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ART. I.—*A new Harmonic Analyzer*; by A. A. MICHELSON and S. W. STRATTON. (With Plate I.)

EVERY one who has had occasion to calculate or to construct graphically the resultant of a large number of simple harmonic motions has felt the need of some simple and fairly accurate machine which would save the considerable time and labor involved in such computations.

The principal difficulty in the realization of such a machine lies in the accumulation of errors involved in the process of addition. The only practical instrument which has yet been devised for effecting this addition is that of Lord Kelvin. In this instrument a flexible cord passes over a number of fixed and movable pulleys. If one end of the cord is fixed, the motion of the other end is equal to twice the sum of the motions of the movable pulleys. The range of the machine is however limited to a small number of elements on account of the stretch of the cord and its imperfect flexibility, so that with a considerable increase in the number of elements the accumulated errors due to these causes would soon neutralize the advantages of the increased number of terms in the series.

It occurred to one of us some years ago that the quantity to be operated upon might be varied almost indefinitely, and that most of the imperfections in existing machines might be practically eliminated. Among the methods which appeared most promising were addition of fluid pressures, elastic and other forces, and electric currents. Of these the simplest in practice is doubtless the addition of the forces of spiral springs.

The principle upon which the use of springs depends may be demonstrated as follows:

Let a (Fig. 1)=lever arm of small springs, s. (but one of which is shown in the fig.)
b=lever arm of large counter-spring, S.
l_o=natural length of small springs.
L_o=natural length of large springs.
$l+x$=stretched length of small springs.
$L+y$=stretched length of large springs.
e=constant of small springs.
E=constant of large springs.
n=number of small springs.
p=force due to one of the small springs.
P=force due to the large spring.

then
$$p=\frac{e}{l_o}\left(l+x-\frac{a}{b}y\right)$$
$$P=\frac{E}{L_o}(L+y)$$
$$a\Sigma p=bP.$$
whence
$$y=\frac{\Sigma x}{n\left(\frac{l}{L}+\frac{a}{b}\right)}$$

From this it follows that the resultant motion is proportional to the algebraic sum of the components, at least to the same order of accuracy as the increment of force of every spring is proportional to the increment of length.

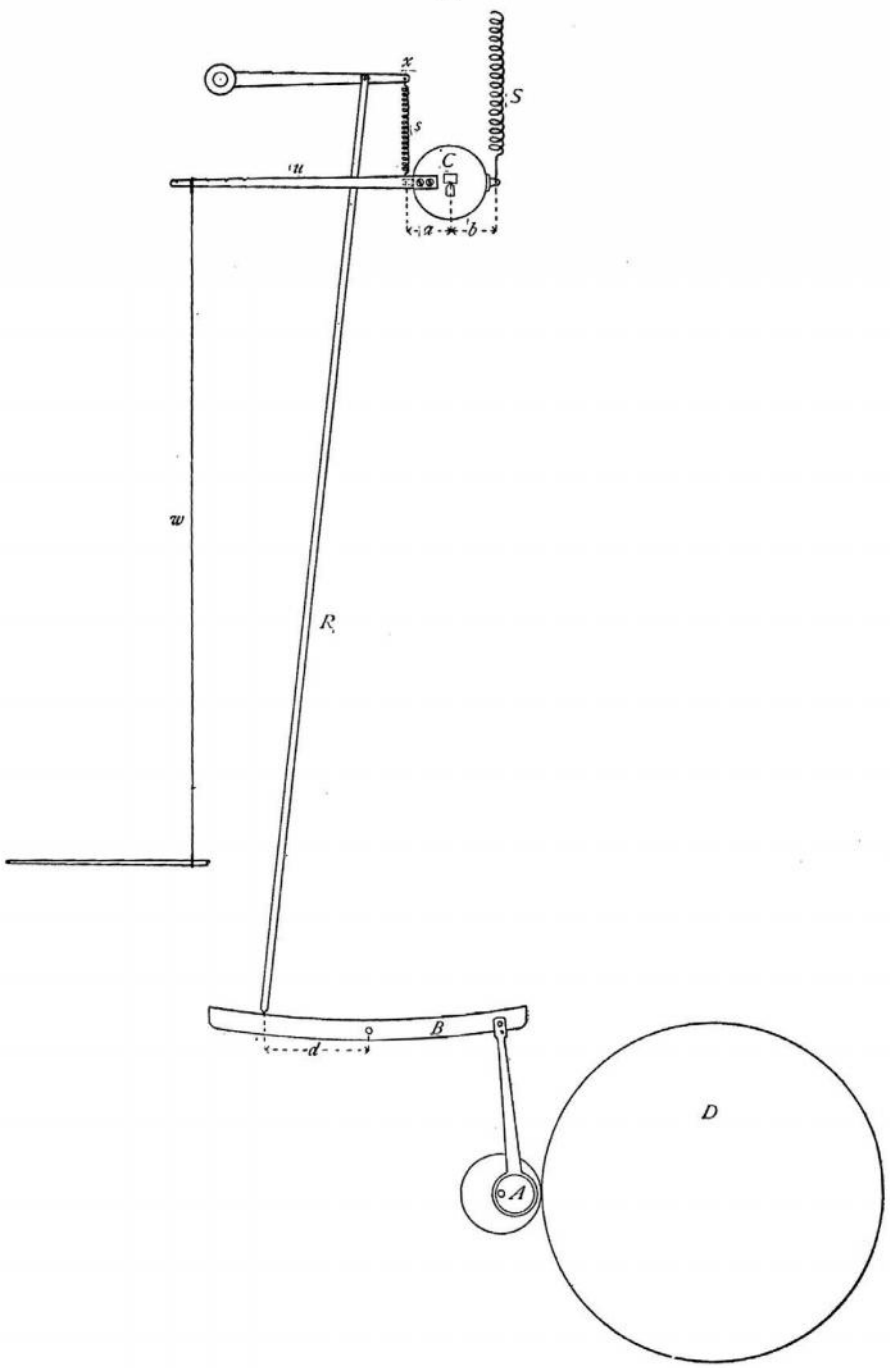
To obtain the greatest amplitude for a given number of elements, the ratios $\frac{l}{L}$ and $\frac{a}{b}$ should be as small as possible, but of course a limit is soon reached, when other considerations enter.

