calculated the area of the triangle MAB, when the excentric anomaly GBM is $= 90^{\circ}$. This area is $= \frac{MB \,.\, AB}{2} = R. \frac{AB}{2} = 463200000$; and, as the whole circular area is to 360°, so is this area 463200000 to $19108'' = 5^{\circ}\ 18'\ 28''$. Now, MAB : DAB :: MB : DC :: R : sin. GD; and therefore, if the given excentric anomaly GD were $= 10^{\circ}$, the triangular space DAB would be $=$ MAB. sin. GD $= 3318''.7 = 55'\ 18''.7$. Therefore GDA $=$ GBD $+$ DAB $= 10^{\circ}\ 55'\ 18''.7$.

With equal ease he solved a second problem—from the given excentric anomaly, to find the true anomaly, that is, the angle GAF. For the cosine BC of the given arch GD, is given in parts of BD $= 100000$; and, when GD is $= 10^{\circ}$, will be $= 98481$. Since also AB, in the same parts, is $= 9264$, and R : cos. ABX ($=$ GBD) :: AB : BX, the line BX will be $= 9124$. Therefore DB $+$ BX $=$ AE $=$ AF $= 109124$; and, since AC $=$ AB $+$ BC $= 107746$, we shall have in the right-angled triangle FAC, AF : AC :: R : cos. GAF $= 9^{\circ}\ 4'\ 56''$. The whole equation therefore, or difference between the true and mean anomalies, will be $= 1^{\circ}\ 50'\ 22''.7$. In the second and third quadrants of excentric anomaly, Af will be $= d$B $\backsim$ BX, and Ac $=$ AB $\backsim$ BC. From the excentric anomaly, to find the true.

168. It was a business of greater difficulty to find the excentric anomaly, from the excentricity and the true anomaly given. Two solutions, however, of this problem were found, and the principles of the first were these. From the true anom. to find the excentric anomaly.

 1. The

Solution first.

1. The parts GD, of the ordinates to the greater axis KP, which are intercepted between the ellipse KHP (fig. 80), and the excentric KEP described upon KP as its diameter, increase in the ratio of the sines of excentric anomaly. For GD : HE :: CD : FE.

2. The tangent MD of the angles DBG, by which the intercepted parts GD of the ordinates are subtended at B, the centre of the orbit, would increase in the ratio of the cosines of excentric anomaly, if GD were supposed to continue invariable.

If BG be produced to meet the tangent in M, we shall have DM : GD :: sin. DGM : sin. DMG; whence DM. sin. DMG = GD. sin. DGM. But the difference between DMG and a right angle is almost insensible; for its complement DBG, under which GD is seen from B, does not even in the orbit of Mars, and at its *maximum*, exceed 8 minutes. Therefore the tangent DM will be as GD. sin. DGM = GD. sin. BGC; or, since GD is supposed to continue invariable, as sin. BGC. But BGC is very nearly equal to BDC, the complement of the excentric anomaly KBD, and cannot at its *maximum* exceed BDC above 8 minutes. Therefore DM will always be very nearly as sin. BDC, that is, as cos. KBD = BC. This tangent therefore, if GD were invariable, would, at the points K, and P, subtend the greatest angle DBM; for there it coincides with GD, and at these points the cosine BC is greatest; and, at L, and the opposite point, 90° remote from K and P, it coincides with BL, and the angle which it subtends vanishes with BC.

3. The tangent of the angle DBG, or, since very small angles are as their tangents, the angle DBG

DBG increases nearly in the compound ratio of the sine and cosine of the excentric anomaly.

The magnitude of the angle DBG depends, 1st, on the magnitude of the intercepted part DG of the ordinate DC, which increases from nothing in K to its greatest amount in L, according to the ratio of the sines; and, 2dly, on the apparent magnitude of the tangent DM, to the eye placed in B; and this apparent magnitude has been seen to decrease from K to L, in the ratio of the cosines. On the first account the angle DBG is = 0, in K; because there the sine of excentric anomaly is = 0: and on the second account DBG is also = 0, in L; because there the cosine of excentric anomaly is = 0. But this angle, at 45° from K and L, has risen to more than half its greatest amount; for the sine of 45° is = 70711, that is, greater than half the radius, and the cosine of 45° is equal to its sine. The rectangle, therefore, under the sine and co-sine of excentric anomaly, is greatest at 45°, and becomes a square equal to half the square of the radius.

4. Let KBD be the excentric anomaly, from the focus A draw to the excentric the line AE parallel to BD, from D and E draw to the axis KP the ordinates DC, EF, meeting the ellipse in G and H, and let BG, AH, be joined; the angle DBG will be equal to the angle EAH. For CD : DG :: FE : EH, and CD : DB :: FE : EA. Therefore DG : DB :: EH : EA; and, since the angles CDB, FEA, are equal, the angle DBG is also equal to the angle EAH.

5. The angle EAH is accurately as sin. KAE. cos. KAH; (for its equal DBG has been proved to be as sin. CBD. cos. KBG), and therefore nearly as sin. KAE. cos. KAE.

This angle EAH, at its greatest amount in 45°

of excentric anomaly, may be found in this manner. When KBL is = 90°, the intercepted part LN of BL is = 429, or 432 (162); and, since BL : LN :: CD : DG, or 100000 : 432 :: 70711 : DG, this intercepted part will, in 45° of excentric anomaly, be found = 315; and consequently CG = 70396. Since, then, CB = 70711, the angle CBG will be = 44° 52′ 19″, and DBG = CBD — CBG = 7′ 41″ = EAH.

Hence the angle EAH at any other point of excentric anomaly, suppose at 30°, will be readily found. For, since sin. 45°. cos. 45° = 5000000000, and DBG at 45° of excentric anomaly = 7′ 41″, the angle EAH at 30° of the same anomaly, will be found by this analogy, sin. 45°. cos. 45° : 7′ 41″ :: sin. 30. cos. 30 : $\frac{\text{sin. } 30.\ \text{cos. } 30}{\text{sin. } 45.\ \text{cos. } 45}$. 461″ = $\frac{4330150000}{5000000000}$. 461″ = $\frac{86603}{100000}$. 461″ = 398″ = 6′ 38″ = EAH.

Suppose then, that it were required to find the excentric anomaly KBE, corresponding to the given true anomaly KAH. The angle EAH, as has now been seen, is = 6′ 38″, and consequently KAE = 30° 6′ 38″. Therefore we shall have in the triangle EAB, sin. BEA = $\frac{\text{AB. sin. BAE}}{\text{BE}}$ = $\frac{9264,\ \text{sin. } 30°\ 6'\ 38''}{100000}$, and BEA = 2° 39′ 49″.7. Whence the required excentric anomaly KBE will be = 32° 46′ 27″.7; and from KBE thus found, the mean anomaly, or area KEA, in parts of the circular area, or KHA, in parts of the elliptical area, will be found as in the former article.

A practical rule, arising from the above investigation, for finding the angle EAH at any point of true anomaly, was this: Multiply the sine of the given true anomaly into the cosine, double the product,

product, cut off the five last figures, and multiply the result into $461''$. The importance of this abbreviation, while logarithms were unknown, is obvious.

169. The other solution was by means of an algebraical analysis. Let the given true anomaly GAF (fig. 78), be $= \varphi$, BD $=$ R, AB $= a$, BC $= x$, BX $= y$. Then AF $=$ DX $=$ R $+ y$, and since R : cos. φ :: R $+ y$: AC, we shall have AC $= \frac{R + y}{R}$ cos. φ, and therefore $x =$ AC — AB $= \frac{R + y}{R}$ cos. $\varphi - a$. Solution second.

But, on account of similar triangles, $a : y ::$ R : $x = \frac{Ry}{a}$. Therefore $\frac{Ry}{a} = \frac{R + y}{R}$ cos. $\varphi - a$: and by the solution of this simple equation, the value of BX $= y$, and consequently of BC $= x$, the cosine of the required excentric anomaly will be found. In general $\frac{Ry}{a} = \frac{R \pm y}{R}$ cos. $\varphi \backsim a$.

For example, let GAF $= \varphi$ be $= 30^\circ$, and consequently its cosine, in parts of BD $= 86603$; and, since AB $= 9264$, the equation will become $\frac{100000y}{9264} = \frac{100000}{100000}\,86603 + \frac{86603}{100000}y - 9264$. That is, $10.79445y - 0.86603y = 86603 - 9264$. Or

$$9.92842y = 77339. \text{ Whence}$$

$$y = 7789, \text{ and } x = \frac{Ry}{a} = 84078.$$

The required angle therefore, GBD, of which this is the cosine, will be $= 32^\circ\ 46'\ 38''$.

170. Thus was Kepler enabled to deduce, from the law which he had with such labour and ingenuity established, the equations of a planet in all points of its orbit: and, by these operations, the

important problem of finding in all cases the relation between the mean and true anomalies, might be considered as practically, though indirectly, solved. For, from the true anomaly given, the excentric anomaly might always be obtained, and from the excentric anomaly the mean; and as these operations might be performed, not only for every degree, but even for every minute and second of true anomaly, tables might be constructed, exhibiting all the possible equations, or differences, between them. But, with respect to the direct solution of the problem—from the mean anomaly given to find the true—he tells us that he found it impracticable, and that he did not believe there was any geometrical or rigorous method of attaining to it. He was obliged therefore to content himself with imploring the assistance of geometers, and proposing to them the problem in the following terms.

Kepler's problem. Having the area of part of a semi-circle given, and a point given in its diameter, to determine an arch of the semi-circle, and an angle at the given point, such that the given area may be comprehended by the lines including the angle, and by the required arch: or, to draw from a given point in the diameter of a semi-circle, a straight line dividing the area of the semi-circle in a given ratio. This is the famous problem peculiarly distinguished by the name of Kepler's problem: and it is obvious, that if the solution were obtained in the semicircle, the position of the line, as AF, which should divide the semi-elliptical area in the same ratio, would immediately follow, from the known relation between the ellipse and circle. This problem has, ever since the time of Kepler, continued to exercise the ingenuity of the ablest geometers; but no solution of it which is rigorously accurate

has

has been obtained. Nor is there much reason to hope that the difficulty will ever be overcome; because, as Kepler remarks, a circular arch, and any of the lines drawn about it, as the sines, are quantities of different kinds; and it transcends the powers of geometry to express their mutual relation with perfect accuracy.

171. When Kepler had thus completed his theory of the planetary motions in longitude, he had the satisfaction to find a new confirmation of his principles in the success with which they represented the whole varieties of observed latitude. Here all the former theories had utterly failed; even his own vicarious theory, though far superior to the rest, had received the most decisive confutation from its false representations of the latitudes; and it shewed the impossibility of justly representing them in any orbit which was exactly circular, because the distances of the planet, both from the earth and the sun, given by an orbit of this form, were in general false. But when the true form of the orbits was discovered, when their mutual ratios and excentricities were determined, and when it was ascertained that their planes intersected each other in a line passing through the centre of the sun, the mutual distances of the planets, the earth and sun, obtained by calculation, were the same with the real distances; the inclinations of the orbits appeared to be invariable; and, without any of the former improbable suppositions of oscillation in three various directions, the geocentric latitudes, calculated with such distances and inclinations, corresponded in the most accurate manner with the latitudes observed. His new theory even enabled him to explain the *phænomenon*, which not only the old astronomers, but T. Brahé himself considered as a paradox, that the observed latitudes were not always

His new theory confirmed by the latitudes.

always the greatest at the oppositions. For, as the observed latitude of a planet would evidently be at its maximum, if the planet were in opposition in the limits, and at the same time at its least distance from the earth; it must necessarily follow, that if the planet were observed, before an opposition for example, at a certain distance from the earth, and with a certain heliocentric latitude, and if that distance should increase towards the opposition in a greater ratio than the sine of the heliocentric latitude, the geocentric latitude towards the opposition would be diminished.

His third law.

172. Another discovery made by Kepler, of equal importance, though not distinguished by the same steady and scrupulous prosecution of principles through their various consequences, which it indeed precluded, but produced chiefly by his extraordinary propensity to trace analogies, was his famous law concerning the mutual relation between the distances of the planets from the sun, and the periods of their revolution. He had determined, as we have seen, by various methods, the distances of every planet from the sun: that the distance of Jupiter, for example, is about five times, and of Saturn more than nine times, greater than the distance of the earth from the sun. But he found also, that the ratios of the times, in which the planetary revolutions were performed, were by no means the same with those of the distances; for the time of the revolution of the earth is but a twelfth part of the time of Jupiter's revolution, and but a thirtieth part of that of Saturn's revolution: and as he had considered the Copernican arrangement of the planets to be defective and unsatisfactory, while no known relation subsisted between those distances and times, the discovery of this relation became to him an object of the greatest

greatest interest, and, indeed, of restless curiosity. His trials for this purpose were various and repeated; he first employed himself in comparing the ratios of the simple distances, or times, with those of the regular solids in geometry, and with the divisions of musical chords; it next occurred to him, on the 8th of March 1618, that, instead of comparing together the simple distances and times, he should compare the numbers expressing their similar powers, such as their squares, or their cubes, &c.; and, lastly, he made the very comparison on which his discovery was founded, between the squares of the times and the cubes of the distances; but, through some error of calculation, no common relation was found between them. Finding it impossible, however, to banish the subject from his thoughts, he tells us that, on the 8th of the following May, he resumed the last of these comparisons, and by repeating his calculations with greater care, found, with the highest delight and even astonishment, that the ratio of the squares of the periodical times of any two planets was constantly and invariably the same with the ratio of the cubes of the greater axes of their orbits, or of their mean distances from the sun. The mean distance, for example, of the earth from the sun to that of Jupiter from the sun, is as 1 to 5.2, and the time of a sidereal revolution of the earth to the time of Jupiter's sidereal revolution, as 365d. 6h. 9′ 11″.2 to 4332d. 8h. 51′ 25″.6. Now, the number which expresses how often the square of the time of Jupiter's revolution contains that of the revolution of the earth, is 140.6874; and this number also expresses, with equal exactness, how often the cube of Jupiter's distance contains the cube of the distance of the earth. This celebrated law received the most perfect confirmation on the discovery of the

the various secondary planets of the system; for these were all found to be subjected to it in their revolutions round their respective primaries; and Newton has demonstrated, that it is the necessary and universal law of all bodies revolving round a centre, and gravitating towards it in the inverse ratio of the squares of their distances from it.

Recapitulation.

173. This, then, is the history of the progress of Kepler, in making those celebrated discoveries, which first gave to astronomy a just title to the name of science. Even at the beginning of his astronomical studies, and when his views were limited to the single object of referring the motions of the planets to the centre of the sun, instead of the centre of the ecliptic, his discoveries were of very great importance: and it is to his ingenuity in prosecuting this original design, that astronomy is indebted for the introduction of many valuable principles, which were either before unknown, or known imperfectly, and improperly applied. Of this kind are his new and various methods for determining the places of the nodes, and the inclinations of the orbits; the doctrine of the invariableness of these inclinations; the principles of the only just and legitimate reductions from any orbit to the ecliptic; and the first accurate determinations of the places and motions, both of the nodes and apsides. In the progress of his studies, and when, on finding the insufficiency of the alteration he had introduced, he reversed the usual process of astronomers, and considered the knowledge of the elements of the terrestrial orbit as the preliminary step for determining the elements of the rest; his discoveries were equally original, and still more valuable. For here, besides a variety of new and ingenious methods for ascertaining

ing the distances of the earth in all points of her orbit from the sun, we find the important consequence was to introduce the principle of the bisection into the solar theory, and to demonstrate that, contrary to the uniform opinion of the ancients, the solar inequalities were no more than those of the planets entirely optical, but partly real : and this principle being universally established, and it appearing to be a general law, that every planet described arches of its orbit, the times of which were as their distances from the sun, the design which he had happily conceived, even in the most early period of his studies, of deriving from this law the whole planetary equations, was not only demonstrated to be possible, but also found, by its success in the solar orbit, to be practicable and advantageous.

But the most interesting and important of Kepler's discoveries, were made in the application of his improved solar theory to the investigation of the theories of the planets. For here, besides ascertaining that the planes of the planetary orbits pass through the sun's centre, we find not only the secret of the true form of the orbit's disclosing itself to his penetration, but also the law investigated of the areas of those orbits being proportional to the times employed in describing them ; and both these discoveries, not only perfectly confirmed by evidence, but successfully applied to the calculation of the equations. As the distances between the earth and the sun, given by the improved solar theory, even in a circular orbit, were, on account of a small degree of excentricity, nearly accurate ; the distances of Mars from the sun, to the determination of which they were employed, must also have been nearly accurate. But when the distances of Mars, thus determined, were employed

employed in deducing the elements of his orbit, and that orbit was supposed to be circular, all the conclusions concerning them were, on account of the greater excentricity, found to be false and inconsistent: and when, after the accurate determination of the elements from other principles, the method of areas was employed to calculate the equations, the errors, both in excess and defect, sometimes arose to more than 8′. Since, therefore, neither of these circumstances could have taken place, had the planet really moved in a circle, and particularly since the method of areas never could have produced so great an error as of 1′, in any point of it, strong suspicions arose that the orbit was not circular; and, notwithstanding the reluctance of Kepler to depart from an opinion so long established, and held as an inviolable maxim, the comparison of the real and observed distances of Mars from the sun, with the distances calculated for a supposed circular orbit, shewed the suspicions to be just; and presented evidence, which could not be resisted, that the orbit in all points between the apsides retired within the circle.

But though it thus appeared that the orbit was not circular, its real form continued undiscovered; and Kepler, in consequence of a precipitate and groundless theory concerning the cause of the deviation from the circle, was led falsely to conclude that it was an oval, coinciding indeed with the circle at the apsides; but whose deviation from it, at 90° of excentric anomaly, counted from either apsis, amounted to no less than 858 parts of its semi-diameter, supposed to contain 100000. The vexatious labour in which this precipitancy engaged him, to describe the oval, to obtain its quadrature, to divide it in any required ratio, and to to derive from it just equations, is almost inconceivable, and what perhaps no other person would have had the fortitude and perseverance to undergo. But it was by this labour that the true form of the orbit was discovered; for, on comparing the distances deduced from observation, in no less than forty different points of anomaly, with those calculated for the same points in the oval orbit, the latter were found to fall as much short of the former, as the former had fallen short of the distances calculated for a circular orbit; and, if the method of areas applied to the circle, had produced equations, erring sometimes more than 8′, in excess as well as in defect, the same method applied to the oval produced equal and contrary errors; that is, of defect where the circle shewed excess, and of excess where the circle shewed defect. It was therefore evident, that the real path of the planet lay precisely in the middle between the circle and the oval; and as in his calculations he had considered the oval as a real ellipse, it followed that the only curve, which could bisect the space intercepted between it and the circle, must be another ellipse.

When Kepler thus found that the distances, given by his oval theory, fell as much short of the observed distances as those given by the circular theory had exceeded them, a fortunate accident discovered to him, that the distances used in the circle were the secants of the optical equations in all the different points of excentric anomaly; and that, if instead of these, he should use the different radii to which they were the secants, such distances would be obtained as should perfectly agree with the distances deduced from observation. But by a mistake committed in their position, that is, in the position of the planet at the time when its

distance

distance was supposed to be just, he again failed in his endeavours to obtain just equations; and, whether he employed the circular areas, or the actual sums of the distances, the true anomalies which he considered as correspondent to them were generally false, and sometimes erred more than 5′ from the point which the planet really occupied. His distances therefore, though proved by observation to be just, seemed to be inconsistent with the elliptical form ascribed to the orbit: for, in fact, by the positions which he had given them, they represented it as a new kind of oval, going beyond the ellipse in the first and fourth quadrants of anomaly, and retiring within it in the second and third; and only differing from the former in this respect, that it deviated less widely from the circle. Accordingly, when rejecting his distances he returned to the ellipse, it was not from perfect conviction of its being the path in which the planet actually moved, but only because no other prospect seemed to remain of applying the principles he had previously established to the derivation of just equations. But by this step of his procedure, the mistake which he had committed in the position of his distances came to be discovered; and the lines which he had substituted for the secants of the optical equations, instead of being inconsistent with the ellipse in which he had supposed the planet Mars to move, were found to lead to the accurate description of it. His speculations, therefore, concerning the elliptical form of the orbit, received the fullest confirmation; the elliptical areas, and the sums of the correspondent diametral distances, were found to be perfectly equivalent; and the just equations derived from them rendered it unquestionable, that this planet both revolves round the sun in an ellipse,

ellipse; and describes round the focus occupied by the sun, areas of its ellipse proportional to the times. By like experiments it was also found, that the same laws regulated the revolutions of all the other planets; and the three discoveries, that the orbits of all the planets are ellipses, in whose common focus the sun is situated; that they describe round the sun areas of their ellipses proportional to the times; and that the squares of the times of their revolutions are proportional to the cubes of the greater axis of their orbits, or of their mean distances from the sun; are justly to be considered as the most important ever made in astronomy. They were, indeed, the foundations of the whole theory of Newton; and it will not perhaps be thought an unjust conclusion from the consideration of them, that no person, in any age, ever soared higher than Kepler, above the common elevation of his contemporaries.

