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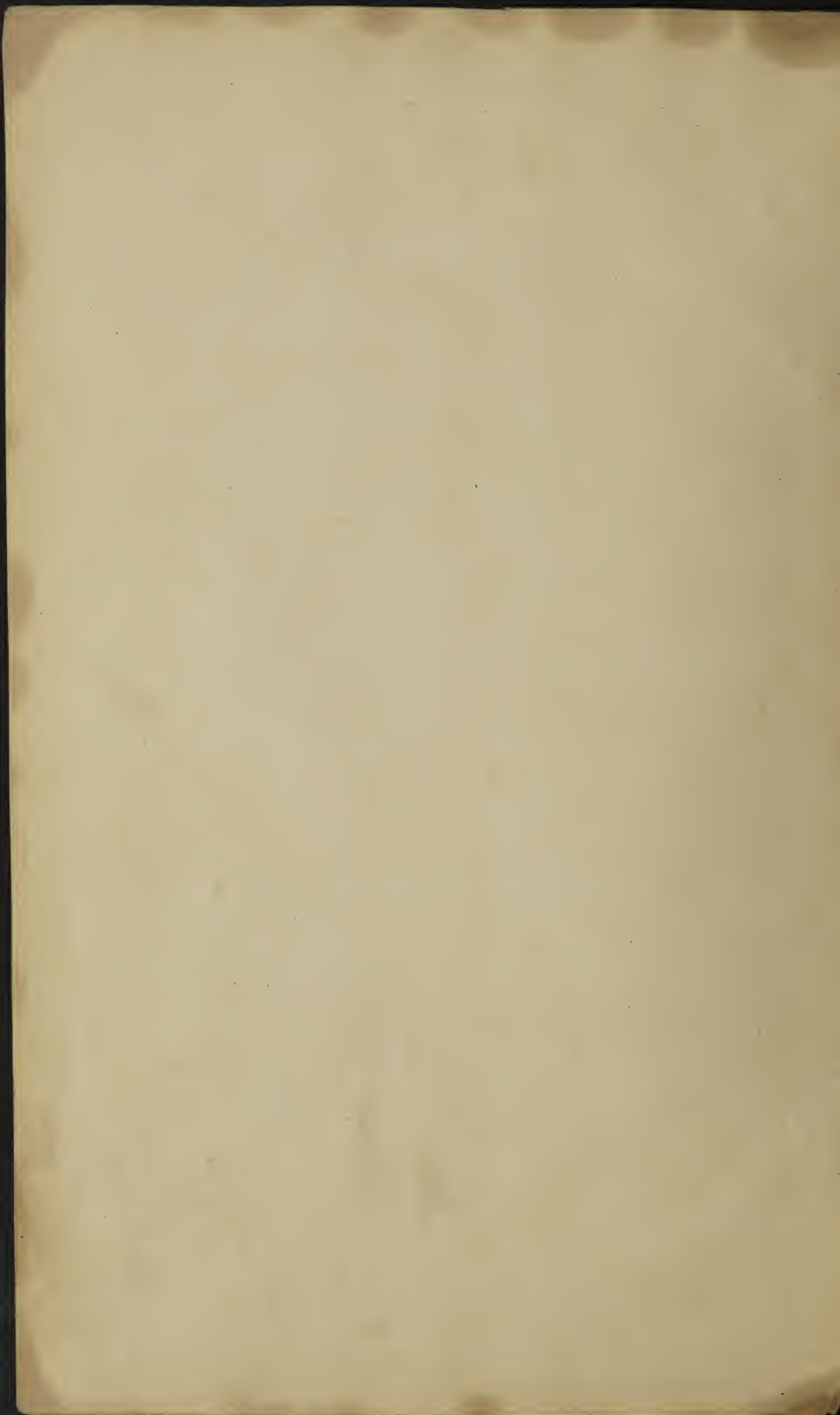
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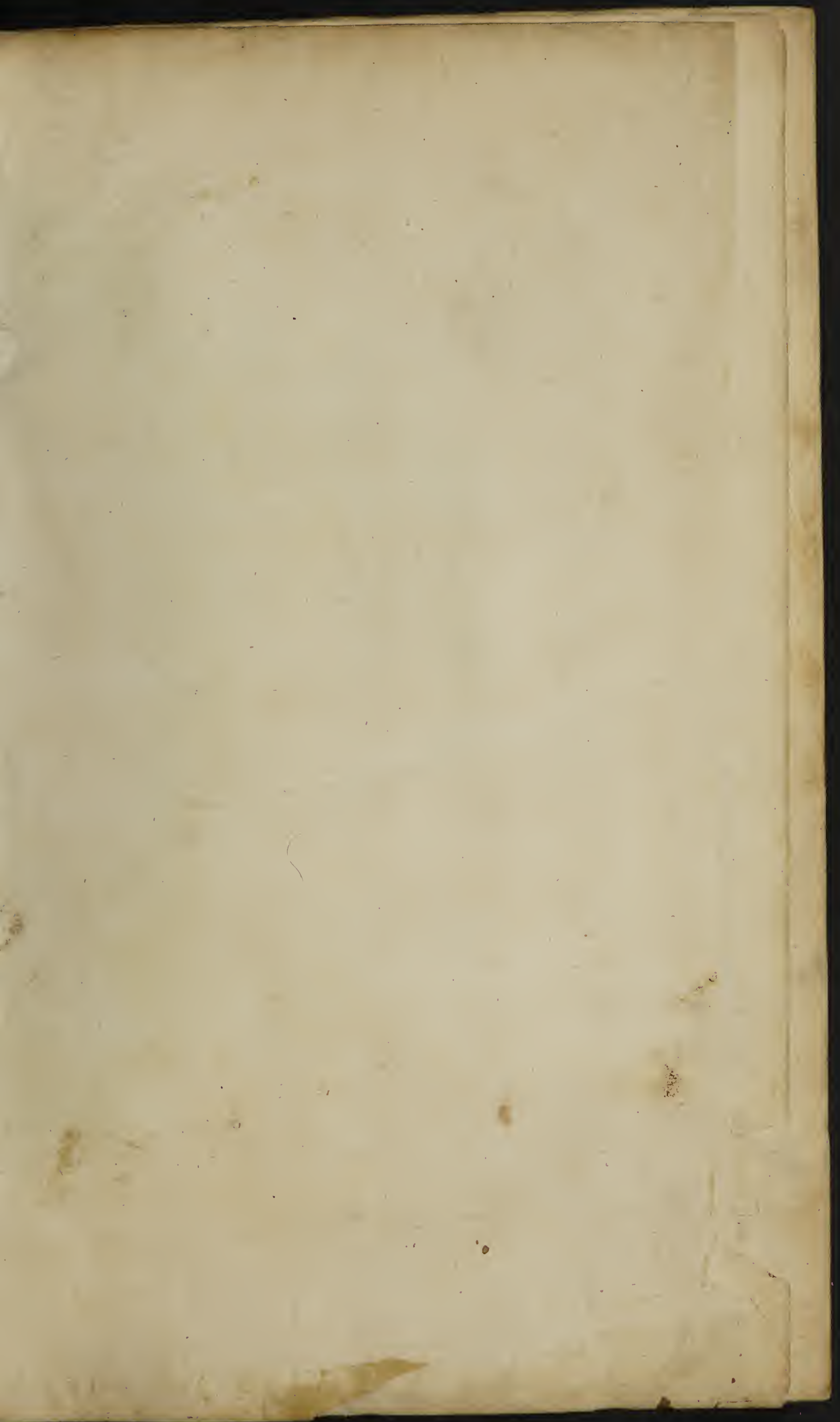
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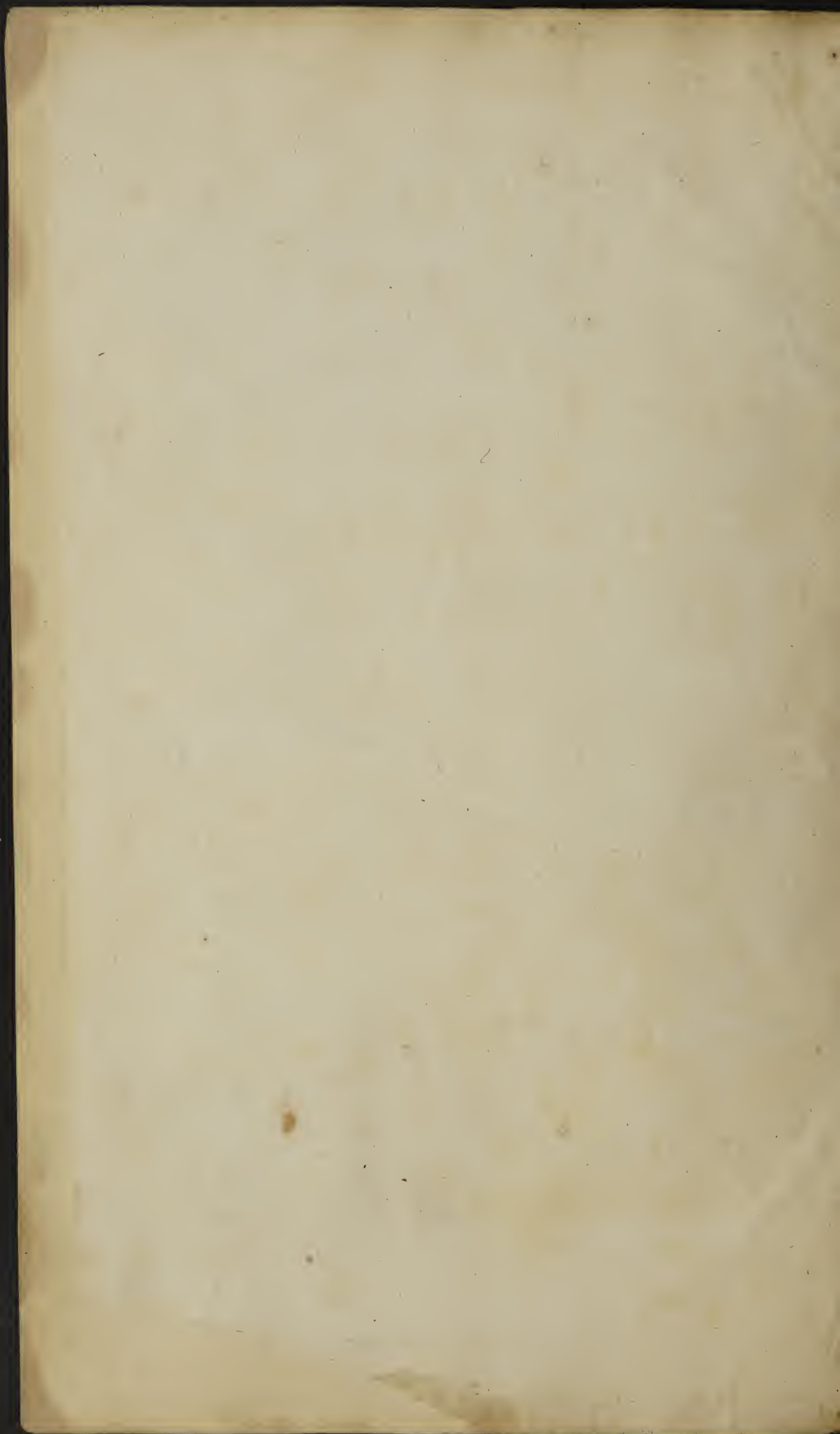
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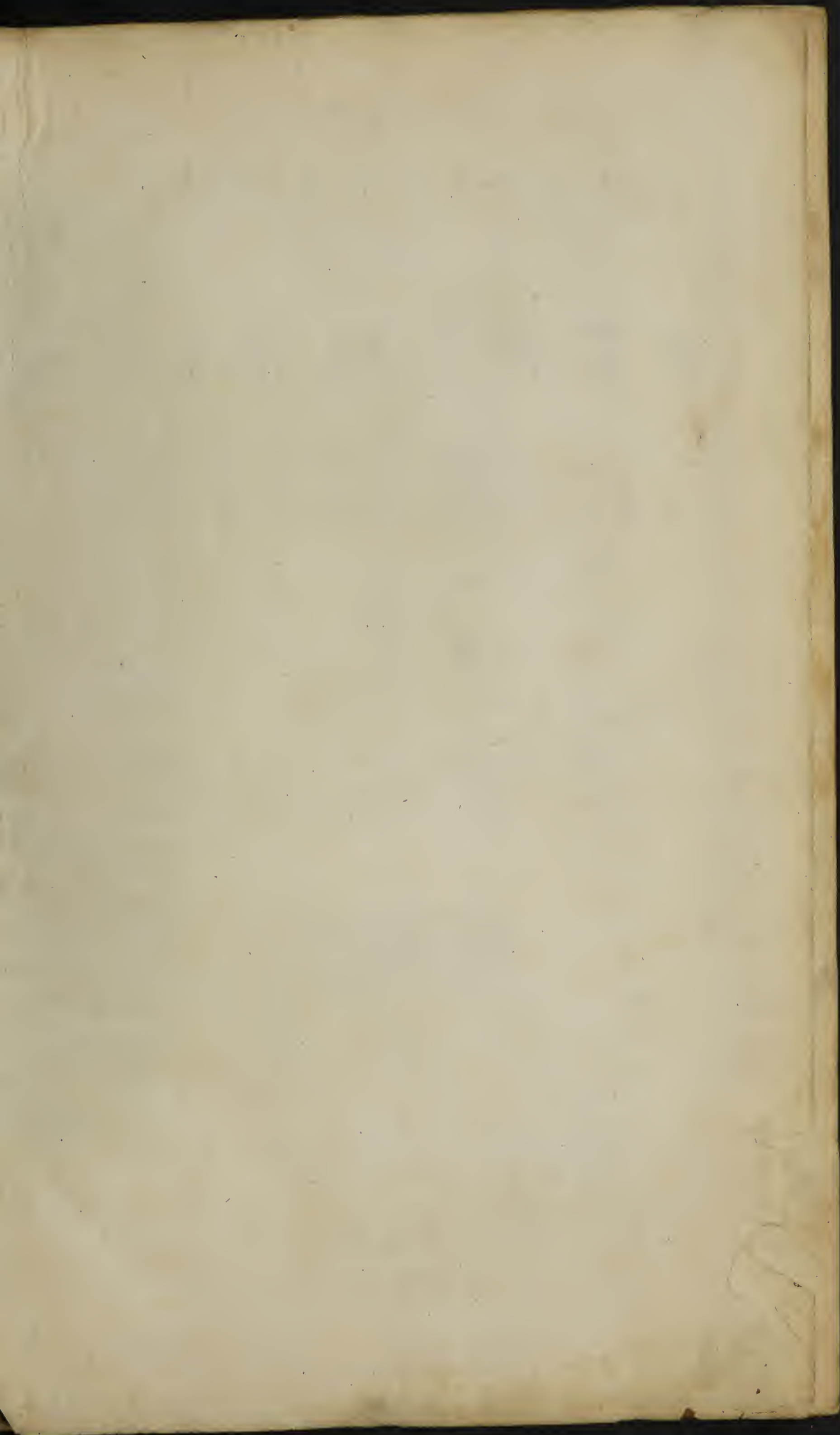
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In 2. Vol. 20
Feb. 12. 27 29
Vol. 2. Collat. & per led.
J. Wright

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THE
THIRD & FOURTH
BOOKS
OF THE
ELEMENTS
OF
ALGEBRA.

Compiled by
JOHN KERSEY.

————— *Si quid novisti rectius istis,
Candidus imperti; si non, his utere mecum.*



LONDON:

Printed by WILLIAM GODDID, for Thomas Passinger
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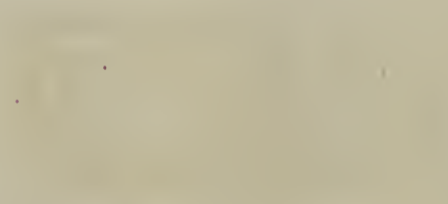
THE HISTORY OF THE

WELLS



By J. H. WELLS

Author of 'The Wells of the County of Wells, N. H.'



Published by the Author, Wells, N. H., 1884.



The C O N T E N T S of the Third Book.

THE Third Book is an Analysis in Species of the choicest of Diophantus's much admired Questions concerning Squares, Cubes, and right-angled Triangles in rational numbers; with other Questions of like nature. To which is also added a brief Exposition upon Monsieur Fermat's Analytical Invention, prefix'd to Monsieur Bachet's Comment upon Diophantus, Printed at Toloze, Ann. 1670. The number of Questions in the Third Book is 130.

The C O N T E N T S of the Fourth Book.

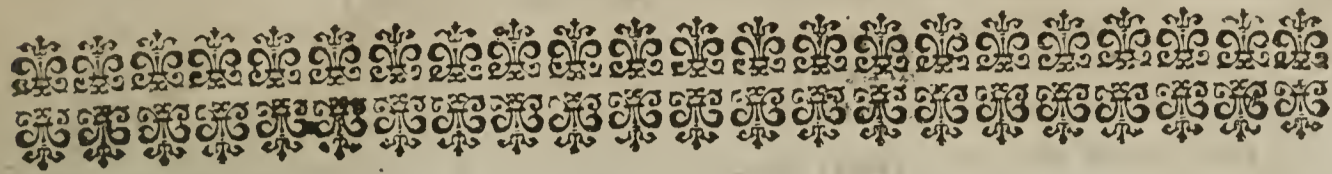
THE Fourth Book is an Introduction to Mathematical Resolution and Composition; where, the excellent use of the Algebraical or Analytical Art is evidently shewn, both in finding out the Solutions of Geometrical Plane Problems, (viz. of such whose Delineations require only the drawing of Right and Circular lines;) as also in discovering Synthetical Demonstrations from the Analytical steps, to prove the truth of those Solutions by a Series of Consequences deduced altogether from things really given or granted. All which are clearly expounded, and copiously exemplified both Geometrically and Arithmetically, by a choice Collection of useful and delightfull Problems. The Fourth Book is divided into Ten Chapters, the Contents whereof are these, viz.

C H A P.

1. The Explication of Characters, &c.
2. The Explication of Axioms.
3. The Explication of Definitions concerning the usual ways of arguing to deduce one Analogy from another.
4. Various Fundamental Theorems demonstrated.
5. A Collection of Canonical Geometrical Effections.
6. Algebraical Fractions Geometrically expounded.
7. } Four Classes of Examples of the Resolution and Composition of
8. } Plane Problems.
9. }
10. }

*Faults to be Corrected in the Third and Fourth Books,
before they be read.*

<i>Pag.</i>	<i>Lin.</i>	<i>Faults,</i>	<i>thus to be Corrected.</i>
7	27	part,	pair.
23	24	To the given difference 60,	The given difference is 60.
55	38	adding 2,	adding 3.
63	10	<i>Chapt.</i>	<i>Book.</i>
67	37	} upon the fourth <i>Book</i> ,	upon <i>Quest. 2. Book IV.</i>
71	48		
104	54	+ $\frac{728}{72}$,	+ $\frac{728}{729}$.
109	49	<u>16098918400</u> ,	<u>160989184000</u> .
214	4	right-angle,	right line.
221	2	greater,	lesser.
228	16	36,	26.
228	19	30,	40.
229	51	and the extremes of,	of the extremes and.



THE
ELEMENTS
OF THE
ALGEBRAICAL ART.

BOOK III.



Mong all the Writers upon the *Algebraical Art*, there hath none been hitherto known more antient, nor any more famous for shewing the admirable force of that Art in solving Arithmetical Problems, than *Diophantus* of *Alexandria*, who lived, (as Authors compute,) above thirteen hundred years ago. He wrote thirteen Books of *Arithmetick*, and one concerning *Multangular numbers*; but of those thirteen, six only are extant, which contain two hundred and eight Questions, many of which are so knotty and abstruse, that they can hardly be solved without the help of *Diophantus*

his peculiar Method; whose Speculations are so sublime, that where there seems to be an impossibility of finding out a single Answer to a Question, he shews how to find out innumerable Answers in rational (or ordinary) numbers. Now to give the ingenious Reader of these *Elements* a delightful Prospect of the rare Inventions of that Prince of Arithmeticians, I have with no small labour framed this Third Book, and therein explain'd the Resolutions of the hardest and choicest of his Questions, in the second, third, fourth, fifth and sixth Books of his *Arithmetick* before-mentioned; as also of divers other subtil Questions invented upon his grounds by *Vieta*, *Bachet*, (the best Commentator hitherto upon *Diophantus*,) *Fermat* and others; among which also divers Questions of my own are inserted, to wit, those which have no citation referring to any Author.

Note, That Δ stands for the word *Triangle*, and \square for a *Square number*; but as to the rest of the Characters used in this third Book, they have already been explain'd in *Chap. I. Book I.* of these *Elements*.

QUEST. I. (This is the 9th of the second Book of *Diophantus*.)

To divide a given square number into two Squares.

RESOLUTION I.

1. Let the square number given to be divided be } 16
2. The Root or side thereof is } 4
3. For the Root or side of the first of the two Squares sought put } a
4. Therefore the first Square is } aa
5. And consequently, (by the first and fourth steps,) the second } $16 - aa$
Square must be equal to }
6. Now let the side of the second Square be feigned to be . . . } $2a - 4$, or $4 - 2a$
7. Therefore the Square of the said feigned side is } $4aa - 16a + 16$
8. Which Square must be equal to $16 - aa$ in the fifth step, hence this following Equation
arise, viz.

$$4aa - 16a + 16 = 16 - aa;$$

A

9. From

9. From which Equation, after due Reduction, the side of the first Square will be made known, viz. } $a = \frac{16}{5}$
 10. And by the ninth and sixth steps, the side of the second Square will likewise be discovered, viz. } $2a - 4 = \frac{12}{5}$

So the sides of the two Squares sought are found $\frac{16}{5}$ and $\frac{12}{5}$; which will solve the Question; for the Square of $\frac{16}{5}$ is $\frac{256}{25}$, and the Square of $\frac{12}{5}$ is $\frac{144}{25}$, both which Squares added together make $\frac{400}{25}$, that is, 16; as was required.

Note. That which is most remarkable in the foregoing Resolution of *Diophantus*, is, his ingenious and peculiar way of feigning the side of a Square to be equated to $16 - aa$ in such manner, that after the Equation is duely reduced, the number represented by a will necessarily be Rational. Now because he makes great use of the like manner of feigning the sides of Squares to be equal to Algebraical quantities, in resolving divers hard Questions, (as will copiously appear in this third Book,) the Learner must endeavour to be very well acquainted with the said Method; for his ease therefore I shall explain it in the following Observations.

Observations upon Quest. 1.

1. Concerning the feigned side $2a - 4$ in the sixth step of the foregoing Resolution, it may be asked, why -4 , and not -5 , -6 , or some other number? to this I answer, There is a necessity that this number be alwayes the side of the Square given to be divided into two Squares; so *Diophantus* feigns the second side to be $2a - 4$, (4 being the side or square Root of 16 the given Square,) to the end that in the Square of $2a - 4$ there may be found the Absolute number 16 , to wit, the Square given to be divided; for then the Square of $2a - 4$ being equated to $16 - aa$, (as in the eighth step of the Resolution,) there will be found ∓ 16 on each part of the Equation, whence by subtracting 16 from each part, there ariseth $5aa = 16a$; and consequently each part of this Equation being divided by a , the Quotients give $5a = 16$, wherefore $a = \frac{16}{5}$.

2. One of the parts of the said feigned side of the second Square must (in this *Quest. 1.*) necessarily have the sign $-$ prefixt to it; so *Diophantus* feigns the said second side to be $2a - 4$, for if it were $2a + 4$, all the parts or terms of its Square would be Affirmative, and consequently no possible Equation would arise; as will easily appear by comparing the Square of $2a + 4$, that is, $4aa + 16a + 16$ to $16 - aa$, whence $5aa + 16a = 0$.

3. The Learner may also demand, why in the sixth step of the foregoing Resolution, the side of the second Square sought is feigned to be $2a - 4$, or $4 - 2a$, and not $a - 4$, or $4 - a$, viz. why is 2 , and not 1 or some other number prefixt to a ? to this I answer, If the side of the first Square sought be assumed or supposed to be a or $1a$, (as it is in the third step of the Resolution,) then the side of the second Square cannot be $a - 4$, or $4 - a$, as will be evident by a due process upon that supposition; for the Square of $a - 4$, or $4 - a$, that is, $aa - 8a + 16$ being equated to $16 - aa$, there will arise, after due Reduction, $a = 4$, and consequently, $a - 4$, or $4 - a$, which was put for the second Square, will be equal to nothing: The like absurdity will follow as often as the numbers prefixt to a in the feigned sides of the two Squares sought are equal to one another, viz. if the first side be feigned to be $5a$, and the second $5a - 4$, or $4 - 5a$; or if the first side be $8a$, and the second $8a - 4$, or $4 - 8a$, &c. from such suppositions a fruitless Equation will ensue, for the side of the second Square will be found equal to nothing. Now for the avoiding of such absurdity, the Learner may take this for a Rule, (the reason whereof will hereafter appear by *Observat. 1.* of the following *Resolution 2.* of this *Quest. 1.*) Let the numbers prefixt to a in the feigned sides of the two Squares sought be any two unequal numbers, viz. if a or $1a$ be put for the first side, the second may be $2a - 4$, or $3a - 4$, &c. Again, if we put $3a$ for the first side, the Square thereof will be $9aa$; and consequently because the Square given to be divided into two Squares is 16 , the second Square shall be equal to $16 - 9aa$, whose side we may feign to be $2a - 4$, or $4 - 2a$, the Square whereof is $4aa - 16a + 16$; this equated to the said $16 - 9aa$, gives after due Reduction $a = \frac{16}{13}$, therefore $3a$ which was put for the side of the first Square shall be $\frac{48}{13}$, and $4 - 2a$ which was put for the side of the second Square will be found $\frac{20}{13}$, and consequently, the two Squares sought shall be $\frac{2304}{169}$ and $\frac{400}{169}$, whose sum makes 16 , as the Question requires.

In which last Example (which is worthy of the Learner's observation) it happens, that in resolving the Positions, the second side is expounded by $4 - 2a$, not by $2a - 4$, although the

the Resolution be justly formed from either of them; for the Square $4aa - 16a + 16$, while a is unknown, may have for its side either $2a - 4$, or $4 - 2a$, and which so ever of these sides be feigned, the same Equation will arise to find out the number a , which, after it is discovered, is to be compared to such of those two feigned sides as will produce a number greater than nothing; so the number a being before found out to be $\frac{16}{13}$, it is manifest that $2a - 4$ is less than nothing, but $4 - 2a$ gives $\frac{20}{13}$, which is the true side of the second Square. So likewise in the Example of *Diophantus*, the side of the second Square cannot be expounded by $4 - 2a$, (although the value of a may be rightly found out from that supposition, as well as from $2a - 4$;) for a being found equal to $\frac{16}{5}$, the said $4 - 2a$ is less than nothing.

4. Here I shall recommend to the Learner one general Observation, viz. The principal scope in feigning the side of a Square to be equated to some Algebraick Quantity wherein the highest unknown Power is aa , must be to feign the said side in such manner, that after due Reduction, either the Absolute numbers may vanish, and consequently an Equation remain between some number of aa , and some number of a , whence by Division the number a will necessarily be Rational; or else, (as hereafter will fully appear in this Book,) that aa may vanish out of each part, and consequently an Equation remain between some number of a , and some Absolute number; hence also the number a will be Rational.

Having explain'd *Diophantus* his Resolution of *Quest. 1.* by Numeral Algebra, I shall in the next place resolve the same by Literal Algebra, whence divers useful Canons will be brought to light.

RESOLUTION 2. of *Quest. 1.* which is here repeated, viz.

To divide a given square number into two Squares.

- | | |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------|
| 1. For the side of the given Square put | d |
| 2. Therefore the said Square is | dd |
| 3. Take any two unequal numbers, suppose s the greater, and r the lesser, (which s and r are to be used instead of the numbers prefixt to a in the foregoing Resolution,) then for the side of the first Square sought put | $sa - d$, or, $d - sa$ |
| 4. And for the side of the second Square sought, put | ra |
| 5. Therefore, from the third step, the first Square is | $ssaa - 2sda + dd$ |
| 6. And from the fourth step, the second Square is | $rraa$ |
| 7. Therefore the summ of those Squares is | $ssaa + rraa - 2sda + dd$ |
| 8. Which summ must be equal to the given Square dd , hence this Equation ariseth, viz. | $ssaa + rraa - 2sda + dd = dd$ |
| 9. Which Equation, after due Reduction, gives | $a = \frac{2sd}{ss + rr}$ |
| 10. Therefore out of the ninth and third steps, the side of the first Square sought is now made known, for it is equal to | $\frac{2rsd}{ss + rr}$ |
| 11. And from the ninth and fourth steps the side of the second Square is also known, for it's equal to | $\frac{ssd - rrd}{ss + rr}$ |

Observations upon the preceding Resolution 2. of *Quest. 1.*

1. The eleventh step of the said Resolution discovers that the known numbers s and r must be unequal, to the end the difference of their Squares may be greater than nothing.

2. After the number a is made known, (as in the ninth step,) it will be manifest that the side of the second Square is to be expounded by $sa - d$, not by $d - sa$; for since a is found equal to $\frac{2sd}{ss + rr}$, (as appears by the ninth step of *Resolut. 2.*) it follows that $sa = \frac{2ssd}{ss + rr}$, and $d - sa = \frac{rrd - ssd}{ss + rr}$, which is less than nothing, for s is greater than r by supposition: But while a is unknown, the side of the second Square may be feigned $d - sa$ as well as $sa - d$, for each of these produceth the same Square $ssaa - 2sda + dd$.

3. The side of the first Square may be feigned sa , and the side of the second $ra - d$, or $d - ra$, for from these Positions the true sides of the two Squares sought will be found

the same as before are express'd in the tenth and eleventh steps of *Resolut.* 2. but in this latter way of feigning the sides, the side of the second Square will be expounded by $d - ra$, not by $ra - d$, for this will be found less than nothing.

4. The tenth and eleventh steps of the foregoing *Resolution* 2: give this

CANON.

Take any two unequal numbers; multiply severally the double of the Product of their multiplication, and the difference of their Squares by the side of the given Square; lastly, divide those Products severally by the sum of the Squares of the two numbers first taken, and the Quotients shall be the sides of the two Squares sought.

An Example in Numbers.

Let the side of the given Square be 4, then take two unequal numbers at pleasure, as 1 and 2; the double Product of their multiplication is 4, the difference of their Squares is 3, then by multiplying the said 4 and 3 severally by 4, (the side of the given Square,) the Products are 16 and 12; these divided severally by 5, (that is, by the sum of the Squares of 1 and 2 the numbers first taken,) give the Quotients $\frac{16}{5}$ and $\frac{12}{5}$, which are the sides of the two Squares sought; for the Squares of $\frac{16}{5}$ and $\frac{12}{5}$ added together make 16, which was given to be divided into two Squares.

5. For as much as (by *Prop.* 47. *Elem.* 1. *Euclid.*) when a Square is equal to two Squares, the sides of those three Squares will make a right-angled Triangle, the preceding *Quest.* 1. may be thus stated, *viz.*

A Rational number being given for the *Hypothensal* of a right-angled Triangle, to find Rational numbers to express the Base and Perpendicular, *viz.* the sides about the right-angle.

This may be solved by the preceding Canon; for if d be put to represent the given Hypothensal, and s and r any two unequal numbers, r being the lesser, these three following numbers will constitute a right-angled Triangle having d for its Hypothensal, *viz.*

Hypothensal,	Base,	Perpendicular.
d	$\frac{ssd - rrd}{ss + rr}$	$\frac{2rsd}{ss + rr}$

6. Moreover, if those three sides of a right-angled Triangle be severally multiplied by the Denominator $ss + rr$, the Products shall also be the sides of a right-angled Triangle, to wit, these following;

Hypoten.	Base,	Perp.
$ssd + rrd$	$ssd - rrd$	$2rsd$

7. And by dividing every one of the three sides last express'd, by their common Factor d , the Quotients will give these three following sides of a right-angled Triangle, *viz.*

Hypoten.	Base,	Perp.
$ss + rr$	$ss - rr$	$2rs$

8. Which three sides last above express'd are in words the following useful Canon, to form a right-angled Triangle in numbers by the help of any two unequal numbers given.

CANON.

Take any two unequal numbers, (suppose s the greater, and r the lesser,) then the sum of their Squares shall be the Hypothensal, the difference of the same Squares shall be one of the sides about the right-angle, and the double Product of the multiplication of the said two numbers, the other side.

The Proof of this Canon.

The Square of $ss + rr$ is	$s^4 + 2ssrr + r^4$,
The Square of $ss - rr$ is	$s^4 - 2ssrr + r^4$,
The Square of $2rs$ is	$+ 4ssrr$.

The first of those three Squares is manifestly equal to the sum of the other two, and therefore the sides of those three Squares, if they be express'd by numbers, shall be the measures of the sides of a right-angled Triangle.

An Example of the said Canon in Numbers.

Take two unequal numbers at pleasure, as 1 and 2; then the sum of their Squares is 5 for the Hypothensal, the difference of the Squares of the same two numbers is 3 for the Base, (that is, either of the sides about the right-angle,) and the double Product of the two numbers

numbers is 4 for the Perpendicular ; but that the numbers 5, 3, 4 may be taken for the measures of the sides of a right-angled Triangle is evident, for the Square of the first is equal to the Squares of the two latter.

9. Three Corollaries deduced from the last preceding Canon:

First, in every right-angled Triangle in such whole numbers which are Prime between themselves, the summ of the Hypothenufal ($ss - rr$) and ($2rs$) one of the sides about the right-angle is a square number, to wit, ($ss - rr + 2rs$) the Square of the summ of (s and r) the two numbers by which the said Triangle may be formed according to the last preceding Canon.

Secondly, the summ of the Hypothenufal ($ss - rr$) and ($ss - rr$) the other of the sides about the right-angle is the double of a square number, to wit, the double of (ss) the Square of (s) the greater of the two numbers by which the Triangle may be formed ; And the excess of the Hypothenufal ($ss - rr$) above the said side ($ss - rr$) is the double of the Square of (r) the lesser of the same two numbers ; therefore,

Thirdly, the three sides of any right-angled Triangle in such Rational whole numbers as are Prime between themselves being severally given, we may find two whole numbers by which the said Triangle may be formed according to the Canon in *Observat.* 8. As, for example, to find two numbers to form these three sides of a right-angled Triangle, to wit, 65, 33, 56, (which are Prime between themselves, for they have no common Divisor but Unity,) I add the Hypothenufal 65 to 33 and 56 severally, and it makes 98 and 121; which latter summ is a Square, and therefore (*per Coroll.* 1.) its Root 11 is the summ of the two numbers sought, and the first summ 98 is the double of the Square 49, whose Root 7 shall be the greater of the two numbers sought, (*per Coroll.* 2.) lastly, by subtracting 7 from 11, the Remainder 4 is the lesser number sought ; whence I conclude, that the right-angled Triangle proposed may be formed out of 7 and 4 ; for the summ of their Squares makes the Hypothenufal 65 ; the difference of the same Squares is 33 one of the sides about the right-angle, and the double Product of 7 and 4, to wit, 56 is the other side.

10. From the two preceding Canons (in *Observat.* 4, and 8.) another may be deduced to solve *Quest.* 1. viz. to divide a given square number into two Squares ; or a Rational number being given for the Hypothenufal of a right-angled Triangle, to find the Base and Perpendicular in Rational numbers.

CANON.

By the foregoing Canon in *Observat.* 8. let a right-angled Triangle be formed out of any two unequal numbers, and call this Triangle the first ; then it shall be, as the Hypothenufal of the said first Triangle is to its Base, so is the given Hypothenufal of a second Triangle desired to its Base ; and as the Hypothenufal of the first Triangle is to its Perpendicular, so is the Hypothenufal of the second to its Perpendicular.

An Example in Numbers.

Let it be required to find the Base and Perpendicular of a right-angled Triangle in numbers whose Hypothenufal shall be 7.

First, by the Canon in *Observat.* 8. I form a right-angled Triangle in numbers, as, $5, 3, 4$
 Then, by the Rule of Three, I find $4\frac{1}{2}$ for the desired Base, $5 \cdot 3 :: 7 \cdot 4\frac{1}{2}$
 thus, $5 \cdot 4 :: 7 \cdot 5\frac{1}{2}$
 Likewise $5\frac{1}{2}$ for the desired Perpendicular, thus, $5 \cdot 4 :: 7 \cdot 5\frac{1}{2}$
 Therefore 7, $4\frac{1}{2}$, $5\frac{1}{2}$ will constitute a right-angled Triangle whose Hypothenufal is 7, as was desired.

11. After the same manner, as many right-angled Triangles in numbers as shall be desired may be found out, which shall have one common Hypothenufal given : As, for example, if three right-angled Triangles in Rational numbers be desired, that 2 may be a Hypothenufal to every one of them, they may be found out thus ;

First, by the Canon in the foregoing *Observat.* 8. let three right-angled Triangles be formed, suppose these, $5, 3, 4$
 $13, 5, 12$
 $17, 15, 8$
 Then,

Then by the Rule of Three, the Bases and Perpendiculars of the three right-angled Triangles sought may be found out thus;

$$\begin{array}{l}
 \text{I. } \left\{ \begin{array}{l} 5 \cdot 3 \text{ :: } 2 \cdot 1\frac{1}{5} \text{ (Base.)} \\ 5 \cdot 4 \text{ :: } 2 \cdot 1\frac{3}{5} \text{ (Perp.)} \end{array} \right. \\
 \text{II. } \left\{ \begin{array}{l} 13 \cdot 5 \text{ :: } 2 \cdot 1\frac{0}{13} \text{ (Base.)} \\ 13 \cdot 12 \text{ :: } 2 \cdot 1\frac{1}{13} \text{ (Perp.)} \end{array} \right. \\
 \text{III. } \left\{ \begin{array}{l} 17 \cdot 15 \text{ :: } 2 \cdot 1\frac{3}{17} \text{ (Base.)} \\ 17 \cdot 8 \text{ :: } 2 \cdot 1\frac{6}{17} \text{ (Perp.)} \end{array} \right.
 \end{array}$$

Whence the desired sides of the three right-angled Triangles having 2 for a common Hypothenufal are found to be these, viz.

Hypoth.	Bases.	Perp.
2	$1\frac{1}{5}$	$1\frac{3}{5}$
2	$1\frac{0}{13}$	$1\frac{1}{13}$
2	$1\frac{3}{17}$	$1\frac{6}{17}$

12. But note well, that in the search of the Triangles last mentioned, the preparatory right-angled Triangles first found out by the Canon in the preceding *Observat.* 8. must not be like, (that is, such as have proportional Sides,) for it will not be difficult to apprehend, that if from them, other Triangles be deduced by the Rule of Three in such manner as before hath been shewn, there will be but one right-angled Triangle found out, when many are desired to have a common Hypothenufal: That your labour therefore may not be in vain, the preparatory right-angled Triangles must be unlike; to which end they must be formed from pairs of numbers expressing different Reasons, and such, that the two numbers by which any one of the preparatory Triangles is formed, must not be in such proportion to one another as the sum is to the difference of two numbers by which any other of the preparatory Triangles is formed. As, for example, if a right-angled Triangle be formed from 1 and 2, then another right-angled Triangle must not be formed from 2 and 4, 3 and 6, &c. because each of these pairs of numbers expressing the same Reason as 1 and 2 will produce a right-angled Triangle like to the first; nor from 3 and 1, 6 and 2, &c. because 3 having such proportion to 1, likewise 6 to 2, as the sum of 1 and 2 to their difference, those pairs also will produce right-angled Triangles like to the first. But that two right-angled Triangles formed from pairs of numbers expressing the same Reason, or from two such pairs, that one number of the one pair hath such proportion to its yoak-fellow, as the sum of the two numbers of the other pair hath to their difference, are like, I prove thus;

First, let a right-angled Triangle be formed from two numbers s and r , so the three sides will be these, viz.	} Hyp. Base, Perp.
Then let a second right-angled Triangle be formed from ds and dr , which have the same proportion to one another as s and r ; so the three sides will be these, viz.	} $ss \mp rr, ss - rr, 2sr$
Again, let a third right-angled Triangle be formed from $s \mp r$ and $s - r$, viz. the sum and difference of the two numbers by which the first Triangle was formed, so the three sides will be these, viz. . . .	} $ddss \mp ddr, ddss - ddr, 2dds$
	} $2ss \mp 2rr, 4sr, 2ss - 2rr$

Now I say, that the second Triangle is like to the first, for the sides of the second are the Products of the sides of the first multiplied by the common Factor dd . The third Triangle is also like to the first, for the sides of the third are the doubles of the sides of the first, and consequently Proportionals to them, but in this order, viz. As the Hypothenufal of the first is to its Base, so is the Hypothenufal of the third to its Perpendicular; and, As the Hypothenufal of the first is to its Perpendicular, so is the Hypothenufal of the third to its Base.

13. By the help of the preceding Canon in *Observat.* 8. as many right-angled Triangles in whole numbers as shall be desired, and which shall have a common Hypothenufal, may be found out in manner following, viz.

Let it be required to find out three right-angled Triangles in whole numbers, which shall have one common Hypothenufal.

First,

First, by the Canon in the foregoing *Observat.* 8. with respect }
 also to the *Note* in the last preceding Observation, let three unlike }
 right-angled Triangles be formed, suppose these, }
 5 , 3 , 4
 13 , 5 , 12
 17 , 15 , 8

Secondly, multiply severally the three sides of the first Triangle }
 5, 3, 4, by 221, that is, the Product of the second and third Hy- }
 pothenusals 13 and 17; so the three Products shall be the sides }
 of a right-angled Triangle, to wit, }
 1105, 663, 884

Thirdly, multiply severally the three sides of the second right- }
 angled Triangle 13, 5, 12, by 85, that is, the Product of the first }
 and third Hypothenusals 5 and 17; so the three Products shall be }
 the sides of this right-angled Triangle, viz. }
 1105, 425, 1020

Lastly, multiply severally the three sides of the third right- }
 angled Triangle 17, 15, 8, by 65, that is, the Product of the first and }
 second Hypothenusals 5 and 13; so the three Products shall be also }
 the sides of a right-angled Triangle; to wit, }
 1105, 975, 520

From the premisses it is manifest that three right-angled Triangles are found out in whole numbers, having 1105 for a common Hypothenusal, and by the same Method you may find out as many as you please.

14. But the smaller the numbers are that express the sides of those preparatory right-angled Triangles the better, and therefore I think it not amiss in this place to shew, how to find out all the unlike right-angled Triangles in whole numbers orderly enumerated, according as their Hypothenusals increase in greatness, so, as that the greatest Hypothenusal may not exceed a given number, suppose 180: To which end,

First, I extract the square Root of 180, and find it falls between 13 and 14, and consequently a right-angled Triangle formed from 14 and 1, will have its Hypothenusal greater than 180; therefore all the pairs of whole numbers, which have the greater number of each part, either 14 or greater than 14, will be unfit for our present search.

Secondly, I subtract 169 the Square of the said 13 from 180 the given limit, and the Remainder is 11, whose square Root falls between 3 and 4; whence 'tis evident that a right-angled Triangle formed from 13 and 4 will have a Hypothenusal greater than 180, but 13 and 3 will give an Hypothenusal less than 180; and therefore I proceed to make an orderly choice of pairs of whole numbers, from the first pair 2 and 1, until I come to 13 and 3 inclusive, and no farther, in this manner, viz.

Thirdly, I write in the first Column of the following Table a Series of whole numbers proceeding from 1, according to the natural order of numbers, as, 1, 2, 3, 4, 5, &c. then at the top of the second Column I set 2 and 1 for the first pair, that done, I combine every number following or standing underneath 2 in the first Column, with every one of the numbers that stands above such following number, except in these two Cases, viz. First, when two numbers so combined are such, that their sum and difference have the same proportion to one another as the two numbers of any pair already set in Column 2. then the two numbers so combined are to be cast out of Column 2. As, for example, because the sum of 3 and 1, to wit, 4, is to their difference 2, as 2 to 1, which 2 and 1 make the first pair already set in Column 2; I omit the writing of the pair 3 and 1 in the second Column: And for the same Reason the pair 5 and 1 is not inserted in the second Column; for the sum of 5 and 1, to wit, 6, is to their difference 4, as 3 to 2, which 3 and 2 made the second pair before written in Column 2. and in like manner all other pairs causing that effect are to be excluded out of the second Column. Again, when two numbers combined as aforesaid happen to be in the same proportion as the two numbers of any pair already set in the second Column, then also the two numbers so combined are to be excluded out of the said Column 2; so 4 and 2 having the same Reason as the first pair 2 and 1, are not inserted in the second Column: the reason of excluding all pairs in those two Cases is, for that they would produce right-angled Triangles like to others before produced, which is contrary to the import of the Proposition. So at length I find only thirty-two pairs of numbers that are fit to be inserted in the said second Column.

Fourthly, from every one of those thirty-two pairs of numbers in the second Column, (the last of which pairs is 13 and 2,) I form a right-angled Triangle (by the Canon in the foregoing *Observat.* 8.) and insert those Triangles into Column 3, among which I find five, to wit, those formed from the pairs 10 and 9; 11 and 8; 11 and 10; 12 and 7;

12 and 7; and 12 and 11, whose Hypothenusals exceed 180 the prescribed limit, and therefore I cast away those five Triangles, and transferr the rest, which are 27 in multitude, into the fourth Column, in such order as the Hypothenusals do increase in greatnes. So 27 unlike right-angled Triangles are found out, which are all that can be given in whole numbers, so as that the greatest Hypothenusal may not exceed 180, as was required. But for further illustration of the premisses view the following Table.

A Table whose fourth Column contains 27 unlike right-angled Triangles in numbers, orderly enumerated according as their Hypothenusals increase in greatnes.

		H . B . P	H . B . P	H . B . P	
1	2, 1	5 . 3 . 4	5 . 3 . 4	5 . 3 . 4	8
2	3, 2	13 . 5 . 12	13 . 5 . 12	13 . 5 . 12	12
3	4, 1	17 . 15 . 8	17 . 15 . 8		
4	4, 3	25 . 7 . 24	25 . 7 . 24	25 . 7 . 24	16
5	5, 2	29 . 21 . 20	29 . 21 . 20		
6	5, 4	41 . 9 . 40	37 . 35 . 12	41 . 9 . 40	20
7	6, 1	37 . 35 . 12	41 . 9 . 40		
8	6, 5	61 . 11 . 60	53 . 45 . 28	61 . 11 . 60	24
9	7, 2	53 . 45 . 28	61 . 11 . 60		
10	7, 4	65 . 33 . 56	65 . 33 . 56		
11	7, 6	85 . 13 . 84	65 . 63 . 16	85 . 13 . 84	28
12	8, 1	65 . 63 . 16	73 . 55 . 48		
13	8, 3	73 . 55 . 48	85 . 13 . 84		
14	8, 5	89 . 39 . 80	85 . 77 . 36		
15	8, 7	113 . 15 . 112	89 . 39 . 80	113 . 15 . 112	32
16	9, 2	85 . 77 . 36	97 . 65 . 72		
17	9, 4	97 . 65 . 72	101 . 99 . 20		
18	9, 8	145 . 17 . 144	109 . 91 . 60	145 . 17 . 144	36
19	10, 1	101 . 99 . 20	113 . 15 . 112		
20	10, 3	109 . 91 . 60	125 . 117 . 44		
21	10, 7	149 . 51 . 140	137 . 105 . 88		
22	10, 9	181 . 19 . 180	145 . 17 . 144	181 . 19 . 180	40
23	11, 2	125 . 117 . 44	145 . 143 . 24		
24	11, 4	137 . 105 . 88	149 . 51 . 140		
25	11, 6	157 . 85 . 132	157 . 85 . 132		
26	11, 8	185 . 57 . 176	169 . 119 . 120		
27	11, 10	221 . 21 . 220	173 . 165 . 52	221 . 21 . 220	44
28	12, 1	145 . 143 . 24			
29	12, 5	169 . 119 . 120			
30	12, 7	193 . 95 . 168			
31	12, 11	265 . 23 . 264		265 . 23 . 264	
32	13, 2	173 . 165 . 52			

15. By inspecting the preceding Table we may perceive that the unlike right-angled Triangles in Column 5. which are formed from 2 and 1; 3 and 2; 4 and 3; 5 and 4; &c. viz. from pairs of such whole numbers as differ by Unity, have these properties, namely;

First, their Bases 3, 5, 7, 9, 11, 13, &c. which you see standing under B in the fifth Column are in an Arithmetical Progression proceeding from the Base 3 of the Primitive right-angled Triangle 5, 3, 4 by the common difference 2.

Secondly, if an Arithmetical Progression be formed from 8 as the first and least Term, and the common difference of the Terms be 4; as this Progression 8, 12, 16, 20, &c. (which is placed in the last Column of the Table,) then 8 the first Term added to the first Hypothenusal 5, makes the second Hypothenusal 13 standing under H in the fifth Column; also 12 the second Term of the same Progression added to 13 the second Hypothenusal, gives 25 the third Hypothenusal in the same Column; and 16 the third Term added to 25 the third Hypothenusal, gives 41 the fourth Hypothenusal, and so forwards continually.

continually. In like manner, 8 the first Term of the same Progression added to 4 the first Perpendicular, gives 12 the second Perpendicular, standing under P in the said fifth Column; also 12 the second Term added to 12 the second Perpendicular, gives 24 the third Perpendicular; and 16 the third Term added to 24 the third Perpendicular, makes 40 the fourth Perpendicular; and so forwards perpetually. So that by the help of the Primitive right-angled Triangle 5, 3, 4, and the said Progression 8, 12, 16, 20, 24, &c. innumerable unlike right-angled Triangles may be found out by Addition only.

Thirdly, the difference between the Hypothensal and Perpendicular of every one of the said Triangles in Column 5. is Unity.

Fourthly, the Base is equal to the summ of the two numbers forming the Triangle.

Fifthly, the summ of the Hypothensal and Perpendicular is a Square, whose side is equal to the Base, or summ of the two numbers forming the Triangle; therefore,

Sixthly, if the summ of the Hypothensal and Perpendicular be multiplied into the Base, the Product shall be a Cube, whose side is equal to the Base.

Seventhly, the difference of the Hypothensal and Base is equal to the double of the Square of the lesser of the two numbers forming the Triangle.

The certainty of all the said Properties will be apparent, if you form right-angled Triangles from these following pairs of numbers, and compare those Triangles to one another, according to the import of the said Properties:

$$\begin{array}{l}
 a, a+1, \text{ the first pair;} \\
 a+1, a+2, \text{ the second pair;}
 \end{array}
 \parallel
 \begin{array}{l}
 a+2, a+3, \text{ the third pair;} \\
 a+3, a+4, \text{ the fourth pair, \&c.}
 \end{array}$$

QUEST. 2. (Quest. 10. Lib. 2. Diophant.)

To divide (13) a number given, which is compos'd of two Squares, (9 and 4,) into two other Squares.

RESOLUTION 1.

1. The side or square Root of 9 the greater Square given is } 3
2. The side of the lesser Square 4 is } 2
3. Let the side of the first of the two Squares sought be } $a+2$
- assumed to be }
4. And let the side of the second Square sought be feigned } $2a-3$, or, $3-2a$
5. Therefore from the third step the first Square desired is } $aa+4a+4$
6. And from the fourth step the second Square sought is } $4aa-12a+9$
7. Therefore the summ of the two Squares sought is } $5aa-8a+13$
8. Which summ last express't must be equal to the given num- } $5aa-8a+13 = 13$
- ber 13, hence this Equation ariseth, viz. }
9. And that Equation, after due Reduction, gives } $a = \frac{8}{5}$
10. Therefore from the ninth and third steps, the side of } $\frac{18}{5}$
- the first Square sought is made known, viz. }
11. And from the ninth and fourth steps, the side of the } $\frac{1}{5}$
- second Square sought is likewise discovered, viz. . . . }

So the sides of the two Squares sought are found $\frac{18}{5}$ and $\frac{1}{5}$; for $\frac{18}{5}$ the Square of $\frac{18}{5}$, added to $\frac{1}{5}$ the Square of $\frac{1}{5}$, makes $\frac{32}{5}$, that is, 13; as was required.

This Question is of the same nature with the foregoing, and deserves to be ranked among the most excellent Problems; for it affords divers admirable Canons concerning the construction of Right-angled Triangles, and is of great use for the understanding of many of Diophantus's Questions, especially in his fifth Book; I shall therefore first explain the preceding Numeral Resolution of the Question, and afterwards resolve the same by Literal Algebra.

Observations upon Quest. 2.

1. It is evident by the foregoing Resolution of Diophantus, That after $a+2$ and $2a-3$, or $3-2a$ are feigned to be the sides of the two Squares sought, the summ of those Squares, that is, $5aa-8a+13$, is equated to the given number 13. viz. $5aa-8a+13 = 13$; which Equation, if there were not the same Absolute number 13 in each part, could not be reduced to an Equation between some number of aa and some number of a , and consequently the number a would not be Rational, unless by meer chance:

Whence then comes it to pass, that the same Absolute number 13 is found in each part of the said Equation? If the Operation be well examin'd, it will appear that the numbers 2 and 3 in the feigned sides of the two Squares sought are the sides of the two given Squares 4 and 9; which 2 and 3 are the only numbers that can be used in the said feigned sides, to cause the number 13 to be found in the summ of their Squares.

2. As to the Signs to be prefixt before the given sides 2 and 3 in the feigned sides of the two Squares sought, they must necessarily be either both —, or one of them +, and the other —, to the end that in the summ of the feigned Squares there may be some number of a with the sign — prefixt; whence it will follow that the said number of a may be transferr'd to the other part of the Equation with the sign +, and then the Absolute numbers vanishing by Subtraction, because they are one and the same number as hath been shewn in the preceding *Observat.* 1. there will remain an Equation between some number of aa and some number of a ; whence by due Division the number a will be Rational.

3. The numbers to be prefixt before a in the feigned sides of the two Squares sought, may be variously chosen according to divers particular Rules that might be given, among which I shall recommend but two to the Learner's practice: The first Rule is this;

Let two unequal numbers be taken to be prefixt before a in the feigned sides, but with this Caution, *viz.* That the greater of the two numbers taken may not have the same proportion to the lesser as the summ of the sides of the two Squares given in the Question hath to their difference: As, if the two Squares given be 4 and 9, whose sides are 2 and 3, the greater of the two numbers taken must not be to the lesser as 5 to 1, because 5 is the summ, and 1 the difference of the said 2 and 3. Suppose therefore that 5 and 3 be taken; then let the first feigned side be $3a + 2$, (3 being the lesser of the said two numbers taken, and 2 the lesser of the sides of the two Squares given,) and let the second feigned side be $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers taken, and 3 the side of the greater Square given:) Now if from those feigned sides the Operation be prosecuted like as in the preceding Resolution of *Quest.* 2. an Equation will rightly ensue to find out two Squares different from those given, but such as being added together shall make the same summ as those given.

The second Rule is this; Let two unequal numbers be taken with this Caution, *viz.* That they be not in the same Reason (or Proportion) as the sides of the two Squares given: As, if the two Squares given in the Question be 9 and 4, whose sides are 3 and 2, then the two numbers taken must not be 3 and 2, 6 and 4, 9 and 6, nor any numbers in the same Reason: Suppose therefore that 5 and 4 be chosen; then for the side of the first Square sought put $4a - 2$, or $2 - 4a$, (4 being the lesser of the said two numbers chosen, and 2 the lesser of the sides of the two Squares given,) and for the side of the second Square sought put $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers before chosen, and 3 the greater of the sides of the two Squares given;) then if from the said feigned sides the Operation be prosecuted like as in the foregoing Resolution of *Quest.* 2. an Equation will ensue, to find out two Squares different from those given, but such as being added together shall make the same summ as those given. The reason of these two Cautions will hereafter appear.

The preceding Observations may suffice for explication of the Resolution of *Quest.* 2. by Numeral *Algebra*; I shall in the next place shew how to resolve the same by Literal *Algebra*, and among various ways that might be used, I shall chuse but two, which correspond with the Rules before given in *Observat.* 3. and do produce divers excellent Canons.

RESOLUTION 2. of *Quest.* 2. which is here repeated, *viz.*

To divide a number given which is compos'd of two known Squares, into two other Squares.

1. For the side of the greater Square given, put . . . $\succ d$
2. And for the side of the lesser Square given, put . . . $\succ b$
3. Therefore the greater Square is . . . $\succ dd$
4. And the lesser Square is . . . $\succ bb$
5. Take two unequal numbers, s the greater, and r the lesser, with this Caution, *viz.* that s be not in such proportion to r , as $d + b$ to $d - b$; which numbers s and r are to be used instead of the numbers that were prefixt before the unknown number a in the

the foregoing Numeral Resolution of this Question; and the reason of the Caution will be shewn in the sixteenth step of this Resolution.

6. For the side of the first of the two Squares sought, }
 put } $ra + b$
 7. And for the side of the second Square sought put } $sa - d$, or, $d - sa$
 8. Therefore from the sixth step the first Square }
 sought is } $r^2aa - 2rba - bb$
 9. And from the seventh step the second Square }
 sought is } $ssaa - 2sda - dd$
 10. Therefore the summ of those two Squares is } $ssaa - r^2aa - 2rba - 2sda - bb - dd$
 11. But the said summ must be equal to $dd + bb$ the summ of the two Squares given in the Question, and before express'd in the third and fourth steps; hence the following Equation ariseth, viz.

$$ssaa - r^2aa - 2rba - 2sda - bb - dd = bb + dd.$$

12. Which Equation, after due Reduction, gives } $a = \frac{2sd - 2rb}{ss - rr}$

13. Therefore from the twelfth and sixth steps, the side of the first Square sought is now made known, and found equal to this following Quantity,

$$\frac{2rsd + ssb - rrb}{ss + rr}$$

14. And from the twelfth and seventh steps, the side of the second Square sought is likewise known, and found equal to

$$\frac{ssd - rrd - 2rsb}{ss + rr}, \text{ or, } \frac{2rsb + rrd - ssd}{ss + rr};$$

That is to say, The former of those two Quantities express'd Fraction-wise shall be the side of the second Square when $ssd - rrd$ is greater than $2rsb$, but the latter of those Quantities shall be the said side when $ssd - rrd$ is less than $2rsb$. For if $ssd - rrd$ be greater than $2rsb$, then by subtracting $2rsb$ from $ssd - rrd$, the Remainder is the same with the Numerator of the first of the two Fractions above express'd; but if $2rsb$ be greater than $ssd - rrd$, then by subtracting $ssd - rrd$ from $2rsb$, the Remainder is the same with the Numerator of the latter of the said Fractions; therefore the side of the second Square may be express'd thus,

$$\frac{ssd - rrd \text{ or } 2rsb}{ss + rr}$$

That is to say, If the difference between $ssd - rrd$ and $2rsb$ be divided by $ss + rr$, the Quotient shall be the side of the second Square sought.

From the premisses ariseth this following

CANON 1.

15. Take two unequal numbers, with this Caution, viz. That the greater may not have the same proportion to the lesser, as the summ of the sides of the two Squares given hath to the difference of the same sides: Multiply the double Product of the multiplication of those two numbers first taken by each of the said two sides given, and reserve the Products; multiply also the difference of the Squares of the said two numbers first taken by each of the said two sides given, and reserve these Products; then add the greater of the two first reserved Products to the lesser of the two latter, and reserve the summ for a Dividend; take also the difference between the lesser of the two first Products and the greater of the two latter for a second Dividend; lastly, divide severally the said Dividends by the summ of the Squares of the two numbers first taken, so shall the Quotients be the sides of the two Squares sought.

Example 1. Where the number given is compos'd of two unequal Squares.

Let it be required to divide 13 which is compos'd of two Squares, 9 and 4, into two other Squares.

- The side of the greater Square given is } 3
 The side of the lesser Square given is } 2
 Take two unequal numbers, with respect to the Caution in the Canon, } 2 and 1
 as these, }
 Then by using those four numbers as the Canon doth direct, the sides } $\frac{18}{5}$ and $\frac{1}{5}$
 of the two Squares sought will be found these, }
 The

The Squares of which sides being added together make 13, as was required.

Example 2.

Let it again be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares different from those found out in Example 1.

The sides of the two Squares given are } 3 and 2

Take two unequal numbers with respect to the Caution in the foregoing Canon 1. as these, } 4 and 3

Then by using the four numbers last before exprest, as the said Canon doth direct, the sides of the two Squares sought will be found these, . . . } $\frac{86}{25}$ and $\frac{27}{25}$

The Proof.

The Square of $\frac{86}{25}$ is $\frac{7396}{625}$,

The Square of $\frac{27}{25}$ is $\frac{729}{625}$,

The summ of those Squares is $\frac{8125}{625}$, or 13.

Example 3. Where the given number is compos'd of two equal Squares.

Let it be required to divide 2, which is compos'd of two equal Squares, 1 and 1, into two unequal Squares.

The side of either of the Squares given is } 1

Take in this Case any two unequal numbers, as } 1 and 2

Then by working with those three numbers according to the direction of Canon 1. the sides of the two Squares sought will be found these, . . . } $\frac{7}{5}$ and $\frac{1}{5}$

The Squares of which sides being added together make 2, as may easily be proved.

16. Now that the necessity of the Caution prescribed in the foregoing Canon 1. about chusing the unequal numbers s and r may appear, I shall prove, That if s the greater of them hath the same proportion to r the lesser, as $d+b$ the summ of the sides of the two unequal Squares given in Quest. 2. hath to $d-b$ the difference of the same sides, then the said Canon will produce the same sides d and b for the sides of the two Squares sought, and consequently the Operation in such Case will be in vain. First, it is manifest by the thirteenth step, that one of the sides found out by the Canon is $\frac{2rsd + s^2b - r^2b}{ss + rr}$; so that if we prove this side to be equal to d the side of the greater

of the two Squares given, then consequently the other side found out by the Canon, that is, the side exprest by the fourteenth step, shall be equal to the side of the lesser of the two Squares given; for the summ of the Squares found out is equal to the summ of those given.

17. Let it therefore be supposed that } $d+b . d-b :: s . r$

18. And then, we are to demonstrate that } $\frac{2rsd + s^2b - r^2b}{ss + rr} = d$

Demonstration.

19. By supposition in the seventeenth step, } $d+b . d-b :: s . r$

20. Therefore by comparing the Rectangle of the extremes to the Rectangle of the means, } $rd + rb = sd - sb$

21. And by adding sb to each part of the last Equation, this ariseth, } $sb + rd + rb = sd$

22. And by subtracting rd from each part, it makes . . . } $sb + rb = sd - rd$

23. And by resolving the last Equation into Proportionals, this Analogy ariseth, viz. } $s+r . s-r :: d . b$

24. And by drawing $s-r$ as a common Factor into the two first Terms of that Analogy, this ariseth,

$$ss - rr . ss + rr - 2rs :: d . b.$$

25. Therefore, by comparing the Product of the extremes in the last Analogy to the Product of the means, this Equation ariseth, viz.

$$ssb - rrb = ssd + rrd - 2rsd.$$

26. Whence by equal Addition of $2rsd$, this Equation ariseth, viz.

$$2rsd + ssb - rrb = ssd + rrd.$$

27. Wherefore by dividing each part of the last Equation by $ss + rr$, this ariseth, viz.

$$\frac{2rsd + ssb - rrb}{ss + rr} = d. \text{ Which was to be demonstrated.}$$

Resolution 3.

RESOLUTION 3. of Quest. 2. which is here repeated; viz.

To divide a given number which is compos'd of two known Squares, into two other Squares.

1. For the side of the greater Square given, put $\succ d$
2. And for the side of the lesser Square given, put $\succ b$
3. Therefore the greater of those Squares shall be $\succ dd$
4. And the lesser $\succ bb$
5. Take two unequal numbers, s the greater, and r the lesser, with this Caution, viz. That s may not be in such proportion to r , as d to b ; which s and r do represent the numbers to be prefixt to the unknown number a , according to the second Rule before mentioned in *Observat. 3. Resolut. 1. Quest. 2.* and the reason of the Caution will be shewn in the sixteenth step of this Resolution.
6. Then for the side of the first Square sought, put $\succ ra - b$, or, $b - ra$
7. And for the side of the second Square sought, put $\succ sa - d$, or, $d - sa$
8. Therefore from the sixth step the first Square sought is $\succ rraa - 2rba + bb$
9. And from the seventh step the second Square sought is $\succ ssaa - 2sda + dd$
10. Therefore the summ of the Squares in the eighth and ninth steps is

$$ssaa + rraa - 2rba - 2sda + bb + dd.$$

11. But the said summ must be equal to the two Squares given, to wit, dd and bb , hence therefore ariseth the following Equation, viz.

$$ssaa + rraa - 2rba - 2sda + bb + dd = bb + dd.$$

12. Which Equation, after due Reduction, gives $\int a = \frac{2rb + 2sd}{ss + rr}$

13. Therefore from the twelfth and sixth steps the side of the first Square sought is now made known, and equal to one of these two Quantities, to wit,

$$\frac{2rsd + rrb - ssb}{ss + rr}, \text{ or, } \frac{ssb - rrb - 2rsd}{ss + rr};$$

That is to say, the former of those two Quantities exprest Fraction-wise shall be the side of the first Square sought, when $ssb - rrb$ is less than $2rsd$; but the latter shall be the said side when $ssb - rrb$ is greater than $2rsd$. For if $ssb - rrb$ be less than $2rsd$, then by subtracting $ssb - rrb$ from $2rsd$, the Remainder will be the same with the Numerator of the first of the two Quantities above exprest Fraction-wise; but if $ssb - rrb$ be greater than $2rsd$, then by subtracting $2rsd$ from $ssb - rrb$, the Remainder will be the same with the Numerator of the latter of the said Quantities: Therefore the side of the first Square sought may be exprest thus,

$$\frac{ssb - rrb \text{ or } 2rsd}{ss + rr};$$

That is to say, If the difference between $ssb - rrb$ and $2rsd$, be divided by $ss + rr$ the Quotient shall be the side of the first Square sought.

14. But from the twelfth and seventh steps the side of the second Square will be found equal to this known Quantity, viz.

$$\frac{2rsb + ssd - rrd}{ss + rr}.$$

From the premisses ariseth this following

CANON 2.

15. Take two unequal numbers, with this Caution, viz. That the greater may not have the same proportion to the lesser, as the side of the greater of the two Squares given hath to the lesser side: Multiply the double Product of the multiplication of the two unequal numbers first taken by each of the said two sides given, and reserve the Products; multiply also the difference of the Squares of the two numbers first taken, by each of the said two sides given, and reserve these Products; then take the difference between the greater of the two first reserved Products and the lesser of the two latter for a Dividend; take also the summ of the lesser of the two first Products and the greater of the two latter for a second Dividend; lastly, divide each of those Dividends by the summ of the Squares of the two numbers first taken, so shall the Quotients be the sides of the two Squares sought.

Example 1.

Example 1. Where the number given to be divided is compos'd of two unequal Squares.

Let it be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares.

The side of the greater Square given is } 3
 The side of the lesser Square given is } 2
 Take two unequal numbers, with respect to the Caution in Canon 2. } 2 and 1
 as these, }
 Then by using those four numbers according to the direction of Canon 2. } 6 and $\frac{17}{5}$
 the sides of the two Squares sought will be found these, viz. }
 The Squares of which sides $\frac{6}{5}$ and $\frac{17}{5}$ being added together make $\frac{36}{25}$ or 13, as was required.

Example 2.

Let it be again required to divide 13, which is compos'd of 9 and 4 into two other Squares different from those found out in *Example 1.*

The sides of the two given Squares, 9 and 4, are } 3 and 2
 Take two unequal numbers with respect to the Caution in Canon 2. } 4 and 1
 as these, }
 Then by working with those four numbers as the said Canon 2. doth } 6 and $\frac{61}{17}$
 direct, the sides of the two Squares sought will be found these, }
 The Squares of which sides are $\frac{36}{289}$ and $\frac{3721}{289}$, whose sum makes $\frac{3757}{289}$, that is, 13, as was required.

Example 3. Where the number given to be divided is compos'd of two equal Squares.

Let it be required to divide 18, which is compos'd of two equal Squares, 9 and 9, into two unequal Squares.

The side of either of the Squares given is } 3
 Take in this Case any two unequal numbers, as } 1 and 2
 Then by using those three numbers according to the direction of the }
 foregoing Canon 2. the sides of the two Squares sought will be found } $\frac{3}{5}$ and $\frac{21}{5}$
 these, viz. }
 The Squares of which sides are $\frac{9}{25}$ and $\frac{441}{25}$, whose sum makes $\frac{450}{25}$, that is 18, as was required.

16. Now that the necessity of the Caution prescribed in the foregoing Canon 2. about chusing the unequal numbers s and r may appear, I shall prove, That if s the greater of them hath the same proportion to r the lesser, as d the side of the greater of the two Squares given in *Quest. 2.* hath to b the side of the lesser of the same Squares, then the said Canon will produce the same sides d and b for the sides of the two Squares sought, and consequently the Operation in such Case will be in vain: First, it is manifest by the fourteenth step, that one of the sides found out by the Canon is $\frac{ssd - rrd + rsb}{ss + rr}$;

so that if we prove this side to be equal to d the side of the greater of the two Squares given, then consequently the other side found out by the Canon, that is, the side express'd in the thirteenth step shall be equal to the side of the lesser of the two Squares given, for the sum of the Squares found out is equal to the sum of those given.

17. Let it therefore be supposed that } $d \cdot b :: s \cdot r$

18. And then we are to demonstrate that } $\frac{ssd - rrd + 2rsb}{ss + rr} = d$

Demonstration.

19. By supposition in the sixteenth step, } $d \cdot b :: s \cdot r$

20. Therefore by comparing the Rectangle of the means }
 to the Rectangle of the extremes, } $sb = rd$

21. And by drawing $2r$ into each part of the last Equation, }
 this ariseth, viz. } $2rsb = 2rrd$

22. And by adding ssd to each part of the last Equation, }
 it gives } $ssd + 2rsb = ssd + 2rrd$

23. And by subtracting rrd from each part of the last }
 Equation; there remains } $ssd - rrd + 2rsb = ssd - rrd$

24. Where-

24. Wherefore by dividing each part of the last Equation } $\frac{ssd - rrd + 2rsb}{ss + rr} = d$
 by $ss + rr$, there ariseth }
 Which was to be proved.

Observations upon the preceding Resolutions 2, and 3. of Quest. 2.
 by Literal Algebra.

1. If z be put equal to $\sqrt{bb + dd}$: that is, the square Root of the number compos'd of two Squares given in Quest. 2. that Question may be stated thus, viz.

Two Rational numbers, b and d , being given for the Base and Perpendicular of a right-angled Triangle whose Hypothenufal is z , Rational or Irrational; to find out other Rational numbers to express the Base and Perpendicular of a second right-angled Triangle whose Hypothenufal shall be z likewise.

The Base and Perpendicular of the Triangle sought shall be given either by Canon 1. in the fifteenth step of Resolution 2. of Quest. 2. or by Canon 2. in the fifteenth step of Resolution 3. and may be express'd by Letters, as before in the thirteenth and fourteenth steps of Resolution 2. or by the thirteenth and fourteenth steps of Resolution 3. viz.

	By Canon 1.		By Canon 2.		
}	Hyp.	$z, (or, \sqrt{bb + dd}:)$	}	Hyp.	$z, (or, \sqrt{bb + dd}:)$
	Perp.	$\frac{2rsd + ssb - rrb}{ss + rr}$		Perp.	$\frac{ssb - rrb \cup 2rsd}{ss + rr}$
	Base,	$\frac{ssd - rrd \cup 2rsb}{ss + rr}$		Base,	$\frac{2rsb + ssd - rrd}{ss + rr}$

2. If the Bases and Perpendiculars of those two right-angled Triangles above-express'd, which I call ΔK and ΔL , be well examined, another way will be discovered to find out the same Bases and Perpendiculars by the help of the Bases and Perpendiculars of two like right-angled Triangles whose Hypothenusals are b and d . For,

First, it is manifest by Observat. 5. Resolut. 2. Quest. 1. of this Book, that these three following numbers will constitute a right-angled Triangle, which hath b for an Hypothenufal, viz.

Hyp.	Base,	Perp.	
b	$\frac{ssb - rrb}{ss + rr}$	$\frac{2rsb}{ss + rr}$	($\Delta M.$)

Likewise these three following numbers will constitute a right-angled Triangle, having d for an Hypothenufal, viz.

Hyp.	Base,	Perp.	
d	$\frac{ssd - rrd}{ss + rr}$	$\frac{2rsd}{ss + rr}$	($\Delta N.$)

Which two Triangles last before express'd, to wit, ΔM and ΔN , are like, for each of them is like to a right-angled Triangle whose three sides are $ss + rr$, $ss - rr$, and $2rs$; Now I say, if the Perpendiculars and Bases of the two right-angled Triangles K and L before express'd in Observat. 1. be well viewed, it will be evident, that they are deduced from the two like right-angled Triangles M and N before express'd in this Observat. 2. which have b and d for Hypothenusals. For, first, the Perpendicular of ΔK is compos'd of the Base of ΔM and the Perpendicular of ΔN ; secondly, the Base of ΔK is equal to the difference between the Perpendicular of ΔM and the Base of ΔN ; thirdly, the Perpendicular of ΔL is equal to the difference between the Base of ΔM and the Perpendicular of ΔN ; lastly, the Base of ΔL is compos'd of the Perpendicular of ΔM and the Base of ΔN .

3. Hence therefore another Canon comes to light, to solve as well the preceding Quest. 2. as also the following Proposition, (which is Prop. 47. in pag. 35. of Vieta's Works,) viz.

From two right-angled Triangles given to deduce a third right-angled Triangle, such, that the Square of the Hypothenufal of the third may be equal to the Squares of the Hypothenusals of the first and second.

This Proposition may be solved by the following

CANON 3.

CANON 3.

First, (by the Canon in *Observat.* 10. *Resolut.* 2. of the preceding *Quest.* 1.) find out two like right angled Triangles in numbers, such, that their Hypothenusals may be the sides of the two Squares given in the foregoing *Quest.* 2. then, for the Perpendicular of the third right-angled Triangle sought, take the summ of the Base of the first and Perpendicular of the second; for the Base of the third, take the difference between the Perpendicular of the first and Base of the second; and the Hypothenusal of the third shall be the square Root of the summ of the Squares of the Hypothenusals of the first and second.

Or thus:

For the Perpendicular of the third right-angled Triangle sought, take the difference between the Base of the first and Perpendicular of the second; for the Base of the third, take the summ of the Perpendicular of the first and Base of the second; and the Hypothenusal of the third shall be the same as before is exprest.

Example 1. in Numbers.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two other Squares.

The sides of 4 and 9 the two Squares given are	2 and 3
Find a first right-angled Triangle in numbers whose Hypothenusal shall be 2, the side of 4 the lesser of the two Squares given, as,	Hyp. Base. Perp. 2, 3/2, 5/2
Find likewise a second right-angled Triangle like to the first, and such, that its Hypothenusal may be 3 the side of the greater Square given, as,	3, 9/5, 12/5
Then by using the Bases and Perpendiculars of those two like right-angled Triangles as <i>Canon</i> 3. doth direct, this third right-angled Triangle will be found out, whose Hypothenusal is equal to $\sqrt{4+9}$: that is, $\sqrt{13}$, and consequently the Base and Perpendicular are the sides of the two Squares sought,	$\sqrt{13}$, 1/5, 18/5
Or, according to the latter part of <i>Canon</i> 3. the sides of the two Squares sought will be found the Base and Perpendicular of this third right-angled Triangle whose Hypothenusal is $\sqrt{13}$, that is, $\sqrt{4+9}$: as before;	$\sqrt{13}$, 17/5, 6/5

Example 2.

Let it be required to divide 25, which is composed of two Squares, 9 and 16, into two other Squares.

The sides of 9 and 16 the two given Squares are	3 and 4
Find a first right-angled Triangle in numbers, whose Hypothenusal shall be 3 the side of the lesser Square given, as,	Hyp. Base, Perp. 3, 16/3, 21/3
Find likewise a second right-angled Triangle like to the first, and such, that its Hypothenusal shall be 4 the side of the greater Square given, as,	4, 48/13, 20/13
Then by using the Bases and Perpendiculars of the two like right-angled Triangles last found out, according to the direction of <i>Canon</i> 3. this third right-angled Triangle will be dicovered, whose Hypothenusal is 5, that is, $\sqrt{9+16}$: and consequently the Base and Perpendicular are, the sides of the two Squares sought,	5, 33/13, 56/13
Or, according to the latter part of <i>Canon</i> 3. the sides of the two Squares sought will be found the Base and Perpendicular of this third right-angled Triangle, whose Hypothenusal is 5, that is, $\sqrt{9+16}$: as before,	5, 63/13, 16/13

4. If every one of the three sides of the two right-angled Triangles K and L before exprest in *Observat.* 1. having z for a common Hypothenusal, Rational or Irrational, be multiplied by $ss+rr$, the Products shall be also the sides of two right-angled Triangles like to the two former respectively; which Products or sides shall be these, viz,

Hyp.

$$\begin{array}{l} \text{Hyp. } zss \text{ } \vdash \text{ } zrr \\ \text{Perp. } \frac{zrsd \text{ } \vdash \text{ } ssb \text{ } - \text{ } rrb}{\text{Base, } \frac{ssd \text{ } - \text{ } rrd \text{ } \infty \text{ } 2rsb} \end{array}$$

$$\begin{array}{l} \text{Hyp. } \frac{zss \text{ } \vdash \text{ } zrr}{\text{Perp. } \frac{ssb \text{ } - \text{ } rrb \text{ } \infty \text{ } 2rsd} \\ \text{Base, } 2rsb \text{ } \vdash \text{ } ssd \text{ } - \text{ } rrd \end{array}$$

Now if the two right-angled Triangles last express'd be well examined, it will appear, that each of them may be deduced from two right-angled Triangles, one of which hath for its Hypothenuſal $ss \text{ } \vdash \text{ } rr$, Base $ss \text{ } - \text{ } rr$, and Perpendicular $2rs$, (or $2rs$ may be called the Base, and $ss \text{ } - \text{ } rr$ the Perpendicular;) but of the other the Hypothenuſal is $z = \sqrt{bb \text{ } \vdash \text{ } dd}$: Rational or Irrational, the Base is b , and the Perpendicular is d , (or d may be called the Base, and b the Perpendicular;) I ſay, from theſe two laſt mentioned Triangles each of the two former may be deduced in ſuch manner as is directed in the following Canon, which is the ſame with that rais'd by *Vieta* in ſolving *Prop. 46.* in pag. 34. of his Works, viz.

From two right-angled Triangles given, to form a third right-angled Triangle.

CANON.

For the Hypothenuſal of the third right-angled Triangle, take the Product of the multiplication of the Hypothenuſals of the two right-angled Triangles given: for the Perpendicular, the ſumm of the Product of the Base of the firſt into the Perpendicular of the ſecond, and the Product of the Base of the ſecond into the Perpendicular of the firſt: and for the Base, take the difference between the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

Or thus:

For the Hypothenuſal of the third right-angled Triangle, take (as before) the Product of the multiplication of the Hypothenuſals of the firſt and ſecond right-angled Triangles given: for the Perpendicular, the difference between the Product of the Base of the firſt into the Perpendicular of the ſecond, and the Product of the Base of the ſecond into the Perpendicular of the firſt: laſtly, for the Base, take the ſumm of the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

An Example in Numbers. Hyp. Base, Perp.

Let there be two right-angled Triangles given in numbers,	}	5	3	4
ſuppoſe theſe,		13	5	12
Then from thoſe Triangles, theſe two are deduced by the	}	65	33	56
two Canons laſt before expreſſ'd, viz.		65	63	16

Note 1. If the two right-angled Triangles given be unlike, then either of thoſe Canons will form a third right-angled Triangle; but if like, then the firſt only will take place: for when the two right-angled Triangles given are like, then the difference of the Products mentioned in the latter Canon are equal to nothing, as will be evident to every diligent Reader.

Note 2. If from any right-angled Triangle taken } H . . . B . . P
twice, ſuppoſe from theſe two, } H . . B . . P
A third right-angled Triangle be deduced according to } HH . BB ∞ PP . 2BP
the firſt Canon, as theſe, }

Then the angle at the Baſe of the third right-angled Triangle ſo deduced, viz. the angle oppoſite to the ſide 2BP ſhall be equal to the double of the angle at the Baſe of the firſt right-angled Triangle, viz. of the angle oppoſite to the ſide P; or elſe equal to the Complement of the ſaid double angle unto two right-angles, when the ſaid double exceeds a right-angle.

Likewiſe, if from two unlike right-angled Triangles, } H . . B . . P
ſuppoſe from theſe, } h . . b . . p

A third right-angled Triangle be deduced according } Hh . Bb ∞ Pp . Ep \vdash Pb
to the firſt Canon, as theſe, }

Then the angle at the Baſe of this third right-angled Triangle, viz. the angle oppoſite to the ſide Ep \vdash Pb ſhall be equal to the ſumm of the angles at the Baſes of the firſt and ſecond right-angled Triangles, viz. of the angles oppoſite to the ſides P and p; or elſe equal to the Complement of the ſaid ſumm unto two right-angles, when that ſumm exceeds a right-angle.

The converse of this rare Speculation is demonstrated by *Andersonus*, in *Theorem. 2.* of *Vieta's* mysterious Doctrine of Angular Sections; and likewise by *Herigonius* at the latter end of the First Tome of his *Cursus Mathematic.*

QUEST. 3.

To divide a given square number into two such Squares, that one of them may consist within given limits.

Let it be required to divide 16 into two such Squares, that one of them may be greater than 10, but less than 11.

Or thus:

A Rational number 4 being given for the Hypotenusal of a right-angled Triangle, to find the Base and Perpendicular in such Rational numbers, that one of them may be greater than $\sqrt{10}$, but less than $\sqrt{11}$.

RESOLUTION.

1. For the given Hypotenusal 4, (which is the side of the given Square 16) put d
2. For $\sqrt{10}$ the lesser of the prescribed limits, put f
3. For $\sqrt{11}$ the greater of the prescribed limits, put g
4. Let two unequal numbers be represented by s and r
5. Then the sides about the right-angle of a right-angled Triangle whose Hypotenusal is d , will be found equal to these Quantities, (by the Canon in *Observat. 5. Resolut. 2. Quest. 1.*) viz. $\frac{2rsd}{ss+rr}$ and $\frac{ssd \text{ or } rrd}{ss+rr}$
6. But since this Question requires that one of those sides, suppose $\frac{2rsd}{ss+rr}$, may be greater than f , yet less than g , the said numbers s and r cannot be any two unequal numbers; and therefore I shall here shew a way to chuse them, so as that they may cause the said side to agree with the said limits: To which end, first, a number at pleasure may be taken for one of the said numbers s and r , as, $1=r$; and then to search out limits for the chusing of s , I proceed in this manner, viz. I put a instead of s while it is unknown, and then since $1=r$, the before-mentioned side $\frac{2rsd}{ss+rr}$ will be express'd thus, $\frac{2da}{aa+1}$, where the number a only is unknown: Now,
7. Let it be supposed (according to the import of the Question) that $\frac{2da}{aa+1} \sqsubset f$
8. Let it also be supposed that $\frac{2da}{aa+1} \supset g$
9. Then by multiplying each part of the supposition in the seventh step by $aa+1$, it follows that $2da \sqsubset faa+f$
10. Therefore by comparing the latter part of the ninth step to the former, $faa+f \supset 2da$
11. And by dividing each part in the last step by f , it follows, that $aa+1 \supset \frac{2da}{f}$
12. And by subtracting 1 from each part, $aa \supset \frac{2da}{f} - 1$
13. Likewise by equal subtraction of $\frac{2da}{f}$ it follows, that $aa - \frac{2da}{f} \supset -1$
14. And by adding the Square of half the Coefficient $\frac{2d}{f}$ to each part in the thirteenth step, $aa - \frac{2da}{f} + \frac{dd}{ff} \supset \frac{dd}{ff} - 1$
15. And by extracting the square Root out of each part in the fourteenth step, $a - \frac{d}{f} \supset \sqrt{\frac{dd-ff}{ff}}$
16. Wherefore by adding $\frac{d}{f}$ to each part in the fifteenth step, it follows that $a \supset \frac{d}{f} + \sqrt{\frac{dd-ff}{ff}}$

Again, 17.

17. Again, because $\frac{d}{f} = a$, (as well as $a = \frac{d}{f}$,) may be the side of the Square in the first part of the fourteenth step, it thence follows that
18. And by adding a to each part in the seventeenth step,
19. And by equal subtraction of $\sqrt{\frac{dd-ff}{ff}}$: it follows that
20. Wherefore by comparing the latter part of the nineteenth step to the first, it's evident that
21. Again, by supposition in the eighth step,
22. And consequently,
23. Whence by arguing in like manner as before, with f , from the ninth step to the sixteenth, it will appear that
24. Again, by arguing with g in like manner as before with f , from the seventeenth step to the twentyeth, it will be evident that
25. Now because a was put instead of s , the sixteenth and twenty-third steps give a Canon for limiting the number s , when $r = 1$; viz.

CANON 1.

$$s \Rightarrow \frac{d + \sqrt{dd-ff}}{f} \quad (2.039, \&c.)$$

$$s \Leftarrow \frac{d + \sqrt{dd-gg}}{g} \quad (1.880, \&c.)$$

26. Again, the twentyeth and twenty-fourth steps give another Canon for limiting the number s , (which was represented by a in the preceding argumentation,) when $r = 1$; viz.

CANON 2.

$$s \Leftarrow \frac{d - \sqrt{dd-ff}}{f} \quad (0.490, \&c.)$$

$$s \Rightarrow \frac{d - \sqrt{dd-gg}}{g} \quad (0.531, \&c.)$$

27. Therefore, if 1 be put for r , and there be given (as before in the first, second and third steps,) $4 = d$; $\sqrt{10} = f$; and $\sqrt{11} = g$; then by Canon 1. s may be any number less than $2\frac{2}{5}$, but greater than $1\frac{2}{5}$; and consequently, if $1 = r$, and $s = 2$, (which is within the last mentioned limits of s ;) then the sides of the two Squares sought (being expounded according to the two Quantities in the fifth step of the Resolution of this Quest. 3.) shall be $\frac{16}{5}$ and $\frac{12}{5}$, viz.

$$\frac{16}{5} = \frac{2rsd}{ss+rr}; \text{ and, } \frac{12}{5} = \frac{ssd-rrd}{ss+rr}.$$

Therefore the two Squares sought are $2\frac{16}{5}$ and $1\frac{12}{5}$, whose sum is $4\frac{28}{5}$; that is, 16; and one of those Squares, to wit, $2\frac{16}{5}$, or $10\frac{6}{5}$, is greater than 10, but less than 11; as was required.

Again, if $1 = r$; $4 = d$; $\sqrt{10} = f$; $\sqrt{11} = g$; (as before,) then by Canon 2. s may be any Fraction greater than $\frac{4}{5}$, but less than $\frac{11}{5}$; and consequently, if $1 = r$. and $s = \frac{1}{2}$, (which is within the last mentioned limits of s ;) then the sides of the two Squares sought will be found the same as before, viz.

$$\frac{16}{5} = \frac{2rsd}{rr+ss}; \text{ and, } \frac{12}{5} = \frac{rrd-ssd}{rr+ss}.$$

Again, if $1 = r$, and $s = \frac{13}{25}$, (which is also within the limits of s discovered by Canon 2.) then the sides of the two Squares sought will be found these, to wit, $\frac{1300}{397}$ and $\frac{212}{397}$; whose

whose Squares are $\frac{1620000}{137609}$ and $\frac{821244}{137609}$, which added together make 16; and the first of those Squares is greater than 10, but less than 11; as was required.

Note, That the manner of searching out limits in this and divers following Questions, is agreeable to the method of resolving Quadratick Equations in Sect. 5, 7, 9. Chap. 15. Book 1.

QUEST. 4.

(This is the fifth of the fourth Book of Vieta's Zetetics; 'tis also resolved by Bachet in his Comment upon the twelfth of the fifth Book of Diophantus; but I shall wave their ways of Resolution, and deduce one from Canon 1. in the fifteenth step of Resolut. 2. of the preceding second Question of this Book.)

To divide a given number which is compos'd of two Squares, into two other Squares, that one of the Squares sought may consist within given limits.

Preparation.

Because the following Resolution of this Question presupposeth each of the prescribed limits to be greater than the lesser of the two Squares given, I shall in the first place shew how from the given limits, when they are not qualified as aforesaid, to infer others, each of which shall be greater than the lesser of the two Squares given, and then the following Resolution will solve the Question propos'd according to any possible limits whatever.

Case 1. When the lesser of the given limits is equal to the lesser of the given Squares.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 4, but less than 5: Here instead of 4 the lesser limit, (which is equal to the lesser Square given,) we may take $4\frac{1}{2}$ or any number between 4 and 5, (5 being the greater limit given:) then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose sum shall be 13; and one of them shall be greater than $4\frac{1}{2}$, but less than 5, and consequently greater than 4, but less than 5; as was required.

Case 2. When the lesser limit is less than the lesser Square given, but the greater limit exceeds the same; viz. When the lesser Square given falls between the given limits.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 5: Here instead of 3 we may take $4\frac{1}{2}$, or any number between 4 the lesser Square given, and 5 the greater limit: then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose sum shall be 13; and one of them shall be greater than $4\frac{1}{2}$, but less than 5, and consequently greater than 3, but less than 5; as was required.

Case 3. When the greater of the two limits given is equal to the lesser of the two Squares given.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 4: First, subtract 3 and 4 severally from 13, so each of the Remainders 10 and 9 is greater than 4 the lesser Square given, and therefore by the following Resolution two Squares may be found out whose sum shall be 13; and one of them less than 10, but greater than 9, and consequently the other Square shall be greater than 3, but less than 4; as was required.

Case 4. When each of the two limits given is less than the lesser Square given.

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 1, but less than 2: First, subtract the said limits 1 and 2 severally from 13 the number given to be divided, so each of the Remainders 12 and 11 is greater than 4 the lesser Square given; and therefore by the following Resolution two Squares may be found out whose sum shall be 13; and one of them less than 12, but greater than 11, and consequently the other Square shall be greater than 1, but less than 2; as was required.

Now let it be desired to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 6, but less than 7.

RESO-

RESOLUTION.

1. For 2, the side or square Root of 4 the lesser of the two Squares given, put $> b$
2. For 3, the side of 9 the greater Square given, put $> d$
3. For $\sqrt{6}$, that is, the square Root of the lesser of the two limits given, put $> f$
4. For $\sqrt{7}$, that is, the square Root of the greater of the two limits, put $> g$
5. Let two unequal numbers be represented by $> s$ and r
6. Now if s be greater than r , and be to r in any Reason (or Proportion) except that which $d + b$ hath to $d - b$, then by Canon 1. in the fifteenth step of Resolut. 2. of the preceding Quest. 2. the sides of two Squares different from dd and bb , but such, whose sum is equal to $bb + dd$, shall be equal to these two following Quantities, (which are exprest also in the thirteenth and fourteenth steps of the said Resolut. 2. Quest. 2.) viz.

$$\frac{2rsd + ssb - rrb}{ss + rr} \quad \text{and} \quad \frac{ssd - rrd}{ss + rr} \approx 2rsb$$

7. But because it's desired that one of those two Quantities or sides last above exprest, suppose, $\frac{2rsd + ssb - rrb}{ss + rr}$ may be greater than f , but less than g , the two unequal numbers s and r must be chosen so as that they may cause the said side to agree with the said limits. To which end, first, a number at pleasure may be taken for one of the said numbers s and r , as $1 = r$, and then to search out limits for the chusing of s , I proceed in this manner, viz: I put a instead of s while it is unknown, and then since $1 = r$, the before-mentioned side $\frac{2rsd + ssb - rrb}{ss + rr}$ will stand thus, $\frac{2da + baa - b}{aa + 1}$, where the number a only is unknown: Now,

8. Let it be supposed that $\frac{2da + baa - b}{aa + 1} < f$

9. Let it also be supposed that $\frac{2da + baa - b}{aa + 1} > g$

10. Then by multiplying each part in the eighth step by $aa + 1$, it follows that $2da + baa - b < faa + f$

11. And by adding b to each part in the tenth step, $2da + baa < faa + f + b$

12. And by subtracting baa from each part in the eleventh step, it follows that $2da < faa - baa + f + b$

13. By supposition in the first and third steps, (agreeable to the Preparation to the Resolution of this Question,) f is greater than b , suppose therefore $c = f - b$

14. Then from the twelfth and thirteenth steps it follows, that $2da < caa + f + b$

15. And by subtracting $f + b$ from each part in the fourteenth step, it's manifest that $2da - f - b < caa$

16. And by dividing each part of the fifteenth step by c , $\frac{2d}{c}a - \frac{f + b}{c} < aa$

17. And by subtracting $\frac{2d}{c}a$ from each part of the sixteenth step, $-\frac{f + b}{c} < aa - \frac{2d}{c}a$

18. And by adding the Square of half the Coefficient $\frac{2d}{c}$ to each part of the seventeenth step, it follows that $\frac{dd}{cc} - \frac{f + b}{c} < aa - \frac{2d}{c}a + \frac{dd}{cc}$

19. And by extracting the square Root out of each part of the eighteenth step, $\sqrt{\frac{dd - fc - bc}{cc}} < a - \frac{d}{c}$

20. And by adding $\frac{d}{c}$ to each part of the 19th step, it follows, that $\frac{d}{c} + \sqrt{\frac{dd - fc - bc}{cc}} < a$

21. Wherefore by comparing the quantity a in the latter part of the 20th step to the sum of the quantities in the first part, it is found that $a > \frac{d}{c} + \sqrt{\frac{dd - fc - bc}{cc}}$

22. Again,

22. Again, because $\frac{d}{c} - a$ (as well as $a - \frac{d}{c}$ in the latter part of the nineteenth step,) may be the square Root of the Square which is the latter part of the eighteenth step, it follows, that
23. And by adding a to each part of the twenty-second step,
24. Wherefore by subtracting $\sqrt{\frac{dd - fc - bc}{cc}}$ from each part of the twenty-third step, it's evident that
25. Again, by supposition in the ninth step,
26. Whence by multiplying each part by $aa + 1$,
27. And by subtracting baa from each part of the twenty-sixth step,
28. And by adding b to each part of the twenty-seventh step,
29. By supposition in the first and fourth steps, (agreeable to the Preparation to the Resolution of this Question,) g is greater than b ; suppose therefore,
30. Then from the twenty-eighth and twenty-ninth steps it follows, that
31. Whence, by arguing in like manner as before from the fourteenth step to the twenty-first, it will appear that
32. Again, by arguing in like manner as before from the twenty-second step to the twenty-fourth, it will be evident that
33. Then out of the twenty-first and thirty-first steps, after a, c and n are exchanged for $s, f - b$ and $g - b$, for these are equal to those, as appears by the Positions in the seventh, thirteenth and twenty-ninth steps, the following Canon I. ariseth for limiting the number s , when $r = 1$; viz.

CANON 1.

$$s \supset \frac{d + \sqrt{dd + bb - ff}}{f - b} \quad (12.562, \&c.)$$

$$s \sqsubset \frac{d + \sqrt{dd + bb - gg}}{g - b} \quad (8.439, \&c.)$$

Again, out of the twenty-fourth and thirty-second steps, after a, c and n are exchanged for $s, f - b$ and $g - b$, (as before,) another Canon ariseth for limiting the number s , when $r = 1$; viz.

CANON 2.

$$s \sqsubset \frac{d - \sqrt{dd + bb - ff}}{f - b} \quad (0.788, \&c.)$$

$$s \supset \frac{d - \sqrt{dd + bb - gg}}{g - b} \quad (0.852, \&c.)$$

Therefore if 1 be taken for the value of r , and there be given, $2 = b$; $3 = d$; $\sqrt{6} = f$; and $\sqrt{7} = g$; (as before in the first, second, third and fourth steps,) then by Canon 1. above-express'd, s may be any number between $12\frac{162}{1000}$ and $8\frac{439}{1000}$; and consequently, if $r = 1$ and $s = 9$, (which value of s is within the limits last before-mentioned, then the sides of the two Squares sought (being expounded according to the two Quantities in the sixth step of the Resolution of this Quest. 4.) shall be $\frac{127}{41}$ and $\frac{122}{41}$, viz.

$$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{107}{41}, \text{ and } \frac{ssd - rrd \approx 2rsb}{ss - rr} = \frac{102}{41}.$$

Therefore the two Squares sought are $\frac{11442}{1681}$ and $\frac{10404}{1681}$, whose sum is $\frac{21846}{1681}$, that is, 13; and the first of those Squares is greater than 6, but less than 7, as was required.

Again, if $1 = r$; $2 = b$; $3 = d$; $\sqrt{6} = f$; $\sqrt{7} = g$; (as before,) then by Canon 2. s may be any Fraction between $\frac{780}{1000}$ and $\frac{852}{1000}$; and consequently, if $1 = r$ and $\frac{2}{5} = s$, (which value of s is within the last mentioned limits,) then the sides of the two Squares sought will be found $\frac{1324}{533}$ and $\frac{1326}{533}$, viz.

$$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{1391}{533}, \text{ and } \frac{ssd - rrd \approx 2rsb}{ss - rr} = \frac{1326}{533}.$$

Therefore the two Squares sought are $\frac{1234881}{284089}$ and $\frac{1758276}{284089}$, whose sum is $\frac{2993157}{284089}$, viz. 13; and the first of those Squares is greater than 6, but less than 7, as was required.

Again, if $1 = r$, and $s = \frac{4}{5}$, (which value of s is also within the limits discovered by Canon 2.) then the sides of the two Squares sought being expounded as before, will be found $\frac{107}{41}$ and $\frac{102}{41}$; which are the same with those before-found in the Example of Canon 1.

Note. If 1 be put equal to r , and the number s be taken by Canon 2. then because in this case s is less than r , the Algebraical Rules of $+$ and $-$ in adding, subtracting, &c. must be observed to resolve the aforefaid literal values of the sides of the Squares sought into numbers, as in the two last Examples.

QUEST. 5. (Quæst. 11. Lib. 2. Diophant.)

To find two square numbers whose difference shall be equal to a given number, suppose 60, (or d .)

RESOLUTION.

1. To the given difference 60, that is, } d
2. Let some number whose Square is less than the given difference be represented by } b
3. For the side of the lesser Square sought put } a
4. And for the side of the greater Square sought put } $a + b$
5. Therefore the lesser Square is } aa
6. And the greater Square is } $aa + 2ba + bb$
7. And the difference of those Squares is } $2ba + bb$
8. But the said difference must be equal to the given difference d , therefore } $2ba + bb = d$
9. Which Equation, after due Reduction, makes known the value of the side of the lesser Square, viz. } $a = \frac{d - bb}{2b}$
10. And from the ninth and fourth steps, the value of the side of the greater Square is also discovered, viz. } $a + b = \frac{d + bb}{2b}$

The two last steps give the following

CANON 1.

Take any square number less than the given difference, and subtract it from the said difference; then divide the Remainder by the double of the side of the Square first taken, and the Quotient shall be the side of the lesser of the two Squares sought; lastly, this side added to the side of the Square first taken, gives the side of the other Square sought.

So if two Squares be desired whose difference shall be 60, I take a square number less than 60, as 36; this subtracted from that leaves 24, which divided by 12 the double of the square Root of 36, gives the Quotient 2, which shall be the side of the lesser Square sought; and then by adding 6 the square Root of the said 36, to the side 2, the sum 8 is the side of the greater Square sought; lastly, the Squares of the said sides 2 and 8, to wit, 4 and 64 will solve the Question, for their difference is 60, as was required.

Observations upon Quest. 5.

1. It is evident by the two last steps of the preceding Resolution, that the values of the sides of the two Squares sought are } $\frac{d - bb}{2b}$ and $\frac{d + bb}{2b}$
 Now

Now if we suppose $bc = d$, then those sides will be converted } $\frac{bc + bb}{2b}$ and $\frac{bc - bb}{2b}$
 into these, viz. }

Which last mentioned sides or Quotients, after the common } $\frac{1}{2}c + \frac{1}{2}b$ and $\frac{1}{2}c - \frac{1}{2}b$
 Factor b is cast away, will be reduced to these, to wit, }

Hence ariseth this elegant Canon, often used by *Diophantus* to find out two Squares in a given difference, viz.

CANON 1.

Take two such unequal numbers that the Product of their multiplication may be equal to the given difference; then half the summ and half the difference of those two numbers shall be the sides of the two Squares sought.

As, for example, if two Squares be desired whose difference shall be 60, I take two such numbers (10 and 6) which being mutually multiplied make 60; then half the summ of 10 and 6 and half their difference are 8 and 2 the sides of the two Squares sought, and consequently the Squares themselves are 64 and 4, whose difference is 60; as was required.

Again, instead of 10 and 6 taken as before, we may take 30 and 2, for the Product of these is equal to the given difference 60; then half the summ of 30 and 2, and half their difference, give 16 and 14, whose Squares 256 and 196 have 60 for their difference; as was required.

After the same manner, Fractions being admitted, innumerable pairs of Squares may be found out, such, that the difference of each pair shall be equal to one and the same number given: For if the given number be divided by a number taken at pleasure, half the summ, and half the difference of the Divisor and Quotient shall be the sides of two Squares whose difference is equal to the given number.

2. But for farther illustration of the truth of the preceding Canon 2. let c and b represent two unequal numbers, and suppose c to be the greater; then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is $\frac{1}{4}cc + \frac{1}{2}cb + \frac{1}{4}bb,$
 The Square of $\frac{1}{2}c - \frac{1}{2}b$ is $\frac{1}{4}cc - \frac{1}{2}cb + \frac{1}{4}bb,$
 The difference of those Squares is $+ cb.$

Whence it is manifest, That the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of half the summ of the said two numbers, and the lesser is the Square of half their difference. Wherefore the truth of the foregoing Canon 2. doth evidently appear.

3. *Vieta* useth the following Canon (which differs but little from the preceding Canon 2.) to find out two Squares in a given difference.

CANON 3.

Take two such unequal numbers, that the Product of their multiplication may be equal to a quarter of the given difference of two Squares sought; then the summ and difference of those two numbers first taken shall be the sides of the desired Squares.

As, for example, if it be desired to find out two Squares whose difference shall be 60; first, I take $\frac{1}{4}$ of the said 60, to wit, 15; then I chuse two such unequal numbers that the Product of their multiplication may make 15, as 5 and 3; lastly, the summ, and difference of 5 and 3, give 8 and 2 for the sides of two Squares whose difference is 60.

The truth of this Canon 3. may be demonstrated thus; let c and b represent two unequal numbers, and suppose c to be the greater, then

The Square of $c + b$ is $cc + 2cb + bb,$
 The Square of $c - b$ is $cc - 2cb + bb,$
 The difference of those Squares is $+ 4cb.$

Whence it is manifest, That the quadruple of the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of the summ of those numbers, and the lesser Square is the Square of the difference of the same two numbers. Wherefore the truth of Canon 3. is evident.

4. If a Square be equal to two Squares, then (by *prop. 47. Elem. 1. Euclid.*) the sides of those three Squares will constitute a right-angled Triangle, viz. the greatest side shall be the Hypothensal, and the other two the sides about the right-angle; whence it follows, that the Square of one of the sides about the right-angle is equal to the difference of the Squares of the other two sides: And therefore if any Rational number be given for

for one of the sides about the right-angle of a right-angled Triangle, the other side about the right-angle and the Hypotenusal shall be given also in Rational numbers by the help of any of the three preceding Canons: As, for example, if 4 be given for the Base; the Square thereof is 16, then by any of the said Canons find out two Squares whose difference may be 16, such are 25 and 9, (and innumerable other pairs of Squares;) therefore their Square Roots or sides, viz. 5 and 3, shall be the desired Hypotenusal and Perpendicular. Whence it is evident, that by the like Operation innumerable right-angled Triangles may be found out in Rational numbers, which shall have one common Base (or Perpendicular) prescribed.

QUEST. 6.

To find two such numbers, that the Product of their multiplication may be equal to a given number, suppose d , and that the Square of half the summ of the said numbers may be greater than a number given, suppose b .

RESOLUTION.

1. For one of the numbers sought put } a
2. Then by dividing the given Product d by a , the Quotient } $\frac{d}{a}$
shall be the other number sought, to wit, }
3. Therefore half the summ of those two numbers is . . . } $\frac{aa + d}{2a}$
4. Now suppose it be desired that the Square of the said half } $\frac{aa + d}{2a} \leq \sqrt{b}$
summ may be greater than the given number b , then it }
necessarily follows, that the summ it-self must be greater than }
the square Root of b , viz. }
5. Therefore from the fourth step, by multiplying each part } $aa + d \leq 2a\sqrt{b}$
by $2a$, it follows, that }
6. That is, (because $a\sqrt{4b} = 2a\sqrt{b}$,) } $aa + d \leq a\sqrt{4b}$
7. And by subtracting d from each part of the sixth step, it } $aa \leq a\sqrt{4b} - d$
follows that }
8. And by subtracting $a\sqrt{4b}$ from each part of the seventh step, } $aa - a\sqrt{4b} \leq -d$
9. And by adding a quarter of the Square of the known Co- } $aa - a\sqrt{4b} + b \leq b - d$
efficient $\sqrt{4b}$, to wit, b , to each part of the eighth step, it }
follows that }
10. And by extracting the square Root out of each part of } $a - \sqrt{b} \leq \sqrt{b - d}$
the ninth step, }
11. Wherefore by adding \sqrt{b} to each part of the tenth step, } $a \leq \sqrt{b} + \sqrt{b - d}$
12. Again, because $-a + \sqrt{b}$ (as well as $a - \sqrt{b}$) may be } $-a + \sqrt{b} \leq \sqrt{b - d}$
the side of the Square $aa - a\sqrt{4b} + b$ in the first part of }
ninth step, it thence follows that }
13. And by adding a to each part of the twelfth step, . . . } $\sqrt{b} \leq a - \sqrt{b - d}$
14. And by subtracting $\sqrt{b - d}$ from each part of the } $\sqrt{b} - \sqrt{b - d} \leq a$
thirteenth step, }
15. Wherefore from the fourteenth step, by comparing the } $a \geq \sqrt{b} - \sqrt{b - d}$
latter part to the former, }
16. The eleventh and fifteenth steps give limits for the choice of (a) one of the two }
numbers sought by this sixth Question, when it requires that the Square of half the summ }
of the same numbers may be greater than a given number; and the premisses afford }
this following

CANON 1.

For one of the numbers sought take any number greater than $\sqrt{b} + \sqrt{b - d}$: or less than $\sqrt{b} - \sqrt{b - d}$: then divide d the given Product of the multiplication of the two numbers sought, by the number first taken, so shall the Quotient be the other number sought.

An Example in Numbers.

Suppose } $d = 128$, and $b = 192$;
Thence it follows, that . . . } $\sqrt{b} + \sqrt{b - d} = 21.856, \&c.$
Also, } $\sqrt{b} - \sqrt{b - d} = 5.856, \&c.$
D Therefore

Therefore according to the direction of the preceding Canon, I take for one of the two numbers sought some number greater than $21\frac{8}{1000}$, or less than $51\frac{8}{1000}$, as the number 2; by this I divide the given number $128 = d$, and the Quotient gives 64 for the other number sought; which two numbers, 2 and 64, will solve the Question, as will be evident by

The Proof.

The Product of the multiplication of 2 and 64 makes the given number 128 (or d), and the Square of half the sum of 2 and 64, viz. the Square of 33 is 1089, which is greater than 192, (or b), as was required. But to the end there may be a possibility of solving the Question proposed, the Canon above-express'd doth shew there is a necessity that the number d must not exceed the number b .

17. The preceding Resolution of *Quest. 6.* presupposeth it to be desired that the Square of half the sum of the two numbers sought may be greater than a number given; but if it were desired that the said Square might be less than a number given, then \sqsupset being used instead of \sqsubset in the said Resolution, there would at length arise this following Canon to solve the said *Question* in the latter Case.

CANON 2.

For one of the numbers sought take any number less than $\sqrt{b} + \sqrt{b-d}$: but greater than $\sqrt{b} - \sqrt{b-d}$: then divide d the given Product of the multiplication of the two numbers sought, by the number first taken, and the Quotient shall be the other number sought.

An Example of this Canon.

Suppose (as before) $d = 128$, and $b = 192$,

Thence it follows, that $\sqrt{b} + \sqrt{b-d} = 21.856$, &c.

Also, $\sqrt{b} - \sqrt{b-d} = 5.856$, &c.

Therefore (according to the latter Canon) I take for one of the two numbers sought some number between $51\frac{8}{1000}$ and $21\frac{8}{1000}$, as 16; by this I divide the given number 128 (or d), and the Quotient gives 8 for the other number sought; which two numbers, 16 and 8, will solve the Question when it requires that the Square of half their sum may be less than the given number 192, as may easily be proved: For the Product of the said 16 and 8 makes the given number 128, and the Square of half the sum of 16 and 8, viz. the Square of 12, is 144, which is less than the given number 192; as was required.

QUEST. 7.

To find two square numbers in a given difference, and that one of those Squares may be greater or less than a given number.

1. Let it be required to find two such square numbers, that their difference may be equal to a given number, suppose d ; and that the greater Square may exceed a given number, suppose b .

RESOLUTION.

It is manifest by Canon 2. of the foregoing *Quest. 5.* That if two numbers be taken, such, that the Product of their multiplication is equal to the given difference of two Squares sought, then half the sum and half the difference of the numbers so taken shall be the sides of those Squares: Therefore if two numbers be found out, such, that their Product is equal to the given difference d , and that the Square of half the sum of the same numbers is greater than the given number b , then that Square shall be the greater of the two Squares required, and the Square of half the difference of the said numbers shall be the lesser Square required: But two such numbers may be found out by the first Canon of the preceding sixth Question, and consequently this seventh Question may be solved by the following

CANON 1.

Take some number greater than $\sqrt{b} + \sqrt{b-d}$: or less than $\sqrt{b} - \sqrt{b-d}$: then divide d the given difference of the two Squares sought by the number first taken, and reserve the Quotient; lastly, half the sum and half the difference of the said Quotient and number first taken shall be the sides of the two Squares sought.

An

An Example in Numbers.

Suppose $d = 128$, and $b = 192$,
 Thence it follows that $\sqrt{b} + \sqrt{b-d} = 21.856$, &c.
 Also, $\sqrt{b} - \sqrt{b-d} = 5.856$, &c.

Therefore according to the direction of the Canon, I take some number greater than $21\frac{856}{1000}$, or less than $5\frac{856}{1000}$, as 2; then by this 2 I divide 128, (to wit, d) and the Quotient is 64; lastly, half the sum of the said 2 and 64 is 33, and half their difference is 31, which 33 and 31 are the sides of two Squares that will solve the Question proposed, as will be evident by

The Proof.

The Squares of 33 and 31 are 1089 and 961; the difference of these is equal to the given difference 128, (to wit, d ;) and the greater Square 1089 is greater than 192, (or b ;) as was required.

2. In like manner, if it were required to find out two Squares whose difference shall be equal to a given number d , and the greater Square less than a given number b ; the sides of the said Squares may be found out by this following

CANON 2.

Take some number less than $\sqrt{b} + \sqrt{b-d}$: but greater than $\sqrt{b} - \sqrt{b-d}$: then divide d the given difference of the Squares sought, by the number so taken, and reserve the Quotient; lastly, half the sum and half the difference of the said Quotient and number first taken shall be the sides of the two Squares sought.

An Example in Numbers.

Suppose $d = 128$, and $b = 192$,
 Thence it follows, that $\sqrt{b} + \sqrt{b-d} = 21.856$, &c.
 Also, $\sqrt{b} - \sqrt{b-d} = 5.856$, &c.

Therefore according to the direction of the last preceding Canon, I take some number between $5\frac{856}{1000}$ and $21\frac{856}{1000}$, as 16; then by this I divide the given number 128, (to wit, d ;) and the Quotient is 8; lastly, half the sum of the said 16 and 8 is 12, but half their difference is 4; which 12 and 4 are the sides of two Squares that will solve the Question, as will be evident by

The Proof.

The Squares of 12 and 4 are 144 and 16; the difference of these is equal to the given difference 128, (or d ;) and the greater Square 144 is less than 192, (or b ;) as was required.

3. But if it were required to find out two Squares in a given difference d , and that the lesser Square might be greater than a given number g ; they may be discovered by the help of the preceding Canons of this seventh Question, in this manner, *viz.*

Let it be required to find two Squares whose difference shall be 24 (or d ;) and that the lesser Square may be greater than 12, (or g ;) Here the scope must be to find out two such Squares that their difference may be 24, and that the greater Square may exceed 36, that is, $24 + 12$, and then the lesser Square will consequently exceed 12. Therefore,

Suppose $d = 24$, and $g = 12$,
 Suppose also $b = 36 = d + g$,
 Thence it follows, that $\sqrt{b} + \sqrt{b-d} = 9.46$, &c.
 Also, $\sqrt{b} - \sqrt{b-d} = 2.53$, &c.

Then (according to the first Canon of this seventh Question) I take some number greater than $9\frac{46}{100}$, or less than $2\frac{53}{100}$, as 2; by this I divide 24, (to wit, d ;) and the Quotient is 12; then half the sum of 2 and 12 is 7, and half their difference is 5; which 7 and 5 are the sides of two Squares 49 and 25, whose difference is 24, (to wit, d ;) and the lesser Square 25 is greater than 12, (or g ;) as was required. But for the greater evidence, let ff be put for the lesser Square found out, and hh for the greater; then

By Construction, $hh = d + ff$,
 Also by Construction, $hh = d + g (= b)$,
 Therefore $ff = g$. Which was to be proved.
 D 2 4. Lastly,

4. Lastly, if it were desired to find out two Squares in a given difference d , and that the lesser Square might be less than a given number g ; let the summ of those two given numbers, (to wit, $d + g$) be called b (as before,) and then by the latter of the two preceding Canons of this seventh Question find out two Squares that their difference may be equal to the given difference d , and that the greater Square may be less than the summ b , so shall the lesser Square be less than the given number g .

QUEST. 8.

[This is the twelfth of the second Book of Diophantus, and the seventh of the fourth Book of Vieta's Zetetics.]

Two numbers being given, suppose 192 and 128, to find a third, which added to each of those given may make each summ to be a Square.

RESOLUTION I.

1. For the number sought put a
2. Then the Question requires that each of these two summis } $192 + a = \square$
may be a square number, viz. } $128 + a = \square$
3. Now that Duplicate Equality (for so Diophantus calls it) may be resolved thus, viz. First, subtract $128 + a$ from $192 + a$, and the Remainder 64 is the difference both of the given numbers and likewise of the two Squares sought; then (by the preceding seventh Question) find two such square numbers that their difference may be 64, and that the greater Square may exceed 192 the greater number given; such are the Squares 289 and 225, whose sides are 17 and 15.
4. Then equate $192 + a$ to 289, thus, } $192 + a = 289$
5. Or equate $128 + a$ to 225, thus, } $128 + a = 225$
6. Lastly, from either of those Equations in the fourth and fifth steps, the number a sought will be also made known, viz. } $a = 97$

I say 97 will solve the Question; for if it be added to 192 and 128 severally, the summ 289 and 225 are Squares, as was required: And out of the premisses well examined, respect also being had to the preceding seventh Question, there will arise this following Canon to find out innumerable Answers to the Question proposed.

CANON.

7. Take any number greater than the summ, or less than the difference of the square Roots of the two numbers given; divide the difference of the two numbers given, by the number first taken, and reserve the Quotient; then from the Square of half the summ of that Quotient and the number taken, subtract the greater of the two numbers given; or, from the Square of half the difference between the said Quotient and the number first taken, subtract the lesser number given; so shall either of the Remainders (for they are equal to one another) be the numbers sought.

An Example of the Canon.

8. Let there be two numbers given to find a third, according } 96
to Quest. 8. as, } 8
9. Their difference is } 88
10. Also } $\sqrt{96} + \sqrt{8} = 12.626, \text{ \&c.}$
11. And } $\sqrt{96} - \sqrt{8} = 6.969, \text{ \&c.}$
12. Let a number be taken, either greater than $12\frac{626}{1000}$, or }
less than $6\frac{969}{1000}$, such is } 4
13. Divide 88 in the ninth step, by 4 in the twelfth, and the }
Quotient is } 22
14. The half of the summ of 4 and 22 in the twelfth and }
thirteenth steps is } 13
15. The Square of the said 13 is } 169
16. From that Square subtract the greater of the two numbers }
given in the eighth step, to wit, } 96
17. So the Remainder is the number sought, to wit, } 73

The Proof.

$$96 + 73 = 169, \text{ whose } \sqrt{\text{ is } 13}$$

$$8 + 73 = 81, \text{ whose } \sqrt{\text{ is } 9}$$

In like manner, if it were desired to find out some number signified by (*a*) that $10a + 54$ may make a Square, also that $10a + 6$ may make a Square, the preceding Canon will give innumerable values of *a*, among which 1 will be found a true value; for if $a = 1$, then $10a + 54 = 64$, and $10a + 6 = 16$.

Observations upon Quest. 8.

1. *Diophantus* in resolving this Question makes an entrance into one of his peculiar subtilties, which he calls a Duplicate Equality, (an ingenious Invention variously used by him, as divers knotty Questions in this Book will make manifest;) the principal hinge whereof depends on the finding of two Squares whose difference shall be equal to the difference of two Algebraick Quantities, each of which is propos'd to be found equal to some known square number: As, in the preceding Resolution of this Quest. 8. two Squares, to wit, 289 and 225 are found out; whose difference 64 is equal to the difference of the two Algebraick Quantities $192 + a$ and $128 + a$, each of which, according to the import of the Question, is to be found equal to some square number, and therefore the number *a* sought must be such as will cause that effect.

2. But that the reason of the Operation in resolving the Duplicate Equality in this eighth Question may clearly appear, two things are to be proved, viz.

First, That the greater of the two square numbers found out in the third step of the Resolution must necessarily exceed the greater of the two numbers given in the Question, and the lesser Square exceed the lesser number given. To prove this, first, it is intended that the desired number *a* should be affirmative, that is, greater than nothing; but if any square number not greater than 192 were set in the place of 289 in the Equation in the fourth step, viz. $192 + a = 289$; or any square number not greater than 128, in the place of 225 in the Equation in the fifth step, viz. $128 + a = 225$; the value of *a* would be less than nothing; and therefore the necessity of the before-mentioned qualification of the said Squares is apparent.

Secondly, That when two square numbers are found out, such, that their difference is equal to the difference of the two numbers given, and that the greater Square exceeds the greater of those given numbers; then consequently the lesser Square shall exceed the lesser number, and these two last mentioned excesses or differences shall be equal to one another. The truth of this consequence will be evident by the following

THEOREM.

If two square numbers, suppose *dd* the greater and *ff* the lesser, have the same difference as two other numbers, suppose *b* the greater and *c* the lesser, and that the greater Square *dd* exceeds the greater number *b*; then the lesser Square *ff* shall exceed the lesser number *c*, and the former excess $dd - b$ shall be equal to the latter $ff - c$. For,

By supposition	>	$dd - ff = b - c$	□	o
Therefore by adding <i>ff</i> to each part, it follows, that		>	$dd = ff + b - c$		
But by supposition	>	$dd = b$		
Wherefore, by subtracting <i>b</i> from each part of the Equation in the last step but one, that which the Theorem affirms		}	$dd - b = ff - c$		
is manifest, viz.	}			

And consequently either of those equal excesses or differences (in the last Equation) shall be a number to solve the Question proposed; for it is manifest that if the former excess $dd - b$ be added to *b*, and the latter excess $ff - c$ to *c*, the summs will be Squares, to wit, *dd* and *ff*.

Another manner of resolving the foregoing Quest. 8. which is here repeated, viz.

To find a number which added severally to 128 and 192, may make the summs to be Squares.

RESOLUTION 2.

1. For the number sought put $aa - 128$ whereby part of the Question is satisfied; for if $aa - 128$ be added to 128, it makes a Square, to wit, *aa*; let therefore the number sought be feigned to be } $aa - 128$

2. But

2. But the Question requires also, that if the number sought be added to 192, the summ may be a Square; add therefore $aa - 128$ to 192, so the summ $aa - 128 + 192$, that is, $aa + 64$ must be equal to a square number, viz. } $aa + 64 = \square$
3. It remains therefore to equate the said $aa + 64$ to a Square, whose side (to the end the value of aa may be greater than 128, as the number $aa - 128$ assumed in the first step requires) may be feigned to be $a +$ any Absolute number less than $2\sqrt{\frac{aa+64}{2}}$, or $-a +$ any Absolute number greater than $2\sqrt{\frac{aa+64}{2}}$, (which limits are discovered by the following third way of resolving this Question;) let therefore the said side be feigned to be $a + 2$, and then the Square of this side being equated to $aa + 64$ as the second step requires, this Equation ariseth, viz.

$$aa + 4a + 4 = aa + 64.$$

4. Which Equation, after due Reduction, makes known the value of a , viz. } $a = 15$

Therefore by the first and fourth steps the number sought will be found 97, which will solve the Question; for if 97 be added to 128 and 192 severally, the summs are Squares, to wit, 225 and 289; and the limits in the third step for feigning the side of one of the Squares sought do shew that the Question is capable of innumerable Answers.

A third manner of resolving the preceding Quest. 8. which is here repeated, viz.

To find a number, which added first to a given number (f), and then to a greater given number (b), may make the summs to be Squares.

RESOLUTION 3.

1. For the difference of the two given numbers b and f put d , viz. } $d = b - f$
suppose }
2. And for the number sought put $aa - f$, (a representing a number unknown,) whereby the first part of the Question is satisfied; for if $aa - f$ be added to f it makes a Square, to wit, aa ; let therefore the number sought be feigned to be } $aa - f$
3. But the Question requires also, that if the number sought be added to b it may make a Square; add therefore $aa - f$ to b , and it makes $aa + b - f$, that is, $aa + d$, (for d was put equal to $b - f$), which must be equal to a Square, viz. } $aa + d = \square$
4. It remains then to equate $aa + d$ to some Square, whose side may be feigned to be either $a + e$, or $-a + u$, (which numbers, a , e and u are all yet unknown;) First, let the side of the said Square be feigned to be $a + e$, so its Square being equated to $aa + d$, this Equation ariseth, viz.

$$aa + 2ae + ee = aa + d.$$

5. Which Equation, after it is duly reduced to find what a is equal to, gives } $a = \frac{d - ee}{2e}$
6. But by the second step } $a \square \sqrt{f}$
7. Therefore by the fifth and sixth steps } $\frac{d - ee}{2e} \square \sqrt{f}$
8. And by multiplying each part of the seventh step by $2e$, it follows, that } $d - ee \square 2e\sqrt{f}$, or $e\sqrt{4f}$
9. And by adding ee to each part of the eighth step, it gives } $d \square ee + e\sqrt{4f}$
10. And by adding f , that is, $\frac{1}{4}$ of the Square of the known Coefficient $\sqrt{4f}$ in the ninth step, to each part thereof, it follows, that } $d + f \square ee + e\sqrt{4f} + f$
11. And by extracting the square Root out of each part of the tenth step, } $\sqrt{d + f} \square e + \sqrt{f}$
12. And by subtracting \sqrt{f} from each part of the eleventh step, } $\sqrt{d + f} - \sqrt{f} \square e$
13. And by comparing the latter part of the twelfth step to the first part, it's manifest that } $e \square \sqrt{d + f} - \sqrt{f}$
14. But by the first step, } $b = d + f$
15. And consequently } $\sqrt{b} = \sqrt{d + f}$

16. Where-

16. Wherefore, by setting \sqrt{b} in the place of $\sqrt{d+f}$ } . $e \Rightarrow \sqrt{b} - \sqrt{f}$
 in the thirteenth step, it's evident that }
 17. And because by the fifth step, } $ee \Rightarrow d$
 18. Therefore } $e \Rightarrow \sqrt{d}$
 19. Again, forasmuch as the side of the Square mentioned in the fourth step may be feigned to be $-a + u$, let the Square of $-a + u$ be equated to $aa + d$, as the third step requires, so this Equation ariseth, viz.

$$aa + d = aa - 2ua + uu.$$

20. Which Equation, after due Reduction, to find } $a = \frac{uu - d}{2u}$
 out what a is equal to, gives }
 21. But by the second step, } $a \sqsubset \sqrt{f}$
 22. Therefore from the twentyeth and twenty-first } $\frac{uu - d}{2u} \sqsubset \sqrt{f}$
 steps, }
 23. And by arguing to find out limits for u , in like }
 manner as before for e from the seventh to the }
 sixteenth step *inclusive*, it will at length appear, that } $u \sqsubset \sqrt{b} + \sqrt{f}$
 24. And because by the twentyeth step } $uu \sqsubset d$
 25. Therefore } $u \sqsubset \sqrt{d}$
 26. Now suppose } $192 = b$
 } $128 = f$
 27. And consequently } $64 = b - f = d$
 28. And from the sixteenth and twenty-sixth steps } $e \Rightarrow 2\frac{1}{1000}, \&c. (\sqrt{b} - \sqrt{f})$
 it follows that }
 29. And from the twenty-third and twenty-sixth } $u \sqsubset 25\frac{1}{1000}, \&c. (\sqrt{b} + \sqrt{f})$
 steps, }
 30. Likewise from the eighteenth and twenty-sixth } $e \Rightarrow 8 (\sqrt{d})$
 steps, }
 31. And from the twenty-fifth, and twenty-seventh } $u \sqsubset 8$
 steps }

But of the limits found out in the four last preceding steps, the two former only are necessary for chusing the numbers e and u , for the two latter not being so strict as the two former are useless. Then after the number e or u is duly chosen according to the said limits in the twenty-eighth and twenty-ninth steps, the number a will be discovered either by the fifth, or by the twentieth step: and lastly, the number sought by the second and twenty-sixth steps. All which will be farther illustrated by the following Canon and Examples.

CANON.

32. Take any number less than the difference, or greater than the sum of the square Roots of the two numbers given in the Question: then divide the difference between the Square of the number first taken, and the difference of the two numbers given, by the double of the number taken; and from the Square of the Quotient subtract the lesser of the two numbers given, so the Remainder shall be the number sought.

Example 1.

- Let there be two numbers given to find a third, } $b = 192$, and $f = 128$
 according to *Quest. 8.* as, }
 Thence it follows, that } $b - f = 64 = d$
 Also, } $\sqrt{b} - \sqrt{f} = 2.542, \&c.$
 And } $\sqrt{b} + \sqrt{f} = 25.170, \&c.$
 Now according to the *Canon*, take some number less }
 than $2\frac{1}{1000}$, and call it e , as } $e = 1$
 Thence it follows, that } $\frac{d - ee}{2e} = 31\frac{1}{2} = a$
 And lastly, } $aa - f = 864\frac{1}{4}$ sought.

The Proof.

$192 + 864\frac{1}{4} = 1056\frac{1}{4}$, whose $\sqrt{\quad}$ extracted is $32\frac{1}{2}$.
 $128 + 864\frac{1}{4} = 992\frac{1}{4}$, whose $\sqrt{\quad}$ extracted is $31\frac{1}{2}$.

Example 2.

Example 2.

Again, the same things being given as in Example 1. take }
 some number greater than $25\frac{1}{1000}$, and call it u , as . . . } $u = 32$
 Thence it follows that } $\frac{uu-d}{2u} = 15 = a$
 And lastly, } $aa-f = 97$ sought.

The Proof.

$192 + 97 = 289$, whose $\sqrt{\quad}$ is 17.
 $128 + 97 = 225$, whose $\sqrt{\quad}$ is 15.

Note. In divers of Diophantus's Questions, where Algebraick Quantities are to be equated to Squares, there is great use of finding out Limits, (after the manner delivered in the last preceding Resolution,) to direct how to feign the sides of the said Squares, so, as that their values in numbers may be greater than nothing; and therefore for the more ample Illustration of that Method I have framed the five Questions next following, in the Resolutions whereof, the industrious Learner will meet with no difficulty, if he be well exercis'd in the manner of resolving Quadratick Equations according to Sect. 5, 7, 9. Chap. 15. Book 1. as also in the Observations upon the first Question of this third Book.

QUEST. 9.

To find a number, call it a , that shall be less than 3, and cause $aa + 12$ to be a square number.

RESOLUTION.

1. Put Letters for the given numbers, viz. } $d = 12$
 } $f = 3$
2. Then the Question requires that $aa + d$ may be equal to a Square, but its Side (or Root) must be so feigned that the value of d may be less than f , and greater than nothing; to which end the said Side may be feigned to be either $a + e$, or $-a + u$; (which a, e and u do represent numbers yet unknown;) First therefore supposing the said Side to be $a + e$, the Square thereof is $aa + 2ae + ee$, which must be equated to $aa + d$ above-mentioned; hence this following Equation ariseth, viz.

$$aa + 2ae + ee = aa + d.$$

3. Which Equation, after due Reduction to find out the value of a , gives } $a = \frac{d-ee}{2e}$
4. And since the Question requires that a may be less than 3, } $a < f$
 viz. }
5. Therefore from the third and fourth steps } $\frac{d-ee}{2e} < f$
6. And by multiplying each part of the fifth step by $2e$, } $d-ee < 2fe$
 it follows that }
7. And by adding ee to each part of the sixth step, . . . } $d < ee + 2fe$
8. And by adding the Square of half the Coefficient $2f$ } $d + ff < ee + 2fe + ff$
 to each part of the seventh step, it gives }
9. And by extracting the Square Root out of each part } $\sqrt{d+ff} < e+f$
 of the eighth step, }
10. And by subtracting f from each part of the ninth step, } $\sqrt{d+ff} - f < e$
11. Wherefore from the tenth step, by comparing the latter } $e < \sqrt{d+ff} - f$
 part to the first, }
12. And since by the third step $ee < d$, therefore, . . . } $e < \sqrt{d}$
13. Again, for as much as the Side of the Square mentioned in the second step may be feigned to be $-a + u$, let the Square of $-a + u$ be equated to $aa + d$ as the Question requires, so this Equation ariseth, viz.

$$aa + d = aa - 2ua + uu.$$

14. Which Equation, after due Reduction to find out the value of a , gives } $a = \frac{uu-d}{2u}$

15. And

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$$\frac{2}{3}ab - \frac{1}{2}aa \Big/ \frac{4}{9}aa bbb + \frac{13}{12}a^4b - a^5 \left(\frac{2}{3}ab + \frac{13}{8}aaa + \frac{1}{2}aab + \frac{3}{4}aaa \right)$$

$$\frac{1}{2} \frac{2}{3}$$

$$0 \quad a \quad 0 \quad + \frac{13}{12}a^4b + \frac{1}{9}aaab - a^5$$

$$\frac{13}{12}a^4b - \frac{13}{16}a^5$$

$$0 \quad + \frac{1}{3}aaab - \frac{3}{16}a^5$$

$$+ \frac{1}{3}aaab - \frac{1}{4}a^4b$$

$$0 \quad - \frac{3}{16}a^5 + \frac{1}{4}a^4b$$

$$- \frac{3}{16}a^5 + \frac{1}{4}a^4b$$

$$\frac{3}{6} \frac{4}{3} \frac{2}{3} \Big/ \frac{13}{12} \left(\frac{39}{24} \frac{13}{8} \right)$$

$$\frac{1}{2} \frac{13}{8} \frac{2}{3} \Big/ \frac{1}{3} \left(\frac{3}{6} \right)$$

$$\frac{aab}{aa}$$

Div of

$$\frac{abc}{b}$$

$$\frac{a^2}{a^2}$$

$$\frac{ab+ac-a}{a}$$

$$\frac{2b-2x}{3x}$$

$$b+c-1$$

$$\frac{b-2}{3}$$

$$\begin{array}{r} 2abd+3bd \\ 3b-1 \end{array}$$

$$\frac{abd^2 + caa - 3aa}{baa - daa + aa}$$

$$\frac{4sd}{2c}$$

$$\frac{2cd}{1c}$$

$$\begin{array}{r} 2ad+sd \\ 3b-1 \end{array}$$

$$\begin{array}{r} 2ba+c-3 \\ b-d+1 \end{array}$$

$$\frac{27ab}{9ad}$$

$$\frac{3ab}{4ad}$$

$$\frac{16gh}{8gh} \quad 2$$

$$\frac{18a^4}{16aa}$$

$$8aa$$

~~27ab~~

$$\frac{30b^4c^4d}{5bbcc d}$$

$$30b^4c^4$$

$$\begin{array}{r} 28bb^4c^4 + 16bb^4 \\ 28bb \end{array}$$

$$\begin{array}{r} 2baa-3caa \\ 2b-3c \end{array}$$

$$\frac{2b-3c}{aa}$$

$$\begin{array}{r} c+d \mid ac+ad+bc+bd \mid a+b \\ \hline ac+ad \\ \hline +bc+bd \\ \hline bc+bd \end{array}$$

$$\begin{array}{r} a+b \mid aa-bb \mid a-b \\ \hline aa+ab \\ \hline -ab-bb \\ \hline -ab-bb \end{array}$$

15. And because the Question requires a to be less than 3, } $a \supset f$
viz. }
 16. Therefore from the fourteenth and fifteenth steps it } $\frac{uu - d}{2u} \supset f$
 follows that }
 17. And by continuing the Process to find out Limits for u , }
 in like manner as before for e from the fifth step to the }
 eleventh inclusive, it will at length appear that } $u \supset \sqrt{d + ff} : + f$
 18. And since by the sixteenth step, $uu \sqsubset d$, therefore } $u \sqsubset \sqrt{d}$

From the first, eleventh, twelfth, seventeenth, eighteenth, third, and fourteenth steps the following Canon is deduced, by the help whereof innumerable Answers may be found out to the Question proposed.

CANON.

19. Take any number (e) between $\sqrt{d + ff} - f$ and \sqrt{d} , that is, between $1\frac{582}{1000}$, &c. and $3\frac{464}{1000}$, &c. Or any number (u) between \sqrt{d} and $\sqrt{d + ff} + f$, that is, between $3\frac{464}{1000}$, &c. and $7\frac{582}{1000}$, &c. Then divide the difference between the Square of the number taken and d , or 12, by the double of the number taken, so the Quotient shall be the number a sought.

Example 1.

Let there be two numbers given in such manner as before } $d = 12$, and $f = 3$
 is supposed in this Quest. 9. *viz.* }
 Then according to the first limits in the Canon take some }
 number between $1\frac{582}{1000}$ and $3\frac{464}{1000}$ as 2, and call this e , } $e = 2$
viz. suppose }
 And then by the latter part of the Canon, } $\frac{d - ee}{2e} = 2 = a$ sought.

Which number 2, to wit, a , will solve the Question, for it is less than 3; and $aa + 12$ that is, 16, is a Square, as was required.

Example 2.

Again, the same things being given as in Example 1. take }
 some number between $3\frac{464}{1000}$ and $7\frac{582}{1000}$, (according to the } $u = 4$
 latter limits in the Canon,) as 4, and call this u , *viz.* suppose }
 And then you will find } $\frac{uu - d}{2u} = \frac{1}{2} = a$
 Which Fraction $\frac{1}{2}$, to wit, a , will solve the Question, for it is less than 3; and }
 $aa + 12$, that is, $12\frac{1}{4}$, is a Square as was required.

QUEST. 10.

To find out a number; call it a , that $aa - 60$ may be greater than $5a$, but less than $8a$.

RESOLUTION.

1. Put Letters for the given numbers, as } $b = 60$
 } $c = 5$
 } $d = 8$
 2. Then the Question requires that $aa - b$ may be greater }
 than ca , yet less than da ; first then let us suppose } $aa - b \sqsubset ca$
 3. Thence it follows, by adding b to each part, that . . . } $aa \sqsubset ca + b$
 4. And by subtracting ca from each part in the third step } $aa - ca \sqsubset b$
 5. And by adding the Square of half the known Coeffici- }
 ent c to each part of the fourth step, it follows that } $aa - ca + \frac{1}{4}cc \sqsubset b + \frac{1}{4}cs$
 6. And by extracting the square Root out of each part }
 of the fifth step, it gives } $a - \frac{1}{2}c \sqsubset \sqrt{b + \frac{1}{4}cc}$
 7. Wherefore by adding $\frac{1}{2}c$ to each part of the sixth step, } $a \sqsubset \frac{1}{2}c + \sqrt{b + \frac{1}{4}cc}$
 8. Again, let us suppose, as the Question also requires, that } $aa - b \supset da$
 9. Whence by arguing in like manner as before from the }
 second step to the seventh inclusive; saving that instead }
 of \sqsubset there, \supset is to be used in this latter argumenta- }
 tion, it will at length appear that } $a \supset \frac{1}{2}d + \sqrt{b + \frac{1}{4}dd}$

E

10. Thus,

10. Thus, (by the seventh and eighth steps) limits are discovered, within which any number may be taken for the value of a the number sought, viz.

$$a \sqsubset \frac{1}{2}c + \sqrt{b + \frac{1}{4}cc} : (10 \frac{6394102}{100000000}, \&c.)$$

$$a \supset \frac{1}{2}d + \sqrt{b + \frac{1}{4}dd} : (12 \frac{7177978}{100000000}, \&c.)$$

As, for example, if $a = 12$, which is within the said Limits, then $aa - 60 = 84$; also $5a = 60$, and $8a = 96$: But 84 (that is, $aa - 60$) is greater than 60 , (that is, $5a$,) and less than 96 , (that is, $8a$;) and therefore the number 12 , (that is, a ,) doth manifestly solve the Question proposed.

QUEST. II.

To find out a number, call it a , that shall be greater than $10 \frac{6394102}{100000000}$, but less than $12 \frac{7177978}{100000000}$, and cause $aa - 60$ to be equal to some square number.

RESOLUTION.

1. Put Letters for the given numbers, as, $\left. \begin{array}{l} b = 60, \\ f = 10 \frac{6394102}{100000000}, \\ d = 12 \frac{7177978}{100000000}. \end{array} \right\}$
2. Then, (according to the import of the Question,) $aa - b$ must be equal to some Square, but the side thereof must be so feigned that the value of a may be greater than f , but less than d ; to which purpose, the said side may be feigned to be $a - e$, or $e - a$, (which a and e do represent numbers unknown,) and then the Square of the said $a - e$, or $e - a$ being equated to $aa - b$ above mentioned, gives this Equation, viz.

$$aa - b = aa - 2ae + ee.$$
3. Which Equation, after due Reduction to find out the value of a , gives $\left. \begin{array}{l} a = \frac{ee + b}{2e} \end{array} \right\}$
4. But according to the Question, $\left. \begin{array}{l} a \sqsubset f \end{array} \right\}$
5. Therefore from the third and fourth steps, $\left. \begin{array}{l} \frac{ee + b}{2e} \sqsubset f \end{array} \right\}$
6. And by multiplying each part of the fifth step by $2e$, it follows that $\left. \begin{array}{l} ee + b \sqsubset 2fe \end{array} \right\}$
7. And by subtracting b from each part of the sixth step, $\left. \begin{array}{l} ee \sqsubset 2fe - b \end{array} \right\}$
8. And by equal subtraction of $2fe$ from each part of the seventh step, $\left. \begin{array}{l} ee - 2fe \sqsubset -b \end{array} \right\}$
9. And by adding ff , that is, the Square of half the known Coefficient $2f$ in the eighth step, to each part, it follows that $\left. \begin{array}{l} ee - 2fe + ff \sqsubset ff - b \end{array} \right\}$
10. And by extracting the square Root out of each part of the ninth step, $\left. \begin{array}{l} e - f \sqsubset \sqrt{ff - b} : \end{array} \right\}$
11. Wherefore by adding f to each part of the tenth step, it's evident that $\left. \begin{array}{l} e \sqsubset f + \sqrt{ff - b} : \end{array} \right\}$
12. Again, because $f - e$, (as well as $e - f$;) may be the side of the Square $ee - 2fe + ff$ in the first part of the ninth step, it thence follows, that $\left. \begin{array}{l} f - e \sqsubset \sqrt{ff - b} : \end{array} \right\}$
13. And by adding e to each part of the twelfth step, $\left. \begin{array}{l} f \sqsubset e + \sqrt{ff - b} : \end{array} \right\}$
14. And by subtracting $\sqrt{ff - b}$ from each part of the thirteenth step, $\left. \begin{array}{l} f - \sqrt{ff - b} \sqsubset e \end{array} \right\}$
15. Wherefore from the fourteenth step, by comparing the latter part to the former, 'tis manifest that $\left. \begin{array}{l} e \supset f - \sqrt{ff - b} : \end{array} \right\}$
16. Again, because the Question requires $\left. \begin{array}{l} a \supset d \end{array} \right\}$
17. It follows from the third and sixteenth steps, that $\left. \begin{array}{l} \frac{ee + b}{2e} \supset d \end{array} \right\}$
18. Whence by arguing in like manner as before from the fifth step to the fifteenth *inclusive*, saving that d is to be used here, instead of f there, and \supset instead of \sqsubset , it will at length appear that $\left. \begin{array}{l} e \supset d + \sqrt{dd - b} : \\ e \sqsubset d - \sqrt{dd - b} : \end{array} \right\}$

From the eleventh, eighteenth, fifteenth, first and third steps the following Canon ariseth, which will find out innumerable Answers to the Question proposed.

CANON.

CANON.

19. Take any number (e) greater than $f + \sqrt{ff - b}$: but less than $d + \sqrt{dd - b}$: (that is, any number between $17\frac{213}{1000}$, &c. and $22\frac{304}{1000}$, &c.) or any number greater than $d - \sqrt{dd - b}$: but less than $f - \sqrt{ff - b}$: (that is, any number between $2\frac{61}{1000}$, &c. and $3\frac{24}{1000}$, &c.) Then $\frac{ee - b}{2e}$ shall be equal to (a) the number sought.

Examples.

First, for the number e take 22 which is within the former limits in the Canon; then $\frac{ee - b}{2e}$ gives $12\frac{4}{11}$ for the number a sought by the Question: For if from the Square of $12\frac{4}{11}$, to wit, $\frac{18496}{121}$, you subtract the given number 60, (or b), the Remainder $\frac{11236}{121}$ is a Square whose side is $\frac{106}{11}$; and the said $12\frac{4}{11}$ (that is, a) is greater than $10\frac{61}{1000}$, &c. but less than $12\frac{71}{1000}$, &c. as the Question requires.

Again, for the number e take 3 which is within the latter limits in the Canon; then $\frac{ee - b}{2e}$ gives $11\frac{1}{2}$ (the number a), which will likewise solve the Question proposed: For if from the Square of $11\frac{1}{2}$ you subtract 60, there will remain a Square, to wit, $\frac{229}{4}$, whose side is $\frac{15}{2}$; and the said $11\frac{1}{2}$ (that is, a) is greater than $10\frac{61}{1000}$, &c. but less than $12\frac{71}{1000}$, &c. as was required.

QUEST. 12.

To find out a number, call it a , that shall be greater than $2\frac{1}{2}$, (a number given,) and cause $aa - 4a + 2$ to be equal to some square number.

RESOLUTION.

1. Put letters for the given numbers, as, $\dots \dots \dots \left\{ \begin{array}{l} b = 4\frac{1}{2} \\ d = 2\frac{1}{2} \\ f = 2. \end{array} \right.$
2. Then the Question requires that $aa + ba + f$ may make a square number, but its side must be so feigned that the value of a may be greater than d . Now to cause those effects, the said side may be feigned to be $a - e$, or $e - a$, (which e and a do represent numbers yet unknown,) and then the Square of $a - e$ or $e - a$, that is, $aa - 2ae + ee$, being equated to $aa + ba + f$, gives this Equation, viz.

$$aa + ba + f = aa - 2ae + ee.$$
3. Which Equation, after due Reduction to find out the value of a , gives $a = \frac{ee - f}{2e + b}$
4. And because the Question requires $a \sqsupseteq d$
5. It follows from the third and fourth steps, that $\frac{ee - f}{2e + b} \sqsupseteq d$
6. And by multiplying each part of the fifth step, by $2e + b$, $ee - f \sqsupseteq 2de + db$
7. And by adding f to each part of the sixth step, $ee \sqsupseteq 2de + db + f$
8. And by subtracting $2de$ from each part of the seventh step, $ee - 2de \sqsupseteq db + f$
9. And by adding the Square of half the known Coefficient $2d$ in the eighth step, to each part, it's manifest that $ee - 2de + dd \sqsupseteq db + f + dd$
10. And by extracting the square Root out of each part of the ninth step, $e - d \sqsupseteq \sqrt{db + f + dd}$
11. Wherefore by adding d to each part of the tenth step, it's evident that $e \sqsupseteq \sqrt{db + f + dd} + d$
12. And consequently by resolving the latter part of the eleventh step into numbers, according to the Positions in the first, $e \sqsupseteq 6\frac{112}{1000}$, &c.
13. The third step also shews that $ee \sqsupseteq f$, and consequently $e \sqsupseteq \sqrt{f}$, ($1\frac{414}{1000}$, &c.)

But this latter limit for the chusing of e is useless, for if e be greater than $6\frac{772}{1000}$, &c. as appears by the twelfth step, it is evidently greater than $1\frac{414}{1000}$, &c.

14. Lastly, from the eleventh, twelfth, third and first steps the following Canon ariseth, which will find innumerable Answers to the Question proposed.

CANON 1.

Take any number greater than $\sqrt{db + f + dd} : + d$, (*viz.* greater than $6\frac{772}{1000}$, &c.) and call the number taken e . Then $\frac{ee - f}{2e + b}$ shall be equal to the number a sought.

15. But if it were desired to find a number a that might be less than $2\frac{1}{2}$, and greater than nothing, and make $aa + 4a + 2$ to be a square number, then the same Positions and Procefs being made as before, saving that \sqsupset is to be used instead of \sqsubset from the fourth step to the twelfth inclusive, at length there would arise this following

CANON 2.

Take any number (e) greater than \sqrt{f} , but less than $\sqrt{db + f + dd} : + d$: (*viz.* any number between $1\frac{414}{1000}$, &c. and $6\frac{772}{1000}$, &c.) Then $\frac{ee - f}{2e + b}$ will give the number a sought.

An Example of the first Canon.

For the number e take 8 which exceeds $6\frac{772}{1000}$, &c. as the first Canon doth direct. Then $\frac{ee - f}{2e + b}$ gives $3\frac{1}{10}$ for the number a sought; for 'tis greater than $2\frac{1}{2}$ (or d), and $aa + 4a + 2$ makes a Square, to wit, $24\frac{1}{10}$, whose side is $4\frac{2}{5}$, as was required.

Note, That $a + u$ might be feigned to be the side of the Square mentioned in the second step, and thence limits would be discovered to chuse the number u , by which the number a would consequently be made known; but I leave the search of these latter limits as an exercise for the Learner.

QUEST. 13.

To find out a number, call it a ; that shall be greater than 1, but less than 4, and make $121 + 45a - 9aa$ to be a square number.

RESOLUTION.

1. First put Consonants to represent the numbers given in the Question, as,

}	$b = 1$
	$d = 4$
	$f = 11$
	$ff = 121$
	$g = 45$
$h = 9$	
2. Then the Question requires that $ff + ga - haa$ may make a square number, whose side must be so feigned that the value of a may be greater than b , but less than d : To which purpose the said side may be feigned to be $f + ea$, or $f - ua$; (where a, e, u do represent numbers unknown:) First then let the said side be feigned $f + ea$, and let its Square $ff + 2fea + ecaa$ be equated to $ff + ga - haa$ above-mentioned, so this following Equation ariseth, *viz.*

$$ff + 2fea + ecaa = ff + ga - haa.$$
3. Which Equation, after due Reduction to find out the value of a , gives

}	$a = \frac{g - 2fe}{b + ee}$
	$a \sqsubset b$
4. And because the Question requires
5. It follows from the third and fourth steps, that

}	$\frac{g - 2fe}{b + ee} \sqsubset b$
	$g - 2fe \sqsubset bh + bee$
6. And by multiplying each part in the fifth step, by the Denominator $b + ee$, it follows, that
7. And by subtracting bh from each part in the sixth step,

}	$g - 2fe - bh \sqsubset bee$
---	------------------------------

8. And

8. And by adding $2fe$ to each part in the seventh step } $g - bh \sqsubset bce + 2fe$
 9. And by dividing every quantity in the eighth step } $\frac{g - bh}{b} \sqsubset ce + \frac{2fe}{b}$
 by b , that bce may be freed from its Coefficient b , }
 it follows that }
 10. And by adding $\frac{ff}{bb}$, that is, the Square of half }
 the Coefficient $\frac{2f}{b}$ to each part of the ninth step, } $\frac{ff + bg - bbb}{bb} \sqsubset ce + \frac{2f}{b}e + \frac{ff}{bb}$
 it gives }
 11. And by extracting the square Root out of each } $\sqrt{\frac{ff + bg - bbb}{bb}} \sqsubset e + \frac{f}{b}$
 part of the tenth step, }
 12. And by subtracting $\frac{f}{b}$ from each part of the } $\sqrt{\frac{ff + bg - bbb}{bb}} - \frac{f}{b} \sqsubset e$
 eleventh step, }
 13. Therefore from the twelfth step, by comparing } $e \sqsubset \sqrt{\frac{ff + bg - bbb}{bb}} - \frac{f}{b}$
 the latter part to the first, it's manifest that . . . }
 14. Again, because the Question requires . . . } $a \sqsupset d$
 15. And by the third step, } $a = \frac{g - 2fe}{b + ce}$
 16. It follows from the fourteenth and fifteenth steps, } $\frac{g - 2fe}{b + ce} \sqsupset d$
 that }
 17. Whence by arguing in like manner as before }
 from the fifth step to the thirteenth inclusive, it } $e \sqsubset \sqrt{\frac{ff + dg - ddb}{dd}} - \frac{f}{d}$
 will at length appear that }
 18. Again, let the side of the Square mentioned in the second step be feigned to be $f - ua$,
 and then the Square of $f - ua$ being equated to $ff + ga - haa$, (as the Question
 requires) this following Equation ariseth, viz.

$$ff - 2fua + uaa = ff + ga - haa.$$

19. Which Equation gives this value of a , viz. . . } $a = \frac{g + 2fu}{b + uu}$

20. But the value of a last mentioned must (as the Question requires) be greater than b ,
 and less than d ; and if the Process be continued from the last preceding step, to find
 out limits for u in like manner as before for e from the third step to the seventeenth
 inclusive, it will at length appear that

$$u \sqsupset \frac{f}{b} + \sqrt{\frac{ff + bg - bbb}{bb}},$$

$$u \sqsubset \frac{f}{d} + \sqrt{\frac{ff + dg - ddb}{dd}}.$$

Now after any number is taken for the value of e within the limits in the thirteenth and
 seventeenth steps, the number a required by the Question will be discovered by the third
 and first steps. Or, after any number is taken for the value of u within the limits in the
 preceding twentieth step, the number a sought will be made known by the nineteenth
 and first steps. All which will be made manifest by the following Canon and Examples.

CANON.

21. Take any number less than $\sqrt{\frac{ff + bg - bbb}{bb}} - \frac{f}{b}$; but greater than

$$\sqrt{\frac{ff + dg - ddb}{dd}} - \frac{f}{d}, \text{ (that is, any number between } \frac{1}{1000}, \text{ \& c. and } \frac{382}{1000}, \text{ \& c.)}$$

and call the number taken e ; then $\frac{g - 2fe}{b + ce}$ shall be the number a sought. Or take any

number less than $\frac{f}{b} + \sqrt{\frac{ff + bg - bbb}{bb}}$, but greater than $\frac{f}{d} + \sqrt{\frac{ff + dg - ddb}{dd}}$,

(that is, any number between $23\frac{529}{1000}, \text{ \& c. and } 5\frac{882}{1000}, \text{ \& c.})$ and call the number

taken u ; then $\frac{g + 2fu}{b + uu}$ will give the number a sought.

Examples.

Examples.

Suppose $e = 1$, which is within the first limits in the Canon, then $\frac{g - 2fe}{b + ee} = \frac{22}{10} = a$ the number sought: For $\frac{22}{10}$ is greater than 1, but less than 4; and if $a = \frac{22}{10}$, then $121 + 45a - 9aa = \frac{12682}{100}$, which is a Square, (for its side is $\frac{1132}{10}$) as the Question requires.

Again, suppose $e = \frac{1}{2}$, (which is likewise within the first limits,) then $\frac{g - 2fe}{b + ee} = \frac{16}{5} = a$ the number sought: For $\frac{16}{5}$ is greater than 1, but less than 4; and if $a = \frac{16}{5}$, then also $121 + 45a - 9aa$ makes a Square, to wit, $\frac{1221}{25}$, whose side is $\frac{111}{5}$.

Again, suppose $e = 18$, which is within the latter limits in the Canon, then $\frac{g - 2fe}{b + ee} = \frac{42}{37} = a$ the number sought: For if $a = \frac{42}{37}$, then $121 + 45a - 9aa$ makes a Square, to wit, $\frac{222622}{1369}$, whose side is $\frac{472}{37}$; and $\frac{42}{37}$ (or a) is greater than 1, but less than 4, as the Question requires.

QUEST. 14. (Quest. 13. Lib. 2. Diophant.)

To find a number, that if it be subtracted first from 192, and then from 64, each Remainder may be a Square.

RESOLUTION.

1. For the number sought put } a
2. Which number must be such, that each of these Quantities (or Remainders) may make a Square, viz. } $192 - a = \square$
 $64 - a = \square$
3. Now to resolve that Duplicate Equality; first, (by Canon 2. Quest. 7. of this Book 3.) find out two such square numbers that their difference may be equal to 128, that is, the difference of the two given numbers 192 and 64, or the difference between the two Algebraick Quantities $192 - a$ and $64 - a$, and that the greater Square may be less than 192, (the greater of the two numbers given in the Question;) but two such Squares are 144 and 16.
4. Then from either of these Equations, } $192 - a = 144$
 $64 - a = 16$
5. One and the same value of a , that is, the number sought will be discovered, viz. } $a = 48$
7. I say 48 will solve the Question, as will be evident by

The Proof.

$$\left. \begin{aligned} 192 - 48 &= 144, \\ 64 - 48 &= 16, \end{aligned} \right\} \text{ which are Squares, as was required.}$$

The premisses give this following

CANON.

8. First, (by the second Canon of the seventh Question of this third Book,) find out two square numbers in the same difference with the numbers given, and that the greater Square may be less than the greater number given; whence consequently, (as will appear by the following Theorem,) the lesser Square shall be less than the lesser number given: Then from the greater number given subtract the greater Square, or from the lesser number subtract the lesser Square, so shall either of those Remainders (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by this following

THEOREM.

9. If two Square numbers have the same difference as two other numbers, and that the greater number exceeds the greater Square, then the lesser number shall exceed the lesser Square, and the excess of the greater number above the greater Square shall be equal to the excess of the lesser number above the lesser Square. To make this manifest, let dd and gg represent two square numbers, whereof dd is the greater; also b the greater, and c the lesser of two other numbers; Then, (according to the import of the Theorem,)

10. Suppose } $b - c = dd - gg \sqsubset 0$
 11. And } $b \sqsubset dd$
 12. Then by adding c to each part of the Equation in the } $b = c + dd - gg$
 tenth step, it follows that }
 13. Therefore by subtracting dd from each part of the last } $b - dd = c - gg$
 Equation, (which subtraction appears to be possible by the }
 eleventh and tenth steps,) that which the Theorem affirms }
 is manifest, viz. }
14. And consequently either of those two equal Excesses or Remainders which make the last Equation, shall be a number that will solve the Question proposed; for it is manifest, that if the first Excess $b - dd$ be subtracted from b , and the latter Excess $c - gg$ from c , the Remainders will be Squares, to wit, dd and gg .

To solve the foregoing 14th Question after another manner, viz.

To find a number, that if it be subtracted first from 9, and then from 21, each Remainder may be a Square.

RESOLUTION.

1. It is evident, that if $9 - aa$ be subtracted from 9, there will } $9 - aa$
 remain a Square, to wit, aa , therefore for the number sought put }
 2. And then if $9 - aa$ be subtracted from 21, the Remainder must } $aa + 12 = \square$
 likewise make a Square, therefore }
 3. It remains to equate $aa + 12$ to some Square, whose side must be so feigned that the value of a may be less than 3; (for by the first step $aa > 9$, and consequently $a > 3$;) But to cause that effect, the said side may be feigned to be either $a +$ any absolute number between $1\frac{1}{10000}$, &c. and $3\frac{4}{10000}$, &c. or else $-a +$ any absolute number between $3\frac{4}{10000}$, &c. and $7\frac{1}{10000}$, &c. (which limits are found out by the ninth Question of this third Book;) let therefore the said side be feigned $a + 2$, and then by equating the Square of $a + 2$ to $aa + 12$ before-mentioned in the second step, this Equation ariseth, viz.

$$aa + 4a + 4 = aa + 12.$$

4. Which Equation gives } $a = 2.$

Therefore from the fourth and first steps the number sought is 5; for if it be subtracted from 9 and 21 severally, it leaves the Squares 4 and 16.

From this latter Resolution of Quest. 14. (respect being had to the foregoing ninth Question,) a Canon may be deduced to find out innumerable Answers to the said fourteenth Question; but I leave it to the Learners exercise.

QUEST. 15. (Quæst. 14. Lib. 2. Diophant.)

To find a number, from which if 27 and 15 (two numbers given) be severally subtracted, each Remainder may be a Square.

RESOLUTION 1.

1. For the number sought put } a
 2. Which number must be such that each of these Quantities or } $a - 27 = \square$
 Remainders may make a Square, viz. } $a - 15 = \square$
 3. Now to resolve that Duplicate Equality, find out (by Canon 2. of the preceding Quest. 5.) two square numbers whose difference may be 12, that is, the difference of the two given numbers 27 and 15; but here is no need of limiting either of the said Squares: Suppose then the said Squares are found 16 and 4,

$$4. \text{ Then from either of these Equations, } \dots \dots \dots \left\{ \begin{array}{l} a - 27 = 4 \\ a - 15 = 16 \end{array} \right.$$

5. The number a sought will be discovered, viz. } $a = 31$

I say 31 will solve the Question; for if from 31 you subtract 27 and 15 severally, the Remainders are Squares, to wit, 4 and 16.

From the premisses there ariseth this following

CANON.

C A N O N.

6. First, (by the preceding fifth Question,) find two square numbers that shall have the same difference as the two numbers given; then add the lesser Square to the greater number, or the greater Square to the lesser number; so shall either of those summs (for they are equal to one another) be the number sought.

The truth of which Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by the following

T H E O R E M.

7. If two square numbers, suppose dd the greater and gg the lesser, have the same difference as two other numbers, suppose b the greater and c the lesser; then the sum of the lesser Square and the greater number shall be equal to the sum of the greater Square and the lesser number: For,

By supposition, } $b - c = dd - gg$

And by adding gg to each part, } $b - c + gg = dd$

Wherefore by adding c to each part of the last Equation, } $b + gg = c + dd$
that which the Theorem affirms is manifest, viz. }

8. And consequently either of those two equal summs in the last Equation shall be a number to solve the Question proposed: For it is evident, that if b be subtracted from the first summ $b + gg$, and c from the latter summ $c + dd$, the Remainders are Squares, to wit, gg and dd .

To solve the foregoing Quest. 15. after another manner, viz.

To find a number, from which if 27 and 15 be severally subtracted, each Remainder may be a Square.

R E S O L U T I O N 2.

1. It is evident that if 27 be subtracted from $aa + 27$, the Remainder will be a Square, to wit, aa ; therefore for the number sought put } $aa + 27$
2. But then 15 being subtracted from the said $aa + 27$, the Remainder must likewise be equal to a Square, therefore . . . } $aa + 12 = \square$
3. It remains to equate $aa + 12$ to some Square, whose side may be feigned either $a +$ any absolute number less than $\sqrt{12}$, or $3\frac{4}{3}$, &c. or else $-a +$ any absolute number greater than the said $3\frac{4}{3}$, &c. (which limits are found out in like manner as in the foregoing Quest 9.) Let therefore the said side be feigned $a + 3$, and then by equating the Square of $a + 3$, that is, $aa + 6a + 9$ to $aa + 12$, the value of a will thence be found $\frac{1}{2}$, and consequently the number sought, (which in the first step was put $aa + 27$) shall be $27\frac{1}{4}$, which will solve the Question: For if from $27\frac{1}{4}$ the given numbers 27 and 15 be severally subtracted, the Remainders will be Squares, to wit, $\frac{1}{4}$ and $\frac{42}{4}$.

But if this second manner of resolving Quest. 15. be formed by Literal Algebra, (like to the third manner of resolving the preceding Quest. 8.) there will arise this

C A N O N.

Take any number greater than the summ, or less than the difference of the square Roots of the two numbers given; then divide the difference between the Square of the number taken and the difference of the given numbers by the double of the number taken; lastly, to the Square of that Quotient add the greater of the numbers given, so shall the summ be the number sought.

A third way of solving the preceding Quest. 15.

Let the Positions in the first and second steps of the preceding Resolut. 2. be resumed; then since $aa + 12$ must be equal to a Square, 'tis evident that 12 is the difference between that Square and aa ; therefore by the preceding fifth Question find two Squares whose difference may be 12; such are 16 and 4, the lesser of which shall be the value of aa ; therefore $aa + 27$ which was put for the number sought will be found 31, as before in the first Resolution of this Question.

QUEST. 16.

QUEST. 16.

To find a number, that if 12 be added to it; and 8 subtracted from the same; as well the Summ as the Remainder may be a Square.

RESOLUTION.

1. For the number sought put } a
2. Then each of these Quantities must make a Square, viz. } $a + 12 = \square$
 $a - 8 = \square$
3. Now to resolve that Duplicate Equality, first, subtract $a - 8$ from $a + 12$; and the Remainder is 20; this is equal as well to the summ of the given numbers as to the difference of the two Squares sought: Then (by the second Canon of the fifth Question of this third Book) find two Squares whose difference shall be 20; such are 36 and 16.
4. Then from either of these Equations } $a + 12 = 36$
 $a - 8 = 16$
5. The number a sought will be made known, viz. . . . } $a = 24$

Which number found out, to wit, 24, will solve the Question; for if it be increased with 12, and lessened by 8, the Summ and Remainder are Squares, to wit, 36 and 16. The substance of the Resolution is contain'd in this following

CANON.

6. First, (by the second Canon of the fifth Question aforegoing) find out two square numbers whose difference shall be equal to the summ of the two numbers given; then subtract the number given to be added (whether it be the greater or the lesser of those given) from the greater Square, or add the number given to be subtracted to the lesser Square; so as well the Remainder as the Summ (for they are equal to one another) shall be the number sought.

The certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by this following

THEOREM.

7. If there be two square numbers, suppose dd the greater, and gg the lesser, whose difference is equal to the summ of two other numbers, b and c ; then the excess of the greater Square above either of the two numbers, shall be equal to the summ of the lesser Square and the other number.
8. For, by supposition } $dd - gg = b + c$
9. Whence by adding gg to each part, it follows that . . . } $dd = gg + b + c$
10. Therefore by subtracting b from each part of the last Equation, this ariseth, viz. } $dd - b = gg + c$
11. Likewise, by subtracting c from each part of the Equation in the ninth step, there remains } $dd - c = gg + b$

Wherefore from the two last Equations, the truth of the Theorem, and consequently of the Canon is manifest.

QUEST. 17.

To find a number, that if it be added to, and subtracted from a given square number, suppose 4, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number sought put } a
2. Which number must be such, that if it be added to and subtracted from 4, as well the Summ as the Remainder may make a Square, viz. } $4 + a = \square$
 $4 - a = \square$
3. To resolve that Duplicate Equality, first, (after the manner of Example 3. Canon 1. Resolut. 2. Quest. 2. of this Book,) divide 8 the double of the given Square 4 into two unequal Squares, $\frac{1}{2} \frac{26}{5}$ and $\frac{4}{5}$.

F

4. Then

4. Then from either of these Equations, $\left. \begin{array}{l} 4 + a = \frac{126}{25} \\ 4 - a = \frac{4}{25} \end{array} \right\}$
 5. The number sought will be discovered, viz. $\left. \begin{array}{l} a = \frac{26}{25} \end{array} \right\}$

Which number $\frac{26}{25}$ will solve the Question proposed; for if it be added to, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{126}{25}$ and $\frac{4}{25}$, whose sides are $\frac{14}{5}$ and $\frac{2}{5}$. By the like Operation (the substance whereof is contained in the following Canon,) you may find out innumerable Answers to this 17th Question.

CANON.

6. Divide the double of the given Square into two unequal Squares, (by the preceding Quest. 2.) then from the greater of the two Squares found out subtract the given Square, or from this subtract the lesser of those two, so shall either of the Remainders, (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in this 17th Question, will be evident by the following

THEOREM.

7. If two square numbers, suppose cc the greater, and dd the lesser, be equal to the double of a square number, as $bb + bb$, or $2bb$; then the excess of the greater of those unequal Squares above one of the equal Squares shall be equal to the excess of the other of the equal Squares above the lesser of the unequal Squares.

8. For by supposition $\left. \begin{array}{l} cc + dd = bb + bb \\ . . . cc \sqsupset dd \\ . . . cc \sqsupset bb \end{array} \right\}$
 9. Also by supposition $\left. \begin{array}{l} . . . dd \sqsupset bb \end{array} \right\}$
 10. Therefore $\left. \begin{array}{l} . . . dd \sqsupset bb \end{array} \right\}$
 11. And $\left. \begin{array}{l} . . . dd \sqsupset bb \end{array} \right\}$

12. And by subtraction of bb from each part of the Equation in the eighth step, $\left. \begin{array}{l} cc + dd - bb = bb \\ \end{array} \right\}$
 13. Wherefore by subtracting dd from each part of the last Equation, (which subtraction the 10th and 11th steps do shew to be possible,) that which the Theorem asserts is manifest, viz. $\left. \begin{array}{l} cc - bb = bb - dd \end{array} \right\}$

14. And consequently either of the Excesses (or Remainders) which make the last preceding Equation shall be the number a sought: For if the first Excess $cc - bb$ be added to bb , and the latter Excess $bb - dd$ subtracted from bb , the Summ and Remainder are Squares, to wit, cc and dd .

QUEST. 18.

To find a number, that if a given Square 9 be added to that number, and from another given Square 4 the same number sought be subtracted, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number sought put $\left. \begin{array}{l} a \\ \end{array} \right\}$
 2. Then the Question requires that $\left. \begin{array}{l} 9 + a = \square \\ 4 - a = \square \end{array} \right\}$
 3. To resolve that Duplicate Equality, first (by the preceding Quest. 4.) divide 13 the summ of the given Squares 9 and 4 into two such other Squares that one of these found may exceed 9 the Square given to be added, but two such Squares are $\frac{324}{25}$ and $\frac{1}{25}$, whose summ is 13, and the greater of them exceeds 9.

4. Then from either of these Equations $\left. \begin{array}{l} 9 + a = \frac{324}{25} \\ 4 - a = \frac{1}{25} \end{array} \right\}$
 5. The number (a) sought is discovered, viz. $\left. \begin{array}{l} a = \frac{22}{25} \end{array} \right\}$

Which number found out, to wit, $\frac{22}{25}$, will solve the Question; for if it be added to 9, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{324}{25}$ and $\frac{1}{25}$, whose sides are $\frac{18}{5}$ and $\frac{1}{5}$. By the like Operation (the substance whereof is express in the following Canon,) you may find out innumerable Answers to this 18th Question, because (by Quest. 4. of this Book) the summ of two Squares may be divided into as many pairs of Squares as you please, such, that one of each pair shall consist within given limits.

CANON.

CANON.

7. First, (by Quest. 4. of this Book,) divide the summ of the two Squares given into two such Squares, that the greater of these found out may exceed the Square given to be added; then from the greater of the two Squares found out subtract the Square given to be added; or, from the other Square given subtract the other Square found out; so shall either of the Remainders (for they are equal to one another) be the number sought, The certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question proposed will be manifest by this following

THEOREM.

8. If the summ of two square numbers, suppose bb the greater, and cc the lesser, be found equal to the summ of two other unequal Squares, dd and ff , and that the greater of of the two former exceeds either of the two latter, then the other of the two latter shall exceed the lesser of the two former, and one excess shall be equal to the other. For,
9. By supposition } $bb + cc = dd + ff$
 10. By supposition also } $bb > dd$
 11. Therefore } $cc > ff$
 12. And by subtracting dd from each part of the Equation in } $bb + cc - dd = ff$
 the ninth step, this ariseth, viz. }
 13. Wherefore by subtracting cc from each part of the last } $bb - dd = ff - cc$
 Equation, (which subtraction the tenth and eleventh steps do }
 shew to be possible) that which the Theorem affirms is ma- }
 nifest, viz. }
 14. And consequently the truth of the Canon and Resolution of the Duplicate Equality in this 18th Question is evident.

QUEST. 19.

To find a square number, that if it be increased or lessened by its side, may make a Square.

RESOLUTION.

1. For the side of the Square sought put a
 2. Therefore the Square it self is aa
 3. Then the Question requires $aa + a = \square$
 $aa - a = \square$
 4. Which Duplicate Equality differs but little from that in the foregoing seventeenth Question, and may be resolved thus: First, (after the manner of Example 3. Canon 1. Resolut. 2. Quest. 2. of this Book, divide 2, which is compos'd of two Squares, 1 and 1, suppos'd to be prefixt to aa and aa in the Duplicate Equality in the third step; into two unequal Squares, $\frac{4}{2}a$ and $\frac{2}{2}a$; then multiply each of these by aa , and equate the greater Product $\frac{4}{2}aa$ to $aa + a$, or the lesser Product $\frac{2}{2}aa$ to $aa - a$, so from either of those Equations one and the same value of a will be discovered, viz. $a = \frac{2}{2}a$, which is the side of the Square sought; for if $\frac{2}{2}a$ be added to and subtracted from its Square $\frac{4}{2}a$, the Summ and Remainder are Squares, to wit, $\frac{12}{2}a$ and $\frac{2}{2}a$; whose sides are $\frac{3}{2}a$ and $\frac{1}{2}a$. It is also easie to be perceived that this Question is capable of innumerable Answers.

QUEST. 20.

To find two numbers in a given Reason, suppose the greater to the lesser as 2 to 1; and that the Square of the summ of the two numbers being added to each of them may make square numbers.

RESOLUTION.

1. For the lesser number sought put a
 2. Then the greater, to the end it may be to the lesser as 2 to 1, } $2a$
 shall be }
 3. Therefore the summ of the two numbers sought is $3a$
 4. And the Square of their summ is $9aa$
 5. To

5. To which Square if the two numbers $2a$ and a be severally } $9aa + 2a = \square$
 added, each sum must be a Square, therefore . . . } $9aa + a = \square$

6. Which Duplicate Equality, (according to *Diophantus's* Method, before explain'd and demonstrated in the Observations upon the first manner of solving the eighth Question of this Book,) may be resolved thus; *viz.* First, subtract $9aa + a$ from $9aa + 2a$, and the Remainder a is the difference of the two Squares that are to be equated to those Algebraick Quantities; then find two Squares whose difference may be equal to the said difference a , but with this Caution, that in each of those Squares there may be found $9aa$, to the end that when the greater Square is equated to $9aa + 2a$, or the lesser to $9aa + a$, the said $9aa$, after due Reduction, may vanish, and an Equation remain between some number of a and some known Rational number, whence the value of a will be expressible by some known number either Affirmative or Negative. Now to find out two such Squares, let two numbers be taken, such, that (according to *Canon 2. Quest. 5.* of this Book,) the Product of their Multiplication may make a , and that the half of their sum may consist of $3a +$ some absolute number, and the half of their difference of $2a -$ some absolute number, (for then it will follow that as well in the Square of the said half sum, as in the Square of the said half difference there will be found $9aa$;) whence we may infer, that the sum of the said two numbers must consist of $a +$ some absolute number, and their difference of $6a -$ some absolute number; to which purpose, divide 1 by 6 , (1 because the difference is $1a$, and 6 because it is the double of the square Root of 9 which is prefixt to aa ;) so the Quotient is $\frac{1}{6}$, and consequently $6a$ multiplied by $\frac{1}{6}$ makes a ; whence 'tis evident that the only two numbers fit for the purpose aforesaid are $6a$ and $\frac{1}{6}$; whose half sum is $3a + \frac{1}{12}$, and half difference $3a - \frac{1}{12}$, (or $-3a + \frac{1}{12}$;) therefore the Squares of the said half sum and half difference are $9aa + \frac{1}{4}a + \frac{1}{144}$ and $9aa - \frac{1}{4}a + \frac{1}{144}$; now let the greater of those Squares be equated to $9aa + 2a$, and the lesser to $9aa + a$, so these two following Equations will arise,

$$viz. \begin{cases} 9aa + 2a = 9aa + \frac{1}{2}a + \frac{1}{36} \\ 9aa + a = 9aa - \frac{1}{2}a + \frac{1}{36} \end{cases}$$

7. Then from either of those Equations, after due Reduction, the value of a will be found to be $\frac{1}{216}$, and consequently, (from the first and second steps,) the numbers sought are $\frac{1}{216}$ and $\frac{1}{216}$, which will solve the Question proposed: For first, the greater hath such proportion to the lesser as 2 to 1 ; and if the Square of the sum of $\frac{1}{216}$ and $\frac{1}{216}$ be added to them severally, the two sums will be Squares, to wit $\frac{1}{3184}$ and $\frac{1}{3184}$, whose sides are $\frac{1}{72}$ and $\frac{1}{72}$.

8. In like manner, to resolve this Duplicate Equality, *viz.*

$$If \begin{cases} 4aa + 3a - 1 = \square \\ 4aa - a - 1 = \square \end{cases} \text{ what is } a = ?$$

First, I find the difference of those two Algebraick Quantities to be $4a$; then I search out two Quantities that being mutually multiplied may make $4a$, and that as well in half their sum as in half their difference there may be found $2a$, (that is, the square Root of $4aa$;) so by working as before is directed, I find $4a$ and 1 to be the only two Quantities agreeing with those conditions: Then the Square of half the sum of $4a$ and 1 , *viz.* the Square of $2a + \frac{1}{2}$ being equated to $4aa + 3a - 1$ will give $a = \frac{1}{4}$; or, the Square of half the difference of $4a$ and 1 , *viz.* the Square of $2a - \frac{1}{2}$ being equated to $4aa - a - 1$, gives $a = \frac{1}{4}$, as before.

QUEST. 21.

To find two numbers in a given Reason, suppose the greater to the less as 3 to 2 , and that the sum of the numbers being added to each of their Squares, may make Squares.

RESOLUTION.

1. For the lesser number put } $2a$
2. Then the greater (to the end that both numbers may be in the }
 Reason prescribed,) shall be } $3a$
3. Therefore their sum is } $5a$

4. Which

4. Which summ added to the Square of each of the said two numbers $2a$ and $3a$, must (as the Question requires) make a Square, therefore

$$\left. \begin{aligned} 4aa + 5a &= \square \\ 9aa + 5a &= \square \end{aligned} \right\}$$

5. Now in order to resolve that Duplicate equality, it must first be reduced to another, wherein there may be equal square numbers prefixt to aa , which may be done thus; Divide 9 the greater of the two Squares that are prefixt to aa by the lesser 4, and then by the Quotient $\frac{9}{4}$ multiply that Algebraick Quantity where the said Divisor 4 is prefixt to aa , viz. $4aa + 5a$ by $\frac{9}{4}$, and it produceth $9aa + \frac{45}{4}a$ to be equated to a Square; so now instead of the Duplicate equality in the fourth step, this ariseth,

$$\text{viz. } \left\{ \begin{aligned} 9aa + \frac{45}{4}a &= \square \\ 9aa + 5a &= \square \end{aligned} \right.$$

6. In which Duplicate equality last above express't, there are equal Squares, to wit, 9 and 9 prefixt to aa , and therefore it may be resolved after the manner before shewn in the 20th Question, and when the value of a is discovered in this latter duplicate equality, it will necessarily constitute the former duplicate equality in the fourth step; for as a Square multiplied by a Square produceth a Square, so conversly, a Square divided by a Square gives the Quotient a Square. In order then to resolve the said duplicate equality in the fifth step, subtract $9aa + 5a$ from $9aa + \frac{45}{4}a$, and the Remainder is $\frac{40}{4}a$, this must be esteem'd the Product made by the mutual multiplication of two quantities to be taken with such Caution, that as well in half their summ as in half their difference there may be found $3a$, (because the square Root of $9aa$ in each of the two quantities to be equated to a Square is $3a$;) so by considering well that Caution, and what hath been said to the like purpose in the sixth step of the Resolution of the foregoing 20th Question, you will find that $6a$ and $\frac{25}{4}$ are the only two numbers, that being mutually multiplied make $\frac{40}{4}a$, and have $3a$ as well in half their summ as in half their difference: Therefore let the Square of half the summ of the said $6a$ and $\frac{25}{4}$, viz. the Square $9aa + \frac{25}{4}a + \frac{625}{16}$ be equated to $9aa + \frac{45}{4}a$; or, let the Square of half the difference of the said $6a$ and $\frac{25}{4}$, viz. the Square $9aa - \frac{25}{4}a + \frac{625}{16}$, be equated to $9aa + 5a$; so from either of those Equations, one and the same value of a will be discovered, viz. $a = \frac{125}{3744}$, and consequently $2a$ and $3a$, which in the first and second steps were put for the numbers sought, will be discovered to be $\frac{125}{1872}$ and $\frac{125}{1248}$, which will solve the Question: For, first, the greater is in proportion to the lesser as 3 to 2; and if their summ be added to their Squares severally, the two summs made by such addition will be Squares, to wit, $\frac{600625}{3504384}$ and $\frac{272625}{1557504}$, whose sides are $\frac{775}{1872}$ and $\frac{525}{1248}$.

7. But the Duplicate equality in the fourth step may be reduced to another wherein there shall be equal square numbers prefixt to aa by this following Operation, which differs from that in the sixth step, viz. because 9 times 4 makes the same Product as 4 times 9, and because a Square multiplied by a Square produceth a Square, let $4aa + 5a$ (in the fourth step) be multiplied by 9, and $9aa + 5a$ by 4; so there will necessarily be found $9aa$ in each Product, and this following duplicate equality comes now to be resolved instead of that in the fourth step,

$$\text{viz. } \left\{ \begin{aligned} 36aa + 45a &= \square \\ 36aa + 20a &= \square \end{aligned} \right.$$

8. Lastly, this Duplicate equality having equal square numbers prefixt to aa , may be resolved like that in the preceding fifth step, and at length the value of a will be found $\frac{125}{3744}$, as before.

QUEST. 22.

To find two such square numbers, that if to the Product of their multiplication a given number (d) be added, the summ may be a Square.

RESOLUTION.

1. For one of the Squares sought take any known square number which may be represented by bb
2. And for the other Square sought put aa
3. Then the Product of their multiplication is $bbaa$
4. To which Product the given number d being added, the summ is $bbaa + d$
5. Which

An Example in Numbers.

Let the number given to be divided be } $33 = b$
 Let two numbers be taken, such, that the summ of their Squares } $2 = c$
 may be less than 33, as } $3 = d$
 Then (by the Canon) the side of the Square sought, (which } $2 = \frac{b - cc - dd}{2c + 2d}$
 side is represented by a in the Resolution,) shall be . . . }
 Also, one of the desired parts of 33 is } $12 = 2ca + cc$
 And the other is } $21 = 2da + dd$

The Proof.

$\left\{ \begin{array}{l} 4 = aa \text{ the Square sought;} \\ 12 + 21 = 33 \text{ the number given;} \\ 4 + 12 = 16 \\ 4 + 21 = 25 \end{array} \right\}$ which are Squares ; as was required.

QUEST. 24.

To find two such numbers, that their summ may make a Square: Also, that each number being added to the Square of the other number, may make a Square.

RESOLUTION.

1. For the summ of the two numbers sought assume some Square, as } ee
 2. And for the first of the two desired numbers put } a
 3. Therefore the other shall be } $ee - a$
 4. Which added to the Square of the first number a , makes the summ } $aa - a + ee$
 5. Which summ last express, the Question requires to be a Square, and such it will be, if we suppose $e = \frac{1}{2}$; for then the said $aa - a + ee$ (in the fourth step) will be equal to $aa - a + \frac{1}{4}$, which is the Square of $a - \frac{1}{2}$, or $\frac{1}{2} - a$: Now therefore let the Resolution be renewed thus,
 6. For the summ of the two numbers sought, (instead of ee) put } $\frac{1}{4}$
 7. And for the first number put (as before) } a
 8. Therefore the other shall be } $\frac{1}{4} - a$
 9. Which latter number added to the Square of the first, makes the summ } $aa - a + \frac{1}{4}$
- But the summ last express is manifestly a Square, whose side is $a - \frac{1}{2}$, or $\frac{1}{2} - a$; therefore two of the conditions in the Question are satisfied.
10. Again, if to the Square of the second number $\frac{1}{4} - a$, viz. to $aa - \frac{1}{2}a + \frac{1}{16}$, the first number a be added, the summ is $aa + \frac{1}{2}a + \frac{1}{16}$, which the Question likewise requires to be a Square, and so it is, for 'tis the Square of $a + \frac{1}{4}$; but if the last mentioned summ had not happened to have been a Square, then a Square might have been feigned equal to it, according to the method in divers preceding Questions of this Book 3.

The premisses discover this following

THEOREM.

11. If the Fraction $\frac{1}{4}$ be divided into any two parts, each part increased with the Square of the other part shall make a Square.
- By the help therefore of this Theorem, innumerable Answers to the Question proposed may be found out.

An Example.

Let two Fractions be taken whose summ makes $\frac{1}{4}$, as $\frac{1}{6}$ and $\frac{1}{12}$; I say these will solve the Question: For first, their summ is a Square; secondly, the first Fraction $\frac{1}{6}$ increased with $\frac{1}{144}$, (the Square of the latter Fraction $\frac{1}{12}$) makes the Square $\frac{1}{36}$; likewise the latter Fraction $\frac{1}{12}$ increased with $\frac{1}{36}$, the Square of the first Fraction $\frac{1}{6}$) makes a Square, to wit, $\frac{1}{9}$.

Moreover, by the help of the said Theorem, this following Question may be solved, viz.

12. To find two numbers in a given Reason, suppose the greater to the lesser as 3 to 2; and that each number being added to the Square of the other number, may make a Square.

Divide the Fraction $\frac{1}{4}$ into two such parts, that the greater may be to the lesser as 3 to 2, so you will find $\frac{3}{20}$ and $\frac{1}{10}$, which will solve the Question: For first, by

by Construction they are in the Reason prescribed; secondly, $\frac{3}{20}$ with $\frac{1}{100}$ (the Square of $\frac{1}{10}$) makes the Square $\frac{4}{25}$; and lastly, $\frac{1}{10}$ with $\frac{2}{400}$ (the Square of $\frac{1}{20}$) makes the Square $\frac{4}{400}$.

13. Another manner of solving the Question last proposed may be this, viz.

For the two numbers sought in the given Reason of 3 to 2, put $> 3a$ and $2a$
 Then, since the Question requires, that each number being added to the Square of the other number may make a Square, this Duplicate equality ariseth, viz. $\left. \begin{aligned} 9aa + 2a &= \square \\ 4aa + 3a &= \square \end{aligned} \right\}$

Which Duplicate equality may be resolved like that in the foregoing twenty-first Question; so the value of a will be found $\frac{1}{100} \frac{6}{80}$, and consequently $3a$ and $2a$ give $\frac{1}{100} \frac{18}{80}$ and $\frac{1}{100} \frac{12}{80}$ for the numbers sought: For first, they are in proportion as 3 to 2; secondly, if to the Square of the first you add the second number, it makes the Square $\frac{1}{100} \frac{36}{6400}$, whose side is $\frac{1}{100} \frac{6}{80}$; lastly, if to the Square of the second number you add the first, it makes the Square $\frac{1}{100} \frac{4}{6400}$, whose side is $\frac{1}{100} \frac{2}{80}$.

QUEST. 25. (Quæst. 29. Lib. 2. Diophant.)

To find two such square numbers, that each being increased with the Product of their multiplication may make a Square.

RESOLUTION.

1. First, (by the preceding Quest. 5.) find out two Squares that may differ by unity, such are $\frac{2}{16}$ and $\frac{1}{16}$, and take the lesser for one of the Squares sought, as $\frac{1}{16}$
2. Then for the other Square sought assume aa
3. Therefore the Product of their multiplication is $\frac{2}{16}aa$

Which Product $\frac{2}{16}aa$ being added to the second Square aa doth manifestly make a Square, to wit, $\frac{14}{16}aa$; but if the said Product $\frac{2}{16}aa$ be added to the first Square $\frac{1}{16}$ it must also make a Square, therefore $\frac{2}{16}aa + \frac{1}{16}$ not being a Square, must be equated to a Square, the side whereof may be variously feigned, let it be $\frac{3}{4}a - 1$; then the Square of $\frac{3}{4}a - 1$ being equated to $\frac{2}{16}aa + \frac{1}{16}$ gives $a = \frac{2}{24}$, and consequently $aa = \frac{1}{376}$ is the second Square sought. I say, $\frac{1}{16}$ and $\frac{1}{376}$ are two Squares, which will solve the Question, as will appear by

The Proof.

The two Squares found out are $\frac{1}{16}$ and $\frac{1}{376}$
 The Product of their multiplication is $\frac{1}{9216}$
 Which Product added severally to $\frac{1}{16}$ and $\frac{1}{376}$, makes these Squares, $\frac{1621}{9216}$ and $\frac{1221}{9216}$
 The sides of which last mentioned Squares are $\frac{12}{96}$ and $\frac{11}{96}$

After the same manner you may easily find out two such Squares, that if the Product of their multiplication be subtracted from them severally, the Remainders may be Squares.

QUEST. 26. (Quæst. 35. Lib. 2. Diophant.)

To find three such numbers, that the Square of every one of them being added to the summ of the three numbers, may make a Square.

RESOLUTION.

1. If to the Product of the multiplication of any two unequal numbers the Square of half their difference be added, the summ shall be a Square, to wit, the Square of half the summ of the two numbers multiplied: Therefore by the help of this Theorem numbers proper for the Resolution of the Question proposed may be taken, viz. Take some number at pleasure, as 12, and divide it thrice into two such numbers that the Product of the two numbers of each pair may make 12; such are these three pairs of numbers, viz. 1, 12. | 2, 6. | 3, 4; then take the half-difference of the two numbers of each pair, so you will find the three half-differences to be these, $\frac{11}{2}$, 2 and $\frac{1}{2}$. Now by the Theorem above-mentioned, if to the Product 12 the Squares of the said three half-differences be severally added, every one of the summs will be a Square; therefore

2. For

2. For the summ of the three numbers sought put } $12aa$
3. And for the first number } $\frac{1}{2}a$
4. And for the second number } $2a$
5. And for the third number } $\frac{1}{2}a$
6. Therefore the summ of those three numbers is } $8a$
7. Which summ must be equal to $12aa$ in the second step, therefore . . . } $8a = 12aa$
8. Which Equation, after due Reduction, gives } $a = \frac{2}{3}$
9. Therefore from the eighth, third, fourth and fifth steps the three numbers sought are these, viz. } $\frac{1}{3}, \frac{2}{3}, \frac{1}{3}$
10. Which three numbers will solve the Question, for if their Squares be severally added to their summ, the three summs will be Squares; to wit, $\frac{16}{9}, \frac{64}{9}$ and $\frac{4}{9}$. It is also evident, that the Question may be extended to four, five, or as many numbers as shall be desired, by this following

CANON.

11. Take some number at pleasure, and divide it into two numbers as many times as there be numbers desired by the Question, but so, as that the Product of the two numbers of each pair may make the number first taken; then divide the summ of the half-differences of the two numbers of each pair by the number taken, and multiply the Quotient by the said half-differences severally; so the Products shall be the numbers sought.

The certainty of this Canon depends upon the Theorem assumed in the Resolution; which Theorem may be demonstrated thus;

Let two numbers be assumed, as, } a the greater;
 } e the lesser.
 The Product of their multiplication is } ae
 The half-difference of the said two numbers is } $\frac{1}{2}a - \frac{1}{2}e$
 The Square of the said half-difference is } $\frac{1}{4}aa - \frac{1}{2}ae + \frac{1}{4}ee$
 To which Square if ae the Product above-mentioned be added, } $\frac{1}{4}aa + \frac{1}{2}ae + \frac{1}{4}ee$
 the summ is }
 Which summ is the Square of $\frac{1}{2}a + \frac{1}{2}e$, to wit, half the summ of the two numbers a and e first taken, as the Theorem affirmed.

QUEST. 27. (pars Quest. 9. Lib. 3. Diophant.)

To find three Squares in Arithmetical proportion, and such, that the half of their summ may exceed the greatest of the three Squares.

RESOLUTION.

1. For the least of the three Squares sought put } aa
2. And to the end the mean Square may exceed the least, let the side of the mean Square be $a + 1$, therefore the mean Square is } $aa + 2a + 1$
3. Therefore the excess of the mean Square above the least is . . . } $2a + 1$
4. And by adding the said excess to the mean Square; the summ must be equal to the greatest, which summ is } $aa + 4a + 2$
5. Therefore $aa + 4a + 2$ must be equated to some Square, with this condition, that $\frac{1}{2}aa + 3a + \frac{1}{2}$, which is the half-summ of the three Squares in the first, second and fourth steps, must (according to the Question) exceed the greatest of the said three Squares; supposing therefore $\frac{1}{2}aa + 3a + \frac{1}{2} \sqsupseteq aa + 4a + 2$, it will thence follow (by arguing after the manner of searching out limits in divers of the foregoing Questions) that $a \sqsupseteq 1 + \sqrt{2}$, that is, $a \sqsupseteq 2.414$, &c.

Therefore $aa + 4a + 2$ (in the fourth step) must be so equated to a Square, that the value of a may be greater than $2\frac{414}{1000}$, &c. Now to cause that effect, innumerable values of a may be found out by the first Canon of the twelfth Question of this Book. Suppose therefore a be found equal to $\frac{11}{10}$, (as in the Example of that Canon) then from the first, second and fourth steps, these three Squares will be discovered, to wit, $\frac{121}{100}, \frac{261}{100}$ and $\frac{441}{100}$, which will solve the Question.

And because a Square multiplied by a Square produces a Square, by multiplying severally the said three Squares found out by the common Denominator 100, the Products

961, 1681 and 2401 will be Squares, whose sides are 31, 41 and 49; also the difference of the first and second is equal to the difference of the second and third, (each difference being 720;) therefore the said three Squares are in Arithmetical proportion; and the half of their summ is manifestly greater than every one of them: Therefore all the conditions in the Question are satisfied.

QUEST. 28. (Quest. 9. Lib. 3. Diophant.)

To find three numbers in Arithmetical proportion, and such, that the summ of every two of them may make a Square.

RESOLUTION.

1. By the preceding twenty-seventh Question find three Squares in Arithmetical proportion, and such, that the half of their summ may exceed every one of them, (the reason of which condition will be evident by the following eighth, ninth and tenth steps,) such are these three Squares, whose sides are 31, 41 and 49, } 961 ; 1681 ; 2401
2. For the three numbers sought put } $a ; e ; \gamma$
3. Then equate the summ of the first and second numbers to the least of the three Squares before found, viz. suppose } $a + e = 961$
4. Likewise equate the summ of the first and third numbers to the mean Square, and it makes } $a + \gamma = 1681$
5. Equate also the summ of the second and third numbers to the greatest Square, and it gives } $e + \gamma = 2401$
6. The summ of the three last Equations is } $2a + 2e + 2\gamma = 5043$
7. The half of the said summ is } $a + e + \gamma = 2521\frac{1}{2}$
8. Then by subtracting the third Equation from the seventh, there remains } $\gamma = 1560\frac{1}{2}$
9. And by subtracting the fourth Equation from the seventh, there remains } $e = 840\frac{1}{2}$
10. And by subtracting the fifth Equation from the seventh, there remains } $a = 120\frac{1}{2}$

I say these three numbers, $120\frac{1}{2}$, $840\frac{1}{2}$, $1560\frac{1}{2}$ will solve the Question; for the difference between the first and second, to wit, 720, is equal to the difference between the second and third; therefore they are in Arithmetical proportion, and the summ of every two of them makes a Square,

$$\text{viz. } \left\{ \begin{array}{l} 120\frac{1}{2} + 840\frac{1}{2} = 961, \text{ whose } \sqrt{\text{ is } 31;} \\ 120\frac{1}{2} + 1560\frac{1}{2} = 1681, \text{ whose } \sqrt{\text{ is } 41;} \\ 840\frac{1}{2} + 1560\frac{1}{2} = 2401, \text{ whose } \sqrt{\text{ is } 49.} \end{array} \right.$$

QUEST. 29. (Quest. 12. Lib. 3. Diophant.)

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three summs shall be Squares.

This Question may be resolved divers ways; I shall here shew three of my own, and if the curious Reader desire to see more variety, he may consult Vieta's Zetet. 7. lib. 5. also Bachet's Comment, and the Solution of Slusius in pag. 177. of his Mesolabum, printed in 1668.

RESOLUTION 1.

1. First, by Quest. 22. of this Book) I seek two such Squares, that if to the Product of their multiplication the given number 12 be added, the summ shall be a Square; such are these Squares 1 and 4, which may be taken for two of the three numbers sought, } 1 and 4
2. Then for the third number sought I put } a
3. Now (according to the Question) the Product of the multiplication of the first and third numbers being increased with 12 must make a Square; also, the Product of the

the second and third numbers together with 12 must make a Square; it remains therefore to resolve this Duplicate equality,

$$\text{viz. } \begin{cases} 1a + 12 = \square \\ 4a + 12 = \square \end{cases}$$

4. But before the said Duplicate equality can be resolved, it must be reduced to another that shall have equal numbers of a , to which purpose I multiply the first of the two quantities to be equated, to wit, $1a + 12$, by 4, (which is prefixt to the latter of those two quantities) and it produceth $4a + 48$ to be equated to a Square, so this Duplicate equality ariseth, (instead of the former,)

$$\text{viz. } \begin{cases} 4a + 48 = \square \\ 4a + 12 = \square \end{cases}$$

5. Now to resolve this latter Duplicate equality, (and consequently the former,) I proceed according to the first manner of solving the preceding Quest. 8. viz. First, the difference between $4a + 48$ and $4a + 12$ is 36; then I seek two such Squares that their difference may be 36, and that the greater of them may exceed 48; but two such Squares are 100 and 64, (found out by Canon 1. of the preceding Quest. 7.)

6. Then from either of these Equations, $\begin{cases} 4a + 48 = 100 \\ 4a + 12 = 64 \end{cases}$

7. The third number sought is discovered, viz. $\begin{cases} \\ \\ \end{cases} \begin{cases} a = 13 \end{cases}$

I say, the numbers 1, 4 and 13 will solve the Question proposed; for if 12 be added to the Product of the first and second, likewise to the Product of the first and third, and lastly to the Product of the second and third, the three summs will be Squares, to wit, 16, 25 and 64. The premisses shew how to solve the Question by innumerable Answers.

Another way of resolving Quest. 29. which is here repeated, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three summs shall be Squares.

RESOLUTION 2.

