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# DESCRIPTION OF A MACHINE

FOR

RESOLVING BY INSPECTION CERTAIN IMPORTANT FORMS

OF

# TRANSCENDENTAL EQUATIONS.

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From the Transactions of the Cambridge Philosophical Society.

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CAMBRIDGE:

PRINTED BY J. SMITH, PRINTER TO THE UNIVERSITY.

M.DCCC.XXXII.

# XV. *Description of a Machine for resolving by Inspection certain important Forms of Transcendental Equations.*

By Sir J. F. W. HERSCHEL,

MEMBER OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *May* 7, 1832.]

(1.) In the course of a conversation with Mr. Babbage on the subject of applying machinery to the performance of numerical computations it occurred to me that, seeing the perfection with which every description of wheelwork and rectilinear or parallel motion can now be executed, almost any combination of circular functions involving as well the arc itself and its multiples and submultiples, as their sines, cosines, chords, &c. *might* be represented by the motion of a point, or by the difference of motions of two points, regulated by mechanism, with almost perfect precision, and that it would therefore need nothing more than to mark the arrival of such a point at some definite line or circle, or to cause the machine in some way or other to come to rest when such difference should attain a given magnitude, or when any other assigned condition should be fulfilled, to obtain a solution of the equation expressive of that condition; which solution should be limited only, in point of exactness, by the precision of the workmanship and the accuracy attainable in hitting the coincidence and reading off the result. This idea (the principle of which, it will be observed, is entirely independent of all numerical calculations, or of any application of wheelwork to perform such calculations, and turns entirely on the modifications which an uniform motion communicated to the *primum mobile* can be made to receive in passing through a train of wheels, levers, &c. and on the accuracy with which circles and straight lines can be graduated and read off) I mentioned as it occurred, and presently illustrated it by application to the well known equation between the excentric and mean anomalies in the elliptic motion of a planet.

(2.) The combination of movements which then suggested itself for this purpose, though theoretically correct, was practically liable to many objections. It happened however that I was at that time engaged in investigating the elliptic orbits of some of the most remarkable double stars; and in the course of that enquiry had continual occasion for the numerical resolution of cases of this equation in every state of the data. Finding the preliminary trials requisite for establishing a rapid conveyance of the successive approximations consume a great deal of time, even more than the approximations themselves when once effectually entered upon, I set myself to consider whether some simple contrivance free from such objections might not be found which would give me by inspection at least a first approximation to the solution, and thus prove of immediate practical utility. After one or two failures from attempting to unwind a thread from the circumference of a wheel revolving on a *fixed* center, and after rejecting as impracticable Newton's mechanical solution of the problem by the rolling of a wheel upon a plane, it occurred to me that a wheel will revolve uniformly and unwind a thread from it with a uniform motion just as well whether it revolve on its own center as an axis, or be carried round, bodily, by the attachment of its center to the end of a revolving arm, whose length may either be permanent or adjustable by a slider. Hence arose the following construction which seems to be as simple as the nature of the problem admits.

(3.) In figure 1, $C$ is a *horizontal axis* at right angles to the plane of the paper, on which is firmly fixed at right angles, or in the plane of rotation of the axis, a cross piece, having a bevelled groove $PQ$, in which the slider $SR$ moves freely backwards and forwards until arrested and fastened by a clamping screw. Projecting forwards from the slider $SR$, and firmly rivetted into it is a short pin $B$, on which, as on an axis, the *excentric wheel DEF* is set, and permitted to turn freely until also arrested and clamped. These clampings are both easily performed by means of a male screw cut on the end of the pin $B$ on which is fitted a female screw cut in the cross head, or *clamping key* $mn$. When this screw is loosened the slider $SR$ can move in its groove and the wheel $DEF$ revolve on the pin $B$, but when tightened these motions are rendered impossible, and the whole apparatus, cross-piece, slider, and excentric wheel become part of the axis $C$, and revolve with it as one mass. Thus we are enabled to adjust, first, the distance of the center $B$ from the axis, and secondly, the position of a given point in the circumference of the excentric wheel with respect to a horizontal line.