NOTES.

A.

AMONG the lunar inequalities manifested by simple observation, might have been mentioned the unequal motion of the apsides discovered by J. Horrox, in consequence of his attentive observations of the lunar diameter. He found that, when the distance of the sun from the moon's apogee was about 45°, and 225°, the apogee was more advanced by 25°, than when that distance was about 135° and 315°. The apsides therefore of the moon's orbit were sometimes progressive and sometimes regressive, and required an equation of 12° 30′, sometimes additive to their mean place, and sometimes subtractive from it. But this inequality was too late in being discovered to have any effect, either in the formation of the ancient theories, or in Kepler's alteration of them: and though it appears to have been discovered also by d'Arzachel, an Arabian astronomer, about the year 1080, in Spain, by different means, the knowledge of it had not reached either T. Brahé, or Kepler.

Inequality of the motion of the lunar apsides discovered by D'Arzachel and Horrox.

B.

Various improvements of this complicated theory were found necessary; and of these, one proposed by Chris. Severinus, commonly called Longomontanus, the assistant of T. Brahé, may serve for an example. In this, the centre of the orbit is B (fig. 17), the distance BC of which from the centre of the earth is not, as according to Ptolemy, an half, but three-fourth parts of the excentricity of the equant; and EL, the semi-diameter of the smaller epicycle, is equal to the remaining fourth part. The centre E of this epicycle is supposed to move uniformly round B, the centre of the orbit, *in consequentia*, in the time of a tropical revolution of either inferior planet; and its diameter KL is always parallel to the line of apsides; so that, when E coincides with A, K is the perigee of the epicycle; but L is the perigee when E coincides with D. The centre of the greater epicycle HMN revolves in the circumference of this smaller one; in the orbit of Ve-

Theory by Longomontanus of the inferior planets.

nus,

nus, with double, and in the orbit of Mercury, with triple the velocity of the centre E; but in such a manner as that, when E coincides with A, the apogee of the orbit, the centre of the greater epicycle is found in K, and, in this nearer part of the smaller epicycle, moves *in antecedentia*. In the orbit of Venus, therefore, when the mean anomaly AE, or ABE, is about 90°, the centre of the greater epicycle will always be in the apogee of the smaller, and in its perigee, always when E coincides with the apsides A, D; but, in the orbit of Mercury, as the centre of HMN revolves with triple the velocity of E, it will be in the perigee of the smaller epicycle, not only when E coincides with A the apogee, but also at 60° before the arrival of E at D, the perigee of the orbit, and again at 60° after passing it. The inferior planet moves in the circumference of the greater epicycle, *in consequentia* in the upper part, and *in antecedentia* in the lower; and it comes to its greatest digressions, in the points M and N, when lines drawn from C the earth are tangents to the epicycle. This theory was supposed to be simpler than the Ptolemaic, and to account equally well, not only for the first inequalities, but for all the varieties of the greatest digressions; and particularly to shew the reason why they did not, in the orbit of Mercury, come to their maximum at D, the perigee of the orbit, as might have been expected, but at two points on each side of it. It also avoided an error which, when the distances of the planets came to be more accurately considered, rendered this part of the Ptolemaic theory inadmissible; for this theory brought the inferior planets, especially Mercury, so near the earth, that their position was totally inconsistent with all observations of their diurnal parallaxes.

C.

General view of the second inequalities.

A more general view of the whole circumstances of the second inequalities may be thus exhibited. Let ABC (fig. 25), be the orbit of the earth, and DEF the orbit of Mars. If we draw a line GQ parallel to AD, it is evident that Mars, in order to appear stationary, must describe the arch DQ, while the earth describes AG; and that, in order to appear direct, or advancing in his orbit, he must describe a greater arch than DQ. But since, in fact, he describes an arch DH less than DQ, while the earth describes AG, he must be left behind by the earth, and to an observer in G appear retrograde. When again the contemporary arches EK and BL, though unequal, shall be so obliquely situated with respect to each other, as to be comprehended between the parallels BE and LK, Mars will, in this interval, appear stationary. Finally, when

when the contemporary arches FM and CN become yet more oblique, in their relative positions, so that the visual rays CF and NM cross each other, before they reach the orbit of Mars, the motion of the planet will appear direct; for NM will be directed to a point more advanced than F. In like manner, if we suppose DEF to be the orbit of the earth, and ABC that of Venus, the changes of appearance will be equally evident. If the lines AD, GH, which bound the contemporary arches AG, DH, should meet beyond the orbit of the earth, then HG will mark a point O, in the zodiac, less advanced than P the point marked by DA, and during the time of describing these arches, therefore, the planet will appear retrograde. The lines again EB and KL, being parallel, will mark the same point in the heavens, and the planet in describing the arch BL will appear stationary. But FC and MN will cross each other between the contemporary arches which they include; and, as MN therefore will mark a point in the zodiac more advanced than FC, the planet, in describing CN, will appear direct.

D.

On the supposition of circular orbits and concentrical to the sun, the modern solution is to this effect. Let ATK (fig. 26), be the orbit of the earth, and BPM that of the planet; let S be the sun, A the earth, and B the planet, when it first appears stationary: and, as it will continue stationary, while the visual rays proceeding from it continue parallel to AB, let TP be the last of such parallel rays. Let V be the velocity of the earth, and v that of the planet, in their orbits; and then, since the arches AT and BP are described in the same time, they will be to each other as these velocities; that is, AT : BP :: V : v. From the points A and B draw to the orbits the tangents AD, BD, meeting each other in D; and since these, during the time of describing the arches AT and BP, are to sense confounded with the arches, we shall have AD : BD :: AT : BP; that is, AD : BD :: V : v. Make AS $= a$, BS $= b$, and the required angle BAS of elongation $= x$; and, since BS : AS :: sin. x : sin. ABS, this last will be $= \frac{a. \text{ sin. } x}{b}$. But sin. ABS = cos. ABD; and, since sin. ^{2}ABD $= r^2 -$ cos. ^{2}ABD, we shall have sin. ABD $= \sqrt{r^2 - \frac{a^2. \text{ sin. } ^2x}{b^2}}$; and, since also, cos. SAB = sin. BAD, we shall have sin BAD $= \sqrt{r^2 - \text{sin. } ^2x}$. But sin. ABD : sin. BAD :: AD : BD :: V : v; and therefore

To determine the points of station.

V :

$$V : v :: \sqrt{r^2 - \frac{a^2 . \sin.^2 x}{b^2}} : \sqrt{r^2 - \sin.^2 x}.$$ Whence

$$V^2 (r^2 - \sin.^2 x) = v^2 (r^2 - \frac{a^2 . \sin.^2 x}{b^2}),$$ and

$$V^2 r^2 b^2 - v^2 r^2 b^2 = V^2 b^2 \sin.^2 x - v^2 a^2 \sin.^2 x.$$ Whence

$$\sin.^2 x = r^2 b^2 (\frac{V^2 - v^2}{V^2 b^2 - v^2 a^2}),$$ and

$$\sin. x = \pm \sqrt{b^2 r^2 (\frac{V^2 - v^2}{V^2 b^2 - v^2 a^2})} = \pm br \sqrt{\frac{V^2 - v^2}{V^2 b^2 - v^2 a^2}}$$

Since in the planetary system $V : v :: \sqrt{b} : \sqrt{a}$ we shall have

$$b : a :: r^2 - \frac{a^2 . \sin.^2 x}{b^2} : r^2 - \sin.^2 x;$$ whence

$$b^3 r^2 - ab^2 r^2 = (b^3 - a^3) \sin.^2 x;$$ and

$$\sin.^2 x = \frac{b^3 r^2 - ab^2 r^2}{b^3 - a^3};$$ or making $r = a = 1$,

$$\sin.^2 x = \frac{b^3 - b^2}{b^3 - 1} = \frac{b^2}{b^2 + b + 1};$$ and $\sin. x = \pm \frac{b}{\sqrt{b^2 + b + 1}}$

If V were $= v$, we should have $\sin. x = 0$; and, consequently, the points of station would be in the conjunctions and oppositions. For, since AD : BD :: V : v, it is impossible that AD can be equal to BD, except when both are parallel, and therefore infinite. If v were greater than V, and at the same AS $= a$, so related to BS $= b$, that $V^2 b^2$ were greater than $v^2 a^2$, the planet never would appear stationary.

E.

The argument of latitude is formed by subtracting the longitude of the node from the longitude of the planet, reckoning both on the planet's orbit. Thus, if A (fig. 30), be the place of the point of Aries in the ecliptic, an arch Ba = BA is to be computed in the orbit, and this arch BA will be the longitude of the node, while PBa is the longitude of the planet, and the argument of latitude is BP = PBa — Ba. When the longitude of the node is greater than the longitude of the planet, 360° are to be added to the last. Thus, if the planet were in p with the longitude pa = 20, while Ba the longitude of the node was = 60, the argument would be 360° + pa — Ba = 380° — 60° = 320° = pDPB; and the nearest distance from the node would be pB = 40°.

Though the rule thus expressed is general, it may perhaps be be thought more easy in practice to find pB directly, by subtracting ap from aB; for pB would thus also be $= 60° - 20° = 40°$.

The chief calculations relating to the second inequalities of a planet, according to the principles of the Copernican system, with the addition of the reductions now employed, will be easily understood by the following description. Let S (fig. 31, 32), be the sun in the centre of the ecliptic, T the place of the earth in the orbit CDT, and P the place of a planet in its orbit APB. By drawing from the planet in P the perpendicular PQ to the plane of the ecliptic, the angle PSQ will be its heliocentric latitude, and PTQ its geocentric latitude; SP and TP will be its real distances in its orbit from the sun, and from the earth; and SQ, TQ its distances in the plane of the ecliptic, called the reduced or shortened distances. The heliocentric longitude in the orbit would be marked by the line SP, and the geocentric longitude by the line TP; but these longitudes reduced to the ecliptic are L and O, marked by the lines SQ, TQ. The angle SQT, therefore, which is the difference of these reduced heliocentric and geocentric longitudes, is the annual parallax; the difference TSQ between the heliocentric longitudes of the planet and the earth is the commutation, and the difference STQ between the geocentric longitudes of the sun and the planet is the elongation. In the triangle STQ, therefore, with any three of its six parts, (that is, its sides and angles) given, the other three may always readily be found. Calculations relating to the second inequalities;

Suppose, for example, that the distance of a planet in its orbit from the sun, and its heliocentric longitude, also in the orbit, together with the longitude of the earth, were found for any particular time, by means of some theory of their first inequalities, and that it were required to calculate for the same time its geocentric longitude. Since the inclination CBP (fig. 30), of the orbit is given, and invariable, and the argument PB of latitude may be formed from the given longitude of the node, the reduction to the ecliptic will easily be found, and thence also the longitude CBA of the planet reckoned on the ecliptic; and therefore in the triangle SQT, the commutation TSQ, which is the difference between this longitude and that of the earth, will be found. From the inclination CBP and the argument of latitude, the heliocentric latitude PSQ, or CSP (fig. 30), will also be found by the analogy R : sin. CBP :: sin. BP : sin. PSC, that is, sin. PSQ = sin. inclination, sin. argument latitude; and, therefore, in the right-angled plane triangle PQS, the reduced distance SQ will be = SP. cos. PSQ; for R : cos. PSQ :: SP : SQ. If, then, we assume the semi-diameter ST of particularly of the geocentric longitude, of the earth's orbit = 1; we shall with SQ, in like parts, and the included angle TSQ, the difference of the longitudes, find the angle of elongation STQ, by the usual analogy $ST + SQ : ST \backsim SQ :: \frac{1}{2}(SQT + STQ) : \frac{1}{2}(SQT \backsim STQ)$; and this angle added to, or subtracted from, the longitude of the sun TS, which is always six signs greater than the longitude of the earth, will give the required geocentric longitude TQ. This is the longitude in the ecliptic, and which observations give, while the geocentric longitude in the orbit is marked by TP; and it is evident that the operation now exemplified, is the same with the calculation of the equation of the orbit, in the Ptolemaic theory, excepting only in what relates to the reductions. In forming the angle of commutation, the longitude of a superior planet is always to be subtracted from the longitude of the earth; and the longitude of the earth is always to be subtracted from that of an inferior planet, because this angle always increases by the excess of the greater velocity above the least.

and latitude.

The geocentric latitude of a planet PTQ, for any given time, may be calculated by this analogy sin. TSQ : sin. STQ :: tan. PSQ : tan. PTQ; that is, sin. commutation : sin. elongation :: tan. heliocentric latitude : tan. geocentric latitude. For, in the triangle PTQ,

TQ : QP :: R : tan. PTQ; and in the triangle PSQ
QP : SQ :: tan. PSQ : R; and therefore
TQ : SQ :: tan. PSQ : tan. PTQ. But in the tria. SQT,
TQ : SQ :: sin. TSQ : sin. STQ; and therefore
sin. TSQ : sin. STQ :: tan. PSQ : tan. PTQ.

Distance found by observations of latitude in opposi. and conjunc.

At oppositions of the superior planets, and conjunctions of the inferior, their distances from the sun, if the inclinations of the orbits and the longitudes of the nodes are given, may be found by observations of their geocentric latitudes. Let the earth be in B (fig. 29), and C the planet observed at its opposition, with the latitude CBD. Therefore, its place, in the ecliptic, is the point D of the line DBA, passing through the centres of the earth and sun; and if the inclination of the orbit, and the argument of latitude are given, the angle CAB of heliocentric latitude will be found; and with the two angles CAB, CBA, of the triangle CAB, and the side AB, which is the distance of the earth from the sun, given by the solar theory, we may easily find also the side AC required.

F.

The usual method of representing the vicissitudes of the seasons, according to the Copernican principles, is as follows.

Let

Let ABCD (fig. 34), be the orbit of the earth, S the sun, PE the axis of the earth, ÆQ the equator, and TR, *tr*, the tropics of the terrestrial globe. It is manifest that, if the axis PE were perpendicular to the plane of the ecliptic, this great circle would coincide with the equator; and, as the rays of the sun must always illuminate one hemisphere of the earth, that, by their direction in the plane of the equator, they would constantly shine from pole to pole, and there could be no variation, either of the seasons, or of the length of day and night. But, since the planes of the equator and the ecliptic intersect each other in an angle of about 23° 30′, the axis PE will be inclined to the latter plane in an angle of about 66° 30′; and, because this angle continues constant during the whole annual revolution, and the direction of the axis invariable, the rays of the sun will be very seldom in the direction of the plane of the equator, and the effects will be very different. When the earth is in the position D, suppose about the 21st of June, with the axis so inclined to the ecliptic that its north pole P is nearest the sun, the solar rays will be perpendicular to the earth's surface at the tropic of Cancer T; and, in the course of the diurnal rotation, every place under this tropic will have the sun in its zenith. In this position also, as the solar rays illuminate a whole hemisphere of the earth, the light will extend as far beyond the north pole P as the distance between the tropic and the equator, and will fall as far short of the other pole; the whole region within the arctic circle, consisting of 23° 30′ round P, will enjoy light, while a region of equal extent round the opposite pole E will be involved in darkness; and the greater portion of all the parallels between the equator and the arctic circle will be illuminated, while the greater portion of all between the equator and the antarctic circle will be darkened. Again, in the position A, 90° distant from D, because the axis PE invariably retains the same direction, both its poles will be equally distant from the sun, and his light will extend to both; a ray drawn from him to the earth will be perpendicular to its surface at the intersection of the equator with the ecliptic; all the points of the equator will successively, in the course of the diurnal rotation, have the sun in their zenith; and all its parallels being equally divided by the circle which terminates the light, the days in all places of the earth will be equal to the nights. Farther, in the position B, diametrically opposite to D, about the 21st of December, the whole appearances will be contrary to those of the 21st of June; for, since the direction of the earth's axis remains invariable, the pole P, which was formerly inclined to the sun, will be now as much withdrawn from him; his rays will fall as much short

Vicissitudes of the seasons.

short of this pole, as, in the position D, they went beyond it; the places under *tr*, the tropic of Capricorn, will have the sun in their zenith; and to all the countries on the north of the equator it will be winter. Finally, in the position C, opposite to A, where both poles are again equally distant from the sun, the appearances at A will return; a ray from the sun perpendicular to the earth's surface, will be in the planes of both the equator and the ecliptic; and those inhabitants of the earth, whose nearest pole is about to incline towards the sun, will have spring; while it will be autumn to those whose nearest pole is about to be withdrawn from him.

G.

Method of observing oppositions.

As the oppositions of the superior planets, and the conjunctions of the inferior, are the conditions of principal importance in the formation of their theories, it will not be improper to give in this place some account of the procedure of astronomers in observing them. One of D. Cassini's methods, for example, consists in determining the right ascension and declination of the planet, by means of a comparison with some remarkable fixed star, and thence calculating its longitude in the ecliptic. If the difference between this and the real place of the sun be precisely 180°, the time and point, in which the planet was observed, are the time and point of the opposition; but, if it be greater or less, the excess or defect is to be noted; and, as it arises from the motion of the sun or earth *in consequentia*, and the motion of the planet, which, toward the oppositions, is always *in antecedentia*, or retrograde, it will consist of the sum of both motions. Consequently, by comparing it with the sum of their motions in 24 hours, in the parts of their orbits which they then describe, the interval between the time of the observation and the opposition will be discovered, and the point and time of the last determined. It ought however to be observed that, in order to obtain proper accuracy, the precession, aberration, nutation, refraction, and diurnal parallax, are all to be taken into the account.

Example—At the opposition of Saturn in 1773, the zenith distances of this planet, and of the star β in Cancer, were taken at the observatory in Milan, by M. de Cæsaris, with a sextant of six feet radius; while, at the same instants, M. Reggio observed their transits over the meridian, with a transit instrument of five feet radius, whose plane had been accurately placed in the meridian by means of a great number of correspondent altitudes. The position of the fixed star had been previously determined by Mr. Bradley and de la Caille.