1. For the given number 12 put b
2. For the three numbers sought put a, e, u
3. Then supposing b to be the difference of two Squares, search out those Squares (by the fifth Question of this Book) and let cc represent the greater, and xx the lesser, therefore $cc - b = xx$
4. Now the Question requires, that if b be added to the Product of the multiplication of a and e (the first and second numbers sought) the sum must be a Square, therefore by supposing that Square to be cc above found, this Equation ariseth, viz. $ae + b = cc$
5. Therefore from the last Equation by equal subtraction of b , $ae = cc - b$
6. And because by Construction in the third step $xx = cc - b$
7. Therefore from the fifth and sixth steps $ae = xx$
8. Therefore by dividing each part of the last Equation by a , it gives $e = \frac{xx}{a}$
9. Again, (by the fifth Question) find two other Squares whose difference shall be equal to b , suppose dd the greater Square, and zz the lesser, therefore $dd - b = zz$
10. Then the Product of the multiplication of the first and third numbers sought, with b added, makes $au + b$, which must be equal to some Square, let it be dd before found, therefore $au + b = dd$
11. Therefore by equal subtraction of b from the last Equation, $au = dd - b$
12. And because by Construction in the ninth step $zz = dd - b$
13. Therefore from the two last Equations $au = zz$
14. Therefore by dividing each part of the last Equation by a , it gives $u = \frac{zz}{a}$

15. Now since by the second, eighth, fifth, fourteenth and eleventh steps, the positions for the three numbers sought are $a, \frac{xx}{a}, (\text{or } \frac{cc - b}{a}) \frac{zz}{a}, (\text{or } \frac{dd - b}{a})$ it is evident,

evident, that if b be added to the Product of the multiplication of the first and second numbers, the sum is a Square, to wit, cc . Likewise, if b be added to the Product of the first and third numbers the sum is a Square, to wit, dd ; but if b be added to the Product of the second and third numbers the sum must also be a Square; therefore

$$\frac{xxzz}{aa} + b \text{ must be equal to a Square whose side we may feign to be } \frac{xz}{a} + t,$$

$$\text{or } \frac{xz}{a} - t, \text{ and consequently } a \text{ will be found equal to } \frac{2xzt}{b \text{ or } tt};$$

But x, z, b and t are known numbers, therefore a the first number sought is known also; and from the eighth and fourteenth steps the second and third numbers will be discovered.

From this second Resolution of *Quest. 29.* it will not be difficult to deduce the following

CANON.

First, supposing the given number (b) to be the difference of two Squares, find out (by the second Canon of the foregoing fifth Question) two pair of Squares in that difference, and let the side of the lesser Square of the one pair be called (x ;) and the side of the lesser Square of the other pair, (z ;) then take some square number whose side may be called (t ;) and let the difference between (tt and b) be called (g ;) then divide the double of the solid Product of the three sides x, z, t , viz. $2xzt$, by (g ;) and the Quotient shall be one of the three numbers sought; lastly, multiply severally the sides x and z by g , and divide the first Product by $2zt$, and the latter by $2xt$, so the Quotients shall be the two other numbers sought. Compare this Canon with the two following Examples.

$b = 12$	12	}	given in the Question.
$x = 2$	2	}	found out according to the directions in the Canon.
$z = \frac{13}{10}$	$\frac{1}{2}$		
$t = \frac{2}{2}$	5		
$g = \frac{29}{4}$	13	}	found out by the Canon to solve the Question.
$\frac{2xzt}{g} = \frac{4}{5}$	$\frac{10}{13}$		
$\frac{gx}{2zt} = 5$	$\frac{26}{5}$		
$\frac{gz}{2xt} = \frac{162}{80}$	$\frac{13}{40}$		

I say the number given being 12, the Question may be solved by these three numbers $\frac{4}{5}, 5, \frac{162}{80}$; likewise by these, $\frac{10}{13}, \frac{26}{5}, \frac{13}{40}$, as may easily be proved.

A third way of resolving Quest. 29. which is here repeated, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three sums shall be Squares.

RESOLUTION 3.

1. For the given number 12 put } b
 2. For the three numbers sought put } a, e, u
 3. Then, according to the Question, $ae + b$ must be equal to a Square, } $ae + b = cc$
let it be some known Square cc , therefore }
 4. Therefore from the last Equation by equal subtraction of b , . . . } $ae = cc - b$
 5. And by dividing each part of the last Equation by a , it gives . } $e = \frac{cc - b}{a}$
 6. Thus, the first of the three numbers sought being a , and the second $\frac{cc - b}{a}$, it is evident, that if b be added to the Product of their multiplication, the sum is a Square, to wit, cc .
 7. Again, according to the conditions in the Question, $au + b$ must } $au + b = dd$
be equal to a Square, let it be some known Square dd , therefore }
 8. Therefore by equal subtraction of b , } $au = dd - b$
9. And

9. And by dividing each part of the last Equation by a , $\therefore u = \frac{dd - b}{a}$

10. Thus, the first of the three numbers sought being a , and the third $\frac{dd - b}{a}$, it is evident, that if b be added to the Product of their multiplication the sum is a Square; to wit, dd . But the Question requires also, that if to the Product of the second and third numbers the given number b be added, the sum must be a Square, therefore let the second and third numbers, to wit, $\frac{cc - b}{a}$ and $\frac{dd - b}{a}$ be mutually multiplied, and to the Product add b , so the sum will be $\frac{baa + ddcc + bb - bdd - bcc}{aa}$, which

must be equal to a Square, the side whereof may be feigned to be either $\frac{dc + b}{a}$ or $\frac{dc \in b}{a}$; first let the side be $\frac{dc + b}{a}$, then its Square being equated to the sum above-mentioned, after due Reduction of that Equation it will appear that $a = c + d$.

11. Therefore, the Equation last express'd, to wit, $a = c + d$ being compared with the Quantities in the second, fifth and ninth steps, the three numbers sought, to wit, a, c, u will be found equal to these known Quantities, viz.

$$c + d, \quad \frac{cc - b}{c + d}, \quad \frac{dd - b}{c + d}$$

12. Again, for as much as the side of the Square to be equated to $\frac{baa + ddcc + bb - bdd - bcc}{aa}$

(the sum above-mentioned in the tenth step) may be feigned to be $\frac{dc \in b}{a}$, (as well as $\frac{dc + b}{a}$;) let the Square of $\frac{dc \in b}{a}$ be equated to the said sum, then by proceeding as before, three other numbers capable of solving the Question will be found equal to these, viz.

$$c \in d, \quad \frac{cc - b}{c \in d}, \quad \frac{dd - b}{c \in d}$$

From the premisses two excellent Canons are deducible to solve the forgoing Quest. 29.

CANON 1.

13. Subtract the given number from two Squares severally, then divide each of the Remainders by the sum of the sides of the same Squares, so shall the two Quotients and the said sum of the sides be three numbers which will solve the Question.

For example, let the given number be 12, subtract it from the Squares 36 and 64, the Remainders are 24 and 52, which being severally divided by 14, (the sum of 6 and 8, which are the sides of the said Squares 36 and 64,) the Quotients $1\frac{2}{7}$ and $3\frac{2}{7}$, with the said 14, are three numbers to solve the Question, as will be evident by

The Proof.

$$\left. \begin{array}{l} 14 \times 1\frac{2}{7} + 12 = 36 \\ 14 \times 3\frac{2}{7} + 12 = 64 \\ 1\frac{2}{7} \times 3\frac{2}{7} + 12 = \frac{20}{49} \end{array} \right\} \text{Which are Squares, as was required.}$$

CANON 2.

14. Subtract the given number from two Squares severally, then divide each of the Remainders by the difference of the sides of the same Squares, so shall the two Quotients and the said difference be three numbers, which will solve the Question.

For example, let the given number be 12, subtract it from the Squares 36 and 64, and divide each of the Remainders 24 and 52, by 2, (which is the difference of 6 and 8, the sides of the said Squares 36 and 64,) so the Quotients 12 and 26, with the said 2, are three numbers that will solve the Question, as will appear by

The Proof.

$$\left. \begin{array}{l} 2 \times 12 + 12 = 36 \\ 2 \times 26 + 12 = 64 \\ 12 \times 26 + 12 = 324 \end{array} \right\} \text{Which are Squares, as was required.}$$

Bachet

Bachet in his Comment upon *Quest. 12. Lib. 3. Dioph.* (which is the same with the preceding 29th Question) delivers two Canons, one of which is the same with *Canon 2.* above exprest, and the other is this following

CANON 3.

15. Subtract the given number from two Squares severally, divide the Remainders severally by the difference of the sides of the same Squares; then shall the two Quotients and their double summ lessened by the aforefaid difference be three numbers which will solve the Question propounded.

For example, let the given number be 12, subtract it from two Squares, suppose 36 and 64; the Remainders 24 and 52 being divided severally by 2, (the difference of the sides of the said Squares 36 and 64) give the Quotients 12 and 26, which are two of the three numbers sought; then from 76 (the double summ of the said Quotients) subtracting 2 (the before-mentioned difference,) the Remainder 74 shall be the third number sought. By this Operation it is evident that the two first numbers are the same with those found out by *Canon 2.* but the third numbers are different: I say the three numbers 12, 26 and 74 will solve the Question, as may easily be proved.

But to manifest the certainty of *Canon 3.* both its Operation and Demonstration may be symbolically exprest in this manner, viz.

Operation.

- 16. Take two unequal numbers, as } d the greater;
- 17. Their difference is } c the lesser.
- 18. Take any number less than the Square of c , as, } $d - c$
- 19. Subtract the number b from the Squares of c and d severally, } b
- so the Remainders are } $cc - b$ and $dd - b$
- 20. Divide each of those Remainders by $d - c$, and the Quo- } $\frac{cc - b}{d - c}$ and $\frac{dd - b}{d - c}$
- tients are } $\frac{2cc + 2dd - 4b}{d - c}$
- 21. The double summ of those Quotients is } $\frac{cc + dd + 2cd - 4b}{d - c}$
- 22. From that double summ subtract $d - c$, and the Remain- } $\frac{cc + dd + 2cd - 4b}{d - c}$
- der is } $\frac{cc - b}{d - c}$ and $\frac{dd - b}{d - c}$
- 23. Thus, the three numbers found out by *Canon 3.* last afore-going to solve *Quest. 29.* are equal to these, viz.

$$\frac{cc - b}{d - c}, \quad \frac{dd - b}{d - c}; \quad \frac{cc + dd + 2cd - 4b}{d - c}.$$

Now I say, if every two of those three numbers be mutually multiplied, and to the Products severally the number b be added, the three summs shall be Squares.

Demonstration.

24. By *Canon 1. Resolut. 3. Quest. 29.* if the Product of the multiplication of the two first numbers, to wit, of $\frac{cc - b}{d - c}$ and $\frac{dd - b}{d - c}$ be increased with b , the summ will be a Square; it remains to prove, that if the Product of the multiplication of the first and third numbers be increased with b , the summ shall also be a Square; likewise that the Product of the multiplication of the second and third numbers increased with the number b , makes a Square. But if the number b be added to the Product of the multiplication of the first and third numbers, the summ is $\frac{cccc + ccdd + 2cccd - 4ccb - 4cdb + 4bb}{dd + cc - 2cd}$,

which is a Square, whose side is $\frac{cc + cd - 2b}{d - c}$; And if b be added to the Product of the

multiplication of the second and third numbers, the summ is : . $\frac{dddd + ddcc + 2dddc - 4ddb - 4cdb + 4bb}{dd + cc - 2cd}$, which is a Square whose side is

$\frac{dd + cd - 2b}{d - c}$. Therefore the truth of *Canon 3.* is evident.

By the help of the second and third Canons last before exprest, *Bachet* extends the preceding *Quest. 29.* to four numbers, as I shall shew in the next Question.

QUEST. 30.

QUEST. 30. (Bachet in Quest. 12. Lib. 3. Diophant.)

To find four such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 3, be added, the four summs shall be Squares.

RESOLUTION.

1. The number given in the Question is 3
2. Let two Squares be feigned from $a + 2$ two such known numbers that their difference may be a Square, and that each of them may exceed the given number 3. To which end let the side of one of those Squares be feigned $a + 2$, and let the other side be $a + 6$, for the difference of 2 and 6 is 4, (a square number,) then will the Squares of the said sides $a + 2$ and $a + 6$ be these,

$$\text{viz. } \begin{cases} aa + 4a + 4, & (\text{the Square of } a + 2.) \\ aa + 12a + 36, & (\text{the Square of } a + 6.) \end{cases}$$

3. From each of those Squares subtract the given number 3, so the Remainders will be these,

$$\text{viz. } \begin{cases} aa + 4a + 1 \\ aa + 12a + 33 \end{cases}$$

4. Divide the said Remainders severally by 4, (the difference of the aforesaid sides $a + 2$ and $a + 6$) and let the Quotients be assumed for two of the four numbers sought,

$$\text{viz. } \begin{cases} \frac{1}{4}aa + a + \frac{1}{4}, & (\text{the first number.}) \\ \frac{1}{4}aa + 3a + 8\frac{1}{4}, & (\text{the second number.}) \end{cases}$$

5. The double summ of the said Quotients (assumed in the last step for the first and second numbers sought) is $aa + 8a + 17$, from which subtract 4, (the difference before-mentioned) and let the Remainder be assumed for the third number,

$$\text{viz. } aa + 8a + 13, \quad (\text{the third number.})$$

6. If the Construction hitherto be compared with the third Canon in the third Resolution of the foregoing 29th Question, it will thence be evident, that if to the Product made by the multiplication of every two of those three numbers before assumed in the fourth and fifth steps, there be added the given number 3, the three summs will be Squares.

7. For the fourth number sought assume the difference of the sides $a + 2$ and $a + 6$ before-mentioned, to wit, 4, (the fourth number.)

8. Then by comparing the Construction in the second, third, fourth and seventh steps with Canon 2. in the third Resolution of the foregoing twenty-ninth Question, it will be manifest, that if to the Product made by the multiplication of every two of these three numbers, to wit, the first, second and fourth numbers before assumed in the fourth and seventh steps, there be added the given number 3, the three summs will be Squares.

9. It remains to make the Product of the multiplication of the third and fourth numbers, the given number 3 being added, equal to a Square; but the third number $aa + 8a + 13$, multiplied by the fourth number 4, gives the Product $4aa + 32a + 52$, to which adding 2, the summ is $4aa + 32a + 55$, which must be equal to a Square, the side whereof may be feigned to be $2a - 10$, then the Square thereof being equated to $4aa + 32a + 55$, the value of a will thence be found equal to $\frac{1}{8}$, by the help whereof, recourse being had to the fourth, fifth and seventh steps, the four numbers sought will be found these, viz. $\frac{242}{256}, \frac{2617}{256}, \frac{1177}{64}$ and 4, which will solve the Question. For if the first be multiplied by every one of the other three, and the Products be severally increased with the given number 3, the three summs will be these Squares, to wit, $\frac{248241}{65536}, \frac{342222}{16384}, \frac{441}{64}$, whose sides are $\frac{221}{256}, \frac{183}{128}$ and $\frac{21}{8}$. Also, if the second be multiplied by the third and fourth numbers severally, and each Product be increased with 3, these two Squares will arise, to wit, $\frac{3122361}{16384}$ and $\frac{2802}{64}$, whose sides are $\frac{1762}{128}$ and $\frac{53}{8}$; lastly, the Product of the third number multiplied by the fourth being added to 3 makes the Square $\frac{1222}{16}$, whose side is $\frac{11}{4}$.

QUEST. 31. (Quest. 15. Lib. 3. Diophant.)

To find three such numbers, that if the Product of the multiplication of every two of them be lessened by the third, the three Remainders shall be Squares.

RESOLUTION.

1. For the first number put 2. And

2. And for the second number put $a + 4$ some known square } $a + 4$
 number, as }
 3. Then the Product of the multiplication of the first and second } $aa + 4a$
 numbers is }
 4. From which Product if $4a$ be subtracted, it is evident the }
 Remainder will be a Square, to wit, aa , therefore for the } $4a$
 third number we may put $4a$, (and so one of the conditions }
 in the Question will be satisfied) }
 5. Then (from the second and fourth steps,) the Product of the }
 multiplication of the second and third number is $4aa + 16a$, }
 from which the first number a being subtracted, there remains } $4aa + 15a = \square$
 $4aa + 15a$, which (according to the Question) must be }
 equal to a Square, *viz.* }
 6. Also (from the first and fourth steps,) the Product of the }
 multiplication of the first and third numbers is $4aa$, from }
 which subtracting the second number $a + 4$, there remains } $4aa - a - 4 = \square$
 $4aa - a - 4$, which (according to the Question) must be }
 equal to a Square, *viz.* }
 7. So in the two last preceding steps we are fallen upon a Duplicate equality, which (upon }
 the grounds before demonstrated) may be resolved thus, *viz.* First, the difference of }
 the two Algebraical quantities to be equated to two Squares, (by subtracting the lesser }
 from the greater) is manifestly $16a + 4$; then two Squares must be found, such, that }
 their difference may be $16a + 4$, and that $4aa$ may be in each of those Squares: }
 Therefore (agreeable to Canon 2. of Quest. 5. of this Book,) two numbers are to be }
 taken, that being mutually multiplied will produce $16a + 4$; moreover, that $2a$ may }
 be found as well in the half-sum as in the half-difference of the numbers taken; but }
 the only two numbers that will agree with those conditions are $4a + 1$ and 4 , whose }
 half-sum is $2a + \frac{1}{2}$, and their half-difference is $2a - \frac{1}{2}$. }
 8. Then by equating the Square of the said $2a + \frac{1}{2}$ to $4aa + 15a$ (in the fifth step,) }
 or the Square of $2a - \frac{1}{2}$ to $4aa - a - 4$ (in the sixth step,) from either of those }
 Equations the value of a will be found $\frac{1}{4}$, which is the first of the three numbers sought, }
 and consequently from the second and fourth steps, the second and third numbers are }
 $\frac{5}{4}$ and 5 , which three numbers will solve the Question; for if from the Product of }
 every two of them the other be subtracted, the three Remainders are Squares, to wit, }
 $\frac{25}{16}$, 25 and 1 .

QUEST. 32.

To find three numbers, such, that if to the Square of every one of them the summ of the other two be added, the three summs may be Squares.

RESOLUTION.

1. Take any number of $a + 1$ some known number; as $a + 1$, for the side of a Square, then the Square of $a + 1$ is $aa + 2a + 1$; now if for the first number sought we put a , for the second $2a$, and for the third 1 , then it is evident that the Square of the first number, together with the summ of the second and third, makes a Square, to wit, $aa + 2a + 1$, whereby one of the conditions in the Question is satisfied; therefore,

For the three numbers sought put } $a, 2a, 1$

2. Then (according to the import of the Question,) the Square }
 of the second number, together with the summ of the first } $4aa + a + 1 = \square$
 and third must make a Square, therefore }
 3. Likewise the Square of the third number, together with the }
 summ of the first and second must make a Square, therefore } $+ 3a + 1 = \square$
 4. So we are fallen upon a Duplicate equality, which differs from any of the preceding }
 Forms, but (upon the same foundation by which those have been resolved) this may }
 be resolved thus; First, supposing the former of the two quantities to be equated to }
 a Square to exceed the latter, (for here we may indifferently take either of them for }
 the greater,) their difference, by subtracting $3a + 1$ from $4aa + a + 1$, is manifestly }
 $4aa =$

$4aa - 2a$; this difference, (as in most of the Duplicate equalities hitherto) must be esteem'd the Product made by the mutual multiplication of two quantities, or factors; but here these two factors must be such, that as well in half their summ as in half their difference there may be found 1, that is, the side of the known Square, (or absolute number) in each of the two quantities to be equated, to the end that when the Square of the said half-summ is equated to $4aa - 2a + 1$; or the Square of the said half-difference to $3a + 1$, the square number 1, by due Reduction of either of those Equations, may vanish. Now to find out two factors qualified as aforesaid, first, take 2, the double of the side of the known Square 1 in each of the two quantities to be equated, with — prefixt, (because — is prefixt to $2a$ in the difference $4aa - 2a$ above-mentioned,) for part of the first of the two desired factors; then divide $2a$, (which is part of the said difference,) by 2 the double of the side of the said known Square 1, and take the Quotient a for the latter factor; then divide $4aa$, (the other part of the said difference $4aa - 2a$) by the said latter factor a , and the Quotient $4a$ connected with — 2 first taken, shall be the compleat first factor: So two factors or quantities to agree with the conditions above mentioned are found to be $4a - 2$ and a ; for the Product of their multiplication is $4aa - 2a$, and 1 is found both in half their summ and half their difference: Then by equating the Square of half the summ of the said factors $4a - 2$ and a , viz the Square of $\frac{5}{2}a - 1$, to $4aa - 2a + 1$; or by equating the Square of half the difference of the same factors, viz the Square of $\frac{3}{2}a - 1$, to $3a + 1$, the value of a will be discovered, viz.

5. From either of these Equations, after due Reduction, $\left. \begin{array}{l} \frac{25}{4}aa - 5a + 1 = 4aa - 2a + 1 \\ \frac{9}{4}aa - 3a + 1 = 3a + 1 \end{array} \right\} \begin{array}{l} a \\ a \end{array}$
6. The same value of a will be discovered, viz. $\therefore a = \frac{8}{3}$

Therefore by the sixth and first steps, these three numbers are found out, to wit, $\frac{8}{3}$, $\frac{14}{3}$ and 1, which will solve the Question proposed: For $\frac{64}{9}$ the Square of the first number, together with $\frac{22}{9}$ the summ of the second and third, makes a Square, to wit, $\frac{100}{9}$; also $\frac{224}{9}$ the Square of the second, together with $\frac{22}{9}$ the summ of the first and third, makes a Square, to wit, $\frac{256}{9}$; and lastly, 1 the Square of the third, with $\frac{24}{3}$ the summ of the first and second makes a Square, to wit, 9. By what hath been said in the first step of the Resolution this Question is capable of innumerable Answers.

QUEST. 33.

To find a number less than 2 a number given, and such, that if it be multiplied by two given numbers severally, suppose by 8 and 6, and if to each of the Products a given square number, suppose 4, be added, the summs may be square numbers.

RESOLUTION.

1. For the number sought put $\dots \dots \dots \therefore a$
2. Then if that position be prosecuted according to the conditions in $\left. \begin{array}{l} 8a + 4 = \square \\ 6a + 4 = \square \end{array} \right\}$ the Question, this Duplicate equality will arise, to wit, $\dots \dots \dots$
3. Which kind of Duplicate equality Diophantus useth in divers Questions, and because the Resolution thereof is a very subtil invention, I have framed this Question purposely to explain it.

First, observe well these three numbers, $\dots \dots \dots \therefore 8a + 4, 6a + 4$ and 4.

Then seek what proportion the excess of the greatest of those three numbers above the mean hath to the excess of the mean above the least; so you will find that the former excess is to the latter as 1 to 3. For the excess of $8a + 4$ above $6a + 4$ is $2a$, and the excess of $6a + 4$ above 4 is $6a$; but $2a$ is to $6a$ as 1 to 3. Therefore the former excess is to the latter as 1 to 3, and consequently the former excess is one third part of the latter.

4. Now the principal scope in resolving the said Duplicate equality is to find out two square numbers with this condition, that the excess of the greater above the less may have such proportion to the excess of the lesser above 4 (the Square given in the Question) as 1 to 3; to wit, as the difference of the numbers 8 and 6, which are prefixt to a in the Duplicate equality, is to 6 the lesser number prefixt: For when two such Squares are found out, then if the greater be equated to $8a + 4$, or the lesser to $6a + 4$,

one and the same value of a will come forth. But to find out the said two Squares I proceed thus :

5. The least of the three Squares above mentioned, to wit, that given in the Question, by the help whereof the other two are to be found out, is 4
6. And to the end the mean Square may exceed the least, let the side of the mean Square be feigned $e + 2$, (2 being the side of the given Square 4;) therefore the mean Square it self is $ee + 4e + 4$
7. Therefore the excess of the mean Square above the least is manifestly $ee + 4e$
8. But by what hath been said before, the excess of the greatest Square above the mean must be $\frac{1}{3}$ part of the excess of the mean Square above the least; therefore (from the last step) the excess of the greatest Square above the mean shall be $\frac{1}{3}ee + \frac{4}{3}e$
9. Therefore by adding the last mentioned excess, to wit, $\frac{1}{3}ee + \frac{4}{3}e$ to the mean Square in the sixth step, the summ will be the greatest of the said three Squares, to wit, $\frac{4}{3}ee + \frac{14}{3}e + 4$
10. Which $\frac{4}{3}ee + \frac{14}{3}e + 4$ must be equated to a Square, but the value of e must be subject to a Determination thus found out; viz. Forasmuch as the two greatest of the three Squares above mentioned must be such, that when the greatest is equated to $8a + 4$, or the mean to $6a + 4$, the value of a may be less than 2, (according to the conditions in the Question;) therefore such a square number must be found out equal to the said $\frac{4}{3}ee + \frac{14}{3}e + 4$, that when 4 is subtracted from the said square number, $\frac{1}{3}$ part of the Remainder may be less than 2. Therefore from $\frac{4}{3}ee + \frac{14}{3}e + 4$ subtract 4, and the Remainder is $\frac{4}{3}ee + \frac{14}{3}e$, whereof $\frac{1}{3}$ is $\frac{4}{9}ee + \frac{14}{9}e$, which must be less than 2; therefore
11. Suppose $\frac{4}{9}ee + \frac{14}{9}e = 2$
12. Thence, by multiplying all by 6, it follows, that $ee + 4e = 12$
13. And by adding the Square of half the Coefficient 4 to each part, there ariseth $ee + 4e + 4 = 16$
14. And by extracting the square Root out of each part of the last step, $e + 2 = 4$
15. Therefore by equal subtraction of 2, it is manifest that $e = 2$
16. Thus we have found that $\frac{4}{3}ee + \frac{14}{3}e + 4$ must be equated to a Square, with this condition, that the value of e may be less than 2. Now to cause that effect, the side of the said Square may be feigned $-2 +$ any number of e greater than $3\frac{1}{3}e$, therefore let the said side be feigned $3\frac{1}{3}e - 2$, then the Square of $3\frac{1}{3}e - 2$ being equated to the said $\frac{4}{3}ee + \frac{14}{3}e + 4$, the value of e will thence be found $\frac{21}{11}$.
17. Now if $e = \frac{21}{11}$
18. Then consequently the Square of $3\frac{1}{3}e - 2$, that is, the greater of the two Squares sought, will be $\frac{2304}{121}$
19. And the Square of $e + 2$ (which in the sixth step was put for the side of the lesser of the two Squares sought,) will be $\frac{1849}{121}$
20. Which two Squares, to wit, $\frac{2304}{121}$ and $\frac{1849}{121}$, (whose sides are $\frac{48}{11}$ and $\frac{43}{11}$;) together with 4, (the square number given in the Question,) are such, that the excess of the greatest above the mean is $\frac{1}{3}$ part of the excess of the mean above the least, (according to the scope designed in the fourth step.) Now if the greater Square $\frac{2304}{121}$ be equated to $8a + 4$, or the lesser Square $\frac{1849}{121}$ to $6a + 4$, from either of those Equations the value of a , to wit, the number sought by the Question will be found $\frac{242}{42}$: For first, it is less than 2, also eight times that number, together with 4, makes the Square $\frac{2304}{121}$; and six times the same number $\frac{442}{42}$, together with 4, makes the Square $\frac{1849}{121}$. It is also evident by the sixteenth step, that as many numbers as one will may be found out to solve the Question proposed.
21. But for the greater evidence of the infallibility of the method of resolving this Duplicate equality, I shall demonstrate the same in manner following, viz.

Suppose

Suppose $\left\{ \begin{array}{l} r = 8 \\ s = 6 \\ c = 4 \\ d = \frac{2104}{121} \\ f = \frac{182}{121} \\ a = \frac{422}{242} \end{array} \right\}$ two Multipliers given in the Question,
 a square number given in the Question,
 two square numbers found out according to the direction
 in the fourth step of the Resolution,
 the number sought.

22. Suppose also, according to the Construction in the Resolution, that the excess of d above f hath such proportion to the excess of f above c , as the excess of r above s hath to s , viz. as, $d-f \cdot f-c :: r-s \cdot s$.

23. Then according to the Construction in the twentieth step of the Resolution, let these two Equations be instituted, viz. $ra + c = d$,
 $sa + c = f$.

24. Now since the Conclusion of the Resolution (in the said twentieth step) takes it for granted, that one and the same value of a , (to wit, the number sought) will be given by either of those two Equations, we must prove that these two Quotients are equal to one another, viz. $\frac{d-c}{r} = \frac{f-c}{s}$.

Demonstration.

25. Forasmuch as by Construction in the 22^d step, $\therefore d-f \cdot f-c :: r-s \cdot s$
 26. Therefore by Composition of Reason, $\therefore d-c \cdot f-c :: r \cdot s$
 27. Therefore alternately $\therefore d-c \cdot r :: f-c \cdot s$
 28. But if four numbers be Proportionals, the Reason of the first to the second is equal to the Reason of the third to the fourth, therefore $\frac{d-c}{r} = \frac{f-c}{s}$
 Which was to be demonstrated.

Observat. 1. upon Quest. 33.

In the Duplicate-equality used in the preceding Quest. 33. both the numbers of a are affirmative; but if they were both negative, or one of them affirmative and the other negative, the Resolution would differ very little from the former, as will appear by the two following Questions.

QUEST. 1.

1. Let it be required to find out the number signified by a in this Duplicate equality, viz. $4 - 2a = \square$
 $4 - 3a = \square$

RESOLUTION.

2. First, these three numbers are to be considered $4, 4-2a, 4-3a$
 Then because the excess of 4 above $4-2a$, hath such proportion to the excess of $4-2a$ above $4-3a$, as 2 to 1, let 4 be considered as the greatest of three Squares, and find the other two, with this condition, that the excess of 4 above the mean may be the double of the excess of the mean above the least; to which end,
 3. Let the greatest of the said three Squares be 4
 4. And to the end the mean Square may be less than the greatest, let the side of the mean Square be $2-e$, therefore the mean Square shall be $ee - 4e + 4$
 5. Therefore the excess of the greatest Square above the mean is $4e - ee$
 6. Therefore, according to the condition prescribed in the second step, the half of the excess in the fifth step shall be the excess of the mean Square above the least, to wit, $2e - \frac{1}{2}ee$
 7. Which half excess, to wit, $2e - \frac{1}{2}ee$ being subtracted from the mean Square in the fourth step, the Remainder shall be equal to the least Square, to wit, $\frac{3}{2}ee - 6e + 4$
 8. Therefore $\frac{3}{2}ee - 6e + 4$ must be equated to a Square, but the value of e must be subject to a Determination thus found out, viz. Forasmuch as the least of the three Squares above mentioned must be such, that when it is equated to $4-3a$ (in the first step) the value of a may be greater than nothing, it is evident the said least Square must necessarily be less than 4; Supposing therefore $\frac{3}{2}ee - 6e + 4 = 4$ from this supposition,

sition, (by arguing in like manner as before from the eleventh step to the sixteenth,) we shall find $e \supset 4$; therefore $\frac{2}{3}ee - 6e + 4$ must be equated to a Square, so, as the value of e may be less than 4. Now to cause that effect, the side of the said Square may be feigned to be $2 -$ any number of e greater than $\frac{2}{3}e$, let therefore the said side be feigned $2 - 2e$, then its Square being equated to $\frac{2}{3}ee - 6e + 4$ will give $e = \frac{4}{3}$, and consequently the mean and least Squares sought will be $\frac{2}{3}e$ and $\frac{4}{9}$, the former of which being equated to $4 - 2a$, or the latter to $4 - 3a$, from either of those Equations the value of a , or the number sought, will be found $\frac{2}{3}$; which will solve *Quest.* 1. before propounded, as will be evident by

The Proof.

If $\frac{2}{3}$ (or a) be multiplied by 2 and 3 severally, and if the Products be severally subtracted from 4, the two Remainders will be $\frac{2}{3}$ and $\frac{4}{9}$, which are Squares, as was required.

QUEST. 2.

1. Let it be required to find out the number signified by a , } $4 + 2a = \square$
 in this Duplicate equality, viz. } $4 - 3a = \square$

RESOLUTION.

2. First, these three numbers are to be considered, . . . } $4 + 2a$, 4, and $4 - 3a$
 Then, because the excess of $4 + 2a$ above 4, hath such proportion to the excess of 4 above $4 - 3a$, as 2 to 3, let 4 be assumed to be the mean of three Squares, and find out the other two, with this condition, that the excess of the greatest above the mean may be $\frac{2}{3}$ of the excess of the mean above the least; to which end,
 3. Let the mean Square be } 4
 4. And to the end the least of the three Squares may }
 be less than the mean, let the side of the least Square } $ee - 4e + 4$
 be $2 - e$, therefore the least Square shall be . . . }
 5. Therefore the excess of the mean Square above the } $-ee + 4e$
 least is }
 6. Therefore according to the condition prescribed in the } $-\frac{2}{3}ee + \frac{2}{3}e$
 second step, $\frac{2}{3}$ of the last mentioned excess shall be equal }
 to the excess of the greatest Square above the mean, to wit, }
 7. Which last excess added to the mean Square 4, will } $-\frac{2}{3}ee + \frac{2}{3}e + 4$
 give the greatest Square, to wit, }

Therefore $-\frac{2}{3}ee + \frac{2}{3}e + 4$ must be equated to a Square, but the value of e must be subject to a Determination thus found out; Forasmuch as the greatest of the three Squares required must be such, that when it is equated to $4 + 2a$ (in the first step) the value of a may be greater than nothing, it is evident that the said greatest Square must be greater than 4.

Suppose therefore $-\frac{2}{3}ee + \frac{2}{3}e + 4 \square 4$, thence it will follow, that $e \supset 4$; therefore the said $-\frac{2}{3}ee + \frac{2}{3}e + 4$ must be equated to a Square, with this caution, that the value of e may be less than 4. Now to find out such a Square, the value of e may be $\frac{2}{3}$, (found out by the Canon of *Quest.* 13. of this Book, *mutatis mutandis*,) whence the greatest and least Squares sought will be $\frac{2}{3}e$ and $\frac{4}{9}$; the former of which being equated to $4 + 2a$, or the latter to $4 - 3a$, from either of those Equations the number a will be found $\frac{2}{3}$, which will solve the Question, as will be manifest by

The Proof.

If $\frac{2}{3}$ (or a) be multiplied first by 2, and then by 3; also, if the first Product be added to 4, and the latter Product be subtracted from 4, the Summ and Remainder will be $\frac{2}{3}$ and $\frac{4}{9}$, which are Squares, as was required.

Observat. 2. upon the preceding Quest. 33.

By the same artifice that hath been used in solving the said *Quest.* 33. this following Duplicate equality may be resolved, viz.

1. Let it be required to find a number, call it a , that } $10a + 9 = \square$
 shall make } $5a + 4 = \square$

2. First,

2. First, by multiplying $5a + 4$ by 9, the Product is $45a + 36$; likewise $10a + 9$ multiplied by 4 produceth $40a + 36$; so the Duplicate equality propounded is reduced to this,

$$\text{viz. } \begin{cases} 45a + 36 = \square \\ 40a + 36 = \square \end{cases}$$

3. Which latter Duplicate equality being of the same kind with that in the foregoing Quest. 33. may be solved by innumerable Answers; but for the greater evidence, the search may be made as before, viz.

4. Let these three numbers be considered, $\triangleright 45a + 36, 40a + 36, 36$

Then because the excess of the greatest of those three numbers above the mean, hath such proportion to the excess of the mean above the least as 1 to 8, let 36 be assumed for the least of three Squares, and search out the other two, with this caution, That the excess of the greatest above the mean may be $\frac{1}{8}$ of the excess of the mean above the least, to which end,

5. Let the least of those three Squares be $\triangleright 36$

6. And to the end the mean Square may be greater than the least, let the side of the mean Square be $e + 6$, therefore the mean Square it self shall be $ee + 12e + 36$

7. Therefore the excess of the mean Square above the least is $ee + 12e$

8. Therefore $\frac{1}{8}$ of that excess (which according to the Caution given as above, must be the excess of the greatest Square above the mean) shall be $\frac{1}{8}ee + \frac{3}{2}e$

9. Which last excess, to wit, $\frac{1}{8}ee + \frac{3}{2}e$, being added to the mean Square in the sixth step, the sum is equal to the greatest Square, to wit, $\frac{1}{8}ee + \frac{3}{2}e + 36$

10. Therefore $\frac{1}{8}ee + \frac{3}{2}e + 36$ must be equated to a Square, which square number when 'tis found out must be equated to $45a + 36$, and therefore the said Square must be greater than 36; but from any affirmative value of e whatsoever, the said $\frac{1}{8}ee + \frac{3}{2}e + 36$ will be manifestly greater than 36. Therefore here being no need of any limit for the value of e , the side of the said Square may be variously feigned; let then the said side be $\frac{3}{2}e - 6$, the Square thereof will be $\frac{9}{4}ee - 18e + 36$, which being equated to $\frac{1}{8}ee + \frac{3}{2}e + 36$, the value of e will thence be found 28; and consequently (from the ninth and sixth steps) the greatest and mean Squares sought will be 1296 and 1156, whose sides are 36 and 34: Then,

11. From this Equation, $\triangleright 45a + 36 = 1296$

12. Or from this Equation, $\triangleright 40a + 36 = 1156$

13. The number a will be discovered, to wit, $\triangleright a = 28$

I say the number 28 will solve the Question, as will be evident by

The Proof.

If 28 (or a) be multiplied by 10 and 5 severally; also, if to the former Product 9 be added, and to the latter Product 4, the two summs will be 289 and 144, which are Squares, as was required.

Observat. 3. upon Quest. 33.

1. In the Duplicate equalities used in the preceding Quest. 33. and the two Observations thereon, the two Algebraick Quantities given to be equated to two Squares, do consist of two unequal numbers of a and of two known square numbers, either equal or unequal; which kind of Duplicate equality you have seen exactly resolved by Rational numbers: But neither Diophantus, nor any Author that I have met with, delivers a Rule to resolve in all cases a Duplicate equality consisting of unequal numbers of a , and of absolute numbers which are not Squares. Monsieur Bachet indeed (in his Comment upon Quest. 45. Book 4. of Diophantus) shews how to resolve the last mentioned Duplicate equality in two Cases, which I shall here explain.

2. The first Case is, when in a Duplicate equality of the kind last mentioned, the difference of the two Algebraick Quantities propos'd to be equated is such, that if it be multiplied or divided by some known number, and if the Product or Quotient be subtracted from the lesser of the two Quantities, the Remainder is a square number. As, for example,

3. Let

3. Let it be required to find out the number signified by a in this } $3a + 13 = \square$
 Duplicate equality, viz. } $a + 7 = \square$
4. Because $2a + 6$, the difference of the two Algebraick Quantities propos'd, being divided by 2 gives the Quotient $a + 3$, which subtracted from $a + 7$ leaves a Square, to wit, 4; the Duplicate equality propos'd is explicable by the method used in the preceding *Quest.* 33. as I shall here make manifest.

First, let these three numbers be considered, to wit, . . . } $3a + 13, a + 7, 4$

Then, by continuing the search in such manner as hath been shewn in solving the foregoing *Quest.* 33. you may find out (among innumerable values of a) $a = 29$, which will solve the Duplicate equality propos'd; for if 29 be multiplied by 3, and 13 be added to that Product, the sum makes a Square, to wit, 100; also, if 7 be added to 29, it makes the Square 36.

5. The second Case is, when the difference of the two Algebraick Quantities which are to be equated to two Squares is such, that if it be multiplied or divided by some known number, and the Product or Quotient be subtracted from the lesser of those two Algebraick Quantities, there remains a negative known number, which taken affirmatively hath such proportion to the said Multiplier or Divisor, as a square number to a square number. For example,

6. Let it be required to find a number, suppose it to be a , that } $6a + 25 = \square$
 will make } $2a + 3 = \square$

7. Because $4a + 22$, the difference of those two Algebraick Quantities, being divided by 2 gives the Quotient $2a + 11$, which subtracted from the lesser of the said two Quantities, leaves -8 ; and the number 8 to the Divisor 2, is as 4 to 1, that is, as a square number to a square number; the Duplicate equality propounded may be resolved thus, viz.

8. Let these three numbers be considered, to wit, . . . } $6a + 25, 2a + 3, \text{ and } -8$

Then forasmuch as $4a + 22$, to wit, the difference of the two greater of those three numbers, is the double of $2a + 11$ which is the difference of the two lesser of the said three numbers, seek two Squares, that the excess of the greater above the lesser may be the double of the excess of the lesser above -8 ; to which end you may proceed thus, viz.

9. For the lesser of the two Squares put } aa
 10. Then the excess of aa above -8 is } $aa + 8$
 11. The double whereof is } $2aa + 16$
 12. Which added to the lesser Square makes a sum equal to the } $3aa + 16$
 greater, to wit, }

Now the said $3aa + 16$ must be equated to a Square, with this caution, That the square number found out must exceed 25; to the end that when the said Square is equated to $6a + 25$, the value of a may be greater than nothing: But to cause that effect, innumerable Squares may be found out, (by the method before-delivered in divers Questions of this Book,) such are 64 and 16; for whether you equate 64 to $6a + 25$, or 16 to $2a + 3$, one and the same value of a , to wit, $6\frac{1}{2}$, will be discovered to solve the Duplicate equality propos'd in the sixth step.

QUEST. 34. (*Quest.* 17. *Lib.* 3. *Diophant.*)

To find three such numbers, that the Product of the multiplication of every two of them, with the sum of the same two numbers, may make a Square.

RESOLUTION.

1. Forasmuch as the Product of the multiplication of two Squares whose sides differ by unity, being added to the sum of the said Squares, will make a Square, (by the *Theorem* at the end of the Resolution of this Question,) let there be put for the first and second numbers } 4 and 9

So is one of the conditions in the Question satisfied; for 4 multiplied by 9 produceth 36, to which if 13 (the sum of 4 and 9) be added, the sum is a Square, to wit, 49.

2. Let the third number be } a

3. Then

- 3. Then the Product of the multiplication of the second and third numbers is $9a$, to which adding their summ $a + 9$, it makes $10a + 9$, which (according to the Question) must be equal to a Square, viz. $10a + 9 = \square$
- 4. Likewise the Product of the multiplication of the first and third numbers is $4a$, which with their summ $a + 4$ makes $5a + 4$, which (according to the Question) must be equal to a Square, viz. $5a + 4 = \square$
- 5. So in the two last steps we are falln into a Duplicate equality, which may be solved by innumerable values of a , as hath been shewn in the second Observation upon the foregoing Quest. 33. of this Chapt. For example, take that value of a there found; to wit, $28 (= a)$ for the third number sought by this Question; I say $4, 9$ and 28 will solve this 34th Question, as will be evident by

The Proof.

The three numbers found out are 4, 9, 28

Now according to the Question,

I.		$4 \times 9, + 9 + 4 = 49$	} Which are Squares, as was required.
II.		$9 \times 28, + 9 + 28 = 289$	
III.		$4 \times 28, + 4 + 28 = 144$	

The first step of the Resolution of this 34th Question is grounded upon this

THEOREM.

- 6. If two numbers differ by unity, the Product made by the multiplication of their Squares, together with the summ of their Squares shall be a Square.
 - The truth of this Theorem may be demonstrated thus,
 - 7. Let there be two numbers which differ by 1 (or unity,) as } a , and $a + 1$
 - 8. Then their Squares are } aa
 $aa + 2a + 1$
 - 9. The summ of those Squares is } $2aa + 2a + 1$
 - 10. The Product of the multiplication of the said Squares is } $aaaa + 2aaa + aa$
 - 11. The summ of the said Summ and Product in the } $aaaa + 2aaa + 3aa + 2a + 1$
 ninth and tenth steps is
 - 12. Which Aggregate is a Square whose side is } $aa + a + 1$
- As will easily appear by multiplying the said side by it self. Therefore the truth of the Theorem is manifest.

QUEST. 35. (Quest. 18. Lib. 3. Diophant.)

This Question is the same with the foregoing 34th, which is here repeated, and solved after another manner.

To find three such numbers, that the Product of the multiplication of every two of them, being added to the summ of the same two numbers, may make a Square.

RESOLUTION.

- 1. Let the first number be } a
- 2. And let the second number be } 3
- 3. Then the Product of their multiplication added to their summ, } $4a + 3$
 makes
- 4. Which $4a + 3$ must (according to the Question) be equal to } $4a + 3 = 25$
 some Square, let it be 25 , therefore
- 5. Therefore from that Equation, } $a = 5\frac{1}{2}$
- 6. So we have found two numbers, to wit, $5\frac{1}{2}$ and 3 , which will satisfie one of the conditions in the Question, for the Product of their multiplication with their summ makes 25 , which is a Square. It remains to find a third number, which must be such, that the Product of the second and third numbers being added to their summ may make a Square; also, that the Product of the first and third numbers being added to their summ, may make a Square: Now to find out the said third number, Diophantus begins again, thus,
- 7. Let the first number be (as before it was found) } $5\frac{1}{2}$

8. And

8. And let the second number be (as before it was assumed,) . . . } 3
 9. Then for the third number put } a
 10. And since (according to the Question) the Product of the multiplication of the second and third numbers, with their summ, must make a Square; therefore, from the eighth and ninth steps, } $4a + 3 = \square$
 11. Also the Product of the first and third numbers with their summ must be a Square; therefore, from the seventh and ninth steps, } $6\frac{1}{2}a + 5\frac{1}{2} = \square$
 12. So in the two last steps we are falln into a Duplicate equality, but 'tis not resolvable by any of the preceding Rules of *Diophantus*; he frames therefore the Positions a-new, wherein his scope is to find such numbers of a in the two Algebraick Quantities to be equated to two Squares, that shall be in proportion one to the other as a square number to a square number, and then he shews how to resolve this new kind of Duplicate equality, which hath not hitherto happened. First, if we examine whence 4 and $6\frac{1}{2}$ (to wit, the numbers prefixt before a in the tenth and eleventh steps) do proceed, we shall find that they arise from the addition of unity to each of the numbers 3 and $5\frac{1}{2}$ first found; (for by multiplying a into those numbers severally, and by adding a to each Product, there ariseth $4a$ and $6\frac{1}{2}a$ above exprest.) Therefore the next search must be to find two such numbers, that being severally increased with unity, the one summ may be to the other as a square number to a square number: And because (by the Theorem in the following first Observation upon this Question,) if we add unity to each of two numbers whereof the greater exceeds the quadruple of the lesser by 3, the two summs will be in the Reason of a Square to a Square; therefore,
 13. Let the first of the three numbers sought be } a
 14. Then (by the said Theorem) the second number shall be . . . } $4a + 3$
 15. Now according to the Question, the Product of the first and second numbers together with their summ must be a Square, } $4aa + 8a + 3 = \square$
 therefore from the two last preceding steps }
 16. The side of which Square may be feigned $2a - 3$ — any known number whose Square is greater than 3, let therefore the said side be $2a - 3$, then its Square being equated to $4aa + 8a + 3$, the value of a will be found $\frac{1}{10}$ for the first number, and consequently $\frac{4}{10}$ ($= 4a + 3$) shall be the second number.

So we have found two numbers which will answer the first part of the Question, and moreover they are fit to raise a Duplicate equality that will be explicable by a Rational number: Therefore now an effectual Resolution may be formed thus;

17. Let the first number be } $\frac{1}{10}$
 18. And the second number } $\frac{4}{10}$
 19. And let the third number be } a
 20. Then according to the Question, the Product of the multiplication of the second and third numbers, together with their summ, must be equal to a Square, therefore from the 18th and 19th steps, } $\frac{4}{10}a + \frac{4}{10} = \square$
 21. Also according to the Question, the Product of the first and third numbers, with their summ, must be a Square; therefore, } $\frac{1}{10}a + \frac{1}{10} = \square$
 from the 17th and 19th steps, }
 22. Now because the numbers drawn into a , in the Duplicate equality exprest in the two last preceding steps, are (by Construction) in the Reason of a Square to a Square, for $\frac{1}{10} \cdot \frac{1}{10} :: 4 : 1$, and consequently (by *Prop. 19. Elem. 7. Euclid.*) $\frac{1}{10} \times 1 = \frac{1}{10} \times 4$. Therefore by multiplying the Algebraick quantity in the twentieth step by 1, and that in the twenty-first step by 4, the numbers of a in the Products will be equal to one another, for the first Product will be $\frac{4}{10}a + \frac{4}{10}$, and the latter $\frac{4}{10}a + \frac{4}{10}$; hence a new Duplicate equality is formed,
 viz. } $\frac{4}{10}a + \frac{4}{10} = \square$
 } $\frac{4}{10}a + \frac{4}{10} = \square$
 23. Which Duplicate equality being of the same kind with that explained in the preceding eighth Question, may be solved by innumerable Answers.

But I shall exhibit only one Answer for an Example. First then, because the difference of the two Algebraick quantities in the Duplicate equality last before exprest is 3, let two Squares be found out (by *Canon 1. Quest. 7. of this Book,*) whose difference shall be

be 3, and that the greater Square may exceed $\frac{42}{10}$; such are the Squares $\frac{784}{100}$ and $\frac{484}{100}$, whose sides are $\frac{28}{10}$ and $\frac{22}{10}$: Then,

24. From either of these Equations, } $\frac{28}{10}a + \frac{42}{10} = \frac{784}{100}$
 $\frac{22}{10}a + \frac{12}{10} = \frac{484}{100}$

25. The same value of a will be discovered for the third number sought, viz. } $a = \frac{7}{10}$

Thus three numbers are found out, to wit, $\frac{3}{10}$, $\frac{42}{10}$ and $\frac{7}{10}$, which will solve the Question, as will be evident by

The Proof.

$$\left. \begin{aligned} \frac{3}{10} \times \frac{42}{10}, + \frac{3}{10} + \frac{42}{10} &= \frac{176}{100} \\ \frac{42}{10} \times \frac{7}{10}, + \frac{42}{10} + \frac{7}{10} &= \frac{784}{100} \\ \frac{3}{10} \times \frac{7}{10}, + \frac{3}{10} + \frac{7}{10} &= \frac{100}{100} \end{aligned} \right\} \text{Which are Squares, as the Question requires.}$$

Observations upon Quest. 35.

1. If the Resolution of this Question be well examined, it will appear, that the forming of the Duplicate equality in the twentieth and twenty-first steps, where the numbers prefixt to a have such Reason to one another as a square number to a square number; agreeable to the Scope before-mentioned in the twelfth step, doth depend upon this following

THEOREM.

If there be two such numbers, that the greater exceeds the quadruple of the lesser by 3, and if unity be added to each number, the summs shall have such Reason between themselves as a Square to a Square, viz. the greater sum shall be to the lesser as 4 to 1. Which Theorem may be easily demonstrated, thus,

Suppose $\left\{ \begin{array}{l} a \\ 4a + 3 \\ a + 1 \\ 4a + 4 \end{array} \right\}$ Two numbers, whereof the greater exceeds the quadruple of the lesser by 3.
 $a + 1 =$ The first number increased with unity.
 $4a + 4 =$ The second number increased with unity.

I say $4a + 4$ hath such proportion to $a + 1$, as a Square to a Square, for,

$$4a + 4 . a + 1 :: 4 . 1.$$

In like manner, if there be two numbers whereof the greater exceeds nine times the lesser by 8, as 17 and 1, then if you add 1 to each number, the summs shall be to one another as a square number to a square number, viz. the greater sum shall be to the lesser as 9 to 1; the like is to be understood of other Squares.

2. After the two numbers prefixt before a in the Duplicate equality formed in the twentieth and twenty-first steps of this Question, are found such, that they have such Reason one to the other as a Square to a Square; then may any two square numbers in that Reason be used as is directed in the twenty-second step: So instead of 4 and 1 there taken, we may take 100 and 25, which have the same Reason between themselves as $\frac{28}{10}$ and $\frac{22}{10}$; For,

$$\frac{28}{10} . \frac{22}{10} :: 100 . 25,$$

Therefore, $\frac{28}{10} \times 25 = \frac{22}{10} \times 100.$

Then by multiplying the Algebraick quantity in the twentieth step of the Resolution by 25, and that in the twenty-first step by 100, the following Duplicate equality (being that which Diophantus useth in solving this Question) will arise,

viz. $\left\{ \begin{array}{l} 130a + 105 = \square \\ 130a + 30 = \square \end{array} \right.$

Hence, by the Canon in the seventh step of Resolut. 1. Quest. 8. (among innumerable values of a that might be found out,) you may find $a = \frac{7}{10}$ (as before) for the third number sought.

QUEST. 36. (Quest. 20. Lib. 3. Diophant.)

To find two numbers, that the Product of their multiplication increased severally with each of them, and also with their sum, may make three Squares.

RESOLUTION.

1. For one of the numbers put } a
 2. And for the other, } $4a - 1$
 3. Then their Product is } $4aa - a$
- I
- Whence

Whence it is evident, that if the first number a be added to the said Product, the sum is a Square, to wit, $4aa$.

4. It remains, that the second number and the sum of both being severally added to the said Product may make a Square; but the second number added to the Product makes $4aa + 3a - 1$; and the sum of both numbers, together with their Product, makes $4aa + 4a - 1$; therefore

$$\left. \begin{array}{l} 4aa + 3a - 1 = \square \\ 4aa + 4a - 1 = \square \end{array} \right\}$$

5. Which Duplicate equality may be resolved by the method before explained in the preceding twentieth and twenty-first Questions. For the difference of those two Algebraick quantities which are to be equated to Squares is a , which is to be divided into two such quantities that the Product of their multiplication may make a , and that both in the half-sum and in the half-difference of those two quantities there may be found $2a$; but such are the quantities $4a$ and $\frac{1}{4}$, whose Product is a ; also the half of their sum is $2a + \frac{1}{8}$, and the half-difference is $2a - \frac{1}{8}$: then by equating the Square of $2a + \frac{1}{8}$ to $4aa + 4a - 1$, or the Square of $2a - \frac{1}{8}$ to $4aa + 3a - 1$, from either of those Equations the value of a will be found $\frac{65}{224}$. Therefore the first number shall be $\frac{65}{224}$, and the second $\frac{36}{224}$; which numbers will solve the Question, as may easily be proved.

QUEST. 37. (Quest. 22. Lib. 3. Diophant.)

To find four such numbers, that every one of them being added to, and subtracted from the Square of the sum of them all, as well the four sums as the four remainders shall be Squares.

RESOLUTION.

1. In every right-angled Triangle, if the Square of the Hypotenusal be increased or lessened by the quadruple of the Area, that is, by the double Product of the multiplication of the two sides about the right-angle, it makes a Square, (which Theorem is made manifest at the end of the Resolution.) Therefore the chief scope is to find four right-angled Triangles in numbers having equal Hypotenusals: But those may be found out thus;

First, (by the Canon in *Observat.* 8. *Resolut.* 2. of *Quest.* 1. of this *Book*;) find out two unlike right-angled Triangles in numbers, such are these,

$$\begin{array}{ccc} 5 & , & 4 & , & 3 \\ 13 & , & 12 & , & 5 \end{array}$$

2. Then multiply the three sides of the first Triangle by the Hypotenusal of the second, also multiply the three sides of the latter Triangle by the Hypotenusal of the first; so the Products will give these two right-angled Triangles having equal Hypotenusals,

viz.
$$\begin{array}{ccc} 65 & , & 52 & , & 39 \\ 65 & , & 60 & , & 25 \end{array}$$

3. By the help of the two unlike right-angled Triangles first found, to wit, 5, 4, 3 and 13, 12, 5, the Canon in *Observat.* 4. upon *Resolut.* 2, and 3. of *Quest.* 2. of this *Book* will give two other right-angled Triangles unlike to those in the second step, but having the same Hypotenusal 65, to wit, these;

$$\begin{array}{ccc} 65 & , & 56 & , & 33 \\ 65 & , & 16 & , & 63 \end{array}$$

4. Then assume a to represent a number unknown, and let it be multiplied by every one of the sides of those four Triangles having 65 for a common Hypotenusal, so the Products will be these,

$$\begin{array}{ccc} 65a & , & 52a & , & 39a \\ 65a & , & 60a & , & 25a \\ 65a & , & 56a & , & 33a \\ 65a & , & 16a & , & 63a \end{array}$$

5. Now for the sum of the four numbers sought by the Question put } . $65a$
 6. Therefore the Square of the said sum is } $4225aa$
 7. Then for the first number sought, take the quadruple of the Area of the first of the four Triangles in the fourth step, *viz.* multiplying $52a$ by $39a$, take the double of that Product for the first number, to wit, } $4056aa$

8. In like manner for the second number take the double Product } 3000aa
of 60a by 25a, that is, }
9. And for the third number take the double Product of 56a by } 3696aa
33a, that is, }
10. And for the fourth number take the double Product of 63a by } 2016aa
16a, that is, }
11. The summ of the four numbers express'd in the seventh, eighth, } 12768aa
ninth and tenth steps is }
12. Which summ must be equal to 65a, which in the fifth step was } 12768aa = 65a
assum'd for the summ of the four numbers sought, hence this }
Equation, }
13. Which Equation reduced, gives } $a = \frac{65}{12768}$

14. Therefore from the thirteenth, seventh, eighth, ninth and tenth steps the four numbers required will be found these, viz. $\frac{17136600}{163021824}$, $\frac{12675000}{163021824}$, $\frac{15615600}{163021824}$, $\frac{8517600}{163021824}$, which four numbers will solve the Question, as will be evident to him that will take the pains of forming the Proof.

But because the Resolution of this Question is chiefly grounded upon a Theorem taken for granted in the first step, I shall here demonstrate the same

THEOREM.

15. If the Square of the Hypotenusal of a right-angled Triangle be increased or lessened by the quadruple of the Area, (that is, the double Product of the multiplication of the sides about the right-angle,) the summ, as also the remainder shall be a Square. For,

If a and e = } the sides about the right-angle of a right-angled Triangle,

Then $2ae$ = } the double Product of those sides,

Also aa and ee = } the Squares of those sides,

And $aa + ee$ = } the Square of the Hypotenusal, (per 47. Prop. 1. Elem. Euclid.)

Hence that which the Theorem asserts is manifest,

viz. $\begin{cases} aa + ee + 2ae = \square, & \text{whose side is } a + e, \\ aa + ee - 2ae = \square, & \text{whose side is } a - e. \end{cases}$

It is also evident from the premisses, that this 37th Question may be extended to five, six, or as many numbers as shall be desired; but first of all, so many numbers as are required, so many right-angled Triangles, in numbers must be found out having equal Hypotenusals; which Triangles in whole numbers may be readily discovered by the method delivered in *Observat.* 13. upon *Resolut.* 2. of *Quest.* 1. of this *Book*.

QUEST. 38.

[This is Quest. 20. of the fourth Book of Vieta's *Zetetics*, and the same with Quest. 3. in Bachet's *Comment upon the fourth Book of Diophantus.*]

Two cube-numbers being given, such, that the double of the lesser exceeds the greater; to find two other cube-numbers whose difference shall be equal to the difference of the given Cubes. (But how to perform this when the double of the lesser Cube is less than the greater, I shall hereafter shew in *Quest.* 42.)

RESOLUTION.

1. Let the sides of the given Cubes be . . . $\begin{cases} d & \text{the greater;} \\ b & \text{the lesser.} \end{cases}$
2. Then the Cubes of those sides are . . . d^3 and b^3
3. And the difference of the said Cubes is $d^3 - b^3$
4. For the side of the lesser Cube sought put $a - d$
5. And for the side of the greater Cube sought } $\frac{dd}{bb} a - b$
put }
6. Therefore the greater Cube sought is . . . $\frac{d^6}{b^6} aaa - \frac{3d^4}{b^3} aa + 3dda - b^3$
7. And the lesser Cube sought is . . . $aaa - 3daa + 3dda - d^3$
8. Therefore the difference of the two Cubes } $\frac{d^6 a^3 - b^6 a^3}{b^6} - \frac{3d^4}{b^3} aa + 3daa + d^3 - b^3$
sought is }
. I 2 9. Which

9. Which difference must be equal to the difference of the given Cubes, therefore,

$$\frac{d^6 a^3 - b^6 a^3}{b^6} = \frac{3d^4}{b^3} aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

10. From that Equation, after due Reduction, this ariseth, } $a = \frac{3d^4 b^3 - 3db^6}{d^6 - b^6}$
 viz.

11. And by reducing the Fraction in the latter part of the last preceding Equation into its least Terms, by the common Divisor for $d^3 - b^3$, it gives } $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the first, fourth, fifth and eleventh steps, the sides of the Cubes sought will be found equal to these known quantities,

$$\text{viz. } \left\{ \begin{array}{l} \frac{2db^3 - d^4}{d^3 + b^3} = \text{the lesser side,} \\ \frac{2bd^3 - b^4}{d^3 + b^3} = \text{the greater side.} \end{array} \right.$$

13. The same sides will be produced, if you put $a - b$ for the side of the greater of the Cubes sought, and $\frac{bb}{dd} a - d$ for the lesser side, (instead of the Positions in the fifth and fourth steps,) and it's evident that each of the sides found out in the twelfth step will be greater than nothing, if $2b^3$ exceeds d^3 , (that is, if the double of the lesser of the two Cubes given exceeds the greater, as the Question presupposeth.

The twelfth step affords this following

CANON.

14. Multiply the excess of the double of the lesser of the two Cubes given above the greater, by the side of the greater; multiply also the excess of the double of the greater Cube above the lesser, by the side of the lesser: then divide each of those Products by the sum of the given Cubes, and the Quotients shall be the sides of the Cubes sought.

Examples in Numbers.

15. Let two Cubes be given, such, that the double of the } $125 = d^3$, and $64 = b^3$
 lesser exceeds the greater, as,

16. The sides of which Cubes are } $5 = d$, and $4 = b$

17. Then by the Canon, } $\frac{2db^3 - d^4}{d^3 + b^3} = \frac{2}{63}$ } the sides of the Cubes sought.
 $\frac{2bd^3 - b^4}{d^3 + b^3} = \frac{248}{63}$

18. The Cubes of those sides $\frac{2}{63}$ and $\frac{248}{63}$ are $\frac{122}{250047}$ and $\frac{112522922}{250047}$, whose difference is 61, which is equal to the difference of the two given Cubes 125 and 64.

19. In like manner, if these two Cubes be given, to wit, 1728 and 1000, whose difference is 728, the foregoing Canon will give $\frac{408}{341}$ and $\frac{1270}{341}$, the sides of two Cubes whose difference is 728.

Observations upon Quest. 38.

First, the chief scope in the Resolution of this Question is, to raise an Equation between some number of aaa and some number of aa , that a may be found equal to a Rational number; to which purpose, the side of one of the Cubes sought may be feigned to be a — one of the sides of the given Cubes, and the other side sought some number of a — the other side given; but this latter number of a must be such as will cause equal numbers of a to arise in the Cubes of those feigned sides, that when the lesser of the feigned Cubes is subtracted from the greater, the numbers of a may vanish, and then the Remainder being equated to the difference of the given Cubes, this difference will likewise vanish; (because 'tis also found in the difference of the feigned Cubes,) and an Equation remain between some number of aaa and some number of aa . Now to cause that effect, supposing (as before in the Resolution) d to represent the side of the greater Cube given, and b the side of the lesser, we may put $a - d$ for the side of the lesser of the Cubes sought, and then the greater side must necessarily be $\frac{dd}{bb} a - b$; or $a - b$ may be put for the greater

side

side sought, and then the lesser must be $\frac{bb}{dd}a - d$; from either of which ways of framing the Positions there will arise, after due Reduction, an Equation between aaa and aa ; whence a will be found equal to a Rational number. All which will be manifest to him that diligently examines the preceding Resolution.

Secondly, if two pairs of Cubes which shall have equal differences be desired in whole numbers, they may easily be found out by the help of the foregoing Canon, in this manner, viz. let d and b represent the sides of two such Cubes; that the double of the Cube of the lesser side b exceeds the Cube of the greater side d , then the said Canon gives this Equation,

$$\left. \begin{array}{l} + \text{ Cube of } \frac{2bd^3 - b^4}{d^3 + b^3} \\ - \text{ Cube of } \frac{2db^3 - d^4}{d^3 + b^3} \end{array} \right\} = \text{Cube of } d - \text{Cube of } b.$$

Now to contract that Equation, suppose f , h and g to be equal to the Numerators and common Denominator, so that Equation will be converted into this, viz.

$$\frac{f^3}{g^3} - \frac{h^3}{g^3} = d^3 - b^3.$$

Whence, by multiplying every Term by the Denominator g^3 , this Equation is produced, viz.

$$f^3 - h^3 = g^3d^3 - g^3b^3.$$

That is, $\left\{ \begin{array}{l} + \text{ Cube of } f \\ - \text{ Cube of } h \end{array} \right\} = \text{Cube of } gd - \text{Cube of } gb.$

In which last Equation, if instead of f , h and g , you take $2bd^3 - b^4$, $2db^3 - d^4$, and $d^3 + b^3$, which were before supposed equal to f , h and g respectively, this following Equation will arise, viz.

$$\left\{ \begin{array}{l} + \text{ Cube of } \frac{2bd^3 - b^4}{2db^3 - d^4} \\ - \text{ Cube of } \frac{2db^3 - d^4}{2db^3 - d^4} \end{array} \right\} = \left\{ \begin{array}{l} + \text{ Cube of } \frac{d^4 + db^3}{bd^3 + b^4} \\ - \text{ Cube of } \frac{bd^3 + b^4}{bd^3 + b^4} \end{array} \right.$$

Which last Equation gives this following

CANON.

First, take two such Cubes in whole numbers that the double of the lesser may exceed the greater, and multiply the excess of the double of the greater above the less by the side of the lesser Cube; secondly, multiply the excess of the double of the lesser Cube above the greater by the side of the greater Cube; thirdly, multiply the sum of the same Cubes by the side of the greater; fourthly, multiply the sum of the said Cubes by the side of the lesser; then the difference of the Cubes of the first and second Products shall be equal to the difference of the Cubes of the third and fourth Products.

An Example in Numbers.

Let two such Cubes be taken, that the double of the lesser exceeds the greater, as, $\left. \begin{array}{l} 125 = ddd \\ 64 = bbb \end{array} \right\}$
 The sides of which Cubes are $\left. \begin{array}{l} 5 = d \\ 4 = b \end{array} \right\}$

Then by working according to the directions of the last preceding Canon, the four Products, that is, the sides of the four Cubes sought, in their least terms will be found these, to wit, $\left. \begin{array}{l} 248, 5, 315, 252 \end{array} \right\}$

Which four numbers will satisfy the Proposition; for the difference between the Cubes of 248 and 5 is equal to the difference between the Cubes of 315 and 252, as may easily be proved.

Hence it is easie to find four Cubes in whole numbers, such, that the sum of two of them shall be equal to the sum of the other two; for if two pairs of Cubes be found out by the last preceding Canon, such, that the first pair hath the same difference as the latter, then the sum of the greater Cube of the first pair and the lesser of the latter, shall be equal to the sum of the lesser Cube of the first pair and the greater of the latter.

Thirdly, *Albert Girard* (in his Comment in *Simon Stevin's* Arithmetick, upon the 19th of the fifth Book of *Diophantus*,) observes, but doth not demonstrate, that the Cubes found out by the preceding Quest. 38. are always less than the Cubes given; which property, since it will be useful in the following Quest. 39. I shall here demonstrate.

Suppose

Suppose $\left\{ \begin{array}{l} \dots d = 5 \\ \dots b = 4 \\ \dots d^3 = 125 \\ \dots b^3 = 64 \\ d^3 - b^3 = 61 \\ \dots 2b^3 \sqsubset d^3 \end{array} \right\}$ the sides of two Cubes, such, that the double of the lesser exceeds the greater.
 $\left. \begin{array}{l} \dots d^3 = 125 \\ \dots b^3 = 64 \end{array} \right\}$ the Cubes of those sides.
 $\left. \begin{array}{l} d^3 - b^3 = 61 \\ \dots 2b^3 \sqsubset d^3 \end{array} \right\}$ the difference of the same Cubes.

Then by the Canon in Sect. 14. Quest. 38. the sides of two Cubes whose difference is equal to the difference of the given Cubes, whose sides are $d (= 5)$ and $b (= 4)$ will be found these that follow, to wit,

$$\frac{2db^3 - d^4}{d^3 + b^3} = \frac{5}{63}; \text{ and } \frac{2bd^3 - b^4}{d^3 + b^3} = \frac{248}{63}.$$

Now because the Cubes given and found out have equal differences, if it be proved that the greater Cube found out is less than the greater Cube given, then consequently the lesser Cube found out shall be less than the lesser Cube given: But that the side of the greater Cube found out, is less than the side of the greater Cube given, (and by consequence the greater Cube found out less than the greater Cube given,) I prove thus,

The greater side found out (as before) is $\dots \left\{ \frac{2bd^3 - b^4}{d^3 + b^3} \right.$

Therefore we must demonstrate that $\dots \left\{ \frac{2bd^3 - b^4}{d^3 + b^3} \right\} \sqsupset d$

Demonstration.

By supposition, $\dots \left\{ \begin{array}{l} d \sqsubset b \\ dbb \sqsubset b^3 \\ b^3 + dbb \sqsubset 2b^3 \\ d^3 \sqsupset 2b^3 \\ d^3 \sqsupset b^3 + dbb \\ bd^3 \sqsupset b^4 + db^3 \\ bd^3 \sqsupset d^4 \\ 2bd^3 \sqsupset d^4 + b^4 + db^3 \\ 2bd^3 - b^4 \sqsupset d^4 + db^3 \\ \frac{2bd^3 - b^4}{d^3 + b^3} \sqsupset d \end{array} \right.$
 Therefore by multiplying d and b severally by bb , it follows, that
 And by adding b^3 to each part,
 By supposition
 Therefore from the two last preceding steps,
 And by multiplying each part in the last step by b ,
 But by multiplying each number in the first step of this Demonstration by d^3 ,
 Therefore by comparing the sum of the numbers in the first parts of the two last preceding steps, to the sum of those in the latter parts,
 And by subtracting b^4 from each part of the last preceding step,
 Wherefore by dividing each part of the last step by $d^3 + b^3$, it's manifest that

Which was to be demonstrated.

Having proved that the greater of the two sides found out by the Canon before discovered for resolving Quest. 38. is less than the greater of the two sides given, it follows, that the Cube of that side found out is less than the Cube of the greater side given, and that the lesser Cube found out is less than the lesser Cube given, (because by Construction the two Cubes found out have the same difference as the Cubes given.) Therefore the truth of the property before affirmed is manifest.

Fourthly, if two pairs of numbers have equal differences, the lesser number of the lesser pair shall have lesser Reason (or Proportion) to the greater number of the same pair, than the lesser number of the greater pair hath to the greater number of this pair. To make this manifest,

Suppose $\left\{ \begin{array}{l} b = 8 \\ c = 6 \\ b - c = 2 \\ d = 5 \\ b - d = 3 \\ c - d = 1 \end{array} \right\}$ two unequal numbers taken at pleasure.
 $\left. \begin{array}{l} b - c = 2 \\ d = 5 \end{array} \right\}$ the difference of those numbers.
 $\left. \begin{array}{l} b - d = 3 \\ c - d = 1 \end{array} \right\}$ a number less than $c (= 6)$ the lesser of the two numbers first taken.
 $\left. \begin{array}{l} b - d = 3 \\ c - d = 1 \end{array} \right\}$ two numbers whose difference is equal to the difference of the numbers first taken.

Now I say that the Reason (or Proportion) of $c - d$ to $b - d$ is less than that of c to b ; therefore,

The Proposition to be demonstrated, is, that $\dots \left\{ \frac{c - d}{b - d} \right\} \sqsupset \frac{c}{b}$

Demon-

Demonstration.

By supposition, } $c \supset b$
 Therefore by multiplying c and b severally by d , it follows } $dc \supset bd$
 that }
 And by adding bc to each part in the last step, } $bc + dc \supset bc + bd$
 And by subtracting bd from each part, } $bc + dc - bd \supset bc$
 And by subtracting dc from each part in the last preceding } $bc - bd \supset bc - dc$
 step, } $bc - bd \supset c$
 And by dividing each part of the last step by $b - d$, } $\frac{bc - bd}{b - d} \supset c$
 Wherefore by dividing each part of the last preceding step } $\frac{c - d}{b - d} \supset \frac{c}{b}$
 by b , }
 Which was to be demonstrated.

Fifthly and lastly, from the preceding 3^d and 4th Observations we may deduce this
COROLLARY.

If two cube-numbers be given, such, that the double of the lesser exceeds the greater, then (by the help of the preceding Canon in Sect. 14. Quest. 38.) two cube-numbers may be found out, whose difference shall be equal to the difference of the given Cubes, and the double of the lesser of the Cubes found out shall be less than the greater of them.

For if two given Cubes, (which I shall call the first pair) be such that the double of the lesser exceeds the greater, we may by the said Canon find out a second pair of Cubes, whose difference shall be equal to the difference of the first pair, and (by Observat. 3.) the Cubes of the second pair shall be less than those of the first pair, (viz. the greater Cube of the second pair shall be less than the greater Cube of the first pair, and the lesser Cube less than the lesser;) and the lesser Cube of the second pair shall have less Reason or proportion to the greater Cube of the same pair, than the lesser Cube of the first pair hath to the greater of the same pair, (by Observat. 4.) But if the double of the lesser Cube of the second pair doth yet happen to exceed the greater of the same pair, then by the help of the second pair of Cubes and the said Canon, we may find a third pair of Cubes, whose difference shall be equal to the common difference of the first and second pairs; and by proceeding in like manner, the double of the lesser of the two Cubes found out will at length necessarily be less than the greater, because (as before hath been proved,) the lesser Cube of each pair found out hath less proportion to the greater of the same pair, than the lesser of the next precedent pair (by which the latter were found out) hath to the greater.

QUEST. 39.

Two cube-numbers being given, such, that the double of the lesser is either greater or less than the greater, to divide the difference of the given Cubes into two Rational cube-numbers.

Preparation.

1. When the double of the lesser of the given Cubes exceeds the greater, two others must be found out, (according to the directions following the Corollary in Observat. 5. Quest. 38.) such, that the difference of these Cubes may be equal to the difference of those given, and that the double of the lesser of the Cubes found out may be less than the greater. Then two cube-numbers being given or found out, such, that the double of the lesser is less than the greater, their difference may be divided into two Rational cube-numbers by the following Resolution, (which is the same in substance with that of the 1st of the 4th Book of Vieta's Zetetics, and of the first Question of Bachet in his Comment upon the fourth Book of Diophantus.)

RESOLUTION.

- 2. Let the sides of the given Cubes (qualified as above } d the greater, and b the lesser.
 is supposed) be } d^3 and b^3
- 3. Therefore the Cubes of those sides are } $d^3 - b^3$
- 4. And the difference of the said Cubes is } $d - a$
- 5. For the side of one of the Cubes sought put } a

6. And

6. And for the side of the other Cube sought } $\frac{dd}{bb}a - b$
 put }
 7. Therefore the first Cube is } $-a^3 + 3daa - 3dda + d^3$
 8. And the latter Cube is } $\frac{d^5}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$
 9. Therefore the summ of those Cubes is } $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3$
 10. Which summ must be equal to the difference of the given Cubes; therefore,

$$\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

11. Which Equation, after due Reduction, gives } $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the first, fifth, sixth and eleventh steps, the sides of the two Cubes sought will be found equal to these known Quantities,

Viz.
$$\left\{ \begin{array}{l} \frac{d^4 - 2db^3}{d^3 + b^3} = \text{the first side,} \\ \frac{2bd^3 - b^4}{d^3 + b^3} = \text{the other side.} \end{array} \right.$$

13. The same sides will be produced, if instead of the Positions in the fifth and sixth steps, there be put $a - b$ and $d - \frac{bb}{dd}a$ for the sides of the Cubes sought. And 'tis evident that each of the sides found out in the twelfth step will be greater than nothing if $2b^3$ be less than d^3 , that is, if the double of the lesser of the two Cubes given be less than the greater, as is supposed in the Preparation to the Resolution of this Question. The sides in the twelfth step, being exprest by words, will give this

CANON.

14. Multiply the excess of the greater of the two given Cubes above the double of the lesser by the side of the greater; multiply also the excess of the double of the greater Cube above the lesser by the side of the lesser; then divide each of those Products by the summ of the said Cubes, and the Quotients shall be the sides of the Cubes sought.

Example 1. in Numbers.

15. Let two such Cubes be given, that the double of the } $8 = d^3$ and $1 = b^3$
 lesser is less than the greater, as, }
 16. The sides of those Cubes are } $2 = d$ and $1 = b$

17. Then by the Canon,
$$\left\{ \begin{array}{l} \frac{d^4 - 2db^3}{d^3 + b^3} = \frac{4}{3} \\ \frac{2bd^3 - b^4}{d^3 + b^3} = \frac{1}{3} \end{array} \right\} \text{the sides of the Cubes sought.}$$

The Cubes of those sides $\frac{4}{3}$ and $\frac{1}{3}$, are $\frac{64}{27}$ and $\frac{1}{27}$, whose summ is $\frac{65}{27}$, which is equal to the difference of the two given Cubes 8 and 1, as was required by the Question.

Example 2.

18. Let it be required to divide 61, which is the difference of these Cubes 125 and 64, into two Rational cube-numbers.

Here, because the double of the lesser Cube exceeds the greater, the Canon above exprest in Sect. 14. is of no force; therefore by the help of the given Cubes, (according to the directions following the Corollary in *Observat. 5. Quest. 38.*) two other Cubes must be found out, such, that their difference may be equal to 61, to wit, the difference of the given Cubes 125 and 64, and that the double of the lesser of the two Cubes found out may be less than the greater of them: But two such Cubes are these, viz. $\frac{122222222}{250047}$ and $\frac{12222222}{250047}$, whose sides are $\frac{248}{63}$ and $\frac{2}{63}$; then using these Cubes as the Canon in the preceding Sect. 14. of this Quest. 39. doth direct, you will find $\frac{122222222}{960946371}$ and $\frac{2782680016}{960946371}$ for the sides of the Cubes required; for the summ of the Cubes of the said sides is 61, which is equal to the difference of the given Cubes, 125 and 64.

19. Hence

19. Hence it is easie to find four cube numbers, the greatest of which shall be equal to the summ of the other three; for when the difference of two given Cubes is divided into two rational Cubes, these two, together with the lesser of the two given Cubes shall make the greater Cube given. But if four such Cubes be desired in whole numbers, they may be readily found out by the following Canon, which is raised by the like manner of arguing as was before used in the second Observation upon the preceding Quest. 38.

CANON.

20. First, take two Cubes in whole numbers, with this caution, That the double of the lesser may be less than the greater, and multiply the excess of the greater Cube above the double of the lesser by the side of the greater; secondly, multiply the excess of the double of the greater Cube above the less by the side of the lesser; thirdly, multiply the summ of the said Cubes by the side of the greater; fourthly, multiply the summ of the same Cubes by the side of the lesser: Then the summ of the Cubes of the first, second and fourth Products shall be equal to the Cube of the third Product.

An example in Numbers.

- 21. Let two such Cubes be taken, that the double of the lesser } $8 = ddd$
is less than the greater, as, } $1 = bbb$
- 22. The sides of which Cubes are } $2 = d$
. } $1 = b$
- 23. Then by the last preceding Canon, the sides of the four }
Cubes sought, in their least terms will be found these, viz } $4, 5, 6, 3.$
- 24. I say the summ of the Cubes of 4, 5 and 3 is equal }
to the Cube of 6, viz. } $64 + 125 + 27 = 216$

QUEST. 40.

To divide any cube-number, suppose 8, into three Rational cube-numbers.

RESOLUTION.

Take any cube-number less than the given Cube 8, as 1; then (by the preceding Quest. 39.) divide 7 the difference of those Cubes into two Cubes, suppose into these, $\frac{64}{27}$ and $\frac{125}{27}$, whose summ 7 is equal to the difference of the given Cubes 8 and 1.

Therefore $\left\{ \begin{array}{l} \frac{64}{27} + \frac{125}{27} = 8 - 1, \\ \frac{64}{27} + \frac{125}{27} + 1 = 8. \end{array} \right.$

Whereby 'tis manifest that three Cubes, to wit, $\frac{64}{27}$, $\frac{125}{27}$ and 1 are found out, whose summ is equal to the given Cube 8, as was required.

Hence you may easily perceive a way to divide a given Cube into any odd number of Cubes; as to divide a Cube into five Cubes, first divide the given Cube into three Cubes, and then divide one of those three into three Cubes, so the other two, with the three Cubes last found out are five Cubes, whose summ is equal to the Cube first given; in like manner you may divide a Cube into 7, 9, 11, 13, &c. Cubes. But there is not any Rational cube-number that can be divided into two Rational Cube numbers; which negative Proposition the Learned Dr Wallis hath demonstrated.

QUEST. 41.

[This is the 19th of the 4th Book of Vieta's Zeteticks, and the same with Quest. 2. of Bachet in his Comment upon Quest. 2. of the 4th Book of Diophantus.]

Two cube-numbers being given, to find two other cube-numbers whose difference shall be equal to the summ of the given Cubes.

RESOLUTION.

- 1. Let the sides of the given Cubes be } d the greater,
. } b the lesser.
- 2. Therefore the Cubes of those sides are } d^3 and b^3
- 3. For the side of the greater Cube sought put } $a + d$
- 4. And for the side of the lesser Cube sought put } $\frac{dd}{bb}a - b$

K

5. There;

5. Therefore the greater Cube sought is . . . $a^3 + 3daa + 3dda + d^3$
 6. And the lesser Cube sought is . . . $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$
 7. Therefore the difference of the Cubes sought } $\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3$
 is
 8. Which difference must be equal to the summ of the given Cubes, therefore;
 $\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3 = d^3 + b^3.$

9. Which Equation, after due Reduction, gives } $a = \frac{3db^3}{d^3 - b^3}$

10. Therefore from the first, third, fourth and tenth steps, the sides of the two Cubes sought will be found equal to these known quantities,

Viz. } $\frac{2db^3 + d^4}{d^3 - b^3} =$ the greater side,
 } $\frac{2bd^3 + b^4}{d^3 - b^3} =$ the lesser side.

The same sides will be produced, if instead of the Positions in the third and fourth steps there be put $d + \frac{bb}{dd}a$ and $a - b$; and those sides above-express'd by Letters give this

CANON.

11. Add the greater of the two Cubes given to the double of the lesser, and multiply the summ by the side of the greater Cube; add also the lesser Cube to the double of the greater, and multiply this summ by the side of the lesser Cube; lastly, divide each of those Products by the difference of the given Cubes, and the Quotients shall be the sides of the Cubes sought.

An Example in Numbers.

Let two Cubes be given, as, } $8 = ddd$, and $1 = bbb$
 The sides whereof are } $2 = d$, and $1 = b$

Then by the Canon, } $\frac{2db^3 + d^4}{d^3 - b^3} = \frac{20}{7}$ } the sides of the Cubes sought;
 } $\frac{2bd^3 + b^4}{d^3 - b^3} = \frac{17}{7}$

The Cubes of those sides $\frac{20}{7}$ and $\frac{17}{7}$ are $\frac{8000}{343}$ and $\frac{4913}{343}$, whose difference 9 is equal to the summ of the given Cubes 8 and 1.

12. Hence 'tis easie to find out four cube-numbers, the greatest of which shall be equal to the summ of the other three: For when by this Question two Cubes are found out, having their difference equal to the summ of two given Cubes, the lesser of the two Cubes found out, together with the two given Cubes shall be equal to the greater of the Cubes found out. But if four such Cubes be desired in whole numbers, they may be readily found out by the following Canon, which is rais'd by the like manner of arguing as was before used in *Observat. 2. Quest. 38.*

CANON.

13. First take any two Cubes in whole numbers, add the greater to the double of the lesser and multiply the summ by the side of the greater; secondly, add the lesser Cube to the double of the greater, and multiply this summ by the side of the lesser; thirdly, multiply the difference of those Cubes by the side of the greater; fourthly, multiply the said difference by the side of the lesser Cube: Then the summ of the Cubes of the three latter Products shall be equal to the Cube of the first Product.

An Example in Numbers.

Let two Cubes in whole numbers be taken at pleasure, as } $8 = ddd$, and $1 = bbb$
 The sides of those Cubes are } $2 = d$, and $1 = b$

Then by the Canon last afore-going, the sides of the } $20, 17, 14, 7$
 four Cubes sought will be found these, to wit, }

I say the Cube of 20 is equal to the summ of the } $8000 = 4913 + 2744 + 343$
 Cubes of 17, 14 and 7, viz. }

QUEST. 42.

Two cube-numbers being given, such, that the double of the lesser is less than the greater, to find out two other Cubes whose difference shall be equal to the difference of the given Cubes. (But how this is to be done when the double of the lesser Cube exceeds the greater, hath already been shewn in Quest. 38.)

RESOLUTION.

1. Let there be two Cubes given, to wit, $\left. \begin{array}{l} ddd = 8 \\ bbb = 1 \end{array} \right\}$
2. By the Canon in Sect. 14. of the foregoing Quest. 39. find out two Cubes whose summ shall be equal to the difference of the given Cubes, such are these, $\left. \begin{array}{l} ccc = \frac{64}{27} \\ ggg = \frac{128}{27} \end{array} \right\}$

Therefore by that Construction, . . . $\rightarrow ddd - bbb = ccc + ggg = 8 - 1 = 7.$

3. By the Canon in Sect. 11. of the foregoing Quest. 41. find two Cubes whose difference shall be equal to the summ of the Cubes ccc and ggg, (found out in the preceding second step,) such are these,

$$\left\{ \begin{array}{l} kkk = \frac{2024284625}{6128487}, \text{ whose side is } \frac{1265}{183}. \\ lll = \frac{1981385216}{6128487}, \text{ whose side is } \frac{1256}{183}. \end{array} \right.$$

Therefore by this Construction, . . . $\rightarrow kkk - lll = ccc + ggg = 7.$

4. Therefore from the second and third steps, (per 1. Axiom. 1, Elem. Euclid.)

$$kkk - lll = ddd - bbb = 7.$$

I say kkk and lll, that is, $\frac{2024284625}{6128487}$ and $\frac{1981385216}{6128487}$, (whose sides are $\frac{1265}{183}$ and $\frac{1256}{183}$;) will solve the Question proposed; for their difference 7 is equal to the difference of the given Cubes 8 and 1.

Note. Although by the preceding Resolutions of this and Quest. 38. innumerable pairs of cube-numbers may be found out, such, that the difference of each pair shall be equal to the difference of two Cubes given, yet neither of those Resolutions will find out all the pairs of Cubes that have the same difference with two given Cubes; for example, if the Cubes 1728 and 1000 be given, whose difference is 728, the Canon in the 14th step of the foregoing Quest. 38. will not find out the Cubes 729 and 1, whose difference is 728; although that Canon, with the help of the Resolution of this Quest. 42. will find out innumerable pairs of Cubes, such, that the difference of each pair shall be 728.

QUEST. 43.

To divide a given number 28 compos'd of two cube numbers 27 and 1, into two other Rational cube-numbers.

[This Question was propounded in 1657. by Mons. Fermat, (as appears by an Epistolical Commerce printed at Oxford in 1658.) but his way of solving it came not to light, till it was publish'd (after his death) among other his Analytical Inventions, by way of Supplement to Mons. Bachet's Comment upon Diophantus, printed at Tholose in 1670. yet the very same way of solving this Question was found out long before by our Learned Dr John Wallis, (though, it seems, not timely enough to have been inserted in the little Book above-mentioned,) and likewise by my self, before I had seen or heard of any Solution to the said Question, in such manner as here it follows.]

RESOLUTION.

1. Let the Cubes 27 and 1, whose summ makes the given number 28, $\left. \begin{array}{l} ddd = 27 \\ bbb = 1 \end{array} \right\}$ be represented by ddd and bbb, viz.
2. By the Canon in Sect. 11. of the foregoing Quest. 41. find two cube-numbers whose difference may be equal to 28 the summ of the given Cubes 27 and 1, that is, (ddd and bbb;) such are these Cubes,

$$\text{viz. } \left\{ \begin{array}{l} ggg = \frac{658503}{17576}, \text{ whose side is } \frac{87}{26}. \\ ccc = \frac{166375}{17576}, \text{ whose side is } \frac{55}{26}. \end{array} \right.$$

Therefore, $\rightarrow ggg - ccc = ddd + bbb = 27 + 1 = 28.$

K 2

3. By

3. By the foregoing *Quest.* 39. find out two Cubes whose summ shall be equal to 28 the difference of the two Cubes *ggg* and *ccc*, such are these,

$$kkk = \frac{253452325273412980702625}{986482093704101505552}$$

$$lll = \frac{22762660963735440852831}{986482093704101505552}$$

The sides of which Cubes are these, to wit, $\left\{ \begin{array}{l} k = \frac{63284705}{21446828} \\ l = \frac{28340511}{21446828} \end{array} \right.$

Therefore, $ggg - ccc = kkk + lll$.

4. But by Construction in the 2^d step, $ggg - ccc = ddd + bbb = 27 + 1 = 28$.

5. Therefore from the two last Equations, (per 1. *Axiom.* 1. *Elem.* Euclid.) $\left. \begin{array}{l} \\ \\ \end{array} \right\} kkk + lll = ddd + bbb = 27 + 1 = 28$.

Whence it is manifest that the two Cubes found out, to wit, *kkk* and *lll*, (which with their sides are before severally express'd by numbers in the third step,) will solve the Question, for their summ makes 28, which is the summ of the given Cubes 27 and 1. And because by the help of the known Cubes *ggg* and *ccc* in the second step, divers pairs of Cubes having the same difference with the said *ggg* and *ccc* may be found out, (by the 38th or 42^d Question foregoing:) Therefore by the help of any of the pairs of Cubes so found out, their difference may be divided into two Cubes whose summ shall be equal to the summ of the given Cubes 27 and 1.

Another Example.

Let it be required to divide 9, which is compos'd of the Cubes 8 and 1, into two other Cubes.

RESOLUTION.

1. Let the Cubes 8 and 1, whose summ makes the given number 9, be represented by *ddd* and *bbb*, viz. $\left\{ \begin{array}{l} ddd = 8 \\ bbb = 1 \end{array} \right.$

2. By the Canon in *Sect.* 11. *Quest.* 41. of this Book, find out two Cubes whose difference may be equal to 9, the summ of the given Cubes 8 and 1; such are these Cubes,

Viz. $\left\{ \begin{array}{l} ggg = \frac{8000}{343}, \text{ whose side is } \frac{20}{7} \\ ccc = \frac{4913}{343}, \text{ whose side is } \frac{17}{7} \end{array} \right.$

Therefore, $ggg - ccc = ddd + bbb = 8 + 1 = 9$.

3. Then by the preceding 39th Question divide the difference of the Cubes *ggg* and *ccc* into two rational Cubes, viz. divide 9 the difference of the Cubes $\frac{8000}{343}$ and $\frac{4913}{343}$ into two Cubes: But here because the double of the lesser Cube $\frac{4213}{343}$ exceeds the greater $\frac{8000}{343}$, two Cubes must first be found out, (by the help of the foregoing *Quest.* 38.) that the difference of these may be equal to the difference of those, and that the double of the lesser of the Cubes found out may be less than the greater; such are these Cubes,

Viz. $\left\{ \begin{array}{l} mmm = \frac{6695590842626239}{738542637646471}, \text{ whose side is } \frac{188479}{90391} \\ nnn = \frac{48707103808000}{738542637646471}, \text{ whose side is } \frac{36520}{90391} \end{array} \right.$

4. Now forasmuch as the double of the lesser of the two Cubes last found out is less than the greater, we may by the help of the preceding *Quest.* 39. divide 9 the difference of those Cubes *mmm* and *nnn*, (and likewise of *ggg* and *ccc*) into two rational Cubes, whose sides will be found these,

Viz. $\left\{ \begin{array}{l} k = \frac{1243617733990094836481}{609623835676137297449} \\ l = \frac{487267171714352336560}{609623835676137297449} \end{array} \right.$

5. Therefore by Construction in the two last preceding steps, $ggg - ccc = kkk + lll$

6. But by Construction in the second step, $ggg - ccc = ddd + bbb = 8 + 1 = 9$

7. Therefore from the two last Equations, $kkk + lll = ddd + bbb = 8 + 1 = 9$

Thus two Cubes (whose sides *k* and *l* are above express'd in numbers) are found out, which added together make 9, the summ of the given Cubes 8 and 1, as was required.

QUEST. 44.

To divide the double of any given cube-number into four cube-numbers.

For example, let it be required to divide 54 the double of the Cube 27, into four cube-numbers.

RESOLUTION.

1. For the given Cube 27 put ddd , viz. suppose } $ddd = 27$
2. Therefore the double of that Cube is } $2ddd = 54$
3. Take any cube-number less than the given Cube 27, as 1, for } $bbb = 1$
which put bbb , viz. suppose }
4. By the foregoing Quest. 43. find two Cubes whose sum may be equal to $ddd + bbb$,
(to wit, $27 + 1$), suppose those which solved the said Quest. 43. in Example 1.

$$\text{viz. } \begin{cases} kkk = \frac{253452325273412980702625}{9864820937041015055552} \\ lll = \frac{22762660963735440852831}{9864820937041015055552} \end{cases}$$

The sides of which Cubes are these, to wit, } $k = \frac{62284702}{21446828}$
 } $l = \frac{22342511}{21446828}$

Therefore by that Construction, } $ddd + bbb = kkk + lll = 28$

5. By Quest. 39. of this Book divide 26 the difference of the Cubes 27 and 1, to wit, $ddd - bbb$ into two Cubes, suppose into these,

$$\text{viz. } \begin{cases} rrr = \frac{421875}{21952}, \text{ whose side is } \frac{75}{28} \\ sss = \frac{148877}{21952}, \text{ whose side is } \frac{53}{28} \end{cases}$$

Therefore by this Construction, } $ddd - bbb = rrr + sss = 26$

6. Therefore by adding together the Equations } $2ddd = kkk + lll + rrr + sss = 54$
in the fourth and fifth steps, this will arise, viz. }

Therefore four Cubes are found out, to wit, kkk , lll , rrr and sss , which with their sides are before express'd in numbers in the fourth and fifth steps, and the sum of those Cubes makes 54, which is equal to the double of the Cube 27 first given, as was required by the Question.

QUEST. 45. (Quest. 17. Lib. 4. Diophant.)

To find out three numbers whose sum may make a Square; and that the second number added to the Square of the first may make a Square; also, that the third number added to the Square of the second may make a Square; and lastly, that the first number added to the Square of the third may make a Square.

RESOLUTION.

1. For the first number sought put a — any known number, as, . . . } $a - 1$
2. The Square thereof is } $aa - 2a + 1$
3. To which Square if $+ 4a$ be added, (to wit, the double of $+ 2a$, } $aa + 2a + 1$
but with the contrary sign $-$,) it makes a Square, to wit, . . . }
4. Therefore for the second number put } $4a$
Whereby one of the conditions in the Question is satisfied; for the second number $4a$ added to the Square of the first number $a - 1$ makes the Square $aa + 2a + 1$, whose Root is $a + 1$.
5. Then form a Square from $4a + 1$, (which is the sum of the second number $4a$ and the known number 1 in the first assumed number $a - 1$, but with the contrary sign $-$,) so the Square of $4a + 1$ will be $16aa + 8a + 1$; from which subtract the Square of the second number $4a$, to wit $16aa$, and put the Remainder $8a + 1$ for the third number: Whence it is evident, that if this third number be added to the Square of the second, the sum is a Square, whereby another of the conditions in the Question is satisfied.
6. From the first, fourth and fifth steps the sum of the three numbers sought is $13a$, which according to the Question must be a Square, let it therefore be equated to some Square, viz. suppose $13a = 169aa$, whence $a = 13aa$; now according to this value

value of a , the first number which was put $a - 1$ will be $13aa - 1$, the second number which was put $4a$ will be $52aa$, and lastly, the third number which was assumed $8a + 1$ will be $104aa + 1$. It remains that the Square of the third number $104aa + 1$, to wit, $10816aaaa + 208aa + 1$, added to the first number, $13aa - 1$ may make a Square; but it makes $10816aaaa + 221aa$, this therefore must be equated to a Square, or the same divided by aa gives $10816aa + 221$ to be equated to a Square, whose side, to the end that a may be greater than $\sqrt{\frac{1}{13}}$, and consequently $13aa$ greater than 1 , may be feigned to be $104a +$ any known number less than $4\frac{2}{10}$, or $104a -$ any known number greater than $60\frac{1}{10}$; let therefore the side of the said Square be feigned $104a + 1$, whence the Square it self is $10816aa + 208a + 1$, which being equated to the aforefaid $10816aa + 221$, will give $a = \frac{2}{3}$. Therefore the positions being resolved, the first number will be $\frac{26621}{2704}$, the second $\frac{127300}{2704}$, the third $\frac{317304}{2704}$; which three numbers will solve the Question, for their summ is $\frac{511225}{2704}$ the Square of the side $\frac{715}{32}$; also the Square of the first number, to wit, $\frac{1341097641}{7311616}$ added to the second makes the Square $\frac{1266436841}{7311616}$ from the side $\frac{42022}{2704}$; moreover the Square of the second, to wit, $\frac{24743290000}{7311616}$ added to the third makes the Square $\frac{25601280016}{7311616}$ from the side $\frac{166004}{2704}$; lastly, the Square of the third, to wit, $\frac{100681828416}{7311616}$ added to the first makes the Square $\frac{100780851600}{7311616}$ from the side $\frac{217460}{2704}$.

QUEST. 46.

To find three numbers, that as well the summ of every two, as of all three; may make a Square.

RESOLUTION.

1. Let b represent any known number, and a some number unknown, then from $a +$ some even number of b , (for avoiding Fractions) as from $a + 2b$ form a Square, which will be $aa + 4ba + 4bb$
2. Then for the first number sought put the two first terms of the said Square, as $aa + 4ba$
3. Then take the half of the said $4ba$, to wit, $2ba$; and prefixing the sign $-$ to it, it makes $-2ba$, to which add bb the Square of half the Coefficient $2b$, and take the summ for the second number sought, to wit, $-2ba + bb$
4. Subtract bb in the said second number from $4bb$ part of the Square first formed, and add the Remainder $3bb$ to $-2ba$, to wit, the same multitude of ba as is in the second number; but with a contrary sign, and put this summ for the third number sought, to wit, $+2ba + 3bb$
5. Then from the premisses it necessarily follows, that the summ of the first and second numbers (in the second and third steps) makes a Square, to wit, $aa + 2ba + bb$
6. And the summ of the second and third numbers (in the third and fourth steps) is a Square, to wit, $. . . + 4bb$
7. Also the summ of all the three numbers is a Square, to wit, $aa + 4ba + 4bb$
8. It remains that the summ of the first and third numbers make a Square, but it makes $aa + 6ba + 3bb$, which must be equated to a Square, yet so as the value of a may be less than $\frac{1}{2}b$, to the end that the second number $-2ba + bb$ may be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $-a +$ any number between $\sqrt{3bb}$ and $3b$, (as may be collected from the Canon in Sect. 15. Quest. 12. of this Book :) Let therefore the said side be $-a + 2b$, and then its Square $aa - 4ba + 4bb$ being equated to $aa + 6ba + 3bb$ (the summ of the first and third numbers) this Equation ariseth, to wit, $aa + 6ba + 3bb = aa - 4ba + 4bb$.
9. Whence after due Reduction, the value of a is made known, viz. $a = \frac{1}{10}b$
10. Therefore, the positions in the second, third and fourth steps being resolved according to that value of a , the three numbers sought are discovered, to wit, $\frac{41}{100}bb$, $\frac{4}{5}bb$ and $\frac{16}{5}bb$.

Hence this

CANON.

CANON.

Take any square number, then $\frac{4}{100}$ of that Square, also $\frac{2}{3}$ of the same Square, and $\frac{2}{5}$ thereof, will give three numbers to solve the Question.

As, for example, if 10 be taken for the side of a Square, then these three numbers will be found out by the Canon, to wit, 41, 80 and 320, which will solve the Question: For the summ of 41 and 80 makes the Square 121, whose side is 11; also the summ of 80 and 320 makes the Square 400, whose side is 20; and the summ of 320 and 41 makes the Square 361, whose side is 19; lastly, the summ of all the three numbers 41, 80 and 320 makes the Square 441, whose side is 21. In like manner you may find out as many Answers in whole numbers as you please, by taking 20, 30, 40 or 50, &c. for the side of a Square, and then taking such parts thereof as the Canon directs.

QUEST. 47. (Quest. 23. Lib. 4. Diophant.)

To find three numbers, that if they be severally added to the Solid produced by their continual multiplication, every one of the three summs may be a Square.

(I shall wave Diophantus's Resolution, and use that of Fermat in his Observation upon this Question, which is much easier.)

RESOLUTION.

1. Let a Square be formed from a — any known number, as from $a - 1$, whose Square is $aa - 2a + 1$
2. Then for the Solid of the three numbers sought put the two first terms of that Square, to wit, $aa - 2a$
3. And for the first number sought put the last term of the said Square, to wit, 1

Whence one of the conditions is satisfied; for if the said first number 1 be added to $aa - 2a$, (that is, the Solid of all the three numbers,) the summ is a Square, to wit, that first formed.

5. For the second number put $2a$

This added to the said Solid $aa - 2a$ makes the Square aa , whereby another of the conditions in the Question is satisfied.

5. Then divide $aa - 2a$, (the Solid of all the three numbers) by $2a$ the Product of the first and second, so the Quotient is

$$\frac{1}{2}a - 1, \text{ (the third number.)}$$

6. Which third number added to the Solid of all the three must also make a Square, but it makes $aa - \frac{1}{2}a - 1$.

7. Therefore $aa - \frac{1}{2}a - 1$ must be equated to a Square, yet so, as the value of a may be greater than 2, to the end that the third number $\frac{1}{2}a - 1$ may be greater than nothing: But to cause that effect, the side of the said Square may be feigned a — any number less than 2, but greater than $\frac{1}{4}$; or a — any number greater than 2; let then the said side be feigned $a - 3$, whose Square equated to $aa - \frac{1}{2}a - 1$, will give $a = \frac{20}{9}$. According to which value, the Positions being resolved, the first number sought is 1, the second $\frac{40}{9}$, and the third $\frac{10}{9}$, which will solve the Question: For the solid Product of their multiplication one into another, to wit, $\frac{400}{81}$, taking to it severally the said three numbers, makes the Squares $\frac{1200}{81}$, $\frac{4000}{81}$ and $\frac{400}{81}$.

QUEST. 48. (Quest. 31. Lib. 4. Diophant.)

To find four square numbers, whose summ added to the summ of their sides may make a number given, suppose 12.

RESOLUTION.

Forasmuch as (by the first Proposition in the following Observation upon this Quest.) every Square increased with his side and $\frac{1}{4}$ of unity makes a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square; therefore the summ of the four Squares sought together with four times $\frac{1}{4}$ will make four Squares; but the given number 12 increased with four times $\frac{1}{4}$, to wit, 1, makes 13. Therefore we must divide 13 into four Squares; then if from every one of their sides we subtract $\frac{1}{2}$, there will remain the sides of the four Squares sought. But 13 is compos'd of two Squares 4 and 9; therefore

therefore (by the first Question of this Book) each of these may be divided into two Squares, viz. 4 into $\frac{64}{25}$ and $\frac{16}{25}$, and 9 into $\frac{144}{25}$ and $\frac{36}{25}$: now the four Roots of those Squares are $\frac{8}{5}$, $\frac{4}{5}$, $\frac{12}{5}$ and $\frac{6}{5}$, from each of which Roots if $\frac{1}{2}$ be subtracted there will remain the sides of the four Squares sought, to wit, the sides $\frac{11}{10}$, $\frac{7}{10}$, $\frac{11}{10}$ and $\frac{11}{10}$, whose Squares $\frac{121}{100}$, $\frac{49}{100}$, $\frac{121}{100}$ and $\frac{121}{100}$ will solve the Question: For if their summ 7 be added to 5 the summ of their sides, it makes the given number 12; which was required.

Observations upon Quest. 48.

The preceding Resolution depends upon two Propositions, viz.

First, if any square number be increased with its side and $\frac{1}{4}$ of unity the summ will be a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square. This may be demonstrated thus;

Let there be a Square	>	aa
Then to that Square add its side and $\frac{1}{4}$ of unity, to wit,	>	a + $\frac{1}{4}$
So the summ makes this Square,	>	aa + a + $\frac{1}{4}$
Whose Root is	>	a + $\frac{1}{2}$
From which Root if you subtract	>	$\frac{1}{2}$
The Remainder is the side of the first Square, to wit,	>	a

Therefore the Proposition is manifest.

Secondly, the said Resolution takes this Proposition for granted, viz. That any given whole number increased with 1 may be divided into four Squares; how this may be generally done *Diophantus* doth not shew: But 'tis evident, that if a given square number increased with 1 makes a Square, or a number composed of two Squares, then the summ may easily be divided into four Squares, or into as many as you please, (by the first or second Question of this Book.) But if 13 be given, how shall 13 increased with 1, that is 14, which is neither a Square nor composed of two Squares, be divided into four Squares? This at first sight seems to be a very hard Task, but if the matter be narrowly considered, the difficulty will soon vanish; for 14 is compos'd of three Squares, to wit, 1, 4, 9; wherefore if one of these be divided into two Squares, then consequently 14 is divided into four Squares. But *Fermat* in his Observation upon this *Quest.* 31. of the fourth Book of *Diophantus*, affirms that every whole number is either a Square, or else compos'd of two, three or four Squares, and he there promiseth to give the Demonstration of this and other abstruse Mysteries in Numbers in a particular Treatise. *Bachet* confesseth he could not demonstrate the same, but gives Examples of the certainty thereof in all whole numbers from 1 to 120, and saith he had made experiment of all whole numbers to 325. If this *Prop.* be granted, then the Question may be easily extended to five, six, or as many Squares as you please, without any Determination. But if two Squares only be desired, whose summ with the summ of their sides may make a Square, then after 2 is added to the quadruple of the given number, the summ must be compos'd of two Squares: And if three Squares be sought, then after 3 is added to the quadruple of the given number the summ must be compos'd of three Squares, which conditions are manifest from the first Proposition above express'd.

QUEST. 49.

[This is the 34th of the fourth Book of *Diophantus*, and the 13th of the fifth Book of *Vieta's Zetetics.*]

To divide a given number x into two such parts, that if the first part be increased with a given number b , and the other part with a given number d ; the Product made by the multiplication of the two summs one into the other may be a square number.

RESOLUTION.

1. For the first part sought put > : : a - b
 2. Therefore the other part, to the end the summ of the parts may }
make x , shall be } x - a + b
 3. Then (according to the Question) adding b to the first part, }
the summ is } : : a
 4. And adding d to the second part, the summ is > x - a + b + d
5. There-

5. Therefore the Product made by the multiplication of those }
 two summs one into the other will be } $xa - aa - ba + da$
 6. That is } $x + b + d$ into $a, -aa$
 7. Which Product must be equated to a Square, whose side may be feigned sa , and then
 the Square of sa , to wit, $ssaa$, being equated to the Product in the sixth step, this
 Equation ariseth, *viz.*

$$x + b + d \text{ into } a, -aa = ssaa.$$

8. Which Equation, after due Reduction, gives } $a = \frac{x + b + d}{ss + 1}$
 9. Therefore from the eighth and first steps, the first part sought }
 will be made known, to wit, } $\frac{x + d - ssb}{ss + 1}$
 10. And from the eighth and second steps the second part sought }
 will be discovered, to wit, } $\frac{ssx + ssb - d}{ss + 1}$

But to the end that each of the two parts in the ninth and tenth steps may be greater than nothing, the number s cannot be taken at pleasure, but within the limits hereafter discovered, *viz.*

11. Forasmuch as the numerator in the ninth step requires that : } $x + d \leq ssb$
 12. Therefore by dividing each part in the last step by b , . . . } $\frac{x + d}{b} \leq ss$
 13. That is, } $ss \geq \frac{x + d}{b}$
 14. Therefore by extracting the square Root out of each part in }
 the last step, } $s \geq \sqrt{\frac{x + d}{b}}$
 15. Again, the Numerator in the tenth step shews, that . . . } $ssx + ssb \leq d$
 16. Therefore by dividing each part by $x + b$, it follows that } $ss \leq \frac{d}{x + b}$
 17. Therefore by extracting the square Root out of each part in }
 the last step, } $s \leq \sqrt{\frac{d}{x + b}}$
 18. Thus in the fourteenth and seventeenth steps it is discovered, that for the number s
 we may take any number between $\sqrt{\frac{x + d}{b}}$ and $\sqrt{\frac{d}{x + b}}$, and then the two desired
 parts whose summ is equal to x the number to be divided will be such as are before
 exprest in the ninth and tenth steps.

An Example in Numbers.

Suppose $\left\{ \begin{array}{l} 4 = x \\ 12 = b \\ 20 = d \end{array} \right\}$ numbers given in the Question;
 Whence $\left\{ \begin{array}{l} \frac{4}{3} = s \end{array} \right\}$ a number chosen within the limits in the eighteenth step.

Then by the help of those known numbers, the ninth and tenth steps will give $\frac{24}{25}$ and $\frac{26}{25}$
 the two parts sought, whose summ is 4, (or x ;) the former of which parts increased
 with 12 (or b ;) and the latter with 20 (or d ;) make the two summs $\frac{324}{25}$ and $\frac{276}{25}$;
 these multiplied one by the other produce a Square whose side is $\frac{432}{25}$. Therefore the
 Question is solved, and manifestly capable of innumerable Answers.

QUEST. 50. (Quest. 35. Lib. 4. Diophant.)

To divide a given number into three numbers, such, that if the Product of the multi-
 plication of the first into the second be increased and lessened by the third, as well the summ
 as the remainder may be a square number.

RESOLUTION.

1. Let the given number be } 6
 2. For the third number sought put } a
 3. For the second number sought put some known number less than the given }
 number 6, as } 2
 4. Therefore, because all the three numbers sought must make 6, the first }
 number shall be } $4 - a$
 L 5. Then

5. Then (according to the Question) the Product of the first and second numbers sought, together with the third, must make a Square, viz. $8 - a = \square$
 6. Also the same Product lessened by the third number must make a Square, viz. $8 - 3a = \square$

7. So in the two last steps we are falln upon a Duplicate equality; but 'tis inexplicable; *Diophantus* therefore seeks out another Duplicate equality wherein the numbers prefixt to *a* may have such proportion to one another as a square number hath to a square number, for then it will be resolvable like that Duplicate equality which hath been already explain'd in *Quest. 35.* of this *Book.* First, then instead of 2 which was assumed for the second number sought, some other number less than 6 must be taken, such, that if it be increased and lessened by unity, the summ may be to the remainder as a square number to a square number; (for if the rise of 3 and 1, which are prefixt to *a* in the Duplicate equality above exprest, be examined, it will appear that 3 ariseth by adding 1 to 2; and 1 ariseth by subtracting 1 from the same number 2.) Therefore let *e* represent some number to be taken instead of 2 for the second number sought; then $e + 1$ must be to $e - 1$; as a Square to a Square, suppose as *bb* to *dd*.

Therefore, as $e + 1 . e - 1 :: bb . dd$

8. Therefore by comparing the Product of the extremes to the Product of the means, $dde + dd = bbe - bb$
 9. Whence after due Reduction, $e = \frac{bb - dd}{bb - dd}$
 10. But the value of *e* must be less than 6, which I shall call *f*, therefore $\frac{bb - dd}{bb - dd} = f$
 11. Therefore by multiplying each part by *f*, $bb - dd = f \times bb - ad$:
 Hence this Canon to find out the number *e*, viz.

12. Take any two square numbers whose summ may be less than the Product of their difference multiplied into the number given in the Question; then divide the summ of the said Squares by their difference, so the Quotient shall be the number to be put for the second number sought by the Question proposed, for it shall be less than the number given in the Question; and if it be increased with 1 and lessened by 1, the summ shall be to the remainder as the greater of the Squares taken is to the lesser. Therefore I take the Squares 4 and 1 and divide their summ 5 by their difference 3, so there ariseth $\frac{5}{3}$ for the second number. Now let the Positions be renewed thus, viz.

13. The number given to be divided into three numbers is 6
 14. For the second number put $\frac{5}{3}$
 15. And for the third number a
 16. Then the summ of the second and third numbers subtracted from 6 (the summ of all three) leaves the first number $\frac{12}{3} - a$
 17. Now (according to the Question) the Product of the first and second numbers together with the third, must make a Square, viz. $\frac{52}{9} - \frac{2}{3}a = \square$
 18. And the same Product lessened by the third number must leave a Square, viz. $\frac{52}{9} - \frac{8}{3}a = \square$

19. So in the two last steps there is a Duplicate equality wherein the numbers prefixt to *a* have such proportion one to another as a Square to a Square, for $\frac{2}{3}$ is to $\frac{8}{3}$ as 1 to 4; and therefore this Duplicate equality may be resoloved like that in the preceding *Quest. 35.* in this manner, viz. Forasmuch as a square number multiplied by a Square produceth a Square, therefore to take away the Fractions, multiply all the Quantities in the 17th and 18th steps by the Denominator 9, so this Duplicate equality ariseth,

$$\text{viz. } \begin{cases} 65 - 6a = \square \\ 65 - 24a = \square \end{cases}$$

20. Then the former of those two Equations multiplied by 4 produceth $260 - 24a = \square$, so at length this Duplicate equality remains to be resoloved,

$$\text{viz. } \begin{cases} 260 - 24a = \square \\ 65 - 24a = \square \end{cases}$$

21. Now the difference of these two Equations being 195, I seek by *Canon 2. Quest. 7.* of this *Book* two such square numbers that their difference may be 195, and that the greater Square may be less than 260. But the only pair of Squares in whole numbers so

so qualified are 196 and 1, the greater of which being equated to $260 - 24a$, or the lesser to $65 - 24a$, will give $a = \frac{2}{3}$ for the third number sought; and consequently, by the Positions in the fourteenth and sixteenth steps, the first and second numbers are $\frac{1}{3}$ and $\frac{2}{3}$, which three numbers will solve the Question, as is evident by the Proof, for their sum is 6; also if $\frac{2}{9}$ the Product of the first and second be increased with the third number $\frac{2}{3}$ it makes the Square $\frac{4}{9}$; but if the same Product be lessened by the said $\frac{2}{3}$ it leaves the Square $\frac{1}{9}$.

QUEST. 51. (Quest. 36. Lib. 4. Diophant.)

To find three numbers, whereof the third may be such a Fraction of unity, that if the first number takes from the second such part or parts as the Fraction expresseth, the sum may be to the remainder in a given Reason, suppose as b to d . Also, that the second number taking the same part or parts from the first, the sum may be to the remainder in a given Reason, suppose as f to g . But the Product made by the mutual multiplication of the first term of each Reason must exceed the Product of the latter terms one into the other, viz. bf must be greater than dg .

Preparation.

Let a and e represent the first and second numbers, and u the third, or Fraction sought; then because to take any part or parts of a number, the number must be multiplied by the Fraction expressing the parts, the Question may be stated thus, viz.

1. If $a + ue . e - ue :: b . d$,
2. And $e + ua . a - ua :: f . g$.

What are the numbers, a, e, u ?

RESOLUTION.

3. By comparing the Product of the extremes to the Product of the means in the first Analogy, this Equation is produced, to wit,

$$da + due = be - bue.$$

4. Likewise from the latter Analogy this Equation is produced, viz.

$$ge + gna = fa - fua.$$

5. From the Equation in the third step by transposition of due , this ariseth,

$$da = be - bue - due.$$

6. And by dividing each part of the last Equation by d , this ariseth,

$$a = \frac{be - bue - due}{d}$$

7. Then by exchanging a in the fourth step, for that which is equal to a in the last Equation, and multiplying all into d , the Equation in the fourth step will be converted into this, viz.

$$\left. \begin{array}{l} dge + bgue - bgue \\ - dguue \end{array} \right\} = \left\{ \begin{array}{l} bfe - 2bfue - dfue \\ + bfue + dfue \end{array} \right.$$

8. From which Equation, after due Reduction, this following Equation ariseth, wherein u only is unknown, viz.

$$\frac{bg + 2bf + df}{bg + dg + bf + df} u - uu = \frac{bf - dg}{bg + dg + bf + df}$$

9. Which last Equation may be resolved (by the Canon in Sect. 10. Chap. 15. Book 1.) in this manner, viz. half the Coefficient drawn into u in the said Equation is

$$\frac{\frac{1}{2}bg + bf + \frac{1}{2}df}{bg + dg + bf + df}$$

10. The Square of the said half Coefficient is

$$\frac{\frac{1}{4}bbgg + bbff + \frac{1}{4}ddff + bbfg + bdf + \frac{1}{2}bdfg}{bg + dg + bf + df}$$

11. Then to reduce the known Absolute quantity which solely possesseth the latter part of the Equation in the eighth step to the same Denominator with the Square in the tenth step, I multiply as well the Numerator as the Denominator of the said Absolute quantity, by its Denominator $bg + dg + bf + df$, and it makes

$$\frac{bbfg + bbff + bdf - bdgg - ddgg - ddfg}{bg + dg + bf + df}$$

12. Which Fraction last above express being subtracted from the Fraction in the tenth step will leave this that follows, to wit,

$$\frac{\frac{1}{4}bbgg + \frac{1}{4}ddff + dagg + \frac{1}{2}bdfg + bagg + ddfg}{bg + dg + bf + af} \text{ into } \frac{bg + dg + bf + df}{}$$

13. The square Root of the Fraction in the twelfth step is

$$\frac{\frac{1}{2}bg + \frac{1}{2}df + dg}{bg + dg + bf + df}$$

14. Which square Root being added to and subtracted from the half-Coefficient in the ninth step, the summ and remainder shall be the two values of u in the Equation in the eighth step, viz.

$$u = 1; \text{ also, } u = \frac{bf - dg}{bg + dg + bf + df}$$

The latter of which values of u , to wit, $\frac{bf - dg}{bg + dg + bf + df}$ is the Fraction sought by the Question.

15. Then according to the said lesser value of u , the compound quantity $\frac{b - bu - du}{d}$ into which e is multiplied in the latter part of the Equation in the sixth step will be reduced into this fractional quantity, to wit, $\frac{bbg + bdg + ddg}{bdg + ddg + bdf + ddf}$, which multiplied into any number taken at pleasure for the value of e , will give the number a ; and therefore to find out a and e in whole numbers we may take the Numerator of that fractional quantity for a , and the Denominator for e , or any two whole numbers in the same proportion with the said Numerator and Denominator, and the lesser of the two values of u before found for the Fraction sought.

An Example in Numbers.

16. Let there be given $\left. \begin{array}{l} 3 = b \\ 1 = d \\ 5 = f \\ 1 = g \end{array} \right\}$

17. Then the three numbers sought will be these, viz. $\left. \begin{array}{l} 16 = bbg + 2bdg + ddg \\ 24 = bdg + ddg + bdf + ddf \\ \frac{7}{12} = \frac{bf - dg}{bg + dg + bf + df} \end{array} \right\}$

18. Which three numbers, to wit, 16, 24 and $\frac{7}{12}$ will solve the Question, as will be manifest by

The Proof.

19. If the first number 16 receive $\frac{1}{12}$ of the second number 24, that is, 14, the summ 30 will be to the remainder 10 as 3 to 1, (viz. as b to d ;) Again, if the second number 24 take $\frac{2}{3}$ parts of the first number 16, to wit, $9\frac{2}{3}$, the summ $14\frac{2}{3}$ will be to the remainder $5\frac{1}{3}$ as 5 to 1, (to wit, as f to g ;) Or instead of 16 and 24 you may take 2 and 3, or any two numbers in the same Proportion.

20. Again, if $b = 3$. $d = 2$. $f = 4$. and $g = 5$. then the literal quantities in the preceding 17th step will give these three numbers, 125, 90 and $\frac{7}{12}$ to solve the Question; or instead of 125 and 90 you may take 25 and 18, or any two numbers in the same Proportion.

Note. If in reducing the Equation in the seventh step, bf were supposed either equal to dg or less than dg , there would come forth an Equation wherein the value of u would be either unity or greater than unity; but according to the import of the Question it ought to be less than unity, and such is the lesser value of u in the Equation in the eighth step, where bf is supposed greater than dg , as the Determination annexed to the Question requires.

QUEST. 52. (Quæst. 41. Lib. 4. Diophant.)

To find two numbers, that the Product of their multiplication may be to their summ in a given Reason (or Proportion,) suppose as r to s .

RESO.

RESOLUTION.

1. For the first number put } a
 2. And for the second } e
 3. Then their summ is } $a + e$
 4. And the Product of their multiplication is } ae
 5. But according to the Question the Product must be to the summ }
as r to s , therefore } $ae . a + e :: r . s$
 6. Therefore by comparing the Product of the extremes to the Pro- }
duct of the means this Equation ariseth, } $sae = ra + re$
 7. And by subtracting re from each part of that Equation, this }
ariseth, viz. } $sae - re = ra$
 8. And by dividing each part of the last Equation by $sa - r$, this }
ariseth, viz. } $e = \frac{ra}{sa - r}$
- Which Equation gives this

CANON.

9. For the first number sought take any number greater than the Quotient that ariseth by dividing the former term of the given Reason by the latter; then multiply the first number so taken by the latter term, and from the Product subtract the former term; lastly, by the Remainder divide the Product made by the multiplication of the first number into the former term, and the Quotient shall be the second number sought. For example, if two numbers be desired that their Product may be to their summ as 3 to 2, (that is, as r to s in the Resolution,) you may take any number greater than $\frac{2}{2}$, as 2, for the first number; then (by the Canon) the other number will be found 6; which numbers 2 and 6 are such; that their Product 12 is to their Summ 8, as 3 to 2, as was desired. After the same manner you may find out innumerable Answers to the Question.
10. But if it were desired to find out two numbers that the Product of their multiplication might be equal to their summ, and that the summ or Product might be a square number; the numbers may be found out thus, viz.

- For the first number sought put } a
 And for the second number } e
 Then according to the Question, } $ae = a + e$
 Therefore by transposition of e , } $ae - e = a$
 Therefore each part of the last Equation being divided by }
 $a - 1$, there will arise } $e = \frac{a}{a - 1}$
 Which last Equation multiplied by a gives } $ae = \frac{aa}{a - 1} (= a + e)$

Which $\frac{aa}{a - 1}$ must (according to the Question) be a square number. But the Numerator aa is a Square; it remains then to equate the Denominator $a - 1$ to some square number, let it be dd , viz. suppose $a - 1 = dd$, whence $a = dd + 1$; according to which value of a , the number e which was before found equal to $\frac{a}{a - 1}$, will be $\frac{dd + 1}{dd}$. Therefore for the two numbers sought take $dd + 1$ and $\frac{dd + 1}{dd}$, which in words give this

CANON.

For one of the numbers sought take any square number increased with unity; then divide that summ by the square number taken, and the Quotient shall be the other number sought.
 As, for example, you may take $4 + 1$; that is, 5 for one of the numbers sought; then dividing 5 by 4 (the Square taken,) the Quotient $\frac{5}{4}$ shall be the other number sought. So 5 and $\frac{5}{4}$ will solve the Question last proposed; for their Product is $\frac{25}{4}$, their Summ also is $\frac{25}{4}$, which is a square number as was required.

QUEST. 53.

To find two numbers, that their difference may be equal to the difference of their Squares, and that the summ of the Squares of the two numbers may be a Square.

RESO-

RESOLUTION.

1. For the greater number put } a
2. And for the lesser } e
3. Then their difference is } $a - e$
4. And the difference of their Squares is } $aa - ee$
5. Therefore according to the Question, } $aa - ee = a - e$
6. Therefore by dividing each part of that Equation by $a - e$, the }
Quotient gives } $a + e = 1$
7. Therefore by transposition of a , } $e = 1 - a$
8. Now taking $1 - a$ instead of e , the two numbers sought are . } a and $1 - a$
9. The difference of which numbers is either $2a - 1$ or $1 - 2a$, and the same is the difference of their Squares: But the summ of their Squares must make a Square; therefore $2aa - 2a + 1$ must be equated to a Square, yet so, as the value of a may be less than 1. Now to cause that effect the said side may be feigned $-1 +$ any multitude of a greater than $2a$; let therefore the said side be feigned $3a - 1$, then the Square of $3a - 1$ being equated to the said $2aa - 2a + 1$ the value of a will be found $\frac{4}{7}$, which subtracted from 1 leaves $\frac{3}{7}$; therefore $\frac{4}{7}$ and $\frac{3}{7}$ are the numbers sought. For as well their difference as the difference of their Squares is $\frac{1}{7}$; and the summ of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$ makes a Square, to wit, $\frac{25}{49}$.

From the premisses ariseth this

CANON.

10. Take any number greater than 2; then divide the excess of that number above unity, by the excess of half the Square of the same number above unity, and the Quotient shall be one of the numbers sought, which subtracted from unity leaves the other number sought. As, for example, take the number 3; then dividing the excess of 3 above 1, that is, 2, by the excess of half the Square of 3 above 1, that is, by $\frac{7}{2}$; the Quotient $\frac{4}{7}$ is one of the numbers sought, which subtracted from 1 leaves $\frac{3}{7}$ for the other number.
11. The same Question may be propounded thus, *viz.* To find a right-angled Triangle in Rational numbers, that the difference of the sides about the right-angle may be equal to the difference of the Squares of the same sides. For solving this Question, take any two numbers found out by the said Canon, as $\frac{4}{7}$ and $\frac{3}{7}$ for the sides about the right-angle, whence the Hypotenusal (to wit, the square Root of the summ of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$) is $\frac{5}{7}$.

Moreover, from the foregoing Resolution of *Quest.* 53. we may deduce this

THEOREM.

12. If unity be divided into any two parts, the difference of the parts is equal to the difference of the Squares of the same parts: And if the Product made by the mutual multiplication of the parts be subtracted from each of them, each Remainder will be a Square: Also the excess of the greater part above its Square is equal to the excess of the lesser part above its Square, and each excess is equal to the Product of the parts. This will easily be manifested by these two numbers a and $1 - a$, whose summ is unity.

QUEST. 54. (Quest. 45. Lib. 4. Diophant.)

To find three numbers, that the excess of the greatest above the mean may be to the excess of the mean above the least in a given Reason, suppose as 3 to 1; and that the summ of every two of the three numbers may be a Square.

RESOLUTION.

1. Forasmuch as the summ of the mean and least of the three numbers must }
be a Square, let it be } 4
2. Therefore the mean is greater than 2; for if we should put it 2, the least }
would be also 2, which is absurd; therefore for the mean let there be put } $a + 2$
3. Therefore from the said positions the least number is } $2 - a$
4. And the excess of the mean above the least is } $2a$
5. But the excess of the greatest above the mean must be the triple of the }
excess of the mean above the least; therefore from the last step the } $6a$
excess of the greatest above the mean is }

6. Which

6. Which last mentioned excess, to wit, $6a$, being added to the mean $a-2$, gives the greatest of the three numbers, to wit, $7a-2$
7. But the Question requires two things more, to wit, that the greatest with the mean may make a Square; and that the greatest with the least may make a Square; hence ariseth this Duplicate equality to be resolved, to wit, $8a-4 = \square$ and $6a-4 = \square$
8. Which Duplicate equality hath already been resolved in the preceding Quest. 33. of this Book, where the number $\frac{422}{242}$, (among innumerable other numbers that might be discovered, and each to be less than 2, as the third step of the preceding Resolution requires,) was found out for the value of a . Therefore $\frac{422}{242}$ being taken for the value of a , and the positions in the sixth, second and third steps resolved accordingly, the three numbers sought will be found these, to wit, $\frac{2338}{484}$, $\frac{1878}{484}$ and $\frac{58}{484}$, which will solve the Question. For first, the excess of the greatest above the mean, to wit, $\frac{2460}{484}$, is the triple of $\frac{1820}{484}$ the excess of the mean above the least; secondly, the summ of the greatest and mean is the Square $\frac{2216}{484}$, whose side is $\frac{96}{22}$; thirdly, the summ of the greatest and least is the Square $\frac{296}{484}$, whose side is $\frac{86}{22}$; lastly, the summ of the mean and least is the Square $\frac{936}{484}$, whose side is $\frac{44}{22}$, that is, 2.

QUEST. 55. (Quest. 2. Lib. 5. Diophant.)

To find three numbers in Geometrical proportion, that every one of them increased with a given number d may make a Square.

RESOLUTION.

1. First, seek a Square which added to the given number d may make a Square, and whose $\frac{1}{4}$ part may exceed d ; suppose it be found bb , let this be put for one of the extreme Proportionals sought, to wit, bb
2. For the other extreme put aa
3. Then because the Product of the extremes is equal to the Square of the mean, therefore the Square of the mean is $bbaa$, whose Root is the mean ba it self, to wit, ba
4. By Construction in the first step the first extreme bb increased with the given number d makes a Square; but according to the Question the other extreme and the mean being severally increased with the same given number d must also make a Square, whence this Duplicate equality ariseth,

viz. $\begin{cases} aa + d = \square \\ ba + d = \square \end{cases}$

5. Now to resolve this Duplicate equality I proceed as in former Questions, viz. First, the difference of those two Equations is $aa - ba$, which is equal to the Product of a into $a - b$; then if the Square of half the summ of a and $a - b$ be equated to $aa + d$, or the Square of half the difference of a and $a - b$ to $ba + d$, from either of those Equations the value of a will be made known: But half the difference of the said a and $a - b$ is $\frac{1}{2}b$, whose Square is $\frac{1}{4}bb$; let this be equated to $ba + d$, and it will be

$ba + d = \frac{1}{4}bb$

6. Whence, after due Reduction, the value of a will be discovered, viz. $a = \frac{\frac{1}{4}bb - d}{b}$
7. And by multiplying each part of the last Equation by b , it gives $ba = \frac{1}{4}bb - d$
- From the premisses ariseth the following Canon to find out the three Proportionals sought:

CANON.

8. Take for one of the extreme Proportionals sought such a square number that if it be increased with the given number it may make a Square, and that a quarter of the first Square may exceed the given number; then from a quarter of the first Square subtract the given number and there will remain the mean Proportional; lastly, divide the mean by the side of the Square first taken, and the Square of the Quotient shall be the other extreme.

An Example in Numbers.

Suppose $19 = d$ the number given in the Question; then find a Square that if it be increased with 19 may make a Square, and that a quarter of the Square found out may exceed

exceed the said 19, (or that the side of the said Square may exceed $\sqrt{76}$.) But such is the Square 81, (found out after the manner of resolving the ninth Question of this Book,) for 81 increased with 19 makes the Square 100, also $\frac{1}{4}$ of 81 is greater than 19; therefore 81 shall be the first of the three Proportionals sought. Then by the Canon above exprest, the other two will be found $\frac{5}{4}$ and $\frac{25}{1296}$. I say, 81, $\frac{5}{4}$ and $\frac{25}{1296}$ will solve the Question: For first they are continual Proportionals, in regard the Product of the extremes is equal to the Square of the mean; secondly, the first Proportional 81 increased with the given number 19 makes the Square 100; thirdly, the mean Proportional $\frac{5}{4}$ increased with the said 19 makes the Square $\frac{25}{4}$; lastly, the third Proportional $\frac{25}{1296}$ increased with the said 19 makes the Square $\frac{25+649}{1296}$, whose side is $\frac{157}{36}$. Therefore the Question is solved, and manifestly capable of innumerable Answers.

QUEST. 56. (Quest. 7. Lib. 5. Diophant.)

To find two numbers, that the Product of their multiplication added to the sum of their Squares may make a Square.

RESOLUTION.

1. For one of the numbers sought take any known number, as } b
2. For the other number put } a
3. The Square of the first is } bb
4. The Square of the second is } aa
5. The Product of the multiplication of the two numbers is } ba
6. Therefore the sum of the said Squares and Product is } $aa + ba + bb$
7. Which sum must be equal to a Square, the side whereof may be feigned to be $a - d$, any known number greater than b , let it be $a - d$, and then the Square of $a - d$ being equated to the said sum, this Equation ariseth, viz. } $aa + ba + bb = aa - 2da - dd$
8. Which Equation after due Reduction gives } $a = \frac{dd - bb}{2d + b}$
9. Therefore from the first, second and eighth steps the two numbers sought are equal to these known numbers, viz. } b and $\frac{dd - bb}{2d + b}$
10. But to avoid Fractions multiply those two numbers severally by the Denominator $2d + b$, and take the Products for two numbers to solve the Question, viz. } $2db + bb$ and $dd - bb$

The Proof.

11. The Square of $dd - bb$ is } $dddd - 2ddbb + bbbb$
12. The Square of $2db + bb$ is } $4dabb + 4dbbb + bbbb$
13. The Product of $dd - bb$ into $2db + bb$ is } $2bddd - 2dbbb + bbdd - bbbb$
14. The sum of the said Squares and Product is } $d^4 + b^4 + 3dabb + 2db^3 + 2bd^3$
15. Which sum is a Square whose Root is } $dd + db + bb$

From the tenth step ariseth

CANON 1.

16. Take any two unequal square numbers, then their difference shall be one of the numbers sought; and the lesser Square increased with the double Product of the multiplication of the sides of those Squares shall be the other number sought.

Moreover, because the Product of the multiplication of the sum of any two numbers into their difference, is equal to the difference of their Squares, therefore from the preceding Canon ariseth

CANON 2.

17. Take any two unequal numbers, then the Product of their sum multiplied into their difference shall be one of the two numbers sought, and the double Product made by the mutual multiplication of the two numbers first taken, together with the Square of the lesser number shall be the other number sought.

From

From the premisses 'tis evident that the Question is capable of innumerable Answers in whole numbers, of which (for the Learners exercise) I shall exhibit six, with their Proofs, in the following Table.

	d, b	s, r	ss	rr	sr	qq	q
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

18. The numbers under d and b in this Table are six pairs of numbers taken at pleasure, by which the latter of the two preceding Canons gives six pairs of numbers under s and r to solve Quest. 56. As, for example, if 2 and 1 be taken, then Canon 2. gives 5 and 3, (that is, s and r) to solve the Question: For 25 and 9, the Squares of 5 and 3, together with 15 the Product of 5 and 3, (that is, $ss + rr + sr$) make the Square 49, (that is qq ,) whose side is 7, (to wit, q .)

But for a further Proof, you may observe from the fourteenth and fifteenth steps of the Resolution, that every number standing under q is equal to its respective $dd + db + bb$; so 7 in the Coluinel of q is equal to the Squares of 2 and 1 standing under d, b , together with the Product of 2 into 1. The like is to be understood of the other five Answers in the Table.

Observat. 1. upon the foregoing Quest. 56.

Whereas it is taken for granted in the tenth step of the preceding Resolution of Quest. 56. that two numbers in the same Reason (or Proportion) with those found out to solve the said Question will likewise satisfie the same, I shall here demonstrate the truth thereof.

Suppositions.

1. Let two numbers capable of solving Quest. 56. suppose } a and b
 5 and 3 , be represented by }
2. And let their Squares be signified by } aa and bb
3. Then according to the import of Quest. 56. } $aa + ba + bb = \square$
4. Let two other numbers having the same Proportion to }
one another as a to b be represented by d and c , viz. } $a . b :: d . c$
suppose }
5. Then add cd the Product of c and d to cc and dd }
the Squares of c and d , so the summ is } $dd + cd + cc$
6. Now we must demonstrate that } $dd + cd + cc = \square$ (a Square.)

Demonstration.

7. By Prop. 17. Elem. 7. Euclid. } $a . b :: aa . ba$
8. And by Prop. 11. Elem. 8. } $aa . ba :: ba . bb$
9. Therefore out of the two last preceding Ana- }
logies, } $a . b :: aa . ba :: ba . bb$
10. And by the like argumentation, } $d . c :: dd . cd :: cd . cc$
11. And because by supposition in the fourth step, } $a . b :: d . c$
12. Therefore out of the ninth, tenth and ele- }
venth steps, (per Prop. 11. Elem. 5.) } $aa . ba :: ba . bb :: dd . cd :: cd . cc$
13. Therefore alternately (per Prop. 13. Elem. 7.) } $aa . dd :: ba . cd :: bb . cc$
14. Therefore (per Prop. 12. Elem. 7.) } $aa + ba + bb . dd + cd + cc :: aa . dd$
15. And because by supposition } aa and dd are Squares.
16. And by supposition in the third step, } $aa + ba + bb = \square$
17. Therefore from the fourteenth, fifteenth and }
sixteenth steps; (per Prop. 24. Elem. 8.) } $dd + cd + cc = \square$

Which was to be proved.

M

Observat. 2.

Observat. 2. upon Quest. 56.

Albert Girard in pag. 618. of Simon Stevin's Arithmetick printed in the French Tongue at Leyden, in 1625. doth from the said seventh Question of the fifth Book of Diophantus deduce this following

T H E O R E M.

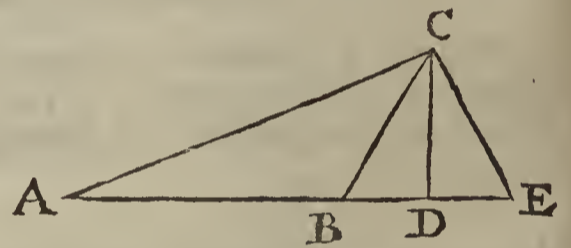
If a plain Triangle be made of three such sides, that the sum of the Squares of two of those sides, together with the Rectangle (or Product of the multiplication) of the same two sides, is equal to the Square of the third side; then the angle opposite to such third side hath for its measure exactly 120 degrees. But if the said Rectangle (or Product) be subtracted from the sum of the said Squares, and the Remainder be equal to the Square of the third side, then the angle opposite to such third side shall have for its measure infallibly 60 degrees.

This may easily be demonstrated by Prop. 12, & 13. Elem. 2. Euclid. but waving the Demonstration, I shall explain the Theorem by Numbers.

First then, if three numbers be desired to express the measures of the sides of a plain Triangle that shall have one angle whose measure is 120 degrees, the preceding Table will furnish you with six such Triangles; for in every rank of numbers in that Table; the three numbers which answer to s, r and q will constitute the Triangle desired. As, for example, 5, 3 and 7; likewise, 8, 7, 13 | 16, 5, 19, &c. will express the Quantities of the sides of Triangles, in every one of which, the measure of the angle opposite to the greatest side is exactly 120 degrees.

Otherwise, without the help of the said Table, if two unequal numbers be taken at pleasure, as $2 = d$, and $1 = b$, then these three following numbers shall express the measures of the sides of a Triangle having an angle of 120 degrees, viz.

$$\left\{ \begin{array}{l} 2db + bb = 5 = AB, \\ dd - bb = 3 = BC = BE = EC, \\ dd + db + bb = 7 = AC. \end{array} \right.$$



Which three numbers, (as is manifest by the preceding Resolution of Quest. 56.) have this property, viz. the sum of the Squares of the two lesser numbers together with the Product of their multiplication is equal to the Square of the third or greatest number.

Moreover, if three unequal numbers be desired to express the Quantities of the sides of a plain Triangle that shall have for the measure of one of its angles exactly 60 degrees, you may readily find them out by the help of the preceding Table: For the numbers answering to $s + r, r$, and q are the three numbers desired; so from the first rank of numbers in the Table, you may take

$$\left\{ \begin{array}{l} 8 = s + r = AE, \\ 3 = r = EC = EB = BC, \\ 7 = q = AC. \end{array} \right.$$

Which three numbers 8, 3, 7 are the measures of the sides of a Triangle having one of its angles, (to wit, that opposite to 7 or q ,) exactly 60 degrees.

Otherwise, without the help of the said Table, if two unequal numbers be taken at pleasure, as $2 = d$, and $1 = b$, then these three following numbers shall be the measures of the sides of a Triangle having one angle of 60 degrees, to wit, that opposite to the side $dd + db + bb$.

$$\left\{ \begin{array}{l} 2db + dd = 8 = AE, \\ dd - bb = 3 = EC, \\ dd + db + bb = 7 = AC. \end{array} \right.$$

In which Triangle the Square of the side $dd + db + bb$ is equal to the sum of the Squares of the other two sides $2db + dd$ and $dd - bb$, less by the Rectangle (or Product) of the same two sides, as is evident by the following

Proof.

$$\begin{array}{l} \text{The Square of } 2db + dd \text{ is} \quad > 4d^2b + 4b^2d + d^4 \\ \text{The Square of } dd - bb \text{ is} \quad > d^4 - 2d^2b + b^4 \end{array}$$

The

The sum of those Squares is $2dabb + 4bddd + 2dddd + bbbb$
 The Product of $2db + dd$ into $dd - bb$ is $> 2bddd + dddd - 2dbbb - ddbb$
 Which Product being subtracted from the
 said sum of the Squares, leaves $dddd + bbbb + 3dabb + 2dbbb + 2bddd$
 Which Remainder is the Square of $dd + db + bb$, as was affirmed.

QUEST. 57. (Quest. 8. Lib. 5. Diophant.)

To find three right-angled Triangles in Rational numbers that shall have equal Area's:

RESOLUTION.

1. First, by either of the Canons in Sect. 16, 17. of the foregoing Quest. 56. find out two numbers } s the greater,
 capable of solving that Question, suppose } r the lesser.
2. Therefore, (according to the import of the said Quest. 56.) the Squares of s and r , with the Product of s into r is equal to some Rational square number, let it be qq , whence $ss + sr + rr = qq$
3. Therefore by Construction in the first and second steps, $\sqrt{ss + sr + rr} = q$, (Rational.)
4. By the Canon in Observat. 8. Resolut. 2. Quest. 1. form a right-angled Triangle from $\sqrt{ss + sr + rr}$: (that is, q) and r , so the three sides will be expressible by these Rational numbers,

viz. $\left\{ \begin{array}{l} ss + sr + 2rr = \text{Hypothensal,} \\ ss + sr = \text{Base,} \\ 2qr = \text{Perpendicular.} \end{array} \right.$

5. In like manner form a second right-angled Triangle from $\sqrt{ss + sr + rr}$: (that is, q) and s , so the three sides will be expressible by these Rational numbers,

viz. $\left\{ \begin{array}{l} 2ss + sr + rr = \text{Hypothensal,} \\ sr + rr = \text{Base,} \\ 2qs = \text{Perpendicular.} \end{array} \right.$

6. Likewise, form a third right-angled Triangle from $\sqrt{ss + sr + rr}$: (that is, q) and $s + r$, so the three sides will be expressible by these Rational numbers,

viz. $\left\{ \begin{array}{l} 2ss + 3sr + 2rr = \text{Hypothensal,} \\ sr = \text{Base,} \\ 2qs + 2qr = \text{Perpendicular.} \end{array} \right.$

7. I say those three Triangles will solve this 57th Question: For first, by Construction they are right-angled Triangles; secondly, all the sides are expressible by Rational numbers, for s , r and q are Rational numbers by Construction; thirdly, if in every one of those three Triangles the Base be multiplied by the Perpendicular, every one of the three Products will manifestly be equal to $2qrss + 2qsrr$; therefore the halves of those Products, that is, the Area's of those three right-angled Triangles are equal to one another, as was required.

8. And since the foregoing Quest. 56. gives innumerable whole numbers answering to s , r and q ; you may find out as many Ternions of right-angled Triangles in whole numbers as shall be desired to solve this 57th Question. But to have the nine sides of every Ternion expressible by whole numbers in the least Terms, every pair of the said numbers s , r must be the least Terms of a Reason; that is, two such numbers as have no common Divisor besides unity.

9. The premisses give the following Canon to find out innumerable Ternions of right-angled Triangles in whole numbers to solve the Question proposed; after the Rational whole numbers represented by s , r and q are first found out by either of the Canons of Quest. 56.

CANON.

$ss + sr + 2rr = h = \text{Hypoth.}$
 $ss + sr = b = \text{Base,}$
 $2qr = p = \text{Perp.}$

$$\begin{array}{l} 2ss + sr + rr = h = \text{Hypoth.} \\ sr + rr = b = \text{Base,} \\ 2qs = p = \text{Perp.} \end{array} \left. \vphantom{\begin{array}{l} 2ss + sr + rr = h = \text{Hypoth.} \\ sr + rr = b = \text{Base,} \\ 2qs = p = \text{Perp.} \end{array}} \right\} \text{ of Triangle II.}$$

$$\begin{array}{l} 2ss + 3sr + 2rr = H = \text{Hypoth.} \\ sr = B = \text{Base,} \\ 2qs + 2qr = P = \text{Perp.} \end{array} \left. \vphantom{\begin{array}{l} 2ss + 3sr + 2rr = H = \text{Hypoth.} \\ sr = B = \text{Base,} \\ 2qs + 2qr = P = \text{Perp.} \end{array}} \right\} \text{ of Triangle III.}$$

Observations upon the Canon last foregoing.

Divers properties, besides the equality of Area's, in the three right-angled Triangles found out by the said Canon, do present themselves to your view, and are worthy of Observation. The principal Properties are these four, viz.

$$\begin{array}{l} 1. \quad h + b = b + b. \\ 2. \quad H + B = 2b + 2b. \\ 3. \quad H - B = b + b = h + b. \\ 4. \quad P = p + p. \end{array}$$

That is to say, in words,

1. The summ of the Hypothensal and Base of the first Triangle, is equal to the summ of the Hypothensal and Base of the second.
2. The summ of the Hypothensal and Base of the third Triangle, is equal to the double summ of the Bases of the first and second.
3. The excess of the Hypothensal above the Base of the third Triangle is equal to the summ of the Hypothensal and Base of the second, and likewise to the summ of the Hypothensal and Base of the first.
4. The Perpendicular of the third Triangle is equal to the summ of the Perpendiculars of the first and second.

By the first of those three Triangles is meant that which hath the shortest Hypothensal; by the second, that whose Hypothensal is next greater than the shortest; and by the third, that which hath the longest Hypothensal; in which order they are set in the Table. But the better to explain the Canon and Properties, I shall resume the Table belonging to *Quest. 56.* and call it *Table I.* whence six Answers in whole numbers to this *Quest. 57:* are deduced, and inserted in the following *Table II.*

Table I. brought from Quest. 56.

	<i>d, b</i>	<i>s, r</i>	<i>ss</i>	<i>rr</i>	<i>sr</i>	<i>qq</i>	<i>q</i>
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

Table II. deduced from Tab. I. by the Canon in Sect. 9. Quest. 57.

	<i>h.</i>	<i>b.</i>	<i>p.</i>	<i>h.</i>	<i>b.</i>	<i>p.</i>	<i>H.</i>	<i>B.</i>	<i>P.</i>
1	58	40	42	74	24	70	113	15	112
2	218	120	182	233	105	208	394	56	390
3	386	336	190	617	105	608	802	80	798
4	1418	1320	518	2458	280	2442	2969	231	2960
5	1082	840	682	1537	385	1488	2186	264	2170
6	2657	2145	1568	3922	880	3822	5426	624	5390

The Construction of *Table I.* hath already been express'd in *Quest. 56.* whence the latter *Table* is deduced according to the Canon in *Sect. 9.* of this *57th* Question, and contains six Answers to it; the first of which is to be understood thus,

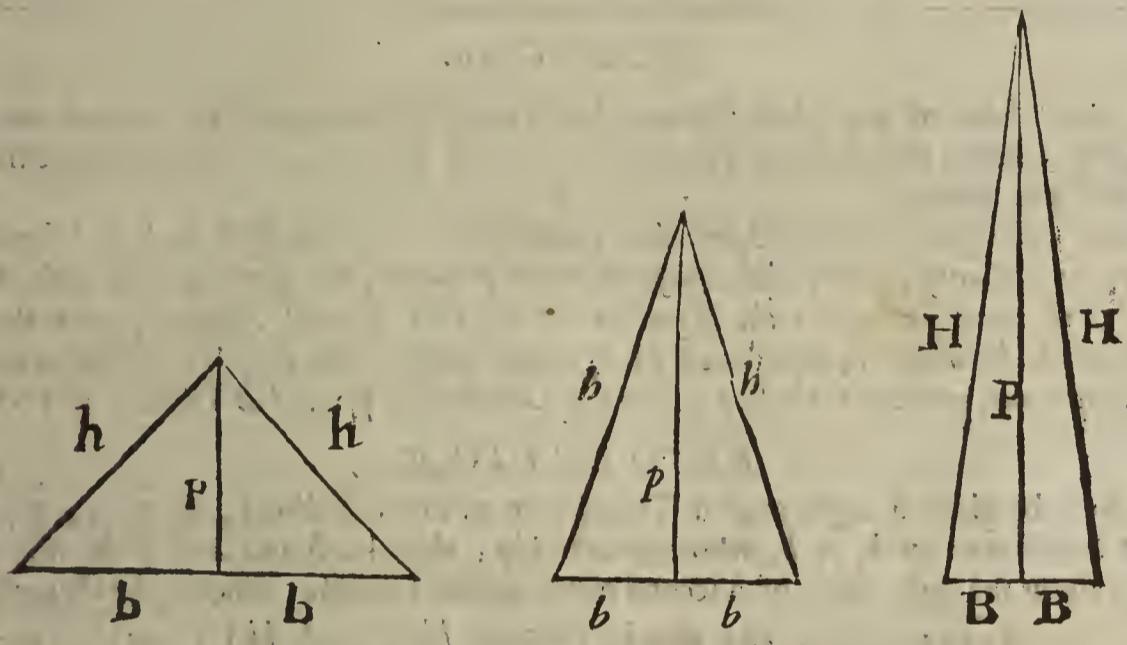
In the first rank, under . . . $h, b, p.$ | $h, b, p.$ | $H, B, P.$
 you will find . . . $58, 40, 42.$ | $74, 24, 70.$ | $113, 15, 112.$

Which three Triangles have equal Area's, and such other properties as before have been declared, viz. $h + b = h + b$, &c. as will easily appear by comparing the numbers answering to those Equations. The like is to be understood of the other five Answers.

QUEST. 58.

[This is Probl. 29. in pag. 131. of the Introduction to Algebra, translated out of High Dutch into English in 1668. by Tho. Brancker, M. A.]

To find three equicrural Triangles equal to one another in Area, and that the Perimeters of two of those Triangles may be equal to one another; and that the sides and Perpendiculars of every one of those three Triangles may be expressible by rational numbers.



RESOLUTION.

1. By the Canon in the ninth step of the Resolution of the foregoing Quest. 57. find out three right-angled Triangles in rational numbers, and equal to one another in Area; such are the three Triangles in any one of the six ranks of numbers in Table II. belonging to the said Quest. 57. for example, take those in the first rank,

viz. $\begin{cases} h, b, p. & | & h, b, p. & | & H, B, P. \\ 58, 40, 42. & | & 74, 24, 70. & | & 113, 15, 112. \end{cases}$

2. Then (as is evident by the Diagram belonging to this Question) the sides of the three equicrural Triangles desired shall be these following,

viz. $\begin{cases} h, h, 2b. & | & h, h, 2b. & | & H, H, 2B. \\ 58, 58, 80. & | & 74, 74, 48. & | & 113, 113, 30. \end{cases}$

3. And the Perpendiculars falling upon the Bases, viz. upon $2b, 2b, 2B$ of those three equicrural Triangles, are these, $\begin{cases} p, p, P. \\ 42, 70, 112. \end{cases}$

Which three equicrural Triangles in rational numbers above express in the second step will solve the Question; as will be evident by

The Proof.

By Construction in the first step the three right-angled Triangles $h, b, p.$ | $h, b, p.$ | H, B, P are equal to one another in Area, therefore their double Area's are equal to one another; but the said double Area's are the Area's of the three equicrural Triangles $h, h, 2b.$ | $h, h, 2b.$ | $H, H, 2B$; and therefore the Area's of those three equicrural Triangles are equal between themselves,

viz. $\begin{cases} bp = bp = BP, \\ 40 \times 42 = 24 \times 70 = 15 \times 112 = 1680. \end{cases}$

5. Moreover, by the first property in the Observations upon the Canon for resolving Quest. 57. this Equation is manifest,

viz. $\begin{cases} h + b = h + b, \\ 58 + 40 = 74 + 24 = 98. \end{cases}$

6. There

6. Therefore the double of the first part of that Equation shall be equal to the double of the latter part,

$$\text{viz. } \begin{cases} 2h + 2b = 2b + 2b, \\ 116 + 80 = 148 + 48 = 196. \end{cases}$$

7. But the said double summs (if you view the Diagram belonging to this Question) are manifestly equal to the Perimeters of the two equicrural Triangles $h, h, 2b$ and $b, b, 2b$, therefore those two equicrural Triangles are equal to one another in their Perimeters as well as in their Area's, and each Area is equal to the Area of the third equicrural Triangle $H, H, 2B$; also all their sides and Perpendiculars are express'd by rational numbers, as the Question required. In like manner five Answers more to this 5th Question may be collected from Table II. in Quest. 57. and 'tis evident from the premisses, that innumerable Ternions of equicrural Triangles in rational whole numbers may be found out to solve the said Quest. 58.

QUEST. 59.

The three sides of any plain right-angled Triangle being given in rational numbers; to find out another right-angled Triangle in rational numbers, which shall have the same Area with the former.

[Mons. de Fermat, in his Observation upon Quest. 8. of the fifth Book of Diophantus, gives a Canon to solve the Question above proposed, but shews not the rise thereof; I shall therefore resolve the Question at large by Literal Algebra, upon the same grounds by which it is resolved by Numeral Algebra in pag. 11. of his Analytical Inventions, prefixed to the late Edition of Diophantus printed at Tholose in 1670.]

RESOLUTION.

Let there be given a right-angled Triangle in rational numbers, as 3, 4, 5, which may be represented by b, p, h , whose Area is $\frac{1}{2}bp$; then let b and $a + p$ be assumed for the sides about the right-angle of a second right-angled Triangle, whence the Hypothensal will be $\sqrt{aa + 2pa + pp + bb}$: that is, (because $hh = pp + bb$) $\sqrt{aa + 2pa + hh}$: and the Area of this latter Triangle is $\frac{1}{2}ba + \frac{1}{2}bp$: Now if this latter Area be divided by the

former Area $\frac{1}{2}bp$; and if by the square Root of the Quotient $\frac{a}{p} + 1$, viz. by $\sqrt{\frac{a}{p} + 1}$: the three sides of the second Triangle be severally divided, the Quotients shall be the three sides of a third right-angled Triangle, whose Area is equal to the Area of the right-angled Triangle first given: For if the sides of the second right-angled Triangle, to wit, $b, a + p$ and $\sqrt{aa + 2pa + hh}$: be severally divided by $\sqrt{\frac{a}{p} + 1}$: the Quotients $\frac{b}{\sqrt{\frac{a}{p} + 1}}$, $\frac{a + p}{\sqrt{\frac{a}{p} + 1}}$ and $\frac{\sqrt{aa + 2pa + hh}}{\sqrt{\frac{a}{p} + 1}}$ are the sides of a third right-angled Triangle

whose Area $\frac{ba + bp}{\frac{2a}{p} + 2}$ is equal to $\frac{1}{2}bp$ the Area of the first right-angled Triangle; (for $\frac{2a}{p} + 2$ into $\frac{1}{2}bp$ makes $ba + bp$:) So that if $aa + 2pa + hh$ and $\frac{a}{p} + 1$ were square numbers, then the Question were solved: But how to make those two Algebraick quantities to be square numbers, the following Resolution shews.

1. Let there be given, as before, the three sides of a right-angled Triangle, as $\begin{matrix} 3 & 4 & 5 \\ b & p & h \end{matrix}$
2. Then for one of the sides about the right-angle of a second Triangle put b
3. And for the other side about the right-angle, $a + p$
4. Then because the sum of the Squares of those two sides must be equal to the Square of the Hypothensal, the Square of the Hypothensal of the second right-angled Triangle shall be $aa + 2pa + pp + bb$
5. That is, (because $hh = pp + bb$), $aa + 2pa + hh$

6. From

6. From the second and third steps the Area of the second right-angled Triangle is $\frac{1}{2}ba + \frac{1}{2}bp$
7. Which Area divided by $\frac{1}{2}bp$, the Area of the first right-angled Triangle, gives the Quotient $\frac{a}{p} + 1$
8. Now according to the scope before-mentioned, each of the quantities in the fifth and seventh steps must be equated to a Square, so we are fallen upon this Duplicate equality, viz. $aa + 2pa + hb = \square$
 $\dots \frac{a}{p} + 1 = \square$
9. In order to resolve that Duplicate equality, I multiply the said $\frac{a}{p} + 1$ by the Square hb , to the end there may be the same known Square hb in each Equation, so it produceth $\frac{hb}{p}a + hb = \square$

10. Then the difference of the said $aa + 2pa + hb$ and $\frac{hb}{p}a + hb$, viz. the difference of the two Squares sought is $aa + \frac{2pp - hb}{p}a$
11. Which difference is equal to the Product of the multiplication of these two quantities, to wit, $a + \frac{2pp - hb}{p}$ and a
12. The half-sum of the two quantities last mentioned is $a + \frac{2pp - hb}{2p}$

13. Then the Square of the said half-sum being equated to $aa + 2pa + hb$, (assumed to be the greater of the two quantities in the eighth step) gives this Equation, viz.
 $aa + 2pa + hb = aa + \frac{2pp - hb}{p}a + \frac{4ppp - hbhb - 4hbpp}{4pp}$

14. Which Equation, after due Reduction, gives $a = \frac{pppp - \frac{1}{4}hbhb - 2hbpp}{hbp}$
15. Therefore from the second, third and fourteenth steps the two sides about the right-angle of the second right-angled Triangle sought are b and $\frac{pppp - \frac{1}{4}hbhb - hbpp}{hbp}$

16. The sum of the Squares of those two sides, by taking $bb + pp$ instead of the Factor hb , and $bbbb + 2bbpp + pppp$ instead of the Factor $hbhb$, will be found $\frac{\frac{1}{8}b^8 + \frac{1}{8}p^8 + \frac{1}{8}b^4p^4 + \frac{3}{4}b^6pp + \frac{3}{4}bbp^6}{b^4pp + 2bbp^4 + p^6}$

17. The square Root of the sum of the Squares in the last step is the Hypotenusal of the second right-angled Triangle sought, to wit, $\frac{\frac{1}{4}bbbb + \frac{1}{4}pppp + \frac{3}{2}bbpp}{bbp + ppp}$
18. Which Hypotenusal, by taking hb instead of $bb + pp$; and $hbhb$ instead of $bbbb + 2bbpp + pppp$, may be expressed thus, $\frac{\frac{1}{4}hbhb + bbpp}{hbp}$

19. Therefore from the fifteenth and eighteenth steps, the three sides of the second right-angled Triangle sought are these, to wit,
 b , $\frac{pppp - \frac{1}{4}hbhb - hbpp}{hbp}$ and $\frac{\frac{1}{4}hbhb + bbpp}{hbp}$.

20. Therefore the Area of the said second right-angled Triangle is $\frac{ppppb + \frac{1}{4}hbhb - hbppb}{2hbp}$
21. Which Area divided by $\frac{bp}{2}$ the Area of the right-angled Triangle first given, gives the Quotient $\frac{pppp - \frac{1}{4}hbhb - hbpp}{hbpp}$
22. The square Root of that Quotient is $\frac{pp - \frac{1}{2}hb}{hp}$

23. By which square Root, if the three sides of the second right-angled Triangle before expressed in the nineteenth step be severally divided, the Quotients will be the three sides of the third right-angled Triangle sought, to wit,

$$1. \left\{ \begin{array}{l} \frac{hpb}{pph - \frac{1}{2}hbhb} \\ \frac{pppp - \frac{1}{4}hbhb - hbpp}{pph - \frac{1}{2}hbhb} \end{array} \right. , \left(\text{or } \frac{hpb}{pp - \frac{1}{2}hb} \right)$$

$$2. \frac{pppp - \frac{1}{4}hbhb - hbpp}{pph - \frac{1}{2}hbhb}$$

$$3. \frac{\frac{1}{4}hbhb + bbpp}{pph - \frac{1}{2}hbhb}$$

24. But

24. But $bbpp - pppp = hbpp$, therefore instead of subtracting $hbpp$ in the Numerator of the second of the three sides last before express'd, we may subtract $bbpp - pppp$, whence that side may be express'd thus $\frac{\frac{1}{4}hbhh - bbpp}{pph - \frac{1}{2}hbh}$, and consequently the three sides of the third right-angled Triangle sought shall be these, to wit,

$$\frac{hbpb}{pph - \frac{1}{2}hbh}, \quad \frac{\frac{1}{4}hbhh - bbpp}{pph - \frac{1}{2}hbh}, \quad \frac{\frac{1}{4}hbhh + bbpp}{pph - \frac{1}{2}hbh}.$$

25. Or, (to avoid Fractions) we may multiply the Numerator and Denominator of every one of the three sides last above express'd, by 4, so these three following sides (which are of the same value with those) will be produced for the third right-angled Triangle sought, to wit,

$$\frac{4hbpb}{4pph - 2hbh}, \quad \frac{hbhh - 4bbpp}{4pph - 2hbh}, \quad \frac{hbhh + 4bbpp}{4pph - 2hbh}.$$

Hence ariseth,

CANON 1.

26. Let b, p, h represent the three sides of any right-angled Triangle given in rational numbers, whereof the Hypothensal is h , and p the greater of the two sides about the right-angle; then from hb and $2bp$ form a right-angled Triangle, (by the Canon in *Observat.* 8. upon the Resolution by literal Algebra of the first Question of this Book;) that done, divide severally the three sides found out by $4pph - 2hbh$, so there will arise the three sides of a right-angled Triangle whose Area shall be equal to the Area of the given right-angled Triangle b, p, h .

27. Again, by supposition, $hb + pp = bb + 2pp$, and $p < b$, therefore $pp - bb = 2pp - hb$; whence by multiplying each part into $2h$ it follows that $2pph - 2bbh = 4pph - 2hbh$; Therefore instead of the Divisor $4pph - 2hbh$ in Canon 1. we may take $2pph - 2bbh$, and so there will arise the following Canon, (which is the same with *Monf. Fermat's* in the place before cited.)

CANON 2.

$$\frac{4hbpb}{2pph - 2bbh}, \quad \frac{hbhh - 4bbpp}{2pph - 2bbh}, \quad \frac{hbhh + 4bbpp}{2pph - 2bbh}.$$

In words thus,

28. Let b, p, h represent the three sides of any right-angled Triangle given in rational numbers, viz. h the Hypothensal, p the greater side about the right-angle, and b the lesser; then from hb and $2bp$ form a right-angled Triangle, and divide severally the three sides found out by $2pph - 2bbh$, so the Quotients shall be the three sides of a right-angled Triangle whose Area is equal to the Area of the given right-angled Triangle b, p, h .

An Example in Numbers.

Let there be given the three sides of a right-angled Triangle } $b \quad p \quad h$
 in rational numbers, to wit, } $3 \quad 4 \quad 5$
 Then by either of the Canons these three sides of a right- } $\frac{42}{70}, \frac{1200}{70}, \frac{1201}{70}$
 angled Triangle will be found out, to wit, }
 Which latter Triangle hath the same Area with the former, to wit, 6.

Again,

Let there be given this right-angled Triangle in numbers, } $b \quad p \quad h$
 to wit, } $5 \quad 12 \quad 13$
 Then by either of the preceding Canons this right-angled } $\frac{14161}{3094}, \frac{42260}{3094}, \frac{42261}{3094}$
 Triangle will be found out, to wit, }
 Which latter Triangle hath the same Area with the former, to wit, 30.

From the premisses it is evident, that from any right-angled Triangle given in rational numbers another of the same Area may be found out; and from the second a third; and from the third a fourth, &c. all which right-angled Triangles shall have one common Area, to wit, that of the given right-angled Triangle, and all their sides expressible by rational numbers.

29. Moreover, from the nineteenth step of the preceding Resolution, several Canons may be deduced, to shew how by the help of any right-angled Triangle given in rational numbers,

numbers, to find out another, whose Area divided by the Area of the given Triangle will give the Quotient a square number. But I shall exhibit only this following

CANON.

Let h represent the Hypotenusal, p the greater side about the right-angle, and b the lesser in any right-angled Triangle given in rational numbers; then from hb and $2bp$ form a right-angled Triangle, and divide severally the three sides found out by hbp ; so there will come forth the three sides of a right-angled Triangle, whose Area divided by the Area of the given Triangle, will give the Quotient a square number whose Root is

$$\frac{pp - \frac{1}{2}hb}{hp} \text{ or } \frac{2pp - hb}{2hp}$$

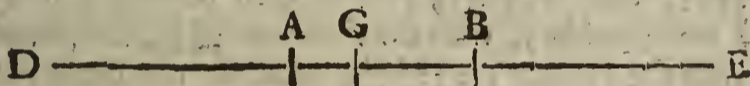
An Example in Numbers.

Let there be given the three sides of a right-angled Triangle	}	b	p	h
in numbers, to wit,		3	4	5
Then by the last Canon, this right-angled Triangle will be	}	3	$\frac{42}{400}$	$\frac{1206}{400}$
found out, to wit,				
The Area of the latter Triangle is	}		$\frac{147}{800}$	
Which Area divided by 6 the Area of the given right-angled				$\frac{49}{1600}$
Triangle 3, 4, 5 gives a square number, to wit,	}			$\frac{7}{40}$
The square Root whereof is				

QUEST. 60. (Quest. 12. Lib. 5. Diophant.)

To divide unity into two such parts, that if to each part a given number, suppose 6, be added, the two summs may be square numbers. But the summ of the double of the given number and unity must either be a square number, or else composed of two Squares.

RESOLUTION.



Let AB be 1, and $AD = BE$ be the given number 6; therefore $DE = 13$. Now we must divide AB , to wit, unity, into two parts, suppose in the point G , that GD and GE may be square numbers; so that in effect we must divide DE , that is, 13 into two such Squares that one of them may be greater than 6, but less than 7. But that may be done by the fourth Question of this Book, where 13 is divided into the Squares $\frac{11442}{1681}$ and $\frac{12424}{1681}$, whose sides are $\frac{107}{41}$ and $\frac{111}{41}$, and each of those Squares is greater than 6, but less than 7; therefore taking each Square from 7, the remainders $\frac{1142}{1681}$ and $\frac{1162}{1681}$, (that is GA and GB ,) are the desired parts of unity: For if to each of those parts the given number 6 be added, the two summs will be the Squares $\frac{11442}{1681}$ and $\frac{12424}{1681}$.

It is also evident, that instead of unity, any number may be given to be divided; provided that the summ of this number and of the double of the other number given make a Square, or else a number composed of two Squares.

QUEST. 61. (Quest. 13. Lib. 5. Diophant.)

To divide unity into two such parts, that to the one adding 2, (a number given,) and to the other 6, (another number given,) the summs may be square numbers. But the summ of the given numbers with unity must either make a square number, or else a number compos'd of two Squares.

RESOLUTION.



Let AB be 1, $AD = 2$, $BE = 6$; therefore $DE = 9$. Now we must divide AB , to wit, 1 into two parts, suppose in the point G , that GD and GE may be square numbers: So that in effect we must divide DE , that is, 9 into two such Squares that one may be greater than 2, but less than 3; or that one Square may be greater than 6, but less than 7. But the third Question of this Book shews how to find out two Squares so qualified, as the Square $\frac{2216}{3809}$ for GD , and the Square $\frac{12222}{3809}$ for GE ; the sides of

N which

which Squares are $\frac{8}{3}$ and $\frac{1}{3}$: then subtracting 2 from the Square $\frac{70}{2809}$, and 6 from the Square $\frac{1}{2809}$, the remainders $\frac{14}{2809}$ and $\frac{1}{2809}$ are the desired parts GA, GB of unity. For if the former part be added to 2, and the latter to 6, the summs are Squares, to wit, $\frac{70}{2809}$ and $\frac{1}{2809}$, as was required.

QUEST. 62. (Quest. 14. Lib. 5. Diophant.)

To divide unity into three such parts, that if to every one of them a given number 3 be added, the three summs may be Squares. But the summ of the triple of the given number and unity must either be a square number, or else a number compos'd of two or three Squares.

RESOLUTION.

It is easie to apprehend that the summ of the three Squares sought makes 10, and that the scope of the search must be to find out three Squares, every one of which may fall between 3 and 4, and that their summ may be 10; for then the three excesses of those Squares above 3 will be the three desired parts of unity. First then, forasmuch as 10 is compos'd of two Squares 9 and 1, I divide 10 into two other Squares whereof one may be greater than 3, but less than 4; such are the Squares $3\frac{2}{369}$ and $6\frac{4}{369}$, whose sides are $\frac{7}{37}$ and $\frac{2}{37}$; for the summ of those Squares is 10, and the first Square $3\frac{2}{369}$ is between 3 and 4: Then I take the Fraction $\frac{2}{369}$, (to wit, the excess of the first Square above 3) for the first of the three desired parts of unity; for if to that Fraction the given number 3 be added, it makes the Square $3\frac{2}{369}$.

Then I divide the latter of the Squares first found, to wit, $6\frac{4}{369}$, whose side is $\frac{2}{37}$, into two such Squares that one may be greater than 3, but less than $3\frac{1}{4}$, or less than $3\frac{1}{4}$ for if such Square be less than $3\frac{1}{4}$, it will necessarily be less than $3\frac{4}{369}$, because $\frac{1}{4}$ is less than $\frac{4}{369}$. But the sides of two such Squares are $\frac{1}{1073}$ and $\frac{1860}{1073}$, therefore the Squares themselves are $3\frac{1}{1151329}$ and $3\frac{1860}{1151329}$; from each of which if 3 be subtracted, the remainders $\frac{1}{1151329}$ and $\frac{1860}{1151329}$, with the Fraction $\frac{2}{369}$ first found, that is, (in the same Denominator with the former) $\frac{2}{1151329}$ shall be the three desired parts of unity to solve the Question.

After the same manner any number given instead of unity may be divided into three such parts, that a number given being added to every one of them may make three Squares. But the summ of the number given to be divided, and the triple of the number to be added, must be either a Square, or a number compos'd of two or three Squares.

As, if it were desired to divide 2 into three parts, that each part increased with 4 may make a Square: First, forasmuch as 14 the summ of 2 and the triple of 4 is compos'd of three Squares 1, 4 and 9; let 10 the summ of 9 and 1 (two of those three Squares) be divided into two other Squares that the first may exceed 4 the number given to be added, but be less than 6; to the end the excess may be less than 2 the number given to be divided; then add the other of the two Squares found out to 4, (the other of the three Squares before mentioned, whose summ made 14,) and divide the summ into two such Squares that each may be greater than 4; lastly, from each of these two Squares last found out, as also from the first Square before found, subtract 4, so the Remainders shall be the desired parts of 2. But the Operation I leave to the Learner's exercise.

QUEST. 63. (Quest. 15. Lib. 5. Diophant.)

To divide unity into three such parts, that if the first be increased with 2, the second with 3, and the third with 4, the three summs may be square numbers. But the summ of the three numbers given and unity must either be a Square, or compos'd of two or three Squares.

RESOLUTION.

First we must divide 10 (the summ of the three numbers given with unity) that the first may exceed 2, the second 3, and the third 4. To which end, first, (by Quest. 4. of this Book) divide 10 into two Squares that one may fall between 2 and 3; such are the Squares $\frac{1}{841}$ and $\frac{6}{841}$, whose Roots are $\frac{1}{29}$ and $\frac{2}{29}$; then from the first Square $\frac{1}{841}$ subtract 2, and take the Remainder $\frac{1}{841}$ for one of the desired parts of unity.

It remains to divide the other Square $\frac{6}{841}$ into two other Squares, that one may fall between

between 3 and 4: But two such Squares will be found $\frac{2823401}{707281}$ and $\frac{2624400}{707281}$; whose sides are $\frac{1701}{841}$ and $\frac{1620}{841}$; then from the latter of the said Squares subtract 3, and from the former, 4; so the two Remainders $\frac{202157}{707281}$ and $\frac{64227}{707281}$ with $\frac{167}{841}$ before found, that is, (in the same Denominator with the two former) $\frac{140447}{707281}$ shall be the three desired numbers, which will solve the Question. For first, their sum makes unity; moreover if 2 be added to $\frac{140447}{707281}$, that is, $\frac{167}{841}$, (the first number) the sum is the Square $\frac{1349}{841}$; and if 3 be added to the second number $\frac{202157}{707281}$, it makes the Square $\frac{2624400}{707281}$; lastly, if 4 be added to the third number $\frac{64227}{707281}$, it makes the Square $\frac{2823401}{707281}$.

QUEST. 64. (Quest. 16. Lib. 5. Diophant.)

To divide a given number 10 into three numbers, that the sum of every two may be a Square; but the double of the given number must be either a Square, or else compos'd of two or three Squares.

RESOLUTION.

Forasmuch as the three numbers sought are to be such that the sum of the first and second must make a Square, also the sum of the first and third must make a Square, likewise the sum of the second and third must make a Square; therefore these three Squares are equal to the said three numbers twice taken. And because the sum of the three numbers is 10, therefore twice their sum, to wit, 20 shall be the sum of the three Squares. We must therefore divide 20 into three Squares, each of which may be less than 10; (for every one of these Squares must be equal to two of the three numbers, and consequently less than 10 the sum of all the three numbers.) But 20 is compos'd of two Squares 16 and 4; therefore we may take 4 (which is less than 10) for one of the three Squares sought, and then divide 16 into two Squares, one of which may fall between 10 and 6, for then the other will also be less than 10, (because both must make 16.) But (by Quest. 4. of this Book) the sides of two such Squares will be found $\frac{480}{169}$ and $\frac{476}{169}$; wherfore the three Squares sought are 4, $\frac{230400}{28561}$ and $\frac{226176}{28561}$, which subtracted severally from 10 leave three Remainders, 6, $\frac{25210}{28561}$ and $\frac{2034}{28561}$ for the three desired numbers, whose sum is 10, and every two of them added together makes a Square, as was desired.

But to make it more clearly evident that three numbers so found out will solve the Question, let *bb*, *cc*, *dd* represent three square numbers, then

Suppose	}	$bb + cc + dd = 20$
And consequently	}	$\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd = 10$
Then from 10, that is, from $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd$ subtract	}	$\frac{1}{2}cc + \frac{1}{2}dd - \frac{1}{2}bb$
<i>bb</i> , <i>cc</i> and <i>dd</i> severally, so the three Remainders are	}	$\frac{1}{2}bb + \frac{1}{2}dd - \frac{1}{2}cc$
	}	$\frac{1}{2}bb + \frac{1}{2}cc - \frac{1}{2}dd$

I say those three Remainders shall be three numbers to solve the Question; for the sum of the first and second makes the Square *dd*, the sum of the second and third makes the Square *bb*, and the sum of the first and third makes the Square *cc*.

QUEST. 65. (Quest. 17. Lib. 5. Diophant.)

To divide a given number, suppose 10, into four numbers, that the sum of every three may make a Square.

RESOLUTION.

Forasmuch as the sum of every three of the four numbers sought must make a Square, therefore the four Squares sought are equal to the four desired numbers thrice taken. But the four numbers thrice taken make 30, therefore 30 must be divided into four such Squares that every one of them may be less than 10, (for every one of the four Squares must be equal to three of the numbers sought, and consequently be less than 10 the sum of all four.) But 30 is compos'd of four Squares, 16, 9, 4 and 1; two of which, to wit, 9 and 4 may be taken for two of the Squares sought, and then 17 (the sum of the other two Squares 16 and 1) must be divided into two Squares that one may be less than 10, but greater than 7, and then the other will be also less than 10; but the sides of two such Squares will (by Quest 4. of this Book) be found $\frac{17}{7}$ and $\frac{47}{17}$, and the Squares themselves are $\frac{289}{49}$ and $\frac{2209}{289}$, each of which is less than 10. Therefore the four Squares sought

sought are 9, 4, $\frac{2704}{189}$ and $\frac{2209}{289}$, which subtracted severally from 10, leave the Remainders 1, 6, $\frac{186}{289}$ and $\frac{681}{289}$ for the four numbers sought, whose summ makes 10, and every three of them added together make the Squares $\frac{209}{289}$, 9, 4 and $\frac{2704}{289}$, whose sides are $\frac{47}{17}$, 3, 2 and $\frac{52}{17}$.

Note. If the quadruple of the number given be a whole number, this Question may be extended to five numbers, or as many as you please: for every whole number is compos'd of four Squares, which may be divided into any multitude of Squares within any possible limits, by the help of the third Question of this Book.

QUEST. 66. (A Lemma, used in the following Quest. 67.)

To find three such Cube-numbers, that if from every one of them a number given, suppose 1, be subtracted, the summ of the Remainders may be a Square.

RESOLUTION.

1. For the side of the first Cube put $a \mp$ any absolute number, as, $a \mp 1$
2. Then take some square number, as 9, and from its $\frac{1}{3}$, to wit, from 3 subtract 1 the absolute number in the side of the first Cube, and let the Remainder 2 be connected by \mp with $-a$ for the side of the second Cube; the reason whereof will appear in *Observation* 1. upon this Question, $-a \mp 2$
3. Let the side of the third Cube be any known number whose Cube exceeds the number given in the Question, to wit, the side 2
4. Then the Cubes of those three sides are these, to wit,

1.	$aaa \mp 3aa \mp 3a \mp 1,$
2.	$-aaa \mp 6aa \mp 12a \mp 8,$
3.	$\dots \dots \dots \mp 8.$
5. From every one of which Cubes subtract 1 the number given in the Question, and add the Remainders together, so the summ will be $9aa - 9a \mp 14$, which must be equated to a Square; but the side thereof must be so feigned that the value of a may be less than 2, to the end that $-a \mp 2$, or $2 - a$ the side of the second Cube may be greater than nothing. Now to cause that effect, the said side may be feigned $3a -$ any absolute number between $\sqrt{14}$ and $\sqrt{32} \mp 6$, (which limits are found out after the method before delivered in divers Questions of this Book;) let therefore the side of the said Square be feigned $3a - 4$, and then the Square of $3a - 4$ being equated to $9aa - 9a \mp 14$ above mentioned, this Equation ariseth, to wit,

$$9aa - 9a \mp 14 = 9aa - 24a \mp 16.$$
6. From which Equation, the value of a will be made known, viz. $a = \frac{2}{5}$
7. Therefore from the sixth, first, second and third steps the sides of the three Cubes sought are $\frac{47}{17}$, $\frac{28}{17}$ and 2; wherefore the Cubes themselves are $\frac{4213}{3375}$, $\frac{2128}{3375}$ and 8, which will solve the Question. For if from every one of those Cubes the given number 1 be subtracted, the summ of the three Remainders in its least terms is $\frac{224}{289}$, which is a Square, as was required.

Observations upon Quest. 66.

1. Two things are remarkable in the preceding positions for the sides of the Cubes sought; First, in the first side $a \mp 1$ there is put $\mp a$, and in the second side $-a \mp 2$ there is put $-a$, to the end that in adding together the Remainders mentioned in the fifth step, $\mp aaa$ and $-aaa$ may destroy one another: Secondly, the unities in the second side must be chosen with such Caution that the number prefixt to aa in the summ of the said three Remainders may be a Square, for if $9aa$ had not been a Square, then $9aa - 9a \mp 14$ could not have been equated to a Square. Therefore that the number prefixt to aa in the said summ of the Remainders may infallibly come forth a square number, the number of unities to be connected with $-a$ by \mp in the side of the second Cube, must be the excess of one third part of some square number above the unit or unities in the side of the first Cube: The reason whereof may be thus manifested, Suppose $a \mp b$, and $-a \mp d$, and 2, to be the sides of the three Cubes sought; then from the Cubes of those sides subtracting the given number 1 severally, the summ of the Remainders is $3baa \mp 3daa \mp 3bba - 3dda \mp bbb \mp ddd \mp 5$; which summ cannot be equated to a Square, unless the number signified by $3b \mp 3d$, which is multiplied into aa be a square number;

number; suppose therefore $3b + 3d = ff$, whence by due Reduction, $d = \frac{1}{3}ff - b$; which shews that d the number of unities in $-a + d$ the side of the second Cube, must be the excess of one third part of some square number, above b the unit or unities in $a + b$ the feigned side of the first Cube, as was directed in the second step of the Resolution.

2. It is easie to apprehend that this Question may be extended to as many numbers as you please: For having put (as before) $a + 1$ for the side of the first Cube, and (for the reasons before given) $-a + 2$ for the side of the second Cube, you may put 2 for the side of a third Cube, 3 for the side of a fourth, &c. provided that the Cubes of these absolute numbers 2, 3, &c. do every one of them exceed the number given to be subtracted; then from the summ of all those Cubes subtracting the given number, the summ of the Remainders may be equated to a Square, because by Construction the number prefixt to aa is a Square.

QUEST. 67. (Quest. 18. Lib. 5. Diophant.)

To find three numbers, that if they be severally added to the Cube of their summ, the three summs made by those additions may be Cubes.

RESOLUTION.

1. For the summ of the three numbers sought put . . . $\} a$
2. Then the Cube of their summ is $\} aaa$
3. For the three numbers sought put $\} 7aaa, 26aaa$ and $63aaa$
4. Whence it is manifest, that if each of the three last mentioned quantities assumed for the three numbers sought, be increased with aaa which was put for the Cube of their summ, there will come forth the Cubes $8aaa, 27aaa$ and $64aaa$; but the summ of these three quantities $7aaa, 26aaa$ and $63aaa$ must be equal to a ; therefore $96aaa = a$, and consequently $96aa = 1$: where if 96 were a square number, the value of a would be expressible by a rational number, and consequently the Question solved. Whence therefore comes 96? examine the Positions and you will find that by subtracting 1 (to wit, unity) from the three Cubes 8, 27 and 64, the remainders 7, 26 and 63 added together make 96. Therefore we must seek three such Cubes that if 1 be subtracted from every one of them, the summ of the three remainders may be a Square: But the sides of three such Cubes are $\frac{1}{15}, \frac{2}{15}$ and 2, (found out by the preceding Quest. 66.) and the Cubes themselves are $\frac{1}{3375}, \frac{8}{3375}$ and 8, from every one of which if 1 be subtracted, the three remainders will be $\frac{1}{3375}, \frac{7}{3375}$ and 7, whose summ in its least terms is the Square $\frac{3}{25}$, whose side is $\frac{3}{5}$. Now by the help of those remainders, let the work be renewed thus, viz.

5. For the summ of the three numbers sought put . . . $\} a$
6. Whence the Cube of their summ is $\} aaa$
7. Then for the first number put $\} \frac{1}{3375}aaa$
8. For the second $\} \frac{8}{3375}aaa$
9. And for the third $\} 7aaa$
10. Then the summ of the three numbers last express'd is . . . $\} \frac{3}{25}aaa$
11. Which summ must be equal to a , which in the fifth step was put for the summ of the three numbers sought, therefore $\} \frac{3}{25}aaa = a$
12. Which Equation, after due Reduction, discovers the summ of the three numbers sought, viz. $\} a = \frac{1}{18}$

Therefore from the twelfth, seventh, eighth and ninth steps, the three numbers sought will be $\frac{1}{157464}, \frac{8}{157464}, \frac{2}{157464}$, whose summ is $\frac{1}{18}$, the Cube whereof is $\frac{1}{5832}$; which being added to every one of the said three numbers, the summs will be Cubes, to wit, $\frac{1}{157464}, \frac{8}{157464}, \frac{2}{157464}$, whose sides are $\frac{1}{54}, \frac{2}{54}$ and $\frac{3}{54}$; therefore the Question is satisfied, and by the help of the preceding Quest. 66. may be extended to four, five or as many numbers as shall be desired.

QUEST. 68.

To find two Cube-numbers, that if their difference be increased with a given number, suppose 2, it may make a Square, and that the side of the greater Cube may be less than 1 a number given.

RESO.

RESOLUTION.

1. For the side of the lesser Cube sought put } a
2. For the side of the greater Cube put a + one third part of some square number; but such third part must be less than 1 the prescribed limit, therefore let the side of the greater Cube be . . . } $a + \frac{1}{3}$
3. Therefore the greater Cube is } $aaa + aa + \frac{1}{3}a + \frac{1}{27}$
4. And the lesser Cube is } aaa
5. Therefore the difference of the said Cubes is } $aa + \frac{1}{3}a + \frac{1}{27}$
6. To which difference add 2 the number first given in the Question, and the sum is } $aa + \frac{1}{3}a + \frac{52}{27}$
7. Which sum must be equal to a Square, the side whereof must be so feigned that the value of a may be less than $\frac{2}{3}$, for then $a + \frac{1}{3}$ (the side of the greater Cube) will be less than 1, as the Question requires. Now to cause that effect, the side of the said Square may be feigned to be either $a +$ any known number between $\frac{2}{1000}$ and $\frac{4}{1000}$, or else $-a +$ any known number between $\frac{4}{1000}$ and $\frac{2}{1000}$, (which limits may be found out by the method directed in the preceding *Quest.* 12.) let therefore the said side be feigned $a + \frac{4}{3}$, and then the Square of $a + \frac{4}{3}$ being equated to the sum in the sixth step, this Equation ariseth, *viz.*

$$aa + \frac{8}{3}a + \frac{16}{9} = aa + \frac{1}{3}a + \frac{52}{27}$$
8. Which Equation after due Reduction gives } $a = \frac{1}{9}$
9. Therefore from the eighth, first and second steps the sides of the two Cubes sought are } $\frac{1}{9}$ and $\frac{4}{9}$
10. The Cubes of which sides $\frac{1}{9}$ and $\frac{4}{9}$, *viz.* $\frac{1}{729}$ and $\frac{64}{729}$ will solve the Question; for if to their difference $\frac{63}{729}$ you add 2, the sum $\frac{1458}{729}$ is a Square, and $\frac{4}{9}$ the side of the greater Cube is less than 1, as was required.

Example. 2.

Let it be required to find two Cube-numbers whose difference added to 1458 (a given number) may make a Square, and that the side of the greater Cube may be less than 9 (a number given.)

Resolution.

1. For the side of the lesser Cube put } a
2. For the side of the greater Cube put a + one third part of some square number, but such third part must be less than 9 the limit above prescribed; therefore let the said side be . . . } $a + 3$
3. Therefore the greater Cube is } $aaa + 9aa + 27a + 27$
4. And the lesser Cube is } aaa
5. Therefore the difference of those Cubes is } $+9aa + 27a + 27$
6. To which difference adding the number first given in this *Example 2.* to wit, } 1458
7. The sum will be } $+9aa + 27a + 1485$
8. Which sum must be equal to a Square, the side whereof must be so feigned that the value of a may be less than 6, for then $a + 3$ (which in the second step was assumed for the side of the greater Cube sought) will be less than the prescribed limit 9; Now to cause that effect, the side of the said Square may be feigned to be $3a +$ any number between 26 and 39, or $-3a +$ any number between 38 and 63; suppose therefore the said side be feigned $3a + 36$, then the Square of $3a + 36$ being equated to the sum in the seventh step, this Equation ariseth, *viz.*

$$9aa + 216a + 1296 = 9aa + 27a + 1485$$
9. Which Equation after due Reduction gives } $a = 1$
10. Therefore from the ninth, first and second steps of this second Example, the sides of the Cubes sought are . . . } 1 and 4
11. The Cubes of which sides 1 and 4, *viz.* 1 and 64 will solve the Question; for if their difference 63 be added to 1458 the number given in *Example 2.* it makes the Square 1521, whose side is 39; and 4 the side of the greater Cube is less than 9, as was required.

Example 3.

Example 3.

Again, the same numbers 1458 and 9 being given as before in Example 2. the side of the Square mentioned in the eighth step may be feigned to be $-3a + 48$, (which is within the limits there exprest,) and then the Square of $-3a + 48$ being equated to $9aa - 27a - 1485$, (before exprest in the seventh step,) after due Reduction the sides of two Cubes to solve Quest. 68. as it is before propos'd in Example 2. will be found $\frac{22}{5}$ and $\frac{22}{5}$; therefore the Cubes themselves are $\frac{22^3}{125}$ and $\frac{22^3}{125}$, whose difference $\frac{12^3}{125}$ added to the given number 1458 makes a Square, to wit, $\frac{40^2}{25}$, whose side is $\frac{20}{5}$; and $\frac{22}{5}$ the side of the greater Cube is less than the prescribed number 9.

QUEST. 69.

To find two such cube-numbers, that if each of them be subtracted from a given squared cube-number, the sum of the remainders may be a Square.

RESOLUTION.

1. Let the given squared cube-number be } $dddddd$ or d^6
2. The Root or side whereof is } d
3. For the side of the first Cube sought put } a
4. The Cube thereof is } aaa
5. For the side of the other Cube sought put $-a +$ } $-a + dd$
the cube-root of the given squared Cube, viz. . . . }
6. The Cube thereof is } $-aaa + 3ddaa - 3d^4a + d^6$
7. Then by subtracting the Cube in the fourth step from } $+d^6 - aaa$
the given squared Cube in the first, there will remain }
8. And by subtracting the Cube in the sixth step from } $+aaa - 3ddaa + 3d^4a$
the given squared Cube in the first, there will remain }
9. The sum of those remainders (in the seventh and } $-3ddaa + 3d^4a + d^6$
eighth steps) is }
10. Which sum must be equal to a Square, the side whereof (in regard d^6 is a Square) we may feign to be either $ea + ddd$, or $ea - ddd$, (where e represents a number yet unknown, and to be chosen according to the limit hereafter discovered:) First then let the said side be feigned $ea + ddd$, and then the Square of $ea + ddd$ being equated to the sum in the ninth step, will give

$$-3ddaa + 3d^4a + d^6 = ecaa + 2ed^3a + d^6$$

11. Which Equation after due Reduction gives . . . } $a = \frac{3d^4 - 2ed^3}{3dd + ee}$
 12. Therefore from the eleventh, first, second, third } $\frac{3d^4 - 2ed^3}{3dd + ee}$ and $\frac{eedd + 2ed^3}{3dd + ee}$
and fifth steps the sides of the two Cubes sought }
will be found equal to these quantities, viz. . . . }
 13. Again, forasmuch as the side of the Square mentioned in the tenth step may be feigned to be $ea - ddd$ (as well as $ea + ddd$,) let the said side be $ea - ddd$, and then its Square being equated to the sum in the ninth step, this Equation ariseth, viz.
 $-3ddaa + 3d^4a + d^6 = ecaa - 2ed^3a + d^6$
 14. Which Equation after due Reduction gives . . . } $a = \frac{3d^4 + 2ed^3}{3dd + ee}$
 15. Therefore from the fourteenth, first, second, third } $\frac{3d^4 + 2ed^3}{3dd + ee}$ and $\frac{ddee - ed^3}{3dd + ee}$
and fifth steps the sides of the two Cubes sought will }
be found equal to these quantities, viz. . . . }
- The two quantities exprest by letters in the twelfth step will give

CANON

16. Supposing d to be the side or Root of the squared cube-number given, take some known number, with this Caution, That its double may be less than the triple of d , and call the number taken e , then the sides of the two Cubes sought shall be these, viz,

$$\frac{3d^4 - 2ed^3}{3dd + ee} \text{ and } \frac{eedd + 2ed^3}{3dd + ee}$$

Ans

An Example in Numbers.

Let there be given any squared cube-number, as } $64 = d^6$
 The Root or side whereof is } $2 = d$
 Then take a number for e , according to the Caution in the Canon, as . . . } $2 = e$

Then by the Canon you will find $\left\{ \begin{array}{l} \frac{3d^4 - 2ed^3}{3dd + ee} = 1 \\ \frac{eedd + 2ed^3}{3dd + ee} = 3 \end{array} \right\}$ the sides of the Cubes sought.

The Cubes of which sides 1 and 3, viz. 1 and 27 will solve the Question proposed; for if each of those Cubes be subtracted from the given squared Cube 64, the sum of the remainders 63 and 37 makes a Square, to wit, 100.

The two quantities express'd by letters in the fifteenth step will give

CANON 2.

17. Supposing d to be the side or Root of a given squared cube-number, take some known number with this Caution, That it be greater than the double of d , and call the number taken e , then the sides of the two Cubes sought shall be these, viz.

$$\frac{3d^4 + 2ed^3}{3dd + ee} \quad \text{and} \quad \frac{deee - 2ed^3}{3dd + ee}$$

An Example of Canon 2. in Numbers.

Let there be given any squared Cube, as } $1 = d^6$
 The side or Root whereof is } $1 = d$
 Then take a number for e , according to the Caution in Canon 2. as . . . } $4 = e$

Then by Canon 2. you will find $\left\{ \begin{array}{l} \frac{3d^4 + 2ed^3}{3dd + ee} = \frac{11}{19} \\ \frac{deee - 2ed^3}{3dd + ee} = \frac{1}{19} \end{array} \right\}$ the sides of the Cubes sought.

The Cubes of which sides $\frac{11}{19}$ and $\frac{1}{19}$, viz. $\frac{1331}{6859}$ and $\frac{1}{6859}$ will solve Quest. 69. for if each of those Cubes be subtracted from 1 the given squared Cube, the sum of the remainders makes a Square, to wit, $\frac{422}{361}$, whose side is $\frac{22}{19}$.

QUEST. 70.

To find three such cube-numbers, that if every one of them be subtracted from a given Cube, suppose 1, the sum of the three remainders may be a Square.

RESOLUTION.

1. First, by the foregoing Quest. 68. find two such cube-numbers, that their difference being added to 2, (the double of the Cube given in this Question,) the sum may be a Square, and that the greater of those two Cubes may be less than the given Cube 1. But two such Cubes are $\frac{8}{27}$ and $\frac{1}{27}$, whose sides are $\frac{2}{3}$ and $\frac{1}{3}$, (found out in the first Example of Quest. 68.) for if the difference of the said Cubes be added to 2, the sum is $\frac{16}{9}$, which is a square number whose side is $\frac{4}{3}$.

2. Then for the side of the first of the three Cubes sought put $a - \frac{2}{3}$, ($\frac{2}{3}$ being the side of the greater of the two Cubes found out in the first step,) $\left. \begin{array}{l} a - \frac{2}{3} \\ \dots \dots \dots \end{array} \right\} a - \frac{2}{3}$
3. For the side of the second Cube sought put $-a + 1$, (1 being the side of the Cube given in the Question,) $\left. \begin{array}{l} -a + 1 \\ \dots \dots \dots \end{array} \right\} -a + 1$
4. And let the side of the third Cube be the side of the lesser of the two Cubes found out in the first step, to wit, $\left. \begin{array}{l} \dots \dots \dots \\ \dots \dots \dots \end{array} \right\} \frac{1}{3}$
5. Then from the second step the first Cube will be $\dots \dots \dots \left. \begin{array}{l} \dots \dots \dots \\ \dots \dots \dots \end{array} \right\} aaa - \frac{22}{9}aa + \frac{16}{27}a - \frac{8}{27}$
6. And from the third step, the second Cube will be $\dots \dots \dots \left. \begin{array}{l} \dots \dots \dots \\ \dots \dots \dots \end{array} \right\} -aaa + 3aa - 3a + 1$
7. And from the fourth step the third Cube will be $\dots \dots \dots \left. \begin{array}{l} \dots \dots \dots \\ \dots \dots \dots \end{array} \right\} \frac{1}{27}$

8. Then according to the Question subtract those three Cubes severally from the given Cube 1, so the three remainders shall be these, to wit,

$$\begin{array}{l} 1. \quad | \quad -aaa + \frac{22}{9}aa - \frac{16}{27}a + \frac{22}{27} \\ 2. \quad | \quad +aaa - 3aa + 3a \\ 3. \quad | \quad \dots \dots \dots + \frac{22}{27} \end{array}$$

9. The

9. The summ of the said three remainders is $\rightarrow -\frac{2}{3}aa - \frac{62}{27}a - \frac{162}{81}$
 10. Which summ must be equal to a Square, whose side, to the end the value of a may be greater than $\frac{2}{9}$, but less than 1, (as the Positions in the second and third steps do require) may be feigned to be either $\frac{12}{9} -$ any number of a between $\frac{2000}{1000}a$ and $\frac{8000}{1000}a$, or else $\frac{12}{9} -$ any number of a greater than $3\frac{1000}{1000}a$, (which limits may be found out by the method before delivered in *Quest.* 13. of this *Book*;) suppose therefore the said side to be $\frac{12}{9}a - \frac{12}{9}$, then the Square of $\frac{12}{9}a - \frac{12}{9}$ being equated to the summ of the three remainders in the ninth step, this Equation ariseth, *viz.*

$$-\frac{2}{3}aa - \frac{62}{27}a - \frac{162}{81} = \frac{1}{9}aa - \frac{26}{27}a - \frac{162}{81}.$$

11. Which Equation after due Reduction will give . . . $\rightarrow a = \frac{12}{16}$

12. Therefore from the eleventh, second, third and fourth steps, the sides of the three Cubes sought will be these, $\left\{ \begin{array}{l} \frac{22}{144}, \frac{22}{144}, \text{ and } \frac{16}{144} \end{array} \right.$

13. The Cubes of which sides are these $\rightarrow \frac{1488877}{2985984}, \frac{12683}{2985984}, \frac{4096}{2985984}$

Which three Cubes will solve the *Question*, as will appear by

The Proof.