About a year ago a machine was constructed on this principle with twenty elements and the results obtained* were so encouraging that it was decided to apply to the Bache Fund for assistance in building the present machine of eighty elements.

Fig. 1 shows the essential parts of a single element. s is one of eighty small springs attached side by side to the lever C, which for greater rigidity has the form of a hollow cylinder, pivoted on knife edges at its axis. S is the large counter-spring. The harmonic motion produced by the excentric A, is communicated to x by the rod R and lever B, the amplitude of the motion at x depending on the adjustable distance d.

* Paper read before the National Academy of Science, April, 1897.

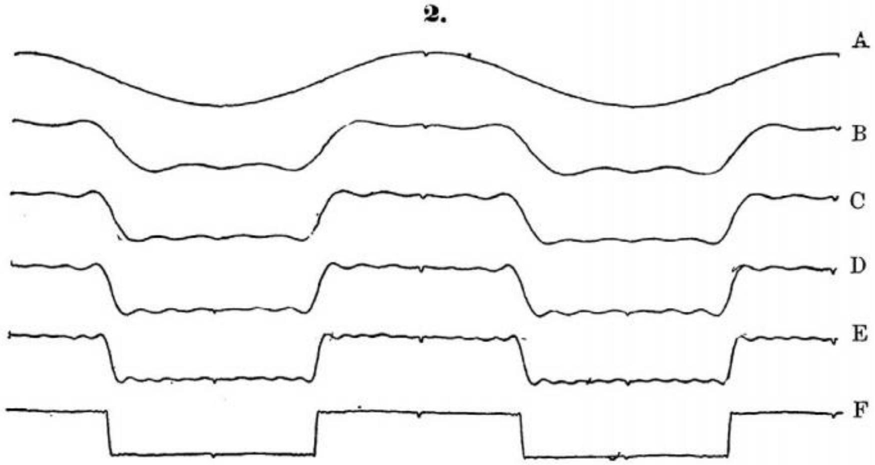
1.



The resultant motion is recorded by a pen connected with u by a fine wire w. Under the pen a slide moves with a speed proportional to the angular motion of the cone D. (Plate I.)

To represent the succession of terms of a Fourier series the excentrics have periods increasing in regular succession from one to eighty. This is accomplished by gearing to each excentric a wheel, the number of whose teeth is in the proper ratio. These last are all fastened together on the same axis and form the cone D. (Plate I.)

2.



A, one term; B, five terms; C, nine terms; D, thirteen terms; E, twenty one terms; F, seventy-nine terms.

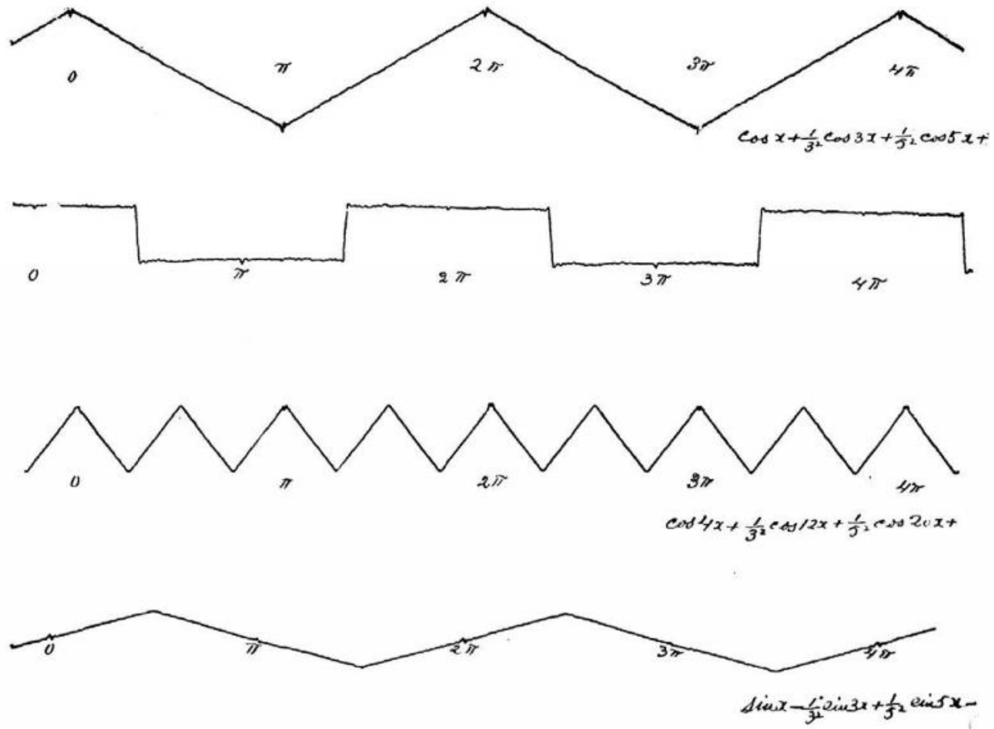
Turning the cone will produce at the points (x) motions corresponding to $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, etc., up to $\cos 80\theta$, and whose amplitudes depend on the distances d. The motion of the elements may also be changed from sine to cosine by disengaging the cone and turning all of the excentrics through 90° by means of a long pinion which can be thrown in gear with all of the excentric wheels at once.

The efficiency and accuracy of the machine is well illustrated in the summation of Fourier series shown in the accompanying figures.

Figure 2 shows the dependence of the accuracy of a particular function on the number of terms of the series. Figures 3, 4, 5, 6 and 7 are illustrations of a number of standard forms, and 8, 9 and 10 illustrate the use of the machine in constructing curves representing functions which scarcely admit of other analytical expression.