(4.) The axis $C$ also carries an *index-arm CH* furnished either with a simple *index* like a clock's hand, or if accuracy be required with a vernier adapted to subdivide the graduations of an *index-circle AH* concentric with $C$, and divided into degrees, &c. Around the excentric wheel $DEF$ a thread of some fine and inextensible material (such as dentist's silk or very delicate silver wire) is wound. For this purpose the edge of the excentric wheel must be turned truly cylindric, leaving at either side a slight elevation like a parapet to prevent the thread from slipping off. In the front edge of this elevation must be cut a small notch $F$, in which the end of the thread is to be carried off the edge of the wheel, and fastened on a pin $P$ stuck (like the tuning pegs in the handle of a violin) into the face of the wheel, so as to allow of lengthening or shortening the thread a little by winding it more or less on this peg. The other end of the thread where it leaves the circumference of the wheel at $E$ hangs down vertically, and has suspended to it a vertical, straight, divided scale $MN$, the divisions of which are marked and read off by the intersection of its fiducial line $MKN$ with the horizontal straight edge $IKJ$ at $K$, for which reading off a microscope moveable on $IJ$, and furnished with a micrometer, *might* be used should such a degree of precision be required. The straight edge $IJ$ is itself also graduated for a purpose to be presently explained, and might in like manner be read off microscopically.

(5.) The axis $C$ also carries on it a barrel $a\,b\,c$, round which and round a fixed smooth pin $x$ a string makes two or three coils (embracing both the barrel and pin in each coil), after which it is attached at both its ends to a weight $w$, which must be such as to afford a sufficient tension to the string to produce a smooth uniform friction on the barrel, and thereby to prevent the axis from turning without the continual application of a moving power, and to bring it at once to rest, without jerks, dragging, or recoil, when the power ceases to act. The power may be applied either at the circumference of the barrel by the hand, or by a handle fixed on the axis $C$, and not represented in the figure. The barrel, weight, and pin $x$, are supposed to lie *behind* the plane of the index circle $AH$, which is that of the paper—the rest of the apparatus *before* it.

(6.) Let unity (1) represent the radius of the excentric wheel, *plus* that of the thread, and $e$ the length of the interval $CB$ between the centers of the index circle and excentric—or the *excentricity* of the latter. Then if we draw $AC$ parallel to the horizon, and put $u$ for the angle $ACB$, the perpendicular $BG$ will $= e . \sin u$. Now, were the center of the excentric wheel preserved constantly on the level of the line $AC$, and that wheel itself merely made to revolve uniformly by the rotation of $C$ about $B$ instead of $B$ about $C$, the part of the thread which would be wound up on its circumference by this rotation from the commencement of the motion would be represented by $1 \times u = u$. This then would be the quantity by which the vertical scale would, on that supposition, be raised above its original position. But in the actual case, not only is the thread so wound on the wheel, but the center $B$ of the wheel being raised above $AC$ by $e \sin u$, carries up with it the wheel, thread, and scale, all by the same quantity. Therefore the total elevation of the scale due to both causes acting at once will be $u + e . \sin u$. If then we put $A$ for this elevation, there will subsist between $u$ and $A$ the transcendental relation

$$u + e . \sin u = A,$$

which is that of the problem of the excentric and mean anomalies in the elliptic motion of a planet, the arcs being reckoned from the aphelion. If we would reckon them from the perihelion, we have only to wrap the thread the other way round the excentric wheel, when the equation expressing the relation between $A$ and $u$ becomes

$$u - e . \sin u = A.$$

(7.) It appears from what has been said, that the values of $u$ being read off on the index circle at $H$, those of $A$ will be read off on the vertical scale at $I$. The zeros of both readings correspond to the horizontal position of the line $CB$, which position having been once well ascertained and verified, may be afterwards at any time recovered by a small level $PS$ screwed on the cross-piece $PQ$, and adjusted accordingly. Both zeros are adjustable—that of the index circle by the stiff-friction motion of the index-arm $CH$, and that of the scale, first, coarsely by loosening the excentric wheel on its center, and winding or unwinding the thread on its circumference, secondly, more delicately by screwing or unscrewing the peg $p$ to which its end is fastened.