By subtracting severally the said three Cubes found out from 1, (the Cube given in the *Question*,) the three remainders will be these,

$$\frac{2837107}{2985984}, \quad \frac{2966371}{2985984}, \quad \frac{2981888}{2985984}.$$

The summ of those remainders is $\rightarrow \frac{8785296}{2985984}$

Which summ reduced to its least terms by the common Divisor 144, will be $\rightarrow \frac{61002}{20736}$

Which $\frac{61002}{20736}$ is a Square, whose side is $\frac{242}{144}$; therefore the *Question* is solved; and (as is evident by the tenth step) capable of innumerable Answers, the positions in the second, third and fourth steps standing unaltered.

Observations upon the preceding Resolution of Quest. 70.

1. The chief scope in the said Resolution is, to form the positions for the sides of the three Cubes sought in such manner, that when the said Cubes are severally subtracted from the Cube given in the *Question*, there may be a possibility of equating the summ of the three remainders to a Square, which summ (as you see in the ninth step) is $-\frac{2}{3}aa - \frac{62}{27}a - \frac{162}{81}$, which could not be equated to a Square if $\frac{162}{81}$ were not an affirmative square number; I shall therefore shew how the said $\frac{162}{81}$ doth necessarily become an affirmative square number by the preceding Operation.

2. If the subtraction of every one of the three feigned Cubes in the fifth, sixth and seventh steps from the given Cube 1, as also the adding of the remainders together be well examined, it will appear, that by adding the Cube $\frac{64}{729}$ to 1, and by subtracting the Cube $\frac{1}{729}$ from 1, and then by adding that summ and remainder together, their summ is $\frac{162}{81}$, which (in regard by Construction the greater of the said Cubes, to wit, $\frac{64}{729}$ is added to 1, and the lesser $\frac{1}{729}$ is subtracted from 1) is the same with the summ that will arise by adding the difference of those Cubes unto 2, (the double of 1.) For if the greater of two numbers be added unto, and the lesser be subtracted from a third number, the summ and remainder added together will make the same summ that ariseth by adding the difference of those two numbers to the double of the said third number: But by Construction in the first step of the Resolution, the said Cubes $\frac{64}{729}$ and $\frac{1}{729}$ are found such that their difference added to 2 makes a Square, to wit, $\frac{162}{81}$. Whence it is manifest that the Algebraick quantity $-\frac{2}{3}aa - \frac{62}{27}a - \frac{162}{81}$ is capable of being equated to a Square, and that variously, as you see in the tenth step of the Resolution.

Example 2.

Let it be required to find three such Cube-numbers, that if every one of them be subtracted from a given Cube, suppose 729, the summ of the three remainders may be a square number.

Resolution.

1. First, by *Quest.* 68. find two such Cubes that if their difference be added to 1458, to wit, the double of the given Cube 729, the summ may be a Square, and that the side of the greater of those two Cubes may be less than 9 the side of the given Cube 729: But two such Cubes are 64 and 1, (found out in the second *Example* of *Quest.* 68.)

O

for

for if their difference 63 be added to the prescribed number 1458, the summ 1521 is a Square whose side is 39.

2. Then for the side of the first of the three Cubes sought }
let there be put $a - 4$, (4 being the side of the Cube 64, }
the greater of the two Cubes found out in the first step,) } $\cdot a - 4$
3. For the side of the second Cube sought put $-a + 9$, }
(9 being the side of the given Cube 729,) . . . } $- a + 9$
4. Let the side of the third Cube be 1, to wit, the side of }
the lesser of the two Cubes found out in the first step, . } $\cdot \cdot \cdot 1$
5. Then (according to the Question) subtract severally the Cubes of those three sides
(assumed in the three last steps) from the given Cube 729, and add the three remainders
together, so the summ will be

$$- 15aa + 195a + 1521.$$

6. Which summ must be equal to a Square, whose side, to the end the value of a may be greater than 4, but less than 9, as the second and third steps require, may be feigned to be either $39 +$ any number of a between $\frac{2}{1000}a$ and $\frac{1}{1000}a$, or else $39 -$ any number of a between $\frac{1}{1000}a$ and $\frac{2}{1000}a$, (which limits may be found out by the method delivered in *Quest.* 13. of this Book.) Suppose therefore the said side be feigned $39 + a$, then the Square thereof being equated to the summ in the fifth step, this Equation will arise, to wit,

$$aa + 78a + 1521 = - 15aa + 195a + 1521.$$

7. Which Equation after due Reduction gives . . . } $a = \frac{117}{16}$
 8. Therefore from the seventh, second, third and fourth }
steps the sides of the three Cubes sought are these, . } $\frac{11}{16}, \frac{27}{16}$ and 1 (or $\frac{16}{16}$)
 9. And consequently the Cubes themselves are . . . } $\frac{1418877}{4096}, \frac{12681}{4096}, \frac{4096}{4096}$ (or 1)
- Which three Cubes will solve the Question, as will be evident by

The Proof.

By subtracting every one of the said three Cubes in the ninth step from the given Cube 729, the three remainders will be these,

$$\frac{2837107}{4096}, \frac{2966301}{4096}, \frac{2981888}{4096}.$$

- The summ of those remainders is . . . } $\frac{3121296}{4096}$
Which summ being reduced to its least terms by the }
common Divisor 16, will be . . . } $\frac{112981}{4096}$
Which is a Square, whose side is . . . } $\frac{141}{16}$

Example 3.

1. Again, the same things remaining as before in the second Example from the first to the sixth step, we may feign the side of the Square mentioned in the said sixth step to be $39 - 10a$, and then the Square of $39 - 10a$ being equated to $- 15aa + 195a + 1521$ will give $a = \frac{123}{23}$.
2. Therefore from the second, third and fourth steps of *Example 2.* the sides of three other Cubes to solve the Question propounded in the said second Example will be found these, to wit, $\frac{103}{23}, \frac{12}{23}$, and 1 (or $\frac{23}{23}$)
3. And consequently the Cubes themselves are $\frac{1062727}{12167}, \frac{1728}{12167}$, 1 (or $\frac{12167}{12167}$)
4. Which three Cubes being severally subtracted from the given Cube 729, the summ of the three remainders in its least terms will be $\frac{1108809}{329}$, which is a Square, whose side is $\frac{1023}{23}$, as was required in *Example 2.*

QUEST. 71.

[*Another way of solving the preceding Quest. 70. when the given Cube is a squared Cube, or the Cube of a Square.*]

Let it be required to find three Cube-numbers, such, that if every one of them be subtracted from a given squared Cube-number, suppose 64, the summ of the three remainders may be a Square.

RESO-

RESOLUTION.

1. First, by the foregoing *Quest.* 69. find two such cube-numbers, that if each of them be subtracted from the given squared Cube 64, the summ of the remainders may be a Square; such are the Cubes 1 and 27, (found out in the Example of *Canon* 1. of the said *Quest.* 69.) for if each of them be subtracted from 64, the summ of the remainders makes the Square 100.
2. Then for the side of the first of the three Cubes sought }
 put $a + 1$ either of the sides of the two Cubes found out } : $a + 1$
 in the first step, *viz.* }
3. For the side of the second Cube put $-a + 4$, (4 being }
 the side of the given squared Cube 64,) } $-a + 4$
4. Let the side of the third Cube, be the side of the other of }
 the two Cubes found out in the first step, to wit; . . . } . . . 3
5. Therefore from the second step the first Cube is . . . } $aaa + 3aa + 3a + 1$
6. And from the third step, the second Cube is . . . } $-aaa + 12aa - 48a + 64$
7. And from the fourth step, the third Cube is . . . } $+ 27$
8. Then (according to the Question) subtract those three Cubes severally from the given squared Cube 64, so the three remainders will be these, to wit,

1.	-aaa - 3aa - 3a + 63,
2.	+aaa - 12aa + 48a,
3. + 37.

9. The summ of which remainders is $> -15aa + 45a + 100$.
10. Which summ is to be equated to a Square, but the side thereof must be so feigned that the value of a may be less than 3, to the end the side $a + 1$ in the second step may be less than 4, because the Cube of $a + 1$ must be subtracted from the Cube of 4. Now to cause that effect, the side of the said Square may be feigned $10 +$ any number of a less than $2\frac{1}{4}a$, or $10 -$ any number of a greater than $6\frac{2}{3}a$; (which limits may be found out by the method before delivered in *Quest.* 13. of this Book :) Let therefore the said side be feigned $a + 10$, then the Square thereof, to wit, $aa + 20a + 100$ being equated to the summ of the remainders in the ninth step, this Equation ariseth, *viz.*

$$aa + 20a + 100 = -15aa + 45a + 100.$$

11. Which Equation, after due Reduction, gives . . . } $a = \frac{22}{16}$
12. Therefore from the eleventh, second, third and fourth }
 steps, the sides of the three Cubes sought will be found } $\frac{41}{16}$, $\frac{39}{16}$ and 3
 these, to wit; }
13. Therefore the Cubes themselves are } $\frac{68221}{4096}$, $\frac{59319}{4096}$ and 27
 Which three Cubes will solve the Question, as will be evident by

The Proof.

By subtracting the said three Cubes severally from 64, (the squared Cube given in the Question,) the three remainders will be these;

$$\frac{193223}{4096}, \quad \frac{202825}{4096}, \quad \frac{151552}{4096}.$$

The summ of those remainders is } $\frac{547600}{4096}$

Which summ being reduced to its least terms by the com- }
 mon Divisor 16, will be } $\frac{34225}{256}$

Which $\frac{34225}{256}$ is a Square whose side is $\frac{185}{16}$, therefore the Question is solved.

Example. 2.

Let it be required to find three such Cube-numbers, that if every one of them be subtracted from 1, the summ of the three remainders may be a Square.

Resolution.

1. First, by the preceding *Quest.* 69. find two such Cube-numbers that if each of them be subtracted from 1, to wit, the given squared Cube, the summ of the three remainders may be a Square; such are the Cubes $\frac{1}{64}$ and $\frac{27}{64}$, for if each of them be subtracted from 1, the summ of the remainders will be $\frac{25}{16}$, which is a Square.

O 2

2. Then

2. Then for the side of the first of the three Cubes sought put $a + \frac{1}{4}$ the side of one of the two Cubes found out in the first step, to wit, $a + \frac{1}{4}$
3. And for the side of the second Cube put $-a + 1$, (1 being the side of the given squared Cube,) $-a + 1$
4. And let the side of the third Cube be the side of the other of the two Cubes found out in the first step, to wit, $\frac{3}{4}$
5. Then (according to the Question) subtract severally the Cubes of those three sides assumed in the three last foregoing steps from 1, (the given squared Cube,) and add the three remainders together, so the summ will be

$$- \frac{1}{4}aa + \frac{4}{16}a + \frac{1}{64} \text{ (or } \frac{2}{16}.)$$

6. Which summ must be equal to a Square, whose side must be so feigned that the value of a may be less than $\frac{3}{4}$, to the end the side $a + \frac{1}{4}$ in the second step of this second Example may be less than 1, for then every one of the three remainders of the subtraction mentioned in the fifth step will be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $\frac{2}{4} +$ any number of a less than $\frac{2}{3}a$, or else $\frac{2}{4} -$ any number of a greater than $\frac{1}{3}a$; therefore let the said side be feigned $\frac{1}{2}a + \frac{2}{4}$, then the Square thereof being equated to the summ of the three remainders in the fifth step, from that Equation you will find

$$a = \frac{2}{64}$$

7. Therefore from the sixth, second, third and fourth steps, the sides of the three Cubes sought will be $\frac{4}{64}$, $\frac{3}{64}$ and $\frac{4}{64}$.

8. And consequently the Cubes themselves are

$$\frac{64}{262144}, \frac{27}{262144}, \frac{16}{262144}$$

Which three Cubes will solve the Question before propounded in Example 2. as will be manifest by

The Proof.

By subtracting every one of the said three Cubes in the eighth step from 1, (the given squared Cube, the three remainders will be these, to wit,

$$\frac{193223}{262144}, \frac{202825}{262144}, \frac{151552}{262144}$$

The summ of those remainders is $\frac{547600}{262144}$
 Which summ being reduced to its least terms by the common Divisor 16, will be $\frac{34225}{16384}$
 Which is a Square, whose side is $\frac{185}{128}$; therefore the Question is solved.

Example 3.

1. The same things being supposed as in the first step of Example 2. we may vary the sides assumed in the second, third and fourth steps, in this manner, viz.

$$\left. \begin{array}{l} a + \frac{1}{4} \\ -a + 1 \\ \frac{3}{4} \end{array} \right\} \text{ we may assume } \left\{ \begin{array}{l} a + \frac{3}{4} \\ -a + 1 \\ \frac{1}{4} \end{array} \right.$$

3. Then by subtracting severally the Cubes of the three sides above-express on the right hand from 1, and adding the three remainders together, the summ will be

$$- \frac{2}{4}aa + \frac{2}{16}a + \frac{2}{16}$$

4. Which summ must be equal to a Square, whose side must be so feigned that the value of a may be less than $\frac{1}{4}$, to the end the side $a + \frac{3}{4}$ may be less than 1. Now to cause that effect we may feign the said side to be $\frac{2}{4} +$ any number of a less than $\frac{2}{4}a$, or $\frac{2}{4} -$ any number of a greater than $10a$; suppose therefore the said side be feigned $\frac{1}{2}a + \frac{2}{4}$, then the Square thereof being equated to the summ of the three remainders before mentioned in the third step, from that Equation you will find

$$a = \frac{7}{88}$$

5. Therefore from those three assumed sides which are placed on the right hand in the second step of this third Example, the sides of the three Cubes sought will be these, to wit,

$$\frac{67}{88}, \frac{87}{88} \text{ and } \frac{22}{88}$$

6. And consequently the Cubes themselves are

$$\frac{300763}{681472}, \frac{618103}{681472}, \frac{10643}{681472}$$

Which three Cubes will solve the Question before-propos'd in Example 2. for if every one of them be subtracted from the given squared Cube 1, the summ of the three remainders in its least terms will be $\frac{42241}{30976}$, which is a Square whose side is $\frac{221}{176}$.

QUEST. 72.

To find four cube-numbers, such, that if they be severally subtracted from a given squared cube-number, suppose 64, the summ of the four remainders may be a square number.

RESOLUTION.

1. By the foregoing Quest. 71. find three such Cubes, that if they be severally subtracted from the given squared Cube 64, the summ of the three remainders may be a Square; such are the Cubes $\frac{68221}{4096}$, $\frac{22312}{4096}$ and 27, whose sides are $\frac{41}{16}$, $\frac{32}{16}$ and 3, (found out in Example 1. of Quest 71.) for if the said Cubes be severally subtracted from 64, the summ of the three remainders in its least terms will be $\frac{34222}{256}$, which is a Square, whose side is $\frac{185}{16}$.
2. Then for the side of the first of the four Cubes sought put $a-3$, (which 3 is the side of one of the three Cubes found out in the first step,)
3. For the side of the second Cube put $a-4$, (4 being the Cube-root of the given squared Cube 64,)
4. Let the side of the third Cube be another of the sides of the said three Cubes found out in the first step, viz. $\frac{41}{16}$.
5. And let the side of the fourth Cube be the side of the remaining Cube in the first step, to wit, $\frac{32}{16}$.
6. Therefore (from the second step) the first Cube sought is $aaa-9aa-27a+27$
7. The second Cube is $-aaa-12aa-48a+64$
8. The third Cube is $\frac{68221}{4096}$
9. And the fourth Cube is $\frac{22312}{4096}$
10. Then those four Cubes being severally subtracted from the given squared Cube 64, the four remainders will be these, to wit,

1.	$-aaa-9aa-27a+\frac{11552}{4096}$,
2.	$+aaa-12aa-48a,$
3.	$\frac{123223}{4096},$
4.	$\frac{20282}{4096}.$

11. The summ of those remainders is $-21aa+21a+\frac{34222}{256}$.
12. Which summ must be equated to a Square, the side whereof must be so feigned that the value of a may be less than 1, to the end the side $a-3$ in the first step may be less than 4, (because the Cube of the said $a-3$ must be subtracted from the Cube of 4,) and if the value of a be less than 1, it will be much less than 4, as the third step requires. Now to cause a to be less than 1, the side of the feigned Square may be $\frac{185}{16}-a$ any number of a less than $\frac{185}{16}a$, or else $\frac{185}{16}-a$ any number of a greater than $23\frac{1}{8}a$, (which limits may be discovered by the method in Quest. 13. of this Book;) therefore we may feign the said side to be $\frac{1}{2}a+\frac{185}{16}$, whose Square being equated to the summ of the four remainders in the eleventh step, this Equation ariseth, viz.

$$\frac{1}{4}aa+\frac{185}{16}a+\frac{34222}{256}=-21aa+21a+\frac{34222}{256}$$

13. Which Equation duly reduced will give $a=\frac{151}{340}$
14. Therefore from the thirteenth, second, third, fourth and fifth steps, the sides of the four Cubes sought are discovered to be these, to wit,

$$\frac{4684}{1360}, \frac{4836}{1360}, \frac{3485}{1360}, \frac{3315}{1360}$$

15. And consequently the four Cubes are these, to wit,

$$\frac{102766285504}{2515456000}, \frac{113099029056}{2515456000}, \frac{42326109125}{2515456000}, \frac{36429280875}{2515456000}$$

Which four Cubes will solve the Question propos'd, as will be manifest by

The Proof.

By subtracting those four Cubes severally from the given squared Cube 64 (or

$$\frac{16028218400}{2515456000}, \frac{47890154944}{2515456000}, \frac{118663074875}{2515456000}, \frac{124559903125}{2515456000}$$

The summ of those four remainders is $\frac{3422236021440}{2515456000}$
 Which summ being reduced to its least terms by the
 common Divisor 1360, will be $\frac{256864722}{1849600}$
 Which is a Square, whose side is $\frac{16027}{1360}$

Therefore

Therefore the Question is solved; and if the method of resolving this and the preceding *Quest.* 71. be well examined, it will not be difficult to apprehend how to find out as many Cubes as shall be desired, which being severally subtracted from a given squared cube-number, or from the Cube of a given Square, the summ of the remainders may be a Square.

QUEST. 73.

To find two numbers, that if each of them be subtracted from the Cube of their summ, the remainders may be Cubes.

RESOLUTION.

First, making choice of some square number as 4, I put $4a$ for the summ of the two numbers sought, the Cube whereof will be $64aaa$; then for the first number I put $56aaa$, and for the other number $37aaa$, for these two being severally subtracted from $64aaa$; the remainders will be $8aaa$ and $27aaa$, which are manifestly Cubes, whereby one part of the Question is satisfied: It remains that the summ of the said assumed numbers $56aaa$ and $37aaa$ be equal to $4a$, viz. $93aaa = 4a$, whence by dividing each part by a , there ariseth $93aa = 4$. Now if the said 93 were a square number, then the value of a would be a rational number, and consequently the Question solved.

But 93 not being a Square, we must enquire whence it ariseth, and by examining the Operation it will appear, that the two Cubes 8 and 27 having been subtracted severally from 64, the remainders are 56 and 37, the summ whereof makes 93 before mentioned. So that our first scope must be to find two such Cubes that if each of them be subtracted from the squared Cube 64, the summ of the remainders may be a Square: But such are the Cubes 1 and 27, (found out by *Canon* 1. of the foregoing *Quest.* 69.) for if each of them be subtracted from 64, the summ of the remainders 63 and 37 makes the Square 100: therefore I begin the Resolution a-new thus; viz.

1. For the summ of the two numbers sought I put } . . . $4a$
2. The Cube thereof is } . . . $64aaa$
3. Then for one of the two numbers sought I put } . . . $56aaa$
4. And for the other number sought I put } . . . $37aaa$
5. Which two numbers being severally subtracted from $64aaa$ the }
remainders will be Cubes, to wit, } $8aaa$ and $27aaa$
6. But the summ of the numbers assumed in the third and fourth steps }
must be equal to $4a$ in the first step, therefore } $100aaa = 4a$
7. Which Equation duly reduced, gives } . . . $a = \frac{4}{93}$
8. Therefore from the seventh, third and fourth steps the two num- }
bers sought are } $\frac{63}{125}$ and $\frac{27}{125}$

Which numbers will solve the Question propos'd, as will be manifest by

The Proof.

9. The summ of the two numbers found out, to wit, $\frac{63}{125}$ and $\frac{27}{125}$, is } . . . $\frac{4}{93}$
10. Therefore the Cube of their summ is } $\frac{64}{125}$
11. From which $\frac{64}{125}$ subtract each of the numbers $\frac{63}{125}$ and $\frac{27}{125}$, so }
the remainders are Cubes, to wit, } $\frac{1}{125}$ and $\frac{27}{125}$

Another Example.

1. First, I take some square number, as 1, then I search out two such Cubes that each of them being subtracted from 1, (the Cube of the Square first taken,) the summ of the remainders may be a Square; such are the Cubes $\frac{1231}{6859}$ and $\frac{212}{6859}$, whose sides are $\frac{11}{19}$ and $\frac{2}{19}$, found out in the Example of *Canon* 2. *Quest.* 69.) for if each of those Cubes be subtracted from 1, the summ of the remainders $\frac{528}{6859}$ and $\frac{6347}{6859}$ will be $\frac{11875}{6859}$, or in its least terms $\frac{625}{361}$, which is a Square whose side is $\frac{25}{19}$; then by the help of those Cubes and remainders I form the Resolution as before in the first Example, viz.

2. For the summ of the two numbers sought I put a or $1a$, }
(1 being the square number first taken,) } $1a$
3. The Cube of the said summ is } $1aaa$
4. Then for one of the two numbers sought I put } $\frac{528}{6859}aaa$
5. And for the other number sought, } $\frac{6347}{6859}aaa$
6. Which two numbers being severally subtracted from $1aaa$, the }
remainders will be Cubes, to wit, } $\frac{1231}{6859}aaa$ and $\frac{212}{6859}aaa$

7. But

7. But the summ of the two numbers in the fourth and fifth steps } $\frac{622}{361}aaa = 1a$
 must be equal to $1a$ in the second step, therefore }
 8. Which Equation duly reduced gives } $a = \frac{12}{25}$
 9. Therefore from the eighth, fourth and fifth steps, the two numbers } $\frac{5528}{15625}$ and $\frac{6347}{15625}$
 sought, in their least terms, will be found }
 Which numbers will solve the Question propos'd, as will be evident by

The Proof.

The summ of the two numbers found out in the ninth step, to wit, } $\frac{12}{25}$
 of $\frac{5528}{15625}$ and $\frac{6347}{15625}$ is in the least terms }
 The Cube of that summ is } $\frac{6852}{15625}$
 From which Cube if you subtract severally the said numbers found } $\frac{11}{25}$ and $\frac{8}{25}$
 out, the remainders will be the Cubes of these sides, to wit, . . . }

QUEST. 74. (Quest. 19. Lib. 5. Diophant.)

To find three such numbers, that if every one of them be subtracted from the Cube of their summ, the three remainders may be Cubes.

[The text of Diophantus in the Resolution of this Question is so obscure, that it affords not any satisfactory Answer; I shall therefore shew how to solve it by two different ways of my own, by the latter of which this Question may be extended to four, five, or as many numbers as shall be desired.]

RESOLUTION.

1. First, take any square number, as 1 , then search out three such Cubes that if they be severally subtracted from the Cube of the said square number 1 , the summ of the three remainders may be a Square: But three such Cubes are $\frac{148877}{2985984}$, $\frac{19683}{2985984}$ and $\frac{4096}{2985984}$, whose sides are $\frac{12}{144}$, $\frac{27}{144}$ and $\frac{64}{144}$; (found out in Example 1. of the preceding Quest. 70.) for if those Cubes be severally subtracted from 1 (or unity,) the summ of the three remainders $\frac{2837107}{2985984}$, $\frac{2966301}{2985984}$ and $\frac{2981888}{2985984}$ will be $\frac{61009}{20736}$, which is a Square, whose side is $\frac{247}{144}$. Now by the help of those three preparatory Cubes and remainders, I proceed thus,
 2. For the summ of the three numbers sought I put a , or $1a$, (1 being the square number first taken,) } $1a$
 3. Therefore the Cube of the said summ is } $1aaa$
 4. Then for the first of the three numbers sought I put $\frac{2837107}{2985984}aaa$, (the said $\frac{2837107}{2985984}$ being one of the three remainders before mentioned in the first step,) } $\frac{2837107}{2985984}aaa$
 5. In like manner having multiplied the second remainder into aaa , I put the Product for the second number sought, to wit, } $\frac{2966301}{2985984}aaa$
 6. Likewise multiplying the third remainder into aaa , I put the Product for the third number sought, to wit, } $\frac{2981888}{2985984}aaa$
 7. Which three numbers in the three last steps being severally subtracted from $1aaa$; the three remainders will (by the Construction in the first step) be Cubes, to wit,
 $\frac{148877}{2985984}aaa$, $\frac{19683}{2985984}aaa$, $\frac{4096}{2985984}aaa$.

8. But the summ of the three numbers in the fourth, fifth and sixth steps must be equal to $1a$ in the second step, whence this Equation ariseth,

$$\frac{61009}{20736}aaa = 1a.$$

9. Which Equation duly reduced gives } $a = \frac{144}{247}$
 10. Therefore from the ninth, second, fourth, fifth and sixth steps, the three numbers sought will be made known, to wit, these,

$$\frac{2837107}{15069223}, \frac{2966301}{15069223}, \frac{2981888}{15069223}.$$

Which three numbers will solve the Question, as will be evident by

The Proof.

The summ of the said three numbers is $\frac{2785226}{15069223}$, which reduced to } $\frac{144}{247}$
 it smallest terms by the common Divisor 61009 , makes }
 The Cube of the said summ is } $\frac{2985224}{15069223}$
 From

From which Cube subtracting severally the three numbers found out in the tenth step, the remainders will be these three Cubes, to wit,

$$\frac{148877}{15069223}, \quad \frac{19683}{15069223}, \quad \frac{4096}{15069223}.$$

The sides of which Cubes are these, viz.

$$\frac{53}{247}, \quad \frac{27}{247}, \quad \frac{16}{247}. \quad \text{Therefore the Question is solved.}$$

But because the operation in finding out the three numbers, as also the three Cubes with their sides as aforesaid will be exceeding laborious by reason of long fractions, unless some Compendiums be used, I shall give a Canon deducible from the premisses to lessen the work, respect being first had to these following

Preparatory Directions.

1. First, the Rules for multiplying and dividing Fractions in *Sect. 22, 26. of Chap. 6. Book 1.* must be diligently observed, that the Products and Quotients may come out in the smallest terms.

2. Secondly, when one, two or more numbers are to be severally multiplied by some number, and the Products are to be severally divided by the same number, that multiplication and division may be quite omitted, for the numbers first propos'd to be multiplied will be the same with the Quotients that arise by the said multiplication and division. Moreover, when one, two or more numbers are to be severally multiplied by some number, and the Products are to be divided by some number greater or less than that multiplying number, reduce the said Multiplier and Divisor into the least terms (when they are not such already) by their greatest common Divisor, and take the Quotients for a new Multiplier and Divisor instead of those first prescribed: As, if 41, 39 and 48 be to be severally multiplied by 32, and the Products be to be severally divided by 16, I first reduce the said 32 and 16 to the smallest terms in the same Reason by the common Divisor 16, so the Quotients or new terms will be 2 and 1; then multiplying 41, 39 and 48 severally by 2, (instead of 32,) and dividing the Products severally by 1, (instead of 16,) the Quotients will be 82, 78 and 96, which are found out much speedier and in smaller terms than those that would be found out by multiplying the said 41, 39 and 48 by 32, and dividing the Products by 16 as was first prescribed. This Rule will oftentimes be very useful in the fourth branch of the following Canon.

3. Thirdly, let the square number first taken in the first step of the foregoing Resolution of this *Quest. 74.* be called bb , and its side b .

4. Fourthly, let the other square number which is equal to the summ of the three remainders found out in the said first step of the Resolution be called cc , and its side c .

These things premised, I proceed to the

CANON.

1. Divide the known number b by the known number c , and call the Quotient a , which is now a known number.

2. Divide the Cube of b by c , and let the Quotient be called d , which known number is the summ of the numbers sought by the Question.

3. Reduce the numbers a and d to their smallest common Denominator.

4. Reduce likewise the sides of the preparatory Cubes (found out in the first step of the Resolution) to their smallest common Denominator, then multiply severally the Numerators of those sides by the Numerator of the number a , and divide the Products severally by the said common Denominator of the sides of the said preparatory Cubes, and reserve the Quotients for Dividends.

5. Divide severally those Dividends reserved, by the Denominator of a or d , (for these were above reduced to a common Denominator,) so shall the Quotients be the sides of the Cubes sought.

6. Lastly, by subtracting severally the Cubes of the sides last found out, from the Cube of the summ of the numbers sought, (which summ was above found by the second step of the Canon,) the remainders shall be the numbers sought, and the smallest that have a common Denominator with the Cubes found out in the fifth step of the Canon.

This Canon with the preceding preparatory Directions may be practically illustrated by the Examples of the preceding *Quest. 73.* and of this and the following 75 and 76 Questions.

Example 2.

Example 2.

Let it be required to find three such numbers, that if every one of them be subtracted from the Cube of their sum, the three remainders may be Cubes.

1. First take some square number, as 9, then find three such Cubes, that if they be severally subtracted from 729 (the Cube of the said Square 9) the sum of the three remainders may be a Square: But three such Cubes are $\frac{148877}{4096}$, $\frac{12681}{4096}$ and $\frac{4026}{4096}$, (or 1,) whose sides are $\frac{7}{16}$, $\frac{2}{16}$ and 1, (found out in the second Example of *Quest.* 70.) for if those Cubes be severally subtracted from 729, (or $\frac{2281888}{4096}$,) the sum of the three remainders $\frac{2817107}{4096}$, $\frac{2266101}{4096}$ and $\frac{2231888}{4096}$ being reduced to its least terms will be $\frac{142081}{256}$, which is a square number whose side is $\frac{121}{16}$.

2. Then by proceeding according to the foregoing preparatory Directions and Canon, the numbers and Cubes sought will be found to be the same as were before found out in the first Example of this 74th Question.

Example 3.

1. Taking again the same square number 9 as in the second Example, I seek three other Cubes, that every one of them being subtracted from 729 (the Cube of the said Square 9) the sum of the three remainders may be a Square: But three such Cubes are $\frac{1022727}{12167}$, $\frac{1728}{12167}$ and 1, whose sides are $\frac{10}{23}$, $\frac{12}{23}$ and 1, (or $\frac{23}{23}$,) found out in the third Example of *Quest.* 70. for if those three Cubes be severally subtracted from the said 729, the sum of the three remainders in its least terms will be $\frac{1108809}{329}$, which is a Square, whose side is $\frac{1051}{23}$.

2. Then by proceeding according to the preparatory Directions and the Canon, (which follow the first Example of this 74th Question,) the sides of the three Cubes sought will be found $\frac{10}{23}$, $\frac{12}{23}$ and $\frac{23}{23}$, and the three numbers sought are these, to wit, $\frac{7721016}{43243551}$, $\frac{888011}{43243551}$ and $\frac{8822226}{43243551}$, whose sum in its least terms is $\frac{207}{351}$, from the Cube whereof if the said three numbers be severally subtracted, the three remainders will be Cubes, whose sides are those above found out. Therefore the Question is solved.

QUEST. 75. (Another way of solving the preceding Quest. 74.)

To find three such cube-numbers, that if every one of them be subtracted from the Cube of their sum, the remainders may be Cubes.

RESOLUTION.

1. First take some square number, as 4, then find three such Cubes that if they be severally subtracted from 64; (the Cube of the said Square 4,) the sum of the three remainders may be a Square: But three such Cubes are $\frac{68221}{4096}$, $\frac{22112}{4096}$ and 27, whose sides are $\frac{4}{16}$, $\frac{2}{16}$ and 3, (found out in the first Example of *Quest.* 71.) for if those Cubes be severally subtracted from the said 64, the three remainders will be these, to wit, $\frac{12222}{4096}$, $\frac{20282}{4096}$ and $\frac{11112}{4096}$, whose sum in its least terms is $\frac{34221}{256}$, which is a Square whose side is $\frac{185}{16}$.

2. Then by proceeding according to the preparatory Directions and Canon which follow the first Example of the preceding *Quest.* 74. the sides of the three Cubes sought will be found these, to wit,

$$\frac{82}{185}, \quad \frac{78}{185}, \quad \frac{96}{185}$$

3. And consequently the Cubes themselves are

$$\frac{551368}{6331625}, \quad \frac{474552}{6331625}, \quad \frac{884736}{6331625}$$

4. And the three numbers sought are these,

$$\frac{1545784}{6331625}, \quad \frac{1622600}{6331625}, \quad \frac{1212416}{6331625}$$

Which will solve the Question, as will be manifest by

The Proof.

5. The sum of the said three numbers is $\frac{4380800}{6331625}$, which reduced to its smallest terms by the common Divisor 34225, gives $\frac{128}{185}$

6. The Cube of the said sum is $\frac{2097112}{6331625}$

From which Cube if you subtract severally the three numbers before found out in the fourth step; the remainders will be the three Cubes above express'd in the third step.

Another Example.

1. First take some square number, as 1, then find three such Cubes that if they be severally subtracted from 1 (the Cube of the Square first taken) the summ of the three remainders may be a Square: But three such Cubes are $\frac{202762}{681472}$, $\frac{618102}{681472}$ and $\frac{10648}{681472}$, whose sides are $\frac{67}{88}$, $\frac{87}{88}$ and $\frac{22}{88}$, (found out in the third Example of *Quest.* 71.) for if those Cubes be severally subtracted from 1, (the Cube of the Square first taken,) the summ of the three remainders $\frac{280709}{681472}$, $\frac{22969}{681472}$ and $\frac{670824}{681472}$ being reduced to its least terms will be $\frac{48241}{30976}$, which is a Square whose side is $\frac{221}{176}$.

2. Then by proceeding according to the preparatory Directions and Canon (which follow the first Example of the preceding *Quest.* 74.) the sides of the three Cubes sought will be found these, to wit, $\frac{134}{221}$, $\frac{174}{221}$ and $\frac{44}{221}$; and the three numbers sought will be these, to wit, $\frac{1041672}{10793861}$, $\frac{181122}{10793861}$ and $\frac{1166122}{10793861}$, which will solve the said 75th Question, as will appear by the Proof.

QUEST. 76.

To find four such numbers, that if every one of them be subtracted from the Cube of their summ, the four remainders may be Cubes.

RESOLUTION.

1. First take some square number, as 4, then find four such Cubes that if they be severally subtracted from 64 (the Cube of the said Square 4) the summ of the four remainders may be a Square: But four such Cubes are these $\frac{102766281204}{2515456000}$, $\frac{111099029016}{2515456000}$, $\frac{42226122122}{2515456000}$ and $\frac{16422280872}{2515456000}$, whose sides are $\frac{4684}{1360}$, $\frac{4836}{1360}$, $\frac{2482}{1360}$ and $\frac{3212}{1360}$, (found out in the Example of the said fifth Question;) for if those Cubes be severally subtracted from the said 64, the four remainders will be these, to wit, $\frac{2822228426}{2515456000}$, $\frac{47820114944}{2515456000}$, $\frac{118661014812}{2515456000}$ and $\frac{124119903122}{2515456000}$; which remainders being added together, the summ in its least terms is $\frac{22864722}{1849600}$, which is a Square whose side is $\frac{16027}{1360}$.

2. Then by proceeding according to the preparatory Directions and Canon, (which follow the first Example of the foregoing *Quest.* 74.) the sides of the four Cubes sought will be found these, to wit,

$$\frac{9368}{16027}, \frac{9672}{16027}, \frac{6970}{16027} \text{ and } \frac{6630}{16027}.$$

3. Therefore the four Cubes sought are these,

$$\frac{822130284032}{4116771011683}, \frac{904792232448}{4116771011683}, \frac{338698873000}{4116771011683}, \frac{291434247000}{4116771011683}.$$

4. Which Cubes being severally subtracted from the Cube of $\frac{16027}{1360}$, (which by the second branch of the Canon will be found out for the summ of the four numbers sought,) the remainders will be the numbers sought, to wit, these,

$$\frac{465783187968}{4116771011683}, \frac{949304599000}{4116771011683}, \frac{383121239552}{4116771011683}, \frac{996479225000}{4116771011683}.$$

Which four numbers solve the Question, as will appear by the Proof.

From the manner of solving this and the preceding 75th Question, it is easie to apprehend, how five, six or as many numbers as shall be desired, may be found out, which being severally subtracted from the Cube of their summ, may leave as many Cubes: But so many numbers as are desired, so many preparatory Cubes must be first found out, such, that if they be severally subtracted from the Cube of some square number chosen at pleasure, the summ of the remainders may be a Square; which preparatory Cubes may be found out by the method before delivered in *Quest.* 71, and 72.

QUEST. 77. (Quest. 20. Lib. 5. Diophant.)

To find three such numbers that if the Cube of their summ be subtracted from every one of them, the remainders may be Cubes.

RESOLUTION.

1. For the summ of the three numbers sought put } a
2. And let the three numbers be } $2aaa, 9aaa, 28aaa$
3. It remains that their summ $39aaa$ be equated to a , whence $39aa = 1$; where if 39 were a Square the Question would be solved by Rational numbers. But 39 is not

not a Square, whence therefore is it produced? Examine the Positions, and you will find that 1 being added severally to the three Cubes 1, 8 and 27, the sum of those three additions makes 39. We must therefore search out three Cubes whose sum increased with 3 may make a Square, to which end

4. For the sides of the three Cubes put } e ; $-e + 3$; and 1
5. Then the sum of the Cubes of those three sides increased }
with 3, makes } $9e^2 - 27e + 31$
6. Which sum is to be equated to a Square, but the side thereof must be so feigned that the value of e may be less than 3; now to cause that effect the side may be variously feigned within limits easie to be discovered from the method in divers preceding Questions of this Book, let it be $3e - 7$, then the Square of $3e - 7$ being equated to $9e^2 - 27e + 31$ will give $e = \frac{6}{5}$; therefore the sides of the three Cubes are $\frac{6}{5}$, $\frac{2}{5}$ and 1, and the Cubes themselves are $\frac{216}{125}$, $\frac{8}{125}$ and 1, by the help whereof the work is to be renewed thus, viz.
7. Add 1 to every one of the three Cubes before found, and the sums will be $\frac{341}{125}$, $\frac{814}{125}$ and $\frac{215}{125}$; then instead of $2aaa$, $9aaa$ and $28aaa$ (in the second step) put for the three numbers sought,

$$\frac{341}{125}aaa, \frac{814}{125}aaa, \frac{215}{125}aaa.$$

8. Then the sum of those three numbers being equated to a ,
(which in the first step was put for the sum of the three numbers sought,) gives this Equation, to wit, } $\frac{289}{25}aaa = a$
9. Whence, after due Reduction, } $a = \frac{5}{17}$

Therefore from the ninth and seventh steps the three numbers sought are $\frac{341}{4913}$, $\frac{814}{4913}$ and $\frac{215}{4913}$; for if from every one of them, the Cube of their sum, to wit, $\frac{125}{4913}$ be subtracted, there will remain the Cubes $\frac{216}{4913}$, $\frac{8}{4913}$, $\frac{1}{4913}$, whose sides are $\frac{6}{17}$, $\frac{2}{17}$, $\frac{1}{17}$.

From the premisses it is evident that the Question is capable of innumerable Answers, and may easily be extended to four, five, or as many numbers as you please.

QUEST. 78. (Quest. 21. Lib. 5. Diophant.)

To find three such numbers that their sum may be a Square, and that if to the Cube of the said sum the three numbers be severally added, the three sums may be square numbers.

RESOLUTION.

1. For the sum of the three numbers sought, that it may be }
a Square, put } aa
2. Then for the first number put } $3aaaaaa$
3. For the second, } $8aaaaaa$
4. And for the third, } $15aaaaaa$
5. Whence it is evident that every one of them added to the Cube of their sum makes a Square; But the sum of the three numbers (in the second, third and fourth steps) must be equal to aa which was first put for their sum; therefore $26aaaaaa = aa$, and consequently; (by dividing each part by aa ,) $26aaaa = 1$. In which last Equation if 26 were a squared square number, the value of a would be a rational number: Whence therefore comes 26? Examine the Positions, and you will find that 'tis the sum of the three numbers 3, 8 and 15, every one of which increased with 1 makes a Square; therefore the scope of our search must be to find three numbers, every one of which increased with 1 may make a Square, and that the sum of the said three numbers may be a Biquadrate: To which end let the three numbers be $aaaa - 2aa$; $aa + 2a$ and $aa - 2a$; for every one of these increased with 1 makes a Square, and their sum makes a Biquadrate, to wit, $aaaa$; and 'tis evident the value of a may be any number greater than 2, (for the third number $aa - 2a$ shews that aa must be greater than $2a$, and consequently a greater than 2.) Suppose therefore $a = 3$, whence the three numbers $aaaa - 2aa$; $aa + 2a$; $aa - 2a$, will be 63, 15 and 3; now with the help of these three numbers the work may be renewed thus, viz.
6. Let the sum of the three numbers sought be (as before) . . . } aa
7. And the first number } $63aaaaaa$
8. The second } $15aaaaaa$
9. And the third } $3aaaaaa$

10. Therefore the summ of the three numbers is } 81 aaaaaa
 11. Which summ must be equal to aa, viz. } 81 aaaaaa = aa
 12. Which Equation after due Reduction gives } a = $\frac{1}{3}$

Therefore from the twelfth, seventh, eighth and ninth steps the three numbers sought are $\frac{63}{729}$, $\frac{15}{729}$ and $\frac{3}{729}$, which will solve the Question; for their summ is a Square, to wit, $\frac{1}{9}$. Also the Cube of the said summ is $\frac{1}{729}$, to which if the three numbers be severally added there will come forth three square numbers, $\frac{64}{729}$, $\frac{16}{729}$ and $\frac{4}{729}$, for their sides are $\frac{8}{27}$, $\frac{4}{27}$ and $\frac{2}{27}$.

QUEST. 79. (Quest. 22. Lib. 5. Diophant.)

To divide a given number, suppose 2, into three such numbers, that every one of them subtracted from the Cube of their summ, viz. from 8, may leave a Square.

RESOLUTION.

Forasmuch as every one of the three numbers sought being less than 2 is to be subtracted from 8, each remainder shall be greater than 6, but less than 8; and the summ of the three remainders, to wit, the three Squares sought makes 22: for the summ of the three desired numbers, to wit, 2, subtracted from three times 8 leaves 22; we must therefore divide 22 into three such Squares that every one of them may be greater than 6, but less than 8. But 22 is composed of three Squares 9, 9 and 4; therefore, first, (by Quest 4. of this Book) let 13 (the summ of 9 and 4) be divided into two Squares, that one may be between 6 and 8, such are the Squares $\frac{49}{625}$ and $\frac{36}{625}$, whose sides are $\frac{7}{25}$ and $\frac{6}{25}$; then the lesser of those Squares, to wit, $\frac{36}{625}$ added to 9 makes $\frac{585}{625}$, which must also be divided into two Squares that each may be between 6 and 8; but by the said Quest. 4. the sides of two such Squares will be found $\frac{1232}{4525}$ and $\frac{1186}{4525}$, which with $\frac{6}{25}$, (that is, $\frac{1248}{4525}$) before found are the sides of the three Squares sought; therefore the three Squares themselves are $\frac{153710404}{20475625}$, $\frac{140778225}{20475625}$ and $\frac{151971121}{20475625}$, whose summ makes 22, and every one of them is greater than 6, but less than 8; therefore those three Squares severally subtracted from 8, leave the three desired parts of 2, to wit, $\frac{19024596}{20475625}$, $\frac{23026772}{20475625}$ and $\frac{7829872}{20475625}$.

QUEST. 80. [This is the 12th of the 4th Book of Vieta's Zeteticks.]

To find three right-angled Triangles in rational numbers, that the Solid of the Perpendiculars may be to the Solid of the Bases as a square number to a square number.

Note. By the Solid of three numbers is meant the Product made by their multiplication one into another, as, the Solid of 2, 3 and 4 is 24, that is, 2 x 3 x 4.

RESOLUTION.

1. Let h, b, p represent the Hypothensal, Base and Perpendicular of any right-angled Triangle in numbers given or found out by the Canon in Observat. 8. Resolut. 2. Quest. 2. of this Book,

$$\text{viz. } \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ h & . & b & . & p \end{cases}$$

2. Then from h and b (by the Canon above-mentioned) form a second right-angled Triangle, and let $2hb$ be called the Base, so the three sides will be these,

$$\text{viz. } \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ hb + bb & . & 2hb & . & hb - bb \end{cases}$$

3. Again, from h and p form a third right-angled Triangle; and let $2hp$ be called the Base, so the three sides will be these,

$$\text{viz. } \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ hb + pp & . & 2hp & . & hb - pp \end{cases}$$

4. I say the Solid of the Perpendiculars of those three right-angled Triangles is to the Solid of their Bases as a Square to a Square, to wit, as pp to $4hb$; which I prove thus,

The Perpendicular of the first right-angled Triangle is p ; the second Perpendicular is $hb - bb$, that is, pp , (for by Construction in the first step, $hb = bb + pp$, whence $hb - bb = pp$;) and the third Perpendicular is $hb - pp$, that is, bb . So that the three Perpendiculars are p, pp and bb , which multiplied one into another will produce

$$pppbb = \text{the Solid of the three Perpendiculars.}$$

Again,

Again, the Bases of the same three right-angled Triangles are $b, 2hb, 2hp$, which multiplied one into another will produce

$$4hbhb = \text{the Solid of the three Bases.}$$

Now because bbp is a common Factor in those two Solids, they shall be in such proportion one to another as the Quotients that arise by dividing the said Solids by bbp , viz.

$$ppbb \cdot 4hbhb :: pp \cdot 4hb.$$

Which was to be proved.

An Example in Numbers.

Let the first right-angled Triangle be $\left. \begin{array}{l} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{array} \right\} \begin{array}{l} h ; b ; p \\ 5 ; 3 ; 4 \end{array}$

Then by Construction in the second step, the second right-angled Triangle shall be $\left. \begin{array}{l} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{array} \right\} \begin{array}{l} 34 ; 30 ; 16 \end{array}$

And by Construction in the third step the third right-angled Triangle is $\left. \begin{array}{l} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{array} \right\} \begin{array}{l} 41 ; 40 ; 9 \end{array}$

Which three Triangles will solve the Question; for the Solid of the Perpendiculars 4, 16, 9 hath such proportion to the Solid of the Bases 3, 30, 40 as the Square of 4 to the Square of 10.

Note. Instead of any one of the Triangles thus found out, you may take another like Triangle, as instead of 34, 30, 16, you may take 17, 15, 8, which with the other two Triangles will solve the Question.

QUEST. 81. (Quæst. 13. Lib. 4. Zetet. Viet.)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, less by the Product of the Bases, may be a Square.

RESOLUTION.

1. Let h, b, p represent the Hypothensal, Base and Perpendicular of a right-angled Triangle, in numbers so given or found out that $2p$ may be greater than b ,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \quad \text{Base,} \quad \text{Perp.} \\ h \quad \cdot \quad b \quad \cdot \quad p \end{array} \right.$$

2. Then from $2p$ and b let a second right-angled Triangle be formed, and let apb be called the Perpendicular, so the three sides will be these,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \quad \text{Base,} \quad \text{Perp.} \\ 4pp + bb \quad \cdot \quad 4pp - bb \quad \cdot \quad 4pb \end{array} \right.$$

3. Then divide the sides of the said second right-angled Triangle severally by b the Base of the first, so there will arise these following sides of a third right-angled Triangle,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \quad \text{Base,} \quad \text{Perp.} \\ \frac{4pp + bb}{b} \quad \cdot \quad \frac{4pp - bb}{b} \quad \cdot \quad 4p \end{array} \right.$$

I say the first and third right-angled Triangles will solve the Question; for if the Product of their Perpendiculars p and $4p$, to wit, $4pp$ be lessened by the Product of their Bases b and $\frac{4pp - bb}{b}$, that is, by $4pp - bb$, the remainder will be a Square, to wit, bb ; which was required.

An Example in Numbers.

Let the first right-angled Triangle be $\left. \begin{array}{l} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{array} \right\} \begin{array}{l} h ; b ; p \\ 5 ; 4 ; 3 \end{array}$

Then the other shall be $\left. \begin{array}{l} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{array} \right\} \begin{array}{l} 13 ; 5 ; 12 \end{array}$

Which Triangles will solve the Question; for the Product of the Perpendiculars 3 and 12, to wit, 36, exceeds 20 the Product of the Bases, by the Square 16.

Note. If the two Triangles found out by this Question be severally multiplied or divided by the same number, they will produce two other Triangles to perform the same effect.

QUEST. 82. (Quæst. 14. Lib. 4. Zetet. Viet.)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, together with the Product of the Bases, may make a Square.

RESO-

RESOLUTION.

1. Let h, b, p represent the Hypothensal, Base and Perpendicular of a right-angled Triangle, in numbers so given or found out that p may exceed $2b$,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \\ h \end{array} \right. \quad \left\{ \begin{array}{l} \text{Base,} \\ b \end{array} \right. \quad \left\{ \begin{array}{l} \text{Perp.} \\ p \end{array} \right.$$

2. Then from p and $2b$ form a second right-angled Triangle, and let $4bp$ be called the Base; so the three sides will be these,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \\ pp - 4bb \end{array} \right. \quad \left\{ \begin{array}{l} \text{Base,} \\ 4bp \end{array} \right. \quad \left\{ \begin{array}{l} \text{Perp.} \\ pp - 4bb \end{array} \right.$$

3. Then divide the sides of the said second right-angled Triangle severally by p the Perpendicular of the first, so there will arise these following sides of a third right-angled Triangle,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \\ \frac{pp - 4bb}{p} \end{array} \right. \quad \left\{ \begin{array}{l} \text{Base,} \\ 4b \end{array} \right. \quad \left\{ \begin{array}{l} \text{Perp.} \\ \frac{pp - 4bb}{p} \end{array} \right.$$

4. I say the first and third right-angled Triangles will solve the Question; for if to $pp - 4bb$ the Product of the Perpendiculars p and $\frac{pp - 4bb}{p}$, you add $4bb$, to wit, the Product of the Bases b and $4b$, the sum will be the Square pp . Which was required.

An Example in Numbers.

The first right-angled Triangle may be } 13, 5, 12
Then the other Triangle sought shall be } $\frac{61}{3}, 20, \frac{11}{3}$

Which Triangles will solve the Question; for the Product of the Perpendiculars 12 and $\frac{11}{3}$, to wit, 44 increased with 100 the Product of the Bases 5 and 20, makes the Square 144.

Note. If the two Triangles found out by this Question be multiplied or divided by the same number, they will produce two other Triangles performing the same effect. So, if 13, 5, 12 and $\frac{61}{3}, 20, \frac{11}{3}$ be multiplied severally by 3, there will be produced 39, 15, 36 and 61, 60, 11; where 396 the Product of the Perpendiculars, with 900 the Product of the Bases makes the Square 1296, whose Root is 36.

QUEST. 83. (Quæst. 15. Lib. 4. Zetet. Viet.)

To find three right-angled Triangles in rational numbers, that the Solid of the Hypothensals may be to the Solid of the Bases, as a square number to a square number.

RESOLUTION.

1. Let h, b, p represent the Hypothensal, Base and Perpendicular of a right-angled Triangle, in numbers so given or found out that $2b$ may exceed p ,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \\ h \end{array} \right. \quad \left\{ \begin{array}{l} \text{Base,} \\ b \end{array} \right. \quad \left\{ \begin{array}{l} \text{Perp.} \\ p \end{array} \right.$$

2. Then from $2b$ and p form a second right-angled Triangle, and let $4bp$ be called the Base, so the three sides will be these,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \\ 4bb - pp \end{array} \right. \quad \left\{ \begin{array}{l} \text{Base,} \\ 4bp \end{array} \right. \quad \left\{ \begin{array}{l} \text{Perp.} \\ 4bb - pp \end{array} \right.$$

3. Then by the help of those two right-angled Triangles, find out a third, by the Canon in *Observat.* 4. upon *Resolut.* 2, and 3. of *Quest.* 2. viz. for the Hypothensal of the third right-angled Triangle take the Product of the Hypothensals of the first and second; the Base shall be the Product of the Bases of the first and second, less by the Product of the Perpendiculars; and the Perpendicular shall be equal to the sum of the Product of the Base of the first into the Perpendicular of the second, and the Product of the Perpendicular of the first into the Base of the second; so the three sides of the third right-angled Triangle will be these,

$$\text{viz. } \left\{ \begin{array}{l} \text{Hypoth.} \\ 4bbh - ppb \end{array} \right. \quad \left\{ \begin{array}{l} \text{Base,} \\ ppp \end{array} \right. \quad \left\{ \begin{array}{l} \text{Perp.} \\ 4bbh - 3bpp \end{array} \right.$$

I say those three right-angled Triangles will solve the Question; for the Solid of the Hypothensals is to the Solid of the Bases as the Square of $4bbh - ppb$ is to the Square of $2bpp$.

An

An Example in Numbers.

The first right-angled Triangle may be > 5, 3, 4
 Then the second in its least terms will be found > 13, 12, 5
 And the third in its least terms is > 65, 16, 63

Which three Triangles will solve the Question; for the Solid of the Hypothenusals is to the Solid of the Bases, as the Square of 65 to the Square of 24.

Otherwise thus;

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, so given or found out that p may exceed $2b$,

$$\text{viz. } \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ & b & p \end{cases}$$

2. Then from p and $2b$ form a second right-angled Triangle, and let $4bp$ be called the Base, so the three sides will be these,

$$\text{viz. } \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ & pp + 4bb & 4pb & pp - 4bb \end{cases}$$

3. Then by the help of those two right-angled Triangles find out a third by the latter Canon in *Observat.* 4. before mentioned, *viz.* for the Hypothenusal of the third right-angled Triangle take the Product of the multiplication of the Hypothenusals of the first and second; for the Base, take the summ of the Product of the Bases and the Product of the Perpendiculars; and for the Perpendicular take the difference of these two Products, to wit, the Product of the Base of the first into the Perpendicular of the second, and the Product of the Perpendicular of the first into the Base of the second. So the three sides of the third right-angled Triangle will be found these,

$$\text{viz. } \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ & pph + 4bbh & ppp & 3bpp + 4bbb \end{cases}$$

An Example in Numbers.

Let the sides of the first right-angled Triangle be > 13, 5, 12
 Then the second in its least terms will be found > 61, 60, 11
 And the third in its least terms shall be > 793, 432, 665

Which three Triangles will solve the Question; for the Solid of the Hypothenusals is to the Solid of the Bases, as the Square of 793 to the Square of 360.

QUEST. 84. (Quest. 24. Lib. 5. Diophant.)

To find three square numbers, that if they be severally added to the solid Product made by their continual multiplication, the three summs may be Squares.

RESOLUTION.

1. For the Solid of the three Squares sought put > aa
 2. Then find three Squares, every one of which increased with unity may make a Square; which Squares may easily be found out by the help of three unlike right-angled Triangles; for if the Square of one of the sides about the right-angle be divided by the Square of the other side, the Quotient will be a Square, which increased with 1 will make a Square: As, if there be exposed these three unlike right-angled Triangles, to wit, 3, 4, 5. | 5, 12, 13. | 8, 15, 17; then in the first Triangle the Square of the Base 3 divided by the Square of the Perpendicular 4 gives the Square $\frac{9}{16}$, which increased with 1, that is, $\frac{25}{16}$, makes the Square $\frac{34}{16}$; the reason whereof is manifest, for by Construction the Numerators 9 and 16 added together make a Square, wherefore the whole Fraction $\frac{34}{16}$ shall be a Square. In like manner in the second Triangle the Square of the Base 5 divided by the Square of the Perpendicular 12 gives the Square $\frac{25}{144}$, which increased with 1, that is, $\frac{149}{144}$, makes the Square $\frac{174}{144}$. And in the third Triangle the Square of the Base 8 divided by the Square of the Base 15, gives the Square $\frac{64}{225}$, which increased with 1, makes the Square $\frac{289}{225}$. Thus three Squares are found out, to wit, $\frac{25}{16}$, $\frac{25}{144}$ and $\frac{64}{225}$, every one of which increased with 1 makes a Square. Now multiply those Squares severally by aa , and take the Products $\frac{25}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ for the three Squares sought; for every one of them added to aa , (which was first put for the Solid made by their continual multiplication,) makes a Square: It remains that the Solid of the said $\frac{25}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ be equal to aa ; But their Solid by continual multiplication is $\frac{1}{3} \frac{1}{4} \frac{2}{5} aaaaaa$,

$\frac{14400}{518400} aaaaaa$, this equated to aa gives $\frac{14400}{518400} aaaaaa = aa$; whence by dividing each part by aa there ariseth $\frac{14400}{518400} aaaaa = 1$, and by extracting the square Root out of each part it gives $\frac{120}{720} a = 1$; in which last Equation if $\frac{120}{720}$ were a Square, then the value of a would be expressible by a rational number, and consequently the Question were solved. Whence therefore comes $\frac{120}{720}$? Examine the work, and you will find that the Numerator 120 is the Solid of the Bases 3, 5 and 8 of the three Triangles first expos'd; (for 14400 is the Solid of the three Squares 9, 25 and 64, and therefore the square Root of 14400, to wit, 120 is the Solid of the sides of those three Squares;) and the Denominator 720 is the Solid of the three Perpendiculars 4, 12 and 15; (for 518400 is the Solid of the three Squares 16, 144 and 225, and therefore the square Root of 518400, to wit, 720 is the Solid of the three Roots of those Squares.) We must therefore find three such right-angled Triangles, that the Solid of their Perpendiculars may be to the Solid of their Bases as a Square to a Square. But by the precedent *Quest.* 80. three such Triangles may be found out; as these, 4, 3, 5. | 8, 15, 17. | 9, 40, 41; the Solid of whose Bases 4, 8, 9, to wit, 288, is to the Solid of their Perpendiculars 3, 15, 40, to wit, 1800, as 144 to 900; that is, as a Square to a Square; then with these Triangles let the work be renewed as before, *viz.*

3. For the Solid of the three Squares sought put } aa
4. Then divide the Squares of the Bases of the three right-angled Triangles last found out, by the Squares of the Perpendiculars, and multiplying the Quotients severally by aa , put the Products for the three Squares sought, to wit, } $\frac{16}{9}aa, \frac{64}{225}aa, \frac{81}{1600}aa$
5. The Solid of those three Squares equated to aa , gives } $\frac{82944}{3240000} aaaaaa = aa$
6. Which Equation, after due Reduction, gives } $a = \frac{2}{3}$
7. Wherefore from the sixth and fourth steps the three Squares sought are $\frac{100}{9}, \frac{16}{9}$ and $\frac{81}{36}$, which mutually multiplied one into another make the Solid $\frac{25}{4}$, to which if the three Squares themselves be severally added, the summs will also be Squares, to wit, $\frac{625}{36}, \frac{289}{36}$ and $\frac{1681}{36}$; for their sides are $\frac{25}{6}, \frac{17}{6}$ and $\frac{41}{6}$. Therefore the Question is solved, and manifestly capable of innumerable Answers.

QUEST. 85. (*Quest.* 25. *Lib.* 5. *Diophant.*)

To find three such Squares, that if they be severally subtracted from the solid Product made by their continual multiplication, the three remainders may be Squares.

RESOLUTION.

1. For the solid Product of the three Squares sought put . . . } aa
2. Then search out three Squares, every one of which subtracted from unity may leave a Square; but three such Squares may be found out by the help of three unlike right-angled Triangles; for if the Square of one of the sides about the right-angle be divided by the Square of the Hypotenusal, the Quotient shall be a Square, which subtracted from 1 will leave a Square: Let therefore three unlike right-angled Triangles be expos'd, as 3, 4, 5. | 12, 5, 13. | 15, 8, 17; then by dividing the Squares of 4, 5 and 8, which I shall here call Bases, (for it matters not which of the sides about the right-angle be called the Base,) by the Squares of the Hypotenusals 5, 13 and 17, the Quotients will be the Squares $\frac{16}{25}, \frac{25}{169}$ and $\frac{64}{289}$, every one of which subtracted from 1 leaves a Square. Then multiply every one of these Squares by aa and assume the Products to be the three Squares sought, to wit, $\frac{16}{25}aa, \frac{25}{169}aa$ and $\frac{64}{289}aa$; for every one of these subtracted from aa (which was first put for the Solid of the three desired Squares) leaves a Square. It remains that the solid Product of the said $\frac{16}{25}aa, \frac{25}{169}aa$ and $\frac{64}{289}aa$ be equal to aa ; but the said Solid by continual multiplication will be found $\frac{25600}{1105} aaaaaa$, therefore $\frac{25600}{1105} aaaaaa = aa$, whence after due Reduction there will arise $\frac{160}{1105} a = 1$; in which last Equation if $\frac{160}{1105}$ were a Square, then the value of a would be expressible by a rational number. Whence therefore comes $\frac{160}{1105}$? Examine the work, and you will find that the Numerator 160 is the solid Product of the Perpendiculars 4, 5 and 8 of the three Triangles first expos'd; and the Denominator 1105 is the Solid of the Hypotenusals 5, 13 and 17. We must therefore find three such right-angled Triangles that the Solid of the Hypotenusals may be to the Solid of their Bases as a Square to a Square: But three such right-angled Triangles may be found out by the preceding *Quest.* 83. suppose these, 5, 3, 4. | 13, 12, 5. | 65, 16, 63; here I shall call 3, 12 and 16 the Bases, by the help whereof the work may be renewed thus, *viz.*

3. For

3. For the Solid of the three Squares sought put } aa
4. Then divide the Squares of the Bases of the three Triangles }
last found out, by the Squares of the Hypothenusals, and }
multiply the Quotients severally by aa , and put the Pro- } $\frac{2}{25}aa, \frac{144}{169}aa, \frac{225}{4225}aa$
ducts for the three Squares sought, to wit, }
5. The Solid of those three Squares being equated to aa , gives } $\frac{17850625}{17850625}aaaaaa = aa$
Which Equation, after due Reduction, will give } $a = \frac{65}{24}$
6. According to which value of a , the Positions in the fourth step being resolved, the three Squares sought will be found $\frac{162}{64}, \frac{25}{4}, \frac{4}{9}$; for these mutually multiplied make the Solid $\frac{4225}{376}$, from which if every one of the said three Squares be subtracted, the remainders will be Squares, to wit, $\frac{162}{36}, \frac{625}{376}, \frac{44}{64}$, whose sides are $\frac{12}{6}, \frac{25}{24}, \frac{2}{8}$.

QUEST. 86. (Quest. 26. Lib. 5. Diophant.)

To find three Squares, that the Solid or Product made by their continual multiplication being subtracted from every one of them, the three remainders may be Squares.

RESOLUTION.

The Resolution of this Question depends upon the Lemma used in the last preceding Question, for as there, so here, three right-angled Triangles are first to be found out, that the Solid of their Hypothenusals may be to the Solid of their Bases as a Square to a Square. Then instead of $\frac{2}{25}aa, \frac{144}{169}aa$ and $\frac{225}{4225}aa$, assumed in the fourth step of the preceding Quest 85. to be the three Squares sought, let $\frac{25}{9}aa, \frac{162}{144}aa$ and $\frac{4225}{256}aa$ be put for the three Squares sought in this 86th Question; where observe, that the Numerators of the three latter Squares which are multiplied into aa , are the same with the Denominators of the former, and the Denominators of the latter the same with the Numerators of the former; by which conversion it will follow, that if from every one of the latter Squares, viz. from $\frac{25}{9}aa, \frac{162}{144}aa$ and $\frac{4225}{256}aa$ you subtract aa , which (as before in Quest. 85.) is to be taken for the Solid of the three Squares sought, the three remainders will be Squares: Then the Solid of the three assumed Squares, that is, of $\frac{25}{9}aa, \frac{162}{144}aa$ and $\frac{4225}{256}aa$, being equated to aa , gives $\frac{17850625}{331776}aaaaaa = aa$; whence, after due Reduction, $a = \frac{25}{65}$, according to which value of a , the Positions being resolved will give $\frac{64}{169}, \frac{25}{25}, \frac{2}{4}$ for the three Squares sought; for the Solid made by their continual multiplication is $\frac{125}{4225}$, which subtracted from every one of the three Squares, leaves the Squares $\frac{1024}{4225}, \frac{1}{169}$ and $\frac{1521}{16900}$, whose sides are $\frac{32}{65}, \frac{1}{13}, \frac{12}{130}$.

QUEST. 87. (Quest. 30. Lib. 5. Diophant.)

To find three Squares, that if to the summ of every two of them, a given number, suppose 15, be added, the three summs may be Squares.

RESOLUTION.

1. For one of the Squares sought take any square number }
at pleasure, as } 9
2. Then we must find two other Squares, such, that each of them added to 24 may make a Square; for since 9 one of the three Squares added to the given number 15 makes 24, it will not be difficult to conceive from the tenor of the Question, that each of the other two Squares taking to it 24 must make a Square. Now to find out those two Squares, divide 24 by each of two sides about the right-angle of some right-angled Triangle, as 3 and 4; so the Quotients 8 and 6 shall also be the sides about the right-angle of a right-angled Triangle, because they are in the same proportion with the former; for by Construction $8 \times 3 = 4 \times 6 = 24$, therefore $3 \cdot 4 :: 6 \cdot 8$; and consequently the halves of 3 and 4, to wit, $\frac{3}{2}$ and 2; and the halves of 8 and 6, to wit, 4 and 3 shall be also the sides about the right-angle of a right-angled Triangle. These things premised,
3. Let the side of one of the two Squares be the difference }
between $3a$ and $\frac{2}{a}$, to wit, } $3a - \frac{2}{a},$ or $\frac{2}{a} - 3a$
4. And the side of the other Square the difference between }
 $4a$ and $\frac{2}{a}$, to wit, } $4a - \frac{2}{a},$ or $\frac{2}{a} - 4a$
5. There-

5. Therefore the Square of the side in the third step is . $\triangleright 9aa - 12 + \frac{4}{aa}$
 6. And the Square of the side in the fourth step is . . $\triangleright 16aa - 12 + \frac{4}{aa}$
 7. The former of those two Squares increased with 24 } $9aa + 12 + \frac{4}{aa}$
 makes a Square, to wit, }
 8. And the latter Square in the sixth step increased with 24 } $16aa + 12 + \frac{4}{aa}$
 makes a Square, to wit, }
 9. The sides of which two Squares in the two last steps are $\triangleright 3a + \frac{2}{a}$ and $4a + \frac{2}{a}$
 10. It remains that the summ of the Squares in the fifth and } $25aa - 9 + \frac{22}{aa}$
 sixth steps, together with the given number 15 may make }
 a Square, but it makes }
 11. Which summ must be equated to a Square, viz. either }
 to $25aa$, or to $\frac{22}{aa}$, let it first be equated to $25aa$, } $25aa - 9 + \frac{22}{aa} = 25aa$
 viz. suppose }
 12. Which Equation, after due Reduction, gives . . . $\triangleright a = \frac{2}{6}$
 13. Therefore from the twelfth, third and fourth steps the sides of the second and third
 Squares sought are $\frac{1}{10}$ and $\frac{22}{15}$, and consequently 9, $\frac{22}{10}$ and $\frac{22}{25}$ are the three Squares
 sought; for every two with 15 make Squares, to wit, $\frac{22}{100}$, $\frac{22}{225}$, $\frac{22}{36}$, whose sides
 are $\frac{22}{10}$, $\frac{22}{15}$, $\frac{22}{6}$.

But if $25aa - 9 + \frac{22}{aa}$ be equated to $\frac{22}{aa}$, then the value of a will be $\frac{2}{3}$, yet the
 same Solution will be found as before, because here we must conceive $\frac{2}{a} - 3a$ and $\frac{2}{a} - 4a$
 to be the sides of the two Squares, which sides being resolved according to the latter value
 of a , to wit, $a = \frac{2}{3}$, there will come forth (as before) $\frac{22}{15}$ and $\frac{1}{10}$.

This Question is capable of innumerable Answers upon a double ground; for first, the
 first square may be any known square number at pleasure; then the sides of the second and
 third Squares may be variously feigned from divers numbers, which may be the sides about
 the right-angle of unlike right-angled Triangles; as instead of 3 and 4 we may take
 8 and 15, 5 and 12, and innumerable others.

QUEST. 88. (Quest. 32. Lib. 5. Diophant.)

To find three Squares that the summ of their Squares may make a Square; or, (which
 is the same thing) to find three numbers that the summ of their squared Squares may be
 a Square.

RESOLUTION.

1. For one of the square numbers sought put $\triangleright aa$
 2. And for the two others put $\triangleright bb$ and cc
 3. Then the summ of the Squares of those three Squares is $\triangleright aaaa + bbbb + cccc$
 4. Which summ must be equated to a Square, let its side be $aa - dd$, the Square whereof
 equated to the said summ gives
 $aaaa + bbbb + cccc = aaaa - 2ddaa + dddd.$
 5. Which Equation, after due Reduction, gives . . . $\triangleright aa = \frac{dddd - bbbb - cccc}{2dd}$
 6. In which last Equation, because the Numerator and Denominator are not perfect Squares,
 the value of a is not expressible by a rational number; but to cause it to be rational,
 we may discover by the said Fraction that a certain number and two Squares must be
 found out, such, that if from the Square of the number the summ of the two Squares
 be subtracted, the remainder may be to the double of the said number as a square num-
 ber to a square number: Now to find out such a number and two Squares, let $rr + ss$
 be put for the number, and rr for one Square, and ss for the other; then from the Square
 of $rr + ss$, that is, from $rrrr + 2rrss + ssss$ subtract the summ of the Squares of
 rr and ss , that is, $rrrr + ssss$, and the remainder $2rrss$ must be to the double of the
 number $rr + ss$, that is, to $2rr + 2ss$, as a Square to a Square; therefore also
 the

the halves of $2rrss$ and $2rr + 2ss$, that is, $rrss$ and $rr + ss$ shall be as a Square to a Square, for the Proportion is not changed: And since $rrss$ is a Square, whose side is rs , it remains only to make $rr + ss$ a Square; but such it will be if r and s represent the sides about the right-angle of a right-angled Triangle whose three sides are expressible by rational numbers, for then $rr + ss$ will be the Square of the Hypotenusal. Therefore from the premisses the following Canon is deducible to solve the Question proposed.

CANON.

Take for two of the three Squares sought the Squares of the sides about the right-angle of some right-angled Triangle in rational numbers; then divide the Product made by the mutual multiplication of those two Squares, by the Square of the Hypotenusal, and there will come forth the third Square sought.

As, for example, let there be exposed the right-angled Triangle 3, 4, 5, then two of the Squares sought shall be 9 and 16, (to wit, the Squares of 3 and 4 the sides about the right-angle;) and if the Product of 9 into 16, that is, 144, be divided by 25 (the Square of the Hypotenusal,) the Quotient $\frac{144}{25}$ shall be the third Square sought. I say 9, 16 and $\frac{144}{25}$ are three Squares, which will solve the Question, for the summ of their Squares makes $\frac{23125}{625}$, which is a Square, whose side is $\frac{481}{25}$.

QUEST. 89. (Quest. 33. Lib. 5. Diophant.)

A certain Vintner made a mixture of two sorts of Wines, whereof one cost eight pence the quart, and the other five pence; at which prices the whole mixture was worth a square number of pence, unto which 60 being added the summ would also make a square number, whose side was the number of quarts contained in the mixture. The Question is, to find the number of quarts of each sort of Wine in the mixture.

RESOLUTION.

1. For the price of the whole mixture put $\dots \dots \dots > aa - 60$
2. Then if 60 be added to that total price the summ will be the Square aa , whose Root (as the Question requires) must be equal to the number of quarts of both sorts of Wine in the mixture, to wit, $\dots \dots \dots \left. \begin{matrix} \\ \\ \end{matrix} \right\} a$
3. From the premisses we may rightly infer, that $aa - 60$ (the total cost of the mixture) is greater than $5a$, but less than $8a$; (that is, greater than the Product of the multiplication of the total number of mixed quarts by the price of the worser sort of Wine in the mixture, but less than the Product of the same number of quarts multiplied into the dearer sort of Wine:) But if $aa - 60$ be greater than $5a$, and less than $8a$, then the value of a (by Quest 10. of this Book) is greater than $10\frac{63}{100000000}$, &c. but less than $12\frac{11222}{100000000}$, &c. Therefore $aa - 60$ (which the Question requires to be a Square) must be equated to some Square whose side must be so feigned that the value of a may be within those limits: Now to cause that effect, the side of the said Square may be feigned $-a +$ any absolute number between $17\frac{231}{10000}$, &c. and $22\frac{804}{10000}$, &c. or $a -$ any absolute number between $2\frac{61}{10000}$, &c. and $3\frac{142}{10000}$, &c. (as hath been shewn in Quest. 11. of this Book.) Let then the said side be feigned $-a + 22$, whole Square $aa - 44a + 484$ equated to $aa - 60$ will give $a = 12\frac{4}{11}$ for the desired number of quarts in the whole mixture.
4. Then from $12\frac{4}{11}$, the Square of the said $12\frac{4}{11}$, subtract 60, and the remainder $\frac{11236}{121}$ is the price of the whole mixture, which is a Square whose side is $\frac{106}{11}$; and because $\frac{11236}{121}$ is the number of pence expressing the value of the mixture, it must be equal to the Product of 8 multiplied by a certain part of $12\frac{4}{11}$, (the number of quarts in the mixture,) together with the Product of 5 multiplied by the remaining part of $12\frac{4}{11}$; we must therefore divide $12\frac{4}{11}$ into two such parts, that if the one be multiplied by 5, and the other by 8, the summ of the two Products may make $\frac{11236}{121}$; but that may be done thus,
5. For one of the desired parts of $12\frac{4}{11}$ put $\dots \dots \dots > \dots \dots e$
6. Then the other shall be $\dots \dots \dots > 12\frac{4}{11} - e$
7. And if the former part be multiplied by 5, and the latter by 8, $\left. \begin{matrix} \\ \\ \end{matrix} \right\} 98\frac{18}{11} - 3e$
the summ of the Products will be $\dots \dots \dots \left. \begin{matrix} \\ \\ \end{matrix} \right\}$
8. Which summ must be equated to $\frac{11236}{121}$, viz. $\dots \dots \dots > 98\frac{18}{11} - 3e = \frac{11236}{121}$
9. Which

Q2

9. Which Equation, after due Reduction, makes known one of the desired parts, to wit, $e = \frac{244}{121}$
10. Which subtracted from $12\frac{4}{11}$, leaves the other part, to wit, $\frac{1222}{121}$
11. I say the total mixture of Wine might be compos'd of $\frac{244}{121}$ quarts of five pence the quart, and $\frac{1222}{121}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $12\frac{4}{11}$ quarts is $\frac{11222}{121}$ pence, which is a square number whose side is $\frac{106}{11}$; and if to the said Square $\frac{11222}{121}$ you add 60, the sum is also a Square, to wit, $\frac{12422}{121}$, whose side $12\frac{4}{11}$ is the number of quarts in the mixture.
12. But because the side of the Square to be equated to $aa - 60$ may be feigned $a - 3$, any absolute number between $2\frac{63}{1000}$, &c. and $3\frac{34}{1000}$, &c. let the said side be $a - 3$, the Square whereof equated to $aa - 60$ will give $a = 11\frac{1}{2}$ for the number of quarts in the mixture; then the Square of $11\frac{1}{2}$ is $\frac{222}{4}$, from which subtracting 60, the remainder $\frac{222}{4}$ is the square number of pence expressing the value of the mixture. Now the said $11\frac{1}{2}$ is to be divided into two such numbers that if one of them be multiplied by 5, and the other by 8, the sum of the Products may make the Square $\frac{222}{4}$; but two such numbers (by working as before) will be found $\frac{72}{12}$ and $\frac{22}{12}$.

I say again, the mixture may be compos'd of $\frac{72}{12}$ quarts of five pence the quart, and $\frac{22}{12}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $11\frac{1}{2}$ quarts, is $\frac{222}{4}$ pence, which is a Square, to which if you add 60 the sum is also a Square, to wit, $\frac{222}{4}$, whose side $11\frac{1}{2}$ is the number of quarts in the whole mixture. From the premisses 'tis evident that the Question is capable of innumerable Answers in rational numbers.

QUEST. 90.

To find a right-angled Triangle in rational numbers, that one of the sides about the right-angle may be to the Area in a given Reason, suppose as r to s .

RESOLUTION.

1. For the Triangle sought let a right-angled Triangle be formed from two numbers, viz. a the greater and e the lesser, so the three sides will be these, to wit, $aa + ee; aa - ee; 2ae$
2. The Area of that Triangle is $aae - aee$
3. Then (according to the Question) let these four quantities be supposed to be Proportionals, viz. $r . s :: aa - ee . aae - aee$.
4. And because if the two latter terms of that Analogy be severally divided by $aa - ee$, the Quotients are 1 and ae , therefore that Analogy may be reduced to this, viz. $r . s :: 1 . ae$.
5. And by comparing the Product of the extremes to the Product of the means, this Equation ariseth, viz. $rae = s$
6. And by dividing each part of the last Equation by ra , this ariseth $e = \frac{s}{ra}$
Hence this

CANON.

7. Take any number at pleasure, which may be called a , then divide s the latter term of the given Reason, by the Product of the first term r multiplied into the number a , and call the Quotient the number e ; lastly, from the said numbers a and e form a right-angled Triangle, and it shall be that which is sought.

An Example in Numbers.

Let it be required to find a right-angled Triangle, such, that one of the sides about the right-angle may be to the Area as 1 to 10.

Suppose, $\left. \begin{matrix} r = 1 \\ s = 10 \end{matrix} \right\}$ the terms of the given Reason;

Then by the Canon, $\left. \begin{matrix} a = 2 \\ e = 5 = \frac{s}{ra} \end{matrix} \right\}$ taken at pleasure,

Lastly, from 5 and 2 form a right-angled Triangle, and the three sides will be 29, 21 and 20, which Triangle will solve the Question; for 21, one of the sides about the right-angle, is to the Area 210, as 1 to 10. Which was required.

Likewise

Likewise by the Canon, this right-angled Triangle, to wit, 101, 99 and 20 will be found to solve the Question, for 99 is to the Area 990, as 1 to 10; and innumerable right-angled Triangles in Fractions may be found to perform the same effect.

QUEST. 91.

To find a right-angled Triangle in rational numbers, that the Hypothenufal may be to the Area in a given Reason; suppose ae r to s .

RESOLUTION.

1. For one of the sides about the right-angle put a
2. And for the other side about the right-angle put e
3. Then the square Root of the sum of the Squares of those sides shall be the Hypothenufal, to wit, $\sqrt{aa + ee}$
4. And the Area of the said Triangle is $\frac{1}{2}ae$
5. Now according to the Question; the Hypothenufal must be to the Area as r to s , therefore from the third and fourth steps this Analogy ariseth, viz.

$$r \cdot s :: \sqrt{aa + ee} : \frac{1}{2}ae.$$

6. But the Squares of those Proportionals are also Proportionals, therefore

$$rr \cdot ss :: aa + ee \cdot \frac{1}{4}aee.$$

7. And from the last Analogy, by comparing the Product of the multiplication of the extremes to the Product of the means, this Equation ariseth, viz.

$$\frac{1}{4}rraee = ssa + ssee.$$

8. From which Equation; by transposition of $ssee$, this ariseth,

$$\frac{1}{4}rraee - ssee = ssa.$$

9. And by dividing each part of the last Equation by $\frac{1}{4}rraa - ss$, there will arise $ee = \frac{ssaa}{\frac{1}{4}rraa - ss}$

10. In which last Equation the Numerator $ssaa$ is a Square whose side is sa , and if the Denominator were a Square, then the whole Fraction would be also a Square, and consequently the side thereof, to wit, the number e would be rational; it remains therefore to equate the Denominator $\frac{1}{4}rraa - ss$ to a Square, to which end, let the side thereof be feigned $\frac{1}{2}ra - b$; then the Square of $\frac{1}{2}ra - b$ being equated to $\frac{1}{4}rraa - ss$, this Equation ariseth, viz.

$$\frac{1}{4}rraa - ss = \frac{1}{4}rraa - rba + bb.$$

11. Whence, after due Reduction, you will find $a = \frac{ss + bb}{rb}$

12. Now if we suppose r , s and b to represent known rational numbers, then a , e and $\sqrt{aa + ee}$: which in the three first steps were put for the three sides of the right-angled Triangle sought, will also (from the eleventh, ninth and tenth steps,) be expressible by rational numbers, to wit, these,

$$\frac{ss + bb}{rb}, \quad \frac{2sss + 2sbb}{rss - rbb}, \quad \frac{ssss + 2ssbb + bbbb}{rssb - rbbb}.$$

13. Or the two first of the three sides last exprest may be reduced to the same Denominator with the third, and then the three sides of the right-angled Triangle sought will be these, to wit,

$$\frac{ssss - bbbb}{rssb - rbbb}, \quad \frac{2ssb + 2sbbb}{rssb - rbbb}, \quad \frac{ssss + 2ssbb + bbbb}{rssb - rbbb}.$$

Which three sides, if they be exprest by words, will give this.

CANON.

14. Take for b any number less than s the latter term of the given Reason; then from the numbers s and b form a right-angled Triangle, and multiply the three sides severally by the Hypothenufal; lastly, divide those three Products severally by the Product made by the multiplication of the difference of the Squares of the two numbers s and b , (which formed the said Triangle,) into the Product of b the lesser of the same two numbers and r the first term of the given Reason; so shall the Quotients be the three sides of a right-angled Triangle, which will solve the Question proposed.

An

An Example in Numbers.

Let it be required to find out a right-angled Triangle whose Hypothenuſal may be to the Area as 2 to 3.

Suppoſe $\left. \begin{matrix} r = 2 \\ s = 3 \\ b = 1 \end{matrix} \right\}$ the Terms of the given Reaſon, leſs than 3, (or s .)

Then form a right-angled Triangle from 3 and 1, (to wit, s and b ,) and the three ſides will be 10, 8 and 6; theſe multiplied ſeverally by the Hypothenuſal 10 will produce 100, 80 and 60, which divided ſeverally by the Product which answers to $ss - bb$ into rb , that is, by 16, will give $\frac{25}{4}$, $\frac{20}{4}$ and $\frac{15}{4}$ for the Triangle ſought; for the Hypothenuſal $\frac{25}{4}$, or $\frac{12.5}{1}$, is to the Area $\frac{15.20}{2}$, as 2 to 3. Which was required.

In like manner, if it were deſired to find a right-angled Triangle whose Hypothenuſal might be to the Area as 3 to 2; then by the Canon, the three ſides will be found $\frac{25}{9}$, $\frac{20}{9}$ and $\frac{15}{9}$.

QUEST. 92.

To find a right-angled Triangle in rational numbers, that the ſumm of all the three ſides may be to the Area in a given Reaſon, ſuppoſe as r to s .

RESOLUTION.

1. For the right-angled Triangle ſought let a Triangle be formed from any two numbers, ſuppoſe from a the greater and e the leſſer, ſo the three ſides will be theſe, $\left. \begin{matrix} aa + ee, aa - ee, 2ae \end{matrix} \right\}$
2. Then the Area will be $\left. \begin{matrix} aae - eee \end{matrix} \right\}$
3. And the ſumm of the three ſides is $\left. \begin{matrix} 2aa + 2ae \end{matrix} \right\}$
4. Now (according to the Queſtion) the ſumm of the three ſides muſt be to the Area, as r to s , therefore,

As $r . s :: 2aa + 2ae : aae - eee.$

5. Or, by dividing each of the two latter terms of that Analogy by a , this arifeth, viz. $\left. \begin{matrix} As r . s :: 2a + 2e . aae - eee. \end{matrix} \right\}$

6. Whence, by comparing the Product of the extremes to the Product of the means, this Equation arifeth, viz. $\left. \begin{matrix} rea - ree = 2sa + 2se \end{matrix} \right\}$

7. Therefore by due tranſpoſition, $\left. \begin{matrix} rea - 2sa = ree + 2se \end{matrix} \right\}$

8. And by dividing all in the laſt Equation by re , this arifeth, $\left. \begin{matrix} aa - \frac{2s}{r}a = \frac{ree + 2s}{r} \end{matrix} \right\}$

9. Which laſt Equation being reſolved by the Canon in Sect. 8. Chap. 15. Book 1. the value of a will be diſcovered, viz. $\left. \begin{matrix} . . . a = e + \frac{2s}{re} \end{matrix} \right\}$

Hence this

CANON.

10. Take any number at pleaſure, which may be called e ; then to the number e add the Quotient that arifeth by dividing the double of the latter term of the given Reaſon, by the Product of the firſt term multiplied into the number e , and call the ſumm the number a ; laſtly, from the ſaid numbers a and e form a right-angled Triangle, and it ſhall be that which is ſought.

An Example in Numbers.

Let it be required to find out a right-angled Triangle, that the ſumm of all the three ſides may be to the Area as 1 to 5.

Suppoſe $\left. \begin{matrix} r = 1 \\ s = 5 \end{matrix} \right\}$ the Terms of the given Reaſon, then

By the Canon, $\left. \begin{matrix} e = 2 \\ a = 7 = e + \frac{2s}{re} \end{matrix} \right\}$ taken at pleaſure,

Then form a right-angled Triangle from 7 and 2, and the three ſides will be 53, 45, 28, which will ſolve the Queſtion, for the ſumm of all the three ſides, to wit, 126, is to the Area 630, as 1 to 5. Which was required to be done.

Likewiſe if a right-angled Triangle be formed from 11 and 1, (to wit, a and e , found out by the Canon,) the three ſides will be 122, 120, 22, whose ſumm 264 is to the Area 1320, as 1 to 5.

Again,

Again, if a right-angled Triangle be formed from 11 and 10, the three sides will be 221, 21 and 220, whose sum 462 is to the Area 2310, as 1 to 5.

Lastly; you may find out as many right-angled Triangles in Fractions as you please to solve the Question. Also, upon the same ground it will not be difficult to find out innumerable Isosceles-Triangles, in every one of which the Perimeter shall be to the Area in a given Reason.

QUEST. 93.

To find a right-angled Triangle in rational numbers, that as well the Hypothenufal as the difference of the sides about the right-angle may be a Square.

RESOLUTION.

1. For the Triangle sought let a right-angled Triangle be formed from two numbers, suppose from a and e , and let a be the greater; so the three sides will be these, viz. } $aa + ee, aa - ee, 2ae$
2. Now (according to the Question) as well the Hypothenufal as the difference of the sides about the right-angle must be a Square, so we are fallen upon this Duplicate equality; viz. } $aa + ee = \square$
 } $2ae - aa + ee = \square$
3. The difference of those two quantities is } $2aa - 2ae$
4. Which difference is equal to the Product of these two quantities, to wit, } $2a - 2e$ and a
5. The half-sum of those two Factors in the last step is } $\frac{1}{2}a - e$
6. Then the Square of the said half-sum being equated to the greater of the two quantities in the second step, this Equation ariseth, viz. } $aa + ee = \frac{1}{4}aa - 3ae + ee$
7. Whence after due Reduction there will arise } $12e = 5a$
8. And by reducing the last Equation into Proportionals, it shall be

As 12 . . 5 :: a . . e.

Hence this

CANON.

9. If from 12 and 5, or any two numbers in that proportion, a right-angled Triangle be formed, it will solve the Question.

As, for example, in the right-angled Triangle 169, 119 and 120, which is formed from 12 and 5, the Hypothenufal 169 is a Square; also the difference of the sides about the right-angle; to wit, 1 is a Square. The same effect will be produced in a right-angled Triangle formed from any two numbers which have such proportion one to another as 12 to 5.

QUEST. 94.

To find a right-angled Triangle; that one of the sides about the right-angle may be a Square, which added to a given multiple, suppose the triple, of the Square of the difference of the sides about the right-angle may make a Square.

RESOLUTION.

1. For one of the sides about the right-angle, that it may be a Square, put } 4
2. And for the other put } a
3. The sum of their Squares must make a Square, to wit, the Square of the Hypothenufal; therefore } $aa + 16 = \square$
4. The difference of the sides about the right-angle is . . . } $a \approx 4$
5. The triple of the Square of that difference is } $3aa - 24a + 48$
6. To which (according to the Question) add 4, the square number first assumed for one of the sides about the right-angle, and the sum must be equal to a Square, viz. . . . } $3aa - 24a + 52 = \square$
7. So in the third and sixth steps we are fallen upon a Duplicate equality; but the numbers prefixt to aa in the quantities to be equated, are not Squares; neither are the two known numbers in the same quantities both Squares; for 52 is not a Square; whereby the said Duplicate equality is inexplicable; but if the said 52 were a Square, then the Duplicate equality

equality might be resolved; therefore instead of the square number 4 which in the first step was assumed for one of the sides about the right-angle, we must seek such a Square that if it be added to the triple of its Square the sum may be a Square. Suppose therefore that Square sought to be ee , this added to the triple of its Square makes $3eeee + ee$ to be equated to a Square, the side whereof may be variously feigned, let it be $ee + e$, then the Square of $ee + e$, to wit, $eeee + 2eee + ee$ being equated to $3eeee + ee$, after due Reduction the value of e will be found 1, and ee is also 1. So we have found a Square, to wit, 1, which added to the triple of its Square makes the Square 4; therefore now the Resolution may be renewed thus, viz.

8. For one of the sides about the right-angle put the Square . . . } 1
9. And for the other side } a
10. The summ of their Squares must be equal to a Square, viz. } $aa + 1 = \square$
11. The difference of the sides about the right-angle is . . . } $a \sim 1$
12. The triple of the Square of that difference is } $3aa - 6a + 3$
13. To which adding the side 1 in the eighth step, the summ }
must be equal to a Square, viz. } $3aa - 6a + 4 = \square$
14. Also from the tenth step, } . . . $aa + 1 = \square$
15. So in the two last steps we have a new Duplicate equality }
which may be resolved thus; first, to the end there may be }
one and the same known square number in each of the two }
quantities to be equated to Squares, I multiply the quantity in }
the fourteenth step, to wit, $aa + 1$ by 4, and it makes $4aa + 4$; }
now each of these quantities is to be equated to a Square, viz. }
} $3aa - 6a + 4 = \square$
} $4aa + 4 = \square$
16. The difference of those two quantities is } $aa + 6a$
17. Which difference is equal to the Product of these two Fa- }
ctors, to wit, } $\frac{2}{3}a + 4$, and $\frac{1}{2}a$
18. Half the summ of those Factors is } $\frac{1}{12}a + 2$
19. The Square of the said half-summ is } $\frac{1}{144}aa + \frac{1}{3}a + 4$
20. Which Square equated to $4aa + 4$, (the greater of the two quantities in the fifteenth }
step,) will after due Reduction give $a = \frac{624}{407}$.
21. Therefore from the twentieth, ninth and eighth steps the sides about the right-angle }
are $\frac{624}{407}$ and 1, the summ of whose Squares is $\frac{1}{163649}$, whose square Root $\frac{121}{407}$ is }
the Hypothensal sought.

I say $\frac{624}{407}$, 1 and $\frac{121}{407}$ are the sides of a right-angled Triangle, which will solve the Question; for one of the sides about the right-angle is a Square, to wit, 1, and if this be added to the triple of the the Square of the difference of the sides about the right-angle, it makes the Square $\frac{1}{163649}$, whose side is $\frac{121}{407}$. From the premisses it is evident that innumerable right-angled Triangles may be found to solve the Question.

QUEST. 95.

To find out a right-angled Triangle in rational numbers, that the Square of one of the sides about the right-angle may be equal to the other of the same sides.

[This is Problem. 15. in pag. 80. of the Introduction to Algebra before cited in Quest. 58. but I shall resolve it after another manner.]

RESOLUTION.

1. For one of the sides about the right-angle put ra , (r repre- }
sents some known number, and a some number unknown,) } ra
2. Then (according to the Question) the Square of that side must }
be the other of the sides about the right-angle, to wit, . . . } r^2aa
3. The summ of the Squares of those sides is } $rrrraaaa + r^2aa$
4. Which summ must be equal to the Square of the Hypothensal, and therefore it remains }
to equate the said $rrrraaaa + r^2aa$ to a Square, to which end take some known number }
, greater than r , and then the side of the said Square may be feigned $sa - r^2aa$, the }
Square whereof being equated to $rrrraaaa + r^2aa$, and due Reduction made, you }
will find

$$a = \frac{ss - rr}{2sr}$$

5. But

5. But s and r were assumed to represent two known numbers whereof s is the greater, therefore from the premisses the three sides of the Triangles sought shall be known also, and may be expressed thus,

$$\left. \begin{array}{l} \frac{ss - rr}{2sr}, \text{ or } \frac{2ssr - 2srr}{4ssr} \\ \frac{ssss - 2ssr + rrr}{4ssr} \\ \frac{ssss - rrr}{4ssr} \end{array} \right\} \begin{array}{l} \text{The sides about the right-angle.} \\ \\ \text{The Hypotenusal.} \end{array}$$

6. Moreover, because the Square of $\frac{2sr}{ss - rr}$ is $\frac{4ssr}{ssss - 2ssr + rrr}$; and the sum of the Squares of the two last Fractions is equal to the Square of $\frac{2ssr + 2srr}{ssss - 2ssr + rrr}$, (as will easily appear by Multiplication and Addition;) therefore the three quantities last expressed shall also be the sides of a right-angled Triangle to solve the Question proposed, viz.

$$\left. \begin{array}{l} \frac{2sr}{ss - rr}, \text{ or } \frac{2ssr - 2srr}{ssss - 2ssr + rrr} \\ \frac{4ssr}{ssss - 2ssr + rrr} \\ \frac{2ssr + 2srr}{ssss - 2ssr + rrr} \end{array} \right\} \begin{array}{l} \text{The sides about the right-angle.} \\ \\ \text{The Hypotenusal.} \end{array}$$

7. If the three sides of the Triangle sought, as they be above expressed in the fifth and sixth steps be compared together, it will be easie to deduce from thence this following

CANON.

First, form a right-angled Triangle from any two unequal numbers; then multiply the three sides of that Triangle severally by either of the sides about the right-angle; lastly, divide severally those three Products by the Square of the other of the sides about the right-angle; so the three Quotients shall be the sides of the right-angled Triangle sought.

Examples in Numbers.

First, find out three numbers to express the sides of a right-angled Triangle, suppose these, } 3, 4, 5
 Then multiply those three sides severally by 4, the greater of the sides about the right-angle, and the three Products will be these, viz. } 12, 16, 20
 Lastly, divide those three Products severally by 9, the Square of the lesser of the sides about the right-angle; so the Quotients will be the sides of a right-angled Triangle to solve the Question proposed, to wit, these, } $\frac{4}{3}, \frac{16}{9}, \frac{20}{9}$

Again,

Let the right-angled Triangle first found out be here repeated, } 4, 3, 5
 to wit, }
 Then multiply those three sides severally by 3, the lesser of the sides about the right-angle, and the Products will be these, } 12, 9, 15
 Lastly, divide those three Products severally by 16, the Square of the greater of the sides about the right-angle; so the Quotients will give these three sides of a right-angled Triangle to solve the Question, to wit, } $\frac{3}{4}, \frac{9}{16}, \frac{15}{16}$

QUEST. 96. (Quest. 1. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the sides about the right-angle being severally subtracted from the Hypotenusal may leave cube-numbers.

RESOLUTION.

1. First, form a right-angled Triangle from two unequal numbers, suppose from a and e , and let a be the greater, } Hypoth. Base, Perp.
 so the three sides will be these, viz. } $aa + ee, aa - ee, 2ae$
 R } 2. Then

2. Then by subtracting the Base from the Hypothensal, the remainder is $2ee$, which should be a Cube, but it is not; yet it shews that e (to wit, one of the numbers from which the desired right-angled Triangle is to be formed,) must be such that the double of its Square may make a Cube. Now that we may chuse the number e with that condition, let $2ee$ be equated to the Cube $bbbee$, viz. suppose $2ee = bbbee$, (where bbb may represent any known cube-number,) whence after due Reduction the value of e will be made known, viz.

$$e = \frac{2}{bbb}$$

3. It remains, that the Perpendicular subtracted from the Hypothensal may leave a Cube; but the remainder is the Square $aa + ee - 2ae$, which is not a Cube, but if its Root $a - e$ were a Cube, then that Square would be a Cube; (for a Cube multiplied into a Cube produceth a Cube, by Prop. 3, & 4. Elem. 9. Euclid.) Let therefore the said Root $a - e$ be equated to some Cube, viz. suppose $a - e = ddd$; hence, and from the third step it follows, that

$$a = e + ddd = \frac{2}{bbb} + ddd$$

Now, if from $\frac{2}{bbb} + ddd$ and $\frac{2}{bbb}$ (the values of a and e .) a right-angled Triangle be formed, it will solve the Question. Hence this

C A N O N.

4. Divide 2 by any cube-number, and reserve the Quotient; then to the said Quotient add any cube-number; lastly, from the sum and Quotient form a right-angled Triangle and it shall solve the Question proposed.

As, for example, I divide 2 by the Cube 8, and reserve the Quotient $\frac{1}{4}$ for one of the numbers by which the right-angled Triangle is to be formed; then to the Quotient $\frac{1}{4}$ I add some Cube, as 1, and the sum is $\frac{5}{4}$; lastly, from $\frac{5}{4}$ and $\frac{1}{4}$ I form a right-angled Triangle and find the sides $\frac{13}{8}$, $\frac{12}{8}$, and $\frac{5}{8}$, which will solve the Question; for the sides about the right-angle, to wit, $\frac{12}{8}$ and $\frac{5}{8}$ being severally subtracted from the Hypothensal $\frac{13}{8}$, the remainders are the Cubes $\frac{1}{8}$ and 1.

But after one right-angled Triangle is found out to solve the Question, if you multiply or divide every one of the sides thereof by one and the same cube-number, the Products or Quotients will give another right-angled Triangle to solve the Question: As, if the three sides before found out, to wit, $\frac{13}{8}$, $\frac{12}{8}$ and $\frac{5}{8}$ be severally multiplied by the Denominator 8, they will produce 13, 12 and 5, which will solve the Question. Likewise, if 13, 12 and 5 be severally multiplied by 8, there will be produced the right-angled Triangle 104, 96 and 40, where the differences between the Hypothensal and the other two sides are Cubes, to wit, 8 and 64. The reason is evident; for,

First, by Construction $\frac{13}{8}$, $\frac{12}{8}$, $\frac{5}{8}$ are the sides of a right-angled Triangle that will solve the Question, viz. . . .

And because a Cube multiplied by a Cube produceth a Cube, those two differences or Cubes multiplied by the Cube 8 shall necessarily produce these differences or Cubes, viz. . . .

Wherefore the right-angled Triangle 13, 12, 5 shall necessarily solve the Question, as well as $\frac{13}{8}$, $\frac{12}{8}$, $\frac{5}{8}$.

QUEST. 97. (This is a Lemma, used by Dioph. in resolving the following Quest. 98.)

To find a right-angled Triangle and a Square in rational numbers, that if the Area of the Triangle be subtracted from the Square, the remainder multiplied by a given number (d) may produce a square number.

RESOLUTION.

- | | | |
|--------------------------------------------------------------------------------------------|---|------------------------------------|
| 1. Let a right-angled Triangle be formed from a and $\frac{1}{a}$; | } | $aa + \frac{1}{aa} =$ the Hypoth. |
| so the three sides will be | | $aa - \frac{1}{aa} =$ the Perpend. |
| | | $2 =$ the Base. |
| 2. Therefore the Area, (by multiplying the Perpendicular into half the Base,) is | } | $aa - \frac{1}{aa}$ |

3. Then

3. Then feign the side of the Square sought to be $\rightarrow a + \frac{2d}{a}$
4. Therefore the Square it self is $\rightarrow aa + 4d + \frac{4dd}{aa}$
5. From which Square, the Area above express'd being subtracted, }
there will remain $\rightarrow 4d + \frac{4dd + 1}{aa}$
6. That is, $\rightarrow \frac{4daa + 4dd + 1}{aa}$
7. Then the said remainder being multiplied into the given num- }
ber (d) produceth $\rightarrow \frac{4ddaa + 4ddd + d}{aa}$
8. Which Product must (according to the Question) be a Square : But the Denominator aa is a Square, it remains therefore to equate the Numerator to a Square, viz. $4ddaa + 4ddd + d$ must be equated to a Square, the side whereof may be variously feigned, let it be $2da + d$; and then the Square of $2da + d$ being equated to the said $4ddaa + 4ddd + d$, this Equation ariseth, to wit,
 $4ddaa + 4dda + dd = 4ddaa + 4ddd + d.$
9. Which Equation after due Reduction gives $\rightarrow a = \frac{4dd + 1 - d}{4d}$

From the ninth, first and third steps ariseth this

CANON.

10. From $\frac{4dd + 1 - d}{4d}$ and $\frac{4d}{4dd + 1 - d}$ form a right-angled Triangle, which shall be that sought by the Question; and the side of the Square sought shall be $\frac{4dd + 1 - d}{4d} + \frac{8dd}{4dd + 1 - d}$.

An Example in Numbers.

Suppose $5 = d$ the number given; then form a right-angled Triangle from $\frac{4}{5}$ and $\frac{2}{5}$, so the Hypothensal will be $\frac{11}{5}$, the Perpendicular $\frac{3}{5}$, and the Base 2 ; moreover, the side of the Square sought will be $\frac{11}{5}$, and the Square it self $\frac{121}{25}$, or (in the same Denominator with the said Hypothensal and Perpendicular) $\frac{529}{625}$: which Square and Triangle will solve the Question; for if the Area of the Triangle, to wit, $\frac{11}{10}$ be subtracted from the said Square $\frac{529}{625}$, the remainder $\frac{419}{625}$, that is, $\frac{11 \cdot 38}{5^2 \cdot 5}$ multiplied into the given number 5 , produceth the Square $\frac{1532}{125}$, whose Root is $\frac{39}{5}$.

Another Example.

Suppose $3 = d$ the number given in the Question; then let a right-angled Triangle be formed from $\frac{1}{3}$ and $\frac{2}{3}$, (which numbers are discovered by the preceding Canon,) so the Hypothensal will be $\frac{5}{3}$, the Perpendicular $\frac{4}{3}$, and the Base 2 ; moreover, the side of the Square sought will be found $\frac{5}{3}$, and the Square it self $\frac{25}{9}$; from which Square subtracting the Area of the Triangle, to wit, $\frac{4}{3}$, the remainder $\frac{17}{9}$, or in its least terms $\frac{17}{9}$, multiplied into the given number 3 , produceth the Square $\frac{17}{3}$, whose side is $\frac{\sqrt{51}}{3}$.

Note. Instead of 2 which is prefixt to d in the Numerator of the Fraction $\frac{2d}{a}$ in the third step of the preceding Resolution, you may take the half of any square number and prefix it to d for a Numerator, over the Denominator a ; as, $4\frac{1}{2}d$, $8d$, $12\frac{1}{2}d$, &c. and then by prosecuting the work as before from the third step to the end of the Resolution, various Answers to the Question from one and the same given number will be discovered.

QUEST. 98. (Quest. 3. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the Area thereof increased with a given number, suppose 5 , may make a Square.

RESOLUTION.

1. Let the sides of some known right-angled Triangle, as $5, 4$ and 3 }
be severally multiplied by a , and take the Products to represent } $5a, 4a, 3a$
the sides of the Triangle sought, to wit, }

R 2

2. Then

2. Then the Area thereof increased with 5 (the number given in the Question) makes $6aa + 5$
3. Which summ must be equal to a Square, suppose it be $9aa$, therefore $6aa + 5 = 9aa$
4. Whence by equal subtraction of $6aa$, there remains $5 = 3aa$
5. Let each part of the last Equation be multiplied by 5, and it produceth $25 = 15aa$
6. Now if 15 which is prefix'd to aa were a Square, then the value of a would be rational: Whence therefore comes 15? Examine the work, and you will find, that by subtracting 6 the Area of the Triangle 5, 4, 3 from the square number 9, and then by multiplying the remainder 3 into the given number 5, there is produced 15; whereby it is manifest that the scope of our search must be to find a right-angled Triangle and a square number, that the Area of the Triangle being subtracted from the Square and the remainder multiplied by the given number 5 may make a Square. But the preceding 97th Question shews how to find out such a Triangle and Square; take if you please those there found in the first Example, to wit, the right-angled Triangle whose Hypotenusal is $\frac{332401}{14400}$, Perpendicular $\frac{331151}{14400}$, and Base 2; and the Square $\frac{682276}{14400}$, whose side is $\frac{826}{120}$: Then renew the Resolution thus,
7. For the three sides of the Triangle sought put $\frac{332401}{14400}a$, $\frac{331151}{14400}a$, $2a$
8. The Area thereof increased with 5 makes $\frac{331151}{14400}aa + 5$
9. Which summ must be equated to the Product of $\frac{682276}{14400}$ (the Square before found) multiplied into aa , so this Equation ariseth, $\frac{331151}{14400}aa + 5 = \frac{682276}{14400}aa$
10. Therefore by subtracting $\frac{331151}{14400}aa$ from each part of that Equation this remains, to wit, $5 = \frac{351125}{14400}aa = \frac{14045}{576}aa$
11. And by multiplying each part of the last Equation into 5 it produceth $25 = \frac{70225}{576}aa$
12. And by extracting the square Root out of each part of the last Equation, there will arise $5 = \frac{265}{24}a$
13. Whence by dividing each part by $\frac{265}{24}$, the value of a will be made known, viz. $a = \frac{24}{53}$
14. Wherefore from the thirteenth and seventh steps the three sides of the Triangle sought will be found these, to wit, $\frac{332401}{31800}$, $\frac{331151}{31800}$, $\frac{48}{53}$, the Area whereof is $\frac{331151}{70225}$, to which adding 5, the summ will be the Square $\frac{682276}{70225}$, whose Root is $\frac{826}{265}$. Therefore the Question is solved.

Vieta, in the 9th of the 5th Book of his *Zetetics*, shews how to find out a right-angled Triangle whose Area increased with a number compos'd of two Squares may make a Square; whereby 'tis probable he thought this Question to be applicable only to a number compos'd of two Squares; because *Diophantus* propos'd the given number 5, which is compos'd of two Squares; but 'tis evident from the precedent Resolution that the Question may be extended to any given number whatsoever: And for greater illustration, let it be required to find out a right-angled Triangle whose Area increased with 3 may make a Square.

First a right-angled Triangle is to be sought, and also a Square, that the Area of the right-angled Triangle being subtracted from the Square and the remainder multiplied by the given number 3, the Product may be a square number: But by the latter Example of the preceding 97th Question such a Triangle and Square are found out, viz. the Triangle whose Hypotenusal is $\frac{84817}{10404}$, the Perpendicular $\frac{82225}{10404}$, and the Base 2; and the Square $\frac{21025}{10404}$, whose side is $\frac{1025}{102}$: then for the three sides of the right-angled Triangle sought let there be put $\frac{84817}{10404}a$, $\frac{82225}{10404}a$ and $2a$, the Area whereof is $\frac{82225}{10404}aa$, to which adding 3 it makes $\frac{82225}{10404}aa + 3$; which summ equated to $\frac{21025}{10404}aa$, ($\frac{21025}{10404}$ being the Square above found,) will give $\frac{17}{40}$ for the value of a , by which $\frac{17}{40}$ the three sides above put being resolved, you will find the three sides of the right-angled Triangle sought to be these, to wit, $\frac{84817}{24480}$, $\frac{16445}{4896}$ and $\frac{17}{40}$; the Area of which Triangle is $\frac{3282}{2304}$, to which adding 3 it makes the Square $\frac{10201}{2304}$, whose side is $\frac{101}{48}$.

QUEST. 99. (Quest. 6. Lib. 6. Diophant.)

To find a right-angled Triangle, that the Area thereof increased with one of the sides about the right-angle may make a given number, suppose n .

RESOLUTION.

1. For the sides about the right-angle of the Triangle sought put a and e
2. Then is the Area $\frac{1}{2}ae$
3. Which increased with one of the sides about the right-angle must make the given number n , hence this Equation, $\frac{1}{2}ae + a = n$
4. From which Equation, after due Reduction, you will find $e = \frac{n-a}{\frac{1}{2}a}$
5. And because $aa + ee$ must be equal to the Square of the Hypotenusal, it follows from the first and fourth steps, that the Square of the Hypotenusal must be equal to the sum of the Squares of a and $\frac{n-a}{\frac{1}{2}a}$; that is,

$$\frac{\frac{1}{4}aaaa + aa - 2na + nn}{\frac{1}{4}aa}$$

6. Of which fractional quantity the Denominator $\frac{1}{4}aa$ is a Square, whose side is $\frac{1}{2}a$; if therefore the Numerator were a Square the whole Fraction would be a Square: It remains then to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to some Square; to which end, let its side be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the said Numerator, this Equation ariseth, viz.

$$\frac{1}{4}aaaa + aa - 2na + nn = \frac{1}{4}aaaa - naa + nn.$$

7. Whence the value of a will be made known, viz. $a = \frac{2n}{n+1}$
8. And according to that value of a , the value of e will also be made known by the fourth step, viz. $e = n-1$
9. And because the square Root of $aa + ee$ is equal to the Hypotenusal, therefore the square Root of the sum of the Squares of the two quantities in the latter parts of the Equations in the seventh and eighth steps, will give for the Hypotenusal sought $\frac{nn+1}{n+1}$
10. Therefore from the seventh, eighth and ninth steps the three sides of the right-angled Triangle sought are these, to wit,

$$\frac{2n}{n+1}, n-1, \left(\text{or } \frac{nn-1}{n+1} \right) \text{ and } \frac{nn+1}{n+1}.$$

Which three sides if they be express'd by words will give the following Canon, which is the same with that delivered by *Fermat* in his Observation upon the sixth Question of the sixth Book of *Diophantus*.

CANON.

11. When the given number is greater than unity let a right-angled Triangle be formed from those two numbers, and then divide the three sides severally by the sum of the given number and unity; so shall the Quotients be the three sides of the right-angled Triangle sought.

An Example in Numbers.

Let it be required to find a right-angled Triangle whose Area increased with one of the sides about the right-angle may make 7.

First (as the Canon directs) let a right-angled Triangle be formed from the given number 7 and 1, so the three sides will be these, 50, 48, 14

Then divide those three sides severally by 8, (to wit, $7+1$), so the Quotients shall be the sides of the right-angled Triangle sought, to wit, $6\frac{1}{4}$, 6, & $1\frac{1}{4}$

I say $6\frac{1}{4}$, 6, and $1\frac{1}{4}$ are the sides of a right-angled Triangle, which will solve the Question; for the Area $\frac{21}{4}$ increased with $\frac{1}{4}$ (one of the sides about the right-angle) makes $\frac{22}{4}$, that is, 7, as was required.

But because the said Canon takes not place unless the given number exceed unity, I shall in the next place explain *Diophantus's* Resolution of this Question, by which way, whatever the given number be, a right-angled Triangle may be found out to solve the Question proposed.

Another

Another way of resolving the foregoing 99th Question.

1. Let h, b, p represent the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle known in numbers, then multiply thoſe three ſides ſeverally by a , (which repreſents a number unknown,) and take the Products for the three ſides of the Triangle ſought, to wit, ha, ba, pa
2. The Area of which Triangle increaſed with one of the ſides about the right-angle, ſuppoſe with ba , muſt be equal to the given number n , therefore $\frac{1}{2}bpa + ba = n$
3. Which Equation divided by $\frac{1}{2}bp$ gives $aa + \frac{b}{\frac{1}{2}bp}a = \frac{n}{\frac{1}{2}bp}$
4. Now to the end that the value of a in the laſt Equation may be a rational number, the Square of half the Coefficient which is drawn into a , together with the abſolute quantity which poſſeſſeth the latter part of the Equation muſt make a Square, (as is evident by the Canon in *ſect. 6. Chap. 15. Book 1.*) therefore $\frac{1}{4}bb + \frac{1}{2}bpn = \square$
5. And becauſe the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only to equate the Numerator to a Square, viz. $\frac{1}{4}bb + \frac{1}{2}bpn = \square$
6. Or, to avoid Fractions, let the ſaid $\frac{1}{4}bb + \frac{1}{2}bpn$ be multiplied by 4, and then $bb + 2bpn = \square$
7. Which laſt Equation ſhews, that in order to the ſolving of the Queſtion propoſed, a right-angled Triangle muſt firſt be found, ſuch, that the Square of one of the ſides about the right-angle, together with the Product of the quadruple Area multiplied by the given number n , may make a Square. Now to find out ſuch a Triangle,
8. For one of the ſides about the right-angle put e
9. And for the other ſide put ſome ſquare number, as, 1
10. Then the quadruple of the Area is $2e$
11. Which multiplied by the given number n , ſuppoſe by $\frac{1}{2}$, makes e
12. To which Product add the Square of one of the ſides about the right-angle, to wit, the Square of 1, which is alſo 1, and the ſumm muſt be equal to a Square, viz. $e + 1 = \square$
13. Alſo the ſumm of the Squares of the ſides about the right-angle muſt be equal to a Square, to wit, the Square of the Hypothenuſal, therefore $ee + 1 = \square$
14. Now in order to reſolve the Duplicate equality in the two laſt ſteps, firſt the difference of the two quantities which are to be equated to Squares is $ee - e$
15. Which difference is equal to the Product of the multiplication of theſe two quantities, or Factors, to wit, $2e - 2$ and $\frac{1}{2}e$
16. The half-ſumm of thoſe two Factors is $\frac{5}{4}e - 1$
17. The Square of which half-ſumm being equated to $ee + 1$, will give $e = \frac{4e}{9}$
18. And becauſe e and 1 were put for the ſides about the right-angle, therefore the ſquare Root of the ſumm of the Squares of $\frac{4e}{9}$ (that is e) and 1, ſhall be the Hypothenuſal, to wit, $\frac{4e}{9}$; ſo we have found out a right-angled Triangle whoſe three ſides are $\frac{4e}{9}$, $\frac{4e}{9}$ and 1, which are fit for renewing the ſearch of the right-angled Triangle ſought by the Queſtion, in this manner, viz.
19. For the three ſides of the right-angled Triangle ſought put $\frac{4e}{9}a, \frac{4e}{9}a, \text{ and } a$
20. The Area of which Triangle increaſed with one of the ſides about the right-angle, ſuppoſe with a , muſt be equal to the given number $\frac{1}{2}$, therefore $\frac{2e}{9}aa + a = \frac{1}{2}$
21. Which Equation being reſolved by the Canon in *ſect. 6. Chap. 15. Book 1.* will give $a = \frac{1}{10}$
22. According to which value of a , the three ſides in the nineteenth ſtep being reſolved, there will be produced the three ſides of the right-angled Triangle ſought, to wit, $\frac{4e}{30}, \frac{4e}{30}, \frac{1}{30}$

I ſay $\frac{4e}{30}, \frac{4e}{30}$ and $\frac{1}{30}$ are the three ſides of a right-angled Triangle which will ſolve the Queſtion,

Question; for the Area $\frac{6}{30}$ increased with $\frac{2}{30}$, (one of the sides about the right-angle,) makes $\frac{8}{30}$, that is, $\frac{4}{15}$; as was required.

QUEST. 100.

To find a right-angled Triangle in rational numbers, that the Area subtracted from one of the sides about the right-angle may leave a given number; let the given number be n .

RESOLUTION.

1. For the sides about the right-angle of the Triangle sought }
 put } a and e
2. Then is the Area } $\frac{1}{2}ae$
3. Which subtracted from one of the sides about the right- }
 angle, suppose from a , leaves } $a - \frac{1}{2}ae = n$
4. Whence, after due Reduction, } $e = \frac{a-n}{\frac{1}{2}a}$
5. And because $aa + ee$ must be equal to the Square of the }
 Hypotenusal, it follows from the first and fourth steps, }
 that the Square of the Hypotenusal must be equal to the } $\frac{1}{4}aaaa + aa - 2na + nn$
 sum of the Squares of a and $\frac{a-n}{\frac{1}{2}a}$, that is, } $\frac{1}{4}aa$

6. And since the Denominator $\frac{1}{4}aa$ is a Square, whose side is $\frac{1}{2}a$, it remains only to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to a Square, whose side may be feigned either $\frac{1}{2}aa - n$ or $\frac{1}{2}aa + n$; first then, let the side of the said Square be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the said $\frac{1}{4}aaaa + aa - 2na + nn$, this Equation ariseth, viz.

$$\frac{1}{4}aaaa + aa - 2na + nn = \frac{1}{4}aaaa - naa + nn.$$

7. From which Equation the value of a will be made known, }
 viz. } $a = \frac{2n}{1+n}$
8. According to which value of a , the number e will be }
 discovered from the fourth step, viz. } $e = 1-n$
9. And the square Root of the sum of the Squares of $\frac{2n}{1+n}$ }
 and $1-n$ shall be the Hypotenusal, to wit, } $\frac{1+n^2}{1+n}$

10. Therefore from the three last preceding steps, the three sides of the right-angled Triangle sought are these, to wit,

$$\frac{2n}{1+n}, 1-n, \left(\text{or } \frac{1-n^2}{1+n}, \right) \text{ and } \frac{1+n^2}{1+n}.$$

11. But if instead of $\frac{1}{2}aa - n$, which in the sixth step was feigned for the side of a Square, we assume $\frac{1}{2}aa + n$, and equate the Square of this side to the before-mentioned $\frac{1}{4}aaaa + aa - 2na + nn$, there will arise $a = \frac{2n}{1-n}$; according to which value, the sides of the Triangle sought will be found these, to wit,

$$\frac{2n}{1-n}, \frac{1-n^2}{1-n}, \frac{1+n^2}{1-n}.$$

The two last steps give this

CANON.

12. When the given number is less than unity, let a right-angled Triangle be formed from unity and the given number; then divide the three sides severally by the sum or difference of unity and the given number; so shall the Quotients be the sides of the Triangle sought.

As, for example, if it be desired to find out a right-angled Triangle, that the Area subtracted from one of the sides about the right-angle may leave $\frac{4}{7}$, the Canon will discover the sides of two Triangles, to wit, $\frac{62}{77}, \frac{32}{77}, \frac{56}{77}$ and $\frac{62}{21}, \frac{32}{21}, \frac{56}{21}$, each of which Triangles will satisfy your desire; for in the first Triangle the Area $\frac{16}{77}$ subtracted from $\frac{56}{77}$, (one of the sides about the right-angle,) leaves the given number $\frac{4}{7}$; likewise in the latter Triangle, the Area $\frac{16}{21}$ subtracted from $\frac{56}{21}$ leaves $\frac{4}{7}$.

But how to solve this Question when the given number is any number whatever, I shall hereafter shew by Fermat's method, in Quest. 130. of this Book.

QUEST. 101. (Quæst. 12. Lib. 6. Diophant.)

To find a right-angled Triangle, that as well the difference of the sides about the right-angle as the greater of the same sides may be a Square; and that the Area, with the lesser of the sides about the right-angle may make a Square.

The Resolution of this Question depends upon three Lemma's, which I shall first explain.

LEMMA 1.

1. If a right-angled Triangle be formed from two numbers whereof the greater is the double of the lesser, as well the difference of the sides about the right-angle as the greater of the same sides shall be a Square. Moreover, if the Area of the said Triangle be multiplied by the Square of a Fraction having unity for its Numerator, and the lesser of the two numbers by which the said Triangle was formed for a Denominator, the Product increased with the lesser of the sides about the right-angle, will make a Square containing nine times the Square of the lesser of the two numbers by which the said Triangle was formed.

To make this manifest, let a right-angled Triangle be formed from a and $2a$, so the three sides will be these, to wit, $5aa; 3aa, 4aa$

Whence 'tis evident, first, that the difference of the sides about the right-angle is a Square, to wit, aa ; secondly, that the greater of the sides about the right-angle is a Square, to wit, $4aa$; and lastly, if the Area $6aaaa$ be multiplied by the Square of $\frac{1}{a}$, that is, by $\frac{1}{aa}$, the Product $6aa$ increased with $3aa$, (that is, the lesser of the sides about the right-angle,) makes the Square $9aa$, which contains nine times the Square of the lesser of the two numbers by which the said right-angled Triangle was formed. Which was to be shewn.

LEMMA 2.

2. Two numbers being given whose sum is a Square, to find innumerable Squares, every one of which being multiplied by one of the given numbers, and taking to the Product the other number, may make a Square.

Let there be two given numbers 6 and 3, and let it be desired to find a Square, such that if it be multiplied by 6, and 3 be added to the Product, the sum may make a Square.

Let the side of the Square sought be $a + 1$
 Whence the Square it self is $aa + 2a + 1$
 Which multiplied by 6 produceth $6aa + 12a + 6$
 To which Product add 3 and it makes $6aa + 12a + 9$

Which $6aa + 12a + 9$ is to be equated to a Square, whose side, (because the absolute number 9 is a Square,) may be variously feigned; let it then be $3a - 3$, the Square whereof, to wit, $9aa - 18a + 9$ being equated to the said $6aa + 12a + 9$ will give $a = 10$; wherefore $a + 1$ the side of the Square sought is 11, and the Square it self is 121, which multiplied by 6 produceth 726, to which adding 3 it makes 729, which is a Square whose side is 27. It is also evident, that instead of 121 innumerable other Squares may be found out to perform the like effect, because the side of the Square to be equated to $6aa + 12a + 9$ may be feigned infinitely.

In like manner if it were desired to find a Square which multiplied into 3, and taking 6 to the Product, may make a Square; let the Square sought be feigned $aa + 2a + 1$, this multiplied by 3 and the Product increased with 6, gives $3aa + 6a + 9$ to be equated to a Square whose side may be feigned $3a - 3$, whence $a = 4$, and therefore $a + 1$ the side of the Square sought is 5.

LEMMA 3.

3. If two numbers b and c whose sum makes not a Square be given, such, that by multiplying one of them, suppose b , by a given Square dd , and by adding to the Product the other number c , the sum $bdd + c$ makes a Square; we may find innumerable other Squares instead of the given Square dd to produce the like effect.

Suppose $\left. \begin{matrix} b = 2, \\ c = 8, \\ d = 2, \\ dd = 4; \end{matrix} \right\}$ whence $bdd + c = 16$.

Now

Now it is required to find another Square instead of dd , or 4 , that if the Square found out be multiplied by 2 , to wit, b ; and the Product be increased with 8 , to wit, c , the summ may be a Square.

For the side of the Square sought put a the side of the given Square dd , viz. $\left. \begin{array}{l} \text{Square } dd, \text{ viz.} \\ \text{Whence the Square sought is} \\ \text{Which multiplied by } 2, \text{ (to wit, } b,) \text{ produceth} \\ \text{To which Product add } 8, \text{ (to wit, } c,) \text{ and the summ is} \end{array} \right\} \begin{array}{l} \dots\dots\dots a+2 \\ \dots\dots\dots aa+4a+4 \\ \dots\dots\dots 2aa+8a+8 \\ \dots\dots\dots 2aa+8a+16 \end{array}$

Which summ is to be equated to a Square, the side whereof (in regard 16 is a Square) may be variously feigned, let it then be $2a-4$, the Square whereof being equated to the said $2aa+8a+16$ will give $a=12$; wherefore $a+2=14$ the side of the Square sought, and the Square it self is 196 , which multiplied by 2 , and taking 8 to the Product makes the Square 400 , whose side is 20 . I shall now proceed to

The RESOLUTION of the preceding QUEST. 101.

4. Let a right-angled Triangle be formed from two such numbers, that the greater may be the double of the lesser, as from a and $2a$, so the three sides will be these, to wit, $\left. \begin{array}{l} \dots\dots\dots 5aa, 3aa, 4aa \end{array} \right\}$

Whence it is evident that the two first parts of the Question are satisfied, for as well the difference of the sides about the right-angle as the greater of them is a Square. It remains that we see whether the Area of the Triangle, with the lesser of the sides about the right-angle makes a Square: But it makes $6aaaa+3aa$, and by dividing all by aa there ariseth $6aa+3$ to be equated to a Square, which is possible to be done, because the summ of 6 and 3 is a Square; for if a be 1 , then aa is also 1 , and consequently $6aa+3$ makes the same Square as $6+3$, to wit, 9 . Since then a is found 1 , a right-angled Triangle formed from 1 and 2 (that is, a and $2a$) will solve the Question. Wherefore the sides of the Triangles sought are $5, 3, 4$.

5. Having found out one Square for the value of aa , to wit, 1 , we may by the help of the preceding Lemma 2. find out innumerable other Squares to perform the same effect; so instead of 1 , the Square 121 is found out in the Example of the said Lemma. And therefore let a Triangle be formed from 11 and 22 , and the sides will be $605, 363, 484$, which will solve the Question; for the difference of the sides about the right-angle is the Square 121 , and the greater of the sides about the right-angle, to wit, 484 , is a Square whose side is 22 ; also the Area added to the lesser side makes 88209 , which is a Square whose side is 297 .

6. But for removing an Obstruction which the Learner may meet with, let a right-angled Triangle be formed from $2a$ and $4a$, whence the sides are $20aa, 12aa$ and $16aa$; where as well the difference of the sides about the right-angle, as the greater of them is a Square. But the Area increased with the lesser side makes $96aaaa+12aa$, and by dividing all by aa it makes $96aa+12$, which must be equated to a Square. But how shall that be done, since 96 which is multiplied into aa is not a Square, neither is the absolute number 12 a Square, nor is the summ of 96 and 12 a Square? This knot, though it seems to be a very hard one, may yet by the help of the last clause of the preceding Lemma 1. be easily untied: For since 96 is the Area of a right-angled Triangle formed from 2 and 4 , and 12 is the lesser of the sides about the right-angle, it will appear by Lemma 1. that if 96 be multiplied by $\frac{1}{4}$, and the Product 24 be increased with 12 it shall necessarily make a Square, to wit, 36 ; and consequently by Lemma 3. we may find out innumerable Squares instead of $\frac{1}{4}$, every one of which being multiplied by 96 , and taking 12 to the Product will make a Square. As, for example,

For the side of a Square to be found out instead of $\frac{1}{4}$, let there be put a the square Root of $\frac{1}{4}$, viz. $\left. \begin{array}{l} \dots\dots\dots a-\frac{1}{2} \\ \text{Then the Square of } a-\frac{1}{2} \text{ is} \\ \text{Which multiplied by } 96 \text{ produceth} \\ \text{To which Product adding } 12 \text{ it makes} \end{array} \right\} \begin{array}{l} \dots\dots\dots a-\frac{1}{2} \\ \dots\dots\dots aa-a+\frac{1}{4} \\ \dots\dots\dots 96aa-96a+24 \\ \dots\dots\dots 96aa-96a+36 \end{array}$

Which $96aa-96a+36$ must be equated to a Square, (the side whereof in regard 36 is a Square,) may be variously feigned, let it then be $4a-6$, the Square whereof being equated to $96aa-96a+36$ will give $a=\frac{2}{3}$; therefore $a-\frac{1}{2}$, (that is, $\frac{1}{6}$) will give $\frac{1}{6}$ for the side of the Square sought and the Square it self is $\frac{1}{36}$, by which if you multiply

multiply 96, and to the Product add 12, it makes the Square $\frac{224}{25}$, whose side is $\frac{18}{5}$. Now forasmuch as the said side $\frac{18}{5}$ is to be taken for the value of a in the $2a$ and $4a$ by which the Triangle was first formed, let a Triangle be now formed from $\frac{18}{5}$ and $\frac{6}{5}$, and the three sides are $\frac{18}{5}$, $\frac{6}{5}$, $\frac{24}{5}$; where 'tis evident that the difference of the sides about the right-angle, as also the greater of them is a Square: But the Area is $\frac{6}{25}$, to which if you add the lesser side $\frac{6}{5}$, it makes the Square $\frac{81}{25}$, whose side is $\frac{9}{5}$.

QUEST. 102.

To find a right-angled Triangle, and a square number, such, that if the Square be multiplied by the lesser of the sides about the right-angle, and to that Product there be added the Product made by the multiplication of the Area of the said Triangle into the difference of the sides about the right-angle, the sum may be a Square. Moreover, that the greater of the sides about the right-angle may be a Square.

CANON.

1. By the preceding Quest. 101. find a right-angled Triangle, that as well the difference of the sides about the right-angle, as the greater of the same sides, may be a Square: Moreover, that the Area increased with the lesser of the said sides may make a Square; so shall such Triangle be that which is sought by this Quest. 102. and the difference of the sides about the right-angle shall be the first Square sought.

But that this Canon will solve the Question proposed, I demonstrate thus;

2. Suppose $\left. \begin{array}{l} b = \text{the Hypothenufal;} \\ b = \text{the greater side,} \\ p = \text{the lesser side,} \end{array} \right\} \text{ about the right-angle.}$
3. Suppose also that by the preceding Quest. 101. $\left. \begin{array}{l} b-p, \\ b, \\ \frac{1}{2}bp+p, \end{array} \right\} \text{ are three square numbers.}$
 b, b, p are found such, that
4. I say $b-p$ is such a square number, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product made by the multiplication of the Area $\frac{1}{2}bp$, into $b-p$ the difference of the sides about the right-angle, the sum shall be a Square: For,
5. The Product of $b-p$ into p is $bp - pp$
6. To which if you add the Product of the Area into the difference of the sides about the right-angle, to wit, . . . $\frac{1}{2}bbp - \frac{1}{2}bpp$
7. The sum is $bp - pp + \frac{1}{2}bbp - \frac{1}{2}bpp$
8. Which sum is a square number, for it is equal to the Product of $b-p$ multiplied into $\frac{1}{2}bp + p$: But each of these Factors $b-p$ and $\frac{1}{2}bp + p$ is a Square by Construction; wherefore the Product of their multiplication, to wit, $bp - pp + \frac{1}{2}bbp - \frac{1}{2}bpp$ is a Square.

An Example in Numbers.

9. Take any right-angled Triangle which will solve the preceding Quest. 101. as, $\left. \begin{array}{l} 5 \\ 3 \\ 4 \end{array} \right\}$

In which Triangle the difference of the sides about the right-angle, to wit, 1, is such a Square that if it be multiplied by 3 the lesser side, and the Product be increased with 6, to wit, the Product of the Area multiplied by the difference of the sides about the right-angle it makes the Square 9. Moreover, the greater side about the right-angle, to wit, 4, is a Square; as was required.

10. But the same right-angled Triangle 5, 3, 4 being retained, we may instead of the Square 1, (to wit, the difference of the sides about the right-angle) find out innumerable Squares, (by the help of Lemma 2. in Quest. 101.) every one of which shall solve this Question, and be within given limits if need require. So if it were desired to find out a Square greater than 6, and such as together with the said right-angled Triangle 5, 3, 4 may solve this 102^d Question, the said Lemma 2. will discover the Squares 25 and 361, (among innumerable others,) which are such, that if each of them be multiplied by 3, (the lesser of the sides about the right-angle of the said Triangle,) and the Products 75 and 1083 be severally increased with 6, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Squares 81 and 1089, whose sides are 9 and 33.

11. And

11. And in like manner, by the help of any other right-angled Triangle which will solve the preceding *Quest.* 101. as the Triangle 605, 363, 484, we may find out innumerable Answers to this *Quest.* 102. For, first, the difference of the sides about the right-angle, to wit, 121 is a Square, and such, that if it be multiplied by the lesser side 363, and to the Product 43923 there be added 10629366, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Square 10673289, whose side is 3267. And therefore by the help of the third *Lemma* which belongs to the Resolution of the preceding *Quest.* 101. you may instead of the Square 121 find out infinite other Squares to perform the same thing, and each Square shall be within given limits if need require.

QUEST. 103. (Quest. 13. Lib. 6. Diophant.)

To find a right-angled Triangle, that the Area thereof being increased severally with each of the sides about the right-angle may make Squares.

RESOLUTION.

1. Let the Hypothensal, Base and Perpendicular of some right-angled Triangle in numbers be represented by $h ; b ; p$
2. Then multiply those sides severally by a , and suppose the Products to be the sides of the right-angled Triangle sought, to wit, $ha ; ba ; pa$
3. Then, (according to the Question,) the Area of the Triangle in the last step being increased with each of the sides about the right-angle must make a Square, hence this Duplicate equality ariseth to be resolved, viz. $\frac{1}{2}bpaa + ba = \square$
 $\frac{1}{2}bpaa + pa = \square$
4. Now suppose $\frac{1}{2}bpaa + ba = eeaa$
5. Whence, after due Reduction, you will find $a = \frac{b}{ee - \frac{1}{2}bp}$
6. According to which value of a , the latter of the two quantities in the third step being resolved, instead of that quantity this that follows ariseth to be equated to a Square, viz. $\frac{bpee + \frac{1}{2}bbbp - \frac{1}{2}bbpp}{eeee - bpee - \frac{1}{4}bbpp} = \square$
7. But because the Denominator of the Fraction last express'd is a Square, whose side is $ee - \frac{1}{2}bp$, it remains only to equate the Numerator to a Square: And because a Square divided by a Square gives the Quotient a Square, therefore if we suppose b in the said Numerator $bpee + \frac{1}{2}bbbp - \frac{1}{2}bbpp$ to be a square number, then the said Numerator divided by b gives $pee + \frac{1}{2}bbp - \frac{1}{2}bpp$, that is, $pee + \frac{1}{2}bp \times b - p$ to be equated to a Square. So that the matter is reduced to this, we must find out a right-angled Triangle, such, that the greater of the sides about the right-angle, suppose b , may be a square number; and we must also find another square number, suppose ee , that may be greater than the Area of the said Triangle, (as may be infer'd from the Denominator of the Fraction in the fifth step,) and such, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the sides about the right-angle, the sum may make a Square: But such a right-angled Triangle and Square, the preceding *Quest.* 102. shews how to find out. Suppose then the Hypothensal, Base and Perpendicular of the said Triangle so found out to be h, b, p ; and the Square to be ee , all which being known in numbers, the number represented by a shall consequently be known from the fifth step: And lastly, if you multiply the numbers h, b, p severally by the number a , the Products shall be the three sides of the Triangle sought. From the premisses there ariseth the following

CANON.

First, by the Canon of the preceding *Quest.* 102. find out a right-angled Triangle, (whose three sides in the Resolution of this *Quest.* 103. are represented by h, b, p ;) and besides, a Square, call it ee , that may be greater than the Area of the said Triangle, and such that if it be multiplied by the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the same sides, the sum may be a Square; then divide the greater of the sides about the right-angle of the said Triangle, by the excess of the said Square ee above the Area, and by the Quotient multiply severally the three sides h, b, p , so shall the Products be the three sides of the Triangle sought.

An Example in Numbers.

It hath been shewn in Sect. 10. of the preceding Quest. 102. that the right-angled Triangle 5, 3, 4 and the Square 25 will solve that Question; and besides, the said Square 25 is greater than 6 the Area of the said Triangle; therefore according to the directions of the Canon above-delivered, I divide 4 the greater of the sides about the right-angle, by 19 the excess of the said Square 25 above 6 the Area of the said Triangle, and the Quotient is $\frac{4}{19}$, by which I multiply severally 5, 3, 4 (the sides of the Triangle first found) and there comes forth $\frac{20}{19}$, $\frac{12}{19}$, $\frac{16}{19}$, which shall be the sides of the Triangle sought; for the Area is $\frac{26}{361}$; to which if we add severally $\frac{12}{19}$ and $\frac{16}{19}$, (the sides about the right-angle,) there will be made the Squares $\frac{324}{361}$ and $\frac{400}{361}$, whose sides are $\frac{18}{19}$ and $\frac{20}{19}$.

After the same manner, the same right-angled Triangle 5, 3, 4 and the Square 361, (found out also by the preceding Quest. 102.) will discover $\frac{22}{355}$, $\frac{12}{355}$ and $\frac{16}{355}$ for the three sides of another right-angled Triangle to solve this Question, as may easily be proved. And because innumerable right-angled Triangles and Squares may be found out to solve the said Quest. 102. infinite Answers may be given to this.

QUEST. 104. (Quest. 15. Lib. 6. Diophant.)

To find a right-angled Triangle, such, that if from its Area the Hypothensal and one of the sides about the right-angle be severally subtracted, each remainder may be a Square.

The Resolution of this and the following 105th Question depends upon a Lemma, which I shall here premise and demonstrate.

LEMMA.

1. If a right-angled Triangle be formed from two square numbers, or from two numbers in proportion one to another as a square number to a square number; I say first, the Square of the difference of those two square numbers being multiplied by the Product of their multiplication will produce a Square less than the Area of the said Triangle; secondly, if from the solid Product made by the multiplication of these three numbers, to wit, the square number above produced less than the Area, the Hypothensal, and that side about the right-angle which is the double Product of the multiplication of the two square numbers forming the Triangle, there be subtracted the solid Product made of these three numbers, to wit, the Area, the said side about the right-angle, and the excess by which the Hypothensal exceeds the same side, the remainder shall be a square number; thirdly and lastly, the summ of those two solid Products shall also be a square number.

Demonstration.

2. Let a right-angled Triangle be formed from two square numbers, suppose *bb* the greater, and *cc* the lesser, whose sides are *b* and *c*, so the three sides of the said Triangle will be these,

$$\text{viz. } \begin{cases} bbbb + cccc = \text{the Hypothensal,} \\ bbbb - cccc = \text{the Base,} \\ 2bbcc = \text{the Perpendicular.} \end{cases}$$

3. The Area of the said Triangle is } $b^6cc - bbc^6$

4. The Product of the multiplication of *bb* and *cc*, }
to wit, *bbcc*, being multiplied by the Square of }
bb - cc, produceth this Square, to wit, } $b^6cc - 2b^4c^4 + bbc^6$

5. Which Square is less than the Area } $b^6cc - bbc^6$

6. For it is evident that } $bbbc - bccc \Rightarrow bbbc + bccc$

7. Therefore by multiplying each part into *bbbc* }
- bccc, it necessarily follows that } $b^6cc - 2b^4c^4 + bbc^6 \Rightarrow b^6cc - bbc^6$

Which was affirmed in the first part of the Lemma.

8. In the next place, if these three following quantities be multiplied one into another,

$$\text{viz. } \begin{cases} b^6cc - 2b^4c^4 + bbc^6, & (\text{the Square in the fourth step,}) \\ b^4 + c^4, & (\text{the Hypothensal,}) \\ 2bbcc, & (\text{the Perpendicular;}) \end{cases}$$

The solid Product of their multiplication }
will be } $2b^{12}c^4 - 4b^{10}c^6 + 4b^8c^8 - 4b^6c^{10} + 2b^4c^{12}$

9. And

9. And if these three following quantities be multiplied one into another, to wit,

$$b^6cc - bbc^6, \quad (\text{the Area,})$$

$$+ 2bbcc, \quad (\text{the Perpendicular,})$$

$$bbbb - cccc - 2bbcc, \quad (\text{the excess of the Hypothenufal above the Perpendicular,})$$

The solid Product of their multiplication will be $> 2b^{12}c^4 - 4b^{10}c^6 - 2b^4c^{12} + 4b^6c^{10}$.

10. Which latter solid Product subtracted from the former, } $4b^8c^8 - 8b^6c^{10} + 4b^4c^{12}$
leaves this Square, to wit, }

The side whereof is $> 2b^4c^4 - 2bbc^6$.

Wherefore the second part of the Lemma is manifest.

11. Lastly, the summ of the two solid Products in the } $4b^{12}c^4 - 8b^{10}c^6 + 4b^8c^8$
eighth and ninth steps makes this Square, to wit, }

The side whereof is $> 2b^6cc - 2b^4c^4$.

Wherefore the Lemma is every way proved.

An Example in Numbers.

Suppose $4 = bb$, and $1 = cc$; then let a right-angled Triangle be formed from 4 and 1, so the three sides will be 17, 15, 8. Now I say, first, that if 9 the Square of the difference between the said 4 and 1, be multiplied by 4 the Product of 4 and 1, there will be produced the square number 36, which is less than 60 the Area of the said Triangle.

Secondly, if from 4896, which is the solid Product made by the multiplication of these three numbers, to wit, 36 the Square before found, the Hypothenufal 17, and 8 the double Product of 4 and 1; there be subtracted 4320, which is the solid Product of these three numbers, to wit, the Area 60; the said double Product 8, and 9 the excess of the Hypothenufal above the said 8; there will remain the Square 576, whose side is 24.

Thirdly and lastly, the summ of the said solid Products 4896 and 4320 makes the Square 9216, whose side is 96.

Now followeth the RESOLUTION of QUEST. 104: before proposed.

1. Let the Hypothenufal, Base and Perpendicular of some } h, b, p
right-angled Triangle in numbers be represented by . . . }

2. Then multiply those sides severally by a , and suppose the } ha, ba, pa
Products to be the sides of the Triangle sought; to wit, }

3. Then, (according to the Question,) if the Area of the } $\frac{1}{2}bpaa - ha = \square$
Triangle in the last step be lessened by the Hypothenufal } $\frac{1}{2}bpaa - pa = \square$
and one of the sides about the right-angle severally, each }
remainder must be a Square; hence this Duplicate equal- }
ity ariseth to be resolved, viz. }

4. Now in order to resolve that Duplicate equality, suppose $> \frac{1}{2}bpaa - pa = ecaa$

5. Whence after due Reduction you will find } $\dots a = \frac{p}{\frac{1}{2}bp - ee}$

6. According to which value of a , if the former of the } $hpee, - \frac{1}{2}bpp \times h - p = \square$
two quantities in the third step be resolved, instead of } $\frac{1}{4}bbpp - bpee + eccc$
that quantity, (to wit, $\frac{1}{2}bpaa - ha$) this that follows }
ariseth to be equated to a Square, viz. }

7. But, because the Denominator of the Fraction last express'd is a Square, for its side is $\frac{1}{2}bp - ee$, it remains only to equate the Numerator to a Square. We must therefore inquire into the rise of the Numerator, so we shall find that it imports the search of a right-angled Triangle h, b, p in numbers; and a square number ee less than the Area of the said Triangle; (for the Denominator of the Fraction in the fifth step of the Resolution shews that $\frac{1}{2}bp$ must exceed ee .) Moreover, the said Triangle and Square must be such, that if from the solid Product made by the multiplication of the said Square, the Hypothenufal, and one of the sides about the right-angle, there be subtracted the solid Product made by the multiplication of the Area, the said side about the right-angle, and the excess of the Hypothenufal above the same side, the remainder may be a Square. But the last preceding Lemma shews how to find out such a Triangle and Square. Suppose then the Hypothenufal, Base and Perpendicular of the said Triangle so found out, to wit, h, b, p , and the Square ee , to be all known in numbers, then the number represented by a shall consequently be known from the fifth step of the Resolution; and lastly, if you multiply the numbers h, b, p severally by the number a , the Products shall be the three sides of the Triangle sought. From the premises there ariseth the following

CANON.

CANON.

8. First, let a right-angled Triangle be formed from two square numbers, or from two numbers which have such proportion to one another as a square number to a square number; then multiply the Square of the difference of those two numbers by the Product of their multiplication and reserve the Product, which is a square number, and may be called *ee*; that done, divide that side about the right-angle which is the double Product of the two numbers that formed the said Triangle, by the excess of its Area above the Square *ee* before reserved, and by the Quotient multiply severally the three sides of the same Triangle, so shall the Products be the three sides of the right-angled Triangle sought.

An Example in Numbers.

First, I form a right-angled Triangle from the square numbers 4 and 1, so the three sides are 17, 15, 8; then I multiply 9 (the Square of the difference between 4 and 1) by 4, the Product of 4 into 1, and there comes forth the Square 36, (to wit, *ee*;) then I divide 8, (to wit, that side about the right-angle of the said Triangle which is the double Product of 4 into 1,) by 24, which is the excess of 60 the Area of the said Triangle above the Square 36, (to wit, *ee*;) and the Quotient is $\frac{1}{3}$, by which I multiply severally 17, 15, 8, (the sides of the right-angled Triangle first found,) and the Products $\frac{17}{3}$, $\frac{15}{3}$ and $\frac{8}{3}$ are the sides of the right-angled Triangle sought: For, if from the Area $\frac{60}{3}$, the Hypotenusal $\frac{17}{3}$, and the side $\frac{8}{3}$ be severally subtracted, there will remain the Squares 1 and 4.

This Question is capable of innumerable Answers in a double respect; for first, instead of 4 and 1 we may take any two square numbers, or any two numbers which are in such proportion to one another as a square number to a square number, for the forming of a right-angled Triangle as the Canon directs: secondly, the same right-angled Triangle 17, 15, 8 being retained, we may instead of the Square 36, to wit, *ee*, find infinite others, every one of which shall be less than the Area 60; and such, that if it be multiplied into 136, (to wit, *hp*;) and from the Product there be subtracted 4320, (to wit, $\frac{1}{2}bpp \times \overline{h-p}$;) the remainder shall be a square number. (The finding out of which Squares may easily be deduced from Lemma 3. in the preceding Quest. 101.)

QUEST. 105.

To find a right-angled Triangle, that if its Area be subtracted from the Hypotenusal, and from one of the sides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let the Hypotenusal, Base and Perpendicular of some right-angled Triangle in numbers be represented by b, b, p
2. Then multiply those sides severally by *a*, and assume the Products to be the sides of the right-angled Triangle sought, to wit, ha, ba, pa
3. Then, (according to the Question,) each of these two quantities must be equated to a Square, viz. $ha - \frac{1}{2}bpaa = \square$
 $pa - \frac{1}{2}bpaa = \square$
4. Now in order to resolve that Duplicate equality, suppose $pa - \frac{1}{2}bpaa = eeaa$
5. Whence after due Reduction you will find $a = \frac{p}{ee + \frac{1}{2}bp}$
6. According to which value of *a*, if the former of the two quantities in the third step be resolved, instead of that quantity, (to wit, $ha - \frac{1}{2}bpaa$;) this that follows ariseth to be equated to a Square, viz. $\frac{hpee + \frac{1}{2}bpp \times \overline{h-p}}{eeee + bpee + \frac{1}{4}bbpp} = \square$
7. But because the Denominator of the Fraction last express'd is a Square, for its side is $ee + \frac{1}{2}bp$, it remains only to equate the Numerator to a Square; and the Numerator well examined, shews that we must find a right-angled Triangle *h, b, p* in numbers, and a square number *ee*, such, that if to the solid Product made by the multiplication of the said Square, the Hypotenusal and one of the sides about the right-angle, there be added the solid Product made by the multiplication of the Area, the said side about the right-angle, and the excess of the Hypotenusal above the same side, the sum may be a Square: But the Lemma prefix'd to the Resolution of the preceding Quest. 104. shews how to find out such a Triangle and Square: Suppose then the Hypotenusal, Base

Base and Perpendicular of the said Triangle so found out, to wit, h, b, p , and the Square ee to be all known in numbers, then consequently the number a shall be known also from the fifth step of the Resolution: And lastly, the numbers h, b, p being multiplied severally by the number a will give the three sides of the Triangle sought. From the premisses there ariseth the following

CANON.

8. First, let a right-angled Triangle be formed from two square numbers, or from two numbers which have such proportion to one another as a square number to a square number; then multiply the Square of the difference of those two numbers by the Product of their multiplication, and reserve the Product, which is a square number, and may be called ee ; then divide that side about the right-angle which is the double Product of the two numbers that formed the said Triangle, by the summ of its Area and the Square ee before reserved; lastly, by the Quotient multiply severally the three sides of the Triangle first formed: So shall the Products be the three sides of the right-angled Triangle sought.

An Example in Numbers.

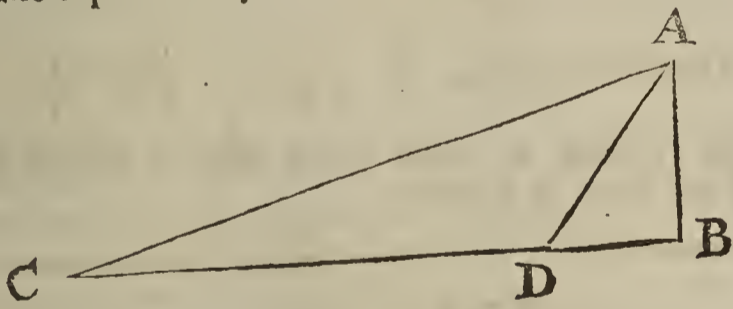
First, a right-angled Triangle being formed from the square numbers 4 and 1, the three sides will be 17, 15, 8; then I multiply 9 the Square of the difference between 4 and 1, by 4 the Product of 4 into 1, and it produceth the Square 36, (to wit, ee ;) then I divide the side 8, by 96 which is the summ of the Area 60 and the Square 36, (to wit, ee ;) so the Quotient is $\frac{1}{12}$; lastly, the sides 17, 15, 8 being multiplied severally by $\frac{1}{12}$ will give $\frac{17}{12}$, $\frac{15}{12}$ and $\frac{8}{12}$ for the right-angled Triangle sought. For if from the Hypotenusal $\frac{17}{12}$, and from the side $\frac{8}{12}$, the Area $\frac{1}{12}$ be subtracted, there will remain the Squares 1 and $\frac{1}{4}$.

After the same manner, among innumerable right-angled Triangles that might be found out by the said Canon to solve this Question, these three will be discovered, to wit, $\frac{41}{36}, \frac{40}{36}, \frac{2}{36} | \frac{27}{45}, \frac{65}{45}, \frac{72}{45} | \frac{257}{240}, \frac{255}{240}, \frac{32}{240}$; every one of which Triangles besides that first found, to wit, $\frac{17}{12}, \frac{15}{12}, \frac{8}{12}$ is express'd by smaller numbers than that Triangle found out by Fermat's method, in the following Quest. 127.

This Question is also capable of innumerable Answers upon another ground, as may easily be collected from what hath been said at the latter end of the preceding Quest. 104.

QUEST. 106. (Quest. 18. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, suppose ABC right-angled at B, that one of its acute-angles BAC being cut into two equal parts by the line AD, the said line AD may be also expressible by a rational number.



RESOLUTION.

1. Let the Hypotenusal, Base and Perpendicular of some right-angled Triangle known in numbers be represented by } $h ; b ; p$
2. Then multiply those sides severally by a , (which represents a number unknown,) and put } $ha = AD$
 $ba = BD$
 $pa = AB$
 $b - ba = DC$
3. Now forasmuch as (per 3. Prop. 6. Elem. Euclid.) these lines in the preceding Diagram are Proportionals, viz. } $BD . AB :: DC . CA$
4. Therefore from the premisses these numbers shall be also Proportionals, viz. } $ba . pa :: b - ba . p - pa$
5. And because (per 47. Prop. 1. Elem. Euclid.) the Square of AB together with the Square of BC is equal to the Square of CA; therefore the Square of pa together with the

the Square of b , (for $ba + b - ba = b$,) shall be equal to the Square of $p - pa$, whence this Equation ariseth, to wit,

$$ppaa + bb = ppaa - 2ppa + pp.$$

6. Which Equation, after due Reduction, gives $\zeta a = \frac{pp - bb}{2pp}$

$$\left\{ \begin{array}{l} \frac{ppp - pbb}{2pp} = AB \\ \frac{2bpp}{2pp} = BC \\ \frac{ppp + pbb}{2pp} = CA \\ \frac{hpp - hbb}{2pp} = AD \\ \frac{bpp - bbb}{2pp} = BD \\ \frac{bpp + bbb}{2pp} = DC \end{array} \right.$$

7. Therefore from the first, second and sixth steps, all the lines sought shall now be known in rational numbers, viz. . . .

8. And by multiplying all the Fractions in the last step by the common Denominator $2pp$, the Products will give these numbers which will also solve the Question, and may serve as a Canon for that purpose, viz.

$$\left\{ \begin{array}{l} ppp - pbb = AB \\ 2bpp = BC \\ ppp + pbb = CA \\ hpp - hbb = AD \\ bpp - bbb = BD \\ bpp + bbb = DC \end{array} \right.$$

An Example in Numbers.

Take any right-angled Triangle in numbers, as 5, 3, 4, then by putting $5 = b$, $3 = p$ and $4 = p$, (the greater of the sides about the right-angle,) you will find

$$\left\{ \begin{array}{l} ppp - pbb = 28 = AB \text{ the Perpendicular,} \\ 2bpp = 96 = BC \text{ the Base,} \\ ppp + pbb = 100 = CA \text{ the Hypothenufal,} \\ hpp - hbb = 35 = AD \text{ the line bisecting the acute-angle opposite to the Base,} \\ bpp - bbb = 21 = BD \\ bpp + bbb = 75 = DC \end{array} \right\} \text{ the segments of the Base.}$$

The Proof.

First, these numbers are Proportionals, $\left\{ \begin{array}{l} 21 \quad . \quad 28 \quad :: \quad 75 \quad . \quad 100 \\ BD \quad . \quad AB \quad :: \quad DC \quad . \quad CA \end{array} \right.$

Therefore (per Prop. 3. Elem. 6. Euclid.) the angle BAD is equal to the angle DAC. The rest of the Proof is obvious.

QUEST. 107.

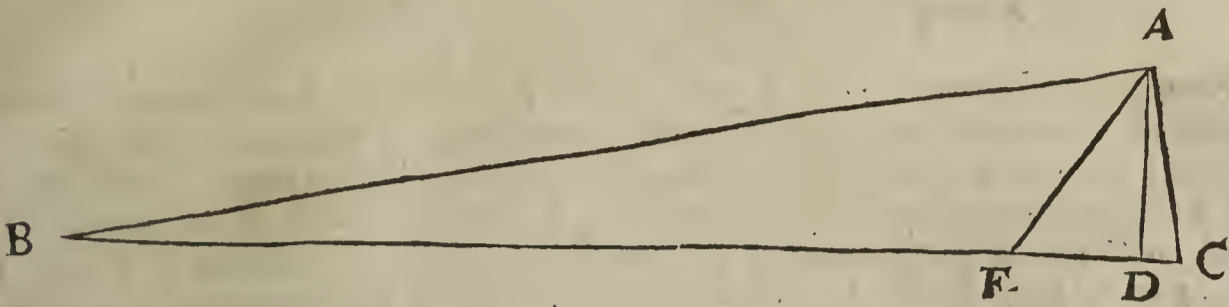
To find out an oblique-angled Triangle, whose three sides, as also the Perpendicular, and a line cutting one of the angles into two equal parts may be express'd severally by rational numbers.

[Jac. de Billy in probl. 5. cap. 4. of the latter part of his Diophant. redivi. printed at Lyons in 1670. resolves this Question briefly by Numeral Algebra; but to the end a Canon may be raised to solve the Question proposed, I shall form the Resolution thereof by Literal Algebra.]

Preparation.

Let there be an oblique-angled Triangle ABC; then supposing AD to be perpendicular to the Base BC, and the line AE to cut the angle BAC into two equal angles EAB

EAB and EAC, let it be required to find out rational numbers to express the quantities of the sides AB, BC, AC; as also of AD, AE, BE, ED, DC.



RESOLUTION.

1. First, let b, p, h represent the Base, Perpendicular and Hypothenufal of any right-angled Triangle known in numbers, and suppose the Perpendicular p to be greater than the Base b , then

Put $\left. \begin{array}{l} b = DE \text{ the Base;} \\ p = AD \text{ the Perp.} \\ h = AE \text{ the Hyp.} \end{array} \right\} \text{ of } \triangle ADE.$

2. Secondly, making p , (that is, AD) to be the Perpendicular of a second right-angled Triangle, suppose ADB, find out the Base DB and the Hypothenufal BA in rational numbers, which may be done thus, viz. Forasmuch as the Square of the Perpendicular AD is equal to the difference of the Squares of the Hypothenufal BA and the Base DB, let the Square of the given number p be esteem'd to be the difference of two square numbers, and find out the Squares themselves, then put the side of the lesser Square for DB, and the side of the greater for BA; but the said Squares must be found out with this Caution, that DB the side of the lesser Square may have greater proportion to DE, (that is, b) than $2pp$ hath to $pp - bb$; as may easily be inferr'd from the Canon of the preceding Quest. 106. where it appears, that when the angle DAC in the Diagram belonging to that Question is equal to the angle DAB, then the Base BC hath such proportion to BD, as $2pp$ to $pp - bb$; but in the Diagram of this Question the angle EAB must be greater than the angle EAD, in regard by supposition the angle EAB is equal to the angle EAC. Now since by Quest. 7. of this Book, innumerable pairs of Squares having one common difference may be found out, such, that the side of one Square of each pair shall be greater or less than a given number, let us suppose the sides of two square numbers to be found out agreeable to the said Caution,

viz. $\left\{ \begin{array}{l} b + d = DE + EB = DB, \\ d = EB, \\ g = BA. \end{array} \right.$

3. Thirdly, the next scope is to find out EC and AC in rational numbers, which must have the same proportion one to another as EB and BA; (for by supposition the angle EAB is equal to the angle EAC, and therefore (per Prop. 3. Elem. 6. Euclid.) $EB \cdot BA :: EC \cdot AC$.) Moreover, the Square of $EC - ED$, that is, of DC, together with the Square of AD must be equal to the Square of AC. But EB and BA were before found out in numbers, to wit, $d = EB$, and $g = BA$; now to find out two numbers in the same proportion as d and g , multiply these severally by a (which represents a number yet unknown,) and put the Products da and ga for EC and AC; whence these are Proportionals,

viz. $\left\{ \begin{array}{l} d \cdot g :: da \cdot ga, \\ EB \cdot BA :: EC \cdot AC. \end{array} \right.$

4. And because the Square of $EC - ED$, that is, the Square of DC, together with the Square of AD makes the Square of AC, therefore in the letters of the Resolution, the Square of $da - b$ together with the Square of p must be equal to the Square of ga ; hence this Equation ariseth, viz.

$$ddaa - 2bda + bb + pp = ggaa.$$

- 5. And because (by the first step of the Resolution,) $hh = bb + pp$
- 6. Therefore $ddaa - 2bda + hh = ggaa$
- 7. Whence, after due Reduction, this following Equation ariseth, viz. $aa + \frac{2bd}{gg - da} a = \frac{hh}{gg - da}$
- 8. Which

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8. Which last Equation being resolved by the Canon in Sect. 6. Chap. 15. Book 1. the value of a will be made known, viz.

$$a = \sqrt{\frac{bbdd + gggh - ddbb}{gg - dd}} : - \frac{bd}{gg - dd}$$

9. Now if the Numerator $bbdd + gggh - ddbb$ be a rational square number, then the value of a is manifestly rational, for the Denominator is a Square, whose side is $gg - dd$; but that the said Numerator is a rational Square I prove thus: In the Triangle BAE obtuse-angled at E , the Square of BA less by the sum of the Square of BE and twice the rectangle made of BE and ED is equal to the Square of AE , (per 12. prop. 2. Elem. Euclid.) therefore in the letters of the Resolution,

$$gg - dd - 2db = hb.$$

10. But if $gg - dd - 2db$ instead of hb be multiplied into $gg - dd$, and to the Product there be added $bbdd$, then instead of the aforesaid Numerator $bbdd + gggh - ddbb$, this following Square will arise, to wit,

$$gggg - 2ggdd - 2ggdb + dddd + 2dddb + ddbb;$$

Whose side is $gg - dd - db$.

11. Therefore from the eighth and tenth steps, the value of a is expressible by a rational number, viz.

$$a = \frac{gg - dd - 2db}{gg - dd}$$

12. Therefore from the third and eleventh steps, $\frac{dgg - ddd - 2ddb}{gg - dd} = EC$

13. And from the third and eleventh steps, $\frac{ggg - gdd - 2gdb}{gg - dd} = AC$

14. And by subtracting b from the quantity in the twelfth step, (viz. ED from EC ,) it gives $\frac{dgg - ddd - bgg - bdd}{gg - dd} = DC$

15. Lastly, if the three Fractions in the three last preceding steps, as also b, p, h, d, g , be severally multiplied by the Denominator $gg - dd$, there will come forth the following quantities in Integers, which may serve as a Canon to solve the Question proposed; provided that the numbers b, p, h, d, g be first found out agreeable to the Caution before prescribed.

CANON.

$ggd - ddd - 2ddb = EC$	}	in the Diagram belonging to this Question.
$ggd - ddd = EB$		
$2ggd - 2ddd - 2ddb = BC$		
$ggg - gdd = BA$		
$ggg - gdd - 2gdb = AC$		
$ggp - ddp = AD$		
$ggb - ddb = AE$		
$ggb - ddb = DE$		
$ggd - ddd - ggb - bdd = DC$		
$ggd + ggb - ddd - ddb = DB$		

An Example in Numbers.

First, take any right-angled Triangle in whole numbers, as the Triangle 18, 24, 30; then

$$\text{Put } \left\{ \begin{array}{l} b = 18 \text{ ED,} \\ p = 24 \text{ AD,} \\ h = 30 \text{ AE.} \end{array} \right.$$

Secondly, making 24, (to wit, $p = AD$) to be the Perpendicular of a second right-angled Triangle, as well as of the first ADE , find out the Base DB , and the Hypotenusal AB in rational numbers; but for the reason before given, the number of the Base DB must have greater proportion to the number of the Base DE , than $2pp$ to $pp - bb$, viz. in this Example, greater proportion than 32 to 7, and consequently DB must exceed DE taken $4\frac{2}{7}$ times: But by supposition $DE = 18$; therefore DB must exceed $82\frac{2}{7}$. Now because the Square of the Perpendicular AD is equal to the difference of the Squares of AB and DB , therefore 576 the Square of the Perpendicular 24 ($= AD$) being taken for the difference of two Squares, find out the Squares severally, but with this condition, that the side of the lesser of them may exceed $82\frac{2}{7}$: But two such Squares (among innumerable other pairs of Squares that may be found out by the seventh Question of

of this Book,) are 20449 and 21025, whose sides are 143 and 145; therefore 143 = DB, | 145 = AB, and 125 = EB, (that is, DB — DE;) then

$$\text{Put } \dots \dots \dots \left\{ \begin{array}{l} d = 125 = EB, \\ g = 145 = BA. \end{array} \right.$$

Lastly, by the help of the numbers before found out for the values of b, p, h, d and g , the preceding Canon will discover rational numbers, which reduced to their least terms by their greatest common Divisor, will give the whole numbers here-under exprest, for the measures of the sides of the oblique-angled Triangle sought; as also of the Perpendicular, and the line cutting the angle opposite to the Base into two equal parts, and of the segments of the Base made as well by the Perpendicular as by the line bisecting the said angle, viz.

$$\left. \begin{array}{l} EC = 125 \\ EB = 750 \\ BC = 875 \\ BA = 870 \\ AC = 145 \\ AD = 144 \\ AE = 180 \\ DE = 108 \\ DC = 17 \\ DB = 858 \end{array} \right\}$$

agreeable to the Diagram and Canon belonging to this Question.

The Proof.

First, these numbers are Proportionals, viz $\left\{ \begin{array}{l} 750 : 870 :: 125 : 145 \\ EB : BA :: EC : AC \end{array} \right.$

Therefore, (per 3. prop. 6. Elem. Euclid.) the angle EAB is equal to the angle EAC. The rest of the Proof is obvious.

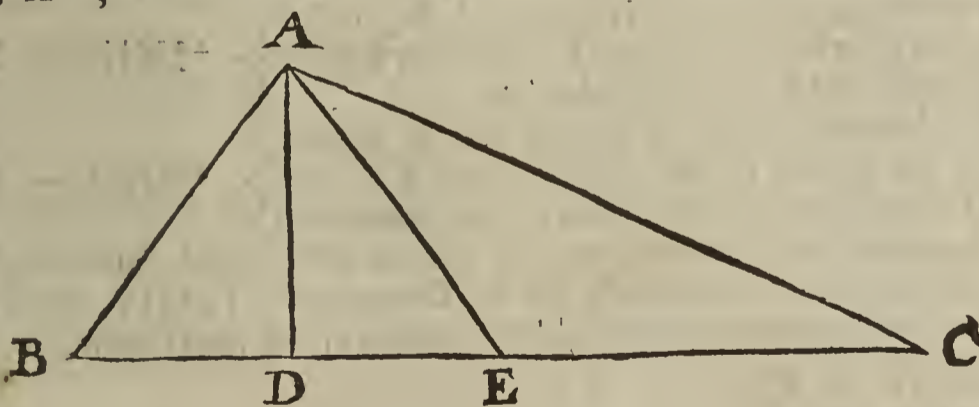
QUEST. 108.

To find out an oblique-angled Triangle, whose three sides, as also the Perpendicular, and a line drawn from the angle opposite to the Base, and cutting the Base into two equal parts, may be exprest severally by rational numbers.

[Jac. de Billy in his Appendix to the Problem cited in the preceding 107th Question resolves this also, but very briefly; I shall therefore form the Resolution thereof at large by Numeral Algebra, by the steps whereof the more curious Analyst may easily resolve it by Specious Algebra, but the Canon thence arising will be exceeding tedious.]

Preparation.

Let there be an oblique-angled Triangle ABC; then supposing AD to be perpendicular to the Base BC, and the line AE to cut the Base BC into two equal parts in the point E, let it be required to find out rational numbers to exprest the quantities of the sides AB, BC, AC; as also of the lines AD, AE, BD, DE.



RESOLUTION.

1. First, take any right-angled Triangle in numbers, as 3, 4, 5; then;

$$\text{Put } \dots \dots \dots \left\{ \begin{array}{l} 3 = BD \text{ the Base,} \\ 4 = AD \text{ the Perpendicular,} \\ 5 = AB \text{ the Hypotenusal,} \end{array} \right\} \text{ of the right-angled Triangle BDA.}$$

2. Then for the distance between the foot of the Perpendicular and the middle of the Base BC, put a , viz. suppose $\dots \dots \dots \left\{ \begin{array}{l} a = DE. \end{array} \right.$

T 2

3. And

3. And because by supposition, $BE = EC$, if to a you add 3, } $a + 3 = BE = EC$
 (to wit, BD ,) the sum shall be equal to half the Base, *viz.* }
 4. And because $DE + EC = DC$, the sum of the two } $2a + 3 = DC$
 Equations in the second and third steps shall be equal to the }
 greater segment of the Base made by the Perpendicular, *viz.* }
5. And because the Square of DC together with the Square of AD is equal to the Square of AC , the Square of $2a + 3$ together with the Square of 4 must make a Square, *viz.*

$$4aa + 12a + 25 = \square.$$

6. And because the Square of DE together with the Square of AD is equal to the Square of AE , the Square of a together with the Square of 4 must be equated to a Square, *viz.*

$$aa + 16 = \square.$$

7. So in the two last steps we are fall upon a Duplicate equality, which, in regard the Squares 25 and 16 are unequal, I reduce to another that shall have equal known Squares, *viz.* (after the manner used in divers preceding Questions of this Book,) I divide the greater Square 25 by the lesser 16, and by the Quotient $\frac{5}{4}$ I multiply the quantity in the sixth step, to wit, $aa + 16$, so the Product $\frac{5}{4}aa + 25$ is to be equated to a Square, and therefore this Duplicate equality comes now to be resolved,

$$\textit{viz.} \begin{cases} 4aa + 12a + 25 = \square \\ \frac{5}{4}aa + 25 = \square \end{cases}$$

8. Now in order to resolve the Duplicate equality last express, first, by subtracting the lesser quantity from the greater, I find their difference to be

$$\frac{3}{4}aa + 12a.$$

9. Then I search out two quantities, the Product of whose multiplication may make the said difference $\frac{3}{4}aa + 12a$; and that as well in the difference as in the sum of the same quantities there may be found 10, (to wit, the double of the side of the Square 25,) so I find those two quantities to be

$$\frac{5}{2}a + 10 \text{ and } \frac{2}{3}a.$$

10. Then the Square of half the difference of the two quantities last express, *viz.* the Square of $\frac{1}{2}(\frac{5}{2}a + 10 - \frac{2}{3}a) + 5$ being equated to $\frac{3}{4}aa + 25$, gives

$$\frac{3}{4}aa + 25 = \frac{1}{10} \frac{17682}{2400} aa + \frac{1}{32} a + 25.$$

11. Which Equation, after due Reduction, will discover the } $a = \frac{422600}{142311} = DE$
 number a , *viz.* }

12. Then by adding the Square of the said number a to 16, } $\frac{710756}{142311} = AE$
 and extracting the square Root of the sum, there ariseth }

13. And by adding 3 (to wit, BD) to the number a , (to } $\frac{852533}{142311} = BE = EC$
 wit, DE ,) the sum shall be the measure of half the Base }
 BC , *viz.* }

14. Therefore the double of the number in the last step is } $\frac{1705066}{142311} = BC$
 the measure of the Base BC , *viz.* }

15. And by adding the number a in the eleventh step to half } $\frac{1278133}{142311} = DC$
 the Base in the fourteenth, the sum is the greater segment }
 of the Base made by the Perpendicular, *viz.* }

16. Then by adding the Square of the said greater segment } $\frac{1399165}{142311} = AC$
 DC to 16 the Square of the Perpendicular AD , and }
 extracting the square Root of the sum, there comes forth }

17. Lastly, by multiplying the numbers in the first, eleventh, twelfth, thirteenth, fourteenth, fifteenth and sixteenth steps severally by the Denominator 142311, there will come forth these following whole numbers for the measures of the lines sought, *viz.*

- 426933 = BD
- 569244 = AD
- 711555 = AB
- 425600 = DE
- 710756 = AE
- 852533 = $BE = EC$
- 1705066 = BC
- 1278133 = DC
- 1399165 = AC

} in the Diagram belonging to this Question.

The Proof is easie to be made, by comparing the summ of the Squares of the numbers answering to the sides about the right-angle of every right-angled Triangle in the Diagram, to the number answering to the Square of the Hypothensal of such Triangle respectively.

QUEST. 109. (Quest. 21. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the Area thereof increased with one of the sides about the right-angle may make a Square; and that the summ of all the three sides may be a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and $a+1$; then divide the three sides severally by $a+1$, and take the Quotients for the sides of the Triangle sought, viz.
2. The summ of those three sides is
3. Which summ reduced to its least terms (by dividing the Numerator by the Denominator according to the general method of Division in Sect. 9. Chap. 5. Book 1.) will be
- Therefore (according to the Question) the said $4a+2$ must be equal to a Cube, which in the following ninth step I shall shew how to find out.
4. Moreover, the Area together with one of the sides about the right-angle of the Triangle in the first step must make a Square: But the Area (by multiplying half the Base into the Perpendicular) will be found
5. And one of the sides about the right-angle (to wit, the Perpendicular) is
6. Which side reduced to the same Denominator with the Area, (by multiplying as well the Numerator $2a+1$, as the Denominator $a+1$, by $a+1$.) will be
7. Now if the side in the last step be added to the Area in the fourth, the summ will be
8. Which summ reduced to its least terms, (by dividing the Numerator by the Denominator) will be
9. Therefore (according to the Question) $2a+1$ must be equated to a Square; and before in the third step it was found that $4a+2$ must be equated to a Cube; now because $4a+2$ is the double of $2a+1$, we must find out a Cube that may be the double of a Square; but the Cube 8 is the double of the Square 4, therefore let $4a+2$ be equated to 8, or $2a+1$ to 4; whence you will find $a = \frac{3}{2}$, and consequently $a+1 = \frac{5}{2}$. Wherefore according to the Positions in the first step, let a right-angled Triangle be formed from $\frac{3}{2}$ and $\frac{5}{2}$, and divide the three sides severally by $\frac{1}{2}$, so there will come forth the sides of the right-angled Triangle sought, to wit, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{4}{2}$, which solve the Question; for the Area $\frac{3}{2}$ together with $\frac{3}{2}$ (to wit, one of the sides about the right-angle) makes the Square 4; and the summ of all the three sides makes a Cube, to wit, 8.

$$\left\{ \begin{array}{l} \frac{2aa+2a+1}{a+1} = \text{the Hypoth.} \\ \frac{2aa+2a}{a+1} = \text{the Base;} \\ \frac{2a+1}{a+1} = \text{the Perpend.} \\ \frac{4aa+6a+2}{a+1} \\ 4a+2 \\ \frac{2aaa+3aa+a}{aa+2a+1} \\ \frac{2a+1}{a+1} \\ \frac{2aa+3a+1}{aa+2a+1} \\ \frac{2aaa+5aa+4a+1}{aa+2a+1} \\ 2a+1 \end{array} \right.$$

It is also evident by the premisses, that the Question is capable of innumerable Answers; in regard you may find out as often as you please a Cube and a Square, such, that the former shall be the double of the latter: As, by equating aaa to $8aa$, it will give the Cube 512, and the Square 256, the former of which is the double of the latter.

QUEST. 110. (Quest. 23. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the summ of all the three sides may be a Square; and that the said summ increased with the Area may make a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and 1, so the three sides will be these, to wit,
2. The

$$\left\{ \begin{array}{l} aa+1; aa-1; 2a \\ 2. The \end{array} \right.$$

2. The summ of all the three sides is $2aa + 2a$, which must be equal to a Square, let it therefore be equated to $eeaa$, } $2aa + 2a = eeaa$
viz. suppose } $a = \frac{2}{ee - 2}$
3. Whence, after due Reduction, you will find } $a = \frac{2}{ee - 2}$
4. Therefore by squaring the last Equation, there ariseth . } $aa = \frac{4}{eeee - 4ee + 4}$
5. Which last Equation doubled, is } $2aa = \frac{4}{eeee - 4ee + 4}$
6. And by adding the double of the Equation in the third step, to wit, $2a = \frac{4}{ee - 2}$ to the Equation in the fifth step, the summ gives this Equation, to wit,

$$2aa + 2a = \frac{4ee}{eeee - 4ee + 4}$$
7. The latter part of which Equation is manifestly a Square, and by Construction 'tis equal to the summ of all the three sides of the Triangle in the first step. It remains that the said summ with the Area make a Cube; but from the first step the Area is $aaa - a$, and according to the value of a in the third step, the Area $aaa - a$ will be reduced to this, to wit,

$$\frac{8ee - 2eeee}{eeeeee - 6eeee + 12ee - 8}$$
8. To which add the summ of all the three sides, to wit, the latter part of the Equation in the sixth step, that is, $\frac{4ee}{eeee - 4ee + 4}$, (which, by multiplying the Numerator and Denominator severally by $ee - 2$, will be reduced to $\frac{4eeee - 8ee}{eeeeee - 6eeee + 12ee - 8}$;) and there will come forth this summ, to wit,

$$\frac{2eeee}{eeeeee - 6eeee + 12ee - 8}$$
9. Which summ last produced must be equal to a Cube; and because by Construction the Denominator is a Cube, to wit, the Cube of $ee - 2$, it remains that we equate the Numerator $2eeee$ to some Cube; or by dividing $2eeee$ by eee it gives $2e$ to be equated to a Cube, which is easie to be done, for we may put $2e$ equal to any known cube-number, as ddd ;
10. Suppose therefore } $2e = ddd$
11. Then because the Denominator of the Fraction in the third step shews that } $e \sqsupseteq \sqrt{2}$
12. And consequently by doubling each part, } $2e \sqsupseteq \sqrt{8}$
13. It follows from the tenth and twelfth steps, that } $ddd \sqsupseteq \sqrt{8}$
14. Again, one of the sides about the right-angle is by Construction in the first step $aa - 1$, therefore } $aa \sqsupseteq 1$
15. But by the fourth step, } $\frac{4}{eeee - 4ee + 4} = aa$
16. Therefore from the two last steps, } $\frac{4}{eeee - 4ee + 4} \sqsupseteq 1$
17. And by multiplying each part in the sixteenth step by the Denominator $eeee - 4ee + 4$, 'tis manifest that } $4 \sqsupseteq eeee - 4ee + 4$
18. Whence, by adding $4ee$ to each part, } $4 + 4ee \sqsupseteq eeee + 4$
19. And by subtracting 4 from each part in the last step } $4ee \sqsupseteq eeee$
20. And by dividing each part in the nineteenth step by ee , } $4 \sqsupseteq ee$
21. And by extracting the square Root out of each part in the 20th step, 'tis evident that } $2 \sqsupseteq e$
22. And by doubling each part in the twenty-first step, } $4 \sqsupseteq 2e$
23. But by supposition in the tenth step, } $ddd = 2e$
24. Therefore from the two last steps, } $4 \sqsupseteq ddd$
25. And consequently, } $ddd \sqsupseteq 4$
26. Thus in the 13th and 25th steps it is found that the cube-number represented by ddd must be greater than the square Root of 8, but less than 4; and then the half of the cube-number taken within those limits shall be the number e , which being known, the number a will also be discovered by the third step: Lastly, a right-angled Triangle formed from the number a and unity, shall be that which is sought. From the premises ariseth this

CANON.

CANON.

27. First, take any cube-number greater than the square Root of 8, (*viz.* greater than $\sqrt[3]{1000}$, &c.) but less than 4; then take the half of that cube-number, and call it *e*; that done, divide 2 by the excess of the Square of the number *e* above 2, and call the Quotient *a*; lastly, let a right-angled Triangle be formed from the number *a* and 1; so shall this Triangle be that which is sought.

An Example in Numbers.

First, I take some cube-number within the limits prescribed in the Canon, as $\frac{27}{16}$, whose half, to wit, $\frac{27}{32}$ is the number *e*; then I divide 2 by the excess of the Square of $\frac{27}{32}$ above 2, so the Quotient $\frac{512}{217}$ is the number *a*; lastly, from $\frac{512}{217}$ and 1 I form a right-angled Triangle and find the three sides to be these; to wit, $\frac{302213}{47089}$, $\frac{211055}{47089}$ and $\frac{222208}{47089}$, which solve the Question: For the summ of those three sides make the Square $\frac{746426}{47089}$, whose side is $\frac{864}{217}$; and the said summ of the sides together with the Area $\frac{110108160}{10218513}$ makes the Cube $\frac{272027722}{10218513}$, whose side is $\frac{648}{217}$.

QUEST. III. (Quest. 24. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the summ of all the three sides may be a Cube; and that the said summ increased with the Area may make a Square.

RESOLUTION.

1. For the summ of all the three sides of the Triangle sought put } *s*
2. And for the Area } *a*
3. Then because the double of the Area is equal to the Product made by the mutual multiplication of the sides about the right-angle, the said Product is } $2a$
4. For one of the sides about the right-angle put } $\frac{1}{e}$
5. Then because the Product of the said sides is $2a$, 'tis manifest that by dividing $2a$ by $\frac{1}{e}$, the Quotient shall be the other side, to wit, } $2ae$
6. Therefore from the fourth and fifth steps the summ of the sides about the right-angle is } $2ae + \frac{1}{e}$
7. Which summ subtracted from *s* the summ of all the three sides, leaves the Hypotenusal, to wit, } $s - 2ae - \frac{1}{e}$
8. The Square of the said Hypotenusal is

$$ss + 4aee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e}.$$

9. And the summ of the Squares of the sides about the right-angle, to wit, of the Squares of $\frac{1}{e}$ and $2ae$ is $\frac{1}{ee} + 4aee.$

10. But the Square of the Hypotenusal is equal to the summ of the Squares of the sides about the right-angle, therefore from the eighth and ninth steps this Equation ariseth, *viz.* $ss + 4aee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e} = \frac{1}{ee} + 4aee.$

11. From which Equation, after due Reduction in order to find out the value of *e*, by the letters *s* and *a*, there will arise this following Equation, *viz.*

$$\frac{ss + 4a}{4sa} e - ee = \frac{2s}{4sa}.$$

12. And by resolving the last Equation according to the Canon in Sect. 10. Chap. 15. Book 1. the two values of *e* will be found these, to wit,

$$e = \frac{\frac{1}{2}ss + 2a}{4sa} + \sqrt{\frac{\frac{1}{4}ssss - 4aa - 6ssa}{16ssaa}};$$

$$e = \frac{\frac{1}{2}ss + 2a}{4sa} - \sqrt{\frac{\frac{1}{4}ssss - 4aa - 6ssa}{16ssaa}};$$

13. But 'tis evident that those values of *e* will not be rational unless the Numerator $\frac{1}{4}ssss - 4aa - 6ssa$ be equal to a Square. Moreover, the Question requires that the

the summ of all the three sides with the Area, to wit, $s + a$ may be equal to a Square; so we are faln upon this Duplicate equality,

$$\text{viz. } \begin{cases} \frac{1}{4}sss + 4aa - 6ssa = \square \\ s + a = \square \end{cases}$$

14. Now if in that Duplicate equality any known square number be taken for the value of s , then we may discover the number a . But the Question requires s , (that is, the summ of all the three sides of the Triangle sought,) to be a Cube, therefore some number which is both a Square and a Cube must be taken for the value of s ; let therefore the squared Cube 64 be put equal to s , and then the said Duplicate equality will be resolved into this,

$$\text{viz. } \begin{cases} 4aa - 24576a + 4194304 = \square \\ a + 64 = \square \end{cases}$$

15. Or you may divide the first of those two quantities by 4, then instead of that first quantity the Quotient is to be equated to a Square, and so this following Duplicate equality ariseth,

$$\text{viz. } \begin{cases} aa - 6144a + 1048576 = \square \\ a + 64 = \square \end{cases}$$

16. But because in the Duplicate equality last express'd, the square numbers 1048576 and 64 are unequal, I divide the greater of them by the less, and by the Quotient 16384 I multiply $a + 64$, and then the Product $16384a + 1048576$ is to be equated to a Square instead of $a + 64$; so at length this Duplicate equality remains to be resolved

$$\text{viz. } \begin{cases} aa - 6144a + 1048576 = \square \\ 16384a + 1048576 = \square \end{cases}$$

17. Now in order to resolve that Duplicate equality, first, supposing the former of the two quantities to be equated to be the greater, their difference is $aa - 22528a$, then according to the method used in divers preceding Questions of this Book, two such numbers are to be found out that the Product of their multiplication may make the said difference $aa - 22528a$, and that as well in the half of the summ as in the half of the difference of the said two numbers there may be found 1024, which is the side of the Square 1048576. But two such numbers are $11a$ and $\frac{1}{11}a - 2048$; the half-difference of these is $\frac{10}{11}a + 1024$, and the Square of $\frac{10}{11}a + 1024$ being equated to $16384a + 1048576$ will give $a = \frac{22424}{225}$, to wit, the Area of the Triangle sought.

Since then the value of a is found to be $\frac{22424}{225}$, and s was before assumed to be 64; according to these values of a and s , the twelfth step will discover the two values of e to be $\frac{25}{448}$ and $\frac{2}{176}$; by either of which if you resolve the positions in the fourth and fifth steps, you will find the sides about the right-angle to be $\frac{4032}{225}$ and $\frac{4400}{225}$; therefore the Hypothenufal is $\frac{2268}{225}$, and the summ of all the three sides is the Cube 64, to which if you add the Area $\frac{22424}{225}$ it makes the Square $\frac{22824}{225}$, whose side is $\frac{232}{15}$.

QUEST. 112. (Quæst. 25. Lib. 6. Diophant.)

To find a right-angled Triangle in rational numbers, that the Square of the Hypothenufal may be compos'd of some Square and its side; and that the Square of the said Hypothenufal being divided by one of the sides about the right-angle may give a number compos'd of a Cube and its side.

RESOLUTION.

1. For one of the sides about the right-angle put } a
2. And for the other, } aa
3. The summ of their Squares is equal to the Square of the Hypothenufal, } $aaaa + aa$
to wit, }
4. Now 'tis evident that $aaaa + aa$ is compos'd of the Square $aaaa$ and its side aa ; but if the said $aaaa + aa$ be divided by a , (which was put for one of the sides about the right-angle,) it gives the Quotient $aaa + a$, which is compos'd of a Cube with its side; so that it remains only to equate $aaaa + aa$ to a Square, that is, to find out a right-angled Triangle that one of the sides about the right-angle may be equal to the Square of the other of the same sides: But the preceding 95th Question shews how to find out such a Triangle; take if you please that in the first Example of the said Question, to wit, $\frac{4}{3}$, $\frac{16}{9}$, $\frac{20}{9}$, which will solve this 112th Question; for the Square of the Hypothenufal $\frac{20}{9}$, viz. $\frac{400}{81}$, is compos'd of the Square $\frac{256}{81}$ and its side $\frac{16}{9}$. Moreover, if the

if the said $\frac{400}{81}$ be divided by $\frac{4}{3}$, (one of the sides about the right-angle,) it gives the Quotient $\frac{120}{27}$, which is compos'd of the Cube $\frac{64}{27}$ and its side $\frac{4}{3}$.

A Prospect of all Diophantus's kinds of Duplicate Equality, shewing also at first sight in which of the preceding Questions they are resolved.

I. The first kind of Duplicate Equality is, when each of two Quantities to be equated to square numbers consists of an unknown Root or number a , with some absolute or known number, and the numbers prefixt to the Root a are equal to one another; as in the five following Examples.

$$1. \left\{ \begin{array}{l} \text{If} \dots : a + 192 = \square \\ \text{And} \dots : a + 128 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 8. of this Book 3.}$$

$$2. \left\{ \begin{array}{l} \text{If} \dots : 10a + 54 = \square \\ \text{And} \dots : 10a + 6 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 8.}$$

$$3. \left\{ \begin{array}{l} \text{If} \dots : 192 - a = \square \\ \text{And} \dots : 64 - a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 14. See the like in Quest. 50.}$$

$$4. \left\{ \begin{array}{l} \text{If} \dots : a - 27 = \square \\ \text{And} \dots : a - 15 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 15.}$$

$$5. \left\{ \begin{array}{l} \text{If} \dots : a + 12 = \square \\ \text{And} \dots : a - 8 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 16.}$$

II. The second kind of Duplicate Equality is, when each of two Quantities to be equated to Squares consists of an unknown Root a , and of one and the same known square number. Also, when the numbers prefixt to a in both Quantities are unequal, and the absolute numbers are unequal Squares; as in the six following Examples.

$$1. \left\{ \begin{array}{l} \text{If} \dots : 4 + a = \square \\ \text{And} \dots : 4 - a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 17.}$$

$$2. \left\{ \begin{array}{l} \text{If} \dots : 9 + a = \square \\ \text{And} \dots : 4 - a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 18.}$$

$$3. \left\{ \begin{array}{l} \text{If} \dots : 8a + 4 = \square \\ \text{And} \dots : 6a + 4 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 33.}$$

$$4. \left\{ \begin{array}{l} \text{If} \dots : 4 - 2a = \square \\ \text{And} \dots : 4 - 3a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Observat. 1. Quest. 33.}$$

$$5. \left\{ \begin{array}{l} \text{If} \dots : 4 + 2a = \square \\ \text{And} \dots : 4 - 3a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Observat. 1. Quest. 33.}$$

$$6. \left\{ \begin{array}{l} \text{If} \dots : 10a + 9 = \square \\ \text{And} \dots : 5a + 4 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Observat. 2. Quest. 33.}$$

III. The third kind of Duplicate Equality is, when each of two Quantities to be equated to Squares consists of some number of a , and an absolute number not a Square, but the numbers prefixt to a have such proportion to one another as a square number to a square number; as in the three following Examples.

V

1. If

$$1. \left\{ \begin{array}{l} \text{If} \dots a + 12 = \square \\ \text{And} \dots 4a + 12 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 29.}$$

$$2. \left\{ \begin{array}{l} \text{If} \dots \frac{22}{10}a + \frac{42}{10} = \square \\ \text{And} \dots \frac{12}{10}a + \frac{2}{10} = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 35.}$$

$$3. \left\{ \begin{array}{l} \text{If} \dots \frac{52}{9} - \frac{2}{3}a = \square \\ \text{And} \dots \frac{52}{9} - \frac{2}{3}a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 50.}$$

IV. The fourth kind of Duplicate Equality is, when two Quantities to be equated to Squares are diversly compos'd of aa and a , or of aa , a and absolute numbers; in which cases, to the end the Duplicate Equality propos'd may be explicable by rational numbers, these two things are requisite to be found therein; viz. First, either the numbers prefixt to aa , or the absolute numbers must be rational Squares: Secondly, the difference of the Quantities propos'd must be either some sole number of a , or compos'd of some number of a and an absolute number, or of some numbers of aa and a ; as in the following Examples.

$$1. \left\{ \begin{array}{l} \text{If} \dots aa + a = \square \\ \text{And} \dots aa - a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 19.}$$

$$2. \left\{ \begin{array}{l} \text{If} \dots 9aa + 2a = \square \\ \text{And} \dots 9aa + a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 20.}$$

$$3. \left\{ \begin{array}{l} \text{If} \dots 4aa + 5a = \square \\ \text{And} \dots 9aa + 5a = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 21. See the like in Quest. 22.}$$

$$4. \left\{ \begin{array}{l} \text{If} \dots 4aa + 3a - 1 = \square \\ \text{And} \dots 4aa - a - 1 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 20.}$$

$$5. \left\{ \begin{array}{l} \text{If} \dots 4aa + 15a = \square \\ \text{And} \dots 4aa - a - 4 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 31.}$$

$$6. \left\{ \begin{array}{l} \text{If} \dots 4aa + a + 1 = \square \\ \text{And} \dots 3a + 1 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 32.}$$

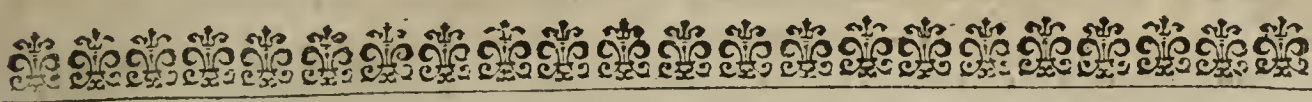
$$7. \left\{ \begin{array}{l} \text{If} \dots aa + d = \square \\ \text{And} \dots ba + d = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 55.}$$

$$8. \left\{ \begin{array}{l} \text{If} \dots aa + 2pa + hb = \square \\ \text{And} \dots \frac{a}{p} + 1 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 59.}$$

$$9. \left\{ \begin{array}{l} \text{If} \dots 3aa - 6a + 4 = \square \\ \text{And} \dots 4aa + 4 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 94. See the like in Quest. 93.}$$

$$10. \left\{ \begin{array}{l} \text{If} \dots ee + 1 = \square \\ \text{And} \dots e + 1 = \square \\ \text{What is the number } e? \end{array} \right\} \text{ Resolved in Quest. 99.}$$

$$11. \left\{ \begin{array}{l} \text{If} \dots aa - 6144a + 1048576 = \square \\ \text{And} \dots a + 64 = \square \\ \text{What is the number } a? \end{array} \right\} \text{ Resolved in Quest. 111.}$$



*A brief Exposition upon Monsieur Fermat's Analytical
 Invention, inserted in the late Edition of Monsieur
 Bachet's Comment upon Diophantus, printed at
 Toloze, Anno 1670.*

JAC. de BILLY, who collected the said Invention out of *Fermat's* Letters, divides it into three Parts. The first is an Improvement of one of *Diophantus's* kinds of Duplicate Equality; (to wit, the fourth and last kind in my preceding Prospect, but the sixth with *Bachet* in his Comment upon *Quest. 24. Book VI. Diophant.*) for whereas *Diophantus's* method of resolving that kind of Duplicate Equality, finds out but one value or two at the most of the Root sought, and sometimes the value found out is Negative, *viz.* less than nothing, *Fermat's* method finds out innumerable affirmative values.

The second part shews how to resolve two kinds of Triplicate Equality, by reducing them to a Duplicate Equality of the same kind with that above mentioned.

The third part shews how to equate a Quantity compos'd of five Terms to a Square; likewise a Quantity of four Terms to a Square or Cube, and for the most part to find out innumerable affirmative values of the Root sought.

These three parts I shall explain in order; and according to my usual method put a ; (instead of N by *Diophantus*) for a Root or number unknown, aa for the Square of that Root, aaa for the Cube, &c.

Concerning the First part of Fermat's Analytical Invention.

Fermat's Rule to find out innumerable affirmative values of the Root sought, in a Duplicate equality of the kind before mentioned, is this,

First, by the vulgar method of *Diophantus*, (explain'd in divers preceding Questions of this Book,) find out one value of the sought Root, (a ;) it matters not whether it be affirmative or negative; then to a joyn that value with its own sign, whether it be $+$ or $-$, and take the sum for a new Root instead of a ; then according to the said new Root let a new Duplicate equality be deduced from the first, and find out the value of a in the new Duplicate equality by the vulgar method. That done, connect this latter value to the first before found, with their own signs, and it will give a second value of the Root sought in the first Duplicate equality. In like manner by the help of this second value you may find out a third, and from the third a fourth, and so infinitely.

Note. After one value of the Root sought is found out in the vulgar way, there will always certainly arise a known Square in each of the two new Algebraick quantities to be equated to Squares, in the second, third, or any following Duplicate equality deduced from the first as the Rule doth direct, the reason whereof will be evident to him that diligently examines the Operation. But when those two known Squares are unequal, the lesser must be reduced to the greater, (in such manner as before hath been shewn in *Quest 21*, and *111.* of this Book,) to the end there may be one and the same known square number in each of the two quantities to be equated, and then the difference of the said quantities will be composed of some number of aa and some number of a ; which kind of difference is absolutely necessary in the use of *Fermat's* Rule before express'd. All which will be made manifest by the following Questions *113*, *114*, *115*, *116.*

QUEST. 113.

To find a number, that if to the Product of its Square multiplied by 32 you add unity, and from the sum subtract eight times the number sought, the remainder may be a Square. Also, that eight times the number sought, together with unity, may make a Square.

V 2.

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In like manner, by taking or substituting $a + \frac{188714}{1697809}$ for a new Root instead of a , and proceeding as before, you may find out a third number, and from the third a fourth, &c. to solve the Question proposed; but the operation in finding out Answers by this method is so excessively laborious, that for the most part he that hath found out two Answers, will hardly have patience to find out a third.

QUEST. 114.

To find a number, that if to the double of its Square you add unity, and from that sum subtract four times the number sought, the remainder may be a Square. Also, that if from unity you subtract the double of the number sought, the remainder may be a Square.

Let a be put for the number sought, and then the Question may be stated thus, viz.

1. If $2aa - 4a + 1 = \square$
2. And $-2a + 1 = \square$ } what is the number a ?

RESOLUTION.