	Time by the Clock.
On the 27th of Feb. 1773, the star crossed the meridian at - - -	9h. 6′ 41″
——— the centre of Saturn crossed at	11 50 19.8
Apparent zenith distance of the star -	35° 31′ 2″.3
——— of Saturn's centre -	35 31 40.3
On the 28th of Feb. the transit of the star was made at - - - -	9h. 2′ 54″.7
——— and of Saturn's centre at	11 46 15.6
Apparent zenith distance of the star -	35° 31′ 2″.3
——— of Saturn's centre -	35 29 50.5
Apparent noon on the 27th at - -	11h. 47′ 20″
——— and on the 28th at -	11 47 18.2
Interval according to the clock -	23 59 58.2
This interval in seconds - -	86398″.2
Interval between apparent noon and the transit of Saturn, on the 27th -	43379″.8
Diurnal motion of the sun in long. at the part of his orbit which he then occupied, by calcul.	3610″.8
Diurnal motion of Saturn, concluded from his transits, on the 26th, 27th, and 28th -	285″.2
Sun's longitude on the 27th at noon, calculated for the meridian of Milan, from De la Caille, - -	11s. 9° 15′ 30″
His motion therefore in longitude, during the interval between his transit and that of Saturn on the 27th, was found by this analogy, 86398″.2 : 43379″.8 :: 3610″.8 : 1813″ =	30′ 13″
Whence the sun's longitude at the time of Saturn's transit on the 27th, was -	11s. 9° 45′ 43″
Supposing then the right ascension and declination of β in Cancer to have been accurately determined for 1750, by applying the corrections for precession, aberration, and nutation, the right ascension of this star, at the time of the observations, will be - -	121° 3′ 25″.5
And its north declination, corrected for refraction . - -	9° 51′ 57″.3
But the observed interval, by the clock, between the star's transits, on the 27th and 28th, is - - -	23h. 56′ 13″.4
And between the transits of the star and Saturn, on the 27th, - -	2 43 38.8

Therefore, as 23h. 56′ 13″.4 : 2h. 43′ 38″.8, so is a compleat revolution of the star, or 360°, to 41° 1′ 3″.2, the difference between

between the right ascension of the star and Saturn, at the time of Saturn's transit; so that the right ascension of Saturn, at his transit on the 27th, was 162° 4′ 28″.7

In like manner, his right ascension on the 28th, was - - - 162° 0′ 2″.5

As the differences between the zenith distances of the star and Saturn, at the times of their transits, did not amount to 2′, and the correct declination of the star had been determined, the declination of Saturn may be safely deduced from it, without particularly calculating his refraction.

Saturn's declination therefore, on the 27th, was North - - - -	9° 51′ 19″.8
—— And on the 28th, also North	9 58 9.6

Therefore his apparent longitude, at his transit on the 27th, was -	5 s. 9° 43′ 26″.1
and on the 28th, - -	5 9 38 41.7
and the difference - - -	4′ 44″.4

This is Saturn's change of apparent longitude in the time of his own revolution to the meridian; but, as his motion is retrograde, and the time of his revolution 4′ shorter than that of the apparent solar revolution, in the present circumstances, 0″.8 is to be added to this difference; so that Saturn's variation of longitude at present, and in the time of the present solar apparent revolution, is - - 4′ 45″.2

Since, then, the apparent longitude of the sun, at the time of Saturn's transit on the 27th, was 11s. 9° 45′ 43″, and that of Saturn 5s. 9° 43′ 26″.1; and the difference, instead of amounting to six signs, or 180°, precisely, falls 2′ 16″.9 short; and since, also, as the earth's motion is direct, and that of the planet retrograde, the arch 2′ 16″.9 must continue to increase; it follows that the opposition was now past. Some previous instant therefore must be calculated, in which the sun's longitude was less, and that of Saturn greater; and such an instant that, by correcting the present longitudes of both, in the proper ratio of the motions of each, two longitudes may be found, whose difference shall be 180° precisely. To find this instant there is evidently given the following analogy; as the sum of the motions of the sun and planet, during the time of the sun's present apparent revolution, is to the time of the sun's apparent revolution, so is the sum of the motions of both bodies beyond the point of opposition, to the time elapsed since the opposition, or from the opposition to the observation; that is, 60′ 10″.8 + 4′ 45″.2 : 23h. 59′ 58″.2 :: 2′ 16″.9 : to the time required; or 3896″ : 86398″.2 ::

:: 136″.9 : 3035″.9 = 50′ 35″.9; and this subtracted from 11h. 50′ 19″.8, the time of Saturn's transit on the 27th, gives 10h. 59′ 43″.9 for the time of the opposition; and, since the sun's motion in 50′ 35″.9, according to its present rate, is 2′ 6″.88, and Saturn's motion 10″.02, also at its present rate, the longitude of the sun at this time will be found = 11s. 9° 43′ 36″.12, and that of Saturn 5s. 9° 43′ 36″.12, differing 180° precisely.

The time however, which is now found, is only time reckoned by the clock; and the clock, on the 27th of Febr. at noon, was 12′ 40″ slower than the sun, and 12′ 41″.8 slower than the sun on the 28th at noon. The interval therefore between apparent noon, on the 27th, and the instant of opposition reckoned by the clock, is 10h. 59′ 43″.9 + 12′ 40″ = 11h. 12′ 23″.9; and, since the clock fell back 1′.8 in the time of the whole revolution of the sun to the meridian, it must, in this interval, have fallen back 0″.8. The apparent time, therefore, of the opposition, reckoned on the meridian of Milan, was 11h. 12′ 24″.7 of the 27th of February 1773. It ought however to be noticed, that the place in which the planet appears, is not that which it really occupies; for, in the present example, the very small equations of aberration and nutation were not applied to the planet; and, if they had, its place would have been found a very small degree more advanced.

It is obvious that the same method is equally applicable to conjunctions.

By means of the comparison of very distant oppositions, or conjunctions, astronomers have principally determined the mean motions of the planets in longitude, or the times of their tropical revolutions. From these, again, the times of their sidereal revolutions are deduced; for, if the arch of precession be given, and called p, and the sidereal revolution be denoted by 360°, the arch described by the planet in the time t of its tropical revolution will be = 360° — p; and Mean motions.

$$360 - p : t :: p : \frac{pt}{360 - p}$$; and this portion of time added

to t will give the time of a sidereal revolution. The procedure is the same with respect to anomalistical and synodical revolutions; and the mean diurnal, annual, and secular motions of a planet likewise follow from the time of the tropical revolution. For this time is to 360° as the Julian year, or 365d. 6h. to the mean annual motion; and this again, divided by 365d. 6h. gives the diurnal motion; and multiplied by 100, the secular; that is, the arch which may be found to remain after the compleat revolutions are rejected.

The

The following table of the most important of these motions is given by M. de la Lande. The secular motions of Saturn, Jupiter, and the Moon, are those only which take place in the present century; for the revolutions of the two last are found to be performed in less time, and of Saturn in more, than they required in former ages; and both the secular and diurnal motions respect the equinoctial points.

	Tropical Revolutions.				Sidereal Revolutions.			
	d.	h.	′	″	d.	h.	′	″
Mercury	87	23	14	25.9	87	23	15	37.0
Venus -	224	16	41	30.4	224	16	49	12.7
Sun - -	365	5	48	45.5	365	6	9	11.2
Mars -	686	22	18	27.3	686	23	30	43.3
Jupiter -	4330	8	58	27.3	4332	8	51	25.6
Saturn -	10749	7	21	50	10761	14	36	42.5
Moon -	27	7	43	4.6	27	7	43	11

	Secular Motions.				Diurnal Motions.		
	s.	°	′	″	°	′	″
Mercury -	2	14	12	10	4	5	32.570376
Venus -	6	19	12	12	1	30	7.806488
Sun -	0	0	46	10	0	59	8.330458
Mars -	2	1	42	10	0	31	26.656536
Jupiter -	5	6	27	30	0	4	59.281314
Saturn - -	4	23	14	30	0	2	0.565914
Moon -	10	7	53	35	13	10	35.02847

The tropical revolution of the newly discovered planet employs 83y. 52d. 4h. the sidereal 83y. 150d. 18h. The secular motion is 2s. 13° 16′ 55″, and the mean diurnal motion 0′ 42″.678026.

H.

In order to demonstrate that the great error in all the former theories, whether excentric or concentric, was, their being founded on observations, not of real, but of mean oppositions, it was necessary for him previously to shew, that, as far as the first inequalities were concerned, they were perfectly, or very nearly, equivalent in their effects. In those cases, however, where the first inequalities were believed to be purely optical, as in the orbits of the sun and moon, their coincidence has been already pointed out (31), and the repetition of Kepler's reasoning becomes unnecessary. But, even in the planetary orbits, where the first inequalities were thought to be partly real, the difference in the effect of the theories is very inconsiderable. For, if, instead of bisecting, like

like Ptolemy, the excentricity of Mars, we should introduce into the excentric theory the correction made by T. Brahé, and make DE (fig. 10) = 7560, and DT = 12600, and thence calculate the equation EBT for 45° of anomaly; we shall find $EBD = \frac{\text{sin. } AEB \cdot DE}{DB} = 3° 4' 52''$, and from the analogy DB + DT : DB — DT :: tan.$\frac{1}{2}$ADB : tan.$\frac{1}{2}$(DTB — DBT) that DBT is = 4° 24' 4"; and therefore the whole equation EBT = 7° 28' 56". Or, if in the concentric theory, (fig. 36) where EQ is = 16380, and QR = 3780, and where, since BAE = 45°, the angle EQR is a right angle, we calculate the equation RAE; we shall find by drawing QS, RT, perpendicular to AC, that ES, the sine of 45°, will in the same parts be = 70711, RT = EQ + ES = 87091, and AT = AS — QR = 66931; so that BAR = $TRA = \frac{R \cdot TA}{RT} = 37° 32' 37''$, and the equation RAE = BAE — BAR = 7° 27' 23"; that is, only 1' 23" less than the former. Equivalency of theories.

When the mean anomaly is 90°, the equation by the former theory will be 11° 25' 58", and by the latter 11° 23' 53"; still differing only 1' 55".

Every proof therefore of inaccuracy of principle, is applicable to both theories; and their difference of form has no effect to remove or diminish the errors.

I.

If the first inequalities of the planets could be considered as purely optical, and the centre of uniform motion as coincident with the centre of the orbit, as in the simple solar hypothesis, it would be of no consequence whether their theory were founded on mean, or apparent oppositions. For if, in the concentric theory, we describe on the centre A (fig. 50), the arch BD of the orbit, and on B, with the distance BC, determined by Copernicus, describe an epicycle, and on D, with the same distance, another epicycle; and if the places of the planets in the epicycles be C and F, such that DF may be always parallel to ABC, here supposed to be the line of apsides; this scheme will represent the theory, founded on mean oppositions: and the planet's path through the heavens will be the circular arch CF, similar and equal to BD, and described upon the centre E, found in AB, by taking AE = BC. The theory, again, founded on apparent oppositions, will be represented by removing the line of apsides

sides from the position ABC to the position AHK, and describing on the points B, H, D, of the orbit a different epicycle, in which, however, the planet so continues to move, that BL, DM, are always parallel to AHK. But though the path of the planet through the heavens is thus removed from CF to LM, it is still a circular arch LKM, similar and equal to CF or BD: and its centre N is found in AH, by taking AN = HK.

In the excentric form of Ptolemy, on the centre *a*, with the distance *ae* = AB, describe the orbit *edf*, produce the line of apsides *ea* to *b*, so as to make *ab* = BC or AE, and from *a* draw *af*, making the angle *eaf* = BAD, and join *fb*. This will represent the theory formed on mean oppositions in the excentric form; and in all respects produce the same effects with the concentric: for *ebf* = CAF, *eb* = CA, and *fb* = FA. The new theory again of Kepler will be represented in this form by making *ead* = BAH, drawing through *d* a new line of apsides *da*, producing it till *ac* become = HK or AN, and joining *ce*, *cf*. In this form the substitution makes no change whatever in the planet's path.

Effects to be expected from Kepler's proposed theory.

In both forms the effect of the substitution is the same. The planet will not in either be represented in the same points of the zodiac, for the path in the concentric form is displaced, and in the excentric form it appears to be displaced: the equation *afc* = MAD, will be different from *afb* = FAD; and in *d* and K no equation will take place, in the same manner as in the ancient theory no equation took place at *e* and C. But though the appearances at *b* and *c* are different, the calculations for the appearances to the eye at *b* may be made with as much accuracy as to the eye at *c*; by means of lines of apsides properly drawn through both, and just excentricities investigated. Indeed it is of no consequence what position may be given to the line of apsides, excepting only that it pass through the points of the greatest and least distance from the eye: for, as the variations of velocity are supposed to be purely optical, *e* will be the point of the greatest retardation to the spectator in *b*, and *d* the point of greatest retardation to the spectator in *c*.

But in the planetary orbits, where the first inequalities are partly real, the translation of the eye from *c* to *b*, or of the planet's path from LM to CF, is not a matter of such indifference; and the different appearances at these two points cannot be represented to both with equal accuracy. For the position of the aphelion does not in this case admit of being varied according to the position of the observer, but is determined to the particular point D (fig. 51), of greatest distance from the sun in A, as being that in which alone the

greatest

greatest physical retardations can happen: and the line of apsides must pass from the sun through the centres of both the orbit and the equant to D, because otherwise the greatest optical and physical retardations could not be accumulated in their proper place. If the eye, for example, were translated from A, the sun, to G, which Copernicus supposed to be the centre of the ecliptic, a line GBF, drawn through B, the centre of the orbit, would mark the point of the greatest optical retardation in F, while the point of the greatest physical retardation would continue as before in D: and consequently neither F nor D would be the points of greatest apparent retardation, but some intermediate point E, found by drawing from G the line GC through the centre of the equant; and in this line it would be necessary to place the centre of the orbit, to avoid the absurdity of having the optical and physical retardations in separate positions.

The theory then for the planets, which is formed on mean oppositions, is represented by the orbit MNO, described upon the centre L; where GLCN is the line of apsides passing neither through the point O of greatest optical, nor M of greatest physical retardation, but through a point N, intermediate to both: and when Kepler proposed to substitute for this a theory formed on apparent oppositions, his proposal was to substitute for MNO another orbit DEF, described on the centre B; where ABCD is the line of apsides, and the greatest optical retardation of consequence united with the greatest physical retardation in the same point D. Yet he shews that by this substitution the ancient equations for the point G and excentricity GC, would not be near so much deranged or varied as T. Brahé had supposed; and to produce full conviction on this subject, he employs T. Brahé's own elements in the particular orbit of the planet Mars.

According to this astronomer, the apogee E of Mars was in 4s. 23° 32′ 16″, and the angle AGC, which is the difference between this longitude and that of the solar apogee, = 47° 59′ 15″, GC was = 19763 in parts of BD = 100000, and AG the solar excentricity, in the same parts = 2346. Consequently by resolving the triangle AGC, the apogee D of the new orbit was found to be in 4s. 29° 0′ 3″, and the excentricity AC = 18034; and dividing this in B in the ratio of GL = 12352 to LC = 7411, AB was found = 11271, and BC = 6763: and B was the centre of the new orbit DEF. The point of this orbit where the greatest differences upon the appearances from G take place, will be found by drawing through C, the centre of the equant, the line CVY at right angles to CG, and cutting both orbits in VY. For if we join LV and BY, and draw to CV the line BZ, parallel

 to

to GC, we shall have in the right-angled triangle LCV, with LV and LC, the line CV = 99725, and in the right-angled triangle BZC, with ZBC = ACG, and BC, the line BZ = 6732, and CZ = 644. Consequently, in the triangle BZY, also right-angled, with BY and BZ, we shall have ZY = 99733; and therefore CY = 100417; and VY = CY — CV = 692.

On the first inequalities,

The line VY measures the quantity by which the planet at the point Y would be displaced by the substitution of the new orbit DEF, and at the instant of time marked by CVY; and VGY will be the difference produced by the substitution on the former equation of first inequality seen from G. To find this difference, we have in the right-angled triangle GCV, from GC and CV given, the angle CGV = 78° 47′ 30″, and in the right-angled triangle GCY, from GC and CY, the angle CGY = 78° 51′ 54″; and therefore VGY = 4′ 24″. This is the greatest difference on the old equations of first inequality that the new theory could produce; for, though VY is not the greatest distance of the orbits, that distance is least obliquely viewed from the point G.

and on the second.

But the effects which the proposed theory would produce on the ancient equations of second inequality, were by no means thus inconsiderable. Through the centres TL, of both orbits (fig. 49), draw the line TLS, and from E, the centre of the equant, draw towards the parts nearest to the earth's orbit, the line ERS, cutting the new orbit in R, and the old in S; and marking the same instant of time on both. When the earth is in the point a, where this line cuts the terrestrial orbit, the planet will be seen in the same point of the zodiac, according to either theory. But, when the earth is on either side of ERS, the intercepted part RS, which is very nearly equal to TL, will subtend at the eye a greater or a less angle, according to the circumstances in which it is viewed; and this angle will evidently be greatest at the point I, where a circle passing through R, S, touches the earth's orbit.

To find therefore the point of contact I, it is necessary to determine RS, both in magnitude and position, and to draw from A the line AX perpendicular to ES. In the triangle ELS, with EL = 7411, and ALS = 47° 59′ 16″, the side ES will be found = 105123, and the angle LES = 44° 59′ 10″. But LET was found = 5° 27′ 47″, and therefore TER = 50° 26′ 57″; and with this angle, and ET = 6763, ER will be found, in the triangle ETR = 104170; and therefore RS = ES — ER = 953, and ESL = 3° 0′ 6″. Again, in the right-angled triangle AXE, with AEX = 44° 59′ 10″, and AE = 19763, AX will be found = 13971, and EX = 13978.

13978. The semi-diameter AI also of the earth's orbit will be found = 656565, in parts of TR, or LS = 100000, from AB = 2346.

Since the circle which passes through R, S, must touch the earth's orbit in I, its centre will be in the semi-diameter AI produced, and at the point Y of its intersection with ZY, a perpendicular to RS bisecting it in Z. Join RY, and from A draw $A\varphi$ parallel to ER, and meeting ZY in φ. Then $Z\varphi = AX$, and $A\varphi = ZX$; and, since $SX = (ES - EX) = 91145$, and $RZ = \frac{1}{2}RS = 476.5$, we shall have $ZX = A\varphi = 90668$. Make $ZY = x$, and $AY = y$. Then

$y^2 = AI^2 + 2AI.IY + IY^2$. But

$IY = RY = \sqrt{RZ^2 + x^2}$ Therefore

$y^2 = AI^2 + 2AI.\sqrt{RZ^2 + x^2} + RZ^2 + x^2$. Again

$y^2 = A\varphi^2 + Y\varphi^2$, and $Y\varphi = Z\varphi \backsim x$. Therefore

$y^2 = A\varphi^2 + x^2 - 2xZ\varphi + Z\varphi^2$. Therefore

$$AI^2 + RZ^2 + x^2 + 2AI.\sqrt{RZ^2 + x^2} = A\varphi^2 + x^2 - 2xZ\varphi + Z\varphi^2. \text{ Or}$$

$$AI^2 + RZ^2 + 2AI.\sqrt{RZ^2 + x^2} = A\varphi^2 - 2xZ\varphi + Z\varphi^2.$$

By resolving this equation Kepler finds $x = ZY = 25772$. His solution is subjoined, as an example of his patience in calculation. By the substitution of known values, the first equation, $y^2 = AI^2 + RZ^2 + x^2 + 2AI.\sqrt{RZ^2 + x^2}$ becomes

$y^2 = 4.310.974.527 + 2\,(97.876.383.536.363. + 4.310.747.475.\,x^2)^{\frac{1}{2}} + x^2$; and, by a like substitution, the second equation $y^2 = A\varphi^2 + x^2 - 2xZ\varphi + Z\varphi^2$ becomes

$y^2 = 8.415.875.065 - 27942x + x^2$. Therefore

$$4.310.974.527 + 2(97.876.383\ 536.363 + 4.310.747.475.x^2)^{\frac{1}{2}} = 8.415.875.065 - 27942\,x. \text{ Therefore}$$

$$2(97.876.383.536.363 + 4.310.747.475\,x^2)^{\frac{1}{2}} = 4.104.900.538 - 27942\,x. \text{ Or}$$

$$(97.876.383.536.363 + 4.310.747.475\,x^2)^{\frac{1}{2}} = 2.052.450.269 - 13971\,x.$$

And, by involving both sides of the equation

$$4.310.747.475.x^2 + 97.896.383.536.363 = 19.503.841.x^2 - 57.349.565\,416.398.x + 421.252.106.718.172.361.$$

$$\text{Whence } 4.115.558.634.x^2 + 57.349.565.416.398.x = 421.156.228.334.645.988.$$

$$\text{Whence again } x^2 + 13934.x = 1.023.328.690; \text{ and } x = 25772 = ZY.$$

Having thus found the semi-diameter ZY of the required circle, the positions of I and S, and the angle RIS, will be easily determined. For Zφ ∽ ZY = φY = 11801; and, since Aφ = ZX = 90665.5, and AφY a right-angle, the angle φAY will be found = 7° 30′ 10″. Since also ESL = 3° 0′ 6″, and SL, or ABM, the line of the solar apsides, is 3 s. 5° 30′ of longitude, SE or φA will be in 3 s. 8° 30′ 6″ of longitude; and, adding φAY = 7° 30′ 10″, YA will be in 3s. 16° 0′ 16″; that is, I the position of the earth, when the greatest difference of second inequality takes place, is in 9 s. 16° 0′ 16″. The difference itself RIS, is easily found, with the sides RZ, ZY, of the right-angled triangle RZY; for from these RYZ = RIS, comes out = 1° 3′ 32″. The mean longitude also of the planet in R is easily found: for AE, according to T. Brahé, is in 4s. 23° 32′ 16″; and since SEL = 44° 59′ 10″, its supplement SEF is = 135° 0′ 50″ = 4 s. 15° 0′ 50″; and therefore ER in 9 s. 8° 33′ 6″. The true longitude, after applying the equation of first inequality, would be 8 s. 27°.