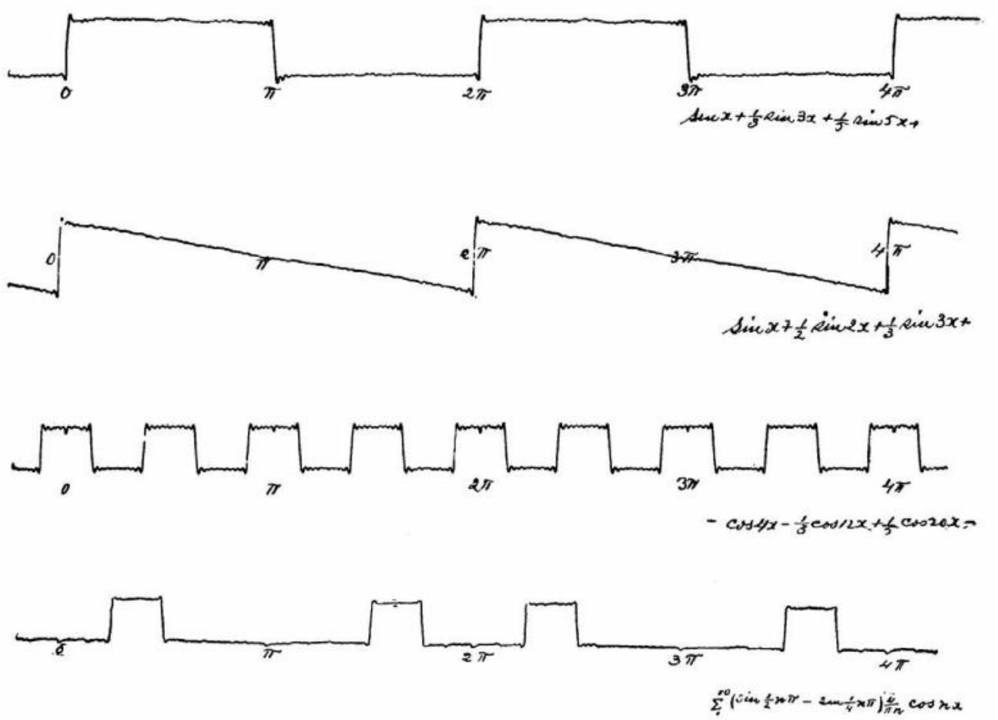
The machine is capable not only of summing up any given trigonometrical series but can also perform the inverse process

3.



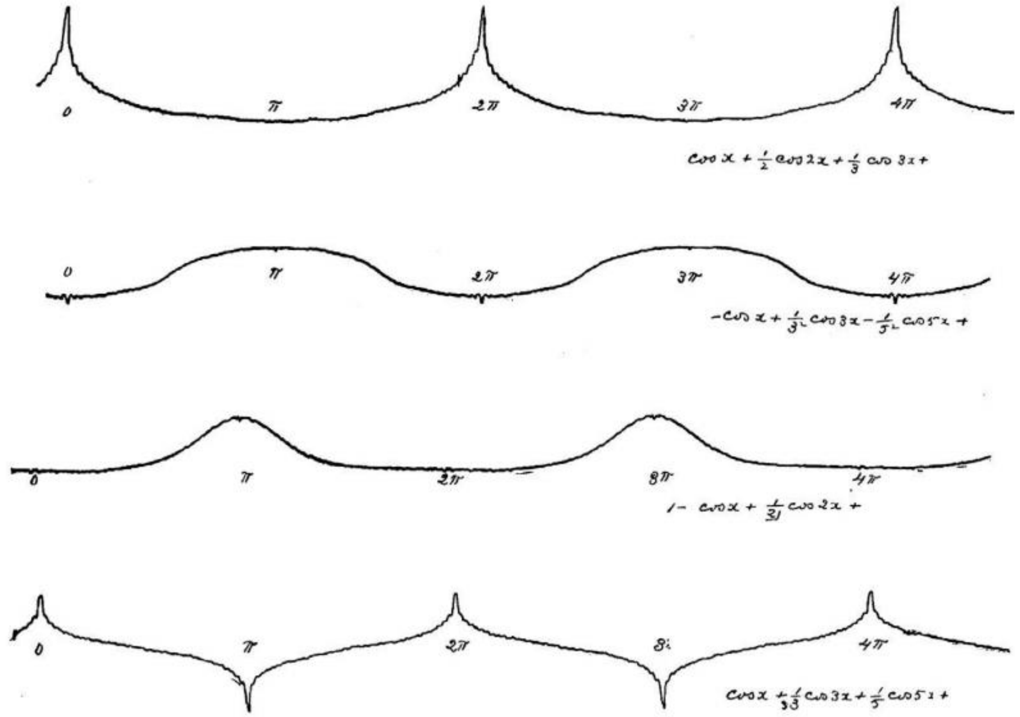
cos x + 1/3² cos 3x + 1/5² cos 5x +
cos 4x + 1/3² cos 12x + 1/5² cos 20x +
sin x − 1/3² sin 3x + 1/5² sin 5x −

4.