3. If that Duplicate equality be resolved by the vulgar method, the only value of a will be found -4 , but this being less than nothing, I search out an affirmative value of a by Fermat's Rule, thus;

4. Instead of a I take for a new Root $a - 4$
5. Then instead of $2aa$ I take the double of the Square of the new Root $a - 4$, viz. $2aa - 16a + 32$
6. To which double Square I add unity, (as the Question requires,) and it makes this sum, $2aa - 16a + 33$
7. Then from that sum I subtract four times the new Root $a - 4$, (instead of $4a$ in the Duplicate equality in the first step,) viz. I subtract $+ 4a - 16$
8. So this remainder must be equal to a Square, viz. $2aa - 20a + 49 = \square$
9. Again, if from unity, the double of the new Root $a - 4$ be subtracted, the remainder must be equal to a Square, viz. $-2a + 9 = \square$
10. So in the eighth and ninth steps there is a new Duplicate equality deduced from that in the first and second steps; but because in the new Duplicate equality the known square numbers 49 and 9 are unequal, I divide 49 by 9, and by the Quotient $\frac{49}{9}$ I multiply $-2a + 9$ in the ninth step, and it gives to be equated to a Square $-\frac{28}{9}a + 49 = \square$
11. The eighth and tenth steps give this new Duplicate equality to be resolved, in which there is the same known square number, (to wit, 49, as Fermat's Rule requires,) viz. $2aa - 20a + 49 = \square$
 $-\frac{28}{9}a + 49 = \square$
12. Now to resolve the last preceding Duplicate equality, according to the vulgar method, I take the difference of the two quantities to be equated, which, by supposing the first quantity to be the greater, is $2aa - \frac{28}{9}a$
13. Then I search out two such quantities, that being mutually multiplied may make the said difference, and that as well in half the sum as in half the difference of the said two quantities there may be found 7, (the square Root of 49,) so I find the two quantities to be $\frac{126}{41}a - 14$ and $\frac{41}{63}a$
14. Half the difference of those two quantities is $\frac{5227}{5106}a - 7$
15. Then the Square of the said half-difference being equated to $-\frac{28}{9}a + 49$ (in the eleventh step) will give $a = \frac{161233426}{39150049}$
16. From which value of a I subtract 4, because the new Root was assumed to be $a - 4$, and the remainder is $\frac{133242}{39150049}$

I say $\frac{133242}{39150049}$ shall be a value of the Root a in the Duplicate equality in the first and second steps, and therefore will solve the Question proposed; for if $a = \frac{133242}{39150049}$, then $2aa - 4a + 1$ makes a Square whose side is $\frac{27466831}{39150049}$; also, $1 - 2a$ makes a Square whose side is $\frac{13327}{6257}$. Thus although a number less than nothing, to wit, -4 was found out for the first value of the Root a , yet by the help thereof an affirmative Root or number is found out to solve the Question proposed, and from that second Root you may find out a third.

QUEST. 115.

QUEST. 115.

To find a right-angled Triangle in rational numbers, that the summ of the sides about the right-angle may make a Square. Also, that either of the same sides being added to the Square of the other of them may make a Square: And that one of the said sides about the right-angle, together with a given multiple (suppose the triple) of the other of them may make a Square.

RESOLUTION.

1. It is manifest by the Theorem in *Quest. 24.* of this *Book*, that if the Fraction $\frac{1}{4}$ be divided into any two parts, then either part added to the Square of the other makes a Square; therefore, to solve the first part of the Question infinitely, put for the sides about the right-angle . . .
2. The summ of the Squares of those sides shall be equal to the Square of the Hypothensal, therefore . . .
3. And because the Question requires that one of the sides about the right-angle, together with the triple of the other, may make a Square, let $\frac{1}{4} - a$ be added to $3a$, and then the summ must be equal to a Square, *viz.* . . .
4. So in the second and third steps there is a Duplicate equality, which, (by the method in divers preceding Questions) may be reduced to this having equal absolute Squares, *viz.* . . .
5. Which Duplicate equality last exprest being resolved by the vulgar method gives but one single value of a , to wit, unity, (as hath been shewn in the preceding *Quest. 113.*) But according to the Positions in the first step of the Resolution of this Question, the number signified by a ought to be less than $\frac{1}{4}$, and consequently $1 = a$ will not solve this Question by affirmative numbers; for if a be 1, then $\frac{1}{4} - a$, (which was put for one of the sides about the right-angle) will be less than nothing. Therefore to search out another value of a , I proceed according to *Fermat's Rule* thus, *viz.* Instead of a , I take for a new Root $a + 1$, by which the Duplicate equality in the fourth step will be reduced to this that follows, (as before hath been shewn in *Quest. 113.*)

$$viz. \begin{cases} 32aa + 56a + 25 = \square \\ + \frac{200}{9}a + 25 = \square \end{cases}$$

6. Now if this Duplicate equality be resolved in the vulgar way, it will give $a = -\frac{1502072}{1697809}$, therefore $\frac{1502072}{1697809} (= a + 1)$ shall be a value of a , (besides 1,) in the Duplicate equality in the fourth step; and because the said $\frac{1502072}{1697809}$ is less than $\frac{1}{4}$, it shall be the first of the sides about the right-angle of the Triangle sought; then since $\frac{1}{4} - a$ was put for the other side, this latter, by subtracting the former from $\frac{1}{4}$, shall be $\frac{242872}{6791236}$, and by reducing the first side to the same Denominator with the latter, the two desired sides about the right-angle are $\frac{714236}{6791236}$ and $\frac{242872}{6791236}$, therefore the Hypothensal is $\frac{1201861}{6791236}$; which right-angled Triangle will solve the Question: For, first, the summ of the sides about the right-angle makes a Square, to wit, $\frac{1}{4}$; secondly, if $\frac{714236}{6791236}$ the first side about the right-angle, be added to the Square of the latter side $\frac{242872}{6791236}$, it makes a Square whose side is $\frac{242872}{6791236}$; thirdly, if $\frac{242872}{6791236}$ the latter side about the right-angle, be added to the Square of the first side $\frac{714236}{6791236}$, it makes a Square, whose side is $\frac{242872}{6791236}$; and lastly, if $\frac{242872}{6791236}$ the latter side about the right-angle, be added to the triple of the first side $\frac{714236}{6791236}$, it makes a Square, whose side is $\frac{1721}{2606}$.

QUEST. 116.

To find a right-angled Triangle, that the Square of the Area being added to the summ of the sides about the right-angle may make a Square.

RESOLUTION.

1. For the sides about the right-angle put } a and 1
2. Then the summ of the Squares of those sides must be equal to a Square, to wit, the Square of the Hypothensal, therefore } . . . $aa + 1 = \square$
3. And the Square of the Area being added to the summ of the sides about the right-angle must make a Square, *viz.* . . . } $\frac{1}{4}aa + a + 1 = \square$
4. So

4. So in the two last steps there is a Duplicate equality qualified as *Fermat's Rule* presupposeth, but if it be resolved in the ordinary way, it gives no value of a either affirmative or negative, I therefore reduce that Duplicate equality to another that shall have equal numbers of aa , by multiplying $\frac{1}{4}aa + a + 1$ in the third step by the Square 4, so it produceth $aa + 4a + 4$, and now this following Duplicate equality comes to be resolved,

$$\text{viz. } \begin{cases} aa + 1 = \square \\ aa + 4a + 4 = \square \end{cases}$$

5. Which Duplicate equality last express being resolved in the vulgar way, (explain'd in *Quest.* 20; 31. of this *Book*,) will give this negative value of a , viz.

$$a = -\frac{1}{8}$$

6. But that value of a being less than nothing, I renew the work according to *Fermat's method*, viz. instead of a I take for a new Root

$$a = \frac{1}{8}$$

7. By which new Root $a = \frac{1}{8}$ I form this Duplicate equality out of that in the second and third steps, (after the manner before explain'd in *Quest.* 113.)

$$\begin{cases} aa - \frac{1}{4}a + \frac{1}{64} = \square \\ aa - \frac{1}{4}a + \frac{28}{64} = \square \end{cases}$$

8. Then I reduce that new Duplicate equality to another that shall have in each quantity one and the same absolute square number, (as *Fermat's Rule* requires,) so this ariseth, viz.

$$\begin{cases} 64aa - 240a + 289 = \square \\ 18496aa - 69360a + 289 = \square \end{cases}$$

9. Now if the Duplicate equality last express be resolved in the vulgar way, the value of a will again be found negative; and if I should assume a third Root and proceed as before, the work would be intollerably tedious, and perhaps produce another negative value. This difficulty leads me to another way by which *Fermat* hath solved divers knotty Problems, but it seems to me to depend more upon chance than Art; however, I shall try whether it will find out an affirmative value of a to solve the Duplicative equality in the eighth step or not.

First then, supposing the latter of the two quantities in the eighth step to be the greater, their difference is $18432aa - 69120a$, then I seek two such numbers that being mutually multiplied may make 18432, and that their summ may make 272, (to wit, the double of 136 the side of the Square 18496 which is prefixt to aa in the eighth step,) so I find 144 and 128, which are very luckily rational, and therefore fit for the present purpose; then these two quantities $144a - 540$ and $128a$ being mutually multiplied will produce the above-mentioned difference, and in half the summ of those Factors there will be found 136a the side of the Square 18496aa in the eighth step: Then by equating the Square of half the summ of the said Factors $144a - 540$ and $128a$, viz. the Square of $136a - 270$, to $18496aa - 69360a + 289$ (in the eighth step,) there will thence arise $\frac{24081}{4080}$ for the value of a ; wherefore $a = \frac{1}{8}$, (which was assumed for the new Root in the sixth step instead of a ,) gives $\frac{24081}{4080}$ for the value of a in the first Duplicate equality in the second and third steps, that is, one of the sides about the right-angle of the Triangle sought; but the other of the said sides was assumed to be 1, therefore the Hypotenusal shall be $\frac{24082}{4080}$; and the three sides of the right-angled Triangle sought are $\frac{24081}{4080}$, 1, $\frac{24082}{4080}$; for if the summ of the sides about the right-angle, that is, of $\frac{24081}{4080}$ and 1 be added to the Square of the Area it makes a Square, whose side is $\frac{24082}{4080}$. And if you desire a second right-angled Triangle to solve the Question, you may put $a = \frac{24081}{4080}$ for a new Root, and proceed as before.

Concerning the second Part of Fermat's Analytical Invention.

Diophantus doth not so much as mention a Triplicate Equality, but *Fermat* shews how to solve two kinds thereof. The first is, when three quantities to be equated to Squares are such, that every one of them is compos'd of some number of a , either affirmative or negative, and one and the same known affirmative square number. The second kind of Triplicate equality is, when three quantities to be equated to Squares are such, that every one of them is compos'd of the same affirmative number of aa , and some number of a either affirmative or negative, without any absolute number. How each of these kinds of Triplicate equality may be resolved the following Questions will make manifest.

QUEST. 117.

QUEST. 117.

To find a number, that if it be multiplied by three given numbers severally, suppose by 1, 2, 5, and to every one of the Products one and the same given Square, suppose 1, be added, the three summs may be Squares.

RESOLUTION.

1. For the number sought put } a
2. Then the Question requires that every one of these three summs }
may make a Square, viz. } $a + 1 = \square$
 $2a + 1 = \square$
 $5a + 1 = \square$
3. Now to resolve that Triplicate equality, first form a Square from any number of $a + 1$ the side of the given Square 1, as from $a + 1$, whose Square is $aa + 2a + 1$; then divide $aa + 2a$ (the two first terms of that Square) by any one of the three numbers prefixt to a in the said Triplicate equality, as by 1, which is tacitly prefixt to a in the first quantity $a + 1$, and take the Quotient $aa + 2a$ for a new Root instead of a , (which was put for the number sought,) whereby the first part of the Question is solved indefinitely; for if $aa + 2a$ be put for the number sought, then unity added to it makes a Square, to wit, $aa + 2a + 1$. Then multiply the said $aa + 2a$ by 2 which is prefixt to a in the second quantity $2a + 1$, and it produceth $2aa + 4a$, to which add the given Square 1, and it makes $2aa + 4a + 1$ to be equated to a Square. Again, multiply $aa + 2a$ by 5 which is prefixt to a in the third quantity $5a + 1$, and it makes $5aa + 10a$; to this add unity and it makes $5aa + 10a + 1$, to be equated to a Square; hence the following Duplicate equality ariseth,

$$\text{viz. } \begin{cases} 2aa + 4a + 1 = \square \\ 5aa + 10a + 1 = \square \end{cases}$$

4. This Duplicate equality being resolved by the vulgar way, will give $a = -6$, by which value of a if $aa + 2a$ be expounded, it makes 24; (for the Square of -6 is $+36$, to which if you add -12 the double of -6 , it makes $+24$;) wherefore 24 is the number sought, and will solve the Question: For if unity be added first to 24, then to 48 the double of 24, and lastly, to 120 the quintuple of 24, the three summs are Squares, to wit, 25, 49, 121.

If you desire another number besides 24 to solve the Question proposed, you may assume $a = -6$ for a new Root of the Duplicate equality last resolved, and thence (by the method before explained in the first Part) find out a second number to solve that Duplicate equality, and consequently the Question.

Note. When in a Triplicate equality of the first kind before defined, the greatest of the three numbers of a is equal to the summ of the other two, then in such case that Triplicate equality, although it may be possible in it self, is inexplicable by the method of resolving the preceding Quest. 117. As, for example, if these three quantities be proposed to be equated to as many Squares, viz. $5a + 1$; $16a + 1$; $21a + 1$; where the greatest number of a is equal to the summ of the other two, (for $21 = 16 + 5$;) and the value of the Root a is 3, according to which, those three quantities being expounded will give these three Squares, 16, 49, 64; it will be in vain to seek out any Answer to that Triplicate equality by Fermat's Rule, for it will produce an absurd or fruitless Equation, as you will find upon tryal.

QUEST. 118.

To find a number, that if it be multiplied by three given numbers, suppose by 3, 4, 5, and the Products be severally subtracted from 1 a given Square, the three remainders may be Squares.

RESOLUTION.

1. For the number sought put } a
2. Then the Question requires that these three Remainders may }
be Squares, viz. } $1 - 3a = \square$
 $1 - 4a = \square$
 $1 - 5a = \square$
3. This Triplicate equality may be resolved like that in the foregoing Quest. 117. For first, I form a Square from 1 — any number of a , as from $1 - 3a$, whose Square is $9aa - 6a + 1$; then I divide $9aa - 6a$ (the two first terms of that Square,) by 3 which

which is prefixt to a in the first quantity $1 - 3a$, and the Quotient is $3aa - 2a$, this with its contrary signs is $-3aa + 2a$, which I put for a new Root instead of a (the number sought,) whereby the first part of the Question is satisfied infinitely; for if the triple of $-3aa + 2a$ be subtracted from the given Square 1, the remainder is a Square, to wit, $9aa - 6a + 1$. Then I multiply the said new Root $-3aa + 2a$ by 4 and 5 severally, (which are prefixt to a in the second and third quantities of the Triplicate equality in the second step,) and subtracting the Products $-12aa + 8a$ and $-15aa + 10a$ severally from the given Square 1, the remainders $12aa - 8a + 1$ and $15aa - 10a + 1$ are to be equated to Squares, so it remains only to solve this following Duplicate equality,

$$\text{viz. } \begin{cases} 12aa - 8a + 1 = \square \\ 15aa - 10a + 1 = \square \end{cases}$$

4. This Duplicate equality being solved in the ordinary way, gives $a = \frac{6}{11}$, by which value of a , the new Root $-3aa + 2a$ being expounded will give $\frac{24}{121}$ for the value of a in the Triplicate equality in the second step. Wherefore $\frac{24}{121}$ will solve the Question; for if its triple, quadruple and quintuple be severally subtracted from unity, the three remainders are Squares, to wit, $\frac{4}{121}, \frac{2}{121}, \frac{1}{121}$.

QUEST. 119.

To find a number, as also four other numbers in Geometrical proportion continued, that if from these Proportionals severally the first number be subtracted, the three remainders may be Squares.

RESOLUTION.

1. For the first number sought put } $a - 1$
2. Then multiply a into any four known numbers continually proportional, as into 1, 2, 4, 8, and assume the Products to be the four Proportionals sought, viz. } $a, 2a, 4a, 8a$
3. Then subtract the number in the first step from those four Proportionals severally, and every one of the remainders must make a Square; but the first remainder is manifestly the Square 1, it remains therefore to resolve this Triplicate equality, viz. . . . } $\begin{cases} a + 1 = \square \\ 3a + 1 = \square \\ 7a + 1 = \square \end{cases}$

Now to resolve that Triplicate equality you may take $aa + 2a$ for a new Root instead of a , whereby the first part of the Triplicate equality will be solved indefinitely, for if $aa + 2a$ be increased with 1 it makes a Square, to wit, $aa + 2a + 1$; then the two other parts of the said Triplicate equality (by the like Operation as was used in Quest. 117.) will be converted into this following Duplicate equality,

$$\text{viz. } \begin{cases} 3aa + 6a + 1 = \square \\ 7aa + 14a + 1 = \square \end{cases}$$

4. Which Duplicate equality being resolved in the vulgar way gives $a = -12$, whence $aa + 2a$ (the new Root) will be found 120; (for the Square of -12 is $+144$, to which if you add -24 , that is, $2a$, it makes 120.) Therefore the first number sought by the Question is 119, (that is, $a - 1$), and the four numbers required to be in continual proportion are 120, 240, 480, 960; (which answer to $a, 2a, 4a, 8a$, in the second step:) for if 119 be subtracted from those four Proportionals severally, the remainders are the Squares 1, 121, 361, 841.

QUEST. 120.

Three square numbers Geometrically proportional being given, to find a number, that if it be added to those Proportionals severally, the three sums may be Squares.

RESOLUTION.

1. Let the three given Squares in continual proportion be : : . } $1, 4, 16 \div$
2. For the number sought put } a
3. Then the Question requires } $\begin{cases} a + 1 = \square \\ a + 4 = \square \\ a + 16 = \square \end{cases}$

X

4. Now

4. Now in regard the three known Squares in that Triplicate equality are unequal, they must be reduced to the same Square, which may be done thus, viz. Because a Square multiplied by a Square produceth a Square, multiply the first quantity $a + 1$ by 64, (the Product of 4 into 16,) and it makes $64a + 64$ to be equated to a Square. Again, multiply the second quantity $a + 4$ by 16, (the Product of 1 into 16,) and it gives $16a + 64$ to be equated to a Square. Likewise, multiply the third quantity $a + 16$ by 4, (the Product of 1 into 4,) and it produceth $4a + 64$ to be equated to a Square. So this Triplicate equality comes forth to be resolved,

$$\text{viz. } \begin{cases} 64a + 64 = \square \\ 16a + 64 = \square \\ 4a + 64 = \square \end{cases}$$

5. This Triplicate equality having in every one of its three quantities the same Square 64 may be resolved (like that in the foregoing Questions 117, 119.) thus, viz. First, form a Square from any number of $a + 1$ the side of the known Square 64, as from $2a + 8$, the Square whereof is $4aa + 32a + 64$; then divide $4aa + 32a$ by 4, which is prefixt to a in the third quantity $4a + 64$, and take the Quotient $aa + 8a$ instead of the Root a , (which was put for the number sought;) whence the last part of the Triplicate equality in the fourth step is solved indefinitely, for four times $aa + 8a$ together with 64 makes a Square, to wit, $4aa + 32a + 64$. Again, by taking the said $aa + 8a$ instead of the Root a , the first and second parts of the said Triplicate equality will be reduced to this Duplicate equality,

$$\text{viz. } \begin{cases} 64aa + 512a + 64 = \square \\ 16aa + 128a + 64 = \square \end{cases}$$

6. Which Duplicate equality being resolved in the vulgar way gives $a = \frac{204}{105}$, whence $aa + 8a$ (the new Root) will be found $\frac{247226}{11025}$, which is the number sought by the Question; for if 1, 4, 16 be severally added to the said $\frac{247226}{11025}$, the three summs will be the Squares of these sides $\frac{229}{105}$, $\frac{526}{105}$, $\frac{224}{105}$.

I shall now proceed to the second kind of Triplicate equality, and explain it by Questions!

QUEST. 121. (Probl. 4. cap. 1. part. 2. Dioph. redivivi.)

To find a number, that if it be multiplied by every one of three given numbers Geometrically proportional, suppose by 1, 2, 4, and the Products be added severally to the Square of the number sought, the three summs may be Squares.

RESOLUTION.

1. For the number sought put a
 2. Then multiply that number a by the three given Proportionals 1, 2, 4, and to the Products severally add the Square of the said number a , so (according to the import of the Question) these three summs must make Squares, viz.

$$\begin{cases} aa + a = \square \\ aa + 2a = \square \\ aa + 4a = \square \end{cases}$$

 3. Now that Triplicate equality being of the second kind before defined, must be reduced to a Triplicate equality of the first kind thus, viz. Take $\frac{1}{a}$ instead of the Root a , (the number sought,) whence the first quantity $aa + a$ will be converted into $\frac{1}{aa} + \frac{1}{a}$; the second quantity $aa + 2a$ will be reduced to $\frac{1}{aa} + \frac{2}{a}$; and the third quantity $aa + 4a$ to $\frac{1}{aa} + \frac{4}{a}$. Then because a Square multiplied by a Square produceth a Square, if every one of the three new terms $\frac{1}{aa} + \frac{1}{a}$, $\frac{1}{aa} + \frac{2}{a}$ and $\frac{1}{aa} + \frac{4}{a}$, which are to be equated to Squares, be multiplied by the Square aa , that Triplicate equality will be reduced to this,

$$\text{viz. } \begin{cases} 1 + a = \square \\ 1 + 2a = \square \\ 1 + 4a = \square \end{cases}$$

4. This Triplicate equality falling under the first form may be resolved like that in the preceding Quest. 117. thus, viz. Instead of a I take for a new Root $aa + 2a$, which increased with 1 makes a Square, to wit, $aa + 2a + 1$, whereby the first part of the Triplicate

Triplicate equality is solved indefinitely; then by multiplying $aa + 2a$ by 2 and 4 severally, and adding 1 to each Product, this Duplicate equality ariseth,

$$\text{viz. } \begin{cases} 2aa + 4a + 1 = \square \\ 4aa + 8a + 1 = \square \end{cases}$$

5. Whence $a = -\frac{20}{7}$, by which if $aa + 2a$ (the new Root) be resolved it makes $\frac{120}{49}$; this divided by 1, because $\frac{1}{a}$ was taken instead of a in the first Triplicate equality, gives $\frac{42}{120}$, which will solve the Question; for if $\frac{42}{120}$ be multiplied severally by 1, 2, 4, and the Products be added severally to the Square of the said $\frac{42}{120}$, the three summs will be Squares, whose sides are $\frac{21}{120}$, $\frac{112}{120}$, $\frac{161}{120}$.

QUEST. 122.

To find three square numbers, whose sum added to their three sides severally, may make Squares.

RESOLUTION.

1. Divide some square number, suppose 121, into three such Squares, that the greatest may exceed the sum of the other two, such are 4, 36, 81; then for the three Squares sought put $4aa, 36aa, 81aa$
2. The sides of those Squares are $2a, 6a, 9a$
3. And the sum of the same Squares is $121aa$
4. But if the said sum $121aa$ be added to the three sides in the second step severally, every one of the three summs must make a Square; hence this Triplicate equality ariseth, viz. $121aa + 2a = \square$, $121aa + 6a = \square$, $121aa + 9a = \square$
5. Which Triplicate equality; (by the method delivered in the preceding Quest. 121.) will be reduced to this Triplicate equality, viz. $121 + 2a = \square$, $121 + 6a = \square$, $121 + 9a = \square$
6. Now to resolve the Triplicate equality last exprest, I take (according to the Rule before given) $2aa + 2a$ for a new Root instead of a , whence the first part of the said Triplicate equality will be solved indefinitely, and the second and third parts will be converted into this Duplicate equality,

$$\text{viz. } \begin{cases} 12aa + 132a + 121 = \square \\ 18aa + 198a + 121 = \square \end{cases}$$

7. Which Duplicate equality being resolved in the ordinary way will give $a = -\frac{22}{47}$; according to which, the new Root $2aa + 2a$ will be found $\frac{6220}{2209}$, and this divided by 1, (because $\frac{1}{a}$ was put instead of a , in reducing the Triplicate equality in the fourth step to that in the fifth,) gives $\frac{220}{62920}$ for the value of a in the Triplicate equality in the fourth step; according to which last mentioned value, the three sides in the second step are $\frac{44}{62920}$, $\frac{132}{62920}$, $\frac{1288}{62920}$, whose Squares will solve the Question; for if those sides be severally added to the sum of their Squares, the three summs will be Squares, whose sides are $\frac{22462}{62920}$, $\frac{22241}{62920}$, $\frac{42211}{62920}$.

Again, if this Triplicate equality were proposed to be resolved, viz. $aa + 2a = \square$, $4aa + 3a = \square$, $16aa + 6a = \square$

It may be reduced to this by putting $\frac{1}{a}$ for a , $1 + 2a = \square$, $4 + 3a = \square$, $16 + 6a = \square$

And that last exprest, (by what hath been said in Quest. 120.) may be reduced to this, $64 + 128a = \square$, $64 + 48a = \square$, $64 + 24a = \square$

And this Triplicate equality being resolved like those in Quest. 117, 120. will give $\frac{120}{329}$ for the value of a in either of the two last preceding Triplicate equalities; and lastly, by dividing 1 by $\frac{120}{329}$, it gives $\frac{220}{110616}$ for the value of a in the Triplicate equality proposed in the eighth step.

QUEST. 123. (Probl. 9. in cap. 1. part. 2. Dioph. redivivi.)

To find a number, that if it be multiplied by three given numbers in Arithmetical Progression, and the Products be subtracted from the Square of the number sought, the three remainders may be Squares. But the following Resolution presupposeth, (for the reason before given in the Note at the end of Quest. 117.) that the greatest of the three numbers given is not equal to the sum of the other two. Let therefore the given numbers be 3, 4, 5.

RESOLUTION.

1. For the number sought put $\rightarrow a$
2. Then (according to the import of the Question) this Triplicate equality must be resolved, viz. $\left\{ \begin{array}{l} aa - 3a = \square \\ aa - 4a = \square \\ aa - 5a = \square \end{array} \right.$
3. But that (according to the Rule in Quest. 121.) will be reduced to this, viz. $\left\{ \begin{array}{l} 1 - 3a = \square \\ 1 - 4a = \square \\ 1 - 5a = \square \end{array} \right.$

In which Triplicate equality last express the value of the Root a was by Quest. 118: found $\frac{1-24}{1-21}$, but because in the reduction of the Triplicate equality in the second step, $\frac{1}{a}$ was taken for a new Root instead of a , we must divide 1 by the said $\frac{1-24}{1-21}$, so there ariseth $\frac{1-21}{24}$ for the number sought by the Question; for if $\frac{1-21}{24}$ be multiplied by 3, 4, 5 severally, and the Products be severally subtracted from the Square of $\frac{1-21}{24}$, the three remainders will be the Squares of $\frac{1-7}{24}$, $\frac{1-5}{24}$ and $\frac{1-3}{24}$.

Note 1. Sometimes, when four, five or more quantities are to be equated to Squares, they may be resolved by the method before explain'd: As, for example,

If this Quadruplicate equality be propos'd to be resolved, $\left\{ \begin{array}{l} 20a + 64 = \square \\ 12a + 16 = \square \\ 8a + 4 = \square \\ 2a + 1 = \square \end{array} \right.$

You may (by the method in the preceding Quest. 120.) reduce that four-fold equality to this, viz. $\left\{ \begin{array}{l} 20a + 64 = \square \\ 48a + 64 = \square \\ 128a + 64 = \square \\ 128a + 64 = \square \end{array} \right.$

Which last Quadruplicate equality is in effect but a Triplicate equality, for there are two Terms which happen to be the same, to wit, $128a + 64$; and therefore you may resolve that Triplicate equality by the method before delivered.

Note 2. Sometimes also by the preceding method of resolving a Triplicate equality of the first kind, you may resolve one of Diophantus's kinds of Duplicate equality, (to wit, that explain'd in the preceding Quest. 33.) more easily than by his method, as will appear by the following Quest. 124, 125, 126.

QUEST. 124. (The same with the foregoing Quest. 33.)

To find a number less than 2, and such, that if it be multiplied severally by two numbers given, suppose by 6 and 8, and to each of the Products there be added the same given Square 4, the two summs may be Squares.

RESOLUTION.

1. For the number sought put $\rightarrow a$
2. Then the Question requires this Duplicate equality to be resolved, $\left\{ \begin{array}{l} 6a + 4 = \square \\ 8a + 4 = \square \end{array} \right.$
3. To which end you may proceed thus; First, (according to the method of resolving a Triplicate equality before delivered,) form a Square from $a + 2$, (2 being the side of the given Square 4;) so that Square will be $aa + 4a + 4$; then take $\frac{1}{6}$ part of $aa + 4a$, ($\frac{1}{6}$ part, because 6 is prefixt to a in the first part of the given Duplicate equality,) and it is $\frac{1}{6}aa + \frac{2}{3}a$, which is to be assumed for a new Root instead of a . Whence 'tis evident, that if the given Square 4 be added to six times $\frac{1}{6}aa + \frac{2}{3}a$, it makes a Square, to wit, $aa + 4a + 4$, whereby the first part of the Question is solved indefinitely. Then multiply the new Root $\frac{1}{6}aa + \frac{2}{3}a$ by 8, and to the Product add 4,

so

so there comes forth $\frac{4}{3}aa - \frac{1}{3}a - 4$ to be equated to a Square, the side whereof must be so feigned that $\frac{1}{6}aa - \frac{2}{3}a$ (the new Root) may be less than 2; but if $\frac{1}{6}aa - \frac{2}{3}a$ be less than 2, it will follow that a is less than 2. Wherefore let $\frac{4}{3}aa - \frac{1}{3}a - 4$ be equated to a Square, so, as that a may be less than 2; to which end the side of the said Square may be feigned $2 - \frac{1}{3}a$ any number of a greater than $3\frac{2}{3}a$; let therefore the said side be feigned $4a - 2$, and then the Square of $4a - 2$ being equated to $\frac{4}{3}aa - \frac{1}{3}a - 4$, will give $a = \frac{1}{3}$, by which, if the new Root $\frac{1}{6}aa - \frac{2}{3}a$ be resolved, it makes $\frac{1}{12}$ for the number sought by the Question proposed: For first, it is less than 2; secondly, six times $\frac{1}{12}$ together with 4 makes a Square, to wit, $\frac{1}{2}$, whose side is $\frac{1}{2}$; and lastly, eight times $\frac{1}{12}$ together with 4 makes the Square $\frac{1}{3}$, whose side is $\frac{1}{3}$.

Again, if this Duplicate equality were proposed, $\dots \left\{ \begin{array}{l} 4 - 6a = \square \\ 4 - 8a = \square \end{array} \right.$

You may put $\frac{2}{3}a - \frac{1}{6}aa$ for a new Root instead of a , and then proceed as before.

Again, if this Duplicate equality were proposed, $\dots \left\{ \begin{array}{l} 4 + 6a = \square \\ 4 - 8a = \square \end{array} \right.$

You may take $\frac{1}{6}aa - \frac{2}{3}a$ for a new Root instead of a , and then proceed as before.

QUEST. 125. (Probl. 10. in cap. 1. part. 2. Dioph. redivivi.)

To find a number, as also three other numbers in Geometrical proportion, that if from these Proportionals severally the first number be subtracted, the three remainders may be Squares.

RESOLUTION.

1. For the first number sought put $\dots \left\{ \begin{array}{l} 2a - 1 \end{array} \right.$
2. And for the three continual Proportionals sought put $\dots \left\{ \begin{array}{l} a, 2a, 4a \end{array} \right.$
3. From which if you subtract the first number sought, the remainders are $\dots \left\{ \begin{array}{l} 1 - a, 1, 1 + 2a \end{array} \right.$
4. Among which remainders the mean is a Square, wherefore it remains to equate each of the two extremes to a Square, viz. $\left\{ \begin{array}{l} 1 - a = \square \\ 1 + 2a = \square \end{array} \right.$

Now to resolve that Duplicate equality, you may take $2aa + 2a$ for a new Root instead of a , whence $1 + 2a$ will be converted into the Square $4aa + 4a + 1$, and $1 - a$ into $1 - 2a - 2aa$, which must be equated to a Square, but with this Caution, That the new Root $2aa + 2a$ may be less than 1, and that $4aa + 4a$ may exceed 1, and consequently that the value of a may fall between $\frac{2}{3}$ and $\frac{1}{3}$; to which end, the side of the said Square may be feigned $1 - 2a$, whose Square $4aa - 4a + 1$ being equated to $1 - 2a - 2aa$, gives $a = \frac{1}{3}$; therefore the new Root $2aa + 2a$ is $\frac{2}{3}$, and the first number sought, which was represented by $2a - 1$, is consequently $\frac{2}{9}$, and the three desired Proportionals are $\frac{2}{9}, \frac{4}{9}, \frac{8}{9}$; which will solve the Question, as may easily be proved.

QUEST. 126.

To find two numbers, such, that if their sum be increased and lessened, as well by their difference as the difference of their Squares, the sums and remainders may be Squares.

RESOLUTION.

1. If unity be divided into any two parts their difference is equal to the difference of their Squares, (as hath been shewn in Quest. 53. of this Book,) therefore for the two numbers sought put $\dots \left\{ \begin{array}{l} \frac{1}{2} + a \text{ and } \frac{1}{2} - a \end{array} \right.$
2. Whence their difference, as also the difference of their Squares is $\dots \left\{ \begin{array}{l} 2a \end{array} \right.$
3. It remains therefore that the sum of the two numbers in the first step, to wit, unity, being increased and lessened by $2a$, the sum and remainder may be Squares; hence this Duplicate equality ariseth, viz. $\left\{ \begin{array}{l} 1 + 2a = \square \\ 1 - 2a = \square \end{array} \right.$
4. Now to resolve that Duplicate equality, you may take $\frac{1}{2}aa + a$ for a new Root instead of a , whence $1 + 2a$ will be converted into the Square $aa + 2a + 1$, and $1 - 2a$ into $1 - 2a - aa$; which must be equated to a Square, but with this Caution, That the new Root $\frac{1}{2}aa + a$ may be less than $\frac{1}{2}$, and consequently a less than $\sqrt{2} - 1$, that

that is, less than $\frac{4}{10}$; but to cause that effect, the side of the said Square may be feigned to be $1 - 2a - aa$ any number of a greater than a , as $1 - 3a$, the Square whereof being equated to the said $1 - 2a - aa$ will give $a = \frac{2}{5}$, therefore the new Root $\frac{1}{2}aa - a$ is $\frac{1}{2} \cdot \frac{4}{25}$, according to which, the Positions in the first step being expounded, will give $\frac{4}{50}$ and $\frac{1}{50}$ for the numbers sought: For if to their sum, which is $\frac{1}{10}$, you add and subtract their difference $\frac{3}{50}$, the sum and remainder are Squares, to wit, $\frac{4}{25}$ and $\frac{1}{25}$; and these will also come forth when the difference of the Squares of the two numbers $\frac{4}{50}$ and $\frac{1}{50}$ is added to and subtracted from their sum $\frac{1}{10}$, because the difference of the two numbers is equal to the difference of their Squares.

Concerning the third Part of Fermat's Analytical Invention.

I. The Scope of this third Part is chiefly to shew how to equate a quantity compos'd of five Terms, viz. of some numbers of $aaaa$, aaa , aa , a with an absolute (or known) number, to a Square; as also to equate a quantity compos'd of four Terms to a Square or Cube; and that in such manner, that for the most part innumerable values of the unknown Root a may be found out.

II. In equating a quantity compos'd of five Terms to a Square, one of these two things is absolutely necessary, viz. either the first Term must be a Biquadrate, or else the last Term, to wit, the absolute number, must be a rational Square. Likewise, in equating a quantity consisting of four Terms to a Square, one of the extremes must be a Square. And lastly, in equating a quantity of four Terms to a Cube, one of the extremes must be a Cube.

III. When a quantity compos'd of five Terms is given to be equated to a Square, and the first Term is a Biquadrate, but the absolute number, that is, the last Term, hath not a rational square Root, then the side of the Square must be feigned so, as that in the Square it self there may be found the same numbers of $aaaa$, aaa and aa as are in the quantity given to be equated, to the end that those three first Terms may by Reduction destroy one another, and consequently an Equation remain between some number of a and an absolute number, whence the value of the Root a , if it hath any possible value, may be expressible by some rational number either affirmative or negative; but how to feign the said side so as to cause that effect, the following Proposition and Canon will shew; where to evidence the certainty thereof, I shall assume b, c, d to stand for Coefficients or known numbers prefixt to aaa , aa and a , and n for the absolute number, (or last Term,) which in this case is supposed to have no rational square Root.

P R O P.

1. Let this quantity be given to be equated to a Square, viz. $aaaa + baaa + caa + da + n = \square$

C A N O N.

2. When $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, then let the side of the Square sought be feigned $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$
 3. But if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, then let the side of the Square be feigned $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers.

4. Let this quantity be given to be equated to a Square, viz. $aaaa + 4aaa + 6aa + 2a + 7 = \square$
 5. Here, because $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, viz. half 6 exceeds $\frac{1}{8}$ of the Square of 4, the first part of the Canon gives this feigned side of a Square, viz. $aa + 2a + 1$
 6. Then by equating the Square of the said side $aa + 2a + 1$ to the given quantity $aaaa + 4aaa + 6aa + 2a + 7$, the value of a , after due Reduction, will be found 3, by which if the given quantity be resolved, it makes the Square 256, whose side is 16.
 7. Again, let this quantity be given to be equated to a Square, $aaaa + 4aaa + 2aa - 6a + 11 = \square$
 8. Here, because $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, viz. $\frac{1}{8}$ of the Square of 4 exceeds the half of 2, the latter part of the Canon gives this feigned side of a Square, viz. $1 - 2a - aa$

9. Then

9. Then by equating the Square of the said feigned side $1 - 2a - aa$ to the given quantity $aaaa + 4aaa + 2aa - 6a + 11$, the value of the Root a will be found 5, by which if the given quantity be resolved it makes the Square 1156, whose side is 34.

IV. When a quantity compos'd of five Terms is to be equated to a Square, and the first Term is not a perfect Biquadrate, but the last Term, that is, the absolute number is a Square, then the side of a Square must be feigned so, as that in the Square it self there may be found the same numbers of aa, a and absolute number, as are in the quantity given to be equated; to the end that those three last Terms by due Reduction may vanish out of each part, and an Equation remain between some numbers of $aaaa$ and aaa , whence the value of the Root a , if it hath any possible value, may be expressible by some rational number either affirmative or negative. But how to feign the said side so as to cause such an effect, the following Proposition and Canon will shew; where to evidence the certainty thereof, I put b, c, d to stand for the Coefficients or known numbers prefixt to aaa, aa, a ; also, rr (whose side is r) for the last Term, which in this Case is a rational square number, and f , which is prefixt to $aaaa$, stands for a number not a Square.

P R O P.

1. Let this quantity be given to be equated to a Square, viz. $faaaa + baaa + caa + da + rr = \square$

C A N O N.

2. When $4crr$ exceeds dd , then let the side of the Square sought be feigned $\frac{4crr - dd}{8rrr}aa + \frac{d}{2r}a + r$
 But if dd exceeds $4crr$, then let the side of the Square be feigned $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rrr}aa$

Examples in Numbers.

3. Let this quantity be given to be equated to a Square, viz. $10aaaa + 4aaa + 19aa + 6a + 9 = \square$
4. Here, because $4crr$ exceeds dd , viz. four times $19 \times 9 \times 9$ exceeds the Square of 6, the first part of the Canon gives this feigned side of a Square, viz. $3aa + 3a + 3$
5. Then by equating the Square of the said side $3aa + 3a + 3$ to the given quantity $10aaaa + 4aaa + 19aa + 6a + 9$, the value of the Root a will be found 2, according to which the said given quantity being expounded makes the Square 289, whose side is 17.
6. Again, let this quantity be given to be equated to a Square, viz. $2aaaa + 3aaa + 3aa + 6a + 1 = \square$
7. Here, because dd exceeds $4crr$, viz. the Square of 6 exceeds four times $3 \times 1 \times 1$, the latter part of the Canon gives this feigned side of a Square, viz. $1 + 3a - 3aa$
8. Then by equating the Square of the said side $1 + 3a - 3aa$, to the given quantity $2aaaa + 3aaa + 3aa + 6a + 1$, the value of the Root a will be found 3, according to which, the said given quantity being expounded makes the Square 289.

V. When a Quantity compos'd of five Terms given to be equated to a Square is such, that the first Term is a Biquadrate, and the last Term (that is, the absolute number) hath a rational square Root, then the side of a Square to be equated to such Quantity may be varied six several ways, (including into this number the two last preceding Canons,) by every one of which the value of the Root a may oftentimes be found out, and express'd by a rational number either affirmative or negative. To evidence this, I shall (as before) put b, c, d to stand for the Coefficients, or known numbers prefixt to aaa, aa and a ; and rr (whose side is r ;) for the rational square number which is the last Term. As, for example,

Let this Quantity be propos'd to be equated to a Square, viz. $aaaa + baaa + caa + da + rr$
 1. Then to the end that $aaaa + da + rr$ may be found in a Square to be equated to the quantity propos'd, the side of that Square may be feigned $aa + \frac{d}{2r}a + r$

2. Or,

- 2. Or, to cause the same effect, the side of the said Square may be feigned } $r + \frac{d}{2r}a - aa$
- 3. Again, that $caa + da + rr$ may be found in a Square to be equated to the quantity proposed, the side of such Square, when $4crr$ exceeds dd , may (agreeable to the Canon in the preceding Sect. 4.) be feigned . . . } $\frac{4crr - dd}{4rrr}aa + \frac{d}{2r}a + r$
- 4. But to cause the same effect, if dd exceeds $4crr$, then then let the side of the Square be feigned } $r + \frac{d}{2r}a - \frac{dd - 4crr}{4rrr}aa$
- 5. Again, that $aaaa + baaa + rr$ may be found in a Square to be equated to the quantity proposed, let the side of the Square be feigned } $aa + \frac{1}{2}ba + r$
- 6. Or, to cause the same effect, the side of the said Square may be feigned } $aa + \frac{1}{2}ba - r$
- 7. Again, that $aaaa + baaa + caa$ may be found in a Square to be equated to the quantity proposed, the side of that Square, when $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, may (agreeable to the Canon in Sect. 2.) be feigned } $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$
- 8. But, to cause the same effect, if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, let the side of the said Square be feigned } $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers of the preceding sides of Squares express'd by Letters.

Let this quantity be proposed to be equated to a Square, viz. } $aaaa + 4aaa + 10aa + 20a + 1 = \square$

1. Then supposing b, c, d, r , to stand for 4, 10, 20, 1, the first of the preceding literal sides will give } $aa + 10a + 1$

The Square of which side $aa + 10a + 1$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, there will arise, after due Reduction, $16aaa = 92aa$; whence by dividing each part by $16aa$, there comes forth $\frac{22}{4}$ for the value of the Root a , according to which, the proposed quantity being expounded will give the Square $\frac{140625}{256}$, whose side is $\frac{375}{16}$.

2. Again, the third literal side gives . . . } $1 + 10a - aa$

The Square of which side $1 + 10a - aa$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{11}{3}$, according to which, the same quantity being resolved, makes the Square of $\frac{218}{9}$.

3. Again, because in the quantity proposed, dd exceeds $4crr$, the fourth literal side gives . . } $1 + 10a - 45aa$

The Square of which side $1 + 10a - 45aa$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{22}{53}$, according to which, the same quantity being expounded makes the Square of the side $\frac{224106}{64009}$.

4. Again, the fifth literal side gives . . . } $aa + 2a + 1$

The Square of which side $aa + 2a + 1$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -4$, according to which, the same quantity being expounded makes the Square 81.

5. Again, the sixth literal side gives . . . } $aa + 2a - 1$

The Square of which side $aa + 2a - 1$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -3$, according to which the said quantity being resolved makes the Square 4.

6. Lastly, because $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, the seventh literal side gives } $aa + 2a + 3$

The Square of which side $aa + 2a + 3$ being equated to the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = 1$, according to which the same quantity being resolved makes the Square 36.

VI. Sometimes, when a quantity compos'd of five Terms is equated to a Square formed from one of the eight literal sides before exprest in Sect. V. no value of the Root a either affirmative or negative can thence be discovered. As, for example,

If this quantity be propos'd to be equated to a Square, viz. } $aaaa + 18aaa + 120aa + 351a + 1519 = \square$
 The Canon in Sect. III. (or the seventh literal side in Sect. VI.) will give this feigned side of a Square, viz. . . . } $aa + 9a + 19\frac{1}{2}$
 The Square of which side $aa + 9a + 19\frac{1}{2}$ is } $aaaa + 18aaa + 120aa + 351a + \frac{1}{4}$
 Which Square being equated to the quantity propos'd will give this fruitless and absurd Equation, viz. } $1138\frac{1}{4} = 0$

VII. When negative Terms are intermingled with affirmative, in a quantity compos'd of five Terms given to be equated to a Square, the side of the Square, (when such an Equation is possible,) shall be one of the eight literal sides before exprest in Sect. V. saving that one, and sometimes two of its signs $+$ must be changed into $-$. As, for example,

If $aaaa - 8aaa + 28aa - 40a + 4$ be given to be equated to a Square, it may be variously done, in regard the extremes $aaaa$ and 4 are Squares. First then, I imagine all the Terms of the propos'd quantity to be affirmative, so it will be $aaaa + 8aaa + 28aa + 40a + 4$; now to feign the side of a Square, that $aaaa + 8aaa + 4$ may by due Reduction vanish out of each part, the fifth literal side in Sect. V. being resolved into numbers will give $aa + 4a + 2$ for the feigned side; but here two of its signs $+$ must be changed into $-$, that in its Square there may be found $aaaa - 8aaa + 4$ to destroy $aaaa - 8aaa + 4$ in the quantity given to be equated, to which end, the said side $aa + 4a + 2$ must be converted into $aa - 4a - 2$, and then the Square of this side being equated to the propos'd quantity $aaaa - 8aaa + 28aa - 40a + 4$ will give $a = \frac{7}{2}$.

Again, to feign the side of a Square to be equated to the same given quantity $aaaa - 8aaa + 28aa - 40a + 4$, so, as to destroy the first, second and last Terms in each part, the first of the literal sides in Sect. V. being resolved into numbers, gives $aa + 10a + 2$ for the feigned side, if all the Terms of the given quantity were affirmative; but that $aaaa - 8aaa + 4$ may vanish out of each part, the said side $aa + 10a + 2$ must be changed into $aa - 10a + 2$, and then the Square of this side being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = \frac{12}{3}$.

Again, to feign the side of a Square to be equated to the same given quantity $aaaa - 8aaa + 28aa - 40a + 4$, in such manner that $aaaa - 40a + 4$ may vanish out of each part, the first of the eight literal sides in Sect. V. being resolved into numbers, gives $aa + 10a + 2$ for the feigned side, if all the Terms of the given quantity were affirmative; but that $aaaa - 40a + 4$ may be expunged out of each part, the said side $aa + 10a + 2$ must be changed into $2 - 10a - aa$, and then the Square of this side being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = -\frac{12}{7}$. Other sides might likewise be feigned, as is evident by the foregoing Sect. V. and to him that is a little exercis'd in this Method it will not be difficult to change the Signs in any possible case.

VIII. When a quantity consisting of five Terms is to be equated to a Square, and one or more values of the Root a are found out, either affirmative or negative, by the Rules before given, you may from every one of those Primitive Roots or values, find out other values of the Root a , even as many as you please, which latter, Fermat calls Derivative Roots of the first, second, third, &c. degree. As, for example,

To find a Derivative Root of the first degree out of the quantity $aaaa + 4aaa + 10aa + 20a + 1$, before propos'd in Sect. V. to be equated to a Square, take one of its Primitive Roots there found out, to wit, -3 , and connect it to a , so it makes $a - 3$, then instead of a take $a - 3$ for a new Root, according to which the propos'd quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa - 8aaa + 28aa - 40a + 4$, as here you see;

aaaa	aaaa	-	12aaa	+	54aa	-	108a	+	81
4aaa		+	4aaa	-	36aa	+	108a	-	108
10aa				+	10aa	-	60a	+	90
20a						+	20a	-	60
1								+	1
Summ,	aaaa	-	8aaa	+	28aa	-	40a	+	4

This summ must be equated to a Square, whose side (as before hath been shewn in Sect. 7.) may be feigned $aa - 4a - 2$, the Square whereof being equated to the said summ will give $a = \frac{7}{2}$; but because the new Root was put $a - 3$, out of $\frac{7}{2}$ subtract 3, and there will remain $\frac{1}{2}$ for a second value of the Root a in the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$, which second value may be called a Derivative Root of the first degree.

In like manner by the help of the said second value $\frac{1}{2}$ you may find out a third, by joyning $+\frac{1}{2}$ to a , and taking $a + \frac{1}{2}$ for a new Root, according to which, the given quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa + 6aaa + \frac{3}{2}aa + \frac{5}{2}a + \frac{2}{16}$ to be equated to a Square, the side whereof may be feigned $aa + 3a + \frac{1}{4}$, (agreeable to the fifth literal side in Sect. 5.) the Square whereof being equated to the given quantity, there will thence arise $a = -11$, therefore the new Root $a + \frac{1}{2}$ gives $a = -\frac{21}{2}$ for a third value of the Root a in the given quantity, that is, a Derivative Root of the second degree.

Nor will the Operation be otherwise to find out a fourth value, or derivative Root of the third degree, by putting for a new Root $a - \frac{21}{2}$, for according to this, every member of the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$ being resolved, there will come forth $aaaa - 38aaa + \frac{1021}{2}aa - \frac{5221}{2}a + \frac{134682}{16}$, this summ is to be equated to a Square, whose side may be feigned $aa - 19a - \frac{167}{4}$, and the Square of this side being equated to the said summ will give $a = \frac{273}{46}$, from which if you subtract $\frac{21}{2}$, (because the new Root was put $a - \frac{21}{2}$;) there will remain $\frac{125}{23}$ for a fourth value of the Root a , that is, a derivative Root of the third degree, out of the quantity first proposed to be equated to a Square.

Lastly, as by the help of one of the primitive Roots of the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$ other Roots have been derived, so by the help of any one of the rest of the primitive Roots of the same quantity, found out in the Examples of Sect. 5. you may proceed to find out other derivative Roots, but sometimes you will meet with fruitless Equations.

IX. A quantity compos'd of four Terms may be equated to a Square, when either the absolute number, that is, the last Term is a Square, or the first Term a Biquadrate.

First, let $20aaa + 5aa + 40a + 16$ be given to be equated to a Square. Feign the side so, that $40a + 16$ may vanish out of each part, to which purpose, let the side be $5a + 4$, (5 being the Quotient that ariseth by dividing 40 in the $4a$, by 8 the double of the side of the given Square 16;) then by equating the Square of $5a + 4$ to the given quantity $20aaa + 5aa + 40a + 16$ you will find $a = 1$, according to which, that quantity being resolved makes the Square 81. Now to find a second value of the Root a , you may put for a new Root $a + 1$, according to which, the given quantity $20aaa + 5aa + 40a + 16$ will be converted into $20aaa + 65aa + 110a + 81$ to be equated to a Square, the side whereof, (that $110a + 81$ may vanish out of each part,) may be feigned $\frac{2}{9}a + 9$, whence after due Reduction, there will arise $a = -\frac{181}{81}$; therefore $a + 1$ (which was put for the new Root) gives $1 - \frac{181}{81}$, that is, $\frac{31}{81}$ for a second value of the Root a , (or a derivative Root of the first degree,) and by putting $a + \frac{31}{81}$ for a new Root you may find out a third value, and so infinitely.

Secondly, an Example where the first Term is $aaaa$ may be this, viz. Let $aaaa + 4aaa - 3aa + 2a$ be given to be equated to a Square. That $aaaa + 4aaa$ may vanish out of each part, feign the side of a Square to be $aa + 2a$, (2 in the $2a$ being the half of 4 prefixt to aaa in the given quantity;) then the Square of $aa + 2a$ being equated to $aaaa + 4aaa - 3aa + 2a$ will give $a = \frac{2}{7}$; and to find out a second value of a you may put $a + \frac{2}{7}$ for a new Root, and proceed as in former Examples.

Thirdly,

Thirdly, although some intermediate Term be omitted in a quantity compos'd of four Terms, such quantity may be equated to a Square: As, to equate $5aaa - 16aaa + 24aa + 16$ to a Square, you may feign its side to be $3aa + 4$, (3 in the $3aa$ being the Quotient that ariseth by dividing 24 which is prefixt to aa in the given quantity, by 8 the double of the side of the given Square 16,) and thence the value of a will be found 4; then you may put $a + 4$ for a new Root to find out a second value of a .

In like manner, if $aaaa + 60aa + 8ca + 500$ be propos'd to be equated to a Square, you may feign its side to be $aa + 30$, (30 being the half of 60 which is prefixt to aa in the propos'd quantity,) whence you will find $a = 5$, and for a derivative Root you may put $a + 5$.

X. A Quantity compos'd of four Terms may be equated to a Cube, when either the absolute number, (that is, the last Term,) or the first Term is a perfect Cube.

First, let $2aaa + aa + 3a - 1$ be given to be equated to a Cube. That the two last Terms may vanish out of each part, feign the side of a Cube to be $a - 1$, (1 being the side of the given Cube 1, and a being the Quotient that ariseth by dividing $3a$ in the given quantity by 3 the triple Square of 1, the cubick Root of the given Cube 1,) then the Cube of $a - 1$, that is, $aaa - 3aa + 3a - 1$ being equated to the given quantity $2aaa + aa + 3a - 1$, will give $a = 2$; then to find out a second value of a you may put $a + 2$ for a new Root.

Secondly, but if the first Term be a rational Cube; as, if $8aaa + 24aa + 2a + 48$ be given to be equated to a Cube; that the first and second Terms may vanish out of each part, feign the side of a Cube to be $2a + 2$, ($2a$ being the side of the Cube $8aaa$, and 2 being the Quotient that ariseth by dividing 24 which is prefixt to aa , by 12 the triple Square of 2 the cubick Root of 8 in $8aaa$;) then the Cube of $2a + 2$ being equated to the given quantity $8aaa + 24aa + 2a + 48$, will give $a = \frac{20}{11}$; whence you may find out derivative Roots as before.

XI. If the first Term of a Quantity compos'd of four Terms given to be equated to a Cube be a rational Cube, and the last Term, to wit; the absolute number be also a Cube, then that given Quantity may be equated to a Cube in a threefold manner.

As, for example, if $aaa + 2aa + 4a + 1$ be propos'd to be equated to a Cube; first, that the first and last Terms may vanish out of each part, feign the side of the Cube to be $a + 1$, (which is compos'd of the cubick Roots of aaa and 1; then the Cube of $a + 1$ being equated to the quantity propos'd will give $a = 1$. Secondly, that the first and second Terms may vanish out of each part, you may feign the side to be $a + \frac{2}{3}$, (a being the side of the Cube aaa , and $\frac{2}{3}$ being the Quotient that ariseth by dividing 2 which is prefixt to aa , by 3 the triple Square of the cubick Root of 1 which is prefixt to aaa ;) whence $a = -\frac{10}{72}$. And lastly, that the third and fourth Terms may vanish, you may feign the side to be $\frac{4}{3}a + 1$, (1 being the side of the given Cube 1; and $\frac{4}{3}a$ being the Quotient that ariseth by dividing $4a$ by 3 the triple Square of the cubick Root of the given Cube 1,) whence $a = \frac{20}{37}$; and by the help of those three primitive Roots you may find out derivatives, in like manner as before.

XII. Sometimes when a Quantity compos'd of four Terms, whereof one or both the extremes are Cubes, is to be equated to a Cube, no value of the Root a either affirmative or negative can be found out by any of the Rules before delivered.

As, if $4aaa + 3aa + 3a - 1$ be given to be equated to a Cube, its side can only be feigned $a + 1$, the Cube whereof being equated to the given quantity will give $3aaa = 0$; and therefore the given quantity cannot be equated to a Cube.

Also, if $aaa + 2aa + 3a + 1$ be to be equated to a Cube, there can but one primitive Root be found out, although there be a threefold way of feigning the side of the Cube according to Sect. XI. which primitive Root will be discovered from the feigned side $a + \frac{2}{3}$, but neither of the other two ways will prove effectual.

I shall now add a few Questions to illustrate the foregoing Third Part of Fermat's Invention, and so conclude this Book.

QUEST. 127.

(The same with the foregoing Quest. 105. but resolved after another manner.)

To find a right-angled Triangle, that the Area being subtracted as well from the Hypothensal as from one of the sides about the right-angle, each remainder may be a Square.

RESOLUTION:

1. Let h, b, p represent the Hypothensal, Base and Perpendicular of a right-angled Triangle in numbers. Divide those three sides severally by a , and put the Quotients for the three sides of the Triangle sought, viz. } $\frac{h}{a}; \frac{b}{a}; \frac{p}{a}$
2. Then by subtracting the Area $\frac{\frac{1}{2}bp}{aa}$, as well from $\frac{b}{a}$, (one of the sides about the right-angle, as from the Hypothensal $\frac{h}{a}$, each remainder must be equal to a Square, and by multiplying each remainder by the Denominator aa ; this Duplicate equality ariseth, viz. } $ba - \frac{1}{2}bp = \square$
 $ba - \frac{1}{2}bp = \square$
3. Let the first of those two quantities be equated to some Square, viz. suppose } $ba - \frac{1}{2}bp = bb$
4. Whence, after due Reduction, } $a = b + \frac{1}{2}p$
5. Therefore by multiplying b into $b + \frac{1}{2}p$, (instead of a ,) the latter of the two quantities in the second step, will be converted into this quantity, which must be equated to a Square, viz. } $bb + \frac{1}{2}bp - \frac{1}{2}bp = \square$
6. Now since h, b, p , were put for the Hypothensal, Base and Perpendicular of a right-angled Triangle, the quantity in the fifth step shews that a right-angled Triangle must be found out such, that if the Hypothensal be multiplied into the summ of one of the sides about the right-angle and half the other side, and the Product be lessened by the Area, the remainder must be a Square: But such a right-angled Triangle, by Fermat's method, (before explained,) may be found out thus, viz.
7. Form a right-angled Triangle from two numbers taken at pleasure, as from $a+1$ and 1 , so the three sides will be these, viz. . . . } $aa + 2a + 2 = \text{Hypoth.}$
 $aa + 2a = \text{Base,}$
 $+ 2a + 2 = \text{Perpend.}$
8. The Product of the Hypothensal into the summ of the Base and half the Perpendicular is . . . } $aaaa + 5aaa + 9aa + 8a + 2$
9. The Area is } $. . + aaa + 3aa + 2a$
10. Which subtracted from the said Product leaves this quantity to be equated to a Square, viz. } $aaaa + 4aaa + 6aa - 6a + 2 = \square$
11. Feign the side of that Square, (according to the 7th literal side in the foregoing Sect. 5. Part 3.) } $aa + 2a + 1$
12. Then the Square of the said side $aa + 2a + 1$ being equated to the quantity in the tenth step will give $a = -\frac{1}{2}$, therefore $a + 1$ and 1 the numbers forming the Triangle shall be $\frac{1}{2}$ and 1 , or (in Integers in the same Reason) 1 and 2 ; by which, if a right-angled Triangle be formed, one of the sides about the right-angle will be less than nothing, to wit, -3 ; (for the Square of 2 is to be subtracted from the Square of 1 , because 1 and 2 answer to $a+1$ and 1 the numbers that formed the Triangle in the seventh step;) to cause therefore all the sides to be affirmative, the work must be renewed in manner following, viz.
13. Let a right-angled Triangle be formed from $a+1$ and 2 , so the three sides will be these, viz. . . . } $aa + 2a + 5 = \text{Hypoth.}$
 $aa + 2a - 3 = \text{Base,}$
 $+ 4a + 4 = \text{Perpend.}$
14. The Product of the Hypothensal into the summ of the Base and half the Perpendicular is } $aaaa + 6aaa + 12aa + 18a - 5$
15. The Area is } $. . + 2aaa + 6aa - 2a - 6$
16. Which being subtracted from the said Product, leaves this quantity to be equated to a Square, viz. } $aaaa + 4aaa + 6aa + 20a + 1 = \square$

17. The

17. The side of that Square may be variously feigned, according to the preceding Sect. 5.) let it be the second literal side in that Sect. viz. $1 + 10a - aa$
 18. Then the Square of the said side $1 + 10a - aa$ being equated to the quantity in the sixteenth step will give $a = \frac{22}{6}$; therefore $a - 1$ and 2 shall be $\frac{22}{6}$ and 2 , that is, in Integers in the same Reason, 29 and 12, by which if you form a right-angled Triangle, the three sides will be 985, 697, 696, that is, b, b, p ; then according to the Positions in the first step, divide every one of those sides by 1045, that is, by $b + \frac{1}{2}p = a$, (as appears by the fourth step,) so the Quotients $\frac{985}{1045}, \frac{697}{1045}, \frac{696}{1045}$ shall be the sides of a right-angled Triangle to solve the Question: For if the Area be subtracted from the Hypothensal $\frac{985}{1045}$ and the Base $\frac{697}{1045}$ severally, the remainders will be the Squares of these sides $\frac{882}{1045}$ and $\frac{627}{1045}$.

And because the quantity in the sixteenth step is capable of being equated to innumerable Squares, (according to the Method before explained,) the Question is also capable of innumerable Answers; but in larger numbers than those, that may be found out by the foregoing Quest. 105. which is the same with this.

QUEST. 128. (Probl. 1. in cap. 1. part. 1. Dioph. redivivi.)

To find a right-angled Triangle, that if the double of its Area be subtracted from every one of the three sides, the remainders may be Squares

RESOLUTION.

1. Let b, p, h represent the Base, Perpendicular, and Hypothensal of a right-angled Triangle. Divide those sides severally by a , and assume the Quotients to be the three sides of the Triangle sought, viz. $\frac{b}{a}, \frac{p}{a}, \frac{h}{a}$
2. Then by subtracting the double Area $\frac{bp}{aa}$ from every one of those three sides, the remainders must be Squares, and multiplying the remainders severally by the Denominator aa , this Triplicate equality ariseth to be resolved, viz. $ba - bp = \square$
 $pa - bp = \square$
 $ha - bp = \square$
3. Now in order to resolve that Duplicate equality, let the first of its three quantities be equated to some Square, viz. suppose $ba - bp = bb$
4. Whence, after due Reduction to find out the value of a , you will discover $a = b + p$
5. Then by multiplying $b + p$, instead of a , into p , the second of the three quantities in the second step (to wit, $pa - bp$) will be converted into this quantity, which is manifestly a Square, viz. $bp + pp - bp = \square = pp$
6. And by multiplying $b + p$, instead of a , into h , the third quantity in the second step will be converted into this quantity to be equated to a Square, viz. $hb + hp - bp = \square$
7. Thus the Triplicate equality in the second step is reduced to a Duplicate equality in the fifth and sixth steps, and because the first of the two quantities in that Duplicate equality happens to be a Square, to wit, pp , it remains only to equate $hb + hp - bp$ (in the sixth step,) to a Square, which discovers the Scope of our search must be this, viz. to find a right-angled Triangle, such, that if the Hypothensal be multiplied by the summ of the sides about the right-angle, and the Product be lessened by the double of the Area, the remainder must be a Square: But such a right-angled Triangle may be found out thus, viz.
8. Form a right-angled Triangle from two numbers taken at pleasure, as from $a + 1$ and 2 , so the three sides will be these, viz. $aa + 2a + 5 = \text{Hyp.}$
 $aa + 2a - 3 = \text{Base,}$
 $+ 4a + 4 = \text{Perp.}$
9. The Product of the Hypothensal into the summ of the Base and Perpendicular is $aaaa + 8aaa + 18aa + 32a + 5$
10. The double Area is $+ 4aaa + 12aa - 4a$
11. Which subtracted from the said Product, leaves this quantity to be equated to a Square, viz. $aaaa + 4aaa + 6aa + 36a + 17 = \square$

12. Feign

12. Feign the side of that Square according to the Canon in the } $aa + 2a + 1$
 preceding Sect. 3. Part 3. and it will be }
 13. Then the Square of the said side $aa + 2a + 1$ being equated to the quantity in the
 eleventh step, will give $a = -\frac{1}{2}$; therefore $a + 1$ and 2, the numbers forming the
 Triangle in the eighth step, shall be $\frac{1}{2}$ and 2, or, (in Integers in the same Reason,)
 1 and 4, by which if a right-angled Triangle be formed, one of the sides about the
 right-angle will be less than nothing, to wit, -15 ; (for the Square of 4 is to be
 subtracted from the Square of 1, because 1 and 4 answer to $a + 1$ and 2, the num-
 bers that formed the Triangle in the eighth step, where $a + 1$ was supposed to exceed 2.)
 To cause therefore all the sides to be affirmative, the work must be renewed thus, viz.

14. Form a right-angled Triangle from $a + 1$ } $aa + 2a + 17 = \text{Hyp.}$
 and 4, so the three sides will be these, viz. } $aa + 2a - 15 = \text{Base,}$
 } $+ 8a + 8 = \text{Perp.}$
 15. The Product of the Hypothensal into the } $aaaa + 12aaa + 30aa + 156a - 119$
 summ of the Base and Perpendicular is . . . }
 16. The double Area is } $. . . + 8aaa + 24aa - 104a - 120$
 17. Which subtracted from the said Product, } $aaaa + 4aaa + 6aa + 260a + 1 = \square$
 leaves this quantity to be equated to a Square, }
 viz. }
 18. The side of that Square may be variously }
 feigned, (according to the preceding Sect. V.) } $1 + 130a - aa$
 let it be the second literal side in that Sect. }
 viz. }

19. The Square of the said side $1 + 130a - aa$ being equated to the quantity in the
 seventeenth step, will give $a = \frac{4223}{66}$, therefore $a + 1$ and 4 shall be $\frac{4229}{66}$ and 4;
 or, (in Integers in the same Reason,) 4289 and 4223; from which, a right-angled
 Triangle being formed, the three sides will be 18465217, 18325825, 2264592;
 that is, b, b, p : Then according to the Positions in the first step, divide those three sides
 severally by 20590417, that is, by $b + p = a$, (as is evident by the fourth step;) so
 the Quotients $\frac{18465217}{20590417}, \frac{18325825}{20590417}, \frac{2264592}{20590417}$ shall be the sides of a right-angled
 Triangle to solve the Question, as may easily be proved.

Note 1. Although the Question be truly solved, yet 'tis evident that it was by chance
 that the Triplicate equality in the second step came to be reduced to a single equality; for
 if the quantity to be equated to a Square in the fifth step, had not happened to have been
 a Square, there would have been an inexplicable Duplicate equality.

Note 2. It is easie to perceive by the second, third and fourth steps; that instead of bp
 the double Area, the Product of bp multiplied by any square number may be given in the
 Question: As, if it were required to find out a right-angled Triangle, that $4bp$, that is,
 eight times the Area being subtracted from every one of the three sides may leave Squares,
 you need only to multiply the Denominator 20590417 of the three sides before found
 by 4, without altering the Numerators; or, if $9bp$, that is, eighteen times the Area, were
 prescribed, then to multiply the Denominator by 9.

QUEST. 129. (Probl. 1. in cap. 2. part. 1. Dioph. redivivi.)

To find a right-angled Triangle, that the Product of the Hypothensal into one of the
 sides about the right-angle, being subtracted from every one of the three sides, may leave
 Squares.

RESOLUTION.

1. Let b, b, p represent the Hypothensal, Base and Perpendicular } $\frac{b}{a}; \frac{b}{a}; \frac{p}{a}$
 of a right-angled Triangle, and for the three sides of the Triangle }
 sought put }
 2. Then from $\frac{bp}{aa}$, (the Product of the Hypothensal into the Per- } $ba - bp = \square$
 pendicular,) subtract every one of the three sides, and the re- } $ba - bp = \square$
 mainders must be Squares; therefore also those remainders multi- } $pa - bp = \square$
 plied into the Denominator aa must make Squares; hence this }
 Triplicate equality ariseth, viz. }

3. Now

3. Now in order to resolve that Triplicate equality, let the first of its three quantities be equated to some Square, viz. suppose } $ha - hp = hb$
4. Whence, after due Reduction to find out the value of a , you will discover } $a = h + p$
5. Then by multiplying $h + p$, instead of a , into b , the second of the three quantities in the second step will be reduced to this to be equated to a Square, viz. } $hb + bp - hp = \square$
6. Likewise by multiplying $h + p$, instead of a , into p , the third quantity in the second step will be reduced to this quantity, which is manifestly a Square, viz. } $hp + pp - hp = \square = pp$
7. Thus the Triplicate equality in the second step is reduced to a Duplicate equality in the fifth and sixth steps; and because the latter quantity in that Duplicate equality happens to be a Square, to wit, pp , it remains only to equate the former, that is, $hb + bp - hp$, to a Square; which shews that a right-angled Triangle must be found, such, that if the summ of the Hypothensal and Perpendicular be multiplied by the Base, and from the Product you subtract the Product of the Hypothensal into the Perpendicular, the remainder may be a Square: But such a right-angled Triangle may be found out thus, viz.
8. Form a right-angled Triangle from $a + 2$ and 1 , (2 and 1 being numbers taken at pleasure,) so the three sides will be these, viz. } $aa + 4a + 5 = \text{Hyp.}$
} $aa + 4a + 3 = \text{Base,}$
} $+ 2a + 4 = \text{Perp.}$
9. The summ of the Hypothensal and Perpendicular being multiplied by the Base, produceth } $aaaa + 10aaa + 36aa + 54a + 27$
10. The Product of the Hypothensal and Perpendicular is } $2aaa + 12aa + 26a + 20$
11. Which latter Product being subtracted from the former, leaves to be equated to a Square, } $aaaa + 8aaa + 24aa + 28a + 7 = \square$
12. Feign the side of that Square according to the Canon in the preceding Sect. III. Part 3. } $aa + 4a + 4$
and it will be
13. Then the Square of the said side being equated to the quantity in the eleventh step will give $a = -\frac{2}{4}$, and therefore $a + 2$ and 1 , which were the numbers forming the Triangle in the eighth step, shall be -1 and 4 , but one of these being negative, the work must be renewed; and now a right-angled Triangle may be confidently formed from $a - 1$ and 4 , which Triangle being used like the former in the ninth, tenth and eleventh steps, at length there will remain $aaaa - 4aaa + 6aa - 260a + 1$ to be equated to a Square, the side whereof may be variously feigned, let it be $1 - 130a + aa$, then the Square of this side being equated to the said $aaaa - 4aaa + 6aa - 260a + 1$, will give $a = 66$; therefore $a - 1$ and 4 the numbers forming the Triangle shall be 65 and 4 , by which if you form a right-angled Triangle, the three sides will be found 4241 , 4209 , 520 , that is, h, b, p . Then according to the Positions in the first step, divide those three numbers severally by 4761 , that is, by $h + p = a$, as appears by the fourth step, and the Quotients $\frac{4241}{4761}, \frac{4209}{4761}, \frac{520}{4761}$ shall be the sides of a right-angled Triangle to solve the Question, as may easily be proved.

QUEST. 130. (Probl. 39. in cap. 1. part. 1. Dioph. redivivi.)

To find a right-angled Triangle, whose Area subtracted from one of the sides about the right-angle, may leave a given number, suppose z , (or n .)

RESOLUTION.

1. Let h, b, p represent the Hypothensal, Base and Perpendicular of a right-angled Triangle; then multiply those sides severally by a , and put the Products for the sides of the Triangle sought, viz. } ha, ba, pa
2. The Area of which Triangle being subtracted from one of its sides about the right-angle, suppose from ba , must leave a remainder equal to the given number z , (or n ,) therefore } $ba - \frac{1}{2}bpaa = n = z$
3. Which Equation divided by $\frac{1}{2}bp$, gives } $\frac{b}{\frac{1}{2}bp}a - aa = \frac{n}{\frac{1}{2}bp}$
4. Now

4. Now that the value of a in the last Equation may be expref-
 fible by a rational number, it is evident by the Canon in *Sett.* 10.
Chap. 15. *Book* 1. that if the absolute quantity in the latter
 part of that Equation be subtracted from the Square of half
 the Coefficient which is drawn into a , the remainder must
 have a rational square Root, therefore } $\frac{\frac{1}{4}bb - \frac{1}{2}bpn}{\frac{1}{4}bbpp} = \square$
5. And because the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only
 to equate the Numerator to a Square, therefore } $\frac{1}{4}bb - \frac{1}{2}bpn = \square$
6. Or, to avoid Fractions, let the said $\frac{1}{4}bb - \frac{1}{2}bpn$ be multiplied
 by 4, and then it remains to equate } $bb - 2bpn = \square$
7. Whence it appears, that in order to solve the Question proposed, a right-angled Triangle
 must first be found, such, that if from the Square of one of its sides about the right-
 angle, the Product of the quadruple of the Area multiplied by the given number n
 be subtracted, the remainder may be a Square: But such a right-angled Triangle may
 be found out thus, *viz.*
8. Form a right-angled Triangle from two
 numbers taken at pleasure, as from $a - 1$
 and 4, so the three sides will be these, } $aa - 2a + 17 = \text{Hyp.}$
viz. } $aa - 2a - 15 = \text{Base,}$
 } $+ 8a - 8 = \text{Perp.}$
9. The Square of the Base is } $aaaa - 4aaa + 26aa + 60a + 225$
10. The Product of the quadruple of the
 Area into the given number 2 is } $. . + 32aaa - 96aa - 416a + 480$
11. Which Product being subtracted from
 the said Square of the Base, leaves this
 quantity to be equated to a Square, *viz.* } $aaaa - 36aaa + 70aa + 476a - 255$
12. Feign the side of the desired Square to be } $aa - 18a - 127$
13. Then the Square of the said side being equated to the quantity in the eleventh step,
 will give $a = -4$. But because this value is negative, let $a - 4$ be put for a new
 Root, and according to that let all the members of the quantity in the eleventh step be
 resolved; so this new quantity $aaaa - 52aaa + 598aa - 2068a + 1521$ comes
 forth to be equated to a Square, the side whereof may be feigned $39 - \frac{1024}{39}a + aa$,
 whence $a = \frac{278226}{1560}$; therefore $a - 4$ (the new Root) shall be $\frac{67222}{390}$, and therefore
 $a - 1$ and 4 the numbers that formed the Triangle in the eighth step shall be $\frac{67602}{390}$
 and 4; or in Integers in the same proportion, 67609 and 1560. Wherefore the
 preparatory Triangle formed from those numbers, and agreeable to the Scope mentioned
 in the seventh step, shall be 4573410481, 4568543281, 210940080; For if its
 quadruple Area be multiplied by the given number 2, and the Product be subtracted from
 the Square of the second side, there will remain a Square, whose side is 4125146321.
14. Now let those three sides of the preparatory Triangle be taken for the values of
 b, b, p , and the given number 2 for n ; in the Equation in the third step, then that
 Equation being resolved will give $a = \frac{2242462}{12046111064720031}$; and if this number be
 multiplied into the three sides of the preparatory Triangle it will give the Triangle sought,
 whose three sides consist of these three Numerators, 25347953801344222,
 25320977530297822, 1169127377676960, having 12046111064720031
 for a common Denominator.

Note. This Question is very easily solved when the given number is less than unity;
 by the Canon of the preceding *Quest.* 100. of this *Book.*



T H E
E L E M E N T S
O F T H E
A L G E B R A I C A L A R T .

B O O K I V .

C H A P . I .

Concerning the Scope of this fourth Book, and the Signification of Characters, Abbreviations and Citations used therein.



THE Design of this Fourth Book is, to shew the excellent Use of the *Algebraical Art* in the Resolution and Composition of Plane Problems, to wit, such as may be solved or effected by drawing only Right (or straight) and Circular Lines. In pursuance of that Design I have divided this Book into Ten Chapters, whereof the first Six are Preparatory to the rest, which contain Four *Classes* or Forms of Examples, shewing how to find out as well Theorems, as Geometrical Effects of Plane Problems, with their Demonstrations, by the Steps of Algebraical Resolution. All which I have endeavour'd to render clear and intelligible to such Readers as are competently exercis'd in the first Six Books of *Euclid's Elements*, and in the First and Second Books of these *Algebraical Elements*.

The Explication of the Signs or Characters.

+	} Signifies {	More.
-		Less.
×		Into, or By.
::		Proportionals.
∴		Continual Proportionals.
√		The Square Root, or Side of a Square.
=		Equal
>		Greater.
<		Lesser.
		Parallel.
∠		A plain Angle in general.
∟		A Right-angle.
⊥		Perpendicular.
○		A Circle.
□	A Square.	
◻	A long Square.	
△	A plain Triangle in general.	

Z

Examples,

Examples, shewing more at large the Signification of the foregoing Characters.

- $a + b$. . . } Signifies the sum of the right lines or numbers represented by a and b .
 $a - b$. . . } The excess by which the right line or number a exceeds the right line or number b ; or, it imports that the latter quantity b is subtracted, or to be subtracted from the former Quantity a .
 $a \times b$, or ab } The Rectangle or Product made by the multiplication of the right line or number a , by the right line or number b .
 aa . . . } The Square of the right line or number signified by a .
 $\frac{aa}{c}$. . . } The right line arising by the Application of the Square of the right line a , to the right line c ; or, the Quotient arising by the Division of the Square of the number a , by the number c .
 $\frac{ab}{c}$. . . } The right line or number arising by the Application or Division of the Rectangle or Product of a into b , by c .
 $a . b :: c . d$, viz. As a is to b ; so c to d .
 $a , b , c ::$, viz. As a is to b ; so b to c .
 \sqrt{ab} . . . } The square Root of the Product of a into b ; or, the side of a Square equal to the Rectangle ab .
 $\sqrt{aa + bb}$. . . } The square Root universal of $aa + bb$; or, the side of a Square equal to the sum of the Squares aa and bb .
 $\sqrt{aa - bb}$. . . } The square Root universal of $aa - bb$; or, the side of a Square equal to the excess of the Square aa above the Square bb .
 $a = b$. . . } The line or number a is equal to the line or number b .
 $a = 5c$. . . } The line or number a is equal to five times the line or number c .
 $a = \frac{1}{2}d$. . . } The line or number a is equal to half the line or number d .
 $a \sqsupset f$. . . } The line or number a is greater than the line or number f .
 $a \sqsubset g$. . . } The line or number a is less than the line or number g .
 $AB \parallel CD$. . . } The line AB is parallel to the line CD .
 $\sphericalangle ABC$. . . } The angle ABC . Observe here, that when an angle is express'd by three letters, the middle letter stands at the angular point.
 $\sphericalangle A$. . . } The angle A .
 $\sphericalangle ABC$ is \perp } The angle ABC is a right-angle.
 $AB \perp BC$. . . } The right-line AB is perpendicular to the right-line BC .
 $ABCD$ is \odot } $ABCD$ is a Circle. Observe here, that the first letter towards the left hand is usually set at the Center.
 $\square AD$. . . } Signifies either the Square AD when the letters A and D stand at the opposite angles of the Square; or else, the Square of the right-line AD , when A and D stand at the ends of the side of the Square.
 $\square A$. . . } The Square of the right-line A .
 $\square AB, BC$ } The long Square, or Rectangle, made of the right-lines AB and BC .
 $\square AB, C$ } The Rectangle of the right-lines AB and C .
 $\square A, B$. . . } The Rectangle of the right-lines A and B .
 $\triangle ABC$. . . } The Triangle ABC .

Explication of Abbreviations and Citations.

- $Probl.$. . . } Problem.
 $Suppos.$. . . } Suppositions.
 $Req.$. . . } It is (or, let it be) required.
 $Prepar.$. . . } Preparation.
 $Constr.$. . . } Construction.
 $Req. demonstr.$ } It is (or, let it be) required to Demonstrate.
 $Conclus.$. . . } Conclusion.
 $Coroll.$. . . } Corollary.
 $Annot.$. . . } Annotation.
 $Explicat.$. . . } Explication.
 $Per Prop. 11.$ } By the 11th Proposition of the fifth Book of *Euclid's* Elements.
 $Elem. 5.$ }

Per Defn. 29. } By the 29th Definition of the first Book of *Euclid's* Elements.
 Elem. 1. }

Per Ax. 1. } By the first Axiom of the second Chapter of this fourth Book.
 Chap. 2. }

Note. In the handling of every Proposition, whether it be a Theorem or Problem, I proceed from the beginning to the end by Steps numbred in the Margin by 1, 2, 3, 4, 5, &c. that by referring to preceding Steps, the rise of the following may be apparent: So when 'tis said, [Therefore out of, or, From 3^o. and 4^o.] it imports, that the thing asserted or inferr'd is manifest by the third and fourth Steps of the Proposition in hand. If any other Abbreviations occur, their meaning will be obvious to every intelligent Reader.

C H A P. II.

The explication of Axioms, or common notions, upon which the force of Inferences or Conclusions, about the Equality, Majority and Minority of Quantities compared to one another, doth chiefly depend.

Axiom 1.

1. IF each of two Quantities be equal to a third, those two are equal between themselves.

Explicat.
 If $AB = EF,$ A ——— B
 And $CD = EF,$ E ——— F
 Then $AB = CD;$ per Ax. 1. C ——— D

That is to say,

If A B be equal to E F, and C D be equal to E F, then A B is equal to C D, by the first *Axiom* of Chap. 2.

Axiom 2.

2. Quantities which are equal to equal Quantities, are also equal between themselves.

Explicat.
 If $\begin{cases} C = D, \\ A = C, \\ B = D, \end{cases}$ A ——— C ———
 Then $A = B;$ per Ax. 2. B ——— D ———

Axiom 3.

3. That which is greater or less than one of two equal Quantities, is also greater or less than the other.

Explicat.
 If $B = C,$ B ———
 And $A \sqsubset B,$ A ———
 Then $A \sqsubset C;$ per Ax. 3. C ———

That is to say,

If B be equal to C, and A be greater than B, then A is greater than C, by Ax. 3.

Axiom 4.

4. If one of two equal Quantities be greater or less than a third, the other of those two shall be also greater or less than the same third.

Explicat.
 If $A = B,$ A ———
 And $A \sqsubset C,$ C ———
 Then $B \sqsubset C;$ per Ax. 4. B ———

Z 2

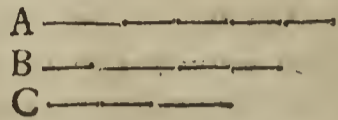
Axiom 5:

Axiom 5.

5. That which is greater than the greater of two Quantities, is also greater than the lesser; and that which is less than the lesser of two Quantities, is also less than the greater.

Explicat.

If $B \sqsubset C$,
 And $A \sqsubset B$,
 Then $A \sqsubset C$; per Ax. 5.

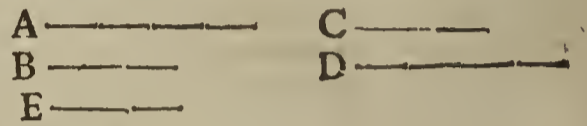


Axiom 6.

6. The exchanging of equal Quantities doth not alter equality.

Explicat.

If $A + B = C + D$,
 And $E = B$,
 Then $A + E = C + D$; per Ax. 6.

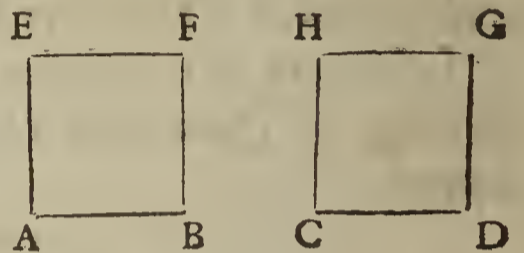


Axiom 7.

7. Interpretation doth not change equality.

Explicat.

Suppos. $\left\{ \begin{array}{l} \square AF = \square CG, \\ AF \text{ is } \square AB, \\ CG \text{ is } \square CD, \\ \square AB = \square CD; \text{ per Ax. 7.} \end{array} \right.$



That is to say,

The Square AF is equal to the Square CG, by supposition.

AF is the Square of the side AB, by supposition.

CG is the Square of the side CD, by supposition.

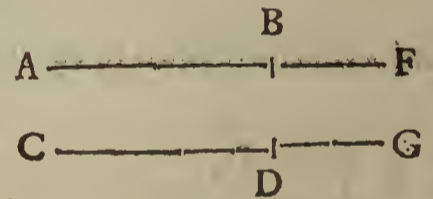
The Square of the side AB is equal to the Square of the side CD, by the seventh Axiom of Chap. 2.

Axiom 8.

8. If to equal Quantities you add equal Quantities, or one and the same Quantity; the wholes shall be equal.

Explicat.

If $AB = CD$,
 And $BF = DG$,
 Then $AB + BF = CD + DG$,
 That is, $AF = CG$; per Ax. 8.

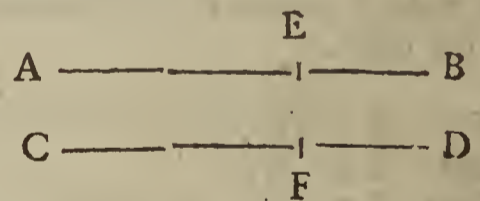


Axiom 9.

9. If from equal Quantities you take away equal Quantities, or one and the same Quantity, the Quantities remaining shall be equal to one another.

Explicat.

If $AB = CD$,
 And $AE = CF$,
 Then $AB - AE = CD - CF$,
 That is, $EB = FD$; per Ax. 9.

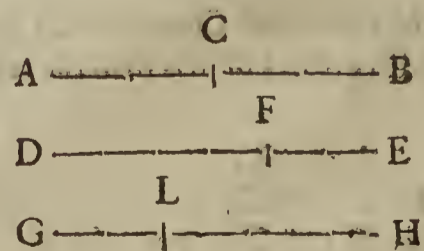


Axiom 10.

10. If from a whole the half be taken away, half will remain; and if more than half be taken away, less than half will remain; but if one third part be taken away, two thirds will remain, &c.

Explicat.

If $AC = \frac{1}{2} AB$,
 Then $CB = \frac{1}{2} AB$, per Ax. 10.
 If $DF = \frac{1}{2} DE$,
 Then $FE = \frac{1}{2} DE$, per Ax. 10.
 If $GL = \frac{1}{3} GH$,
 Then $LH = \frac{2}{3} GH$, per Ax. 10.



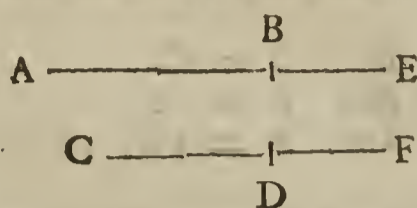
Axiom 11.

Axiom 11.

11. If to unequal quantities equal quantities be added, the wholes are unequal.

Explicat.

If $AB \sqsubset CD$,
And $BE = DF$,
Then $AE \sqsubset CF$; per *Ax. 11.*

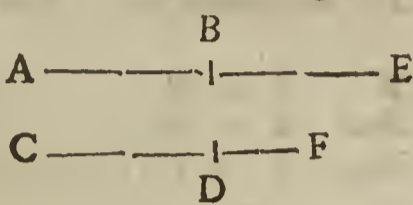


Axiom 12.

12. If to equal quantities you add unequal quantities, the wholes are unequal.

Explicat.

If $AB = CD$,
And $BE \sqsubset DF$,
Then $AE \sqsubset CF$; per *Ax. 12.*

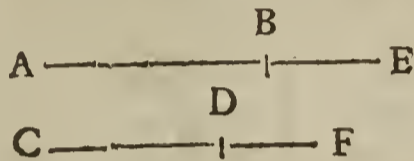


Axiom 13.

13. If to unequal quantities unequal quantities be added, the greater to the greater, and the less to the less, the wholes are unequal, to wit, the former the greater, and the latter the lesser.

Explicat.

If $AB \sqsubset CD$,
And $BE \sqsubset DF$,
Then $AE \sqsubset CF$; per *Ax. 13.*

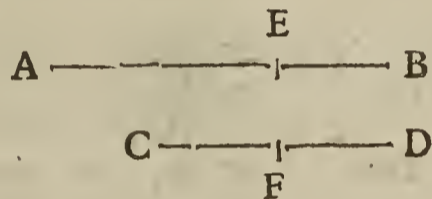


Axiom 14.

14. If from unequal quantities equal quantities or one and the same quantity be taken away, the remainders will be unequal.

Explicat.

If $AB \sqsubset CD$,
And $EB = FD$,
Then $AE \sqsubset CF$; per *Ax. 14.*

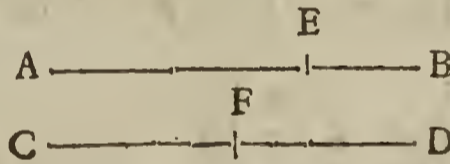


Axiom 15.

15. If from equal quantities unequal quantities be taken away, the remainders are unequal.

Explicat.

If $AB = CD$,
And $AE \sqsubset CF$,
Then $EB \supset FD$; per *Ax. 15.*

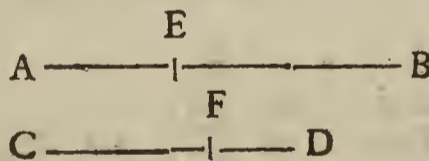


Axiom 16.

16. If from unequal quantities unequal quantities be taken away, from the greater the lesser, and from the lesser the greater, the remainders are unequal; to wit, the former the greater, and the latter the lesser.

Explicat.

If $AB \sqsubset CD$,
And $CF \sqsubset AE$,
Then $EB \sqsubset FD$; per *Ax. 16.*

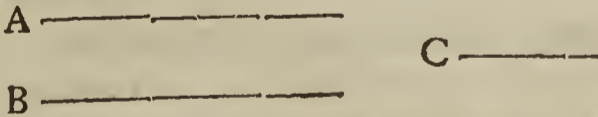


Axiom 17.

17. Quantities which are the doubles of one and the same quantity, or of equal quantities, are equal between themselves. Conceive the same of triples, quadruples, &c.

Explicat.

If $A = 2C$,
And $B = 2C$,
Then $A = B$; per *Ax. 17.*

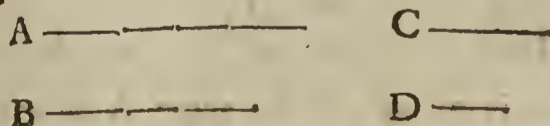


Axiom 18.

18. The double of the greater of two quantities is greater than the double of the lesser.

Explicat.

If $\left\{ \begin{array}{l} C \sqsubset D, \\ A = 2C, \\ B = 2D, \end{array} \right.$
Then $A \sqsubset B$; per *Ax. 18.*



Axiom 19.

Axiom 19.

19. That which is the double of one of two equal quantities, is also the double of the other.

Explicat.

If $B = C,$	A _____	B _____
And $A = 2B,$		
Then $A = 2C;$ per Ax. 19.		C _____

Axiom 20.

20. If one of two equal quantities be the double of a third, the other of those two is also the double of the same third.

Explicat.

If $A = B,$	A _____	C _____
And $A = 2C,$		
Then $B = 2C;$ per Ax. 20.	B _____	

Axiom 21.

21. Quantities which are the halves of one and the same quantity, or of equal quantities, are equal between themselves. Understand the same of thirds, fourths, &c.

Explicat.

If $A = \frac{1}{2}C,$	A _____	C _____
And $B = \frac{1}{2}C,$		
Then $A = B;$ per Ax. 21.	B _____	

Axiom 22.

22. The half of the greater of two quantities is greater than the half of the lesser.

Explicat.

If	{	$C \sqsubset D,$	A _____	C _____
		$A = \frac{1}{2}C,$		
		$B = \frac{1}{2}D,$	B _____	D _____
Then		$A \sqsubset B;$ per Ax. 22.		

Axiom 23.

23. That which is the half of one of two equal quantities is also the half of the other.

Explicat.

If $B = C,$	A _____	B _____
And $A = \frac{1}{2}B,$		
Then $A = \frac{1}{2}C;$ per Ax. 23.		C _____

Axiom 24.

24. If one of two equal quantities be the half of a third, the other of those two shall be the half of the same third.

Explicat.

If $A = B,$	A _____	C _____
And $A = \frac{1}{2}C,$		
Then $B = \frac{1}{2}C;$ per Ax. 24.	B _____	

What hath been said in the eight last preceding Axioms concerning the double and the half, may be also understood of the triple, quadruple, quintuple, &c. and of thirds, fourths, fifths, &c.

Axiom 25.

25. Every whole is greater than its part.

Axiom 26.

26. All right angles are equal between themselves.

Explicat.

If $\sphericalangle A$ be $\perp,$	A \perp	B \perp
And $\sphericalangle B$ be $\perp,$		
Then $\sphericalangle A = \sphericalangle B;$ per Ax. 26.		

That is to say, if the angle A be a right-angle, and the angle B be a right-angle; then the angle A is equal to the angle B.

Axiom 27.

Axiom 27.

27. If one of two or more equal angles be a right-angle, every one of the rest of those equal angles is also a right-angle.

Explicat.

If $\sphericalangle A = \sphericalangle B = \sphericalangle C$,
 And $\sphericalangle A$ be \perp , $A \perp, B \perp, C \perp$.
 Then $\sphericalangle B$ and $\sphericalangle C$ are \perp ; *per Ax. 27.*

That is to say, If the angles A, B and C be equal to one another, and the angle A be a right-angle, then the angles B and C are also right-angles. *Per Axiom. 24. Chap. 2.*

Axiom 28.

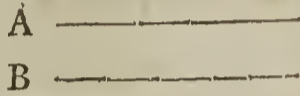
28. Every whole is equal to all its parts taken together.

Axiom 29.

29. If a quantity be neither greater nor less than another quantity, those quantities are equal between themselves.

Explicat.

If A be neither \sqsubset nor \sqsupset than B,
 Then $A = B$; *per Ax. 29.*



C H A P. III.

The explication of Definitions, concerning the ways of arguing used by Mathematicians, to infer one Analogie from another.

THE ways of arguing about *Reasons*, or *Proportions*, are principally six, which are explain'd in this Chapter in such order as they are express'd by the 12th, 13th, 14th, 15th, 16th and 17th Definitions at the beginning of the fifth Book of *Euclid's Elements*; to which six ways of reasoning, six others are also here inserted as Annotations, being the *Scholies* of *Clavius* and *Herigonius* upon such Propositions of *Euclid's Elements* as are hereafter cited. All which are very useful in Mathematical Resolution and Composition, as will appear in the following 7th, 8th, 9th and 10th Chapters.

Definition I.

1. *Alternate Reason* is the comparing of the Antecedent to the Antecedent; and the Consequent to the Consequent.

Explicat.

If this Analogy be propos'd, $a : b :: c : d$
 $6 : 4 :: 12 : 8$
 Then alternately, or by permutation, $a : c :: b : d$ *Per Prop. 15. Elem. 5.*
 $6 : 12 :: 4 : 8$

That is to say, If *a* hath such Reason (or Proportion) to *b*, as *c* to *d*; then alternately, or by permutation of Reason, as *a* is to *c*, so shall *b* be to *d*.

But note diligently, that in this first way of arguing, all the four Proportionals in the Analogy propounded must necessarily be Quantities of one and the same kind, that is, either all Lines, or all Planes, &c. For although it may properly be said, as the line *a* is to the line *b*, so is the Plane *c* to the Plane *d*; yet it cannot be thence infer'd by Alternate Reason, that the line *a* is to the Plane *c*, as the line *b* to the Plane *d*, because there is no Proportion between a Line and a Plane, which are quantities of different kinds: But in all the following ways of arguing, the two first Proportionals may be of one kind, and the two latter of another, as is manifest by the Demonstrations in the fifth Book of *Euclid's Elements*.

Defini-

Definit. II.

2. *Inverse Reason* is the taking of the Consequent as the Antecedent, to compare it to the Antecedent as if it were the Consequent.

Explicat

If $\left\{ \begin{array}{l} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{array} \right.$
 Then inversly, $\left\{ \begin{array}{l} b . a :: d . c \\ 4 . 6 :: 8 . 12 \end{array} \right.$

Per Coroll. prop. 4. Elem. 5.

Definit. III.

3. *Composition of Reason* is the taking of the Antecedent and Consequent both as one, to compare it to the same Consequent.

Explicat.

If $\left\{ \begin{array}{l} a . b :: c . d \\ 6 . 12 :: 4 . 8 \end{array} \right.$
 Then by Composition, $\left\{ \begin{array}{l} a+b . b :: c+d . d \\ 18 . 12 :: 12 . 8 \end{array} \right.$

Per prop. 18. Elem. 5.

Annot. 1.

4. *Composition of Reason converse* is the taking of the Antecedent and Consequent both as one, to compare it to the same Antecedent.

Explicat.

If $\left\{ \begin{array}{l} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{array} \right.$
 Then by Compos. converse, $\left\{ \begin{array}{l} a+b . a :: c+d . c \\ 10 . 6 :: 20 . 12 \end{array} \right.$

Per Schol. 1. Clavii in prop. 18. Elem. 5.

Annot. 2.

5. *Composition of Reason contrary* is the comparing of the Antecedent to the Antecedent and Consequent taken both as one.

Explicat.

If $\left\{ \begin{array}{l} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{array} \right.$
 Then by Compos. contrary, $\left\{ \begin{array}{l} a . a+b :: c . c+d \\ 6 . 10 :: 12 . 20 \end{array} \right.$

Per Schol. 2. Clavii in prop. 18. Elem. 5.

Annot. 3.

6. *Composition of Reason inversly contrary* is the comparing of the Consequent to the Antecedent and Consequent taken both as one.

Explicat.

If $\left\{ \begin{array}{l} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{array} \right.$
 Then by Composition inversly contrary, $\left\{ \begin{array}{l} b . a+b :: d . c+d \\ 4 . 10 :: 8 . 20 \end{array} \right.$

Per Schol. 2. Herigon. in prop. 18. Elem. 5.

Definit. IV.

7. *Division of Reason* is the comparing of the excess whereby the Antecedent exceeds the Consequent, to the same Consequent.

Explicat.

If $\left\{ \begin{array}{l} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{array} \right.$
 Then by Division, $\left\{ \begin{array}{l} a-b . b :: c-d . d \\ 2 . 4 :: 4 . 8 \end{array} \right.$

Per prop. 17. Elem. 5.

But in this way of arguing by Division of Reason, 'tis manifest that the Antecedent must necessarily be greater than the Consequent.

Annot. 1.

Annot. 1.

8. *Division of Reason converse* is the comparing of the Consequent to the excess whereby the Antecedent exceeds the Consequent.

Explicat.

If $\left\{ \begin{array}{l} a : b :: c : d \\ 9 : 4 :: 18 : 8 \end{array} \right.$

Then by Division converse, $\left\{ \begin{array}{l} b : a-b :: d : c-d \\ 4 : 5 :: 8 : 10 \end{array} \right.$

Per Schol.
1. Clavii
in prop. 17.
Elem. 5.

Annot. 2.

9. *Division of Reason contrary* is the comparing of the Antecedent, to the excess whereby the Consequent exceeds the Antecedent.

Explicat.

If $\left\{ \begin{array}{l} a : b :: c : d \\ 4 : 6 :: 8 : 12 \end{array} \right.$

Then by Division contrary, $\left\{ \begin{array}{l} a : b-a :: c : d-c \\ 4 : 2 :: 8 : 4 \end{array} \right.$

Per Schol.
2. Clavii
in prop. 17.
Elem. 5.

But here 'tis manifest that the Consequent must be greater than the Antecedent.

Annot. 3.

10. *Division of Reason inversly contrary* is the comparing of the excess whereby the Consequent exceeds the Antecedent, to the same Antecedent.

Explicat.

If $\left\{ \begin{array}{l} a : b :: c : d \\ 4 : 6 :: 8 : 12 \end{array} \right.$

Then by Division inversly contrary, $\left\{ \begin{array}{l} b-a : a :: d-c : c \\ 2 : 4 :: 4 : 8 \end{array} \right.$

Per Schol.
2. Heri-
gon. in
prop. 17.
Elem. 5.

Definit. V.

11. *Conversion of Reason* is the comparing of the Antecedent to the excess by which the Antecedent exceeds the Consequent.

Explicat.

If $\left\{ \begin{array}{l} a : b :: c : d \\ 6 : 4 :: 12 : 8 \end{array} \right.$

Then by converse Reason, $\left\{ \begin{array}{l} a : a-b :: c : c-d \\ 6 : 2 :: 12 : 4 \end{array} \right.$

Per Coroll.
prop. 19.
Elem. 5.

Definit. VI.

12. *Reason of equality* is, when more than two quantities in one Rank, and as many in another are such, that if two to two be compared, they are in the same Reason; and it also happens, that as the first is to the last in the first rank of Quantities, so is the first to the last in the latter rank. Or otherwise, 'tis a comparison of the extremes to one another, the mean quantities being taken away.

But there are two ways of arguing by Reason of equality, to wit, one when the Proportion is Ordinate, the other when the Proportion is Inordinate or Disturbed; both which are explain'd in the two following Definitions.

Definit. VII.

13. *Ordinate proportion* is, when in the first rank of quantities, as the Antecedent is to the Consequent; so in the latter rank is the Antecedent to the Consequent: and when in the first rank as the Consequent is to some other, so in the latter rank is the Consequent to some other.

A 2

Explicat.

Explicat.

If to these quantities propounded, $\left\{ \begin{array}{l} A,4 \cdot B,6 \cdot C,12 \cdot D,8 \\ E,10 \cdot F,15 \cdot G,30 \cdot H,20 \end{array} \right.$

These Analogies do happen, $\therefore \left\{ \begin{array}{l} A,4 \cdot B,6 :: E,10 \cdot F,15 \\ B,6 \cdot C,12 :: F,15 \cdot G,30 \\ C,12 \cdot D,8 :: G,30 \cdot H,20 \end{array} \right.$

Per prop.
22. Elem.

Then by Reason of equality, $\therefore \prec A,4 \cdot D,8 :: E,10 \cdot H,20$

1.

That is to say, when the proportion in both ranks of quantities propounded is ordinate, (according to *Defin. 7.*) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C, D; so shall the first be to the last in the second rank of quantities E, F, G, H.

Definit. VIII.

14. *Inordinate proportion* is, when three quantities standing in one rank and three in another do afford these Analogies; viz. as the first quantity in the first rank is to the second in the same rank; so is the second quantity in the second rank to the third in the same rank: and as the second quantity in the first rank, is to the third in the same rank; so is the first quantity in the second rank to the second in the same rank.

Explicat.

If to these quantities propounded, $\left\{ \begin{array}{l} A,4 \cdot B,6 \cdot C,3 \\ D,20 \cdot E,10 \cdot F,15 \end{array} \right.$

These Analogies do happen; $\therefore \left\{ \begin{array}{l} A,4 \cdot B,6 :: E,10 \cdot F,15 \\ B,6 \cdot C,3 :: D,20 \cdot E,10 \end{array} \right.$

Per prop.
23. Elem.
5.

Then by reason of Equality, $\therefore \prec A,4 \cdot C,3 :: D,20 \cdot F,15$

That is to say, when the proportion in both ranks of quantities propounded is inordinate, (according to *Defin. 8.*) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C; so shall the first be to the last in the latter rank of quantities, D, E, F.

C H A P. IV.

Various fundamental Theorems frequently used in Mathematical Resolution and Composition.

Theorem I.

A Rectangle (or right-angled Parallelogram) comprehended under any right-line and the difference of any two right-lines, is equal to the difference of two Rectangles comprehended under the first line and each of the two latter.

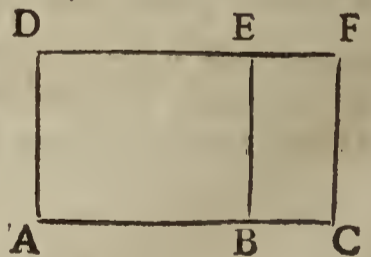
Suppos.

1. AD is a right-line,
2. AC and BC are right-lines,
3. ABC is a right-line,
4. AB = AC - BC.

5. . . Req. demonstr. . . . $\therefore \square AD, AB = \square AD, AC - \square AD, BC.$

Preparat.

6. Make $\square AF$ to be contain'd under AD and AC,
7. Make BE \perp AC,



Demon-

Demonstration.

8. By Constr. in 6°, and 7°. (and } $\square AE + \square BF = \square AF$
per prop. 1. Elem. 2.)
9. Therefore, by subtracting $\square BF$ } $\square AE = \square AF - \square BF$
 from each part of that Equation,
10. That is, (*per Ax. 7. Chap. 2.*) } $\square AD, AB = \square AD, AC - \square AD, BC$
 Which was to be demonstrated. BE

Illustration Algebraical.

Let three right-lines be represented by $\left. \begin{array}{l} a \dots \\ b \dots \\ c \dots \end{array} \right\} \begin{array}{l} 3 \\ 6 \\ 2 \end{array}$

Suppose also $\left. \begin{array}{l} b \square c \\ ab - ac \end{array} \right\} \begin{array}{l} \\ 18 - 6 = 12 \end{array}$

Then if the first line be multiplied by the difference of the second and third, that is, $a \times b - c$, the Product will be

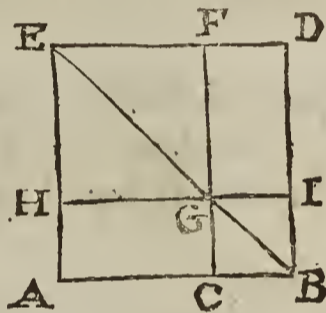
Which Product is manifestly the difference between the Product of the first line a into the second b , and the Product of the first line a into the third c , according to the tenour of *Theorem 1.*

Theorem II.

If a right-line be cut into any two parts, the Square described upon the whole line is equal to the Squares described upon the parts, and to twice the Rectangle comprehended under the parts.

Suppos.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. $AC + CB = AB$.



4. \therefore Req. demonstr. $\therefore \square AB = \square AC + \square CB + 2\square AC, CB$.

Prepar.

5. Upon AB describe the $\square AD$, (*per prop. 46. Elem. 1.*)
6. Draw the Diameter EB
7. Draw CF \parallel AE (or BD,) and cutting EB in G, (*per prop. 31. Elem. 1.*)
8. By the point G draw HGI \parallel AB, (or ED.)

Demonstration.

9. By Constr. in 5°, $\square AD$ is $\square AB$.
10. Therefore, (*per 29. Defin. 1. Elem.*) $\sphericalangle A, \sphericalangle AED, \sphericalangle D, \sphericalangle DBA$ are \perp .
11. And out of 7°, 8°, and 10°; (*per 29. prop. 1. Elem.*) $\sphericalangle EHG, \sphericalangle EFG, \sphericalangle HGF$ are \perp .
12. And out of 5°, (*per 29. defin. 1. Elem.*) $AE = AB = BD = DE$.
13. Therefore out of 10°, and 12°; (*per prop. 5, & 32. Elem. 1.*) $\sphericalangle AEB$ is $\frac{1}{2} \perp$.
14. Likewise, out of 10°, and 12°; $\sphericalangle DEB$ is $\frac{1}{2} \perp$.
15. Likewise out of 11°, and 13°; $\sphericalangle HGE$ is $\frac{1}{2} \perp$.
16. Likewise out of 11°, and 14°; $\sphericalangle FGE$ is $\frac{1}{2} \perp$.
17. Therefore out of 13°, and 15°; (*per prop. 6. Elem. 1.*) $HE = HG$.
18. Likewise out of 14°, and 16°; $EF = FG$.
19. And from 7°, and 8°; (*per prop. 34. Elem. 1.*) $EF = HG$.
20. Wherefore out of 11°, 17°, 18°, 19°; (*per 29. defin. 1. Elem.*) HF is $\square HG$, or $\square AC$.
21. And in the same respect, CI is $\square CB$.
22. Therefore from 21°, $CG = CB$.
23. And from 22°, (*per 36. prop. 1. Elem.*) $\square AG$, (or $\square AC, CG$,) = $\square AC, CB$.

A 2

24. But

24. But (per 43. prop. 1. Elem.) $\therefore \square AG = \square GD.$
 25. Therefore out of 23°, and 24°; } $\square GD = \square AC, CB.$
 (per Ax. 1. Chap. 2.) }
 26. But (per Ax. 28. Chap. 2.) $\therefore \square AD = \square HF + \square CI + \square AG + \square GD.$
 27. Wherefore, out of 5°, 20°, 21°; } $\square AB = \square AC + \square CB + 2 \square AC, CB.$
 24°, 25°, 26°; (per Ax. 7. Chap. 2.) }
 Which was to be dem.

Coroll. 1.

Hence it is manifest that the Parallelograms which are about the Diameter of a Square, are also Squares themselves.

Coroll. 2.

It also appears that the Diameter of any Square divides its angles into two equal parts.

Illustration Algebraical.

Suppose a right-line to be cut into two parts, to wit, $\therefore a$ and b . . . | 6 and 2
 Then the summ of the parts is $\therefore a + b$. . . | 8
 Which summ multiplied by it self produceth the Square }
 of the whole line, to wit, $\therefore aa + 2ba + bb$ | 64

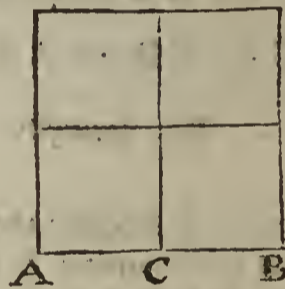
Which Product or Square doth manifestly consist of the Squares of the parts a and b , and twice the Product (or Rectangle) of the same parts; according to the tenour of the preceding Theor. 2.

Theorem III.

A Square described upon any right-line is equal to four times the Square of the half of the same line; and consequently, a quarter of the former Square is equal to the latter.

Suppos.

1. AB is a right-line;
2. AC = CB, therefore
3. AB = AC + CB = 2AC or 2CB.



4. . . . Req. demonstr. . . . $\therefore \square AB = 4 \square AC, \text{ (or } 4 \square CB, \text{)} \text{ Also,}$
 $\therefore \frac{1}{4} \square AB = \square AC, \text{ or } \square CB.$

Demonstration.

5. By supposition, $\therefore AC = CB.$
6. Therefore, (per Sch. of prop. 46. El. 1.) $\therefore \square AC = \square CB.$
7. And out of 5°, (per prop. 36. Elem. 1.) $\therefore \square AC = \square AC, CB.$
8. And out of 7°, (per Ax. 17. Chap. 2.) $\therefore 2 \square AC = 2 \square AC, CB.$
9. And out of 6°, (per Ax. 8. Chap. 2.) $\therefore 2 \square AC = \square AC + \square CB.$
10. And out of 8°, and 9°, (per Ax. 8.) $\therefore 4 \square AC = \square AC + \square CB + 2 \square AC, CB.$
11. But per Theor. 2. of this Chap. $\therefore \square AB = \square AC + \square CB + 2 \square AC, CB.$
12. Wherefore out of 10°, and 11°; } $\square AB = 4 \square AC, \text{ (or } 4 \square CB. \text{)}$
 (per Ax. 1.) }
 Concluf. } 13. And out of 12°, (per Ax. 21.) } $\frac{1}{4} \square AB = \square AC, \text{ (or } \square CB. \text{)}$
 Chap. 2.) }
 Which was to be dem.

Illustration Algebraical.

Let a right-line be represented by $\therefore 2a$ | 10
 The half thereof is $\therefore a$ | 5
 The Square of the whole line $2a$ is $\therefore 4aa$ | 100
 The Square of the half, to wit, of a is $\therefore aa$ | 25

The first of those Squares is evidently equal to four times the latter, and consequently a quarter of the former is equal to the latter; as is affirmed by Theor. 3.

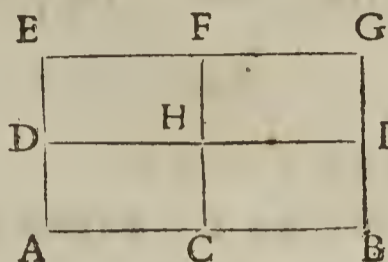
Theor. IV.

Theorem IV.

A Rectangle (or long Square) comprehended under any two unequal right-lines is equal to four times the Rectangle comprehended under the half of each of those lines; and consequently a quarter of the first Rectangle is equal to the latter.

Suppos.

1. AB and AE are right-lines,
2. AC = CB = 1/2 AB,
3. AD = DE = 1/2 AE.



4. . . Req. demonstr. ∴ { $\square AB, AE = 4 \square AC, AD.$ Also,
 $\frac{1}{4} \square AB, AE = \square AC, AD.$

Prepar.

5. Make $\square AG$ to be contain'd under AB and AE.
6. Draw $CF \parallel AE$, (or BG .) Likewise, $DI \parallel EG$, (or AB .)

Demonstration.

7. By Constr. in 5°, . . . } AG is $\square AB, AE$.
8. Therefore (per 30. defn. } $\angle A, \angle E, \angle G, \angle B$ are \perp
 1. Elem.) . . . }
9. And because by Constr. } $CF \parallel AE \parallel BG$.
 in 6°, . . . }
10. Likewise by Constr. in 6° } $DI \parallel EG \parallel AB$.
11. Therefore out of 8°, 9° } AH, DF, HG, CI are \square .
 and 10°; (per prop. 29. }
 Elem. 1.) . . . }
12. And because by Suppos. } $AC = CB$. Also $AD = DE$.
 in 2° and 3°, . . . }
13. Therefore out of 11° } $\square AH = \square DF = \square HG = \square CI$.
 and 12°, (per prop. 36. }
 Elem. 1.) . . . }
14. And from 13°, (per Ax. } $\square AH + \square DF + \square HG + \square CI = 4 \square AH$.
 8. Chap. 2.) . . . }
15. But (per Ax. 28. Ch. 2.) } $\square AH + \square DF + \square HG + \square CI = \square AG$.
16. Therefore out of 14° and } $\square AG = 4 \square AH$.
 15°, (per Ax. 1.) . . . }
17. That is, (per Ax. 7. } $\square AB, AE = 4 \square AC, AD$.
 Chap. 2.) . . . }
18. And consequently from } $\frac{1}{4} \square AB, AE = \square AC, AD$.
 17°, (per Ax. 21.) . . . }

} Concluf.

Which was to be dem.

Illustration Alegebraical.

Let a right-line be represented by	2a	6
And another right-line by	2b	4
The half of the former line is	a	3
And the half of the latter is	b	2
The Product or Rectangle of the two whole lines is	4ab	24
The Product of the half of each line is	ab	6

The first of those Products is evidently equal to four times the latter, and consequently a quarter of the former is equal to the latter, according to the tenour of Theor. 4.

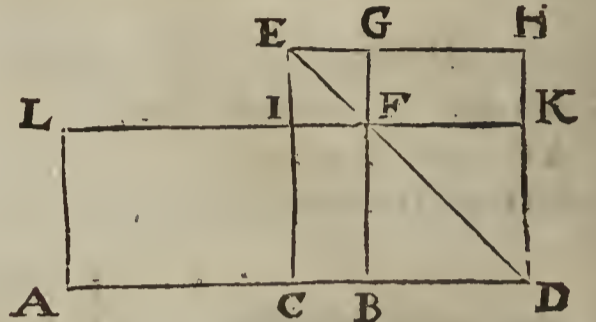
Theor. V.

Theorem V.

If a right-line be cut into any two unequal parts, the Square of the difference of the parts is equal to the Squares of the parts, less by twice the Rectangle (or long Square) comprehended under the parts: Also, the Square of half the difference of the said parts is equal to a quarter of each of the Squares of the parts, less by half the Rectangle of the parts.

Suppos.

1. A B is a right-line.
2. A C and C B are parts of A B.
3. A C \neq C B.



Prepar.

4. Produce A B to D, so, that A C = C D, then 'tis manifest that B D is the difference of the parts A C and C B; for C D (or A C) - C B = B D.
5. Upon C D describe the \square C H; (per prop. 46. Elem 1.)
6. Draw the Diameter E D.
7. By the point B draw B G \parallel C E (or D H,) and cutting E D in F.
8. By the point F draw L F K \parallel A D, or E H.
9. By the point A draw A L \parallel C E.

10. . . Req. demonstr. . . $\left\{ \begin{array}{l} \square B D = \square A C + \square C B - 2 \square A C, C B. \text{ Also,} \\ \square \frac{1}{2} B D = \frac{1}{4} \square A C + \frac{1}{4} \square C B - \frac{1}{2} \square A C, C B. \end{array} \right.$

Demonstration.

11. By Constr. in 4° and 5°, } C H is \square C D or \square A C.
12. Therefore (per Coroll. 1. } B K is \square B D.
13. Likewise . . . } I G is \square E G or \square C B.
14. And (per prop. 43. } \square C F = \square F H.
15. Therefore (per Ax. 8. } \square C G = \square I H.
16. Again, (by Constr. in } C E = A C.
17. And 'tis evident that } C B = C B.
18. Therefore from 16° and } \square C E, C B (or \square C G) = \square A C, C B.
19. Therefore out of 15° and } \square C G + \square I H = 2 \square A C, C B.
20. And because (per Ax. } \square B K + \square C G + \square I H = \square C H + \square I G.
21. Therefore out of 19° } \square B K + 2 \square A C, C B = \square C H + \square I G.
22. Therefore from 21° } \square B K = \square C H + \square I G - 2 \square A C, C B.
23. Therefore out of 22° } \square B D = \square A C + \square C B - 2 \square A C, C B.

Conclus. 1. Which was to be dem.

24. Moreover, out of 23°, } $\frac{1}{4} \square B D = \frac{1}{4} \square A C + \frac{1}{4} \square C B - \frac{1}{2} \square A C, C B.$
25. And per The. 3. of this Ch. } $\frac{1}{4} \square B D = \square \frac{1}{2} B D$
26. Therefore out of 24° } $\square \frac{1}{2} B D = \frac{1}{4} \square A C + \frac{1}{4} \square C B - \frac{1}{2} \square A C, C B.$

Conclus. 2. Which was also to be dem.

Illustra-

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, to wit,	}	a and b . . .	16 and 10
Suppose also		$a \sqsupset b$. . .	
Then the difference of the parts is	}	$a - b$. . .	6
And half the difference of the parts is		$\frac{1}{2}a - \frac{1}{2}b$. . .	
The Square of the whole difference is	}	$aa + bb - 2ba$	36
The Square of the half difference is		$\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ba$	

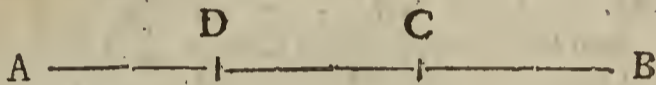
Which two Squares do manifestly prove the certainty of what is affirmed in the foregoing Theor. 5.

Theorem VI.

If a right-line be cut into any two unequal parts, the Square of the whole line together with the Square of the difference of the parts is equal to twice the Squares of the parts; and consequently half the Square of the whole line together with half the Square of the difference of the parts is equal to the summ of the Squares of the parts.

Suppos.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. $AC \sqsupset CB$.



Prepar.

4. From CA cut off CD \cong CB, thence it follows that AD ($= AC - CB$) is the difference of the parts AC and CB.

5. . . . Req. demonstr. . . . } $\square AB + \square AD = 2 \square AC + 2 \square CB$. Also, $\frac{1}{2} \square AB + \frac{1}{2} \square AD = \square AC + \square CB$.

Demonstration.

6. By Theor. 2. of this Chapt. . . } $\square AB = \square AC + \square CB + 2 \square AC, CB$.
7. And by Theor. 5. of this Chapt. . } $\square AD = \square AC + \square CB - 2 \square AC, CB$.
8. Therefore out of 6° and 7°, (per Ax. 8. Chap. 2.) . . . } $\square AB + \square AD = 2 \square AC + 2 \square CB$.
9. And consequently, (per Ax. 21. Chap. 2.) . . . } $\frac{1}{2} \square AB + \frac{1}{2} \square AD = \square AC + \square CB$.

} Concluf.

Which was to be dem.

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, to wit,	}	a and b . . .	6 & 4
Suppose also,		$a \sqsupset b$	
Then the summ of the parts is	}	$a + b$. . .	10
And the difference of the parts is		$a - b$. . .	
The Square of the whole line, that is, the Square of the summ of the parts, is	}	$aa + bb + 2ba$	100
The Square of the difference of the parts is		$aa + bb - 2ba$	
The summ of those Squares is	}	$2aa + 2bb$. . .	104

Which summ; (according to the tenour of the preceding Theor. 6.) is manifestly equal to twice the summ of the Squares of the parts.

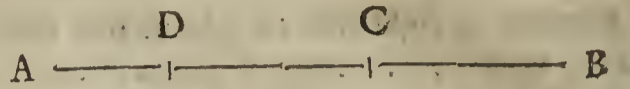
Theorem VII.

If a right-line be cut into any two unequal parts, the Square of the whole line is equal to four times the Rectangle (or long Square) comprehended under the parts, together with the Square of the difference of the parts: Also, the Square of half the said right-line, (or of half the summ of the parts,) is equal to the Rectangle of the parts together with a quarter of the Square of the difference of the parts.

Suppos.

Suppos.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. AC \sqsupset CB.



Prepar.

4. From CA cut off CD = CB, whence AD (= AC - CB) is the difference of the parts AC and CB.

5. . . . *Req. demonstr.* . . . $\left\{ \begin{array}{l} \square AB = 4 \square AC, CB + \square AD. \text{ Also,} \\ \square \frac{1}{2} AB = \square AC, CB + \frac{1}{4} \square AD. \end{array} \right.$

Demonstration.

6. By *Theor. 5.* of this *Chapt.* . . . $\square AD = \square AC + \square CB - 2 \square AC, CB.$
7. Therefore by adding $4 \square AC, CB$ to each part of that Equation this ariseth, (*per ax. 8. ch. 2.*) . . . $4 \square AC, CB + \square AD = \square AC + \square CB + 2 \square AC, CB.$
8. But *per Theor. 2.* of this *Chap.* . . . $\square AB = \square AC + \square CB + 2 \square AC, CB.$
9. Therefore from 7^o and 8^o, (*per Ax. 1. Chap. 2.*) . . . $\square AB = 4 \square AC, CB + \square AD.$
Which was to be dem.
10. Moreover, from 9^o, (*per Ax. 21. Chap. 2.*) . . . $\frac{1}{4} \square AB = \square AC, CB + \frac{1}{4} \square AD.$
11. And by *Theor. 3.* of this *Cha.* . . . $\frac{1}{4} \square AB = \square \frac{1}{2} AB.$
12. Therefore from 10^o and 11^o, (*per Ax. 1.*) . . . $\square \frac{1}{2} AB = \square AC, CB + \frac{1}{4} \square AD.$
Which was also to be dem.

Illustration Algebraical.

Suppose a right-line to be cut into two parts, to wit,	} a and b . . .	6 and 4
Suppose also	} a \sqsupset b	
Then the summ of the parts is	} a + b . . .	10
And the difference of the parts is	} a - b . . .	2
The Square of the summ of the parts, that is, the Square of the whole line is	} aa + bb + 2ba	100
And the Square of the difference of the parts is	} aa + bb - 2ba	4
To which Square of the difference if you add the quadruple Product of the parts, to wit,	} . . . + 4ba	96

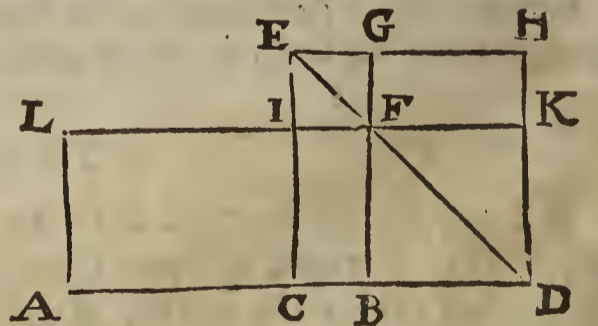
The summ (according to the tenour of *Theor. 7.*) makes the Square of the whole line, to wit, $aa + bb + 2ba.$

Theorem VIII.

If a right-line be cut into any two unequal parts, the Rectangle (or long Square) comprehended under the whole line and the difference of the parts is equal to the difference of the Squares of the parts. Also, the Rectangle under half the said right-line, (that is, half the summ of the parts) and half the difference of the parts, is equal to a quarter of the difference of the Squares of the parts.

Suppos.

1. AB is a right-line.
2. AC and CB are parts of AB.
3. AC \sqsupset CB.



Prepar.

4. Produce AB to D, so, that AC = CD, thence it follows that BD is the difference of the parts AC and CB; for CD (or AC) - CB = BD.

5. Upon

5. Upon CD describe the \square CH.
6. Draw the Diameter ED.
7. By the point B draw BG \parallel CE (or DH,) and cutting ED in F.
8. By the point F draw LFK \parallel AD, or EH.
9. By the point A draw AL \parallel CE.
10. . . Req. demonstr. . . $\left. \begin{array}{l} \square AB, BD = \square AC - \square CB. \text{ Also;} \\ \square \frac{1}{2}AB, \frac{1}{2}BD = \frac{1}{4}\square AC - \frac{1}{4}\square CB. \end{array} \right\}$

Demonstration.

11. By Constr. in 5° , . . . } CH is \square CD.
12. Therefore out of 6° and 11° , } IG = \square IF, and BK = \square BD.
- (per Cor. 1. The. 2. of this Cha.) }
13. And from 12° , (per 29. defin. } BF = BD.
1. Elem) }
14. By Constr. in 7° and 8° , . . . } CF is \square .
15. Therefore from 14° , (per 34. } BF = CI; and IF = CB.
- prop. 1. Elem.) }
16. And from 13° and 15° , (per } BD = CI.
- Ax. 1.) }
17. By Constr. in 4° and 5° , . . . } DH = CA.
18. Therefore out of 16° and 17° , } \square BD, DH = \square CI, CA.
- (per 36. prop. 1. Elem.) }
19. That is, (per Ax. 7. Chap. 2.) } \square BH = \square AI.
20. Therefore by adding \square CF to } Gnomon, ICDHG = AF = \square AB, BD (BF.)
- each part of the Equation in 19°, }
21. But 'tis manifest (per Ax. 9. } Gnomon, ICDHG = \square CH - \square IG.
- Chap. 2.) that . . . }
22. Therefore from 20° and 21° , } \square AB, BD = \square CH - \square IG.
- (per Ax. 1.) }
23. And because from $4^\circ, 5^\circ, 11^\circ$, } \square CD (or \square AC) - \square CB (\square IG) = \square CH - \square IG,
- and 15° ; . . . }
24. Therefore from 22° and 23° , } \square AB, BD = \square AC - \square CB. Conclus. 1.
- (per Ax. 1.) }
- Which was to be dem.
25. Moreover, from 24° , (per } $\frac{1}{4}\square$ AB, BD = $\frac{1}{4}\square$ AC - $\frac{1}{4}\square$ CB.
- Ax. 21. Chap. 2.) }
26. And by Theor. 4. of this Chapt. } $\frac{1}{4}\square$ AB, BD = \square $\frac{1}{2}$ AB, $\frac{1}{2}$ BD.
27. Therefore out of 25° and 26° , } \square $\frac{1}{2}$ AB, $\frac{1}{2}$ BD = $\frac{1}{4}\square$ AC - $\frac{1}{4}\square$ CB. Conclus. 2.
- (per Ax. 1.) }
- Which was also to be dem.

Illustration Algebraical.

Suppose a right-line to be cut into two parts, to wit,	a and b	6 and 4
Suppose also	$a - b$	
Then the sum of the parts is	$a + b$	10
And the difference of the parts is	$a - b$	2
Therefore the Rectangle (or Product) of the sum and	$aa - bb$	20
difference of the parts is		

Which Rectangle (according to the tenour of Theor. 8.) is manifestly equal to the difference of the Squares of the parts a and b . And by multiplying $\frac{1}{2}a + \frac{1}{2}b$ into $\frac{1}{2}a - \frac{1}{2}b$, the latter part of the said Theorem will be also manifest.

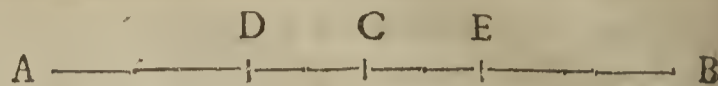
Theorem IX.

If a right-line be cut into any two unequal parts, the greater part shall be equal to half the whole line, together with half the difference of the parts: And, the lesser part shall be equal to half the whole line less by half the difference of the parts.

B b Suppos.

Suppos.

1. AB is a right-line,
2. AE and EB are parts of AB,
3. AE \sqsupset EB.



Prepar.

4. From AB cut off AD = EB, thence it follows that DE is the difference of the parts AE and EB; for DE = AE - AD (EB.)
5. Divide DE into two equal parts in C; therefore DC = CE = $\frac{1}{2}$ DE.
6. \therefore . . . *Req. demonstr.* . . . $\left. \begin{array}{l} AE = \frac{1}{2} AB + \frac{1}{2} DE. \\ EB = \frac{1}{2} AB - \frac{1}{2} DE. \end{array} \right\}$

Demonstration.

7. Because by Constr. in 4 $^{\circ}$, \therefore . . . $\left. \begin{array}{l} AD = EB. \\ DC = CE = \frac{1}{2} DE. \end{array} \right\}$
8. And by Constr. in 5 $^{\circ}$, . . .
9. Therefore the summ of the Equations in 7 $^{\circ}$ and 8 $^{\circ}$, gives (*per Ax. 8. Chap. 2.*) . . . $\left. \begin{array}{l} AC = CB = \frac{1}{2} AB. \\ AE = \frac{1}{2} AB + \frac{1}{2} DE. \\ EB = \frac{1}{2} AB - \frac{1}{2} DE. \end{array} \right\}$ Which was to be Dem.
10. And the summ of the Equations in 8 $^{\circ}$ and 9 $^{\circ}$ gives . . .
11. And the Equation in 8 $^{\circ}$ subtracted from the Equation in 9 $^{\circ}$, gives . . .

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, } a and b | 6 and 4
 to wit, }
 Suppose also } $a \sqsupset b$
 Then half the whole line, that is, half the summ of the } $\frac{1}{2}a + \frac{1}{2}b$ | 5
 parts is }
 And half the difference of the parts is } $\frac{1}{2}a - \frac{1}{2}b$ | 1

Now (according to the import of *Theor. 9.*) the summ of the said half summ and half difference doth manifestly make a the greater part: And the excess of the said half summ above the said difference is manifestly equal to b the lesser part.

CHAP. V.

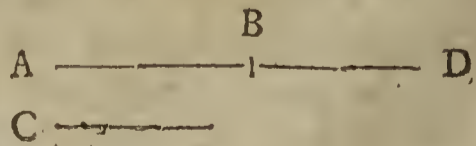
A Collection of Canonical Geometrical Effections, frequently used in the Construction of Plane Problems; more especially of those whose Solutions are found out by the Algebraical Art.

AS all Arithmetical Operations are compris'd under five kinds, to wit, *Addition, Subtraction, Multiplication, Division* and the *Extraction of Roots*, so all those Geometrical Constructions which are formed according to Canons deduced from the Algebraical Resolutions of Problems, do principally depend upon the like kinds of Operations, or Effections; but how these Geometrical Effections, (or the Arithmetick of Geometry) may be perform'd, so far as is necessary to the Construction of Plane Problems, to wit, such as may be solved by drawing only right-lines and describing the Circumferences of Circles, I shall shew in this Chapter, the Contents whereof are extracted out of the first six Books of *Euclid's Elements*, wherein I presuppose the Reader to be competently versed.

Problem I.

To add a given right-line to a right-line given.

Let AB and C be two right-lines given to be added together, *viz.* let it be required to find out a right-line which shall be equal to both the given right-lines taken together as one right-line,



Con-

Construction.

By prop. 2. Elem. 1. produce (or continue) the given line AB to D, that BD may be equal to C, so is AD the right-line sought; for by Construction $AD = AB + BD$, and $BD = C$, therefore (per Ax. 6. Chap. 2.) $AD = AB + C$, as was required;

Probl. II.

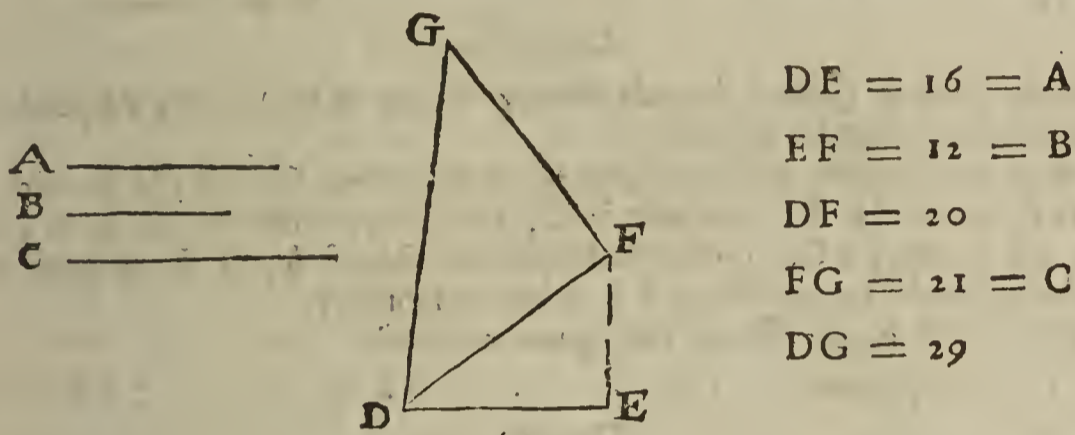
Two or more Squares being given, to find a Square equal to them all.

Suppos.

1. A, B, C are the sides of three Squares given.

Req. to find

2. DG a right-line, such, that $\square DG = \square A + \square B + \square C$.



Constr.

3. Make $DE = A$.
4. Make $EF \perp DE$.
5. Make $EF = B$.
6. Draw DF .
7. Make $FG \perp DF$.
8. Make $FG = C$.
9. Draw DG , which shall be the side of the Square required.
10. . . . Req. demonstr. . . $\square GD = \square A + \square B + \square C$.

Demonstration.

11. Because by Constr. in 7° and 4° , $\angle DFG = \angle DEF$.
12. Therefore, (per prop. 47. Elem. 1.) $\square DG = \square FG + \square FD$.
13. Likewise . . . $\square FD = \square DE + \square EF$.
14. Therefore from 12^o and 13^o, $\square DG = \square DE + \square EF + \square FG = \square A + \square B + \square C$.
(per Ax. 6. Chap. 2.) . . .

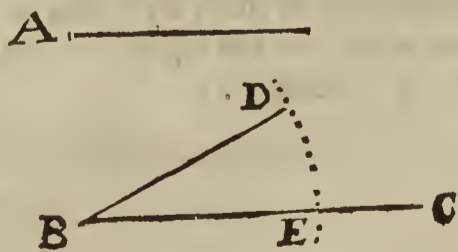
Which was to be done.

After the same manner of Construction, as many Squares as one will may be added into one. But if Planes of any other kind, as Long-Squares, Rhombs, Rhomboids, Triangles, &c. be given to be added, they must first be transformed into Squares, which may be done by Prop. 14. Elem. 2. or by various ways delivered in the practical Geometry of divers Mathematicians, and then they may be added together as before.

Probl. III.

To subtract or cut off a right-line given from a greater right-line given.

The subtracting or cutting off one right-line from another, to wit, a lesser from a greater, is perform'd by Prop. 3. Elem. 1. For, if two unequal right-lines be given, suppose BC the greater, and A the lesser, then by describing a Circle from B as a Center, with the distance or Semidiameter BD equal to the lesser line A, the right-line BE = BD or A will be cut off from the greater line BC, and consequently, EC is the excess whereby BC exceeds A or BE.



B b 2

Probl. IV.

Probl. IV.

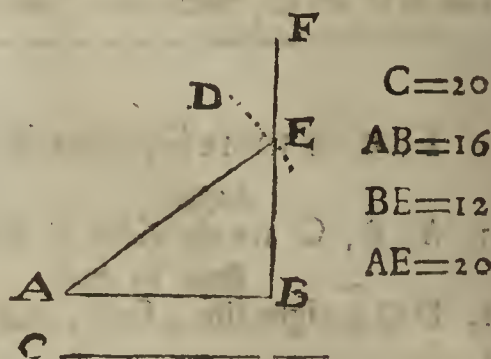
Two unequal Squares being given, to find a Square equal to the excess whereby the greater exceeds the less.

Suppos.

- 1. C and AB are the sides of two Squares given,
- 2. $C \sqsupset AB$.

Req. to find

- 3. BE a right-line, such, that $\square BE = \square C - \square AB$.



Construction.

- 4. Upon the point B, (one of the ends of the given line AB,) erect a Perpendicular, and draw it forth at length, as BF.
- 5. From A as a Center, at the distance of the given line C, describe the arch DE, to cut the Perpendicular BF, suppose in E; for by supposition the line C is greater than AB, and therefore a Circle described upon the Center A, at the distance of C shall necessarily cut the Perpendicular BF produced infinitely.
- 6. I say BE shall be the side of the Square required.
- 7. . . . Req. demonstr. . . . $\square BE = \square C - \square AB$.

Demonstration.

- 8. By Constr. in 4° and 5°, . . . $\angle ABE$ is \perp , and $C = AE$.
 - 9. Therefore (per prop. 47. Elem. 1.) $\square AB + \square BE = \square AE = \square C$.
 - 10. Therefore (per Ax. 9. Chap. 2.) $\square BE = \square C - \square AB$.
- Which was to be done.

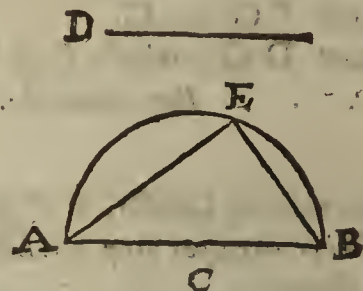
Another Construction of Probl. 4.

Suppos.

- 11. AB and D are the sides of two Squares given.
- 12. $AB \sqsupset D$.

Req. to find

- 13. EB a right-line, such, that $\square EB = \square AB - \square D$.



Construction.

- 14. Upon the given line AB as a Diameter, describe the Semicircle CAEB, and inscribe $AE = D$, which is possible to be done, for by supposition $AB \sqsupset D$. Lastly, draw EB which shall be the side of the Square required.
- 15. . . . Req. demonstr. . . . $\square EB = \square AB - \square D$.

Demonstration.

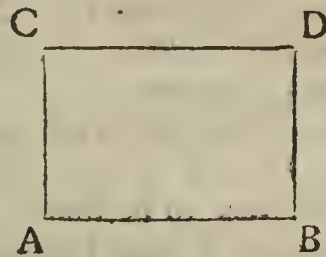
- 16. By Constr. in 14°, and per prop. 31. } $\angle AEE$ is \perp , and $AE = D$.
 - Elem. 3. }
 - 17. Therefore, per prop. 47. Elem. 1. $\square AE$ (or $\square D$) $+ \square EB = \square AB$.
 - 18. Therefore per Ax. 9. Chap. 2. $\square EB = \square AB - \square D$.
- Which was to be done.

Note. If many Squares be given to be subtracted from a Square given, those to be subtracted must first be added together, by the preceding Probl. 2. of this Chapt. and then subtraction may be made by either of the two foregoing Constructions of Probl. 4. But if Planes which are not Squares be to be subtracted, they must first be reduced to Squares, by Prop. 14. Elem. 2.

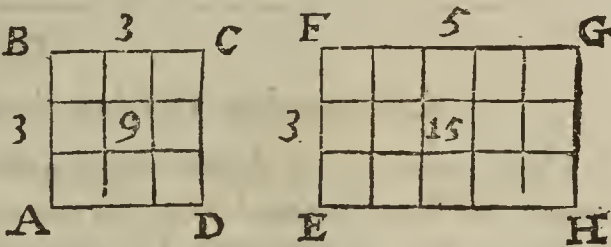
Probl. V.

Concerning Geometrical Multiplication.

1. A right-line is said to be multiplied by a right-line, when a right-angled Parallelogram, whether it be a Square or a Long-square, is comprehended under one of the said right-lines as a length, and the other as a breadth. As, if the right-line AC be conceived to be moved along the line AB, so, as that AC always makes a right-angle with the line AB, until the point C be come to the point D, and the point A to the point B; then the right-angled Parallelogram ACDB is described by such moving of the line AC, and imports the same thing with the Product of the multiplication of the line AB by the line AC. Which Product, or right-angled Parallelogram, is also usually called a Rectangle.



2. A Rectangle is also implied by the Product of the multiplication of any two numbers; for the Area of a Rectangle is equal to the Product made by the multiplication of the number expressing the measure of one of the sides about the right-angle; by the number expressing the measure of the other side about the same angle. As in the Rectangle or long-Square EG, if its length EH or FG be 5 feet; and the breadth EF or HG, 3 feet, then the Product of the multiplication of 5 by 3, to wit, 15, imports the Area, or number of square feet contain'd in the said Rectangle. Likewise, if AB or AD the side of the Square AC be 3 feet, the Area is 9 square feet. All which evidently appears by the Diagrams here in view.

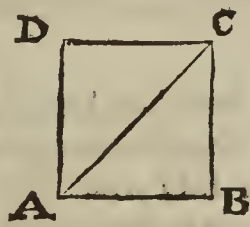


3. If a right-line be to be multiplied by a whole number, it may be done by Addition, (by *Probl. 1.* of this *Chapt.*) As, if the line AB be to be multiplied by 3, it implies only the producing or continuing of the said line to such a point D, that the whole line AD may be equal to the triple of the given line AB.



4. But if a right-line AD be to be multiplied by a Fraction, or (which is of the same import,) if it be required to cut off some segment, as $\frac{2}{3}$ parts from AD; first, (*per Schol. of Prop. 10. Elem. 6.*) divide the line AD into three equal parts, which suppose to be AB, BC, CD, then the segment AC which is compos'd of two of those three parts is manifestly $\frac{2}{3}$ of the line AD.

5. If a Square be to be multiplied by a whole number, the side of the Square sought may be found out by addition of Squares, as before in *Probl. 2.* So if the Square of the right-line AB, or BC be to be doubled, or multiplied by 2, it implies only the finding of the right-line AC, (by *Probl. 2.* of this *Chapt.*) whose Square is equal to the double of the Square of AB or BC. For if $AB = BC$, and $BC \perp AB$, then (*per prop. 47. Elem. 1.*) $\square AC = \square AB + \square BC = 2 \square AB$ or $2 \square BC$. The same thing also may be done by the way delivered in the following *Sect. 6.*



$AB = 3$
 $BC = 3$
 $AC = \sqrt{18}$

6. If a Square be to be multiplied by a whole number, or by any Fraction whatever, whether it be a proper or improper Fraction, that is, if it be required to find the side of a Square that shall be equal to any prescribed Multiple, or to any part or parts of a given Square, it may be done thus: Let the right-line A be the side of a Square given, and let it be required to find a Square which shall contain five times the given Square whose



$A = 2$
 $5A = 10$
 $5 \square A = 20$
 $M = \sqrt{20}$

side

side is A. To effect this, find (per *Probl. 9.* of this *Chapt.*) a mean Proportional between the given side A, and a right-line equal to five times A; which mean suppose to be the right-line M: I say the line M is the Square required; which I prove thus,

Req. demonstr. $\square M = 5 \square A.$

Demonstration.

By Constr. these are Proportionals, *viz.* } $A : M :: M : 5A$
 Therefore, (per *prop. 17. Elem. 6.*) } $\square A, 5A = \square M$
 But (per *prop. 1. Elem. 2.* respect being had to the } $\square A, 5A = 5 \square A$
 last preceding Diagram,) }
 Therefore from the two preceding Equations, } $\square M = 5 \square A.$
 (per *Ax. 1.*) }

Which was to be done.

In like manner a mean Proportional between A and $\frac{2}{3}A$ shall be the side of a Square equal to $\frac{2}{3}\square A$. Also a mean Proportional between $3\frac{1}{2}A$ (or $\frac{7}{2}A$) and A shall be the side of a Square equal to $\frac{7}{2}\square A$, or $3\frac{1}{2}\square A$. The same thing may be effected by *Probl. 11.* of this *Chapt.* the proportion of the Square given to the Square sought being first exprest by two right-lines, by the help of the foregoing *Sect. 3,* or *4.* of this *Probl. 5.*

Probl. VI.

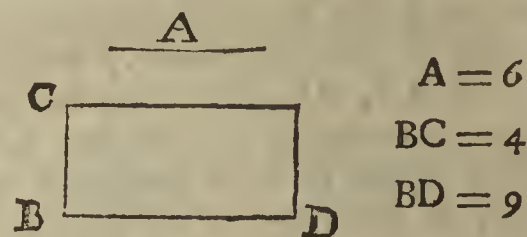
Concerning Geometrical Division, or Application.

That Geometrical Effection which answers to *Division* in Arithmetick is called *Application*, the Scope whereof when 'tis exercis'd about the Construction of Plane Problems is only this, *viz.* A Rectangle, (or right-angled Parallelogram) being given, as also a right-line, to find out another right-line, such, that a Rectangle contain'd under the line found out, and the line given shall be equal to the Rectangle first given; which Effection (or Construction) is called the Application of a given Rectangle to a right-line given, and the right-line arising out of the Application is called the *Parabola*, or the Geometrical Quotient, which is found out by the *Rule of Three* in right-lines by the following 7th or 8th *Problems* of this *Chapter*: For as the line given is to either of the sides about the right-angle of the given Rectangle, so is the other side about the same angle, to the line sought, to wit, the Geometrical Quotient.

This will be made manifest by the two following Examples, in the first whereof the Rectangle given is a Square, in the latter a long-Square.

Suppos. 1.

1. A is the side of a Square given,
2. BC is a right-line given.



Req. to find

3. BD, a right-line, such, that $\square BD, BC = \square A.$

Construction.

4. By *Probl. 7.* of this *Chap.* let it be made as BC to A, so A to a third Proportional, which suppose to be the line BD, therefore, } $BC . A :: A . BD.$
5. I say BD is the right-line sought by the *Probl.* propounded; it remains then to prove that a Rectangle contain'd under the right lines BD and BC is equal to the Square of line A.

Prepar.

6. Let a Rectangle be made of the lines BD and BC, as $\square CD$, (per *prop 46. Elem. 1.*) Then
7. *Req. demonstr.* $\square CD = \square A.$

Demonstration.

8. By Constr. in 4^o, } $BC . A :: A . BD.$
9. Therefore, (per *prop. 17. Elem. 6.*) } $\square A = \square BC, BD.$
10. But by Constr. in 6^o, } $\square CD = \square BC, BD.$
11. Therefore (per *Ax. 1. Chap. 2.*) } $\square CD = \square A.$

Which was to be done.

Suppos. 2.

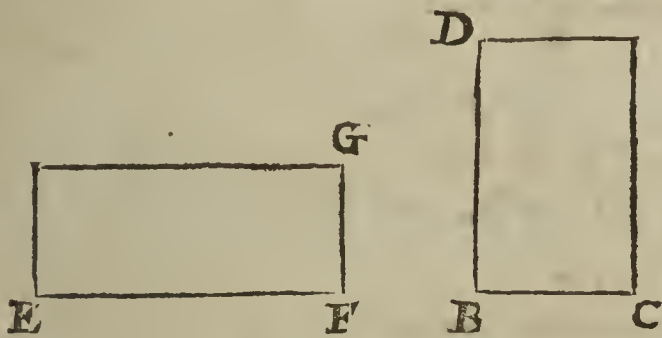
Suppos. 2.

12. EG is a \square whose sides EF, FG are given severally.

13. BC is a right-line given.

Req. to find

14. BD a right-line, such, that $\square BC, BD = \square EG$.



- EF = 12
- FG = 5
- BC = 6
- BD = 10

Construction.

15. By *Probl. 8.* of this *Chapt.* let it be made }
 as BC to EF, so FG to a fourth Proportio- } $BC : EF :: FG : BD$
 nal, which suppose to be the line BD, therefore, }
16. I say BD is the right-line sought; it remains therefore to prove that a Rectangle contain'd under BC and BD is equal to the given Rectangle EG.

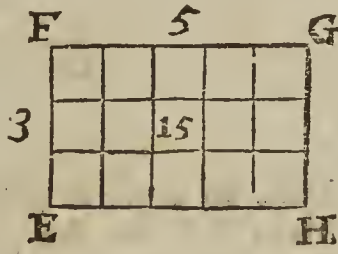
Prepar.

17. Let a Rectangle be made of the lines BC and BD, as $\square CD$, (*per prop. 46. Elem. 1.*) Then
18. . . . *Req. demonstr.* $\square CD = \square EG$.

Demonstration.

19. By *Constr.* in 15°, } $BC : EF :: FG : BD$
 20. Therefore, (*per prop. 16. Elem. 6.*) . . . } $\square EF, FG = \square BC, BD$
 21. But by *Constr.* in 17°, } $\square CD = \square BC, BD$
 22. Therefore from the two last preceding Equa- } $\square CD = \square EF, FG$, or $\square EG$.
 tions, (*per Ax. 1. Chap. 2.*) }
- Which was to be done.

23. From the premisses 'tis evident that Geometrical Application answers to Division in Arithmetick, for the Rectangle applied is correspondent to the Dividend, and the right-line to which the Rectangle is applied answers to the Divisor, and the right-line arising out of the Application, the Quotient: Therefore, if the Area of a Rectangle and either of its sides be given, the other side is also given; for if the Area be divided by the given side, the Quotient is the other side. So if FG, or EH, one of the sides of the Rectangle EG, be 5, and FE, or GH, the other side, 3, the Area 15 divided by 5, (to wit, FG,) gives 3 for FE. Likewise the Area 15 divided by 3, (to wit, FE,) gives 5 for FG, or EH.



Probl. VII.

Unto two right-lines given to find a third Proportional.

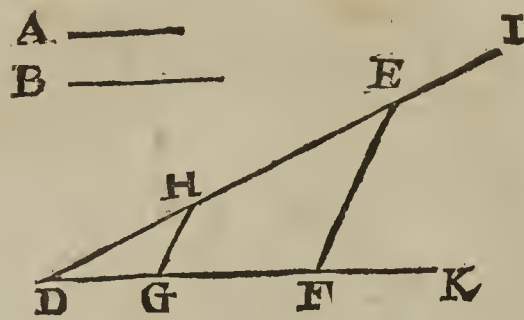
Suppos.

1. A and B are two right-lines given.

Req. to find

2. HE a right-line, such, that : : : : A . B :: B : HE.

A —



A = 4
B = 6
HE = 9

Construction.

3. Make an angle at pleasure, as $\angle IDK$.
4. Make $DG = A$. Also $GF = B$, and $DH = B$.
5. Draw the right-line HG .
6. By the point F draw $FE \parallel GH$.
7. I say, HE is the third Proportional sought.
8. . . . *Req. demonstr.* $A : B :: B : HE$.

Demonstration.

9. Because by *Constr.* in 6°, $\triangleright HG \parallel EF$ (in $\triangle DEF$.)
 10. Therefore, (*per prop. 2. Elem. 6.*) $\triangleright DG : GF :: DH : HE$.
 11. Therefore out of 4° and 10°, by exchanging } $A : B :: B : HE$.
equal right-lines, }
- Which was to be done.

Probl. VIII.

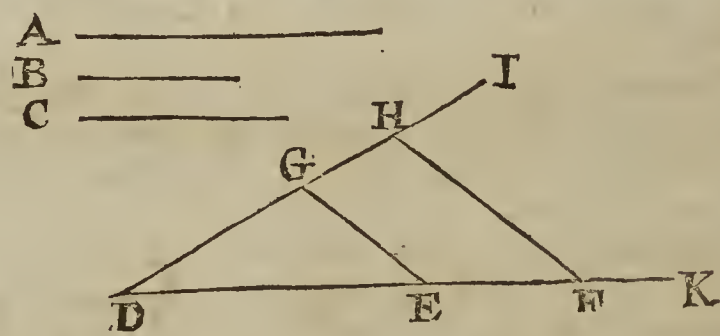
Unto three right-lines given to find a fourth Proportional.

Suppos.

1. A, B, C are three right-lines given.

Req. to find

2. GH a right-line, such, that $A : B :: C : GH$.



A = 27
B = 18
C = 24
GH = 16

Construction.

3. Make an angle at pleasure, as $\angle IDK$.
4. Make $DE = A$, also $EF = B$, and $DG = C$.
5. Draw the right-line EG .
6. By the point F draw $FH \parallel EG$.
7. I say, GH is the fourth Proportional required.
8. . . . *Req. demonstr.* $A : B :: C : GH$.

Demonstration.

9. Because by *Constr.* in 6°, $\triangleright EG \parallel FH$ (in $\triangle DHF$.)
 10. Therefore, (*per prop. 2. Elem. 6.*) $\triangleright DE : EF :: DG : GH$.
 11. Therefore out of 4° and 10°, by exchanging } $A : B :: C : GH$.
equal right-lines, }
- Which was to be done.

Con-

Concerning the extraction of the Square Root.

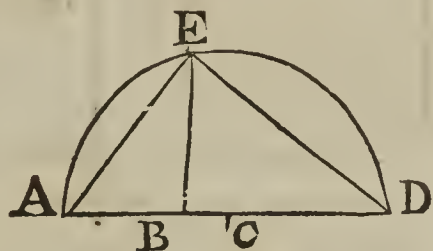
Probl. IX.

Between two right-lines given to find a mean Proportional; or,
To transform a Long-square into a Square.

Suppos.

1. A B and B D are two right-lines given.

Req. to find



- AB = 9
- BD = 16
- BE = 12
- AD = 25
- AC = 12½

2. BE a right-line, such, that $AB \cdot BE :: BE \cdot BD$
3. Whence consequently, (per prop. 17. Elem. 6.) $\square BE = \square AB, BD$.

Construction.

- 4. Joyn the given lines A B and B D so together in the point B, that they may make one right-line, as A D.
- 5. Upon A D as a Diameter, describe the Semicircle C A E D.
- 6. Upon the point B where the given lines A B and B D are joyned together, erect a Perpendicular and extend it to the Circumference of the Semicircle C A E D, as B E, which shall be the mean Proportional required.
- 7. . . . Req. demonstr. . . . $AB \cdot BE :: BE \cdot BD$

Prepar.

8. Draw the right-lines, A E and D E.

Demonstration.

- 9. By prop. 31. Elem. 3. $\angle AED$ is \perp .
- 10. And by Constr. in 6°, $BE \perp AD$.
- 11. Therefore from 9° and 10°, (per Coroll. of prop. 8. Elem. 6.) $AB \cdot BE :: BE \cdot BD$

Which was to be done.

Corollary:

- 12. Hence 'tis manifest, that a right-line drawn in a Circle from any point of the Diameter perpendicularly, and extended to the Circumference, is a mean proportional between the two segments of the Diameter which are made by the same Perpendicular.
- 13. Moreover, because from the preceding eleventh step, (per prop. 17. Elem. 6.) $\square BE = \square AB, BD$; therefore, if A B and B D be the sides of a Rectangle, a mean proportional B E between those sides shall be the side of a Square equal to that Rectangle or Long-square.
- 14. Here also by the way, the Learner may take notice, that a mean proportional number between two given numbers, is found out by extracting the square Root of the Product made by the mutual multiplication of the two numbers given: As, if it be required to find out a Mean between 16 (or B D,) and 9 (or A B,) the Rectangle or Product of 16 into 9 is 144, whose square Root 12, (or B E,) is the mean proportional sought; for $16 \cdot 12 :: 12 \cdot 9$.

Probl. X.

To find a right-line, to which a given right-line may be in the proportion of two Squares given.

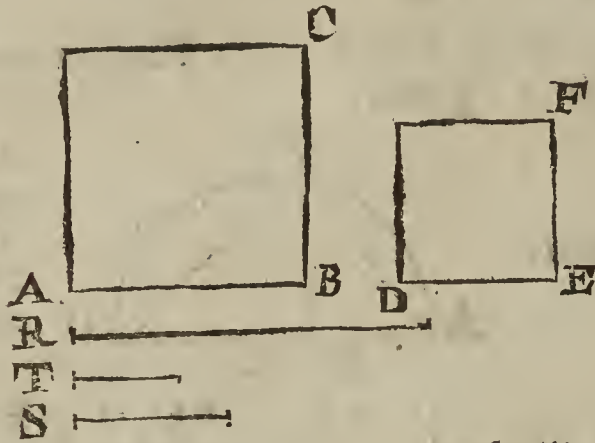
Suppos.

- 1. A C is a Square given, whose side is A B.
- 2. D F is a Square given, whose side is D E.
- 3. R is a right-line given.

C c

Req.

Req. to find
 4. S a right-line, such, that . . . $\square AC . \square DF :: R : S.$



$AB = 6$
 $\square AB = 36$
 $DE = 4$
 $\square DE = 16$
 $R = 9$
 $T = 2\frac{2}{3}$
 $S = 4$

Construction.

5. By *Probl. 7.* of this *Chapt.* let it be made as AB to DE , so DE to a third proportional, which suppose to be the line T , therefore . . .
 $AB : DE :: DE : T.$
 6. Again, (by the foregoing *Probl. 8.*) let it be made as AB to T , so R to a fourth proportional, which suppose to be the line S , therefore . . .
 $AB : T :: R : S.$

I say the line S is that required by the *Problem*, therefore

7. . . . *Req. demonstr.* . . . $\square AC . \square DF :: R : S.$

Demonstration.

8. Because by *Construction* in 5°, . . . $AB : DE :: DE : T.$
 9. Therefore, (per *Coroll. Prop. 20. Elem. 6.*) . . . $AB : T :: \square AB . \square DE,$
 10. And because by *Constr.* in 6°, . . . $AB : T :: R : S.$
 11. Therefore from 9° and 10°, (per *Prop. 11. Elem. 5.*) . . . $\square AB . \square DE :: R . S.$
 12. That is, (per *Ax. 7. Chap. 2.*) . . . $\square AC : \square DF :: R . S.$
 Which was to be done.

Probl. XI.

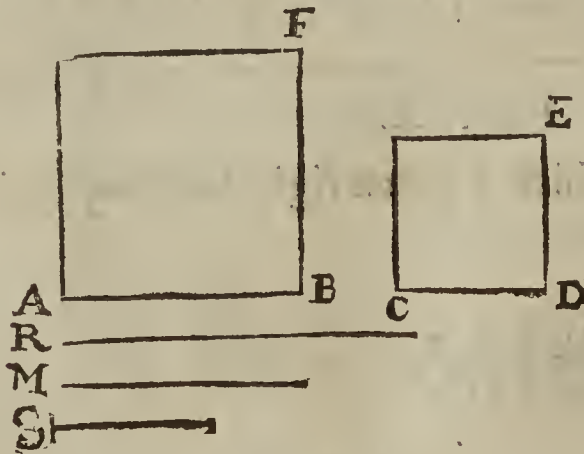
A Square being given, to find out another Square greater or less than that given, according to a Proportion given.

Suppos.

1. AF is a Square given, whose side is AB .
2. R and S are two right-lines expressing the given Proportion.

Req. to make

3. CE a Square, such, that . . . $R : S :: \square AF . \square CE.$



$AB = 6$
 $\square AB = 36$
 $R = 9$
 $S = 4$
 $M = 6$
 $CD = 4$
 $\square CD = 16$

Con-

Construction.

4. By *Probl. 9.* of this *Chapt.* find a mean proportional between the given lines *R* and *S*, which mean suppose to be the right-line *M*, therefore
- $$R \cdot M :: M \cdot S.$$
5. Again, (by *Probl. 8.* of this *Chapt.*) let it be made as *R* to *M*, so *AB* (the side of the given Square,) to a fourth proportional, which suppose to be the right-line *CD*, therefore
- $$R \cdot M :: AB \cdot CD.$$
6. Upon the line *CD* describe a Square, as *CE*, which shall be that required by the *Problem*, therefore
7. . . . *Req. demonstr.* $R \cdot S :: \square AF \cdot \square CE.$

Demonstration.

8. Because by *Constr.* in 4°, $R \cdot M :: M \cdot S.$
 9. Therefore, (per *Coroll. prop. 20. Elem. 6.*) $\square R \cdot \square M :: R \cdot S.$
 10. And because by *Constr.* in 5°, $R \cdot M :: AB \cdot CD.$
 11. Therefore, (per *prop. 22. Elem. 6.*) $\square R \cdot \square M :: \square AB \cdot \square CD.$
 12. Therefore out of 9° and 11°, (per *prop. 11. Elem. 5.*) $R \cdot S :: \square AB \cdot \square CD.$
 13. That is, (per *Ax. 7. Chap. 2.*) $R \cdot S :: \square AF \cdot \square CE.$
- Which was to be done.

Probl. XII.

The mean of three proportional right-lines being given, as also the difference of the extremes, to find the extremes.

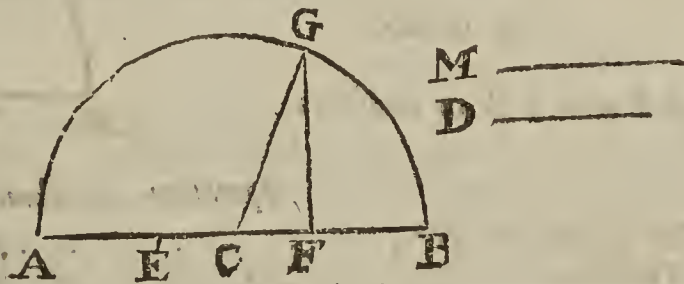
Suppos.

1. *M* = the mean of three proportionals is given.
 2. *D* = the difference of the extremes is given.

Req. to find

3. *AF* and *FB* two right-lines, such, that $AF - FB = D$; Also, that
 4. $AF \cdot M :: M \cdot FB.$

- | | |
|----------------|----------------|
| <i>M</i> = 12 | <i>AF</i> = 18 |
| <i>D</i> = 10 | <i>FB</i> = 8 |
| <i>EF</i> = 10 | <i>AE</i> = 8 |
| <i>FG</i> = 12 | <i>CA</i> = 13 |
| <i>EC</i> = 5 | <i>CG</i> = 13 |
| <i>CF</i> = 5 | <i>CB</i> = 13 |



Construction.

5. Make $EF = D$ (the given difference.)
 6. Upon the point *F* erect $FG \perp FE.$
 7. Make $FG = M$ (the given mean.)
 8. Divide *EF* into two equal parts in *C*.
 9. Draw the right-line *CG*.
 10. From *C* as a Center, at the distance of *CG* describe the Semicircle *CAGB*
 11. Draw the Diameter *AECFB*.
 12. I say *AF* and *FB* are the extreme proportionals required.
 13. . . . *Req. demonstr.* $\left\{ \begin{array}{l} AF - FB = D. \\ AF \cdot M :: M \cdot FB. \end{array} \right.$

Demonstration.

- 14. Because by *Construction* in 10° , CAGB }
 is a Semicircle whose Center is C; therefore, } $CA = CB = CG.$
 (*per defin. 15. Elem. 1.*) }
- 15. And because by *Constr.* in 8° , } $CE = CF.$
- 16. Therefore the Equation in 15° being sub- }
 tracted from that in 14° , gives (as is evident } $EA = FB.$
 by the Diagram) }
- 17. It is also evident by the Diagram that . . . } $AF - EA = EF.$
- 18. Therefore out of 16° and 17° , (*per Ax. 6.* } $AF - FB = EF.$
Chap. 2.) }
- 19. But by *Constr.* in 5° , } $D = EF.$
- Conclus. 1. 20. Therefore from 18° and 19° , (*per Ax. 1.*) } $AF - FB = D.$
 Which was to be dem.
- 21. Again, because (*per Coroll. Probl. 9. of this* } $AF . FG :: FG . FB.$
Chapt.) } $M = FG.$
- 22. And by *Constr.* in 7° , }
- Conclus. 2. 23. Therefore from 21° and 22° , by exchanging } $AF . M :: M . FB.$
 equal right-lines, }

Therefore the *Problem* propounded is satisfied; and out of the premisses the following *Theorem* is deducible, for the solving of the same *Probl. 12.* Arithmetically.

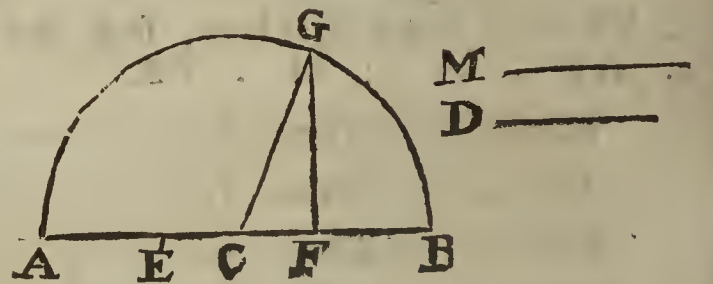
Theorem.

- 24. If half the difference of the extremes of three Proportionals, be added to the side (or Square Root) of that Square which is equal to the Square of half the difference of the extremes together with the Square of the mean, the summ shall be the greater extreme: But if the said half difference be subtracted from the said side, the remainder shall be the lesser extreme. Hence,

The Arithmetical solution of Probl. 12.

Suppos.

- 25. AF, FG, FB are $\div\div$, viz: $AF . FG :: FG . FB.$
- 26. $AF \sqsubset FB.$
- 27. $FG = 12.$ } Given.
- 28. $EF = 10 = AF - FB.$ }



Req. to find

- 29. AF and FB in numbers.

Operation Arithmetical.

- 30. By *Suppos.* in 28° , } $EF = 10.$
- 31. Therefore } $\frac{1}{2} EF = CF = 5 = CE.$
- 32. And consequently, } $\square CF = 25.$
- 33. Also from 27° , } $\square FG = 144.$
- 34. The summ of the Equations in 32° and 33° , }
 gives (*per prop. 47. Elem. 1.*) } $\square CG = 169.$
- 35. The square Root of each part of the Equation }
 in 34° gives } $CG = 13 = CA.$
- 36. Therefore the summ of the Equations in 31° }
 and 35° gives } $AF = 18.$ } Sought.
- 37. And by subtracting the Equation in 31° }
 from that in 35° , } $EA = FB = 8.$ }

The Proof.

- 38. It is manifest that these are Proportionals, }
 for the Product of the extremes is equal to the } $18 . 12 :: 12 . 8.$
 Square of the mean, viz. } $AF . FG :: FG . FB.$

39. Also,

39. Also the mean proportional is 12, and the difference of the extremes is 10, as was prescribed.

Coroll. 1.

40. From the premisses it may be inferr'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Equation so constituted, that the Square of an unknown right-line less by a Rectangle contain'd under that unknown line and some known right-line, is equal to a known Plane; the said unknown line shall be given by the preceding Geometrical Construction of *Probl. 12.* But here is to be noted, that if the said known Plane be not a Square, it must first be reduced to a Square, by *Probl. 9.* of this *Chapt.* or by *prop. 14. Elem. 2.*

41. As, in this Equation $aa - da = mm;$

If we suppose aa to represent an unknown Square whose side is a , also da a Rectangle contain'd under the said unknown side a and some known right-line represented by d , and that the said unknown Square aa less by the said Rectangle da is equal to some known Square, as mm , whose side is m ; then the said unknown side or right-line a may be found out by the Geometrical Construction of the preceding *Probl. 12.*

42. For (by *prop. 14. Elem. 6.*) the Equation above propos'd in 41° may be resolved into these Proportionals,

$$a - d . m :: m . a;$$

Of which three Proportionals the mean m is known by supposition, as also d the difference of the extremes a and $a - d$, (for a exceeds $a - d$ by d ;) therefore by the Construction of the foregoing *Probl. 12.* the extremes shall be given severally, the greater whereof is the line a sought.

43. Moreover, if in the Equation above propounded, to wit, $aa - da = mm$, we suppose $m = 12$, and $d = 10$, then the quantity of the line a shall be also given in number; for by the first part of the preceding *Theorem* in 24°,

$$a = \frac{1}{2}d + \sqrt{\frac{1}{4}dd + mm} = 18.$$

Coroll. 2.

44. It also follows from the preceding Construction of *Probl. 12.* that if by the Algebraical Resolution of a Geometrical Problem, an Equation be found out, such, that the Square of a right-line sought, together with a Rectangle contain'd under that unknown line and some known right-line, is equal to a known Plane; the said unknown line shall be given by the Geometrical Construction of the said *Probl. 12.* But if (as before hath been said) the said known Plane be not a Square, it must first be reduced to a Square.

45. As, in this Equation, $aa + da = mm;$

If we suppose aa to represent an unknown Square whose side is a , also da a Rectangle contain'd under the said unknown side a , and some known right-line represented by d ; and that the said unknown Square aa together with the said Rectangle da is equal to some known Square, as mm , whose side is m ; then the said unknown side, or right-line a , may be found out by the Geometrical Construction of the preceding *Probl. 12.*

46. For the Equation above propounded in 45° may be resolved into these Proportionals, *viz.*

$$a + d . m :: m . a;$$

Of which three Proportionals the mean m is known by supposition, as also d the difference of the extremes $a + d$ and a , (for $a + d$ exceeds a by d ;) therefore by the Construction of the foregoing *Probl. 12.* the extremes shall be given severally, the lesser whereof is the line a sought.

47. Lastly, if in the Equation above propounded in 45°, to wit, $aa + da = mm$, we suppose $m = 12$ and $d = 10$, then the quantity of the line a shall be also given in number; for by the latter part of the preceding *Theorem* in 24°,

$$a = \sqrt{\frac{1}{4}dd + mm} - \frac{1}{2}d = 8.$$

Probl. XIII.

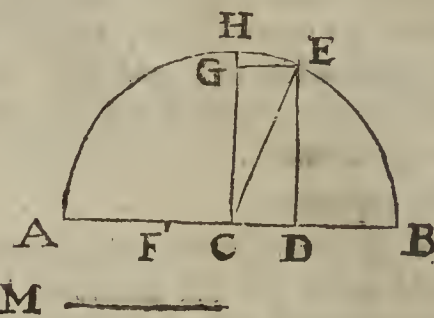
The mean of three proportional right-lines being given, as also the sum of the extremes, to find the extremes. But the given mean must not exceed half the given sum of the extremes.

Suppos.

1. M = the mean of three Proportionals is given.
2. AB = the sum of the extremes is given. Also M not $\leq \frac{1}{2} AB$.

Req. to find

3. AD and DB two right-lines; such, that $AD + DB = AB$. Also, that
4. $AD \cdot M :: M \cdot DB$.



- $M = 12$
- $AB = 26$
- $AC = 13$
- $CE = 13$
- $CB = 13$
- $AD = 18$
- $DB = 8$
- $DE = 12$
- $CD = 5$
- $CF = 5$

Construction.

5. Divide AB into two equal parts in C .
6. From C as a Center, with the distance CA , or CB , describe the Semicircle $CAHB$.
7. Upon the Center C erect $CH \perp AB$.
8. From CH cut off $CG = M$, (the given mean Proportional,) which is possible to be done, for by the Determination annex'd to the *Problem* propounded, M not $\leq CH$, or $\frac{1}{2} AB$.
9. By the point G draw $GE \parallel AB$, which GE will either touch the Semicircle in H , when $M = CH = \frac{1}{2} AB$; or else cut the Semicircle, when $M > CH$, (or $\frac{1}{2} AB$,) suppose then in this Example that $M > CH$, and consequently that GE cuts the Circumference, as in E .
10. From the point E let fall $ED \perp AB$, then shall AD and DB be the extreme Proportionals required; for first their sum, in regard ADB is a right-line, (to wit, the Diameter of the Semicircle $CAHB$,) is equal to AB the given sum; it remains to prove that as AD is to M , so M to DB , therefore
11. . . . Req. demonstr. $AD \cdot M :: M \cdot DB$.

Demonstration.

12. Because by *Construction* in 5° and 6° , . . . } $CAEB$ is a Semicircle.
13. And by *Constr.* in 10° , . . . } $DE \perp AB$.
14. Therefore from 12° and 13° , (per *Coroll.* in 12° } $AD \cdot DE :: DE \cdot DB$.
- of *Probl. 9. Chap. 5.*) . . . } .
15. And because by *Constr.* in $7^\circ, 9^\circ, 10^\circ$, . . . } DG is \square .
16. Therefore (per *prop. 34. Elem. 1.*) . . . } $DE = CG$.
17. But by *Constr.* in 8° , . . . } $M = CG$.
18. Therefore out of 16° and 17° , (per *Ax. 1.*) } $DE = M$.
19. Wherefore from 14° and 18° , (by taking M } $AD \cdot M :: M \cdot DB$.
- instead of DE ,) . . . } .

Which was to be done.

20. The reason of the Determination annex'd to the *Problem*, to wit, that the line prescribed for the mean must not exceed half the line given for the sum of the extremes, will be evident by this that follows. First, if the right-line M be less than CH , or $\frac{1}{2} AB$, and at the distance of M a line be drawn parallel to the Diameter AB , as the parallel GE , it will necessarily cut the Semicircle, as in E , in which case the extreme Proportionals, to wit, the segments of the Diameter which are made by the falling of the Perpendicular ED , will always be unequal. Secondly, if the line M be equal to CH , or $\frac{1}{2} AB$, and at the distance of M or CH a line be drawn parallel to the Diameter AB , such parallel will touch the Semicircle in H , and consequently HC which is perpendicular to AB will be a mean between AC and CB , in which case, the three Proportionals AC, CH and CB are equal to one another, for each of them is the Semidiameter of the Semicircle. Lastly, if the line M be greater than CH , or $\frac{1}{2} AB$, then 'tis easie to perceive, that a right-line drawn parallel to the Diameter AB at the distance of such line M cannot possibly either touch or cut the Semicircle, but will lye altogether without the same, and consequently such line M cannot be a mean pro-

proportional between any two segments of the Diameter; for a mean proportional line between two extremes is a right-line within a Circle, drawn perpendicularly to the Diameter, and extended only unto (not beyond) the Circumference: And therefore that there may be a possibility of solving this 13th Problem, the line given for the mean Proportional must not be longer than half the line given for the summ of the extremes; which is the Determination annex'd to the Problem.

The premisses being well understood, it will not be difficult to apprehend how the following Theorem is thence deducible; for the solving of Probl. 13. Arithmetically.

Theorem.

21. If half the summ of the extremes of three Proportionals be increased with the side (or square Root) of that Square which is equal to the excess whereby the Square of the said half summ exceeds the Square of the mean; the said half summ so increased shall be equal to the greater extreme: But if the said half summ be lessened by the side (or square Root) aforesaid, the remainder shall be the lesser extreme. Hence,

The Arithmetical solution of Probl. 13.

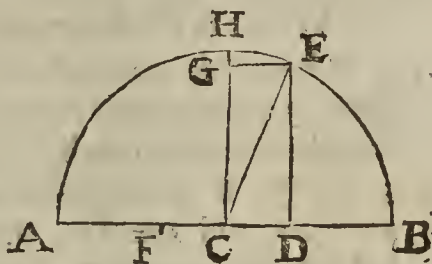
Suppos.

22. AD, DE, DB are $\div\div$, viz. AD . DE :: DE . DB.

23. DE = 12. } Given,
24. AB = 26 = AD + DB: }

Req. to find

25. AD and BD in numbers.



Operation Arithmetical. M ———

- 26. By Suppos. in 24°, } AB = 26.
- 27. Therefore, } $\frac{1}{2}$ AB = CB = 13 = CE = CA.
- 28. And consequently, } \square CE = 169.
- 29. And from 23°, } \square DE = 144.
- 30. Therefore by subtracting the Equation in 29°, }
from that in 28°, there will remain (per prop. }
47. Elem. 1.) } \square CD = 25.
- 31. And consequently, by extracting the square }
Root out of each part of the last Equation, } CD = 5.
- 32. Therefore by adding together the Equations }
in 27° and 31°, } AD = 18. } Sought.
- 33. And by subtracting the Equation in 31° from }
that in 27°, } DB = 8. }

Coroll.

34. From the premisses it may be inferr'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Equation so constituted, that a Rectangle contain'd under some known right-line and a right-line sought, less by the Square of the same line sought, is equal to a known Plane; then the said right-line sought shall be given by the preceding Geometrical Construction of Probl. 13. But here is to be noted, that if the said known Plane be not a Square, it must first be reduced to a Square, by Probl. 9. of this Chapt. or by Prop. 14. Elem. 2.

35. As, if this Equation be propos'd, $sa - aa = mm$;

We may suppose a in that Equation to represent a right-line unknown, aa the Square of that line, s a right line known, and that sa the Rectangle contain'd under those lines, less by the said unknown Square aa , is equal to some known Square, as mm , whose side is m ; then shall the said unknown right line a be given by the preceding Construction of Probl. 13.

36. For the Equation before propos'd in 35°, may be resolv'd into these Proportionals, viz.

$$s - a . m :: m . a;$$

Of which three Proportionals the mean m is known by supposition, as also s the summ of

- of the extremes $s - a$ and a ; therefore, by the foregoing *Construction* of *Probl. 13.* the extremes shall be given severally, either of which may be taken for the line a sought.
37. For (viewing the Diagram belonging to *Probl. 13.*) } $AD \cdot DE :: DE \cdot DB.$
 let us suppose that }
 38. Then putting } $m = DE.$
 39. Also } $s = AB.$
 40. And } $a = AD.$
 41. It follows that } $s - a = DB = AB - AD.$
 42. And } $a \cdot m :: m \cdot s - a.$
 43. Wherefore, by comparing the Rectangle of the } $sa - aa = mm.$
 extremes to the Square of the mean, this Equation }
 is produced, }

Which is the same with the Equation before propos'd in 35° .

44. Again, the same Equation will arise if we put $a = DB$, the *Suppositions* in 37° , 38° and 39° remaining unalter'd; for,
 45. Then it will follow that } $s - a = AD.$
 46. And } $s - a \cdot m :: m \cdot a.$
 47. That is, } $AD \cdot DE :: DE \cdot DB.$
 48. Therefore from 46° , } $sa - aa = mm.$

Which is the same with the Equation before produced in 43° .

49. Whence 'tis manifest that the right line a sought in all Quadratick Equations of the same form with that before propos'd in 35° , may be either of two right lines, represented in the Diagram by AD and DB , for which cause such Equation is called *Ambiguous*.
 50. Lastly, if in the Equation above propounded, to wit, $sa - aa = mm$, we suppose $m = 12$, and $s = 26$, then the quantity of the line a shall be also given in number; for by the preceding *Theorem* in 21° ,

$$a = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - mm} = 18;$$

Or, $a = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - mm} = 8.$

Probl. XIV.

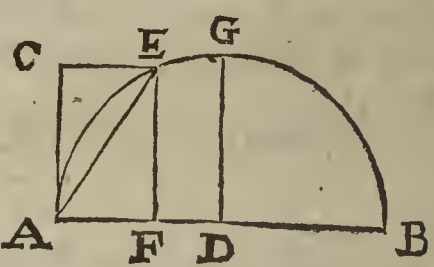
To divide a right-line given into two parts, such, that another right-line may be a mean Proportional between the parts. But the line given for the mean must not exceed half the line given to be divided.

Suppos.

1. AB is a right line given to be cut into two parts.
2. AC is a right line given to be a mean Proportional between the parts.
3. AC not $\leq \frac{1}{2} AB$.

Req. to find

4. AF and FB such parts of AB that $AF + FB = AB$; Also, that
5. $AF \cdot AC :: AC \cdot FB$.



- $AB = 26$
- $AC = 12$
- $AD = 13$
- $DB = 13$
- $AF = 8$
- $FE = 12$
- $FB = 18$

Construction.

6. Upon A one of the ends of AB , erect $AC \perp AB$.
7. Divide AB into two equal parts in D .
8. From D as a Center, with the distance DA , or DB , describe the Semicircle $DAEB$.
9. By the point C draw $CE \parallel AB$, so shall CE (by what hath been said in 20° of *Probl. 13.*) either touch or cut the Semicircle $DAEB$, for by *Suppos.* AC not $\leq \frac{1}{2} AB$, (or DG ;) suppose then in this Example that $AC = DG$, or DA , and consequently that CE cuts the Semicircle as in E .
10. From the point E let fall $EF \perp AB$, then shall AF and FB be the desired parts of AB , between which parts, AC is a mean Proportional, as I shall in the next place demonstrate.
11. . . . *Req. demonstr.* $AF \cdot AC :: AC \cdot FB$.

Preparat.

12. Draw the right lines AE and EB .

Demon-

Demonstration.

13. Because by *Constr.* in 8° , $DAEB$ is a Semicircle, } $\angle AEB$ is \perp .
- therefore (per *prop.* 31. *Elem.* 3.) }
14. And because by *Constr.* in 10° , } $EF \perp AB$.
15. Therefore from 13° and 14° , (per *Coroll.* *prop.* 8. } $AF \cdot FE :: FE \cdot FB$,
Elem. 6.) }
16. And because by *Constr.* in 6° , 9° and 10° , . . . } CF is \square .
17. Therefore, (per *prop.* 34. *Elem.* 1.) } $AC = FE$.
18. Therefore from 17° and 15° , by taking AC in- } $AF \cdot AC :: AC \cdot FB$.
stead of FE ; }

Which was to be done.

Note. This Problem may be solved Arithmetically in the same manner as the preceding *Probl.* 13.

Probl. XV.

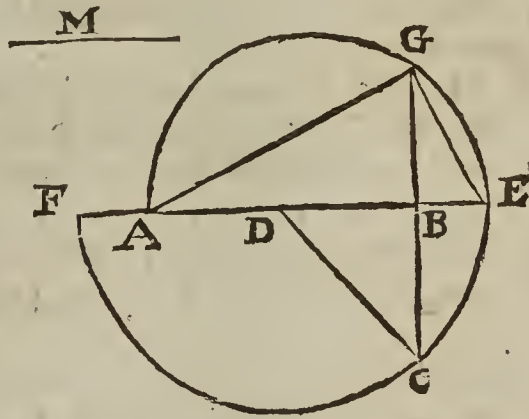
The mean of three Squares in continual proportion being given, as also a Square equal to the difference of the extremes, to find out the extremes.

Or thus;

The Base of a right-angled Triangle being given, as also a mean proportional between the Hypothenufal and Perpendicular, to find the Triangle.

Suppos.

1. AB = the Base of a right-angled Triangle is given.
2. M = a mean proportional between the Hypothenufal and Perpendicular is given:



Req. to find out the Triangle.

3. Making the given Base AB the first of three } $AB : M :: M : BC$.
Proportionals, and the given Mean M the }
second, find out a third, (per *Probl.* 7. of this }
Chapt.) which suppose to be BC , therefore, }
4. Upon B one of the ends of the given Base AB , make $BC \perp AB$, whence $\angle ABC$ is \perp .
5. Divide AB into two equal parts in D .
6. Draw the right-line DC .
7. From D as a Center, at the distance of DC , describe the Semicircle $DFCE$ having $FADBE$ for its Diameter.
8. Upon AE as a Diameter describe the Semicircle AGE .
9. Continue CB to the Circumference in G .
10. Draw the right lines AG and EG .
11. I say ABG is the right-angled Triangle required; now we must shew that it will satisfy the *Problem* propounded. First then, by *Supposition* AB is equal to the given Base, but that the angle ABG is a right-angle, and that the given line M is a mean proportional between the Hypothenufal AG and the Perpendicular BG , the following *Demonstration* will make manifest.

12. *Req. demonstr.* : . . . } $\angle ABG$ is \perp ; Also,
 } $AG \cdot M :: M \cdot BG$

Demonstration.

13. Because by *Constr.* in the fourth step, . . . } $\angle ABC$ is \perp ,
14. And by *Constr.* in 9° , } CBG is a right-line;
15. Therefore, (per *Coroll.* *Prop.* 13. *Elem.* 1.) } $\angle ABG$ is \perp . Which was to be Dem,
16. Again, because by *Constr.* in 7° , . . . } $DF = DE = DC$,
17. And by *Constr.* in 5° , } $DA = DB$;
18. Therefore by subtracting the Equation } $AF = BE$,
in 17° from that in 16° , }
D d } \neq . And

19. And by adding AB to each part of the last } preceding Equation, } . . . FB = AE.
20. Again, out of 4° and 7° (per Coroll. Probl 9. } of this Chapt. } FB . BC :: BC . BE.
21. Therefore from 20°, (per prop. 17. Elem. 6.) } □FB, BE = □BC.
22. And from 19°, (per prop. 1. Elem. 6.) by } taking BE as a common altitude, . . . } □FB, BE = □AE, BE.
23. Therefore out of 21° and 22° (per Ax. 1. } Chap. 2.) } . . . □BC = □AE, BE.
24. Again, because by Constr. in 8°, . . . } AGE is a Semicircle.
25. Therefore, (per prop. 31. Elem. 3.) . . . } ∠AGE is ⊥.
26. And because by what hath been proved in 15°, } GB ⊥ BA.
27. Therefore from 25° and 26°, (per Coroll. 2. } prop. 8. Elem. 6.) } AE . GE :: GE . BE.
28. Therefore out of 27°, (per prop. 17. Elem. 6.) } □GE = □AE, BE.
29. But it hath been proved in 23°, that . . . } □BC = □AE, BE.
30. Therefore from 28° and 29°, (per Ax. 1.) } □BC = □GE.
31. And consequently, } BC = GE.
32. Again, out of 25° and 26°, (per prop. 8. El. 6.) } △ABG and △GBE are like.
33. Therefore from 32°, (per prop. 4. Elem. 6.) } AB . AG :: BG . GE.
34. And from 33°, (per prop. 16. Elem. 6.) } □AG, BG = □AB, GE.
35. And out of 31°, by taking AB for a com- } mon altitude, } □AB, BC = □AB, GE.
36. Therefore from 34° and 35°, (per Ax. 1.) } □AB, BC = □AG, BG.
37. But by Constr in 3°, and per prop. 17. Elem. 6. } □AB, BC = □M.
38. And consequently from 36° and 37°, (per } Ax. 1.) } □AG, BG = □M.
39. Therefore out of 38°, (per prop. 14. Elem. 6.) } AG . M :: M . BG.
- Which was to be dem. And therefore the Problem is satisfied.

Coroll. 1.

40. From the premisses it may be inferr'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in continued proportion, such, that the greater extreme is a Square unknown, the mean a known Square, and the lesser extreme the excess whereby that unknown Square exceeds some known Square; then the side of the said unknown Square shall be given by the Geometrical Construction of the preceding Probl. 15.

As, for example, let there be propos'd the following Analogy, where *aa* represents a Square unknown, and *a* its side; also *mm* and *bb* two known Squares, whose sides are *m* and *b*;

$$aa - bb . mm :: mm . aa.$$

41. Then, (because, by prop. 22. Elem. 6. the sides of proportional Squares are proportionals also,) from the Analogy propos'd this ariseth, viz.

$$\sqrt{aa - bb} : m :: m . a.$$

42. Which three Proportionals last express being well consider'd, it will be manifest that the greater extreme, to wit, *a*, may be esteem'd the Hypothenufal of a right-angled Triangle whose Base is *b*, and Perpendicular $\sqrt{aa - bb}$: the lesser extreme: Now in that right-angled Triangle the Base *b* is given by supposition, as also *m* a right line which is a mean proportional between the said Hypothenufal and Perpendicular, and therefore the Hypothenufal and Perpendicular shall be given severally by the preceding Construction of Probl. 15. which Hypothenufal shall be the line represented by *a* in the Analogy propos'd.

43. It is also manifest, that if the Terms of the Analogy propos'd for an Example in 40°, be suppos'd to represent numbers, then by comparing the Rectangle of the the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, viz.

$$aaaa - bbaa = mmmm;$$

Where,

where, if we suppose $bb = 6$; also $mm = \sqrt{27}$, and a to stand for some number unknown, that Biquadratick Equation may be exprest thus, viz.

$$aaaa - 6aa = 27.$$

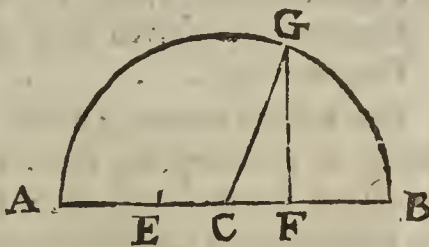
44. In which last Equation, the unknown number a may be found out, either by an Arithmetical Operation deducible from the preceding Geometrical Construction of *Probl. 15.* or else after the said Equation is resolved into these three continual Proportionals,

$$aa - 6 \cdot \sqrt{27} :: \sqrt{27} \cdot aa,$$

The greater extreme aa may first be made known after the manner of the Arithmetical Solution of the foregoing *Probl. 12.* and then the square Root of that number found out for the value of aa shall be the number a sought. All which will be manifest by the following Operation and Diagram.

Suppos.

- 45. $AF, FG, FB \div \div$; viz. $AF \cdot FG :: FG \cdot FB.$
- 46. $AF \sqsupset FB.$
- 47. $\square FG = 27.$
- 48. $AF - FB = 6 = EF.$ } given.



Req. to find out

49. AF the greater extreme, signified by aa in the Analogy before exprest in 44°.

Operation Arithmetical.

- 50. By *Supposition* in 48°, } $EF = 6.$
- 51. Therefore } $\frac{1}{2}EF = CF = 3 = CE.$
- 52. And consequently, } $\square CF = 9.$
- 53. By *Suppos.* in 47°, } $\square FG = 27.$
- 54. The summ of the two last Equations gives } $\square CG = 36.$
(*per prop. 47. Elem. 1.*) }
- 55. The square Root of the last Equation gives } $CG = 6 = CA!$
- 56. The summ of the Equations in 51° and 55°, } $AF = 9 = aa.$
gives }
- 57. Wherefore, } $\sqrt{9} = 3 = a.$
- 58. Moreover, out of the premisses ariseth the following *Canon*, for the Arithmetical Resolution of all Biquadratick Equations falling under the same form with that before exprest in 43°, and here-under repeated, where a represents a number sought, but b and m two numbers given; viz.

If $aaaa - bbaa = mmmm,$

Then $a = \sqrt{\frac{1}{2}bb + \sqrt{\frac{1}{4}bbbb + mmmm}}:$

Coroll. 2.

59. From the premisses also it may be inferr'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in continual proportion, such, that the lesser extreme is a Square unknown, the mean a known Square, and the greater extreme is composed of the said unknown Square and some known Square; then the side of the said unknown Square shall be given by the Geometrical Construction of the foregoing *Probl. 15.* As, for example,

If this Analogy be propos'd, where aa represents a Square unknown, and a its side; also, mm and bb two known Squares, whose sides are m and b ,

$$aa + bb \cdot mm :: mm \cdot aa.$$

60. Then (*per prop. 22. Elem. 6.*) these also shall be Proportionals,

$$\sqrt{aa + bb} \cdot m :: m \cdot a.$$

61. Which three last preceding Proportionals being well examined, it will be manifest that the greater extreme, to wit, $\sqrt{aa + bb}$: may be esteem'd the Hypothensal of a right-angled Triangle whose Base is b , and Perpendicular a , (the lesser extreme.) Now in that right-angled Triangle the Base b is given by supposition, as also m a right-line which is a mean Proportional between the said Hypothensal and Perpendicular, and therefore the Hypothensal and Perpendicular shall be given severally by the preceding Construction of *Probl. 15.* which Perpendicular shall be the line represented by a in the Analogy propos'd in 59°.

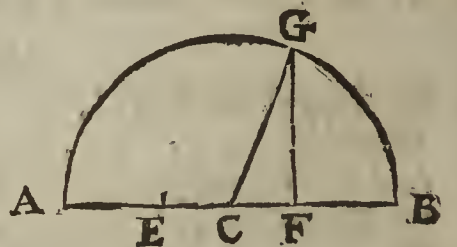
62. It is also manifest, that if the Terms of the Analogy in 59° be supposed to represent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, viz.

$$aaaa + bbaa = mmmm.$$

Where, if we suppose $bb = 6$, also $mm = \sqrt{135}$, and a to represent some number unknown; that Biquadratick Equation may be exprest thus, } $aaaa + 6aa = 135$;
 63. In which last Equation the unknown number a may be found out either by an Arithmetical Operation, deducible out of the preceding Geometrical Construction of *Probl. 15.* or else after the said Equation is resolved into these three continual Proportionals, } $aa + 6 : \sqrt{135} :: \sqrt{135} : aa.$

The lesser extreme aa may first be discovered after the manner of the Arithmetical Solution of the foregoing *Probl. 12.* of this *Chapt.* and then the square Root of that number found out for the value of aa shall be the number a sought. All which will be manifest by the following Operation and Diagram.

- Suppos.*
 64. $AF, FG, FB, \div \div \div$ viz. $AF \cdot FG :: FG \cdot FB.$
 65. $AF \sqsupset FB.$
 66. $\square FG = 135.$
 67. $AF - FB = 6 = EF.$



- Req. to find out*
 68. FB the lesser extreme, which answers to aa in the Analogy before exprest in 63°.

Operation Arithmetical.

69. By *Suppos.* in 67°, } $EF = 6.$
 70. Therefore } $\frac{1}{2}EF = CF = 3 = CE,$
 71. And consequently, } $\square CF = 9.$
 72. By *Suppos.* in 66°, } $\square FG = 135;$
 73. The summ of the two last preceding Equations } $\square CG = 144.$
 gives (*per prop. 47. Elem. 1.*) }
 74. The square Root of the Equation in 73° gives } $CG = 12 = CB.$
 75. And by subtracting the Equation in 70°, } $FB = 9 = aa.$
 from that in 74°, }
 76. Wherefore } $\sqrt{9} = 3 = a.$
 77. Moreover, out of the premisses ariseth the following *Canon*, for the Arithmetical Resolution of all Biquadratick Equations falling under the same Form with that before exprest in 62°, and here-under repeated; where a represents a number sought, but b and m two given numbers; viz.

If $aaaa + bbaa = mmmm,$
 Then $a = \sqrt{\sqrt{\frac{1}{4}bbbb} + mmmm} - \frac{1}{2}bb :$

Coroll. 3.

78. From the premisses also it may be inferr'd, That if the Algebraical Resolution of a Geometrical *Problem* discovers an Analogy consisting of three Planes in continual proportion, such, that the mean is a Square known, and in each of the extremes one and the same unknown Square is found Affirmative, (that is, with the Sign + prefixt to it,) together with some known Square having either + or - prefixt to it, or when in each of the extremes one and the same unknown Square is found negative, (that is, with the sign - set before it,) together with some known Square having the sign + prefixt to it; then in either of those Cases, the side of that unknown Square may be found out Geometrically by the preceding *Construction of Probl. 15,* as will be manifest by the four following Examples.

Example 1.

79. } Suppose $aa + bb : mm :: mm : aa + cc,$
 } And consequently, $\sqrt{aa + bb} : m :: m : \sqrt{aa + cc} :$ In

In which Analogies, let aa represent an unknown Square whose side is a , but bb , mm and cc three known Squares, whose sides are b , m and c ; suppose also that b is greater than c , and consequently $aa + bb < aa + cc$. I say then, that the unknown side a shall be given by the preceding *Construction* of *Probl. 15*. For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb - cc$, which by *Suppos.* is given; therefore the side of a Square equal to that difference, to wit, $\sqrt{bb - cc}$: may be esteem'd the Base of a right-angled Triangle, by the help of which given Base, and of the right-line m , which by *Supposition* is a given mean Proportional between the Hypothenufal $\sqrt{aa + bb}$: and the Perpendicular $\sqrt{aa + cc}$: the Hypothenufal and Perpendicular shall be given severally (by the said *Probl. 15*.) and may be represented by h and p , whose Squares are hh and pp ; then by equating hh to the greater extreme $aa + bb$, or pp , to $aa + cc$, the side a will be found equal to $\sqrt{hh - bb}$: or $\sqrt{pp - cc}$: which Roots are given, and equal to one another.