The solution of this problem is perfectly the same with that given long afterwards by Sir Isaac Newton, in his *Arithmetica Universalis*, Problem 45.

Yet these laborious investigations are not the only evidences of the scrupulous exactness, with which Kepler examined every part of the ancient theory, before he adventured to introduce his own. For, out of deference to T. Brahé, he minutely traced all the possible effects of the proposed substitution, in every one of the three systems; and the trouble given him by the complicated form of T. Brahé's system and theory was very great. But Kepler had given to T. Brahé, in his dying moments, a promise to this purpose, and not only the present investigations, but the two following parts of his commentary, are proofs how sacredly he fulfilled it. The present investigations also are not only given in the excentric forms, but even in the concentric.

I.

To communicate just notions of Kepler's scrupulous attention and candour, it would be necessary to transcribe his whole book. In the present examination these three examples must suffice.

On the 13th of April 1589, at 12h. 5′, the longitude of Mars, deduced by T. Brahé with great attention from an observation, was 7s. 3° 58′ 31″, and, after a correction for parallax, stated at 7s. 3° 57′ 11″. The time was 1h. 30′ before the opposition; and, as the diurnal motion found by the observations

servations of some successive days was 22′, the longitude in opposition ought to have been 7s. 3° 55′ 49″. The table makes it 7s. 3° 58′ 10″.

On the 10th of November 1597, at 8h. 30′, the observed longitude of Mars was doubly marked, first in 3s. 3° 30′, and next in 3s. 4° 1′. A mean of both is 3s. 3° 45′ 30″. The time assigned for the opposition is 3d. 5h. 5′ later, in which interval the motion *in antecedentia*, taken from Magini's tables, is 1° 15′. The longitude in opposition ought therefore to have been 3s. 2° 30′ 30″. In the table it is 3s. 2° 28′. The cause of the uncertainty, marked with respect to this observation, is evident from its date. T. Brahé had been obliged to withdraw from Huenna; and, though his other instruments were left with his assistants, had found means to carry off with him (his radius) one of the most valuable. The loss to astronomy, by his removal at this time, was great. The opposition of 1597 was of the utmost consequence for detecting the parallax of Mars, and very few so favourable occur in the course of a whole life-time.

Examination of the mean oppositions on which T Brahe formed hi theory.

On the 13th of January, at Prague, the right ascension of Mars, determined by distances, at 11h. 50′, was

	° ′ ″
1st. from the bright star in the first of Gemini	134 23 39
2d. from Cor leonis, or Regulus - -	134 27 37
3d. from Pollux - - - -	134 23 18
4th. from the third star in the wing of Virgo	134 29 48
Consequently, a mean of all the four determinations is - - - - -	134 24 33

Whence the longitude of Mars, at 11h. 40′ of mean time, reduced to the meridian of Uraniburg, was 4s. 10° 38′ 46″. On the 24th of January, at the same hour, his longitude is stated at 4s. 6° 18′ 0″. His diurnal motion therefore was 23′ 44″, and consequently his longitude in opposition, on the 19th of January O. S. is rightly marked 4s. 8° 18′ 45″. From these differences of the right ascensions it appears, that considerable uncertainties may arise from the observations themselves, as well as from their application, unless they have been made with great attention, and the observer favoured with all possible advantages. These, in the present case, had been deficient; for, though many of T. Brahé's instruments had by this time arrived in Bohemia, several of the largest were wanting; and such as had arrived were not set up with sufficient accuracy, nor freed from the derangements which they had suffered in the transportation. But indeed it was found, that right ascensions, deduced from distances, were in general liable to some uncertainty; and Longomontanus admitted,

 that

that differences of 3′, among such right ascensions, had not been unfrequent even at Huen.

K.

T. Brahé's observations for discovesing the parallax were chiefly these. On the 26th of December 1582, at 8h. 28′ 30″, the planet was found, by distances from the 2d and 7th of Gemini and from Oculus Tauri, to be in 3s. 17° 38′ 10″ of longitude; and on the same day, by distances from Cor leonis, at 19h. 15′, to be in 3s. 17° 28′ 30″; so that the motion, in 10h. 46′ 30″, was 9′ 40″ retrograde. The altitude in the first observation was 40° 50′, where refraction was supposed to have no effect, but in the second it was only 14° 4′.

But by other observations, of the 27th, 29th, and 30th, the motion in longitude was found to be 25′ in 24h. 21′; and therefore, in 10h. 46′ 30″, it ought to have been 11′ 30″ instead of 9′ 40″.

The effect of parallax, if sensible, was to place the planet at 8h. 28′ 30″ more to the east than it ought to be, and at 19h. 15′ more to the west; and consequently the apparent retrograde motion ought to have been greater than 11′ 30″; whereas it is 1′ 50″ less. But the effect of refraction is contrary to that of parallax; and T. Brahé, in his tables, reckons at 13° of altitude the refraction of a fixed star 4′, and that of the sun 8′. Even the last of these refractions will occasion but a very small variation of the planet's longitude; because the whole sign of Cancer, at setting, was very oblique to the horizon of Huenna, and the refraction would affect the latitude chiefly. The refraction therefore in longitude cannot be stated at more than 3′; and this added to 9′ 40″ gives 12′ 40″ for the apparent retrograde motion in 10h. 46′ 30″, when cleared of refraction. Its excess above 11′ 30″ is 1′ 10″, the sum of the parallaxes in longitude given by the present observations.

Attempts to discover the parallax of Mars.

About the 16th and 17th of January 1583, the retrograde motion of Mars was precisely in a line passing from Cor leonis to the bright star in the foot of Ericthonius; and, by taking the distances from these stars at 10h 30′ on the 16th, and 10h. 36′ on the 17th, the motion in 24h. 6′ was found to be 14′ 30″. Therefore, in the interval of 9h. 30′, between 7h. 30′, and 17h. on the 16th, it ought to have been 5′ 37″.5. But by actual observation it was found to be 6′. No more therefore than 22″.5 is left for the sum of the parallaxes in longitude; and to this hardly any addition could be made upon account of refraction, because, though the altitude in the last observation was only 15°, the signs of Cancer and Leo were

were very oblique to the horizon, and the altitudes of Mars and Cor leonis much the same.

On the 17th, at 5h. 20′ and at 15h., the variation of distance from the same fixed stars was 7′ in the included interval of 9h. 40′; and in the interval of 9h. 18′ between 7h. 34′ and 16h. 52′ it was 8′. But the real motion, which was 14′ 30″ in 24h. 6′, ought to have been 5′ 53″.4 in the first interval, and 5′ 40″.8 in the second. The first sum therefore of the parallaxes was 1′ 6″.6, and of the second 2′ 19″.2.

Lastly, On the 18th, the apparent motion in longitude was 5′ 30″ in the interval of 7h. 53′, between 8h. 52′ and 16h. 45′. But, from observations of the 18th and 19th, the real motion appeared to be 10′ 30″ in 22h. 11′; and therefore it was no more than 3′ 43″.8 in the interval 5h. 30′; so that all which is left for the sum of the parallaxes is only 1′ 46″.2.

Finding the results from these observations, when examined in a direct manner, so inconsiderable, and so far below the common opinion, Kepler proceeded to examine them by a converse method; and, since the solar parallax was then estimated at 3′, he supposed the horizontal parallax of Mars in his present position to be 4′, and proceeded to investigate what sum this would give for the parallax in longitude in the former interval of 9h. 40′ on the 17th of January. The sun's longitude at 5h. 20′ was 10s. 7° 22′, and 10s. 7° 31′ at 15h. and therefore with the obliquity of the ecliptic given,

	° ′	° ′
His right ascensions were found to be	309 47 &	309 56
To which, adding the apparent time converted into degrees - -	79 0	225 0
The right ascension of the meridian was	28 45	174 56
Longitude of the culminating point of the ecliptic - - -	30 56	174 29
Its declination - - -	11 50	2 12
Its altitude - - -	45 55	36 18
Angle of the ecliptic with the meridian	69 29	66 31
Altitude of the Nonagesimal - -	49 12	42 20
Distance between the culminating and orient points of the ecliptic -	108 45	61 31
Distance between the culminating point and the Nonagesimal - -	18 45	28 29
Longitude of the Nonagesimal - -	49 42	202 53
Longitude of Mars - - -	90 0	90 10
Distance of Mars from the Nonagesimal	40 19	112 43
Correspondent parallaxes of longitude	1 57.5	2 37.8

 The

The sum of these parallaxes is 4′ 35″.3. As therefore the real motion at this time was 6′, it must have been increased to 10′ 35″.3 by the sum of the parallaxes; and as it was increased no farther than to 7′, it is impossible that the horizontal parallax should be so great as 4′.

Other observations made by T. Brahé for the same purpose, in the years 1585, 1587, &c. were examined with equal care, and the results were found to be equally unsatisfactory. Some gave a parallax so inconsiderable as was thought far below the truth, some gave none at all, and other results were directly repugnant to the possibility of its existence.

Kepler next details his own obs rvations on this subject, and nothing can exceed the candour and fairness with which they are related, or the ardour and patience with which they were made. In his investigation he employed the method of latitudes; and, as the longitudes of Mars and Arcturus were at that time nearly the same, his observations were chiefly of distances from this fixed star.

On the 17th and 19th of February 1604, his observations chiefly served to prove that Mars was stationary, and that his real distance from Arcturus was 29° 18′. But on the 22d, when Os leonis was in the meridian, it was 29° 15′, and when Cor Scorpii was in the meridian, it was 29° 19′; so that a variation of about 4′ 15″ on the latitude had taken place in the interval; which, as Mars was still stationary, argued an horizontal parallax of no less than 9′. But, when Os leonis was in the meridian, the altitude of Mars was only 12° 30′; and, if we apply to the planet T. Brahé's refractions of the fixed stars, the variation of latitude must be diminished 2′ 18″, so that it will become 1′ 57″, and give an horizontal parallax of 4′ 11″. But if we apply his solar refractions, the variation of latitude must be diminished 4′ 36″, that is, the result would be contradictory to every degree of parallax.

The observations of the 26th were also unfavourable to parallax, and the variation of distance from Arcturus amounted at the utmost to no more than 1′.

L.

The observations of the planet in the node, or near to it, were these. 1. On the 4th of March 1590, at 7h. 10′, he found from the right ascension of Mars = 22° 35′ 10″, and his declination = 9° 26′ north, that the longitude was 0s 24° 22′ 56″, and the latitude 3′ 12″ south. As the effects of parallax and refraction seemed here to be equal, they were neglected. 2. On the 23d of January 1592, at 10h. 15′, Mars was found in 0s. 11° 34′ 30″ of longitude, with 2′ of south latitude;

latitude; and here also the refraction was neglected, the altitude being 25° in the meridian. But as the elongation from the sun was 60°, and therefore the horizontal parallax the same with that of the sun, which was reckoned at 3′, a change was thereby supposed to be produced upon the latitude of nearly 2′; for the circle of latitude nearly coincided with the meridian, and the planet was concluded to be actually in the node. 3. On the 10th of December 1593, Mars was also in the node; for, after corrections for parallax and refraction, the latitude was no greater than 45″ north. 4. On the 27th of October 1595, at 12h. the latitude of Mars, cleared of parallax and refraction, was 2′ 20″ south; and on the 28th, after like corrections, it was 25″ north; and Kepler concluded that the planet had been in the node at 0h. on the 28th.

Determination of the longitude of the nodes.

These observations were mutually confirmed by reckoning 687 days, the time of a tropical revolution, backwards from 0h. on the 28th of October 1595; by which means we are brought to the 10th of December 1593, the date of the third observation, and thence another period of 687 days will bring us to the 23d of January 1592, when the planet was actually in the node; and a third period of 687 days will terminate at noon, on the 7th of March 1590, when the arch of 3′ 12′ south latitude is found from the diurnal motion in latitude to have actually been described.

In like manner, on the 3d of June 1595, at 19h. 10′, when the altitude of Mars was 8°, his longitude was 7s. 13° 36′ 40′, and latitude 3′ 16″ north. The parallax was insensible, because the distance was greater than that of the sun; but the refraction was great, and chiefly affected the latitude, so that instead of 3′ 16″ north, it may be reckoned at 2′ 3″, or even more minutes south. Also, on the 15th of April 1589, the observed latitude, at the time of the least distance of the planet from the earth, was 1° 7′ 0″ north, and after 21 days it decreased to 0° 6′ 40′ north. The rate of decrease was not indeed uniform, but became gradually slower; the error however will be insensible, though we should calculate the time when the planet was in the ecliptic by the common analogy 60′ 20″ : 21d. :: 6′ 40′ : 2′ 8″. The planet therefore was in the node on the 9th of May, or near the end of the 8th. Hence, reckoning three times 687 days backward, we are brought to the 29th of December 1594, towards the end of that day, when the planet ought to be again found in the node. Accordingly, on the 3d of January 1595, that is, five days after, the observed latitude at 19h. 10′ was found to be 3′ 16″ south. Nor is any error committed by now giving five days to the motion of 3′ 16″ in latitude, and in 1589 giving only 2d. 8h. to the motion of 6′ 40″ in latitude, for in 1594 the

the planet was near the conjunction, and in 1589 near the opposition.

M.

Apparent oppositions.

The attention of Kepler in calculating the times and points of his apparent oppositions, was the same which we have seen employed in his examination of T. Brahé's mean oppositions. To prevent the possibility of suspecting that he had suited the observations to his theory, he gives a particular account of his treatment of every one, noticing their minutest circumstances, and especially pointing out those which might introduce the least ambiguity or uncertainty; whether they arose from the imperfection of the observation, or from the inaccurate doctrines taught concerning parallax and refraction. In treating his own observations, he even describes the whole procedure of making them; and his attention and fairness, in the conclusions deduced from all, certainly was never exceeded.

N.

It is impossible to convey a just notion of the difficulties which attended the formation of Kepler's theory, or of his indefatigable patience in it, except by a repetition of his procedure. This perhaps will be acceptable as an object of curiosity, and an example of the management of calculations, before the discovery of logarithms.

The oppositions he employed were those of the years 1587, 1591, 1593, and 1595; in which the observed longitudes were 5s. 25° 43′ 0″; 8s. 26° 43′ 0″; 11s. 12° 16′ 0″; and 1s. 17° 31′ 40″; and the mean longitude, calculated from T. Brahé's tables, 6s. 0° 47′ 40″; 9s. 5° 43′ 55″; 11s. 9° 55′ 4″; and 1s. 7° 14′ 9″. To clear these of every degree of foreign motion, they were reduced to the year 1587, by corrections taken from the same tables for the precession; and these corrections were, for the first interval from 1587 to 1591, 3′ 37″; for the second, from 1587 to 1593, 5′ 30″: and 7′ 18″ for the third interval from 1587 to 1595. They therefore became

		s.	°	′	″		s.	°	′	″
App. long.	AF =	5	25	43	0;	AE =	8	26	39	43;
Mean long.	CF =	6	0	47	40;	CE =	9	5	40	18;

		s.	°	′	″		s.	°	′	″
App. long.	AD =	11	12	10	30;	AG =	1	17	24	22
Mean long.	CD =	11	9	49	34;	CG =	1	7	6	51.

In

In the first assumption for the angles FAH, EAH, &c. of true anomaly, he supposed that, in 1587, the longitude of the aphelion H was 4s. 28° 44′ 0″; and for the angles FCH, ECH, &c. of mean anomaly, that, besides this assumption for the longitude of the aphelion, an addition of 3′ 16″ was to be made to T. Brahé's mean longitudes. But (for some reason which I have not discovered) though he uses this correction in forming the angle FCH, and also in computing the angles AFC, AEC, &c. of equation, he does not use it in forming the other angles of anomaly ECH, KCD, GCK. The mean anomalies thus formed are these; for, since

	s. ° ′ ″		s. ° ′ ″		s. ° ′ ″		s. ° ′ ″
CH is in -	4 28 44 0	CH	4 28 44 0	CH	4 28 44 0	CH	4 28 44 0, and
CF in -	6 0 50 56	CE	9 5 40 18	CD	11 9 49 34	CG	1 7 6 51,
HCF =	32 6 56	HCE =	126 56 18		6 11 5 34		8 8 22 51
		KCE =	53 3 42	KCD =	11 5 34	KCG =	68 22 51

The angles of equation were thus computed. Since

	s. ° ′ ″		s. ° ′ ″		s. ° ′ ″		s. ° ′ ″
CF is in -	6 0 50 56	CE	9 5 43 34	CD	11 9 52 50	CG	1 7 10 17, and
AF in -	5 25 43 0	AE	8 26 39 23	AD	11 12 10 30	AG	1 17 24 22
AFC =	5 7 56	AEC =	9 4 11	ADC =	2 17 40	AGC =	10 14 15.

By assuming AC = 10000, the lines AF, AE, AD, AG, will be found in like parts, by resolving the triangles of which AC is the common base. For AF $= \frac{\text{sin. FCH}}{\text{sin. AFC}} = \frac{53163}{8945} = 59433$; and AE $= \frac{\text{sin. KCE}}{\text{sin. AEC}} = 50703$; and in like manner AD = 48052; and AG = 52302.

To discover if the points D, E, F, G, be situated in the circumference of the same circle, we have to find the sum of the angles at F and D; by resolving the triangles whose bases are FE, ED, DG, GF, and the point A their common vertex. The angles at A, and therefore their half supplements are given. For, since

	s. ° ′ ″		s. ° ′ ″		s. ° ′ ″		s. ° ′ ″
AF is in -	5 25 43 0	AE	8 26 39 23	AD	11 12 10 30	AG	1 17 24 22
AE in .	8 26 39 23	AD	11 12 10 30	AG	1 17 24 22	AF	5 25 43 0
FAE	= 90 56 23	EAD	= 75 31 7	DAG	= 65 13 52	GAF	= 128 18 38; and
the ½ suppl.	= 44 31 48		52 14 27		57 23 4		25 50 41; and
the tangents -	98373		129093		156271		48438.