(8.) The division of the vertical scale must be into 360 equal parts or degrees, the whole length from 0 to 360 being that of the circumference of a circle whose radius is 1. This length may be determined (better than by any attempt to measure the radius) by turning the axis $C$ round through one complete revolution, and noting by temporary marks or dots on the scale the points intersected on the fiducial line, by the horizontal straight edge $IJ$ at the two extremities of its motion. The interval between these dots is the value of 360 parts required, and must be subdivided accordingly. It is evident that whatever be the value of $e$ in the equation $u+e \, . \sin u = A$ an increase of 360° in $u$ must correspond to 360° increase in $A$, so that the accuracy of this process is, theoretically speaking, independent of the position of $B$ on the slider—but practically it is preferable, before executing this most essential of all the preliminary operations, to bring the center of the excentric as nearly as possible to coincidence with the axis $C$, because in that position any slight deviation from horizontality in the line $IJ$ will not influence the result.

(9.) The perfect horizontality of this line is however of material import to the correct performance of the machine. It is easily verified and secured by a spirit level and screw adjustments at one or both ends. The straight edge *IJ* itself should be graduated into equal parts, corresponding to decimals and centesimals of the radius (1), to obtain which it is only requisite to measure the length of 360° on the vertical scale as before obtained, and divide the result by $2\pi = 2 \times 3{\cdot}14159$, &c. It is likewise convenient that the straight edge should have a screw motion in a horizontal direction, by which its error of zero may be destroyed without altering its horizontal position.

(10.) The fiducial line of the vertical scale, or that whose intersection with the straight edge marks the values of *A*, ought to be exactly vertical when the scale hangs freely. This condition is not indeed essential to correct performance, provided it have been graduated in its inclined position, but, if satisfied, it greatly facilitates other essential adjustments, and may therefore be regarded as one of those which must be gone through. It is very easy, all that is needed being to make the lower end of the vertical scale terminate in a narrow tail-piece of lead, which being flexible, we may by bending it a little one way or other *tilt* the center of gravity of the scale, so that the fiducial line shall coincide in direction with a fine plumb line.

(11.) The values of *e* may be read off in two ways, either, first, by a graduation on the slider itself, (in which the fixed part may serve as a vernier to the moveable one,) or on the straight edge. In the former case the graduation must be, like that of the straight edge, into decimals and centesimals of the unit radius determined as above described, and the zero point must be ascertained as follows. Set the straight edge *IJ* and the slider both horizontal by their respective levels, adjust the index hand *H* to 0°, and bring the center *B* of the excentric wheel as nearly as may be over *C*, the center of the horizontal axis; then make very slowly one complete revolution of that axis, noting carefully at its commencement, and end, and at every 30°, the reading off of the straight edge *IJ*, where it is intersected by the fiducial line of the vertical scale, (which should always be allowed to attain perfect rest free from lateral oscillations). If the point of intersection be found not to have varied on the straight edge, it is evident that the coincidence of *B* and *C* must have been perfect; but if otherwise, its extreme variations will mark out the diameter of the small circle which *B* continues to describe about *C*. This must be destroyed by shifting the place of *B* on the slider, and if needed, by altering the place of the slider itself on the axis (which may possibly have been originally erroneous, so as not to allow of the line described by *B* in its groove, passing through *C* at all). As soon as this is done, and the invariability of the above-mentioned intersection ascertained, a fine line must be drawn directly across the slider and its groove at each end, and thus the zero points of the divisions both of the slider and its verniers at each end are secured. The divisions should be carried along the whole length of the cross piece and along both edges of the slider, on one forwards, and on the other backwards, by which means the machine is equally adapted for positive and negative values of $e$. If the cross-piece be long enough, the machine will of course serve for values of $e$ equal to or greater than unity as well as less.

(12.) If we would read off $e$ on the horizontal straight edge, we must first set the index hand to 0, and then to 180°—the difference of the readings is equal to $2e$, being the diameter of the circle described by *B* transferred to the straight edge by perpendiculars to the horizon.

(13.) The only adjustment of any degree of delicacy is that of the horizontal situation of $BC$, or of the cross-piece level, the line $BC$ being imaginary and intangible. Good workmanship will of course ensure a very near approach to parallelism between this line and the two sides of the cross-piece; but if an error be still supposed to exist it may be detected by the following process: make $e = 1$, or set the slider to 1·00, and then beginning at 0° of the index read off the value of $A$, corresponding to equal small increments of $u$ (for example from degree to degree) round a complete semicircle to 180°. Then, since we have

$$\frac{dA}{du} = 1 + \cos u, \text{ and } \frac{d^2A}{du^2} = -\sin u,$$

a comparison of differences and second differences will readily enable us to perceive whether any appreciable deviation from the true position exists. It is only by the differences of readings that an error in the zero of $u$ can be separated from one in that of $A$, (in which is included that of the index reading at $H$), as the form of the equation $u + e \,.\, \sin u = A$ will easily make evident.