Example 2.

$$80. \quad \left. \begin{array}{l} \text{Suppose } \dots \dots \dots aa - bb \dots mm :: mm \dots aa - cc, \\ \text{And consequently, } \sqrt{aa - bb} : m :: m : \sqrt{aa - cc} : \end{array} \right\}$$

In which Analogies, if (as before in *Example 1*.) we suppose aa to represent an unknown Square whose side is a ; also bb , mm and cc three known Squares, whose sides are b , m and c , and b to be greater than c , whence consequently $aa - cc < aa - bb$; then the unknown side a shall be given by the preceding *Construction* of *Probl. 15*. For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb - cc$, which by *Supposition* is given; then the side of a Square equal to that difference, to wit, $\sqrt{bb - cc}$: may be esteem'd the Base of a right-angled Triangle, by the help of which given Base and of the right-line m , which by *Supposition* is a given mean Proportional between the Hypothenufal $\sqrt{aa - cc}$: and the Perpendicular $\sqrt{aa - bb}$: the Hypothenufal and Perpendicular shall be given severally by the said *Probl. 15*. and may be represented by h and p , whose Squares are hh and pp ; then by equating hh to $aa - cc$ (the greater extreme,) or pp to $aa - bb$; the unknown side or right-line a will be found equal to $\sqrt{hh + cc}$: or $\sqrt{pp + bb}$: which Roots are given, and equal to one another.

Example 3.

$$81. \quad \left. \begin{array}{l} \text{Suppose } \dots \dots \dots aa + bb \dots mm :: mm \dots aa - cc, \\ \text{And consequently, } \sqrt{aa + bb} : m :: m : \sqrt{aa - cc} : \end{array} \right\}$$

Example 4.

$$82. \quad \left. \begin{array}{l} \text{Suppose } \dots \dots \dots bb - aa \dots mm :: mm \dots cc - aa, \\ \text{And consequently, } \sqrt{bb - aa} : m :: m : \sqrt{cc - aa} : \end{array} \right\}$$

By what hath been said in the Explication of *Examples 1*. and *2*. it will not be difficult to conceive how to find out in like manner the unknown Root or right-line represented by a in the third and fourth *Examples*, where b , m and c are suppos'd to be right-lines severally given.

83. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in any of the three preceding *Corollaries* of *Probl. 15*. such known Squares or Rectangles must first of all be converted into a simple Square, per *Probl. 2. Chap. 4*.

Probl. XVI.

The mean of three Squares in Continual proportion being given, as also a Square equal to the sum of the extremes, to find out the extremes.

Or

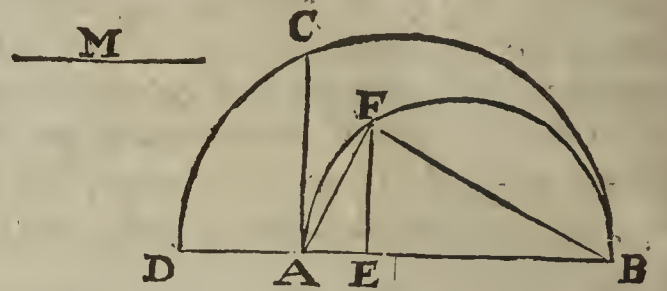
Or thus,

The Hypothenufal of a right-angled Triangle being given, as also a mean Proportional between the Base and Perpendicular, to find out the Triangle. But a right-angle arising out of the Application of the Square of the given Mean to the given Hypothenufal must not exceed half the Hypothenufal.

Suppos.

- 1. AB = the Hypothenufal of a right-angled Triangle is given.
- 2. M = a mean proportional between the Base and Perpendicular is given.
- 3. $\frac{\square M}{AB}$ not $\square \frac{1}{2} AB$.

Req. to find out the Triangle.



Construction.

- 4. Upon the point A, (one of the ends of the given Hypothenufal AB,) erect $AC \perp AB$, and make $AC = M$ the given mean Proportional.
- 5. Let it be made (per Probl. 7. of this Chapt.) as AB to AC, so AC to a third Proportional, which suppose to be found AD, therefore, } $AB \cdot AC :: AC \cdot AD$
That is, (because $M = AC$,) . . . } $AB \cdot M :: M \cdot AD$.
- 6. Upon AB as a Diameter describe the Semicircle AFB.
- 7. Upon DB as a Diameter compos'd of AB and AD as one right-line, describe the Semicircle DCB.
- 8. By Probl. 14. of this Chapt. divide the given Hypothenufal AB into two such parts, that the line AD may be a mean proportional between the parts; which is possible to be done, for by Construction in 5°, $AD = \frac{\square M}{AB}$, and by Supposition in 3°, agreeable to the Determination annex'd to the Problem, $\frac{\square M}{AB}$ or AD not $\square \frac{1}{2} AB$; suppose then the line AB to be cut in E, into two such parts AE and EB, that EF is a mean proportional between AE and EB, and that EF is equal to AD, therefore,

$AE \cdot EF \text{ (or AD,)} :: EF \text{ (or AD,)} \cdot EB.$

- 9. Draw the lines AF and BF, then shall AFB be the Triangle fought. Now we must shew that it will satisfy the Problem. First then, by supposition AB is equal to the given Hypothenufal; but that the angle AFB is a right-angle, and that the given right-line M is a mean proportional between AF and BF, (to wit, the Base and Perpendicular,) the following Demonstration will make manifest.
- 10. . . Req. demonstr. . . } $\angle AFB$ is \perp ; also,
. . . } $AF \cdot M :: M \cdot FB.$

Demonstration.

- 11. Because by Constr. in 6°, . . . } AFB is a Semicircle.
- 12. Therefore, (per prop. 31. Elem. 3.) . . . } $\angle AFB$ is \perp . Which was to be Dem.
- 13. Again, because by Constr. in 8°, . . . } $EF = AD$.
- 14. Therefore, by taking in AB as a common altitude, it follows from 13°, (per prop. 1. Elem. 6.) that . . . } $\square EF, AB = \square AD, AB$.
- 15. And because by Constr. in 6° and 8°, . . . } $FE \perp AB$.
- 16. And it hath been proved in 12°, . . . } $\angle AFB$ is \perp .
- 17. Therefore out of 15° and 16° (per prop. 8. Elem. 6.) . . . } $\triangle AEF$ and $\triangle AFB$ are like.
- 18. And consequently, (per prop. 4. Elem. 6.) } $EF \cdot AF :: FB \cdot AB$.
- 19. And from 18°, (per prop. 17. Elem. 6.) } $\square EF, AB = \square AF, FB$.
- 20. Therefore from 14° and 19°, (per Ax. 1.) } $\square AD, AB = \square AF, FB$.
- 21. And because by Constr. in 5°, . . . } $AB \cdot M :: M \cdot AD$.
- 22. And consequently (per prop. 17. Elem. 6.) } $\square AD, AB = \square M$.
- 23. Therefore out of 20° and 22°, (per Ax. 1.) } $\square AF, FB = \square M$.

24. There-

24. Therefore from 23°, (per prop. 14. Elem. 6.) . . . > AF . M :: M . FB.
Which was to be Dem. And therefore the Problem is satisfied.

Coroll. 1:

25. From the premisses it may be inferr'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in Continual proportion, such, that one of the extremes is an unknown Square, the mean a known Square, and the other extreme the excess whereby some known Square exceeds the said unknown Square; then the side of the said unknown Square shall be given by the preceding Geometrical Construction of Probl. 16.

As, for example, supposing *aa* to represent an unknown Square, whose side is *a*; also *mm* and *hb* two known Squares, whose sides are *m* and *h*, let this Analogy be propos'd, viz. } $hb - aa . mm :: mm . aa.$
26. Then (per prop. 22. Elem. 6.) the sides of } $\sqrt{hb - aa} : m :: m . a.$
proportional Squares are Proportionals also, }
therefore }

27. Which three continual Proportionals last express'd being well observed, it will be manifest that the extremes, to wit, $\sqrt{hb - aa}$: and *a* may be esteem'd the Base and Perpendicular of a right-angled Triangle, whose Hypotenusal is *h*. Now in that right-angled Triangle, the Hypotenusal *h* is given by Supposition, as also *m* a right-line, which is a mean Proportional between the Base and Perpendicular; and therefore the Base and Perpendicular shall be given severally by the preceding Construction of Probl. 16. either of which right lines, viz. either the Base or Perpendicular so found out may be taken for the line represented by *a* in the Analogy propos'd in 25°. For, viewing the Diagram belonging to Probl. 16.

28. Suppose > $\angle AFB = \perp$ in $\triangle AFB.$
29. Suppose also > $FA . M :: M . FB:$
30. And consequently (per prop. 22. Elem. 6.) > $\square FA . \square M :: \square M . \square FB:$
31. Then put > $mm = \square M.$
32. Also, > $hb = \square AB.$
33. And > $aa = \square FB.$

34. Then from 28°, 32° and 33°, it will be manifest (per prop. 47. Elem. 1.) that } $hb - aa = \square FA (= \square AB - \square FB.)$

35. Also from 30°, 31°, 33°, 34°, . . . > $hb - aa . mm :: mm . aa.$

36. Which Analogy is the same with that propos'd in 25°, and it will also be produced by putting $aa = \square FA$, (the Positions in 31° and 32° remaining unalter'd;)
For then it follows that > $hb - aa = \square FB (= \square AB - \square FA.)$

37. And because by Suppos. in 30°, . . . > $\square FB . \square M :: \square M . \square FA.$

38. Therefore from the premisses, . . . > $hb - aa . mm :: mm . aa.$
Thus it appears, that from either of the two right-lines found out by Probl. 16. one and the same Analogy may be constituted, in which respect 'tis said to be Ambiguous.

39. It is also manifest, that if the Terms of the Analogy in 25° be suppos'd to represent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this Biquadratick Equation will be produced, viz. } $hbaa - aaaa = mmmm.$

Where, if we suppose $hb = 5$, also $mm = 2$, and *a* to stand for some number unknown, that Biquadratick Equation may be express'd thus, viz. } $5aa - aaaa = 4$

40. In which last Equation the unknown number *a* may be found out either by an Arithmetical Operation deducible from the preceding Geometrical Construction of Probl. 16. or else, after the said Equation is resolv'd into these three Continual Proportionals, . . . } $5 - aa . 2 :: 2 . aa.$

The two values of *aa* may be found out after the manner of the Arithmetical Solution of the foregoing Probl. 13. of this Chapt. and consequently the square Root of each number

number found out for the value of aa being extracted, there will arise two numbers, each of which may be taken for the number a in the Equation in 39° . All which will be evident by the following Operation and Diagram.

Suppos.

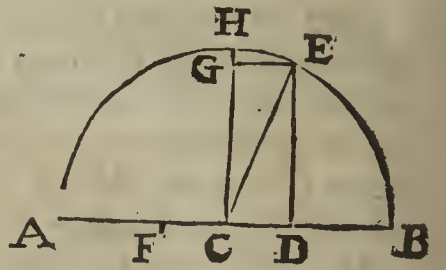
41. $AD, DE, DB \div \div, \text{viz. } AD \cdot DE :: DE \cdot DB.$

42. $DE = 2$, whence $\square DE = 4.$

43. $AD + DB = 5 = AB.$

Req. to find

44. AD and DB in numbers, (signified by aa in the Equation in 39° .)



Operation Arithmetical.

45. By *Suppos.* in 43° , $\sqrt{\quad} \dots AB = 5.$

46. Therefore, $\sqrt{\frac{1}{2}AB} = CE = 2\frac{1}{2} = AC = CB.$

47. And consequently, $\square CE = 6\frac{1}{4}.$

48. By *Supposition* in 42° , $\square DE = 4.$

49. And by subtracting the Equation in 48° from } . . . $\square CD = 2\frac{1}{4}.$

that in 47° , }

50. The square Root of the last Equation gives . . . $CD = 1\frac{1}{2}.$

51. The summ of the Equations in 46° and 50° , gives } . . . $AD = 4 = aa.$

52. And consequently, $\sqrt{4} = 2 = a.$

53. Again, by subtracting the Equation in 50° , from } . . . $DB = 1 = aa.$

that in 46° , }

54. And consequently, $\sqrt{1} = 1 = a.$

Whence 'tis manifest, that the number signified by a in the Analogy in 40° (or in the Equation in 39° .) may be either 2 or 1.

55. Moreover, out of the premisses ariseth the following *Canon*, for the Arithmetical Resolution of all Biquadratic Equations falling under the same Form with that before propos'd in 39° , and here-under repeated; where a represents a number sought, but b and m two numbers given, and subject to the Determination annex'd to *Probl. 16. viz. $\frac{mm}{b}$* must not be greater than $\frac{1}{2}b$, and consequently mm not greater than $\frac{1}{2}bh$.

If . . . $hb aa - aaaa = mmmm.$

Then, . . . $a = \sqrt{\frac{1}{2}bh + \sqrt{\frac{1}{4}hbhb - mmmm}}:$

Or, . . . $a = \sqrt{\frac{1}{2}bh - \sqrt{\frac{1}{4}hbhb - mmmm}}:$

Coroll. 2.

56. From the premisses it may be also inferr'd, That if the Algebraical Resolution of a Geometrical *Problem* discovers an Analogy consisting of three Planes in Continual proportion, such, that the mean is a known Square, and also one of the extremes is the excess of some unknown Square above a known Square, and the other extreme is the excess of some known Square above the said unknown Square; or, when one of the extremes is the excess of a known Square above an unknown Square, and the other is the summ of the same unknown Square and a known Square; in each of those Cases, the side of that unknown Square shall be given by the preceding *Construction* of *Probl. 16.*

Example 1.

57. . . . $\left\{ \begin{array}{l} \text{Suppose } \dots aa - dd \dots 4mm :: 4mm \dots cc - aa, \\ \text{And consequently, } \sqrt{aa - dd} \dots 2m :: 2m \dots \sqrt{cc - aa} : \end{array} \right.$

In which Analogies, if we suppose aa to represent an unknown Square whose side is a ; but dd , $4mm$ and cc three known Squares, whose sides are d , $2m$ and c ; the unknown side or right line a shall be given by the preceding *Construction* of *Probl. 16.* For the summ of the extremes in the first of those Analogies is $cc - dd$, which is given by *Supposition*; then the side of a Square equal to that summ, to wit, $\sqrt{cc - dd}$: may be esteem'd the Hypotenusal of a right-angled Triangle, by the help of which given Hypotenusal and the right-line $2m$, which by *Supposition* is a given mean Proportional between the Base and Perpendicular, (represented by $\sqrt{aa - dd}$: and $\sqrt{cc - aa}$;) the Base and Perpendicular shall be given severally by the said *Probl. 16.* and may be represented by f and g ,

f and g , whose Squares are ff and gg ; then by equating ff to $aa - dd$, or gg to $cc - aa$, one value of the right-line a sought shall be given; again, by equating gg to $aa - dd$, or ff to $cc - aa$, another value of the right-line a shall be given: So two lines are found out, either of which may be taken for the line a sought, and therefore the Analogy above-propos'd in *Example 1.* and all others of the same kind are said to be Ambiguous.

Example 2.

58. . . . } Suppose $bb - aa . mm :: mm . aa + cc,$
 } And consequently, $\sqrt{bb - aa} . m :: m . \sqrt{aa + cc}:$

By what hath been said in the Explication of *Example 1.* 'tis very easie to conceive how to find out in like manner the unknown Root or right-line represented by a in *Example 2.* where b, m and c are suppos'd to be right-lines severally given.

83. Note. When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in either of the three preceding *Corollaries* of *Probl. 16.* such known Squares or Rectangles must first of all be converted into a simple Square, per *Probl. 2. Chap. 4.*

C H A P. VI.

The manner of finding out such Right-lines and Squares as are represented by Algebraical Fractions, the Quantities constituting those Fractions being given severally.

Probl. I.

Suppos.

1. $a = \frac{bb}{c}$. An Equation propounded.

2. b and c are right lines given severally.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into these Proportionals, viz. } $c : b :: b . a$

In which Analogy, the two first Terms are right lines given by *Supposition*, therefore a third Proportional to them shall be given also, (per *Probl. 7. Chap. 5.*) which third Proportional is the line a sought.

Probl. II.

Suppos.

1. $a = \frac{bd}{c}$. An Equation propos'd.

2. b, c and d are right lines given severally.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this } $c : b :: d : a$
 Analogy, viz. }

In which Analogy, the three first Terms are right lines given by *Supposition*, therefore (per *Probl. 8. Chap. 5.*) the fourth Proportional, to wit, the line a shall be given also.

Probl. III.

Suppos.

1. $a = \frac{bb + bd}{c - f}$. An Equation propos'd.

2. b, c, d and f are right lines given severally.

E e

Req.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this } $c - f . b + d :: b : a$
 Analogy, }
- In which Analogy the three first Terms are given; for first, c and f being given by *Supposition*, a right line equal to $c - f$ shall be given also, (*per Probl. 3. Chap. 5.*) secondly, b and d being given by *Supposition*, a right line equal to $b + d$ shall be given, (*per Probl. 1. Chap. 4.*) and lastly, the fourth Proportional line a sought shall be given, *per Probl. 8. Chap. 5.*

Probl. IV.

Suppos.

1. $a = \frac{bb - dd}{c + f}$. An Equation propos'd.

2. b, c, d and f are right lines given.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this } $c + f . b + d :: b - d . a$
 Analogy, }
4. In which Analogy the three first Terms are given, and therefore the fourth proportional line a sought shall be given also by *Probl. 8. Chap. 5.*
5. In like manner, if } $a = \frac{bb + 2bc + cc}{d + f}$
 6. Then this Analogy will discover the line a sought, } $d + f . b + c :: b + c . a$
viz. } $a = \frac{bb - 2bc + cc}{d + f}$
 7. Again, if } $a = \frac{bb - 2bc + cc}{d + f}$
 8. Then supposing $b < c$, the line a shall be given } $d + f . b - c :: b - c : a$
 by this Analogy, *viz.* }

Probl. V.

Suppos.

1. $a = \frac{bc + df}{g}$. An Equation propos'd.

2. b, c, d, f, g are right lines given.

Req. to find the line a .

Construction.

3. First reduce df to a Rectangle, which shall have } $b . d :: f : h$
 b for one of its sides, *viz.* let it be made (*per* }
Probl. 8. Chap. 5.) as b to d , so f to a fourth Pro- }
 portional, which may be called h ; therefore . . . }
4. Therefore by comparing the Rectangle of the } $bh = df$
 extremes to the Rectangle of the means, . . . }
5. Then by setting bh in the place of df in the Equa- } $a = \frac{bc + bh}{g}$
 tion propos'd, it will be converted into this, *viz.* }
6. Which last Equation may be resolved into this } $g . b :: c + h . a$
 Analogy, }

But in the Analogy last express'd, the three first Terms are given by *Supposition* and *Construction*, therefore (*per Probl. 8. Chap. 5.*) the fourth proportional line a sought shall be given also. The same line a may be found out divers other ways, as the industrious Learner will easily perceive.

Probl. VI.

Suppos.

1. $a = \frac{bcd f}{gcd - ghk}$. An Equation propos'd.

2. b, c, d, f, g, h and k are right lines given.

Req. to find the line a .

Constru-

Construction.

3. First, reduce bc to a Rectangle which shall have g for one of its sides, viz. let it be made (per *Probl. 8. Chap. 5.*) as g to b , so c to a fourth Proportional, call it l , therefore, } $g : b :: c : l$.
4. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means; } $gl = bc$.
5. Therefore from 1° and 4°, by setting gl in the place of bc in the Equation propos'd, it gives } $a = \frac{gldf}{gcd - gbk} = \frac{ldf}{cd - bk}$.
6. Again, reduce bk to a Rectangle that shall have d for one of its sides, viz. let it be made as d to b ; so k to a fourth, which may be called m , therefore } $dm = bk$.
7. Therefore from 5° and 6°, by setting dm in the place of bk in the Fraction in the latter part of the 5th step, it gives } $a = \frac{ldf}{cd - dm} = \frac{lf}{c - m}$.
8. Therefore by resolving the latter Fraction in the 7th step into Proportionals, this Analogy ariseth, viz. } $c - m . l :: f . a$.

In which Analogy the three first Terms are given by *Supposition* and *Construction*, therefore the fourth Proportional, to wit, the line a sought shall be given also, by *Probl. 8. Chap. 5.* of this Book.

Probl. VII.

Suppos.

1. $a = \frac{dbb}{cc}$. An Equation propos'd.
2. b, c, d are right lines given.
Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this Analogy, viz. } $cc . bb :: d . a$.

In which Analogy, the three first Terms are given by *Supposition*, and qualified according to the tenour of *Probl. 10. Chap. 5.* therefore the fourth Term, that is, the line a sought shall be given also by that *Problem*.

Probl. VIII.

Suppos.

1. $a = \frac{bcd}{ff}$. An Equation propos'd.
2. b, c, d, f , are right lines given.
Req. to find the line a .

Construction.

3. First, the Equation propos'd may be resolved into this Analogy, } $ff . bc :: d . a$.
4. Then let ff be reduced to a Rectangle that shall have b for one of its sides; viz. let it be made (per *probl. 7. Chap. 5.*) as b to f , so f to a third Proportional, which may be called g , therefore } $bg = ff$.
5. Then by taking bg instead of ff for the first Term of the Analogy in the third step, that Analogy will be converted into this, } $bg . bc :: d . a$.
6. Whence, by casting away the common altitude b , this Analogy ariseth, } $g . c :: d . a$.

In which last Analogy the three first Terms are right lines given by *Supposition* and *Construction*, therefore the fourth proportional line a shall be given also.

Probl. IX.

Suppos.

1. $a = \frac{2bbc}{bb - cc}$. An Equation propos'd.

2. b and c are right lines given.

Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this Analogy, } $bb - cc . bb :: 2c . a.$
4. Then (by *Probl. 4. Chap. 5.*) find a Square equal to $bb - cc$, which Square may be called dd ; this being set in the place of $bb - cc$, (the first Term of the preceding Analogy,) will give these Proportionals, *viz.* } $dd : bb :: 2c . a.$
5. Reduce dd to a Rectangle that shall have b for one of its sides, *viz.* let it be made as b to d , so d to a third Proportional, which may be called f , therefore } $bf = dd.$
6. Then by taking bf instead of dd , the Analogy in the 4th step will be converted into this, } $bf . bb :: 2c . a.$
7. Whence, by rejecting the common altitude b , this Analogy ariseth, } $f . b :: 2c . a.$

In which last Analogy the three first Terms are right lines given by *Supposition* and *Construction*; therefore the fourth Proportional line a shall be given also, *per Probl. 8. Ch. 5.*

Probl. X.

Suppos.

1. $a = \frac{bbcc - ddf}{dff - bcc}$. An Equation propos'd.

2. b, c, d, f , are right lines given.

Req. to find the line a .

Construction.

3. Let it be made (*per Probl. 8. Chap. 5.*) as c to f , so d to a fourth proportional line, call it g , therefore, } $cg = df.$
4. Then setting cg in the place of df in the Equation propos'd, and expunging c out of the Numerator and Denominator, it gives } $a = \frac{bbc - gdf}{gf - bc}.$
5. Again, let it be made as c to f , so g to a fourth proportional line, call it k , therefore, } $ck = gf.$
6. Then setting ck in the place of gf in the Fraction in the fourth step, and casting c out of the Numerator and Denominator, there ariseth } $a = \frac{bb - kd}{k - b}.$
7. Again, let it be made as k to b , so b to a third proportional line, call it m , therefore, } $km = bb.$
8. Then setting km in the place of bb in the Numerator of the Fraction in the 6th step, there will thence arise } $a = \frac{km - kd}{k - b}.$
9. Again, let it be made as $k - b$ to $m - d$, so k to a fourth Proportional, which shall be the line a sought, *viz.* } $k - b . m - d :: k . a.$
10. From the preceding *Construction* 'tis evident that the desired line a may be found out by these four following Rules of Three, *viz.*

1. $c . f :: d : g.$
2. $c . f :: g : k.$
3. $k . b :: b : m.$
4. $k - b . m - d :: k . a.$

11. But the Numerator and Denominator of the Algebraical Fraction in the Equation propos'd in this *Problem* do manifestly shew, that the lines b, c, d, f must be given with this Caution or Determination, *viz.*

$f = \frac{bc}{d}$; but $f = \sqrt{\frac{bcc}{d}}$.

That

That is to say, if it be made as d to b , so c to a fourth Proportional, the line f must be greater than that fourth Proportional; but if (by *Probl. 11. Chap. 5.*) it be made as d to b , so cc to another Square, then the line f must be greater than the side of that latter Square.

Now if the line f be given within those limits, then k will be greater than b , and m greater than d , as the last Analogy in the 10th step requires. The same limits of f may be easily inferr'd also from the four Analogies in the 10th step.

An Example in Numbers.

If $\left\{ \begin{array}{l} b = 40, \\ c = 24, \\ d = 16, \\ f = 48; \end{array} \right\}$ then by the Analogies in the 10th step you will find $\left\{ \begin{array}{l} g = 32, \\ k = 64, \\ m = 25, \\ a = 24. \end{array} \right.$

Thence it follows that $a = 24 = \frac{bbcc - ddf}{dff - bcc}$, the Equation propos'd.

Probl. XI.

Suppos.
1. $aa = \frac{bbd}{c}$. An Equation propos'd.

2. b, c, d are right lines given.
Req. to find the line a .

Construction.
3. The Equation propos'd may be resolved into this } $c : d :: bb : aa$.
Analogy, }

In which Analogy, the three first Terms are given, and qualified according to the tenour of *Probl. 11. Chap. 5.* therefore the line a shall be given also by that *Problem*.

Probl. XII.

Suppos.
1. $aa = \frac{bdf}{c}$. An Equation propos'd.

2. b, c, d and f are right lines given.
Req. to find the line a .

3. The Fraction $\frac{bdf}{c}$ signifies a Rectangle contain'd under a right line equal to $\frac{bd}{c}$ and the line f , therefore first, according to the *Construction* of *Probl. 2.* of this Chapter, find a right line equal to $\frac{bd}{c}$, which line may be called g , therefore,

$$aa = fg = \frac{bdf}{c}.$$

4. Then by *Probl. 9. Chap. 5.* reduce the Rectangle fg to a Square, which may be called mm , whose side is m , therefore,
 $aa = mm (= fg.)$ And consequently, $m = a$ the line sought

Probl. XIII.

Suppos.
1. $aa = \frac{bbcc}{dd}$. An Equation propos'd.

2. b, c and d are right lines given.
Req. to find the line a .

Construction.
3. The Equation propos'd may be resolved into this } $dd : bb :: cc : aa$.
Analogy, }

4. But the sides of proportional Squares are also Proportional, therefore from 3^o, } $d : b :: c : a$.

In which last Analogy the three first Terms are right lines given, therefore the fourth proportional line a sought shall be given also.

Probl. XIV.

Suppos.

Probl. XIV.

1. $aa = \frac{2bcd}{gg}$. An Equation propos'd.2. b, c, d, f, g are right lines given.Req. to find the line a .

Construction.

3. The Equation propos'd may be resolved into this
Analogy, } $gg \cdot 2bc :: df \cdot aa.$
4. By *Probl. 9. Chap. 5.* reduce $2bc$ to a Square, which
may be called hh ; reduce likewise df to a Square, which
you may call kk , then the preceding Analogy will
be converted into this, } $gg \cdot hh :: kk \cdot aa.$
5. But the sides of proportional Squares are also Pro-
portionals, therefore, from 4° , } $g \cdot h :: k \cdot a.$

In which last Analogy the three first Terms are right lines given, and therefore the fourth proportional line a sought shall be given also.

Many other ways might be shewn to construct (or effect) most of the preceding *Problems* of this Chapter; but for brevity sake, I leave them to be found out by the industrious Learner, who by the help of those before deliver'd will also easily perceive how to solve other *Problems* of like nature: And now having explain'd all such things as are materially necessary by way of Preparation to the Resolution and Composition of *Plane Problems*, I shall proceed to *Examples*, which I have divided into four *Classes* or *Forms*, contain'd in the four following Chapters.

C H A P. VII.

The first Classis of Examples of the Resolution and Composition of Plane Problems, to wit, such whose Construction may be perform'd by drawing only Right and Circular lines.

IN which Examples, the Resolution ends either in an Analogy whose three first Terms are right lines known, and the fourth gives the right line sought; or else it ends in a simple Equation between the right line sought, and one or more right lines known. What is meant by *Mathematical Resolution and Composition*, I have hinted by Definitions in the beginning of *Chap. 1. Book I.* of this Treatise, and now I come to expound and illustrate the same by Examples, after I have recommended a few things by way of Caution and Direction to Learners.

First, Let the Analyst take care to understand the import and meaning of a Problem propounded, lest by too much haste he lose his labour, or be too forward in censuring the Proposer, when the fault is in himself; for many undertake to be Correctors of others, when they themselves have indeed more need of correction.

Secondly, Forasmuch as the most part of Problems propos'd in Geometrical Figures have need of Preparation, let the Analyst endeavour, before he begins the Algebraical Resolution, to find out as much as he can by the Synthetical Method, which proceeds by a Series of Consequences deduced altogether from known Quantities; and sometimes it will be convenient to premise one or more preparatory Propositions to render the Resolution of a Problem propos'd, the more simple and intelligible.

Thirdly, When the Resolution of a Geometrical Problem is begun, the like care must be taken to keep every step thereof in the simplest Terms, for avoiding Equations of higher Powers than the nature of the Problem requires, especially such as exceed Geometrical Dimensions; for example, in the Resolution of a Plane Problem, no Term of any Analogy or Equation ought to exceed two Dimensions, *viz.* every Term must be either a right Line or a Plane, for 'tis improper to introduce Solids in the Resolution or Composition of a Plane Problem.

Fourthly;

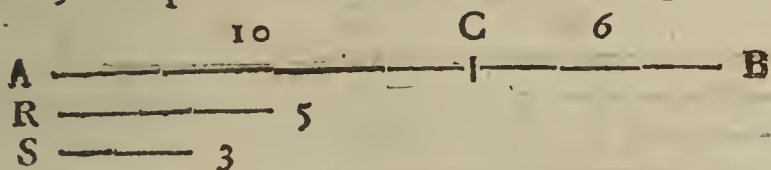
Fourthly, The Scope or aim of the Analyst in solving a Problem must be, first, to find out a Canon to direct how the Construction of the Problem may be effected by the Quantities given, and then the Construction being finish'd to form a Demonstration Synthetically; that may clearly prove the Problem to be fully satisfied. But although a Canon rightly found out by the Algebraick Art bids that only to be done which is possible, yet oftentimes in the Construction even of a Plane Problem, such objections will start up against the possibility of the Construction, as cannot be solved by any thing apparent either in the Canon, or in the proposition of the Problem: As, if a plain Triangle be to be made of three right lines, whereof one is rightly found out by Construction according to the direction of the Canon, and the other two are also discovered by the help of Quantities given and found out, yet before a Triangle can be made of those three right lines, it must be proved, that every two of them being joyned together as one right line, are longer than the third; which Proof may happen to be a more difficult work than the invention of the Canon: And therefore when any doubt ariseth concerning the possibility of any particular Construction, and it doth not clearly appear whether such Construction can be done or not, an industrious Enquiry must be made to discover what is absolutely necessary to be given or granted to make that possible to be done, which the Problem requires, or the Canon bids to be done: Upon which Search, oftentimes one or more Determinations or Cautions will be found necessary to limit the Quantities given in the Problem, that its Construction may meet with no Impediment. Examples of Determinations will appear in divers Problems in this and the following Chapters.

Fifthly, After all necessary Determinations are premis'd, and the Construction of a Problem is finish'd, it remains to demonstrate that the Quantity or Quantities found out by the Construction will satisfy the Problem: But the Demonstration of the Solution of a Plane Problem, if its Construction be Algebraically found out in such manner that no Term of any Analogy or Equation in the Resolution exceeds Geometrical dimensions, may be formed by a repetition of the steps of the Resolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Resolution; and the Demonstration of a Theorem may be formed by the steps of the Algebraick Resolution in a direct order, that is, by proceeding forward from the beginning to the end of the Resolution. All which will be copiously illustrated by the Resolutions and Compositions of Problems in this and the following Chapters.

Sixthly and lastly, I desire the Reader to take notice that in the Resolution of a Problem, I use the small Italian letters, *a, b, c, d, &c.* assuming always some Vowel, as *a, or e, &c.* to represent a line sought; and Consonants, as *b, c, d, &c.* to signify lines given or known: But in the Composition of a Problem, that is, in its Construction and Demonstration, I use the Roman Capital letters, *A, B, C, D, &c.* to express lines known, that the Resolution and Composition may be compared to one another without Confusion.

Problem I.

To divide a given right line into two parts which shall be in a given Reason, that is, one part to the other as two right lines given.



Suppos.

1. $b = AB$ a right line given to be cut into two parts.
2. $\left. \begin{matrix} r = R \\ s = S \end{matrix} \right\}$ the Terms of the given Reason of the parts sought.

Req. to find

3. AC and CB such parts of AB , that $AC + CB = AB$; also,
4. $AC . CB :: R . S.$

Resolution.

5. For one of the parts sought put $a.$
6. Therefore from 1° and 5° the other part is $b - a.$
7. And according to the tenour of the Problem propounded, $r . s :: a . b - a.$
8. There-

8. Therefore by *Composition of Reason converse*, (defin'd in *Seet. 4. Chap. 3.*) } $r + s . r :: b . a$.
 Which last Analogy gives this

C A N O N.

9. As the summ of the Terms of the given Reason is to the first Term ; (that is, which of the two you please,) so is the line given to be divided, to one of the parts desired, which part subtracted from the line given to be divided leaves the other part.

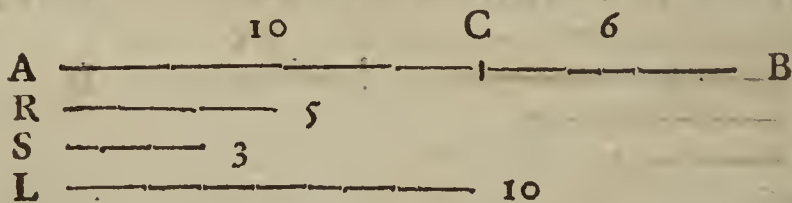
10. *Note.* Although the Analogy in the seventh step may be converted into an Equation, (by comparing the Rectangle of the extremes to the Rectangle of the means,) from which, after due Reduction, the Analogy in the eighth step will arise, yet in Geometrical Demonstrations, which require a Contemplation upon Schemes or Figures, an Analogy in right lines is more simple, and easier to be understood than an Equation between Planes, or Solids; and therefore 'tis more usual with Geometricians in their Argumentations, to proceed as much as is possible from one Analogy to another, by Composition, Division, and other ways of arguing about Proportions, (defin'd in *Chap. 3.* of this *Book*,) that at length an Analogy may arise, when there is a possibility, wherein the three first Terms are given to find a fourth Proportional, which gives the Quantity sought; but there will be very often a necessity of converting an Analogy into an Equation, when known Quantities cannot be otherwise separated from unknown, as will hereafter appear by variety of Examples.

Concerning the Composition of a Geometrical Problem.

11. The *Composition* of a *Problem* consists of two parts, to wit, *Construction*, (or *Delineation*,) and *Demonstration*; the former finds out that which is required to be done or found out, and the latter proves that that which is done or found out will satisfy the *Problem* propounded.

But before the *Construction* be begun, if the *Problem* be not universal, such *Determinations* (or *Cautions*) as are needful to limit the given Quantities, that the *Problem* may be possible must be annexed to it, and the truth and reason of such *Determinations* made manifest; for 'tis the Office of him that undertakes to solve a *Problem* to determine what can, and what cannot be done; and if that which is required be possible, then to shew how, and how many ways it may be done: Now the *Algebraical Art* is an excellent Guide to shew the way leading to those ends; for first, the *Canon* resulting from the *Resolution* doth for the most part discover all such *Determinations* as are necessary to limit the given Quantities that the *Problem* may be possible, and directs also how its *Construction* may be made by working only with given Quantities. And lastly, if no Term of any Analogy or Equation in the *Resolution* exceeds *Geometrical Dimensions*, a *Demonstration* of the *Solution* of the *Problem* may be form'd out of the steps of the *Resolution* in a retrograde order, that is, by returning backwards from the end of the *Resolution* to its beginning. But these things will best appear by Examples, and therefore I shall proceed to

The Composition of Probl. 1.



Suppos.

- 12. A B is a right line given to be cut into two parts.
- 13. R and S are the Terms of the given Reason of the parts sought.

Req. to find

- 14. A C and C B such parts of A B, that $A C + C B = A B$. Also;
- 15. $A C . C B :: R . S$.

Construction.

16. Let it be made (per *Probl. 8. Chap. 5.*) as $R + S$ to R, so A B to a fourth Proportional line, which may be called L, therefore

$$R + S . R :: A B . L.$$

17. From A B cut off $A C = L$, which is possible to be done if A B be greater than L, but

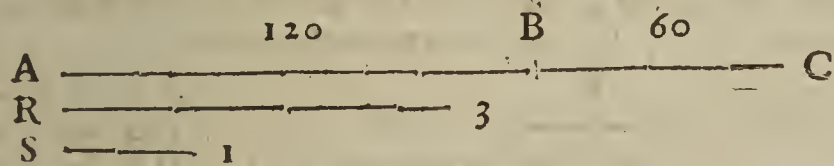
but $R \dashv S$ the first Term of the last preceding Analogy is evidently greater than R the second Term, therefore (per Schol Prop. 14. Elem. 5.) the third Term AB shall be greater than the fourth L ; and consequently $AC = L$ may be cut off from AB . That done, the given line AB is divided in the point C into two parts AC, CB , which will satisfy the Problem For first, $AC \dashv CB = AB$, and that AC is to CB as R to S , I shall demonstrate by a retrograde repetition of the steps of the Resolution in manner following.

18. *Req. demonstr.* $R \cdot S :: AC \cdot CB$.
- Demonstration.*
19. Because by *Constr.* in 16°, $\succ R \dashv S \cdot R :: AB \cdot L$.
20. And by *Constr.* in 17°, $\succ AC = L$.
21. Therefore from 19°, by taking AC instead of L , $\succ R \dashv S \cdot R :: AB \cdot AC$.
That is, in 8°, (the last step of the Resolution,) $\succ r \dashv s \cdot r :: b \cdot a$.
22. Therefore from the Analogy in 21°, by *Division of Reason converse*, (defin'd in Sect. 8. Ch. 3.) $\succ R \cdot S :: AC \cdot CB$. *Conclus.*
That is, in 7°, $\succ r \cdot s :: a \cdot b - a$.
- Which was to be done.

Note. In forming a *Demonstration* by a repetition of the steps of the Resolution in a backward order, it must be observed as a perpetual Rule, That when in the Resolution you pass forward from one step to another by *Composition of Reason*, in the *Demonstration* you are to return backward by *Division of Reason*; and when you pass by *Division of Reason* in the Resolution, you are to return by *Composition of Reason* in the *Demonstration*: also, *Addition* in the one, answers to *Subtraction* in the other. All which will be evident in the following *Problems*.

Probl. II.

To a given right line to add another right line, that the given with the added may have a given Reason to the line added. But the first Term of the Reason must be greater than the latter.



Suppos.

1. $b = AB$ a right line given to be increased.
2. $\left. \begin{matrix} r = R \\ s = S \end{matrix} \right\}$ the Terms of the given Reason.
3. $R \dashv S$.

Req. to find

4. BC a right line, such, that $AB \dashv BC : BC :: R : S$.

Resolution.

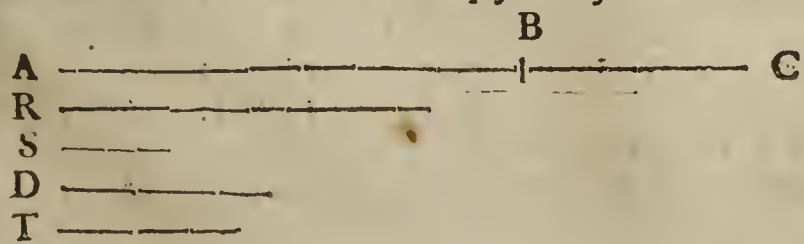
5. For the line sought put $\succ a$.
6. Which added to the given line b makes $\succ b \dashv a$.
7. Then according to the import of the Problem, $\succ r \cdot s :: b \dashv a \cdot a$.
8. Therefore by *Division of Reason*, (defin'd in Sect. 7. Chap. 3.) $\succ r - s \cdot s :: b \cdot a$.

Hence this

Canon.

9. As the difference of the Terms of the given Reason is to the lesser Term, so is the line given to be increased, to the increase sought.

The Composition of Probl. 2.



- $AB = 120$
- $R = 3$
- $S = 1$
- $D = 2$
- $T = 60$
- $BC = 60$

E f

Suppos.

Suppos.

- 10. AB is a right line given.
- 11. R and S are the Terms of a given Reason.
- 12. $R \sqsubset S$.
- 13. $D = R - S$.

Req. to find

- 14. BC a right line, such, that $AB + BC \cdot BC :: R \cdot S$.

Construction.

- 15. Let it be made (*per Probl. 8. Chap. 5.*) as } $D \cdot S :: AB \cdot T$
 D (or $R - S$) to S, so AB to a fourth proportional line, call it T, therefore, . . .
- 16. Let AB be continued to C, so, that . . . } $BC = T$.

Now the line BC (or T) being found out by the help of the given lines, AB, R and S, according to the direction of the *Canon*, we must shew that it will satisfy the *Probl.* therefore,

- 17. . . . *Req. demonstr.* $R \cdot S :: AC \cdot (AB + BC) \cdot BC$.

Demonstration.

- 18. Because by *Constr.* in 15° and 16° , . . . } $R - S \cdot S :: AB \cdot BC$.
- 19. Therefore by *Compos. of Reason*, . . . } $R \cdot S :: AC \cdot BC$.

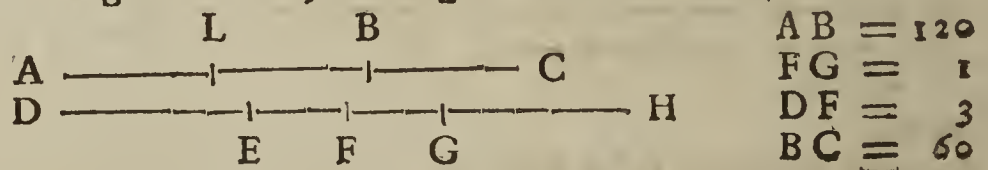
Which was to be done.

Note. In passing from the first step of this Demonstration, (which is the last step in the Resolution,) to the second; the Argumentation is made by *Composition of Reason*, because in passing to the last step of the Resolution from the last but one, it was argued by *Division of Reason*; agreeable to the *Note* at the end of the preceding *Probl. 1.*

Probl. III.

To a given right line to add another right line, that the Difference of the given and added may have a given Reason to their Summ. But the first Term of the Reason must be less than the latter Term.

This Problem hath two Cases; for either the given right line shall exceed the added, or the added the given. First, let the given exceed the added.



Suppos.

- 1. $b = AB$ a right line given.
- 2. $\left. \begin{matrix} r = FG \\ s = DF \end{matrix} \right\}$ the Terms of the given Reason.
- 3. $r \sqsupset s$.

Req. to find

- 4. BC a right line, such, that $AB - BC \cdot AB + BC :: FG \cdot DF$.

Resolution.

- 5. For the line sought to be added put . . . } a .
- 6. Therefore the excess of the given line b above } $b - a$.
 the line sought shall be . . . }
- 7. And the summ of the given line and the line } $b + a$.
 sought shall be . . . }
- 8. Therefore, according to the tenour of the Pro- } $r \cdot s :: b - a \cdot b + a$.
 blem propounded, this Analogy ariseth, . . . }
- 9. Therefore by *Composition of Reason*, . . . } $s + r \cdot s :: 2b \cdot b + a$.
- 10. And by doubling the Consequents, . . . } $s + r \cdot 2s :: 2b \cdot 2b + 2a$.
- 11. And inverfly, . . . } $2s \cdot s + r :: 2b + 2a \cdot 2b$.
- 12. And by *Division of Reason*, . . . } $s - r \cdot s + r :: 2a \cdot 2b$.
- 13. And inverfly, . . . } $s + r \cdot s - r :: 2b \cdot 2a$.
- 14. But a simple quantity is to a simple, as the } $b \cdot a :: 2b \cdot 2a$.
 double of the former to the double of the latter, }
 therefore . . . }

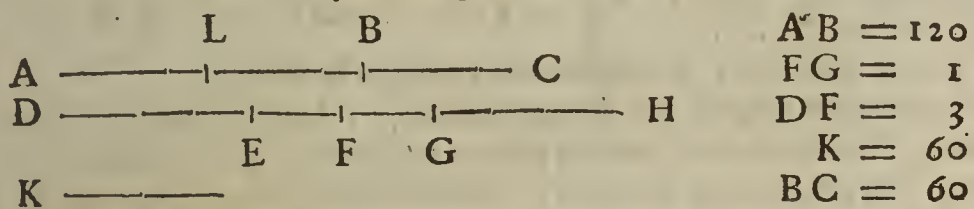
15. And

15. And out of 13° and 14°, (per prop. II. Elem. 5.) $s+r . s-r :: b . a$.
Hence this **CANON.**

16. As the summ of the Terms of the given Reason is to their difference, so is the given line to the line sought. Therefore the line required to be added to the given line, is given also.

17. *Note.* The line sought may easily be discovered by the Analogy in the ninth step; where the three first Terms being known, the fourth is known by Consequence; and since that fourth Term is evidently compos'd of the given line and the line sought, the given line subtracted from that known fourth Proportional shall necessarily give the line sought; whence 'tis manifest, that the Argumentation continued from the ninth step to the end of the Resolution is not of necessity, but only to shew how the line sought may be purely the fourth Proportional of an Analogy whose three first Terms are known, and consequently the line sought is known also: Which way of arguing by Analogies is more proper, (when it may be used,) than that by Equations, as hath before been hinted in Sect. 10. Probl. 1. of this Chapter.

The Composition of Case 1. Probl. 3.



Suppos.

- 18. AB is a right line given.
- 19. FG and DF are the Terms of a given Reason.
- 20. $FG \supset DF$.

Req. to find

21. BC a right line, such, that $AB-BC . AB+BC :: FG . DF$.

Construction.

- 22. Let it be made as $DF+FG$ (that is, DG,) to $DF-FG$, so AB to a fourth proportional line, call it K, therefore, $DF+FG : DF-FG :: AB : K$.
- 23. Let AB be continued to C, so, that $BC = K$.
- 24. Now the line BC, or K being found out by the help of the given lines AB, DF, FG, (according to the direction of the Canon,) we must shew that it will satisfy the Problem, viz. that the difference of AB and BC is to their summ, as FG to DF; but this Analogy, (after I have premis'd a few things to contract the Demonstration,) I shall make manifest by a repetition of the steps of the Resolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Resolution.

Prepar.

- 25. From AB cut off $AL = BC = K$, which is possible to be done, for the first Term of the Analogy in 22° is evidently greater than the second, and therefore (per Schol. Prop. 14. Elem. 5.) the third Term AB shall be greater than the fourth K, or BC; suppose therefore $AL = BC$.
- 26. Thence it follows that $LB = AB - BC$.
- 27. Let DF be continued to H, so, that $DF = FH$.
- 28. From DF cut off $FE = FG$, which is possible to be done, for by Supposition in 20°, $DF \supset FG$; suppose therefore $FE = FG$.
- 29. Therefore by subtracting the last Equation from that in 27°, $DE = GH = DF - FG$.
- 30. *Req. demonstr.* $FG . DF :: AB-BC : AB+BC :: LB . AC$.

Demonstration.

- 31. By Constr. in 22° and 23°, $DF+FG : DF-FG :: AB : BC$.
That is in 15°, (the last step of the Resolution,) $s+r . s-r :: b . a$.
- 32. But there is the same Reason of the double to the double, as of the simple to the simple; therefore, $2AB : 2BC :: AB : BC$.

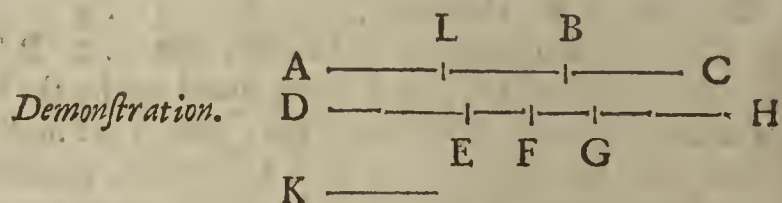
F f 2

That

- That is, in the 14th step, . . . } $2b \cdot 2a :: b \cdot a$
33. Therefore out of 31° and 32°, } $DF+FG \cdot DF-FG :: 2AB \cdot 2BC$
 (per Prop. 11. Elem. 5.) . . . }
- That is, in the 13th step, . . . } $s+r \cdot s-r :: 2b \cdot 2a$
34. Therefore inverfly, . . . } $DF-FG \cdot DF+FG :: 2BC \cdot 2AB$
 That is, in the 12th step, . . . } $s-r \cdot s+r :: 2a \cdot 2b$
35. Therefore by *Compos. of Reason*, } $2DF \cdot DF+FG :: 2AB+2BC \cdot 2AB$
 That is, in the 11th step, . . . } $2s \cdot s+r :: 2b+2a \cdot 2b$
36. Therefore inverfly, . . . } $DF+FG \cdot 2DF :: 2AB \cdot 2AB+2BC$
 That is, in the 10th step, . . . } $s+r \cdot 2s :: 2b \cdot 2b+2a$
27. Therefore by halving the Conse- } $DF+FG \cdot DF :: 2AB \cdot AB+BC$
 quents in the 36th step, . . . }
- That is, in the 9th step, . . . } $s+r \cdot s :: 2b \cdot b+a$
38. Therefore by *Division of Reason*, } $FG \cdot DF :: AB-BC \cdot AB+BC$
 That is, in the 8th step, . . . } $r \cdot s :: b-a \cdot b+a$
39. Therefore from 38° and 36°, } $FG \cdot DF :: LB \cdot AC$
 (per Ax. 6. Chap. 2.) . . . }

Which was to be Dem. Therefore the Problem is satisfied.

30. *Note.* Under every step of this Demonstration, I have set the correspondent step of the Resolution, that the Learner having respect to the *Note* at the end of *Probl. 1.* of this Chapter, may clearly perceive how the Demonstration is form'd out of the steps of the Resolution in a Retrograde order, that is, by returning backwards from the end to the beginning of the Resolution; for the first step in the Demonstration answers to the last in the Resolution, the second in the Demonstration, to the last but one in the Resolution, and so backwards in the Resolution, until the Analogy that was first assumed in the Resolution be positively and infallibly proved to be true. But after the Demonstration is in that manner discovered, the Algebraical steps must be omitted: So when the foregoing Demonstration beginning at the 31th step, is freed from the Analogies express'd by the small *Italian* letters belonging to the Resolution, and contracted by the help of the preparatory Equations in the 26th and 29th steps, respect also being had to the Diagram, there will arise this following



41. Because by *Constr.* in 22° and 23°, } $DG \cdot DE :: AB \cdot BC$
42. And by *prop. 15. Elem. 1.* . . . } $2AB \cdot 2BC :: AB \cdot BC$
43. Therefore, *per prop. 11. Elem. 5.* } $DG \cdot DE :: 2AB \cdot 2BC$
44. And inverfly, } $DE \cdot DG :: 2BC \cdot 2AB$
45. And by *Composition*, } $2DF \cdot DG :: 2AC \cdot 2AB$
46. And inverfly, } $DG \cdot 2DF :: 2AB \cdot 2AC$
47. And by halving the Consequents, } $DG \cdot DF :: 2AB \cdot AC$
48. Wherefore by *Division of Reason*, } $FG \cdot DF :: LB \cdot AC$
 Which in 30° was *Req. dem.*

49. Thus you have seen the first Case of *Probl. 3.* effected and demonstrated Synthetically; or by way of Composition, which argues altogether with known quantities; but the substance of the Composition, to wit, the *Construction* and *Demonstration*, was found out Analytically, or by way of Resolution, which from an Assumption of the quantity sought as if it were known or granted, together with the help of one or more known quantities, proceeds by Consequences, until in Conclusion the quantity so assumed or feigned to be known, is found equal to some quantity certainly known, and is therefore known also.

But it may be objected, that *Demonstrations* formed by the steps of Algebraical Resolution are for the most part rude and prolix; this I grant, but experience shews, that a *Demonstration* so found out may oftentimes be easily contracted, or, at least, give light to find out others more succinēt and elegant. And since my purpose is, to shew the Learner a general and ready way of forming the *Demonstrations* of such *Theorems*, and *Solutions* of *Problems* as he finds out by ALGEBRA, when no Term of any Analogy or Equation

Equation in any step of the Resolution exceeds Geometrical Dimensions, I shall very seldom digress from the steps of the Resolution.

The Resolution of Case 2. Probl. 3.

In this Case the line sought is suppos'd to exceed the given line AB.

- 50. For the given line AB put (as before) $\} b.$
- 51. And for the line sought, $\} a.$
- 52. Therefore the excess of the line sought }
above the given is $\} a - b.$
- 53. And the summ of both lines is . . . $\} a + b.$
- 54. Therefore, according to the tenour of }
Probl. 3. this Analogy ariseth, . . . $\} r . s :: a - b . a + b.$
- 55. Therefore by Composition of Reason, $\} s + r . s :: 2a . a + b.$
- 56. And by doubling the Consequents, $\} s + r . 2s :: 2a . 2a + 2b.$
- 57. And inversly, $\} . 2s . s + r :: 2a + 2b . 2a.$
- 58. And by Division of Reason, . . . $\} s - r . s + r :: 2b . 2a.$
- 59. And because there is the same Reason }
of the simple to the simple, as of the }
double to the double, therefore . . . }
 . $b . a :: 2b . 2a.$
- 60. And out of 58° and 59°, (per prop. 11. }
Elem. 5.) $\} s - r . s + r :: b . a.$

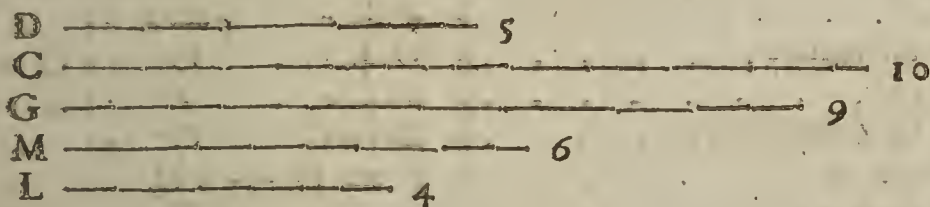
Hence this

CANON.

- 61. As the Difference of the Terms of the given Reason is to their Summ, so is the given line to the line sought. Therefore the line required to be added to the given line, is given also.
- The Composition of this latter Case differing but little from the former, I shall leave it as an exercise to the Learner.

Probl. IV.

The difference of the extremes of three proportional right lines being given, as also the summ of the mean and lesser extreme, to find the Proportionals.



Suppos.

- 1. G, M, L $\frac{d}{c}$; viz. G . M :: M . L.
- 2. G \subset L.
- 3. $d = G - L$ is given.
- 4. $c = M + L$ is given.

Req. to find G, M, L:

Resolution.

- 5. For the lesser extreme Proportional put . . . $\} a.$
- 6. Therefore out of 3° and 5°, the greater extreme is $\} d + a.$
- 7. And from 4° and 5°, the mean is $\} c - a.$
- 8. Therefore according to the tenour of the Problem, $\} d + a . c - a :: c - a . a.$
- 9. Therefore by Composition of Reason, . . . $\} d + c . c - a :: c . a.$
- 10. And alternately, $\} d + c . c :: c - a . a.$
- 11. Wherefore by Composition, $\} d + 2c . c :: c . a.$

Hence this

CANON.

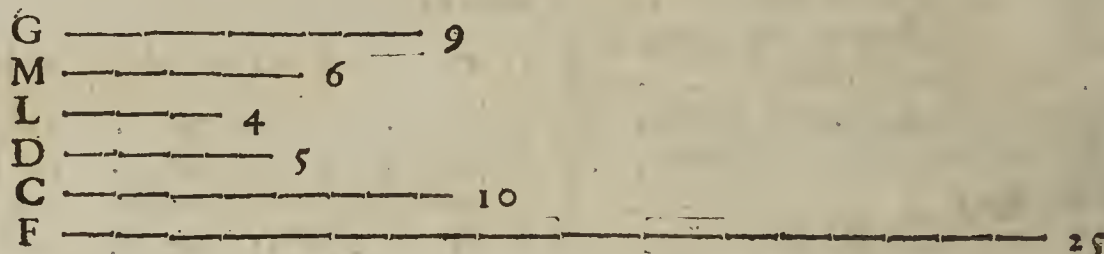
- 12. As the summ of the difference and the extremes of the double summ of the mean and lesser extreme, is to the summ of the mean and lesser extreme, so is the last mentioned summ to the lesser extreme. Therefore the lesser extreme sought is given.

13. After

13. After a Canon is found out by the Algebraical Art, it may be propounded in the form of a Theorem, whose Demonstration may be made by a repetition of the steps of the Resolution in a direct order, (not in a retrograde,) viz. by proceeding from the beginning to the end of the Resolution; as for example, the last preceding Canon may be propos'd in the form of a Theorem, and demonstrated thus.

THEOREM 1.

14. If three right lines be Proportionals, the summ of the mean and lesser extreme, shall be a mean proportional between the lesser extreme, and the summ of the difference of the extremes and the double summ of the mean and lesser extreme.



Suppos.

- 15. $G, M, L \div \div$; viz. $G : M :: M : L$.
- 16. $G \sqsubset L$.

Prepar.

- 17. Make $D = G - L$, therefore $D + L = G$.
- 18. Make $C = M + L$, therefore $C - L = M$.
- 19. Make $F = D + 2C = G + 2M + L$.

20. . . . Req. demonstr. $F : C :: C : L$.

Demonstration.

- 21. By Suppos. in 15°, $G : M :: M : L$
- 22. Therefore out of 17°, 18° and 21°, by exchanging equal right lines, $D + L : C - L :: C - L : L$
That is, in 8°, (the first Analogy in the Resolution,) $d + a : c - a :: c - a : a$
- 23. Therefore from 22°, by Composition of Reason, $D + C : C - L :: C : L$
That is, in 9°, $d + c : c - a :: c : a$
- 24. Therefore alternately, $D + C : C :: C - L : L$
That is, in 10°, $d + c : c :: c - a : a$
- 25. Therefore by Composition, $D + 2C : C :: C : L$
- 26. But by Constr. in 19°, $F = D + 2C$.
- 27. Therefore from 25° and 26°, $F : C :: C : L$.

Which was to be Dem.

But that Demonstration, after the letters of the Resolution are cast away, may be compendiously reduced unto this that follows, respect being had to the Suppositions and Preparation in 15°, 16°, 17°, 18°, 19°.

28. . . . Req. demonstr. $G + 2M + L : M + L :: M + L : L$

Demonstration.

- 29. By Suppos. in 15°, $G : M :: M : L$
- 30. Therefore by Composition, $G + M : M :: M + L : L$
- 31. And alternately, $G + M : M + L :: M : L$
- 32. Wherefore by Compos. $G + 2M + L : M + L :: M + L : L$

Which was to be Dem.

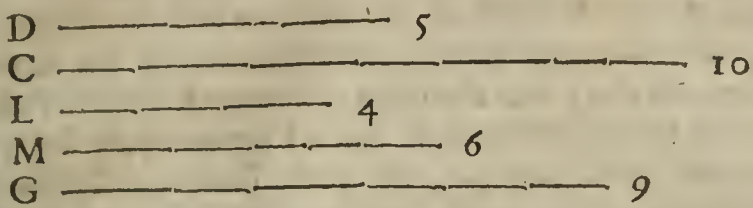
Hence ariseth

THEOREM 2.

33. If three right lines be Proportionals, the summ of the mean and lesser extreme shall be a mean Proportional between the lesser extreme, and the aggregate of the summ of the extremes and double summ of the mean.

The

The Composition of Probl. 4.



Suppos.

- 34. D = the difference of the extremes of three Proportionals is given.
 - 35. C = the summ of the mean and lesser extreme is given.
- Req. to find the Proportionals.

Construction.

- 36. Let it be made (by *Probl. 7. Chap. 5.*) as } $D + 2C . C :: C . L$
 D + 2C to C, so C to a third proportional
 line, which suppose to be found L, therefore,
- 37. By which Analogy, (and *per Schol. Prop. 14.* } $M + L = C$
Elem. 5.) the mean C is greater than L, therefore
 find a right line M equal to C - L, thence
 it follows that
- 38. Find a right line G equal to D + L, therefore, } $G - L = D$.
- 39. So by the help of the given lines D and C, according to the direction of the Canon in the preceding twelfth step, three right lines are found out, to wit, L, M and G, which shall be the three Proportionals required. Now we must shew that they will satisfie the Problem. First then, 'tis manifest by *Construction* in 38°, that the difference of the extremes G and L is equal to the given difference D. Secondly, by *Construction* in 37°, the summ of the mean M and the lesser extreme L is equal to the given summ C. It remains only to prove that the said G, M and L are Proportionals, viz. that as G is to M, so M to L; but this Analogy may be made manifest by a Repetition of the steps of the preceding Resolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Resolution, in manner following.

40. . . . Req. demonstr. G . M :: M . L

Demonstration.

- 41. Because by *Constr.* in 36°, } $D + 2C . C :: C . L$
 That is, in 11°, (the last step of the Resolution,) } $d + 2c . c :: c . a$.
- 42. Therefore by *Division of Reason*, } $D + C . C :: C - L . L$
 That is, in the tenth step, } $d + c . c :: c - a . a$.
- 43. Therefore alternately, } $D + C . C - L :: C . L$
 That is, in the ninth step, } $d + c . c - a :: c . a$.
- 44. Therefore, by *Division of Reason*, } $D + L . C - L :: C - L . L$
 That is, in the eighth step, } $d + a . c - a :: c - a . a$.
- 45. And because by *Constr.* in 38° and 37°, } $G = D + L$. Also, $M = C - L$.
- 46. Therefore out of 44° and 45°, by exchanging } $G . M :: M . L$
 equal right lines, }

Which was to be Dem. And therefore the Problem is satisfied.

Another way of resolving the foregoing Probl. 4.

- 47. The same things being supposed and given as before } $a = M$
 in 1°, 2°, 3° and 4° of this *Probl.* put a for the mean
 proportional sought; viz. suppose }
- 48. Therefore out of 4° and 47° the lesser extreme shall be } $c - a (= L)$
- 49. And by adding d the given difference of the extremes } $d + c - a (= G)$
 to the said lesser extreme c - a, the greater extreme
 shall be }
- 50. Therefore according to the tenour of the *Probl.* this } $d + c - a . a :: a . c - a$
 Analogy ariseth out of 47°, 48°, 49°, }
- 51. Therefore by *Composition of Reason*, } $d + c . a :: c . c - a$
- 52. And alternately, } $d + c . c :: a . c - a$
- 53. And inversly, } $c . d + c :: c - a . a$
- 54. Therefore by *Composition of Reason*, } $d + 2c . d + c :: c . a$
 Which

Which last Analogy gives this *CANON*.

55. As the aggregate of the difference of the extremes and the double summ of the mean and lesser extreme, is to the aggregate of the difference of the extremes and the summ of the mean and lesser extreme; so is the summ of the mean and lesser extreme to the mean. Therefore the difference of the extremes of three Proportionals being given, as also the summ of the mean and lesser extreme, the mean shall be also given by the Canon last exprest. The Demonstration whereof, and the *Composition* of the Problem according to this latter way of Resolution being very easie, I shall leave the same to the Learners exercise.

Probl. V.

The summ of the first and second of three Proportionals being given, as also the summ of the second and third, to find the Proportionals.

Suppos.

1. L, M, N are $\div\div$; viz. L . M :: M . N.
2. $b = L + M$ is given.
3. $c = M + N$ is given.

Req. to find L, M, N.

B		15
C		10
L		9
M		6
N		4

Resolution.

4. Put a for the first Proportional sought, viz. } $a = L$.
5. Therefore out of 2° and 4° , the mean is } $b - a (= M)$.
6. And by subtracting the said mean $b - a$ }
from the given summ c , the remainder gives } $c + a - b (= N)$.
7. Therefore (according to the *Probl.*) these }
must be Proportionals, viz. } $a . b - a :: b - a . c + a - b$.
8. Therefore inversly, } $b - a . a :: c + a - b . b - a$.
9. And by *Composition of Reason*, } $b . a :: c . b - a$.
10. And alternately, } $b . c :: a . b - a$.
11. And inversly, } $c . b :: b - a . a$.
12. Therefore by *Composition of Reason*, } $b + c . b :: b . a$.

Which last Analogy gives this

CANON.

13. As the aggregate of the summ of the first and second Proportionals and summ of the second and third, is to the summ of the first and second; so is the last mentioned summ to the first Proportional.

Therefore if the summ of the first and second of three Proportionals be given, as also the summ of the second and third, the mean shall be also given by the said Canon; whence also this

THEOREM.

14. If three right lines be Proportionals, the summ of the first and second is a mean Proportional, between the first, and the aggregate of the summ of the first and second, and summ of the second and third.

Which Theorem may easily be demonstrated by a repetition of the steps of the Resolution in a direct order, after the manner of demonstrating the Theorem in 14° of the foregoing *Probl. 4.* but for brevity sake I shall leave the Demonstration to the Learners practice, and proceed to the *Composition* of *Probl. 5.*

The Composition of Probl. 5.

Suppos.

15. B = the summ of the first and second of three Proportionals is given.
16. C = the summ of the second and third Proportionals is given.

Req.

Req. to find the Proportionals. [In the preceding Diagram.]

17. By *Probl. 7. Chap. 5.* let it be made as $B \perp C$ to B , so B to a third Proportional, suppose it be L , therefore

$$B \perp C \cdot B :: B \cdot L.$$

18. Make $M = B - L$, whence $L \perp M = B$; but that $B \sqsupset L$, as that effecton requires, is manifest by *Construction* in 17°; for $B \perp C$ the first Term of the Analogy in 17° is greater than B in the second, and therefore (*per Schol. Prop. 14. Elem. 5.*) the third Term, which is also B , shall be greater than L the fourth, therefore 'tis possible to cut off from B the right line L , and a right line will remain, which may be called M .

19. Make $N = C \perp L - B$, which is possible to be done if $C \perp L \sqsupset B$; but that $C \perp L \sqsupset B$, I prove thus,

By *Constr.* in 17°, $B \perp C \cdot B :: B \cdot L.$

Therefore (*per Schol. Prop. 25. Elem. 5.*) $B \perp C \perp L \sqsupset 2 B.$

And consequently, by equal subtraction of B , $C \perp L \sqsupset B.$

Which was to be proved. Therefore 'tis possible from the summ of the right lines C and L to cut off the right line B , and a right line will remain, which may be called N .

20. I say L , M and N are the three Proportionals required. Now we must shew that they will satisfie the Problem.

21. First then, the summ of the right lines L and M is (by *Construction* in 18°,) equal to the given summ B .

22. Secondly, that the summ of the right lines M and N is equal to the given summ C , I prove thus,

By *Constr.* in 18°, $M = B - L.$

And by *Constr.* in 19°, $N = C \perp L - B.$

Therefore by adding the two last Equations together, $M \perp N = C.$

Which was to be proved. It remains to shew that the said three right lines L , M and N are Proportionals, but that will be made manifest by the following Demonstration, which is formed out of the preceding Resolution by a repetition of the steps thereof in a retrograde order, viz. by returning backwards from the end to the beginning of the Resolution.

23. *Req. demonstr.* $L \cdot M :: M \cdot N.$

Demonstration.

24. Forasmuch as by *Construction* in 17°, $B \perp C \cdot B :: B \cdot L.$

That is, in 12°, (the last step of the Resolut.) $b \perp c \cdot b :: b \cdot a.$

25. Therefore by *Division of Reason*, $C \cdot B :: B - L \cdot L.$

That is, in 11°, $c \cdot b :: b - a \cdot a.$

26. Therefore inversly, $B \cdot C :: L \cdot B - L.$

That is, in 10°, $b \cdot c :: a \cdot b - a.$

27. Therefore alternately, $B \cdot L :: C \cdot B - L.$

That is, in 9°, $b \cdot a :: c \cdot b - a.$

28. And because by *Construction* in 18°, $M = B - L.$

29. Therefore out of 27° and 28°, $B \cdot L :: C \cdot M.$

30. And because it hath been proved in 18°, that $B \sqsupset L.$

31. Therefore out of 29°, by *Division of Reason*, $B - L \cdot L :: C - M \cdot M.$

That is, in 8°, $b - a \cdot a :: c - a - b \cdot b - a.$

32. Therefore inversly, $L \cdot B - L :: M \cdot C - M.$

That is, in 7°, $a \cdot b - a :: b - a \cdot b - c + a - b.$

33. But by *Constr.* in 18°, $M = B - L.$

34. And by what hath been proved in 22°, $N = C \perp M.$

35. Therefore out of 32°, 33° and 34°, $L \cdot M :: M \cdot N.$

Which was to be Dem. and therefore the Problem is satisfied.

Another way of resolving the foregoing Probl. 5.

36. The same things being given and supposed, as before in 1°, 2°, 3°, put a for the mean Proportional sought, viz. $a = M.$

37. Therefore out of 2° and 36°, the first Proportional shall be $b - a (= L.)$

38. And out of 3° and 36°, the third Proportional shall be $c - a (= N.)$

G g 39. There-

39. Therefore out of 36°, 37° and 38°, according to the tenour of *Probl. 5.* } $b-a . a :: a . c-a.$
 40. Therefore inverfly, . . . } $a . b-a :: c-a . a.$
 41. And by *Composition of Reason,* . . . } $b . b-a :: c . a.$
 42. And alternly, . . . } $b . c :: b-a . a.$
 43. Wherefore by *Composition of Reason,* . . . } $b+c . c :: b . a.$

Which last Analogy gives this

C A N O N.

44. As the aggregate of the given summ of the first and second Proportionals, and the given summ of the second and third, is to the summ of the second and third; so is the summ of the first and second, to the mean Proportional sought.

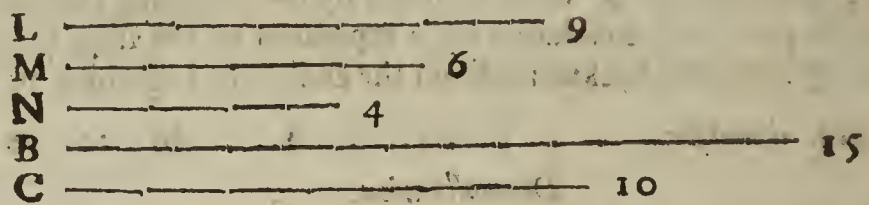
Which Canon, if it be propounded in the form of a Theorem, may be demonstrated by a repetition of the steps of the Resolution in a direct order. But leaving that and the *Composition of Probl. 5.* according to the latter Resolution, to the Learners exercise, I shall demonstrate the following Theorem by a repetition of the steps of the latter Resolution in a retrograde order.

T H E O R E M.

45. If three right lines be such, that the aggregate of the summ of the first and second and summ of the second and third, is to the summ of the second and third; as the summ of the first and second, to the second: those three lines shall be Proportionals, viz. As the first is to the second, so is the second to the third.

Suppos.

46. L, M, N, are three right lines.
 47. $B = L + M$, whence $B - M = L$.
 48. $C = M + N$, whence $C - M = N$.
 49. $B + C : C :: B : M$.



50. . . *Req. demonstr.* L, M, N are $::$, viz. $L : M :: M : N$.

Demonstration.

51. Because by *Suppos.* in 49°, . . . } $B+C . C :: B . M.$
 52. Therefore by *Division of Reason,* . . . } $B : C :: B-M . M.$
 53. And alternately, . . . } $B . B-M :: C . M.$
 54. Therefore by *Division of Reason,* . . . } $M . B-M :: C-M . M.$
 55. And inverfly, . . . } $B-M . M :: M . C-M.$
 56. But by *Suppos.* in 47°, . . . } $L = B - M.$
 57. And by *Suppos.* in 48°, . . . } $N = C - M.$
 58. Therefore out of 55°, 56° and 57°, . . . } $L : M :: M : N.$

Which was to be dem.

Probl. VI.

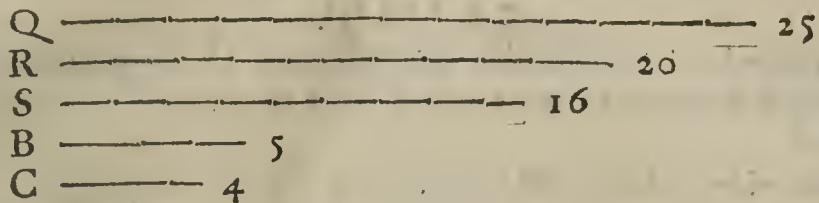
The difference of the greater extreme and mean of three Proportionals being given, as also the difference of the mean and lesser extreme; to find the Proportionals. But the first difference must be greater than the latter.

Suppos.

1. Q, R, S are $::$, viz. $Q : R :: R : S$.
 2. $Q \sqsupset R$.
 3. $b = Q - R$ is given.
 4. $c = R - S$ is given.

Req. to find Q, R, S.

Q



Resolution.

- 5. Put a for the mean Proportional sought, viz. . . . } $a = R.$
- 6. Therefore out of 3° and 5° , the greater extreme shall be } $a + b (= Q.)$
- 7. And out of 4° and 5° , the lesser extreme shall be . . . } $a - c (= S.)$
- 8. Therefore out of 6° , 5° and 7° , this Analogy will arise, }
 (according to the import of the Problem,) viz. } $a + b . a :: a . a - c.$
- 9. Therefore by *Division of Reason*, } $b . a :: c . a - c.$
- 10. And alternately, } $b . c :: a . a - c.$
- 11. And by *Conversion of Reason*, } $b . b - c :: a . c.$
- 12. And inverfly, } $b - c . b :: c . a.$

Which last Analogy gives this

CANON.

13. As the excess by which the given difference of the greater extreme and mean exceeds the given difference of the mean and lesser extreme, is to the difference of the greater extreme and mean; so is the difference of the mean and lesser extreme, to the mean Proportional sought, whence the extremes will be easily discovered.

Which Canon, if it be propounded in the form of a Theorem, may be easily demonstrated by a repetition of the steps of the Resolution in a direct order; but leaving that to the Learner's practice, I shall demonstrate the following Theorem by a retrograde repetition of the steps of the Resolution.

THEOREM I.

14. If three right lines be such, that the excess by which the excess of the first above the second exceeds the excess of the second above the third, be to the excess of the second above the third, as the excess of the first above the second is to the second, then those three right lines shall be Proportionals, viz. As the first is to the second, so the second to the third.

Suppos.

- 15. Q, R, S are three right lines.
- 16. $Q \sqsupset R.$
- 17. $R \sqsupset S.$
- 18. $B = Q - R$, whence $Q = B + R.$
- 19. $C = R - S$, whence $S = R - C.$
- 20. $B - C . C :: B . R.$
- 21. . . . Req. demonstr. Q, R, S are \therefore , viz. $Q . R :: R . S.$

Demonstration.

- 22. Because by *Suppos.* in 20° , } $B - C . C :: B . R.$
- 23. Therefore alternately, } $B - C . B :: C . R.$
- 24. And inverfly, } $B . B - C :: R . C.$
- 25. And by *Conversion of Reason*, } $B . C :: R . R - C.$
- 26. And alternately, } $B . R :: C . R - C.$
- 27. And by *Compos. of Reason*, } $B + R . R :: R . R - C.$
- 28. But by *Supposition* in 18° , } $Q = B + R.$
- 29. Also by *Suppos.* in 19° , } $S = R - C.$
- 30. Therefore out of 27° , 28° and 29° , by exchange }
 of equal right lines, } $Q . R :: R . S.$

Which was to be Dem.

31. The Determination annex'd to *Probl. 6.* to wit, That the given difference of the greater extreme and mean must be greater than the given difference of the mean and lesser extreme, is discovered by the preceding Canon in 13° , and is necessarily to be prescribed for limiting the said differences, that they may be capable of constructing the Problem; as will be manifest by the subsequent

LEMMA.

32. If three right lines be Proportionals, the difference of the greater extreme and mean, is greater than the difference of the mean and lesser extreme.

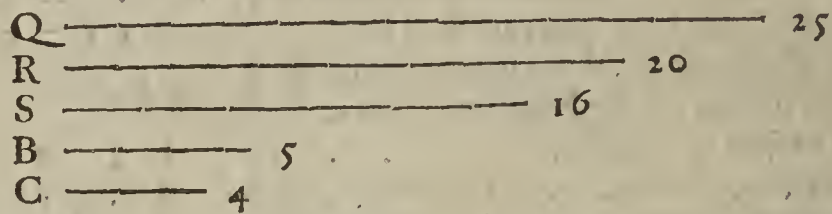
Suppos.

33. Q, R, S are $\div\div$, viz. $Q \cdot R :: R \cdot S$.

34. $Q \sqsupset R$, and consequently, $R \sqsupset S$.

35. $B = Q - R$, whence $B + R = Q$.

36. $C = R - S$, whence $R - C = S$.



37. . . . *Req. demonstr.* . . . $B \sqsupset C$.

Demonstration.

38. Because by *Suppos.* in 33°, } $Q \cdot R :: R \cdot S$.

39. And by *Suppos.* in 35° and 36°, } $B + R = Q$. And $R - C = S$.

40. Therefore out of 38° and 39°, } $B + R \cdot R :: R \cdot R - C$.

41. And by *Division of Reason*, } $B \cdot R :: C \cdot R - C$.

42. But $R \sqsupset R - C$, therefore from 41°, (*per* } $B \sqsupset C$.
Schol. prop. 14. Elem. 5.) }

Which was to be Demonstr.

The Determination being demonstrated, I shall proceed to

The Composition of Probl. 6.

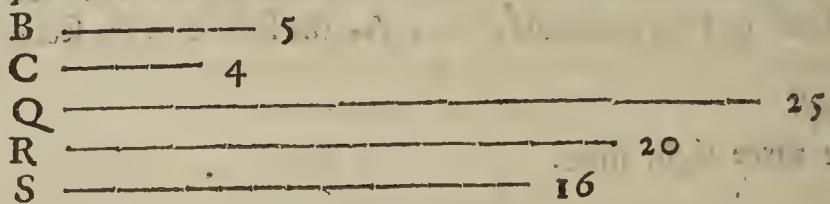
Suppos.

43. $B =$ the difference of the greater extreme and mean of three Proportionals is given.

44. $C =$ the difference of the mean and lesser extreme is given.

45. $B \sqsupset C$. (*Determination.*)

Req. to find the Proportionals. }



Construction.

46. By *Probl. 8. Chap. 5.* let it be made as $B - C$ to B , fo C to a fourth Proportional which may be called R , therefore,

$$B - C \cdot B :: C \cdot R.$$

which fourth Proportional R shall be greater than the third C , because the second B is greater than the first $B - C$.

47. Make $Q = R + B$; whence, $Q - R = B$.

48. Make $S = R - C$; whence, $R - S = C$, which effect is possible, for by the Analogy in 46°, it is manifest that $R \sqsupset C$.

49. I say Q, R and S are the three Proportionals required: Now we must shew that they will satisfy the Problem; First then, by *Construction* in 47°, the excess of Q above R is equal to the given difference B ; secondly, by *Constr.* in 48°, the excess of R above S is equal to the given difference C . So it remains only to prove that Q, R and S are Proportionals, in this order, viz. $Q \cdot R :: R \cdot S$, but that is made manifest by the subsequent *Demonstration*, which is form'd out of the preceding Resolution, by a repetition of the steps thereof in a retrograde (not in a direct) order.

50. . . . *Req. demonstr.* } $Q \cdot R :: R \cdot S$.

Demonstration.

51. Because by *Constr.* in 46°, which answers to the } $B - C \cdot B :: C \cdot R$
 last step of the Resolution, to wit, 12°, }

52. Therefore inversly, } $B \cdot B - C :: R \cdot C$.

53. And by *Conversion of Reason*, } $B \cdot C :: R \cdot R - C$.

54. And alternately, } $B \cdot R :: C \cdot R - C$.

55. There-

- 55. Therefore by *Composition of Reason*, } $B + R . R :: R . R - C.$
- 56. But by *Constr.* in 47° , } $Q = B + R.$
- 57. And by *Constr.* in 48° , } $S = R - C.$
- 58. Therefore out of $55^\circ, 56^\circ, 57^\circ$, } $Q . R :: R . S.$

Which was to be dem. therefore that is done which was required by *Probl. 6.*

Another way of resolving the foregoing Probl. 6.

- 59. The same things being given and supposed as before }
in $1^\circ, 2^\circ, 3^\circ, 4^\circ$ of *Probl. 6.* for the lesser extreme } $a.$
of the three Proportionals sought, put }
- 60. To which lesser extreme if you add the given dif- }
ference c , it makes the mean, to wit, } $a + c.$
- 61. And by adding the given difference b to the mean }
Proportional, it gives the greater extreme, to wit, } $a + c + b.$
- 62. Therefore according to the import of the Problem, }
these must be Proportionals, *viz.* } $a . a + c :: a + c . a + c + b.$
- 63. Therefore inversly, } $a + c . a :: a + c + b . a + c.$
- 64. And by *Division of Reason*, } $c . a :: b . a + c.$
- 65. And by *altern and inverse Reason*, } $b . c :: a + c . a.$
- 66. Wherefore by *Division of Reason*, } $b - c . c :: c . a.$

Which last Analogy gives this **CANON.**

- 67. As the excess whereby the given difference of the greater extreme and mean exceeds the given difference of the mean and lesser extreme, is to the difference of the mean and lesser extreme; so is the difference last mentioned, to the lesser extreme sought: whence the mean and greater extreme are easily discovered.

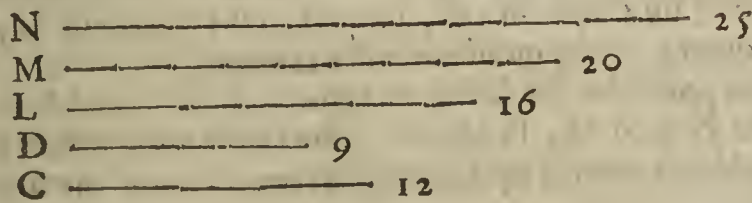
Probl. VII.

The difference of the extremes of three Proportionals being given, as also a right line whose Square is equal to the difference between the Square of the mean and the Square of one of the extremes, to find the Proportionals.

Suppos.

- 1. N, M, L are $\div\div$; *viz.* $N . M :: M . L.$
- 2. $N \sqsupset L.$
- 3. $d = N - L$ is given.
- 4. $c = \sqrt{\square M - \square L}$: is given; and consequently;
- 5. $cc = \square M - \square L$ is given.

Req. to find N, M, L.



Resolution.

- 6. Put a for the lesser extreme sought, *viz.* } $a = L.$
- 7. Therefore out of 3° and 6° , the greater extreme shall be . . . } $a + d (= N.)$
- 8. And out of 6° and 7° , the Rectangle contained under the ex- }
tremes, or the Square of the mean, is equal to . . . } $aa + da.$
- 9. And out of 6° , the Square of the lesser extreme is . . . } $aa.$
- 10. Which Square aa being subtracted from $aa + da$, (to wit, }
from the Square of the mean,) the remainder shall be the diffe- } $da.$
rence of the said Squares, *viz.* }
- 11. But the difference last mentioned must be equal to the given }
difference cc , therefore } $da = cc.$
- 12. Which Equation may be resolved into this Analogy, . . . } $d . c :: c . a.$

Hence

Hence this

CANON.

13. As the given difference of the extremes, is to the given right line whose Square is equal to the difference of the Squares of the mean and lesser extreme; so is the same right line to the lesser extreme sought.

Or thus,

As the given difference of the extremes, is to the given right line whose Square is equal to the difference of the Squares of the mean and greater extreme; so is the same right line to the greater extreme sought.

Hence this

THEOREM.

14. If three right lines be Proportionals, the difference of the Squares of the mean and lesser extreme is equal to the Rectangle contained under the difference of the extremes and the lesser extreme.

Or thus,

The difference of the Squares of the mean and greater extreme, is equal to the Rectangle contained under the difference of the extremes and greater extreme.

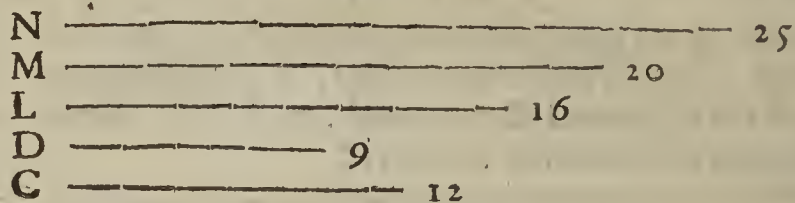
The Composition of Probl. 7.

Suppos.

15. D = the difference of the extremes of three Proportionals is given.

16. C = a right line, whose Square is equal to the difference of the Squares of the mean and lesser extreme is given.

Req. to find the Proportionals:



Construction.

17. By *Probl. 7. Chap. 5.* let it be made as D to C, so C to a third Proportional, which suppose to be the right line L, therefore,

$$D \cdot C :: C \cdot L.$$

18. By *Probl. 2. Chap. 5.* find a right line, as M, such, that its Square may be equal to the Square of L together with the Square of C, therefore;

$$\square M = \square L + \square C;$$

And consequently, $\square M - \square L = \square C.$

19. Make $N = L + D$; whence, $N - L = D.$

20. I say N, M and L are the three Proportionals required. Now we must shew that they will satisfy the Problem.

21. First then, by *Construction* in 19°, the excess of N above L is equal to D the given difference of the extremes.

22. Secondly, the excess by which the Square of the mean M exceeds the Square of the lesser extreme L, is (by *Constr.* in 18°,) equal to the Square of C, to wit, the given difference of the Squares of the mean and lesser extreme.

23. It remains only, to prove that the said three right lines N, M and L are Proportionals, in this order, viz. As N is to M, so M to L; But that is made manifest by the subsequent Demonstration, which is formed by a retrograde repetition of the steps of the preceding Resolution.

18. . . Req. demonstr. N . M :: M . L.

Demonstration.

25. Because by *Construction* in 17°, } $D \cdot C :: C \cdot L.$

That is, in 12°, (the last step of the *Resolut.*) } $d \cdot c :: c \cdot a.$

26. Therefore, per 17. prop. 6. Elem. } $\square D, L = \square C.$

That is, in 11°, } $da = cc.$

27. Again, because by *Constr.* in 19°, } $N = L + D.$

28. Therefore (per prop. 1. Elem. 6.) by drawing L as } $\square NL = \square L + \square D, L$

29. And consequently out of 28° and 26°, by exchange } $\square NL = \square L + \square C.$

30. But

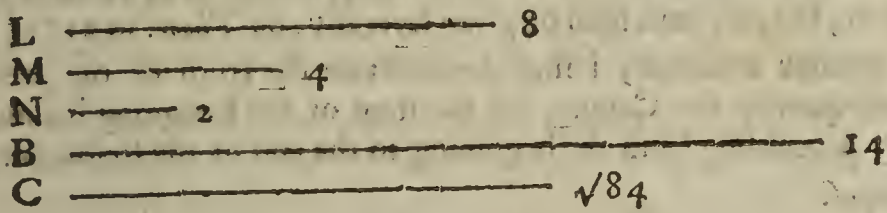
30. But by *Constr.* in 18° , } $\square M = \square L + \square C$.
 31. Therefore out of 29° and 30° , *per* 1. *Ax.* *Chap.* 2. } $\square NL = \square M$.
 32. Therefore out of 31° , *per* 14. *prop.* 6. *Elem.* } $N . M :: M . L$.

Which was to be *Demonstr.* therefore the Problem propounded is satisfied.

Having by the preceding Examples of Resolution and Composition given the Learner a taste of the manner of arguing by Analogies, which is the best way when the nature of a Problem will admit the same, I shall now proceed to Examples of arguing partly by Equations, and partly by Analogies. But it must be remembered, that when in the Resolution you pass from one step to another by Addition, in the Demonstration of the Problem you must return by Subtraction: For Addition in the Resolution requires Subtraction in the Composition, and Subtraction in the one, Addition in the other; also Composition of Reason in the one requires Division of Reason in the other, as before hath been said in the *Note* at the end of *Probl.* 1. of this Chapter.

Probl. VIII.

A right line equal to the summ of three proportional right lines being given, as also a right line whose Square is equal to the summ of the Squares of all the said Proportionals, to find out the Proportionals severally. But the first of those lines given must be greater than the latter, yet not greater than the right line arising out of the Application of the triple Square of the said latter line to the first.



Suppos.

1. L, M, N $\frac{8}{4} :: \frac{4}{2}$, *viz.* L . M :: M . N.
2. $b = L + M + N$ is given.
3. $c = \sqrt{\square L + \square M + \square N}$: is given; therefore,
4. $cc = \square L + \square M + \square N$ is given also.

Req. to find L, M, N.

Resolution.

5. For the mean Proportional sought put a , *viz.* } $a = M$.
 suppose }
6. Therefore from 2° and 5° , the summ of the } $b - a = L + N$.
 extremes shall be }
7. Therefore the Square of the summ of the ex- } $bb + aa - 2ba = \square L + \square N + 2\square LN$.
 tremes is }
8. And the Square of the mean Proportional, or } $aa = \square M = \square LN$.
 the Rectangle of the extremes is }
9. Which Square or Rectangle in 8° , being sub- } $bb - 2ba = \square L + \square M + \square N$.
 tracted from the Square in 7° , leaves the summ }
 of the Squares of all the three Proportionals, *viz.* }
10. But by *Supposition* in 4° , } $cc = \square L + \square M + \square N$.
11. Therefore from 9° and 10° , (*per* *Ax.* 1. } $bb - 2ba = cc$.
Chap. 2.) }
12. And by adding $2ba$ to each part of the E- } $bb = cc + 2ba$.
 quation in 11° , this ariseth, }
13. And by subtracting cc from each part of the } $bb - cc = 2ba$.
 Equation in 12° , }
14. And because (*per* *Theor.* 8. *Chap.* 4.) } $bb - cc = \overline{b + c} \times \overline{b - c}$.
 }
15. Therefore from 13° and 14° , (*per* *Ax.* 1. } $2ba = \overline{b + c} \times \overline{b - c}$.
Chap. 2.) }
16. Therefore by resolving the last preceding } $2b . \overline{b + c} :: \overline{b - c} . a$.
 Equation into Proportionals, it shall be . . . }

From

From the ninth step ariseth

THEOREM 1.

17. If three right lines be Proportionals, the excess whereby the Square of their summ exceeds twice the Rectangle made of that summ and the mean proportional, shall be equal to the summ of the Squares of all the three proportionals.

From the last step of the Resolution ariseth

THEOREM 2.

18. If three right lines be Proportionals, then this Analogy will attend them, viz. As the double summ of all the three Proportionals is to the simple summ increased with the side of a Square equal to the summ of the Squares of the three Proportionals; so is the excess whereby the summ of the three Proportionals exceeds the said side, to the mean Proportional.

Therefore the summ of three Proportionals being given, as also the summ of their Squares, the mean Proportional shall be given also by the preceding Theor. 2. whence the summ of the extremes is consequently given. And lastly, the summ of the extremes being given, as also the mean, the extremes shall be given severally, by *Probl. 13. Chap. 5.*

But to solve this *Probl. 8.* Arithmetically, the following Canon, (deducible from the 13th step,) will be more ready than *Theor. 2.*

CANON.

19. From the Square of the given summ of three Proportionals, subtract the given summ of their Squares, and divide the remainder by the double of the first given summ; so shall the Quotient be the mean Proportional; which subtracted from the summ of all three, leaves the summ of the extremes. And lastly, the summ of the extremes being given, as also the mean, the extremes shall be given severally, by *Theor. in 21° of Probl. 13. Ch. 5.*

But for the greater evidence, I shall demonstrate the truth of the preceding *Theor. 1.* and 2. and consequently the Canon, by the steps of the foregoing Resolution in a direct order, viz. by proceeding from the beginning to the end of the Resolution.

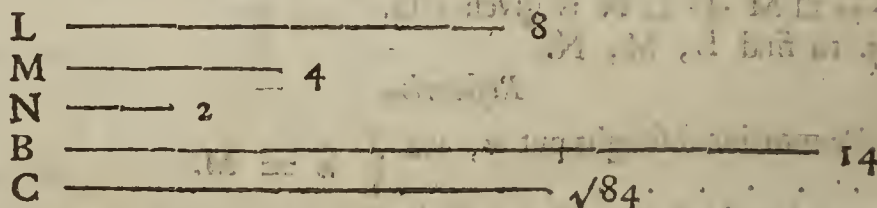
Suppos.

20. $L, M, N :: L \cdot M :: M \cdot N.$

21. $B = L + M + N.$

22. $C = \sqrt{\square L + \square M + \square N}.$

23. $\square C = \square L + \square M + \square N.$



24. . . . *Req. demonstr.* } *Theor. 1.* $\square B - 2 \square BM = \square C.$
 } *Theor. 2.* $2B \cdot B - C :: B - C \cdot M,$

Demonstration.

- | | |
|----------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| 25. By <i>Supposition</i> in 21°, | B = L + M + N. |
| 26. Therefore by subtracting M from each part, | B - M = L + N. |
| 27. And by squaring each part in 26°, this Equation will arise, (per <i>Theor. 5.</i> and 2. of <i>Chap. 4.</i>) | $\square B + \square M - 2 \square BM = \square L + \square N + 2 \square LN.$ |
| 28. And from 20°, (per <i>prop. 17. Elem. 6.</i>) | $\square M = \square LN.$ |
| 29. Therefore by subtracting the Equation in 28° from that in 27°, this will manifest, (per <i>Ax. 9.</i> & 6. <i>Chap. 2.</i>) | $\square B - 2 \square BM = \square L + \square M + \square N.$ |
| 30. But by <i>Suppos.</i> in 23°, | $\square C = \square L + \square M + \square N.$ |
| 31. Therefore from 29° and 30°, (per <i>Ax. 1. Chap. 2.</i>) | $\square B - 2 \square BM = \square C.$ |

Which was *Theor. 1.* to be demonstr.

32. Again, by adding 2 $\square BM$ to each part of the Equation in 31°, this ariseth, . . . } $\square B = \square C + 2 \square BM.$

33. And by subtracting $\square C$ from each part of the }
 Equation in 32°, } $\square B - \square C = 2 \square BM.$
 34. But by *Theor. 8. Chap. 4.* } $\square B - \square C = \square : \overline{B+C} \times \overline{B-C} :$
 35. Therefore from 33° and 34°, (*per Ax. 1.*) } $2 \square BM = \square : \overline{B+C} \times \overline{B-C} :$
 36. Therefore, (*per prop. 14. Elem. 6.*) . . . } $2B \cdot \overline{B+C} :: \overline{B-C} \cdot M.$

Which was *Theor. 2.* to be dem. Therefore the truth of both the preceding Theorems is made manifest; also the Canon in 19° is evident from the Equation in 33°, by Application of each part thereof unto 2 B.

37. I shall in the next place, in order to the Composition of *Probl. 8.* before propounded, demonstrate the Determination annex'd to it for limiting the lines given, that they may be capable of effecting the Problem.

Determination. $\left\{ \begin{array}{l} B \sqsubset C, \\ B \text{ not } \sqsubset \frac{3 \square C}{B}. \end{array} \right.$

That is, the line given for the summ of three Proportionals must be greater than that given right line whose Square is equal to the summ of the Squares of all the three Proportionals, yet not greater than the right line arising out of the Application of the triple of the said Square, to the right line given for the summ of the three Proportionals.

38. The Scope of the Determination is, to remove two Objections that may be brought against the Construction of the Problem (in the following 76th and 87th steps,) unless the given lines be limited as the Determination prescribes, whose first part, to wit, that $B \sqsubset C$ is discovered by the last step of the Resolution, and already demonstrated. The latter part of the Determination, to wit, that the given line B ought not to be greater than $\frac{3 \square C}{B}$, is neither apparent in the proposition of the Problem, nor in either of the Theorems resulting from the Resolution; but that it is a property adherent to three Proportionals, I shall demonstrate by the following *Lemma*, and afterwards shew that it is necessary to make the Problem possible.

LEMMA.

39. If three right lines be Proportionals, their summ shall sometimes be equal to the right line arising out of the Application of the triple summ of all their Squares to their said summ; sometimes less, but never greater than the right line arising by the said Application.
 40. Two Cases are to be demonstrated to prove this *Lemma*, for three Proportionals are either equal between themselves, or unequal. In the first Case, 'tis easie to perceive that the Square of the summ of three Proportionals is equal to the triple summ of their Squares; for supposing N, N, N to represent three Proportionals, their summ is 3N, the Square whereof is $9 \square N$, which is manifestly equal to $3 \square N + 3 \square N + 3 \square N$, to wit, the triple summ of the Squares of the said three Proportionals; and therefore if $9 \square N$, the triple summ of the Squares of the said three equal Proportionals N, N, N, be applied to (or divided by) 3 N the summ of the same Proportionals, the line (or Quotient) arising by that Application shall necessarily be 3 N, (the summ of the said Proportionals.) Therefore the first Case of the *Lemma* is manifest.

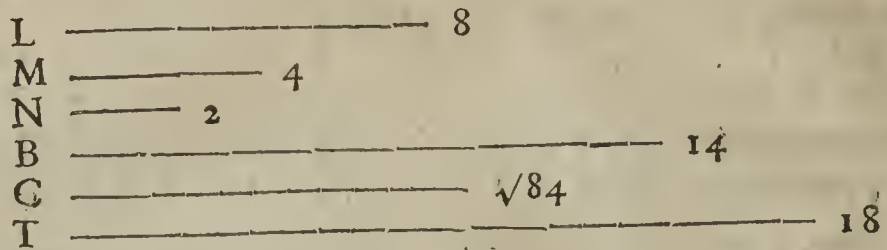
Suppos. in Case 2.

41. L, M, N $\div \div$, viz. L . M :: M . N.
 42. L \sqsubset M; and consequently M \sqsubset N.

Prepar.

43. By *Probl. 1. Chap. 5.* make $B = L + M + N.$
 44. By *Probl. 2. Chap. 5.* make $C = \sqrt{\square L + \square M + \square N} :$
 45. Thence it follows, that $\square C = \square L + \square M + \square N.$
 46. By *Probl. 8. Chap. 5.* let it be made, $B \cdot C :: 3 C$ (to a fourth) T.
 47. Thence it follows, that $T = \frac{3 \square C}{B}.$

H h L —



48. . . . Req. demonstr. $B \supset T$, or $\frac{3 \square C}{B}$.

Demonstration.

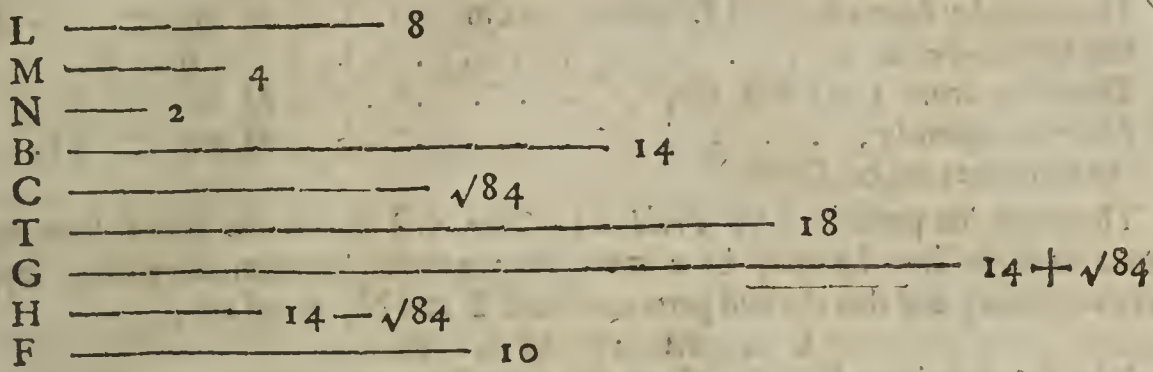
- 49. Because by *Suppos.* in 41° , . . . } $L \cdot M :: M \cdot N$.
- 50. Therefore, (*per prop. 1. Elem. 6.*) } $\square L \cdot \square LM :: L \cdot M :: M \cdot N$.
- by reason of the common altitude L, }
- 51. Likewise by reason of the com- } $\square MN \cdot \square N :: M \cdot N :: L \cdot M$.
- mon altitude N, }
- 52. Therefore out of 50° and 51° , } $\square MN \cdot \square N :: \square L \cdot \square LM$.
- (*per prop. 11. Elem. 5.*) }
- 53. And out of 42° and 52° , by *Di-* } $\square MN - \square N \cdot \square N :: \square L - \square LM \cdot \square LM$.
- vision of Reason*, }
- 54. And inverfly, } $\square N \cdot \square MN - \square N :: \square LM \cdot \square L - \square LM$.
- 55. But from 41° and 42° , } $\square N \supset \square LM$.
- 56. Therefore from 54° and 55° , (*per* } $\square MN - \square N \supset \square L - \square LM$.
- prop. 14. Elem. 5.*) }
- 57. And by adding $\square LM$, to each } $\square LM + \square MN - \square N \supset \square L$.
- part in 56° , }
- 58. And by adding $\square N$ to each part } $\square LM + \square MN \supset \square L + \square N$.
- in 57° , }
- 59. And by adding $\square M$ to each part } $\square LM + \square M + \square MN \supset \square L + \square M + \square N$.
- in 58° , }
- 60. But by *Constr.* in 45° , } $\square C = \square L + \square M + \square N$.
- 61. Therefore from 59° and 60° , } $\square LM + \square M + \square MN \supset \square C$.
- (*per Ax. 3. Chap. 2.*) }
- 62. And because by *Constr.* in 43° , } $L + M + N = B$.
- 63. Therefore from 62° , by reason of } $\square LM + \square M + \square MN = \square BM$.
- the common altitude M, }
- 64. Therefore from 61° and 63° , } $\square BM \supset \square C$.
- (*per Ax. 4. Chap. 2.*) }
- 65. And consequently, } $2 \square BM \supset 2 \square C$.
- 66. And by adding $\square C$ to each part } $\square C + 2 \square BM \supset 3 \square C$.
- in 65° , }
- 67. But it hath been proved in 32° , } $\square C + 2 \square BM = \square B$.
- (in the preceding *Demonstration* }
of *Theor. 1*, and 2.) that . . . }
- 68. Therefore out of 66° and 67° , } $\square B \supset 3 \square C$.
- (*per Ax. 4. Chap. 2.*) }
- 69. Therefore by Application of each } $B \supset \frac{3 \square C}{B}$.
- part in 68° to B, }
- 70. But by *Constr.* in 46° , } $T = \frac{3 \square C}{B}$.
- Therefore from 69° and 70° , (*per* } $B \supset T$ or $\frac{3 \square C}{B}$.
- Ax. 3. Chap. 2.*) }

Which was to be Demonstr.

71. Now because every three Proportionals whatever, are either equal or unequal between themselves, and it hath been shewn, that when they be equal to one another, their sum is equal to the right line arising out of the Application of the triple sum of their Squares to their sum; but when unequal, the sum of the three Proportionals is less than the right line arising by the said Application; it is manifest that the said sum can never be greater than the said right line: And therefore the truth of the preceding *Lemma*, and consequently the reason of the latter part of the *Determination* annex to *Probl. 8.* are evident.

The

The Composition of the foregoing Probl. 8.



Suppos.

- 72. B = the summ of three proportional right lines is given.
- 73. C = a right line, whose Square is equal to the summ of the Squares of the said three Proportionals, is given.
- 74. $B \not\sqsubset C$; yet B not $\sqsubset \frac{3 \square C}{B}$. (Determination.)

Req. to find out the three Proportionals.

Construction.

- 75. By *Probl. 1. Chap. 5.* find a right line, as G, }
equal to the summ of the given right lines B and C, } $G = B + C$.
therefore }
- 76. By *Probl. 3. Chap. 5.* find a right line, as H, }
equal to the excess of B above C; which effecton } $H = B - C$.
is possible, for by *Suppos.* in 74°, $B \not\sqsubset C$, therefore, }
- 77. By *Probl. 8. Chap. 5.* let it be made as 2B to G; }
so H to a fourth proportional line, call it M, therefore } $2B \cdot G :: H \cdot M$.
- 78. Find a right line $F = B - M$, which is possible to be done if B exceeds M, but
that $B \not\sqsubset M$ I prove thus,
- 79. By *Construction* in 75°, } $B + C = G$.
- 80. And by *Constr.* in 76°, } $B - C = H$.
- 81. Therefore (per *Theor. 8. Chap. 4.*) by comparing }
the Rectangles made of the two last Equations, . } $\square B - \square C = \square GH$.
- 82. And by the first part of the last Equation 'tis }
manifest that } $2 \square B \not\sqsubset \square B - \square C$.
- 83. Therefore from 81° and 82°, (per *Ax. 3. Chap. 2.*) } $2 \square B \not\sqsubset \square GH$.
- 84. But from 77°, (per *prop. 16. Elem. 6.*) } $2 \square BM = \square GH$.
- 85. Therefore from 83° and 84°, (per *Ax. 3. Chap. 2.*) } $2 \square B \not\sqsubset 2 \square BM$.
- 86. Therefore by Application of each part in 85° }
to 2B, } $B \not\sqsubset M$.

Which was to be shewn.

- 87. By *Probl. 14. Chap. 5.* divide the line F (before found in 78°) into two such parts,
that the line M may be a mean Proportional between the parts; which is possible to be
done if M be not greater than $\frac{1}{2} F$, but that M is not greater than $\frac{1}{2} F$, I prove thus,
- 88. By *Supposition* in 74°, } $B \not\sqsubset \frac{3 \square C}{B}$.
- 89. Therefore by drawing B into each part, . . . } $\square B \not\sqsubset 3 \square C$.
- 90. And by adding $2 \square B$ to each part in 89°, . . . } $3 \square B \not\sqsubset 2 \square B + 3 \square C$.
- 91. And because by *Suppos.* in 74°, } $B \not\sqsubset C$.
- 92. And consequently, } $3 \square B \not\sqsubset 3 \square C$.
- 93. Therefore by subtracting $3 \square C$ from each part }
in 90°, } $3 \square B - 3 \square C \not\sqsubset 2 \square B$.
- 94. It hath been proved in 81°, that } $\square B - \square C = \square GH$.
- 95. And in 84°, that } $2 \square BM = \square GH$.
- 96. Therefore from the two last preceding Equations, } $\square B - \square C = 2 \square BM$.
- 97. And consequently, } $3 \square B - 3 \square C = 6 \square BM$.
- 98. Therefore from 93° and 97°, (per *Ax. 4. Chap. 2.*) } $6 \square BM \not\sqsubset 2 \square B$.
- 99. And consequently, } $3 \square BM \not\sqsubset \square B$.

100. Therefore by Application of each part in 99° to B, } $3 M$ not $\square B$.
 101. It hath been proved in 86° , that } $B \square M$.
 102. Therefore by subtracting M from each part in 100° , } $2 M$ not $\square B - M$.
 103. But by *Constr.* in 78° , } $F = B - M$.
 104. Therefore from 102° and 103° , } $2 M$ not $\square F$.
 105. And consequently, } M not $\square \frac{1}{2} F$.

Which was to be Dem.

106. Therefore 'tis possible (by *Probl. 14. Chap. 5.*) to cut the line F into two such parts, that the line M may be a mean Proportional between the parts; suppose it therefore done, and that the said parts are found L and N, therefore,

$$L \cdot M :: M \cdot N.$$

107. I say the right lines L, M, N are the three Proportionals sought: Now we must shew that they will satisfie the Problem propounded; First then, by Construction in the last preceding step they are Proportionals. Secondly, that the summ of the said L, M, N is equal to the given summ B, I prove thus,

108. Because by *Constr.* in 106° , } $L + N = F$.
 109. Therefore by adding M to each part, } $L + M + N = F + M$.
 110. But by *Constr.* in 78° , } $B - M = F$.
 111. And consequently, } $B = F + M$.
 112. Therefore from 109° and 111° , (*per Ax. 1. Chap. 2.*) } $L + M + N = B$.

Which was to be proved.

113. It remains only to shew, that the summ of the Squares of the said right lines L, M and N is equal to the Square of the given line C; but that is made manifest by the following Demonstration, which is formed out of the steps of the preceding Resolution, in a retrograde (not in a direct) order.

114. . . . *Req. demonstr.* } $\square L + \square M + \square N = \square C$.

Demonstration.

115. Because by *Constr.* in 77° , } $2 B \cdot G :: H \cdot M$.
 That is, in 16° , (the last step of the Resolution,) } $2 b \cdot b + c :: b - c$.
 116. Therefore from 115° , (*per prop. 16. Elem. 6.*) } $2 \square BM = \square GH$.
 That is, in 15° , } $2ba = b + c \times b - c$.
 117. But from 81° , } $\square B - \square C = \square GH$.
 That is, in 14° , } $bb - cc = b + c \times b - c$.
 118. Therefore from 116° and 117° , (*per Ax. 1. Chap. 2.*) } $\square B - \square C = 2 \square BM$.
 That is, in 13° , } $bb - cc = 2ba$.
 119. And by addition of $\square C$ to each part in 118° , } $\square B = \square C + 2 \square BM$.
 That is, in 12° , } $bb = cc + 2ba$.
 120. And by subtraction of $2 \square BM$ from each }
 part in 119° , } $\square B - 2 \square BM = \square C$.
 That is, in 11° , } $bb - 2ba = cc$.
 121. By *Constr.* in 106° , } $L \cdot M :: M \cdot N$.
 122. And it hath been proved in 112° , that . . . } $L + M + N = B$.
 123. Therefore out of 121° and 122° (*per Theor. 1. in 17^\circ of this Probl. 8.*) } $\square B - 2 \square BM = \square L + \square M + \square N$.
 124. Therefore from 120° and 123° , (*per Ax. 1. Chap. 2.*) } $\square L + \square M + \square N = \square C$.

Which was to be Dem.

Thus you have seen the preceding *Probl. 8.* effected and demonstrated Synthetically, that is, by working and arguing with given Quantities only; but the substance of the Effectation and Demonstration was first found out Analytically.

125. But me-thinks I hear the Learner object, That although the preceding Resolution of *Probl. 8.* be clear and easie, yet the prescribed Determination, especially its latter part, and the reason thereof seems hard to be found out: For answer to this, I say, That the first part of the Determination is apparent in the last step of the Resolution, in which also the latter part of the Determination is tacitly implied, and may thence be easily inferr'd thus;

126. Suppose (as before in the Resolution,) b to represent the given summ of three Proportionals, and cc a given Square equal to the summ of the Squares of those Proportionals, then the mean Proportional, (by the Canon in the 19th step,) will be found equal to $\frac{bb - cc}{2b}$.
127. Which mean Proportional subtracted from b the given summ of all the three Proportionals, gives the summ of the extremes, to wit, $\frac{bb + cc}{2b}$.
- Therefore half the summ of the extremes is $\frac{bb + cc}{4b}$.

128. Now there is given the mean of three Proportionals, as also the summ of the extremes, to find out the extremes severally, which may be done by *Probl. 13. Chap. 5.* of this Book, provided that the mean doth not exceed half the summ of the extremes; that such Effectation therefore may be possible, this following Determination is necessary for limiting the given lines (or numbers) b and c ; viz.

$$\frac{bb - cc}{2b}, \text{ or } \frac{2bb - 2cc}{4b} \text{ not } \sqsubset \frac{bb + cc}{4b}.$$

129. Therefore, each part being multiplied by $4b$, $\dots \dots \dots \succ 2bb - 2cc \text{ not } \sqsubset bb + cc.$
130. And by adding $2cc$ to each part in 129^o, $\dots \dots \dots \succ 2bb \text{ not } \sqsubset bb + 3cc.$
131. And by subtracting bb from each part in 130^o, $\dots \dots \dots \succ bb \text{ not } \sqsubset 3cc.$
132. And by Application of b to each part in 131^o, $\dots \dots \dots \succ b \text{ not } \sqsubset \frac{3cc}{b}.$

Which is the latter part of the Determination added to *Probl. 8.* for limiting the lines given, that they may be capable of effecting or solving the Problem. If therefore any one neglecting this Determination, should rashly give $10 = b$ for the summ of the three continual Proportionals, and $32 = cc$ for the summ of their Squares, the said Determination shews that 'tis impossible to find out three such Proportionals: For although $10 \sqsubset \sqrt{32}$, that is, $b \sqsubset c$, as the first part of the Determination requires, yet $100 \sqsubset 96$, that is, bb is greater than $3cc$, which contradicts the latter part of the Determination: But the better to shew the impossibility, let the Canon in the forgoing 19th step be used to find out a mean Proportional by the help of 10 given for the summ of three continual Proportionals, and of 32 given for the summ of their Squares, so you will find $3\frac{2}{5}$ for the mean Proportional, which subtracted from 10 the summ of all the three Proportionals, leaves $6\frac{2}{5}$ for the summ of the extremes; but 'tis manifest by *Probl. 14. Chap. 5.* that $6\frac{2}{5}$ cannot possibly be divided into two such parts that $3\frac{2}{5}$ may be a mean Proportional between the parts, because the mean $3\frac{2}{5}$ exceeds the half of $6\frac{2}{5}$ the summ of the extremes. This may suffice to shew that Determinations are necessary to be added to such Problems as are not universal, that is, such as cannot be solved or effected by numbers or right lines given at random, but with due restrictions or limitations.

Probl. IX.

The Base and leggs of a plain Triangle which hath unequal acute angles at the Base being severally given, to find the segments of the Base which are made by the falling of the Perpendicular from the angle opposite to the Base.

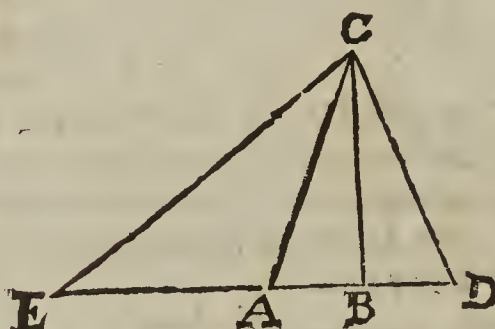
Note. The Geometrical effectation of this Problem is very easie, for if (by *prop. 12. Elem. 1.*) a Perpendicular be let fall upon the Base from the opposite angle, it will divide the Base into the two segments required; but the scope of the Problem is to find out a Theorem whereby the said segments, and consequently the Perpendicular and Area of the Triangle may be discovered Arithmetically.

Suppos.

1. EDC is a Δ , whose angles E and D are acute.
2. $\angle D \sqsubset \angle E$, therefore,
3. EC \sqsubset DC, (per *prop. 19. Elem. 1.*)
4. CB \perp ED.
5. BE and BD are the segments of the Base ED.
6. BE \sqsubset BD.
7. BA = BD.

- 7. $BA = BD$.
- 8. $AE = BE - BD$ (BA) the difference of the segments.
- 9. $b = ED$ the Base is given.
- 10. $c = EC$ the greater legg is given.
- 11. $d = DC$ the lesser legg is given.

Req. to find BE and BD .



ED =	63
EC =	60
DC =	39
EA =	33
BA =	15 = BD
BE =	48
BC =	36
Δ EDC =	1134

Resolution.

- 12. Put a for the difference of the segments of the Base, viz. $\} a = AE$.
- 13. Then (per Theor. 9. Chap. 4.) the greater segment of the Base shall be $\} \frac{1}{2}b + \frac{1}{2}a = BE$.
- 14. Likewise by the same Theor. the lesser segment shall be $\} \frac{1}{2}b - \frac{1}{2}a = BD$.
- 15. Therefore the Square of the greater segment shall be $\} \frac{1}{4}bb + \frac{1}{2}ba + \frac{1}{4}aa (= \square BE.)$
- 16. And the Square of the lesser segment shall be $\} \frac{1}{4}bb - \frac{1}{2}ba + \frac{1}{4}aa (= \square BD.)$
- 17. By 47. prop. 1. Elem. $\} \square EC = \square BE + \square BC$.
- 18. Therefore by equal subtraction of $\square BE$; $\} \square EC - \square BE = \square BC$.
- 19. That is, in the letters belonging to the Resolution, $\} cc - \frac{1}{4}bb - \frac{1}{2}ba - \frac{1}{4}aa = \square BC$.
- 20. In like manner, (per 47. prop. 1. Elem.) $\} \square DC - \square DB = \square BC$.
- 21. That is, in the letters of the Resolution, $\} dd - \frac{1}{4}bb + \frac{1}{2}ba - \frac{1}{4}aa = \square BC$.
- 22. Therefore out of 19° and 21°, (per 1. Axiom. 2. Chap.)

$$cc - \frac{1}{4}bb - \frac{1}{2}ba - \frac{1}{4}aa = dd - \frac{1}{4}bb + \frac{1}{2}ba - \frac{1}{4}aa$$
- 23. Therefore from 22°, by equal addition and subtraction, $\} cc - dd = ba$.
- 24. Therefore from 23°, by Application of each part of that Equation to b , $\} \frac{cc - dd}{b} = a$.
- 25. Moreover, because by Theor. 8. Chap. 4. $\} \frac{cc - dd}{c + d} = \frac{c - d}{c - d} \times \frac{c - d}{c + d}$.
- 26. Therefore out of 23° and 25°, $\} \frac{cc - dd}{c + d} \times \frac{c + d}{c - d} = ba$.
- 27. Which last Equation may be resolved into this Analogy, $\} b \cdot c + d :: c - d \cdot a$.

Out of 24° ariseth

THEOR. 1.

- 28. If the difference of the Squares of the leggs of a plain Triangle which hath unequal acute angles at the Base, be applied to, (or divided by) the Base; the line or number arising shall be equal to the difference of the segments of the Base which are made by the falling of the Perpendicular from the angle opposite to the Base.

Out of 27° ariseth

THEOR. 2.

- 29. As the Base of a plane Triangle which hath unequal acute angles at the Base, is to the sum of the leggs; so is the difference of the leggs, to the difference of the segments of the Base which are made by the falling of the Perpendicular from the angle opposite to the Base.
- 30. Then the Base being given, as also the difference of the segments of the Base, the segments themselves shall be given severally; for (by Theor. 9. Chap. 4.) half the Base, together with half the difference of the segments shall be equal to the greater segment; and half the Base less by half the difference of the segments shall be equal to the lesser segment. Lastly, the square Root of the excess of the Square of the greater leg above the Square of the greater segment of the Base; or the square Root of the excess of the Square of the lesser leg above the Square of the lesser segment, will give the Perpendicular, which multiplied into half the Base, gives the Area of the Triangle.

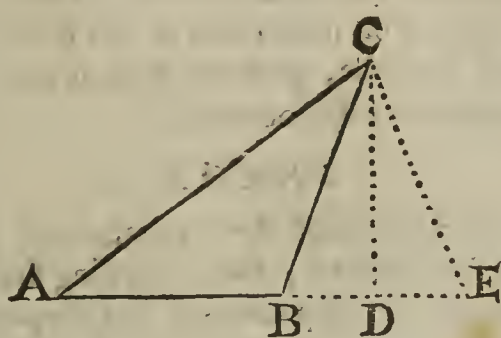
Probl. X.

Probl. X.

The Base and leggs of a plane Triangle obtufangled at the Base being severally given, to find out Arithmetically the distance from the foot of the Perpendicular falling without the Triangle, to each end of the Base: And consequently the Perpendicular and Area of the Triangle.

Suppos.

1. ABC is a Δ obtufangled at B, therefore
2. AC \sqsupset BC.
3. ABE is a right line.
4. CD \perp AE.
5. DE = DB.
6. AE = AB \div 2 BD, (2 DE.)
7. b = AB the Base is given.
8. c = AC the greater legg is given.
9. d = BC the lesser legg is given.



- AB = 33
- AC = 60
- BC = 39
- AE = 63
- BD = 15 = DE
- DC = 36
- $\Delta ABC = 594$

Req. to find DB, and DA.

Resolution.

10. Suppose $\checkmark a = AE.$
11. Then because $\checkmark BE = AE - AB.$
12. Therefore the halves of all in 11 $^{\circ}$ being taken, . . . $\checkmark \frac{1}{2}BE$ (or BD) $= \frac{1}{2}AE - \frac{1}{2}AB.$
13. That is, in the letters belonging to the Resolution, . . $\checkmark \frac{1}{2}a - \frac{1}{2}b = BD.$
14. And because $\checkmark AB \div BD = AD.$
15. Therefore in the letters of the Resolution, . . . $\checkmark b \div \frac{1}{2}a - \frac{1}{2}b = AD.$
16. That is, $\checkmark \frac{1}{2}a \div \frac{1}{2}b = AD.$
17. The Square whereof is $\checkmark \frac{1}{4}aa - \frac{1}{2}ba + \frac{1}{4}bb = \square AD.$
18. And the Square of the Equation in 13 $^{\circ}$ is . . . $\checkmark \frac{1}{4}aa - \frac{1}{2}ba + \frac{1}{4}bb = \square BD.$
19. By 47. prop. 1. Elem. $\checkmark \square AC = \square AD + \square DC.$
20. Therefore by equal subtraction of $\square AD$, . . . $\checkmark \square AC - \square AD = \square DC.$
21. That is, in the letters of the Resolution, . . . $\checkmark cc - \frac{1}{4}aa - \frac{1}{2}ba - \frac{1}{4}bb = \square DC.$
22. In like manner, by 47. prop. 1. Elem. . . . $\checkmark \square BC - \square BD = \square DC.$
23. That is, in the letters of the Resolut. . . . $\checkmark dd - \frac{1}{4}aa + \frac{1}{2}ba - \frac{1}{4}bb = \square DC.$
24. Therefore out of 21 $^{\circ}$ and 23 $^{\circ}$, (per 1. Axiom. 2. Chap.)

$$cc - \frac{1}{4}aa - \frac{1}{2}ba - \frac{1}{4}bb = dd - \frac{1}{4}aa + \frac{1}{2}ba - \frac{1}{4}bb.$$

25. Therefore out of 24 $^{\circ}$, by equal addition and subtraction, $\checkmark cc - dd = ba.$
26. Therefore each part of the last Equation being applied to b, $\checkmark \frac{cc - dd}{b} = a.$
27. Moreover, because by Theor. 8. Chap. 4. . . . $\checkmark \frac{cc - dd}{c + d} = \frac{b + c \times b - c}{c - d} = ba.$
28. Therefore out of 25 $^{\circ}$ and 27 $^{\circ}$, $\checkmark \frac{cc - dd}{c + d} = \frac{b + c \times b - c}{c - d} = ba.$
29. Which last Equation may be resolved into this Analogy, $\checkmark b \cdot c + d :: c - d \cdot a.$

Out of 26 $^{\circ}$ arifeth

THEOR. 1.

30. If the difference of the Squares of the leggs of a plane Triangle obtufangled at the Base, be applied to (or divided by) the Base, the line (or number) arising shall be equal to the summ of the Base and the double of the distance from the obtuse angle to the foot of the Perpendicular, falling upon the Base produced without the Triangle.

Out of 29 $^{\circ}$ arifeth

THEOR. 2.

31. As the Base of a plane Triangle obtufangled at the Base, is to the summ of the leggs; so is the difference of the leggs, to a fourth Proportional, which is compos'd of the Base and the double of the distance from the obtuse angle to the foot of the Perpendicular, falling without the Triangle upon the Base prolonged.

32. Then

32. Then the Base, and the line compos'd of the Base and the double distance from the obtuse angle to the foot of the Perpendicular being severally given by either of the said Theorems, the distance from the foot of the Perpendicular to each end of the Base shall be given also; for out of 13° and 16° of the preceding Resolution it is manifest, that half the said compos'd line together with half the Base, is equal to the distance from the foot of the Perpendicular to the remoter end of the Base; and half the said compos'd line less by half the Base, is equal to the distance from the foot of the Perpendicular to the nearer end of the Base. Lastly, the Square Root of the excess of the Square of the lesser leg above the Square of the distance from the foot of the Perpendicular to the obtuse angle, (or nearer end of the Base,) will give the Perpendicular, which multiplied into half the Base, gives the Area of the Triangle.

Here it will not be improper to add the following

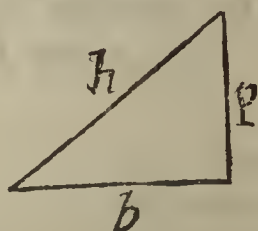
THEOR. 3.

33. If the difference of the Squares of the Hypotenusal and Perpendicular of a right-angled Triangle be applied to the Base, the line arising shall be equal to the Base. Or, As the Base is to the sum of the Hypotenusal and Perpendicular, so is their difference to the Base it self.

This may be demonstrated in manner following.

Suppos.

- 34. h = the Hypotenusal of a right-angled Triangle.
- 35. p = the Perpendicular.
- 36. b = the Base.



Req. demonstr.

37. $\frac{hh - pp}{b} = b$, that is,

38. $b \cdot h + p :: h - p \cdot b$.

Demonstration.

- 39. By 47. prop. 1. Elem. } $hh = bb + pp$.
- 40. Therefore, } $hh - pp = bb$.
- 41. Therefore by Application of each part of the last } $\frac{hh - pp}{b} = b$.
- Equation to b , }
- Which was to be dem.
- 42. Moreover, because by Theor. 8. Chap. 4. . . . } $hh - pp = \square \begin{cases} h+p, \\ h-p. \end{cases}$
- 43. Therefore out of 40° and 42°, (per Ax. 1. } $bb = \square \begin{cases} h+p, \\ h-p. \end{cases}$
- Chap. 2.) }
- 44. Therefore, from the last Equation this Analogy } $b \cdot h + p :: h - p \cdot b$.
- ariseth, }
- Which was to be dem.

From Theor. 1. in 28° of the foregoing Probl. 9. and from Theor. 1, and 3. in 30° and 33° of the preceding Probl. 10. we may deduce the following

COROLLARY.

45. If the three sides of any plane Triangle be severally given in numbers, the kind of every one of the three angles is given also: For supposing any one of the three sides to be called the Base, and the other two the leggs: First, if the leggs be equal to one another, the angles at the Base are equal acute angles; secondly, if the leggs be unequal, and the Quotient that ariseth by dividing the difference of the Squares of the leggs by the Base be less than the Base, then there will be unequal acute angles at the Base; thirdly, if the said Quotient be greater than the Base, then that angle at the Base which is opposite to the greater leg shall be obtuse, and consequently the other two angles acute; and lastly, if the said Quotient be equal to the Base, then the angle at the Base opposite to the greater leg is a right angle.

Or more easily, thus;

If the Square of one of the three sides of a plain Triangle be equal to the sum of the Squares of the other two, then the angle opposite to that side is a right angle; if greater, obtuse; if lesser, acute; as is manifest by prop. 48. Elem. 1. and by prop. 12, and 13. Elem. 2. A Lemma,

A Lemma, useful in the following Probl. 11, and 12.

If within a plane Triangle a right line be drawn parallel to the Base, the Perpendicular falling upon the Base from the opposite angle, shall be to the Base, as that segment of the Perpendicular which is intercepted between the said parallel line and the said angle, is to the said parallel line.

Suppos.

1. CEG is a Δ .
2. NM \parallel CE.
3. GL \perp CE.
4. . . . Req. demonstr. LG . CE :: OG . NM.

Demonstration.

5. By Supposition in 2 $^{\circ}$, } NM \parallel CE.
 6. Therefore (per 29. prop. 1. Elem.) . . . } \angle GNM = \angle GCE.
 7. Likewise } \angle GMN = \angle GEC.
 8. Therefore, (per Coroll. prop. 32. Elem. 1.) } Δ CLG and Δ NOG are equiangular.
 9. Likewise } Δ ELG and Δ MOG are equiangular.
 10. Therefore (per prop. 4. Elem. 6.) . . . } LC . LG :: ON . OG.
 11. Therefore alternately, } LC . ON :: LG . OG.
 12. And by the like argumentation, . . . } LE . OM :: LG . OG.
 13. Therefore out of 11 $^{\circ}$ and 12 $^{\circ}$, (per } LC . ON :: LE . OM.
 11. prop. 5. Elem.) }
 14. Therefore (per 12. prop. 5. Elem.) . . } LC . ON :: LC+LE . ON+OM.
 15. But it hath above been proved in 11 $^{\circ}$, that } LC . ON :: LG . OG.
 16. Therefore out of 14 $^{\circ}$ and 15 $^{\circ}$, (per } LC+LE . ON+OM :: LG . OG.
 11. prop. 5. Elem.) }
 17. That is, (viewing the Fig.) } CE . NM :: LG . OG.
 18. Therefore alternely, } CE . LG :: NM . OG.
 19. Therefore inverfly, } LG . CE :: OG . NM.
- Which was to be Dem.

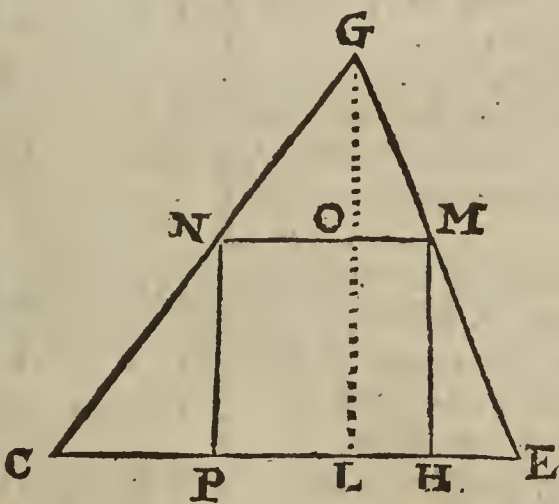
Probl. XI.

To inscribe a Square in a Triangle given.

Note. If the given Triangle be obtufangled, one of the sides of the Square to be inscribed must necessarily be a segment of that side of the Triangle which is opposite to the obtuse angle, for otherwise, (as may easily be perceived) all the angular points of the Square cannot lye in the sides of the Triangle; but if the Triangle given be right-angled or acute-angled, the Square may stand upon any one of the three sides as a Base.

Suppos.

1. CEG is a Δ given.
2. \angle GCE is not obtuse.
3. \angle GEC is not obtuse.
4. GL \perp CE.
5. $b = CE$ the Base is given.
6. $p = LG$ the Perpendic. is given.
7. . . . Req. to inscribe \square PNMH in the Δ CEG.



- CE = 182
- CG = 195
- EG = 169
- GL = 156
- CL = 117
- LE = 65
- NP = 84
- NM = 84

Resolution.

8. Supposing PNMH to be the Square required, }
 put a to represent its side, viz. } $a = LO$, or NM, or NP.
9. It is manifest by the foregoing Lemma, that . } LG . CE :: OG . NM.
10. That is, in the letters belonging to the Resolut. } $p . b :: p - a . a$.
11. Therefore by Composition of Reason, . . . } $p + b . b :: p . a$.

Hence this

C A N O N.

12. As the summ of the Perpendicular and Base is to the Base, so is the Perpendicular to the side of the inscribed Square sought.

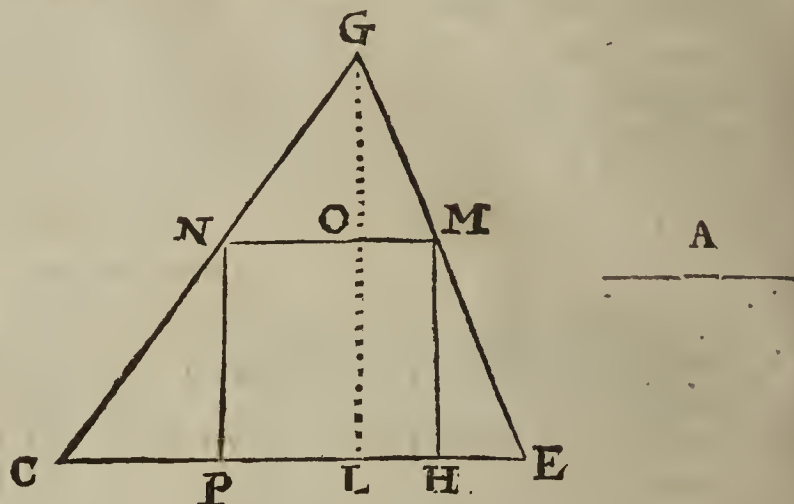
The Composition of Probl. 11.

Suppos.

- 13. CEG is a Δ given.
- 14. $\angle GCE$ is not obtuse.
- 15. $\angle GEC$ is not obtuse.

Req.

To inscribe \square PNMH in Δ CEG.



Construction.

- 16. First, make $GL \perp CE$.
- 17. Then making $LG \perp CE$ the first of four Proportionals, CE the second, and LG the third; find (per Probl. 8. Chap. 5.) a fourth Proportional, which suppose to be the line A, therefore,
 $LG \perp CE \cdot CE :: LG \cdot A$.
- 18. From LG cut off $LO = A$, which is possible to be done, for in the Analogy in 17° the first Proportional is manifestly greater than the second, therefore the third Proportional shall be greater than the fourth, (per Schol. prop. 14. Elem. 5.) that is, $LG \perp A$; therefore 'tis possible to cut off from LG a segment equal to A, as LO.
- 19. By the point O draw $NQM \parallel CE$.
- 20. Also draw NP and MH $\parallel OL$.
- 21. I say PNMH is the Square required.
- 22. . . . Req. demonstr. that PNMH is a \square ;

Demonstration.

- 23. By Constr. in 17°, } $LG \perp CE \cdot CE :: LG \cdot A$.
- 24. And by Constr. in 18°, } $LO = A$.
- 25. Therefore out of 23° and 24°, } $LG \perp CE \cdot CE :: LG \cdot LO$.
- That is, in 11°, } $p \perp b \cdot b :: p \cdot a$.
- 26. Therefore from 25°, by Di- } $LG \cdot CE :: LG - LO \cdot LO$.
- vision of Reason, } $p \cdot b :: p - a \cdot a$.
- 27. It is manifest by the Figure that } $LG - LO = OG$.
- 28. Therefore out of 26° and 27°, } $LG \cdot CE :: OG \cdot LO$.
- 29. By the Lemma prefixt to this } $LG \cdot CE :: OG \cdot NM$.
- Problem, }
- 30. Therefore out of 28° and 29°, } $OG \cdot LO :: OG \cdot NM$.
- (per 11. prop. 5. Elem.) }
- 31. Therefore (per 14. prop. 5. El.) } $LO = NM$.
- 32. By Constr. in 19° and 20°, } PNMH, PNOL and LOMH are Parallelograms.
- and by defin. 35. Elem. 1. } $PN = MH = LO$. Also, $NM = PH$.
- 33. Therefore (per prop. 34. El. 1.) } $PN = NM = MH = PH$.
- 34. And from 31° and 33°, } $\angle GLC = \perp = \angle GLE$.
- 35. And because by Constr. in 16°, } $\angle PNM = \perp = \angle NMH = \angle MHP = \angle HPN$.
- 36. Therefore from 32° and 35°, } (per prop. 29, & 34. Elem. 1.)
- (per prop. 29, & 34. Elem. 1.) } \square
- 37. Therefore from 34° and 36°, } PNMH is a \square .
- (per defin. 29. Elem. 1.) }

Which was to be Dem.

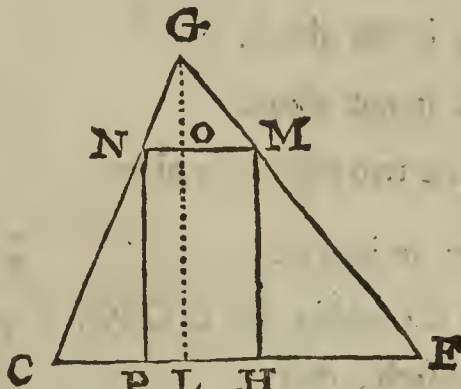
Probl. XII.

Probl. XII.

Within a given Triangle to inscribe a long Square, whose sides shall be in a given Reason.

Suppos.

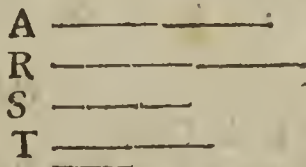
1. CEG is a Triangle given.
2. $\angle GCE$ is not obtuse.
3. $\angle GEC$ is not obtuse.
4. $GL \perp CE$.
5. $b = CE$ the Base is given.
6. $p = LG$ the Perpendicular is given.
7. r and s are the Terms of a given Reason.



- CE = 140
- CG = 130
- EG = 150
- GL = 120
- LE = 90
- LC = 50
- PN = 84
- NM = 42
- R = 2
- S = 1
- T = 60
- A = 84

Req. to inscribe

8. PNMH a \square within the $\triangle CEG$, and such, that
9. $PN \cdot NM :: R \cdot S$.



Resolution.

10. Supposing PNMH to be the Rectangle sought, } $a = PN$, or LO .
11. Therefore out of 6° , and 10° , } $p - a = OG$.
12. It is manifest by the Lemma prefixt to the foregoing Probl. 11. that } $LG \cdot CE :: OG : NM$.
13. That is, in the letters belonging to the Resolution, } $p \cdot b :: p - a \cdot \frac{bp - ba}{p}$.
14. It is required in 9° , that } $R \cdot S :: PN \cdot NM$.
15. That is, } $r \cdot s :: a \cdot \frac{bp - ba}{p}$.
16. Therefore the two latter terms of the last Analogy being drawn into p , this Analogy will arise, } $r \cdot s :: pa \cdot bp - ba$.
17. Now to avoid an Equation between Solids, (for 'tis improper to introduce Solids in arguing about a plane Problem,) let it be made as r to s , so p to a fourth Proportional, which suppose to be t , therefore } $r : s :: p : t$.
18. Therefore from the Analogies in 16° and 17° , this will arise, (per 11. prop. 5. Elem.) } $p \cdot t :: pa \cdot bp - ba$.
19. And by drawing a as a common altitude into p and t severally, this Analogy will be manifest, (per 1. prop. 6. Elem.) } $p \cdot t :: pa \cdot ta$.
20. Therefore from 18° and 19° this Analogy will arise, (per 11. prop. 5. Elem.) } $pa \cdot bp - ba :: pa \cdot ta$.
21. Therefore out of 20° , (per 14. prop. 5. Elem.) } $bp - ba = ta$.
22. Therefore out of 21° , by addition of ba to each part, } $bp = ba + ta$.
23. Which last Equation gives this following Analogy, (per 14. prop. 6. Elem.) } $b + t \cdot b :: p \cdot a$.

Out of 17° and 23° ariseth this

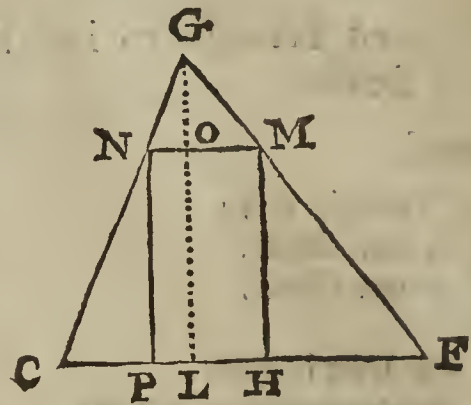
CANON.

24. Let it be made as R to S , so the Perpendicular GL to a fourth Proportional T ; also let it be made as the summ of the Base CE and the fourth Proportional T to the Base CE ; so the Perpendicular GL to a fourth Proportional A , which shall be equal to the altitude PN of the Rectangle required to be inscribed. Lastly, as R to S , so A to NM , to wit, that side of the Rectangle which must be parallel to the Base CE .

The Composition of Probl. 12.

Suppos.

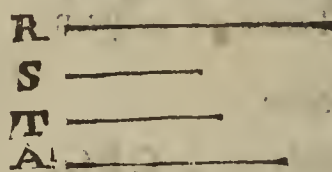
- 25. CEG is a Δ given.
- 26. $\angle GCE$ is not obtuse.
- 27. $\angle GEC$ is not obtuse.
- 28. R and S are two right lines given.



CE =	140
CG =	130
EG =	150
GL =	120
LE =	90
LC =	50
PN =	84
NM =	42
R =	2
S =	1
T =	60
A =	84

Req. to inscribe

- 29. PNMH a \square within the Δ CEG, and such, that
- 30. $PN : NM :: R : S$.



Construction.

- 31. First, make $GL \perp CE$.
- 32. Then making R the first of four Proportionals, S the second, and GL the third, find a fourth Proportional, which suppose to be T, therefore
 $R \cdot S :: GL \cdot T$.
- 33. Again, making $CE + T$ the first of four Proportionals, CE the second, and GL the third, find a fourth Proportional, let it be A, therefore
 $CE + T : CE :: GL : A$.
- 34. From GL cut off $LO = A$, which is possible to be done; for in the Analogy in 33° the first Proportional is manifestly greater than the second, therefore (per Schol. of 14. prop. 5. Elem.) the third shall be greater than the fourth, therefore $GL > A$.
- 35. By the point O draw $NOM \parallel CE$, also by the points N and M draw NP and $MH \parallel OL$.
- 36. I say PNMH is the right-angled Parallelogram required to be inscribed. Now we must shew that PN is to NM as R to S; also that the angles NPH, MHP, NMH and PNM are right angles. First, that the side PN is to the side NM, as the line R to the line S, I shall make manifest by a retrograde repetition of the steps of the foregoing Resolution of Probl. 12.

37. . . . Req. demonstr. $R \cdot S :: PN \cdot NM$.

Demonstration.

- 38. Because by Construction in 33°, $CE + T : CE :: GL : A$.
That is, in 23°, (the last step of the Resolut.) $b + c : b :: p : a$.
- 39. And by Constr. in 34°, $LO = A$.
- 40. Therefore out of 38° and 39°, $CE + T : CE :: GL : LO$.
- 41. Therefore (per 16. prop. 6. Elem.) $\square GE, GL = \square CE, LO + \square T, LO$.
That is, in 22°, $bp = ba + ta$.
- 42. Therefore out of 41°, by equal subtraction of $\square CE, LO$, $\square CE, GL - \square CE, LO = \square T, LO$.
That is, in 21°, $bp - ba = ta$.
- 43. From 42°, (per 7. prop. 5. Elem.) this subsequent Analogy will be manifest,
 $\square GL, LO \cdot \square CE, GL - \square CE, LO :: \square GL, LO \cdot \square T, LO$.
That is, in 20°, $pa \cdot bp - ba :: pa \cdot ta$.
- 44. And by reason of the common altitude LO this Analogy is manifest, (per 1. prop. 6. El.)
 $GL \cdot T :: \square GL, LO \cdot \square T, LO$.
That is, in 19°, $p \cdot t :: pa \cdot ta$.
- 45. Therefore from the Analogies in 43° and 44°, (per 11. prop. 5. Elem.)
 $GL \cdot T :: \square GL, LO \cdot \square CE, GL - \square CE, LO$.
That is, in 18°, $p \cdot t :: pa \cdot bp - ba$.
- 46. By Constr. in 32°, $GL \cdot T :: R \cdot S$.
That is, in 17°, $p \cdot t :: r \cdot s$.

47. There-

47. Therefore from 45° and 46° , (per 11. prop. 5. Elem.)
 $R \cdot S :: \square GL, LO \cdot \square CE, GL - \square CE, LO.$
 That is, in 16° , $r \cdot s :: pa \cdot bp - ba.$
48. It is manifest by the Figure, that $GL - LO = OG.$
49. Therefore by taking CE as a common altitude, $\square CE, GL - \square CE, LO = \square CE, OG.$
50. Therefore out of 47° and 49° , by exchanging equal Rectangles, this Analogy is manifest, $R \cdot S :: \square GL, LO : \square CE, OG.$
51. By the Lemma placed next before Probl. 11. it's manifest that $GL \cdot CE :: OG \cdot NM.$
52. And consequently, (per 16. prop. 6. El.) $\square GL, NM = \square CE, OG.$
53. Therefore out of 50° and 52° , by exchanging equal Rectangles, $R \cdot S :: \square GL, LO \cdot \square GL, NM.$
54. Therefore, the common altitude GL being cast away, $R : S :: LO \cdot NM.$
55. And because by Constr. in 35° , (and by 35. defin. 1. Elem.) PNOL is a Parallelogram, therefore (per 24. prop. 1. Elem.) $PN = LO.$
56. Therefore out of 54° and 55° , by setting PN in the place of LO, $R \cdot S :: PN \cdot NM.$

Which was to be Demonstr.

But that the angles of the inscribed Parallelogram PNMH are right angles, is easie to prove out of the Construction, after the same manner as in the preceding Probl. 11. from 31° to 37° .

Probl. XIII.

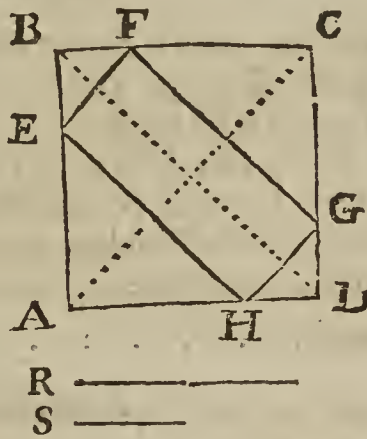
Within a given Square to inscribe a long Square, whose sides shall be in a given Reason, and parallel to the Diameters of the Square.

Suppos.

1. ABCD is a \square given.
2. $b = BC$ (or BA) is given.
3. r and s are the Terms of a given Reason.

Req. to inscribe

4. EFGH a \square within the \square ABCD, so, as that the points E, F, G and H may lye in the sides of the \square ABCD; also, that



- $R = 2$
 $S = 5$
 $AB = 28 = BC$
 $BF = 8 = BE$
 $DG = 8 = DH$
 $FC = 20 = CG$
 $AE = 20 = AH$
 $EF = 4\sqrt{8} = HG$
 $FG = 10\sqrt{8} = EH$

5. $EF \parallel AC \parallel HG$; likewise, $FG \parallel BD \parallel EH$. Also;
6. $EF \cdot FG :: r \cdot s.$

Preparat.

7. Suppose EFGH to be the \square required, then according to the tenour of the Problem, $EF \parallel AC.$
8. Therefore (per 29. prop. 1. Elem.) $\angle BEF = \angle BAC.$
9. Likewise, $\angle BFE = \angle BCA.$
10. Therefore equiangular are $\triangle BEF$ and $\triangle BAC.$
11. Therefore (per 4. prop. 6. Elem.) $BA \cdot BC :: BE \cdot BF.$
12. Therefore alternately, $BA \cdot BE :: BC \cdot BF.$
13. But by Supposition in 1 $^\circ$, $BA = BC.$
14. Therefore (per 14. prop. 5. Elem.) $BE = BF.$
15. And because by Suppos. in 1 $^\circ$, $\angle EBF = \angle C.$
16. Therefore out of 14 $^\circ$ and 15 $^\circ$, (per 47. prop. 1. Elem.) $2 \square BF = \square EF.$
17. And by the like argumentation, $2 \square FC = \square FG.$

Resoln.

Resolution.

- 18. Suppose } $a = BF$, or BE .
- 19. Therefore out of 2° and 18° , } $b - a = FC$, or CG .
- 20. The Square of the segment in 18° is } $aa = \square BF$.
- 21. The Square of the segment in 19° is } $bb - 2ba + aa = \square FC$.
- 22. Therefore out of 20° and 16° , } $2aa = \square EF$.
- 23. Also out of 21° and 17° , } $2bb - 4ba + 2aa = \square FG$.
- 24. Therefore out of 22° , } $\sqrt{2aa} = EF$.
- 25. And out of 23° , } $\sqrt{2bb - 4ba + 2aa} = FG$.
- 26. Therefore according to the condition in the Problem,

$$r \quad s \quad :: \quad \sqrt{2aa} \quad \sqrt{2bb - 4ba + 2aa}.$$

- 27. But their Squares also are Proportionals, therefore

$$rr \quad ss \quad :: \quad 2aa \quad 2bb - 4ba + 2aa.$$

- 28. And because the two latter Terms of the last Analogy are in the same Reason as their halves, therefore

$$rr \quad ss \quad :: \quad aa \quad bb - 2ba + aa.$$

- 29. But the Roots of the four last Proportionals are also }
Proportionals, therefore } $r \quad s \quad :: \quad a \quad b - a$,

- 30. Therefore inverfly, } $s \quad r \quad :: \quad b - a \quad a$.

- 31. Therefore by *Composition of Reason*, } $s + r \quad r \quad :: \quad b \quad a$ (BF.)

- 32. Also out of 29° , by *Compof. of Reason*, } $r + s \quad s \quad :: \quad b \quad FC$.

From the premisses arifeth this **CANON.**

- 33. By *Probl. 1.* of this *Chapt.* divide the side of the given Square into two segments that may be one to the other in the given Reason; then find out the side of a Square equal to the double Square of the greater segment, also the side of a Square equal to the double Square of the lesser segment; so shall those sides be the length and bredth of the desired Rectangle, (or long Square,) which may be inscribed within the given Square by this following

Construction.

- 34. First divide BC into two parts in F , that may }
be one to the other as R to S , (*per Probl. 1.* of } $R \quad S \quad :: \quad BF \quad FC$.
this *Chapt.*) *viz.* suppose it be made as . . . }
- 35. Then let $BE = BF$; also $DG = DH = BF$, and draw the right lines $EF, FG,$
 GH and HE , so shall $EFGH$ be the long Square given to be inscribed: Now we
must shew that it will satisfie the Problem propounded; first, that EF is to FG as R to S ,
I shall demonstrate by a retrograde repetition of the steps of the preceding Resolution.

- 36. . . . *Req. demonstr.* } $R \quad S \quad :: \quad EF \quad EG$.

Demonstration.

- 37. Because by *Constr.* in 34° , } $R \quad S \quad :: \quad BF \quad FC$.
- 38. Therefore (*per prop. 22. Elem. 6.*) } $\square R \quad \square S \quad :: \quad \square BF \quad \square FC$.
- 39. And by taking the doubles of the two latter }
Terms of the last preceding Analogy, . . . } $\square R \quad \square S \quad :: \quad 2\square BF \quad 2\square FC$.
- 40. By *Constr.* in 35° , } $BE = BF$.
- 41. And by *Suppos.* } $\sphericalangle EBF = \sphericalangle$.
- 42. Therefore from 40° and 41° , (*per prop. 47.* }
Elem. 1.) } $\square EF = 2\square BF$.
- 43. Likewise, } $\square FG = 2\square FC$.
- 44. Therefore out of $39^\circ, 42^\circ$ and 43° , by ex- }
change of equal quantities, } $\square R \quad \square S \quad :: \quad \square EF \quad \square FG$.
- 45. Therefore from 44° , (*per prop. 22. Elem. 6.*) } $R \quad S \quad :: \quad EF \quad FG$.

Which was to be Dem.

It remains to demonstrate that the quadrilateral Figure $EFGH$ is right-angled at E, F, G and H .

- 46. . . . *Req. demonstr.* . . . $\sphericalangle EFG = \sphericalangle = \sphericalangle FGH = \sphericalangle GHE = \sphericalangle HEF$.
Demons.

Demonstration.

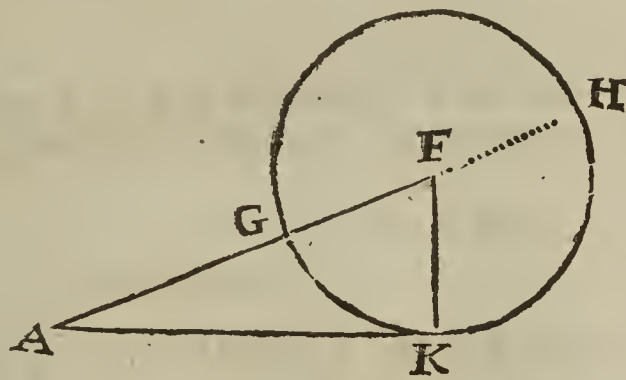
- 47. Because by Constr. in 35° , } $\therefore BF = BE.$
- 48. And by Suppos. } $\angle EBF = \perp.$
- 49. Therefore (per Coroll. prop. 32. Elem. 1.) . . . } $\angle BFE = \frac{1}{2}\perp.$
- 50. Likewise } $\angle CFG = \frac{1}{2}\perp.$
- 51. Therefore from 49° and 50° , (per Coroll. } $\angle EFG = \perp.$
prop. 13. Elem. 1.) }
- 52. And by the like argumentation, } $\angle FGH, \angle GHE, \angle HEF$ are $\perp.$
Which was to be Dem. Therefore the Problem is satisfied.

Probl. XIV.

In a right-angled Triangle, one of the sides about the right angle being given, as also the summ of the other side about the right angle and the Hypothenufal, to find out the Triangle. But the said summ must be greater than the given side, as is manifest by prop. 22. Elem. 1.

Suppos.

- 1. $\triangle AKF$ is right-angled at K.
- 2. $b = AK$ is given.
- 3. $c = AF + FK$ is given.
- 4. $c > b$.



- $AK = 12$
- $KF = 5$
- $FA = 13$

Req. to find $\triangle AKF$.

Resolution.

- 5. For the difference between the Hypothenufal AF and the Perpendicular FK, put a .
 - 6. Then from 3° and 5° , (per Theor. 9. Chap. 4.) the Hypothenufal shall be $\frac{1}{2}c + \frac{1}{2}a$.
 - 7. And by the same Theorem, the Perpendicular KF shall be $\frac{1}{2}c - \frac{1}{2}a$.
 - 8. Therefore from 6° the Square of the Hypothenufal shall be $\frac{1}{4}cc + \frac{1}{4}aa + \frac{1}{2}ca$.
 - 9. And from 7° the Square of the Perpendicular is $\frac{1}{4}cc + \frac{1}{4}aa - \frac{1}{2}ca$.
 - 10. The difference of those Squares is ca .
 - 11. But (per prop. 47. Elem. 1.) the difference of the Square of the Hypothenufal and the Square of one of the sides about the right angle is equal to the Square of the other side about the right angle, therefore from 10° and 2° , $ca = bb$.
 - 12. Therefore, by resolving the last preceding Equation into Proportionals, $c \cdot b :: b \cdot a$.
 - 13. Therefore out of 12° and 6° , the Hypothenufal is now given, $\frac{1}{2}c + \frac{bb}{2c} = AF$.
viz.
 - 14. And from 12° and 7° , the desired side about the right angle is also given, viz. $\frac{1}{2}c - \frac{bb}{2c} = FK$.
- Out of 12° ariseth this

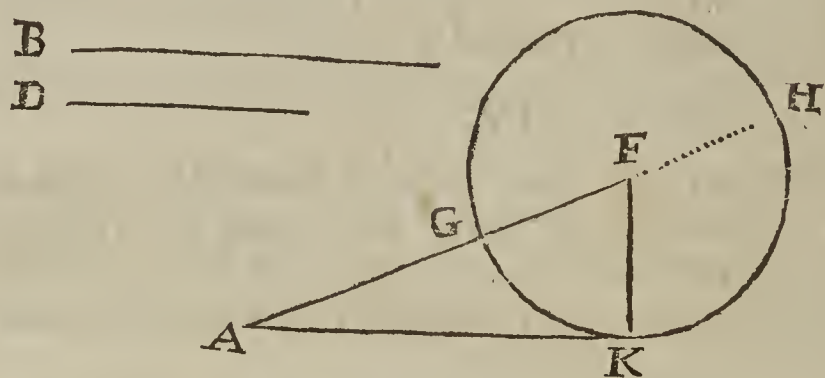
THEOREM.

- 15. One of the sides about the right angle of a right-angled Triangle is a mean Proportional between the summ and difference of the Hypothenufal and the other side about the right-angle. And consequently (per prop. 17. Elem. 1.) the Rectangle of the said summ and difference is equal to the Square of the first mentioned side, or mean Proportional.
- 16. Therefore, the Base of a right-angled Triangle, (that is, which of the two sides about the right angle you please,) being given, as also the summ of the Hypothenufal and Perpendicular, the difference of the Hypothenufal and Perpendicular shall be given also. For by the Theorem above exprest, (which is the same with Prop. 36. Elem. 3. when the

the line there mentioned to cut the Circle passeth through the Center,) the said difference is a third Proportional proceeding from the said Base and Summ.

Therefore also, the said Hypothenufal and Perpendicular shall be given severally, (per Theor. 9. Chap. 4.) for they shall be equal to the given Quantities before exprest in 13° and 14° of this Problem.

The Composition of Probl. 14.



AK = 12 = B
 KF = 5 = FH
 FA = 13
 AH = 18
 D = 8 = AG

Suppos.

- 17. B = the Base of a right-angled Triangle is given.
- 18. AH = the summ of the Hypothenufal and Perpendicular is given.
- 19. AH \square B.

Req. to find out the Δ .

Construction.

- 20. Let it be made as AH to B, so B to a third } AH : B :: B : D.
 Proportional, call it D, therefore . . . }
- 21. From AH cut off AG = D, which is possible to be done if AH \square D; but by
 Supposition AH \square B, therefore from the Analogy in 20°, AH \square D.
- 22. Divide GH into two equal parts in F, therefore FG = FH.
- 23. From F as a Center, at the distance of FG or FH describe the Circle FGKH;
 and (per prop. 17. Elem. 3.) draw the right line AK to touch the said Circle in K;
 draw also the Semidiameter FK; then shall AKF be the Triangle sought. Now we
 must shew that it will satisfie the Problem.
- 24. First then, by Construction in 23°, FH = FK, therefore AF \perp FK = AH the
 given summ of the Hypothenufal and Perpendicular. It remains to shew that AK
 is equal to the given Base B, and that the angle AKF is a right angle; therefore,
- 25. . . Req. demonstr. . . . AK = B. Also, \angle AKF = \perp .

Demonstration.

- 26. Because by Constr. in 20° and 21°, . . . } AH : B :: B : D (AG.)
 - 27. Therefore, (per prop. 17. Elem. 6.) . . . } \square AH, AG = \square B.
 - 28. And because by Constr. in 23°, AK toucheth } \square AH, AG = \square AK.
 \odot FGKH in K, therefore (per prop. 36. Elem. 3.) }
 - 29. Therefore out of 27° and 28°, (per Ax. 1. } \square AK = \square B.
 Chap. 2.) . . . }
 - 30. Therefore, (per Schol. prop. 48. Elem. 1.) } . AK = B. Which was to be dem.
 - 31. And because by Constr. in 23°, AK toucheth } FK \perp KA. Therefore, \angle AKF = \perp .
 \odot FGKH in K, and FK is Radius, therefore, }
 (per prop. 18. Elem. 3.) . . . }
- Which was also to be Dem. Therefore the Problem is satisfied.

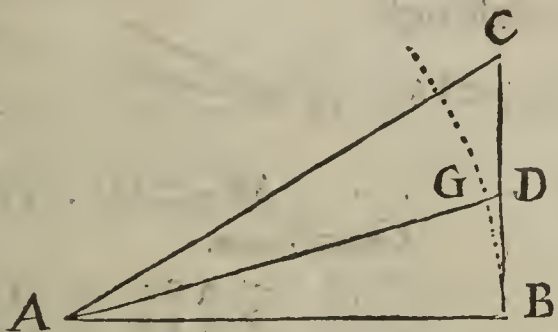
Probl. XV.

The Radius (or Semidiameter) of a Circle being given, as also the Tangent of an arch, to find the Tangent of the double of that arch. But the given Tangent must be less than the Radius.

Suppos.

Suppos.

1. $\triangle ABC$ is right-angled at B.
2. $\angle BAD = \angle DAC = \frac{1}{2} \angle CAB$.
3. $r = AB$ the Radius of a Circle, whose Center is A, is given.
4. $t = BD$ the Tangent of the arch BG, or angle DAB, is given.
5. $BD \perp AB$.



Req. to find BC the Tangent of the angle CAB, that is, the Tangent of $2 \angle DAB$.

Resolution.

6. Put $a = BC$.
7. Therefore out of 4° and 6° , $a - t = DC$.
8. And out of $1^\circ, 3^\circ$ and 6° , (per 47. prop. 1. Elem.) $\sqrt{rr + aa} = AC$.
9. And because by Supposition $\angle BAD = \angle DAC$, therefore (per 3. prop. 6. Elem.) $DC \cdot BD :: AC \cdot AB$.
10. That is, in the letters of the Resolution, $a - t \cdot t :: \sqrt{rr + aa} \cdot r$.
11. But their Squares also shall be Proportionals, (per 22. prop. 6. Elem.) viz. $aa - 2ta + tt \cdot tt :: rr + aa \cdot rr$.
12. Therefore from the Analogy in the 11th step, by Division of Reason, $aa - 2ta \cdot tt :: aa \cdot rr$.
13. Therefore alternly, $aa - 2ta \cdot aa :: tt \cdot rr$.
14. And by reason of the common altitude a , in the two first Terms of this Analogy, it is manifest (per 1. prop. 6. Elem.) that $aa - 2ta \cdot aa :: a - 2t \cdot a$.
15. Therefore from the two last preceding Analogies, (per 11. prop. 5. Elem.) $a - 2t \cdot a :: tt \cdot rr$.
16. Therefore inverfly, $rr \cdot tt :: a \cdot a - 2t$.
17. And converfly, $rr \cdot rr - tt :: a \cdot 2t$.
18. And by doubling the Antecedents, (per Schol. Clavii in prop. 22. Elem. 5.) $2rr \cdot rr - tt :: 2a \cdot 2t$.
19. And inverfly, $rr - tt \cdot 2rr :: 2t \cdot 2a$.
20. But (per 15. prop. 5. Elem.) $t \cdot a :: 2t \cdot 2a$.
21. Therefore out of 19° and 20° , (per 11. prop. 5. Elem.) $rr - tt \cdot 2rr :: t \cdot a$.

Hence this

THEOREM.

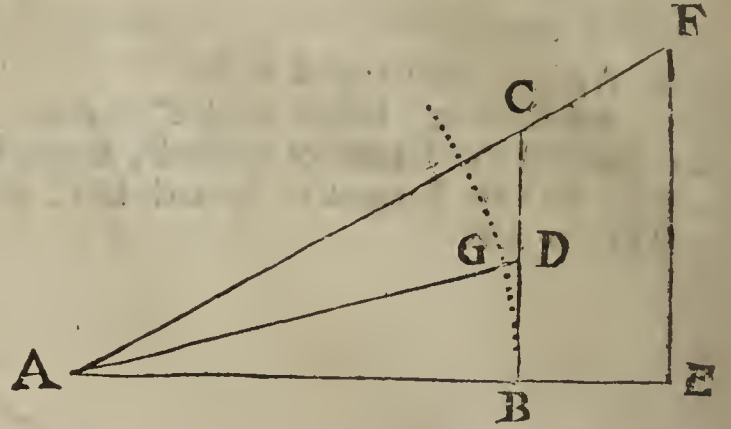
22. As the excess whereby the square of the Radius exceeds the square of the Tangent of an arch less than half a Quadrant, is to the double square of the Radius; so is the said Tangent, to the Tangent of the double of that arch.

Therefore the Radius being given, as also the Tangent of an arch less than 45° degrees, the Tangent of the double of that arch is given also, without extracting any Root.

This Theorem was found out by the Learned Dr. John Pell, who thereby hath clearly shew'd the falsity of Longomontanus's pretended quadrature of the Circle, as appears by the Controversie between them, (printed at Amsterdam in 1647.) where the Demonstrations of divers eminent Mathematicians to manifest the truth of the Theorem are inserted, which may also be easily demonstrated by the steps of the preceding Resolution, by proceeding in a direct order from the beginning to the end thereof; but leaving that Demonstration to the Learner's practice, I shall give another which here follows.

Suppos.

- 23. $\angle ABC$ and $\angle AEF$ are right angles.
- 24. ACF and ABE are right lines.
- 25. $\angle DAB$ less than half a right angle.
- 26. $\angle DAB = \angle DAC = \frac{1}{2} \angle FAE$.



- 27. . . . Req. demonstr. . . . $\square AB - \square BD : 2 \square AB :: BD . BC$.

Demonstration.

- 28. By Supposition, $\angle ABC = \perp = \angle AEF$.
- 29. By Suppos. also, $\angle FAE = 2 \angle DAB$.
- 30. Therefore, (per Theor. 2. angular. sect. Viet.) $AE . EF :: \square AB - \square BD . 2 \square AB, BD$.
- 31. And because from 23° and 24° $\triangle AEF$ and $\triangle ABC$ are like, it's manifest (per 4. prop. 6. Elem.) that . . . $AE . EF :: AB . BC$.
- 32. Therefore from the two last preceding Analogies, (per 11. prop. 5. Elem) . . . $\square AB - \square BD . 2 \square AB, BD :: AB . BC$.
- 33. But by reason of the common altitude $2 AB$, this Analogy is manifest, (per 1. prop. 6. Elem.) viz. . . . $2 \square AB, BD . 2 \square AB :: BD . AB$.
- 34. Wherefore from the two last preceding Analogies, (per 23. prop. 5. Elem agreeable to Definit. 8. of Inordinate proportion in the foregoing Chap. 3.) . . . $\square AB - \square BD : 2 \square AB :: BD . BC$ Which was to be dem.

Illustration by Numbers.

- 35. Let the Radius AB be 100000.
- 36. Then supposing the arch BG or angle DAB to be 15. degrees, its Tangent BD (by the Trigonometrical Canon) is 26795.
- 37. Now to find out BC , the Tangent of 30. degrees, (the double of the given angle,) according to the preceding Theorem, multiply the given Tangent 26795, by the double Square of the Radius, viz. by 20000000000.
- 38. The Product is to be reserved for a Dividend, to wit, 53590000000000.
- 39. Then from the Square of the Radius, viz. from 10000000000.
- 40. Subtract the Square of the given Tangent 26795, viz. 717972025.
- 41. The remainder is a Divisor, to wit, 9282027975.
- 42. Lastly, divide the Dividend in the 38th step, by the Divisor in the 41th, and the Quotient, neglecting the remainder, is 57735.

Which 57735 is the desired Tangent of 30. degrees, the double of the given angle or arch 15. degrees, and exactly agrees with the Trigonometrical Canon, the Radius being assumed 100000.

Probl. XVI.

The Radius of a Circle, and the Tangents of two arches being given severally, to find the Tangents of the summ and difference of those arches. But when the Tangent of the summ is required, the Rectangle of the given Tangents must be less than the Square of the Radius.

In finding out the Demonstration of the Theorem in the 22th step of the foregoing Probl. 15. I discovered the two following Theorems, to solve this Probl. 16, which I have not met with in any Author.

THEOR. I.

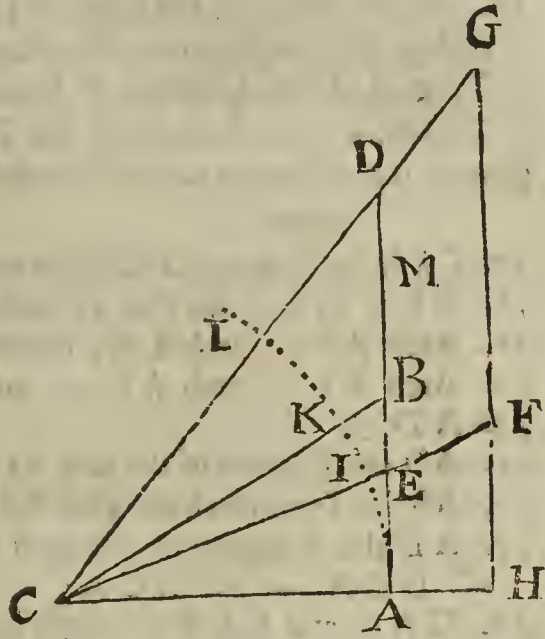
THEOR. I.

As the excess whereby the Square of the Radius exceeds the Rectangle of the Tangents of two arches, is to the Square of the Radius; so is the sum of those Tangents, to the Tangent of the sum of the said arches.

Therefore the Radius of a Circle, and the Tangents of two arches whose sum is less than a Quadrant, being given severally, the Tangent of the sum of those arches shall be given also.

Suppos.

1. CA is the Radius of a Circle whose Center is C.
2. AL an arch less than a Quadrant.
3. Arch LK = arch AI, therefore
4. Arch AL = arch AK + arch AI.
5. AD ⊥ CA.
6. AB the Tangent of the arch AK, or ∠BCA.
7. AE the Tangent of the arch AI, or ∠ECA.
8. AD the Tangent of the arch AL, or of ∠BCA + ∠ECA.
9. CAH and CDG are right lines.
10. HG ⊥ CAH.
11. BM = AE, therefore AM = BA + AE.



12. □CA - □BA, AE . □CA :: AM . AD.
BA + AE

Req. demonstr.

13. Because by *Supposition*, } ∠CAD = ∠ = ∠CHG.
14. Also by *Suppos.* in the right-angled Triangles } ∠GCH = ∠BCA + ∠ECA.
- GCH, BCA and ECA, }
15. Therefore, (per *Theor. 2. Angular. section. Viet.*)

$$CH . HG :: \square CA - \square BA, AE . \square AM, CA.$$

16. And because by *Suppos.* in 5°, 9° and 10°, ΔCAD and ΔCHG are equiangular; therefore, (per 4. prop. 6. Elem.)

$$CH . HG :: CA . AD.$$

17. Therefore from the two last preceding Analogies, (per 11. prop. 5. Elem.)

$$\square CA - \square BA, AE . \square AM, CA :: CA . AD.$$

18. But by reason of the common altitude CA this following Analogy is manifest; (per 1 prop. 6. Elem.)

$$\square AM, CA . \square CA :: AM . CA.$$

19. Therefore from the two last preceding Analogies, (per 23. prop. 5. Elem. agreeable to *Defin. 8.* of Inordinate proportion, in the foregoing *Chap. 3.*)

$$\square CA - \square BA, AE . \square CA :: AM . AD.$$

Which was to be dem.

Illustration by Numbers.

20. Let the Radius CA be } 100000.
21. Then supposing the arch AK to be 30. degrees, its Tangent }
AB by the Trigonometrical Canon is } 57735.
22. Likewise supposing the arch AI to be 20. degrees, its }
Tangent AE is } 36397.
- Now find out AD the Tangent of the arch AL, 50. degrees, viz. the Tangent of the
sum of the given arches AK, 30. degrees, and AI, (or KL,) 20. degrees, the Ope-
ration, according to the preceding *Theor. 1.* will be as it here follows, viz.
23. Multiply the sum of the given Tangents 57735 and }
36397, to wit, } 94132.
24. By the Square of the Radius, viz. by } 10000000000.
22. The Product shall be a Dividend, to wit, } 94132000000000.
26. Then from the said Square of the Radius in the 24th step, }
subtract the Product of the given Tangents, to wit, . . . } . . . 2101380795.

27. The remainder is a Divisor, to wit, } 7898619205.
 28. Lastly, divide the Dividend in the 25th step, by the Divi- }
 for in the 27th and the Quotient, (neglecting the remainder,) is } 119175.

Which 119175 is the desired Tangent of 50. degrees, the summ of the given arches 30, and 20. degrees, and exactly agrees with the Trigonometrical Canon, the Radius being assumed 100000.

THEOR. 2.

29. As the summ of the Square of the Radius and the Rectangle of the Tangents of two arches is to the Square of the Radius; so is the difference of those Tangents, to the Tangent of the difference of the said arches.

Therefore, the Radius and the Tangents of two arches being severally given; the Tangent of the difference of those arches shall be given also.

Suppos.

- 30. CA is the Radius of a Circle whose Center is C
- 31. AL is an arch less than a Quadrant.
- 32. Arch AI = arch LK, therefore
- 33. Arch AI = arch AL — arch AK.
- 34. AD ⊥ CA.
- 35. AD the Tangent of the arch AL or ∠DCA.
- 36. AB the Tangent of the arch AK or ∠BCA.
- 37. AE, the Tangent of the arch AI, viz. of ∠DCA — ∠BCA.
- 38. CAH and CEF are right lines.
- 39. HF ⊥ CAH.

Req. demonstr.

40. □CA + □AD, AB . □CA :: BD . AE.
 DA-AB

Demonstration.

- 41. Because by *Supposition* } ∠CAD = ∠ = ∠CHG.
- 42. Also by *Suppos.* in the right-angled Triangles }
 FCH, DCA, BCA, } ∠FCH = ∠DCA — ∠BCA.
- 43. Therefore, (per *Theor. 1. Angular. section. Viet.*)

CH . HF :: □CA + □AD, AB . □BD, CA; (that is, DA — AB x CA.)

44. And because by *Suppos.* in 34°, 38° and 39°, Δ CAE and Δ CHF are equian-
 gular, therefore, (per 4. prop. 6. Elem.)

CH . HF :: CA . AE.

45. Therefore from the two last preceding Analogies, (per 11. prop. 5. Elem.)

□CA + □AD, AB . □BD, CA :: CA . AE.

46. But by reason of the common altitude CA, this following Analogy is manifest,
 (per 1. prop. 6. Elem.)

□BD, CA . □CA :: BD . CA.

47. Therefore from the two last preceding Analogies, (per 23. prop. 5. Elem. agreeable
 to *Defin. 8.* of Inordinate proportion, in the foregoing *Chap. 3.*)

□CA + □AD, AB . □CA :: BD . AE.

Which was to be dem. DA-AB

Illustration by Numbers.

48. Let the Radius CA be } 100000.

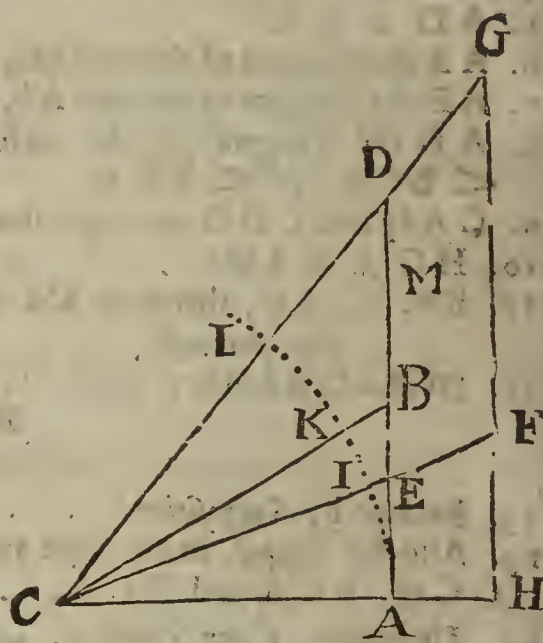
49. Then supposing the arch AL to be 50. degrees, its Tangent }
 AD by the Trigonometrical Canon is } 119175.

50. Likewise, supposing the arch AK to be 30. degrees, its }
 Tangent is } 57735.

Now to find out AE the Tangent of the arch AI, 20. degrees, viz. the Tangent of the
 difference of the given arches AL, 50. degrees, and AK, 30. degrees, the Operation,
 according to the preceding *Theor. 2.* will be as followeth, viz.

51. Multiply the difference of the given Tangents 119175 }
 and 57735, to wit, } 61440.

52. By



- 52. By the Square of the Radius, viz. by } . . . 10000000000.
- 53. The Product shall be a Dividend, to wit, } 614400000000000.
- 54. Then to the said Square of the Radius, add the Product } . . . 6880568625.
- of the given Tangents 119175 and 57735, viz. }
- 55. The sum is a Divisor, to wit, } . . . 16880568625.
- 56. Lastly, divide the Dividend in the 53th step by the Divisor } 36396.
- in the 55th, and the Quotient, neglecting the remainder, will be }

Which 36396 is the desired Tangent of 20. degrees, the difference of the given arches 50, and 30. degrees, and nearly agrees with the Trigonometrical Canon, the Radius being assumed 100000. But for more exactness, the Radius and Tangents given ought to consist of more places than those before exprest in 48°, 49° and 50°.

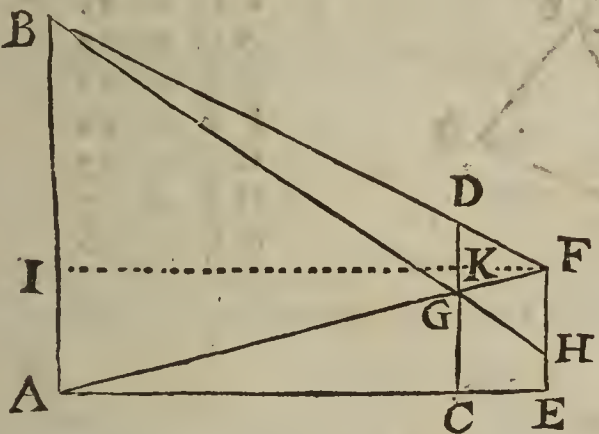
Probl. XVII.

To find the altitude of a Tower, when the foot thereof is inaccessible.

Suppos.

- 1. EA = AC + CE is an horizontal line.
- 2. CA part of EA is accessible only at C.
- 3. AB the height of a Tower.
- 4. CD and EF two staves perpendicular to EA, and at such a distance one from the other, that FDB is a right line.
- 5. FA a right line cutting CD in G.
- 6. BH a right line cutting CD in G.
- 7. FI = and || EA, and cuts CD in K.
- 8. $b = DG,$
- 9. $d = CE,$ } are given severally.
- 10. $f = FH;$ }

Req. to find AC and AB.



- DG = 24
- CE = 30
- FH = 25
- AC = 720
- AB = 600

Preparat.

- 11. Because in $\triangle BFH,$ $DG \parallel FH,$ therefore, $\triangle BFH$ } and $\triangle BDG$ are like, therefore (per prop. 4. Elem. 6.) } $BF \cdot BD :: FH \cdot DG.$
- 12. Likewise because in $\triangle FBI,$ $DK \parallel BI,$ therefore } $BF \cdot BD :: FI \cdot IK.$
- 13. Therefore out of 11° and 12°, (per pro. 11. Ele 5.) } $FH \cdot DG :: FI \cdot IK.$
- 14. That is, (because $AE = IF,$ and $AC = IK,$) } $FH \cdot DG :: AE \cdot AC.$

Resolution.

- 15. Put } $a = AC.$
- 16. Therefore from 9° and 15°, } $a + d = AE.$
- 17. Then from the 8th, 10th, 14th, 15th and 16th steps } this Analogy ariseth, viz. } $f \cdot b :: a + d \cdot a.$
- 18. Therefore by Division of Reason, } $f - b \cdot b :: d \cdot a.$
- 19. Again, in order to find AB, add $d,$ that is, CE, } to $\frac{db}{f-b},$ (which by the 18th step is manifestly } $\frac{fd}{f-b} = AE.$
- equal to $a,$) that is, AC, and it makes }
- 20. Then because $\triangle FBA$ and $\triangle FDG$ are like, (for } $FK \cdot GD :: FI \cdot AB.$
- $DG \parallel BA,$) this Analogy will be manifest (by the } $\frac{FK}{FI} = \frac{GD}{AB}.$
- Lemma prefixt to Probl. 11. of this Chapt.) viz. }

21. That

21. That is, by exchange of equal right lines, $\} CE \cdot GD :: AE \cdot AB.$
 22. That is, in the letters belonging to the Resolution, $\} d \cdot b :: \frac{fd}{f-b} \cdot \frac{fb}{f-b} (AB.)$

From the 18th, 21th and 22th steps ariseth this

CANON.

23. As the excess whereby FH exceeds DG, is to DG, so is CE to AC; and so is FH to AB. The Demonstration of this Canon being very easie to be deduced from the premisses, I shall only illustrate it by

An Example in Numbers.

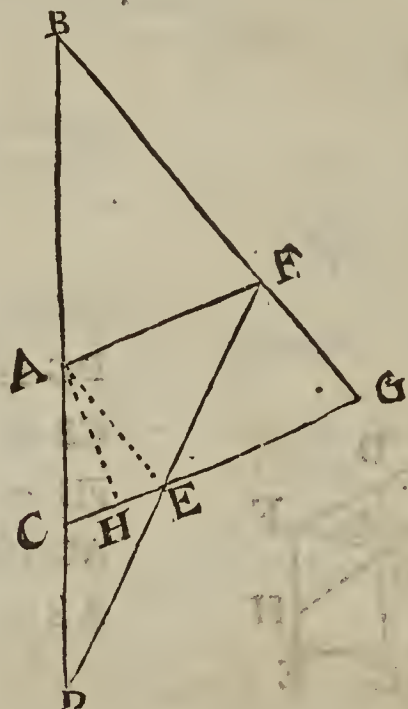
Suppose $\} \begin{matrix} DG = b = 24 \\ CE = d = 30 \\ FH = f = 25 \end{matrix} \} \text{ Inches.}$

Then by the Canon,

I. $\} \begin{matrix} FH - DG & \cdot & DG & :: & CE & \cdot & AC \\ \hline 1 & \cdot & 24 & :: & 30 & \cdot & 720. \end{matrix}$
 II. $\} \begin{matrix} FH - DG & \cdot & DG & :: & FH & \cdot & AB \\ \hline 1 & \cdot & 24 & :: & 25 & \cdot & 600. \end{matrix}$

Probl. XVIII.

To find the length of a right line AB, when there is access to one of its ends, suppose to A.



- GC = 51
- CA = 26
- AF = 34 = GE
- FG = 25 = AE
- AH = 24
- AB = 52
- FB = 50
- CE = 17
- HE = 7
- HC = 10

Preparat.

1. From the given point A measure two equal distances, as $AC = CD$; and so, as $DCAB$ may be a right (or straight) line; measure likewise DE and EF, so as DE may be equal to EF and DEF a straight line, making an angle ADF at pleasure: then measure FA, which shall be parallel to a line drawn from C to E, (per 2. prop. 6. Elem.) for $DC \cdot CA :: DE \cdot EF$; then measure CEG, that is, the parallel CE produced to such a Station G, that GFB may be a straight line, measure also GF. Now of the Trapezium CAFG the two sides AF and CG being parallels, and every one of the four sides CA, AF, FG, GC, given by measure as above, in Yards or what equal parts you please, the length of AB, also of FB may be found out in the same kind of parts, by the following Resolution.

Suppos.

$\} \begin{matrix} CAB \text{ and } GFB \text{ are right lines.} \\ AF \parallel CG. \\ b = CG \\ c = CA \\ d = AF \\ f = FG \end{matrix} \} \text{ given severally.}$

Req. to find AB and FB.

Resol-

Resolution.

- 3. Put } $a = AB.$
- 4. Then because $\triangle BCG$ and $\triangle BAF$ are equi- }
angular, for $AF \parallel CG$, this Analogy is mani- }
fest, (*per 4. prop. 6. Elem.*) } $CG \cdot CB :: AF \cdot AB.$
- 5. That is, in the letters of the Resolution, . } $b \cdot c + a :: d \cdot a.$
- 6. Therefore alternately, } $b \cdot d :: c + a \cdot a.$
- 7. Therefore by *Division of Reason*, } $b - d \cdot d :: c \cdot a.$
- 8. And by the like argumentation this Analogy }
will arise, } $b - d \cdot d :: f \cdot FB.$

The Analogies in the seventh and eighth steps afford this

THEOR. 1.

- 9. In a Trapezium $CAEG$ having two parallel sides AF, CG ; As the difference of those two parallel sides is to the lesser of them; so is each of the two other sides not parallels, to its continuation to the point of their concurrence, *viz.* $As, CE \cdot AF :: CA \cdot AB :: FG \cdot FB.$
- 10. The Demonstration whereof is manifest; for drawing $AE \parallel FG$, the Triangles CAE and ABF shall be like; and because $AE \parallel FG$ and $AF \parallel EG$, it follows (*per prop. 34. Elem. 1.*) that $AE = FG$, also $AF = EG$, and $CE = CG - AF$. Wherefore $CE \cdot AF :: CA \cdot AB :: FG \cdot FB$, as the Theorem affirms.

An Example in Numbers.

- 11. Suppose } $AF \parallel CG$ in the Trapezium $CAFG$; also;
} $CG = 51.$
} $CA = 26.$
} $AF = 34 = GE.$
} $FG = 25 = EA.$
} $CE = 17 = CG - AF.$

- 12. Then by the Theorem before express'd in 9° ,

$$CE \cdot (CG - AF) \cdot AF :: \begin{cases} CA \cdot AB. \\ 26 \cdot 52 \\ GE \cdot FB. \\ 25 \cdot 50 \end{cases}$$

Here by the way, the Learner may observe, that the four sides of a Trapezium having two parallel sides being severally given, the Area of that Trapezium shall be given also by this following

THEOR. 2.

- 13. If a Trapezium hath two parallel sides, the Rectangle or Product made by the multiplication of half the summ of the parallel sides, into the shortest distance between the said Parallels, shall be equal to the Area of the Trapezium.

- 14. . . . *Suppos.* } $AF \parallel CG$ in the Trapezium $CAFG.$
} $CG \perp AE.$
} AC not $\parallel FG.$

Prepar.

- 15. Make $AH \perp CG$; also $GE = AF$, and draw AE .
- 16. . . . *Req. demonstr.* . . . } $AH \times \frac{1}{2}CG + \frac{1}{2}AF = \text{Trapez. } CAFG.$
} That is, $AH \times EG + \frac{1}{2}CE = \text{Trapez. } CAFG.$

Demonstration.

- 17. By *Supposition* in 14° , } $AF \parallel CG.$
- 18. And by *Constr.* in 15° , } $GE = AF$ and $AH \perp HG.$
- 19. Therefore out of 17° and 18° , (*per 33, & 35. prop. 1. Elem.*) } $AH \times EG = \text{Parallelogr. } EAFG.$
- 20. And (*per Schol. 41. prop. 1. Elem.*) } $AH \times \frac{1}{2}CE = \triangle CAE.$
- 21. Therefore from the two last Equations,
 $AH \times EG + \frac{1}{2}CE = EAFG + \triangle CAE = \text{Trapez. } CAFG.$
Which was to be Demonstr.

An

An Example in Numbers.

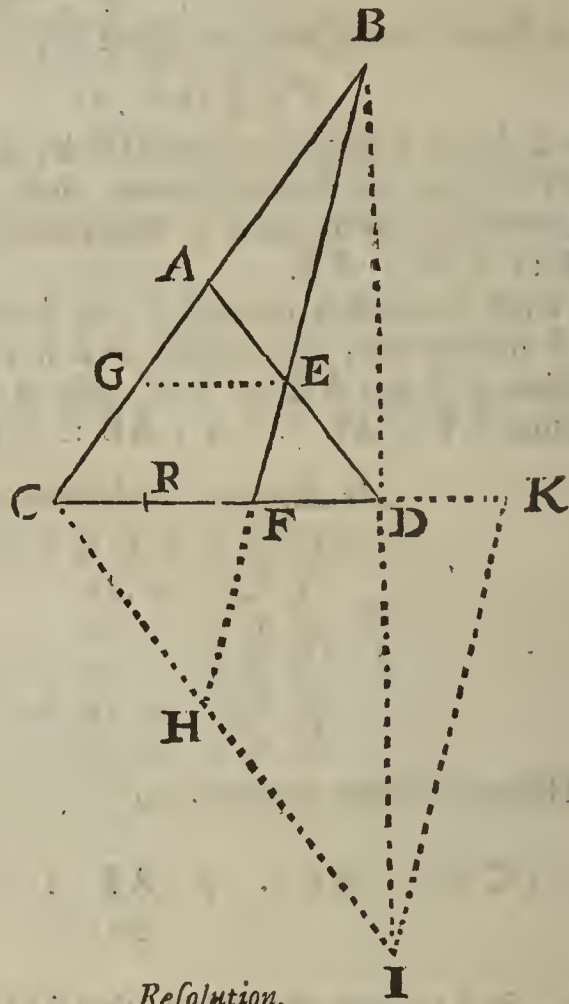
Let the same things be given before as in the 11th step, whence the three sides of the Triangle CEA will be 17, 26, 25, and by the help of these, the Perpendicular AH will be found 24, (by Theor. in 29° and 30° of *Probl. 9. Chap. 7.*) which multiplied by $42\frac{1}{2}$, that is, half the sum of the parallel sides AF, CG, gives 1020 for the Area of the Trapezium CAFG.

Probl. XIX.

Another way to measure a distance, when we can come to one end thereof.

Suppos.

- 1. { AB a distance accessible only at A.
- { CAB a right line.
- { $b=CA$
- { $c=AD$
- { $d=DC$ } given severally.
- { $f=AE$
- { $g=CF$
- { $GE \parallel CD$.
- { BEF a right line.



- CA = 210
- AD = 210
- CD = 252
- AE = 90
- CF = 152
- CK = $354\frac{2}{3}$
- AB = $204\frac{6}{11}$

Req. to find AB.

Resolution.

- 2. Put } $a = AB$.
- 3. Then because $\triangle ADC$ and $\triangle AEG$ are } $AD \cdot AE \quad :: \quad DC \cdot GE$.
- equiangular, (for $GE \parallel CD$), it shall be, } $(per\ prop.\ 4.\ Elem.\ 6.)$
- 4. That is, in the letters of the Resolution, } $c \cdot f \quad :: \quad d \cdot \frac{df}{c}$.
- 5. Likewise in the same $\triangle ADC$ and $\triangle AEG$, } $AD \cdot AE \quad :: \quad CA \cdot GA$.
- 6. That is, in the letters of the Resolution, } $c \cdot f \quad :: \quad b \cdot \frac{bf}{c}$.
- 7. Again, in the equiangular Triangles CBF } $CF \cdot CA + AB \quad :: \quad GE \cdot GA + AB$.
- and GBE, }
- 8. That is, } $g \cdot b + a \quad :: \quad \frac{df}{c} \cdot \frac{bf}{c} + a$.
- 9. Which Analogy gives this Equation, viz. } $\frac{bfg}{c} + ga = \frac{bdf}{c} + \frac{dfa}{c}$.
- 10. Therefore by subtracting $\frac{dfa}{c}$ from each } $\frac{bfg}{c} + ga - \frac{dfa}{c} = \frac{bdf}{c}$.
- part, (for g , or CF , is greater than $\frac{df}{c}$ or } EG), }
- 11. And by subtracting $\frac{bfg}{c}$ from each part } $g - \frac{df}{c}$ into $a = \frac{bdf - bfg}{c}$.
- of the last Equation, this ariseth, }
- 12. That is, } $g - \frac{df}{c}$ into $a = \frac{df - fg}{c}$ into b .

13. There-

13. Therefore by resolving the last preceding Equation } into Proportionals, this Analogy ariseth, viz. . . . } $g - \frac{df}{c} \cdot \frac{df - fg}{c} :: b \cdot a.$
14. And by multiplying each of the two first Terms } of that Analogy by c , this ariseth, viz. . . . } $gc - df : df - fg :: b \cdot a.$
15. Let fm be made equal to gc , viz. As f to c , so } g to a fourth, which may be called m , then setting } fm in the place of gc in the first Term of the Analogy } in the 14th step, this ariseth, viz. . . . } $fm - df \cdot df - fg :: b \cdot a.$
16. Therefore by dividing the first and second Terms of } the last preceding Analogy by f , this ariseth, viz. } $m - d \cdot d - g :: b \cdot a.$
- Which Analogy last before exprest affords this

CANON.

17. Let it be made as f to c , so g to a fourth, which may be called m ; (that is, in the Diagram, as AE to AD , or as CH to CI , so CF to CK ;) then let it be made, as $m - d$ to $d - g$, so b to a the distance sought; (that is, in the Diagram, as KD to DF , or as ID to DB , so CA to AB .)

The Demonstration whereof, (in regard CI is parallel to AD , and IK to HFB ,) is manifest (per prop. 2, & 4. Elem. 6.) the Proportionals in the Canon being compared to the respective lines in the Diagram belonging to this Problem. Which Canon may be demonstrated also by proceeding according to the steps of the Resolution, in a direct order, viz. from the beginning to the end; and although the Demonstration in this way be prolix, yet since 'tis certain, and may serve to improve the younger Analyst's skill in the Resolution of Plane Problems, I shall here form it at large.

Prepar.

18. Let it be made as } $AD \cdot AE :: CF \cdot CR.$
That is, } $c \cdot f :: g \cdot \frac{fg}{c}.$

Then the same things being supposed and made as in the first and seventeenth steps,

19. . . Req. demonstr. } $KD \cdot DF :: CA \cdot AB.$

Demonstration.

20. Forasmuch as $\triangle ADC$ and $\triangle AEG$ } are like, it shall be (per prop. 4. Ele. 6.) } $AD \cdot AE :: DC \cdot GE.$
That is, in the fourth step, } $c : f :: d : \frac{df}{c}.$

21. Again, in the same $\triangle ADC$ and } $\triangle AEG$, } $AD : AE :: CA : GA.$
That is, in 6^o, } $c : f :: b : \frac{bf}{c}.$

22. And because $\triangle CBF$ and $\triangle GBE$ } are like, it shall be } $CF \cdot CA + AB :: GE \cdot GA + AB.$
That is, in 8^o, } $g \cdot b + a :: \frac{df}{c} \cdot \frac{bf}{c} + a.$

23. Therefore from the 22th step, this Analogy is manifest, (per 16. prop. Elem. 6.) viz. } $\square GA, CF + \square CF, AB = \square CA, GE + \square GE, AB.$

That is, } $\frac{bfg}{c} + ga = \frac{bdf}{c} + \frac{dfa}{c}.$
in 9^o, }

24. And by subtracting $\square GE, AB$ from each part of the Equation in the 23th step, this remains, viz. } $\square GA, CF + \square CF, AB - \square GE, AB = \square CA, GE.$

That is, } $\frac{bfg}{c} + ga - \frac{dfa}{c} = \frac{bdf}{c}.$
in 10^o, }

25. And by subtracting $\square GA, CF$ from each part of the Equation in the 24th step, this remains, viz. } $\square CF, AB - \square GE, AB = \square CA, GE - \square GA, CF.$

That is, } $g - \frac{df}{c}$ into $a = \frac{bdf - bfg}{c}.$
in 11^o, }

That is, } $g = \frac{df}{c}$ into $a = \frac{df - fg}{c}$ into $b.$
in 12^o, }

Now that $\square CR, CA$ may be set in the place of $\square GA, CF$ in the latter part of the Equation in the 25th step, I shall by the four following steps prove that $\square CR, CA = \square GA, CF$. First then,

26. By the 21th step, $AD \cdot AE :: CA \cdot GA$.

27. And by the 18th step, $AD \cdot AE :: CF \cdot CR$.

28. Therefore from the 26th and 27th steps, (per 11. prop. 5. Elem.) $CA \cdot GA :: CF \cdot CR$.

29. Therefore from the 28th step, (per 16. prop. El. 6.) $\square CR, CA = \square GA, CF$.

30. Therefore by setting $\square CR, CA$ in the place of $\square GA, CF$ in the latter part of the Equation in the 25th step, that Equation will be converted into this, viz.

$$\square CF, AB - \square GE, AB = \square CA, GE - \square CR, CA.$$

That is, in 12°, $g - \frac{df}{c}$ into $a = \frac{df - fg}{c}$ into b .

31. Therefore by resolving the Equation in the 30th step into Proportionals, this Analogy arifeth, viz.

$$CF - GE \cdot GE - CR :: CA \cdot AB.$$

That is, in 13°, $g - \frac{df}{c} \cdot \frac{df - fg}{c} :: b \cdot a$.