Since, then, we have in the triangle AFE, and the rest, the analogies AF + AE : AF — AE :: tan. ½ suppl. FAE : tan. ½ (AEF — AFE) &c. the calculation for the angles at F and D will be

AF + AE = 110136; AE + AD = 98775; AD + AG = 100354; AG + AF = 111735;
AF — AE = 8730; AE — AD = 2651; AG — AD = 4250; AF — AG = 7131;

$\frac{AF - AE}{AF + AE} = 7926$; $\frac{AE - AD}{AE + AD} = 2684$; $\frac{AG - AD}{AD + AG} = 4235$; $\frac{AF - AG}{AG + AF} = 6382$;

$$\tan.\ \tfrac{1}{2}(AEF - AFE) = \tan.\ \tfrac{1}{2}\ \text{suppl.}\ FAE\left(\frac{AF - AE}{AF + AE}\right) = 98373.7926 = 7797;$$

$$\tan.\ \tfrac{1}{2}(ADE - AED) = \tan.\ \tfrac{1}{2}\ \text{suppl.}\ EAD\left(\frac{AE - AD}{AE + AD}\right) = 129093.2684 = 3462;$$

$$\tan.\ \tfrac{1}{2}(ADG - AGD) = \tan.\ \tfrac{1}{2}\ \text{suppl.}\ DAG\left(\frac{AG - AD}{AD + AG}\right) = 156271.4235 = 6618;$$

$$\tan.\ \tfrac{1}{2}(AGF - AFG) = \tan.\ \tfrac{1}{2}\ \text{suppl.}\ GAF\left(\frac{AF - AG}{AG + AF}\right) = 48438\ 6382 = 3091.$$ Whence

$\tfrac{1}{2}(AEF - AFE) = 4° 27' 30''$ - and $\tfrac{1}{2}(ADE - AED) = 1° 59' 4''$
$\tfrac{1}{2}(AGF - AFG) = 1\ 47\ 59$ - $\tfrac{1}{2}(ADG - AGD) = 3\ 47\ 10.$ So that the

sum of both differences is = 6 15 29 - sum of both differences = 5 46 14.

But, when the points D, E, F, G, are in a circle, the first sum, which is the defect of the angles at F below half the sum of the angles at the bases FE, FG, ought to be equal to the last sum, which is the excess of the angles at D above half the sum of the angles at the bases DE, DG. Since then, in the present case, the defect at F is greater by 29′ 15″ than the excess at D, the points D, E, F, G, are not in a circle; and some of the things assumed are false.

Kepler informs us, that it required various repetitions of the same labour before he discovered that the excess at D would come out equal to the defect at F, by an addition of 3′ 20″ to the longitude of the aphelion; which of consequence will be 4s. 28° 47′ 20″; so that the mean anomalies will now become HCF = 32° 3′ 36″, KCE = 53° 7′ 2″, KCD = 11° 2′ 14″, and KCG = 68° 19′ 31″; but the angles of equation, and the angles at A continue unaltered. The tangents therefore of the half supplements of the angles at A continue, but the lines proceeding from A will be altered; for $AF = \frac{\text{sin. HCF}}{\text{sin. AFC}}$, will now be $= \frac{53081}{8935} = 59341$, and in the same manner AE = 50740, AD = 47815, AG = 52281. Consequently, as was intended, the half differences of the angles at the bases will be altered; for now

$$\text{tan.}\ \tfrac{1}{2}\,(AEF - AFE) = \text{tan.}\ \tfrac{1}{2}\ \text{suppl.}\ FAE\left(\frac{AF - AE}{AF + AE}\right) = 98373.7813 = 7686;$$

$$\text{tan.}\ \tfrac{1}{2}\,(ADE - AED) = \text{tan.}\ \tfrac{1}{2}\ \text{suppl.}\ EAD\left(\frac{AE - AD}{AE + AD}\right) = 129093.2968 = 3831;$$

$$\text{tan.}\ \tfrac{1}{2}\,(ADG - AGD) = \text{tan.}\ \tfrac{1}{2}\ \text{suppl.}\ DAG\left(\frac{AG - AD}{AD + AG}\right) = 156271.4462 = 6973;$$

$$\text{tan.}\ \tfrac{1}{2}\,(AGF - AFG) = \text{tan.}\ \tfrac{1}{2}\ \text{suppl.}\ GAF\left(\frac{AF - AG}{AG + AF}\right) = 48438.6325 = 3064.$$ Whence

$\tfrac{1}{2}\,(AEF - AFE) = 4° 23′ 41″$	-	$\tfrac{1}{2}\,(ADE - AED) = 2° 11′ 37″$	
$\tfrac{1}{2}\,(AGF - AFG) = 1\ \ 45\ \ 18$	-	$\tfrac{1}{2}\,(ADG - AGD) = 3\ \ 59\ \ 10$	
Sum of both = 6 8 59		Sum of both = 6 10 47.	The excess

at D being now greater by 1′ 48″ than the defect at F, it follows, that the points D, E, F, G, though more nearly than before, are not yet precisely in a circle.

The former excess at D was less by 29′ 15″ than the defect at F, and the present excess at D is 1′ 48″ greater; and therefore the addition to the longitude of the aphelion H has been too great. But, instead of again repeating the whole tedious process, it will be sufficient to employ the common rule of proportion. When the error was 29′ 15″, the sum of all the four half differences was 12° 1′ 43″; and now when the error is 1′ 48″, their sum is 12° 19′ 46. The difference is 18′ 3″; and the difference of the errors, as they are in contrary directions, is 31′ 3″. As therefore 31′ 3″, the difference of the errors, to 18′ 3″, the excess of the present sum above the former false one, so will be the present error 1′ 48″, to the excess of the present sum above its just amount. This will be found = 1′ 2″; and subtracted from 12° 19′ 46″ gives 12° 18′ 44″ for the just sum required; so that the excess at D, and the defect at F, ought each to be = 6° 9′ 22″.

Supposing then that, in consequence of this correction, the points D, E, F, G, are in the circumference of the same circle, it is next to be enquired if its centre B be situated in AC.

Here 1. Since the ½ suppl. of FAE is = 44° 31′ 48″, and of FAG = 25° 50′ 41″, their sum will be = 70° 22′ 29″; whence, subtracting 6° 9′ 22″, which ought to be the defect at F, the remainder will be 64° 13′ 7″ = EFG; and, consequently EBG will be = 128° 26′ 14″.

2. Since EAG = (EAD + DAG =) 140° 44′ 59″, its ½ suppl. will be = 19° 37′ 40″. But, in resolving the triangle EAG, in order to find EG and AEG, a correction must be made upon the sides AE, AG, in consequence of the change on the angles at D and F. This will be found by considering, that since the increase of the sum of all the four half differences, from 12° 1′ 43″ to 12° 19′ 46″, was too great, the variations of AE and AG must have been too great also; and, as the first was increased by 37′, and the second diminished by 21′, that now, since 12° 19′ 46″ is by a correction of 1′ 2″ reduced to 12° 18′ 44″, AE and AG must be again varied, by shortening the first and lengthening the second. As therefore 18′ 3″, the first variation of the sum of the half differences, is to 58′, the sum of the changes produced upon AE and AG, so is 1′ 2″, the second variation of the sum of the half differences, to 3 the sum of the changes now to be made upon AE and AG; and dividing this in the ratio of 37 to 21, AE will become = 50738.7, and AG = 52282.3. Therefore tan. ½ (AEG − AGE) = tan. ½ suppl. EAG

(AG

$(\frac{AG + AE}{AG - AE})$ = tan. 19° 37′ 40″ $(\frac{103021}{1544})$ = tan. 18′ 11″; and AEG = 19° 55′ 51″; as also EG = $\frac{AG.\ sin.\ EAG}{sin.\ AEG}$ = $\frac{52282.63271}{34088}$ = 97041.

3. Since the base EG of the isosceles triangle EBG, and the vertical angle EBG are thus found, the angle BEG at the base is given, and = 25° 46′ 53″; and, therefore, BE = $\frac{EG.\ sin.\ BEG}{sin.\ EBG}$ = $\frac{97041.43494}{78327}$ = 53860.

4. In the triangle BEA, the angle BEA is given, for it is = BEG — AEG = 5° 51′ 2″, and also its ½ suppl. = 87° 4′ 27″. Therefore tan. ½ (BAE — ABE) = tan. ½ suppl. BEA $(\frac{BE - AE}{BE + AE})$ = $\frac{1957200.3121}{104599}$ = 58402 = tan. 30° 17′ 8″; so that BAE = 117° 21′ 37″.

But, since in the second operation the aphelion H was found to be too far advanced in longitude, let it now, in consequence of the last correction, be considered as advanced no more than 3′ 8″, instead of 3′ 20″ beyond the longitude first assumed. Then, since AH is in 4s. 28° 47′ 8″, and AE in 8s. 26° 39′ 23″, CAE or HAE will be = 117° 52′ 5″; that is, greater by 30′ 28″ than BAE; and B is not situated in AC, but on side of it towards E. The suppositions therefore for FAH and FCH, must, one of them, or perhaps both, be false.

But these angles of anomaly cannot be varied by the mere variation of the assumed longitude of the aphelion; because no other position of it will permit the points D, E, F, G, to be situated in the circumference of the same circle; and before it can be farther varied, the mean longitudes, or the position of the lines FC, FE, &c. must be varied. This, therefore, was the next step of Kepler's procedure; and he tells us, it was not till after a great variety of unsuccessful trials, that he found his purpose would be nearly accomplished by the addition of 2′ more to the longitude of the aphelion, and of 30″ at the same time to the mean longitudes. By these additions the mean anomalies FCH, ECH, &c. are all diminished 1′ 30″ each; and we have FCH = 32° 3′ 36″; KCE = 53° 7′ 2″; KCD = 11° 0′ 44″; and KCG = 68° 18′ 1″. The angles again of equation will become AFC = 5° 8′ 26″; AEC = 9° 4′ 41″; ADC = 2° 17′ 10″; and AGC = 10° 13′ 45″; being increased 30″ in the first semi-circle of anomaly, and as much diminished in the second: consequently, the lines

lines drawn from A will be also changed; for now $AF = \frac{\text{sin. HCF}}{\text{sin. AFC}} = 59201$; $AE = \frac{\text{sin. KCE}}{\text{sin. AEC}} = 50775$; $AD = \frac{\text{sin. KCD}}{\text{sin. ADC}} = 47887$; $AG = \frac{\text{sin. KCG}}{\text{sin. AGC}} = 52322$; and, though the angles at A must remain unaltered, being determined by the observations, and therefore also the tangents of their half supplements, the tangents of the half differences will become, tan. $\frac{1}{2}$ (AEF — AFE) = 98373.7662 = 7536; tan. $\frac{1}{2}$ (ADE — AED) = 129093.2927 = 3779: tan. $\frac{1}{2}$ (ADG — AGD) = 156271.4426 = 6917; and tan. $\frac{1}{2}$ (AGF — AFG) = 48438.6168 = 2988. Therefore

$\frac{1}{2}$ (AEF — AFE) =	4° 18′ 36″	- -	$\frac{1}{2}$ (ADE — AED) =	2° 9′ 52″
$\frac{1}{2}$ (AGF — AFG) =	1 42 41	- -	$\frac{1}{2}$ (ADG — AGD) =	3 57 24, and the
Defect at F =	6 1 17		Excess at D =	6 7 16; so that, the dif-

ference being 5′ 49″, the points D, E, F, G, are not in the circumference of a circle.

Let therefore the aphelion be brought back 38″, that is, from 4s. 28° 49′ 40″, to 4s. 28° 48′ 42″, without any farther change on the mean longitudes: the mean anomaly will of consequence be HCF = 32° 2′ 44″; KCE = 53° 7′ 54″; KCD = 11° 1′ 22″; KCG = 68° 18′ 39″; the lines from A will also be AF = 59219, AE = 50769, AD = 47931, AG = 52317; and the quantities $\frac{AF - AE}{AF + AE}$, &c. which in the last operation were 7662; 2927; 4426; 6168; will be 7683; 2875; 4375; 6188; and their respective differences from the former, to wit, 21, 52, 51, 20, being multiplied into the tangents of the half suppl. of the constant angles at A, will give the alterations that ought to be made on the tangents of the half differences, to wit, 21, 67, 80, 10. Of these the first and the last are additive to the correspondent tangents, and the second and third subtractive. Hence the corrections will be found, which ought to be made on the former half-differences; to wit, 41″ +; 2′ 14″ —; 2′ 39″ —; 19″ +. So that now

$\frac{1}{2}$ (AEF — AFE) = 4° 19′ 17″ - - $\frac{1}{2}$ (ADE — AED) = 2° 7′ 38″
$\frac{1}{2}$ (AGF — AFG) = 1 43 0 - - $\frac{1}{2}$ (ADG — AGD) = 3 54 45; and the

Defect at F = 6 2 17 Excess at D = 6 2 23, that is, almost precisely equal, and the points D, E, F, G, therefore, again in a circle.

The operations therefore are to be repeated for the position of B its centre.

1. In consequence of the last assumptions, EFG is = 64° 20′ 9″; and therefore EBG = 128° 40′ 18″, and its $\frac{1}{2}$ suppl. BEG = 25° 29′ 51″. 2dly. In consequence of the changes on AE and AG, the angle AEG becomes = 19° 55′ 54″, and EG = 97103. 3dly. In the triangle BEG, BE = $\frac{\text{sin. BEG}}{\text{sin. EBG}}$. EG = 53866. 4thly. BEA = BEG — AEG = 5° 43′ 57″, and its $\frac{1}{2}$ suppl. = 87° 8′ 1″.5. Therefore $\frac{1}{2}$ (BAE — ABE) = 30° 35′ 22″, and BAE = 117° 43′ 23″.5.

But, since the aphelion is supposed to be in 4s. 28° 48′ 42″, and AE is in 8s. 26° 39′ 23″, the angle CAE is = 117° 50′ 41″; and therefore BA makes with CA an angle of 7′ 17″.5. The difference being so inconsiderable, BA might be safely calculated as actually coinciding with CA; and in that state it would be = $\frac{\text{sin. BEA}}{\text{sin. BAE}}$ = 11283 in parts of BE = 100000.

But BE in parts of CA = 10000 was = 53866, and therefore AC = 18564, in parts of BE = 100000; and the centre B divides the excentricity AC in the ratio of 11283 to 7281.

To remove however all suspicion of error, let the corrections be calculated by the usual methods. In the first assumption, the quantities found were

BE = 53860; AE = 50739; BEA = 5° 51′ 2″; BAE = 117° 21′ 37″; and now by the second,
BE = 53866; AE = 50769; BEA = 5 43 57; BAE = 117 43 23.5; and therefore the differences

6 30 7 5 21 46.5

Since, then, the error of BAE, to wit, $7' \, 17''.5$, is about $\frac{1}{3}$ of the difference between the two results for this angle, all the results from the second assumption are to be varied by a third part of the differences between them and the results from the first; that is, corrections are to be made upon them of $+ 2$; $+ 11$; $- 2' \, 15''$; $+ 7' \, 17''.5$. So that at last BE $= 53868$, AE $= 50780$, BEA $= 5° \, 41' \, 32''$; and hence, calculating BAE, it will be found accurately equal to CAE.

Therefore, finally, $BA = \frac{\text{sin. BEA}}{\text{sin. BAE}} = 11332$, and $BC = 7232$. Or, in the Copernican form of a concentrical circle with two epicycles, the semi-diameter of the greater would be 14998, and of the less 3628, in parts of the semi-diameter of the orbit $= 100000$.

O.

It was a business of no little difficulty to find any pairs of observations, in circumstances fit for Kepler's purpose. The earth's apsides were in 9s. 5° 30′ and 3s. 5° 30′, and no two observations from the earth in these opposite points, made of the planet, either in 6s. 5° 30′, or 0s. 5° 30′, could be found in T. Brahé's whole collection. It became necessary, therefore, to investigate the times in which, while Mars was near to either of these points of quadrature, the angles of commutation KCF, LCF, though not right angles, were equal to each other. When these times were found, T. Brahe's collection was next to be examined, to discover if in them any observations of the planet had been made; and if his ardour had not been indefatigable, no pair with such concurring conditions could have been found.

As the heliocentric places of Mars were required to be in 6s. 5° 30′ and 0s. 5° 30′, and the apogee of his orbit, according to T. Brahé, was in 4s. 23° 30′, the required true anomaly was 42°, or 318°; and T. Brahé's equation, at either of these points, was 8° 15′ 16″. The requisite mean anomaly therefore was 50° 15′ 56″, or 309° 44′ 4″; and Mars was twelve times in the former of these during the course of T. Brahé's observations.

It was therefore to be enquired if any of these twelve times, in which Mars was in 50° 15′ 16″ of mean anomaly, and consequently in a line perpendicular to the line of the earth's apsides, the earth was to be found in two points G, H, or K, L, of her orbit, making with the planet the equal angles of commutation FCG, FCH, or FCK, FCL.

A revolution of Mars is performed in 687 days, and two revolutions

revolutions of the earth in $730\frac{1}{2}$ days. The difference of these periods is $43\frac{1}{2}$; during which time the earth moves through 42° 54′ 23″. If, then, in the space of two years, two places of the earth be required, which shall make equal angles, in equal times, with a perpendicular from Mars to the line of apsides, on each side of it, each place must make an angle with it of 21° 27′. In four years they must form similarly situated angles of 42° 54′; in six years, of 64° 22′; and in eight years, of 85° 48′. No such pair at the distance of eight years were to be found in T. Brahé's register; but a pair, at the distance of six years, was found as proper for the purpose as could reasonably be expected. These were the observations of the 30th of May 1585, at 5h. and of January 20th, 1591, at 0h.; in both which the mean heliocentric longitude of Mars was 6s. 22° 43′ 8″, and therefore the true heliocentric longitude, after applying T. Brahé's equation, 6s. 13° 28′ 16″; and the earth in the former being at K, in 8s. 17° 51′ 46″, and in the latter at L, in 4s. 9° 4′ 46″, the angles of commutation KCF, LCF, were each = 64° 23′ 30″. The observations, indeed, were not actually made at the precise instants now mentioned, and only reduced to them from May 18th and January 22d, by means of the diurnal motions in Magini's tables; because the commutations would not otherwise have been equal. In the former observation, indeed, the distance of the earth from the aphelion, in 9s. 5° 30′ was only 17° 38′, *in antecedentia*; and in the latter, the distance from the perihelion was no less *in consequentia* than 33° 35′; but this was an imperfection which could not be avoided.

The investigation of the distance BC, between the centres of the orbit and the equant, was to this effect.

Join BK, BL, KL; from L draw LP perpendicular to KC produced; draw, from B and C, the perpendiculars BM and CN to KL; and from N draw to BM the line NO parallel to the line of apsides DE. Assume FC = 100000.

Since KFC = 36° 51′, and KCF = 64° 23′ 30″, we shall have the angle FKC = 78° 45′ 30″, and therefore KC = 61148; and, in the same manner, since LFC = 38° 6′, LC will be found = 63186.

Again, since KCL = 128° 47′ 19″, PCL will be = 51° 12′ 41″, and PLC = 38° 47′ 19″; and therefore, since LC = 63186, PL will be found = 49251, and PC = 39583. Therefore PK = 100731; and since $\tan. LKP = \frac{PL}{PK}$, this angle will be found = 26° 3′ 21″, and its secant KL = 112126, and therefore LM = $\frac{1}{2}$KL = 56063. But CLK = 180° − (KCL + LKP) = 25° 9′ 21″, and therefore CN = CL. sin. CLK = 26859, and LN = CL. cos CLK = 57193.

 Therefore

Therefore MN = LN — LM = 1130. Since also, KCE = 17° 38′, and LKC = 26° 3′ 21″, we shall have ONM = LKC — KCE = 8° 25′ 21″. Therefore MO = MN. tan. ONM = 167; and NO = BC = 1143. But BM = CN + MO = 27025; and KB = $\sqrt{KM^2 + BM^2}$ = 62237; and therefore BC, in parts of KB = 100000, will be found = 1837.

Two conditions are indeed assumed in this investigation, viz. that the earth's orbit is a perfect circle, and that the place of its aphelion, as given by T. Brahé, is just. But though neither of these be accurately true, the certainty of the general conclusion will not thereby be affected.

P.

In the above experiment, the mean longitude of Mars, on the 5th of March 1590, at 7h. 10′, was 1s. 4° 38′ 50″, according to the Rhudolphine tables, and the equation 11° 14′ 55″ additive; so that the true heliocentric longitude was 1s. 15° 53′ 45″. To the mean longitude 1′ 36″ was added for the precession during every interval, and the remaining true longitudes became 1s. 15° 55′ 23″; 1s. 15° 56′ 56″; 1s. 15° 58′ 30″; on the application of the proper equations.

The mean places of the earth were, E in 5s. 22° 58′ 46″; F in 4s. 10° 5′ 57″; G in 2s. 27° 13′ 12″; H in 1s. 14° 20′ 25″. The angles of commutation therefore were ECM = 127° 5′ 1″; FCM = 84° 10′ 34″; GCM = 41° 16′ 16″; and HCM = 1° 38′ 5″.