(14.) However truly the cylindrical form of the excentric wheel is attained, if the axis of the cylinder be not parallel to that of the index circle, the thread will wind off an ellipse in place of a circular arc, of which equal portions will not correspond to equal angles of rotation. This then affords a means of detecting and rectifying such want of parallelism in the axes in question; but as its effect would be confounded with that arising from the error in the origin of $u$ just mentioned in the last article, it will be preferable to examine through a whole revolution of the index axis (which should be perfectly horizontal) whether the thread maintains precisely the same distance from either edge of the excentric wheel as it wraps itself round it.

(15.) Supposing all these adjustments well made, the workmanship good, and the graduations such as may be executed, there seems no reason in the nature of the case why the mechanical solution afforded by the apparatus we have above described should not possess equal precision with any astronomical observation, and therefore be available in many instances where extreme nicety of computation is not required, or where several hypothetical ellipses may require to be tried in the calculation of the orbit of a comet or other celestial body. In the enquiries to which I have already alluded I found in fact a very material saving of time and trouble from the use of such an instrument, though constructed in the rudest manner from materials casually at hand.

(16.) The solution of the equation $u + e \, . \sin u = A$ includes that of its derivative forms

$$u + a \, . \sin (u + b) = A,$$

$$\text{and } u + a \, . \sin u + b \, . \cos u = A,$$

where however only sines and cosines of $u$ are involved. If we would introduce tangents, secants, &c. we must have recourse to a modification of our mechanism. For instance, suppose the equation to be resolved were

$$u + p \, . \tan u = A,$$

then, retaining the slider, excentric wheel, and horizontal axis *C*, as in the contrivance already described, let the straight edge *IJ* (fig. 2) instead of being permanently fixed in a horizontal position, be made to revolve on a center *I* vertically below *C*, with half the angular velocity of the index hand, which may be done either by a toothed wheel working into another of twice the diameter, or by catgut wrapping tightly round cylinders in the same proportion, and let the zero of both rotations be in the horizontal positions of *CB* and

$IJ$. Moreover, let the index circle instead of being as before concentric with the axis $C$ be now described about the center $I$, and the index arm with its vernier be connected with that axis as we before supposed to be with the axis $C$.

(17.) Put $u$ for the arc read off on this circle or the angle $XIJ$, also let $e = CB$, $BD = BE = 1$; then since the angle $GCB$ is twice $XIJ$, or $= 2u$, we have

$$MJ = DE + EJ - DEM = DE + BG + KJ - DEM,$$
$$= 2u + e \,.\, \sin 2u + KL \,.\, \tan u - DEM.$$

$DEM$ is the part of the thread included between its point of attachment to the wheel, and the zero of the scale—it is therefore an arbitrary constant. If we put $c$ to represent it, and observe that $KL = CL - CG - GK = CI \,.\, \text{cotan}\, u - CB \,.\, \cos 2u - 1$ $= b \,.\, \text{cotan}\, u - e \,.\, \cos 2u - 1$, (putting $b$ for the constant distance $CI$) we shall have for the expression of $MJ$,

$$MJ = 2u + e \,.\, \sin 2u - c + \{b \,.\, \text{cotan}\, u - e \,.\, \cos 2u - 1\} \,.\, \tan u$$
$$= (2u + b - c) + (e - 2) \,.\, \tan u.$$

If therefore we put $MJ = 2A + b - c$; and take $e = 2p + 1$, the relation between $A$ and $u$ becomes

$$u + p \,.\, \tan u = A,$$

which is the equation proposed.

(18.) In order then to adopt the above construction to any given case of this equation we must set the slider so that the distance $BC$ shall $= 2p + 1$. That is to say, the zero of the scale of the divisions on the slider must commence at a distance from $C$ equal to the unit or radius of the excentric + that of the thread, and its graduation must be into parts *double* of the corresponding parts of $p$. In like manner the graduation of the vertical scale $MJ$ must begin at the point where the revolving straight edge intersects it in its horizontal position (in which also $CB$ must be adjusted to be accurately horizontal,) and the parts of this scale must be double of the corresponding parts of $A$, that is, *double* degrees of $u$. Each part therefore must be one 180th part of the circumference of the circle $DEF$, as measured by the unwinding of the thread in the mode explained in (Art. 8.)