32. And by drawing AD into the first and second Terms of the Analogy in 31°, this will be manifest, (per 1. prop. 6. Elem.)

$$\square CF, AD - \square GE, AD \cdot \square GE, AD - \square CR, AD :: CF - GE \cdot GE - CR.$$

33. Therefore out of 31° and 32°, (per 11. prop. 5. Elem.)

$$\square CF, AD - \square GE, AD \cdot \square GE, AD - \square CR, AD :: CA \cdot AB.$$

That is, in 14°, $gc - df \cdot df - fg :: b \cdot a$.

34. And because from the Construction in 17°, $\square CK, AE = \square CF, AD$.

35. And out of 20°, $\square DC, AE = \square GE, AD$.

36. And out of 18°, $\square CF, AE = \square CR, AD$.

37. Therefore out of 33°, 34°, 35° 36°, by exchange of equal Rectangles, this Analogy arifeth, viz.

$$\square CK, AE - \square DC, AE \cdot \square DC, AE - \square CF, AE :: CA \cdot AB.$$

That is, in 15°, $fm - df \cdot df - gf :: b \cdot a$.

38. And this following Analogy, by reason of the common altitude AE , will be manifest by prop. 1. Elem. 6.

$$\square CK, AE - \square DC, AE \cdot \square DC, AE - \square CF, AE :: CK - DC \cdot DC - CF.$$

39. Therefore out of 37° and 38°, (per 11. prop. 5. Elem.)

$$CK - DC \cdot DC - CF :: CA \cdot AB.$$

That is, in 16°, $m - d \cdot d - g :: b \cdot a$.

40. Therefore from 39°, respect being had to the Diagram,

$$KD \cdot DF :: CA \cdot AB.$$

Which was to be Dem.

Divers other Canons might be deduced from the premisses, which for brevity sake I shall pass over; but to satisfie such Artists as delight in Problems of this kind, because they may be practically applied to measure distances in the field without any Mathematical Instrument to observe angles, I shall illustrate the preceding Canon (in 17°,) by

An Example in Numbers.

$$\text{Suppose } \left. \begin{array}{l} CA = b = 210 \\ AD = c = 210 \\ CD = d = 252 \\ AE = f = 90 \\ CF = g = 152 \end{array} \right\} \text{ Feet.}$$

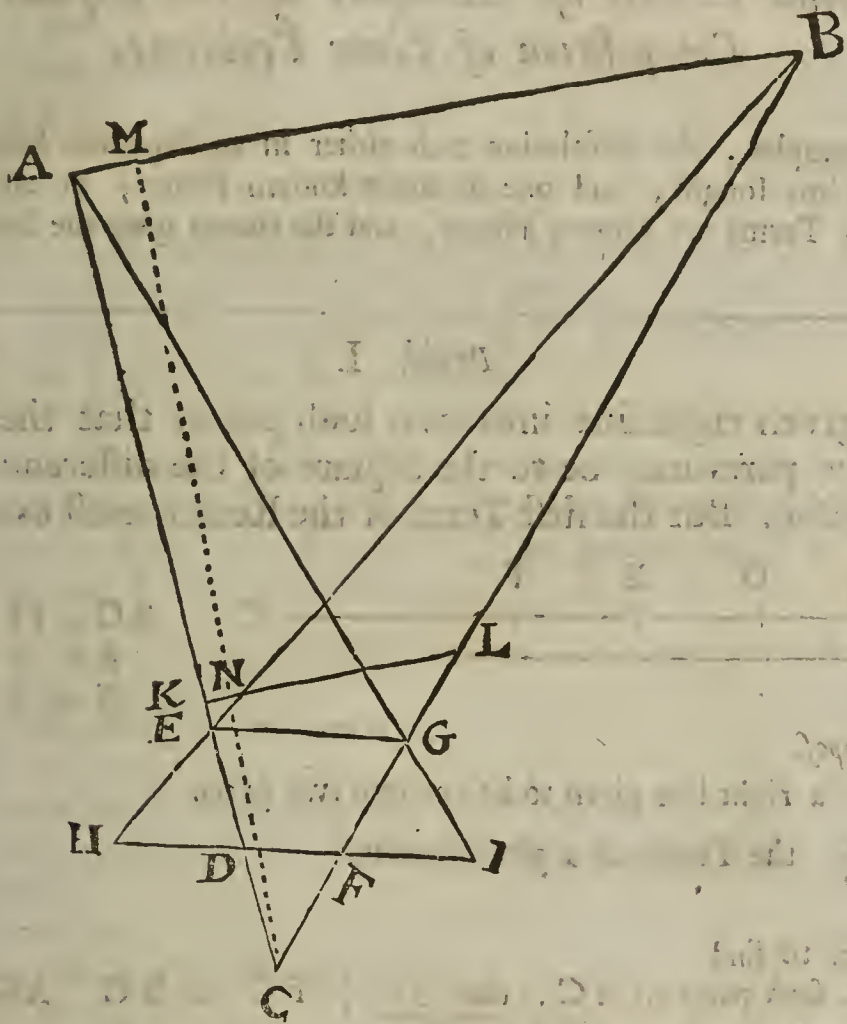
Then by the Canon in 17° of this Probl.

$$\text{I. } \left\{ \begin{array}{l} AE \cdot AD :: CF \cdot CK \\ 90 \cdot 210 :: 152 \cdot 354 \frac{1}{2} \end{array} \right.$$

$$\text{II. } \left\{ \begin{array}{l} CK - CD \cdot DC - CF :: CA \cdot AB \\ \text{or, } KD \cdot DF :: CA \cdot AB \\ 102 \frac{1}{2} \cdot 100 :: 210 \cdot 204 \frac{1}{2} \end{array} \right.$$

Probl. XX.

To find the length of a right line AB, when neither of its ends A or B is accessible.



Make choice of some station C, where you may both see, and have liberty to measure towards A and B; then measure two equal lines CD and DE, in Yards or Feet, &c. so as CDEA may be a right (or straight) line; likewise measure CF and FG equal to one another, and so as CFGB may be a straight line. Measure also EG, which shall be parallel to a line drawn from D to F, (*per 2. prop. 6. Elem.*) for $CD : DE :: CF : FG$. Then supposing DF to be produced to H and I, to wit, such stations that HEB and IGA may be right lines, measure HE and GI; also HF and DI. Now there are given severally the four sides of the Trapezium HEGF, having two parallel sides EG and HF, therefore the length of GB shall be given also by the preceding *Probl. 18*. In like manner, the four sides of the Trapezium DEGI having two parallel sides EG, DI, are given severally, therefore the line EA shall be given also by the said *Probl. 18*. Then EA and GB, also CE and CG being given (as before,) the lines CA and CB shall be given also, by Addition. Then take some Aliquot part of CA, suppose one third part, and measure it from C to K; measure likewise one third part of CB from C to L; so shall KL be parallel to the inaccessible line AB sought, (*per prop. 2. Elem. 6.*) Then measure KL. Lastly, as CK to KL, so is CA to AB; or as CL to LK, so is CB to BA; but CK, KL, CA, CL, CB are given in Yards or Feet, &c. (as before,) and therefore AB the distance sought is given also in the same kind of measure.

If you desire to find the shortest distance from C to AB, to wit, the Perpendicular CM, it may be found by the help of the three sides of the Triangle CAB, whose measures were before discovered. Or if you would draw a line in the field from C towards the inaccessible line AB, that shall fall perpendicularly upon the said AB produced infinitely, you may by the three sides of the Triangle CKL, and by the *Coroll. in 45° of Probl. 10. of this Chap.* discover whether a Perpendicular from C will fall within or without the Triangle CKL, and also find the distance from either end of the Base KL to the foot of the Perpendicular, which distance being measured in the field accordingly, the Perpendicular CN may be drawn, which being produced, shall be also perpendicular to the inaccessible line AB. But how this Problem may be solved by measuring only five right lines, I shall hereafter shew in *Probl. 20. Chap. 9.*

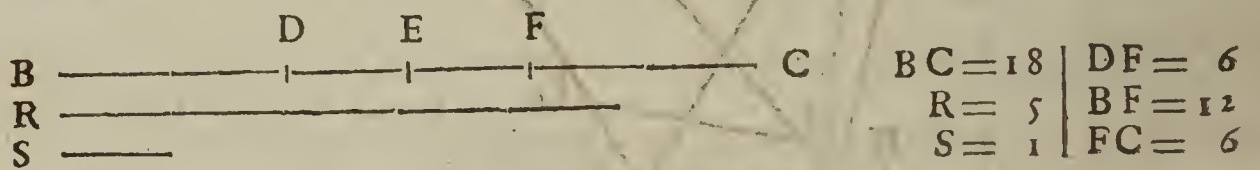
C H A P. VIII.

The second Classis of Examples of the Resolution and Composition of Plane Problems.

IN which Examples, the Resolution ends either in an Equation between the Square of the right line sought, and one or more known Planes; or else in an Analogy whose three first Terms are known Planes, and the fourth gives the Square of the right line sought.

Probl. I.

To cut a given right line into two such parts, that the sum of the Squares of the parts may be to the Square of the difference of the parts in a given Reason. But the first Term of the Reason must exceed the latter.



Suppos.

1. $b = BC$ a right line given to be cut into two parts.
2. $\left. \begin{matrix} r = R \\ s = S \end{matrix} \right\}$ the Terms of a given Reason.
3. $r > s$.

Req. to find

4. BF and FC such parts of BC, that $BF + FC = BC$. Also;
5. $\square BF + \square FC . \square : BF - FC : :: R . S$.

Resolution.

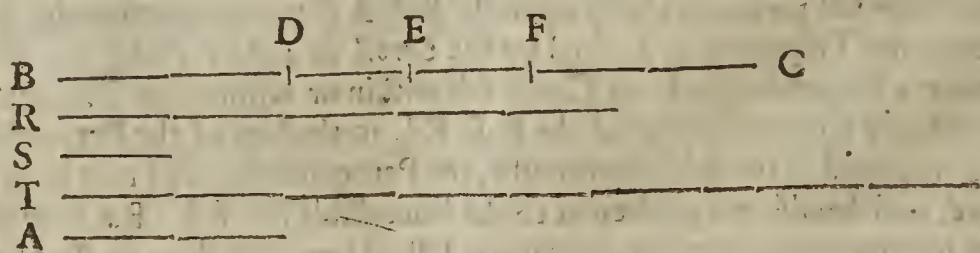
6. Put a for the difference of the parts sought, viz. $a = BF - FC$.
7. Therefore the Square of the difference of the parts is aa .
8. And from 1° and 6°, the sum of the Squares of the parts (per Theor. 6. Chap. 4.) is $\frac{1}{2}bb + \frac{1}{2}aa$.
9. Therefore according to the tenour of the Problem these must be Proportionals, viz. $r . s :: \frac{1}{2}bb + \frac{1}{2}aa . aa$.
10. Whence, by doubling the Antecedents, this Analogy arifeth, $2r . s :: bb + aa . aa$.
11. Therefore by Division of Reason, $2r - s . s :: bb . aa$. Which last Analogy gives this

C A N O N.

12. As the excess whereby the double of the first Term of the given Reason exceeds the latter Term, is to the latter Term, so is the Square of the line given to be cut into two parts, to the Square of the difference of the parts. Therefore the difference of the parts is given, and consequently the parts are given severally by Theor. 9. Chap. 4.

The reason of the Determination annex'd to the Problem is evident by Theor. 5. Chap. 4. which shews, that if a right line be divided into two unequal parts, the sum of the Squares of the parts is greater than the Square of the difference of the parts, by the double Rectangle of the parts.

The Composition of the foregoing Probl. 1.



Suppos.

Suppos.

- 13. BC is a right line given to be cut into two parts.
- 14. R and S are the Terms of a given Reason.
- 15. $R \sqsupset S$.

Req. to find

- 16. BF and FC such parts of BC, that $BF + FC = BC$. Also,
- 17. $\square BF + \square FC . \square : BF - FC : :: R . S$.

Construction.

- 18. Find a right line T that may be equal to the excess of 2R above S, which is possible to be done, for by *Supposition* $R \sqsupset S$; suppose therefore $T = 2R - S$.
- 19. By *Probl. 11. Chap. 5.* let it be made as T to S, so the Square of BC to another Square, whose side suppose to be A, therefore,

$$T . S :: \square BC . \square A.$$

- 20. Divide BC into two equal parts in E, therefore $EB = EC$.
- 21. From EC and EB cut off EF and ED, such parts, that each may be equal to $\frac{1}{2}A$; which is possible to be done, if $EB (= EC)$ be greater than $\frac{1}{2}A$. But that EB or EC is greater than $\frac{1}{2}A$, I prove thus;

By *Suppos.* in 15° , $R \sqsupset S$.
 And consequently, $2R \sqsupset 2S$.
 Therefore by subtracting S from each part, $2R - S \sqsupset S$.
 But by *Constr.* in 18° , $T = 2R - S$.
 Therefore from the two last preceding steps, (*per Ax. 4. Chap. 2.*) $T \sqsupset S$.
 Therefore from the Analogy in 19° , and from the last preceding step, $BC \sqsupset A$.
 And consequently, $\frac{1}{2}BC \sqsupset \frac{1}{2}A$.
 But by *Constr.* in 20° , $\frac{1}{2}BC = EB = EC$.
 Therefore from the two last preceding steps, $EB \text{ or } EC \sqsupset \frac{1}{2}A$.

Which was to be Dem.

- 22. I say BF and FC are the desired parts of BC. For first, their summ is manifestly equal to BC; and by *Constr.* in 20° and 21° the difference between the said parts BF and FC, that is, $BF - FC$ (BD) is equal to DF. But that the summ of the Squares of the parts BF and FC, is to the Square of their difference DF, as R to S, I shall demonstrate by a repetition of the steps of the foregoing Resolution in a backward order.

23. . . . *Req. demonstr.* $R . S :: \square BF + \square FC . \square DF$.

Demonstration.

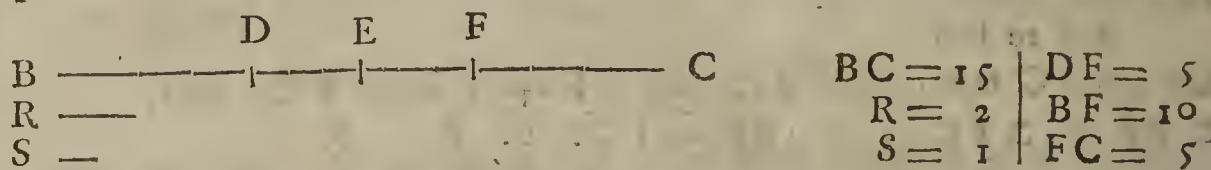
- 24. By *Constr.* in 19° , $T . S :: \square BC . \square A$.
- 25. And by *Constr.* in 18° and 21° , $2R - S = T$. And $DF = A$.
- 26. Therefore from 24° and 25°, by } $2R - S . S :: \square BC . \square DF$.
 exchange of equal quantities, }
- That is, in 11° , $2r - s . s :: bb . aa$.
- 27. Therefore from 26°, by *Composition* } $2R . S :: \square BC + \square DF . \square DF$.
 of Reason, }
- That is, in 10° , $2r . s :: bb + aa . aa$.
- 28. Therefore from 27°, by halving the } $R . S :: \frac{1}{2}\square BC + \frac{1}{2}\square DF . \square DF$.
 Antecedents, }
- That is, in 9° , $r . s :: \frac{1}{2}bb + \frac{1}{2}aa . aa$.
- 29. By *Constr.* in 20° and 21° , BC is the }
 summ, and DF the difference of the }
 parts BF and FC, therefore (*per* } $\square BF + \square FC = \frac{1}{2}\square BC + \frac{1}{2}\square DF$.
Theor. 6. Chap. 4.) }
- 30. Therefore from 28° and 29°, by } $R . S :: \square BF + \square FC . \square DF$.
 exchanging equal quantities, }

Which was to be Demonstr. Therefore that is done which was required by the Problem.

Probl. 11.

Probl. II.

To cut a given right line into two such parts, that the Rectangle of the parts, to the Square of their difference, may have a given Reason.



Suppos.

1. $b = BC$ a right line given to be cut into two parts.
2. $\left\{ \begin{array}{l} r = R \\ s = S \end{array} \right\}$ the Terms of a given Reason.

Req. to find

3. BF and FC such parts of BC , that $BF + FC = BC$. Also,
4. $\square BF, FC \cdot \square : BF - FC : :: R \cdot S$.

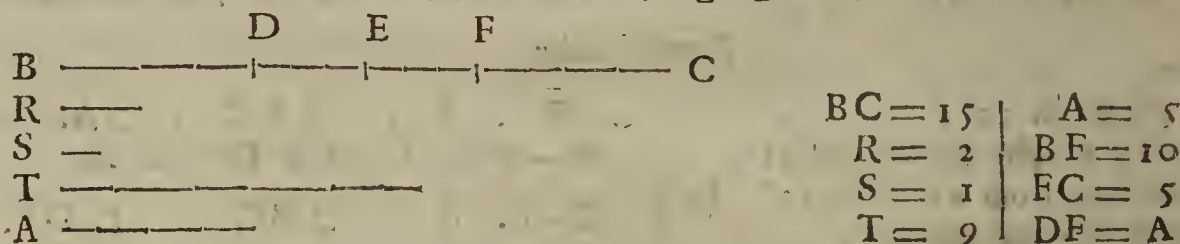
Resolution.

5. Put a for the difference of the parts sought; viz. $a = BF - FC$.
 6. Therefore from 1° and 5° , (per Theor. 9. Chap. 4.) the greater part shall be $\frac{1}{2}b + \frac{1}{2}a (= BF)$.
 7. And by the same Theorem, the lesser part shall be $\frac{1}{2}b - \frac{1}{2}a (= FC)$.
 8. Therefore from 6° and 7° , the Rectangle (or Product) of the parts is $\frac{1}{4}bb - \frac{1}{4}aa$.
 9. And from 5° , the Square of the difference of the parts is aa .
 10. Therefore from 4° , 8° and 9° , according to the tenor of the Problem, this Analogy ariseth, viz. $r \cdot s :: \frac{1}{4}bb - \frac{1}{4}aa \cdot aa$.
 11. Whence, by quadrupling the Antecedents, $4r \cdot s :: bb - aa \cdot aa$.
 11. Therefore by Composition of Reason, $4r + s \cdot s :: bb \cdot aa$.
- Which last Analogy affords this **CANON**.

13. As the summ of the second Term of the given Reason and the quadruple of the first is to the second Term; so is the Square of the line given to be divided into two parts, to the Square of the difference of the parts.

Therefore the difference of the parts is given, and consequently the parts severally, by Theor. 9. Chap. 4.

The Composition of the foregoing Probl. 2.



Suppos.

14. BC is a right line given to be cut into two parts.
15. R and S are the Terms of a given Reason.

Req. to find

16. BF and FC such parts of BC , that $BF + FC = BC$. Also,
17. $\square BF, FC \cdot \square : BF - FC : :: R \cdot S$.

Construction.

18. Find a right line $T = 4R + S$.
19. By Probl. 11. Chap. 5. let it be made as T to S , so the Square of BC to another Square, whose side suppose to be A , therefore, $T \cdot S :: \square BC \cdot \square A$.
20. Divide BC into two equal parts in E , therefore $EC = EB$.
21. From EC and EB cut off EF and ED , such parts, that each may be equal to $\frac{1}{2}A$, which may be done, if $EC (= EB)$ be greater than $\frac{1}{2}A$; but that EC or EB is greater than $\frac{1}{2}A$, I prove thus;

By

By *Construction* in 18° , T \square S.
 Therefore from 19° , BC \square A.
 And consequently, $\frac{1}{2}BC \square \frac{1}{2}A$.
 But by *Constr.* in 20° , $\frac{1}{2}BC = EC = EB$.
 Therefore from the two last preceding steps, EC or EB $\square \frac{1}{2}A$.

Which was to be Dem.

22. I say BF and FC are such parts of BC as will satisfy the Problem. For first, BF + FC = BC; and by *Construction* in 20° and 21° , the difference between the said parts BF and FC, that is, BF - FC (BD) is equal to DF. But that the Rectangle of the said parts BF and FC is to the Square of their difference DF as R to S; the following Demonstration, form'd out of the preceding Resolution by a repetition of its steps in a backward order will make manifest.

23. . . . *Req. demonstr.* R . S :: $\square BF, FC$. $\square DF$.

Demonstration.

24. By *Constr.* in 19° , T . S :: $\square BC$. $\square A$.
 25. And by *Constr.* in 18° and 21° , $4R + S = T$. And DF = A.
 26. Therefore from 24° and 25° , by ex- } $4R + S . S :: \square BC . \square DF$.
 change of equal quantities,
 27. Therefore by *Division of Reason*, $4R . S :: \square BC - \square DF . \square DF$.
 28. And by taking $\frac{1}{4}$ of the Antecedents in 27° , } $R . S :: \frac{1}{4}\square BC - \frac{1}{4}\square DF . \square DF$.
 29. By *Constr.* in 20° and 21° , BC is the } $\square BF, FC = \frac{1}{4}\square BC - \frac{1}{4}\square DF$.
 summ, and DF the difference of the parts }
 BF and FC; therefore by *Theor. 7. Chap. 4.* }
 30. Therefore from 28° and 29° , by ex- } $R . S :: \square BF, FC . \square DF$.
 change of equal quantities,

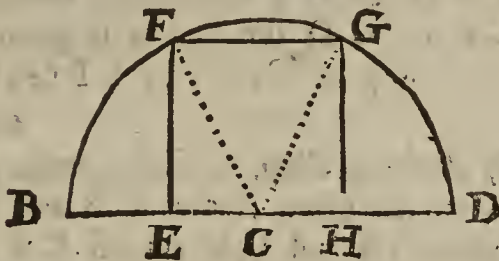
Which was to be Dem. Therefore that is done which was required by the Problem.

A LEMMA, leading to the following Probl. 3.

If a Square, or long Square be inscribed in a Semicircle, the Center of the Semicircle is in the middle of the Base of the Square, or long Square.

Suppos.

1. C B F G D a Semicircle, whose Center is C.
2. E F G H is a Square.
3. . . . *Req. demonstr.* . . . CE = CH.



Prepar.

4. Draw the right lines CF and CG.

Demonstration.

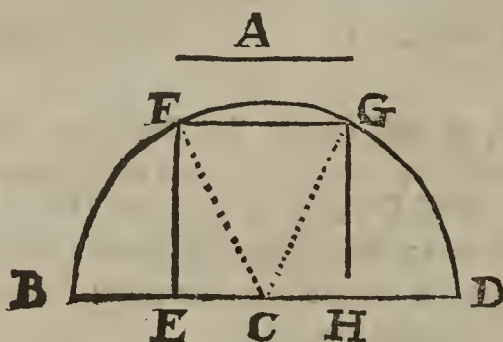
5. By *Defin. 15. Elem. 1.* } CF = CG. |
6. And by *Suppos.* in 2° , } $\angle FEC = \angle = \angle GHC$.
7. Therefore (*per prop. 47. Elem. 1.*) . . . } $\square EF + \square CE = \square CF = \square CG$.
8. Likewise, } $\square HG + \square CH = \square CG = \square CF$.
9. Therefore from 7° and 8° , (*per Ax. 1.*) } $\square EF + \square CE = \square HG + \square CH$.
Chap. 2.) }
 10. But by *Suppos.* } EF = HG.
 11. And consequently, } $\square EF = \square HG$.
 12. Therefore by subtracting $\square EF$ or $\square HG$ } $\square CE = \square CH$.
 from each part of the Equation in 9° , the }
 remainders will be equal, *viz.* . . . }
 13. Therefore } CE = CH.

Which was to be Dem. The same Demonstration may be made when a long Square is inscribed in a Semicircle.

Probl. III.

Probl. III.

To inscribe a Square in a given Semicircle.



$$\begin{aligned}
 CB &= CD = 10 \\
 EF &= EH = \sqrt{80} \\
 CE &= CH = \sqrt{20} \\
 EB &= HD = 10 - \sqrt{20}
 \end{aligned}$$

Suppos.

1. C B F G D is a Semicircle, whose Center is C.
2. $r = CB = CD$ is given.

Req. to inscribe $\square EFGH$.

Resolution.

3. Put a for the side of the Square required, viz. $\dots \} a = EF = EH.$
4. Therefore by the preceding Lemma, $\dots \} \frac{1}{2}a = CE = CH.$
5. And because (per prop. 47. Elem. 1.) $\dots \} \square EF + \square CE = \square CF,$
6. Therefore in the letters of the Resolution, $\dots \} aa + \frac{1}{4}aa = rr.$
7. Therefore, by multiplying the last Equation by 4, $\dots \} 4aa + aa = 4rr.$
8. That is, $\dots \} 5aa = 4rr.$
9. Therefore by taking $\frac{1}{5}$ of the last Equation, $\dots \} aa = \frac{4}{5}rr.$
10. Therefore, by extracting the square Root out of each part of the last preceding Equation, $\dots \} a = \sqrt{\frac{4}{5}rr}.$

Hence this

CANON.

11. The square Root of $\frac{4}{5}$ parts of the Square of the Semidiameter is equal to the side of the Square inscribed in the Semicircle.

The Composition of the foregoing Probl. 3.

Suppos.

12. C B F G D is a Semicircle, whose Center is C.
13. $CB = CD$ the Radius is given.

Req. to inscribe $\square EFGH$.

Construction.

14. By Probl. 9. Chap. 5. find a mean proportional line A } $CB \cdot A :: A \cdot \frac{4}{5}CB.$
between the given Radius CB and $\frac{4}{5}CB$, therefore }
15. From CB and CD cut off CE and CH , such segments, that as well CE as CH may be equal to $\frac{1}{2}A$, and consequently $EH = A$; which Effect is possible, for by Construction in 14^o CB is the greatest of three Proportionals, whereof A is the mean, therefore $CB \sqsupset A$, and because $CD = CB$, therefore also $CD \sqsupset A$; and consequently a segment equal to $\frac{1}{2}A$, as CE or CH may be cut off from CB or CD .
16. Make EF and $HG \perp BD$, and draw FG , so is $EFGH$ the Square required to be inscribed, as will be evident by the following Demonstration, form'd out of the preceding Resolution, by a repetition of its steps in a backward (not direct) order.
17. \dots Req. demonstr. \dots $EFGH$ is a \square .

Preparat.

18. Draw the right lines CF and CG .

Demonstration.

19. By Constr. in 14^o, $\dots \} CB \cdot A :: A \cdot \frac{4}{5}CB,$
20. And by Constr. in 15^o, $\dots \} EH = A.$
21. Therefore from 19^o and 20^o, by exchanging equal } $CB \cdot EH :: EH \cdot \frac{4}{5}CB,$
right lines, $\dots \} \square EH = \frac{4}{5} \square CB.$
22. And from 21^o, (per prop. 17. Elem. 6.) $\dots \} \square EH = \frac{4}{5} \square CB.$
23. But the quintuples of equal quantities are also equal, } $5 \square EH = 4 \square CB.$
therefore from 22^o, $\dots \} \dots$

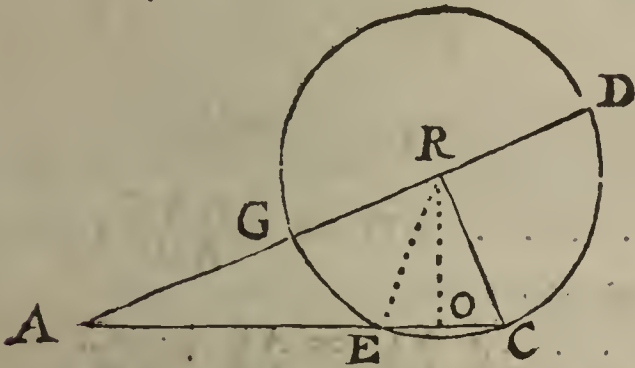
24. That

24. That is } $4 \square EH + \square EH = 4 \square CB.$
 25. And by taking $\frac{1}{4}$ of all in 24°, } $\square EH + \frac{1}{4} \square EH = \square CB = \square CF.$
 26. And because by *Constr.* in 15° , } $CE = \frac{1}{2} EH = CH.$
 27. And consequently, (*per Theor. 3. Chap. 4.*) } $\square CE = \frac{1}{4} \square EH.$
 28. Therefore from 25° and 27°, by exchanging } $\square EH + \square CE = \square CF.$
 equal quantities, } $\angle FEH = \angle = \angle GHE.$
 29. And because by *Constr.* in 16° , } $\square EF + \square CE = \square CF.$
 30. And consequently, (*per prop. 47. Elem. 1.*) } $\square EH + \square CE = \square EF + \square CE.$
 31. Therefore from 28° and 30°, (*per Ax. 1. Ch. 2.*) } $\square EH = \square EF.$
 32. Therefore from 31°, by subtracting $\square CE$ from } $EH = EF.$
 from each part, } $EH = HG.$
 33. But the sides of equal Squares are also equal, } $EF = \text{and } \parallel HG.$
 therefore from 32°, } $EH = \text{and } \parallel FG.$
 34. And by arguing in like manner as in the six last } $EFGH \text{ is equilateral.}$
 preceding steps, it will be manifest that } $EFGH \text{ is right-angled.}$
 35. And because from 33°, 34° and 29°, (*per* } $EFGH \text{ is a } \square.$
Ax. 1. & prop. 28. Elem. 1.) }
 36. Therefore from 35°, (*per prop. 33. Elem. 1.*) }
 37. Therefore from 33°, 34° and 36°, }
 38. And from 29°, and *Coroll. prop. 29. Elem. 1.* }
 39. Therefore from 37° and 38°, (*per def. 29. El. 1.*) }

Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. IV.

The Hypothenufal of a right-angled Triangle being given, as also the summ of the leggs containing the right angle, to find the Triangle. But the summ of the leggs must be greater than the Hypothenufal, yet not greater than the right line arising by application of the double Square of the Hypothenufal to the summ of the leggs.



AC = 169	AE = 119
AR = 156	EC = 50
RC = 65	OE = 25
AD = 221	OC = 25
AG = 91	RO = 60

Suppos.

1. ARC is a Δ right-angled at R.
 2. $b = AC$ the Hypothenufal is given.
 3. $b = AR + RC$, the summ of the leggs is given.
- Req. to find ΔARC .

Resolution.

4. Supposing the leggs about the right angle to be unequal, to wit, $AR > RC$, put a for their difference, *viz.* } $a = AR - RC = AG$
5. Therefore from 3° and 4°, the summ of the Squares of the leggs (*per Theor. 6. Chap. 4.*) is } $\frac{1}{2} bb + \frac{1}{2} aa.$
6. Therefore from 5° and 2°, (*per prop. 47. Elem. 1.*) this Equation ariseth, } $\frac{1}{2} bb + \frac{1}{2} aa = bh.$
7. And by doubling each part of that Equation, } $bb + aa = 2bh.$
8. And by subtracting bb from each part of the last Equation, } $aa = 2bh - bb.$
9. Therefore by extracting the square Root out of each part of the last preceding Equation, } $a = \sqrt{2bh - bb}.$

Hence this

M m

CANON.

CANON.

10. The difference of the leggs about the right angle is equal to the Square Root of the excess whereby the double Square of the Hypothensal exceeds the Square of the sum of the leggs.

Therefore the difference of the leggs is given, and consequently by the given sum and difference of the leggs, the leggs shall be given severally, per Theor. 9. Chap. 4.

11. But in order to the Geometrical Effect of the Problem propounded, the truth and reason of the Determination annex'd to it must be made manifest. First then, the reason of the first part of the Determination, to wit, that the right line given for the sum of the leggs about the right angle must be longer than the line given for the Hypothensal, is evident by prop. 22. Elem. 1. which shews that the sum of every two sides of a plain Triangle is greater (or longer) than the third. The latter part of the Determination is discovered by the Canon, which requires that bb be subtracted from $2hh$, and therefore bb must not exceed $2hh$, and consequently, by dividing as well bb as $2hh$ by b , 'tis manifest that b must not be greater than $\frac{2hh}{b}$; that is, the right line given for the sum of the leggs about the right angle, must not be longer than the right line arising by the Application of the double Square of the Hypothensal to the sum of the leggs. The truth of this Determination will more fully appear by the following

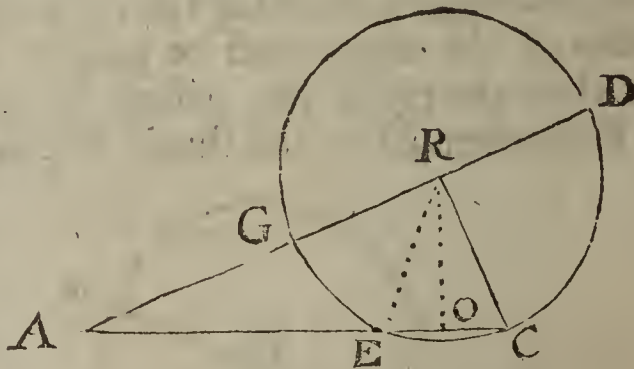
THEOREM.

12. In a right-angled plain Triangle, the sum of the leggs about the right angle is sometimes less than the right line arising by the Application of the double Square of the Hypothensal to the sum of the leggs, and sometimes equal to, but never greater than the said right line.

The leggs about the right angle are either unequal, or else equal between themselves; I shall begin with the first Case.

Suppos. in Case 1.

- 13. $\triangle ARC$ is a \triangle .
- 14. $\angle ARC$ is \perp .
- 15. $AR \neq RC$.
- 16. $RCGD$ is a \odot , whence
- 17. $AD = AR + RC$ (RD.)
- 18. $AG = AR - RC$ (RG.)



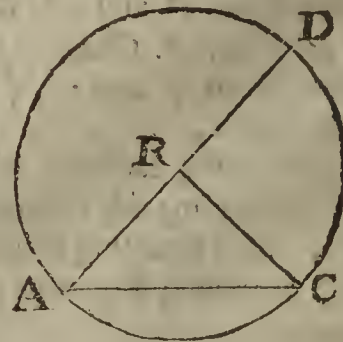
19. . . Req. demonstr. $AD = \frac{2 \square AC}{AD}$.

Demonstration.

- 20. By Supposition in 17°, $AD = AR + RC$.
 - 21. And by Suppos. in 18°, $AG = AR - RC$.
 - 22. Therefore from 20° and 21, (per Theor. 6. Ch. 4.) $\frac{1}{2} \square AD + \frac{1}{2} \square AG = \square AR + \square RC$.
 - 23. And because by Suppos. in 14°, $\angle ARC$ is \perp ; therefore (per prop. 47. Elem. 1.) $\square AC = \square AR + \square RC$.
 - 24. Therefore from 22° and 23°, (per Ax. 1. Ch. 2.) $\frac{1}{2} \square AD + \frac{1}{2} \square AG = \square AC$.
 - 25. And by doubling the last Equation, $\square AD + \square AG = 2 \square AC$.
 - 26. Therefore from 25°, $\square AD = 2 \square AC$.
 - 27. Therefore by Application of each part to AD, $AD = \frac{2 \square AC}{AD}$.
- Which was Case 1. to be Dem.

Suppos. in Case 2.

- 28. $\triangle ARC$ is a \triangle .
- 29. $\angle ARC$ is \perp .
- 30. $AR = RC = RD$.
- 31. $AD = AR + RC$.
- 32. . . Req. demonstr. . . $AD = \frac{2 \square AC}{AD}$.



Demonstr.

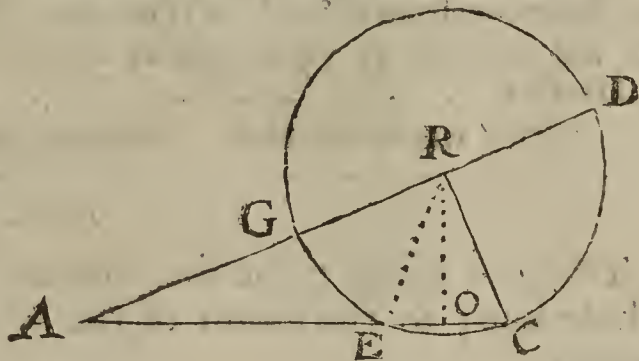
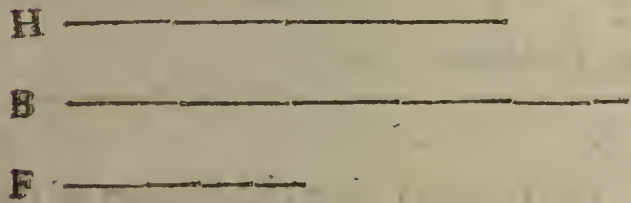
Demonstration.

- 33. Because by *Suppos.* in 30° and 31° , } $AR = RC = \frac{1}{2} AD.$
- 34. Therefore their Squares are also equal, *viz.* } $\square AR = \frac{1}{4} \square AD.$
- 35. Likewise from 33° , } $\square RC = \frac{1}{4} \square AD.$
- 36. The sum of the Equations in 34° and 35° , gives, } $\square AR + \square RC = \frac{1}{2} \square AD.$
- 37. And because by *Suppos.* $\angle ARC$ is \perp , therefore } $\square AR + \square RC = \square AC.$
(*per prop. 47. Elem. 1.*) }
- 38. Therefore from 36° and 37° , (*per Ax. 1. Chap. 2.*) } $\frac{1}{2} \square AD = \square AC.$
- 39. And by doubling the last Equation, } $\square AD = 2 \square AC.$
- 40. Therefore by Application of each part to AD , . . . } $AD = \frac{2 \square AC}{AD}.$

Which was *Case 2.* to be Demonstr.

41. Now because in every right-angled Triangle, the sides about the right angle are either unequal or equal between themselves, and it hath been demonstrated, that when the said sides are unequal, their sum is less than the right line arising by the Application of the double Square of the Hypotenusal to the said sum; but when the said sides are equal to one another, their sum is equal to the said right line; it is evident that the sum of the sides about the right angle can never be greater than the right line arising by the said Application. Therefore the truth of the Theorem is manifest, and consequently the Hypotenusal and the sum of the legs about the right angle must be given with due Caution, according to the import of the Determination annex'd to the Problem, that its Solution may be possible.

The Composition of the foregoing Probl. 4.



Suppos.

- 42. $H =$ the Hypotenusal of a right-angled Triangle is given.
- 43. $B = AD$ the sum of the legs about the right angle is given.
- 44. $AD < H$, but AD not $< \frac{2 \square H}{AD}$. (*Determination.*)

Req. to find the Triangle.

Construction.

- 45. By the *Determination* in 44° AD is not greater than $\frac{2 \square H}{AD}$; suppose then it be granted, or discovered by H and B given in numbers, that AD is less than $\frac{2 \square H}{AD}$, and consequently, (by multiplying each part by AD ,) that $\square AD < 2 \square H$; then it evidently follows, that 'tis possible (*per Probl. 4. Chap. 5.*) to find out a right line F ; such, that its Square shall be equal to $2 \square H - \square AD$; suppose therefore

$$F = \sqrt{2 \square H - \square AD} :$$

- 46. From AD cut off $AG = F$, which may be done, for that AD is greater than F , I prove thus;

- By *Suppos.* in 44°, } $AD < H.$
- Therefore } $\square AD < \square H.$
- And by doubling each part, } $2 \square AD < 2 \square H.$
- And by subtracting $\square AD$ from each part, } $\square AD < 2 \square H - \square AD.$
- But by *Constr.* in 45°, } $\square F = 2 \square H - \square AD.$
- Therefore from the two last preceding steps, (*per Ax. 3. Ch. 2.*) } $\square AD < \square F.$
- Therefore } $AD < F.$

Which was to be Dem.

- 47. Divide GD into two equal parts in R , therefore $RG = RD.$

Mm 2

48. Make

48. Make $RC \perp AR$; also $RC = RD$ or RG , and draw AC .
 49. I say ARC is the Triangle sought. Now we must shew that it will satisfy the Problem; first then by *Construction* in 48° , $RC \perp AR$, therefore the angle ARC is a right angle; secondly, the sum of the legs AR and RC about the right angle, is equal to $AD (= B)$ the given sum of the legs. It remains only to prove that AC is equal to the given Hypothensal H ; but that will be made manifest by the following *Demonstration*, form'd out of the preceding Resolution by a repetition of its steps in a backward (not direct) order.

50. . . . *Req. demonstr.* $AC = H$.

Demonstration.

51. By <i>Constr.</i> in 45° ,	>	$\sqrt{2 \square H - \square AD} = F$.
52. And by <i>Constr.</i> in 46° ,	> $AG = F$.
53. Therefore from 51° and 52° , (<i>per Ax. 1. Chap. 2.</i>)	>	$AG = \sqrt{2 \square H - \square AD}$:
54. But the Squares of equal right lines are also equal, } therefore from 53° , }	>	$\square AG = 2 \square H - \square AD$.
55. Therefore from 54° , by adding $\square AD$ to each part, } 56. And by taking the halves of all in 55° , }	>	$\square AD + \square AG = 2 \square H$, $\frac{1}{2} \square AD + \frac{1}{2} \square AG = \square H$.
57. And because by <i>Constr.</i> in 47° and 48° , AD is the } sum, and AG the difference of the parts AR and } RC , (or RD), therefore (<i>per Theor. 6. Chap. 4.</i>) }	>	$\frac{1}{2} \square AD + \frac{1}{2} \square AG = \square AR + \square RC$.
58. Therefore from 56° and 57° , (<i>per Ax. 1. Cha. 2.</i>)	>	$\square H = \square AR + \square RC$.
59. But by <i>Constr.</i> in 48° , $\angle ARC$ is \perp , and con- } sequently, (<i>per prop. 47. Elem. 1.</i>) }	>	$\square AC = \square AR + \square RC$.
60. Therefore from 58° and 59° , (<i>per Ax. 1. Chap. 2.</i>)	>	$\square AC = \square H$.
61. But the sides of equal Squares are also equal, } therefore }	>	$AC = H$.

Which was to be Dem. Therefore the Problem is satisfied.

Probl. V.

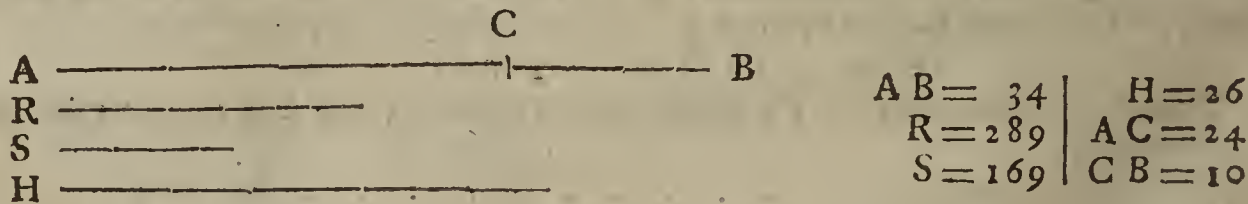
To cut a given right line into two such parts, that the Square of the whole line, to the sum of the Squares of the parts may have a given Reason.

Or thus,

In a right-angled Triangle, the sum of the legs about the right angle being given, as also the Proportion which the Square of the said sum hath to the Square of the Hypothensal, to find the Triangle. But the given quantities must be liable to this

Determination.

The first Term of the given Reason must be greater than the latter Term, yet not greater than the double of the latter Term. For in a right-angled Triangle, the Square of the sum of the legs about the right angle is always greater than the Square of the Hypothensal, but never greater than the double Square of the Hypothensal, as hath been demonstrated in the preceding *Probl. 4.*



Suppos.

1. AB is a right line given to be cut into two parts.
2. R and S are the Terms of a given Reason.
3. $R \sqsupset S$; but R not $\sqsupset 2S$.

Req. to find

4. AC and CB such parts of AB , that $AC + CB = AB$. Also,
5. $\square AB \cdot \square AC + \square CB :: R \cdot S$.

Con-

Construction.

6. By *Probl. 11. Chap. 5.* let it be made, as R to S; so the Square of AB to another Square, whose side suppose to be H; therefore

$$R . S :: \square AB . \square H.$$

7. Then supposing H to be the Hypotenusal of a right-angled Triangle, and AB the sum of the sides about the right-angle, find out the said sides and Triangle by the foregoing *Probl. 4.* For if R be greater than S, but not greater than 2S, according to the import of the Determination added to that Problem, 'tis possible to find out such a right-angled Triangle, and then the sides about the right-angle shall be equal to AC and CB, the parts sought by this *Probl. 5.* Therefore,

8. . . *Req. demonstr.* R . S :: $\square AB . \square AC + \square CB.$

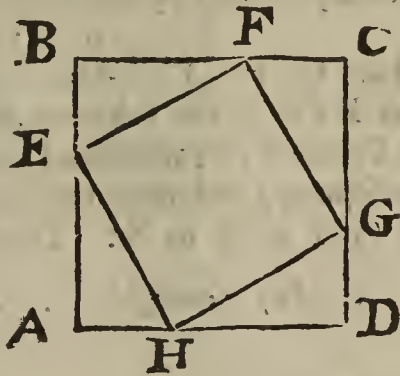
Demonstration.

- 9. Because by *Constr.* in 6°, } R . S :: $\square AB . \square H.$
 - 10. And by *Constr.* in 7°, } $\square AC + \square CB = \square H.$
 - 11. Therefore from 9° and 10°, by exchanging } R . S :: $\square AB . \square AC + \square CB.$
equal quantities, }
- Which was to be Dem. Therefore the Problem is satisfied.

A LEMMA, leading to the following *Probl. 6.*

Suppos.

- 1. ABCD is a Square.
- 2. BF and FC are parts of BC.
- 3. CF = BE = AH = DG; therefore,
- 4. BF = AE = DH = CG.
- 5. EF, FG, GH, HE are right lines.



6. . . *Req. demonstr.* that EFGH is a Square.

Demonstration.

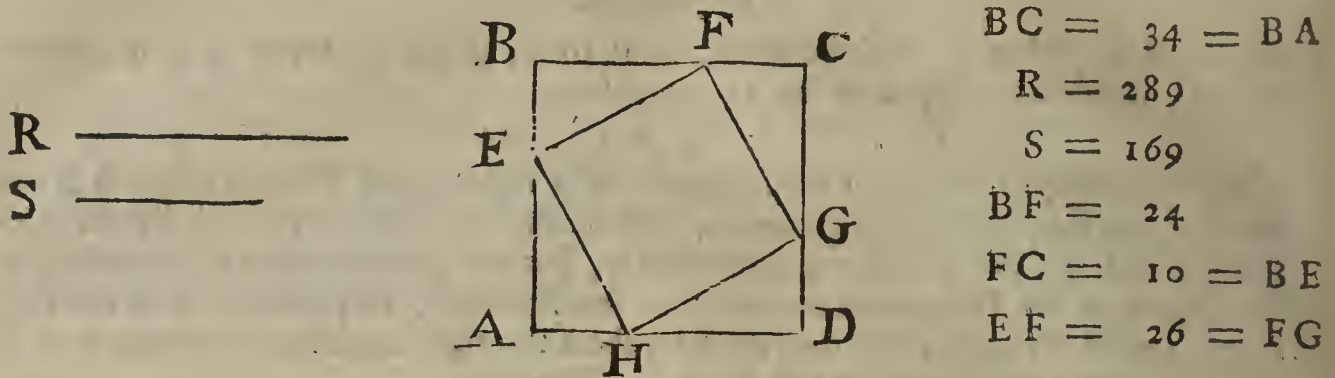
- 7. Because by *Suppos.* in 3° and 4°, } CF = BE; and BF = CG.
- 8. And by *Suppos.* in 1°, } $\angle EBF = \angle FCG.$
- 9. Therefore from 7° and 8°, (*per prop. 4. Elem. 1.*) } EF = FG; and $\angle BEF = \angle GFC.$
- 10. And in like manner, } EF = EH = HG = FG.
- 11. Again, because by *Suppos.* in 1°, } $\angle EBF$ is $\perp.$
- 12. Therefore (*per Coroll. prop. 32. Elem. 1.*) } $\angle BFE + \angle BEF = \perp.$
- 13. But it hath been proved in 9°, that } $\angle GFC = \angle BEF.$
- 14. Therefore from 12° and 13°, (*per Ax. 6. Chap. 2.*) } $\angle BFE + \angle GFC = \perp.$
- 15. And from 14°, (*per Coroll. prop. 13. Elem. 1.*) } $\angle EFG$ is $\perp.$
- 16. And by arguing in like manner as in the five } $\angle FGH = \perp = \angle GHE = \angle GEF.$
last preceding steps, it will be evident that . . . }
- 17. Therefore from 10°, 15° and 16°, (*per Defin. 29. Elem. 1.*) } EFGH is a Square.

Which was to be Dem.

Probl. VI.

In a given Square to inscribe another Square whose angular points may lye in the sides of the given Square; and that the Square given to the Square inscribed may be in a given Reason, suppose as R to S. But R must be greater than S, yet not greater than 2S, as may easily be inferr'd from the preceding *Probl. 5.*

R -----



- Suppos.*
1. ABCD is a Square, whose side AB or BC is given.
 2. R and S are the Terms of a given Reason.
 3. $R \sqsupset S$; but R not $\sqsupset 2S$.

- Req. to inscribe*
4. $\square EFGH$ in the $\square ABCD$. Also, that
 5. $\square ABCD \cdot \square EFGH :: R \cdot S$.

- Construction.*
6. By the foregoing *Probl. 5.* of this *Chapt.* cut BC the side of the given Square into two such parts in F, that the Square of BC, that is, $\square ABCD$, may be in such proportion to the summ of the Squares of the parts BF, FC, as R to S; which may be done, because by *Supposition* $R \sqsupset S$, yet R not $\sqsupset 2S$; suppose therefore
 $R \cdot S :: \square BC \cdot \square BF + \square FC$.
 7. Make $BE = CF = AH = DG$, and draw the right lines EF, FG, GH and HE; so is EFGH the inscribed Square required. Now we must shew that it will satisfy the Problem, to which end, two things are to be proved, *viz.* First, that EFGH is a Square, and then that the Square ABCD hath such proportion to the Square EFGH as R to S: Therefore,

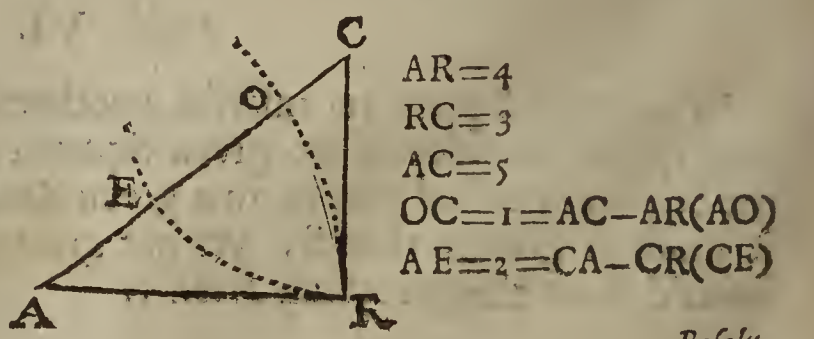
8. . . . *Req. demonstr.* } EFGH is a \square . Also, that
 } $R \cdot S :: \square ABCD \cdot \square EFGH$.

- Demonstration.*
9. By *Suppos.* in 1°, } ABCD is a \square .
 10. Therefore } $BC = BA = AD = DC$.
 11. By *Constr.* in 7°, } $CF = BE = AH = DG$.
 12. Therefore by subtracting the Equations in 11° } $BF = AE = DH = CG$.
 from those in 10°, there will remain }
 13. Therefore out of 9°, 10°, 11° and 12°, by the } EFGH is a \square .
Lemma prefixt before this Problem,)
 14. Again by *Constr.* in 6°, } $R \cdot S :: \square BC \cdot \square BF + \square FC$.
 15. And because by *Suppos.* in 1°, } $\angle EBF$ is \perp .
 16. And consequently, (*per prop. 47. Elem. 1.*) } $\square EF = \square BF + \square BE (\square FC)$.
 17. Therefore from 14° and 16°, by exchanging equal quantities,
 $R \cdot S :: \square BC$ (or $\square ABCD$) $\cdot \square EF$ (or $\square EFGH$.)
 Which was to be *Demonstr.* Therefore the Problem is satisfied.

Probl. VII.

In a right-angled Triangle, the difference between the Hypothensal and each of the sides about the right angle being given, to find the Triangle.

- Suppos.*
1. ARC is a Δ right-angled at R.
 2. $b = OC = AC - AR$ is given.
 3. $d = AE = CA - CR$ is given; whence,
 4. $g = d + b$ is given. And,
 5. $kk = dd + bb$ is given.
- Req. to find ΔARC .*



- AR=4
 RC=3
 AC=5
 OC=1=AC-AR(AO)
 AE=2=CA-CR(CE)

Resolu-

Resolution.

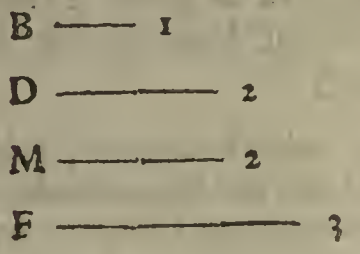
- 6. Put a for the excess of the Hypothenusal above the summ of the given differences, viz. } $a = EO.$
- 7. Therefore from the premisses the Hypothenusal shall be } $a + g (= AC.)$
- 8. And the Base } $a + d (= AR.)$
- 9. And the Perpendicular } $a + b (= CR.)$
- 10. Therefore from 7° the Square of the Hypothenusal is } $aa + 2ga + gg.$
- 11. And from 8° the Square of the Base is } $aa + 2da + dd.$
- 12. And from 9° the Square of the Perpendicular is . } $aa + 2ba + bb.$
- 13. Therefore the summ of all in 11° and 12°, gives the summ of the Squares of the Base and Perpendicular, viz. } $2aa + 2da + 2ba + dd + bb.$
- 14. That is, (because by *Suppos.* in 5°, $kk = dd + bb,$ and from 4°, $2g = 2d + 2b,$) } $2aa + 2ga + kk.$
- 15. Therefore from 10° and 14°, this Equation ariseth, (per prop. 47. Elem. 1.) } $2aa + 2ga + kk = aa + 2ga + gg.$
- 16. Therefore, by subtracting $aa + 2ga$ from each part of that Equation, this ariseth, } $aa + kk = gg.$
- 17. And by subtracting kk from each part of the last Equation, } $aa = gg - kk.$
- 18. But from 4°, } $dd + bb + 2db = gg.$
- 19. And from 5°, } $dd + bb = kk.$
- 20. And by subtracting the Equation in 19° from that in 18°, this remains, } $2db = gg - kk.$
- 21. Therefore from 17° and 20°, (per Ax. 1. Chap. 2.) } $aa = 2db.$
- 22. Therefore by extracting the square Root out of each part of the last Equation, it gives } $a = \sqrt{2db}.$
- 23. Therefore out of 22°, 6°, 7° and 4°, the Hypothenusal is given, to wit, } $d + b + \sqrt{2db} = AC.$
- 24. And from 22°, 6°, 8° and 3°, the Base is also given, viz. } $d + \sqrt{2db} = AR.$
- 25. And from 22°, 6°, 9° and 3°, the Perpendicular is also discovered, viz. } $b + \sqrt{2db} = CR.$

Which three last preceding steps give this

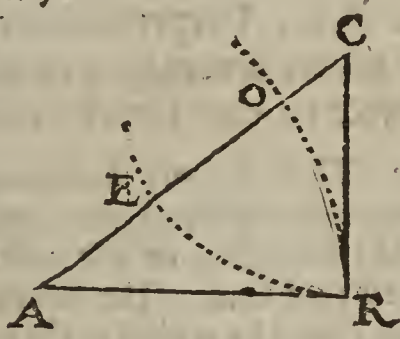
CANON.

- 26. To the summ of the given differences add the square Root of their double Rectangle; so shall the summ of that Addition be the Hypothenusal sought. Then add that square Root to the given differences severally, and these two summs shall be the desired sides about the right angle.

The Composition of Probl. 7.



- AR = 4
- RC = 3
- AC = 5



Suppos.

- 27. B = the excess whereby the Hypothenusal of a right-angled Triangle exceeds one of the sides about the right angle is given.
- 28. D = the excess of the Hypothenusal above the other side about the right angle is given also.

Req. to find out the Triangle.

Construction.

- 29. By *Probl. 9. Chap. 5.* find a mean Proportional line, as M, } $2D \cdot M :: M \cdot B.$
between $2D$ and B , therefore, }
- 30. Then

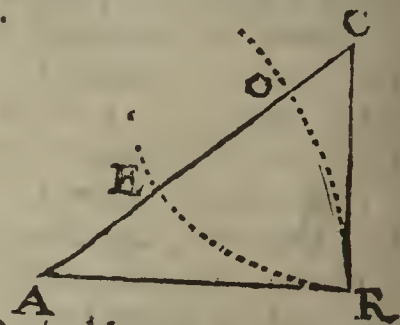
30. Then let a Triangle, as ARC, be made of three right lines equal to these given, to wit, $D+B+M$, $D+M$, $B+M$; which may be done, (*per probl. 22. Elem. 1.*) for the summ of every two of those three lines is manifestly greater than the third; suppose therefore

31. I say ARC is the Triangle required. Now we must shew that it will satisfy the Problem. First then, 'tis manifest that the difference between AC, that is, $D+B+M$, and AR, that is, $D+M$, is equal to the given difference B; also the difference between AC, that is, $D+B+M$, and RC, that is, $B+M$, is manifestly equal to the given difference D. So it remains only to prove that the angle ARC is a right angle, which will be made manifest by the following Demonstration.

$$\left. \begin{aligned} AC &= D+B+M, \\ AR &= D+M, \\ RC &= B+M. \end{aligned} \right\}$$

Prepar.

32. Make $F = B + M$, therefore
 33. From 30° and 32°, $AC = D + F$. Also $RC = F$.
 34. . . Req. demonstr. $\angle ARC = \perp$.



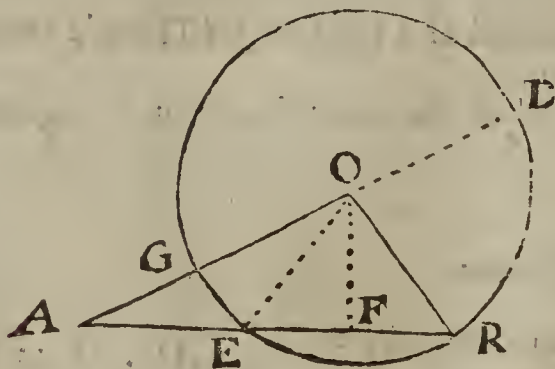
Demonstration.

35. Because by Constr. in 30°, } $AR = D + M$.
 36. Therefore, (*per Theor. 2. Chap. 4.*) . . . } $\square AR = \square D + \square M + 2\square DM$.
 37. By Constr. in 32°, } $F = B + M$.
 38. Therefore from 37°, by drawing $2D$ into } $2\square DF = 2\square DB + 2\square DM$.
 each part, (*per prop. 1. Elem. 6.*) . . . }
 39. But from the Constr. in 29°, it follows } $\square M = 2\square DB$.
 (*per prop. 17. Elem. 6.*) that . . . }
 40. Therefore from 38° and 39°, (*per Ax. 6. } $2\square DF = \square M + 2\square DM$.
 Chap. 2.*) }
 41. Likewise from 36° and 40°, } $\square AR = \square D + 2\square DF$.
 42. By Constr. in 33°, } $RC = F$.
 43. And consequently, } $\square RC = \square F$.
 44. Therefore the summ of the Equations in 41° } $\square AR + \square RC = \square D + 2\square DF + \square F$.
 and 43°, gives (*per Ax. 8. Chap. 2.*) . . . }
 45. By Constr. in 33°, } $AC = D + F$.
 46. And consequently, (*per Theor. 2. Chap. 4.*) } $\square AC = \square D + 2\square DF + \square F$.
 47. Therefore from 44° and 46°, (*per Ax. 1. } $\square AC = \square AR + \square RC$.
 Chap. 2.*) }
 48. Therefore, (*per prop. 48. Elem. 1.*) . . . } $\angle ARC = \perp$.
 Which was to be Dem. Therefore the Problem is satisfied.

Probl. VIII.

The Base, Perpendicular and summ of the leggs of a plain Triangle being given severally, to find the Triangle. But the lines given must be subject to the Determination hereafter declared.

Note. There is more than enough given in this Problem, unless it requires a Triangle that hath either unequal acute angles at the Base, in which Case the Perpendicular falls within the Triangle; or else is obtusangled at the Base, in which latter Case the Perpendicular falls without. The following Resolution handles the first Case, but with a little alteration it may be applied to the latter, as will hereafter appear.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

Preparat.

1. Suppose ARO to be the Triangle sought, having unequal acute angles A and R at the ends of the Base AR; then from the Center O, at the distance of the lesser legg OR, describe the Circle ORGD cutting the greater legg OA in G; so shall AG be the difference of the leggs OA and OR, for OG = OR.
2. Produce AO to the Circumference in D, then is AD equal to the summ of the leggs AO and OR; for OD = OR.
3. Draw the Semidiameter OE, and let fall OF ⊥ ER, so will OF cut ER into two equal parts in F, (per prop. 3. Elem. 3.) These things premised, the Resolution of the Problem propos'd may be formed in manner following.

Suppos.

4. $b = AR$, the Base of $\triangle ARO$ is given.
5. $p = OF$ the Perpendicular is given.
6. $c = AD = AO + OR$, the summ of the leggs is given.

Req. to find the Triangle.

Resolution.

7. Put a for the unknown difference of the leggs, viz. assume $a = AG (= AO - OR.)$
8. Then the difference of the segments of the Base made by the falling of the Perpendicular shall be $\frac{ca}{b}$, for, (by Theor. 2. Probl. 9. Chap. 7.) $b \cdot c :: a \cdot \frac{ca}{b} (A.E.)$
9. Therefore from 4° and 8° , (by Theor. 9. Chap. 4.) the greater segment of the Base shall be $\frac{1}{2}b + \frac{ca}{2b} (= AF.)$
10. And (by the Theorem last mentioned,) the lesser segment shall be $\frac{1}{2}b - \frac{ca}{2b} (= FR.)$
11. Therefore the Square of the greater segment (in 9°) shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca.$
12. And the Square of the lesser segment (in 10°) shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca$
13. By prop. 47. Elem. 1. $\square AF + \square FO = \square AO.$
14. Therefore out of $11^\circ, 13^\circ$ and 5° , the Square of the greater legg AO shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp.$
15. And out of 5° and 12° , the Square of the lesser legg OR shall be $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp.$
16. Therefore from 14° and 15° , the summ of the Squares of the leggs shall be $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp.$

Now because by Theor. 2. Chapt. 4.

$$\square AO + \square RO + 2\square AO, RO = \square : AO + OR : = \square AD;$$

Therefore to the end the quantities in 16° may be made a compleat Square of AD, let the double Rectangle of the leggs AO and RO be found out in this manner, viz.

17. From 6° and 7° , (by Theor. 9. Chap. 4.) the greater legg is $\frac{1}{2}c + \frac{1}{2}a (= AO.)$
18. And (by the Theorem last mentioned,) the lesser legg is $\frac{1}{2}c - \frac{1}{2}a (= RO.)$
19. Therefore the Rectang. of the legs is $\frac{1}{4}cc - \frac{1}{4}aa (= \square AO, RO)$
20. And the double Rectangle of the leggs is $\frac{1}{2}cc - \frac{1}{2}aa (= 2\square AO, RO.)$
21. Therefore the summ of all in 16° and 20° , gives the Square of the summ of the leggs, viz. $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa (= \square AD)$

N n

22. Which

22. Which quantities in 21° must be equal to cc the Square of the given summ of the leggs; hence this Equation,

$$\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa = cc.$$

23. Therefore by subtracting $\frac{1}{2}cc$ from each part of that Equation, this will arise,

$$\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp - \frac{1}{2}aa = \frac{1}{2}cc.$$

24. And by doubling all in 23°, $bb + \frac{ccaa}{bb} + 4pp - aa = cc.$

25. Now to the end that known quantities may be separated from unknown, enquiry must be made, whether $\frac{ccaa}{bb}$ be greater or less than aa . But because (as appears in 8°,) these are Proportionals, *viz.* $b \cdot a :: c \cdot \frac{ca}{b}$, and $c > b$, for the summ of the leggs of any plain Triangle is longer than the Base, therefore (*per prop.* 14. *Elem.* 5.) $\frac{ca}{b}$ shall be greater than a , and consequently the Square of the former greater than the Square of the latter, *viz.* $\frac{ccaa}{bb} > aa$, therefore aa may be subtracted from $\frac{ccaa}{bb}$, and there will remain a quantity greater than nothing. From the premisses therefore it is manifest that $bb + 4pp$ may be subtracted from each part of the Equation above exprest in 24°, and the quantity remaining on each part will be greater than nothing, and the Equation arising by that subtraction will be this,

$$\frac{ccaa}{bb} - aa = cc - bb - 4pp.$$

26. That is, by reducing $\frac{ccaa}{bb} - aa$ into the form of a Fraction,

$$\frac{ccaa - bbaa}{bb} = cc - bb - 4pp.$$

27. Which last Equation may be resolved into this Analogy,

$$bb \cdot cc - bb :: aa \cdot cc - bb - 4pp.$$

28. Therefore by Inverse and Altern Reason,

$$cc - bb \cdot cc - bb - 4pp :: bb \cdot aa.$$

29. But the sides of proportional Squares are also Proportionals, therefore from the last preceding Analogy,

$$\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: b \cdot a.$$

Of which last Analogy the three first Terms are given, therefore the fourth Term, which is the difference of the leggs sought, is given also. Moreover, by the same quantities first given, the line AE , which is the difference of the segments of the Base made by the falling of the Perpendicular, shall be given also; for,

30. By *Theor.* 2. *Probl.* 9. *Chap.* 7. $c \cdot AE :: b \cdot a.$

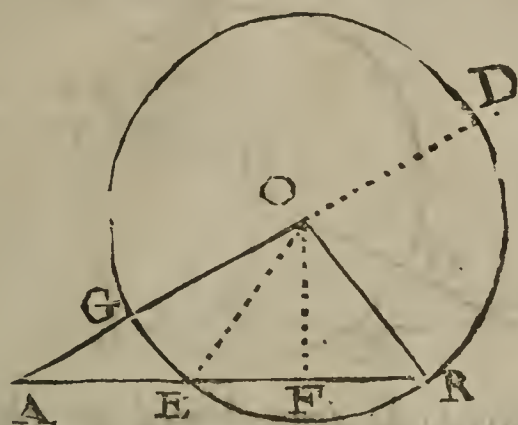
31. Therefore out of 29° and 30°, (by *prop.* 11. *Elem.* 5.)

$$\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: c \cdot AE.$$

32. *Note.* In this Resolution, the Perpendicular is supposed to fall within the Triangle sought, and $\frac{ca}{b}$ (the fourth Proportional of the Analogy in 8°,) represents AE the difference of the segments of the Base made by the falling of the Perpendicular; but when the Perpendicular falls without, as in $\triangle AEO$, where AE is the Base, the said $\frac{ca}{b}$ represents the line AR , which is compos'd of the Base AE , and ER the double of the distance FE from F the foot of the Perpendicular to the obtuse angle at E . But whether the Perpendicular falls within or without the Triangle, the Resolution runs into that Equation before exprest in 22°. Which things being well observed, the Theorems hereafter exprest, (which will be very useful in the following Problems,) will clearly arise out of the preceding Resolution; *viz.*

The Equation in 25° gives

THEOR. I.



THEOR. 1.

33. In a plain Triangle whose leggs are unequal, if the Perpendicular falls within, the Square of the difference of the segments of the Base made by the falling of the Perpendicular, is greater than the Square of the difference of the leggs, by the excess whereby the Square of the summ of the leggs exceeds the summ of the Square of the Base and the Square of the double Perpendicular. But when the Perpendicular falls without the Triangle upon the Base increased, then the Square of the line compos'd of the Base and double distance between the foot of the Perpendicular and the obtuse angle, is greater than the Square of the difference of the leggs, by the excess above-mentioned.

The Analogy in 29° gives

THEOR. 2.

34. As the right line whose Square is equal to the excess whereby the Square of the summ of the leggs of a plain Triangle exceeds the Square of the Base, is to the right line whose Square is equal to the excess of the said Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the Base to the difference of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the excess of the Square of the summ of the leggs above the Square of the Base, is to the excess of the Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the Square of the Base to the Square of the difference of the leggs.

Therefore, if the Base, Perpendicular, and summ of the leggs of a plain Triangle whose leggs are unequal, be given severally, the difference of the leggs shall be given also; and consequently the leggs severally, by *Theor. 9. Chap. 4.*

The Analogy in 31° gives

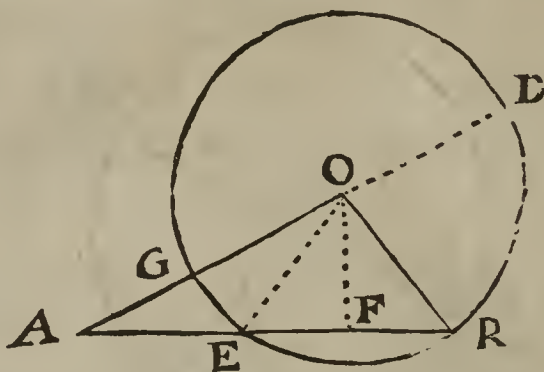
THEOR. 3.

35. As the right line whose Square is equal to the excess whereby the Square of the summ of the unequal leggs of a plain Triangle exceeds the Square of the Base, is to the right line whose Square is equal to the excess of the Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the summ of the leggs to a fourth Proportional, which is less than the Base, when the Perpendicular falls within, for then it is the difference of the segments of the Base made by the Perpendicular. But when the Perpendicular falls without, the said fourth Proportional exceeds the Base, and is compos'd of the Base and double distance between the foot of the Perpendicular and the obtuse angle at the nearer end of the Base. Lastly, when the said fourth Proportional is equal to the Base, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl. 8.* be express'd by numbers, we may discover by *Theor. 3.* above express'd, whether the Triangle sought be acute-angled, or obtuse-angled, or right-angled at the Base, *viz.* of what kind the angles at the Base are.

36. The truth of the three preceding Theorems, when the angles at the Base are acute and unequal, may be demonstrated by the steps of the foregoing Resolution, by proceeding in a direct order from the beginning to the end of the Resolution, in manner following.

Let there be a Triangle propos'd, as ARO, having unequal acute angles at the ends of the Base AR, and let the same things be supposed as before in 1° , 2° and 3° of this Problem.



The three Theorems to be demonstrated are these following, viz.

- 37. $\square AE - \square AG = \square AD - \square AR - 4\square OF.$
- 38. $\sqrt{\square AD - \square AR} : \sqrt{\square AD - \square AR - 4\square FO} :: AR . AG.$
- 39. $\sqrt{\square AD - \square AR} : \sqrt{\square AD - \square AR - 4\square FO} :: AD . AE.$

Demonstration.

- 40. By Theor. 2. in 29° of Probl. 9. Chap. 7. this Analogy } $AR . AD :: AG . AE.$
is manifest, } $b . c :: a . \frac{ca}{b}.$
- 41. And because AR is divided into two unequal parts in } $\frac{1}{2}AR + \frac{1}{2}AE = AF.$
F, and AE is the difference of those parts, therefore (per } $\frac{1}{2}b + \frac{ca}{2b} = AF.$
Theor. 9. Chap. 4.) } $\frac{1}{2}AR - \frac{1}{2}AE = FR.$
- 42. And by the same Theorem, } $\frac{1}{2}b - \frac{ca}{2b} (= FR)$
That is, in 10°, }

- 43. And (by Theor. 2. Chap. 4.) the Square of the Equation in 41° gives
 $\frac{1}{4}\square AR + \frac{1}{4}\square AE + \frac{1}{2}\square AR, AE = \square AF.$
That is, } $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca (= \square AF.)$
in 11°, }

- 44. And (by Theor. 5. Chap. 4.) the Square of the Equation in 42° gives
 $\frac{1}{4}\square AR + \frac{1}{4}\square AE - \frac{1}{2}\square AR, AE = \square FR.$
That is, } $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca (= \square FR.)$
in 12°, }

- 45. And by adding $\square OF$ to each part of the Equation in 43°, it makes (per prop 47. Ele. 1.)
 $\frac{1}{4}\square AR + \frac{1}{4}\square AE + \frac{1}{2}\square AR, AE + \square OF = \square AO.$
That is, } $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp (= \square AO.)$
in 14°, }

- 46. And by adding $\square OF$ to each part of the Equation in 44°, this arifeth, (per pro. 47. El. 1.)
 $\frac{1}{4}\square AR + \frac{1}{4}\square AE - \frac{1}{2}\square AR, AE + \square OF = \square RO.$
That is, } $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp (= \square RO.)$
in 15°, }

- 47. And the summ of the Equations in 45° and 46° gives
 $\frac{1}{2}\square AR + \frac{1}{2}\square AE + 2\square OF = \square AO + \square RO.$
That is, } $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp (= \square AO + \square RO.)$
in 16°, }

It is manifest (per Theor. 2. Chap. 4.) that the Equation in 47° wants only $2\square AO, RO$ to complet the Square of $\square AO + \square RO$, that is, the Square of AD; therefore in order to fill up that Square, I proceed as in the five steps next following.

- 48. By Theor. 9. Chap. 4. } $\frac{1}{2}AD + \frac{1}{2}AG = AO.$
That is, in 17°, } $\frac{1}{2}c + \frac{1}{2}a = AO.$
- 49. And by the same Theorem, } $\frac{1}{2}AD - \frac{1}{2}AG = RO.$
That is, in 18°, } $\frac{1}{2}c - \frac{1}{2}a = RO.$
- 50. And by Theor. 8. Chap. 4. the Rectangle of } $\frac{1}{4}\square AD - \frac{1}{4}\square AG = \square AO, RO.$
the Equations in 48° and 49° is } $\frac{1}{4}cc - \frac{1}{4}aa = \square AO, RO.$
- That is, in 19°, }

51. And

51. And by doubling the Equation in 50°, . . . } $\frac{1}{2} \square AD - \frac{1}{2} \square AG = 2 \square AO, RO.$
 That is, in 20°, } $\frac{1}{2} cc - \frac{1}{2} aa = 2 \square AO, RO.$

52. The summ of the Equations in 47° and 51° makes the Square of the summ of the leggs AO and RO, viz. the Square of AD,

$$\frac{1}{2} \square AR + \frac{1}{2} \square AE + 2 \square OF + \frac{1}{2} \square AD - \frac{1}{2} \square AG = \square AD.$$

That is, }
 in 22°, } $\frac{1}{2} bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2} cc - \frac{1}{2} aa = cc.$

53. And by subtracting $\frac{1}{2} \square AD$ from each part of the Equation in 52°, this remains,

$$\frac{1}{2} \square AR + \frac{1}{2} \square AE + 2 \square OF - \frac{1}{2} \square AG = \frac{1}{2} \square AD.$$

That is, }
 in 23°, } $\frac{1}{2} bb + \frac{ccaa}{2bb} + 2pp - \frac{1}{2} aa = \frac{1}{2} cc.$

54. And by doubling the Equation in 53°,

$$\square AR + \square AE + 4 \square OF - \square AG = \square AD.$$

That is, }
 in 24°, } $bb + \frac{ccaa}{bb} + 4pp - aa = cc.$

55. And because (by prop. 8. Elem. 3.) $\square AE \square AG$, and consequently $\square AE \square AG$, therefore $\square AE - \square AG \square O$; whence it is manifest, that if $\square AR + 4 \square OF$ be subtracted from each part of the Equation in 54°, there will remain on each part a quantity greater than nothing, and the Equation arising by that subtraction will be this that follows, viz.

$$\square AE - \square AG = \square AD - \square AR - 4 \square OF.$$

That is, }
 in 25°, } $\frac{ccaa}{bb} - aa = cc - bb - 4pp.$

And in }
 26°, } $\frac{ccaa - bbaa}{bb} = cc - bb - 4pp.$

Which was Theor. 1. to be demonstr.

Now to pass from the 26th step to the 27th of the preceding Algebraical Resolution, by the lines of the Diagram, some Analogies, not expressd in the Resolution, must be introduced; and in order to their discovery, the Learner may observe, that the Algebraical Fraction $\frac{ccaa - bbaa}{bb}$ in 26° denotes a Plane, which is the fourth Term of an Analogy

whose three first Terms are these three Planes, to wit, bb , $cc - bb$ and aa , which answer to these three Planes, (in the lines of the Diagram,) to wit, $\square AR$, $\square AD - \square AR$ and $\square AG$; therefore, $\square AE - \square AG$ which is correspondent to the said Algebraical Fraction

$\frac{ccaa - bbaa}{bb}$, must likewise be the fourth Term of an Analogy whose three first Terms are the said Planes $\square AR$, $\square AD - \square AR$ and $\square AG$; but how the said Analogy is brought to light, the four steps next following will shew.

56. Because, (as hath been shewn in 40°)

$$AD . AR :: AE . AG.$$

57. Therefore, (per prop. 22. Elem. 6.)

$$\square AD . \square AR :: \square AE . \square AG.$$

58. Therefore by Division of Reason,

$$\square AD - \square AR . \square AR :: \square AE - \square AG . \square AG.$$

59. Therefore inverfly,

$$\square AR . \square AD - \square AR :: \square AG . \square AE - \square AG.$$

60. But it hath been shewn in 55°, that

$$\square AE - \square AG = \square AD - \square AR - 4 \square OF.$$

61. Therefore from 59° and 60° (by exchanging equal quantities) this Analogy ariseth,

$$\square AR . \square AD - \square AR :: \square AG . \square AD - \square AR - 4 \square OF.$$

That is, in 27°,

$$bb . cc - bb :: aa . cc - bb - 4pp.$$

62. Therefore from 61°, by inverse and altern Reason,

$$\square AD - \square AR . \square AD - \square AR - 4 \square OF :: \square AR . \square AG.$$

That is, in 28°,

$$cc - bb . cc - bb - 4pp :: bb . aa.$$

63. But

63. But the sides of proportional Squares are also Proportionals, therefore from 62° this Analogy ariseth,

$$\sqrt{\square AD - \square AR} . \sqrt{\square AD - \square AR - 4\square OF} :: AR . AG.$$

That is, in 29°,

$$\sqrt{cc - bb} . \sqrt{cc - bb - 4pp} :: b . a.$$

Which was *Theor. 2.* to be Demonstr.

64. Again, because by *Theor. 2.* in 29° of *Probl. 9. Chapt. 7.*

$$AD . AE :: AR . AG.$$

65. Therefore from 63° and 64°, (*per prop. 11. Elem. 5.*)

$$\sqrt{\square AD - \square AR} . \sqrt{\square AD - \square AR - 4\square OF} :: AD . AE.$$

Which was *Theor. 3.* to be Demonstr.

66. Again, it hath been shewn in 62°, that

$$\square AR . \square AG :: \square AD - \square AR . \square AD - \square AR - 4\square OF.$$

67. Therefore by *converse Reason*,

$$\square AR . \square AR - \square AG :: \square AD - \square AR . 4\square OF.$$

Which last Analogy affords

THEOR. 4.

68. As the Square of the Base of any plain Triangle whose leggs are unequal, is to the excess whereby the Square of the Base exceeds the Square of the difference of the leggs; so is the excess whereby the Square of the summ of the leggs exceeds the Square of the Base, to the Square of the double Perpendicular.

Therefore, the Base and leggs of any plain Triangle whose leggs are unequal, being severally given in numbers, the Perpendicular falling upon that Base within the Triangle, or without upon the Base increased, shall be given also in numbers.

From the said *Theor. 4.* and *prop. 41. Elem. 1.* 'tis easie to deduce this following

THEOR. 5.

69. The Rectangle made of these two right lines, to wit, the right line whose Square is equal to the excess whereby a quarter of the Square of the Base of a plain Triangle exceeds a quarter of the Square of the difference of the leggs; and the right line whose Square is equal to the excess of a quarter of the Square of the summ of the leggs above a quarter of the Square of the Base, shall be equal to the Triangle.

To make this manifest, let the $\triangle ARO$ be taken as before in the Resolution, then

$$70. \dots \text{Req. demonstr. } \square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \times \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} : = \triangle ARO.$$

Demonstration.

71. By *Theor. 4.* in 68° of this Problem,

$$\square AR . \square AR - \square AG :: \square AD - \square AR . 4\square OF.$$

72. And by taking $\frac{1}{4}$ of every Term of that Analogy,

$$\frac{1}{4}\square AR . \frac{1}{4}\square AR - \frac{1}{4}\square AG :: \frac{1}{4}\square AD - \frac{1}{4}\square AR . \square OF.$$

73. But the sides of proportional Squares are also Proportionals, therefore from the last Analogy,

$$\frac{1}{2}AR . \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} :: \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} . OF.$$

74. Therefore, (*per prop. 16. Elem. 6.*)

$$\frac{1}{2}\square AR, OF = \square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \times \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR}.$$

75. But (*per prop. 41. Elem. 1.*)

$$\frac{1}{2}\square AR, OF = \triangle ARO.$$

76. Therefore from 74° and 75°, (*per Ax. 1. Chapt. 2.*)

$$\square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \times \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} = \triangle ARO.$$

Which was to be Dem.

Hence the following Canons are deducible, to find out the Area of a plain Triangle Arithmetically, without the help of the Perpendicular, the Base and leggs being severally given in numbers, and the leggs unequal between themselves.

CANON I.

77. From a quarter of the Square of the Base subtract a quarter of the Square of the difference of the leggs, and reserve the remainder; then from a quarter of the Square of the summ of the leggs subtract a quarter of the Square of the Base, and reserve the remainder;

remainder; that done, multiply the first remainder by the second, and extract the Square Root of the Product, so shall that square Root be the Area of the Triangle.

78. Again, because by *Theor. 8. Chapt. 4.*

$$\frac{1}{4} \square AR - \frac{1}{4} \square AG = \square \text{ of } \frac{1}{2} AR + \frac{1}{2} AG \times \frac{1}{2} AR - \frac{1}{2} AG.$$

79. Likewise by the same Theorem,

$$\frac{1}{4} \square AD - \frac{1}{4} \square AR = \square \text{ of } \frac{1}{2} AD + \frac{1}{2} AR \times \frac{1}{2} AD - \frac{1}{2} AR.$$

Therefore from 77°, 78° and 79°, by exchanging equal Factors, there will arise

CANON 2.

80. Multiply these four numbers one into another, to wit,

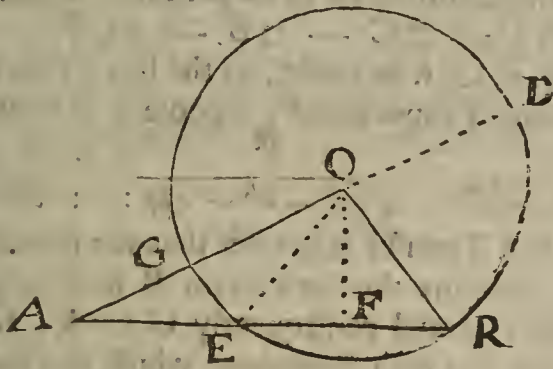
1. The summ of half the Base, and half the difference of the leggs.
2. The excess of half the Base above half the difference of the leggs.
3. The summ of half the summ of the leggs, and half the Base.
4. The excess of half the summ of the leggs above half the Base.

Then extract the square Root of the Product made by the continual multiplication of those four numbers, so shall that square Root be the Area of the Triangle.

81. Again, if we suppose $\left\{ \begin{array}{l} B = \text{the Base,} \\ A = \text{the greater legg,} \\ E = \text{the lesser legg} \end{array} \right\}$ of a plain Triangle.

Then the four numbers above mentioned in *Canon 2.* may be expressd thus, viz.

$$\begin{array}{l} 1. \left| \begin{array}{l} \frac{1}{2} B + \frac{1}{2} A - \frac{1}{2} E, \\ \frac{1}{2} B - \frac{1}{2} A + \frac{1}{2} E, \\ \frac{1}{2} A + \frac{1}{2} E + \frac{1}{2} B, \\ \frac{1}{2} A - \frac{1}{2} E - \frac{1}{2} B. \end{array} \right. \left. \begin{array}{l} \text{Or thus,} \\ \frac{1}{2} B + \frac{1}{2} A + \frac{1}{2} E - E, \\ \frac{1}{2} B + \frac{1}{2} A + \frac{1}{2} E - A, \\ \frac{1}{2} A + \frac{1}{2} E + \frac{1}{2} B, \\ \frac{1}{2} A + \frac{1}{2} E + \frac{1}{2} B - B. \end{array} \right. \end{array}$$



82. Therefore, if the four numbers last before expressd, (to wit, those standing on the right hand,) be multiplied one into another continually, the Product shall be equal to the Square of the Area of the Triangle whose three sides are represented by B, A, E. But if those four numbers be well observed, it will be evident that the number third in order is the half summ of the three sides of the Triangle, and the other three numbers are the Remainers arising by the subtraction of the three sides severally from their half summ; Hence therefore ariseth the vulgar Canon, to find out the Area of any plain Triangle whose three sides are severally given in numbers, viz.

CANON 3.

83. From half the summ of the three sides of any plain Triangle subtract the three sides severally; then multiply the said half summ and the three remainders one into another, according to the Rule of Continual Multiplication, and extract the square Root of the last Product, so shall that square Root be the Area of the Triangle.

Divers other Canons might be raised from the premisses, to find out the Area of a plain Triangle; but 'tis now time to proceed to the Composition of the Problem in hand, and that its Construction may be possible, the lines given must be subject to this

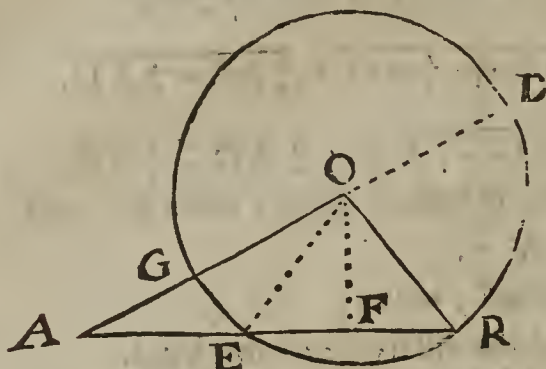
Determination.

84. $c \sqsupset \sqrt{bb - 4pp}$: that is, in words,
The given summ of the leggs must be longer than that right line whose Square is equal to the summ of the Square of the Base and the Square of the double of the Perpendicular.

This Determination doth openly shew it self in *Theor. 2.* in the 34th step of this Problem, and therefore that Theorem having already been demonstrated, the Determination is consequently both true and necessary for limiting the lines given.

The

The Composition of the foregoing Probl. 8.



- B _____
- P _____
- C _____
- H _____
- I _____
- K _____

Suppos.

- 85. B = the Base of a Triangle is given.
- 86. P = the Perpendicular is given.
- 87. C = the summ of the leggs is given.
- 88. $C \sqsupseteq \sqrt{\square B + 4 \square P}$: (Determination.)

Req. to make the Triangle.

Construction.

89. By *Probl. 4. Chap. 5.* find a right line H, such, that its Square may be equal to $\square C - \square B$; which Effecttion is possible, as is evident by the Determination prescribed in 88°, therefore suppose

$$\square H = \square C - \square B.$$

90. Find likewise a right line I, such, that its Square may be equal to $\square C - \square B - 4 \square P$, which Effecttion the Determination shews to be possible, therefore suppose

$$\square I = \square C - \square B - 4 \square P.$$

91. Then by *Probl. 8. Chap. 5.* let it be made, as the line H to the line I, so the line B (the given Base) to a fourth Proportional, suppose it to be the line K, therefore,

$$\frac{H}{I} :: \frac{B}{K}.$$

That is, in 29°, $\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: b : a.$

In which Analogy, the first Term H is greater than the second Term I, (as is evident by *Constr.* in 90° and 91°,) therefore the third Term B shall be greater than the fourth K, (per *Schol. Prop. 14. Elem. 5.*) and consequently $C \sqsupseteq K$, for (by the Determination in 88°) $C \sqsupseteq B$. Thus far that hath been done which is directed by *Theor. 2. in 34°* of this *Probl.* the rest of the Construction follows.

92. Let a Triangle be made of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, which is possible to be done. (per *prop. 22. Elem. 1.*) if $C \sqsupseteq K$, and that the summ of every two of those three lines be longer than the third; but that those lines are so qualified, I prove thus;

First, by what hath been said in 91°, $C \sqsupseteq K$, and consequently $\frac{1}{2}C - \frac{1}{2}K$ is equal to some real right line.

Secondly, the summ of B and $\frac{1}{2}C + \frac{1}{2}K$ is manifestly greater than the third line $\frac{1}{2}C - \frac{1}{2}K$.

Thirdly, the summ of the two lines $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$ makes C, which (by the Determination in 88°) is greater than the third line B.

Fourthly, that the summ of B and $\frac{1}{2}C - \frac{1}{2}K$ is greater than the third line $\frac{1}{2}C + \frac{1}{2}K$ may be proved thus;

It hath been shewn in 91°, that B \sqsupseteq K.

Therefore by adding $\frac{1}{2}C$ to each part, B + $\frac{1}{2}C \sqsupseteq$ K + $\frac{1}{2}C$.

Therefore by subtracting $\frac{1}{2}K$ from each part, B + $\frac{1}{2}C - \frac{1}{2}K \sqsupseteq$ K + $\frac{1}{2}C$.

Which was to be Dem.

Now since it hath been proved that $\frac{1}{2}C - \frac{1}{2}K$ is equal to some real right line, and that the summ of every two of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, is greater than the third, 'tis possible to make a Triangle of those three lines, (per *prop. 22. Elem. 1.*) Suppose then it be done, and that the Triangle so made is ARO, (in the preceding Diagram,) having its Base AR equal to the given Base B, and the greater legg AO equal to $\frac{1}{2}C + \frac{1}{2}K$, and the lesser legg RO equal to $\frac{1}{2}C - \frac{1}{2}K$. I say the Triangle
ARO

A R O will satisfy the Problem propounded; but to render the Demonstration thereof the more easie to Learners, I shall premise a few things in eight steps next following.

93. If the quantities of the given lines B, P and C be express'd by numbers, it will be easie to discover the kind of the Triangle sought, when the leggs are unequal, (as they were suppos'd to be in the Resolution,) by *Theor. 3.* in 35° of this Problem; for if the fourth Proportional found out by that Theorem be less than the Base, the Perpendicular falls within the Triangle; if greater, without; if equal to the Base, upon the end of the Base.

Supposing then it be discovered, that the Perpendicular falls upon A R within the Triangle A R O, from the Center O, at the distance of the lesser legg O R, ($= \frac{1}{2}C - \frac{1}{2}K$) describe the Circle O R G D cutting O A in G; then produce A O to the Circumference in D, draw also the Semidiameter O E, and from the Center O let fall O F perpendicular to E B, therefore (*per prop. 3. Elem. 3.*) F E = F R. Then,

- 94. Because (*per defn. 15. Elem. 1.*) } O D = O R = O G.
- 95. Therefore by adding A O to each part, } A D = A O + O R.
- 96. But by *Constr.* in 92°, } C = A O + O R.
- 97. Therefore from 95° and 96°, (*per Ax. 1. Chap. 2.*) . . . } A D = C.
- 98. Again, by *Constr.* in 92°, } A O = $\frac{1}{2}C + \frac{1}{2}K$.
- 99. Also by *Constr.* in 92°, } O R = $\frac{1}{2}C - \frac{1}{2}K$.
- 100. Therefore, by subtracting the Equation in 99° from that }
in 98°, } A G = K.

101. Now I shall shew that the Triangle A R O, made as before, will satisfy the Problem. First then by *Construction* in 92°, the Base A R is equal to the given Base B; and it hath been proved in 96°, that A O + O R = C the given summ of the leggs. So it remains only to shew, that the Perpendicular O F is equal to the given Perpendicular P; but that is made manifest by the following Demonstration, which is form'd out of the foregoing Resolution, by a repetition of its steps in a backward (not direct) order.

102. . . . *Req. demonstr.* O F = P.

Demonstration.

- 103. By *Constr.* in 91°, H I :: B K.
- That is, in 29°, $\sqrt{cc - bb} . \sqrt{cc - bb - 4pp} :: b . a$.
- 104. Therefore, (*per prop. 22. El. 6.*) $\square H \square I :: \square B \square K$.

Now that the Terms of the last Analogy may be converted into their equivalent quantities expressible by the lines in the Diagram, these seven Equations next following are to be well observed.

- 105. By *Constr.* in 89°, } $\square C - \square B = \square H$.
- 106. And from 97°, } $\square AD = \square C$.
- 107. And from 92°, } $\square AR = \square B$.
- 108. Therefore from 105°, 106°, 107°, . . . } $\square AD - \square AR = \square H$.
- 109. Again, by *Constr.* in 90°, } $\square C - \square B - 4\square P = \square I$.
- 110. Therefore from 105°, 108° and 109°, . . . } $\square AD - \square AR - 4\square P = \square I$.
- 111. And from 100°, } $\square AG = \square K$.

112. Therefore the Terms of the Analogy in 104° being exchanged for their equivalent quantities in 108°, 110°, 107° and 111°, that Analogy will be converted into this, *viz.*

$$\square AD - \square AR . \square AD - \square AR - 4\square P :: \square AR . \square AG.$$

That is, in 28°,

$$cc - bb . . . cc - bb - 4pp :: bb . aa.$$

- 113. Therefore by *altern and inverse Reason*,
- $\square AR . \square AD - \square AR :: \square AG . \square AD - \square AR - 4\square P$.]

That is, in 27°,

$$bb . . . cc - bb :: aa . cc - bb - 4pp.$$

Now to return backwards from the 27th to the 25th step of the Resolution, by the lines of the Diagram, some Analogies not express'd in the Resolution must be introduced, (which are infer'd from the Algebraical Fraction $\frac{cca - bba}{bb}$, as before hath been hinted in 55°,) to wit, the four Analogies next following.

O o

114. By

114. By *Theor.* 2. in 29° of *Probl.* 9. *Chap.* 7. $\triangleright AD \cdot AR :: AE \cdot AG$.

115. Therefore (*per prop.* 22. *Elem.* 6.) $\triangleright \square AD \cdot \square AR :: \square AE \cdot \square AG$.

116. Therefore, by *Division of Reason*,
 $\square AD - \square AR \cdot \square AR :: \square AE - \square AG \cdot \square AG$.

117. And inverfly,
 $\square AR \cdot \square AD - \square AR :: \square AG \cdot \square AE - \square AG$.

118. Therefore from 113° and 114°, (*per prop.* 11, & 14. *Elem.* 5.)
 $\square AE - \square AG = \square AD - \square AR - 4 \square P$.

That is, $\int \frac{ccaa}{bb} - aa = cc - bb - 4pp$.

119. And by adding $4 \square P$, also $\square AG$ to each part, and subtracting $\square AE$ from each part of the Equation in 118°, this will arise,

$$4 \square P = \square AG + \square AD - \square AR - \square AE$$

120. But by *Theor.* 1. in 33° of this Problem,

$$\square AE - \square AG = \square AD - \square AR - 4 \square OF$$

121. Therefore by adding $4 \square OF$, also $\square AG$ to each part, and subtracting $\square AE$ from each part of the last preceding Equation, this will arise,

$$4 \square OF = \square AG + \square AD - \square AR - \square AE$$

122. Therefore from 119° and 121°, (*per Ax.* 1. *Chap.* 2.) $\triangleright 4 \square OF = 4 \square P$.

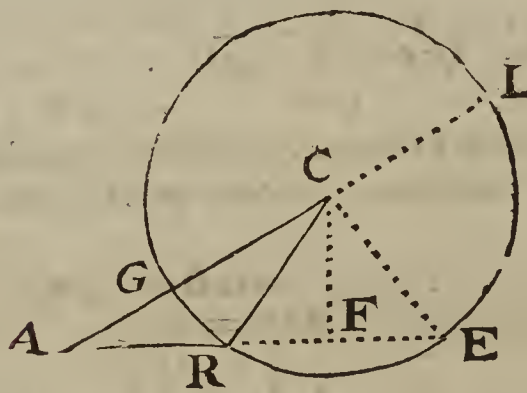
123. Therefore, $\triangleright \square OF = \square P$.

124. Therefore, $\triangleright \square OF = \square P$.

Which was to be Demonstr. Therefore the Problem is satisfied. But for further Illustration, all the preceding Canons and Theorems raised out of the Resolution, may be exemplified by the Numbers placed near the Diagram at the beginning of this Problem.

Probl. IX.

The Base, Perpendicular and difference of the leggs of a plain Triangle being severally given, to find the Triangle. But the Base must exceed the difference of the leggs.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

- Suppose the Triangle ARC obtufangled at R, (the end of the Base AR,) to be that which is sought; then from the Center C, at the distance of the lesser legg CR describe the Circle CRGLE, cutting the greater legg CA in G; so shall AG be the difference of the leggs CA and CR, for CG = CR.
- Produce AC and AR to the Circumference in L and E, then is AL equal to the summ of the leggs AC and CR, for CL = CR.
- Draw the Semidiameter CE, and make CF ⊥ RE, so shall CF cut RE into two equal parts in F, (*per prop.* 3. *Elem.* 3.) Which things being premis'd, the Resolution of the Problem propounded may be resolved in manner following.

Suppos.

- $b = AR$ the Base of $\triangle ARC$ is given.
- $p = CF$ the Perpendicular is given.
- $d = AG$ the difference of the leggs AC and RC is given.

Req. to find the Triangle.

Resolution.

- Put a for the unknown summ of the leggs, viz. assume
 $a = AC + RC = AL$.

8. Then

8. Then from *Theor.* 2. in 34° of the foregoing *Probl.* 8. of this *Chap.* this Analogy arifeth,

$$\sqrt{aa - bb} : \sqrt{aa - bb - 4pp} :: b . d.$$

9. The Squares of which proportional lines shall be Proportionals also, therefore

$$aa - bb . aa - bb - 4pp :: bb . dd.$$

10. Therefore by Conversion of Reason,

$$aa - bb . 4pp :: bb . bb - dd.$$

11. And alternately,

$$aa - bb . bb :: 4pp . bb - dd.$$

12. And by Composition,

$$aa . bb :: 4pp + bb - dd . bb - dd.$$

13. And by inverse and alternate Reason,

$$bb - dd . 4pp + bb - dd :: bb . aa.$$

14. But the sides of proportional Squares are also Proportionals, therefore from the last preceding Analogy,

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: b . a.$$

15. And because by *Theor.* 2. *Probl.* 10. *Chap.* 7.

$$d . AE :: b . a$$

16. Therefore from the two last preceding Analogies, this arifeth,

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: d . AE.$$

The Analogy in 14° gives

THEOR. 1.

17. As the right line whose Square is equal to the excess whereby the Square of the Base of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whose Square is equal to the said excess together with the Square of the double Perpendicular; so is the Base, to the summ of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the excess of the Square of the Base above the Square of the difference of the leggs, is to the summ of the said excess and the Square of the double Perpendicular; so is the Square of the Base, to the Square of the summ of the leggs.

Therefore, the Base, Perpendicular, and difference of the leggs of a plain Triangle being severally given, the summ of the leggs shall be given also by the said *Theor.* 1. And consequently the leggs shall be given severally, by *Theor.* 9. *Chap.* 4.

The Analogy in 16° gives

THEOR. 2.

18. As the right line whose Square is equal to the excess by which the Square of the Base of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whose Square is equal to the said excess together with the Square of the double Perpendicular; so is the difference of the leggs to a fourth Proportional; which exceeds the Base, when the Perpendicular falls without the Triangle, for then 'tis the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle at the nearer end of the Base; but when the Perpendicular falls within, the said fourth Proportional is less than the Base, and is the difference of the segments of the Base made by the Perpendicular: Lastly, when the said fourth Proportional is equal to the Base, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl.* 9. be express'd by numbers, we may discover by *Theor.* 2. above express'd, whether the Triangle sought be obtusangled, acute-angled, or right-angled at the Base.

It is also evident by the 14^{th} and 16^{th} steps of the Resolution, that to the end the Problem propounded may be possible, the given lines must be subject to this

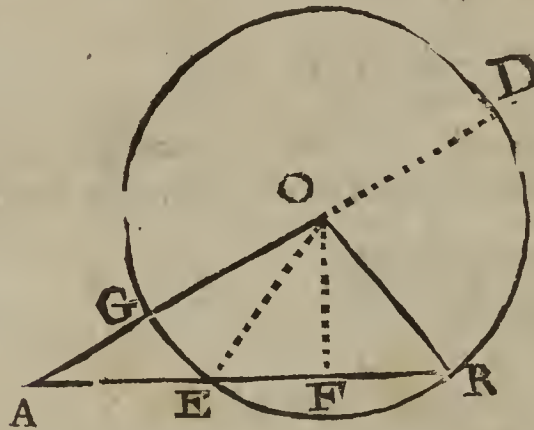
Determination.

The given Base must exceed the given difference of the leggs.

The truth of the preceding Theorems and Determination, as also the Composition of this *Probl.* 9. will be obvious to him that understands what hath been delivered in the foregoing *Probl.* 8. and therefore I shall wave the Composition.

Probl. X.

In a plain Triangle having unequal acute angles at the Base, the Perpendicular, difference of the leggs, and difference of the segments of the Base made by the Perpendicular, being severally given, to find the Triangle. But the lines given must be subject to the Determinations hereafter exprest.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

1. Let the Diagram belonging to the preceding *Probl. 8.* of this *Chapt.* be here repeated, and suppose the $\triangle ARO$ having unequal acute angles A and R at the ends of the Base AR to be the Triangle sought; then respect being had to the Preparatory Construction in the three first steps of the said *Probl. 8.* the Resolution of this *Probl. 10.* may be formed thus;

Suppos.

2. $p = OF$ the Perpendicular of $\triangle ARO$ is given.
 3. $d = AG$ the difference of the leggs is given.
 4. $b = AE$ the difference of the segments of the Base is given.

Req. to find the Triangle.

Resolution.

5. It is manifest that if AE the given difference of the segments FA , FR be esteemed the Base of the $\triangle AEO$ obtusangled at E , then AG shall be the difference of the leggs AO and EO , as well as of AO and RO , (for $EO = RO = OG$,) and OF a common Perpendicular to the two Triangles ARO and AEO ; therefore in $\triangle AEO$, the Base AE , the Perpendicular OF , and AG the difference of the leggs AO and EO being severally given, the said $\triangle AEO$ shall be given by the foregoing *Probl. 9.* of this *Chapt.* For, the sum of the leggs, to wit, $AD = AO + EO = AO + OR$ shall be given by this following Analogy, (according to *Theor. 1.* in 17° of the said *Probl. 9.*) viz.

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: b . AD.$$

6. Then AD and AG the sum and difference of the leggs AO and EO , or of AO and RO , being given, the said leggs shall be given severally by *Theor. 9.* *Chap. 4.*
 7. Moreover, forasmuch as AR in reference to $\triangle AEO$ obtusangled at E , is compos'd of the Base AE and ER , ($= 2FE = 2FR$,) the said AR , which is also the Base of $\triangle ARO$, shall be given by *Theor. 2.* in 18° of the preceding *Probl. 9.* For,

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: d . AR.$$

From the premisses tis manifest that the Base and leggs of the Triangle sought in this *Probl. 10.* may be found out by the foregoing *Probl. 9.* But that there may be a possibility of finding out the Triangle required, the lines given must be liable to these two following Determinations, viz.

Determination 1.

8. The line given for the difference of the segments of the Base made by the Perpendicular falling within the Triangle, must exceed the given difference of the leggs, that is, (in the Figure belonging to this *Probl. 10.*) $AE \sqsupseteq AG$, the truth whereof is manifest by *prop. 8.* *Elem. 3.*

9. Again,

9. Again, because by *Supposition* the Triangle sought hath unequal acute angles at the Base, the Perpendicular falls within, and the Base must necessarily exceed the difference of the segments of the Base made by the Perpendicular; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy before exprest in 7° must exceed the given line A E. Hence,

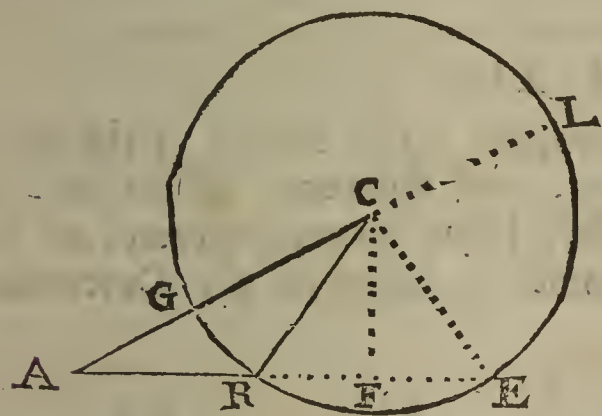
Determination 2.

10. $\frac{\sqrt{4pp + bb - dd} : \times d}{\sqrt{bb - dd}} \sqsubset b.$

Supposing then the given Quantities to be qualified according to the tenour of the Determinations before prescribed, the industrious Learner may easily apply what hath been said in the foregoing 5th, 6th and 7th steps, as well to the Geometrical Effecttion, as to the Arithmetical Solution of this *Probl. 10.*

Probl. XI.

In a plain Triangle obtufangled at the Base, the Perpendicular, difference of the leggs, and the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle, being severally given, to find the Triangle. But the given lines must be subject to the Determinations hereafter exprest.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Let the Diagram belonging to the foregoing *Probl. 9.* of this *Chapt.* be here repeated, and suppose the ΔARC obtufangled at R, (the end of the Base AR,) to be the Triangle sought; then respect being had to the preparatory Construction in 1°, 2° and 3° of *Probl. 9.* the Resolution of this *Probl. 11.* may be formed thus;

Suppos.

2. $p = CF$ the Perpendicular of ΔARC is given.
3. $d = AG$ the difference of the leggs AC and RC is given.
4. $b = AE$ the line compos'd of the Base AR and 2 FE, (or 2 FR,) is given.

Req. to find the Triangle.

Resolution.

5. It is manifest, that if the given line AE be esteemed the Base of the ΔAEC having unequal acute angles at A and E, then AG is the difference of the leggs AC and EC, as well as of AC and RC, (for $EC = RC$,) and CF is a common Perpendicular to the two Triangles AEC and ARC; therefore in ΔAEC , the Base AE, the Perpendicular CF, and AG the difference of the leggs AC and EC, (or RC,) being given severally, the said ΔAEC shall be given by *Probl. 9.* of this *Chapt.* For $AL = AC + EC$, (the summ of the leggs of ΔAEC ,) shall be given by this following Analogy, (according to *Theor. 2.* in 17° of the said *Probl. 9.*) viz.

$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: b . AL.$

6. Then AL and AG the summ and difference of the leggs AC and EC being severally given, the leggs themselves shall be also given severally, by *Theor. 9. Chap. 4.*
7. Moreover, because AR in reference to the ΔAEC is the difference of the segments FA and FE, made by the Perpendicular CF, and is also the Base of the ΔARC required,

required, the said AR shall be given by *Theor. 2.* in *Probl. 18.* of *Probl. 9.* of this *Chap.*
For,

$$\sqrt{bb - dd} : \sqrt{4pp + bb - dd} :: d : AR.$$

From the premisses 'tis evident, that the three sides of the Triangle required by this *Probl. 11.* are discovered by *Probl. 9.* of this *Chapt.* But the lines given must be subject to the following Determinations, that there may be a possibility of finding out a Triangle to satisfy the Problem propounded.

Determination 1.

8. The line given for the summ of the Base and double distance from the foot of the Perpendicular to the obtuse angle, must be longer than the line given for the difference of the leggs, that is, $AE \sqsupset AG$, as may be easily proved; for by *Supposition* $\triangle ARC$ is obtusangled at R, therefore $AE \sqsupset AR$, and consequently AE much greater than AG, for (*per prop. 8. Elem. 3.*) $AR \sqsupset AG$.

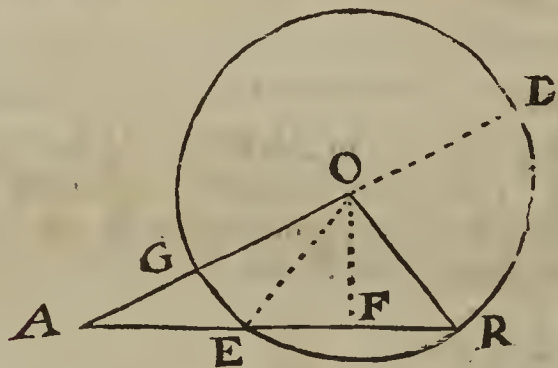
Again, because by *Supposition* the Triangle sought is obtusangled at the Base, the Perpendicular falls without, and the Base shall necessarily be less than the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy in 7°, must be less than the given line AE. Hence,

Determination 2.

$$\frac{\sqrt{4pp + bb - dd} : x \cdot d}{\sqrt{bb - dd} : b.}$$

Probl. XII.

In a plain Triangle having unequal acute angles at the Base, the Perpendicular, summ of the leggs, and difference of the segments of the Base made by the Perpendicular, being severally given, to find the Triangle. But the lines given must be liable to the Determinations hereafter declared.



AR = 21	FR = 6
AO = 17	OF = 8
OR = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Prepar.

1. Let the Diagram belonging to the preceding *Probl. 8.* of this *Chapt.* be here repeated, and suppose the Triangle ARO having unequal acute angles A and R at the ends of the Base AR to be the Triangle sought; then respect being had to the preparatory Construction in the three first steps of the said *Probl. 8.* the Resolution of this *Probl. 12.* may be formed thus;

Suppos.

2. $p = OF$ the Perpendicular of $\triangle ARO$ is given.
3. $c = AD = AO + RO$ the summ of the leggs is given.
4. $b = AE = FA - FR$ the difference of the segments of the Base is given.

Req to find the Triangle.

Resolution.

5. It is evident, that if the given line AE be esteem'd the Base of the $\triangle AEO$ obtusangled at E, then AD is the summ of the leggs AO and EO, as well as of AO and RO, for $EO = RO$, and OF is a common Perpendicular to the two Triangles AEO and ARO; therefore in $\triangle AEO$, the Base AE, the Perpendicular OF, and AD the summ of the leggs AO and EO being severally given, the Triangle

AEO

AEO shall be given by the foregoing *Probl. 8.* of this *Chapt.* For AG, the difference of the leggs AO and EO, shall be given by this following Analogy, (according to *Theor. 2.* in 24° of *Probl. 8.*)

$$\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: b . AG.$$

6. Then AD and AG the summ and difference of the leggs AO and EO, (or RO,) being given severally, the said leggs shall be also given severally, by *Theor. 9. Chap. 4.*

7. Moreover, forasmuch as AR in reference to the $\triangle AEO$ obtusangled at E, is compos'd of the Base AE and ER, ($= 2FE = 2FR,$) the said AR, which is also the Base of $\triangle ARO$ required, shall be given by *Theor. 3.* in 35° of *Probl. 8.*

For,

$$\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: c . AR.$$

From the premisses 'tis manifest, that the Base and leggs of the Triangle sought by this *Probl. 12.* are discovered by the foregoing *Probl. 8.* But that there may be a possibility of finding out the desired Triangle, the given lines must be subject to these two following Determinations, *viz.*

Determination 1.

$$c \sqsupset \sqrt{bb + 4pp} : \text{That is,}$$

8. The line given for the summ of the leggs must exceed that right line whose Square is equal to the summ of the Square of the given Base and the Square of the double of the given Perpendicular.

This Determination doth openly shew it self in the preceding Analogy in 5°, and hath already been demonstrated in *Probl. 8.* of this Chapter.

9. Again, because by *Supposition* the Triangle sought hath unequal acute angles at the Base, the Perpendicular falls within, and the Base must necessarily exceed the difference of the segments of the Base made by the Perpendicular, therefore to the end the given lines may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy in 7°, must exceed the given line AE. Hence,

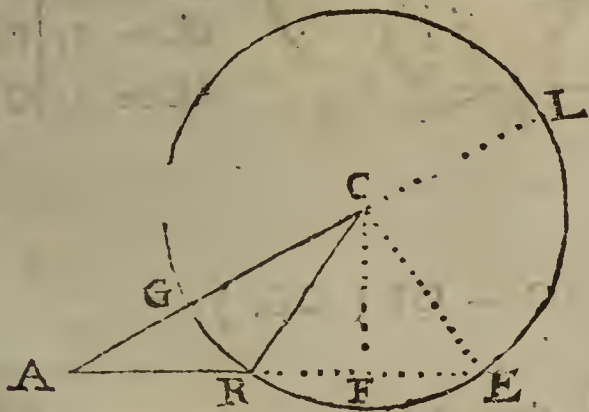
Determination 2.

$$\sqrt{cc - bb - 4pp} : x \cdot c \sqsupset b.$$

$$\sqrt{cc - bb} :$$

Probl. XIII.

In a plain Triangle obtusangled at the Base, the Perpendicular, summ of the leggs, and the line compos'd of the Base and double distance from the foot of the Perpendicular to the obtuse angle, being given severally, to find the Triangle. But the given lines must be subject to the Determinations hereafter declared.



AR = 9	FE = 6
AC = 17	CF = 8
CR = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Preparat.

1. Let the Diagram belonging to the foregoing *Probl. 9.* of this *Chapt.* be here repeated, and suppose the $\triangle ARC$ obtusangled at R, (the end of the Base AR,) to be the Triangle sought; then respect being had to the preparatory Construction in 1°, 2° and 3° of the said *Probl. 9.* the Resolution of this *Probl. 13.* may be formed thus;

Suppos.

2. $p = CF$ the Perpendicular of $\triangle ARC$ is given.
3. $c = AL = AC + RC$ the summ of the leggs is given.

4. $b = AE$

4. $b = AE$ the line compos'd of the Base AR and $2FR$ (or $2FE$) is given.
Req. to find the Triangle.

Resolution.

5. It is manifest, that if the given line AE be esteem'd the Base of the $\triangle AEC$ having unequal acute angles at A and E , then AL is the summ of the leggs AC and EC , as well as of AC and RC , (for $EC = RC$,) and CF is a common Perpendicular to the two Triangles ARC and AEC ; therefore in $\triangle AEC$, the Base AE , the Perpendicular CF , and AL the summ of the leggs AC and EC , (or RC ,) being given severally, the $\triangle AEC$ shall be given by the foregoing *Probl. 8.* of this Chapter. For AG the difference of the leggs shall be given by this following Analogy, (according to *Theor. 2.* in 34° of *Probl. 8.*)

$$\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: b . AG.$$

6. Then AL and AG the summ and difference of the leggs AC and EC being given severally, the leggs shall also be given severally, by *Theor. 9. Chap. 4.*

7. Moreover because AR , in reference to the $\triangle AEC$, is the difference of the segments FA and FE made by the Perpendicular CF , and is also the Base of the $\triangle ARC$ required, the said AR shall be given by *Theor. 3.* in 35° of *Probl. 8.* For,

$$\sqrt{cc - bb} : \sqrt{cc - bb - 4pp} :: c . AR.$$

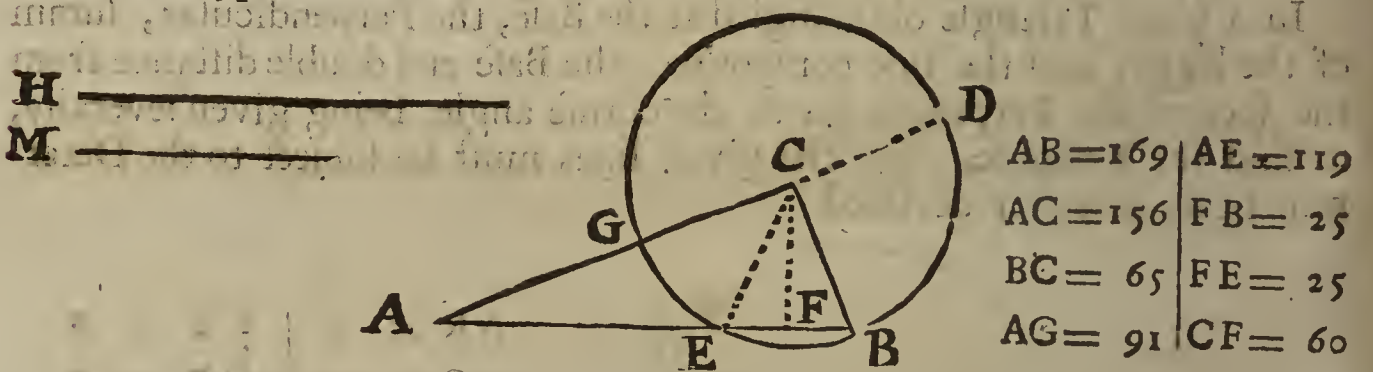
8. From the premisses 'tis evident, that the three sides of the Triangle required by this *Probl. 13.* are discovered by *Probl. 8.* of this *Chapt.* But the lines given must be subject to the following Determinations, *viz.*

Determinat. 1. $c \sqsupset \sqrt{bb + 4pp} :$

Determinat. 2. $\frac{\sqrt{cc - bb - 4pp} \times c}{\sqrt{cc - bb}} \sqsupset b.$

Probl. XIV.

The Hypothenufal of a right-angled Triangle being given, as also a mean Proportional between the Base and Perpendicular, to find the Triangle. But the right line arising by the Application of the Square of the given mean to the given Hypothenufal, must not be greater than half the Hypothenufal.



Suppos.

1. ABC is a \triangle right-angled at C .
2. $AC \sqsupset CB$.
3. $CBGD$ is a \odot , whence $AG = AC - CB$ (CG .)
4. $CF \perp AB$.
5. $b = AB$ the Hypothenufal is given.
6. $m = M$ a right line given, and such, that $AC . m :: m . CB$.

Req. to find out $\triangle ABC$.

Resolut. 1.

7. Because by *Supposition* in 6 $^\circ$ the right line m is a mean Proportional between AC and CB , therefore $mm = \square AC, CB$, that is, $\square AB, CF$, for (*per prop. 41. Elem. 1.*) each of those Rectangles is equal to $2 \triangle ABC$; therefore $\frac{mm}{b} = CF$ the Perpendicular is given; therefore also $\triangle ABC$ shall be given both Geometrically and Arithmetically,

tically, by *Probl. 16. Chap. 5.* But here I shall frame the Resolution and Composition of the same Problem after another manner.

Resolut. 2.

8. Put a for the difference of the leggs sought, *viz.* } $a = AG = AC - CB.$
 9. Then because AD is the summ, and AG the }
 difference of the leggs AC and CB , therefore } $\square M(\square AC, CB) + \frac{1}{4}\square AG = \square \frac{1}{2}AD.$
 (*per Theor. 7. Chap. 4.*) }
 10. Therefore in the letters belonging to the }
 Resolution, the Square of half the summ of } $mm + \frac{1}{4}aa (= \square \frac{1}{2}AD.)$
 the leggs shall be equal to }
 11. Therefore the square Root of that Square of }
 the half summ shall be the half summ of the } $\sqrt{mm + \frac{1}{4}aa} : (= \frac{1}{2}AD.)$
 leggs, to wit, }
 12. Therefore from 8° and 11° , (*by Theor. 9.* }
Chap. 4.) the greater legg is } $\sqrt{mm + \frac{1}{4}aa} + \frac{1}{2}a (= AC.)$
 13. And the lesser legg is } $\sqrt{mm + \frac{1}{4}aa} - \frac{1}{2}a (= CB.)$
 14. Therefore from 12° , (*by Theor. 2. Chap. 4.*) }
 the Square of the greater legg shall be } $mm + \frac{1}{2}aa + a \times \sqrt{mm + \frac{1}{4}aa} :$
 15. And from 13° , (*by Theor. 5. Chap. 4.*) the }
 Square of the lesser legg shall be } $mm + \frac{1}{2}aa - a \times \sqrt{mm + \frac{1}{4}aa} :$
 16. Therefore the summ of all in 14° and 15° }
 gives the summ of the Squares of the leggs, to wit, } $2mm + aa (= \square AC + \square CB.)$
 17. And because by *Suppos.* in $1^\circ < ACB = \perp$, }
 therefore from 5° and 16° , (*per prop. 47. Ele. 1.*) } $2mm + aa = hb (= \square AB.)$
 this Equation ariseth, }
 18. Therefore by subtracting $2mm$ from each }
 part of the last Equation, } $aa = hb - 2mm.$
 19. Therefore by extracting the square Root out }
 of each part of the last Equation, the difference } $a = \sqrt{hb - 2mm} : (= AG.)$
 of the leggs is made known, *viz.* }
 20. Therefore from $12^\circ, 13^\circ, 18^\circ$ and 19° , the leggs shall be given severally, *viz.*

$$\begin{cases} AC = \sqrt{\frac{1}{4}hb + \frac{1}{2}mm} + \sqrt{\frac{1}{4}hb - \frac{1}{2}mm} : \\ CB = \sqrt{\frac{1}{4}hb + \frac{1}{2}mm} - \sqrt{\frac{1}{4}hb - \frac{1}{2}mm} : \end{cases}$$

The Equation in 19° gives

CANON 1.

21. From the Square of the given Hypothen. subtract the double Square of the given mean Proportional, so the square root of the remainder shall be the difference of the leggs sought.

The Equations in 20° give

CANON 2.

22. To and from the Square of half the given Hypothenusal add and subtract half the Square of the given mean Proportional, and reserve the summ and remainder; then extract the square Root out of the said summ and remainder severally; lastly, the summ and difference of the said square Roots shall be the sides about the right angle of the Triangle sought.

Note. If the values of AC and CB (the sides about the right angle) before express in 20° be severally squared, and the Universal square Root extracted out of each Product, there will come forth the Canon delivered in *Sect. 55. Probl. 16. Chap. 5.* for the Arithmetical Resolution of such ambiguous Biquadratic Equations as fall under the Form there expounded.

But that the truth of the preceding Canons may more clearly appear, I shall propound and demonstrate them in the form of Theorems, by a repetition of the steps of the foregoing Resolution.

THEOR. 1.

23. In a right-angled Triangle having unequal leggs about the right angle, the difference of those leggs is equal to a right line whose Square is equal to the excess whereby the Square of the Hypothenusal exceeds the double Square of a mean Proportional between the said leggs.

THEOR. 2.

24. In a right-angled Triangle having unequal leggs about the right angle, the greater legg is equal to the sum of these two right lines, to wit, the right line whose Square is equal to the Square of half the Hypothensal together with half the Square of a mean Proportional between the leggs, and the right line whose Square is equal to the excess whereby the Square of half the Hypothensal exceeds half the Square of the said mean. But the lesser legg is equal to the difference of the said two right lines.

Suppos.

25. ABC is a Δ right-angled at C.

26. AC \perp CB.

27. CBGD is a \odot ; and ACD is a right line, therefore

28. AD = AC + CB, and AG = AC - CB.

29. M is a right line, such, that AC : M :: M : CB;

Req. demonstr.

30. Theor. 1. $AG = \sqrt{\square AB - 2\square M}$:

31. Theor. 2. $\begin{cases} AC = \sqrt{\frac{1}{4}\square AB + \frac{1}{2}\square M} + \sqrt{\frac{1}{4}\square AB - \frac{1}{2}\square M} \\ CB = \sqrt{\frac{1}{4}\square AB + \frac{1}{2}\square M} - \sqrt{\frac{1}{4}\square AB - \frac{1}{2}\square M} \end{cases}$

Demonstration.

32. By Suppos. in 28°, } $AG = AC - CB.$

33. Therefore the Square of that Equation, } $\square AG = \square AC + \square CB - 2\square AC, CB,$

34. By Suppos. in 25° $\angle ACB$ is \perp , there- } $\square AB = \square AC + \square CB.$

35. Therefore from 33° and 34°, (per Ax. 6. } $\square AG = \square AB - 2\square AC, CB,$

36. From 29°, (per prop. 17. Elem. 6.) } $\square M = \square AC, CB.$

37. And consequently, } $2\square M = 2\square AC, CB.$

38. Therefore from 35° and 37°, (per Ax. 6. } $\square AG = \square AB - 2\square M.$

39. But the sides of equal Squares are also } $AG = \sqrt{\square AB - 2\square M}:$

Which was Theor. 1. to be Dem.

40. Again, because by Suppos. in 28°, AD is } $\square \frac{1}{2} AD = \square AC, CB + \frac{1}{4}\square AG.$

41. And because from 29°, (per prop. 17. } $\square M = \square AC, CB.$

42. And by taking $\frac{1}{4}$ of all in 38°, . . . } $\frac{1}{4}\square AB - \frac{1}{2}\square M = \frac{1}{4}\square AG.$

43. Therefore from 40°, 41° and 42°, (per } $\square \frac{1}{2} AD = \frac{1}{4}\square AB + \frac{1}{2}\square M.$

44. But the sides of equal Squares are also } $\frac{1}{2} AD = \sqrt{\frac{1}{4}\square AB + \frac{1}{2}\square M}:$

45. And for the like reason, 'tis manifest from } $\frac{1}{2} AG = \sqrt{\frac{1}{4}\square AB - \frac{1}{2}\square M}:$

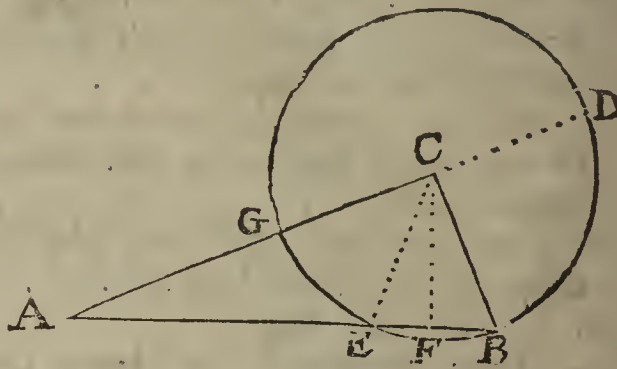
46. Therefore, by taking the sum and difference of the Equations in 44° and 45°, these will arise,

$\begin{cases} \frac{1}{2} AD + \frac{1}{2} AG = \sqrt{\frac{1}{4}\square AB + \frac{1}{2}\square M} + \sqrt{\frac{1}{4}\square AB - \frac{1}{2}\square M} \\ \frac{1}{2} AD - \frac{1}{2} AG = \sqrt{\frac{1}{4}\square AB + \frac{1}{2}\square M} - \sqrt{\frac{1}{4}\square AB - \frac{1}{2}\square M} \end{cases}$

47. And because AD is the sum, and AG the difference of AC and CB; therefore (per Theor. 9. Chap. 4.)

$AC = \frac{1}{2} AD + \frac{1}{2} AG; \text{ and } CB = \frac{1}{2} AD - \frac{1}{2} AG.$

48. There-



48. Therefore from 46° and 47°, (per Ax. 1. Chap. 2.)

$$\begin{cases} AC = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} + \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M} \\ CB = \sqrt{\frac{1}{4} \square AB + \frac{1}{2} \square M} - \sqrt{\frac{1}{4} \square AB - \frac{1}{2} \square M} \end{cases}$$

Which was Theor. 2. to be Demonstr.

In the next place, to the end the Geometrical Effect of the foregoing Probl. 14. may meet with no obstruction, I shall prove the truth of the Determination annex'd to the Problem, by demonstrating this following

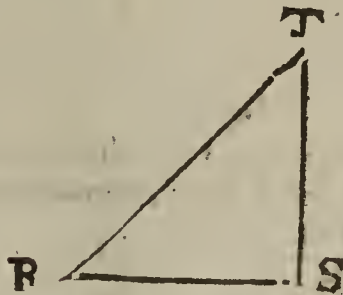
L E M M A.

49. In a right-angled Triangle, if the Square of a mean Proportional between the sides about the right angle, (that is, if the Rectangle of those sides) be applied to the Hypothenufal, the line thence arising shall sometimes be equal to half the Hypothenufal, and sometimes less, but never greater than the said half.

The sides about the right angle are either equal to one another, or else unequal; I shall begin with the first Case.

Suppos. in Case 1.

- 50. RST is a Δ right-angled at S.
- 51. RS = ST.
- 52. M is a right line, such, that
- 53. RS . M :: M . ST.



54. . . Req. demonstr. . . $\frac{\square M}{RT} = \frac{1}{2} RT.$

Demonstration.

- 55. By *Supposition* in 51°, . . . } . . . RS = ST.
- 56. Therefore by drawing ST as a common } $\square RS, ST = \square ST = \square RS.$
- 57. Therefore, (per Ax. 8. Chap. 2.) } $2 \square RS, ST = \square ST + \square RS.$
- 58. And because by *Suppos.* in 50° $\angle S$ is \perp , } $\therefore \square RT = \square ST + \square RS.$
- 59. Therefore from 57° and 58°, (per Ax. 1. } $2 \square RS, ST = \square RT.$
- 60. And consequently, . . . } $\square RS, ST = \frac{1}{2} \square RT.$
- 61. But from 53°, (per prop. 17. Elem. 6.) } $\square RS, ST = \square M.$
- 62. Therefore from 60° and 61°, (per Ax. 1. } $\therefore \square M = \frac{1}{2} \square RT = \square RT, \frac{1}{2} RT.$
- 63. Therefore from 62°, by Application of each } $\frac{\square M}{RT} = \frac{1}{2} RT.$

Which was Case 1. to be Dem.

Suppos. in Case 2.

- 64. ABC is a Δ right-angled at C.
 - 65. AC \square CB.
 - 66. M a right line, such, that . . . AC . M :: M . CB.
- [See the Diagr. in the precedent Page.]

67. . . Req. demonstr. . . $\frac{\square M}{AB} = \frac{1}{2} AB.$

Demonstration.

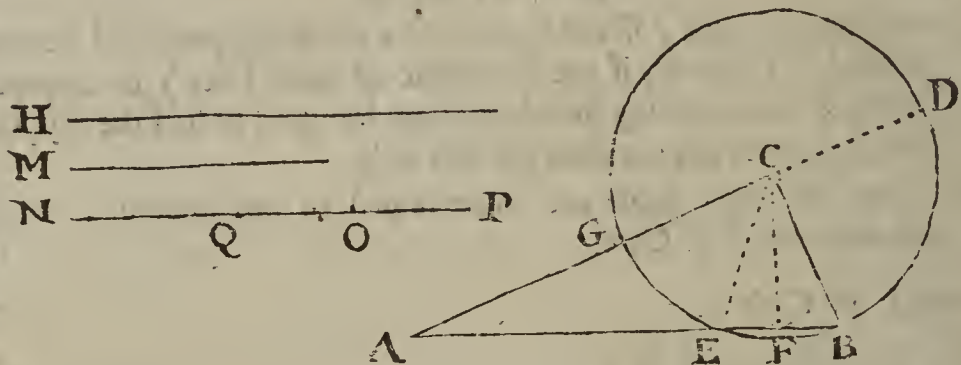
- 68. By the preceding Theor. 1. before demon- } $2 \square M = \square AB.$
- 69. Therefore, by taking the half of each part, } $\square M = \frac{1}{2} \square AB.$
- 70. Therefore from 69°, by Application of each } $\frac{\square M}{AB} = \frac{1}{2} AB.$

Which was Case 2. to be Dem.

71. Now because in every right-angled Triangle, the sides about the right angle are either equal or unequal between themselves, and it hath been demonstrated, that when the said sides are equal to one another, the right line arising by the Square of a mean

Proportional between the said sides to the Hypothenufal is equal to half the Hypothenufal ; but when the said sides are unequal , the said right line is less than half the Hypothenufal , it is evident that the right line arising by the said Application can never be greater than half the Hypothenufal : Therefore the truth of the Lemma is manifest , and consequently , the lines given in *Probl. 14.* must be subject to the Determination annex'd to it , that there may be a possibility of effecting the Problem.

The Composition of the foregoing *Probl. 14.*



Suppos.

- 72. H = the Hypothenufal of a right-angled Triangle is given.
- 73. M = a mean Proportional between the sides about the right angle is given.

74. $\frac{\square M}{H}$ not $\ll \frac{1}{2}H$. (*Determination.*)

Req to find the Triangle.

Construction.

- 75. By *Probl. 2. Chap. 5.* find a right line NO , such, that its Square may be equal to $\frac{1}{4}\square H + \frac{1}{2}\square M$, therefore

$$NO = \sqrt{\frac{1}{4}\square H + \frac{1}{2}\square M}$$

- 76. By the *Determination* in 74°, $\frac{\square M}{H}$ not $\ll \frac{1}{2}H$; suppose then it be granted, or discovered by H and M given in numbers, that $\frac{\square M}{H}$ is less than $\frac{1}{2}H$, and consequently, (by multiplying each part into H ,) that $\square M \ll \frac{1}{2}\square H$; then it evidently follows, that 'tis possible (*per Probl. 4. Chap. 5.*) to find out a right line OP , such, that its Square may be equal to $\frac{1}{4}\square H - \frac{1}{2}\square M$, suppose therefore

$$OP = \sqrt{\frac{1}{4}\square H - \frac{1}{2}\square M}$$

- 77. Make $NP = NO + OP$, then from the *Construction* in 75° and 76°, 'tis manifest that

$$NP = \sqrt{\frac{1}{4}\square H + \frac{1}{2}\square M} + \sqrt{\frac{1}{4}\square H - \frac{1}{2}\square M}$$

- 78. From NO cut off $OQ = OP$, which may be done, for 'tis evident by *Constr.* in 75° and 76°, that $NO \ll OP$, suppose therefore $OQ = OP$, then from 75° and 76° it follows that

$$NQ = NO - OP = \sqrt{\frac{1}{4}\square H + \frac{1}{2}\square M} - \sqrt{\frac{1}{4}\square H - \frac{1}{2}\square M}$$

- 79. Make $AC = NP$; also $CB = NQ$, and $CB \perp AC$; lastly, draw AB .
- 80. I say ABC is the right-angled Triangle required. Now we must shew that it will satisfy the Problem. First then by *Construction* in 79°, $CB \perp AC$, and consequently the angle ACB is a right angle. But that the Hypothenufal AB is equal to the given Hypothenufal H , and that the given right line M is a mean Proportional between AC and CB , (the sides about the right angle,) the following Demonstration will make manifest.

81. . . . *Req. demonstr.* $\left. \begin{array}{l} AB = H. \text{ Also, that} \\ AC \cdot M :: M \cdot CB. \end{array} \right\}$

Demonstration.

- 82. By *Constr.* in 77°, $\left. \begin{array}{l} NP = NO + OP. \\ \square NP = \square NO + \square OP + 2\square NO, OP. \end{array} \right\}$
- 83. Therefore (*per Theor. 2. Chap. 4.*) $\left. \begin{array}{l} NQ = NO - OP. \\ \square NQ = \square NO + \square OP - 2\square NO, OP. \end{array} \right\}$
- 84. By *Constr.* in 78°, $\left. \begin{array}{l} \square NP + \square NQ = 2\square NO + 2\square OP. \end{array} \right\}$
- 85. Therefore (*per Theor. 5. Chap. 4.*) $\left. \begin{array}{l} \square NP + \square NQ = 2\square NO + 2\square OP. \end{array} \right\}$
- 86. Therefore, by adding together the Equations in 83° and 85°, $\left. \begin{array}{l} \square NP + \square NQ = 2\square NO + 2\square OP. \end{array} \right\}$

87. And

87. And from the *Constr.* in 79° , (*per prop.* 47. *Elem.* 1.) } $\square NP + \square NQ = \square AC + \square CB.$
88. Therefore from 86° and 87° , (*per Ax.* 1. *Chap.* 2.) } $2\square NO + 2\square OP = \square AC + \square CB.$
89. But by *Constr.* in 79° , $\angle ACB$ is \perp , therefore (*per prop.* 47. *Elem.* 1.) } $\square AB = \square AC + \square CB.$
90. Therefore from 88° and 89° , (*per Ax.* 1. *Chap.* 2.) } $2\square NO + 2\square OP = \square AB.$
91. But from the *Constr.* in 75° and 76° , (by adding together the double Squares of the Equations there exprest,) 'tis evident that } $2\square NO + 2\square OP = \square H.$
92. Therefore from 90° and 91° , (*per Ax.* 1. *Chap.* 2.) } $\square AB = \square H.$
93. But the sides of equal Squares are also equal, therefore from 92° , } $AB = H.$ Which was to be Dem.
94. Again, because by *Constr.* in 77° and 78° , NP is the sum, and NQ the difference of NO and OP, therefore (*per Theor.* 8. *Chap.* 4.) } $\square NP, NQ = \square NO - \square OP.$
95. But from the *Constr.* in 79° , } $\square NP, NQ = \square AC, CB.$
96. Therefore from 94° and 95° , (*per Ax.* 1. *Chap.* 2.) } $\square AC, CB = \square NO - \square OP.$
97. By *Constr.* in 75° , } $\frac{1}{4}\square H + \frac{1}{2}\square M = \square NO.$
98. And by *Constr.* in 76° , } $\frac{1}{4}\square H - \frac{1}{2}\square M = \square OP.$
99. And from 97° , by subtracting $\square OP$ from each part, } $\frac{1}{4}\square H + \frac{1}{2}\square M - \square OP = \square NO - \square OP.$
100. And from 98° ; by adding $\frac{1}{2}\square M$ to each part, } $\frac{1}{2}\square M + \square OP = \frac{1}{4}\square H.$
101. And by adding the Equation in 100^o to that in 99^o, } $\frac{1}{4}\square H + \square M = \square NO - \square OP + \frac{1}{4}\square H.$
102. And from 101^o, by subtracting $\frac{1}{4}\square H$ from each part, there will remain } $\square M = \square NO - \square OP.$
103. But it hath been proved in 96^o, that } $\square AC, CB = \square NO - \square OP.$
104. Therefore from 102^o and 103^o, (*per Ax.* 1. *Chap.* 2.) } $\square AC, CB = \square M.$
105. Wherefore from 104^o, (*per prop.* 14. *Elem.* 6.) } $AC . M :: M . CB.$

Which was to be Demonstr. Therefore that is done which the Problem required.

Note. The foregoing Problem is the same in effect with this, *viz.* The sum of the Squares and the Rectangle of two right lines being given severally, to find out those lines.

Probl. XV.

The Hypothensal and Area of a right-angled Triangle being given severally, to find out the Triangle. But the right line arising by the Application of the double Area to the Hypothensal, must not be greater than half the Hypothensal.

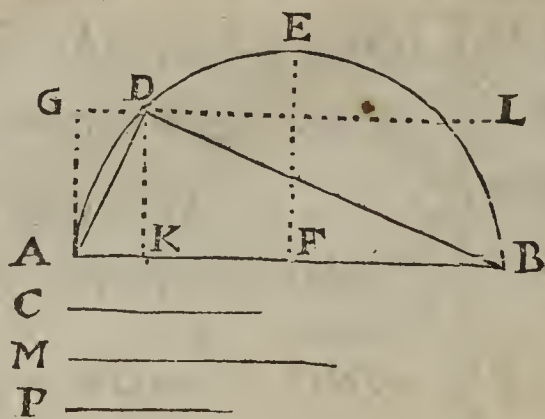
This Problem differs but little from the preceding 14th, for there, the Rectangle of the sides about the right angle, (that is, the double Area,) and Hypothensal are given; but here, half the said Rectangle, (that is, the Area of the right-angled Triangle sought,) and the Hypothensal are given.

Suppos.

1. ABD is a Δ right-angled at D.
2. $b = AB$ the Hypothensal is given.
3. $c = C$ a given right line, whose Square is equal to ΔABD , that is, $\square \frac{1}{2} BD, DA.$

Req. to find out the Triangle.

Resolu-



C _____
M _____
P _____

AB = 169	C = $\sqrt{5070}$
BD = 156	M = $\sqrt{10140}$
AD = 65	P = 60
DK = 60	
BK = 144	
KA = 25	

Resolution.

4. Because (by *prop. 41. Elem. 1.*) the Rectangle of BD into DA is equal to the double Area of $\triangle ABD$, therefore from 3° the said Rectangle or double Area is } 2cc.
5. And out of 2° and 3°, (by *Theor. 1. in 23° of the foregoing Probl. 14. of this Chapt.*) the Square of the difference of the leggs about the right angle shall be } hb - 4cc.
6. And by adding 8cc, (to wit, four Rectangles of the leggs,) to hb - 4cc, (that is, the Square of the difference of the leggs,) the summ of that Addition gives (per *Theor. 7. Chap. 4.*) the Square of the summ of the leggs, to wit, } hb + 4cc.
7. And because (by *Theor. 3. Chap. 4.*) a quarter of the Square of any whole right line is equal to the Square of the half; therefore from 6°, the Square of half the summ of the leggs shall be } $\frac{1}{4}hb + cc.$
8. And consequently from 7°, half the summ of the leggs is } $\sqrt{\frac{1}{4}hb + cc}:$
9. And from 5°, (by *Theor. 3. Chap. 4.*) the Square of half the difference of the leggs is } $\frac{1}{4}hb - cc.$
10. And consequently from 9°, half the difference of the leggs is } $\sqrt{\frac{1}{4}hb - cc}:$
11. Therefore from 8° and 10°, (by *Theor. 9. Chap. 4.*) the leggs shall be given severally, viz.

$$BD = \sqrt{\frac{1}{4}hb + cc} + \sqrt{\frac{1}{4}hb - cc}:$$

$$DA = \sqrt{\frac{1}{4}hb + cc} - \sqrt{\frac{1}{4}hb - cc}:$$

From 6° and 5° ariseth

THEOR. 1.

12. In every right-angled Triangle having unequal sides about the right angle, the Square of the summ of those sides is equal to the Square of the Hypothensal together with the quadruple of the Area: But the Square of the difference of the same sides is equal to the excess whereby the Square of the Hypothensal exceeds the quadruple of the Area. The Equations in 11° give

THEOR. 2.

13. In every right-angled Triangle having unequal sides about the right-angle, if to and from the Square of half the Hypothensal, the Area be added and subtracted severally, and out of the summ and remainder severally the square Root be extracted, the summ and difference of those square Roots shall be equal to the sides about the right angle.

The truth of the Determination annex'd to this *Probl. 15.* hath already been demonstrated in the preceding *Probl. 14.* and the reason thereof will appear in the following Construction.

The Composition of the foregoing *Probl. 15.*

Suppos.

14. AB = the Hypothensal of a right-angled Triangle is given.
15. C is a right line given, whose Square is equal to the Area of that Triangle.
16. $\frac{2 \square C}{AB}$ not $\square \frac{1}{2} AB$.

Req. to find the Triangle.

Constru.

Construction.

17. This Problem might be effected according to the direction of the foregoing Theorem in 13° , but more compendiously thus; First, by *Probl. 2. Chap. 5.* find a right line M, such, that its Square may be equal to $2 \square C$, therefore

$$M = \sqrt{2 \square C}.$$

18. Then by *Probl. 7. Chap. 5.* let it be made as A B to M, so M to a third proportional line, suppose it to be the line P, therefore

$$AB \cdot M :: M \cdot P.$$

19. Upon A B describe the Semicircle F A D B.

20. Make $AG \perp AB$; also $AG = P$; and $GL \parallel AB$, which Parallel GL shall necessarily either touch the Semicircle F A D B, or cut the same; for by *Suppos.* in 16° , $\frac{2 \square C}{AB}$ not $\leq \frac{1}{2} AB$, and by *Constr.* in 17° and 18° , P is equal to $\frac{2 \square C}{AB}$; therefore

P is not greater than $\frac{1}{2} AB$, (= the Semidiameter F E.) But GL was before drawn parallel to A B at the distance of the right line P, (= A G,) and therefore the said Parallel shall either touch the Semicircle in E, or else cut the same. Supposing then the Parallel GL to cut the Semicircle in D, draw the right lines A D and D B, so shall A D B be the right-angled Triangle required. But now we must shew that it will satisfy the Problem.

21. First then by *Construction* in 19° , A B the Base of the Triangle A D B is that which in 14° was prescribed for the Hypothensal of the right-angled Triangle sought; secondly, by *Constr.* in 19° and 20° the angle A D B is in the Semicircle F A D B, and therefore 'tis a right angle, (*per prop. 31. Elem. 3.*) thirdly and lastly, that the Area of the right-angled Triangle A D B is equal to the Square of the given right line C, the following Demonstration will make manifest.

Prepar.

22. From the point D in the Circumference, let fall D K perpendicular to the Diameter A B.

23. . . . *Req. demonstr.* $\triangle A D B = \square C.$

Demonstration.

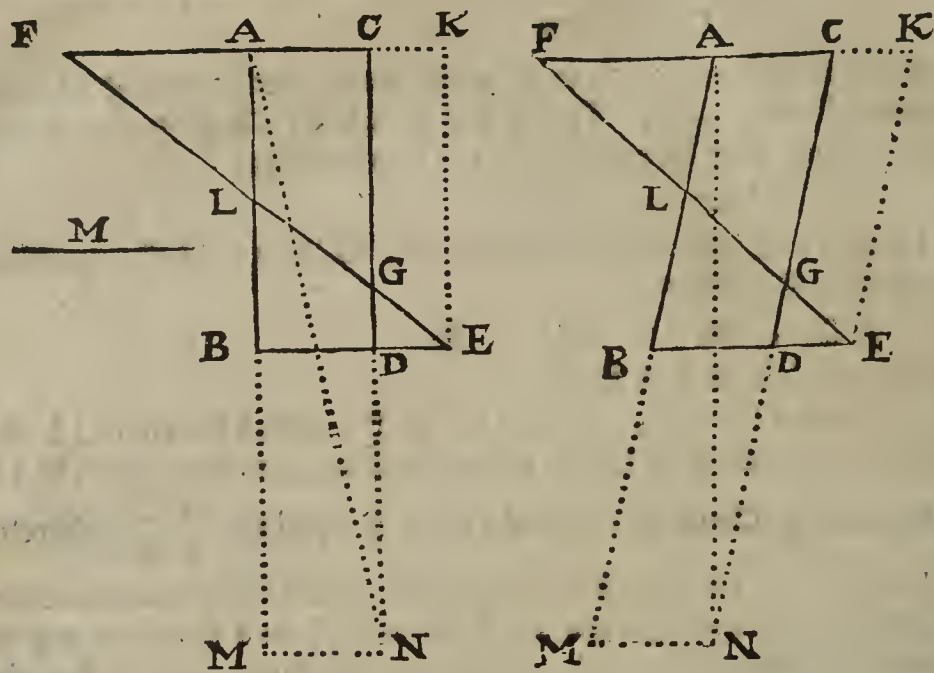
- 24. By *Constr.* in 18° , } $AB \cdot M :: M \cdot P.$
- 25. And from the *Constr.* in 20° and 22° , (*per prop. 34. Elem. 1.*) } $P = AG = DK.$
- 26. Therefore from 24° and 25° , by taking D K } $AB \cdot M :: M \cdot DK.$
instead of P, }
- 27. Therefore from 26° , (*per prop. 17. Elem. 6.*) } $\square AB, DK = \square M.$
- 28. But by *Constr.* in 17° , } $\dots 2 \square C = \square M.$
- 29. Therefore from 27° and 28° , (*per Ax. 1. Chap. 2.*) } $\square AB, DK = 2 \square C.$
- 30. And because by *Constr.* $\angle A D B$ is \perp , there- } $\square AB, DK = 2 \triangle A D B.$
fore (*per prop. 41. Elem. 1.*) }
- 31. Therefore from 29° and 30° , (*per Ax. 1. Chap. 2.*) } $2 \triangle A D B = 2 \square C.$
- 32. Therefore from 31° , (*per Ax. 9. Chap. 2.*) } $\triangle A D B = \square C.$

Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. XVI. (Prop. 164. Lib. 7. Pappi.)

A Parallelogram B A C D being given by Position, from a given point E in B D produced, to draw a right-line E F to concurr with C A produced in F, so as to make the Triangle F C G equal to the given Parallelogram B A C D.

Suppos.



- AB = 75
 - BD = 32
 - DE = 20
 - CK = 20
 - FA = 48
 - EK = 75
 - FK = 100
 - CG = 60
 - GD = 15
 - AL = 36
 - LB = 39
 - EF = 125
- in Fig. 1. }

Suppos.

1. BACD is a Parallelogram given by Position.
2. E is a point given in BD produced.
3. $b = AC = BD$ is given.
4. $d = CD = AB$ is given.
5. $c = DE$ is given.

Req. to find

6. AF a right line to be added to CA in a direct line, that EF being drawn, it may make $\triangle FCG = BACD$.

Prepar.

7. By the given point E draw $EK \parallel DC$, and to concurr with AC produced in K, whence $CK = DE$, and $EK = DC$.
8. Let CD be continued to N, so, that $DN = DC$, whence $CN = 2DC$, and therefore $\triangle ACN = \square AC, CD = BACD$, (per prop. 41. Elem. 1.)

Resolution.

9. Suppose that done which is required, and put
10. Then because by Constr. in 7° $EK \parallel DC$, the Triangles FCG and FKE are equiangular, (per prop. 29. Elem. 1.) therefore (per prop. 4. Elem. 6.) these are Proportionals, viz.
11. By Constr. in 8°,
12. And the Problem requires
13. Therefore from 11° and 12°, (per Ax. 1. Chap. 2.)
14. And because those equal Triangles ACN and FCG have a common angle FCN, the sides about that angle shall be reciprocally proportional, (per prop. 15. Elem. 6.) therefore
15. Therefore, by halving the Antecedents in the last Analogy,
16. But it hath been shewn above in 10°, that
17. Therefore from 15° and 16°, (per prop. 11. Elem. 5.)
18. And by doubling the two latter Terms, their Reason is not alter'd, therefore
19. Therefore from the last Analogy by Division of Reason,
20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes,

$$a = FA.$$

$$FK \cdot FC :: KE \cdot CG.$$

$$a+b+c \cdot a+b :: d \cdot \frac{da+db}{a+b+c}.$$

$$\triangle ACN = BACD.$$

$$\triangle FCG = BACD.$$

$$\triangle ACN = \triangle FCG.$$

$$FC \cdot AC :: CN \cdot CG.$$

$$a+b \cdot b :: 2d \cdot \frac{da+db}{a+b+c}.$$

$$\frac{1}{2}a + \frac{1}{2}b \cdot b :: d \cdot \frac{da+db}{a+b+c}.$$

$$a+b+c \cdot a+b :: d \cdot \frac{da+db}{a+b+c}.$$

$$a+b+c \cdot a+b :: \frac{1}{2}a + \frac{1}{2}b \cdot b.$$

$$a+b+c \cdot a+b :: a+b \cdot 2b.$$

$$c \cdot a+b :: a-b \cdot 2b.$$

$$aa - bb = 2bc.$$

21. There-

- 21. Therefore by adding bb to each part, } $aa = bb + 2bc$ (= $\square FA$.)
- 22. Therefore, by extracting the square Root out of } $a = \sqrt{bb + 2bc} = FA$.
- 23. Again, from 3° and 5° , respect being had to the } $BE (= BD + DE) = b + c$.
- Diagram, }
- 24. Therefore by squaring each part, } $\square BE = bb + cc + 2bc$.
- 25. And by subtracting $\square DE = cc$ from each part, . } $\square BE - \square DE = bb + 2bc$.
- 26. But from 21° , } . . . $\square FA = bb + 2bc$.
- 27. Therefore from 25° and 26° , (*per Ax. 1. Chap. 2.*) } . . . $\square FA = \square BE - \square DE$.
- 28. Therefore by extracting the square Root out of } $FA = \sqrt{\square BE - \square DE}$:
- each part, }

The Equations in 21° and 27° do afford this

THEOREM.

29. If $EC \parallel BE$, and $AB \parallel CD$, and $\triangle FCG = BACD$, then the Square of FA is equal to the Square of BD together with twice the Rectangle of BD into DE . Moreover, the Square of FA is equal to the excess by which the Square of BE exceeds the Square of DE .

Therefore BD and DE being given severally, FA shall be given also, and consequently EF may be drawn to solve the Problem propounded.

But to manifest the truth of the said Theorem, I shall form a Demonstration thereof by a repetition of the steps of the preceding Resolution in a direct order, to which end, let respect be had to the *Diagram, Suppos.* and *Prepar.* at the beginning of the Problem.

30. . . *Req. demonstr.* . . . $\square FA = \square BD + 2\square BD, DE = \square BE - \square DE$.

Demonstration.

- 31. Forasmuch as $\triangle FKE$ and $\triangle FCG$ are } $FK \cdot FC :: KE \cdot CG$.
- equiangular, (for by *Constr.* in 7° , EK } $\triangle ACN = BACD$.
- $\parallel CD$,) therefore (*per prop. 4. Elem. 6.*) } $\triangle FCG = BACD$.
- 32. By *Constr.* in 8° , }
- 33. And by *Suppos.* in 29° , }
- 34. Therefore from 32° and 33° , (*per* } $\triangle ACN = \triangle FCG$.
- Ax. 1. Chap. 2.*) }
- 35. And because $\angle FCN$ is common to } $FC \cdot AC :: CN$ (or $2CD$) $\cdot CG$.
- those equal Triangles ACN and FCG , } $\frac{1}{2}FC \cdot AC :: CD$ (or KE) $\cdot CG$.
- therefore, (*per prop. 15. Elem. 6.*) . }
- 36. Therefore from 35° , by halving the } $FK \cdot FC :: KE \cdot CG$.
- Antecedents, }
- 37. But it hath been shewn in 31° , that . } $FK \cdot FC :: \frac{1}{2}FC \cdot AC$.
- 38. Therefore from 36° and 37° , (*per* } $FK \cdot FC :: FC$ (or $FA - AC$) $\cdot 2AC$.
- prop. 11. Elem. 5.*) }
- 39. And from 38° , by doubling the two } $CK \cdot FC :: FA - AC \cdot 2AC$.
- latter Terms, }
- 40. And from 39° , by *Division of Reason*, } $DE \cdot FA + AC :: FA - AC \cdot 2BD$.
- 41. That is, (as is evident by the Diagram,) } $\square \{FA + AC, \}$ = $2\square BD, DE$.
- 42. Therefore from 41° , (*per prop. 16.* } $\square \{FA - AC, \}$ = $\square FA - \square AC$ ($\square BD$.)
- Elem. 6.*) }
- 43. But by *Theor. 8. Chap. 4.* }
- 44. Therefore from 42° and 43° , (*per* } $\square FA - \square BD = 2\square BD, DE$.
- Ax. 1. Chap. 2.*) }
- 45. Therefore from 44° , by adding $\square BD$ } $\square FA = \square BD + 2\square BD, DE$.
- to each part, }

Which was to be Dem.

- 46. Again, because by *Supposition* in 2° , } $BE = BD + DE$.
- and by the Diagram, }
- 47. Therefore by squaring each part of that } $\square BE = \square BD + \square DE + 2\square BD, DE$.
- Equation, (*per Theor. 2. Chap. 4.*) . . }

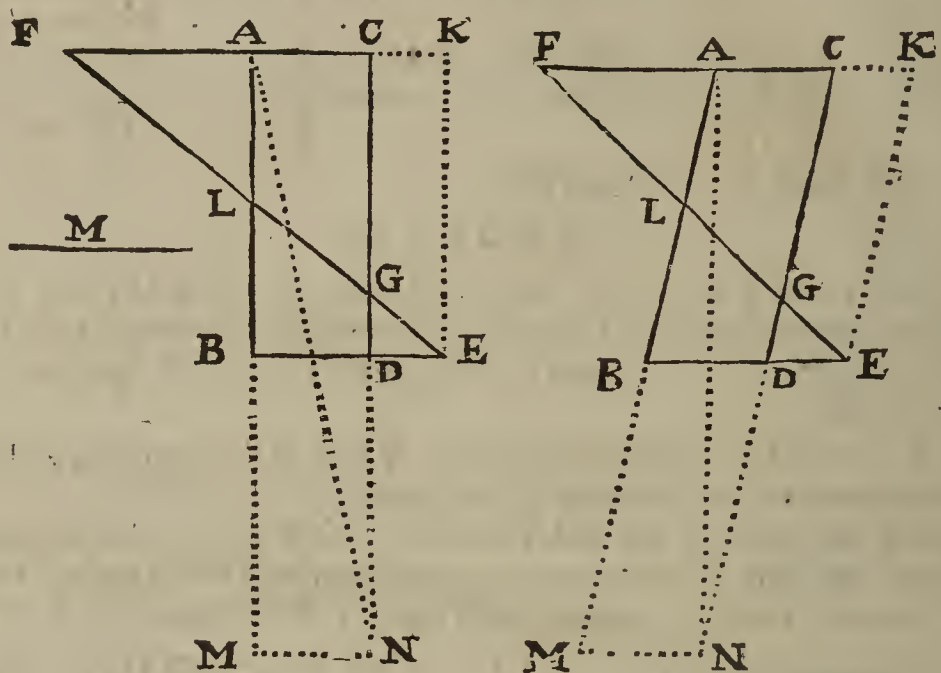
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48. And

48. And from 47°, by subtracting $\square DE$ } $\square BE - \square DE = \square BD + 2\square BD, DE.$
 from each part, }
 49. Therefore from 45° and 48°, (per } $\square FA = \square BE - \square DE.$
 Ax. 1. Chap. 2.) }

Which was also to be Dem. Therefore the truth of the preceding Theorem is manifest.

The Composition of the foregoing Probl. 16.



Suppos.

50. BACD is a Parallelogram given by Position.
 51. E is a point given in BD continued.
 52. . . Req. to draw EF a right line, such, that $\triangle FCG = BACD.$

Construction.

53. By *Probl. 9. Chap. 5.* find a mean proportional line M between BD and $BD + 2DE$, therefore
 $BD \cdot M :: M \cdot BD + 2DE.$
 54. Produce CA to such a point F, that AF may be equal to the line M, (to wit, the mean Proportional found out in 53°,) then draw a right line from E to F, so shall the Triangle FCG be equal to the Rectangle BACD, as was required; the truth whereof will evidently appear by the following Demonstration, form'd out of the foregoing Resolution by a repetition of its steps in a backward (not direct) order. But by way of Preparation, draw EK || and = DC; also make CN = 2CD; draw AN, and produce FC to K.

55. . . Req. demonstr. $\triangle FCG = BACD.$

Demonstration.

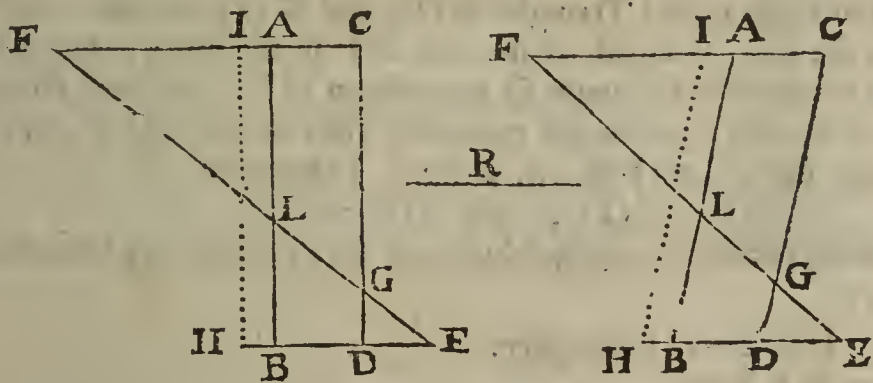
56. By *Constr.* in 53° and 54°, . . } $BD \cdot M$ (or FA) $:: M \cdot BD + 2DE.$
 57. Therefore (per *prop. 17. Ele. 6.*) } $\square FA = \square BD + 2\square BD, DE.$
 58. Therefore from 57°, by sub- } $\square FA - \square BD = 2\square BD, DE.$
 tracting $\square BD$ from each part, }
 59. But by *Theor. 8. Chap. 4.* . . } $\square FA - \square AC$ ($\square BD$) = $\square \begin{cases} FA + AC, \\ FA - AC. \end{cases}$
 60. Therefore from 58° and 59°, } $\square \begin{cases} FA + AC, \\ FA - AC \end{cases} = 2\square BD, DE.$
 (per *Ax. 1. Chap. 2.*) }
 61. Therefore from 60°, (per } $DE(CK) \cdot FA + AC :: FA - AC \cdot 2BD(2AC.)$
prop. 14. Elem. 6.) }
 62. That is, as is evident by the } $CK \cdot FC :: FA - AC \cdot 2AC.$
 Diagram, }
 63. Therefore from 62°, (by *Com- } $FK \cdot FC :: FC(FA + AC) \cdot 2AC.$
pos. of Reason,) }
 64. And from 63°, by halving the } $FK \cdot FC :: \frac{1}{2}FC \cdot AC.$
 two latter Terms, }*

65. But

65. But because $\triangle FKE$ and $\triangle FCG$ are like, (for $EK \parallel DC$), therefore, (per prop. 4. Elem. 6.) } $FK \cdot FC :: KE (CD) \cdot CG.$
 66. Therefore from 64° and 65° , (per prop. 11. El. 5.) } $\frac{1}{2}FC : AC :: CD \cdot CG.$
 67. And from 66° , by doubling the Antecedents, . } $FC \cdot AC :: CN(2CD) \cdot CG.$
 68. And because $\angle FCN$ is common to $\triangle FCG$ and $\triangle ACN$, and it appears in 67° , that the sides about that common angle are reciprocally proportional, therefore (per prop. 15. Elem. 6.) . } $\triangle FCG = \triangle ACN.$
 69. But by Constr. in 8° , } $BACD = \triangle ACN.$
 70. Therefore from 68° and 69° , (by Ax. 1. Ch 2.) } $\triangle FCG = BACD.$
- Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. XVII.

A Parallelogram $BACD$ being given by Position, from a given point E in BD produced, to draw a right line EF to meet with CA produced in F , that the Triangle FCG may have a given Reason to the Parallelogram $BACD$, suppose as HD to BD .



Construction.

1. By the point H draw $HI \parallel BA$ or DC (per prop. 31. Elem. 1.) then by the last preceding Problem draw a right line EF , so as to make the Triangle FCG equal to the Parallelogram $HICD$, so shall $\triangle FCG$ be to $BACD$ as HD to BD , which was required; the truth whereof will be manifest by the following Demonstration.
2. . . . Req. demonstr. $\triangle FCG \cdot BACD :: HD \cdot BD.$

Demonstration.

3. Because (per prop. 1. Elem 6.) } $HICD \cdot BACD :: HD \cdot BD.$
4. And by Constr. in 1° , } $\triangle FCG = HICD.$
5. Therefore from 3° and 4° , } $\triangle FCG \cdot BACD :: HD \cdot BD.$

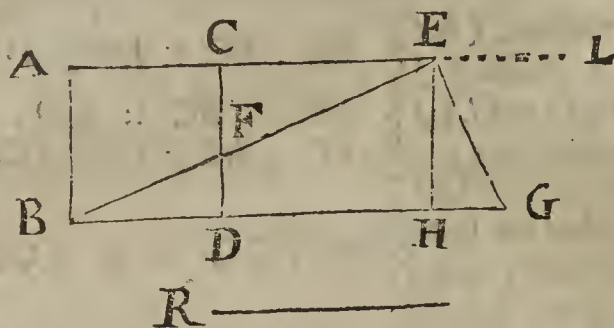
Which was to be Dem.

6. After the same manner, from the given point E a right line may be drawn so as to make the Triangle FCG equal to a given Space, suppose the Square of the right line R , by making the Parallelogram $HICD$ equal to the Square of R , and $\triangle FCG = HICD$: For,

If by Construction $HICD = \square R,$
 And by Constr. $HICD = \triangle FCG.$
 Then it follows (per Ax. 1.) that . . . $\triangle FCG = \square R.$

Probl. XVIII. (Prop. 71. Lib. 7. Pappi.)

A Square $BACD$ (whose side is BD or DC) being given, to draw a right line from the angle B , as BE , that may so cut the side DC , and concurr with the side AC produced towards L , that FE may be equal to a given right line R .



BD = HE =	60
R = FE =	91
BF = EG =	65
DG =	109
DF = HG =	25
FC =	35
DH = CE =	84

Prepar.

1. Suppose that done which is required, viz. that BE is a right line so drawn from the angle B that it cuts DC in F, and concurs with AC produced in E, and makes FE equal to the given right line R.
2. Make EG perpendicular to BE, and let BD be continued until it concurr with EG in G, and from E let fall EH perpendicular to BG, whence it follows (per prop. 8, & 2. Elem. 6.) that

$\triangle BEG$
 $\triangle BHE$
 $\triangle GHE$
 $\triangle BDF$

are like (that is, equiangular) right-angled Triangles, and therefore the sides about the equal angles are Proportionals, (per prop. 4. Elem. 6.)

3. And because the right-angled Triangles BDF and GHE are like, and the side BD (= DC) in the one, is equal to the side HE in the other, and the angle BFD opposite to BD is equal to the angle G opposite to HE, therefore the remaining sides of $\triangle BDF$ shall be also equal to the remaining sides of $\triangle GHE$, viz. each side to its correspondent side, (per prop. 26. Elem. 1.) therefore

$$BF = GE, \text{ and } DF = HG.$$

These things being premised, the Resolution of the Problem may be formed thus:

Suppos.

4. $b = BD = DC = HE$ is given.
5. $d = FE = R$ is given.

Resolution.

6. For DG put a , viz. suppose } $a = DG.$
7. And for BF (= GE) put e , viz. suppose } $e = BF = GE.$
8. Then from 4° and 6°, } $b + a = BG.$
9. And from 7° and 5°, } $e + d = BE.$
10. The Square of the Equation in 7° gives } $ee = \square BF.$
11. And the Square of the Equation in 8° gives } $bb + 2ba + aa = \square BG.$
12. And the Square of the Equation in 9° gives } $ee + 2de + dd = \square BE.$
13. Now because by Constr. in 2°, the Triangle BEG is right-angled at E, and from 3°, $\square BF = \square GE$, therefore from 10°, 11°, 12°, (per prop. 47. Elem. 1.) this Equation ariseth, viz.

$$\square BG = \square BE + \square GE (\square BF.)$$

$$bb + 2ba + aa = ee + 2ed + dd + ee.$$

14. And because from 2°, } $\triangle BEG$ and $\triangle BHE$ are like.
15. Therefore (per prop. 4. Elem. 6.) } $BG : GE :: BE : EH.$
16. That is, in the letters of the Resolution, } $b + a : e :: e + d : b.$
17. Which Analogy being reduced to an Equation, gives } $bb + ba = ee + ed.$
18. And by subtracting the Equation in 17° from that } $ba + aa = ee + ed + dd.$
- in 13°, this remains, }
19. And if instead of $ee + ed$ in the last preceding Equation, } $ba + aa = bb + ba + dd.$
- there be taken $bb + ba$, which in 17° appears to be equal }
to $ee + ed$, then the Equation in 18° will be reduced to }
this }
20. Whence, by subtracting ba from each part, there } $aa = bb + dd.$
- remains }
21. Therefore by extracting the square Root out of each } $a = \sqrt{bb + dd} = DG.$
- part of the last Equation, it gives }

Hence

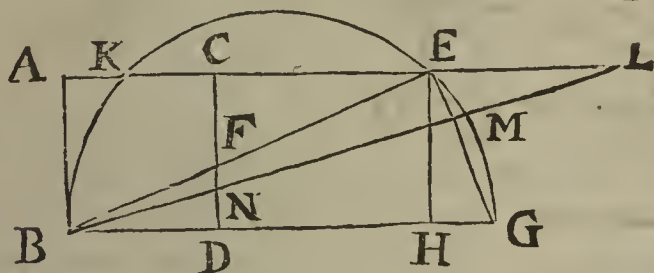
Hence this

T H E O R E M.

22. The right line DG is equal to that right line whose Square is equal to the sum of the Squares of BD and FE. Therefore if BD (or DC) and FE be given severally, then DG is given also, by the help whereof the Problem may be effected.

This Theorem is demonstrated in Prop. 71. of the 7th Book of Pappus's Mathematical Collections, and the truth thereof is also manifest by the foregoing Resolution, wherein the Argumentation is clearly Geometrical as well as Algebraical, and therefore there is no need of any further Demonstration of the said Theorem.

The Composition of the foregoing Probl. 18.



- BD = HE = 60
- R = FE = 91
- BF = EG = 65
- DG = 109
- DF = HG = 25
- FC = 35
- DH = CE = 84

R _____
P _____

Suppos.

- 23. B A C D is a Square given, whose side is B D or D C.
- 24. R is a right line given.
- 25. . . . Req. { To draw a right line from the angle B, as B E, that may so cut the side D C, and concurr with the side A C produced towards L, that F E may be equal to a given right line R.

Construction.

- 26. By *Probl. 2. Chap. 5.* find a right line P, such, that its Square may be equal to $\square BD + \square R$, therefore } $P = \sqrt{\square BD + \square R}$:
- 27. To B D add the line P, so, that B D and P may } $DG = P$, and $BG = BD + P$.
- 28. Upon B G describe the Semicircle B K E G.
- 29. Let A C be continued towards L, so shall the line produced cut the Semicircle B K E G, I say cut it, not touch it, nor lye without it; for $\frac{1}{2} B G$ is greater than B D or B A, as may be proved thus:

Because by *Constr.* in 27°, $DG = P$.
 And by *Constr.* in 26°, $P \square BD$.
 Therefore (per *Ax. 4. Chap. 2.*) $DG \square BD$.
 And by adding B D to each part, $BD + DG \square 2 BD$.
 But by *Constr.* in 27°, $BD + DG = BG$.
 Therefore (per *Ax. 3. Chap. 2.*) $BG \square 2 BD$.
 And consequently, $\frac{1}{2} BG \square BD$ or B A.

Which was to be proved. And therefore A C E, which is parallel to B G at the distance of B A shall necessarily cut the Semicircle B K E G in two points, as in K and E.

- 30. Lastly, draw the right line B E, so shall F E be equal to the given right line R, as was required. But that F E = R, I demonstrate thus;
- 31. . . . Req. demonstr. F E = R.

Demonstration.

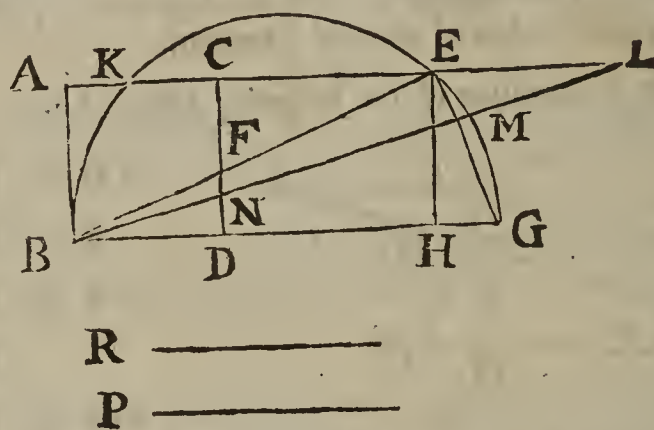
- 32. By *Constr.* in 26°, $\square BD + \square R = \square P$.
- 33. And from the *Constr.* in 27°, $\square DG = \square P$.
- 34. Therefore (per *Ax. 1. Chap. 2.*) $\square DG = \square BD + \square R$.
- 35. But by the *Theor.* in 22° of this Problem, $\square DG = \square BD + \square FE$.
- 36. Therefore from 34° and 35°, (per *Ax. 1. Chap. 2.*) } $\square BD + \square FE = \square BD + \square R$.
- 37. And from 36°, by subtracting $\square BD$ from each part, } $\square FE = \square R$.
- 38. Therefore, F E = R.

Which was to be Dem.

39. But

39. But that FE is the only right line that can be found equal to the given right line R, and in such a Position as the Problem requires, I shall demonstrate in the next place; to which end, let some other line besides BE, as BL, be drawn from B to concur with AC produced beyond E in L: Now if BL doth effect the Problem, then NL must be equal to FE; but NL is greater than FE, as will be made manifest by the following Demonstration.

40. . . . Req. demonstr. NL \sqsupset FE.



Demonstration.

- 41. Because by *Suppos.* \sphericalangle BAL is \perp .
- 42. Therefore (*per prop. 47. Elem. 1.*) \square BL = \square BA + \square AL.
- 43. And in like manner, \square BE = \square BA + \square AE.
- 44. By *Suppos.* in 39°, AL \sqsupset AE, and consequently, \square AL \sqsupset \square AE.
- 45. Therefore from 42°, 43° and 44°, (*per Ax. 4. Chap. 2.*) \square BL \sqsupset \square BE.
- 46. Therefore from 45°, BL \sqsupset BE.
- 47. Again, because by *Suppos.* \sphericalangle BDC is \perp .
- 48. Therefore (*per prop. 47. Elem. 1.*) \square BF = \square BD + \square DF.
- 49. And in like manner, \square BN = \square BD + \square DN.
- 50. And because DF \sqsupset DN, and consequently, \square DF \sqsupset \square DN.
- 51. Therefore from 48°, 49° and 50°, \square BF \sqsupset \square BN.
- 52. Therefore from 51°, BF \sqsupset BN.
- 53. And consequently, BN \supset BF.
- 54. And because it hath been shewn in 46°, that BL \sqsupset BE.
- 55. Therefore from 52°, 53°, 54°, (*per Ax. 16. Chap. 2.*) BL - BN \sqsupset BE - BF.
- 56. That is, (as is evident by the Diagram,) NL \sqsupset FE.

Which was to be Dem. And therefore BL will not effect the Problem propounded. The like Demonstration will hold good in comparing BE to any other right line that shall be drawn from B to cut DC, and to concur with AC produced.

57. Here the Learner may observe, that in resolving a Problem by the Algebraick Art, there may often-times be found out various Equations so constituted, that every one of them may be capable of solving the Problem, but the simplest of those Equations is to be preferr'd before the rest, and chiefly to be aim'd at, though for the most part 'tis much harder to be discovered than those more compounded. As, in the foregoing *Probl. 18.* among various Equations that may be found out to solve the same, that in 21° at the end of the preceding Resolution is the simplest. But who would think, that the way to solve that Problem is to search out the quantity of the line DG, and not rather of one of these lines, to wit, BE, BF, AE, CE, DF? for by any one of these lines, from the consideration of the like right-angled Triangles BAE, FCE, BDF we may come to an Equation more easily than by the line DG, but the Geometrical Construction of such Equation will be much harder than that of the Equation whereby DG is before discovered: And because the Equation resulting upon the search of any of the said five lines, to wit, BF, BE, AE, CE, DF falls under a higher Form than any of the Equations expounded in this Book; I shall refer the more curious Reader for satisfaction concerning the same, to *Pag. 82, 83, 84* of *Renatus des Cartes's Geometry*, set forth by *Fran. van Schooten* in 1659. yet I shall here shew how the quantities of those five lines before mentioned are also deducible from the preceding Resolution.

First then, the same things being supposed as before, draw EG; and make EH \perp BG; then let the Equations in the preceding 17th and 21th steps be here repeated, *viz.*

58. It hath been shewn in 17°, that } $bb + ba = ee + ed.$
 59. And in 21°, that } $a = \sqrt{bb + dd}:$
 60. Therefore if $\sqrt{bb + dd}$: instead of a be drawn }
 into b , the Equation in 58° will be reduced to this, } $bb + b\sqrt{bb + dd} = ee + ed.$
viz. }
 61. Which last Equation may be reduced into these }
 three Proportionals, *viz.* } $e + d, \sqrt{bb + b\sqrt{bb + dd}}, e \div \div.$
 62. Of which three Proportionals, the mean, to wit, $\sqrt{bb + b\sqrt{bb + dd}}$: is given, as
 also d the difference of the extremes $e + d$ and e , therefore the extremes shall be given
 severally, by the *Theor.* in 24° of *Probl.* 12. *Chap.* 5. *viz.*

$$\sqrt{\frac{1}{4}dd + bb + b\sqrt{bb + dd}} : -\frac{1}{2}d = e = BF.$$

$$\sqrt{\frac{1}{4}dd + bb + b\sqrt{bb + dd}} : +\frac{1}{2}d = e + d = BE.$$

63. And because $EH (= DC = DB)$ is a mean Proportional between BH and HG ,
 whose summ is BG ; which mean and summ of the extremes are represented in the
 preceding Resolution by b and $b + a$, whereof b is given in 4°, and a in 21°, for
 'tis there found equal to $\sqrt{bb + dd}$: and consequently $b + a = b + \sqrt{bb + dd}$: there-
 fore by the help of the said given mean b , and the said given summ of the extremes,
 to wit, $b + \sqrt{bb + dd}$: the extremes $BH (= AE)$ and $HG (= DF)$ shall be
 given severally by the *Theor.* in 21° of *Probl.* 13. *Chap.* 5. *viz.*

$$\frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{1}{4}dd} : +\sqrt{\frac{1}{4}dd - \frac{1}{2}bb + \frac{1}{2}b\sqrt{bb + dd}} = BH = AE.$$

$$\frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{1}{4}dd} : -\sqrt{\frac{1}{4}dd - \frac{1}{2}bb + \frac{1}{2}b\sqrt{bb + dd}} = HG = DF.$$

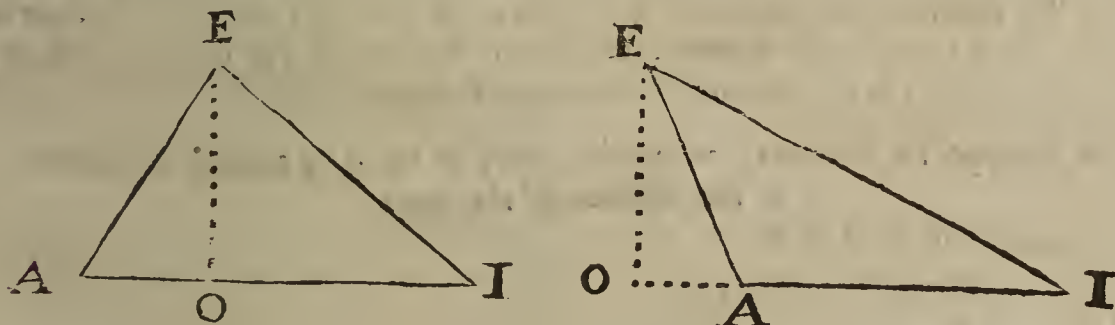
64. And because $CE = DH = BH - BD$, and BH and BD are given as before,
 therefore $CE (= DH)$ is given also, *viz.*

$$\sqrt{\frac{1}{4}bb + \frac{1}{4}dd} : +\sqrt{\frac{1}{4}dd - \frac{1}{2}bb + \frac{1}{2}b\sqrt{bb + dd}} : -\frac{1}{2}b = CE.$$

Lastly, for the better illustration of the premisses, I have calculated whole Numbers
 (placed near the Diagram) to express the Quantities of all the Right lines given and sought
 in this *Probl.* 18.

A L E M M A, leading to the following *Probl.* 19.

If in any oblique-angled plain Triangle, any one of the three sides be called the Base, and
 the other two the leggs; Then, as the Radius, (or total Sine,) is to the Sine complement
 of the angle contain'd under the leggs; so is the double Rectangle of the leggs, to the diffe-
 rence between the summ of the Squares of the leggs, and the Square of the Base.



Suppos.

1. Let EI be the Base, } of the oblique-angled Triangle $A E I$.
2. AE and AI the leggs }
3. $\angle A$ (that is, $\angle E A I$) is contain'd under the leggs AE, AI .
4. $EO \perp AI$.
5. $R =$ the Radius, or total Sine.
6. $Sc. \angle A =$ the Sine complement of the Angle A , (or $\angle E A I$), that is, the Sine of the angle $A E O$.

Req. demonstr.

7. $\left\{ \begin{array}{l} \text{If } \angle A \text{ be acute, then, } R \cdot Sc. \angle A :: 2 \square AE, AI \cdot \square AE + \square AI - \square EI. \\ \text{If } \angle A \text{ be obtuse, then, } R \cdot Sc. \angle A :: 2 \square AE, AI \cdot \square EI - \square AE - \square AI. \end{array} \right.$
Demonstr.

Demonstration.

8. By a vulgar Axiom in the Doctrine of plain Triangles, } R . Sc. $\angle A :: AE . AO$.
9. Therefore by taking the common altitude AI, } R . Sc. $\angle A :: \square AE, AI . \square AO, AI$.
10. And by doubling the two latter Terms, } R . Sc. $\angle A :: 2\square AE, AI . 2\square AO, AI$.
11. And because by *Supposition* in Case 1. $\angle A$ is acute, therefore (per prop. 13. Elem. 2.) } $\square AE + \square AI - \square EI = 2\square AO, AI$.
12. Therefore, from 10° and 11°, by exchanging equal quantities, } R . Sc. $\angle A :: 2\square AE, AI . \square AE + \square AI - \square EI$.
Which was to be Dem.
13. But when $\angle A$ is obtuse, then (per prop. 12. Elem. 2.) } $\square EI - \square AE - \square AI = 2\square AO, AI$.
14. Therefore from 10° and 13°, by exchanging equal quantities, } R . Sc. $\angle A :: 2\square AE, AI : \square EI - \square AE - \square AI$
Which was also to be Dem. Therefore the truth of the *Lemma* is manifest. Hence this

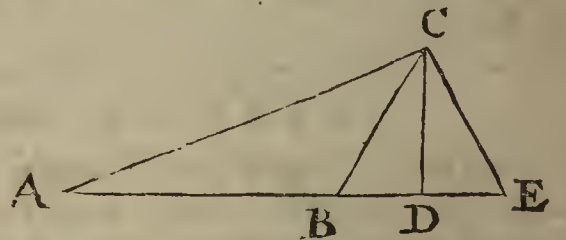
COROLLARY.

15. If in an oblique-angled plain Triangle the three sides be given severally, the angles shall also be given severally, without the help of the Perpendicular; for if the side opposite to an angle sought be called the Base, and the other two sides the leggs; then as the double Rectangle of the leggs is to the difference between the sum of the Squares of the leggs and the Square of the Base; so is the Radius to the Sine complement of the angle opposite to the Base. Which angle is acute when the Square of the Base is less than the sum of the Squares of the leggs; but obtuse when greater.

An Example in Numbers, where the sum of the Squares of the leggs exceeds the Square of the Base.

Suppos. in $\triangle AEC$.

16. $AC = 7$, the Base is given.
17. $AE = 8$, } the leggs are given.
18. $EC = 3$, }
- Req. to find $\angle E$.



Solution Arithmetical.

19. By the preceding Corollary; As 48 the double Rectangle of the leggs, AE, EC , is to 24 the excess of the sum of the Squares of the leggs above the Square of the Base AC ; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whose complement 60. degrees is the measure of the angle E sought.

An Example in Numbers, where the Square of the Base exceeds the sum of the Squares of the leggs.

Suppos. in $\triangle ABC$.

20. $AC = 7$, the Base is given.
21. $AB = 5$, } the leggs are given.
22. $BC = 3$, }
- Req. to find $\angle ABC$.

Solution Arithmetical.

23. By the preceding Corollary; As 30, the double Rectangle of the leggs, AB, BC , is to 15, the excess of the Square of the Base above the sum of the Squares of the leggs; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whose complement 60. degrees subtracted from 180. degrees, leaves 120. degrees for the angle ABC sought.

Note. Because in this second Example the Square of the Base exceeds the sum of the Squares of the leggs, the angle sought is obtuse; and therefore the complement of the angle relating to the Sine which is the fourth Proportional of the before-mentioned Analogy, being subtracted from 180. degrees, leaves the angle sought. But when the sum of the

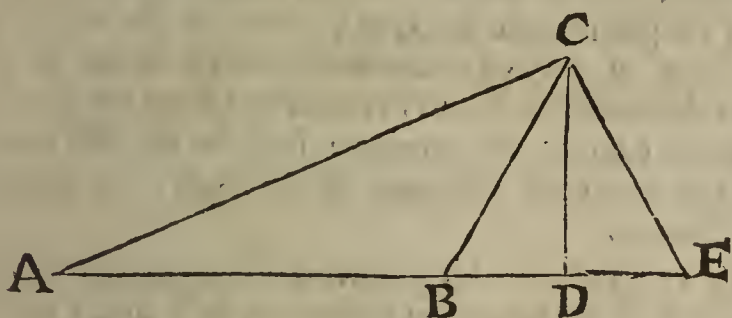
the Squares of the leggs exceeds the Square of the Base, then the complement it self of the angle relating to the said Sine (or fourth Proportional) is the angle sought; as in the first Example.

Probl. XIX.

The Base (that is, any side) of a plain Triangle being given, as also the angle opposite to the Base, and the summ of the sides (or leggs) containing that angle, to find the Triangle. But the given summ of the leggs must exceed the given Base, (*per prop. 22. Elem. 1.*)

Note. Because in this Problem the given angle is not of the same kind with the right lines given, a right line is to be found out, by the help of that angle, which may stand instead of the angle: To which end, let a Circle be described at any distance, and make an angle at the Center equal to the given angle, then from one end of that arch of the Circumference which is the measure of the said angle at the Center, let fall a Perpendicular upon a Diameter drawn to the other end of the said arch, so is the said Perpendicular the Sine of the given angle, and the segment of the Diameter between the foot of the same Perpendicular and the Center of the Circle is the Sine-complement of the given angle. Now instead of the given angle the said Sine-complement may be taken, by the help whereof, and of the Radius (or Semidiameter) of the said Circle, the Resolution of the Problem propounded may be formed in manner following.

The Problem hath three Cases, for the given angle opposite to the Base given is either right, or acute, or obtuse; the first of those Cases hath already been solved in *Probl 4.* of this *Chapt.* I shall therefore begin with the second Case, which supposeth the given angle to be acute, and the leggs containing that angle to be unequal.



- AC = 7
- AE = 8
- EC = 3 = BC
- AB = 5
- ∠E = 60 degrees
- ∠ABC = 120 degrees

Suppos.

1. $b = AC$ the Base of $\triangle ACE$ is given.
2. $c = AE + EC$ the summ of the leggs is given.
3. $\angle E$ opposite to the Base AC is acute, and given.
4. $r =$ the Radius (or total Sine) is given.
5. $d =$ the Sine-complement of $\angle E$ is given.

Req. to find out the Triangle.

Resolution.

6. Suppose ACE to be the \triangle sought, and put a for the difference of the leggs AE, EC ; *viz.* assume } $a = AE - EC.$
7. Therefore from 2° and 6° , (*per Theor. 9. Chap. 4.*) the greater leg shall be } $\frac{1}{2}c + \frac{1}{2}a = AE.$
8. And (by the same *Theor.*) the lesser leg shall be } $\frac{1}{2}c - \frac{1}{2}a = EC.$
9. Therefore the double Product of the leggs is } $\frac{1}{2}cc - \frac{1}{2}aa.$
10. And the summ of the Squares of the leggs is } $\frac{1}{2}cc + \frac{1}{2}aa.$
11. And because the given angle E is acute, the summ of the Squares of the leggs exceeds the Square of the Base, (*per prop. 13. Elem. 2.*) therefore $\frac{1}{2}cc + \frac{1}{2}aa$ exceeds bb , and the excess it self is } $\frac{1}{2}cc + \frac{1}{2}aa - bb.$
12. And from $4^\circ, 5^\circ, 9^\circ$ and 11° , this Analogy is manifest, (by the *Lemma* prefixt before this *Probl.*) *viz.*

$$r \cdot d :: \frac{1}{2}cc - \frac{1}{2}aa \cdot \frac{1}{2}cc + \frac{1}{2}aa - bb.$$

13. Therefore from that Analogy, by Composition of Reason converse;

$$r + d \cdot r :: cc - bb \cdot \frac{1}{2}cc - \frac{1}{2}aa.$$

R r

14: And

Suppos.

- 22. B = the Base of a Triangle is given.
- 23. C = the summ of the leggs is given.
- 24. C \square B. (*Determination.*)
- 25. \angle E = to the angle opposite to the Base is acute, and given.
- 26. R = the Radius or Semidiameter of a Circle is given.
- 27. D = the Sine-complement of the angle E, where the Radius is equal to the line R, is given.

Req. to make the Triangle.

Construction.

- 28. Find a right line H that may be equal to R + D.
- 29. By *Probl. 4. Chap. 5.* find a right line K, such, that its Square may be equal to $2 \square C - 2 \square B$, which effecttion is possible, for by *Supposition* C \square B, therefore

$$\square K = 2 \square C - 2 \square B.$$
- 30. By *Probl. 11. Chap. 5.* let it be made as H to R, so $\square K$ to another Square, whose side suppose to be found F, therefore

$$H . R :: \square K . \square F.$$

That is, in 14° , $r + d . r :: 2cc - 2bb . cc - aa.$

- 31. Find a right line G, such, that its Square may be equal to $\square C - \square F$, which effecttion is possible if C \square F; but that C is greater than F, I prove thus,

By the Theorem in the preceding 21^{th} step, $\square F$ the last Term of the Analogy in 30° is equal to the excess of $\square C$ above the Square of the difference of the leggs, therefore $\square C = \square F +$ the Square of the said difference, whence 'tis manifest that $\square C \square \square F$, and consequently C \square F, therefore 'tis possible to find a right line G, such, that

$$\square G = \square C - \square F (= aa.)$$

Thus far the Construction hath been made according to the direction of *Canon 1.*

- 32. Now let a Triangle be made of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}G$, and $\frac{1}{2}C - \frac{1}{2}G$, which effecttion is possible (*per prop. 22. Elem. 1.*) if C \square G, and the summ of every two of those three lines be greater than the third; but these things may be made manifest thus,

First, it hath been proved in 31° that C \square G, and consequently $\frac{1}{2}C - \frac{1}{2}G$ (one of the above-mentioned three lines,) is greater than nothing, and therefore equal to some real right line.

Secondly, it is manifest that the summ of B and $\frac{1}{2}C + \frac{1}{2}G$ is greater than $\frac{1}{2}C - \frac{1}{2}G$.

Thirdly, the summ of $\frac{1}{2}C + \frac{1}{2}G$ and $\frac{1}{2}C - \frac{1}{2}G$ makes C, which by *Supposition* is greater than B.

Fourthly, that the summ of B and $\frac{1}{2}C - \frac{1}{2}G$ is greater than $\frac{1}{2}C + \frac{1}{2}G$, that is, B \square G, I prove thus;

By <i>Constr.</i> in 30° ,	}	H . R ::	$\square K$.	$\square F$.
That is, as appears by <i>Constr.</i> in 28° ,	}	R + D . R ::	$2 \square C - 2 \square B$.	$\square C - \square G$.
29° and 31° ,	}	D . R ::	$\square C + \square G - 2 \square B$.	$\square C - \square G$.
Therefore by Division of Reason,	}	R . D ::	$\square C - \square G$.	$\square C + \square G - 2 \square B$.
And inversly,	}	R . D ::	$\square C - \square G$.	$\square C + \square G - 2 \square B$.
But R \square D, therefore from the last	}	$\square C - \square G \square$	$\square C + \square G - 2 \square B$.		
preceding Analogy, (<i>per Schol prop. 14.</i>	}	$\square C - \square G \square$	$\square C + \square G - 2 \square B$.		
<i>Elem 5.</i>)	}	$\square C \square$	$\square C + 2 \square G - 2 \square B$.		
And by adding $\square G$ to each part,	}	$2 \square B + \square C \square$	$\square C + 2 \square G$.		
And by adding $2 \square B$ to each part,	}	$2 \square B \square$	$2 \square G$.		
And by subtracting $\square C$ from each part,	}	$\square B \square$	$\square G$.		
And by halving each part,	}	B \square G.			Which was to be Dem.
Therefore,	}				

- 33. Now since it hath been shewn that the summ of every two of these three right lines, B, $\frac{1}{2}C + \frac{1}{2}G$, and $\frac{1}{2}C - \frac{1}{2}G$, is greater than the third, 'tis possible to make a Triangle of those three lines; suppose it therefore done, and that the Triangle so made is MOP, and that MP is equal to B, MO = $\frac{1}{2}C + \frac{1}{2}G$, and OP = $\frac{1}{2}C - \frac{1}{2}G$, then shall MOP be the Triangle required. Now we must shew that it will satisfie the Problem. First then by *Construction*, MP = B the given Base; likewise by *Constr.*

$$Rr . 2 \qquad \qquad \qquad MO + OP$$

MO + OP = C the line prescribed for the summ of the leggs; it remains only to prove that the angle MOP is equal to the given angle E; but that will be made manifest by the following Demonstration, which is formed out of the steps of the foregoing Resolutinn, by returning backwards from the 14th step to the 1th.

Prepar.

See the last preceding Diagram. 34. From the Center O, at the distance of OP, describe the Circle OPRNS, and produce MO to N in the Circumference; therefore MN = MO + OP (ON) = C, and RM = OM - OP (OR) = G.

35 . . Req. demonstr. < MOP = < E.

Demonstration.

- 36. By Constr. in 30°, H . R :: □K . □F.
- 37. Therefore from 36°, 28°, 29°, 31°, by exchange of equal quantities, R + D . R :: 2□C - 2□B . □C - □G.
- That is, in 14°, r + d . r :: 2cc - 2bb . cc - aa.
- 38. And because by Constr. in 33° and 34°, MN = C . MP = B . MR = G.
- 39. Therefore from 37° and 38°, by exchanging equal quantities, . . . R + D . R :: 2□MN - 2□MP . □MN - □MR.
- 40. And by halving the two latter Terms, R + D . R :: □MN - □MP . ½□MN - ½□MR.
- That is, in 13°, r + d . r :: cc - bb . ½cc - ½aa.
- 41. Therefore from 40°, by Division of Reason, D . R :: ½□MN + ½□MR - □MP . ½□MN - ½□MR.
- 42. Therefore inversly, R . D :: ½□MN - ½□MR . ½□MN + ½□MR - □MP.
- That is, in 12°, r . d :: ½cc - ½aa . ½cc + ½aa - bb.
- 43. And because by Theor. 7. Chap. 4. 2□OM, OP = ½□MN - ½□MR.
- 44. And by Theor. 6. Chap. 4. □OM + □OP = ½□MN + ½□MR.
- 45. Therefore from 42°, 43°, 44°, by exchanging equal quantities, . . . R . D :: 2□OM, OP . □OM + □OP - □MP.
- 46. By the last Term of the last preceding Analogy 'tis evident that □OM + □OP = □MP.
- 47. Therefore in Δ MOP, (per prop. 13. Elem. 2.) < MOP is acute.
- 48. Therefore in Δ MOP, (by the Lemma prefixt before this Probl. Rad. Sin.comp.<MOP :: { 2□OM, OP. □OM + □OP - □MP
- 49. Therefore from 45° and 48°, (per prop. 11. Elem. 5.) R . D :: Rad. Sin. comp. < MOP.
- 50. But by Suppos. in 26° and 27°; R . D :: Rad. Sin. comp. < E.
- 51. Therefore from 49° and 50°, (per prop. 11. Elem. 5.) Rad. Sin. comp. < MOP :: Rad. Sin. comp. < E.
- 52. Therefore, (per prop. 14. El. 5.) Sin. comp. < MOP = Sin. comp. < E.
- 53. Therefore from 51° and 52°, (per prop 28. Elem. 3.) < MOP = < E. Which was to be Dem.

After the same manner, the third Case of Probl. 19. (viz. when the given angle is obtuse,) may be Geometrically effected and demonstrated.

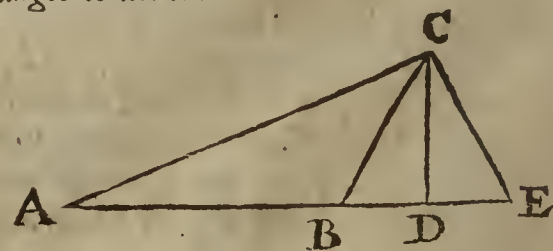
Examples in Numbers, to illustrate the foregoing Resolution of Probl. 19.

Example 1. where the given angle is acute.

Suppos. in Δ AEC.

- 54. . . AC = 7, the Base is given.
- 55. AE + EC = 11, the summ of the leggs is given.
- 56. . . < E = 60 degrees is given.

Req. to find AE and EC severally.



Solution

Solution Arithmetical.