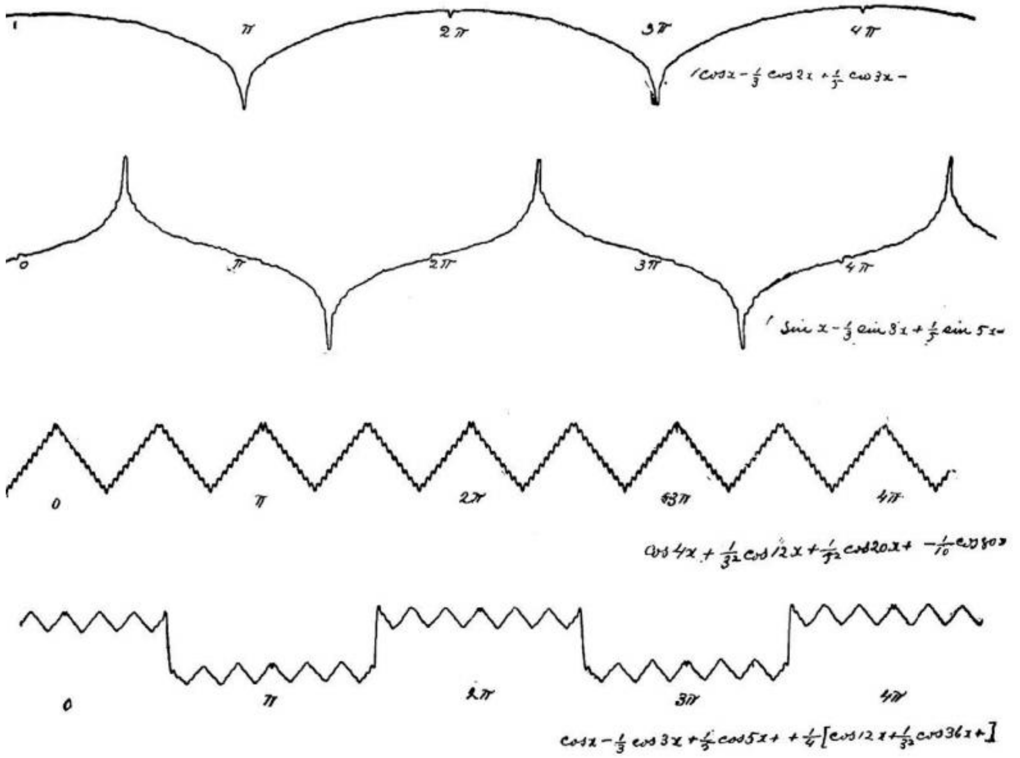
sin x + 1/3 sin 3x + 1/5 sin 5x +
sin x + 1/2 sin 2x + 1/3 sin 3x +
− cos 4x − 1/3 cos 12x + 1/5 cos 20x −

5.



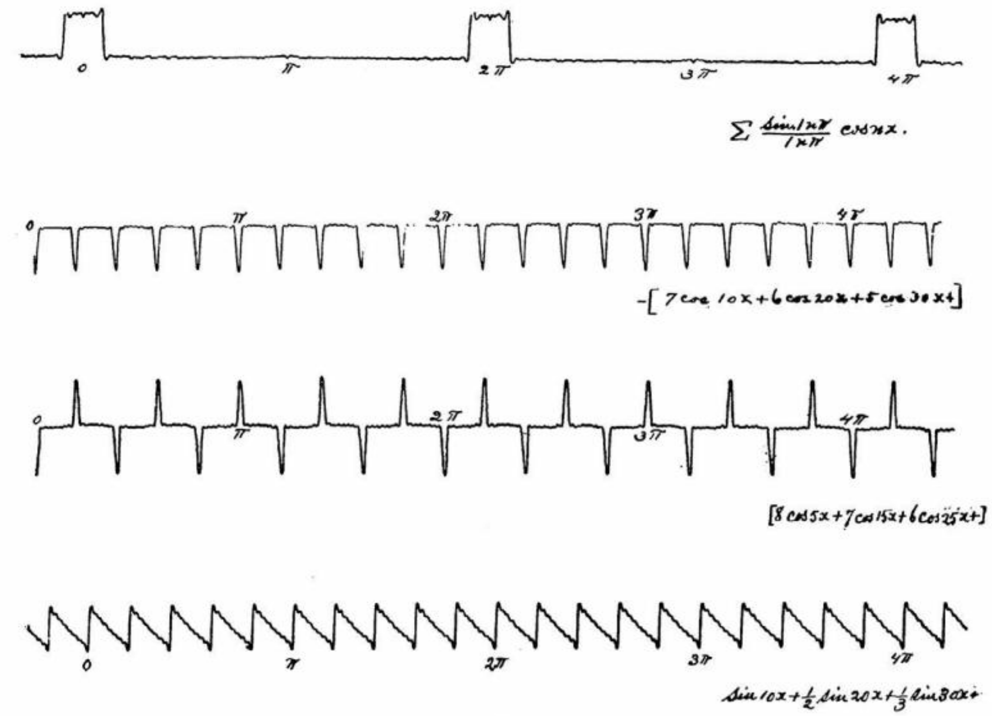
cos x + 1/2 cos 2x + 1/3 cos 3x +
− cos x + 1/3² cos 3x − 1/5² cos 5x +

6.

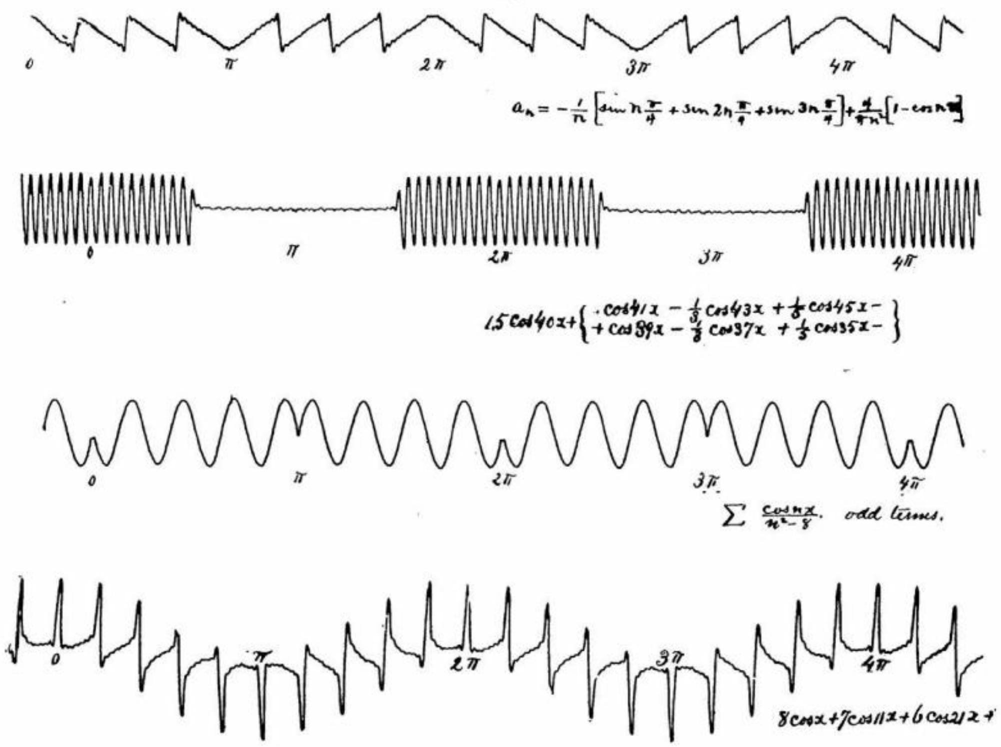


sin x − 1/3 sin 3x + 1/5 sin 5x −
cos 4x + 1/3² cos 12x + 1/5² cos 20x +
cos x − 1/3 cos 3x + 1/5 cos 5x + + 1/4 [cos 12x + 1/3² cos 36x +]