(19.) If the equation proposed were the rather more general one

$$u + p \, . \tan u + q \, . \sec u = A,$$

we might employ the same construction slightly modified by making the straight edge $IJ$ revolve attached to an arm at right angles to the axis of the lower wheel, so that the motion of rotation of $IJ$ shall as before be uniform, and half as swift as that of $CB$, but its direction not passing through the center $Q$ of its revolutions. All other things remaining as in (Art. 16.), let the perpendicular $QI = f$ then will the value of $MJ$ be as before,

$$MJ = 2u + e \sin 2u + (CL - CG - GK.) \, . \tan u,$$

but in this case we have

$$CL = f \, . \operatorname{cosec} u + b \, . \operatorname{cotan} u,$$

so that the value of $MJ$ becomes

$$MJ = 2u + e \, . \sin 2u + \{f \, . \operatorname{cosec} u + b \, . \operatorname{cotan} u + e \, . \cos 2u - 1\} \, . \tan u,$$

$$= (2u + b) + (e - 1) \, . \tan u + f \, . \sec u,$$

that is, putting $MJ = 2A + b$ ; $e = 2p + 1$ ; $f = 2q$,

$$A = u + p \, . \tan u + q \, . \sec u.$$

(20.) It is needless to enlarge on the adjustment of these contrivances, which are merely introduced as specimens of the variations which a trifling change in the construction of our mechanism is capable of making in the form of the equations resolved. I will only observe here, that as ovals can now be turned of almost any excentricity, which shall scarcely deviate perceptibly from the true elliptic figure, so, in all these and similar cases, by substituting elliptic for circular *excentric wheels*, the arc $u$ instead of being directly related to the sines, cosines, tangents, &c., involved in the equations may, without the slightest increase of mechanism, or any additional difficulty in the process of solution, be replaced by a transcendent of that form which depends on the rectification of the ellipse.

(21.) It is almost needless to mention that any mechanical contrivance which converts a uniform motion $u$ into another not uniform, but varying according to any function $\phi(u)$ of the former, affords either a solution of the equation $\phi(u) = A$, or a tabulation of the values of $\phi(u)$, just as we please. In the one case we have only to arrest the motion at equal intervals of the scale on which the graduation of $A$ is engraved, and read off the graduation of that on which $u$ is represented. In the latter the movements must be arrested at equal intervals of $u$, and the values of $A$ read off. Thus the tabulations of the direct and inverse functions proceed, *pari passu*.

(22.) It is not my intention in this paper to enter at large into the general question of the representation of analytical functions by continued motion, though perhaps I may take a future opportunity of so doing. I will only here consider one other case, by which, without any great complication of machinery, the principle I have above adopted may be extended to equations containing several transcendental relations, such as

$$u + p \,.\, \sin m\,u + q \,.\, \sin n\,u = A,$$

and others of the same nature. Suppose, instead of attaching our excentric wheel to a point in the revolving arm $BC$ we attach it to a second revolving arm, whose center of rotation occupies the point which that of the excentric itself occupied in the original construction, and let this second arm have a velocity of rotation in a constant ratio to that of the first, a condition attainable by contrivances to be presently considered. In that case our construction will be as in fig. (4), respecting which figure we will establish the following notation.

$CB = e$; $BB' = e'$; $B'E' = 1$; angle $ACB = \alpha u$; $DBB' = \beta u$; $MK = x$, and $D'E'M = c$,

when we have

$$MK = x = D'E'K - D'E'M = D'E' + B'G - D'E'M,$$
$$= (\alpha + \beta)\, u + B'T + BV - c,$$
$$= (\alpha + \beta)\, u - c + e \,.\, \sin \alpha u + e' \,.\, \sin (\alpha + \beta)\, u,$$

assume therefore $\alpha = m$; $\alpha + \beta = n$; $e = np$; $e' = nq$; $x = nA - c$, and the relation between $u$ and $A$ will be that proposed, viz.