57. Suppose the Radius of a Circle to be } 100000.
 58. Then the Sine-complement of the given angle E, 60. degrees, that is, }
 the Sine of 30. degrees, is } 50000.
 59. Therefore the summ of the Radius and that Sine-complement is . } 150000.
 60. Then (by Canon 1. in the preceding 15th step,) As the said summ 150000 is to the Radius 100000; So is 144, (the excess whereby 242 the double Square of 11 the given summ of the leggs exceeds 98 the double Square of the given Base 7,) to a fourth Proportional 96, which subtracted from 121 the Square of the given summ of the leggs, leaves 25, whose square Root 5 is the difference of the leggs: Therefore (per Theor. 9. Chap. 4.) the leggs themselves, to wit, AE and EC shall be 8 and 3.

Example 2. where the given Angle is obtuse.

Suppos. in $\triangle ABC$.

61. . . . AC = 7, the Base is given.
 62. AB + BC = 8, the summ of the leggs is given.
 63. $\angle ABC = 120$ degrees is given.

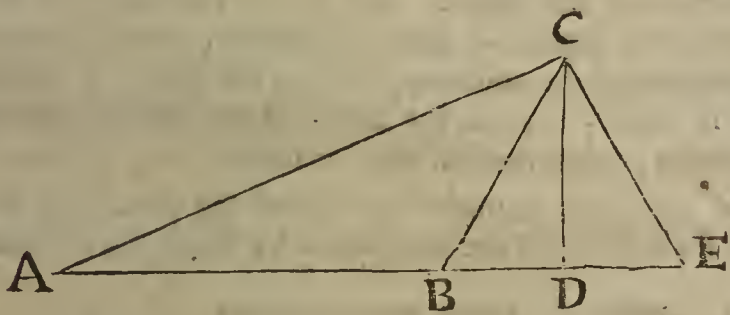
Req. to find AB and BC severally.

Solution Arithmetical.

64. Suppose the Radius of a Circle to be } 100000.
 65. Then the Sine-complement of the given angle ABC 120. degrees, }
 that is, the Sine of 30. degrees, is } 50000.
 66. Therefore the excess of the Radius above the Sine-complement is . } 50000.
 67. Then (by Canon 2. in the preceding 20th step,) As the said excess 50000 is to the Radius 100000, So is 30 (the excess whereby 128 the double Square of 8 the given summ of the leggs exceeds 98 the double Square of the given Base 7,) to a fourth Proportional 60; which subtracted from 64 the Square of the given summ of the leggs, leaves 4, whose square Root 2 is the difference of the leggs sought: Therefore (per Theor. 9. Chap. 4.) the leggs themselves, to wit, AB and BC shall be 5 and 3.

Probl. XX.

The Base of a plain Triangle being given, as also the angle opposite to the Base, and the difference of the sides (or leggs) containing that angle, to find the Triangle. But the given Base must be greater than the given difference.



- AC = 7
 AE = 8
 EC = 3 = BC.
 AB = 5
 $\angle E = 60$ degrees.
 $\angle ABC = 120$ degrees.

Suppos.

1. $b = AC$ the Base of $\triangle ACE$ is given.
 2. $c = AE - EC$ the difference of the leggs is given.
 3. $\angle E$ opposite to the Base AC is acute, and given.
 4. $r =$ the Radius (or total Sine) is given.
 5. $d =$ the Sine-complement of $\angle E$ is given.

Req. to find AE and EC severally.

Resolution.

6. For the summ of the leggs AE and EC, put a , }
 viz. suppose } $a = AE + EC$.
 7. Then out of $1^\circ, 2^\circ, 4^\circ, 5^\circ$ and 6° , (by the Theor. }
 in 21^o of the foregoing Probl. 19.) this follow- }
 ing Analogy will arise, } $r + d . r :: 2aa - 2bb . aa - cc$.

8. Therefore

- 8. Therefore by doubling the Consequents, } $r+d . 2r :: 2aa-2bb . 2aa-2cc.$
(per Schol. Clavii in prop. 22. Elem. 5.) . . . }
- 9. And by halving each of the two latter Terms, } $r+d . 2r :: aa-bb . aa-cc.$
- 10. And inversly, } $2r . r+d :: aa-cc . aa-bb.$
- 11. Therefore by Conversion and Inversion of } $r-d . 2r :: bb-cc . aa-cc.$
 Reason, }

Hence C A N O N 1.

12. When the given angle is acute, let it be made, As the excess of the Radius above the Sine-complement of that angle, is to the double Radius; So the excess of the Square of the given Base above the Square of the given difference of the leggs, to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the square Root of the summ shall be the summ of the leggs sought. Lastly, the summ, as also the difference of the leggs being given, the leggs shall be given severally, by Theor. 9. Chap. 4.

- 13. But when the given angle is obtuse, then by } $r-d . r :: 2aa-2bb . aa-cc.$
 the Theorem in 21° of the preceding Probl. 20. }
 this Analogy will arise, }
- 14. Therefore by doubling the Consequents, . . } $r-d . 2r :: 2aa-2bb . 2aa-2cc.$
- 15. And by halving each of the two latter Terms, } $r-d . 2r :: aa-bb . aa-cc.$
- 16. Therefore inversly, } $2r . r-d :: aa-cc . aa-bb.$
- 17. Therefore by Conversion and Inversion of } $r+d . 2r :: bb-cc . aa-cc.$
 Reason, }

Hence C A N O N 2.

18. When the given angle is obtuse, let it be made, As the summ of the Radius and Sine-complement of that angle, is to the double Radius; So the excess of the Square of the given Base above the Square of the given difference of the leggs to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the square Root of the summ shall be the summ of the leggs sought. Lastly, the summ and difference of the leggs being given, the leggs shall be given severally, by Theor. 9. Chap. 4.

From the preceding Canons in 12° and 18° this following Theorem is deducible, and easie to be demonstrated by the steps of the Resolution in a direct order.

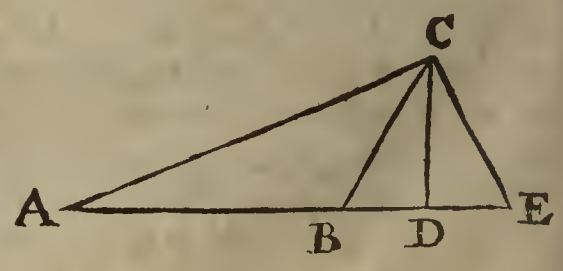
T H E O R E M.

19. If any one of the three sides of an oblique-angled plain Triangle be called the Base, and the other two sides (or leggs) be unequal; Then the excess of the Square of the Base above the Square of the difference of the leggs, shall be to the excess of the Square of the summ of the leggs above the Square of their difference; As the excess of the Radius above the Sine-complement of the angle opposite to the Base, is to the double Radius, when the said angle is acute; but as the summ of the Radius and the Sine-complement is to the double Radius, when the said angle is obtuse.

The Composition of this Probl. 20. may be made like that of the preceding Probl. 19. But waving the Composition, I shall illustrate the Resolution by Examples in Numbers.

Example 1. where the given angle is acute.

- Suppos. in $\triangle AEC.$
- 20. . . . $AC = 7$ the Base is given.
- 21. $AE - EC = 5$ the difference of the leggs is given.
- 22. . . . $\angle E = 60.$ degrees is given.
- Req. to find AE and EC severally.



Solution Arithmetical.

- 23. Suppose the Radius of a Circle to be } 100000.
- 24. Then the Sine-complement of the given angle E, 60. degrees, that is, }
 the Sine of 30. degrees is } 50000.
- 25. Therefore the excess of the Radius above the Sine-complement is . . } 50000.
- 26. The

26. Then by *Canon 1.* (in the preceding 12th step,) As the said excess 50000 is to the double Radius 200000; So is 24, (the excess whereby 49 the Square of the Base exceeds 25 the Square of the difference of the leggs,) to a fourth Proportional 96, which increased with 25 the Square of the difference of the leggs, makes 121, whose square Root 11 is the summ of the leggs sought. Therefore (by *Theor. 9. Chap. 4.*) the leggs themselves, to wit, AE and EC shall be 8 and 3.

Example 2. where the given angle is obtuse.

Suppos. in $\triangle ABC$.

- 27. . . . AC = 7 the Base is given.
 - 28. AB — BC = 2 the difference of the leggs is given.
 - 29. . . $\angle ABC = 120$. degrees is given.
- Req. to find AB and BC severally.

Solution Arithmetical.

- 30. Suppose the Radius of a Circle to be } 100000.
- 31. Then the Sine-complement of the given angle ABC, 120. degrees, }
that is, the Sine of 30. degrees is } 50000.
- 32. Therefore the summ of the Radius and that Sine-complement is . . . } 150000.
- 33. Then (by *Canon 2.* in the preceding 18th step,) As the said summ 150000 is to the double Radius 200000; So is 45, (the excess of 49 the Square of the Base above 4 the Square of the difference of the leggs,) to a fourth Proportional 60, which increased with 4 the Square of the difference of the leggs, makes 64, whose square Root 8 is the summ of the leggs sought. Therefore (by *Theor. 9. Chap. 4.*) the leggs themselves, to wit, AB and BC shall be 5 and 3.

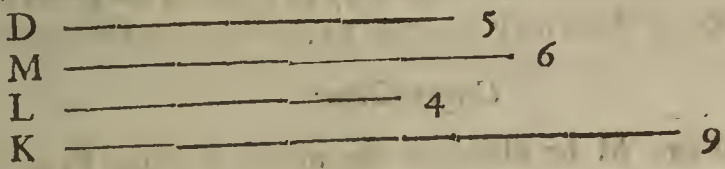
CHAP. IX.

The third Classis of Examples of the Resolution and Composition of Plane Problems.

IN which Examples, the Resolution ends in an Analogy, wherein the Mean of three proportional right lines is given, as also the Difference or else the Summ of the Extremes, to find the Extremes severally.

Probl. I.

To find two right lines, such, that their difference may be equal to a right line given; and that the Rectangle made of the lines found out may be equal to a given Space.



Suppos.

- 1. $d = D$ the difference of two right lines is given.
- 2. $m = M$ a right line given, whose Square is equal to a given Space.

Req. to find

- 3. L and K two right lines, such, that $K - L = D$. Also, that
- 4. $\square KL = \square M$.

Resolution.

- 5. Put a for the lesser of the two right lines sought, viz. sup- } $a = L$.
- pose } $a + d (= K)$
- 6. Therefore from 1^o and 5^o the greater right line sought shall be } $a + d (= K)$
- 7. And from 5^o and 6^o, the Rectangle (or Product of their } $aa + da$.
- multiplication) shall be }

8. But

8. But the said Rectangle must be equal to the given Space, }
 supposed to be equal to the Square of m , therefore . . . } $aa + da = mm.$
 9. Which Equation may be resolved into these Proportionals, }
viz. } $a + d . m :: m . a.$
 10. Of which three Proportionals $a + d$, m and a , the mean m is given, as also d the
 difference of the extremes $a + d$ and a , therefore the extremes severally, which are
 the right lines sought by this Problem, shall be given also, *per Probl. 12. Chap. 5.* whence
 also, (respect being had to the Theorem in 24° of the same *Probl.*) there will arise this
 following

CANON.

11. } $\sqrt{\frac{1}{4}dd + mm} : + \frac{1}{2}d = K,$ } the lines sought.
 } $\sqrt{\frac{1}{4}dd + mm} : - \frac{1}{2}d = L,$ }

That is, in words,

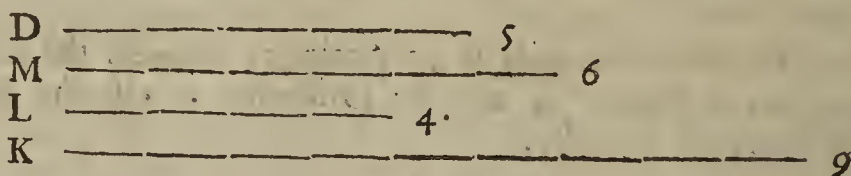
To the Square of half the given difference add the Square equal to the given Space, and
 extract the square Root of the summ: Then adding half the said difference to the said square
 Root, this summ shall be equal to the greater of the two right lines sought; but subtracting
 the said half difference from the said square Root, the remainder shall be equal to the lesser
 line sought.

This Canon may be also Synthetically inferr'd from the given lines, by the help of
Theor. 7, and 9. Chap. 4. For,

12. By considering the things given and sought in 1°, }
 2°, 3° and 4° of this Problem it follows by *Theor. 7.* } $\frac{1}{4}dd + mm = \square : \frac{1}{2}K + \frac{1}{2}L:$
Chap. 4. that }
 13. Therefore by extracting the square Root out of } $\sqrt{\frac{1}{4}dd + mm} = \frac{1}{2}K + \frac{1}{2}L.$
 each part in 12°, }
 14. And from 1° and 3°, } $\frac{1}{2}d = \frac{1}{2}K - \frac{1}{2}L.$
 15. Therefore (by *Theor. 9. Chap. 4.*) the summ and }
 difference of the two last preceding Equations gives } $\sqrt{\frac{1}{4}dd + mm} : + \frac{1}{2}d = K.$
 the two lines sought by this Problem, *viz.* . . . } $\sqrt{\frac{1}{4}dd + mm} : - \frac{1}{2}d = L.$

Thus you see the same Canon is discovered as before.

The Composition of Probl. 1.



Suppos.

16. D a right-line equal to the difference of two right lines sought, is given.
 17. M a right line given, whose Square is equal to a given Space.
Req. to find,
 18. Two such right lines, that their difference may be equal to the given difference D, and
 that the Rectangle made of them may be equal to the Square of the given right line M.

Construction.

19. Let the given right-line M be esteemed the mean of three Proportionals, and the
 given right line D the difference of the extremes; then by *Probl. 12. Chap. 5.* find the
 extremes, the lesser whereof suppose to be L, therefore the greater shall be equal to
 $L + D$, and consequently these shall be Proportionals, *viz.*
 $L + D . M :: M . L.$
 20. Make $K = L + D$, whence $K - L = D.$
 21. I say K and L are the two right lines sought, but that they will satisfy the Problem
 propounded, I prove thus: First by *Construction* in the 20th step, the difference of the
 said right lines K and L is equal to the given difference D; so it remains only to prove
 that the Rectangle made of the said right lines K and L is equal to the Square of the
 given right line M, but that is here-under demonstrated by returning backwards from
 the 9th step, (to wit, the last of the Resolution) to the 8th.

22. . . . *Req. demonstr.* $\square K, L = \square M.$

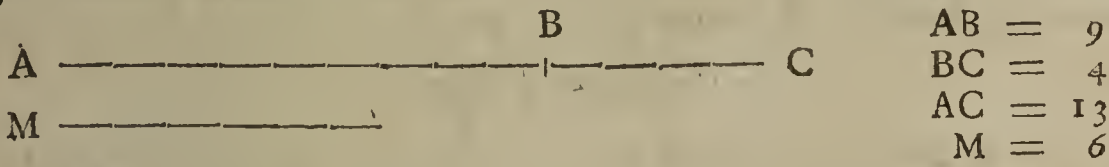
Demonstr.

Demonstration.

23. By Constr. in 19° and 20°, } $K(L + D) \cdot M :: M \cdot L$.
 That is, in 9°, } $a + d \cdot m :: m \cdot a$.
 24. Therefore (per 17. prop. 6. Elem.) } $\square K, L (\square L + \square D, L) = \square M$.
 That is, in 8°, } $aa + da = mm$.
 Which was to be Demonstr. Therefore that is done which the Problem required.

Probl. II.

To cut a given right line into two such parts, that the Rectangle made of the parts may be equal to a Space given. But the side of a Square equal to the given Space must not be greater than half the right line given.



Suppos.

1. $b = AC$ is a right line given to be cut into two parts.
 2. $m = M$ is a right line given, whose Square is equal to a Figure or Space given.
- Req. to find
3. AB and BC such parts of AC , that $AB + BC = AC$. Also,
 4. $\square AB, BC = \square M$.

Resolution.

5. Put a for either of the parts sought, viz. suppose . . . } $a = AB$.
6. Therefore from 1° and 5° the other part sought shall be } $b - a = BC$.
7. And from 5° and 6° the Rectangle of the parts is . . . } $ba - aa$.
8. But the said Rectangle must be equal to the given Space, } $ba - aa = mm$.
 therefore }
9. Which Equation may be resolved into this Analogy, . . . } $b - a \cdot m :: m \cdot a$.
10. But that Analogy doth manifestly consist of three Proportionals, whereof the mean m is given, as also b the summ of the extremes $b - a$ and a ; therefore the extremes severally, (which are the parts required by this Probl.) shall be given also, by Probl. 13. Chap. 5. Whence also this

CANON.

$$11. \dots \dots \dots \left\{ \begin{array}{l} \frac{1}{2}b + \sqrt{\frac{1}{4}bb - mm} = AB, \\ \frac{1}{2}b - \sqrt{\frac{1}{4}bb - mm} = BC. \end{array} \right.$$

That is, in words,

From the Square of half the right line given to be cut into two parts, subtract the Square equal to the given Space, and extract the square Root of the remainder; then add and subtract the said square Root to and from half the right line given to be cut; so the Summ and Remainder shall be the parts required.

This Canon may be also Synthetically inferr'd from the given lines, by the help of Theor. 7, and 9. Chap. 4. For,

12. By considering the things given and sought in 1°, }
 2°, 3° and 4° of this Problem, it follows by } $\frac{1}{4}bb - mm = \square : \frac{1}{2}AB - \frac{1}{2}BC :$
 Theor. 7. Chap. 4. that }
13. Therefore by extracting the square Root out of } $\sqrt{\frac{1}{4}bb - mm} = \frac{1}{2}AB - \frac{1}{2}BC$.
 each part in 12°, }
14. And from 1° and 3°, } $\frac{1}{2}b = \frac{1}{2}AB + \frac{1}{2}BC$.
15. Therefore (by Theor. 9. Chap. 4.) the summ and } $\frac{1}{2}b + \sqrt{\frac{1}{4}bb - mm} = AB$.
 difference of the two last preceding Equations gives } $\frac{1}{2}b - \sqrt{\frac{1}{4}bb - mm} = BC$.
 the lines sought by this Problem, viz. . . . }

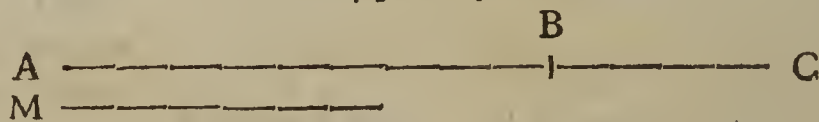
Thus you see the same Canon is discovered as before in 11°; and thereby it evidently appears, that to the end the given lines may be capable of effecting the Problem propos'd, they must be subject to this following

Determination.

16. The side of a Square equal to the given Space must not be greater than half the right line given to be cut into two parts. But that this Determination is necessary, I demonstrate thus;

Forasmuch as by the 9th step of the Resolution the right line m is found to be the mean of three Proportionals whereof the summ of the extremes is b , and in 20° of *Probl. 13. Chap 5.* it hath been proved that the mean of three Proportionals cannot be greater than half the summ of the extremes, it follows, that m must not be greater than $\frac{1}{2}b$. If therefore m happens to be greater than $\frac{1}{2}b$, the Problem propounded cannot possibly be solved; for then mm cannot be subtracted from $\frac{1}{4}bb$, as the Canon directs. But if m be equal to, or less than $\frac{1}{2}b$, then that which the Canon bids to be done is possible; by which also 'tis easie to perceive, that when $m = \frac{1}{2}b$, then the extreme Proportionals (which are equal to the parts required,) will be equal between themselves, that is, each of them will be equal to $\frac{1}{2}b$, to wit, half the line given to be cut into two parts.

The Composition of *Probl. 2.*



Suppos.

- 17. AC is a right line given to be cut into two parts.
- 18. M is a right line given, whose Square is equal to a Space given.
- 19. M not $\llcorner \frac{1}{2} AC$. (*Determination.*)

Req. to find

- 20. AB and BC such parts of AC, that $AB + BC = AC$. Also;
- 21. $\square AB, BC = \square M$.

Construction.

22. By *Probl. 14. Chap. 5.* cut the given right line AC into two such parts in B, that the line M may be a mean Proportional between the parts; which effecton is possible, for by the Determination in 19°, M not $\llcorner \frac{1}{2} AC$; suppose therefore

$$AB \cdot M :: M \cdot BC.$$

23. I say AB and BC are the parts required, for their summ by *Construction* is equal to the given right line AC; and because by *Constr.* also it hath been made, As AB to M, so M to BC, therefore (*per 17. prop. 6. Elem.*) $\square AB, BC = \square M$. Which was required to be done.

Probl. III.

To cut a given right line according to the extreme and mean Reason; that is, into two such parts, that the Rectangle made of the whole line and lesser part may be equal to the Square of the greater.



$$\begin{aligned} AB &= 10 \\ AC &= \sqrt{125} - 5 \\ CB &= 15 - \sqrt{125} \end{aligned}$$

Suppos.

- 1. $b = AB$ is a right line given.
- Req. to find
- 2. AC and CB such parts of AB, that $\square AB, CB = \square AC$.

Resolution.

- 3. Put a for the greater part sought, viz. } $a = AC$.
- 4. Therefore from 1° and 3° the lesser part shall be . . . } $b - a = CB$.
- 5. Therefore from 1° and 4° the Rectangle (or Product) } $bb - ba$.
of the whole line and lesser part is }
- 6. Which Rectangle, (according to the tenour of the Pro- } $bb - ba = aa$.
blem,) must be equal to the Square of the greater part, }
therefore }
- 7. Therefore, by adding ba to each part of that Equation, } $bb = aa + ba$.
- 8. Which last Equation may be resolved into this Ana- } $a \cdot b :: b \cdot b + a$.
logy, }

9. But

9. But that Analogy doth manifestly consist of three Proportionals, whereof the mean b is given, which is also the difference of the extremes a and $b + a$; therefore the extremes, whereof the lesser is equal to the greater part sought, shall be given also, by *Probl. 12. Chap. 5.* Whence also there will arise this following

C A N O N.

10. $\sqrt{bb + \frac{1}{4}bb} - \frac{1}{2}b = AC$, the greater segment.

That is, in words,

To the Square of the given line, add the Square of half the same line; then from the square Root of that summ subtract half the given line, and the remainder shall be the greater part sought, which subtracted from the given line leaves the lesser part.

The Composition of *Probl. 3.*



Suppos.

11. Let AB be a right line given to be cut into two such parts, that the Rectangle made of the whole line and the lesser part may be equal to the Square of the greater.

Construction.

12. Let the given line AB be esteemed the mean, as also the difference of the extremes of three Proportionals; then by *Probl. 12. Chap. 5.* find out the extremes severally, the lesser whereof we may suppose to be the right line L , and consequently the greater extreme is $AB + L$, therefore these are Proportionals, *viz.*

$$L . AB :: AB . AB + L.$$

13. From AB cut off $AC = L$, which is possible to be done, for by *Construction* in 12°, $AB \square L$, because AB is the mean of three Proportionals whose lesser extreme is L . So the given line AB is cut into two parts in C , according to extreme and mean Reason, *viz.* the Rectangle made of the whole line AB and the lesser part CB is equal to the Square of AC the greater part, as will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a backward order.

14. . . . *Req. demonstr.* $\square AB, CB = \square AC.$

Demonstration.

- 15. By *Constr.* in 12° and 13°, } $AC . AB :: AB . AB + AC.$
- That is, in 8°, } $a . b :: b . b + a.$
- 16. Therefore (*per prop. 17. Elem. 6.*) } $\square AB = \square AC + \square AB, AC.$
- That is, in 7°, } $bb = aa + ba.$
- 17. Therefore by subtracting $\square AB, AC$ from } $\square AB - \square AB, AC = \square AC.$
- each part, }
- That is, in 6°, } $bb - ba = aa.$
- 18. And because (as is evident by the Diagram,) } $AB - AC = CB.$
- 19. Therefore, by drawing AB as a common } $\square AB - \square AB, AC = \square AB, CB.$
- altitude into each part of the last Equation, . }
- 20. Therefore from 17° and 19°, (*per Ax. 1.* } $\square AB, CB = \square AC.$
- Chap. 2.*) }

Which was to be Demonstr. Therefore the Problem is satisfied.

C O R O L L. I.

21. From the preceding Resolution, the invention of a right-angled Triangle whose three sides shall be Proportionals is discovered; for if a given right line b be cut according to extreme and mean Reason, and the greater segment be a , 'tis manifest by the 7th step of the foregoing Resolution, that

$$bb = ba + aa.$$

22. And because (by *prop. 48. Elem. 1.*) if a Square be equal to two Squares, the sides of those three Squares will constitute a right-angled Triangle, therefore the square Roots

of the three Planes in the preceding Equation in 21° , viz. b , \sqrt{ba} , a shall be the three sides of a right-angled Triangle, and be Proportionals also, for the Rectangle of the extremes is manifestly equal to the Square of the mean. Hence therefore a Canon is discovered to find out a right-angled Triangle, whose three sides shall be Proportionals.

C A N O N.

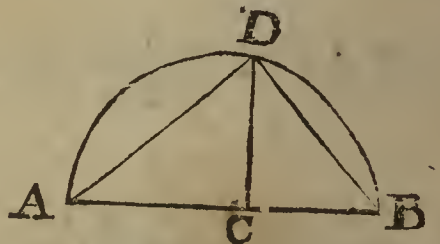
$$\begin{array}{l}
 23. \dots \dots \dots \left\{ \begin{array}{l}
 \text{Hyp.} = b \text{ a right line or number taken at pleasure,} \\
 \text{Base} = \sqrt{bb + \frac{1}{4}bb} : - \frac{1}{2}b. \\
 \text{Perp.} = \sqrt{b \times \sqrt{bb + \frac{1}{4}bb} - \frac{1}{2}b} :
 \end{array} \right.
 \end{array}$$

That is, in words,

Take any right line (or number) for the Hypotenusal of a right-angled Triangle, and cut it into two parts according to extreme and mean Reason; then the greater part shall be one of the sides about the right angle, and a mean Proportional between the greater part and the Hypotenusal shall be the other side about the right angle; and those three sides shall be Proportionals.

Construction.

$$\begin{array}{l}
 \angle ADB \text{ is } \perp. \\
 AB, AD, DB \text{ } \therefore \\
 AC = DB.
 \end{array}
 \left| \begin{array}{l}
 AB = 10 \\
 AD = 7.8615 \\
 DB = 6.1803
 \end{array} \right.$$



24. Take any right line, as AB, and (by the foregoing *Probl. 3.*) cut the same by extreme and mean Reason in C, therefore

$$AB \cdot AC :: AC \cdot CB.$$

25. Then upon the line AB describe the Semicircle ADB, and from C raise $CD \perp AB$. Lastly, draw AD and DB; so shall ADB be a right-angled Triangle whose three sides are Proportionals: For first, the $\angle ADB$ being in the Semicircle, is a right angle, (*per prop. 31. Elem. 3.*) But that the three sides AB, AD, DB are Proportionals, I prove thus;

26. . . . *Req. demonstr.* $AB \cdot AD :: AD \cdot DB.$

Demonstration.

- 27. Because by *Constr.* in 25° ADB is a Semicircle, and $CD \perp AB$, therefore (*per Coroll. prop. 8. Elem. 6.*) $AB \cdot DB :: DB \cdot CB.$
- 28. And because by *Constr.* in 24° , $AB \cdot AC :: AC \cdot CB.$
- 29. Therefore out of 27° and 28° , (*per prop 17. El. 6.*) $\square AB, CB = \square DB = \square AC.$
- 30. And consequently, $DB = AC.$
- 31. And because (by *Constr.* in 25° ,) $\triangle ADB$ is right-angled at D, and $DC \perp AB$, therefore (*per prop. 8. Elem 6.*) $AB \cdot AD :: AD \cdot AC.$
- 32. Therefore from 30° and 31° , $AB \cdot AD :: AD \cdot DB.$

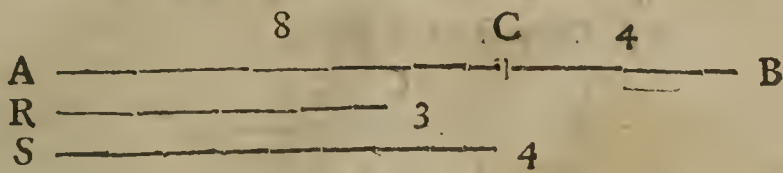
Which was to be Dem.

C O R O L L. 2.

33. From the premisses the way of solving this following Problem is also deducible; viz. To cut a right line given into three such Proportionals, that the Square of the greatest shall be equal to the Squares of the other two: For if it be made as the summ of the three Proportionals AB, AD, DB (being the sides of a right-angled Triangle, and also Proportionals,) is to every one of them; so the right line given to three others; the Problem will be satisfied.

Probl. IV.

To cut a given right line into two such parts, that the Rectangle made of the whole line and one of the parts, to the Square of the other part may have a given Reason.



Suppos.

1. $b = AB$ is a right line given to be cut into two parts.
2. $\left. \begin{matrix} r = R \\ s = S \end{matrix} \right\}$ are the Terms of a given Reason.

Req. to find

3. AC and CB such parts of AB , that $AC + CB = AB$. Also, that
4. $\square AB, CB \cdot \square AC :: R \cdot S$.

Resolution.

5. Put a for one of the parts sought, viz. . . . } $a = AC$.
6. Therefore from 1° and 5° the other part shall be } $b - a = CB$.
7. Therefore the Square of the first part is . . . } $aa (= \square AC.)$
8. And the Rectangle of the given line b and latter part is } $bb - ba (= \square AB, CB)$.
9. Therefore to answer what the Problem requires, these must be Proportionals, viz. . . . } $r \cdot s :: bb - ba \cdot aa$.
10. Now to avoid an Equation between Solids, let it be made, as r to s , so b to a fourth Proportional, which may be called d , therefore . . . } $r \cdot s :: b \cdot d$.
11. Therefore from 9° and 10°, (per prop. 11. Elem. 5.) } $b \cdot d :: bb - ba \cdot aa$.
12. And by drawing $b - a$ as a common altitude into b and d severally, this Analogy is manifest, (per prop. 1. Elem. 6.) } $b \cdot d :: bb - ba \cdot db - da$.
13. Therefore from 11° and 12°, (per prop. 11. Elem. 1.) } $bb - ba \cdot aa :: bb - ba \cdot db - da$.
14. And because the first Term of the last Analogy is equal to the third, the second shall be equal to the fourth, (per prop. 14. Elem. 5.) therefore, } $aa = db - da$.
15. Therefore by adding da to each part of the last Equation, it gives } $aa + da = db$.
16. Which last Equation may be resolved into this Analogy, } $a + d \cdot \sqrt{db} :: \sqrt{db} \cdot a$.

In which Analogy (consisting of three Proportionals,) the mean, to wit, \sqrt{db} is given, as also d the difference of the extremes $a + d$ and a , therefore the extremes severally, (the lesser whereof is one of the Parts sought by this Problem,) shall be given also by *Probl. 12. Chap. 5.* Whence also this

CANON.

17. $\sqrt{\frac{1}{4}dd + db} :: \frac{1}{2}d = AC$.

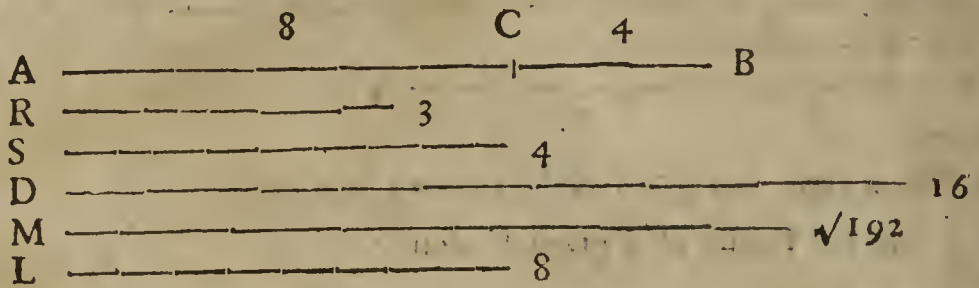
That is, in words,

Let it be made, As R the first Term of the given Reason, to the latter S ; so AB the line given to be cut into two parts, to a fourth Proportional, which may be called D . Then to the Square of half that fourth Proportional, add the Rectangle made of the same Proportional and the given line AB , and from the Square Root of the sum subtract half the said fourth Proportional D : The remainder shall be one of the parts sought, which may be called the first. Lastly, the said first part being subtracted from the given line AB , gives also the other part.

18. *Note.* Whether the first Term of the given Reason be greater or less than the latter, it shall always be; As the first Term is to the latter; so the Rectangle made of the given line and the latter part found out by the Canon; to the Square of the first part.

The

The Composition of Probl. 4.



Suppos.

- 19. A B is a right line given to be cut into two parts.
- 20. R and S are the Terms of a given Reason.

Req. to find

- 21. A C and C B such parts of A B, that $AC \cdot CB = AB$. Also, that
- 22. $\square AB, CB \cdot \square AC :: R \cdot S$.

Construction.

- 23. By *Probl. 8. Chap. 5.* let it be made; as R to S, }
 so A B to a fourth, which may be called D, therefore } $R \cdot S :: AB \cdot D$.
- 24. By *Probl. 9. Chap. 5.* find a mean proportional line, }
 as M, between A B and D, therefore . . . } $AB \cdot M :: M \cdot D$.
- 25. Therefore (*per prop. 17. Elem. 6.*) . . . } $\square M = \square D, AB$.
- 26. Then esteeming the line M to be the mean of three Proportionals, and D the difference of the extremes, find out the extremes, (*per Probl. 12. Chap. 5.*) the lesser whereof suppose to be L, then consequently the greater is equal to $L + D$, therefore these are Proportionals, viz. $L + D \cdot M :: M \cdot L$.

- 27. Therefore, *per Theor.* in 24° of *Probl. 12. Chap. 5.* } $L = \sqrt{\frac{1}{4}\square D + \square M} - \frac{1}{2}D$.
- 28. From A B cut off $AC = L$, which effect is possible, if $AB > L$, but that A B is greater than L, I prove thus,
 First, it is manifest that . . . } $\square AB + \frac{1}{4}\square D + \square D, AB < \frac{1}{4}\square D + \square D, AB$.
 And by extracting the square Root out } $AB + \frac{1}{2}D < \sqrt{\frac{1}{4}\square D + \square D, AB}$:
 of each part, . . . }
 And by subtracting $\frac{1}{2}D$ from each part, } $AB < \sqrt{\frac{1}{4}\square D + \square D, AB} - \frac{1}{2}D$.
 But by *Constr.* in 25° and 27°, . . . } $L = \sqrt{\frac{1}{4}\square D + \square D, AB} - \frac{1}{2}D$.
 Therefore from the two last preceding } $AB < L$. Which was to be Dem.

29. I say the given line A B is cut in C into two parts, according to the tenour of the Problem propounded. Now we must shew, that the Rectangle made of the whole line A B and one of the parts, is to the Square of the other part as R to S. But that Analogy is made manifest by the following Demonstration, formed out of the steps of the preceding Resolution, by returning backwards from the 16th step to the Analogy in the 9th step.

- 30. . . . *Req. demonstr.* $R \cdot S :: \square AB, CB \cdot \square AC$.

Demonstration.

- 31. Because by *Construction* in 26°, . . . } $L + D \cdot M :: M \cdot L$.
- 32. And by *Constr.* in 28°, . . . } $AC = L$.
- 33. Therefore from 31° and 32°, by exchange } $AC + D \cdot M :: M \cdot AC$.
 of equal quantities, . . . }
 That is, in 16°, . . . } $a + d \cdot \sqrt{db} :: \sqrt{db} \cdot a$.
- 34. Therefore, (*per prop. 17. Elem. 6.*) . . . } $\square AC + \square D, AC = \square M$.
- 35. And because by *Constr.* in 24° and 25°, . . . } $\square D, AB = \square M$.
- 36. Therefore from 34° and 35°, (*per Ax. 1. Chap. 2.*) . . . } $\square AC + \square D, AC = \square D, AB$.
 That is, in 15°, . . . } $aa + da = db$.
- 37. Therefore by subtracting $\square D, AC$ from each } $\square AC = \square D, AB - \square D, AC$.
 part of the Equation in 36°, . . . }
 That is, in 14°, . . . } $aa = db - da$.

38. And

38. And from 37°, (*per prop 7. Elem. 5.*) this Analogy is manifest,
- $$\left. \begin{array}{l} \square AB - \square AB, AC \\ \square AC \\ \square AB - \square AB, AC \\ \square D, AB - \square D, AC \end{array} \right\} \text{that is, in } 13^\circ, \left\{ \begin{array}{l} bb - ba . \\ aa :: \\ bb - ba . \\ db - da . \end{array} \right.$$
39. And by reason of the common altitude AB-AC, this following Analogy is manifest, (*per prop. 1. Elem. 6.*)
- $$\left. \begin{array}{l} AB . \\ D \\ \square AB - \square AB, AC \\ \square D, AB - \square D, AC \end{array} \right\} \text{that is, in } 12^\circ, \left\{ \begin{array}{l} b . \\ d :: \\ bb - ba . \\ db - da . \end{array} \right.$$
40. Therefore from 38° and 39°, (*per prop. 11. Elem. 5.*)
- $$AB . D :: \square AB - \square AB, AC . \square AC.$$
- That is, in 11°,
- $$b . d :: bb - ba . aa.$$
41. And because by *Constr.* in 23°,
- $$AB . D :: R . S.$$
- That is, in 10°,
- $$b . d :: r . s.$$
42. Therefore from 40° and 41°, (*per prop. 11. Elem. 5.*)
- $$R . S :: \square AB - \square AB, AC . \square AC.$$
- That is, in 9°,
- $$r . s :: bb - ba . aa.$$
43. And because (*per Diagram,*)
- $$CB = AB - AC.$$
44. And consequently, (*per prop. 1. Elem. 6.*) by drawing AB into each part,
- $$\square AB, CB = \square AB - \square AB, AC.$$
45. Therefore from 42° and 44°, by exchanging equal quantities,
- $$R . S :: \square AB, CB . \square AC.$$
- Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. V.

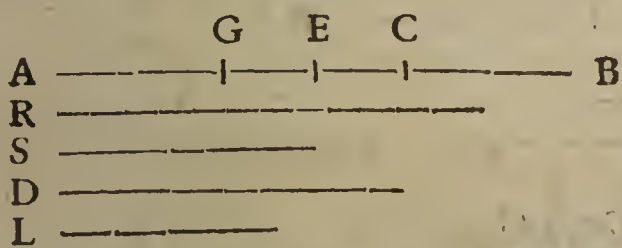
To cut a given right line into two such parts, that the difference of the Squares of the parts, to the Rectangle made of the same parts, may have a given Reason.

Suppos.

1. $b = AB$ is a right line given to be cut into two parts.
 2. $r = R$
 3. $s = S$
- are the Terms of a given Reason.

Req. to find

4. AC and CB, such parts of AB, that $AC + CB = AB$. Also, that
5. $\square AC - \square CB . \square AC, CB :: R . S$.



- AB = 12
- R = 3
- S = 2
- D = 8
- L = 4
- AC = 8
- CB = 4

Resolution.

6. For the difference of the parts sought put } a .
 7. Then out of 1° and 6°, (*by Theor. 9. Chap. 4.*) } the greater part shall be } $\frac{1}{2}b + \frac{1}{2}a$.
 8. And (*by the same Theorem*) the lesser part shall be } $\frac{1}{2}b - \frac{1}{2}a$.
 9. Therefore from 7° and 8°, the Product or Rect- } angle of the parts shall be } $\frac{1}{4}bb - \frac{1}{4}aa$.
 10. And (*from 1° and 6°, per Theor. 8. Chap. 4.*) } the difference of the Squares of the parts is } ba .
 11. Then according to the import of the Problem } propounded, this Analogy arifeth, (*out of 2°, 3°, } 10° and 9°, viz.*) } $r . s :: ba . \frac{1}{4}bb - \frac{1}{4}aa$.
 12. Now to avoid an Equation between Solids, let it } be made, as r to s , so b to a fourth Proportional, } which may be called d , therefore } $r . s :: b . d$.
 13. Therefore out of 11° and 12°, (*per pro. 11. El. 5.*) } $b . d :: ba . \frac{1}{4}bb - \frac{1}{4}aa$
14. And

14. And by drawing a as a common altitude into b and d severally, this following Analogy will be manifest, (per prop. 1. Elem. 6.) viz. $b . d :: ba . da.$
15. Therefore from the 13th and 14th steps, (per prop. 11. Elem. 5.) $ba . \frac{1}{4}bb - \frac{1}{4}aa :: ba . da.$
16. Therefore from the 15th step this Equation aritheth, (per prop. 14. Elem. 5.) viz. $\frac{1}{4}bb - \frac{1}{4}aa = da.$
17. And by multiplying each Term of the Equation in the 16th step by 4, this Equation is produced, viz. $bb - aa = 4da.$
18. Therefore by adding aa to each part, $bb = aa + 4da.$
19. Which Equat. may be resolved into this Analogy, $a + 4d . b :: b . a.$
20. But that Analogy doth manifestly consist of three Proportionals; whereof the mean b is given, as also $4d$ the difference of the extremes $a + 4d$ and a ; therefore the extremes severally, the lesser whereof is the difference of the parts sought by this Problem, shall be given also, by *Probl. 12. Chap. 5.* whence also, (respect being had to the Theorem in 24^o of the same Problem,) there will arise this

CANON.

21. . . . $\sqrt{bb + 4dd} : -2d = a$, the difference of the parts sought.

That is, in words,

Let it be made, as the first Term (whether it be the greater or lesser) of the given Reason, is to the second; so the line given to be divided, to a fourth Proportional. Then add the Square of the double of that Proportional to the Square of the line given to be divided, and from the square Root of the summ subtract the double of the said Proportional, so shall the remainder be the difference of the parts sought. Then by the summ and difference of the parts, the parts shall be given severally by *Theor. 9. Chap. 4.*

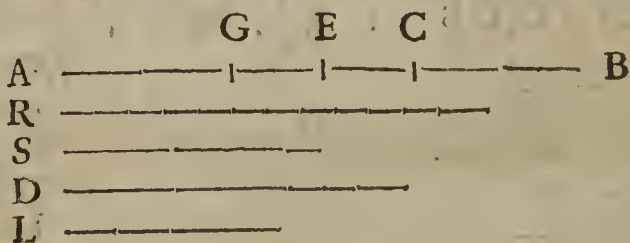
The Composition of Probl. 5.

Suppos.

22. AB is a right line given.
23. R and S are the Terms of a given Reason.

Req. to find

24. AC and CB such parts of AB, that $AC + CB = AB$. Also,
25. $\square AC - \square CB . \square AC, CB :: R . S$.



Construction.

26. By *Probl. 8. Chap. 5.* let it be made, as R to S, so the given line AB to a fourth Proportional, which may be called D, therefore

$$R : S :: AB . D.$$

27. Then esteeming the given line AB to be the mean of three Proportionals, and $4D$ the difference of the extremes, find out the extremes, (per *Probl. 12. Chap. 5.*) the lesser whereof suppose to be L, then consequently the greater is equal to $L + 4D$, and these are Proportionals, viz.

$$L + 4D . AB :: AB . L.$$

Whence, (per *Theor.* in 24^o of *Probl. 12. Chap. 5.*) $L = \sqrt{\square AB + 4\square D} - 2D$.

Thus far the Construction hath been made according to the direction of the Canon in 21^o.

28. Divide the given line AB into two equal parts in E, whence $EA = EB = \frac{1}{2}AB$.

29. From EA and EB cut off EG and EC, such parts, that each may be equal to $\frac{1}{2}L$, which is possible to be done, if $\frac{1}{2}AB \leq \frac{1}{2}L$; but that $\frac{1}{2}AB \leq \frac{1}{2}L$, I prove thus,

First, it is manifest that $\square AB + 4\square AB, D + 4\square D \leq \square AB + 4\square D$.

Therefore by extracting the square Root out of each part, $AB + 2D \leq \sqrt{\square AB + 4\square D}$:

And

And by subtracting $2D$ from each part, . . . } $AB \sqsubset \sqrt{\square AB + 4\square D} - 2D.$
 But by *Constr.* in 27° , . . . } $L \sqsubset \sqrt{\square AB + 4\square D} - 2D.$
 Therefore, (*per Ax. 3. Chap. 2.*) . . . } $AB \sqsubset L$; and $\frac{1}{2}AB \sqsubset \frac{1}{2}L.$

Therefore it is possible to cut off from $\frac{1}{2}AB$, that is, from $EB = EA$, a right line equal to $\frac{1}{2}L$; suppose therefore $EC = EG = \frac{1}{2}L.$

30. I say the given right line AB is cut into two parts in C , as the Problem requires; *viz.*
 The difference of the Squares of the parts AC and CB , is to the Rectangle made of the same parts, as R to S ; which Analogy may be demonstrated by a retrograde repetition of the steps of the preceding Resolution, in manner following.

31. . . . *Req. demonstr.* . . . $R . S :: \square AC - \square CB . \square AC, CB.$

Demonstration.

32. Forasmuch as by *Constr.* in 27° , } $L + 4D . AB :: AB . L.$

That is, in 19° , . . . } $a + 4d . b :: b . a.$

33. Therefore from 32° , (*per prop. 17. Elem. 6.*) } $\square AB = \square L + 4\square DL.$

That is, in 18° , . . . } $bb = aa + 4da.$

34. Therefore from 33° , by subtracting the $\square L$ from each part, } $\square AB - \square L = 4\square DL.$

That is, in 17° , . . . } $bb - aa = 4da.$

35. And by taking a quarter of the Equation in 34° , it gives } $\frac{1}{4}\square AB - \frac{1}{4}\square L = \square DL.$

That is, in 16° , . . . } $\frac{1}{4}bb - \frac{1}{4}aa = da.$

36. Therefore from 35° this following Analogy will be manifest, (*per 7. prop. 5. Elem.*) } $\square AB, L . \frac{1}{4}\square AB - \frac{1}{4}\square L :: \square AB, L . \square DL.$

That is, in 15° , . . . } $ba . \frac{1}{4}bb - \frac{1}{4}aa :: ba . da.$

37. And by reason of the common altitude L , this following Analogy will be manifest, (*per prop. 1. Elem. 6.*) } $AB . D :: \square AB, L . \square DL.$

That is, in 14° , . . . } $b . d :: ba . da.$

38. Therefore from 36° and 37° , (*per prop. 11. Elem. 5.*) } $AB . D :: \square AB, L . \frac{1}{4}\square AB - \frac{1}{4}\square L.$

That is, in 13° , . . . } $b . d :: ba . \frac{1}{4}bb - \frac{1}{4}aa.$

39. And because by *Constr.* in 26° , } $AB . D :: R . S.$

That is, in 12° , . . . } $b . d :: r . s.$

40. Therefore out of 38° and 39° , (*per 11. prop. 5. Elem.*) } $R . S :: \square AB, L . \frac{1}{4}\square AB - \frac{1}{4}\square L.$

That is, in 11° , . . . } $r . s :: ba . \frac{1}{4}bb - \frac{1}{4}aa.$

41. And because by *Constr.* in 29° , } $EC = \frac{1}{2}L = EG.$

42. Therefore by taking $2EC$, that is, GC , instead of L in the Analogy in 40° , this will thence arise, *viz.* } $R . S :: \square AB, GC . \frac{1}{4}\square AB - \frac{1}{4}\square GC.$

43. And because (*per Diagram,*) } $AB = AC + CB.$

44. And by *Constr.* in 28° and 29° , } $GC = AC - CB (AG.)$

45. Therefore from 43° and 44° , (*per Theor. 8. Chap. 4.*) } $\square AC - \square CB = \square AB, GC.$

46. And because by *Constr.* in 29° , } $AC = \frac{1}{2}AB + \frac{1}{2}GC.$

47. And by *Constr.* also in 29° , } $CB = \frac{1}{2}AB - \frac{1}{2}GC.$

48. Therefore from 46° and 47° , (*per Theor. 8. Chap. 4. & Prop. 1. Elem. 6.*) } $\square AC, CB = \frac{1}{4}\square AB - \frac{1}{4}\square GC.$

49. Therefore if instead of the two latter Terms of the Analogy in 42° , you take their equivalent quantities, to wit, the first parts of the Equations in 45° and 48° , this Analogy will arise, *viz.* } $R . S :: \square AC - \square CB . \square AC, CB.$

Which was to be demonstr. Therefore the Problem is satisfied.

Probl. VI.

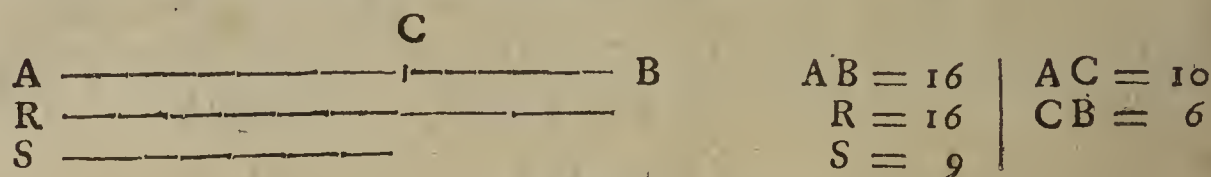
To cut a given right line into two such parts, that the Rectangle made of the whole line and the difference of the parts, to the Square of the lesser part may have a given Reason.

Suppos.

1. $b = AB$ is a right line given to be cut into two parts.
2. $\left. \begin{matrix} r = R \\ s = S \end{matrix} \right\}$ are the Terms of a given Reason.

Req. to find

3. AC and CB such parts of AB , that $AC + CB = AB$. Also,
4. $\square AB \times AC - CB \cdot \square CB :: R \cdot S$.



Resolution.

5. Put a for the greater part sought, *viz.* } $a = AC$.
assume }
6. Therefore out of 1° and 5°, the lesser } $b - a = CB$.
part shall be }
7. And from 5° and 6°, the difference of } $2a - b$.
the parts is }
8. And from 1° and 7°, the Rectangle of the } $2ba - bb$.
given line into the difference of the parts is }
9. And from 6°, the Square of the lesser } $bb + aa - 2ba$.
part is }
10. Therefore from 2°, 8° and 9°, according } $r \cdot s :: 2ba - bb \cdot bb + aa - 2ba$.
to the import of the Problem, these must }
be Proportionals, *viz.* }
11. Therefore inversly, } $s \cdot r :: bb + aa - 2ba \cdot 2ba - bb$.
12. And by Composition, } $s + r \cdot r :: aa \cdot 2ba - bb$.
13. And inversly, } $r \cdot s + r :: 2ba - bb \cdot aa$.
14. Now to avoid an Equation between So- }
lids, let it be made, as r to $s + r$; so b }
to a fourth Proportional, which may be }
called d , therefore } $\dots r \cdot s + r :: b \cdot d$,
15. Therefore from 13° and 14°, (per } $\dots \dots \dots b \cdot d :: 2ba - bb \cdot aa$.
prop. 11. Elem. 5.) }
16. And by drawing $2a - b$ as a common }
altitude into b and d severally, this Analogy }
is manifest, (per *prop. 1. Elem. 6.*) . } $\dots \dots b \cdot d :: 2ba - bb \cdot 2da - db$.
17. Therefore from 15° and 16°, (per } $2ba - bb \cdot 2da - db :: 2ba - bb \cdot aa$.
prop. 11. Elem. 5.) }
18. And from 17°, (per *prop. 14. Elem. 5.*) } $2da - db = aa$.
19. And by adding db to each part of that } $2da = aa + db$.
Equation, }
20. And by subtracting aa from each part, } $2da - aa = db$.
21. Which last Equation may be resolved } $2d - a \cdot \sqrt{db} :: \sqrt{db} \cdot a$.
into this Analogy, }
22. But that Analogy doth manifestly consist of three Proportionals, whereof the mean \sqrt{db} is given, as also $2d$ the summ of the extremes $2d - a$ and a ; therefore the extremes severally, the lesser whereof is the greater part sought, shall be given also, by *Probl. 13. Ch. 5.* And from the premisses and the Theorem in 21° of the same Problem there will arise this following

CANON.

23. $d - \sqrt{dd - db} :: AC$, the greater part sought.

That

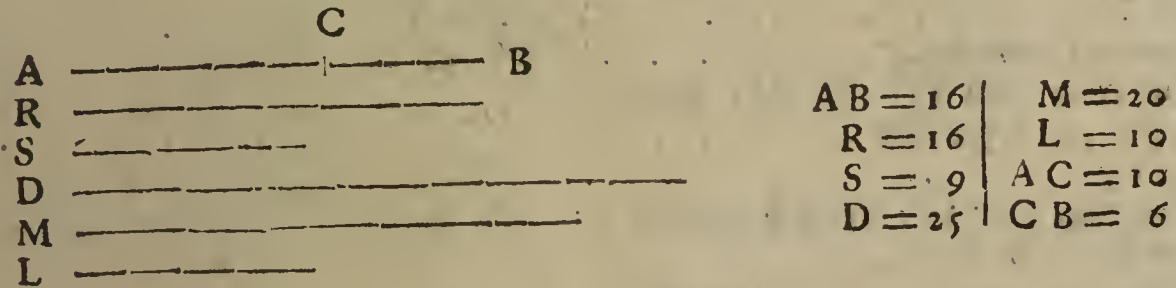
That is, in words,

Let it be made as R the first Term of the given Reason, to R+S the summ of both Terms; so AB the line given to be cut into two parts, to a fourth Proportional, which may be called D. Then from the Square of that fourth Proportional, subtract the Rectangle made of the said fourth Proportional and the given line AB, and extract the square Root of the remainder. Lastly, subtract the said square Root from the said fourth Proportional, and this remainder shall be the greater part sought.

24. Note. Although the Equation found out in 20° may be expounded by either of the two extreme Proportionals mentioned in 22°, yet the lesser of them is only capable of solving the Problem propounded; for the greater extreme is greater than the line given to be cut into two parts, and therefore cannot be equal to either of the parts; which I prove thus,

If \sqrt{db} be the mean of three Proportionals, and $2d$ the summ of the extremes, then (by *Probl. 13. Chap. 5.*) the greater extreme shall be equal to $d + \sqrt{dd - db}$.
 Now we must prove that $d + \sqrt{dd - db} \leq b$.
 First, by *Construction* in 14°, (and by *Schol. prop. 14. Elem. 5.*) 'tis evident that $d \leq b$.
 And consequently, by drawing d into each part, $dd \leq db$.
 Therefore, $\sqrt{dd - db} \leq 0$.
 And consequently, by adding d to each part, $d + \sqrt{dd - db} \leq d$.
 But by *Constr.* in 14°, $d \leq b$.
 Therefore (per *Ax. 5. Chap. 2.*) $d + \sqrt{dd - db} \leq b$.
 Which was to be Dem.

The Composition of the preceding *Probl. 6.*



Suppos.

25. AB is a right line given to be cut into two parts.
 26. R and S are the Terms of a given Reason.

Req. to find

27. AC and CB such parts of AB, that $AC + CB = AB$. Also,
 28. $\square AB \times AC - CB \cdot \square CB :: R \cdot S$.

Construction.

29. By *Probl. 8. Chap. 5.* let it be made, as R to S+R; so AB to a fourth Proportional, which we may suppose to be the line D, therefore $R \cdot S + R :: AB \cdot D$.
 30. By *Probl. 9. Chap. 5.* find a mean proportional line, as M, between AB and D, therefore $AB \cdot M :: M \cdot D$.
 31. Therefore it follows from the last Analogy, (per *prop. 17. Elem. 6.*) that $\square AB, D = \square M$.
 32. Then by *Probl. 14. Chap. 5.* cut the double of the right line D (before found in 29°) into two such parts, that the line M may be a mean Proportional between them; which Effect is possible, if M be not greater than D; but that M is less than D, I prove thus,

By the *Construct.* in 29° 'tis manifest that $D = AB$, and by *Constr.* in 30°, M is a mean Proportional between AB and D, therefore $M = \sqrt{D \cdot AB}$. Therefore 'tis possible to cut $2D$ into two such parts that M may be a mean Proportional between the parts. Suppose then it be done, and that the lesser part is found L, therefore these shall be Proportionals, viz.

$$2D - L \cdot M :: M \cdot L.$$

33. And consequently, (per Theor. in 21° of Probl. 13. Chap. 5.) } $L = D - \sqrt{\square D - \square M}$:

34. From AB cut off AC = L, which is possible to be done if AB \supset L, but that AB \supset L, I prove thus;

First, from the Constr. in 29°, . . . } $D \supset AB$.
 Therefore, by drawing AB into each part, } $\square D, AB \supset \square AB$.
 And by adding $\square D$ to each part, . . . } $\square D + \square D, AB \supset \square D + \square AB$.
 And by subtracting 2 $\square D, AB$ from each } $\square D - \square D, AB \supset \square D + \square AB - 2 \square D, AB$.
 part, . . . }
 And by extracting the square Root out of } $\sqrt{\square D - \square D, AB} \supset D - AB$.
 each part, . . . }
 And by adding AB to each part, . . . } $AB + \sqrt{\square D - \square D, AB} \supset D$.
 And by subtracting $\sqrt{\square D - \square D, AB}$ } $AB \supset D - \sqrt{\square D - \square D, AB}$:
 from each part, . . . }
 But it hath been shewn in 31° and 33°, } $L = D - \sqrt{\square D - \square D, AB}$:
 that . . . }
 Therefore from the two last preceding steps, } $AB \supset L$.

Which was to be Dem. Therefore 'tis possible to cut off from AB a segment equal to L, as AC.

35. I say the given right line AB, in the point C is cut into two parts, AC and CB, which will solve the Problem; viz. the Rectangle made of the whole line AB, and the excess of AC above CB, is to the Square of CB; as the line R to the line S: As will be made manifest by the following Demonstration, form'd out of the foregoing Resolution, by a retrograde repetition of the steps thereof.

36. . . . Req. demonstr. . . . R . S :: $\square AB \times AC - CB . \square CB$.

Demonstration.

37. By Constr. in 32°, . . . } $2D - L . M :: M . L$.
 38. And by Constr. in 34°, . . . } $AC = L$.
 39. Therefore from 37° and 38°, by ex- } $2D - AC . M :: M . AC$.
 changing equal quantities, . . . }
 That is, in 21, . . . } $2d - a . \sqrt{db} :: \sqrt{db} . a$.
 40. Therefore from 39°, (per prop. 17. El. 6.) } $2 \square D, AC - \square AC = \square M$.
 41. But by Constr. in 31°, . . . } $\square D, AB = \square M$.
 42. Therefore from 40° and 41°, (per Ax. 1. } $2 \square D, AC - \square AC = \square D, AB$.
 Chap. 2.) . . . }
 That is, in 20°, . . . } $2da - aa = db$.
 43. And by adding $\square AC$ to each part of the } $2 \square D, AC = \square AC + \square D, AB$.
 Equation in 42°, . . . }
 That is, in 19°, . . . } $2da = aa + db$.
 44. And by subtracting $\square D, AB$ from each } $2 \square D, AC - \square D, AB = \square AC$.
 part in 43°, . . . }
 That is, in 18°, . . . } $2da - db = aa$.

45. And from 44°, (per prop. 7. Elem. 5.) this Analogy will be manifest,
 $2 \square AB, AC, - \square AB$. } that is, in 17°, } $2ba - bb$.
 $2 \square D, AC, - \square D, AB ::$ } $2da - db ::$.
 $2 \square AB, AC, - \square AB$. } $2ba - bb$.
 $\square AC .$ } aa .

46. And by reason of the common altitude 2 AC - AB, this following Analogy is manifest, (per prop. 1. Elem. 6.)
 $2 \square AB, AC - \square AB$. } that is, in 16°, } $2ba - bb$.
 $2 \square D, AC - \square D, AB ::$ } $2da - db ::$.
 AB . } b .
 D . } d .

47. Therefore from 45° and 46°, (per } $AB . D :: 2 \square AB, AC, - \square AB . \square AC$.
 prop. 11. Elem. 5.) . . . }
 That is, in 15°, . . . } $b . d :: 2ba - bb . aa$.
 48. But by Construction in 29°, . . . } $AB . D :: R . S + R$.

That

- That is, in 14°, } $b \cdot d :: r \cdot s + r.$
 49. Therefore from 47° and 48°, (per } $R \cdot S + R :: 2 \square AB, AC - \square AB \cdot \square AC.$
prop. 11. Elem 5.) }
 That is, in 13°, } $r \cdot s + r :: 2ba - bb \cdot aa.$
 50. Therefore inverfly, } $S + R \cdot R :: \square AC \cdot 2 \square AB, AC, - \square AB.$
 That is, in 12°, } $s + r \cdot r :: aa \cdot 2ba - bb.$
 51. Therefore by Division of Reason,

$$\left. \begin{array}{l} S \\ R \\ \square AB + \square AC - 2 \square AB, AC \\ 2 \square AB, AC, - \square AB \end{array} \right\} \text{that is, in 11°}, \left\{ \begin{array}{l} s \\ r \\ bb + aa - 2ba \\ 2ba - bb \end{array} \right.$$

52. Therefore inverfly,

$$\left. \begin{array}{l} R \\ S \\ 2 \square AB, AC, - \square AB \\ \square AB + \square AC - 2 \square AB, AC \end{array} \right\} \text{that is, in 10°}, \left\{ \begin{array}{l} r \\ s \\ 2ba - bb \\ bb + aa - 2ba \end{array} \right.$$

Now the Scope in the six steps next following is to prove, that $\square AB \times AC - CB$ is equal to $2 \square AB, AC - \square AB$, to wit, the third Term of the Analogy in 52°.

53. By *Constr.* in 34°, } $AB = AC + CB.$
 54. And by adding AC to each part, } $AB + AC = 2AC + CB.$
 55. And by subtracting CB from each } $AB + AC - CB = 2AC.$
 part in 54°, }
 56. It is evident by the first part of the } $2AC \square AB.$
 Equation in 44°, that }
 57. Therefore AB may be subtracted } $AC - CB = 2AC - AB.$
 from each part of the Equation in 55° }
 and this Equation between two real }
 right lines will remain, *viz.* }
 58. Therefore by drawing AB as a com- } $\square AB \times AC - CB = 2 \square AB, AC, - \square AB.$
 mon altitude into each part of the last }
 Equation, it will produce (by *Theor* 1. }
Chap. 4.) }

Which was to be shewn. It remains to prove that $\square CB$ is equal to $\square AB + \square AC - 2 \square AB, AC$, to wit, the fourth Term of the Analogy in 52°.

59. By *Constr.* in 34°, } $CB = AB - AC.$
 60. And because the Squares of equal } $\square CB = \square AB + \square AC - 2 \square AB, AC.$
 right lines are also equal, therefore }
 from 59°, (per *Theor.* 5. *Chap.* 4.) }
 Which was to be Demonstr,

61. Lastly, instead of the third and fourth Terms of the Analogy in 52°, their equivalent quantities, to wit, those in the first parts of the Equations in 58° and 60° being taken, this following Analogy ariseth, *viz.*

$$R \cdot S :: \square AB \times AC - CB \cdot \square CB.$$

Which was *Req. demonstr.* in 36°. Therefore the Problem is satisfied.

Probl. VII.

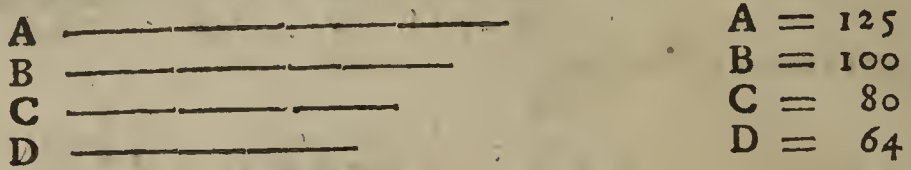
To find two right lines that their summ may be equal to a right line given; and that the difference of the Squares of those two lines, to the Square of the lesser of them, may have a given Reason.

This Problem is the same in effect with the preceding sixth; for the difference of the Squares of the two right lines sought by this Problem, is equal to the Rectangle of the summ and difference of the parts sought by the last preceding Problem.

A LEMMA,

A L E M M A, leading to the following Probl. 8.

If four right lines be in continual proportion, the summ of the means is a mean Proportional between the summ of the first and second, and the summ of the third and fourth Proportionals.



Suppos.

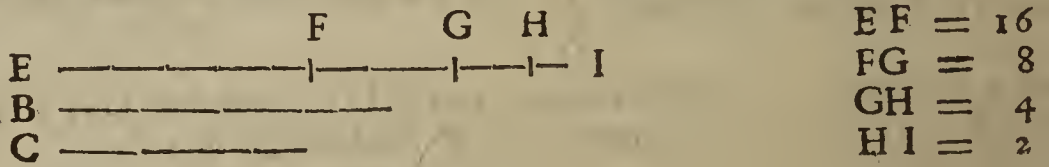
1. A, B, C, D $\div\div$, viz. A . B :: B . C :: C . D.
2. . . *Req. demonstr.* A + B . B + C :: B + C . C + D.

Demonstration:

3. By *Suppos.* in 1°, } A . B :: B . C.
 4. Therefore by Composition of Reason, } A + B . B :: B + C . C.
 5. And alternately, } A + B . B + C :: B . C.
 6. Again, by *Suppos.* } B . C :: C . D.
 7. Therefore by Composition, } B + C . C :: C + D . D.
 8. And alternately, } B + C . C + D :: C . D.
 9. Therefore from 6° and 8°, (*per prop. 11. Elem. 5.*) } B + C . C + D :: B . C.
 10. Likewise from 5° and 9°, (*per prop. 11. Elem. 5.*) } A + B . B + C :: B + C . C + D.
- Which was to be Demonstr.

Probl. VIII.

The summ of the extremes, and summ of the means of four right lines in Continual proportion being given severally, to find the Proportionals. But the first summ must be greater than the latter; the reason whereof is manifest, (*per prop. 25. Elem. 5.*)



Suppos.

1. EF, FG, GH, HI $\div\div$, viz. EF . FG :: FG . GH :: GH . HI.
 2. $b = EF + HI$ is given. Also,
 3. $c = FG + GH = FH$ is given: Therefore,
 4. $d = b + c = EI$ is given.
 5. $b > c$. (*Determination.*)
- Req. to find EF, FG, GH, HI.*

Resolution.

6. By *Suppos.* in 1°, } EF, FG, GH, HI $\div\div$.
7. Therefore by the *Lemma* prefixt } EF + FG . FG + GH :: FG + GH . GH + HI.
8. That is, as is evident by the } EG . FH :: FH . GI.
9. Of which three continual Proportionals the mean FH, that is, c , is given; as also EI, ($= EG + GI$), that is, d , the summ of the extremes. Therefore, by the Theorem in 21° of *Probl. 13. Chap. 5.* the extremes shall be given severally, viz.
10. } $\frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} = EG = EF + FG.$
11. } $\frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} = GI = GH + HI.$
11. Therefore, from 10° and 3° it is manifest, that of these three Proportionals, EF, FG, GH, the summ of the first and second, to wit, EG, is given; also the summ of the second and third, to wit, FH, is given. Likewise of these three Proportionals, FG, GH,

GH, HI, the summ of the first and second, to wit, FH, is given; also GI the summ of the second and third is given; therefore, (according to the Canon in 44° of *Probl. 5. Chap. 7.*) FG and GH shall be given severally by these following Analogies, viz.

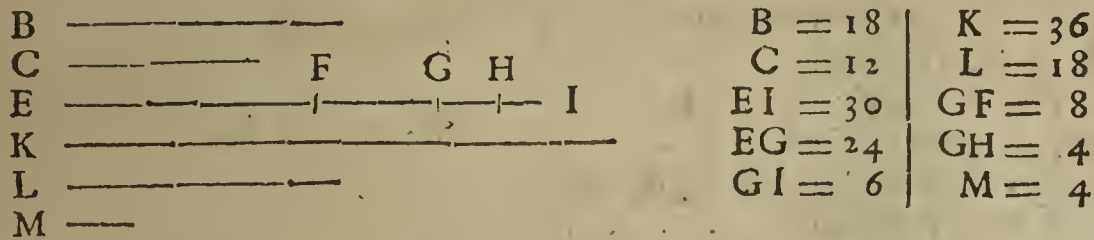
$$12. \left\{ \begin{array}{l} c + \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} : c :: \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} : FG. \\ c + \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} : c :: \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} : GH. \end{array} \right.$$

Which Analogies, respect being had to the Equations in 10°, and to the Diagram, will give this following

CANON.

13. From the Square of half the aggregate of the given summ of the extremes and the given summ of the means, subtract the Square of the summ of the means, and extract the Square Root of the remainder. Then add and subtract that square Root to and from the said half aggregate, and reserve the summ and remainder. Then it shall be, as the summ reserved together with the summ of the means, is to the summ of the means; so the summ reserved to the greater mean sought. Or, as the remainder reserved together with the summ of the means, is to the summ of the means; so the remainder reserved to the lesser mean. Lastly, the summ before reserved being lessened by the greater mean, gives the greater extreme: or, the remainder reserved being lessened by the lesser mean, gives the lesser extreme.

The Composition of the foregoing *Probl. 8.*



Suppos.

- 14. B = the summ of the extremes of four right lines in continual proportion is given.
- 15. C = the summ of the means is given.
- 16. B \square C. (*Determination.*)

Req. to find the four Proportionals severally.

Construction.

- 17. Make EI = B + C.
- 18. Divide EI into two such parts in G, that the line C may be a mean Proportional between the parts; which effect is possible, (*per Probl. 14. Chap. 5.*) if C be not greater than $\frac{1}{2}$ EI, but that C is less than $\frac{1}{2}$ EI, I prove thus;
 - By the Determination in 16°, C \supset B.
 - Therefore by adding C to each part, 2C \supset B + C.
 - But by Construction in 17°, EI = B + C.
 - Therefore, (*per Ax. 3. Chap. 2.*) 2C \supset EI.
 - And consequently, C \supset $\frac{1}{2}$ EI.

Which was to be shewn.

- Therefore 'tis possible to cut EI into two such parts, that C shall be a mean Proportional between them. Suppose then EI to be so cut in G, and that EG is greater than GI; therefore,
- 19. Make } EG . C :: C . GI,
 - 20. Let it also be made (*by Probl. 8. Chap. 5.*) as K } K = EG + C; also, L = GI + C.
 - to C, so EG to a fourth Proportional GF, there- } K . C :: EG . GF.
 - fore, } L . C :: GI . GH.
 - 21. Again, let it be made as L to C, so GI to a } fourth Proportional GH; therefore;
 - 22. Then from EG cut off GF, and from GI cut off GH; (which subtractions are possible, for by Construction in 19°, K is greater than C; therefore from the Analogy in 20°, EG is greater than GF: Likewise by Construction in 19°, L is greater than C, and consequently from the Analogy in 21°, GI is greater than GH;) so the remainders EF and HI are the extreme Proportionals sought. I say EF, FG, GH and HI are four

four continual Proportionals, which will satisfie the Problem propounded. But to make the truth thereof evident, I shall prove three things, viz. First, that FH the summ of the means FG and GH is equal to the given summ C: Secondly, that the summ of the extremes EF and HI is equal to the given summ B: Thirdly and lastly, that the said EF, FG, GH, HI are in continual proportion in this order, viz.

$$EF \cdot FG :: FG \cdot GH :: GH \cdot HI.$$

First, that FH the summ of the means FG and GH is equal to the given summ C, I prove thus;

Preparat.

- 23. To the lines K and C find a third Proportional, as M, (per Probl. 7. Chap. 5.) } $K \cdot C :: C \cdot M.$
- therefore, }
- 24. . . . Req. demonstr. $FH = C.$

Demonstration.

- 25. By Constr. in 18°, } $EG \cdot C :: C \cdot GI.$
- 26. Therefore by Composition of Reason, } $EG + C \cdot C :: C + GI \cdot GI.$
- 27. That is, by exchanging equal right lines } $K \cdot C :: L \cdot GI.$
- according to the Construction in 19°, }
- 28. Therefore alternly, } $K \cdot L :: C \cdot GI.$
- 29. And by drawing C as a common altitude } $K \cdot L :: \square C \cdot \square C, GI$
- into the two latter Terms, }
- 30. But from the Constr. in 23° and 21°, } $\square KM = \square C,$ and $\square L, GH = \square C, GI.$
- (per prop. 17, & 16. Elem. 6.) }
- 31. Therefore from 29° and 30°, by ex- } $K \cdot L :: \square K, M \cdot \square L, GH$
- change of equal Rectangles, }
- 32. But (per prop. 1. Elem. 6.) } $K \cdot L :: \square K, M \cdot \square L, M.$
- 33. That from 31° and 32°, (per prop. 11. } $\square K, M \cdot \square L, GH :: \square K, M \cdot \square L, M.$
- Elem. 5.) }
- 34. Therefore from 33°, (per prop. 14. El. 5.) } $\square L, GH = \square L, M.$
- 35. Therefore from 34°, (per prop. 14. El. 6.) } $L \cdot M :: L \cdot GH.$
- 36. Therefore from 35°, (per prop. 14. El. 5.) } $M = GH.$
- 37. Again, by Constr. in 20°, } $K \cdot C :: EG \cdot GF.$
- 38. And by Constr. in 23°, } $K \cdot C :: C \cdot M.$
- 39. Therefore from 37° and 38°, (per } $2K \cdot 2C :: EG + C \cdot GF + M.$
- Coroll. Herigon. in prop. 12. Elem. 5.) }
- 40. And because by Constr. in 19°, } $K = EG + C.$
- 41. Therefore from 39° and 40°, } $2K \cdot 2C :: K \cdot GF + M.$
- 42. But (per prop. 15. Elem. 5.) } $2K \cdot 2C :: K \cdot C.$
- 43. Therefore from 41° and 42°, (per } $K \cdot GF + M :: K \cdot C.$
- prop. 11. Elem. 5.) }
- 44. Therefore from 43°, (per prop. 14. El. 5.) } $GF + M = C.$
- 45. But it hath been proved in 36°, that . . . } $GH = M.$
- 46. Therefore from 44° and 45, (per Ax. 6. } $GF + GH = C.$
- Chap. 2.) }
- 47. But 'tis evident by the Diagram, that } $GF + GH = FH.$
- 48. Therefore from 46° and 47°, (per } $FH = C.$ Which was to be Dem.
- Ax. 1. Chap. 2.) }

Secondly, that the summ of the extremes EF and HI is equal to the given summ B, I demonstrate thus;

- 49. . . . Req. demonstr. $EF + HI = B.$

Demonstration.

- 50. By Constr. in 17°, } $EI = B + C.$
- 51. And it hath been proved in 48°, that . . . } $FH = C.$
- 52. Therefore by subtracting C or FH } $EI - FH = B.$
- from each part in 50°, }
- 53. But 'tis evident by the Diagram, that . . } $EI - FH = EF + HI.$

45. There-

54. Therefore from 52° and 53° , (*per* } $EF + HI = B$. Which was to be Dem.
Ax. 1. Chap. 2.) }

Thirdly and lastly, that EF, FG, GH, HI are in continual proportion, I demonstrate thus ;

55. . . . *Req. demonstr.* $EF . FG :: FG . GH :: GH . HI$.

Demonstration.

56. By *Constr.* in 20° , } $K . C :: EG . GF$.

57. And by *Constr.* in 19° , } $EG + C = K$.

58. Therefore from 56° and 57° , } $EG + C . C :: EG . GF$.

59. And because it hath been proved in 48° , that . . . } $FH = C$.

60. Therefore from 58° and 59° , } $EG + FH . FH :: EG . GF$.

61. It is manifest by the Diagram, that of the three right lines EF, FG, GH the sum of the first and second is EG , and the sum of the second and third is FH ; therefore the last preceding Analogy is qualified in every respect according to the Theorem in 45° of *Probl. 5. Chap. 7.* whence EF, FG, GH shall be Proportionals, *viz.*

$$EF . FG :: FG . GH.$$

62. Again, by *Constr.* in 21° , } $L . C :: GI . GH$.

63. And by *Constr.* in 19° , } $L = GI + C$.

64. Therefore from 62° and 63° , } $GI + C . C :: GI . GH$.

65. But it hath been proved in 48° , that . . . } $FH = C$.

66. Therefore from 64° and 65° , } $GI + FH . FH :: GI . GH$.

67. Therefore from 66° , in like manner as before } $FG . GH :: GH . HI$
 in 61° , (*per Theor.* in 45° of *Probl. 5. Chap. 7.*) }

68. And from 61° , } $EF . FG :: FG . GH$.

69. Wherefore from 67° and 68° , (*per prop. 11.* } $EF . FG :: FG . GH :: GH . HI$
Elem. 5.) }

Which was to be demonstrated in the last place. Therefore the Problem is satisfied.

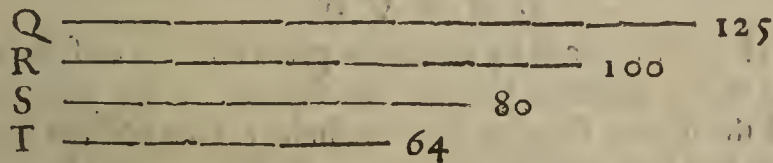
L E M M A, leading to the following *Probl. 9.*

If four right lines be in continual proportion, the difference of the means is a mean Proportional between the difference of the first and second, and difference of the third and fourth Proportionals.

Suppos.

1. Q, R, S, T $::$; *viz.* $Q . R :: R . S :: S . T$.

2. $Q \sqsubset R$. Whence $R \sqsubset S$, and $S \sqsubset T$.



3. . . . *Req. demonstr.* $Q - R . R - S :: R - S . S - T$.

Demonstration.

4. By *Suppos.* in 1° , } $Q . R :: R . S$.

5. Also by *Suppos.* in 2° , } $Q \sqsubset R$.

6. Therefore by Division of Reason, . . . } $Q - R . R :: R - S . S$.

7. And alternly, } $Q - R . R - S :: R . S$.

8. Again, by *Suppos.* } $R . S :: S . T$.

9. Therefore by Division of Reason, . . . } $R - S . S :: S - T . T$.

10. And alternly, } $R - S . S - T :: S . T$.

11. Therefore from 8° and 10° , (*per prop. 11.* } $R - S . S - T :: R . S$
Elem. 5.) }

12. Likewise from 7° and 11° , *per prop. 11. El. 5.* } $Q - R . R - S :: R - S . S - T$.

Which was to be Demonstr.

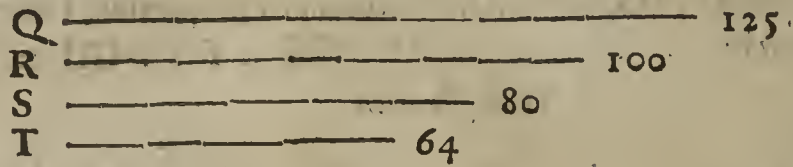
Probl. IX.

The difference of the extremes, and difference of the means of four right lines in Continual proportion being given severally, to find the

V u

Proportio-

Proportionals. But the given difference of the extremes must be greater than the triple difference of the means.



Suppos.

1. $Q, R, S, T \div \div$; *viz.* $Q \cdot R :: R \cdot S :: S \cdot T$.
2. $Q \sqsubset R$, whence $R \sqsubset S$; and $S \sqsubset T$.
3. $b = Q - T$ is given.
4. $c = R - S$ is given.
5. $d = b - c$ is given.
6. $b \sqsubset 3c$. (*Determination.*)
Req. to find Q, R, S, T .

Resolution.

7. By *Suppos.* in 1° and 2°, } $Q, R, S, T \div \div$; also $Q \sqsubset R$.
8. Therefore by the *Lemma* prefixt before this } $Q - R \cdot R - S :: R - S \cdot S - T$.
9. Of which three continual Proportionals in 8° the mean $R - S$, that is, c , is given; as also $d (= Q - T - R)$, the summ of the extremes $Q - R$ and $S - T$; therefore by the Theorem in 21° of *Probl. 13. Chap. 5.* the extremes shall be given severally, *viz.*

$$10. \dots \left\{ \begin{array}{l} \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} = Q - R. \\ \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} = S - T. \end{array} \right.$$

11. Therefore from 4°, 5° and 10° 'tis manifest, that of these three Proportionals Q, R, S , the difference of the first and second, to wit, $Q - R$ is given, also $R - S$ the difference of the second and third is given. Likewise of these three Proportionals R, S, T , the difference of the first and second, to wit, $R - S$ is given, also $S - T$ the difference of the second and third is given; therefore, (according to the Canon in 13. of *Probl. 6. Chap. 7.*) R and S shall be given severally by these following Analogies, *viz.*

$$12. \dots \left\{ \begin{array}{l} \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} : -c \quad . \quad c :: \frac{1}{2}d + \sqrt{\frac{1}{4}dd - cc} : \quad R. \\ c - \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} : \quad c :: \frac{1}{2}d - \sqrt{\frac{1}{4}dd - cc} : \quad S. \end{array} \right.$$

Which Analogies, respect being had to the Equations in 10°, do afford this

C A N O N.

13. From the Square of half the excess by which the given difference of the extremes exceeds the given difference of the means, subtract the Square of the given difference of the means, and extract the square Root of the remainder; then add and subtract that square Root to and from the said half-excess, and reserve the Summ and Remainder. Then it shall be, as the excess of the Summ reserved above the given difference of the means, is to the difference of the means; so is the Summ reserved to the greater mean sought. Or, as the excess of the difference of the means above the Remainder reserved, is to the difference of the means; so is the Remainder reserved to the lesser mean sought. Lastly, if the Summ reserved be added to the greater mean it gives the greater extreme, and if the Remainder reserved be subtracted from the lesser mean it gives the lesser extreme.

But to the end there may be a possibility of effecting the Problem propounded, the given lines must be liable to this

Determination.

14. The given difference of the extremes must be greater than the triple of the given difference of the means.
The truth of this Determination will be made manifest by the following Theorem, and that 'tis necessary, will be evident in 28° of the following Construction of the Problem.

T H E O R E M.

15. If four right lines be in continual proportion, the difference of the extremes is greater than the triple difference of the means.

Suppos.

Suppos.

16. $Q, R, S, T \div \div$; viz. $Q \cdot R :: R \cdot S :: S \cdot T$.

17. $Q \sqsubset R$. Whence $R \sqsubset S$; and $S \sqsubset T$.

18. . . . *Req. demonstr.* . . . $Q - T \sqsubset 3R - 3S$.

Demonstration.

19. By *Suppos.* in 16° and 17°, . . . } $Q, R, S, T \div \div$; also, $Q \sqsubset R$.

20. Therefore by the *Lemma* prefixt before this } $Q - R \cdot R - S :: R - S \cdot S - T$.

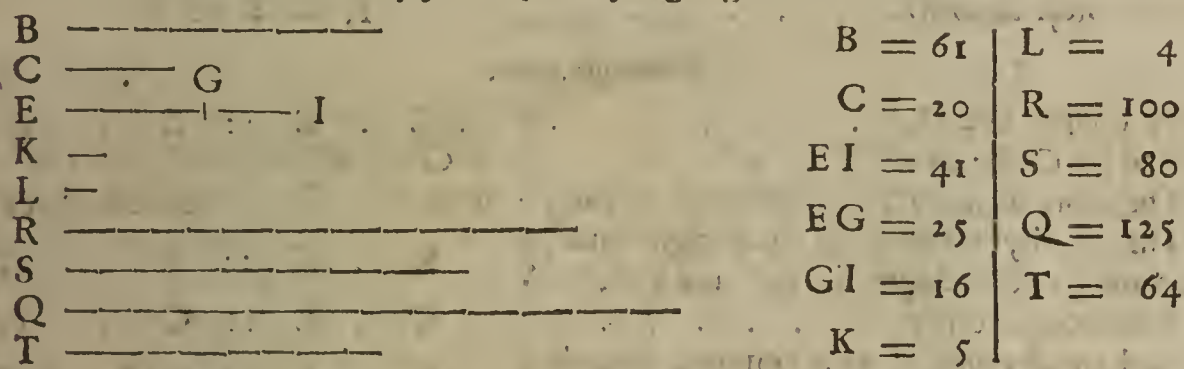
21. But if four quantities be Proportionals, the }
 summ of the extremes is greater than the summ }
 of the means, (*per prop. 25. Elem. 5.*) there- }
 fore from 20°, . . . } $Q + S - R - T \sqsubset 2R - 2S$.

22. And by adding R to each part in 21°, . . . } $Q + S - T \sqsubset 3R - 2S$.

23. Wherefore by subtracting S from each part }
 in 22°, . . . } $Q - T \sqsubset 3R - 3S$.

Which was to be Demonstr.

The Composition of the foregoing Probl. 9.



Suppos.

24. B = the difference of the extremes of four right lines in continual proportion is given.

25. C = the difference of the means is given.

26. $B \sqsubset 3C$. (*Determination*)

Req. to find the Proportionals severally.

Construction.

27. Make $EI = B - C$.

28. Divide EI into two such parts in G that C may be a mean Proportional between the parts, which may be done, (*per Probl. 14. Chap. 5.*) if C be not greater than $\frac{1}{2}EI$. But that C is less than $\frac{1}{2}EI$, I prove thus;

By the *Determination* in 26°, . . . } $3C \sqsupset B$.

Therefore by subtracting C from each part, . . . } $2C \sqsupset B - C$.

But by *Constr.* in 27°, . . . } $EI = B - C$.

Therefore, (*per Ax. 3. Chap. 2.*) . . . } $2C \sqsupset EI$.

And consequently, . . . } $C \sqsupset \frac{1}{2}EI$.

Which was to be shewn. Therefore 'tis possible (*per Probl. 14. Chap. 5.*) to cut EI into two such parts that C may be a mean Proportional between them. Suppose then that EI is so cut in G, and that EG is the greater part, and GI the lesser; therefore,

$$EG \cdot C :: C \cdot GI$$

29. Make $K = EG - C$, which is possible to be done, for by *Constr.* in 28°, $EG \sqsubset C$.

30. Make $L = C - GI$, which is possible to be done, for by *Constr.* in 28°, $C \sqsubset GI$.

31. By *Probl. 8. Chap. 5.* let it be made as K to C, so EG to a fourth Proportional, suppose it be found R, therefore

$$K \cdot C :: EG \cdot R$$

32. Again, let it be made, as L to C, so GI to a fourth Proportional S, therefore

$$L \cdot C :: GI \cdot S$$

33. Make $Q = EG + R$, whence $Q - R = EG$.

34. Make $T = S - GI$, whence $S - T = GI$; but that GI is less than S, as is implied by this Effecttion, I prove thus;

V u 2

By

By *Constr.* in 30° , $L = C - GI$.
 Therefore by adding GI to each part, $L + GI = C$.
 Therefore, $L \supset C$.
 Therefore from the Analogy in 32° , (*per Schol. prop. 14. Elem. 5.*) $GI \supset S$.

Which was to be proved.

35. I say Q, R, S, T (found out by *Constr.* in $33^\circ, 31^\circ, 32^\circ$ and 34° .) are the four continual Proportionals sought. But to make it manifest that they will satisfy the Problem, I shall prove three things, *viz.* First, that the difference of the means R and S is equal to the given difference C : Secondly, that the difference of the extremes Q and T is equal to the given difference B : Thirdly and lastly, that the said Q, R, S, T are in continual proportion in this order, *viz.* $Q \cdot R :: R \cdot S :: S \cdot T$.

First then, that the difference of the means R and S is equal to the given difference C , I demonstrate thus :

Prepar.

36. By *Probl. 7. Chap. 5.* let it be made as K }
 to C , so C to a third proportional line M , } $K \cdot C :: C \cdot M$.
 therefore }
Req. demonstr. $R - S = C$.

Demonstration.

37. By *Constr.* in 28° , } $EG \cdot C :: C \cdot GI$.
 38. And by *Constr.* in 28° , } $EG \sqsubset C$.
 39. Therefore from 37° , by Division of Reason, } $EG - C \cdot C :: C - GI \cdot GI$.
 40. That is, by exchanging equal right lines, } $K \cdot C :: L \cdot GI$.
 according to the *Constr.* in 29° and 30° , }
 41. Therefore alternly, } $K \cdot L :: C \cdot GI$.
 42. And by drawing C as a common altitude } $K : L :: \square C \cdot \square C, GI$.
 into each of the two latter Terms in 41° , }
 43. But from the *Constr.* in 36° and 32° , . . . } $\square K, M = \square C$; and $\square L, S = \square C, GI$.
 44. Therefore from 42° and 43° , by exchange } $K \cdot L :: \square K, M \cdot \square L, S$.
 of equal Rectangles, }
 45. But by reason of the common altitude M , } $K \cdot L :: \square K, M \cdot \square L, M$.
 this Analogy is manifest, }
 46. Therefore from 44° and 45° , (*per prop. 11.* } $\square K, M \cdot \square L, S :: \square K, M \cdot \square L, M$.
Elem. 5.) }
 47. Therefore from 46° , (*per prop. 14. Elem. 5.*) } $\square L, S = \square L, M$.
 48. And from 47° , (*per prop. 14. Elem. 6.*) } $L \cdot M :: L \cdot S$.
 49. Therefore from 48° , (*per prop. 14. Elem. 5.*) } $M = S$.
 50. Again, by *Constr.* in 31° , } $K \cdot C :: EG \cdot R$.
 51. And by *Constr.* in 36° , } $K \cdot C :: C \cdot M$.
 52. Therefore from 50° and 51° , (*per prop. 11.* } $EG \cdot R :: C \cdot M$.
Elem. 5.) }
 53. Therefore alternly, } $EG \cdot C :: R \cdot M$.
 54. Therefore from 38° and 53° , by Division } $EG - C \cdot C :: R - M \cdot M$.
 of Reason, }
 55. And because by *Constr.* in 29° , } $K = EG - C$.
 56. Therefore from 54° and 55° , } $K \cdot C :: R - M \cdot M$.
 57. But by *Constr.* in 36° , } $K \cdot C :: C \cdot M$.
 58. Therefore from 56° and 57° , (*per prop. 11.* } $R - M \cdot M :: C \cdot M$.
Elem. 5.) }
 59. Therefore from 58° , (*per prop. 14. Elem. 5.*) } $R - M = C$.
 60. But it hath been proved in 49° , that } $M = S$.
 61. Therefore from 59° and 60° , (*per Ax. 6.* } $R - S = C$. Which was to be Dem.
Chap. 2.) }

Secondly, that the difference of the extremes Q and T is equal to the given difference B , I demonstrate thus :

62. . . . *Req. demonstr.* $Q - T = B$, *Demonstr.*

Demonstration.

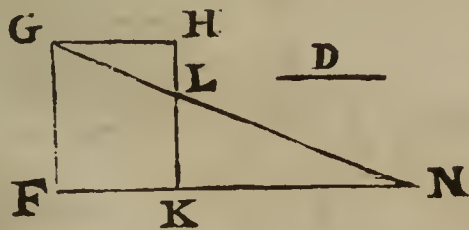
63. By *Constr.* in 33° , } $Q = EG + R$.
 64. And by *Constr.* in 34° , } $T = S - GI$.
 65. And from 61° , } $R \sqsubset S$.
 66. Therefore from $63^\circ, 64^\circ$ and 65° , . . . } $Q \sqsubset T$; and $EG + R \sqsubset S - GI$.
 67. And by subtracting the Equation in 64° }
 from the Equation in 63° , } $Q - T = EG + R + GI - S$.
 68. But by *Constr.* in 28° , } $EI = EG + GI$.
 69. Therefore from 67° and 68° , (*per Ax. 6.* }
Chap. 2.) } $Q - T = EI + R - S$.
 70. Again, by *Constr.* in 27° , } $B - C = EI$.
 71. And it hath been proved in 61° , that . . . } $C = R - S$.
 72. Therefore the summ of the Equations in 70° }
 and 71° gives } $B = EI + R - S$.
 73. Therefore from 69° and 72° , (*per Ax. 1.* }
Chap. 2.) } $Q - T = B$. Which was to be Dem.
 Thirdly and lastly, that Q, R, S, T are in continual proportion I demonstrate thus:
 74. . . . *Req. demonstr.* } $Q . R :: R . S :: S . T$.

Demonstration.

75. By *Constr.* in 31° , } $K . C :: EG . R$.
 76. And by *Constr.* in 29° and 33° , . . . } $EG - C = K$; and $Q - R = EG$.
 77. And it hath been proved in 61° , that . . . } $R - S = C$.
 78. Therefore from $75^\circ, 76^\circ$ and 77° , by ex- }
 changing equal right lines, } $Q - R - \overline{R - S} . R - S :: Q - R . R$.
 79. Therefore from the last preceding Analogy, }
 by the *Theor.* in 14° of *Probl. 6. Chap. 7.* } $Q . R :: R . S$.
 Q, R, S are $\div\div$, *viz.* }
 80. Again, by *Constr.* in 32° , } $L . GI :: C . S$.
 81. And by *Constr.* in 30° and 34° , . . . } $C - GI = L$; and $S - T = GI$.
 82. And it hath been proved in 61° , that . . . } $R - S = C$.
 83. Therefore from $80^\circ, 81^\circ$ and 82° , by ex- }
 changing equal right lines, } $R - S - \overline{S - T} . S - T :: R - S . S$.
 84. Therefore from the last preceding Analogy }
 by the *Theor.* in 14° of *Probl. 6. Chap. 7.* } $R . S :: S . T$.
 R, S, T are $\div\div$, *viz.* }
 85. Wherefore from 79° and 84° , (*per prop. 11.* }
Elem. 5.) } $Q . R :: R . S :: S . T$.
 Which was to be demonstrated in the last place. Therefore the Problem is satisfied.

Probl. X.

A Rectangle FGHK being given by Position, to draw a right line GN from G one of the angles opposite to the Base FK, to cut the Base produced, suppose in N, so as to make the Triangle KLN (lying without the Rectangle) equal to a given Space; suppose the Square of the right line D.



FK = 12	KL = 10
FG = 15	GN = 39
□D = 120	LN = 26
KN = 24	GL = 13

Suppos.

1. FGHK is a \square given.
2. $b = FK$ or GH is given.
3. $c = FG$ or KH is given.
4. $d = D$ the side of a Square given.

Req.

Req. to find

5. KN a right line to be added to FK, so, as FKN may be a strait line, and that GN being drawn it may make $\triangle KLN = \square D$.

Resolution.

- 6. Put a for the desired increase of the Base FK, viz. $\} a = KN.$
- 7. Then because $\triangle NKL$ and $\triangle NFG$ are equi-
angular, these sides are Proportionals, (per prop. 4.
Elem. 6.) viz. $\} FN . FG :: KN . KL.$
- 8. That is, in the letters belonging to the Resolution, $\} a + b . c :: a . \frac{ca}{a+b}$
- 9. And because (per prop. 41. Elem. 1.) . . . $\} \square KN, KL = 2 \triangle KLN.$
- 10. Therefore from 8°, 9°, 4°, 5° and 6°, . . . $\} a \times \frac{ca}{a+b} = 2dd = 2 \triangle KLN.$
- 11. Now to avoid an Equation between Solids,
let it be made as c to d , so $2d$ to a fourth Pro-
portional, call it t , therefore $\} c . d :: 2d . t.$
- 12. Whence, (per prop. 16. Elem. 6.) . . . $\} ct = 2dd.$
- 13. Therefore from 10° and 12°, (per Ax. 1. Ch. 2) $\} a \times \frac{ca}{a+b} = ct.$
- 14. Which Equation may be resolved into these
Proportionals, viz. . . . $\} a . t :: c . \frac{ca}{a+b}$
- 15. And by drawing $a+b$ as a common Factor
into each of the two latter Terms of the last Ana-
logy, this arifeth, . . . $\} a . t :: ca+cb . ca.$
- 16. And by casting away the common Factor c ,
this Analogy arifeth, . . . $\} a . t :: a+b . a.$
- 17. And from the last Analogy, by Division of
Reason, . . . $\} a-t . t :: b . a.$
- 18. Therefore, by comparing the Rectangle of the
extremes to the Rectangle of the means, . . . $\} aa - ta = tb.$
- 19. Which Equation may be resolved into this Ana-
logy, . . . $\} a-t . \sqrt{tb} :: \sqrt{tb} . a.$

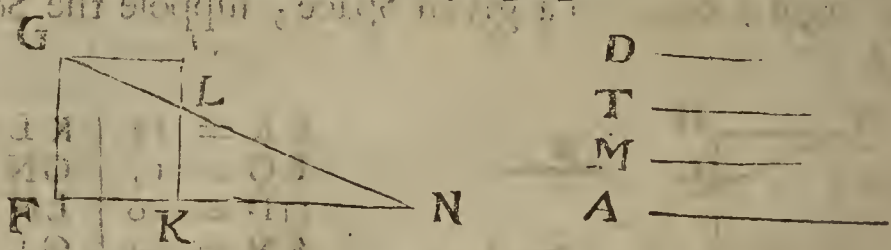
But the last Analogy doth manifestly consist of three Proportionals, whereof the mean, to wit, \sqrt{tb} is given, as also t the difference of the extremes a and $a-t$; therefore the extremes severally, the greater whereof is the desired increase KN, shall be given also, per Probl. 12. Chap. 5. and the Theorem in 24° of the same Probl. gives this following

CANON.

20. . . . $\frac{1}{2}t + \sqrt{\frac{1}{4}tt + tb} = KN$. That is, in words,

Let it be made as FG the altitude of the given Rectangle, to D the side of the given Square; so 2D the double of the same side, to a fourth Proportional, which may be called T. Then to the Square of half that fourth Proportional T, add the Rectangle of T into FK the Base of the given Rectangle, and extract the square Root of the summ. Lastly, that square Root added to half T, gives KN the desired increase of the Base.

The Composition of the foregoing Probl. 10.



Suppos.

- 21. FGHK is a \square given
- 22. D is the side of a Square given.

Req. to find

23. KN a right line to be added to FK, so, as FKN may be a strait line, and that GN being drawn, it may make $\triangle KLN = \square D$.

Constr.

Construction.

- 24. By *Probl. 8. Chap. 5.* let it be made as FG to D, so }
 $2D$ to a fourth proportional line, suppose it be found }
 T; therefore, } $FG \cdot D :: 2D \cdot T.$
- 25. By *Probl. 9. Chap. 5.* find a mean Proportional, }
 as M, between T and FK, therefore } $T \cdot M :: M \cdot FK.$
- 26. Making M to be the mean of three Proportionals, }
 and T the difference of the extremes, find the extremes }
 severally, (*per Probl. 12. Chap. 5.*) the greater }
 whereof suppose to be A, then the lesser shall be }
 $A - T$, therefore } $A - T \cdot M :: M \cdot A.$
- That is, in 19° the last step of the Resolution, . . . } $a - t \cdot \sqrt{tb} :: \sqrt{tb} \cdot a.$
- 27. Produce FK to such a point N, that KN may be equal to the right line A found out }
 in 26° , and draw GN; so shall $\triangle KLN$ be equal to the Square of the given-line D, }
 as was required. But that $\triangle KLN = \square D$, the following Demonstration, form'd }
 out of the preceding Resolution by a repetition of the steps thereof in a backward }
 (not direct) order will make manifest.
- 28. . . . *Req. demonstr.* } $\triangle KLN = \square D.$

Demonstration.

- 29. By *Constr.* in 26° and 27° , . . . } $KN - T \cdot M :: M \cdot KN.$
 That is, in 19° the last step of the }
 Resolution, } $a - t \cdot \sqrt{tb} :: \sqrt{tb} \cdot a.$
- 30. Therefore from 29° , (*per prop. 17.* }
Elem. 6.) } $\square KN - \square T, KN = \square M.$
- 31. Likewise from the *Constr.* in 25° , }
 } $\square T, FK = \square M.$
- 32. Therefore from 30° and 31° , (*per* }
Ax. 1. Chap. 2.) } $\square KN - \square T, KN = \square T, FK.$
 That is, in 18° , } $aa - ta = tb.$
- 33. Therefore from 32° , (*per prop. 14.* }
Elem. 6.) } $KN - T \cdot T :: FK \cdot KN.$
 That is, in 17° , } $a - t \cdot t :: b \cdot a.$
- 34. And from 33° , by Composition }
 of Reason, } $KN \cdot T :: KN + EK \cdot KN.$
 That is, in 16° , } $a \cdot t :: a + b \cdot a.$
- 35. And from 34° , by taking in the }
 the common altitude FG, . . . } $KN \cdot T :: \square FG, KN + \square FG, FK \cdot \square FG, KN.$
 That is, in 15° , } $a \cdot t :: ca + cb \cdot ca.$
- 36. And because $\triangle NKL$ and $\triangle NFG$ }
 are equiangular, therefore (*per* }
prop. 4. Elem. 6.) } $FN \cdot FG :: KN \cdot KL.$
- 37. And consequently, (*per prop. 16.* }
Elem. 6.) } $\square FN, KL = \square FG, KN.$
- 38. Therefore from 35° and 37° , by }
 exchanging equal Rectangles, . . . } $KN \cdot T :: \square FG, KN + \square FG, FK \cdot \square FN, KL.$
- 39. And from 38° , by rejecting the }
 the common altitude $KN + FK$, }
 that is, FN, } $KN \cdot T :: FG \cdot KL.$
 That is, in 14° , } $a \cdot t :: c \cdot \frac{ca}{a+b}.$
- 40. Therefore from 39° , (*per prop. 16.* }
Elem. 6.) } $\square KN, KL = \square FG, T.$
 That is, in 13° , } $a \times \frac{ca}{a+b} = ct.$
- 41. And because from the *Constr.* in }
 24° , (*per prop. 16. Elem. 6.*) . . . } $2 \square D = \square FG, T.$
- 42. Therefore from 40° and 41° , }
 (*per Ax. 1. Chap. 2.*) } $\square KN, KL = 2 \square D.$
 That is, in 10° , } $a \times \frac{ca}{a+b} = 2dd.$
- 43. But (*per prop. 41. Elem. 1.*) } $\square KN, KL = 2 \triangle KLN.$

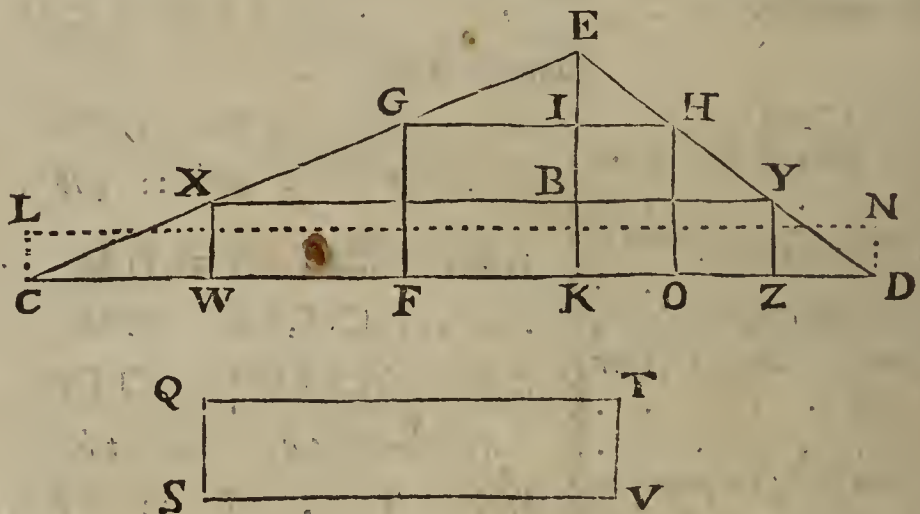
44. There.

44. Therefore from 42° and 43° , (per Ax. 1. Chap. 2.) $\therefore 2 \triangle KLN = 2 \square D$.
 45. Therefore from 44° , (per Ax. 21. Chap. 2.) $\therefore \triangle KLN = \square D$.
 Which was to be demonst. Therefore that is done which the Problem required.

Probl. XI.

In a given Triangle to inscribe a Rectangle equal to a given Rectangle. But the right line arising by the Application of the given Rectangle to the Base of the given Triangle, must not exceed a quarter of the Perpendicular falling upon that Base; and consequently the double of the Rectangle must not be greater than the Triangle.

Note. By the Base of the Triangle given in this Problem, is meant such a side as hath not an obtuse angle at either of its ends within the Triangle; for 'tis easie to conceive, that if the Triangle be obtusangled at the Base, a Rectangle cannot be inscribed within the Triangle, so, as that the Base of a Rectangle may be a segment of the Base of the Triangle, and all the angular points of the Rectangle lye in the sides of the Triangle.



Suppos.

1. CDE is a \triangle given.
2. $b = CD$ the Base is given, and neither $\angle C$ nor $\angle D$ is obtuse.
3. $p = EK$ the Perpendicular is given.
4. $\square ST$, and the sides thereof, to wit, SV and SQ are given.
5. $r = \frac{\square ST}{CD} = CL$ or DN is given; whence, $\square CN = \square ST$.

Req. to inscribe

6. FGHO a \square within the $\triangle CDE$, with condition that
7. $\square FGHO$ may be equal to $\square ST$.

Resolution.

- | | |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <ol style="list-style-type: none"> 8. Put a for the altitude of the Rectangle required, viz. 9. Which altitude subtracted from the Perpendicular EK, leaves EI, therefore from 3° and 8°, 10. It is manifest by the Lemma prefixt before <i>Probl. 11. Chap. 7.</i> that 11. Therefore (from $3^\circ, 9^\circ$ and 2°;) in the letters belonging to the Resolution, 12. And because according to the import of the Problem, 13. Therefore (from $8^\circ, 11^\circ, 2^\circ$ and 5°;) in the letters of the Resolution, 14. Which last Equation is reducible to this Analogy, (per prop. 14. Elem. 6.) 15. Therefore from 11° and 14°, (per prop. 11. Elem. 5.) | $a = FG = KI = OH.$
$p - a = EI.$
$EK . EI :: CD . GH.$
$p . p - a :: b . \frac{bp - ba}{p}.$
$\square FG, GH = \square ST, \text{ or } \square CD, CL.$
$a \times \frac{bp - ba}{p} = br.$
$a . r :: b . \frac{bp - ba}{p}.$
$p . p - a :: a . r.$ |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

16. Which

16. Which last Analogy gives this Equation, (per prop. 16. Elem. 6.) } $pa - aa = pr.$
 17. And that Equation may be resolved into these Proportional, (per prop. 14. Elem. 6.) } $p - a \cdot \sqrt{pr} :: \sqrt{pr} \cdot a.$
 Of which three Proportionals the mean, to wit, \sqrt{pr} is given, as also p the summ of the extremes, therefore the extremes severally, (to wit, KI and EI , or EB and KB , the segments of the Perpendicular EK .) shall be given also, (by *Probl. 13. Chap. 5.*) and the Theorem in 21° of the same *Probl. 13.* gives this following

C A N O N.

18. } $\frac{1}{2}p + \sqrt{\frac{1}{4}pp - pr} = KI (= EB,) \}$ viz. in words,
 } $\frac{1}{2}p - \sqrt{\frac{1}{4}pp - pr} = KB (= EI,)$

First, let it be made as the Base of the given Triangle, to one of the sides of the given Rectangle; so the other side of the same Rectangle, to a fourth Proportional, which may be called r . Secondly, from the Square of half the Perpendicular falling upon the said Base, subtract the Rectangle made of the said Perpendicular and fourth Proportional r . Thirdly, extract the square Root of the remainder. Fourthly, add and subtract the said square Root to and from the said half Perpendicular; the summ and remainder shall be segments of the Perpendicular, either of which may be taken for the altitude of the Rectangle required to be inscribed. Lastly, the Base of the Rectangle sought is equal to the right line arising by the Application of the given Rectangle to the altitude before found.

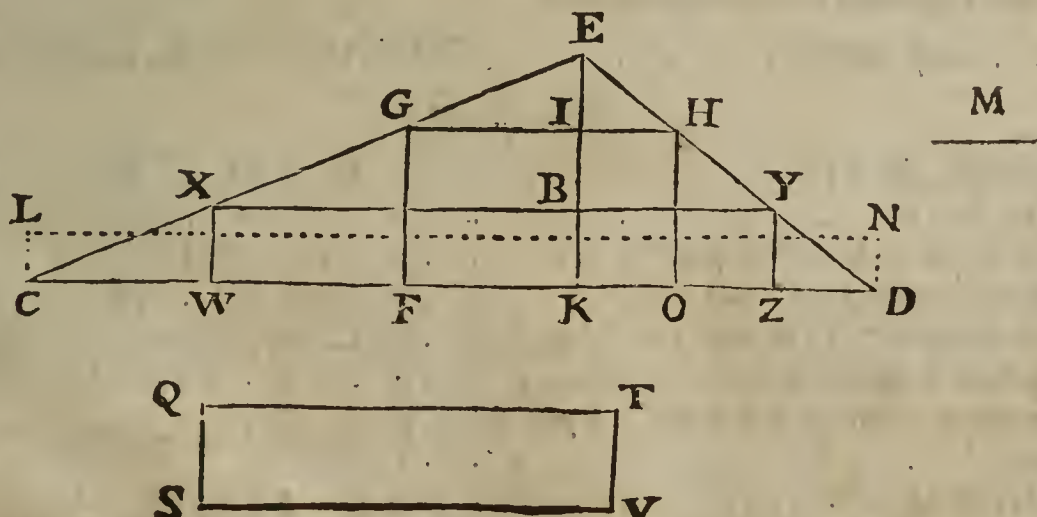
This Canon may be propounded in the form of a Theorem, which may easily be demonstrated by a repetition of the steps of the preceding Resolution in a direct (not retrograde) order; but taking the truth of the Canon for granted, I shall proceed to the Composition of the Problem, for the effecting whereof, 'tis necessary that the given quantities be qualified according to the tenour of this

Determination.

19. The right line arising by the Application of the given Rectangle, to the Base of the given Triangle, must not exceed a quarter of the Perpendicular falling upon that Base; and consequently, the double of the Rectangle must not be greater than the Triangle.

- Which Determination shews it self openly in the Canon, } pr not $\sqsupseteq \frac{1}{4}pp.$
 where it appears, that to the end there may be a possibility }
 of subtracting pr from $\frac{1}{4}pp$, 'tis necessary that . . . }
 And consequently, by dividing each part by p , . . . } r not $\sqsupseteq \frac{1}{4}p.$
 And by doubling each part, . . . } $2r$ not $\sqsupseteq \frac{1}{2}p.$
 And by drawing b into each part, . . . } $2rb$ not $\sqsupseteq \frac{1}{2}pb.$
 But from 2° and 5°, . . . } $2rb = 2 \square ST.$
 And by *prop. 41. Elem. 1.* . . . } $\frac{1}{2}pb = \triangle CDE.$
 Therefore from the three last preceding steps, by exchanging } $2 \square ST$ not $\sqsupseteq \triangle CDE.$
 equal quantities, . . . }
 Therefore from the premisses, both the rise and truth of the Determination are manifest.

The Composition of the foregoing *Probl. 11.*



Suppos.

20. CDE is a \triangle given.
 21. CD the Base is given; and neither $\angle C$ nor $\angle D$ is obtuse.

22. EK

- 22. EK the Perpendicular is given.
- 23. $\square ST$, and the sides thereof, to wit, SV and SQ are given.
- 24. $CL = DN = \frac{\square ST}{CD}$ is given, (per Probl. 8. Chap. 5.)
- 25. CL not $\sqsubset \frac{1}{4} EK$. (Determination.)
Req. to inscribe
- 26. $\square FGH O$ in $\triangle CDE$, so, that $\square FGH O = \square ST$.

Construction.

- 27. By Probl. 9. Chap. 5. find a mean Proportional, as M, between EK and CL, therefore,
 $EK \cdot M :: M \cdot CL$.
- 28. By Probl. 14. Chap. 5. cut EK into two such parts in I, that M (before found) may be a mean Proportional between the parts, which effect is possible if M be not greater than $\frac{1}{2} EK$; but that M is greater than $\frac{1}{2} EK$, I prove thus;
By the Determination in 25° , } CL not $\sqsubset \frac{1}{4} EK$.
Therefore by drawing EK into each part, } $\square EK, CL$ not $\sqsubset \frac{1}{4} \square EK$.
But from the Constr. in 27° , (per prop. 17. Elem. 6.) . . . } $\square EK, CL = \square M$.
Therefore from the two last preceding steps, (per Ax. 4. } $\square M$ not $\sqsubset \frac{1}{4} \square EK$.
Chap. 2.) }
Therefore by extracting the square Root out of each part, . } M not $\sqsubset \frac{1}{2} EK$.

Which was to be proved. Therefore 'tis possible to cut EK into two such parts, that M shall be a mean Proportional between them; suppose then it be done, (per Probl. 14. Chap. 5.) and that the parts are EI and KI, therefore $EI + KI = EK$. Also,

$$EI \text{ (or } EK - KI) \cdot M :: M \cdot KI.$$

That is, in 17° , $p - a \cdot \sqrt{pr} :: \sqrt{pr} \cdot a$.

- 29. Then set either of the said parts of EK, suppose the greater part, from K to I; and by the point I, draw GIH parallel to CD. Lastly, from the points G and H let fall GF and HO Perpendiculars to the Base CD; so shall FGH O be the inclosed Rectangle required. But to make it manifest that the said Rectangle will satisfy the Problem, two things are to be proved, viz. First, that all the angles of the quadrilateral Figure FGH O are right angles; and then that the said Rectangle is equal to the given Rectangle SQT V.
- 30. . . . Req. demonstr. FGH O is a \square .

Demonstration.

- 31. By Constr. in 29° , } $GH \parallel FO$.
- 32. Also by Constr in 29° , } GF and HO are $\perp FO$.
- 33. Therefore, (per defin. 10. Elem. 1.) } $\angle GFO = \perp = \angle HOF$.
- 34. Therefore from $31^\circ, 32^\circ, 33^\circ$, (per prop. 29. Elem. 1.) } $FGHO$ is \square .

Which was to be Demonstr.

It remains to prove that $\square FG, GH$ (that is, $\square FH$) = $\square ST$; but that equality will be manifest by the following Demonstration, form'd out of the preceding Resolution by a retrograde repetition of the steps thereof.

- 35. . . . Req. demonstr. $\square FG, GH$ (that is, $\square FH$) = $\square ST$.

Demonstration.

- 36. By Constr. in 28° , } $EK - KI \cdot M :: M \cdot KI$.
That is, in 17° , } $p - a \cdot \sqrt{pr} :: \sqrt{pr} \cdot a$.
- 37. And from $22^\circ, 29^\circ$ and 34° , } $FG = KI$.
- 38. Therefore from 36° and 37° , } $EK - FG \cdot M :: M \cdot FG$.
- 39. And from 38° , (per prop. 17. Elem. 6.) . . . } $\square EK, FG, - \square FG = \square M$.
- 40. Likewise from the Constr. in 27° , } $\square EK, CL = \square M$.
- 41. Therefore from 39° and 40° , (per Ax. 1. } $\square EK, FG, - \square FG = \square EK, CL$.
Chap. 2.) }
- That is, in 16° , } $pa - aa = pr$.
- 42. Therefore from 41° , (per prop. 14. Elem. 6.) } $EK \cdot EK - FG :: FG \cdot CL$.
That is, in 15° , } $p \cdot p - a :: a \cdot r$.
- 43. By Constr. in 29° , } $GH \parallel CD$.

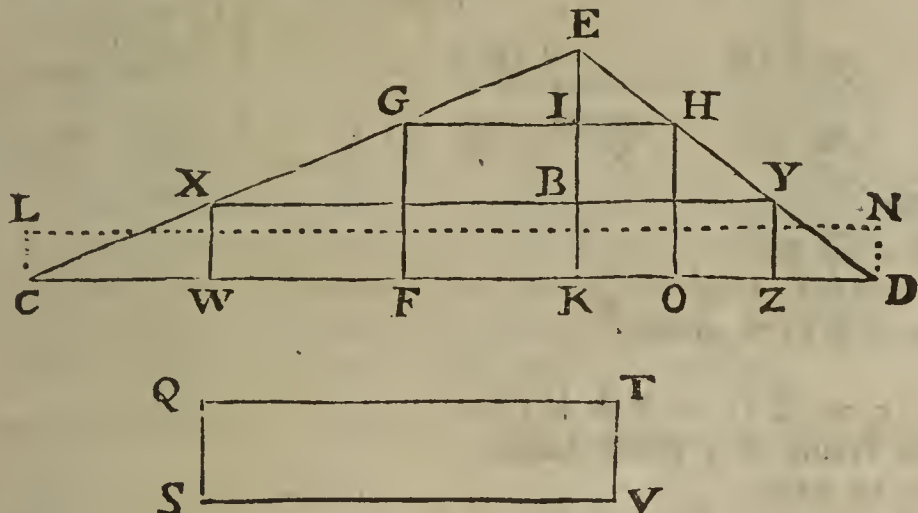
44. There-

44. Therefore by the *Lemma* prefixt before } $EK \cdot EK - FG :: CD \cdot GH.$
Probl. 11. Chap. 7. } EI
 That is, in 11° , } $p \cdot p - a :: b \cdot \frac{bp - ba}{p}.$
 45. Therefore from 42° and 44° , (*per prop. 11.* } $FG : CL :: CD \cdot GH.$
Elem. 5.) }
 46. And from 45° , (*per prop. 16. Elem. 6.*) . . } $\square FG, GH = \square CD, CL.$
 47. But from the *Constr.* in 24° , } $\square ST = \square CD, CL.$
 48. Therefore from 46° and 47° , (*per Ax. 1.* } $\square FG, GH$ (that is, $\square FH$) = $\square ST.$
Chap. 2.) }

Which was to be Demonstr. Therefore that is done which the Problem required.

49. *Note.* If KB be made equal to EI , then shall EB be equal to IK , (by reason of the common intersegment IB ,) and consequently EK is cut in B as well as in I , according to the import of the preceding Construction in 28° . Therefore if by the point B a parallel be drawn to the Base CD , as XY , and from the points X and Y , perpendiculars be let fall upon CD , as XW and YZ , the inscribed $\square WY$, that is, $WXYZ$ shall be also equal to the given Rectangle ST , that is, $SQTV$, and the Demonstration may be formed as before, by taking KB or WX instead of KI . So two Rectangles are inscribed in the given $\triangle CDE$, each of which is equal to the given Rectangle $SQTV$.

Examples in Numbers to illustrate the preceding Resolution of *Probl. 11.*



Suppos.

50. $CD = 168$ the Base }
 51. $CE = 117$ } the leggs } of $\triangle CDE$ are given severally.
 52. $DE = 75$ }
 53. $SV = 84$ } the sides of $\square ST$, therefore $\square ST = 1680.$
 54. $SQ = 20$ }
 55. $\square FH = \square ST = 1680.$
 56. $EK \perp CD.$

Req to find in Numbers,

57. FG or HO , }
 58. GH or FO , } the sides of $\square FH.$

Solution Arithmetical.

59. $EK = 45$, found out by the three sides of $\triangle CDE$ given in $50^\circ, 51^\circ, 52^\circ$, by the help of *Theor. 4.* in 68° of *Probl. 8. Chap. 8.*
 60. $KI = 30$ }
 61. $IE = 15$ } found out by the Canon in 18° of this Problem.
 62. $FG = 30 = KI$, found out in $60^\circ.$
 63. $GH = 56 = FO$, given from 55° and 62° . For $\frac{1680}{30} = 56.$

The Proof.

64. $\square FG, GH = 1680 = \square SV, SQ$, (= $\square ST$.) Also,
 65. . . } 15 (or $45 - 30$) . $56 :: 45 \cdot 168$; that is,
 . . } EI (or $EK - KI$) . $GH :: EK \cdot CD.$

66. Therefore by the converse of the *Lemma* prefixt before *Probl. 11. Chap. 7.*
 $GH \parallel CD$. Also G and H are in CE and DE .

X x 2

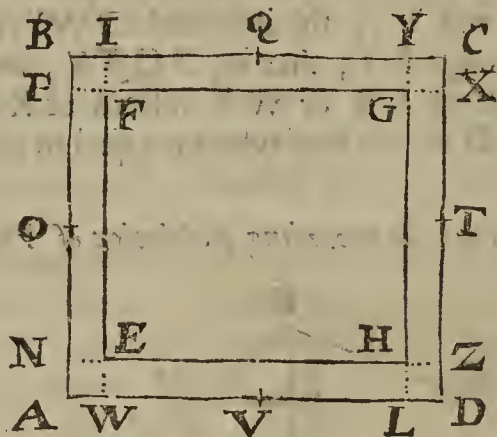
Another

Another Example.

67. Again, the same $\triangle CDE$ and its sides being given in numbers as before in $50^\circ, 51^\circ$ and 52° , you will find (by the like Operation as in *Example 1.*) $XW = 15 = YZ$, and $XY = 112 = WZ$; whence the Area of $\square XZ$ is 1680, which is the same with the Area of $\square ST$ prescribed in *Example 1.* And that the Rectangle XZ or $WXYZ$ is inscribed in $\triangle CDE$, may be proved in like manner as before in 65° and 66° .

Probl. XII.

Within a given Rectangle to make a Rectangle, with this condition, that there may be an equal parallel distance between their sides; and that the Space lying between the sides of both the Rectangles may be to the inscribed Rectangle in a given Reason.



- BC = 440
- BA = 400
- R = 2
- S = 9
- FG = 400
- FE = 360
- FI = 20 = GX

Suppos.

1. ABCD is a \square given.
2. $b = BC = AD$ is given.
3. $c = BA = CD$ is given.
4. $b > c$.
5. $d = b - c = BC - BA$ is given.
6. r and s the Terms of a given Reason.

Req. to make

7. $\square EFGH$ within the $\square ABCD$ in such manner, that
8. $FI = GX = HL = EN$. Also, that
9. $\square BC, BA - \square FG, FE :: \square FG, FE :: r . s$.

Prepar.

10. By viewing the Diagram, and reflecting upon } what is given and required, it will be evident that } $BC = FG + 2GX (2FI)$
11. Likewise, } $BA = FE + 2GX (2FI)$
12. And by subtracting the Equation in 11^o from } that in 10^o, this remains, viz. } $BC - BA = FG - FE (=d)$
13. Whence 'tis manifest that the difference between } the length and breadth of the Rect- } angle required to be inscribed is given; for 'tis equal } to the difference between the length } and breadth of the given $\square ABCD$.

Resolution.

14. Put a for the shorter side of the required Rect- } angle EFGH, viz. } $a = EF = HG$.
15. Therefore from 5^o, 13^o and 14^o, the longer } side shall be } $a + d (= FG.)$
16. Therefore from 14^o and 15^o, the Area of } $\square EFGH$ is equal to } $aa + da (= \square EFGH.)$
17. And from 2^o and 3^o the Area of the given } Rectangle is } $bc (= \square ABCD.)$
18. And by subtracting the Area in 16^o from that } in 17^o, there will remain } $bc - aa - da (= BFGCHDEA.)$
19. Therefore from 9^o, 18^o and 16^o, according to } the tenour of the Problem, this Analogy ariseth, viz. } $r . s :: bc - aa - da . aa + da$.

20. There-

20. Therefore from 19°, by Composition of Reason, } $r+s . s :: bc . aa+d.$
 21. Now to avoid an Equation between Solids, let }
 it be made as $r+s$ to s , so b to a fourth Pro- } $r+s . s :: b . f.$
 portional, call it f , therefore }
 22. Therefore from 20° and 21°, (per prop. 11. El. 5.) } $b . f :: bc . aa+d.$
 23. And this Analogy, by reason of the common } $b . f :: bc . fc.$
 Factor c is evident, (per prop. 1. Elem. 6.) viz. }
 24. Therefore from 22° and 23°, (per prop. 11. } $bc . aa+d :: bc . fc.$
 Elem. 5.) }
 25. Therefore from 24°, (per prop. 14. Elem. 5.) } $aa+d = fc.$
 26. Which Equation may be resolved into these } $a . \sqrt{fc} :: \sqrt{fc} . a+d.$
 Proportionals, viz. }

Of which three Proportionals the mean, to wit, \sqrt{fc} is given, as also d the difference of the extremes $a+d$ and a , therefore per Probl. 12. Chap. 5. the extremes shall be given severally, (which are the sides of the Rectangle required to be inscribed;) and the Theorem in 24° of the said Probl. 12. gives this following

CANON.

27 } $\sqrt{\frac{1}{4}dd + fc} - \frac{1}{2}d = EF.$ } viz. in words,
 } $\sqrt{\frac{1}{4}dd + fc} + \frac{1}{2}d = FG.$ }

Make $r+s$ the summ of the Terms of the given Reason the first of four Proportionals, s the latter of those Terms the second Proportional, BC or AD the longer side of the given Rectangle the third Proportional, and to those three find a fourth, which may be called f . Then to the Square of half the difference between the length and breadth of the given Rectangle, add the Rectangle made of the said fourth Proportional and the said breadth. Then to and from the square Root of that summ, add and subtract the said half difference, so shall the summ and remainder made by that addition and subtraction be the desired length and breadth of the Rectangle to be inscribed; which length or breadth being subtracted from the length or breadth of the given Rectangle, the half of the remainder is the parallel distance between the sides of both the said Rectangles.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 12.

Suppos.

28. $BC = 440$ } the sides of the given Rectangle $ABCD.$
 29. $BA = 400$ }
 30. $R = 2$ } the Terms of the given Reason.
 31. $S = 9$ }

Req. to make

32. $\square EFGH$ within the $\square ABCD$, in such manner, that
 33. $FI = GX = HL = EN.$ Also,
 34. $\square BC, BA - \square FG, FE . \square FG, FE :: R : S :: 2 . 9.$

Solution Arithmetical.

35. $\square BC, BA = 176000$, from 28° and 29°.
 36. $FG = . . 400$, } found out by the Canon in 27°.
 37. $FE = . . 360$, }
 38. $\square FG, FE = 144000$, from 36° and 37°.
 39. $\square BC, BA - \square FG, FE = 32000$, from 35° and 38°.

The Proof.

40. $R . S :: \square BC, BA - \square FG, FE . \square FG, FE.$
 41. $2 . 9 :: \frac{32000}{144000} .$
 42. $FI = 20 \approx GX = HL = EN$ the parallel distance.

Another way of resolving the preceding Probl. 12.

43. The same things being given and required as before, }
 let a be put for the side of a Square equal to the in- } $aa = \square EFGH.$
 scribed Rectangle, therefore }
 44. From 2° and 3° the Area of the given Rectangle is } $bc.$
 45. Therefore the difference of those Rectangles is . } $bc - aa.$

46. There-

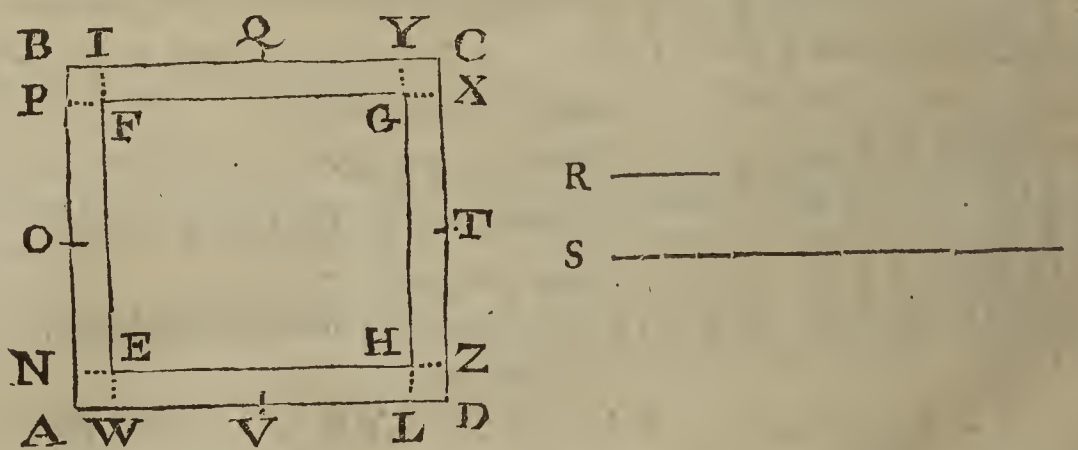
46. Therefore according to the tenour of the Problem }
 this Analogy arifeth, viz. } $r \cdot s :: bc - aa \cdot aa.$
 47. Whence, by Composition of Reason, this Analogy }
 arifeth, which gives the Area of the Rectangle to be } $r - s \cdot s :: bc \cdot aa.$
 inscribed, }
 From the laſt Analogy arifeth **CANON 2.**

As the ſumm of the Terms of the given Reason is to the latter Term, ſo is the Area of the given Rectangle to the Area of the inſcribed Rectangle; therefore the Area of the inſcribed Rectangle is given alſo. Then the Area of the inſcribed Rectangle being given, as alſo the difference of the ſides, (for this difference, as before hath been ſhewn in 13°, is equal to the difference of the ſides of the given Rectangle,) the ſides ſhall be given ſeverally by *Probl. 1.* of this Chapter. And laſtly, the length of the inſcribed Rectangle being ſubtracted from the length of the given Rectangle, or the breadth from the breadth, the half of the remainder is the parallel diſtance between the ſides of both the Rectangles.

This Canon may be exemplified by the numbers given in the preceding *Examp. 1.* And in regard the Composition of this Problem according to either of the ſaid ways of Reſolution will not be difficult to him that underſtands the preceding Problems of this Chapter, I ſhall wave the Composition, and leave it as an exerciſe to the induſtrious Learner.

Probl. XIII.

A Nobleman having made choice of a plot of ground for the making of a Garden of pleaſure, gives direction to a Surveyor to trace out a Rectangle, or long-Square, whoſe length and breadth ſhall be equal to two given right lines BC and BA. Alſo to make another long-Square within the former, in ſuch manner, that there may be an equal parallel diſtance between the ſides of both the ſaid long-Squares. Moreover, the Nobleman's deſign is, that the ſpace lying between the ſides of both the long-Squares ſhall be ſunk perpendicularly, to make a Mote or Ditch whoſe depth ſhall be equal to a given right line S, and the breadth thereof ſuch, that the earth digged out of the intended Ditch being layd upon the ſaid interior long-Square as a Baſe, may be capable of raiſing a rectangular Mount whoſe altitude ſhall be equal to a given right line R. The Queſtion is, to find out the length and breadth of the interior long-Square, as alſo the breadth of the Ditch, that is, the parallel diſtance between the ſides of both the long-Squares.



Suppoſ.

1. $BC = AD,$ } the ſides of $\square ABCD$ are ſeverally given.
2. $BA = CD,$ }
3. $R =$ a right line given for the height of the deſired Mount.
4. $S =$ a right line given for the depth of the deſired Mote or Ditch.

Req. to find

5. $FG,$ or $EH,$ the length of the interior $\square EFGH.$
4. $EF,$ or $HG,$ the breadth of the ſaid $\square EFGH.$

7. $FI =$

7. $FI = GX = HL = EN$ the parallel distance.
 8. $R \times \square EFGH = S \times \text{Plane BFGCDHEA}$.

Construction.

9. By the preceding *Probl. 12*. let a Rectangle or long-Square be made within the given $\square ABCD$, in such manner, that there may be an equal parallel distance between their sides, and that the Space lying between the sides of both Rectangles may have such proportion to the inscribed Rectangle, as the given right line R , (prescribed for the height of the Mount,) hath to the given right line S , (prescribed for the depth of the Ditch.) Now suppose that by the said 12th Problem the $\square EFGH$ is so made within the $\square ABCD$, that the sides of the one keep an equal parallel distance to the sides of the other, *viz.* $FI = GX = HL = EN$; and that as R is to S , so the interval or Plane $BFGCDHEA$, to the $\square EFGH$. Then it will be manifest (*per prop. 34. Elem 11.*) that $R \times \square EFGH$ (which is equal to the Solidity of the Mount,) is equal to $S \times \text{Plane BFGCDHEA}$, (which is equal to the solidity of the Ditch;) as was required.

The quantities of the length and breadth of the inscribed Rectangle (or Base of the Mount,) as also of the parallel distance (or breadth of the Ditch) may be found out in numbers by either of the Canons of the preceding *Probl. 12*. and for the greater evidence, I shall here add

An Example in Numbers, to illustrate the preceding Construction of Probl. 13.

Suppos.

10. $BC = 44^{\circ}$
 11. $BA = 400$ } the sides of the given $\square ABCD$.
 12. $R = 2$ } the given altitude of the Mount to be raised perpendicularly upon $\square EFGH$.
 13. $S = 9$ } the given depth of the Ditch $BFGCDHEA$.

Req. to find out in Numbers,

14. FG and FE the sides of $\square EFGH$. Also,
 15. $FI = GX = HL = EN$ the parallel distance; with condition also, that
 16. $R \times \square FG, FE$ may be equal to $S \times \text{Plane BFGCDHEA}$.

Solution Arithmetical.

17. $FG = 400$
 18. $FE = 360$
 19. $FI = 20$ } found out by the quantities given in $10^{\circ}, 11^{\circ}, 12^{\circ}, 13^{\circ}$, according to the preceding Construction in 9° of this *Probl. 13*.

The Proof.

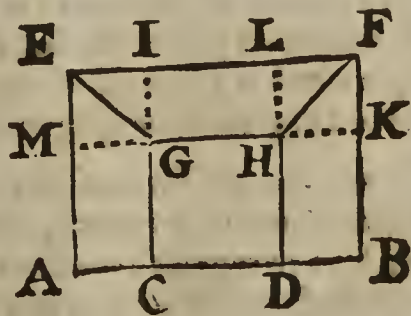
20. $\square FG, FE = 144000$, the Area of $\square EFGH$, *viz.* the Base of the Mount.
 21. $R \times \square FG, FE = 288000$, the Solid content of the Mount.
 22. $\square BC, BA - \square FG, FE = 32000$, the Area of $BFGCDHEA$.
 23. $S \times \text{Area of BFGCDHEA} = 288000$, the Solidity of the Ditch.
 24. $R \times \square FG, FE = S \times \text{Plane BFGCDHEA} = 288000$, as was required.

Probl. XIV.

Within a given Rectangle $A E F B$ to make a Rectangle $C G H D$, with this condition, that after the right lines $E G$ and $H F$ are drawn, the Spaces $C G H D$, $E G H F$, $H D B F$ and $G C A E$ may be equal to one another, and consequently every one of them equal to a fourth part of the given Rectangle $A E F B$.

Suppos.

1. $A E F B$ is a \square given.
 2. $IE = LF = AC = DB$.
 3. $EM = FK = HL = IG$.
 4. $IL = GH = CD$.
 5. $g = AB = EF$ is given.
 6. $f = AE = BF$ is given.



Req.

Req. to make

7. $\square CGHD = EGHF = HDBF = GCAE = \frac{1}{4} \square AEFB.$

Resolution.

8. Put $a = CD = GH = IL.$
 9. Then because $EF - IL = IE + LF,$ 'tis manifest from $2^\circ, 5^\circ$ and $8^\circ,$ that $g - a = IE + LF.$
 10. And because by *Suppos.* in 2° $IE = LF,$ the half of the Equation in 9° gives $\frac{1}{2}g - \frac{1}{2}a = IE = LF.$
 11. And the summ of the Equations in 8° and 10° gives $\frac{1}{2}g + \frac{1}{2}a = IF = EL.$
 12. Then supposing $\square CGHD$ to be equal to $\frac{1}{4} \square AEFB,$ that is, $\frac{1}{4}fg;$ let $\frac{1}{4}fg$ be divided by $a,$ that is, $GH,$ and the Quotient gives $\frac{\frac{1}{4}fg}{a} = HD = BK.$
 13. And because $BF - BK = KF,$ by subtracting $\frac{\frac{1}{4}fg}{a}$ from $f,$ that is, BK from $BF,$ there will remain $f - \frac{\frac{1}{4}fg}{a} = KF = LH.$
 14. Now the Problem requires that $\square IF, KF$ (that is, $EGHF$) $= \frac{1}{4} \square AE, EF.$
 15. That is, in the letters belonging to the Resolution, (as appears by the 11^{th} and 13^{th} steps,) $\frac{1}{2}g + \frac{1}{2}a \times f - \frac{\frac{1}{4}fg}{a} = \frac{1}{4}fg.$
 16. Which last Equation may be resolved into these Proportionals, *viz.* $\frac{1}{2}g + \frac{1}{2}a \cdot \frac{1}{4}g :: f \cdot f - \frac{\frac{1}{4}fg}{a}.$
 17. And by doubling the two first Terms of that Analogy, this ariseth, *viz.* $g + a \cdot \frac{1}{2}g :: f \cdot f - \frac{\frac{1}{4}fg}{a}.$
 18. Whence by Conversion of Reason, $g + a \cdot \frac{1}{2}g + a :: f \cdot \frac{\frac{1}{4}fg}{a}.$
 19. And by drawing a into each of the two latter Terms of the last preceding Analogy, $g + a \cdot \frac{1}{2}g + a :: fa \cdot \frac{1}{4}fg.$
 20. And by dividing each of the two latter Terms of the Analogy in 19° by $f,$ this ariseth, *viz.* $g + a \cdot \frac{1}{2}g + a :: a \cdot \frac{1}{4}g.$
 21. Whence, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation ariseth, *viz.* $aa + \frac{1}{2}ga = \frac{1}{4}gg + \frac{1}{4}ga.$
 22. And by subtracting $\frac{1}{4}ga$ from each part of the Equation in $21^\circ,$ this ariseth, *viz.* $aa + \frac{1}{4}ga = \frac{1}{4}gg.$
 23. Which last Equation may be resolved into these Proportionals, *viz.* $a + \frac{1}{4}g \cdot \frac{1}{2}g :: \frac{1}{2}g \cdot a.$
 24. But of those three continual Proportionals, the mean, to wit, $\frac{1}{2}g$ is given, as also $\frac{1}{4}g$ the difference of the extremes $a + \frac{1}{4}g$ and $a,$ therefore the extremes shall be given severally, (*per Probl. 12. Chap. 5.*) the lesser of which shall be equal to the desired line $CD,$ ($= GH = IL,$) represented by a in the precedent Resolution of this Problem: And the Theorem in 24° of *Probl. 12. Chap. 5.* gives this following

C A N O N.

25. $a = \sqrt{\frac{1}{64}gg} - \frac{1}{8}g = CD = GH = IL$

That is, in words,

To the Square of one eighth part of the Base (that is, either of the sides) of the given Rectangle, add a quarter of the Square of the same Base, and from the square Root of the summ subtract one eighth part of the said Base; the remainder shall be that side of the required Rectangle which is a segment of the Base of the given Rectangle. Whence the rest of the lines in the Diagram belonging to this *Probl. 14.* shall be given also.

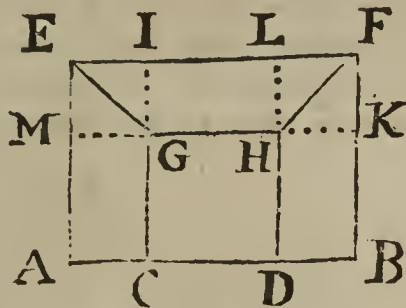
26. It is evident, that no Term of any Analogy or Equation in the foregoing Resolution exceeds the dimensions of a Square, and therefore the forming of the Composition of this Problem by a retrograde repetition of the steps of the Resolution will not be difficult to him that understands the Resolutions and Compositions of the precedent Problems of this

this Chapter; waving therefore the Geometrical Effectation and Demonstration, I shall apply the Canon before exprest to the Arithmetical Solution of the Problem propounded.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 14.

Suppos.

- 27. AEFB is a \square , whose Base is AB, and altitude AE.
- 28. AB = 10 = EF is given.
- 29. AE = 6 = BF is given.
- 30. EM = FK = HL = IG.
- 31. IE = LF = AC = DB.



Req. to find in numbers,

- 32. The quantities of the lines CD, (= IL,) IE, (= LF,) HD, (= BK,) FK, (= HL,) with this condition, that the Area of every one of these four Spaces, to wit, \square CGHD, EGHF, HDBF and GCAE may be equal to a quarter of the Area of the given Rectangle AEFB, viz.

$$\square CGHD = EGHF = HDBF = GCAE = 15 = \frac{1}{4} \square AEFB.$$

Solution Arithmetical.

- 33. From 28° , by the Canon in 25° , you will find $\sqrt{\frac{42}{16}} - \frac{1}{4} = CD = GH = IL.$
- 34. And by subtracting $\sqrt{\frac{42}{16}} - \frac{1}{4}$ from 10, that is, IL from EF, there will remain $\frac{42}{4} - \sqrt{\frac{42}{16}} = IE + LF.$
- 35. And because IE = LF, the half of $\frac{42}{4} - \sqrt{\frac{42}{16}}$ gives $\frac{42}{8} - \sqrt{\frac{42}{64}} = IE = LF = KH.$
- 36. And the summ of the numbers in 33° and 35° makes $\frac{32}{8} + \sqrt{\frac{42}{64}} = IF = EL.$
- 37. Then by dividing 15 the Area of \square CGHD, that is, $\frac{1}{4}$ of 60 the Area of \square AEFB, by $\sqrt{\frac{42}{16}} - \frac{1}{4}$, that is, CD, the Quotient gives $\frac{1}{4} + \sqrt{\frac{3822}{400}} = DH = BK.$
- 38. And by subtracting $\frac{1}{4} + \sqrt{\frac{3822}{400}}$ from 6, that is, BK from BF, there will remain $\frac{21}{4} - \sqrt{\frac{3822}{400}} = FK = HL.$

So in the six last preceding steps the quantities of all the lines sought by *Probl. 14.* are found out in numbers; but that they will satisfie the condition prescribed in 32° , will be evident by

The Proof.

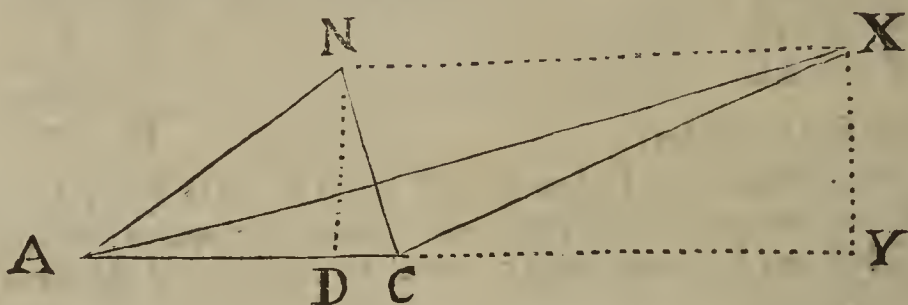
- 39. The Product of $\frac{1}{4} + \sqrt{\frac{3822}{400}}$ into $\frac{1}{4} + \sqrt{\frac{3822}{400}} = 15.$
That is, \square CD into \square DH = \square CGHD.
- 40. The Product of $\frac{32}{8} + \sqrt{\frac{42}{64}}$ into $\frac{21}{4} - \sqrt{\frac{3822}{400}} = 15.$
That is, \square IF into \square FK = \square EGHF.
- 41. The Product of $\frac{1}{4} + \sqrt{\frac{3822}{400}}$ into $\frac{42}{8} - \sqrt{\frac{42}{64}} = \square$ HDBK.
- 42. The Product of $\frac{32}{8} - \sqrt{\frac{3822}{400}}$ into $\frac{42}{8} - \sqrt{\frac{42}{64}} = \triangle$ HKF.
- 43. The summ of the Products in 41° and 42° makes $15 = \square$ HDBK + \triangle HKF = \square HDBF.
- 44. And from 27° , 30° , 31° and 43° , \square GCAE = \square HDBF = 15.
- 45. Therefore from 39° , 40° , 43° and 44° , 'tis evident that \square CGHD = \square EGHF = \square HDBF = \square GCAE = 15 = $\frac{1}{4} \square$ AEFB.

Which was to be done. All which Calculations will be evident to him that understands the Arithmetick of Surd-numbers, handled at large in *Chap. 9. Book II.* of this Treatise.

Probl. XV.

The Base, Perpendicular and Proportion of the legs of a plain Triangle being severally given, to find out the Triangle. But the given lines must be subject to the Determination hereafter exprest.

Note. There is more than enough given in this Problem, unless it requires a Triangle that hath either unequal acute angles, or else an obtuse angle at the Base; in the first of those Cases the Perpendicular falls within the Triangle, in the latter without; but the following Resolution may be applied to each Case.



Suppos.

1. $\triangle ACN$ is acute-angled at the ends of the Base AC .
2. $\triangle ACX$ is obtuse-angled at C , the end of the Base AC .
3. $b = AC$ the Base is given.
4. $p = ND = XY$ the Perpendicular is given.
5. r and s are the given Terms of the Proportion of the legs, viz.
 $r . s :: AN . NC :: AX . XC$.
6. $r = s$.

Req. to find the Triangle.

Resolution.

7. Put a for the distance from the foot of the Perpendicular to the remoter end of the Base, viz. suppose . . . } $a = DA$ or YA .
8. Therefore from 3° and 7°, the distance from the foot of the Perpendicular to the nearer end of the Base is . . . } $b - a$, or $a - b$; viz. DC or YC .
9. The Square of which distance is . . . } $aa - 2ba + bb$ ($= \square DC$ or $\square YC$.)
10. The Square of the distance in 7° is } aa ($= \square DA$ or $\square YA$.)
11. The Square of the given Perpendicular in 4° is . . . } pp ($= \square ND = \square XY$.)
12. By prop. 47. Elem. 1. . . . } $\square DC + \square ND = \square CN$.
13. Likewise, } $\square YC + \square YX$ ($\square ND$) $= \square CX$.
14. Therefore from 9°, 11°, 12° and 13°, the Square of the lesser leg, in the letters of the Resolution, is . . . } $aa - 2ba + bb + pp$ ($= \square CN$ or $\square CX$.)
15. Again, by prop. 47. Elem. 1. . . } $\square DA + \square ND = \square AN$.
16. Likewise, } $\square YA + \square YX$ ($\square ND$) $= \square AX$.
17. Therefore from 10°, 11°, 15° and 16°, the Square of the greater leg is } $aa + pp$ ($= \square AN$ or $\square AX$.)
18. And consequently the greater leg is } $\sqrt{aa + pp}$: ($= AN$ or AX .)
19. And from 14°, the lesser leg is . . } $\sqrt{aa - 2ba + bb + pp}$: ($= CN$ or CX .)
20. Therefore from 5°, 18° and 19°, according to the tenour of the Problem, } $r . s :: \sqrt{aa + pp} : \sqrt{aa - 2ba + bb + pp}$:
21. Therefore from 20°, (per prop. 22. Elem. 6.) } $rr . ss :: aa + pp . aa - 2ba + bb + pp$.
22. Now in order to find out an Equation wherein the highest Power of the line a sought may not exceed a Square; to r and s find a third Proportional, which may be called t , therefore, . . . } $r . s :: s . t$.

23. There-

23. Therefore from 22°, (per Coroll. }
 prop. 20. Elem. 6.) } $rr . ss :: r . t.$
24. Therefore from 21° and 23°, (per }
 prop. 11. Elem. 5.) } $r . t :: aa + pp . aa - 2ba + bb + pp.$
25. Therefore from 24°, by Conversion }
 of Reason, } $r . r - t :: aa + pp . 2ba - bb.$
26. Therefore inverfly, }
 } $r - t . r :: 2ba - bb . aa + pp.$
27. Let it be made as $r - t$ to r , so b to }
 a fourth Proportional, which may be }
 called f , therefore, }
 } $r - t . r :: b . f.$
28. Therefore from 26° and 27°, (per }
 prop. 11. Elem. 5.) } $b . f :: 2ba - bb . aa + pp.$
29. And by drawing $2a - b$ as a com- }
 mon Factor into b and f severally, this }
 Analogy is manifest, (per prop. 1. El. 6) }
 } $b . f :: 2ba - bb . 2fa - fb.$
30. Therefore from 28° and 29°, (per }
 prop. 11. Elem. 5.) } $2ba - bb . aa + pp :: 2ba - bb . 2fa - fb.$
31. Therefore from 30°, (per prop. 14. }
 Elem 5.) this Equation ariseth, . . . }
 } $2fa - fb = aa + pp.$
32. Whence, by adding fb to each part, }
 } $2fa = aa + pp + fb.$
33. And by subtracting aa from each part }
 of the last Equation, this ariseth, . . }
 } $2fa - aa = pp + fb.$
34. Which last preceding Equation may }
 be converted into this Analogy, viz. }
 } $2f - a . \sqrt{pp + fb} :: \sqrt{pp + fb} . a.$
35. But that Analogy doth manifestly consist of three continual Proportionals, whereof the mean; to wit, $\sqrt{pp + fb}$: is given, as also $2f$ the summ. of the extremes $2f - a$ and a ; therefore the extremes shall be given severally, (by Probl. 13. Chap. 5.) either of which may be taken for the line a sought, viz.

$$a = f + \sqrt{ff - pp - fb} = YA;$$

$$\text{Or, } a = f - \sqrt{ff - pp - fb} = DA.$$

36. From 34° and 35° 'tis easie to perceive that $\sqrt{pp + fb}$: cannot be greater than f ; for the mean of three Proportionals never exceeds half the summ of the extremes, (as hath been shewn in 20° of Probl. 13. Chap. 5.) But the said $\sqrt{pp + fb}$: may sometimes be equal to, and sometimes less than f ; to the end therefore there may be a possibility of finding out a Triangle to satisfy the Problem propounded, the given lines must be subject to this following

Determination, $\sqrt{pp + fb}$: not \sqsupset f .

That is, in words,

First, if it be made as r to s , so s to a third Proportional t . Secondly, as the excess of r above t , to r ; so the given Base b to a fourth Proportional f . Then the side of a Square equal to the summ of the Square of the given Perpendicular p and the Rectangle of f into b , must not be greater than f ; for when the said side happens to be greater than f , 'tis impossible to find a Triangle qualified as the Problem requires, by the help of the given lines r, s, b and p .

This Determination is discovered by the three Proportionals in 34°, which are rightly inferr'd from the preceding Resolution; and since the Resolution is clearly Geometrical as well as Arithmetical, I shall take the truth of the Determination for granted.

37. It hath before been declared in 35°, that the distance sought, which is represented by a in the Resolution, may be either of the two right lines or extreme Proportionals found out in the said 35th step; which two right lines will be equal to one another when $\sqrt{pp + fb} = f$, for then each of those lines will be equal to f ; (as is evident by the Equations in 35°;) in which Case, there can but one Triangle be found out to solve the Problem, and that Triangle will always be obtuse-angled at the Base. But when it happens that $\sqrt{pp + fb} < f$, then the said extreme Proportionals, (to wit, the values of a in 35°,) will be unequal between themselves; and in this Case the Problem propounded may be solved by either of those two right lines, or extreme Proportionals,

viz. two different Triangles may be found out wherein these three things will be common, to wit, the Base, the Perpendicular, and the Proportion of the leggs; of which Triangles, that which is formed by the help of the greater of the said two right lines, (or extreme Proportionals,) will always be obtuse-angled at the Base; but the other Triangle form'd by the help of the lesser of the said two right lines will sometimes be obtuse-angled at the Base, sometimes acute-angled, and sometimes right angled. Now to discover which of those three kinds of Triangles will happen, I shall give three Rules, which presuppose the quantities of the given lines to be express'd by Numbers.

Rule I.

38. If $\frac{pp}{b} + b \leq f$; but $\frac{pp}{f} + b$ not $\leq f$; then the lesser value of a in 35° , (that is, $f - \sqrt{ff - pp - fb}$;) is greater than the Base b , and consequently the Triangle form'd by the help of the said lesser value shall be obtuse-angled at the Base.

Rule II.

39. If $\frac{pp}{b} + b > f$; then the lesser value of a in 35° is less than the Base, and consequently the Triangle form'd by the help of the said lesser value shall be acute-angled at the Base.

Rule III.

40. If $\frac{pp}{b} + b = f$; then the lesser value of a in 35° is equal to the Base, and consequently, the Triangle form'd by the help of the said lesser value shall be right-angled at the Base.

The truth of Rule I. may be demonstrated thus;

41. . . . *Suppos.* in Rule I. . . . $\frac{pp}{b} + b \leq f$; but $\frac{pp}{f} + b$ not $\leq f$.
 42. . . . *Req. demonstr.* . . . $f - \sqrt{ff - pp - fb} > b$; as is affirmed in Rule I.

Demonstration.

43. By *Suppos.* in 41°, $\frac{pp}{b} + b \leq f$.
 44. Therefore by multiplying each part by b , . . . $pp + bb \leq fb$.
 45. And by adding ff to each part in 44°, . . . $ff + pp + bb \leq ff + fb$.
 46. And by subtracting pp from each part in 45°,
 (which the *Supposition* in 41°, or the *Determination* in 36° shews to be possible,) it follows that } $ff + bb \leq ff + fb - pp$.
 47. Likewise by subtracting $2fb$ from each part } $ff + bb - 2fb \leq ff - pp - fb$.
 in 46°, }
 48. And by extracting the square Root out of each } $f - b \leq \sqrt{ff - pp - fb}$:
 part in 47°, }
 49. And by adding b to each part in 48°, . . . $f \leq \sqrt{ff - pp - fb} + b$.
 50. Wherefore by subtracting $\sqrt{ff - pp - fb}$ } $f - \sqrt{ff - pp - fb} \leq b$.
 from each part in 49°, }

Which was to be Demonstr.

After the same manner the truth of the preceding second and third Rules may be demonstrated; and from the premisses the following Canon is deducible, for the Arithmetical Solution of the Problem propounded.

C A N O N.

51. Let it be made as r the greater Term of the given Reason, (or Proportion,) to s the lesser; so the same s to a third Proportional, which may be called t : Let it also be made as the excess of r above t to r ; so the given Base AC to a fourth Proportional, which may be called f . Then from the Square of f subtract the Square of the given Perpendicular $ND (= XY)$ together with the Rectangle made of f into the Base AC , and out of the remainder, if any happen, extract the square Root. That done, add the said square Root to f before found, and the sum shall be the distance from the foot of the Perpendicular falling without the Triangle upon the Base continued to the remoter end of the Base; which distance we may suppose to be AY in the following *Fig.* 1, 2, and 3.

and 3. Whence $CY = AY - AC$ is given; and consequently, (*per prop. 47. Elem. 1.*) $CX = \sqrt{\square CY + \square XY}$: is given, and $AX = \sqrt{\square AY + \square YX}$: is given also. Therefore $\triangle ACX$, whose angle ACX is obtuse, is given, which I shall call the first of the two Triangles that will solve the Problem propounded.

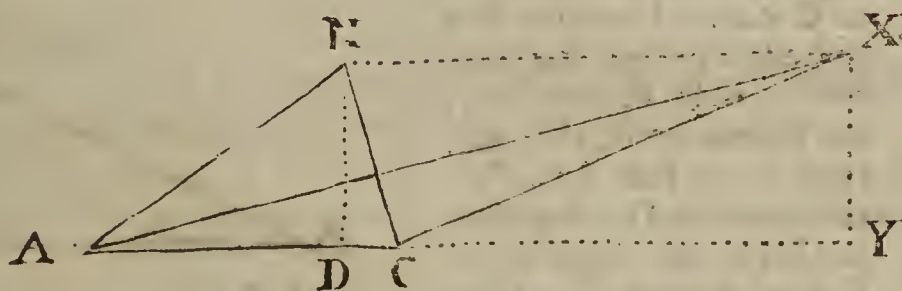
Again, subtract the square Root before found, from the before mentioned fourth Proportional f , and reserve the remainder. Then observe whether the said remainder be less, greater, or equal to the given Base AC ; if less, then the said remainder shall be equal to AD , to wit, the greater segment of the Base AC made by the falling of the Perpendicular ND within the Triangle ANC in *Fig. 1.* Whence $AN = \sqrt{\square AD + \square ND}$: is given. Likewise $CN = \sqrt{\square CD + \square DN}$: is given, and therefore $\triangle ACN$ acute-angled at A and C is given; which I call the latter of the two Triangles that will solve the Problem. But if the remainder before reserved happens to be greater than the given Base AC , then the said remainder shall be the distance from the foot of the Perpendicular falling without the Triangle to the remoter end of the Base, which distance we may suppose to be AD in *Fig. 2.* whence $CD = AD - AC$ is given, and consequently, (*per prop. 47. Elem. 1.*) $CN = \sqrt{\square CD + \square DN}$: is given. Likewise $AN = \sqrt{\square AD + \square DN}$: is given; and therefore in *Fig. 2.* $\triangle ACN$ obtuse-angled at C is given, which shall be the latter of two Triangles that will solve the Problem. But if the remainder before reserved happens to be equal to the given Base AC , then the latter of two Triangles that will solve the Problem shall be right-angled at the Base; as the $\triangle ACN$ right-angled at C , in *Fig. 3.* and consequently, $AN = \sqrt{\square AC + \square CN}$: (*per prop. 47. Elem. 1.*) is given. Therefore in *Fig. 3.* $\triangle ACN$ is given also.

Lastly, when it happens that nothing remains after subtraction is made of the sum of the Square of the given Perpendicular ND and the Rectangle of the given Base AC into the fourth Proportional f , from the Square of the same f ; then f it self shall be the distance from the foot of the Perpendicular falling upon the Base continued to the remoter end of the said Base; which distance we may suppose to be AD in $\triangle ADN$ in *Fig. 4.* Whence $CD = AD - AC$ is given, and consequently, (*per prop. 47. Elem. 1.*) $CN = \sqrt{\square CD + \square DN}$: is given: Likewise $AN = \sqrt{\square AD + \square DN}$: is given. Therefore in *Fig. 4.* $\triangle ACN$ obtuse-angled at C is given, which is the only Triangle in this Case that will solve the Problem.

TRIANGLES in Numbers, to illustrate the preceding Canon of Probl. 15.

FIG. 1.

52. In this *Fig. 1.* the Triangles ACX and ACN , the first of which is obtuse-angled at the Base AC , and the latter acute-angled, have one common Base AC , also equal Perpendiculars ND and XY ; and the legs AX, CX of $\triangle ACX$ have the same Proportion one to the other, as the legs AN, CN of $\triangle ACN$.



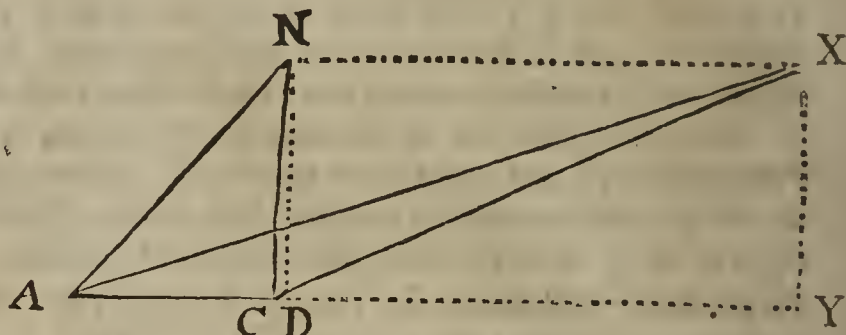
$AC = 39$		$AC = 39$
$AN = 40$		$AX = 8\sqrt{153}$
$CN = 25$		$CX = 5\sqrt{153}$
$ND = 24$		$XY = 24$
$DA = 32$		$YA = 96$
$DC = 7$		$YC = 57$

$AN . CN :: AX . CX :: 8 . 5.$

FIG. 2.

FIG. 2.

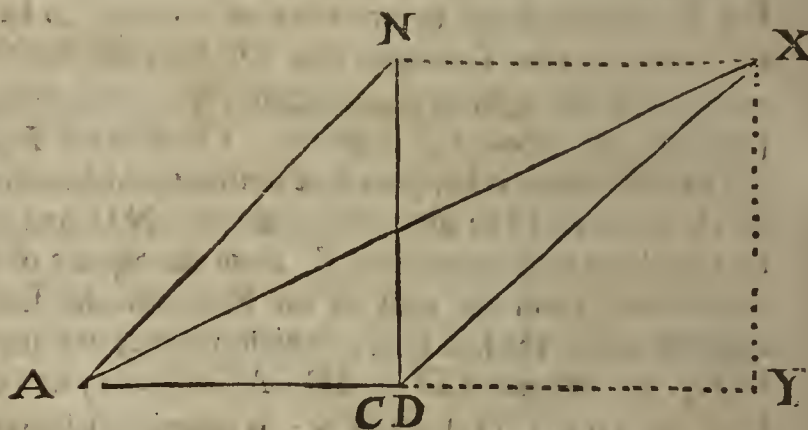
53. In this *Fig. 2.* the Triangles ACX and ACN , each of which is obtuse-angled at the Base AC , have one common Base AC , also equal Perpendiculars ND and XY , and the leggs AX , CX of $\triangle ACX$ have the same Proportion one to the other, as the leggs AN , CN of $\triangle ACN$.



$AC =$	812	$AC =$	812
$AN =$	$4\sqrt{102544}$	$AX =$	$4\sqrt{570256}$
$CN =$	$3\sqrt{102544}$	$CX =$	$3\sqrt{570256}$
$ND =$	960	$XY =$	960
$DC =$	36	$YC =$	2052
$DA =$	848	$YA =$	2864
$AN \cdot CN :: AX \cdot CX :: 4 \cdot 3.$			

FIG. 3.

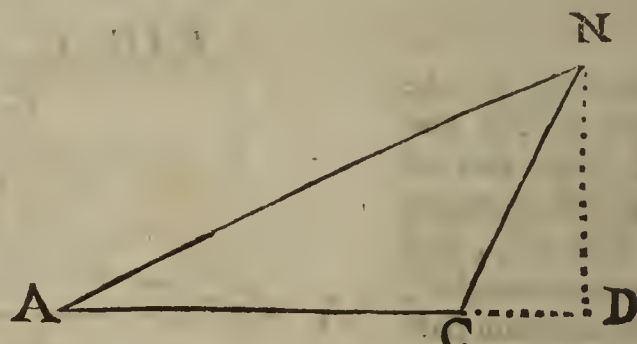
54. In this *Fig. 3.* the Triangles ACX and ACN , the first of which is obtuse-angled at the Base AC , and the latter right-angled, have one common Base AC , also equal Perpendiculars ND and XY ; and the leggs AX , CX of $\triangle ACX$ have the same Proportion one to the other, as the leggs AN , CN of $\triangle ACN$.



$AC =$	8	$AC =$	8	$XY =$	6
$AN =$	10	$AX =$	$5\sqrt{13}$	$YC =$	9
$CN =$	6	$CX =$	$3\sqrt{13}$	$YA =$	17
$AN \cdot CN :: AX \cdot CX :: 5 \cdot 3.$					

FIG. 4.

55. In this *Fig. 4.* the Triangle ACN obtuse-angled at C cannot be matcht with any other Triangle that shall have its Base, Perpendicular, and Proportion of the leggs, equal to the Base AC , Perpendicular ND , and Proportion of the leggs AN , CN of the said $\triangle ACN$.



$AC =$	30	$ND =$	20
$AN =$	$2\sqrt{500}$	$DC =$	10
$CN =$	$1\sqrt{500}$	$AD =$	40
$AN \cdot CN :: 2 \cdot 1.$			

56. It is prescribed by the preceding Determination in 36° , that $\sqrt{pp + fb}$: must not be greater than f , I shall therefore divide the Composition of this *Probl. 15.* into two Cases, viz.

Case 1. when $\sqrt{pp + fb} > f$. Case 2. when $\sqrt{pp + fb} = f$.

The Composition of Case 1. *Probl. 15.*

Suppos.

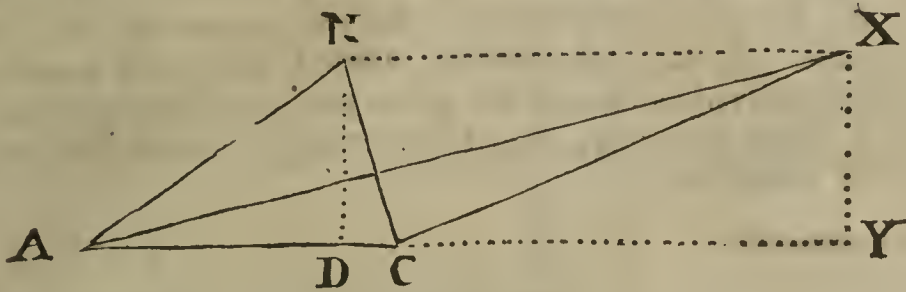
57. B a right line equal to the Base of a plain Triangle is given.

58. P a right line equal to the Perpendicular is given.

59. R and S

59. R and S two right lines expressing the Reason (or Proportion) of the leggs of the same Triangle, are given.

60. $R \sqsubset S$.



Req. to find out the Triangle.

- B _____
- P _____
- R _____
- S _____
- T _____
- F _____
- M _____
- K _____
- L _____

Construction.

- 61. To the given lines R and S find a third Proportional, } $R . S :: S . T$.
(by *Probl. 7 Chap. 5.*) suppose the line T, therefore . }
- 62. Also (by *Probl. 8. Chap. 5.*) let it be made as $R - T$ }
to R; so the given Base B to a fourth Proportional line F, } $R - T . R :: B . F$.
therefore }
- 63. By *Probl. 2. Chap. 5.* find a right line M, such that its } $\square M = \square P + \square F, B$.
Square may be equal to $\square P + \square B, F$, therefore . . . }
- 64. By *Probl. 14. Chap. 5.* divide the double of F into two such parts, that the line M }
may be a mean between them; which Effect is possible, for by *Suppos. in Case 1.* }
(before express in 56°) the line M (that is, $\sqrt{pp + fb}$) is less than F; suppose }
then that $2F$ is cut into two parts, whereof the greater is equal to the line K, and the }
lesser equal to the line L; and that the line M is a mean Proportional between K and L, }
therefore these are Proportionals, viz. }

$$\frac{2F - K}{L} . M :: M . K, \quad || \quad \frac{2F - L}{K} . M :: M . L.$$

Each of which Analogies is correspondent to that in the 34th step of the preceding Resolution, viz.

$$2f - a . \sqrt{pp + fb} : :: \sqrt{pp + fb} : . a.$$

Now by the help of the line K, found out as above, an obtuse-angled plain Triangle to solve the Problem propounded may be made in manner following, viz.

- 65. Make $AC = B$, (the given Base)
- 66. Produce AC to Y, so that AY may be equal to K, which is greater than AC, as may be proved thus;

It is manifest that } $R \sqsubset R - T$.
Therefore from the Analogy in 62^o, (per *Coroll. of 14. prop* }
5. Elem.) } $F \sqsubset B$.
But by *Constr.* in 65^o, } $AC = B$.
Therefore (per *Ax. 3. Chap. 2.*) } $F \sqsubset AC$ (or B.)
And because the greatest of three Proportionals is greater than }
half the sum of the extremes, therefore from 64^o, . . . } $K \sqsubset F$.
Therefore (per *Ax. 5. Chap. 2.*) } AY or $K \sqsubset AC$.

Which was to be demonstr.

- 67. Make $YX \perp AY$, also make $YX = P$, the given Perpendicular.
- 68. Lastly, from A and C (the ends of the Base AC) draw the right lines AX and CX to meet with the top of the Perpendicular YX in X, so the Triangle ACX obtusangled at C, (for as before hath been proved in 66^o, $AY \sqsubset AC$) shall be

one

one of the two Triangles which in *Case 1.* will satisfy the Problem; which I prove thus,
 69. First, by *Construction* in 65° the Base AC is equal to the given Base B; secondly,
 by *Constr.* in 67° the line YX is perpendicular to AC continued, and equal to the given
 Perpendicular P. It remains only to prove that the greater legg AX hath such pro-
 portion to the lesser legg CX, as R to S; which Analogy will be made manifest by
 the following Demonstration, formed out of the preceding Resolution by a repetition
 of its steps in a retrograde order, viz. by returning backwards from the end to the
 beginning of the Resolution.

70. . . . *Req. demonstr.* R . S :: AX . CX.

Demonstration.

71. Forasmuch as by *Constr.* in 64° , $2F - K . M :: M . K.$
 72. And by *Constr.* in 66° , $AY = K.$
 73. Therefore from 71° and 72° , $2F - AY . M :: M . AY.$
 That is, in 34° , the last step of the Resolution, $2f - a . \sqrt{pp + fb} :: \sqrt{pp + fb} . a.$
 74. But from 73° , (*per 17. prop. 6. Elem.*) $2\Box F, AY - \Box AY = \Box M.$
 75. And by *Constr.* in 63° , $\Box P + \Box F, B = \Box M.$
 76. Therefore from 74° and 75° , (*per 1. Ax. Chap. 2.*) $2\Box F, AY - \Box AY = \Box P + \Box F, B.$
 77. And because by *Constr.* in 67° , $YX = P.$
 78. And consequently, $\Box YX = \Box P.$
 79. And by *Constr.* in 65° , $AC = B.$
 80. Therefore out of 76° , 78° and 79° , $2\Box F, AY - \Box AY = \Box YX + \Box F, AC$
 That is, in 33° , $2fa - aa = pp + fb.$
 81. And from 80° , by adding $\Box AY$ to each part, $2\Box F, AY = \Box AY + \Box YX + \Box F, AC.$
 That is, in 32° , $2fa = aa + pp + fb.$
 82. And by subtracting $\Box F, AC$ from each part of the Equation in 81° , $2\Box F, AY - \Box F, AC = \Box AY + \Box YX.$
 That is, in 31° , $2fa - fb = aa + pp.$
 83. And this following Analogy is manifest, (*per prop. 7. Elem. 5.*) for the first and third
 Proportionals are one and the same, and the second and fourth equal one to the other,
 (as hath before been proved in 82° ;) $2\Box AC, AY - \Box AC . \Box AY + \Box YX :: 2ba - bb . aa + pp ::$

$$\left. \begin{array}{l} 2\Box AC, AY - \Box AC . \\ \Box AY + \Box YX :: \end{array} \right\} \text{that is, in } 30^\circ, \left\{ \begin{array}{l} 2ba - bb . \\ aa + pp :: \\ 2ba - bb . \\ 2fa - fb . \end{array} \right.$$

84. And by reason of the common altitude $2AY - AC$ in the two latter Terms of the
 subsequent Analogy, it will be manifest (*per 1. prop. 6. Elem.*) that

$$\left. \begin{array}{l} AC . \\ F :: \\ 2\Box AC, AY - \Box AC . \\ 2\Box F, AY - \Box F, AC . \end{array} \right\} \text{that is, in } 29^\circ, \left\{ \begin{array}{l} b . \\ f :: \\ 2ba - bb . \\ 2fa - fb . \end{array} \right.$$

85. And because the two latter Terms of the Analogy in 84° , are the same, and in the
 same order with the two latter Terms of the Analogy in 83° , therefore from 83° and
 84° (*per 11. prop. 5. Elem.*) these shall be Proportionals, viz.

$$\left. \begin{array}{l} AC . \\ F :: \\ 2\Box AC, AY - \Box AC . \\ \Box AY + \Box YX . \end{array} \right\} \text{that is, in } 28^\circ, \left\{ \begin{array}{l} b . \\ f :: \\ 2ba - bb . \\ aa + pp . \end{array} \right.$$

86. But from the *Constr.* in 62° and 65° , $AC . F :: R - T . R.$
 That is, in 27° , $b . f :: r - t . r.$

87. Therefore from 85° and 86° , (*per 11. prop. 5. Elem.*) these shall be Proportionals, viz.

$$\left. \begin{array}{l} R - T . \\ R :: \\ 2\Box AC, AY - \Box AC . \\ \Box AY + \Box YX . \end{array} \right\} \text{that is, in } 26^\circ, \left\{ \begin{array}{l} r - t . \\ r :: \\ 2ba - bb . \\ aa + pp . \end{array} \right.$$

88. And

88. And from 87°, by Reason Inverse, these are Proportionals, viz.

$$\left. \begin{array}{l} R \cdot T :: \\ \square AY + \square YX \cdot \\ 2 \cdot \square AC, AY - \square AC \cdot \end{array} \right\} \text{that is, in } 25^\circ, \left\{ \begin{array}{l} r \cdot t :: \\ aa + pp \cdot \\ 2ba - bb \cdot \end{array} \right.$$

89. And from 88°, by Conversion of Reason, these are Proportionals, viz.

$$\left. \begin{array}{l} R \cdot T :: \\ \square AY + \square YX \cdot \\ \square AY - 2 \square AC, AY + \square AC + \square YX \cdot \end{array} \right\} \text{which answer to those in } 24^\circ \\ \text{in the Resolution, viz.} \\ r \cdot t :: aa + pp \cdot aa - 2ba + bb + pp.$$

90. And because by Constr. in 61°, } R . S :: S . T.

That is, in 22°, } r . s :: s . t.

91. Therefore (per Coroll. of 20. prop. 6. Elem.) } R . T :: \square R . \square S.

That is, in 23°, } r . t :: rr . ss.

92. Therefore from 89° and 91°, (per 11. prop. 5. Elem.) these shall be Proportionals, viz.

$$\left. \begin{array}{l} \square R \cdot \\ \square S :: \\ \square AY + \square YX \cdot \\ \square AY - 2 \square AC, AY + \square AC + \square YX \cdot \end{array} \right\} \text{which answer to those in } 21^\circ, \\ \text{viz.} \\ rr \cdot ss :: aa + pp \cdot aa - 2ba + bb + pp.$$

93. And because by Constr. in 67°, } YX \perp AY.

94. Therefore (per 47. prop. 1. Elem.) } \square AX = \square AY + \square YX.

95. Likewise } \square CX = \square CY + \square YX.

96. Moreover by Constr. in 66°, and by the Diagr. } CY = AY - AC.

97. And consequently, (per 5. Theor. 4. Chap.) } \square CY = \square AY - 2 \square AC, AY + \square AC.

That is, in 9°, } \square a-b = aa - 2ba + bb.

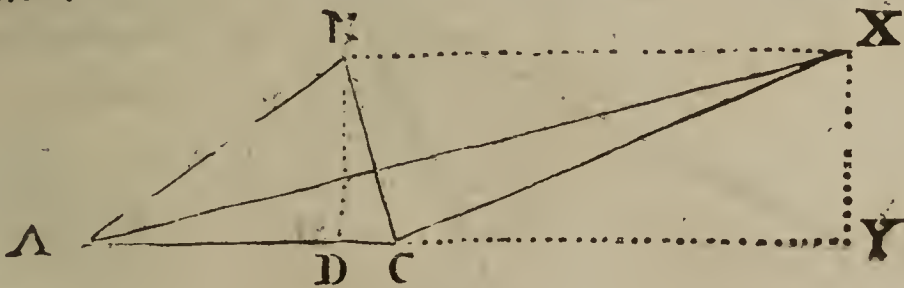
98. Therefore if instead of \square CY in 95°, we take that which in 97° is found equal to \square CY, the Equation in 95° will be reduced to this, viz.

$$\square CX = \square AY - 2 \square AC, AY + \square AC + \square YX.$$

99. Likewise if instead of the third and fourth Proportionals in 92°, we take those Squares which are found equal to them respectively in 94° and 98°, the Analogy in 92° will be reduced to this, viz. \square R . \square S :: \square AX . \square CX.

100. Wherefore (per prop. 22. Elem. 6.) R . S :: AX . CX. Which was to be Dem.

101. Another Triangle to solve the Problem in Case 1. before express in 56°, may by the help of the lesser Root L before found in 64°, be formed thus, viz. Let the lines before given and found out in 57°, 58°, 59°, 60°, 61°, 62°, 63°, 64°, together with the Diagram standing between 60° and 61° be here repeated, then will the Construction be as followeth.



Req. to find out the Triangle.

- B _____
- P _____
- R _____
- S _____
- T _____
- F _____
- M _____
- K _____
- L _____

Z z

Constr.

102. Make $AC = B$: (the given Base.)
 103. Upon AC , continued if need be, make $AD = L$, which lesser Root L , (as before hath been shewn,) will sometimes be greater than the Base; but supposing it be discovered (by *Rule 2.* in 39° of this *Probl.*) that L is lesser than B , or AC , cut off from AC a segment equal to L , as AD .
 104. Make $DN \perp AC$ in the point D , also make $DN = P$ the given Perpendicular.
 105. Lastly, from the ends of the Base AC draw the right lines AN and CN to meet with the top of the Perpendicular DN , in N ; so the Triangle ACN acute-angled at A and C , (for by *Supposition* AD is lesser than AC ;) will satisfy the Problem, as well as the $\triangle ACX$ before found. For first, by *Construction* in 102° the Base AC is equal to the given Base B : Secondly, the Perpendicular DN (by *Constr.* in 104°) is equal to the given Perpendicular P ; and by a repetition of the steps of the Resolution in a backward order, in like manner as before in the preceding Demonstration, saving that L must be used here instead of K , and ND instead of XY ; it may easily be proved that the leggs AN and CN are in the given Reason of R to S .

Moreover, when the lesser Root L is greater than the Base, the Triangle formed by the help of such lesser Root shall be obtuse-angled at the Base, and the Construction and Demonstration in every respect like to that by the greater Root.

But it must be remembred, that when the Perpendicular falls within the Triangle, then the Square of DC is equal to the Square of $AC - AD$; but when it falls without, then the Square of YC is equal to the Square of $YA - AC$: So that before the Construction and Demonstration by the lesser Root be entred upon, it will be requisite to find out the kind of the Triangle, by the help of the three preceding Rules in 38°, 39°, 40°; and when it happens that $r . s :: \sqrt{bb + pp} : p$, then 'tis evident (by 47. *prop. 1. Elem.*) that the Triangle formed by the lesser Root will be right-angled at the Base, and in such Case there is no need of further proof.

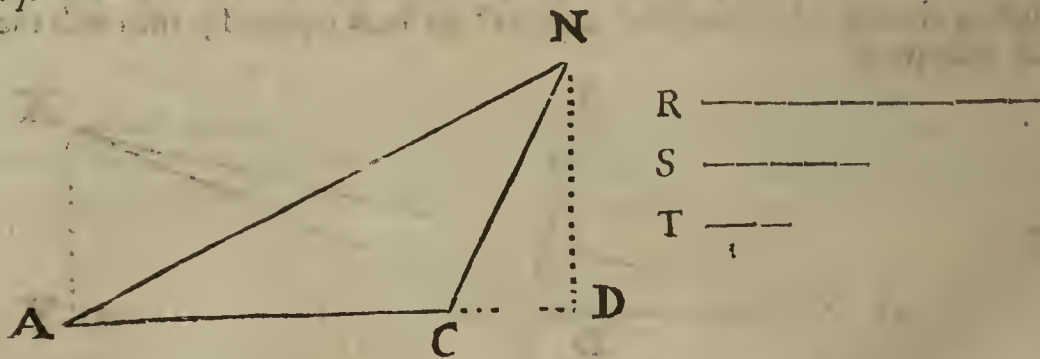
The Composition of Case 2. Probl. 15.

106. Which Case, in the letters belonging to the preceding Resolution presupposeth . . . } $f = \sqrt{pp + fb} = a.$
 107. And consequently, by squaring each part, . . . } $ff = pp + fb = aa.$
 108. That is, in the lines of the ensuing *Constr.* and *Diagr.* } $\square AD = \square DN + \square AD, AC.$

Suppos.

109. $AC =$ the Base of a Triangle is given.
 110. $DN =$ the Perpendicular is given.
 111. R and S two right lines expressing the Reason of the leggs are given.
 112. $R - S$.

Req. to find the \triangle .



Construction.

113. By *Probl. 7. Chap. 5.* let it be made as R to S ; so S to a third Proportional, suppose it be found T , therefore,
 $R . S :: S . T.$
 114. By *Probl. 8. Chap. 5.* let it be made as $R - T$ to R ; so AC to a fourth Proportional, suppose it to be AD , therefore,
 $R - T . R :: AC . AD.$

Which fourth Proportional AD shall necessarily be greater than AC , because R is greater than $R - T$.

115. Make

115. Make $DN \perp AD$ in the point D ; then from A and C , the ends of the given Base AC , draw the right lines AN and CN to meet with the top of the Perpendicular DN in N ; so shall ACN be the Triangle required. For first, the Base AC is equal to the given Base; also the Perpendicular ND is equal to the given Perpendicular. But that the leggs AN and CN are in the given Reason of R to S , it may easily be demonstrated by a backward repetition of the steps of the foregoing Resolution, in like manner as before in the Composition of *Case 1*; with this Caution, That as often as a is found in the Resolution, f must be taken instead of a , because in this second Case f is equal to a ; for since by *Supposition* in 106° , $f = \sqrt{pp + fb}$: it will be evident from 35° , that $f = a$. But in regard the Demonstration of this second Case differs not from that of the following *Probl. 16*. I shall wave it here.

COROLLARY.

116. From the premisses it follows, that the Perpendicular DN of the Triangle ACN formed in *Case 2*: (before express in 106° , 107° and 108° ;) is a mean Proportional between AD and DC the distances from D the foot of the Perpendicular falling without the Triangle to the ends of the Base; and consequently, (*per prop. 6. Elem. 6.*) the Triangles ADN and CDN are equiangular. See the last preceding Diagram, and compare it with this following Demonstration.

117. . . . *Req. demonstr.* $\left\{ \begin{array}{l} AD \cdot DN :: DN \cdot DC. \text{ Also,} \\ \triangle ADN \text{ and } \triangle CDN \text{ are equiangular.} \end{array} \right.$

Demonstration.

118. By *Suppos.* in 108° , $\left\{ \begin{array}{l} \square AD = \square DN + \square AD, AC. \\ \square AD - \square AD, AC = \square DN. \end{array} \right.$

119. Therefore by subtracting $\square AD, AC$ from each part, $\left\{ \begin{array}{l} \square AD - \square AD, AC = \square DN. \end{array} \right.$

120. And from 119°, (*per prop. 14. Elem. 6.*) these are Proportionals, *viz.* $\left\{ \begin{array}{l} AD \cdot DN :: DN \cdot AD - AC. \end{array} \right.$

121. But 'tis evident by the last preceding Diagram, that $\left\{ \begin{array}{l} DC = AD - AC. \end{array} \right.$

122. Therefore from 120° and 121°, . . . $\left\{ \begin{array}{l} AD \cdot DN :: DN \cdot DC. \end{array} \right.$

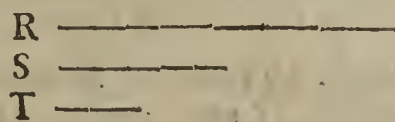
123. Therefore from 122°, (*per prop. 6. Elem. 6.*) $\left\{ \begin{array}{l} \triangle ADN \text{ and } \triangle CDN \text{ are equiangular.} \end{array} \right.$

Which was to be Demonstr.

From the preceding *Corollary* and *Construction* of *Case 2*. the following *Probl. 16*. is deducible.

Probl. XVI.

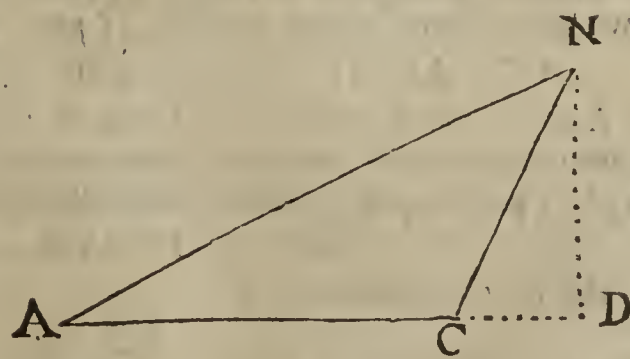
To find a plain Triangle obtuse-angled at the Base, and that the Base may be equal to a right line given. Also, that the Perpendicular falling upon the Base continued may be a mean Proportional between the distances from the foot of the Perpendicular to the ends of the Base: And that the leggs of the Triangle may be in a given Reason, suppose as R to S .



Suppos.

1. $\triangle ACN$ is obtuse-angled at C .
2. AC the Base is given.
3. ACD is a right line.
4. $DN \perp AD$.
5. $AD \cdot DN :: DN \cdot DC$.
6. R and S are right lines given.
7. $R \cdot S :: AN \cdot CN$.

Req. to make the $\triangle ACN$.



Construction.

8. Making R the first of three Proportionals, and S the second, find a third, (*per 7. Probl. 5. Chap.*) let it be T , therefore

$$R \cdot S :: S \cdot T.$$

Zz 2

9. Also

9. Also making $R - T$ the first of four Proportionals, R the second, and the given Base AC the third, find a fourth, (per 8. *Probl. 5. Chap.*) suppose it be AD , therefore

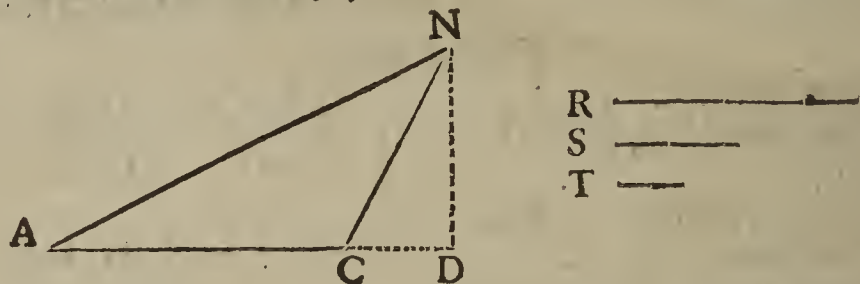
$$R - T . R :: AC . AD.$$

Which fourth Proportional AD shall necessarily be greater than AC , because the second Proportional R is greater than the first $R - T$.

10. Find a mean Proportional, as DN , between AD and DC , (per 9. *Probl. 5. Chap.*) therefore,

$$AD . DN :: DN . DC (= AD - AC.)$$

11. Make $DN \perp AD$ in the point D ; then draw the right lines AN and CN , so shall ACN be the Triangle required. Now we must shew that it will satisfy the Problem: First then, AC the Base is equal to the right line prescribed for the Base, and from the 9th step it is less than AD ; therefore the angle ACN is obtuse: Secondly, the Perpendicular ND , (by *Construction* in 10° ,) is a mean Proportional between AD and DC , (to wit, the two distances from D the foot of the Perpendicular ND , to A and C the ends of the Base AC .) It remains only to prove, That the legs AN and CN of the Triangle ACN , are in such proportion one to the other as R to S ; which Analogy I shall make manifest by the following Demonstration, formed out of the Resolution of the preceding *Probl. 15.* by a backward repetition of the steps of the said Resolution, in *Case 2.* (but respect must be had to the Caution given in 115° of the said *Probl. 15.*)



12. . . . *Req. demonstr.* $R . S :: AN . CN.$

Demonstration.

13. Forasmuch as by *Constr.* in 10° , . . . } $AD . DN :: DN . DC (AD - AC.)$

14. Therefore (per 17. *prop. 6. Elem.*) . . . } $\square AD - \square AD, AC = \square DN.$

15. And from 14° , by equal addition of $\square AD, AC$, } $\square AD = \square DN + \square AD, AC.$

16. And from 15° , by equal addition of $\square AD$, } $2\square AD = \square AD + \square DN + \square AD, AC.$

17. And from 16° , by equal subtraction of } $2\square AD - \square AD, AC = \square AD + \square DN.$
 $\square AD, AC$, }

18. And because in the following Analogy the first and third Terms are one and the same, and the second and fourth equal one to the other, (as hath been proved in 17° .) therefore (per 7. *prop. 5. Elem.*)

$$\left. \begin{array}{l} 2\square AC, AD - \square AC . \\ \square AD + \square DN :: \\ 2\square AC, AD - \square AC . \\ 2\square AD - \square AD, AC . \end{array} \right\} \text{Proportionals.}$$

19. And the subsequent Analogy, by reason of the common altitude $2AD - AC$ in the two latter Terms, will be manifest, (per 1. *prop. 6. Elem.*) viz.

$$AC . AD :: 2\square AC, AD - \square AC . 2\square AD - \square AD, AC.$$

20. And because the two latter Terms of the Analogy in 19° are the same and in the same order with the two latter Terms of the Analogy in 18° , therefore from 18° and 19° ; (per 11. *prop. 5. Elem.*) these shall be Proportionals, viz.

$$AC . AD :: 2\square AC, AD - \square AC . \square AD + \square DN.$$

21. But by *Construction* in 9° ,

$$AC . AD :: R - T . R.$$

22. Therefore from 20° and 21° , (per 11. *prop. 5. Elem.*)

$$R - T . R :: 2\square AC, AD - \square AC . \square AD + \square DN.$$

23. And from 22° , by Reason inverse,

$$R . R - T :: \square AD + \square DN . 2\square AC, AD - \square AC;$$

24. And

24. And from 23°, by Conversion of Reason, these shall be Proportionals, viz.

$$\begin{array}{l} R \cdot S \\ T \cdot R \end{array} :: \left. \begin{array}{l} \square AD + \square DN \\ \square AD + \square DN + \square AC - 2\square AC, AD \end{array} \right\} \text{Proportionals.}$$

25. And because by *Constr.* in 8°, . . . } $R \cdot S :: S \cdot T.$

26. And consequently, (per *Coroll.* of 20. *prop.* 6. *Elem.*) } $R \cdot T :: \square R \cdot \square S.$

27. Therefore from 24° and 26°, (per 11. *prop.* 5. *Elem.*) these shall be Proportionals, viz.

$$\begin{array}{l} \square R \\ \square S \end{array} :: \left. \begin{array}{l} \square AD + \square DN \\ \square AD + \square DN + \square AC - 2\square AC, AD \end{array} \right\} \text{Proportionals.}$$

28. And because by *Constr.* in 11°, . . . } $DN \perp AD.$

29. Therefore, per 47. *prop.* 1. *Elem.* (respect } $\square AN = \square AD + \square DN.$
being had to the Diagram,)

30. Likewise, . . . } $\square CN = \square DC + \square DN.$

31. Again, by *Constr.* in 9°, and by view of the } $DC = AD - AC.$
Diagram,

32. And consequently, (per *Theor.* 5. *Chap.* 4.) } $\square DC = \square AD + \square AC - 2\square AD, AC.$

33. Therefore if instead of $\square DC$ in 30°, we set that which in 32° is found equal to $\square DC$, the Equation in 30° will be reduced to this, viz.

$$\square CN = \square AD + \square AC - 2\square AD, AC + \square DN.$$

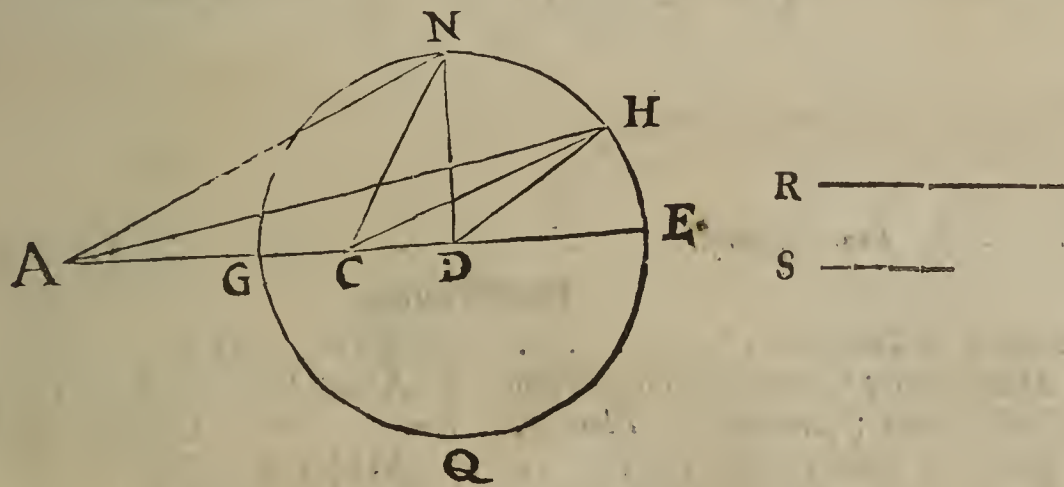
34. Likewise, if instead of the third and fourth Proportionals in 27°, we take those Squares which are found equal to them respectively in 29° and 33°, the Analogy in 27° will be reduced to this, $\square R \cdot \square S :: \square AN \cdot \square CN.$

35. Wherefore . . . $R \cdot S :: AN \cdot CN.$

Which was to be demonstrated. Therefore that is done which the Problem required.

Probl. XVII. (Probl. Apollon. Pergæi.)

Two points (A and C) being given in a Plane, to describe a Circle in the same Plane, that two right lines drawn from those points to concur in any point of the Circumference may have a given Reason; suppose the greater line to the less, as R to S.



Construction.

1. Upon the given line AC as a Base, to wit, the shortest distance between the given points A and C, make (by the preceding *Probl.* 16.) a Triangle ACN obtuse-angled at C, and such, that the Perpendicular ND falling upon AC produced, may be a mean Proportional between AD and DC; also, that the leggs AN and CN may be in the given Reason of R to S. Therefore by that Construction these are Proportionals, viz.

$$\begin{array}{l} AD \cdot DN :: DN \cdot DC \\ R \cdot S :: AN \cdot CN \end{array}$$

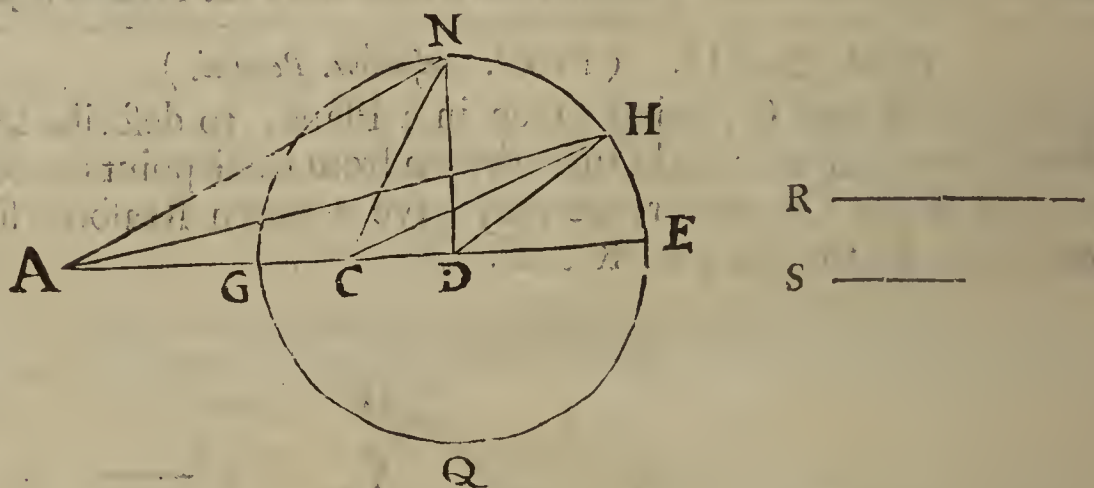
2. Then from the Center D, at the distance of the Perpendicular DN, describe the Circle DGNEQ, which shall necessarily cut AC; for by *Construction* DN is a mean Proportional.

Proportional between DA and DC, which DC being but part of DA is less than DA, therefore the mean, or Semidiameter DN or DG is less than DA, but greater than DC. Now I say the Circle DGNEQ is that which is required by the Problem, and therefore we must shew that if two right lines be drawn from the given points A and C to meet in any point of the Circumference of that Circle, those right lines shall have such proportion one to the other as the given lines R and S; the demonstration whereof I shall divide into three Cases, in regard there may be a threefold position of the point taken in the Circumference; for the point may be either E, or else G, to wit, the ends of that Diameter which lyes in the same straight line with the given line AC; or lastly, the point may be taken in any other part of the Circumference, as H; which Cases I shall demonstrate in their order.

Preparat.