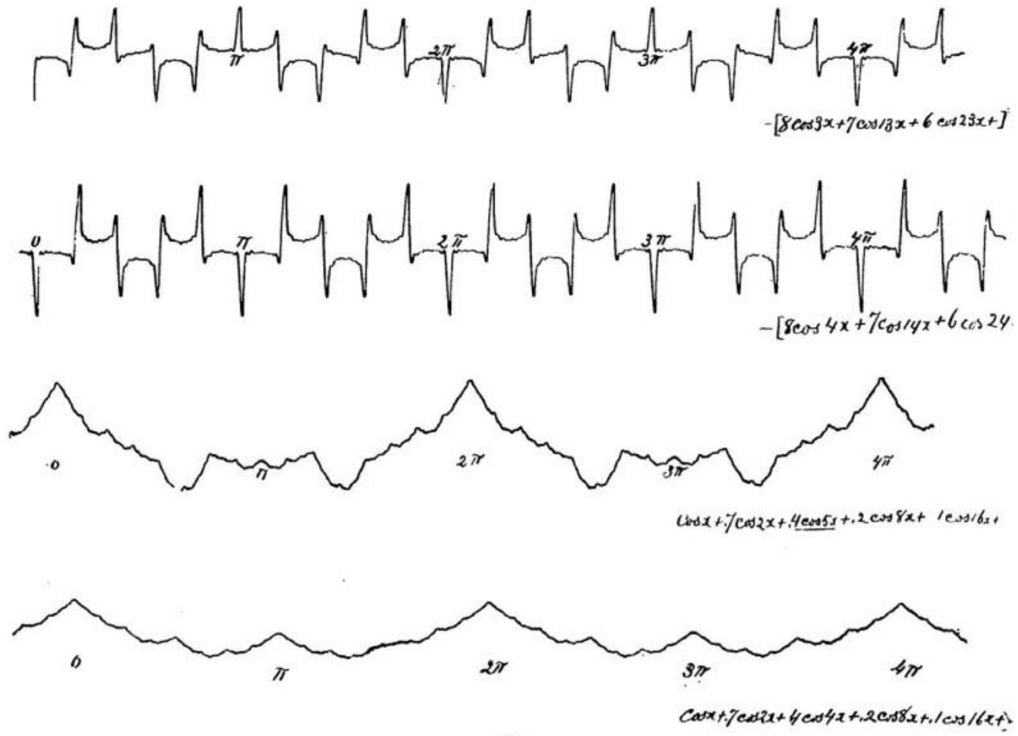
7.


0
π
2π
3π
4π
Σ sin 1nπ / 1nπ cos nx.
−[7 cos 10x + 6 cos 20x + 5 cos 30x +]
[8 cos 5x + 7 cos 15x + 6 cos 25x +]
sin 10x + 1/2 sin 20x + 1/3 sin 30x +

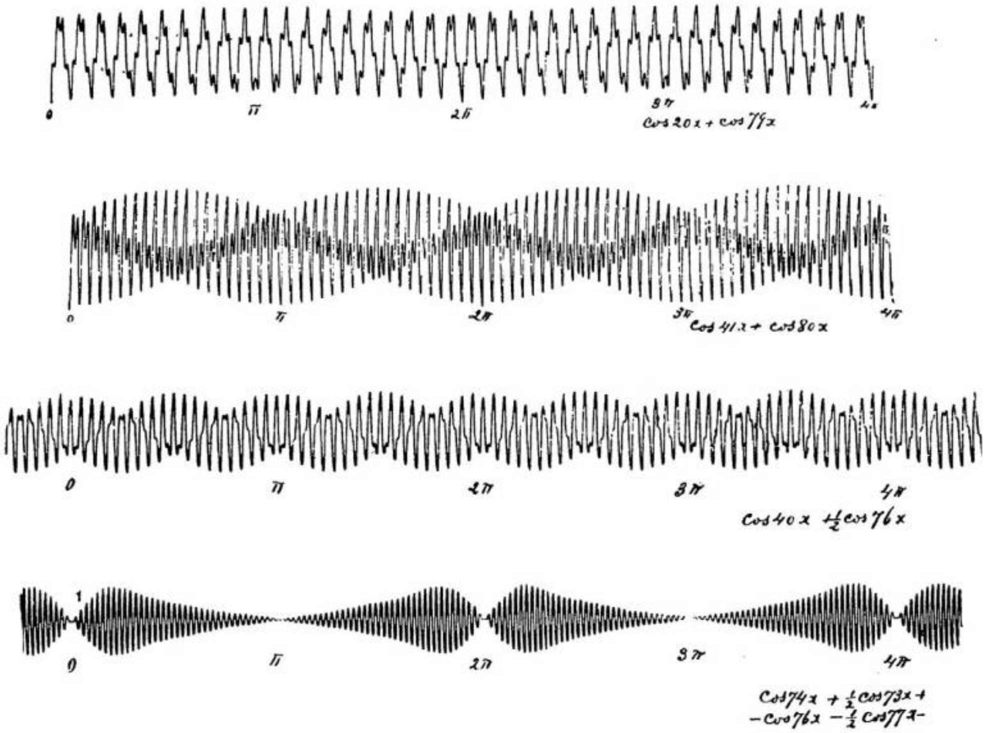
8.