The observed longitudes of Mars were, from E, 0s 25° 6′ 0″; from F, 0s. 19° 19′ 0″; from G, 0s. 3° 35′ 30″; and from H, 1s. 19° 21′ 35″. Consequently the parallaxes were EMC = 20° 47′ 45″; FMC = 35° 46′ 23″; GMC = 42° 21′ 26″; and HMC = 3° 23′ 5″. Whence the elongations came out CEM = 32° 7′ 14″; CFM = 60° 3′ 3″; CGM = 96° 22′ 18″; and CHM = 174° 58′ 50″.

When, with these data, the distances CE, CF, CG, CH, were found, the distance BC of C from the centre of the orbit was to be investigated.

Join BE, BF, BG, and EF, EG, FG. As all the angles about C are equal, and each of them, as ECF consists of 42° 52′ 47″, we shall have in the triangle ECF, with this angle, and the sides CE, CF, the base EF = 49169, and CEF = 69° 18′ 46″: and in the same manner, in the triangles FCG, ECG, the angle CGF will be found = 68° 12′ 6″; and CGE = 46° 39′ 10″

Therefore EGF = CGF — CGE = 21° 33′ 16″, and thence

thence EBF at the centre = 42° 6′ 30″, and BEF = 68° 26′ 44″; and from these, with EF also found, the semi-diameter BF = 66923. Therefore, finally, in the triangle BEC, with the angle BEC = CEF — BEF = 0° 52′ 2″, and the sides BE, CE, the base BC will be found = 1023, that is, = 1530 in parts of BE = 100000; and the angle ECB = 97° 50′ 30″: so that the longitude of the perihelion P is 2s. 15° 8 16″, differing more than 20° from the position given it by T. Brahé. But it is to be remembered, that the mean longitude and equations, taken from his Rhudolphine tables, are so far from being accurate, that if, instead of the observations from E, F, G, we employ any other combination of three, the place of the aphelion will be sometimes more advanced, and sometimes less advanced, than the position given by T. Brahé; and different results will be also found for the distance BC. But all the results agree in placing B between A and C, and making BC nearly one-half of the whole excentricity 3592.

Q.

The heliocentric longitude of Mars, calculated on the principles of the vicarious theory, where the planet is referred to the centre of the sun, and not to C the supposed centre of the earth's orbit, was 1s. 14° 19′ 52″, on the 25th of October 1595, at 5h. 45′; and the times when the planet returned to the same point of longitude, were December 7th, 1593, at 8h. 0′; January 23d, 1592, at 7h. 20′; and March 4th, 1590, at 7h. 10′: so that the heliocentric longitudes, corrected for the precession, were 1s. 14° 18′ 16″, 1s. 14° 16′ 40″, and 1s. 14°. 15′ 4″.

After the observations had been carefully corrected for refraction and diurnal parallax, and the true longitudes of the earth calculated from the Rhudolphine equations, the geocentric longitudes deduced from them, by means of the diurnal motions in Magini's tables, were

	Geoc. long. of Mars.	App. long. of the earth.
	s. ° ′ ″	s. ° ′ ″
1590	0 24 20 0	5 24 0 25
1592	0 9 24 0	4 10 17 8
1593	0 3 4 30	2 25 53 24
1595	1 19 42 0	1 11 41 34

The parallaxes therefore were EMA = 19° 55′ 4″; FMA = 34° 52′ 40″, GMA = 41° 13′ 46″, and HMA = 5° 22′ 8″; and the elongations AEM = 30° 19′ 35″, AFM = 59° 6′ 52″, AGM = 97° 11′ 6″, and AHM = 171° 59′ 34″, Therefore,

by

by assuming AM = 100000, AE = $\frac{\text{sin. EMA}}{\text{sin. AEM}}$ = 67467; AF = $\frac{\text{sin. FMA}}{\text{sin. AFM}}$ = 66632, AG = $\frac{\text{sin. GMA}}{\text{sin. AGM}}$ = 66429, AH = $\frac{\text{sin. HMA}}{\text{sin. AHM}}$ = 67220.

From these distances, and with the angles EAF, FAG, GAH, it is required to determine the excentricity AB of the orbit; for, if this shall come out = 3592, or 3600, according to T. Brahé, the equant and orbit, notwithstanding of the former investigation, must coincide.

The angle EAG is = 88° 7′ 1″, and, with an addition of 3′ 12″ for the precession, it becomes 88° 10′ 13″, and consequently its ½ suppl. 45° 54′ 54″; whence, with AE and AG, the angle AEG will be found = 45° 27′ 22″, and the side EG = 93159. In the same manner, in the triangle AEH, where the angle EAH is = 132° 18′ 51″, but with an addition of 4′ 48″ for the precession, = 132° 23′ 39″, its ½ suppl. will be = 23° 48′ 11″; and therefore, with AE and AH, the angle AHE will be found = 23° 51′ 0″. Lastly, in the triangle AGH, where GAH, also corrected for the precession, is = 44° 13′ 26″, and its ½ suppl. = 67° 53′ 17″, with AG and AH, the angle AHG will be found = 67° 3′ 12″.

Since, then, EHG = AHG — AHE = 43° 12′ 12″, its double EBG will be = 86° 24′ 24″; and BEG = 46° 47′ 48″, and therefore EB = 68141. Since also AEB = BEG — AEG = 1° 20′ 6″, we shall, with AE and BE, find EBP = 68° 26′ 7″, that is, the longitude of P, the perihelion, = 3s. 15° 34′ 18″, AB = 2516, in parts of BE = 100000.

But it was previously known, that the longitude of the perihelion P, instead of 3s. 15° 34′ 18″, ought to be nearly 3s. 5° 30′, as determined by T. Brahé; and therefore, that AE = 67467, was very nearly the earth's mean distance from the sun, and AG = 66429, very nearly the perihelion distance. It follows hence, that their difference 1038, in parts of AM = 100000, or 1539, in parts of BE = 100000, cannot be much less than the true excentricity. But AB now found, by taking in AH to the investigation, is much greater than 1539, found from AE and AG; and some error therefore must evidently have been committed in the observations, and most probably in the observation from H. The geocentric longitude seen from this point, Kepler supposes, ought to have been 1s. 19° 40′, instead of 1s. 19° 42′. By this change AH will become = 67030, and repeating the investigation with this substituted for 67220, AB will be found = 2100, and the longitude of the perihelion = 3s. 10° 36′.

It

It was also possible that an error of 1′ might have been committed in the heliocentric longitude of the planet deduced by the vicarious theory. Kepler therefore repeated the operation with the heliocentric longitude increased 1′; and found the place of the aphelion brought by this change to 9° 7′ 23″, and the excentricity reduced to 1880, by which means it clearly appeared that, with the true place of the aphelion, it would be still further reduced: and in fact, by supposing also the geocentric longitude observed from H, in 1595, diminished only 30″, it was reduced to 1800.

He also repeated the investigation, without any alteration either of the heliocentric, or of any of the geocentric longitudes, by combining the observations from E, F, and G; and found with these 9 s. 8° 51′ for the place of the aphelion, and 2000 for the excentricity AB: and here also, by only supposing the longitude observed from F to be 1′, or 2′, below the truth, and, in consequence of increasing it, also increasing AF, his purpose was again attained, and the aphelion being brought to 9s. 5° 30′, the excentricity AB was a second time reduced to 1800.

He found also by this investigation, that the distance AM of the planet, in this point of the orbit from the sun, amounted to 147443, and even exceeded it, in parts of BE = 100000.

Finally, to ascertain his conclusion, that the excentricity AB of the earth's orbit did not exceed 1800, the half of T. Brahé's excentricity, he calculated, with AM = 147700, AB = 1800, and the heliocentric longitude, in 1595, = 1s. 14° 21′ 7″, the geocentric longitude for the times of observation, on the supposition that the orbit of the earth, between the apsides, retired within the circle, and the results were

Calculated longitudes.					Observed longitudes.			
s.	°	′	″		s.	°	′	″
0	24	21	13	-	0	24	20	0
0	9	23	30	-	0	9	24	0
0	3	2	30	-	0	3	4	30
1	19	42	40	-	1	19	42	0

Hitherto Kepler had submitted to the trouble of performing all his laborious and intricate calculations, in every one of the three several systems, and in both the excentric and the concentric forms. This trouble he submitted to, partly to avoid every possibility of mistaking the truth, and partly to fulfil the dying request of T. Brahé. But, as none of these various hypotheses produced any sensible difference on his conclusions, he henceforth discontinued such multiplication of labour, and confined his investigations to the forms of the Copernican system.

R.

It was known, from the periodical revolution of Mars, that if at any particular time, as February 11th, 1589, at 17h. 13′, he was in 6s. 5° 25′ 14″ of heliocentric longitude, he would, in former or subsequent years, be in the same longitude on May 9th, 1585, at 18h. 11′; March 27th, 1587, at 17h. 42′; and December 30th, 1590, at 16h. 44′. Accordingly, at these several times,

	Helioc. long. Mars. s. ° ′ ″	Geoc. long. Mars. s. ° ′ ″	Longitude Sun. s. ° ′ ″
1585,	6 5 22 2	4 26 54 30	1 28 55 45
1587,	23 38	5 18 12 0	0 16 50 24
1589,	25 14	7 8 48 15	11 3 41 40
1590,	26 50	7 9 47 10	9 19 6 48

With the angles therefore AEM, AFM, &c. of elongation, and the angles EMA, FMA, &c. of parallax, given, the distances of the earth from the sun, in parts of AM = 100000, will be AE = 62227.5; AF = 61675; AG = 60658; AH = 60291.

As the angles EGF, EHF, stand on the same circular arch EF, and are angles at the circumference of the circle EFGH, they must be equal; and, if they do not come out equal, the position of AM has not been properly assumed. These angles therefore are to be calculated, by resolving the four triangles EAG, FAG, EAH, FAH, in which, from the sides EA, FA, GA, HA, now found, and the included angles at A, given by means of the sun's longitudes, taken from the Rhudolphine tables, and corrected for the precession, we shall find

$$\left.\begin{array}{l} EGA = 48^\circ\ 8'\ 59'' \\ FGA = 69\ 37\ \ 0 \end{array}\right\} \text{whence } EGF = 21^\circ\ 28'\ 1''$$

$$\text{and } \left.\begin{array}{l} EHA = 25\ 38\ 30 \\ FHA = 46\ 47\ 36 \end{array}\right\} \text{whence } EHF = 21\ \ 19\ \ 6$$

$$\text{Difference} = 8\ 55$$

Let the operation therefore be repeated with AM advanced 2′ in longitude; and the angles EGF, EHF, will be found, the first = 21° 40′ 9″, and the second = 21° 22′ 14″; so that their difference now rises to 17° 55′, and shews that AM ought to have been drawn back, and not advanced.

Let the operation be again repeated with AM, drawn back 2′, and EGF will be found = 21° 15′ 54″, and EHF = 21° 13′ 54″, differing only 2′. But, though this inconsiderable difference might have been safely neglected, Kepler repeated the whole

whole investigation a fourth time, with the heliocentric longitude of the planet diminished 30″ more; and then the angles EGF, EHF, came out each = 21° 13′.

Since, then, EGF = 21° 13′, its double EBF will be = 42° 26′; and therefore BFE = 68° 47′. Since also, in consequence of the last correction, AE becomes = 62117, and AF = 61525; from these, with EAF = 42° 6′ 37″, EFA will be found = 69° 43′ 31″, and EF = 4418. But EF being the chord of EBF = 42° 26′, is, in parts of BE = 100000, equal to 72379; and consequently AM, in the same parts, = 162818, and AF = 100174. Therefore, with AF and BF, and the included angle AFB = EFA — BFE = 0° 56′ 31″, the angle FAB will be found = 83° 30′, and AB = 1653; so that the aphelion is in 9s. 10° 19′, and the excentricity nearly the half of that determined by T. Brahé; and, with a just place of the aphelion, would evidently rise to 1800.

S.

The times in which the planet would necessarily be found in the same point of longitude, whatever that longitude might be; the longitude of the earth at these times, the geocentric longitude of the planet, and the distances of the earth from the sun, calculated for the excentricity 1800, were all deduced with Kepler's usual attention, and stood as follows:

Times.				Geoc. long. of Mars.
1583,	April	22d at	20h. 6	4s. 1° 29′ 30″
1585,	Mar.	9th	19 40	4 11 48 20
1587,	Jan.	25th	19 10	6 4 41 45
1588,	Dec.	12th	18 45	6 13 35 40
1590,	Oct.	30th	18 10	6 2 57 20

Long. Earth.	Distances.
7s. 12° 10′ 3″	AE = 101049
5 19 41 4	AF = 99770
4 16 5 55	AG = 98612
3 1 44 53	AH = 98203
1 17 28 33	AK = 98770.

The last was the time of actual observation, and as it was solitary, it was used as the epoch from which the other times were reckoned. To reduce the observation of 1583, which was also solitary, to the time required, the horary motions of Magini's tables were employed; and to reduce those of the other years, the horary motions were supplied by various observations of successive days.

The

The first investigation was made with the observations from F and G. The angle FAG, corrected for the precession, was 43° 36′ 45″, and, with the including sides AF, AG, the angle AGF was found = 69° 1′ 41″, AFG = 67° 21′ 34″, and the base FG = 73700. Then, in triangle FMG, with the angle MFG = AFM — AFG = 64° 45′ 41″, the angle MGF = AGM — AGF = 62° 22′ 29″, and the side FG, the side FM was found = 81915. Therefore, in the triangle MFA, with FM, FA, and the included angle AFM, the angle FAM was found = 21° 26′ 32″, and AM = 166208; so that AM was in 5s. 8° 14′ 32″ of longitude, in March 1585.

The second investigation was made in the same manner, with the observations at E and H: it gave AM = 166179, that is, differing only 29 from the former result; and for its position in 1583, 5s. 8° 11′ 31″, which, corrected for the precession, gives 5s. 8° 13′ 8″ for the position in 1585, differing from the former no more than 1′ 24″.

Lastly, in the triangle AKM, with the given angle AKM, the sides AK = 98770, and AM = 166208, the angle AMK was found = 24° 37′ 38″; so that the position of AM in 1590 was in 5s. 8° 19′ 52″; and, by a correction of 4′ 48″ for the precession, 5s. 8° 15′ 4″ for the position in 1585, that is, differing from the first result no more than 32″.

T.

In the ancient hypothesis the orbit of the earth was the circle PQST (fig. 61), described upon the centre C, and the excentricity of the orbit was AC = 3600. But in Kepler's theory, confirmed by so many convincing proofs, this was only the equant of the earth, and the orbit was the circle DHFE, by whose centre B, the excentricity AC was bisected. In this new theory, therefore, the distances of the earth from the sun, which we have seen to be of so great consequence in the investigation of the elements of the other orbits, generally differed from the ancient distances; and they were calculated for every degree of anomaly, in the following manner.

In the line of apsides DE, the aphelion distance is AD = BD + AB = 101800; and the perihelion distance is AE = BD — AB = 98200. These, in the ancient hypothesis, were AP = CP + AC = 103600, and AT = CP — AC = 96400.

At 90° of true anomaly, when the earth is in the point F, the new excentricity AB becomes the sine of the optical equation ABF = 1° 1′ 53″; and therefore AF its co-sine = 99984. In the ancient hypothesis AC = 3600 is the sine of the whole equation ARC, and therefore its co-sine AR = 99934.

When

When the true anomaly DAH is $= 45°$, the distance AH will be found by drawing from the centre B, the line BN, perpendicular to AH, and joining BH. Then, AN = AB. cos. $45° = 1272 =$ BN. Whence BHN $= 43' 45''$, and its co-sine HN $= 99992$, and therefore AH = HN + AN $= 101264$. Whence also, in the opposite degree 225° of true anomaly, by producing AH to meet the orbit again in K, we shall have AK = HN — AN $= 98720$. In the former hypothesis these distances are AQ $= 102514$, and AV $= 97422$, found by drawing from C the line CX perpendicular to QAV.

It likewise appears that the mean distances, though the same as in the ancient hypothesis, do not take place at 90° and 270° of true anomaly; but, as Reinholdus had observed with respect to the planetary orbits, at two points where a parallel to FAG, bisecting AB, meets the orbit DHFE; and that, as Ptolemy had also observed, in all cases where the bisection takes place, the greatest equations do not happen at these points of mean distance, but at F and G, in 90° and 270° of true anomaly.

But, though Kepler expected that these new distances of the earth from the sun, would produce considerable alterations on the elements of the planetary theories, he shews by experiment, that their effect on the ancient solar equations would be altogether inconsiderable.

At the mean distances towards F and G, the effect is altogether insensible. For, whether CA be considered as the sine of the angle CRA to the radius CR, or as the tangent of the angle CFA to the radius AF, the equations will be almost perfectly the same.

In other points, as at the octants, though the equations must be calculated by different methods, their difference is still very inconsiderable. In the ancient theory, the equation at 45° is found by a single operation, to wit, sin. AQC $= \frac{\text{AC. sin. } 45}{\text{CQ}} = 1° 27' 31''$; and this is also its amount at 135° of true anomaly. In the new theory, again, it is found by a double operation; for 1st. sin. AHB $= \frac{\text{AB. sin. } 45}{\text{BH}} = 43' 46''$; and, 2dly, tan. $\frac{1}{2}$ (BCH — BHC) = tan. $\frac{1}{2}$ suppl. CBH $\left(\frac{\text{BH} - \text{BC}}{\text{BH} + \text{BC}}\right)$; whence the physical equation at 45° of anomaly, will be BHC $= 43' 38''$; and at 135°, it will be $= 43' 42''$: and consequently the whole equation AHC, at the first octant, will be $= 1° 27' 24''$, and at the second $1° 27' 28''$.

As

As the differences between the physical and optical equations, in an orbit so little excentrical, were in most cases insensible, Kepler thought it sufficient to calculate the optical equation only, and by doubling it to obtain the whole. So that in both octants it was 1° 27′ 32″.

The disposition of the distances in Kepler's table of them was peculiar. They were calculated for compleat degrees of true anomaly, but the corresponding true anomalies are not given. The anomalies, which are marked in the table, on each side of every distance, are, the one less than the mean anomaly by the real part of the solar equation, and the other less than the true by the optical part of it. Consequently, as Kepler considered these two parts of the equation to be sensibly equal, the true anomaly is the mean between the anomalies marked in the table. The disposition was made in this manner.

Mean anomalies.	Distances.	True anomalies.	Mean of them.
° ′ ″		° ′ ″	°
45 43 45	101265	44 16 15	45
46 44 30	101242	45 15 30	46
47 45 15	101219	46 14 45	47

Thus the anomalies, entitled mean, are not the angles described about C the centre of the equant, but the angles formed at B the centre of the orbit, when the optical equations, as BHA, are added to the angles formed at A: and the anomalies, entitled true, are not the angles described about the sun in A, but angles less than these, and found by subtracting equations, equal to the optical ones, from the angles at A. It is true that when, in consequence of this arrangement, you apply to the table with a given true anomaly, as 44° 16′ 15″, the corresponding distance 101265 is not the radius vector of the earth in 44° 16′ 15″ of true anomaly, but in 45°; and it is therefore too short: or, if you apply to the table with a given mean anomaly, as 45° 43′ 45″, the distance 101265 is too short for this mean anomaly, for it really belongs to the mean anomaly 46° 27′ 30″. But this was done purposely, when Kepler discovered that the orbit between the apsides retired within the circle, that the distances might fall as much short of the circular distances as the limits of the oval required.

U.

1. In calculating equations, according to this natural principle, Kepler generally abridged his labour, by reckoning the physical and optical parts of the equation equal. Indeed, in

in orbits so little excentrical as that of the earth, the difference was almost insensible. For, if on the centre E (fig. 64), with the distance EB we describe the arch BX, to meet EA in X, the sector BEX will represent the optical equation as accurately as the angle BEA. Now this sector differs from the area BEA, that is, from the physical equation, only by the very small area, or triangle, BXA; and, in the solar orbit, this difference, even at the octants, does not exceed 33 seconds. But in more excentrical orbits it is by no means to be neglected.