3. Forasmuch as in the Triangles ADN and CDN the angle at D is common, and the sides about that angle are Proportionals, for by Construction in 1° it hath been made, as $AD \cdot DN :: DN \cdot DC$, therefore (per prop. 6. Elem. 6.) $\triangle ADN$ and $\triangle CDN$ are equiangular.
4. But we must enquire which angles in those like Triangles are equal one to the other. First then, because the angle at D is common, the angle CND in $\triangle CDN$ must be equal either to the angle AND , or to the angle NAD in $\triangle ADN$; but the angle CND being but part of the angle AND cannot be equal to it, therefore $\angle CND = \angle NAD$. Also, $\angle NCD = \angle AND$.
5. In like manner, because by Constr. in 1° and 2°, $AD \cdot DN$ (or DH) $:: DN$ (or DH) $\cdot DC$.
6. Therefore, (per prop. 6. Elem. 6.) $\triangle ADH$ and $\triangle CDH$ are equiangular.
7. And for the like reason as before in 4°, $\angle CHD = \angle HAD$. Also, $\angle HCD = \angle AHD$.

These things premised, I shall proceed to the Demonstration of the three CASES before mentioned.



8. ∴ I. Req. Demonstr. R . S :: AE . CE.

Demonstration.

9. Because by Constr. in 1°, } $AD \cdot DN :: DN \cdot DC$.
10. Therefore by Composition of Reason, } $AD + DN \cdot DN :: DN + DC \cdot DC$.
11. And because (per defn. 15. Elem. 1.) } } $DE = DN$.
12. Therefore from 10° and 11°, } $AD + DE \cdot DN :: DE + DC \cdot DC$.
13. That is, as is evident by the Diagram, } $AE \cdot DN :: CE \cdot DC$.
14. Therefore alternly, } $AE \cdot CE :: DN \cdot DC$.
15. Again, it hath been proved in 3°, that } $\triangle ADN$ and $\triangle CDN$ are equiangular.
16. And in 4°, that } $\angle NCD = \angle AND$.
17. Therefore from 15° and 16°, (per } $AN \cdot DN :: CN \cdot DC$.
- prop. 4. Elem. 6.) }
18. Therefore alternly, } $AN \cdot CN :: DN \cdot DC$.
19. But by Constr. in 1°, } $AN \cdot CN :: R \cdot S$.
20. Therefore from 18° and 19°, (per 11. } $R \cdot S :: DN \cdot DC$.
- prop. 5. Elem.) }

21. But

21. But it hath been proved in 14°, that $\triangleright AE \cdot CE :: DN \cdot DC$
 22. Therefore from 20° and 21°, (per 11. } $R \cdot S :: AE \cdot CE$
prop. 5. Elem.) }
 Which was to be demonstr.
 23. . . . II. Req. Demonstr. $R \cdot S :: AG \cdot CG$

Demonstration.

24. Forasmuch as by Constr. in 1°, . . . $\triangleright AD \cdot DN :: DN \cdot DC$
 25. Therefore by Division of Reason, . . . $\triangleright AD - DN \cdot DN :: DN - DC \cdot DC$
 26. And because (per defin. 15. Elem. 1.) $\triangleright DG = DN$
 27. Therefore from 25° and 26°, . . . $\triangleright AD - DG \cdot DN :: DG - DC \cdot DC$
 28. That is, (as is evident by the Diagram,) $\triangleright AG \cdot DN :: CG \cdot DC$
 29. Therefore alternly, $\triangleright AG \cdot CG :: DN \cdot DC$
 30. But before in 20°, it hath been proved that $\triangleright R \cdot S :: DN \cdot DC$
 31. Therefore from 29° and 30°, (per 11. } $R \cdot S :: AG \cdot CG$
prop. 5. Elem.) }
 Which was to be demonstrated in the second place.
 32. . . . III. Req. Demonstr. $R \cdot S :: AH \cdot CH$

Demonstration.

33. It hath before been proved in 6°, that $\triangleright \triangle ADH$ and $\triangle CDH$ are equiangular.
 34. And in 7°, that $\triangleright \angle HCD = \angle AHD$
 35. Therefore from 33° and 34°, (per 4. } $CH \cdot CD :: AH \cdot HD$
prop. 6. Elem.) }
 36. And alternly, $\triangleright CH \cdot AH :: CD \cdot HD$ (or DN .)
 37. Therefore inversly, $\triangleright AH \cdot CH :: DN \cdot CD$
 38. But it hath been proved in 20°, that $\triangleright R \cdot S :: DN \cdot CD$
 39. Therefore from 37° and 38°, (per 11. } $R \cdot S :: AH \cdot CH$
prop. 5. Elem.) }

Which was to be demonstrated in the last place. Therefore that is done which the Problem required.

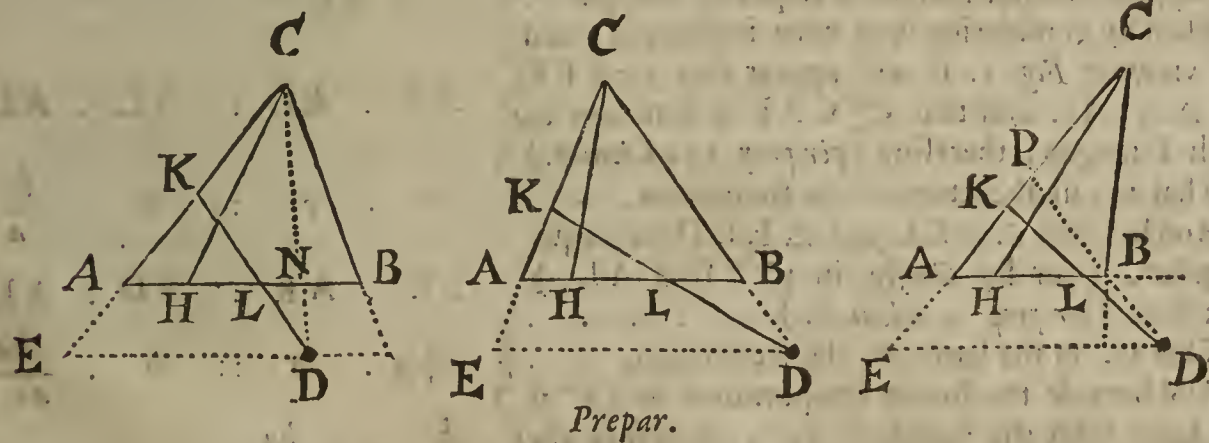
Probl. XVIII.

To divide a given Triangle ABC into two parts which shall be in a given Reason, suppose as AH to HB, by a right line DK drawn from a given point D without the Triangle.

FIG. 1.

FIG. 2.

FIG. 3.



Prepar.

1. By the given point D draw DE parallel to the Base AB, and continue the leggs CA, CB beneath the Base; then the point D will either lye between the Increases of the leggs, as in Fig. 1. or else in one of the said Increases, as in Fig. 2. or lastly, between the Increases of the Base and one of the leggs, as in Fig. 3.
2. Divide the Base AB in H in the given Reason, and draw CH, therefore (per *prop. 1. Elem. 6.*) $\triangle ACH \cdot \triangle HCB :: AH \cdot HB$.
3. In Fig. 1. let a line be drawn from the given point D to C the angle opposite to the Base AB, as the line DC, which will either cut the Base AB in the point H, in which the

the Base is divided in the given Reason, in which Case the Problem is evidently satisfied; or else in some other point N, and then the point H will either lye between N and A, or between N and B; if H lye between N and A, then the desired line of partition to be drawn from D will cut AB and AC; but if H lye between N and B, then the said line of partition will cut AB and BC.

4. In Fig. 2. where the given point D lyes in CB increased, 'tis evident that the line of partition to be drawn from D shall necessarily cut AB and AC.
5. In Fig. 3. the line of partition to be drawn from D will sometimes cut AB and AC, sometimes it may pass by the angular point B and cut AC only, and sometimes it will cut BC and AC; but which of these lines will happen to be cut when the given point D is posited according to the Definition in 1°, relating to Fig. 3. may be discovered by the Rule hereafter given in 34° of this Problem.
6. In every one of those three Cases before defined in 1°, which may happen by the various position of the given point D, the Resolution of the Problem propos'd will be one and the same. Supposing then it be discovered, that the line of partition to be drawn from D must cut AB and AC in each of the three preceding Figures, the Scope of the Resolution is to find a point in AC, as K, to which a right line being drawn from D; as DK, this line DK may cut the Base AB between H and N in Fig. 1. or between H and B in Fig. 2. likewise in Fig. 3. between H and B, (or else pass by the angular point B;) so as to make the Triangle AKL equal to the Triangle ACH, whence it evidently follows that $LKCB = \triangle HCB$, and $\triangle AKL . LKCB :: AH . HB$. These things premised, the Resolution of the Problem propounded may be formed in manner following.

Suppos.

7. ABC is a \triangle given in Fig. 1.
8. D is a point given without the $\triangle ABC$.
9. AH and HB are in a given Reason.
10. $b = AC$ is given.
11. $c = AH$ is given.
12. $g = AE$ is given.
13. $h = ED$ ($\parallel AB$) is given.

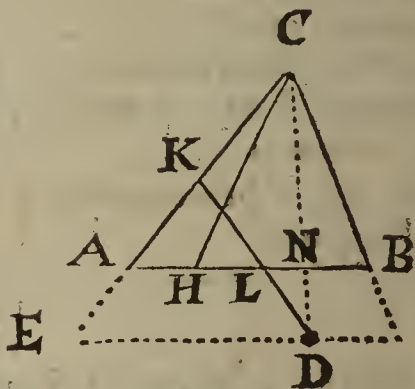
Req. to find

14. AK such a segment of AC, that DK being drawn, it may make
15. $\triangle ALK . LKCB :: AH . HB :: \triangle ACH . \triangle HCB$.

Resolution.

16. Suppose that done which is required, and put . . .
17. Then by considering well what is required, and by viewing Fig. 1. it will appear that $\triangle ALK = \triangle ACH$, and that $\angle CAL$ is common to both Triangles; therefore (per prop. 15. Elem. 6.) . . .
18. That is, in the letters of the Resolution, . . .
19. And because $\triangle AKL$ and $\triangle EKD$ are equiangular, (for by Constr. in 1° $ED \parallel AL$;) therefore (per prop. 4. Elem. 6.) . . .
20. That is, in the letters of the Resolution, . . .
21. And because the fourth Proportional in 17° is the same with the fourth in 19°, therefore the fourth Proportionals in 18° and 20° shall also be equal to one another, viz. . . .
22. Now to avoid an Equation between Solids, let it be made as h to b , so c to a fourth Proportional, which may be called m ; therefore, . . .
23. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, viz. . . .

FIG. 1.



$$\begin{aligned}
 & a = AK. \\
 & AK . AH :: AC . AL. \\
 & a . c :: b . \frac{bc}{a}. \\
 & EK . AK :: ED . AL. \\
 & a+g . a :: b . \frac{ha}{a+g}. \\
 & \frac{bc}{a} = \frac{ha}{a+g}. \\
 & h . b :: c . m. \\
 & hm = bc.
 \end{aligned}$$

34. There-

24. Therefore from 21° and 23°, by exchanging } $\frac{bm}{a} = \frac{ba}{a+g} = \frac{bc}{a}$
 equal quantities, }
 25. Whence 'tis easie to inferr that these are Pro- } $a \cdot m :: b \cdot \frac{ba}{a+g} (= \frac{bc}{a})$
 portionals, viz. }
 26. But it hath been shewn in 20°, that . . . } $a+g \cdot a :: b \cdot \frac{ba}{a+g}$
 27. Therefore from 25° and 26°, (per prop. 11. } $a : m :: a+g : a$
Elem. 5.) }
 28. And from 27°, by comparing the Rectangle of } $aa = ma + mg$
 the extremes to the Rectangle of the means, . . . }
 29. And from 28°, by subtracting *ma* from each } $aa - ma = mg$
 part, }
 30. Which last Equation may be resolved into these } $a - m \cdot \sqrt{mg} :: \sqrt{mg} \cdot a$
 Proportionals, viz. }
 31. Of which three Proportionals, the mean, to wit, \sqrt{mg} is given, as also *m* the dif-
 ference of the extremes *a* and *a - m*; therefore the extremes shall be given severally, (per
Probl. 12. Chap. 5.:) the greater whereof is equal to the desired line AK, which (by the
 Theorem in 24° of the said *Probl. 12. Chap. 5.*) will be found equal to this right line,
 (or number,) viz.

$$\frac{1}{2}m + \sqrt{\frac{1}{4}mm + mg} = AK = a.$$

From which Equation and premisses we may deduce this following

C A N O N.

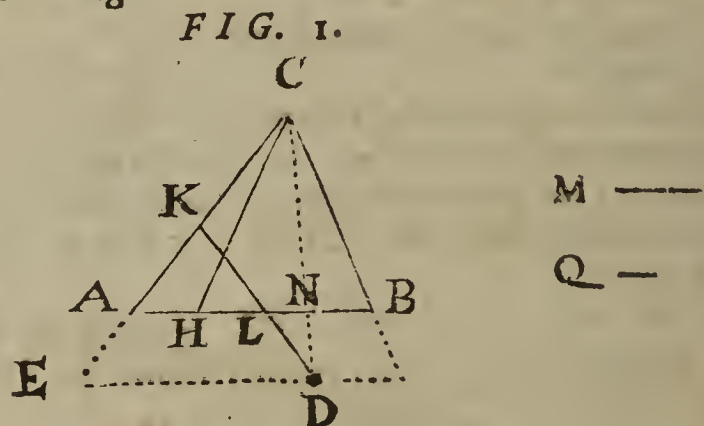
32. Let it be made as ED to AC, so AH to a fourth Proportional, which may be
 called M; then to the half of M add the square Root of the summ of the Square of
 half M and the Rectangle of M into AE, so shall the summ of that addition be the
 value of AK sought.
33. This Canon serves to find out the value of the line AK in every one of the three pre-
 ceding Figures, and when the given point D is posited according to the Definition of the
 first and second Cases in 1°, as in Fig. 1, and 2. it is easie to discover from what hath
 been said in 3° and 4°, which of the sides of the given Triangle ABC will be cut by the
 line of partition to be drawn from D. But when the point D is posited according
 to the Definition of the third Case in 1°, as in Fig. 3. then it may be doubtfull which
 of the sides are to be cut; to remove therefore this ambiguity, observe the following
 Directions, viz. First, draw a right line from the given point D (in Fig. 3.) to pass
 by the angular point B, as DBP; then is EABD a Trapezium, having (by Con-
 struction) two parallel sides AB and ED, and the other two sides EA and DB
 which are not Parallels, being continued will meet in some point in AC, as in P, for
 (by Construction) EAC is a right line. Now if AB, ED and EA be severally
 given in numbers, the line AP shall be also given in number; for putting $g = AE$,
 and $h = ED$, (as before in the Resolution,) also $k = AB$, the line AP (by the
 Theorem in 9° of *Probl. 18. Chap. 7.*) will be found equal to $\frac{gk}{h-k}$. It is also
 manifest, that if a right line be drawn from any point in AC, between P and C, to
 the given point D, the line so drawn must necessarily cut BC, for the line PBD is
 supposed to pass by the angular point B; but if a right line be drawn from any point
 in AC between P and A to the point D, the line so drawn will evidently cut AB.
 From the premisses therefore we may inferr this following

R U L E.

34. If $\frac{1}{2}m + \sqrt{\frac{1}{4}mm + gm}$: the value of AK, be not greater than $\frac{gk}{h-k}$ the value
 of AP in Fig. 3. then the line of partition to be drawn from the given point D, shall
 either pass by the angular point B, as the line DBP, or else cut AB in some
 point between B and A, and AC in some point between P and A. But if the
 said value of AK be greater than the said value of AP, then the line of partition
 will cut BC and AC, and in this latter Case a Parallel is to be drawn by the point
 D to BC, which is to be esteemed the Base, and then the given lines being
 A a a respectively

respectively changed, the line found out by the Canon must be set from C towards A.

35. The Problem propounded needs not any Determination to be annexed to it, either to limit the quantities of the given lines, or the position of the given point without the given Triangle. But because from what hath been said in 1°, 2°, 3°, 4° and 5°, it evidently appears, that in every one of the three preceding Figures the given side AC must be greater than the quantity of the sought line AK, except only in one Case in Fig. 1. when H lyes in the line DC, and is the same with the point N, for then AK is equal to AC; it will be requisite to prove, that in all other cases the side AC is greater than that right line which the Canon finds out for the value of AK, viz. in the letters of the Resolution, that $b \sqsubset \frac{1}{2}m + \sqrt{\frac{1}{4}mm + gm}$: The truth hereof I shall first demonstrate in this following



Prepar.

- 36. Let it be made as ED to AH, so AC to a fourth } ED . AH :: AC . M.
- proportional line, which may be called M, therefore, }
- 37. Let it also be made as ED to AH, so AE to a } ED . AH :: AE . Q.
- fourth proportional line, Q, therefore, . . . }
- 38. Then from those Analogies it follows (per prop. 11. } AE . Q :: AC . M.
- Elem. 5.) that . . . }
- 39. . . . } Req. demonstr. . . . AC $\sqsubset \frac{1}{2}M + \sqrt{\square AE, M + \frac{1}{4}\square M}$:
- That is, in the letters of the Resolution, $b \sqsubset \frac{1}{2}m + \sqrt{gm + \frac{1}{4}mm}$:

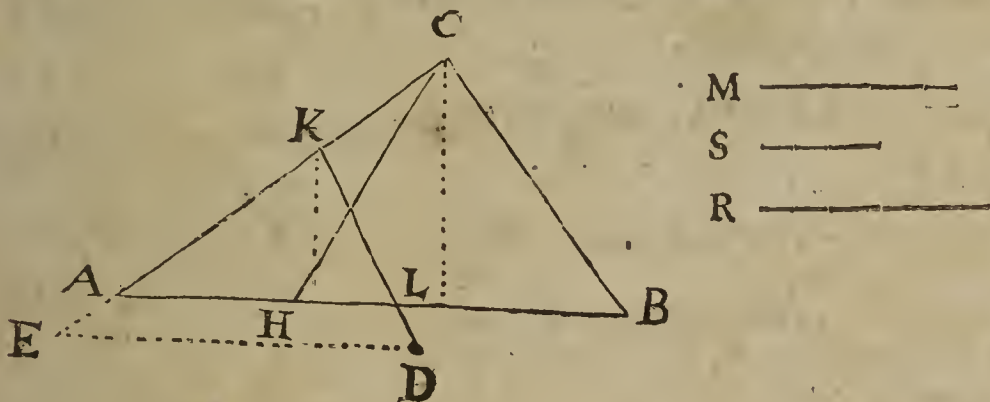
Demonstration.

- 40. By Constr. in 1°, AN || ED, there- } EC . ED :: AC . AN.
- fore in Fig. 1. $\triangle ACN$ and $\triangle ECD$ } are equiangular, and consequently, (per
- prop. 4. Elem. 6.) . . . }
- 41. Therefore from 40°, (per prop. 16. } $\square EC, AN = \square ED, AC$.
- Elem. 6.) . . . }
- 42. By Suppos. in Fig. 1. . . . } AN \sqsubset AH.
- 43. Therefore by drawing EC into each } $\square EC, AN \sqsubset \square EC, AH$.
- part, . . . }
- 44. Therefore from 41° and 43° (per } $\square ED, AC \sqsubset \square EC, AH$.
- Axi. 4. Chap. 2.) . . . }
- 45. And because, (as is evident by Fig. 1.) } AC + AE = EC.
- 46. Therefore from 45°, by drawing } $\square AC, AH + \square AE, AH = \square EC, AH$.
- AH into each part, . . . }
- 47. Therefore from 44° and 46°, (per } $\square ED, AC \sqsubset \square AC, AH + \square AE, AH$.
- Ax. 4. Chap. 2.) . . . }
- 48. From 36° it follows (per prop. 16. } $\square ED, M = \square AC, AH$.
- Elem. 6.) that . . . }
- 49. Likewise from 37°, that . . . } $\square ED, Q = \square AE, AH$.
- 50. And by adding the Equation in 49° } $\square ED, M + \square ED, Q = \square AC, AH + \square AE, AH$
- to that in 48°, it makes . . . }
- 51. Therefore from 47° and 50°, (per } $\square ED, AC \sqsubset \square ED, M + \square ED, Q$.
- Ax. 4. Chap. 2.) . . . }
- 52. Therefore from 51°, by casting a- } AC \sqsubset M + Q.
- way the common altitude ED, . . . }

53. And

53. And from 52° , by taking in the }
 common altitude AC, } $\square AC \sqsubset \square AC, M + \square AC, Q.$
 54. But from 38° it follows (per prop. 16. }
Elem. 6.) that } $\square AE, M = \square AC, Q.$
 55. Therefore from 53° and 54° , (per }
Ax. 6. Chap. 2.) } $\square AC \sqsubset \square AC, M + \square AE, M.$
 56. And from 55° , by subtracting \square }
 AC, M from each part, } $\square AC - \square AC, M \sqsubset \square AE, M.$
 57. And by adding $\frac{1}{4} \square M$ to each part }
 in 56° , } $\square AC + \frac{1}{4} \square M - \square AC, M \sqsubset \square AE, M + \frac{1}{4} \square M.$
 58. From 52° 'tis evident, that . . . }
 $AC \sqsubset \frac{1}{2} M.$
 59. And by *Theor. 5. Chap. 4.* }
 $\square : AC - \frac{1}{2} M : = \square AC + \frac{1}{4} \square M - \square AC, M.$
 60. Therefore from 57° and 59° , (per }
Ax. 4. Chap. 2.) } $\square : AC - \frac{1}{2} M : \sqsubset \square AE, M + \frac{1}{4} \square M.$
 61. And because if one Plane exceeds }
 another, the side of a Square equal to }
 the former shall exceed the side of a }
 Square equal to the latter, therefore }
 from 60° , } $AC - \frac{1}{2} M \sqsubset \sqrt{\square AE, M + \frac{1}{4} \square M} :$
 62. Therefore from 61° , by adding $\frac{1}{2} M$ }
 to each part, } $AC \sqsubset \frac{1}{2} M + \sqrt{\square AE, M + \frac{1}{4} \square M} :$
 Which was to be Dem. The like Demonstration may be applied to *Fig. 2*, and *3*. after
 N is set in the place of B in *Fig. 2*. and C in the place of P in *Fig. 3*.

The Composition of the preceding Probl. 18.



Suppos.

63. $\triangle ABC$ is given.
 64. AH and HB are in a given Reason.
 65. D is a point given without the $\triangle ABC$.
 66. DE is \parallel AB, and given.
 67. AE the Increase of CA continued until it cut DE is given.

Req. to find

68. AK such a segment of AC, that DK being drawn, it may make
 69. $\triangle ALK \cdot LKCB :: AH \cdot HB.$

Construction.

70. Supposing (by what hath been said in $1^\circ, 2^\circ, 3^\circ, 4^\circ, 5^\circ, 33^\circ$ and 34° ;) it be discovered, that the line of partition to be drawn from D must cut the sides AB and AC; draw DE parallel to AB, and cutting CA continued in E.
 71. Then by *Probl. 8. Chap. 5.* let it be made; as ED to }
 AC, so AH to a fourth proportional line M, therefore } $ED \cdot AC :: AH \cdot M.$
 72. Find a mean proportional line, as S, between M and }
 AE, therefore } $M \cdot S :: S \cdot AE.$
 73. Then esteeming the line S to be the mean of three Proportionals, and the line M the difference of the extremes, find out the extremes, (per *Probl. 12. Chap. 5.*) the greater whereof suppose to be the line R, whence the lesser shall be equal to $R - M$, therefore these are Proportionals, viz. }
 $R - M \cdot S :: S \cdot R.$
 That is, in 30° , the last step of the Resolution, . . . } $a - m \cdot \sqrt{mg} :: \sqrt{mg} \cdot a.$

74. From AC cut off AK = R, which may be done, for that AC is greater than R, I prove thus;

By Constr. in 73°, } R - M . S :: S . R.
 Therefore by the Theorem in 24° of Prob. 12. }
 Chap. 5. } $R = \frac{1}{2}M + \sqrt{\square S + \frac{1}{4}\square M}$:
 And because from 72°, (per prop. 17. Elem. 6.) } $\square AE, M = \square S$.
 Therefore from the two last steps, (per Ax. 6. }
 Chap. 2.) } $R = \frac{1}{2}M + \sqrt{\square AE, M + \frac{1}{4}\square M}$:
 But it hath been shewn in 62°, that } $AC < \frac{1}{2}M + \sqrt{\square AE, M + \frac{1}{4}\square M}$:
 Therefore from the two last preceding steps, }
 (per Ax. 3. Chap. 2.) } AC < R. Which was to be Dem.

75. Lastly, draw the line DK, cutting AB in L; then shall the Triangle ALK be to the Trapezium LKCB, as AH to HB; which was required. The truth whereof will be made manifest by the following Demonstration, formed out of the foregoing Resolution by a repetition of its steps in a backward order.

76. . . . Req. demonstr. $\triangle ALK . LKCB :: AH . HB$.

Demonstration.

77. By Constr. in 73°, } R - M . S :: S . R.
 That is, in 30°, (the last step of the Resolution,) } $a - m . \sqrt{mg} :: \sqrt{mg} . a$.
 78. Therefore from 77°, (per prop. 17. Elem. 6.) } $\square R - \square MR = \square S$.
 79. And because by Constr. in 74°, } AK = R.
 80. And from the Constr. in 72°, (per prop. 17. }
 Elem. 6.) } $\square M, AE = \square S$.
 81. Therefore from 78°, 79° and 80°, by ex- }
 changing equal quantities, } $\square AK - \square M, AK = \square M, AE$.
 That is, in 29°, } $.aa - ma = mg$.
 82. Therefore from 81°, by adding $\square M, AK$ }
 to each part, } $\square AK = \square M, AK + \square M, AE$.
 That is, in 28°, } $.aa = ma + mg$.
 83. Therefore from 82°, (per prop. 14. Elem. 6.) } $AK . M :: AK + AE . AK$.
 That is, in 27°, } $a . m :: a + g . a$.
 84. But because $\triangle EKD$ and $\triangle AKL$ are like, }
 (for by Constr. in 70°, ED || AL,) therefore } $ED . AL :: AK + AE . AK$.
 (per prop. 4. Elem. 6.) }
 That is, in 26°, } $b . \frac{ha}{a+g} :: a + g . a$.
 85. Therefore from 83° and 84°, (per prop. 11. }
 Elem. 5.) } $AK . M :: ED . AL$.
 That is, in 25°, } $a . m :: b . \frac{ha}{a+g} (= \frac{bc}{a})$.
 86. But by Constr. in 71°, } $M . AC :: AH . ED$.
 87. Therefore from 85° and 86°, (per prop. 23. }
 Elem. 5.) agreeable to Defn. 8. Chap. 3. con- } $AK . AC :: AH . AL$.
 cerning Inordinate proportion, }
 That is, in 18°, } $a . b :: c . \frac{bc}{a}$.
 88. And because $\triangle ALK$ and $\triangle ACH$ have a }
 common angle, to wit, $\angle KAL$, and (as ap- } $\triangle ALK = \triangle ACH$.
 pears by the Analogy in 87°,) the sides about }
 that angle are reciprocally proportional, there- }
 fore (per prop 15. Elem. 6.) }
 89. Therefore by subtracting $\triangle ALK$ and }
 $\triangle ACH$ severally from the $\triangle ABC$, the } $\text{Trapez. LKCB} = \triangle HCB$.
 remaining spaces shall be equal one to another, }
 viz. }
 90. But (per prop. 1. Elem. 6.) } $\triangle ACH . \triangle HCB :: AH . HB$.
 91. Therefore from 88°, 89° and 90°, by ex- }
 changing equal spaces, } $\triangle ALK . LKCB :: AH . AB$.

Which was to be demonstrated. Therefore that is done which the Problem required.

AHC, whence it evidently follows that $LKCB = \triangle HCB$, and $\triangle AKL$.
 $LKCB :: AH . HB$. These things being premised, the Resolution of the Problem
 propounded may be formed in manner following.

Suppos.

4. $\triangle ABC$ is given.
5. D is a point given within the $\triangle ABC$.
6. AH and HB are in a given Reason.
7. $b = AC$ is given.
8. $c = AH$ is given.
9. $g = AE$ is given.
10. $h = ED$ ($\parallel AB$) is given.

Req. to find

11. AK such a segment of AC, that KDL being drawn, it may make
12. $\triangle AKL . LKCB :: AH . HB :: \triangle ACH . \triangle HCB$.

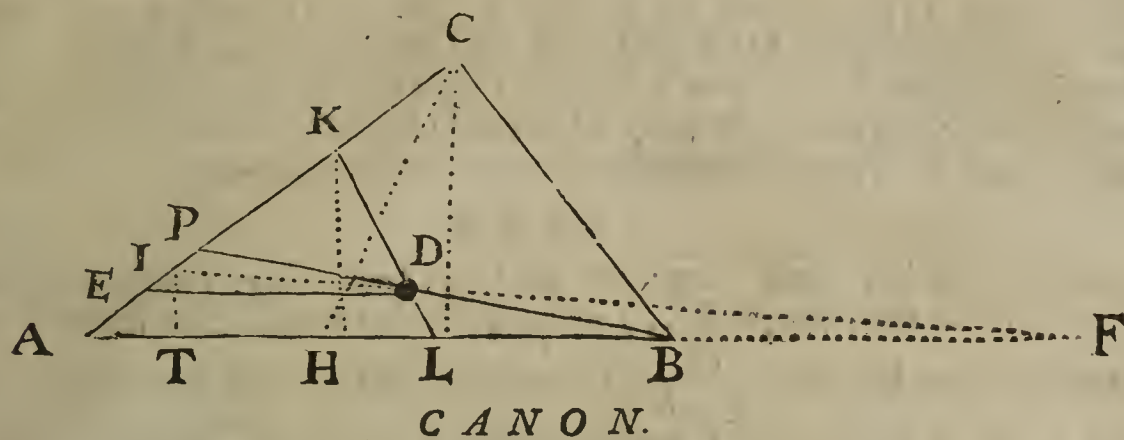
Resolution.

- | | |
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| <p>13. Suppose that done which is required, and put</p> <p>14. Then by considering well what is required, and viewing the Diagram, it will appear that $\triangle AKL = \triangle ACH$, and that the $\angle CAL$ is common to both those Triangles, therefore (<i>per prop. 15. El. 6.</i>)</p> <p>15. That is, in the letters of the Resolution;</p> <p>16. And because $\triangle AKL$ and $\triangle EKD$ are like, (for by <i>Constr.</i> in 1^o, $DE \parallel AL$), therefore (<i>per prop. 4. Elem. 6.</i>)</p> <p>17. That is, in the letters of the Resolution;</p> <p>18. And because the fourth Proportional in 14^o is the same with the fourth in 16^o, therefore the fourth Proportionals in 15^o and 17^o; shall be equal to one another, <i>viz.</i></p> <p>19. Now to avoid an Equation between Solids, let it be made as b to b, so c to a fourth Proportional, which may be called m, therefore</p> <p>20. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, <i>viz.</i></p> <p>21. Therefore from 18^o and 20^o, by exchanging equal Rectangles, 'tis evident that</p> <p>22. Whence 'tis easie to infer that these are Proportionals, <i>viz.</i></p> <p>23. But it hath been shewn in 17^o, that</p> <p>24. Therefore from 22^o and 23^o, (<i>per prop. 11. El. 5.</i>)</p> <p>25. And from 24^o, by comparing the Rectangle of the extremes to the Rectangle of the means,</p> <p>26. From which Equation, after due transposition, this ariseth, <i>viz.</i></p> <p>27. Which last Equation may be resolved into these Proportionals, <i>viz.</i></p> <p>28. Of which three Proportionals, the mean, to wit, \sqrt{mg} is given, as also m the summ of the extremes, therefore the extremes shall be given severally (<i>per Probl. 13. Chap. 5.</i>) the values whereof, by the Theorem in 21^o of the said <i>Probl. 13. Chap. 5.</i> will be found equal to these right lines, (or numbers,) <i>viz.</i></p> | <p>$a = AK$.</p> <p>$AK . AH :: AC . AL$.</p> <p>$a . c :: b . \frac{bc}{a}$.</p> <p>$EK . AK :: ED . AL$.</p> <p>$a - g . a :: h . \frac{ha}{a - g}$.</p> <p>$\frac{bc}{a} = \frac{ha}{a - g}$.</p> <p>$b . b :: c . m$.</p> <p>$bm = bc$.</p> <p>$\frac{bm}{a} = \frac{ha}{a - g} = \frac{bc}{a}$.</p> <p>$a . m :: b . \frac{ha}{a - g} (= \frac{bc}{a})$.</p> <p>$a - g . a :: h . \frac{ha}{a - g}$.</p> <p>$a . m :: a - g . a$.</p> <p>$aa = ma - mg$.</p> <p>$ma - aa = mg$.</p> <p>$m - a . \sqrt{mg} :: \sqrt{mg} . a$.</p> |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

$$\left\{ \begin{array}{l} \frac{1}{2}m + \sqrt{\frac{1}{4}mm - mg} : \\ \frac{1}{2}m - \sqrt{\frac{1}{4}mm - mg} : \end{array} \right\} \text{ the two Roots of the Equation in 26}^{\circ}.$$

Hence this

CANON.



Let it be made as ED to AC, so AH to a fourth Proportional, which may be called M. Then to and from the half of M, add and subtract the square Root of the excess of the Square of half M above the Rectangle of M into AE, so shall the sum and remainder made by that Addition and Subtraction be the two values of AK sought, (represented by *a* in the foregoing Resolution.)

29. But to the end the said values of AK may be Real, that is, greater than nothing, the given lines must be subject to this

Determination. g not $\leq \frac{1}{4}m$, or $\frac{bc}{4b}$; that is, in words,

The line AE must not exceed the right line arising by the Application of the Rectangle made of the lines AC and AH to the quadruple of the line ED.

The truth of this Determination, which is discovered both by the Analogy in 27°, and by the values of *a* in 28°, may be proved thus,

It hath been discovered in 27°, that these are Proportionals, viz. } $m - a \cdot \sqrt{mg} :: \sqrt{mg} \cdot a.$

Of which Proportionals the sum of the extremes is evidently *m*, and the mean is \sqrt{mg} ; therefore (as hath been shewn in 20° of *Probl. 13. Chap. 5.*) } \sqrt{mg} not $\leq \frac{1}{2}m.$

Whence, by squaring each part it follows, that } mg not $\leq \frac{1}{4}mm.$
And by Application of each part to *m*, } g not $\leq \frac{1}{4}m.$

But by *Constr.* in 19°, } $\frac{bc}{b} = m.$

And consequently, by taking $\frac{1}{4}$ of each part, } $\frac{bc}{4b} = \frac{1}{4}m.$

Therefore from the three last steps, } g not $\leq \frac{1}{4}m$, or $\frac{bc}{4b}.$

Which was to be Demonstr.

30. It is also easie to perceive by the two Roots or extreme Proportionals found out in 28°, that if $g = \frac{1}{4}m$, and consequently $mg = \frac{1}{4}mm$, then those Roots will be equal to one another, viz. each Root equal to $\frac{1}{2}m$, which, if it fall within the limits hereafter declared in the Rule in 31° of this Problem, shall be equal to the line AK sought; But if $g > \frac{1}{4}m$, then the said Roots will be unequal, and the Equation in 26° may be expounded by each of those Roots; and sometimes either of them may be taken for the value of the line AK sought, but sometimes only one of the said Roots, and sometimes neither of them. To discover therefore whether there be a possibility of effecting the Problem or not, by the lines given in such manner as before is exprest, the value of the line AP must be enquired; to which end, first, draw a right line that may pass by the angular point B and the given point D, (in the precedent Diagram,) then is EABD a Trapezium having (by *Constr.* in 1°,) two parallel sides AB and ED, and the other two sides AE and BD, which are not Parallels, being continued will meet in some point in AC, as in P. Now if AB, ED and AE be severally given in numbers, the line EP shall be also given in number, for putting $g = AE$, and $b = ED$, (as before in the Resolution,) also $k = AB$; the line EP (by the Theorem in 9° of *Probl. 18. Chap. 7.*) will be found equal to $\frac{gb}{k-b}$, to which adding *g*, that is, AE, it makes $\frac{gk}{k-b}$ for the value of AP.

It is

It is also evident, that if from any point in AC between E and P, a right line be drawn to pass by the given point D, as IDF, it shall necessarily cut the Base AB continued without the Triangle ABC, as in F; but if a right line be drawn from any point in AC between P and C to pass by the point D, as KDL, it shall necessarily cut the Base AB between A and B. From the premisses we may infer this following

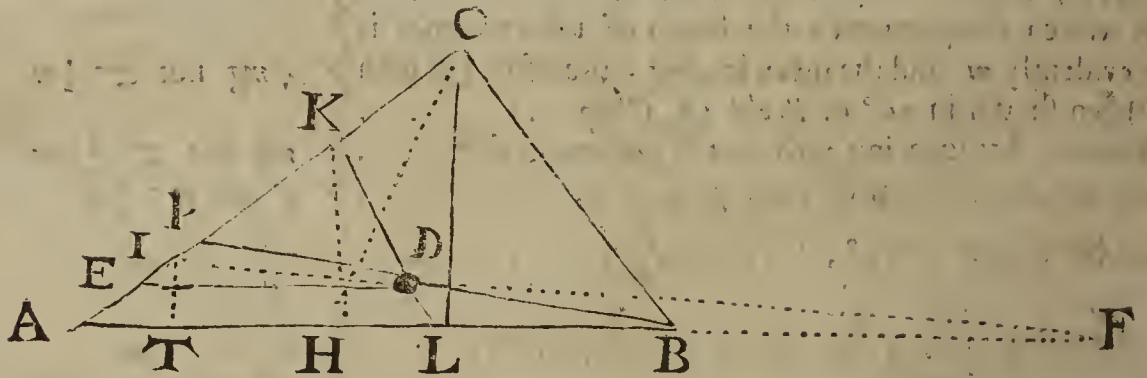
RULE.

31. First, if $\frac{1}{2}m + \sqrt{\frac{1}{4}mm - mg}$: and $\frac{1}{2}m - \sqrt{\frac{1}{4}mm - mg}$: (the Roots before found out in 28° for the values of AK,) be unequal, and each of them less than AC, but neither of them less than $\frac{gk}{k-h}$, (the value of AP,) then two right lines equal to those Roots or values of AK being set from A towards C will end in two points, from which if two right lines be drawn to pass by the given point D, each of them shall divide the given Triangle ABC into two parts in the given Reason.

Secondly, if only one of those Roots or values of AK happens to be less than AC, and not less than the said value of AP, then the given Triangle ABC can be divided only by one right line passing by the given point D, so as to cut both AB and AC, or AC only, to divide the said Triangle into two parts in the given Reason.

Thirdly and lastly, if neither of the said Roots or values of AK happens to be within the limits above express'd, then 'tis impossible to draw a right line that shall pass by the given point D, and cut both AB and AC, or AC only, and pass by the angular point B, to divide the given Triangle into two parts in the given Reason: And if the like impossibility happen after a Parallel is drawn by the given point D to each of the other two sides AC, BC, and tryal made as before, it shall be impossible to perform what the Problem requires; but when there is a possibility, then oftentimes which two sides you please may be cut by the line of partition.

The Composition of the foregoing Probl. 19.



Suppos.

- 32. $\triangle ABC$ is given.
- 33. AH and HB are in a given Reason.
- 34. D is a point given within the $\triangle ABC$.
- 35. DE ($\parallel AB$) is given.
- 36. AE is given, and not greater than $\frac{\square AC, AH}{4 ED}$, agreeable to the preceding Determination in 29°.

Req. to find

- 37. AK such a segment of AC, that KDL being drawn, it may make
- 38. $\triangle ALK \cdot LKCB :: AH \cdot HB :: \triangle ACH \cdot \triangle HCB$.

Construction.

- 39. By the given point D draw $DE \parallel AB$.
- 40. Then supposing it be discovered by the preceding Rule in 31°, that the line of partition which is to pass by the given point D may cut AB and AC, let it be made (by *Probl. 8. Chap. 5.*) as ED to AC, so AH to a fourth proportional line, which may be called M, therefore, $ED \cdot AC :: AH \cdot M$.
- 41. Find a mean proportional line, as S, between M and AE, therefore $M \cdot S :: S \cdot AE$.
- 42. Divide the line M (before found) into two such parts, that the line S may be a mean Proportional.

Proportional between the parts, which may be done (by *Probl. 14. Chap. 5.*) if S be not greater than $\frac{1}{2}M$; but that S is not greater than $\frac{1}{2}M$, I prove thus:

From the foregoing *Constr.* in 40° , it is easie to perceive that . . . } $M = \frac{\square AC, AH}{ED}$.

Whence, by taking $\frac{1}{4}$ of each part, . . . } $\frac{1}{4}M = \frac{\square AC, AH}{4ED}$.

By the Determination in 36° , . . . } $AE \text{ not } \subset \frac{\square AC, AH}{4ED}$.

Therefore from the two last steps, (*per Ax. 3. Chap. 2.*) } $AE \text{ not } \subset \frac{1}{4}M$.

Therefore by drawing M into each part, . . . } $\square M, AE \text{ not } \subset \frac{1}{4}\square M$.

And because from 41° , (*per prop. 17. Elem. 6.*) } $\square S = \square M, AE$.

Therefore from the two last preceding steps, (*per Ax. 4. Chap. 2.*) } $\square S \text{ not } \subset \frac{1}{4}\square M$.

But if one Square exceeds another, the side of the former exceeds also the side of the latter, therefore from the last step, } $S \text{ not } \subset \frac{1}{2}M$.

Which was to be Demonstr. Therefore 'tis possible to cut the line M into two such parts, that the line S may be a mean Proportional between them; suppose then the right line R be found the greater part, and T the lesser, therefore these are Proportionals, viz.

$$\begin{cases} M - R \cdot S :: S \cdot R. \\ M - T \cdot S :: S \cdot T. \end{cases}$$

That is, in 27° , . . . $m - a \cdot \sqrt{mg} :: \sqrt{mg} \cdot a$.

Which two lines R and T do answer to the two Roots or values of AK before express'd in the Canon in 28° , viz.

$$\begin{cases} R = \frac{1}{2}m + \sqrt{\frac{1}{4}mm - mg} \\ T = \frac{1}{2}m - \sqrt{\frac{1}{4}mm - mg} \end{cases} \text{ the values of AK.}$$

43. Now supposing that by the limits in 31° it be discovered that the greater Root or line R is less than AC, and not less than AP, let the line R be set from A towards C, as to K, viz. make $AK = R$, and draw the right line KDL, then shall $\triangle AKL$ be equal to $\triangle ACH$, and consequently $\triangle AKL \cdot LKCB :: AH \cdot HB :: \triangle ACH \cdot \triangle HCB$, as was required.

And if the lesser Root or line T happens to be less than AC, but not less than AP, let the line T be set also from A towards C, (as to I in the following Diagram belonging to the latter Example of this Problem,) and then a right line being drawn from the point in AC where T doth end, to pass by the given point D, it shall likewise divide the given Triangle ABC into two parts in the given Reason.

But if either of the said lines R and T, which are to be set (as before) from A towards C, happens to fall between E and P, then the line of partition to pass by D will cut the Base AB continued without the Triangle ABC; as, if AI be supposed equal to T, and the line ID be drawn and continued till it concurr with the Base AB continued, as in F, then although $\triangle AIF$ be equal to $\triangle ACH$, yet it solves not the Problem, in regard part of $\triangle AIF$ lyes without the given $\triangle ABC$.

It remains to prove that $\triangle ALK \cdot LKCB :: AH \cdot AB$, but this will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a backward (not direct) order.

44. . . . *Req. demonstr.* $\triangle ALK \cdot LKCB :: AH \cdot HB$.

Demonstration.

45. By *Constr.* in 42° , . . . } $M - R \cdot S :: S \cdot R$.

That is, in 27° , (the last step of the Resolution,) } $m - a \cdot \sqrt{mg} :: \sqrt{mg} \cdot a$.

46. Therefore from 45° , (*per prop. 17. Elem. 6.*) } $\square MR - \square R = \square S$.

47. And because by *Constr.* in 43° , . . . } $AK = R$.

48. And from the *Constr.* in 41° , (*per prop. 17. Elem. 6.*) } $\square M, AE = \square S$.

49. Therefore from 46° , 47° and 48° , by exchanging equal quantities, . . . } $\square M, AK - \square AK = \square M, AE$.

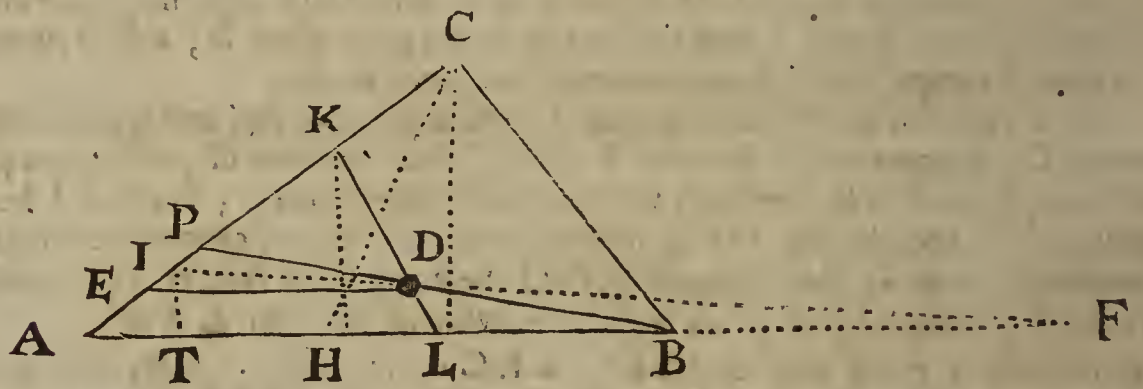
That is, in 26° , . . . } $ma - aa = mg$.

B b b

50. There-

50. Therefore from 49° , by adding $\square AK$,
and subtracting $\square M, AE$ from each part,
That is, in 25° , } $\square AK = \square M, AK - \square M, AE.$
51. Therefore by resolving the Equation in 50°
into Proportionals, } $. aa = ma - mg.$
That is, in 24° , } $AK \cdot M :: AK - AE \cdot AK.$
52. But because $\triangle EKD$ and $\triangle AKL$ are like,
(for by *Constr.* in 39° , $ED \parallel AL$,) therefore
(per prop. 4. *Elem.* 6.) } $a \cdot m :: a - g \cdot a.$
That is, in 23° , } $ED \cdot AL :: AK - AE \cdot AK.$
53. Therefore from 51° and 52° , (per prop. 11.
Elem. 5.) } $b \cdot \frac{ha}{a-g} :: a - g \cdot a.$
That is, in 22° , } $AK \cdot M :: ED \cdot AL.$
54. But by *Constr.* in 40° , } $a \cdot m :: b \cdot \frac{ha}{a-g} (= \frac{bc}{a}.)$
55. Therefore from 53° and 54° , (per prop. 23.
Elem. 5.) agreeable to *Defn.* 8. *Chap.* 3. . } $ED \cdot AC :: AH \cdot M.$
That is, in 15° , } $AK \cdot AC :: AH \cdot AL.$
56. And because $\triangle AKL$ and $\triangle ACH$ have a
common angle, to wit, $\angle KAL$, and (as ap-
pears by the Analogy in 55° ,) the sides about
that angle are reciprocally proportional, there-
fore (per prop. 15. *Elem.* 6.) } $a \cdot b :: c \cdot \frac{bc}{a}.$
57. Therefore by subtracting $\triangle AKL$ and
 $\triangle ACH$ severally from $\triangle ABC$, the re-
maining Spaces shall be equal one to another,
viz. } $\triangle AKL = \triangle ACH.$
58. But (per prop. 1. *Elem.* 6.) } Trapez. $LKCB = \triangle HCB.$
59. Therefore from 56° , 57° and 58° , by ex-
changing equal Spaces, } $\triangle ACH \cdot \triangle HCB :: AH \cdot HB.$
Which was to be demonstrated. Therefore that is done which the Problem required.
 $\triangle AKL \cdot LKCB :: AH \cdot HB.$

An Example in Numbers, to illustrate the precedent Resolution of Probl. 19. in which Example the greater of the two Roots or values of AK before express in 28° , is only capable of solving the Problem.



Suppos.

60. $AB = 231\frac{1}{4}$ the Base }
61. $AC = 185$ } of $\triangle ABC$ are given.
62. $BC = 138\frac{3}{4}$ } the legs }
63. D is a point given within the $\triangle ABC$.
64. $DE \parallel AB$; and $DE = 112$ is given.
65. $EA = 25$ is given.
66. $AH = 94\frac{8}{8}$ }
67. $HB = 136\frac{2}{8}$ } in a given Reason, *viz.* as 560 to 809.
68. CL, KH, IT are each $\perp AB$.

Req. to find in number,

69. AK, such, that the right line KDL being drawn, it may make
70. $\triangle AKL \cdot LKCB :: AH \cdot HB :: 560 \cdot 809.$

Solution

Solution Arithmetical.

71. $AK = 125 =$ the greater of the two Roots in 28° , found out by the Canon there exprest.

The Proof.

72. $AL = 140$; for $\left\{ \begin{array}{l} AK - AE . ED \\ 100 . 112 \end{array} \right. :: AK . AL .$
 73. $CL = 111$, found out by the three sides of $\triangle ABC$ given in $60^\circ, 61^\circ$ and 62° , with the help of *Theor. 4.* in 68° of *Probl. 8. Chap. 8.*
 74. $KH = 75$; for $\left\{ \begin{array}{l} AC . CL \\ 185 . 111 \end{array} \right. :: AK . KH .$
 75. $\frac{1}{2} AB \times CL = 1283\frac{1}{8}$ the Area of $\triangle ABC$.
 76. $\frac{1}{2} AL \times KH = 5250$ the Area of $\triangle AKL$.
 77. $\triangle ABC - \triangle AKL = 7584\frac{1}{8}$ the Area of $LKCB$.
 78. $\left\{ \begin{array}{l} \triangle AKL . LKCB \\ 5250 . 7584\frac{1}{8} \end{array} \right. :: AH . HB :: 560 . 809 .$
 $\left\{ \begin{array}{l} 5250 . 7584\frac{1}{8} \\ 941\frac{1}{48} . 1361\frac{27}{48} \end{array} \right. :: 560 . 809 .$
 Which was to be done.

79. *Note.* In this Example, AI the lesser of the two Roots or values of AK in 28° falls between E and P , and therefore it cannot solve the Problem; for if ID be drawn and continued, it will cut AB continued without the $\triangle ABC$, as in F . But 'tis worth observing, that the $\triangle AIF$ is equal to the $\triangle AKL$, as the following Proof will make manifest.

80. $AI = 31\frac{1}{4} =$ the lesser Root, found out by the Canon in 28° .

81. $EA = 25$, given in 65° .

82. $EI = 6\frac{1}{4} = AI - AE$.

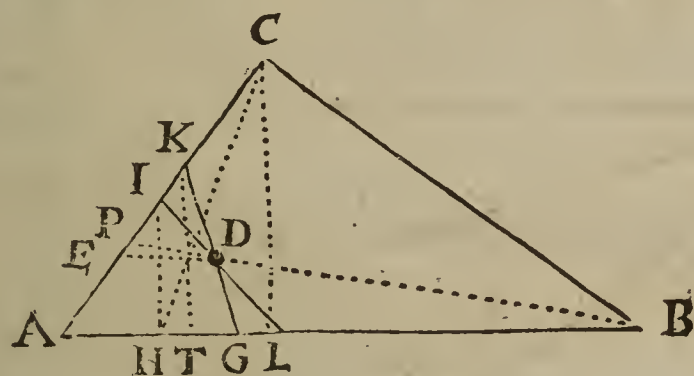
83. $DE = 112$, given in 64° .

84. $AF = 560$; for $\left\{ \begin{array}{l} EI . ED \\ 6\frac{1}{4} . 112 \end{array} \right. :: AI . AF .$

85. $IT = 18\frac{1}{4}$; for $\left\{ \begin{array}{l} AK . KH \\ 125 . 75 \end{array} \right. :: AI . IT .$

86. $\frac{1}{2} AF \times IT = 5250 = \frac{1}{2} AL \times KH$, therefore $\triangle AIF = \triangle AKL$.
 Which was to be proved.

*Another Example in Numbers, referring to the subsequent Figure, where each of the two Roots or values of AK before exprest in 28° is capable of solving the foregoing *Probl. 19.**



Suppos.

87. $AB = 231\frac{1}{4}$ the Base } of $\triangle ABC$ are given.
 88. $AC = 138\frac{1}{4}$ } the leggs }
 89. $BC = 185$ }
 90. D is a point given within the $\triangle ABC$.
 91. $DE \parallel AB$; and $DE = 361\frac{1}{11}$ is given.
 92. $EA = 38$ is given.
 93. $AH = 40$ } in a given Reason, viz. as 32 to 153.
 94. $HB = 191\frac{1}{4}$ }
 95. CL, KT, IH are each $\perp AB$.

Req. to find in numbers,

96. AK and AI , such, that the right lines KDG and IDL being drawn, these Analogies shall ensue, viz.

97. $\left\{ \begin{array}{l} \triangle AKG . GKCB \\ \triangle AIL . LICB \end{array} \right. :: AH . HB :: 32 . 153 .$ Also,
 $\left\{ \begin{array}{l} \triangle AKG . GKCB \\ \triangle AIL . LICB \end{array} \right. :: AH . HB :: 32 . 153 .$

line; measure likewise CE, a distance at pleasure, yet so, as CEB may be a straight line. Again, measure ED; also DF, that is, a distance directly towards B, until you come to the point F, where DB and EA cut one another; measure also FE.

Secondly, in solving this Problem Arithmetically according to the following Resolution, by the help of those five lines measured as above is directed, these are two principal Cases, viz. first, when the angle ACB is acute; secondly, when 'tis obtuse; and in regard the three sides of the Triangle CDE are supposed (as above) to be severally given in numbers, we may (by the Corollary in 45° of *Probl. 10. Chap. 7.*) discover the kind of every one of the angles of the said Triangle. First then, suppose it be found that the angles DCE and CED in the Diagram before express are each of them acute, then if a Perpendicular be let fall from the angle CDE, as DI, upon the Base CE, it will necessarily fall within the said Triangle CDE; likewise if the angle DCE be acute, and the angle CED obtuse, then a Perpendicular from E will fall within the said Triangle upon the opposite side CD; but if the angle DCE be obtuse, as in the following Diagram belonging to the latter Example of this Problem 'tis supposed to be,) then a Perpendicular from D or E will fall without the Triangle DCE.

Thirdly, let a Perpendicular be supposed to fall from E upon DB, as EG; this Perpendicular shall be less than the Perpendicular DI, for EG is manifestly less than the Perpendicular IK, which is less than ID the Hypothensal of the right-angled Triangle IKD.

Fourthly, supposing a Perpendicular to fall from A upon CB produced infinitely, it will either fall upon the point B, or else within or without the Triangle CBA; but in this Example I shall suppose that Perpendicular to be AH, falling within the Triangle CBA. Now by the help of those five lines CD, CE, ED, DF, FE measured in Feet, or what equal parts you please, the length of the inaccessible distance AB may be found out in the same kind of parts, in manner following.

Suppos.

- 1. $b = CE = 152$
- 2. $c = CD = 210$
- 3. $d = DF = 90$
- 4. $f = FE = 100$
- 5. $g = DE = 170$
- 6. $p = DI = 168$
- 7. $k = IE = 26$
- 8. $l = IC = 126$
- 9. $n = EG = 80$

Lines given.

These are consequently given, for they may be found out by the help of the given sides of $\triangle CED$ and $\triangle DFE$, (*per Theor. in 29° and 30° of Probl. 9. Chap. 7.*)

I. *Req.* to find EB and FB.

Resolution.

- 10. Put } $a = EB.$
- 11. Then it follows from 7° and 10°, that . . . } $a + k = IB.$
- 12. The Square of the last Equation gives . . . } $aa + 2ka + kk = \square IB.$
- 13. To which Square adding the Square of p , that is, } $pp = \square DI.$
- 14. The sum makes } $aa + 2ka + kk + pp = \square DB.$
- 15. And because in $\triangle EID$ right-angled at I, . . } $gg = kk + pp = \square DE.$
- 16. Therefore from 14° and 15°, (*per Ax. 6. Ch. 2.*) } $aa + 2ka + gg = \square DB.$
- 17. And by extracting the square Root of the last } $\sqrt{aa + 2ka + gg} = DB.$
Equation, this ariseth,
- 18. By *Supposition* the angle CED is acute, and } $\square DI, EB = \square EG, DB (= 2\triangle DEB)$
consequently (*per Coroll prop. 13. Elem. 1.*) the angle }
DEB is obtuse, therefore (*per prop. 41. Elem. 1.*) }
- 19. That is, in the letters of the Resolution, . . } $pa = n \times \sqrt{aa + 2ka + gg}:$
- 20. Which last preceding Equation may be resolved } $p \cdot n :: \sqrt{aa + 2ka + gg} : a.$
into these Proportionals, viz. }
- 21. The Squares of which Proportionals are also } $pp \cdot nn :: aa + 2ka + gg \cdot aa.$
Proportionals, viz. }
- 22. Now to avoid an Equation between Solids; to } $p \cdot n :: n \cdot q.$
 p and n find a third Proportional, call it q , therefore }

23. There-

- 23. Therefore from the last Analogy this will }
 arise, (per Coroll. of 20. prop. 6. Elem.) . . . } $pp : nn :: p \cdot q.$
- 24. Therefore from the Analogies in 21° and }
 23° this is manifest, (per 11. prop. 5. Elem.) } $p \cdot q :: aa + 2ka + gg \cdot aa.$
- 25. And from 24°, by Division of Reason, . } $p - q \cdot q :: 2ka + gg \cdot aa.$
- 26. Then unto $2k$ and g find a third Proportio- }
 nal, which may be called s , therefore . . . } $2k \cdot g :: g \cdot s.$
- 27. Therefore by resolving that Analogy into an }
 Equation, it gives } $2ks = gg.$
- 28. Suppose } $r = p - q.$
- 29. Then from 25°, 27° and 28°, by exchange }
 of equal quantities, } $r \cdot q :: 2ka + 2ks \cdot aa.$
- 30. Let it be made } $r \cdot q :: 2k \cdot (to\ a\ fourth)\ t.$
- 31. Then from 29° and 30°, it's manifest (per }
 11. prop. 5. Elem.) that } $2k \cdot t :: 2ka + 2ks \cdot aa.$
- 32. And by drawing $a + s$ into $2k$ and t seve- }
 rally, this Analogy is manifest, } $2k \cdot t :: 2ka + 2ks \cdot ta + ts.$
- 33. Therefore from 31° and 32°, (per 11. prop. }
 5. Elem.) } $2ka + 2ks \cdot aa :: 2ka + 2ks \cdot ta + ts.$
- 34. In which last Analogy, the first Term is }
 equal to the third, therefore the second shall }
 be equal to the fourth, viz. } $aa = ta + ts.$
- 35. Therefore by subtracting ta from each part }
 of the last Equation, } $aa - ta = ts.$
- 36. Therefore (by the Canon in 43° of *Probl. 12.* }
Chap. 5.) } $a = \frac{1}{2}t + \sqrt{\frac{1}{4}tt + ts} = EB.$

From 22°, 26°, 28°, 30° and 36°, 'tis easie to deduce this following

CANON 1.

- 37. First, to p and n , (that is, DI and EG,) find }
 a third Proportional, which may be called q ; }
 therefore, } $p \cdot n :: n \cdot q (= \frac{200}{21}.)$
- Secondly, unto $2k$ and g , (that is, 2IE and DE,) }
 find a third Proportional, which may be called s ; }
 therefore, } $2k \cdot g :: g \cdot s (= \frac{2221}{13}.)$
- Thirdly, make } $r = p - q (= \frac{2128}{21}.)$
- Fourthly, let it be made as r to q , so $2k$ to a fourth }
 Proportional, which may be called t ; therefore, } $r \cdot q :: 2k \cdot t (= \frac{1200}{341}.)$
- Fifthly and lastly, by the Canon in 43° of }
Probl. 12. Ch. 5. the quantity of EB (represented }
 by a in the Resolution) will be made known, viz. } $\frac{1}{2}t + \sqrt{\frac{1}{4}tt + ts} = 100 = EB = a.$

The Demonstration of the said Canon is manifest by the preceding Resolution, which is argued Geometrically as well as Arithmetically, and no quantity in any step thereof exceeds the dimensions of a Plane. Moreover, by converting the Analogy in 21° into an Equation according to the vulgar way, the following Canon will arise, which is the same in substance with the former, but not so apt for Demonstration.

CANON 2.

- 38. Multiply k , and the Square of g , severally by the Square of n , and divide each Product by the excess of the Square of p , above the Square of n ; then add the latter Quotient to the Square of the first, and extract the square Root of the sum; lastly, the said square Root added to the first Quotient will give the same value of a , (that is, $100 = EB$) as before.
- 39. Now in order to find FB, first, . . . } $IE + EB = IB = 126.$
- 40. Then } $\sqrt{\square IB + \square ID} = DB = 210.$
- 41. And } $DB - DF = FB = 120.$

II. To find DA.

- 42. The lines CD, DB, BC, DF, CE being given or found out, (as before,) the length of DA may be discovered by the Canon in 17° of *Probl. 19. Chap. 7. viz.*
 First,

First, . . . $DF \cdot DB :: CE \cdot (to\ a\ fourth)\ M \cdot (= 354\frac{2}{3}.)$
 Then, . . . $M - CB \cdot EB :: CD \cdot DA (= 204\frac{2}{11}.)$

III. To find AH and CH.

43. By *Supposition* AH is perpendicular to CH, and each of them may be found out thus,

$$CD \cdot CA :: \begin{cases} DI \cdot AH (= 331\frac{2}{11}.) \\ CI \cdot CH (= 248\frac{2}{11}.) \end{cases}$$

IV. To find AB the distance required.

44. Because in this Example CH happens to be less than CB, (that is, $CE + EB$,) subtract CH from CB, and the remainder is HB, viz.

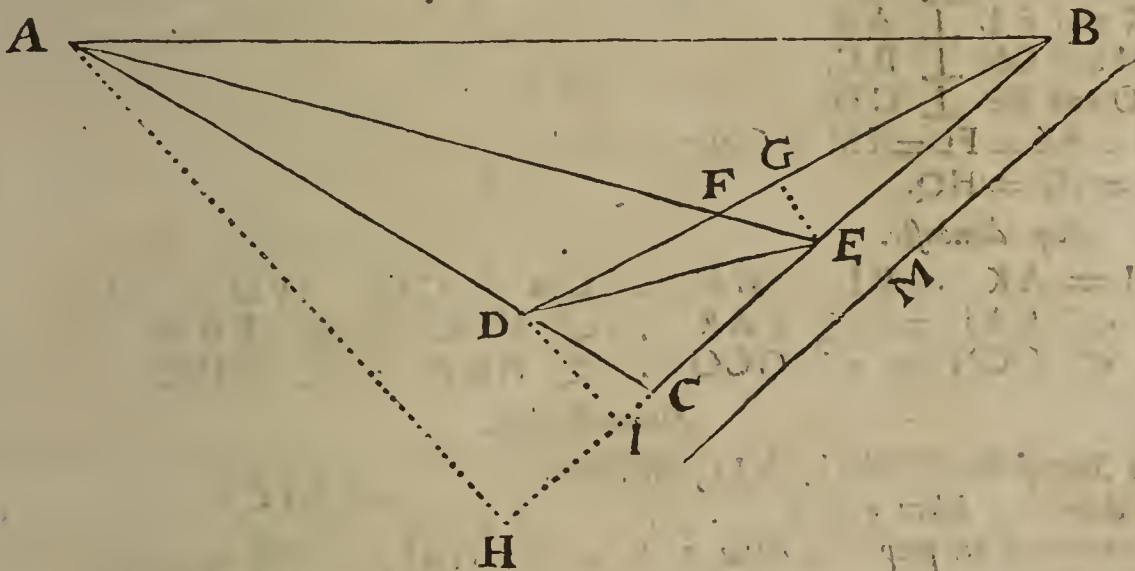
$$CB - CH = HB = 3\frac{2}{11}.$$

Then in $\triangle AHB$ right-angled at H,

$$\sqrt{\square AH + \square HB} = AB = \frac{\sqrt{13309200}}{11}, \text{ that is, } 331,652, \&c.$$

45. *Note.* When CH happens to exceed CB, then the Perpendicular AH will fall without the $\triangle CAB$, upon CB produced; in which Case CB is to be subtracted from CH, and then the Square of the remainder being added to the Square of the Perpendicular AH, the square Root of the sum shall be the desired distance AB. And lastly, when CH happens to be found equal to CB, then the Perpendicular AH is the same with the distance AB.

The Arithmetical Solution of Case 2. Probl. 20. viz. when the angle ACB is obtuse.



- Suppos.*
- $b = CE = 63\frac{2}{3}.$
 - $c = CD = 41\frac{2}{3}.$
 - $d = DF = 60\frac{1}{169}.$
 - $f = FE = 29\frac{1}{169}.$
 - $g = DE = 85.$
 - $p = DI = 40.$
 - $k = IE = 75.$
 - $l = CI = 11\frac{2}{3}.$
 - $n = EG = 20\frac{4}{3}.$
46. $\left. \begin{matrix} b = CE = 63\frac{2}{3}. \\ c = CD = 41\frac{2}{3}. \\ d = DF = 60\frac{1}{169}. \\ f = FE = 29\frac{1}{169}. \\ g = DE = 85. \\ p = DI = 40. \\ k = IE = 75. \\ l = CI = 11\frac{2}{3}. \\ n = EG = 20\frac{4}{3}. \end{matrix} \right\}$ Lines given.
- $\left. \begin{matrix} p = DI = 40. \\ k = IE = 75. \\ l = CI = 11\frac{2}{3}. \\ n = EG = 20\frac{4}{3}. \end{matrix} \right\}$ These are consequently given, for they may be found out by the help of the given sides of $\triangle CED$ and $\triangle DFE$, (*per Theor. 2. in 31° and 32° of Probl. 10. Chap. 7.*)
- Req. to find AB.

Operation Arithmetical.

- First, by *Canon 2.* in 38° of the preceding Resolution you will find by the help of the respective numbers given in 46° , . . . $EB = 82\frac{1}{2}.$
- Secondly, . . . $IE + EB = IB = 157\frac{1}{2}.$
- Thirdly, in $\triangle DIB$ right-angled at I, . . . $\sqrt{\square IB + \square ID} = DB = 162\frac{1}{2}.$
- Fourthly, (by the *Canon* in 17° of *Probl. 19. Chap. 7.*) let it be made,
- $\left. \begin{matrix} DF \cdot DB :: CE \cdot (to\ a\ fourth,) \\ M - CB \cdot EB :: CD \cdot (to\ a\ fourth,) \end{matrix} \right\}$ $M = 169\frac{1}{30807};$ then
 - $\left. \begin{matrix} M - CB \cdot EB :: CD \cdot (to\ a\ fourth,) \\ CD + DA = CA = 187\frac{4}{34809}.$ then
- Fifthly, . . . $CD + DA = CA = 187\frac{4}{34809}.$
- Sixthly,

Sixthly, because $\triangle CDI$ and $\triangle CAH$ are like, these are Proportionals, viz.

$$\begin{cases} CD : CI :: CA : CH (= 52 \frac{26692}{174045}.) \\ CD : DI :: CA : AH (= 180 \frac{10248}{38015}.) \end{cases}$$

Seventhly, $CH + CB = HB = 198 \frac{42061}{116030}.$

Lastly, in the $\triangle AHB$ right-angled at H , $\sqrt{\square MB + \square HA} = AB = 268 \frac{1000}{1000}, \text{ \&c.}$

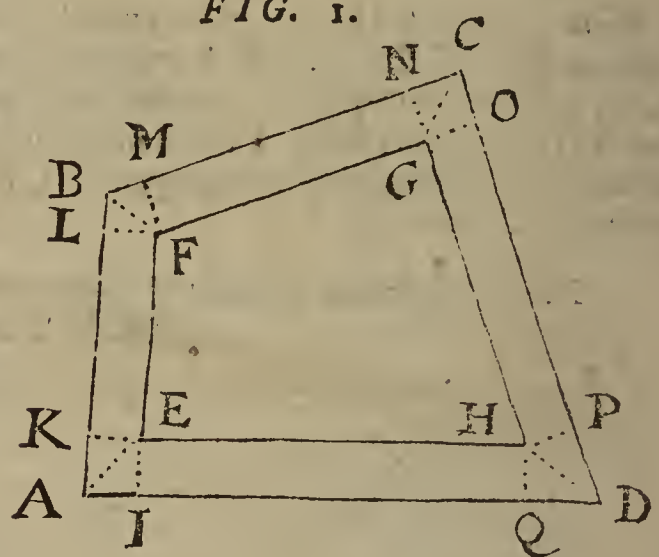
Note. When the angle ACB happens to be a right angle, then after EB and DA are found out in like manner as before in 38° and 42° of this Problem, there will be given CA and CB the sides about the right angle ACB of $\triangle ACB$, therefore (per prop. 47. Elem. 1.) the square Root of the summ of the Squares of those sides shall be the quantity of the Hypothensal AB , to wit, the distance sought.

LEMMA 1. leading to Probl. 21.

Suppos.

1. $ABCD$ is a Trapezium.
2. $EFGH$ is a Trapezium.
3. $EF \parallel AB$.
4. $FG \parallel BC$.
5. $GH \parallel CD$.
6. $EH \parallel AD$.
7. AE, BF, CG, DH are right lines.
8. EI and $HQ \perp AD$.
9. EK and $FL \perp AB$.
10. FM and $GN \perp BC$.
11. GO and $HP \perp CD$.
12. $EI = EK = FL = FM = GN = GO = HP = HQ.$

FIG. 1.



Req. demonstr.

13. $AI = AK \cdot BL = BM \cdot CN = CO \cdot PD = DQ.$
14. $\begin{cases} \angle EAI = \angle EAK \\ \angle GCN = \angle GCO \\ \angle FBL = \angle FBM \\ \angle HDP = \angle HDQ \end{cases}$

Demonstration.

15. By *Suppos.* in $8^\circ EI \perp AD$, therefore } ... $\angle AIE = \perp.$
(per defin. 10. Elem. 1.)
16. Therefore (per prop. 47. Elem. 1.) } $\square AI + \square IE = \square AE.$
17. Likewise, } $\square AK + \square KE = \square AE.$
18. Therefore from 16° and 17° , (per Ax. 1. } $\square AI + \square IE = \square AK + \square KE.$
Chap. 2.)
19. By *Suppos.* in $12^\circ, IE = KE$, and consequently } ... $\square IE = \square KE.$
20. Therefore from 18° and 19° , (per Ax. 9. } ... $\square AI = \square AK.$
Chap. 2.)
21. Therefore from 20° , (per Schol. prop. 48. } ... $AI = AK.$
Elem. 1.)

Which was to be Demonstr.

22. Again, because AE is common to the Triangles AIE and AKE , and the other sides of those Triangles are correspondently equal to one another, (as appears in 12° and 21°), therefore (per prop. 8. Elem. 1.) } $\angle EAI = \angle EAK.$

Which was also to be Demonstr.

And by the like argumentation the truth of the rest of the Equations in 13° and 14° may be demonstrated.

LEMMA 2.

Let the same things be supposed here as before in Lemma 1. then, (respect being had to Fig. 1.)

Req.

1. . . . Req. demonstr. $\left\{ \begin{array}{l} \text{As Radius is to the summ of the Tangents of} \\ \text{So EI (or EK) to AI} \text{---} \text{LB} \text{---} \text{NC} \text{---} \text{PD.} \end{array} \right. \left. \begin{array}{l} \text{A EI,} \\ \text{B FL,} \\ \text{C GN,} \\ \text{D HP;} \end{array} \right.$

Demonstration.

2. By Supposition in 12° of Lemma 1. . . . EI = FL = GN = HP.

3. Therefore these four following Analogies will be manifest by a vulgar Axiom in the Doctrine of plain Triangles, viz.

- Radius . EI :: Tangent \angle AEI . AI.
- Radius . FL, or EI :: Tangent \angle BFL . LB.
- Radius . GN, or EI :: Tangent \angle CGN . NC.
- Radius . HP, or EI :: Tangent \angle DHP . PD.

4. And from those four Analogies this that follows is deducible, (per Schol. prop. 12. Elem. 5.) viz.

As 4 Rad.
 To 4 EI;

So is the summ of the Tangents of these angles, to wit, $\left. \begin{array}{l} \text{A EI,} \\ \text{B FL,} \\ \text{C GN,} \\ \text{D HP;} \end{array} \right\}$

To AI + LB + NC + PD.

5. But (by prop. 15. Elem. 5.) . . . 4 Rad. 4 EI :: Rad. EI.

6. Therefore from the two last preceding Analogies in 4° and 5° this ariseth, (per prop. 11. Elem. 5.) viz.

As Rad.
 To EI;

So is the summ of the Tangents of $\left. \begin{array}{l} \text{A EI,} \\ \text{B FL,} \\ \text{C GN,} \\ \text{D HP;} \end{array} \right\}$

To AI + LB + NC + PD.

7. Therefore alternately, As Radius is to the summ of the Tangents of $\left. \begin{array}{l} \text{A EI,} \\ \text{B FL,} \\ \text{C GN,} \\ \text{D HP;} \end{array} \right\}$
 So EI to AI + LB + NC + PD. $\left. \begin{array}{l} \text{A EI,} \\ \text{B FL,} \\ \text{C GN,} \\ \text{D HP;} \end{array} \right\}$

Which was to be Demonstr.

LEMMA 3.

Let the same things be supposed here as before in Lemma 1. then (respect being had to Fig. 1.)

Req. demonstr.

$$1. \left\{ \begin{array}{l} \square EI \times \overline{AD - AI} \text{---} \square EI \times \overline{AB - LB} \text{---} \\ \square EI \times \overline{BC - NC} \text{---} \square EI \times \overline{CD - PD} \end{array} \right\} = \text{Space AEFBCGHDA.}$$

That is, in words,

The Rectangle made of the parallel distance EI, (= EK,) and the excess by which the summ of the four sides AD, AB, BC, CD of the Trapezium ABCD, exceeds the summ of the four segments AI, LB, NC, PD, is equal to the Interval or Space AEFBCGHDA.

Demonstration.

2. By Suppos. in 8° and 12° of Lemma 1. > EI and HQ \perp IQ. Also, EI = HQ.

3. Therefore (per prop. 27, 33, 34. Elem. 1.) > EH = IQ.

4. It is evident by Fig. and Lemma 1. respect being had to the last Equation, that > IQ = EH = AD - AI - QD (PD.)

5. And by adding AD to each part of the last Equation, > AD + EH = 2AD - AI - QD (PD.)

6. And by arguing as in 4° and 5°, this Equation will be manifest, viz. > AB + EF = 2AB - AK (AI) - LB.

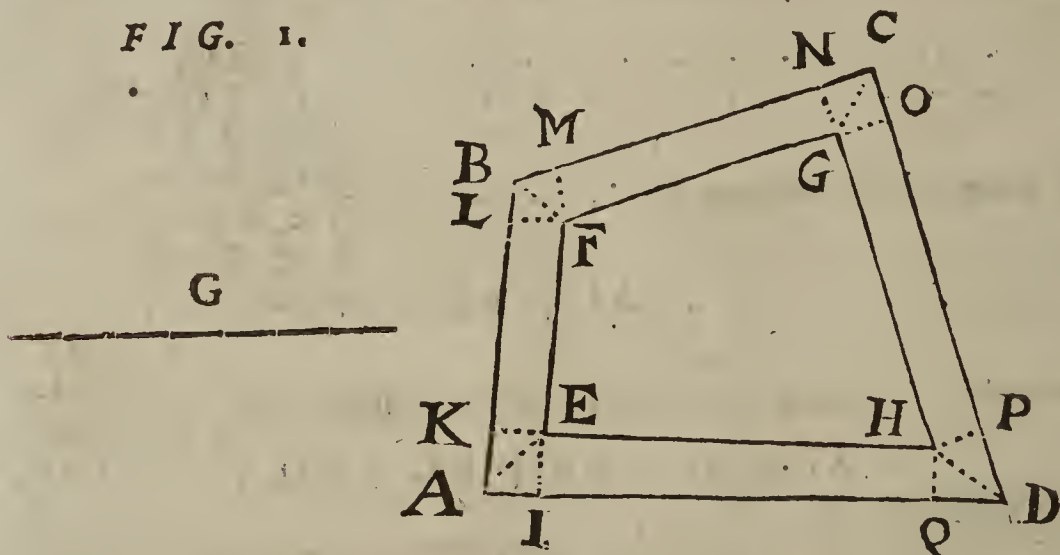
C c c

7. Likewise,

7. Likewise, } $BC + FG = 2BC - BM(LB) - NC.$
 8. Likewise, } $CD + GH = 2CD - CO(NC) - PD(QD.)$
 9. And by comparing the half summ of } $\frac{1}{2}AD + \frac{1}{2}EH +$ } $\left\{ \begin{array}{l} AD - AI + \\ AB - LB + \\ BC - NC + \\ CD - PD. \end{array} \right.$ }
 the first parts of the four last preceding }
 Equations to the half summ of the latter }
 parts, this Equation arifeth; viz. }
 10. By Suppos. the Trapezium AEHD } $\square EI \times \frac{1}{2}AD + \frac{1}{2}EH = \text{Trapez. AEHD.}$ }
 hath two parallel sides AD and EH, }
 and EI \perp AD, therefore (per Theor. 2. }
 in 13° of Probl. 18. Chap. 7.) . . . }
 11. In like manner; (respect being had to } $\square EI \times \frac{1}{2}AB + \frac{1}{2}EF = \text{Trapez. AEFB.}$ }
 the Suppos. in 12° of Lemma 1.) . . . }
 12. Likewise, } $\square EI \times \frac{1}{2}BC + \frac{1}{2}FG = \text{Trapez. BFGC.}$ }
 13. Likewise, } $\square EI \times \frac{1}{2}CD + \frac{1}{2}GH = \text{Trapez. DHGC.}$ }
 14. By viewing Fig. 1, it will be evident, that the summ of the four Trapezia exprest in }
 the four last preceding Equations is equal to the Interval or Space AEFBCGHDA, }
 therefore from 9°, 10°, 11°, 12° and 13°, by exchanging equal quantities this Equation }
 arifeth, viz. }
 $\square EI \times AD - AI + \square EI \times AB - LB +$ } $= \text{Space AEFBCGHDA.}$ }
 $\square EI \times BC - NC + \square EI \times CD - PD$ }
 Which was to be Demonstr.

LEMMA 4.

FIG. 1.



Let the same things be suppos'd here as in Lemma 1. and let the line G be found out by this Analogy, viz.

1. . . . } As the summ of the Tangents of } $\left\{ \begin{array}{l} AEI, \\ BFL, \\ CGN, \\ DHP, \\ \text{Radius;} \end{array} \right.$
 To } $AB + BC + CD + DA,$
 So }
 To a fourth Proportional } $G.$ Then
 2. . . . Req. demonstr. } $\frac{1}{2}G \square EI, \text{ (the parallel distance.)}$

Demonstration.

3. By inverting the Terms of the Analogy demonstrated in 7° of the foregoing Lemma 2. these are Proportionals, viz.

4. . . . } As the summ of the Tangents of } $\left\{ \begin{array}{l} AEI, \\ BFL, \\ CGN, \\ DHP, \\ \text{Radius;} \end{array} \right.$
 To } $AI + LB + NC + PD,$
 So is } $EI.$
 To the parallel distance }
 5. Therefore

5. Therefore from 1° and 4°, (per prop. 11. Elem. 5.)

$$AB + BC + CD + DA . G :: AI + LB + NC + PD . EI.$$

6. But by prop. 15. Elem. 5.

$$AB + BC + CD + DA . G :: \frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA . \frac{1}{2}G.$$

7. Therefore from 5° and 6°, (per prop. 11. Elem. 5.)

$$\frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA . \frac{1}{2}G :: AI + LB + NC + PD . EI.$$

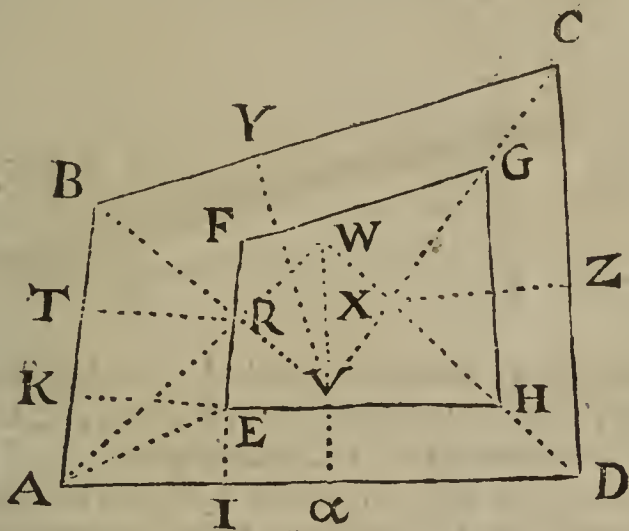
8. In which last Analogy the first Term is greater than the third; for 'tis evident by Fig. 1. and by what hath been demonstrated in Lemma 1. that the double of the first Term is greater than the double of the third, in regard the double of the third is but part of the double of the first: Therefore (per prop. 14. Elem. 5.) the second Term is greater than the fourth, viz.

$$\frac{1}{2}G \sqsupset EI \text{ (the parallel distance.)}$$

Which was to be Demonstr.

L E M M A 5.

FIG. 2.



If the angles A, B, C, D of the Trapezium ABCD be severally divided into two equal parts by the right lines AW, DW, BV, CV; and if from the points W, R, V and X where every two of the said lines that lye next to one another do intersect, four right lines Wα, RT, VY and XZ be drawn perpendicularly upon AD, AB, BC and CD; and if the Perpendicular RT be shorter, or, if not shorter, yet not longer than any one of the other three Perpendiculars VY, XZ and Wα: And lastly, if a Trapezium be made within the before-mentioned Trapezium ABCD, so, as that the sides of the one are parallel to the sides of the other, and every where separated by an equal parallel distance: Then I say that the said parallel distance shall be less than the said Perpendicular RT.

For if it be said that the parallel distance is equal to the Perpendicular RT, then by the point R let ERF be drawn parallel to AB, and finish the Trapezium EFGH so as FG may be parallel to BC, likewise GH || CD and EH || AD; draw also AE: Now if the sides of the interior Trapezium EFGH be every where distant from the sides of the exterior Trapezium by an equal parallel distance, then by Lemma 1. the angle EAI is equal to the angle EAK; but by Supposition in this Lemma 5. $\angle RAI = \angle RAK$, and how short soever the parallel line ERF be drawn, the angle EAI will be but part of, and consequently less than $\angle RAI$, ($= \angle RAK$), as is evident by Fig. 2. therefore the angle EAI cannot be equal to the angle EAK, and consequently it contradicts Lemma 1. before demonstrated. The like contradiction will ensue if a Parallel be drawn to AB at a distance greater than RT. Wherefore I conclude, that if a Trapezium be made within a Trapezium in such manner as is above supposed, the parallel distance shall be less than RT, which is suppos'd to be shorter, or, if not shorter, yet not longer than any one of the three Perpendiculars VY, XZ and Wα before mentioned.

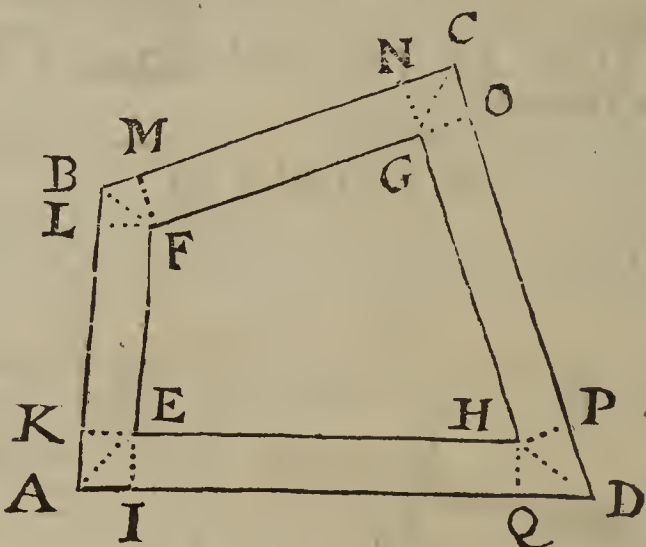
Probl. XXI.

A Trapezium being given by Position, as also the quantities of all its sides and angles severally, to make a Trapezium within the former, in such manner that the sides of the one may be every where separated

from the sides of the other by an equal parallel distance; and that the Space lying between the sides of both the Trapezia may be equal to any possible Space (or Figure) given.

Note. This Problem hath two Cases, viz. First, when each of the Diagonals of the given Trapezium lyes within the same; Secondly, when one of the Diagonals lyes without. I shall handle the first Case only, for this well understood, will be a sufficient light to shew the industrious Analyst how to solve the latter Case, which is briefly done by the Learned *Fran. à Schooten*, in his *Tractat. de concinnandis Demonstrationibus ex Calculo Algebraico.*

FIG. 1.



Suppos.

1. ABCD is a Trapezium given, whose Diagonals AC and BD do lye within the same.
2. c = a right line given, equal to the summ of AB, BC, CD, DA.
3. r = the Radius, or Semidiameter of a Circle is given.
4. s = a right line given equal to the summ of the Tangents of the angles AEI, BFL; CGN and DHP, agreeable to the Radius r ; which angles are the complements of the halves of the given angles of the Trapezium ABCD.
5. b = the side of a given Square equal to the Interval or Space lying between the sides of the given Trapezium ABCD, and the sides of the Trapezium required to be made within the former.

Req. to make

6. EFGH a Trapezium, such, that EF || AB; FG || BC; GH || CD, and EH || AD. Also,
7. Space AEFBCGHDAE = $\square b$, (or bb .) Also,
8. The Perpendiculars EI, FL, GN and HP to be equal between themselves, that there may be an equal parallel distance between the sides of the Trapezium given, and of that required.

Resolution of CASE 1.

9. Suppose that done which is required, and put a for }
the parallel distance, viz. } $a = EI = EK = FM = GO$.
10. Then by the foregoing Lemma 2. this Analogy is manifest, viz.

As Radius, to the summ of the Tangents of }
 $\left. \begin{array}{l} \sphericalangle AEI, \\ \sphericalangle BFL, \\ \sphericalangle CGN, \\ \sphericalangle DHP; \end{array} \right\}$

So is the parallel distance EI, to $AI \div LB \div NC \div PD$.

11. Therefore in the letters of the Resolution,

$$r \cdot s :: a \cdot \frac{sa}{r} \quad (= AI \div LB \div NC \div PD.)$$

12. By the preceding Lemma 3. this Equation is manifest, viz.

$$\left. \begin{array}{l} \square EI \times AD - AI \div \square EI \times AB - LB \div \\ \square EI \times BC - NC \div \square EI \times CD - PD \end{array} \right\} = \text{Space AEFBCGHDA.}$$

13. Therefore in the letters of the Resolution,

$$a \times c - \frac{sa^2}{r} = bb \quad (= \text{Space AEFBCGHDA.})$$

14. Which

14. Which Equation in 13° may be resolved into these Proportionals, viz. $b \cdot c - \frac{sa}{r} :: a \cdot b.$
15. And by drawing r as a common Factor into each of the two first Terms of the last preceding Analogy, this ariseth, viz. $br \cdot cr - sa :: a \cdot b.$
16. Now to avoid an Equation between Solids, which would arise by comparing the Rectangle of the extremes to the Rectangle of the means of the last Analogy, let it be made, as s to r , so b to a fourth Proportional, which may be called d , therefore $s : r :: b \cdot d.$
17. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, viz. $sd = br.$
18. Again, let it be made, as s to r , so c to a fourth Proportional, call it g , therefore $s \cdot r :: c \cdot g.$
19. Therefore from the last Analogy, by comparing the Rectangle of the extremes to the Rectangle of the means, $sg = cr.$
20. Therefore from 15°, 17° and 19°, by exchanging equal Rectangles, this Analogy ariseth, viz. $sd \cdot sg - sa :: a \cdot b.$
21. Therefore from 20°, by casting the common Factor s out of the two first Terms, $d \cdot g - a :: a \cdot b.$
22. From which last Analogy, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation is produced, viz. $ga - aa = bd.$
23. Which Equation may be resolved into these Proportionals, viz. $g - a \cdot \sqrt{bd} :: \sqrt{bd} \cdot a.$
24. But of those three continual Proportionals, the mean, to wit, \sqrt{bd} is given, as also g the summ of the extremes $g - a$ and a , therefore (per Probl. 13. Chap. 5.) the extremes shall be given severally, which by the Theorem in 21° of the said Probl. 13. are equal to these right lines, (or numbers,) viz.

$$\left. \begin{aligned} \frac{1}{2}g + \sqrt{\frac{1}{4}gg - bd} \\ \frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd} \end{aligned} \right\} = \text{the extreme Proportionals in } 23^\circ.$$

Which extreme Proportionals, (or Roots of the Equation in 22°,) are equal one to the other when $\frac{1}{2}g = \sqrt{bd}$, in which Case each of the said extremes is evidently equal to $\frac{1}{2}g$; but if $\frac{1}{2}g < \sqrt{bd}$, then the said extremes are unequal, as they happen to be in the Resolution of this Problem, but the lesser of them only (for the reason hereafter given in 26°) shall be the parallel distance sought. Hence this

CANON.

25. $\frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd} = EI = EK$ the parallel distance. That is, in words, Let it be made, As (s) the summ of the Tangents of the complements of the halves of the angles of the given Trapezium $ABCD$, is to the Radius (r); So (b) the given side of a Square equal to the prescribed Interval or Space between the sides of both the Trapezia, to a fourth Proportional (d); and so (c) the summ of the four sides of the given Trapezium to a fourth Proportional (g). Then subtract the square Root of the excess whereby the Square of half g exceeds the Rectangle made of b and d , from half g ; the remainder shall be $EI (= EK)$ the parallel distance sought.

Which Canon may be propounded in the form of a Theorem, the Demonstration whereof may be easily framed by a repetition of the steps of the foregoing Resolution; by proceeding in a direct order from the beginning to the end thereof; for the Argumentation therein used is clearly Geometrical, as well as Arithmetical: But waving the Demonstration of the Canon, I shall in the next place shew what Determinations are necessary for limiting the given lines, that there may be a possibility of effecting the Problem propounded.

Determinat. i.

26. . . . $\frac{1}{2}g < \sqrt{bd}$; that is; $\frac{cr}{2s} < \sqrt{b \times \frac{br}{s}}$

Although

Although there be a possibility to find out the extreme Proportionals in 24° , (or the two Roots of the Equation in 22°), if $\frac{1}{2}g$ be not less than \sqrt{bd} , yet the parallel distance is not possible unless $\frac{1}{2}g$ be greater than \sqrt{bd} ; for if $\frac{1}{2}g = \sqrt{bd}$, then each of the said extreme Proportionals is equal to $\frac{1}{2}g$; but by the preceding Lemma 4. $\frac{1}{2}g$ is greater than EI or EK (the parallel distance,) therefore neither of those equal extremes can be the parallel distance: But if $\frac{1}{2}g < \sqrt{bd}$, as *Determinat.* 1. requires, then the two extreme Proportionals before mentioned are unequal; for if $\frac{1}{2}g$, that is, half the sum of the extremes of three Proportionals be greater than the mean, to wit, \sqrt{bd} , then the extremes are unequal, and consequently in such Case $\frac{1}{2}g$ is greater than the lesser extreme; therefore agreeable to Lemma 4. the lesser extreme shall be the parallel distance required; but the greater extreme is to be neglected; because 'tis greater than $\frac{1}{2}g$, and consequently doth contradict the said Lemma 4.

Determinat. 2.

27. $\frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd} : \square RT.$

The perpendicular line RT in Fig. 2. prefix'd before the precedent Lemma 5. is supposed to be shorter; or if not shorter, yet not longer than any one of the other three Perpendiculars VY, XZ and Wa. But the quantities of the said four Perpendiculars may be found out in numbers by the Doctrine of Plain Triangles; for in each of the Triangles ARB, BVC, CXD and AWD, the Base is given, as also the angles at the Base, to find the Perpendicular. Which Bases are the given sides of the Trapezium ABCD, and the angles at the Bases are the halves of the given angles of the said Trapezium, therefore the said Perpendiculars are given also, whence it may be known which of them is the shortest. Now because by the foregoing Canon in 25° , and by what hath been said in 26° , the parallel distance is found equal to $\frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd}$: and because by the preceding Lemma 5. the said parallel distance must be less than RT, which is supposed to be the shortest of the said four Perpendiculars, or if not the shortest, yet not longer than any one of the other three, therefore $\frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd}$: must be less than RT; otherwise that cannot be done which the Problem requires.

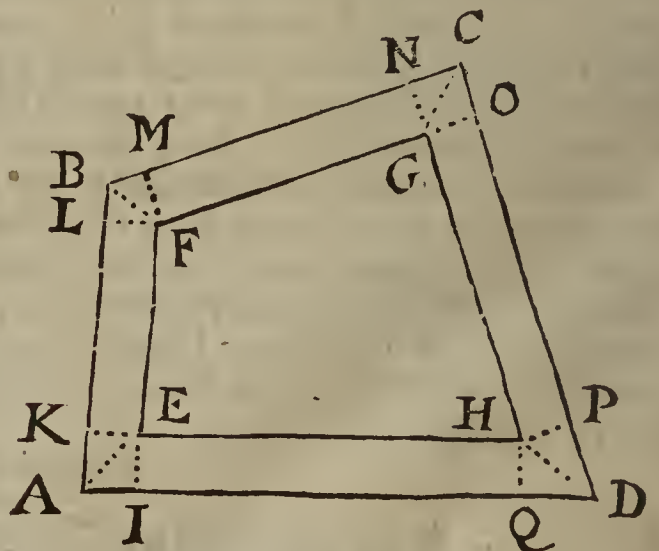
But if the given lines be qualified according to the import of the precedent Determinations, then the parallel distance EI or EK shall be given by the foregoing Canon; and then, after the angles A, B, C, D of the given Trapezium ABCD in Fig. 1. are severally divided into two equal parts by the right lines AE, BF, CG and DH; and after right lines are drawn parallel to the sides AB, BC, CD, DA, at the distance of the said EI or EK, the points where those Parallels do concurr in the lines bisecting the said angles will form the desired Trapezium EFGH: And it will not be difficult to demonstrate, by a repetition of the steps of the foregoing Resolution in a backward order, (in like manner as in divers preceding Problems of this Chapter,) that the Area of the Interval or Space lying between the sides of both the Trapezia is equal to the Square of the given side b. But leaving the Composition of this Problem to the Learners practice, I shall prove the truth of its Solution by an Example in Numbers.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 21.

Suppos.

- 28. ABCD is a Trapezium.
 - 29. EFGH is a Trapezium.
 - 30. $\begin{cases} EH \parallel AD. EF \parallel AB. \\ FG \parallel BC. GH \parallel CD. \end{cases}$
 - 31. AE, BF, CG, DH are right lines.
 - 32. EI and HQ \perp AD.
 - 33. EK and FL \perp AB.
 - 34. FM and GN \perp BC.
 - 35. GO and HP \perp CD.

 - 36. AD = 40
 - 37. AB = 24
 - 38. BC = 29
 - 39. CD = 36
- } are given.



40. $\angle EAI$

- Gr. - Min.
40. \sphericalangle EAI = \sphericalangle EAK = 44 : 6. }
 41. \sphericalangle FBL = \sphericalangle FBM = 55 : 56. } Are given.
 42. \sphericalangle GCN = \sphericalangle GCO = 44 : 43. }
 43. \sphericalangle HDP = \sphericalangle HDQ = 35 : 15. }
44. \sphericalangle AEI = \sphericalangle AEK = 45 : 54. }
 45. \sphericalangle BFL = \sphericalangle BFM = 34 : 4. } These are consequently given, be-
 46. \sphericalangle CGN = \sphericalangle CGO = 45 : 17. } cause the angles at I, K, L, M, N,
 47. \sphericalangle DHP = \sphericalangle DHQ = 54 : 44. } O, P, Q are right angles.
48. $c = 129 = AD + AB + BC + CD$ is given; (from $36^\circ, 37^\circ, 38^\circ, 39^\circ$.)
 49. $bb = 450\frac{111}{1000}$ is given for the Area of the Space AEFBCGHDA.
 50. $r = 100000$ the Radius of a Circle is given.

Req. to find

51. $EI = FL = GN = HP$, (the parallel distance.)

Solution Arithmetical.

- Gr. Min.
52. $103192 =$ Tangent of $45 : 54 = \sphericalangle$ AEI = \sphericalangle AEK.
 53. $67620 =$ Tangent of $34 : 04 = \sphericalangle$ BFL = \sphericalangle BFM.
 54. $100994 =$ Tangent of $45 : 17 = \sphericalangle$ CGN = \sphericalangle CGO.
 55. $141409 =$ Tangent of $54 : 44 = \sphericalangle$ DHP = \sphericalangle DHQ.
 56. $413215 = s =$ the summ of those four Tangents.

Then according to the foregoing Canon in 25° , let it be made,

57. As $\begin{cases} s \\ 413215 \end{cases} \cdot \begin{cases} r \\ 100000 \end{cases} :: \begin{cases} b \\ \sqrt{450\frac{111}{1000}} \end{cases} \cdot \begin{cases} d \\ 5\frac{111}{1000} \end{cases}, \text{ \&c.}$

58. As $\begin{cases} s \\ 413215 \end{cases} \cdot \begin{cases} r \\ 100000 \end{cases} :: \begin{cases} c \\ 129 \end{cases} \cdot \begin{cases} g \\ 31\frac{111}{1000} \end{cases}, \text{ \&c.}$

Lastly, by the precedent Canon in 25° ,

59. $\frac{1}{2}g - \sqrt{\frac{1}{4}gg - bd} : 4$ (very near) = $EI = EK$.

The Proof.

To r, s and EI find a fourth Proportional, (agreeable to the foregoing Lemma 2.) viz.

60. $\begin{cases} r \\ 100000 \end{cases} \cdot \begin{cases} s \\ 413215 \end{cases} :: \begin{cases} EI \\ 4 \end{cases} \cdot \begin{cases} AI + LB + NC + PD \\ 16\frac{111}{1000} \end{cases}$

Then (by Lemma 3.)

$EI \times \begin{cases} AD - AI + AB - LB + \\ BC - NC + CD - PD \end{cases} =$ Space AEFBCGHDA.

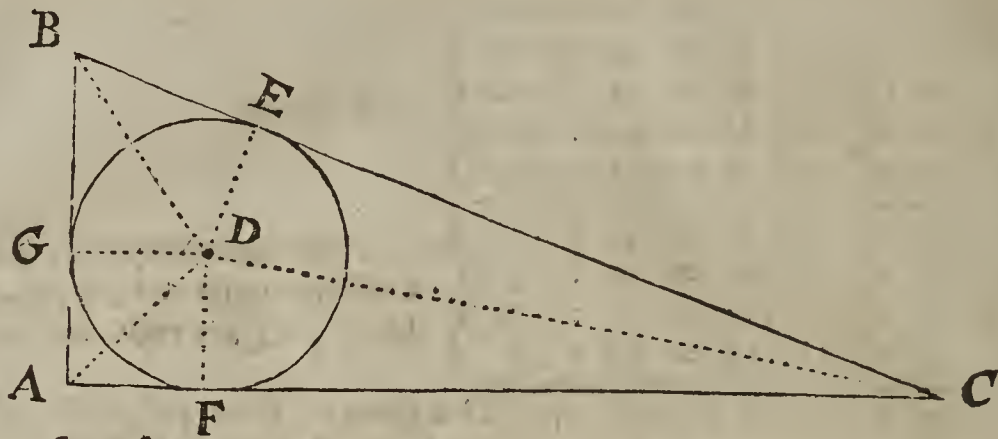
That is, 4 into $129 - 16\frac{111}{1000} = 449\frac{889}{1000}, \text{ \&c.}$

Which Area doth not want $\frac{1}{20}$ of $450\frac{111}{1000}$ the prescribed Area of the Interval AEFBCGHDA, the defect arising from this, that such numbers as were found out near the truth, were assumed to be exactly true, to avoid tediousness of Calculation. Therefore the truth of the Solution of the Problem propounded is evident.

Probl. XXII.

In a right-angled plain Triangle, the Area being given, as also the Perimeter, (that is, the summ of all the three sides,) to find out the Triangle. But the quadruple Area must be less than the Square of the Perimeter.

Suppos.



- AC = 156
- AB = 65
- BC = 169
- DF = 26
- DG = 26
- DE = 26

Suppos.

1. $\triangle ABC$ is right-angled at A.
2. $\square_{\frac{1}{2}} AC, AB =$ the Area of $\triangle ABC$ is given.
3. $AC + AB + BC =$ the Perimeter is given.

Req. to find out AC, AB, BC severally.

Prepar.

4. Supposing ABC to be the Triangle sought, bisect (by *Probl. 9. Elem. 1.*) the angles BAC and ABC by the lines AD and BD meeting in D; and make $DF \perp AC$. $DG \perp AB$. $DE \perp BC$; and draw DC: Then is DF (= DG = DE) the Semidiameter of the inscribed Circle DFGE, which toucheth the Triangle in the points F, G and E, (*per prop. 4. Elem. 4.*)

Resolut. I.

5. From the premisses 'tis easie to perceive, that in any plain Triangle, the Semiperimeter multiplied by the Semidiameter of the inscribed Circle produceth the Area of the Triangle; and consequently the Area divided by the Semiperimeter gives the Semidiameter of the inscribed Circle: As in $\triangle ABC$ before expos'd, the Rectangle (or Product) of $\frac{1}{2} AC$ into DF is equal to the $\triangle ADC$, (*per prop. 41. Elem. 1.*) Likewise, $\square_{\frac{1}{2}} AB, DG (DF) = \triangle ABD$: Also, $\square_{\frac{1}{2}} BC, DE (DF) = \triangle BCD$; and those three Triangles are evidently equal to $\triangle ABC$.
6. Moreover, in every right-angled plain Triangle, if the Diameter of the inscribed Circle be subtracted from the Perimeter, the remainder is equal to the double of the Hypothenufal: As in $\triangle ABC$ before expos'd, if $AF + AG$, (= $DF + DG =$ the Diameter of the inscribed Circle DFGE,) be subtracted from the Perimeter $AC + AB + BC$, there evidently remains $FC + GB + BC = 2 BE + 2 EC = 2 BC$; for $FC = EC$, and $GB = BE$. Therefore,

From 5° and 6° we may deduce

CANON I.

7. Divide the given Area by the Semiperimeter, and subtract the double of the Quotient from the whole Perimeter; the half of the remainder shall be the Hypothenufal; which subtracted from the Perimeter leaves the summ of the sides about the right angle. Then the Hypothenufal being given, as also the summ of the sides about the right angle, the sides shall be given severally, both Geometrically and Arithmetically, by *Probl. 4. Chap. 8.*

Another way to find out the Hypothenufal.

Suppos.

8. $\triangle ABC$ is right-angled at A.
9. $cc = \square_{\frac{1}{2}} AC, AB = \triangle ABC$ is given.
10. $b = AC + AB + BC$ is given.

Req. to find BC the Hypothenufal.

Resolut. II.

11. Put a for the Hypothenufal, viz. } $a = BC$.
12. Therefore from 10° and 11°, the summ of the sides } $b - a (= AC + AB.)$
about the right angle is }
13. And from 11° the Square of the Hypothenufal is . } aa .
14. And from 12°, the Square of the summ of the sides } $aa - 2ba + bb$.
about the right angle is }

15. Therefore

15. Therefore from 9° , 13° and 14° , (by *Theor. 1.* in 12° of *Probl. 15. Chap. 8.*) this Equation ariseth, *viz.* } $aa - 2ba + bb = aa + 4cc.$
 16. From which Equation reduced, the Hypothensal will be made known, *viz.* } $a = \frac{1}{2}b - \frac{2cc}{b}.$

Hence

CANON 2.

17. From the Semiperimeter subtract the Quotient of the double Area divided by the Perimeter, and the remainder shall be the Hypothensal. Then the sides about the right angle shall be given as before in *Canon 1.*
 18. But to the end there may be a possibility of finding out a right-angled Triangle to solve the Problem propounded, it is evident by *Canon 2.* that the given quantities b and cc must be such, that } $\frac{1}{2}b \sqsupseteq \frac{2cc}{b}.$
 19. Whence, by doubling each part, } $b \sqsupseteq \frac{4cc}{b}.$
 20. Therefore from 19° , by multiplying each part into b , } $bb \sqsupseteq 4cc.$
 21. And consequently, } $4cc \sqsupseteq bb.$

Therefore the reason of the Determination added to *Probl. 22.* is manifest, and the Canons may be exemplified by any right-angled Triangle in Rational numbers.

CHAP. X.

The fourth Classis of Examples of the Resolution and Composition of Plane Problems.

IN which Examples, the Resolution ends in an Analogy consisting of three Squares in continual Proportion, whereof the Mean is given; as also a Square equal either to the Difference, or else to the Summ of the Extremes; and therefore the Extremes shall be given severally, by *Probl. 15.* or *16.* of *Chap. 5.*

Probl. I.

The difference of the Squares and the Rectangle of two right lines being given severally, to find out those lines.

Suppos.

1. d = the given side of a Square equal to the difference of the Squares of two right lines.
2. m = the given side of a Square equal to the Rectangle of the same lines.

Req. to find out the lines.

Resolution.

3. For the lesser of the two right lines sought put } $a.$
4. Therefore the Square of the lesser line is } $aa.$
5. And from 1° and 4° , the Square of the greater line is } $aa + dd.$
6. Therefore from 5° the greater line is } $\sqrt{aa + dd}.$
7. And from 3° and 6° the Rectangle made of the two right lines sought is } $a \times \sqrt{aa + dd}.$
8. Which Rectangle (according to the import of the Problem) must be equal to the Square of the given right line m ; therefore, } $a \times \sqrt{aa + dd} = mm.$
9. And that Equation may be resolved into these Proportionals, *viz.* } $\sqrt{aa + dd} : m :: m : a.$

Which Analogy is qualified in every respect like that in 60° of *Probl. 15. Chap. 5.* therefore the lines sought, which are represented by a and $\sqrt{aa + dd}$: shall be given by the Geometrical Construction of the said *Probl. 15.* and their quantities in Numbers shall be given also by the Canon in 77° of the same Problem.

D d d

Probl. II.

Probl. II.

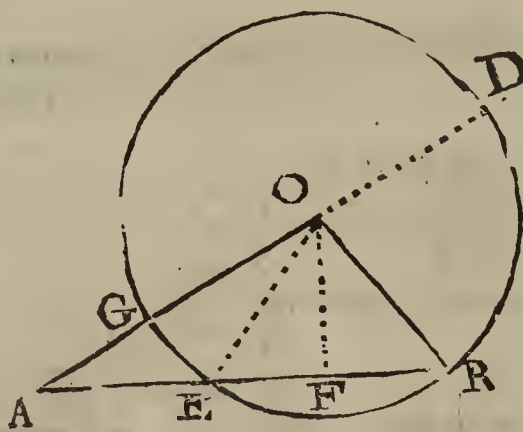
In a plain Triangle, any two sides and the Area being severally given, to find out the Triangle. But eight times the given Area must not exceed the excess by which the Square of the sum of the two given sides is greater than the Square of their difference.

Note. The two given sides of the Triangle sought are either equal one to the other, or unequal; if equal, then the Triangle is either equicrural, or equilateral; each of which is compos'd of two equal right-angled Triangles, having a common Perpendicular falling from the angle contain'd under two equal sides upon the middle of the third side, (or Base;) in which Case, the Problem propounded may be stated thus; viz. The Hypotenusal of a right angled Triangle is given, as also the Area, (that is, half the given Area,) to find the sides about the right angle; which sides may be found out both Geometrically and Arithmetically, by *Probl. 15. Chap. 8.* and then either of those sides about the right angle being doubled, gives the Base of a Triangle, either equicrural or equilateral, to solve the Problem. But if the two sides given be unequal, then the Triangle sought is either right-angled, or else obtuse-angled, or hath unequal acute angles at the Base; if right-angled, then the third side or Base shall be given by *prop. 47. Elem. 1.* by which *Prop.* it may be discovered whether the Triangle sought, when the two given sides are unequal, and express'd by numbers, be right-angled at the Base or not; for if the Square of the lesser side given be subtracted from the Square of the greater, and the Square Root of the remainder be multiplied by half the said lesser side, and if that Product be equal to the given Area, then the Triangle sought is right-angled at the Base. Now when the Triangle sought is neither right-angled at the Base, nor hath equal acute angles at the Base, the Resolution of the Problem propos'd may be formed in manner following.

Suppos.

1. ARO is a Δ whose angles A and R are acute and unequal, also $\angle R < \angle A$, and consequently $AO < RO$, (per *prop. 19. Elem. 1.*)
2. $b = AO$ is given.
3. $k = RO$ is given.
4. $c = b + k = AD$ is given.
5. $d = b - k = AG$ is given.
6. $m =$ a given right line, whose Square is equal to the Triangle ARO.

Req. to find out the Triangle.



- AR = 21
- AO = 17
- RO = 10
- OF = 8
- AE = 9
- EF = 6
- FR = 6
- AF = 15
- AG = 7
- $\square M = 84 = \Delta ARO.$

Resolution.

7. Put a for the Base (or third side) sought, viz. $a = AR.$
8. Then by *Theor. 5. in 69^o of Probl. 8. Chap. 8.* this Equation is manifest, viz.
 $\sqrt{\frac{1}{4}aa - \frac{1}{4}dd} : \text{into } \sqrt{\frac{1}{4}cc - \frac{1}{4}aa} :: mm.$
9. Which Equation may be resolved into these Proportionals, viz.
 $\sqrt{\frac{1}{4}aa - \frac{1}{4}dd} : m :: m : \sqrt{\frac{1}{4}cc - \frac{1}{4}aa}$
10. And their Squares are also Proportionals, viz.
 $\frac{1}{4}aa - \frac{1}{4}dd : mm :: mm : \frac{1}{4}cc - \frac{1}{4}aa.$
11. Which Proportionals being severally quadrupled give these, viz.
 $aa - dd : 4mm :: 4mm : cc - aa.$

12. And

12. And by extracting the square Root out of every one of the four last Proportionals; these also shall be Proportionals, viz.

$$\sqrt{aa - dd} : 2m :: 2m : \sqrt{cc - aa}$$

13. Now forasmuch as the two last Analogies in 11° and 12° are in every respect like to those in 57° of *Probl. 16. Chap. 5.* therefore by the Geometrical Construction of that Problem, respect being had to what hath been deliver'd in the said *Sect. 57.* two right lines may be found out, either of which may be taken for the value of *a*, that is, the Base of the Triangle sought; which right lines are equal to one another when $8mm = cc - dd$, but unequal when $8mm$ is less than $cc - dd$; in which latter Case, two Triangles may be found out having unequal Bases, but each Triangle shall have the same Area and leggs, as will hereafter appear in the Composition of this Problem. But to the end there may be a possibility of effecting the same by the help of the lines given, they must be subject to the following Determination, which is discovered by the Analogy in 11°.

Determination.

14. The octuple of the given Area must not be greater than the excess by which the Square of the sum of the two given sides exceeds the Square of their difference.

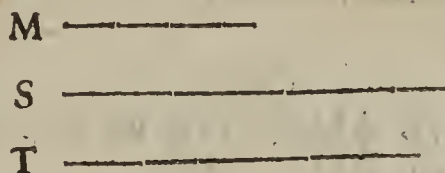
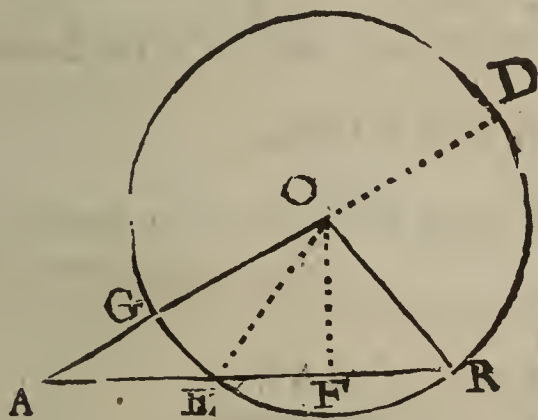
The truth of this Determination will be evident by the following

LEMMA.

15. The octuple of the Area of a plain Triangle having unequal leggs, is never greater than the excess by which the Square of the sum of the leggs exceeds the Square of their difference.

Suppos.

- 16. AR the Base } of $\triangle ARO$.
- 17. AO and OR the leggs } of $\triangle ARO$.
- 18. $AO \perp OR$.
- 19. M is a right line, such, that $\square M = \triangle ARO$.
- 20. $AD = AO + OR$.
- 21. $AG = AO - OR$.
- 22. . . . *Req. demonstr.* . . . $8\square M$, or $8\triangle ARO$, not $\perp \square AD - \square AG$.



Prepar.

23. By *Probl. 4. Chap. 5.* find a right line S, such, that its Square may be equal to $\square AR - \square AG$; therefore,

$$S = \sqrt{\square AR - \square AG}$$

24. Likewise by the same *Probl.* find a right line T, such, that its Square may be equal to $\square AD - \square AR$; therefore,

$$T = \sqrt{\square AD - \square AR}$$

Demonstration.

25. Forasmuch as by *Theor. 5. in 69° of Probl. 8. Chap 8.* (respect being had to the Diagram here in view,)

$$\square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \times \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} : = \triangle ARO.$$

26. And by *Supposition* in 19°, $\square M = \triangle ARO$.

27. Therefore from 25° and 26°, (*per Axiom. 1. Chap. 2.*)

$$\square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \times \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} : = \square M.$$

28. And by resolving the last Equation into Proportionals, this Analogy is manifest, viz.
 $\sqrt{\frac{1}{4} \square AR - \frac{1}{4} \square AG} : M :: M : \sqrt{\frac{1}{4} \square AD - \frac{1}{4} \square AR}$
29. The Squares of which Proportionals are also Proportionals, (per prop. 22. Elem. 6.) viz.
 $\frac{1}{4} \square AR - \frac{1}{4} \square AG : \square M :: \square M : \frac{1}{4} \square AD - \frac{1}{4} \square AR$
30. And by quadrupling all in 29°,
 $\square AR - \square AG : 4 \square M :: 4 \square M : \square AD - \square AR$
31. But the sides of proportional Squares are also Proportionals, therefore from 30°,
 $\sqrt{\square AR - \square AG} : 2 M :: 2 M : \sqrt{\square AD - \square AR}$
32. And because by Constr. in 23°, . . . S = $\sqrt{\square AR - \square AG}$;
33. Also by Constr. in 24°, . . . T = $\sqrt{\square AD - \square AR}$;
34. Therefore from 31°, 32° and 33°, by exchanging equal right lines, this Analogy is manifest, viz.
 $S : 2 M :: 2 M : T$
35. And because the last Analogy consists of three Proportionals, therefore by what hath been said in 20° of Probl. 13. Chap. 5.
 $4 M \text{ not } \sqsubset S \vdash T$
36. And by squaring each part in 35°,
 $16 \square M \text{ not } \sqsubset \square S \vdash \square T \vdash 2 \square S, T$
37. And because by Constr. in 23°, . . . $\square AR - \square AG = \square S$.
38. And by Constr. in 24°, . . . $\square AD - \square AR = \square T$.
39. Therefore by adding the two last Equations together,
 $\square AD - \square AG = \square S \vdash \square T$
40. Therefore from 36° and 38°, by exchanging equal Squares,
 $16 \square M \text{ not } \sqsubset \square AD - \square AG \vdash 2 \square S, T$
41. But from 34°, (per 17. prop. Elem. 1.) . . . $4 \square M = \square S, T$.
42. And consequently, . . . $8 \square M = 2 \square S, T$.
43. Therefore from 40° and 42°,
 $16 \square M \text{ not } \sqsubset \square AD - \square AG \vdash 8 \square M$
44. Wherefore from 43°, by subtracting $8 \square M$ from each part,
 $8 \square M \text{ not } \sqsubset \square AD - \square AG$

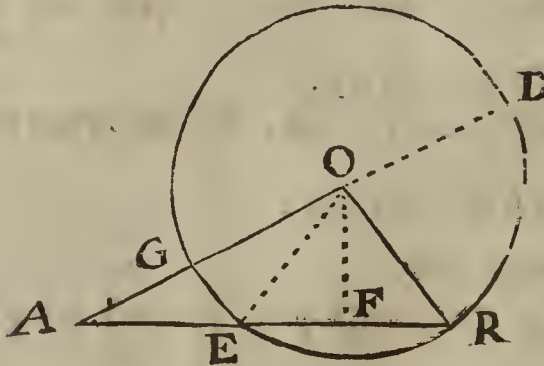
Which was to be Demonstr. Therefore the truth both of the Lemma and Determination is manifest.

The Composition of Probl. 2.

Suppos.

45. AO and OR the leggs of the Triangle ARO are given severally.
 46. AO \sqsubset OR.
 47. AD = AO \vdash OR is given.
 48. AG = AO - OR is given.
 49. M = the given side of a Square equal to $\triangle ARO$.
 50. $8 \square M \text{ not } \sqsubset \square AD - \square AG$. (Determination.)

Req. to find out the Triangle.



- M _____
 H _____
 F _____
 G _____
 L _____
 N _____

Construction.

51. By Probl. 4. Chap. 5. find a right line H, such, that its Square may be equal to $\square AD - \square AG$; therefore,
 $\square H = \square AD - \square AG$.

52. Then

52. Then making H to be the Hypothenuſal of a right-angled Triangle, and 2M a mean Proportional between the Baſe and Perpendicular, find out (*per Probl. 16. Chap. 5.*) the Baſe and Perpendicular; but ſuch a Triangle cannot be found out unleſs it be proved that $\frac{4 \square M}{H}$ is not greater than $\frac{1}{2} H$; for, (agreeable to the Determination annexed to the ſaid *Probl. 16.*) the right line ariſing by the Application of the Square of the given mean Proportional to the given Hypothenuſal, muſt not be greater than half the Hypothenuſal; I ſhall therefore firſt prove that $\frac{4 \square M}{H}$ is not greater than $\frac{1}{2} H$.

Req. demonſtr. $\frac{4 \square M}{H}$ not $\sqsupseteq \frac{1}{2} H$.

Demonſtration.

By the Determination in 50° , $\} 8 \square M$ not $\sqsupseteq \square AD - \square AG$.
 And by *Conſtr.* in 51° , $\} \square H = \square AD - \square AG$.
 Therefore, (*per Ax. 3. Chap. 2.*) $\} 8 \square M$ not $\sqsupseteq \square H$.
 And conſequently, $\} 4 \square M$ not $\sqsupseteq \frac{1}{2} \square H$.
 Wherefore by Application of each part to H, $\} \frac{4 \square M}{H}$ not $\sqsupseteq \frac{1}{2} H$.

Which was to be Demonſtr.

53. Having proved that 'tis poſſible to find out a right-angled Triangle which ſhall have the right line H for a Hypothenuſal, and the double of the right line M for a mean Proportional between the Baſe and Perpendicular, ſuppoſe that (*by Probl. 16. Chap. 5.*) the right lines F and G are found equal to the ſaid Baſe and Perpendicular; therefore by ſuch Conſtruction,

$$F . . . 2M :: 2M . G.$$

Alſo, $\square F + \square G = \square H$.

54. By *Probl. 2. Chap. 5.* find out a right line L, ſuch, that its Square may be equal to $\square AG + \square G$; therefore, $\square L = \square AG + \square G$.

55. Likewise, find out a right line N, ſuch, that its Square may be equal to $\square AG + \square F$; therefore, $\square N = \square AG + \square F$.

56. Now either of the ſaid right lines N and L, (being the two values of *a* in the foregoing Reſolution,) ſhall be the Baſe of a Triangle, to ſatiſſie the Problem propounded; therefore let a Triangle, as ARO be made, ſo, that its Baſe AR may be equal to the right line N, and the leggs equal to the given right lines AO, OR; which may be done, (*by prop. 22. Elem. 1.*) if thoſe three right lines AR, AO and OR be ſuch that every two taken together as one right line, is longer than the third; but that the ſumm of every two of the ſaid right lines is longer than the third, I demouſtrate thus:

57. . . . I. *Req. demonſtr.* AR + AO \sqsupseteq OR.

Demonſtration.

58. Forasmuch as by *Suppoſition* in 46° , AO \sqsupseteq OR.

59. Therefore much more AR + AO \sqsupseteq OR.

60. . . . II. *Req. demonſtr.* AR + OR \sqsupseteq AO.

Demonſtration.

61. By *Conſtr.* in 55° , $\square AG + \square F = \square N$.

62. Alſo by *Conſtr.* in 56° , AR = N, and conſequently, $\square AR = \square N$.

63. Therefore from 61° and 62°, (*per Ax. 1. Chap. 2.*) $\square AR = \square AG + \square F$.

64. Therefore from 63°, $\square AR \sqsupseteq \square AG$.

65. And conſequently, AR \sqsupseteq AG.

66. But by *Suppoſ.* in 48° , AG = AO - OR.

67. Therefore from 65° and 66°, (*per Ax. 3. Ch. 2.*) $\} AR \sqsupseteq AO - OR$.

68. Wherefore by adding OR to each part in 67°, $\} AR + OR \sqsupseteq AO$.

Which was to Demonſtr.

69. III. *Req.*

69. . . . III. Req. demonstr. $AO + OR \sqsubset AR.$

Demonstration.

- 70. By *Construction* in 51° , } $\square AD - \square AG = \square H.$
- 71. Also by *Constr.* in 53° , } $\square F + \square G = \square H.$
- 72. Therefore from 70° and 71° ; (*per Ax. 1.* } $\square AD - \square AG = \square F + \square G.$
- 73. And from 72° , by adding $\square AG$ to each } $\square AD = \square F + \square G + \square AG.$
- 74. And from 73° , by equal subtraction of $\square G$, } $\square AD - \square G = \square F + \square AG.$
- 75. But by *Constr.* in 55° ; } $\square N = \square F + \square AG.$
- 76. Therefore from 74° and 75° , (*per Ax. 1.* } $\square AD - \square G = \square N.$
- 77. And because by *Constr.* in 56° , $AR = N$, } $\square AR = \square N.$
- 78. Therefore from 76° and 77° , (*per Ax. 1.* } $\square AD - \square G = \square AR.$
- 79. And by adding the $\square G$ to each part in 78° , } $\square AD = \square AR + \square G.$
- 80. From 79° 'tis evident, (*per Ax. 25. Chap. 2.* } $\square AD \sqsubset \square AR.$
- 81. And consequently, } $AD \sqsubset AR.$
- 82. And by *Suppos.* in 47° , } $AD = AO + OR.$
- 83. Wherefore from 81° and 82° , (*per Ax. 4.* } $AO + OR \sqsubset AR.$

Which was to be Demonstr.

84. Having proved that of the said three right lines AR , AO and OR every two taken together as one right line is longer than the third, it is possible to make a Triangle of them, (by *prop. 22. Elem. 1.*) suppose it then to be $\triangle ARO$; I say the Triangle so made is that which was required. Now we must shew that it will satisfy the Problem propounded.

85. First then, by *Construction* in 84° , the leggs AO and OR are equal to the two right lines given in 45° to be the leggs of the Triangle sought; and that the $\triangle ARO$ is equal to the Square of the given right line M , the following *Demonstration*, formed by a retrograde repetition of the steps of the *Resolution* foregoing, will make manifest.

86. . . . Req. demonstr. $\triangle ARO = \square M.$

Demonstration.

- 87. By *Construction* in 53° , } $F : 2M :: 2M . G.$
- 88. Therefore, (*per 22. prop. 6. Elem.*) . . . } $\square F . 4\square M :: 4\square M . \square G.$
- 89. By *Constr.* in 55° , } $\square N = \square AG + \square F.$
- 90. Also by *Constr.* in 56° , $N = AR$, and con- } $\square N = \square AR.$
- 91. Therefore from 89° and 90° , (*per Ax. 1.*) } $\square AR = \square AG + \square F.$
- 92. And from 91° , by equal subtraction of $\square AG$, } $\square AR - \square AG = \square F.$
- 93. It hath been proved in 79° , that } $\square AD = \square AR + \square G.$
- 94. And consequently, by subtracting $\square AR$ from } $\square AD - \square AR = \square G.$
- 95. Therefore from 88° , 92° and 94° , by exchanging equal Planes, this Analogy will be manifest, *viz.* $\square AR - \square AG . 4\square M :: 4\square M . \square AD - \square AR.$

That is, in 11° , $aa - dd . 4mm :: 4mm . cc - aa.$

96. And by taking a quarter of all in 95° ,

$$\frac{1}{4}\square AR - \frac{1}{4}\square AG . \square M :: \square M . \frac{1}{4}\square AD - \frac{1}{4}\square AR.$$

That is, in 10° , $\frac{1}{4}aa - \frac{1}{4}dd . mm :: mm . \frac{1}{4}cc - \frac{1}{4}aa.$

97. And because (*per prop. 22. Elem. 6.*) the sides of proportional Squares are also Proportionals, therefore from 96° ,

$$\sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : M :: M : \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR}.$$

That is, in 9° , $\sqrt{\frac{1}{4}aa - \frac{1}{4}dd} : m :: m : \sqrt{\frac{1}{4}cc - \frac{1}{4}aa}.$

98. And

98. And from 97°, (per 17. prop. 6. Elem.)

$$\square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \times \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} : = \square M.$$

That is, in 8°,

$$\sqrt{\frac{1}{4}aa - \frac{1}{4}dd} : \text{ into } \sqrt{\frac{1}{4}cc - \frac{1}{4}aa} : = mm.$$

99. But by *Theor. 5.* in 69° of *Probl. 8. Chap. 8.* respect being had to the last preceding Diagram,

$$\square \text{ of } \sqrt{\frac{1}{4}\square AR - \frac{1}{4}\square AG} : \times \sqrt{\frac{1}{4}\square AD - \frac{1}{4}\square AR} : = \triangle ARO.$$

100. Therefore from 98° and 99°, (per *Axiom. 1. Chap. 2.*) } $\triangle ARO = \square M.$

Which was to be Demonstr.

Therefore one Triangle is found out to solve the Problem; and by the like Construction and Argumentation another Triangle may be made upon the right line L as a Base, which latter Triangle shall have its Area and leggs correspondently equal to the Area and leggs of the first Triangle.

101. But in order to raise an Arithmetical Canon for the finding out of the third side (or Base) of the Triangle sought, first let us suppose the given quantities to be exprest by numbers, and then let the Analogy in 11° be here repeated, viz.

$$aa - dd . 4mm :: 4mm . cc - aa.$$

102. Which Analogy is reducible to this following Equation, viz.

$$ccaa + ddaa - aaaa = 16mmmm + ccdd.$$

103. Now if that Equation be resolved according to the Canon in 55° of *Probl. 16. Chap. 5.* this following Canon will thence arise, whereby the Base of the Triangle sought may be found out Arithmetically.

C A N O N.

$$\sqrt{\frac{1}{2}cc + \frac{1}{2}dd} + \sqrt{\frac{1}{4}ccc + \frac{1}{4}ddd - \frac{1}{2}ccdd - 16mmmm} : = a.$$

Also,

$$\sqrt{\frac{1}{2}cc + \frac{1}{2}dd} - \sqrt{\frac{1}{4}ccc + \frac{1}{4}ddd - \frac{1}{2}ccdd - 16mmmm} : = a.$$

An Example in Numbers, to illustrate the foregoing Resolution of *Probl. 2.*

Suppos.

- 104. $b = 17$ } the leggs of a plain Triangle are given; whence,
- 105. $k = 10$ }
- 106. $c = 27$ the summ of the leggs is given, and
- 107. $d = 7$ the difference of the leggs is given.
- 108. $mm = 84$ the Area of the same Triangle is given.

Req. to find out the Base or third side.

Solution Arithmetical.

109. By the help of the numbers given in 106°, 107°, 108°, and of the Canon in 103°, the numbers 21 and $\sqrt{337}$ will be found out, either of which may be taken for the value of a in the foregoing Resolution, that is, the Base of a Triangle which shall have 17 and 10 for its leggs, and 84 for its Area: For as well from these three sides 21, 17 and 10, as from these, $\sqrt{337}$, 17 and 10, the Area will be found 84, (by *Canon 1.* in 77° of *Probl. 8. Chap. 8.*) And therefore the Problem propounded is solved both Arithmetically and Geometrically.

Probl. III.

The Base, Perpendicular and Rectangle of the leggs of a plain Triangle being given severally, to find out the Triangle. But the given quantities must be subject to the Determinations hereafter exprest.

Suppos.

- 1. b = the Base of a Triangle is given.
- 2. p = the Perpendicular is given.
- 3. r = the side of a Square equal to the Rectangle of the leggs is given.

Req. to find the Triangle.

4. *Vieta,*

4. *Vieta*, in *Probl. 1.* of his Appendix to his *Appollonius Gallus*, shews the Construction of this Problem with great facility, by the help of this Theorem, *viz.* The Rectangle of any two sides containing an angle of a plain Triangle is equal to the Rectangle of the Perpendicular falling from that angle upon the opposite side, and the Diameter of the Circle circumscribing the same Triangle. Afterwards *Marinus Ghetaldus* in his Treatise of Mathematical Resolution and Composition supplies *Vieta's* Construction with Determinations: But it will be difficult for Learners to deduce the Arithmetical Solution from that Geometrical Effecttion; I shall therefore form Resolutions and Compositions to solve this Problem both Arithmetically and Geometrically in all Cases, which are these five, *viz.*

Case 1.	When	$rr = \frac{1}{4}bb + pp.$
Case 2.	When	$rr \sqsubset \frac{1}{4}bb + pp.$
	And	$rr \sqsubset \frac{1}{2}bb.$
Case 3.	When	$rr \sqsupset \frac{1}{4}bb + pp.$
	And	$rr \sqsupset \frac{1}{2}bb.$
Case 4.	When	$rr \sqsubset \frac{1}{4}bb + pp.$
	And	$rr = \frac{1}{2}bb.$
Case 5.	When	$rr \sqsupset \frac{1}{4}bb + pp.$

Now to solve the proposed Problem in every one of those Cases, I shall give a peculiar Resolution, with a Canon and an Example in numbers; and if the given Quantities be exprest by numbers, it will be easie to discern under which of those Cases they fall.

The Resolution of CASE 1.

5. One Triangle may be always easily found out to solve the Problem in *Case 1.* *viz.* when $rr = \frac{1}{4}bb + pp$; for if upon the middle of the given Base, the given Perpendicular be erected, and from the ends of the Base two right lines be drawn to meet at the top of the Perpendicular, then the equicrural Triangle so formed will satisfie the Problem. For in every Isoceles, or equicrural Triangle, the Square of half the Base, (or a quarter of the Square of the whole Base,) together with the Square of the Perpendicular is (per *prop. 47. Elem. 1.*) equal to the Square of either of the equal leggs, that is, the Rectangle of the leggs.

Moreover, when in *Case 1.* it happens that $b \sqsubset 2p$, then besides that equicrural Triangle, another having unequal leggs may be found out to solve the Problem, by this following Resolution, *viz.*

6. For the unknown difference of the leggs put $\} a.$
7. Then by *Theor. 7. Ch. 4.* the Square of the summ of the leggs is $\} aa + 4rr.$
8. Therefore from 6° and 7°, (per *Theor. 2. in 34° of Probl. 8. Chap. 8.*)
 $aa + 4rr - bb . aa + 4rr - bb - 4pp :: bb . aa.$
9. And because by *Suppos.* in *Case 1.* $\} rr = \frac{1}{4}bb + pp.$
10. And consequently, $\} 4rr = bb + 4pp.$
11. Therefore from 8° and 10°, by exchanging equal quantities,
 $aa + bb + 4pp - bb . aa + bb + 4pp - bb - 4pp :: bb . aa .$ Proportionals.
12. Whence, by casting away such equal quantities as destroy one another by reason of contrary signs $+$ and $-$, this Analogy arifeth,
 $aa + 4pp . aa :: bb . aa.$
13. Therefore from 12°, (per *prop. 14. Elem. 5.*) this Equation }
ensues, *viz.* } $aa + 4pp = bb.$
14. And by subtracting $4pp$ from each part of that Equation, }
. } $aa = bb - 4pp.$
15. Therefore by extracting the square Root out of each part }
of the last Equation, } $a = \sqrt{bb - 4pp}:$
16. And out of 7° and 14°, the Square of the summ of the leggs }
will be found equal to } $bb - 4pp + 4rr.$
17. And from 10° and 16°, by exchanging $4rr$ for its equivalent }
quantity $bb + 4pp$, the Square of the summ of the leggs is }
also equal to } $bb - 4pp + bb + 4pp.$

18. Which

18. Which last quantity being first contracted, and then the square Root }
 extracted, the summ of the leggs will be made known, to wit, . . . } $\sqrt{2bb}$.
 The preceding 15th and 18th steps do afford this

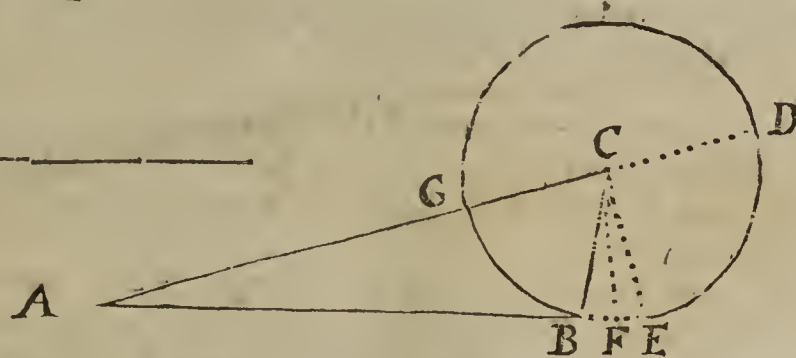
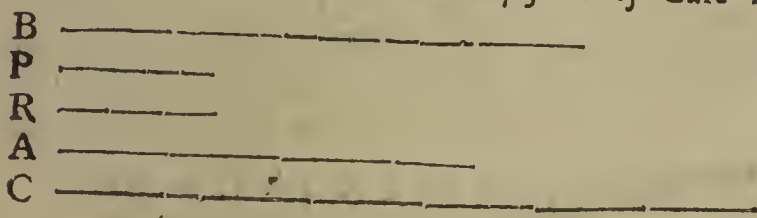
CANON.

19. . . } $\sqrt{bb - 4pp}$: = the Difference } of the Leggs sought.
 $\sqrt{2bb}$ = the Summ }

Then the summ and difference of the leggs sought being given, the leggs shall be given severally, (per Theor. 9. Chap. 4.) and consequently the Triangle.

By which Canon it appears, that the Triangle sought in Case 1. cannot have unequal leggs unless the given Base exceeds the double of the given Perpendicular: For by the Canon, the difference of the leggs is equal to $\sqrt{bb - 4pp}$: which cannot be a real Quantity, (to wit, greater than nothing,) unless $b \geq 2p$.

The Composition of Case 1. Probl. 3.



Suppos.

- 20. B = the Base of a Triangle is given.
- 21. P = the Perpendicular is given.
- 22. R = the side of a Square equal to the Rectangle of the leggs is given.
- 23. $\square R = \frac{1}{4} \square B \dashv \square P$. Also, $B \geq 2P$.

Req. to find the Triangle.

Construction.

24. By Probl. 4. Chap. 5. find a right line A, such, that its Square may be equal to $\square B - 4 \square P$: Which Effecttion is possible, for by Suppos. in 23^o, $B \geq 2P$; therefore,

$$A = \sqrt{\square B - 4 \square P} \text{ (that is, } a = \sqrt{bb - 4pp} \text{)}$$

25. Again, by Probl. 2. Chap. 5. find a right line C, such, that its Square may be equal unto $2 \square B$, therefore, $C = \sqrt{2 \square B} (= \sqrt{2bb})$

Thus far the Construction hath been made according to the direction of the Canon in 19^o.
 26. Now let a Triangle be made of these three right lines, to wit, B, $\frac{1}{2}C \dashv \frac{1}{2}A$, and $\frac{1}{2}C - \frac{1}{2}A$; which Effecttion is possible, (per prop. 22. Elem. 1.) if $C \geq A$, and the summ of every two of those lines be greater than the third: But that those lines are so qualified, I prove thus;

First, from the Construction in 24^o and 25^o it is easie to inferr that $C \geq A$, and consequently $\frac{1}{2}C - \frac{1}{2}A$ is equal to some real right line.

Secondly, it is manifest that the summ of the right lines B and $\frac{1}{2}C \dashv \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C - \frac{1}{2}A$.

Thirdly, that the summ of the two lines $\frac{1}{2}C \dashv \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line B, viz. that $C \geq B$, is evident by Constr. in 25^o.

Fourthly, that the summ of the two lines B and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C \dashv \frac{1}{2}A$, viz. that $B \geq A$, I prove thus;

By Constr. in 24^o, } $\square B - 4 \square P = \square A$.

Therefore by adding $4 \square P$ to each part, . . . } $\square B = \square A \dashv 4 \square P$.

Therefore, (per Ax. 25. Chap. 2.) . . . } $\square B \geq \square A$.

Therefore, } $B \geq A$.

Which was to be demonstrated in the fourth and last place.

Having proved that $\frac{1}{2}C - \frac{1}{2}A$ is a real right line, and that the summ of every two of these three right lines, to wit, B, $\frac{1}{2}C \dashv \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$, is greater than the third,
 E e e 'tis

'tis possible to make a Triangle of those three lines; (*per prop. 22. Elem. 1.*) Suppose then it be made, and that the Triangle found out is ABC , having its Base $AB = B$, (the Base prescribed in the Problem,) and the greater leg $AC = \frac{1}{2}C + \frac{1}{2}A$, and the lesser leg $BC = \frac{1}{2}C - \frac{1}{2}A$: I say the Triangle ABC will satisfy the Problem propounded. But to render the Demonstration thereof the more clear and easie, I shall premise a few things in seven steps next following.

- 27. If the given Quantities be express'd by numbers, the kind of the Triangle sought in *Case 1.* when the leggs are unequal, as they were supposed in the foregoing Resolution, may be discovered by the help of the Canon in 19° of this Problem, and of *Theor. 3.* in 35° of *Probl. 8. Chap. 8.* Supposing then it be discovered that the Perpendicular falls without the Triangle ABC upon AB increased, from C as a Center at the distance of $CB (= \frac{1}{2}C - \frac{1}{2}A)$ describe the Circle $CBGD$ cutting CA in G ; then produce AC and AB to the Circumference in D and E ; draw also the Semidiameter CE , and from the Center C make $CF \perp BE$, whence $FE = FB$, (*per prop. 3. Elem. 3.*)
- 28. Then because (*per Defin. 15. Elem. 1.*) $\} CB = CD.$
- 29. It follows, (by adding CA to each part,) that $\} CA + CB = CD + CA = AD.$
- 30. And because by *Constr.* in 26°, $\} \frac{1}{2}C + \frac{1}{2}A = CA.$
- 31. Also by *Constr.* in 26°, $\} \frac{1}{2}C - \frac{1}{2}A = CB = CD = CG.$
- 32. Therefore the sum of the Equations in 30° $\} C = CA + CB = AD.$
and 31° gives $\}$
- 33. And by subtracting the Equation in 31° from $\} A = CA - CB = AG.$
that in 30°, $\}$
- 34. Now I shall shew that the Triangle ABC made as before will satisfy the Problem. First then by *Constr.* in 26° the Base AB is equal to the prescribed Base B . Secondly, that the Rectangle of the leggs AC and BC is equal to the Square of the given line R , (to wit, the prescribed Rectangle,) I shall here demonstrate.

35. . . . *Req. demonstr.* $\square AC, BC = \square R.$

Demonstration.

- 36. By *Constr.* in 24° and 26°, $\} \square A = \square AB - 4\square P.$
- 37. And from 33°, $\} \square A = \square AG.$
- 38. Therefore from 36° and 37°, (*per Axiom. 1.*) $\} \square AG = \square AB - 4\square P.$
(*Chap. 2.*) $\}$
- 39. And by adding $4\square P$ to each part in 38°, $\} 4\square P + \square AG = \square AB.$
- 40. And by adding $\square AB$ to each part of the last $\} \square AB + 4\square P + \square AG = 2\square AB.$
Equation, $\}$
- 41. But from 32°, 25° and 26°, $\} \square AD = 2\square AB.$
- 42. Therefore from 40° and 41°, (*per Axiom. 1.*) $\} \square AB + 4\square P + \square AG = \square AD.$
(*Chap. 2.*) $\}$
- 43. And by subtracting $\square AG$ from each part $\} \square AB + 4\square P = \square AD - \square AG.$
in 42°, $\}$
- 44. From 32° and 33°, AD is the sum, and AG $\} \square AC, BC = \square AD - \square AG.$
the difference of CA and CB , therefore, (*per* $\}$
Theor. 7. Chap. 4. $\}$
- 45. Therefore from 43° and 44°, (*per Axiom. 1.*) $\} \square AB + 4\square P = 4\square AC, BC.$
(*Chap. 2.*) $\}$
- 46. But from 23° and 26°, $\} \square AB + 4\square P = 4\square R.$
- 47. Therefore from 45° and 46°, (*per Ax. 1.*) $\} 4\square AC, BC = 4\square R.$
- 48. Therefore, (*per Ax. 21. Chap. 2.*) $\} \square AC, BC = \square R.$

Which was to be Demonstr.

Thirdly and lastly, that the Perpendicular CF is equal to the given Perpendicular P , I shall here demonstrate by a retrograde repetition of the steps of the preceding Resolution.

49. . . . *Req. demonstr.* $CF = P.$

Demonstration.

- 50. By *Constr.* in 24°, $\} A = \sqrt{\square B - 4\square P}:$
- 51. And it hath been shewn in 33°, that $\} A = AG.$

52. There.

52. Therefore from 50° and 51°, (per Axiom. 1. Chap. 2.) } $AG = \sqrt{\square B - 4 \square P}:$

53. And because by Constr. in 26° $AB = B$, therefore from 52°, } $AG = \sqrt{\square AB - 4 \square P}:$

. That is, in 15°, } . . $a = \sqrt{bb - 4pp}:$

54. And from 53°, by comparing the Squares of each part one to another, } $\square AG = \square AB - 4 \square P.$

. That is, in 14°, } . . $aa = bb - 4pp.$

55. And by adding $4 \square P$ to each part of the Equation in 54°, } $\square AG + 4 \square P = \square AB.$

. That is, in 13°, } . . $aa + 4pp = bb.$

56. And from 55°, (per prop. 7. Elem. 5.) this Analogy arifeth, viz.

$$\square AG + 4 \square P . \square AG :: \square AB . \square AG.$$

That is, in 12°,

$$aa + 4pp . aa :: bb . aa.$$

57. It is manifest that the Analogy in 56° is equal to this that follows, (for $\square AB - \square AB = 0$. Also, $4 \square P - 4 \square P = 0$.) viz.

$$\left. \begin{array}{l} \square AG + \square AB + 4 \square P - \square AB . \\ \square AG + \square AB + 4 \square P - \square AB - 4 \square P . \\ \square AB . \\ \square AG . \end{array} \right\} \text{Proportionals.}$$

That is, in 11°,

$$\left. \begin{array}{l} aa + bb + 4pp - bb . \\ aa + bb + 4pp - bb - 4pp . \\ bb . \\ aa . \end{array} \right\} \text{Proportionals.}$$

58. From 23° and 26°, } $4 \square R = \square AB + 4 \square P.$

. That is, in 10°, } $4rr = bb + 4pp.$

59. Therefore from 57° and 58°, by exchanging equal quantities, this Analogy arifeth,

$$\left. \begin{array}{l} \square AG + 4 \square R - \square AB . \\ \square AG + 4 \square R - \square AB - 4 \square P . \\ \square AB . \\ \square AG . \end{array} \right\} \text{That is, in 8°}, \left\{ \begin{array}{l} aa + 4rr - bb . \\ aa + 4rr - bb - 4pp . \\ bb . \\ aa . \end{array} \right.$$

60. But it hath been shewn in 47°, that . . . } $4 \square AC, BC = 4 \square R.$

61. Therefore by setting $4 \square AC, BC$ in the place of $4 \square R$ in the Analogy in 59°, this arifeth, viz.

$$\left. \begin{array}{l} \square AG + 4 \square AC, BC - \square AB . \\ \square AG + 4 \square AC, BC - \square AB - 4 \square P . \\ \square AB . \\ \square AG . \end{array} \right\} \text{are Proportionals.}$$

62. And because from 32° and 33°, AD is the sum, and AG the difference of AC and BC , therefore by Theor. 7. Chap. 4.

$$\square AD = \square AG + 4 \square AC, BC.$$

63. Therefore from 61° and 62°, by exchanging equal quantities, this Analogy arifeth, viz.

$$\square AD - \square AB . \square AD - \square AB - 4 \square P :: \square AB . \square AG.$$

64. And from 63°, by Conversion of Reason,

$$\square AD - \square AB . 4 \square P :: \square AB . \square AB - \square AG.$$

65. And from 65°, by altern and inverse Reason,

$$\square AB . \square AB - \square AG :: \square AD - \square AB . 4 \square P.$$

66. But by Theor. 4. in 68° of Probl. 8. Chap. 8.

$$\square AB . \square AB - \square AG :: \square AD - \square AB . 4 \square CF.$$

Ecc 2

67. There-

67. Therefore from 65° and 66° , (*per prop. 11. and 14. Elem. 5.*)
 $4 \square CF = 4 \square P$. And consequently, $CF = P$.

Which was to be Demonstrated in the last place; and therefore *Case 1. Probl. 4.* is solved.

An Example in Numbers, to illustrate the foregoing Resolution of Case 1. Probl. 3.

Suppos.

68. $b = 20$ the Base of a Triangle is given.
 69. $p = 6$ the Perpendicular is given.
 70. $rr = 136$ the Rectangle of the leggs is given.
 71. $rr = \frac{1}{4}bb + pp$; also, } agreeable to *Case 1.*
 72. $b \square = 2p$, }

Req. to find the Triangle.

Solution Arithmetical.

73. $16 =$ the difference of the leggs, is found out of 68° and 69° , by the Canon in 19° of this Problem.

74. $\sqrt{800} =$ the summ of the leggs, is found out of 68° , by the same Canon.

75. $\left\{ \begin{array}{l} \sqrt{200} + 8 = \\ \sqrt{200} - 8 = \end{array} \right\}$ the leggs are found out of 73° and 74° , (*per Theor. 9. Chap 4.*)

The Proof.

76. \square of $\left\{ \begin{array}{l} \sqrt{200}, + 8, \\ \sqrt{200} - 8 \end{array} \right\} = 136$ the given Rectangle. Also,

77. If . . . $\left\{ \begin{array}{l} 20 = \text{the Base} \dots \\ \sqrt{200} + 8, \\ \sqrt{200} - 8 \end{array} \right\}$ the leggs } of a Triangle.

78. Then (*per Theor. 4. in 68° of Probl. 8. Chap. 8.*) the Perpendicular will be found 6, which is the same with the given Perpendicular in 69° .

Moreover, $\sqrt{136}$ and $\sqrt{136}$ shall be the leggs of an equicrural Triangle having the same Base, Perpendicular and Rectangle of the leggs as are before given in 68° , 69° , and 70° .

The Resolution of CASE 2. Probl. 3.

Suppos.

1. $b =$ the Base of a Triangle is given.
 2. $p =$ the Perpendicular is given.
 3. $r =$ a right line given, whose Square is equal to the Rectangle of the leggs of the said Triangle.
 4. $rr \square = \frac{1}{4}bb + pp$, } by *Suppos.* in *Case 2.*
 5. $rr \square = \frac{1}{2}bb$, }

Req. to find the Triangle.

6. The Triangle sought in this second Case, as also in the third, fourth and fifth Cases of the Problem propounded, cannot be equicrural, for in every equicrural Triangle $rr = \frac{1}{4}bb + pp$, which agrees only with the *Suppos.* in *Case 1.* and consequently the leggs of the Triangle sought in all the other Cases are unequal; therefore,

For the difference of the leggs sought put } a .

7. Then from 4° , 5° and 6° , (*per Theor. 7. Chap. 4.*) the } $aa + 4rr$.
 Square of the summ of the leggs shall be }

8. Therefore from the premisses (*per Theor. 2. in 34° of Probl. 8. Chap. 8.*) this Analogy ariseth, *viz.*

$$aa + 4rr - bb \dots aa + 4rr - bb - 4pp :: bb : aa.$$

9. Therefore alternately,

$$aa + 4rr - bb \dots bb :: aa + 4rr - bb - 4pp \dots aa.$$

10. From 4° 'tis evident that } $4rr - bb - 4pp \square = 0$.

11. And from 10° , by adding aa to each part, } $aa + 4rr - bb - 4pp \square = aa$.

12. There-

12. Therefore from the Analogy in 9°, by Division of Reason, (which is evidently possible from the Equation in 11°,) these are Proportionals, viz.

$$aa + 4rr - 2bb . bb :: 4rr - bb - 4pp . aa.$$

13. From 5° 'tis easie to perceive that $4rr - 2bb$ }
is greater than nothing, suppose therefore . . . } $cc = 4rr - 2bb.$

14. It appears also in 10°, that $4rr$ exceeds $bb + 4pp$, }
we may therefore suppose } $dd = 4rr - bb - 4pp.$

15. Then from 12°, 13° and 14°, by exchanging }
equal quantities, this Analogy ariseth, viz. . . . } $aa + cc . bb :: dd . aa.$

16. And from 15°, (per prop. 22. Elem. 6.) . . . } $\sqrt{aa + cc} : . b :: d . a.$

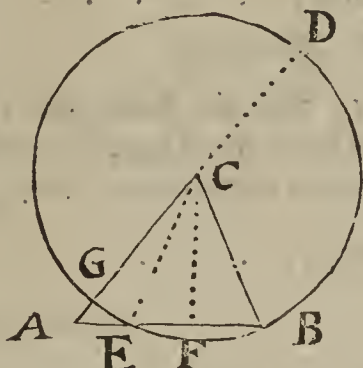
17. Then between b and d find a mean Proportional, }
suppose it to be m , therefore, } $b . m :: m . d.$

18. Therefore from 16° and 17°, by exchanging }
the mean Proportionals, (according to Defn. 8. } $\sqrt{aa + cc} : . m :: m . a.$
Chap. 3. concerning Inordinate proportion,) this }
Analogy ariseth, viz. }

19. Which last Analogy doth evidently consist of three Proportionals, whereof the greater extreme, to wit, $\sqrt{aa + cc}$: is equal to the Hypothensal of a right-angled Triangle, having c for its Base, and a for its Perpendicular: Now in that right-angled Triangle the Base c is given, also m , a mean Proportional between the Hypothensal $\sqrt{aa + cc}$: and the Perpendicular a , is given; therefore the Perpendicular a shall be given also, (per *Probl. 15. Chap. 5.*) which Perpendicular is equal to the difference of the leggs of the Triangle sought by this *Probl. 3. in Case 2.* Then the Rectangle and the difference of the leggs being given, the leggs shall be given severally, by *Probl. 1. Chap. 9.* Therefore the Triangle sought is given also.

The Composition of CASE 2. Probl. 3.

- B _____
- P _____
- R _____
- C —
- D —
- M _____
- A —
- F _____



Suppos.

- 20. B = the Base of a Triangle is given.
- 21. P = the Perpendicular is given.
- 22. R = the side of a Square equal to the Rectangle of the leggs, is given.
- 23. $\square R \sqsubset \frac{1}{4} \square B + \square P,$ } by *Suppos.* in *Case 2.*
- 24. $\square R \sqsubset \frac{1}{2} \square B,$ }

Req. to make the Triangle.

Construction:

25. By *Probl. 4. Chap 5.* find a right line C, such, that its Square may be equal to $4\square R - 2\square B$, (which Effecton is evidently possible from the *Suppos.* in 24°;) therefore,

$$\square C = 4\square R - 2\square B.$$

26. Find likewise a right line D, such, that its Square may be equal to $4\square R - \square B - 4\square P$, (which may be done, for 'tis easie to infer from the *Suppos.* in 23°, that $4\square R \sqsubset \square B + 4\square P$,) therefore,

$$\square D = 4\square R - \square B - 4\square P.$$

27. By

27. By *Probl. 9. Chap. 5* find a mean proportional line M between the given Base B and the line D, (found out in 26°;) therefore,

$$B \cdot M :: M \cdot D.$$

28. Then esteeming the line C to be the Base of a right-angled Triangle, and the line M to be a mean Proportional between the Hypothensal and Perpendicular; find out (by *Probl. 15. Chap. 5.*) the Perpendicular it self, suppose it be the right line A: Then if A be the Perpendicular, and C the Base, the Hypothensal shall be equal to $\sqrt{\square A + \square C}$: and from that Effecttion this Analogy ariseth, viz.

$$\sqrt{\square A + \square C} : M :: M \cdot A.$$

Which line A shall be equal to the difference of the leggs of the Triangle sought.

29. By *Probl. 2. Chap 5.* find a right line F, such, that its Square may be equal to $4\square R + \square A$; therefore,

$$\square F = 4\square R + \square A.$$

Which line F shall be equal to the summ of the leggs of the Triangle sought.

30. Then let a Triangle be made of these three right lines, viz. B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$; which may be done (by *Prop. 22. Elem. 1.*) if F be greater than A, and the summ of every two of those three lines be greater than the third: First, that $F > A$, is evident from the *Constr.* in 29°. Secondly, it is also evident that the summ of B and $\frac{1}{2}F + \frac{1}{2}A$ exceeds $\frac{1}{2}F - \frac{1}{2}A$. Thirdly, that the summ of $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ exceeds the Base B, viz. that $F > B$, I demonstrate thus;

By *Constr.* in 29°, } $4\square R + \square A = \square F.$

And from the *Constr.* in 25°, } $2\square R - \frac{1}{2}\square C = \square B.$

But 'tis evident (per *Axiom. 25. Chap. 2.*) } that } $4\square R + \square A < 2\square R - \frac{1}{2}\square C.$

Therefore from the three last preceding } steps, (per *Ax. 3. Chap. 2.*) . . . } $\square F < \square B.$

And consequently, } $F < B.$

Which was to be Demonstr.

Fourthly and lastly, that the summ of B and $\frac{1}{2}F - \frac{1}{2}A$ is greater than $\frac{1}{2}F + \frac{1}{2}A$; viz. that $B > A$, the following Demonstration, formed by a repetition of the steps of the foregoing Resolution in a retrograde order, will make manifest.

31. . . . *Req. demonstr.* } $B > A.$

Demonstration.

32. Because by *Constr.* in 28°, } $\sqrt{\square A + \square C} : M :: M \cdot A.$

That is, in 18°, (the last step of the Resolution,) } $\sqrt{aa + cc} : m :: m \cdot a.$

33. And by *Constr.* in 27°, } $M \cdot B :: D \cdot M.$

34. Therefore from 32° and 33°, by exchanging the mean Proportionals, (according to *Defin. 8. Chap. 3.*) this Analogy ariseth, viz. } $\sqrt{\square A + \square C} : B :: D \cdot A.$

That is, in 16°, } $\sqrt{aa + cc} : b :: d \cdot a.$

35. But the Squares of proportional lines are also Proportionals, therefore from 34°, (per *prop. 22. Elem. 6.*) . . . } $\square A + \square C \cdot \square B :: \square D \cdot \square A.$

That is, in 15°, } $aa + cc \cdot bb :: dd \cdot aa.$

36. And because by *Constr.* in 25°, } $4\square R - 2\square B = \square C.$

37. Also by *Constr.* in 26°, } $4\square R - \square B - 4\square P = \square D.$

38. Therefore from 35°, 36° and 37°, by exchanging equal quantities, this Analogy ariseth, viz.

$$\square A + 4\square R - 2\square B \cdot \square B :: 4\square R - \square B - 4\square P \cdot \square A.$$

That is, in 12°,

$$aa + 4rr - 2bb \cdot bb :: 4rr - bb - 4pp \cdot aa.$$

39. There-

39. Therefore from 38°, by Composition of Reason,

$$\square A + 4\square R - \square B \cdot \square B :: \square A + 4\square R - \square B - 4\square P \cdot \square A.$$

That is, in 9°,

$$aa + 4rr - bb \cdot bb :: aa + 4rr - bb - 4pp \cdot aa.$$

40. But the first Proportional in 39° is evidently greater than the third, therefore (per prop. 14. Elem. 5.) the second shall be greater than the fourth, viz.

$$\square B \sqsupset \square A; \text{ therefore } B \sqsupset A.$$

Which was to be Demonstr.

41. Now because it hath been demonstrated that $F \sqsupset A$, and consequently $\frac{1}{2}F - \frac{1}{2}A$ is equal to a real right line, and that the sum of every two of these three right lines, to wit, $B, \frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ is longer than the third, 'tis possible to make a Triangle of them (per prop. 22. Elem. 1.) Suppose then it be done, and that the Triangle so made is ABC , having its Base AB equal to the right line B , (the Base prescribed in the Problem;) also $AC = \frac{1}{2}F + \frac{1}{2}A$, and $BC = \frac{1}{2}F - \frac{1}{2}A$; I say the Triangle ABC will satisfy Case 2. Probl. 3. before propounded. But to render the Demonstration thereof the more easie, I shall premise a few things in seven steps next following.

42. First, if the quantities given in the Problem be express'd by numbers, the kind of the Triangle sought in Case 2. shall be known: For the Base was first given, and by what hath been said in 19°, the legs are given also; therefore by the Corollary in 45° of Probl. 10. Chap. 7. it may be discovered whether the Triangle sought be obtuse-angled, acute-angled, or right-angled at the Base. Supposing then it be found that the Perpendicular falls upon the Base AB within the Triangle; from the Center C , at the distance of CB , ($= \frac{1}{2}F - \frac{1}{2}A$.) the lesser leg of the Triangle ABC made as before, describe the Circle $CBGD$ cutting CA in G ; then produce AC to the Circumference in D , and draw the Semidiameter CE ; and from the Center C let fall $CF \perp EB$, therefore $FE = FB$, (per prop. 3. Elem. 3.) Then,

43. Because (per defin. 15. Elem. 1.) . . . } $CD = CB = CG.$

44. Therefore by adding AC to each part, } $AD = AC + CB.$

45. By Constr. in 41°; . . . } $AC = \frac{1}{2}F + \frac{1}{2}A.$

46. Also by Constr. in 41°, . . . } $CB = \frac{1}{2}F - \frac{1}{2}A.$

47. Therefore the summ of the Equations in } $AD = F = AC + CB.$

48. And the Equation in 46° being subtracted } $AG = A = AC - CB.$

Now I shall shew that the Triangle ABC , formed as before is express'd in 41°, will satisfy Case 2. Probl. 3. First then, by Constr. in 41° the Base AB is equal to the prescribed Base B . Secondly, that the Rectangle made of the legs AC and BC is equal to the given Rectangle, that is, the Square of the right line R , I shall here-under demonstrate.

49. . . . Req. demonstr. . . . } $\square AC, BC = \square R.$

Demonstration.

50. By Constr. in 29°, . . . } $\square F = 4\square R + \square A.$

51. And because from 47°, . . . } $\square AD = \square F.$

52. And from 48°, . . . } $\square AG = \square A.$

53. Therefore from 50°, 51° and 52°, by } $\square AD = 4\square R + \square AG.$

54. And from 53°, by subtracting $\square AG$ } $\square AD - \square AG = 4\square R.$

55. It hath been shewn in 47° and 48°, that } $\square AD = 4\square AC, BC + \square AG.$

56. And from 55°, by subtracting $\square AG$ } $\square AD - \square AG = 4\square AC, BC.$

57. There-

57. Therefore from 54° and 56°, (per Axiom. 1. }
 Chap. 2.) } 4 □ AC, BC = 4 □ R.
 58. Therefore from 57°, (per Ax. 21. Chap. 2.) . . }
 Which was to be Demonstr.

Thirdly and lastly, that the Perpendicular CF is equal to the given Perpendicular P, the following Demonstration, form'd out of the steps of the preceding Resolution by a repetition of its steps in a backward order, will make manifest.

59. . . . Req. demonstr. CF = P.

Demonstration.

It hath been proved in 39°, that

60. . □A + 4□R - □B . □B :: □A + 4□R - □B - 4□P . □A.

That is, in 9°,

aa + 4rr - bb . bb :: aa + 4rr - bb - 4pp . aa.

61. And because from 41° and 48°, . . . □AB = □B; and □AG = □A.

62. Therefore from 60° and 61°, by exchanging equal quantities,
 □AG + 4□R - □AB . □AB :: □AG + 4□R - □AB - 4□P . □AG.

63. Therefore from 62°, by alternate Reason,
 □AG + 4□R - □AB . □AG + 4□R - □AB - 4□P :: □AB . □AG.

That is, in 8°,

aa + 4rr - bb . aa + 4rr - bb - 4pp :: bb . aa.

64. It hath been proved in 53°, that □AD = □AG + 4□R.

65. Therefore from 63° and 64°, by exchanging equal quantities,
 □AD - □AB . □AD - □AB - 4□P :: □AB . □AG.

66. Therefore from 65°, by Conversion of Reason,
 □AD - □AB . 4□P :: □AB . □AB - □AG.

67. And from 66°, by Altern and Inverse Reason,
 □AB . □AB - □AG :: □AD - □AB . 4□P.

68. But by Theor. 4. in 68° of Probl. 8.
 □AB . □AB - □AG :: □AD - □AB . 4□CF.

69. Therefore from 67° and 68°, (per prop. 11. Elem. 5.)
 □AD - □AB . 4□CF :: □AD - □AB . 4□P.

70. Therefore from 69°, (per prop. 14. Elem. 5.) . . . } 4□CF = 4□P.

71. Therefore from 70°, (per Ax. 21. Chap. 2.) . . . } □CF = □P.

72. But the sides of equal Squares are also equal, therefore from 71°, }
 Which was to be Demonstr.

73. An Arithmetical Canon to find out the difference of the legs of the Triangle sought in Case 2. may be deduced from the Analogy in 12°, for by comparing the Rectangle of the extremes to the Rectangle of the means of that Analogy, this ariseth, viz.

aaaa + 4rraa - 2bbaa = 4rrbb - bbbb - 4ppbb.

Which Equation being resolved according to the Canon in 77° of Probl. 15. Chap. 5. will give this following

CANON.

74. . . . √(2) : √(4rrrr - 4bbpp + bb - 2rr) : = a.

An Example in Numbers, to illustrate the foregoing Resolution of Case 2. Probl. 3.

Suppos.

75. b = 14 the Base of a Triangle is given.
 76. p = 12 the Perpendicular is given.
 77. rr = 195 the Rectangle of the leggs is given.
 78. rr = $\frac{1}{4}bb + pp$, } (per Suppos. in Case 2.)
 79. rr = $\frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

80. 2 = the difference of the leggs, is found out of 74°, 75°, 76° and 77°.
 81. 28 = the summ of the leggs, is found out of 7°, 77° and 80°.
 82. 15 and 13 = the leggs are found out of 80° and 81°, by Theor. 9. Chap. 4.

The

The Proof.

83. $15 \times 13 = 195$ the given Rectangle; and if
 84. $\left. \begin{array}{l} 14 = \text{the Base} \\ 13 \} \\ 15 \} \end{array} \right\} = \text{the leggs} \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{ of a Triangle, then}$
 85. $12 = \text{the Perpendicular}$ will be found out, (*per Theor. 4. in 68° of Probl. 8. Chap. 8.*) which is the same with the given Perpendicular in 76° .

The Resolution of CASE 3. Probl. 3.

Suppos.

1. $b = \text{the Base of a Triangle}$ is given.
2. $p = \text{the Perpendicular}$ is given.
3. $r = \text{the side of a Square equal to the Rectangle of the leggs}$ is given.
4. $rr \sqsubset \frac{1}{4}bb + pp$, $\left. \begin{array}{l} \\ \\ \end{array} \right\} \text{ (per Suppos. in Case 3.)}$
5. $rr \supset \frac{1}{2}bb$,

Req. to find the Triangle.

The leggs of the Triangle sought in this Case 3. are unequal, for the reason before given in 6° of the Resolution of Case 2. therefore,

6. For the difference of the leggs put $\} a$.
7. Therefore from 3° and 6° , the Square of the summ of the leggs (*per Ther. 7. Chap. 4.*) shall be $\} aa + 4rr$.
8. And from $1^\circ, 2^\circ, 4^\circ, 6^\circ$ and 7° , (*per Theor. 2. in 34° of Probl. 8. Chap. 8.*) this Analogy ariseth, viz. $aa + 4rr - bb . aa + 4rr - bb - 4pp :: bb . aa$.

9. Therefore alternately,
 $aa + 4rr - bb . bb :: aa + 4rr - bb - 4pp . aa$.

10. From 4° 'tis evident that $\} 4rr - bb - 4pp \sqsubset 0$.
11. Whence it follows, by adding aa to each part, $\} aa + 4rr - bb - 4pp \sqsubset aa$.

12. Therefore from the Analogy in 9° , by Division of Reason, (which is evidently possible from the Equation in 11° .) these are Proportionals, viz.
 $aa + 4rr - 2bb . bb :: 4rr - bb - 4pp . aa$.

13. It is evident by the first Term of the last Analogy, that $\} aa + 4rr \sqsubset 2bb$.

14. And 'tis easie to infer from the Suppos. in 5° , that $\} 2bb \sqsubset 4rr$.

15. Therefore from 13° , by subtracting $4rr$ from each part, $\} aa \sqsubset 2bb - 4rr$.

16. It is evident, that if $2bb - 4rr$ be subtracted from aa , the remainder $aa + 4rr - 2bb$ is the same with the first Term of the Analogy in 12° ; therefore if cc be put equal to $2bb - 4rr$, and $aa - cc$ be taken instead of the said first Term $aa + 4rr - 2bb$ that Analogy will be converted into this, viz.
 $aa - cc . bb :: 4rr - bb - 4pp . aa$.

17. And by putting $dd = 4rr - bb - 4pp$, the last preceding Analogy will be converted into this that follows, viz.
 $aa - cc . bb :: dd . aa$.

18. But the sides of proportional Squares are Proportionals also, (*per prop. 22. Elem. 6.*) therefore,
 $\sqrt{aa - cc} : b :: d . a$.

19. Between b and d find a mean Proportional, which may be called m ; therefore,
 $b . m :: m . d$.

20. Therefore from 18° and 19° , by exchanging the mean Proportionals, according to *Defin. 8. Chap. 3.* concerning Inordinate Proportion, this Analogy ariseth, viz.
 $\sqrt{aa - cc} : m :: m . a$.

21. Which three continual Proportionals last exprest being well examined, it will appear that the greater extreme a may be esteem'd the Hypothenusal of a right-angled Triangle whose Base is c , and the Perpendicular is $\sqrt{aa - cc}$: Now in that right-angled Triangle the Base c is given, as also m , a mean Proportional between the Hypothenusal a and the Perpendicular $\sqrt{aa - cc}$: therefore the Hypothenusal, which is equal to the difference of the leggs of the Triangle sought in Case 3. shall be given also, (*per Probl. 15. Chap. 5.*) Then the Rectangle and difference of the leggs being given severally, the summ of the leggs shall be given also, (*by Probl. 1. Chap. 9.*) And lastly, the summ and difference of the leggs being given, the leggs shall be given severally, (*by Theor. 9. Chap. 4.*)

This third Case needs not any other Determination than what is implied in the Suppositions; and the Composition may be formed (by the help of *Probl. 15. Chap. 5.*) in like manner as before in *Case 2.*

22. But for the Learner's fuller satisfaction, an Arithmetical Canon to find out the value of a , to wit, the difference of the leggs of the Triangle sought in *Case 3.* is deducible from the premisses: For the Rectangle of the extremes of the Analogy in 12° being compared to the Rectangle of the means, and it being observed (as is easie to infer from the *Suppos.* in 5° ;) that $2bb \sqsubset 4rr$, this following Biquadratick Equation ariseth, *viz.*

$$aaaa - 2bb - 4rr \times aa = 4rrbb - bbbb - 4ppbb.$$

Which Equation, if it be resolved according to the Canon in 58° of *Probl. 15. Chap. 5.* gives this

C A N O N.

23. . . . $a = \sqrt{bb - 2rr} + \sqrt{4rrrr - 4ppbb} =$ the difference of the leggs.

An Example in Numbers, to illustrate the precedent Resolution of Case 3. Probl. 3.

Suppos.

24. $b = 10185$ the Base of a Triangle is given.
 25. $p = 4752$ the Perpendicular is given.
 26. $rr = 50307696$ the Rectangle of the leggs is given.
 27. $rr \sqsubset \frac{1}{4}bb + pp$, } agreeable to the *Suppos.* in *Case 3.*
 28. $rr \sqsupset \frac{1}{2}bb$,

Req. to find the Triangle.

Solution Arithmetical.

29. $5529 =$ the difference of the leggs is found out of $23^\circ, 24^\circ, 25^\circ$ and 26° .

30. $15225 =$ the summ of the leggs is found out of $7^\circ, 26^\circ$ and 29° .

31. $\left. \begin{matrix} 10377 \\ 4848 \end{matrix} \right\} =$ the leggs are found out of 29° and 30° , (by *Theor. 9. Chap. 4.*)

The Proof.

32. $10377 \times 4848 = 50307696$ the given Rectangle: And, if
 33. $\left. \begin{matrix} 10185 = \text{the Base} \\ 10377 \\ 4848 \end{matrix} \right\} = \text{the leggs}$ } of a Triangle; then, by those three sides,

34. $4752 =$ the Perpendicular will be found out, (per *Theor. 4.* in 68° of *Probl. 8. Chap. 8.*) which is the same with the Perpendicular given in 25° .

The Resolution of CASE 4. Probl. 3.

Suppos.

1. $b =$ the Base of a Triangle is given.
 2. $p =$ the Perpendicular is given.
 3. $r =$ the side of a given Square equal to the Rectangle of the leggs.
 4. $rr \sqsubset \frac{1}{4}bb + pp$, } (per *Suppos.* in *Case 4.*)
 5. $rr = \frac{1}{2}bb$,

Req. to find the Triangle.

6. For the difference of the leggs sought put $> a$.

7. Then proceed as before in the Resolution of *Case 3.* from the 6^{th} step to the 12^{th} , and let the Analogy in that 12^{th} step be here repeated, *viz.*

$$aa + 4rr - 2bb . bb :: 4rr - bb - 4pp . aa.$$

8. Now because by *Suppos.* in 5° , $> 4rr = 2bb$.

7. And consequently from 8° , by subtracting bb from each part, $> 4rr - bb = bb$.

10. Therefore 'tis evident from 8° and 9° , that by exchanging equal quantities the Proportionals in 7° are reducible to these, *viz.*

$$aa . bb :: bb - 4pp . aa.$$

11. But the sides of proportional Squares are also Proportionals, therefore from 10° ,

$$a . b :: \sqrt{bb - 4pp} . a.$$

12. There-

12. Therefore from 11°, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation ariseth, viz.

$$aa = b\sqrt{bb - 4pp}:$$

13. Therefore from 12°, by extracting the square Root out of each part, the difference of the leggs sought is made known, viz.

$$a = \sqrt{b \times \sqrt{bb - 4pp}}:$$

14. And because by *Theor. 7. Chap. 5.* the Square of the summ of any two right lines (or numbers) is equal to four times the Rectangle together with the Square of their difference; therefore from 3°, 6°, 12° and 13°, the summ of the leggs sought is also made known, viz.

$$\sqrt{aa} + \sqrt{4rr} = \sqrt{b\sqrt{bb - 4pp} + 4rr}:$$

The two last preceding Equations give this following

C A N O N.

15. $\left\{ \begin{array}{l} \sqrt{b\sqrt{bb - 4pp}} = \text{the difference of the leggs.} \\ \sqrt{b\sqrt{bb - 4pp} + 4rr} = \text{the summ of the leggs.} \end{array} \right.$

Then the summ and difference of the leggs being given, the leggs shall be given severally by *Theor. 9. Chap. 4.*

By which Canon and Resolution foregoing, the Geometrical Effectation and Demonstration of *Case 4.* is very easie to be made, and therefore I shall leave the same to the Learner's practice.

An Example in Numbers, to illustrate the foregoing Resolution of Case 4. Probl. 3.

Suppos.

- 16. $b = 5$ the Base of a Triangle is given.
- 17. $p = 2$ the Perpendicular is given.
- 18. $rr = 12\frac{1}{2}$ the Rectangle of the leggs is given.
- 19. $rr \sqsubset \frac{1}{4}bb - pp$, } agreeable to the *Suppos.* in *Case 4.*
- 20. $rr = \frac{1}{2}bb$,

Req. to find the Triangle.

Solution Arithmetical.

- 21. $\sqrt{15} =$ the difference of the leggs is found out of 15°, 16° and 17°.
- 22. $\sqrt{65} =$ the summ of the leggs is found out of 15°, 16°, 17° and 18°.
- 23. $\left\{ \begin{array}{l} \sqrt{\frac{65}{4}} + \sqrt{\frac{15}{4}} \\ \sqrt{\frac{65}{4}} - \sqrt{\frac{15}{4}} \end{array} \right\} =$ the leggs, found out of 21° and 22°; (*per Theor. 9. Chap. 4.*)

The Proof.

- 24. $\sqrt{\frac{65}{4}} + \sqrt{\frac{15}{4}}$ into $\sqrt{\frac{65}{4}} - \sqrt{\frac{15}{4}} = 12\frac{1}{2}$ the given Rectangle.
- $\left\{ \begin{array}{l} 5 = \text{the Base of a Triangle.} \\ \sqrt{\frac{65}{4}} + \sqrt{\frac{15}{4}} \end{array} \right\}$ the Leggs;
- 25. And, if $\left\{ \begin{array}{l} \sqrt{\frac{65}{4}} + \sqrt{\frac{15}{4}} \\ \sqrt{\frac{65}{4}} - \sqrt{\frac{15}{4}} \end{array} \right\}$ the Leggs;
- 26. Then 2 = the Perpendicular will be found out, (*per Theor. 4. in 68° of Probl. 8. Chap. 8.*) which is the same with the given Perpendicular in 17°.

The Resolution of CASE 5. Probl. 3.

Suppos.

- 1. $b =$ the Base of a Triangle is given.
- 2. $p =$ the Perpendicular is given.
- 3. $r =$ the side of a Square equal to the Rectangle of the leggs is given.
- 4. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the preceding *Suppos.* in *Case 5.*

Req. to find the Triangle.

5. For the difference of the leggs sought put . . . } a .
6. Then from 3° and 5° , (*per Theor 7. Chap. 4.*) the }
Square of the summ of the leggs is . . . } $aa + 4rr$.
7. And from 1° , 2° , 5° and 6° , (*per Theor. 2. in 34° of Probl. 8. Chap. 8.*) this Analogy ariseth, *viz.*
 $aa + 4rr - bb$. $aa + 4rr - bb - 4pp$:: bb . aa .
8. Therefore alternly,
 $aa + 4rr - bb$. bb :: $aa + 4rr - bb - 4pp$. aa .
9. Therefore inverily,
 aa . $aa + 4rr - bb - 4pp$:: bb . $aa + 4rr - bb$.
10. By quadrupling all in 4° , and subtracting $\frac{1}{4}bb + pp$ }
from each part, it will be evident that . . . } $4rr - bb - 4pp \sqsupset 0$.
11. And by adding aa to each part in 10° , . . . } $aa + 4rr - bb - 4pp \sqsupset aa$.
12. Therefore from 9° , by Conversion of Reason, (which the 11^{th} step shews is possible,) these are Proportionals, *viz.*
 aa . $bb + 4pp - 4rr$:: bb . $2bb - 4rr - aa$.
13. It is evident from the *Suppos.* in 4° , that . . . } $bb + 4pp \sqsubset 4rr$.
14. Therefore for the excess whereby $bb + 4pp$ ex- }
ceeds $4rr$ we may put dd , whence, . . . } $dd = bb + 4pp - 4rr$.
15. By viewing the 11^{th} step it will appear that the }
third Proportional in 8° is less than the fourth, there- }
fore the first is less than the second, *viz.* . . . } $aa + 4rr - bb \sqsupset bb$.
16. Whence, by adding bb to each part, . . . } $aa + 4rr \sqsupset 2bb$.
17. And from 16° , by comparing the latter part to }
the first, 'tis easie to inferr that . . . } $2bb \sqsubset 4rr$.
18. Therefore for the excess whereby $2bb$ exceeds $4rr$, }
we may put cc , whence, . . . } $cc = 2bb - 4rr$.
19. Then from the Analogy in 12° , by exchanging }
equal quantities according to the Positions in 14° }
and 18° , this Analogy ariseth, *viz.* . . . } aa . dd :: bb . $cc - aa$.
20. And because (*per prop. 22. Elem. 6.*) the sides of }
proportional Squares are also Proportionals, there- }
fore from 19° , . . . } a . d :: b . $\sqrt{cc - aa}$:
21. Between b and d find a mean Proportional, which }
may be called m , therefore . . . } b . m :: m . d .
22. Therefore from 20° and 21° , by exchanging the }
mean Proportionals according to *Defin. 8. Chap. 3.* }
this Analogy ariseth, *viz.* . . . } a . m :: m . $\sqrt{cc - aa}$:
23. Which last Analogy doth evidently consist of three continual Proportionals, whereof the extremes a and $\sqrt{cc - aa}$: may be esteem'd the Base and Perpendicular of a right-angled Triangle whose Hypotenusal c is given, as also m a mean Proportional between the Base and Perpendicular; therefore (*per Probl. 16. Chap. 5.*) the Base and Perpendicular shall be given severally, either of which may be taken for the difference of the leggs of a Triangle to satisfie the Problem propounded in *Case 5*. For here, two different Triangles may be always found out that shall have these three things common, to wit, the Base, the Perpendicular, and Rectangle of the leggs, except when it happens that $\frac{bp}{r} = r$, for then the two values of a will be equal to one another, in which Case there can only one Triangle be found out to agree with the preceding *Suppos.* in 1° , 2° , 3° and 4° .

But that there may be a possibility of finding out a Triangle to solve *Probl. 3.* in *Case 5*. the right line represented by $\frac{bp}{r}$ must not be longer than r : Now to make it evident that this Determination is necessary in *Case 5*,

24. . . . *Req. demonstr.* $\frac{bp}{r}$ not $\sqsubset r$.

Demonstr.

Demonstration.

25. First, if c be given for the Hypotenusal, and m for a mean Proportional between the Base and Perpendicular of a right-angled Triangle, (as before is supposed in 23°,) then to find out that Triangle by *Probl. 16. Chap. 5.* this Determination is necessary, viz. $\frac{mm}{c}$ not $\sqsubset \frac{1}{2}c$.
26. Therefore from 25°, by multiplying each part into c , $mm \cdot \text{not} \sqsubset \frac{1}{2}cc$.
27. But from 21°, (*per prop. 17. Elem. 6.*) $bd = mm$.
28. Therefore from 26° and 27°, (*per Ax. 4. Chap. 2.*) bd not $\sqsubset \frac{1}{2}cc$.
29. And because (as is evident in 14°,) $\sqrt{bb + 4pp - 4rr} = d$.
30. Therefore from 28° and 29°, by exchanging equal Factors, $b\sqrt{bb + 4pp - 4rr}$: not $\sqsubset \frac{1}{2}cc$.
31. And because from 18°, $bb - 2rr = \frac{1}{2}cc$.
32. Therefore from 30° and 31°, (*per Ax. 3. Chap. 2.*) $b\sqrt{bb + 4pp - 4rr}$: not $\sqsubset bb - 2rr$.
33. And from 32°, (by dividing each part by b ,) $\sqrt{bb + 4pp - 4rr}$: not $\sqsubset b - \frac{2rr}{b}$.
34. And from 33°, by squaring each part, $bb + 4pp - 4rr$ not $\sqsubset bb - 4rr + \frac{4rrrr}{bb}$.
35. And from 34°, by adding $4rr$ unto, and subtracting bb from each part, $4pp$ not $\sqsubset \frac{4rrrr}{bb}$.
36. And from 35°, by extracting the square Root out of each part, $2p$ not $\sqsubset \frac{2rr}{b}$.
37. And from 36°, by multiplying each part by b , $2bp$ not $\sqsubset 2rr$.
38. And by halving each part in 37°, bp not $\sqsubset rr$.
39. Therefore from 38°, by dividing each part by r , $\frac{bp}{r}$ not $\sqsubset r$. Which was to be Dem.

40. Supposing then that $\frac{bp}{r}$ is not greater than r , the Composition of *Case 5. Probl. 3.* may be easily formed out of the last preceding Resolution, by the help of *Probl. 16. Chap. 5.* in like manner as before in *Case 2.* and an Arithmetical Canon to find the difference of the leggs of the Triangle sought in the said 5th Case may be deduced from the Analogy in the foregoing 12th step: For the Rectangle of the extremes of that Analogy being compared to the Rectangle of the means, this following Biquadratick Equation ariseth, viz.

$$2bb - 4rr \times aa, - aaaa = bbbb + 4ppbb - 4rrbb.$$

Which Equation being resolved according to the Canon in 55° of *Probl. 16. Chap. 5.* gives this

CANON.

41. $a = \sqrt{bb - 2rr + \sqrt{4rrrr - 4bbpp}}$: Also, $a = \sqrt{bb - 2rr - \sqrt{4rrrr - 4bbpp}}$:
42. Now either of those Roots or values of a may be taken for the difference of the leggs of a Triangle to solve *Case 5. Probl. 3.* before propounded; and if it happens that $\frac{bp}{r} \sqsupset r$, (and consequently, $bp \sqsupset rr$,) then those values of a are unequal, and there may be two different Triangles found out to satisfy the said *Case 5.* But when $\frac{bp}{r} = r$, and consequently, $bp = rr$,) then the said values of a are equal to one another, each being equal to $\sqrt{bb - 2rr}$: which shall be the difference of the leggs of a Triangle having a right angle opposite to the Base, (as appears by Canon 1. in 21° of *Probl. 14. Chap. 8.*) Which Triangle, when $\frac{bp}{r} = r$, is the only Triangle that can be found to solve *Case 5. Probl. 3.*

Examples

Examples in Numbers, to illustrate the preceding Resolution of Case 5. Probl. 3.

Examp. 1. Where two Triangles are found out to solve Case 5.

Suppos.

43. $b = 51$ the Base of a Triangle is given.
 44. $p = 12$ the Perpendicular is given.
 45. $rr = 740$ the Rectangle of the leggs is given.
 46. $rr \supseteq \frac{1}{4}bb + pp$, agreeable to the Suppos. in Case 5.
 Req. to find the Triangle.

Solution Arithmetical.

47. $17 =$ the difference of the leggs is found out of $43^\circ, 44^\circ$ and 45° , by the lesser Root in 41° .
 48. $57 =$ the summ of the leggs is found out of $6^\circ, 45^\circ$ and 47° .
 49. 37 and $20 =$ the leggs are found out of 47° and 48° , (per Theor. 9. Chap. 4.)

The Proof.

50. $37 \times 20 = 740$ the given Rectangle.
 51. And if $\left\{ \begin{array}{l} 51 = \text{the Base} \\ 37 \\ 20 \end{array} \right\} = \text{the leggs}$ } of a Triangle; then,
 52. $12 =$ the Perpendicular will be found out of 51° , (per Theor. 4. in 68° of Probl. 8. Chap. 8.) which is the same with the given Perpendicular in 44° .
 Again, from the same things given as before in $43^\circ, 44^\circ, 45^\circ$, another Triangle may be found out by the greater Root in 41° , to solve Case 5. Probl. 3, the sides of which latter Triangle are here-under exprest, viz.

53. $\sqrt{1953} =$ the difference of the leggs.
 54. $\sqrt{4913} =$ the summ of the leggs.
 55. $\sqrt{\frac{4213}{4}} + \sqrt{\frac{1213}{4}}$
 56. $\sqrt{\frac{4213}{4}} - \sqrt{\frac{1213}{4}}$ } the leggs.

The Proof is easie to be made, in like manner as in Example 1.

Examp. 2. Where only one Triangle can be found out to solve Case 5. Probl. 3.

57. $b = 169$ the Base of a Triangle is given.
 58. $p = 60$ the Perpendicular is given.
 59. $rr = 10140$ the Rectangle of the leggs is given.
 60. $rr \supseteq \frac{1}{4}bb + pp$, agreeable to the Suppos. in Case 5.
 Req. to find the Triangle.

Solution Arithmetical.

61. $91 =$ the difference of the leggs is found out of $57^\circ, 58^\circ$ and 59° , by either of the Roots in 41° .
 62. $221 =$ the summ of the leggs is found out of $6^\circ, 59^\circ$ and 61° .
 63. 156 and $65 =$ the leggs are found out of 61° and 62° , (per Theor. 9. Chap. 4.)

The Proof.

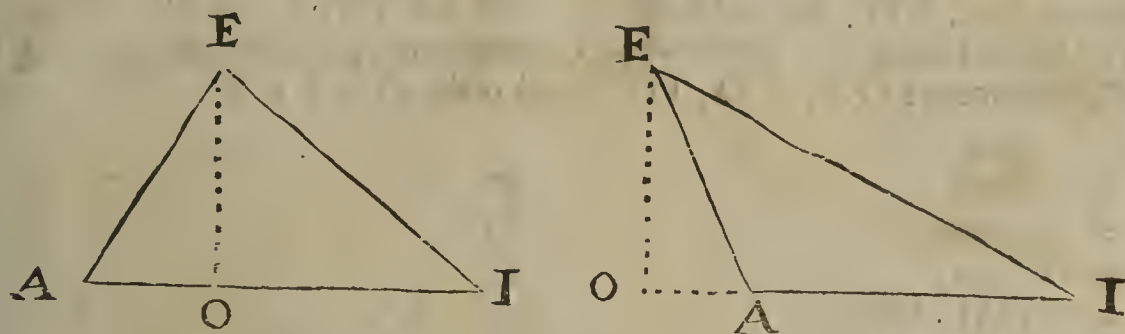
64. $156 \times 65 = 10140$ the Rectangle given in 59° .
 65. And if $\left\{ \begin{array}{l} 169 = \text{the Base} \\ 156 \\ 65 \end{array} \right\} = \text{the leggs}$ } of a Triangle, then
 66. $60 =$ the Perpendicular will be found out of 65° , (per Theor. 4. in 68° of Probl. 8. Chap. 8.) which is equal to the given Perpendicular in 58° .

Note. This last Triangle hath a right angle opposite to the Base, agreeable to what was before hinted in 42° .

A L E M M A, leading to the following Probl. 4.

In any plain Triangle, As the Sine of an angle is to the Radius, (or total Sine;) So is the double Area of the Triangle, (that is, the Rectangle made of the Perpendicular and Base,) to the Rectangle of the leggs containing the said angle.

Suppos.



Suppos.

1. EI the Base
2. $\left. \begin{matrix} AE \\ AI \end{matrix} \right\}$ the leggs } of the oblique-angled $\triangle AIE$.
3. $\angle A$, (that is, $\angle EAI$) is contain'd under the leggs AE, AI.
4. $EO \perp AI$.
5. R = the Radius, (or total Sine.)
6. $S < A =$ the Sine of the angle A.
7. . . . *Req. demonstr.* $S < A \cdot R :: \square EO, AI \cdot \square AE, AI$.

Demonstration.

8. By a vulgar *Axiom* in the Doctrine of plain } $S < A \cdot R :: EO \cdot AE$.
Triangles this Analogy is manifest, *viz.* }
 9. Therefore by drawing AI into each of the two } $S < A \cdot R :: \square EO, AI \cdot \square AE, AI$.
latter Terms of that Analogy, this arifeth, *viz.* }
- Which was to be Demonstr.

Probl. IV.

The Base of a plain Triangle being given, as also the Perpendicular, and angle opposite to the Base, to find the Triangle.

Construction.

Let a Circle be described by a Radius (or Semidiameter) taken at pleasure, and according to the *Note* at the beginning of *Probl. 19. Chap. 8.* find out a right line that shall be the Sine of an angle equal to the given angle. Then (by *Probl. 9. Chap. 5.*) find out a Square equal to a Rectangle made of the given Base and Perpendicular. That done, let it be made, (by *Probl. 11. Chap. 5.*) As the Sine, (found out as above,) to the Radius first assumed; So the said Square to another Square, which Square (or fourth Proportional) found out, shall be equal to the Rectangle of the leggs containing the given angle, (as is evident by the foregoing *Lemma*) Now there is given the Base and Perpendicular, as also a Square equal to the given Rectangle of the leggs, to find out the Triangle; and therefore if those given quantities be exprest by numbers, this *Probl. 4.* may be solved both Geometrically and Arithmetically in all Cases, by the help of the preceding *Probl. 3.*

Note. Some subtil Geometrical Problems, wherein the measures of angles as well as of right lines are given in numbers, may be solved Arithmetically by the Doctrine of Plain Triangles, without the help of *Algebra*; of such kind is the following Problem, with which I shall conclude this Treatise.

Probl. V.

The distances AB, AC, BC between three Towers A, B and C not standing in a straight line, being given severally in Feet. Also a fourth Tower being suppos'd to stand within the Triangle ABC, as at D; and the measures of the angles ADB, BDC and CDA being given severally in Degrees; to find the distance between the fourth Tower D and each of the other three, *viz.* the measures of the three right lines DB, DA and DC in Feet.

Prepar.

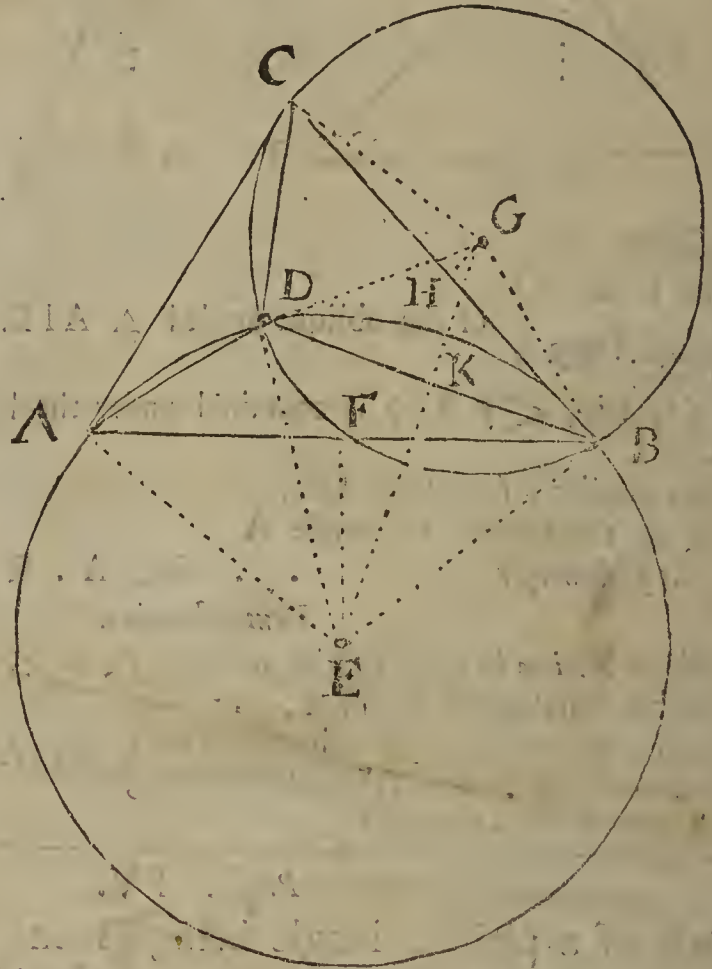
Forasmuch as by *Suppos.* the point D lyes within the Triangle ABC, and consequently the three points A, D, B do not lye in a straight line; let the Circumference of a Circle be supposed to pass by those points, as ADB, whose Center is F; suppose also the

Semi-

Semidiameters EA, ED, EB to be drawn, and $EF \perp AB$. In like manner, supposing G to be the Center of a Circle whose Circumference passeth by the points B, D, C, draw the Semidiameters GB, GD, GC, and make $GH \perp CB$.

<i>Feet.</i>	
AB =	20000
AC =	15000
BC =	18000
EB =	12521.286 †
GB =	9317.445 †
DB =	13888.610 †
DA =	8283.832 †
DC =	8406.944 †

<i>Gr. Min.</i>	
ADB =	127: 0
BDC =	105: 0
CDA =	128: 0
FEB =	53: 0
EBA =	37: 0
HGB =	75: 0
GBC =	15: 0
CBA =	46: 8
GBE =	98: 8
GEB =	33: 41
DBA =	19: 19
DBC =	26: 49



Solution Arithmetical, by the Doctrine of plain Triangles.

First, subtract the given $\angle ADB$ from two right angles, (*viz.* from 180. Degrees,) the remainder shall be the sum of the unknown angles DAB and DBA, (*per prop. 32. Elem. 1.*)

Secondly, forasmuch as (*by prop. 20. Elem. 3.*) $\angle DEB = 2\angle DAB$, and $\angle DEA = 2\angle DBA$; it follows that $\angle AEB = 2\angle DAB + 2\angle DBA$; therefore in $\triangle FEB$ right-angled at F, the $\angle FEB$ (that is, $\frac{1}{2}\angle AEB$) = $\angle DAB + \angle DBA$ is given; and by *Suppos.* $FB = \frac{1}{2}AB$ is given, therefore the Semidiameter $EB = ED = EA$ shall be given also.

Thirdly, by arguing as above in the first and second steps $\angle HGB = \angle HGC$ is given; also $GD = GC = GB$ the Semidiameter of the Circle $GBDC$ is given.

Fourthly, because $EF \perp AB$ and $\angle FEB$ is given as before, therefore $\angle EBA$ the Complement of the $\angle FEB$ to a right angle is given; likewise the $\angle GBC$ the Complement of $\angle HGB$ to a right angle is given; and the $\angle CBA$ is given, for it may be found out by the three given sides AB, AC, BC; therefore $\angle GBE$ the sum of those three angles, EBA, CBA, GBC is given.

Fifthly, in $\triangle GBE$, the sides GB and EB, (to wit, the Semidiameters of the two Circles $GBDC$ and $EADB$,) are given severally, as also the angle GBE comprehended by those sides, therefore the angle GEB is given also.

Sixthly, because the two Triangles EGB and EGD have two sides GB, EB equal to the two sides GD, ED, *viz.* $GB = GD$, and $EB = ED$, also the Base GE common to both those Triangles; the angles contain'd under equal right lines shall be equal, *viz.* $\angle GEB = \angle GED = \frac{1}{2}\angle DEB$: But $\angle GEB (= \frac{1}{2}\angle DEB)$ is given in the fifth step, and (*per prop. 20. Elem. 3.*) $\angle DAB$ is equal to $\frac{1}{2}\angle DEB (= \angle GEB)$, therefore $\angle DAB$ is given. Now in $\triangle ADB$ there is given $\angle DAB$, as also $\angle ADB$ and the side AB, therefore the sides DB and DA, (to wit, two of the Distances sought,) are given also.

Seventhly and lastly, in $\triangle DAC$ there is given DA, as also AC, and $\angle ADC$, therefore DC (the third Distance sought,) is given.

The End of the Fourth and last BOOK.

S O L I D E O G L O R I A.