0
π
2π
3π
4π
a_n = − 1/n [sin n π/4 + sin 2n π/4 + sin 3n π/4] + 4/πn² [1 − cos nπ]
1.5 cos 40x + { cos 41x − 1/3 cos 43x + 1/5 cos 45x − ; + cos 39x − 1/3 cos 37x + 1/5 cos 35x − }
Σ cos nx / n² − 8. odd terms.
8 cos x + 7 cos 11x + 6 cos 21x +

9.


0
π
2π
3π
4π
−[8 cos 3x + 7 cos 13x + 6 cos 23x +]
−[8 cos 4x + 7 cos 14x + 6 cos 24
cos x + .7 cos 2x + .4 cos 5x + .2 cos 8x + .1 cos 16x +
cos x + .7 cos 2x + .4 cos 4x + .2 cos 8x + .1 cos 16x +

10.

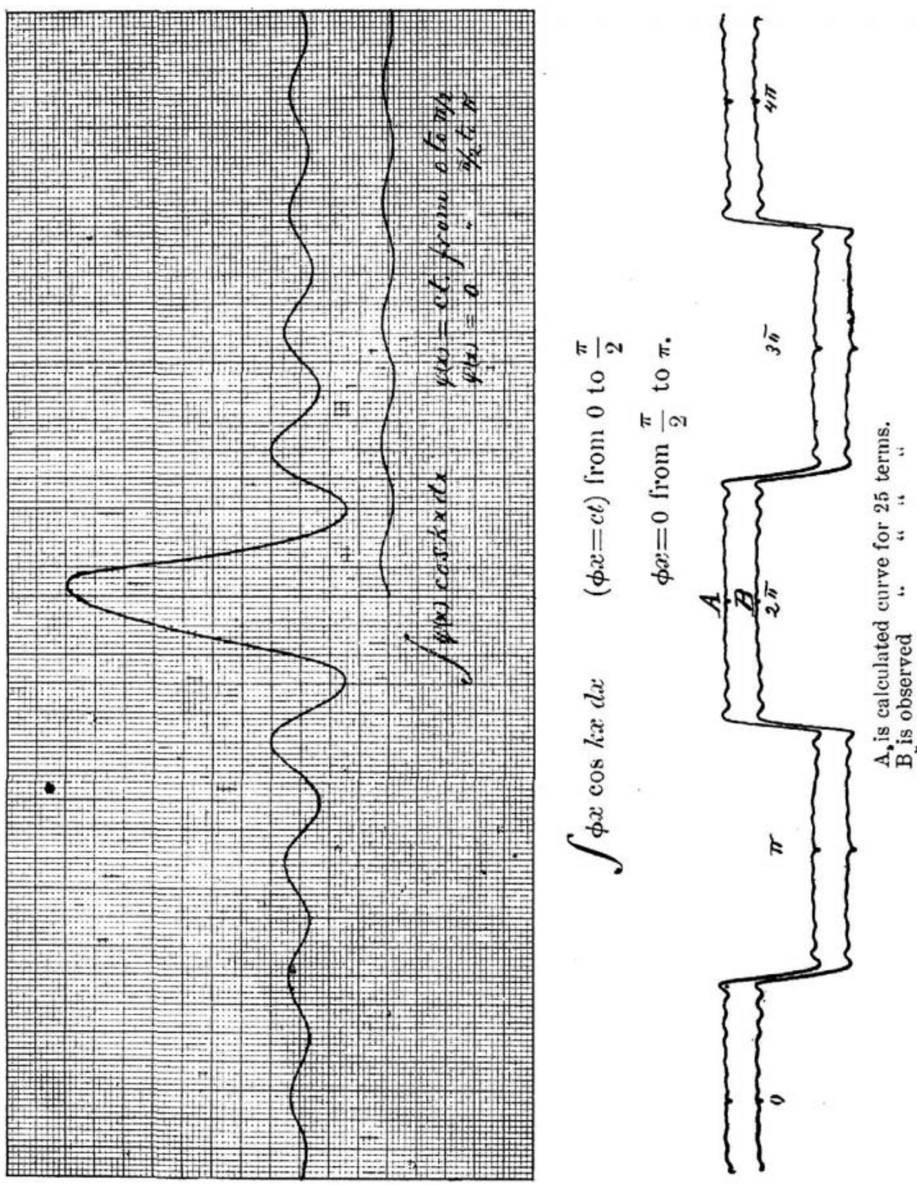

0
π
2π
3π
4π
cos 20x + cos 77x
cos 41x + cos 80x
cos 40x + 1/2 cos 76x
cos 74x + 1/2 cos 73x +
− cos 76x − 1/2 cos 77x −

of finding for any given function the coefficients of the corresponding Fourier series.

Thus if $$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x +$$

we have $$a_k = \frac{2}{\pi}\int_0^{\pi} f(x) \cos kx \, dx$$

11.



On the other hand, if n is the number of an element of the machine and a the distance between any two elements, and the

amplitude d (fig. 1) is proportional to $f(na)$, the machine gives

$$\sum_0^m f(na) \cos n\theta = \sum_0^{\pi} f(x) \cos \frac{m}{\pi} \theta x$$

which is proportional to a_k if $k = \frac{m}{\pi}\theta$. Hence to obtain the integral, the lower ends of the vertical rods R (Plate I) are moved along the levers B to distances proportional to the ordinates of the curve $y = f(na)$.

The curve thus obtained for a_k is a *continuous* function of k which approximates to the value of the integral as the number of elements increases. To obtain the values corresponding to the coefficients of the Fourier series, the angle $\theta = \pi$, or the corresponding distance on the curve, is divided into m equal parts. The required coefficients are then proportional to the ordinates erected at these divisions.

