A NOTE ON THE MEDIANT INEQUALITY

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Abstract

It is well known that if a, b, c and d are four non negative real numbers, and $bd \neq 0$, then the mediant m of the two fractions $q_1 = a/b$ and $q_2 = c/d$, defined by $m = (a+c)/(b+d)$, fulfills $q_1 \leq m \leq q_2$. It turns out that the notion of mediant can be extended to n fractions in positive numbers, and that a generalized mediant inequality holds, a fact that seems to have been first noticed by Cauchy. Actually, the notion of mediant and the mediant inequality can be generalized to fractions of integrable functions $f(x)/g(x)$, as explained in this note. To this end, nothing is needed but extremely elementary mathematics.

Let

$$q_1 = a_1/b_1,\ q_2 = a_2/b_2 \dots q_n = a_n/b_n$$

be n fractions, where a_i is real and non negative, and b_i real and positive, for every $1 \leq i \leq n$. By definition, the *mediant* of $q_1, q_2, \dots, q_n$ is

$$m = \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$

More generally, if $w_1, \dots, w_n$ are n positive real numbers, then the *weighted mediant* of $q_1, \dots, q_n$ with respect to $w_1, \dots, w_n$ is

$$m_w = \frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 b_1 + w_2 b_2 + \cdots + w_n b_n}.$$

We point out that it can be assumed without loss of generality that $\sum_{i=1}^{n} w_i = 1$. Also, it is clear that the mediant m is the weighted mediant with respect to $w_1 = w_2 = \ldots = w_n$.

It should be observed that the notion of mediant is not well defined if the fractions are identified to rational numbers, unless the numbers a_i and b_i are supposed to be coprime. Unfortunately, this somewhat restricts the extent of the mediant inequality (see below).

Another way to circumvent this difficutly is to distinguish between the notion of "ratio" and the notion of "fraction", which should be seen as a pair (a_i, b_i) of integers.

It is also possible to state the mediant inequality without resorting to the notion of mediant at all, or to regard this notion as a convenient way to state this inequality.

Theorem: *With the previous notations, assume that* $q_1 \leq q_2 \leq \cdots \leq q_n$. *Then*

$$q_1 \leq m \leq q_n$$

More generally, if $w_1, w_2, \ldots, w_2$ *are* n *positive numbers, then the weighted median* m_w *of* $q_1, \ldots, q_n$ *fulfills*

$$q_1 \leq m_w \leq q_n$$

Even more generally, if $f(x)$ *and* $g(x)$ *are two measurable real functions over a domain* D*, with* $f(x) \geq 0$ *and* $g(x) > 0$ *for every* x*, then for every positive Lebesgue measure* μ *over* D*, there holds*

$$\inf_{x \in D}(f(x)/g(x)) \leq \frac{\int_D f(x)d\mu}{\int_D g(x)d\mu} \leq \sup_{x \in D}(f(x)/g(x)).$$

Proof: We first prove the second assertion. For every i, there holds

$$q_1 \leq a_i/b_i \leq q_n,$$

hence

$$q_1 w_i b_i \leq w_i a_i \leq q_n w_i b_i.$$

Summing over the indices i leads to

$$q_1 \sum_{i=1^n} w_i b_i \leq \sum_{i=1}^{n} w_i a_i \leq q_n \sum_{i=1}^{n} w_i b_i.$$

The conclusion follows by dividing each member of this equation by $\sum_i w_i b_i$.

Similarly, in order to prove the third assertion, let us put

$$\alpha = \inf_{x \in D}(f(x)/g(x)) \quad \text{and} \quad \beta = \sup_{x \in D}(f(x)/g(x)).$$

For every x, there holds

$$\alpha \leq \frac{f(x)}{g(x)} \leq \beta,$$

hence

$$\alpha \int_D g(x)d\mu \leq \int_D f(x)d\mu \leq \beta \int_D g(x)d\mu.$$

The proof of the third assertion follows. ∎