2. As Kepler observed that, when a circle is divided into 360, or any indefinite number of equal parts by diameters, the sum of the distances of the points of division, from any excentric point A, is greater than the sum of the distances of the same points from the centre B; that is, greater than the area of the circle: so he likewise observed, that if as many lines, making equal angles with one another, be drawn through A to the circumference, the sum of the distances of their extremities from A, will be less than the sum of the distances BD, BY, of the same extremities from the centre B, that is, less than the area of the circle; this area being in both cases considered as equivalent to the sum of the indefinite semi-diameters.

3. To shew the real excess of the sums of the distances from A, in the arches CD, CE, CF, &c. above the areas CDA, CEA, CFA, &c. employed to measure them, he supposed a straight line *cdefghk* (fig. 65), to have been found equal to the circumference CDEFGHK, and its equal parts *cd*, *de*, *ef*, &c. equal to the arches CD, DE, EF, &c. From *c*, *d*, *e*, &c. he drew lines *cb*, *db*, *eb*, &c. each equal to the semi-diameter BC, and perpendicular to *ck*, and by joining the points *b*, *b*, *b*, had a parallelogram *bckb*, double of the triangle by which Archimedes attempted to measure the area of the semi-circle; and small parallelograms *cb*, *db*, &c. double of every particular sector CDB, DEB, &c.

He took also in *cb*, *db*, *eb*, &c. produced when necessary, the distances *ca*, *da*, *ea*, &c. respectively equal to AC, AD, AE, &c.; and joining the points *a*, *a*, *a*, by the curve line *aaa*, had the figure *acka*, double of the sum of the distances AC, AD, AE, &c.; and in which the lines *ca*, *da*, *ea*, were inclined to the curve *aaa*, in the same angles in which the distances AC, AD, AE, &c. stood inclined to the circumference AFK.

Finally, he drew from A, to the diameters DS, ER, &c. the perpendiculars AT, AV, &c. and took in the lines *da*, *ea*, &c. the parts *dm*, *en*, &c. respectively equal to TD, VE, &c. and by joining the points *a*, *m*, *n*, &c. he had another curvilinear figure *ackaqponma* precisely equal to the parallelogram *bckb*; and consequently double of the semi-circular area: so that the lunula comprehended between the two curves was double the excess of the sum of the distances AC, AD, AE, &c. above the sum of the areas CDA, DEA, &c. that is, above the area of the semi-circle

As Kepler could find no accurate quadrature of these areas, he calls upon all the geometricians of the age for their assistance. He observes also, that the breadth of the lunula is greatest towards the points *p* and *q*; and that it seems to be bisected, not by the line *aof*, but by *bbb*. This however he also recommends to all geometricians, as a subject worthy of their investigation.

V.

In shewing how inconsiderable the difference is between the area of a semi-circle, that is, the sum of all the distances of the indefinite number of equal parts of its circumference from the centre, and the sum of the distances of the same parts from the excentric point A (fig. 64), he observes, that this difference is the space *amnopqaa* comprehended between the curves of fig. 65. The greatest breadth of this space is at the point *o*, where the line *fo* = *fb*, is equal to FB the semi-diameter of the orbit; and *fa* is equal to FA = 100432, the secant of the greatest optical equation BFA. From their difference *ao* = 432, the sum of the other excesses *na*, *ma*, *pa*, *qa*, may nearly be inferred.

For, if all the particular excesses, on each side of the greatest *ao*, were to *ao*, as the sum of all the sines of excentric anomaly on each side of the semi-diameter, or *sinus totus*, FB, is to FB, then as FB = 100000 to 11458869, the sum of the sines, so would the excess *ao* = 432 be to 49934, the sum of the excesses of the secants of the optical equations above their correspondent *radii*.

But the excesses *na*, *ma*, *oa*, *pa*, *qa*, are not to one another as the sines of excentric anomaly, but nearly in the duplicate ratio of those sines. The sine of 90°, for example, is double the sine of 30°. But the optical equation at 90° of excentric anomaly, is 5° 19′, and the excess of its secant above the radius 432. The optical equation again at 30° of excentric anomaly, like the sine of 30°, is very nearly the half of 5° 19′. But the excess of the secant of this optical equation above the radius is only 107, that is but a fourth part of 432. Afterwards these excesses become almost insensible; and, by actually computing them, the sum amounts to no more than 7000, which is not fully the fourth part of 49934.

The

W.

The demonstration that the lunula cut off by the ellipse of the oval theory from the plane of the excentric, is but insensibly greater than the area of the semi-circle AZB, is to this purpose·

Let AB (fig. 68), be bisected in Q, draw QR perpendicular to AB, and meeting the excentric in R; join RA, RB, and from S, the centre of the equant, draw to the same circumference the line ST parallel to RB; join BT, AT, and from A as a centre, with the distance AR, describe an arch cutting AT in V, and BT in X, and join AX.

Since R is at an equal distance from both A and B, AR will be the mean distance of the planet from A the sun. But because ST is parallel to BR, and AX = AR, the point X is the position of the planet in its oval orbit, at the time CBR or CST, and when in the excentric it would be found in R; and TX, that part of BT which is intercepted between the arch RX and the excentric, will nearly measure the extreme breadth of the lunula CXKRT. It would be accurately measured by a line from X parallel to QR.

From B draw to AR the perpendicular BZ. Then will XT be double of AZ. For, join RT, and from R draw to BT the perpendicular RN; and from X to AR the perpendicular XO. Then, since ABR = BST, and BT = BR = AR, and BS = AB, the triangle TSB will be similar and equal to RBA. Therefore AB will be parallel and equal to TR; and, since RN is parallel and equal to BZ, AZ will be = TN. Since, also, RN = OX, and BR = AR = AX, and the angles at O and N right, the triangles BRN, AXO, will be similar and equal; and therefore AO = BN; and, consequently AZ = NX. Therefore TN = NX, and TX = 2AZ. Since, then, a perpendicular RN is drawn from a point R in the circumference of a circle to its diameter BT, we shall have $RN^2 = (BT + BN)\ TN$. But $TR^2 = AB^2 = RN^2 + TN^2$; and therefore $AB^2 = (BT + BN)\ TN + TN^2 = 2BT.\ TN = BT.\ TX$. But, according to Archimedes, BT^2 is to BT. TX nearly as the area of the circle to the area of the ellipse; and therefore $BT^2 - BT.\ TX$, or $BT^2 - AB^2$ will be nearly the difference between these areas, that is, nearly equal to twice the lunula CXKRT.

The point X is a point of the orbit described by the assistance of the vicarious theory: and as it falls so far within the ellipse CNK, whose excentricity is BA, and its greater axis CK, the errors of the oval theory, and the propriety of the improvement afterwards made upon it, become evident almost by inspection.

But

But it was not only necessary to find the quadrature of the lunula, but also to divide it in any given ratio by a line drawn from the excentric point A; and one of Kepler's attempts for this purpose was as follows. Let the semi-circle CFK (fig. 64) be, as formerly, extended into the straight line *cfk* (fig. 65), which is divided into the same number of equal parts, by perpendiculars *ca*, *da*, &c. representing the distances from A. Let the quadrant CF be = *cf*; and in *fa* take towards *a* the part *fr*, such that it may be to the longest of the lines *ba*, that is, to the excentricity, as *ba* to the semi-diameter *bc*; and in the same manner in *ea*, the part *es*, which may be to the next longest *ba*, as this also is to the semi-diameter; and so on continually, till all the lines of this kind be determined; so as to represent the breadth of the lunula at all the corresponding points of the semi-circle: and let all the points thus found be joined by the curve *ctsrvxk*.

Since, then, the whole space included between *cfk* and the curve *aoa*, is double of the area of the semi-circle, it seemed probable that the space included between *cfk*, and the curve *crk*, was also double of the lunula; and every portion of it, as *cid*, double of the correspondent portion of the lunula. But not finding any geometrical demonstration of these things, he again implores the assistance of geometers.

X.

In his attempts for the rectification of the oval circumference, Kepler endeavoured to derive some assistance from geometry. For, let BC : BA :: BA : CN (fig. 70), so that CN shall be equal to the extreme breadth of the lunula; and from B, towards N, take BL = $\frac{1}{2}$CN. On the centre L, with the distance LC, describe the circle CQM, touching the excentric in C; and on the centre B, with the distance BN, describe the circle NOM, which consequently will touch the oval CEK in O, the extremity of its conjugate axis BO, and the circle CQM in M, a point of the line of apsides. It is evident that the circle NOM is less than CQM, and CQM less than CHK; and, since the semi-diameters CB, CL, and BN, of these circles are in arithmetical proportion, the circumference CQM will be an arithmetical mean between CHK and NOM.

Since, therefore, the oval circumference COK touches CHK in C and K, and NOM in O, and the point opposite to O, it seems not to differ sensibly in length from the circumference CQM.

It appears however to be somewhat, though very inconsiderably, longer. For the area of a circle, whose semi-diameter

is

is a geometrical mean between the two semi-axes of an ellipse, is equal to the area of the ellipse; and supposing the present oval sensibly equal to an ellipse described on the same axes, its circumference will therefore be longer than that of the circle now mentioned, and probably therefore longer than the circumference CQM. For the semi-diameter of CQM is an arithmetical mean between CB and BN; and therefore longer than a geometrical mean, though, in the present circumstances, very inconsiderably.

Since, then, CN = 858, BL = 429, and BC = 100000, we shall have LC = 99571; and, since 100000 : 99571 :: 360° : 358° 27′ 20″, this will be the length of the whole oval circumference, and therefore its half COK = 179° 13′ 40″. It was not however from this investigation, but from the laborious calculations above mentioned, that Kepler concluded it to be = 179° 14′ 15″.

As the decurtation of the oval circumference is nearly equal to the optical amplification of it, Kepler also observes, that the laborious operation, of first shortening and afterwards enlarging every particular arch of the oval, might perhaps be thought unnecessary; and no doubt it would have been unnecessary, if the decurtation and amplification had in all points of the orbit been the same. But though the decurtation be in all points the same, the optical amplification is variable; and, for example, the defect of the whole oval semi-circumference being 45′ 45″, the decurtation of every degree of it must be = 15″; whereas, at the aphelion, the optical amplification of an arch of 1° does not exceed 1″.

Y.

Though the attempts to be now mentioned for deducing the equations of Mars, in the oval theory, were equally unsuccessful with the rest, it seems impossible to form a just conception, either of the character of Kepler, or of his progress in the prosecution of his discoveries, without considering them.

1. He calculated for the excentricity 9165, all the distances of the planet from the sun, which corresponded to compleat degrees of that kind of anomaly, which is a mean between the true and mean anomalies; and which, as being only useful in calculating distances, he called the distantiary anomaly. This is the angle LAH (fig. 69), for LBH in that figure represents the mean anomaly. The sum of these distances he found = 35924252, and less than 36000000, the sum of 360 semi-diameters; for it is the sum only of 180 lines drawn through A to meet the excentric. He then had

 this

this analogy, as 35924252 to 360°, or the whole periodical time, so is the sum of any given number of distances to the time required; and with a view of obtaining accuracy, each distance employed was a mean of the two by which every arch was terminated.

This procedure however led to an absurdity; for, let B (fig. 71), be the centre of the excentric DCL, used for computing the distances, and where DBC represents the mean anomaly, and let A be the sun, and on A, with the distance AK = BD, describe the circle KHP. Then, if the distantiary anomaly DAC be = 45°, the sum of the distances of all its 46 degrees from A will be = 4867852; and from the analogy 35924252 : 360° :: 4867852 : 48° 46′ 51″, this last will represent the time of describing the arch DC; that is, it will be equivalent to the area DAC. But this is only a very little greater than the sector DBC; for if, with the given angle DAC, and the sides BC, BA, we calculate the angle or sector DBC, it will be found = 48° 42′ 56″. In contradiction therefore to the hypothesis, the planet was represented as moving round A with almost perfect equability.

The cause of the error soon appeared. The distances corresponded to equal divisions of the angle CAD, that is, of the arch KH; and consequently to unequal divisions of the arch DC: and, the divisions of DC being longest where the distances from A were greatest, the distances were too few in number; and therefore the area CAD, supposed to be equivalent to their sum, too small. That his labour, however, might not be wholly lost, he subtracted from the angle CAD = 45° of distantiary anomaly, the difference 3° 46′ 51″, between it and the mean anomaly CBD, and thus obtained the true anomaly EAD = 41° 13′ 9″. Then, supposing that the planet, in the time CBD, described about B an angle EBD = CAD, he thus collected together as many distances in the smaller arch ED, or OD, and distributed them among equal parts of it, as had by mistake been distributed among unequal parts of CD; and thus from the distantiary anomaly CAD and BC, BA, given, he was enabled to calculate every distance CA = EA, and every physical equation BEA = CAE. But, though the lines EB and CA were here falsely supposed parallel, the equations differed very little from those deduced in the original form of the theory (156). For to the

Mean anomalies.	The true anom. were	Vicarious theory.	Differences.
48° 42′ 59″	41° 13′ 9″	41° 21′ 0″	7′ 51″ —
95 15 31	84 44 18	84 39 18	5 0 +
138 42 59	131 20 24	131 4 7	16 17 +

and the planet's motions at the apsides continued too slow, and at the mean distances too rapid.

2. He

2. He calculated with the same excentricity a set of third proportionals, every one of which was to every one of the former distances, as that distance to the semi-diameter of the orbit. These third proportionals were also measures of the times in which the planet describes very small arches of its orbit. For, since the times of describing such arches were as the squares of their distances, that is, as AH^2 to AC^2 (138), and AH : AC :: AC : AF, these times will also be as AH to AF. This procedure however supposed, that the distance AC was both of a just magnitude, and in a just position; or that the orbit, contrary to the theory, was a perfect circle. The effects therefore were what might have been expected. The distances between the apsides being too long, the third proportionals were also too long; and consequently the planet's motions were in these parts too slow, and at the apsides too rapid. For to the

True anom.	The mean anom. were	Vicarious theory.	Differences.
45° 0′ 0″	52° 39′ 40″	52° 53′ 0″	13′ 20″—
90 0 0	100 29 12	100 34 30	5 18 —
135 0 0	142 10 47	142 9 0	1 47 +

On adding together the third proportionals, Kepler found, with some surprize, that their sum was precisely = 36000000; and he proposes the fact in this manner for the investigation of geometers. Let two equal circles CD, KH, be described on the centres B and A; join AB, producing it to meet CD in D, L, and KH in K, P; divide the circumference KHP into 360 equal arches, draw through all the points of division lines AK, AH, AP, &c. cutting the circumference DCL in D, C, L, &c. and make AK : AD :: AD : AG; and AH : AC :: AC : AF; and AP : AL :: AL : AM, &c. It is required to demonstrate, that AG + AF + AM, &c. = 36000000.

3. As, by calculating the distances and third proportionals corresponding to equal divisions of the angle CAD, this angle no longer represented the distantiary anomaly, but instead of it the true, and rendered the orbit in the second attempt a circle; so, for a third attempt he reduced, by the common rule of analogy, his distances from unequal divisions of the excentric, to become the distances for equal divisions of it; and then, with the excentric anomaly CD, or CBD, consisting of any given number, suppose 45, of these divisions, he calculated the angle CAD, and added together the 45 distances of the divisions in CD, in order to give him the time in which CAD was described. But the same mistake was by these means repeated; for CAD being the true, and not the distantiary anomaly, the distances were represented as being

 just,

just, in position as well as magnitude, and the orbit again became a perfect circle, in which the motions, as before, were too rapid at the apsides, and too slow about the mean distances. For the effects were these:

Mean anom.	True anom.	Vicarious theory.	Differences.
48° 38′ 31″	41° 31′ 0″	41° 17′ 6″	13′ 54″ +
95 13 58	84 45 50	84 37 45	8 5 +
138 45 41	131 1 52	131 7 13	5 24 —.

4. He next calculated a set of third proportionals to the distances now described, which corresponded to compleat degrees of excentric anomaly CD. Their sum was found to be = 36384621. But by these the errors were increased. For the sum of the distances was 36075562; and if the first 45 of these gave too long a time for describing the arch CD, the third proportionals belonging to them must have given a time still longer.

5. His next recourse therefore was to the table of true anomalies, calculated on the principles of the vicarious theory, and to the former combination of it with the oval theory, in order to obtain distances, accurate both in position and magnitude (152). Since, then, the distances AH (fig. 69), corresponding to compleat degrees of mean anomaly LBH, or LDG, corresponded of consequence to fractional parts of true anomaly LAK, he reduced them by the common rule of proportion to compleat degrees of true anomaly, and found that the sum of such distances, now corresponding to all the equal angles about the point A, was 35770014. But the procedure continued to be unsuccessful. For, though the sum was less than 360 semi-diameters, the divisions of the angle LAG being equal, those of the oval EK were rendered unequal, and too few distances were allotted to it in the parts towards the aphelion; insomuch, that when their sum was added together to make up the area EKA, it did not come out equal to the assumed mean anomaly EBH. For example, if EBH be equal to 48° 44′, the angle EAH of distantiary anomaly will, from BH and BA also given, be found = 45°, and the distance AH = 105784; but this distance is, by the vicarious hypothesis, carried upwards to AK; and the true anomaly given by that hypothesis is EAK = 41° 22′. When this angle EAK, therefore, is divided into equal parts, it contains only 41 distances, and something more than a third; and the sum of these will not give an area EKA equal to the sector or mean anomaly EBH, but an area much less.

6. A sixth attempt therefore was made, by calculating a set of third proportionals to the semi-diameters, and every one

one of the distances now described; and these therefore were measures of the times in which the planet describes those arches of its orbit, which subtend equal and indefinitely small angles of true anomaly. Their sum was 35692408; and, as this was equivalent to the whole periodical time, or 360°, so the time, or mean anomaly corresponding to any given number of them, was found by the common rule of analogy. The results were these:

True anom.	Mean anom.	Vicarious theory.	Differences.
41° 0′ 0″	48° 24′ 3″	48° 12′ 9″	5′ 1″ +
81 0 0	91 30 39	91 34 8	3 29 —
91 0 0	101 28 10	101 34 7	5 27 —
131 0 0	138 28 5	138 39 28	11 23 —

It is true, that by increasing the excentricity, the errors would be diminished; but the planet would still be too slow about the apsides, and too rapid in the intermediate parts. This method was equivalent to that of art. 156, and in practice more convenient; and it neglected the supposed revolution of the planet in an epicycle, and only considered the variation of its distances from the sun; that is, its librations. But though, on this account, it approached a step nearer to the truth, the errors shewed the librations to be too great: and they clearly appeared to arise from the principles of the theory, and not from the imperfect manner of expressing them in the calculation.

Z.

That elongation of the earth, from a line joining the centres of the planet and the sun, which will most sensibly shew any error committed on the planet's distance from the sun, may be thus investigated.

Let A (fig. 75), be the sun, B the planet, CD the orbit of the earth. From B draw the straight line BE perpendicular to BA, and find in BE a point E, such that a circle described upon it with the semi-diameter BE, shall touch the orbit of the earth. Through the point of contact C, draw ACE, and the line CF perpendicular to AB. The sine CF of the required angle CAB, will be to the distance CA of the earth from the sun, as the excess BF of the distance of the planet from the sun, above the co-sine of CAB, is to the planet's distance CA. For EC (= EB) : EA :: CF : CA; and EC : EA :: BF : BA; and, therefore CF : CA :: BF : BA; and, since BF differs but little from BD, CF : CA :: BD : BA.

If therefore AB, the distance of Mars from the sun, should be = 161000, BD would be = 61000, and CF $= \frac{CA \cdot BD}{BA} = 37882$, = sin. CAB, which angle will consequently be nearly = 22° 15′. At the perihelion it will not exceed 18° 30′, but at the aphelion it will rise to 28°.

TABLE of the distances of Mars from the Sun, now investigated.