Figure 11 gives the approximate value of $\int \varphi(x) \cos kx \, dx$ when $\varphi(x)$ = constant from 0 to a, and is zero for all other values. The exact integral is $\frac{\sin ka}{k}$. The accuracy of the approximation is shown by the following table, which gives the observed and the calculated values of the first twenty coefficients for $a = 4{\cdot}0$.

$$\int_0^a \cos kx \, dx$$

n.	obs.	calc.	△
0	100·0	100·0	0·0
1	65·0	64·0	1·0
2	0·0	0·0	0 0
3	—20·0	—21·0	1·0
4	0·0	0·0	0·0
5	12·5	13·0	—0·5
6	—1·5	0·0	—1·5
7	—9·0	—9·0	0 0
8	0·0	0·0	0·0
9	6 0	7·0	—1·0
10	0·0	0·0	—2·0
11	—6·0	—6·0	0·0
12	0·0	0·0	—0.0
13	4·0	5·0	—1·0
14	—2·0	0.0	—2·0
15	—4·0	4·5	0·5
16	0·5	0·0	0·5
17	8·5	4·0	—0·5
18	—1·0	0·0	—1·0
19	—3·5	—3·0	0·5
20	0·0	0·0	0·0

The average error is only 0·65 of one per cent. of the value of the greatest term.

The accuracy of the result is also shown in curves A and B (fig. 11). The former gives the summation of the calculated terms and the latter of the observed.

Another illustration is given in figure 12 in which

$\varphi(x) = e^{-a^2x^2}$

For $a = {\cdot}1$ the following are the values of the coefficients of the first twelve terms of the equivalent Fourier series.

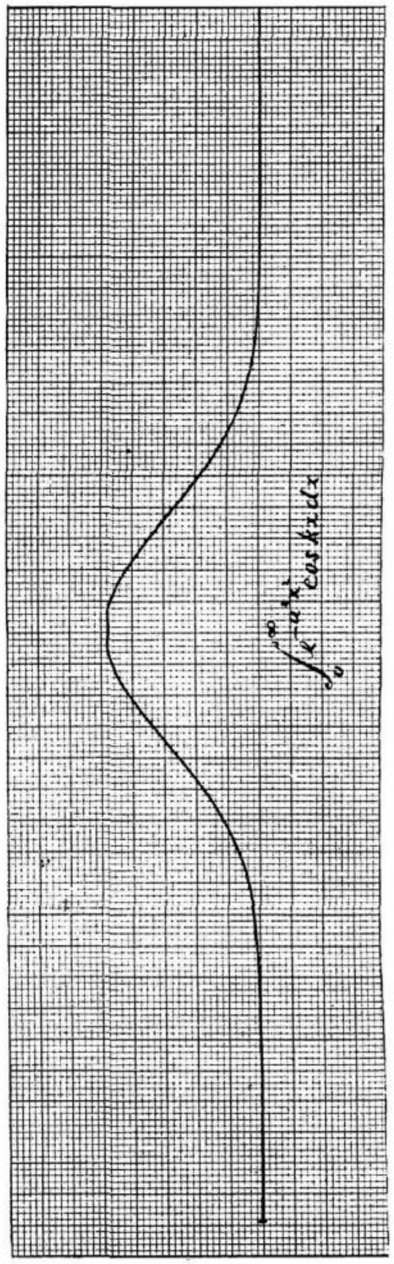
$$\int_0^\infty e^{-a^2x^2} \cos kx\, dx$$

n.	obs.	calc.	$\triangle$
0	100·0	100·0	0·0
1	95·0	96·0	—1·0
2	85·0	86·0	—1·0
3	70·0	70·0	0·0
4	53·0	54·0	—1.0
5	38·0	38·0	0·0
6	25·0	25·0	0·0
7	16·0	15·0	1·0
8	8·8	8·0	—0·8
9	5·0	4·5	0·5
10	3·6	2·0	1·6
11	2·4	1·0	1·4
12	1·6	0·5	1·1

Here the average error is only 0·7 per cent of the value of the greatest term.

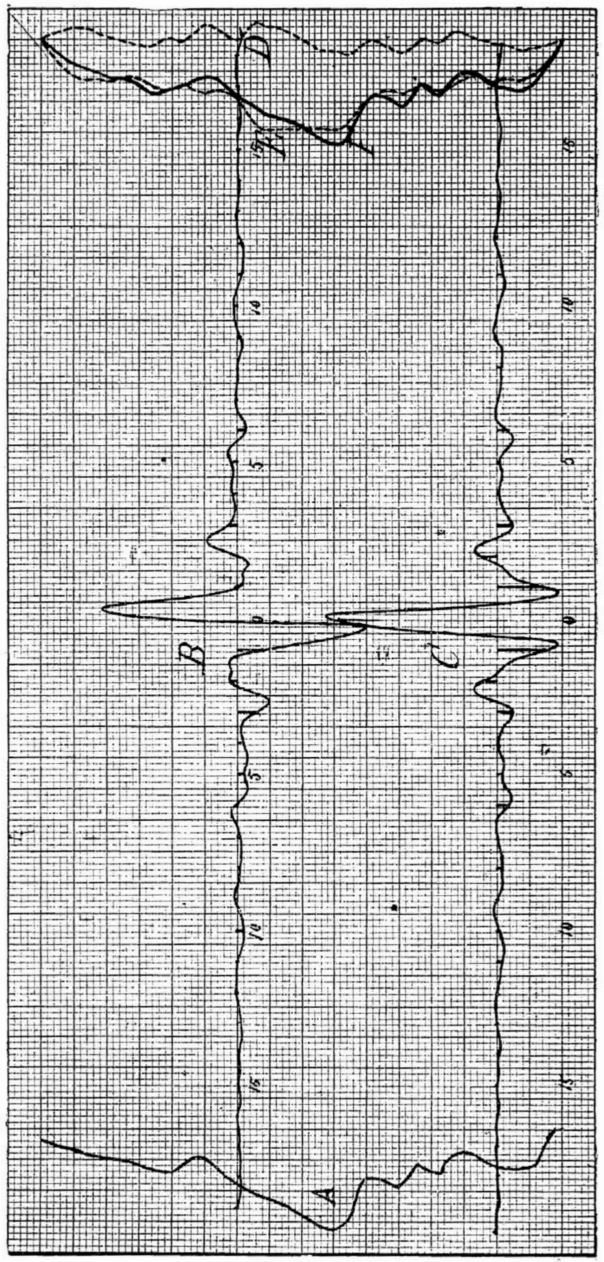
The complete cycle of operations of finding the coefficients of the complete Fourier series (sines and cosines) and their recombination, reproducing the original function, is illustrated in figure 13. *A* is the original curve. *B* and *C* are the values of $\int\varphi(x)\sin kx\,dx$ and $\int\varphi(x)\cos kx\,dx$ respectively. Their in-

12.



$$\int_0^\infty e^{-a^2x^2} \cos kx\, dx$$

13.



$$F = \phi(x) = 15\sin\theta - 75\cos\theta - 10\sin 2\theta + 21\cos 2\theta + 20\sin 3\theta - 24\cos 3\theta + 8\sin 4\theta - 3\cos 4\theta + 10\sin 5\theta - 8\cos 5\theta$$
$$-2\sin 11\theta - 10\cos 11\theta - 5\sin 12\theta + 1\cos 12\theta + 4\sin 13\theta + 0\cos 13\theta + 0\sin 14\theta + 0\cos 14\theta + 1\sin 15\theta + 0\cos 15\theta$$

$$F = \phi(x) = -8\sin 6\theta - 18\cos 6\theta + 4\sin 7\theta - 1\cos 7\theta + 2\sin 8\theta - 8\cos 8\theta - 5\sin 9\theta + 1\cos 9\theta + 6\sin 10\theta - 2\cos 10\theta$$
$$-2\sin 16\theta - 1\cos 16\theta + 1\sin 17\theta + 0\cos 17\theta + 3\sin 18\theta + 3\cos 18\theta - 2\sin 19\theta + 0\cos 19\theta + 0\sin 20\theta + 0\cos 20\theta$$

$$A = \phi(x) \qquad B = \int\phi(x)\sin kx\,dx \qquad C = \int\phi(x)\cos kx\,dx$$

$$D = \sum_0^{20} B\sin n\theta \qquad E = \sum_0^{20} C\cos n\theta \qquad F = D + E$$

tersections with the ordinates midway between the heavier ordinates, give the coefficients of the sine and cosine series respectively. The sums of the first twenty terms are represented by the curves *D* and *E*, and finally the sum of these two curves produces the curve *F*, which agrees sufficiently well with the original to be easily recognizable.

It appears, therefore, that the machine is capable of effecting the integration $\int \varphi(x) \cos kx\, dx$ with an accuracy comparable with that of other integrating machines; and while it is scarcely hoped that it will be used for this purpose where great accuracy is required, it certainly saves an enormous amount of labor in cases where an error of one or two per cent is unimportant.

The experience gained in the construction of the present machine shows that it would be quite feasible to increase the number of elements to several hundred or even to a thousand with a proportional increase in the accuracy of the integrations.

Finally it is well to note that the principle of summation here employed is so general that it may be used for series of any function by giving to the points (p) the motions corresponding to the required functions, instead of the simple harmonic motion furnished by the excentrics. A simple method of effecting this change would be to cut metal templates of the required forms, mounting them on a common axis. In fact the harmonic motion of the original machine was thus produced.

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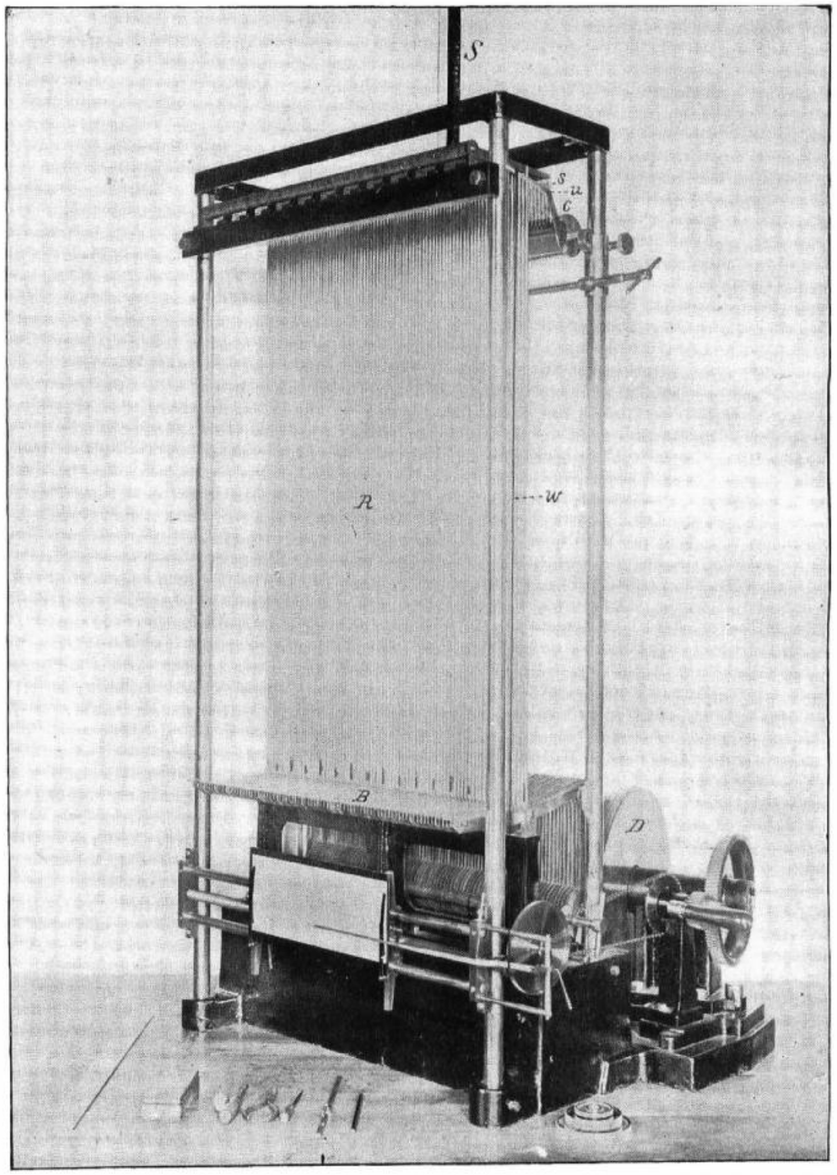

A New Harmonic Analyzer.