		Times.	Long. Sun.	Dist. Sun.	Dist. ♂	Helioc. long.	Cal. Geoc. long.	Observed.	Differences.	Latitudes.
		d. h. ′	s. ° ′			s. ° ′ ″			′ ″	° ′
1582,	Nov.	23 16 0	8 11 41	98345	158852	3 0 42 11	3 26 40 0	3 26 38 30	1 30 +	2 49 N
	Dec.	26 8 30	9 15 4	98226	162104	3 16 7 28	3 17 44 19	3 17 40 30	3 49 +	4 7
	Dec.	30 8 10	9 19 9	98252	162443	3 17 56 32	3 16 6 20	3 16 0 30	5 20 +	4 8
1583,	Jan.	26 6 15	10 16 33	98624	164421	4 0 6 24	3 8 17 57	3 8 20 30	2 23 —	2 52
1584,	Dec.	21 14 0	9 10 16	98207	164907	4 3 51 45	5 1 14 34	5 1 13 30	1 4 +	3 31
1585,	Jan.	24 9 0	10 14 53	98595	166210	4 18 47 8	4 24 3 58	4 24 7 30	3 32 —	4 31
	Feb.	4 6 40	10 26 10	98830	166400	4 23 33 41	4 19 43 52	4 19 47 0	3 8 —	4 28
	Mar.	12 10 30	0 2 16	99858	166170	5 9 23 14	4 11 43 31	4 11 46 0	2 29 —	3 22
1587,	Jan.	25 17 0	10 16 1	98611	166232	5 8 13 40	6 4 41 50	6 4 42 0	0 10 —	3 26
	Mar.	4 13 24	11 24 0	99595	164737	5 24 56 50	5 26 24 41	5 26 25 40	0 59 —	3 38
		10 11 30	11 29 52	99780	164382	5 27 35 54	5 24 5 15	5 24 5 15	0 0	3 29
	Apr.	21 9 30	1 10 48	101010	161027	6 16 44 51	5 15 49 50	5 15 48 20	1 30 +	1 48
1589,	Mar.	8 16 24	11 28 36	99736	161000	6 16 55 14	7 12 14 7	7 12 16 50	2 43 —	2 4
	Apr.	13 11 15	1 3 38	100810	157141	7 4 1 50	7 4 45 0	7 4 43 20	1 40 +	1 10
		15 12 5	1 5 36	100866	156900	7 5 1 41	7 3 58 57	7 3 58 20	0 37 +	1 4
	May	6 11 20	1 25 49	101366	154326	7 15 30 36	6 27 8 17	6 27 7 20	0 57 +	0 7

1591,	May	13 14 0	2 2 10	101467	147891	8 12 7 38	9 2 15 36	9 2 20 0	4 24 —	2 25 S.
	June	6 12 2	2 24 59	101769	144981	8 25 38 48	8 27 11 45	8 27 15 0	3 15 —	3 55
		10 11 50	2 28 47	101789	144526	8 27 56 49	8 25 57 57	8 26 2 36	4 39 —	4 8
		28 10 24	3 15 51	101770	142608	9 8 29 32	8 21 4 21	8 21 10 0	5 39 —	4 45
1593,	July	21 14 0	4 8 26	101498	138376	10 20 1 38	11 17 48 14	11 17 45 45	2 31 —	5 46
	Aug.	22 12 20	5 9 11	100761	138463	11 10 15 25	11 13 9 39	11 13 10 15	0 36 —	6 7
		29 10 20	5 11 54	100562	138682	11 14 37 15	11 11 11 41	11 11 14 0	2 19 —	5 22
	Oct.	3 8 0	6 20 15	99500	140697	0 6 19 39	11 7 49 54	11 7 50 10	0 16 —	3 17
1595,	Sept.	17 16 45	6 4 18	99990	143222	0 22 49 19	1 26 5 45	1 26 7 12	1 27 —	1 42
	Oct.	27 12 30	7 13 59	98851	147890	1 15 35 38	1 18 50 46	1 18 51 15	0 29 —	0 6
	Nov.	3 12 0	7 21 2	98694	148773	1 19 26 33	1 16 18 33	1 16 18 30	0 3 +	0 17 N
	Dec.	18 8 0	9 6 43	98200	154539	2 13 2 29	1 11 39 1	1 11 40 0	0 59 —	1 40

It appears from this table that, when Mars is in Cancer, his calculated geocentric longitudes exceed the observed ones about 4′, and fall as much short of them in the opposite parts of the orbit. But false distances could not produce the same effects on both sides of the same oppositions; and the errors seem rather to arise from an error of 1′ in the position of the line of apsides.

A a.

When Kepler endeavoured to assign some possible causes of a planet's revolution in a circular orbit round the sun, though not situated in its centre (140), one was its libration, according to a certain ratio, in that diameter of its epicycle which passes through the centre of the sun. But the great difficulty of the supposition was that, to the description of a perfect circle, it was necessary that the librations should be all unequal, though corresponding to equal arches of the epicycle. For example, if on the centre A (fig. 63, N° 2), with the distances AN, AR, circles shall be described cutting the diameter GPA in K, L, the libration GK, at the most distant extremity G, will be less than PL, at the nearest extremity P, though the arches GN, PR of the epicycle are equal. But, since it is now found that the orbit can no longer be considered as circular, and that the planet's distances from the sun, corresponding to these equal arches of the epicycle, are AO, AS, and not AK, AL; the more distant libration GO is rendered precisely equal to the least distant one PS. It is true, that still the middle part OS, is greater than either of the extremes GO, PS, though the arches GN, NR, PR, are equal; but Kepler supposed that he could assign a natural and sufficient cause of this exception.

As he had ascribed the revolution of a planet to the impulse of the sun's magnetical fibres, every way extended in straight lines from his centre, and continually carried about by his rotation on an axis; he endeavours also to shew, that the librations of a planet, through the versed sines of the arches of its epicycle, may be ascribed to a like mechanical cause; and that there is no necessity for supposing the planet endowed with animation or intelligence.

His theory on this subject principally consists in applying to the planets Gilbert's doctrine concerning the earth, and which he had also applied to the sun, that every one of them may be considered as a great magnet; and he endeavours to account for the librations now demonstrated to take place, and which were afterwards found to produce an elliptical orbit, by the constant parallelism of that axis, which joins the planet's magnetical poles, to a line perpendicular to the line of apsides.

Let EGH (fig. 76), be a great magnet representing the planet Mars, EF the magnetical axis which joins the attractive pole E, and the repulsive pole F, and whose positions, in all the various points C, D, K, T, of the orbit, are always in a line perpendicular to ABC; and let the axis EF be bisected by the perpendicular diameter GH. It is evident that, when, in the course of the planet's constant revolution, it is found in the apsis C, or T, it will have no tendency either to approach the sun, or to withdraw from him; because the perpendicular GH, coinciding with the line of apsides, passes through the centre of the sun; and the angles ACE, ACF, being equal, the attractive and repulsive forces are *in equilibrio* But when, in the descending semi-circle, the planet comes to the point K, and the magnetical axis is in the line KA, passing through the centre of the sun, the attractive force, because the attractive pole E is nearest the sun, will be at its maximum; and the portion of the diameter through which it librates (as OS in fig. 63), will be greatest; and in the opposite point the repulsive force will in like manner be at at its maximum; and in either case such force may be represented by the whole line EF. It may hence also appear why the extreme librations of the planet, (as GO, PS, in fig. 63), though equal in length, are made in unequal times; both, because magnets attract each other more forcibly at a less than at a greater distance, and because the planet, near the perihelion, is more rapidly impelled forwards by the solar fibres.

In all cases the attractive force will be as the sine of the true anomaly. Let the planet, for example, be in the point D, where the true anomaly is the angle CAD = RDH, and the attractive pole E be inclined, the angle EDA, or EDR, to the line DR, passing through the centre of the sun: and through R draw to EF the perpendicular RL. According to the principles of the balance, the attractive force will, in this case, be represented by the line FL, and the repulsive force by EL; and, since the whole line FE represents either, when at their maximum, if we cut off from FL the part FQ = EL, the line QL will represent the whole attractive force which acts upon the planet in D; or if DE, the half of FE, be supposed to measure this force when at its maximum, DL, the half of QL, will measure it when the planet is in D. But DL is = RM, the sine of the true anomaly RDH; and therefore the force by which the planet is attracted to the sun, in the descending semi-circle, and by which it is in like manner repelled from him in the ascending semi-circle of excentric anomaly, is always as the sine of the true anomaly.

But all observations testify that the libration itself, produced by either of these forces, is not as the sine of the true anomaly, but as the versed sine of the excentric anomaly; and therefore, if this should be also found to be the effect produced by magnetical force, there would be a probable method for accounting for the planetary librations, without having recourse to intelligence or animation. Certainly this seems to be very nearly the case; for, since the sine of every arch RH of true anomaly is the measure of the attractive force in that degree of it, the sum of these sines will be nearly the measure of all the attractions or impressions made by it, in all degrees of anomaly; and the libration is the effect of all these impressions joined together. But the sum of the sines of the arch RH, answering to all the degrees, or indefinitely small equal parts, into which it is divided, is to the sum of the sines of the quadrant ERH, divided into like parts, nearly as HM, the versed sine of RH, to the semidiameter HD. For example, if RH be = 15°, since the sum of the sines of 15° is = 208166, and the sum of the sines of 90° = 5789431, we shall have 5789431 : 208166 :: 100000 : 3594;

3594; now the versed sine of 15° to the same radius, is = 3047. In like manner the sum of 30 sines is = 792598; and 5789431 : 792598 :: 100000 : 13691; and the versed sine of 30° is = 13397: and still in like manner the sum of 60 sines is = 2908017; and 5789431 : 2908017 :: 100000 : 50000, which is also the versed of 60°. These ratios, indeed, differ considerably at the beginning of the quadrant; for 1745 is the sine of 1°; and 5789431 : 1745 :: 100000 : 30; whereas the versed sine of 1° is only 15. But it ought to be considered, that as the librations increase, the difference of the ratios becomes insensible.

This reasoning, indeed, only shews that the librations are nearly as the versed sines of the true anomalies, whereas the observations testify that they are as the versed sines of the excentric anomalies HDN = CBD; which in the first and fourth quadrants are always greater than the true. But it is evident that, by the substitution of the versed sines of excentric anomaly, the reasoning would become less exceptionable; for, as they are greater than the versed sines of true anomaly, so likewise are the last terms in all the above analogies.

It appears, then, from the example of a magnet, that those librations of a planet which produce an elliptical orbit, may with some probability be ascribed to a mechanical cause; without having recourse to the supposition of the planet's being endowed with intelligence; and Kepler also endeavours to prove, that this supposition would render the explication much more difficult, if not impossible.

In this explication the magnetical axis is supposed to be perfectly different from that of the planet's diurnal rotation; and the magnet to be an interior globe, within the exterior surface of the planet, and unaffected by its rotation.

B b.

That the excentric area GDA increases in the same ratio with the sum of the correspondent diametral distances, may easily appear from the principles employed in the fluxionary calculus; and these are the principles from which Kepler evidently derived the proposition.

Let GB (fig. 79), be the diameter of the circle GED, whose circumference is divided into any number (suppose 360), of indefinitely small equal parts GE, EF, EG, GH, HD, &c. so that the arches GE, GF, GG, GH, GD, of excentric anomaly increase uniformly by the common difference EG; the sine DC of the arch GD, will be equal to EG (cos. GE + cos. GF + cos. GG + cos. GH + cos. GD).

Let

Let EP, FQ, GR, HS, DC, be the sines, and BP, BQ, BR, BS, BC, the co-sines of the arches GE, GF, GG, GH, GD; from E, F, G, H, draw to the sines the perpendiculars EV, FX, GY, HZ, and join BE, BF, BG, BH, BD.

It is evident that DC = EP + FV + GX + HY + DZ.

But BE : BP :: GE : EP = GE.BP = GE. cos. GE, (for GE is sensibly = tan. GE)

and BF : BQ :: EF : FV = EF.BQ = GE. cos. GF,
BG : BR :: FG : GX = FG.BR = GE. cos. GG,
BH : BS :: GH : HY = GH.BS = GE. cos. GH,
BD : BC :: HD : DZ = AD.BC = GE. cos. GD.

Therefore DC = GE (cos. GE + cos. GF + cos. GG + cos. GH + cos. GD). The sines therefore of excentric anomaly increase in the ratio of the sum of as many co-sines of excentric anomaly, as there are points of equal division in the arches of which they are the sines.

But the circular area GDA consists of two parts; the sector GBD, and the triangular area ABD.

And the sum of the diametral distances consists also of two parts; the sum of the semi-diameters drawn from B, to the points of equal division in GD, and the sum of the lines BK, BL, BM, BN, BO.

Now the sector GBD increases precisely as the number of the semi-diameters.

And the triangle ABD increases as the sine DC; for it is equal to DC. $\frac{AB}{2}$, and it has now been proved that the sine DC increases as the sum of the co-sines of GD.

Therefore the triangle ABD increases as the sum of the co-sines of GD.

But the lines BK, BL, BM, BN, BO, are the co-sines corresponding to the same divisions of GD, multiplied into AB, and therefore their sum increases in the same ratio with the sum of the co-sines of GD; that is, in the same ratio with the triangle ABD.

Therefore the whole area GDA increases in the same ratio with the sum of the diametral distances corresponding to the equal divisions of GD.

The reason, therefore, was evident why the method of areas was so unsuccessful, while the circumferential distances were considered as the just distances. For the sum of any pair of them drawn from A to opposite points of the excentric near the extremities of the diameter GAT (fig. 78), is indeed nearly equal to the sum of two semi-diameters; but, the sum of AD, A*d*, drawn to the points D, *d*, more distant from G and T, is much greater; and at M and N, 90° from G and

G and T, the excess of the sum of AM + AN, above the sum of two semi-diameters, becomes the greatest possible. The excess therefore of any sum of circumferentials, above the sum of an equal number of semi-diameters, does not increase as the number of the lines summed, and much less as the sines of excentric anomaly increase; for the difference of such sines gradually diminishes, till towards 90° of anomaly it wholly vanishes; whereas, the excess now mentioned increases towards 90° with greatest rapidity. But the circular areas GDA increase, in the part GBD, precisely as the number of the lines summed; and, in the part ABD, precisely as the sines of excentric anomaly; and therefore cannot justly measure those sums of circumferentials which increase in ratios entirely different.

C c.

The reason of Kepler's doubts concerning his conclusion, that the arch of the ellipse ought to terminate in a point F of the ordinate DC, was that by first dividing the excentric into equal arches, and drawing ordinates to the axis GT from the points of division, the ellipse would be divided by these ordinates into unequal arches. But the object which seemed to be proposed in calculating the equations of any orbit, was to estimate and compare together the times in which equal arches of it were described; and therefore, though it was certain that all the distances of the planet from the point A, occupied by the sun, were diametral and not circumferential distances, he questioned if they should not terminate in a point Z of the diameter BD, rather than in a point F of the ordinate DC. It was considered, indeed, as an answer to this difficulty, that, even if the ellipse were divided into equal arches, the ratio between them and their times could not be obtained by the circular areas; for the times, taken together, of describing two such opposite equal arches at the apsides G and T, would as much exceed the times of describing them at K and *k*, at the distance of 90° from the apsides, as the greater axis GA + AT exceeds the less axis KB + B*k*; and therefore the times, taken together, of describing opposite arches, could not be rendered equal, except by diminishing the arches at the apsides, and increasing them in departing from the apsides. But, though this was the very effect produced in the method of division by ordinates; it was not a sufficient solution of the difficulty: and though this consideration made it evident, that the arches at the apsides ought to be the least, it did not make it evident that

that their precise terminations should be in the ordinates drawn through the points of equal division in the excentric.

Kepler therefore farther considered, that if the ellipse GKT should be divided into equal arches, and the elliptical areas GFA, GKA, GKTG, should be used to measure the sums of the distances corresponding to the points of equal division in GF, GFK, GKT, the same error would be committed in the ellipse, which was committed with respect to the circle, when, in deducing the solar equations, he used the circular areas instead of the sums of the circumferential distances. For, as there the areas of the circle were only the sums of the diametral distances, and less than the sum of the circumferential, so here his substitution of the elliptical areas for the sums of the distances of equal arches of the ellipse was employing also too small a measure of those sums; for it was substituting, for example, the distances YF, Y*f*, that is, FB*f*, in place of the greater distances AF + A*f*. Besides, the elliptical areas were unfit measures of the sums of such distances upon another account, that, to wit, they increased precisely in the same ratio with the circular areas; whereas, the increase of the excess of the distances from A, above the distances from the centre B, was made in a very different ratio.

At the beginning, indeed, G or T, of the quadrants of anomaly GK, T*k*, no error could be committed by substituting GBT for AG + AT; but, if at either end of it, K or *k*, we should substitute KB*k* for AK + A*k*, the errors would be KM, *k*N, and at their *maximum*; and the just distance would be to the error as BM to KM. If, therefore, the whole sum of the distances should be measured by the area of the ellipse, a standard thus erring in defect, the defect, when it came to be distributed among the particular distances of every equal arch, would render every one of them, that is, would exhibit the times of describing these equal arches as, too short. Even AG and AT themselves, that is, the times of describing such equal arches at the apsides, would be shortened; for, though they contributed their full quota to the composition of the area, they would not receive it back in the distribution.

But if the ellipse should be divided into unequal arches, by ordinates to the axis drawn through all the points of equal division in the excentric, the errors committed by using the areas of the ellipse, though a defective standard, would by the form of the areas be compleatly remedied. For the indefinitely small arches GD, GF, of the circle and the ellipse, are to each other, at the apsides, sensibly in the ratio of DC to FC, that is, of BM to BK, or of the real

distance

distance to its defective measure; and, at the end of the quadrants, the indefinitely small arches DM and FK are sensibly equal, in the same manner as at the apsides, the standard GBT was equal to AG + AT. As therefore the substitution of the areas shortened the distances at G and T, and added to their length at K and *k*, so now, when the elliptical arches are determined by ordinates drawn from the points of equal division in the circle, the arches of the apsides are shortened where the distances are shortened, and at K and *k* lengthened where the distances are lengthened; and thus the times and arches made to continue in their due proportion.

At the intermediate parts again of the quadrant, Kepler considered that the excess of FA + AF above FB*f*, increases rapidly from small beginnings, till at the angular distance of 45° from the apsides, the velocity of increase comes to its maximum; and from thence the velocity of increase is diminished, till at 90° it ceases to increase. Now the increase of the elliptical arches, determined by ordinates, observes the same rate of progress; for, at the apsides, the arch is to its increment as CF to FD; but as the arch is small, so is its increment; and at the end of the quadrant DM and FK, are nearly equal, and consequently the increments are again diminished. They must therefore, like the excesses, have varied most at 45°.

This argumentation, however, was neither geometrical, nor perfectly satisfactory; nor was it the foundation of Kepler's practice. It only served to confirm a conclusion previously established by experience.

D d.

Example. Let the given excentric anomaly be GBD = 10°.

The first part of the sum of the diametral distances corresponding to the equal divisions of GD, is the sum of the ten semi-diameters drawn to thess points of division, and = 1000000; and the second part is the sum of the ten co-sines of the excentric anomalies, which terminate in these points, multiplied into AB, and = 92108. Consequently the required sum of diametral distances amounts to 1092108. Considering then 36000000 as equivalent to the whole elliptical area, and 1092108 as equivalent to the particular elliptical area GFA, we have this analogy 36000000 : 1296000″, or the area of the circle in seconds, as 1092108 : 39316″ = GDA = 10° 55′ 16″.

But

But GDA is also $= GBD + ABD = GBD + DC. \frac{AB}{2}$ $= 10^\circ + 4632.$ sin. $10^\circ = 4632.17364.8 = 80443380$, in parts of which the area of the circle contains 31415926536, and as this whole area is equivalent to 1296000″, the area ABD will be equivalent to 3318″ = 55′ 18″. The whole area GDA therefore thus computed, is = 10° 55′ 18″, differing only by 2″ from the former result.

FINIS.

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