$$u + p \,.\, \sin mu + q \,.\, \sin nu = A.$$

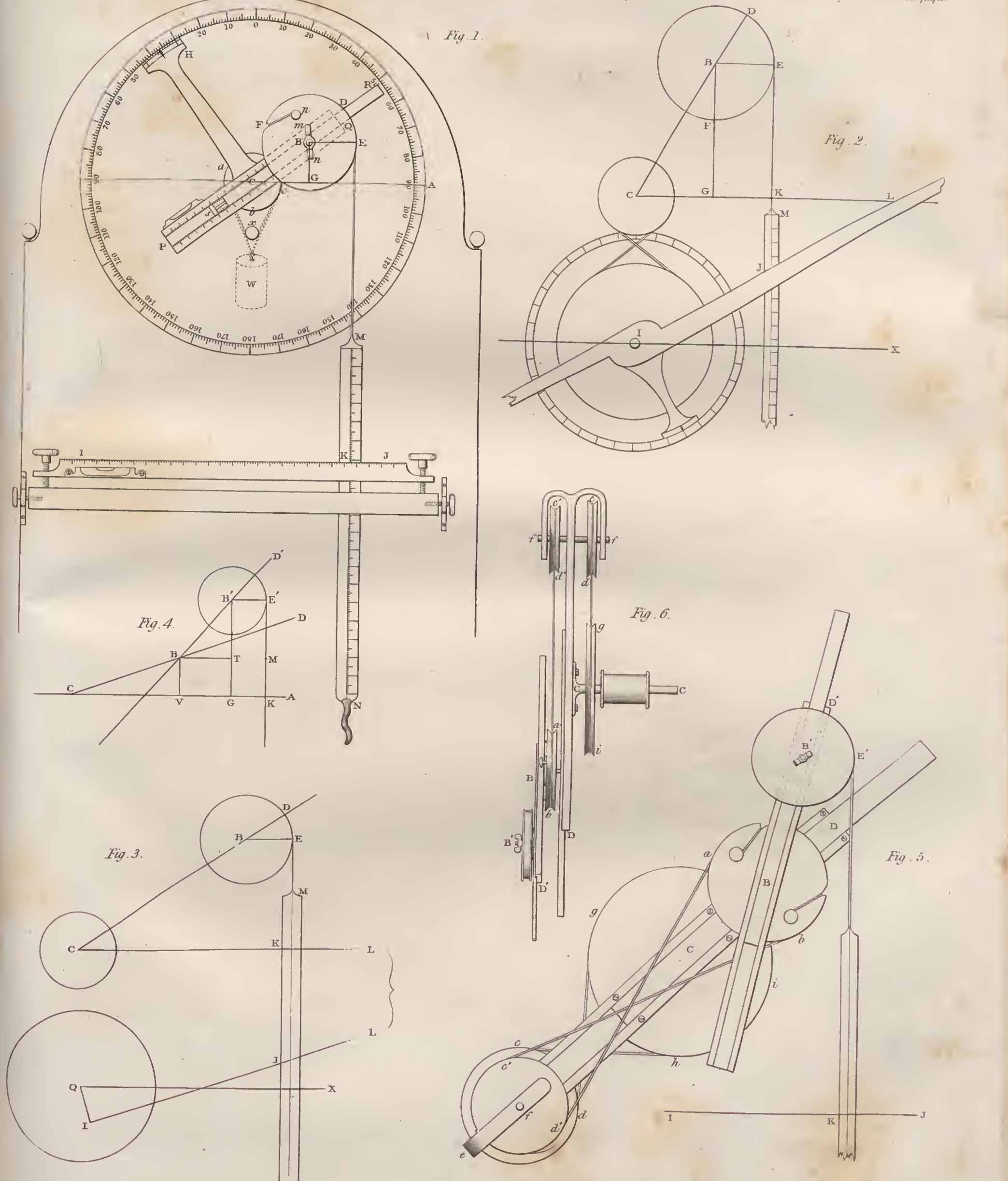
This equation, it will be observed, can always be so prepared as to make $m$ and $n$ integers, or, if we prefer it, fractions, whose denominators are integers. The form however which will require the least apparatus of wheels, and into which it is easily transformed, is the following:

$$A = u + p \,.\, \sin u + q \,.\, \sin nu,$$

in which $n$ is less than unity; for in this state of the equation the first mover may be applied at once to the axis $C$, which, as in the former construction, may carry an index arm reading off $u$ in degrees on a circle concentric with it. The whole difficulty then is reduced to the solution of a mechanical problem. To communicate to the arm $BB'$ revolving on a center $B$, attached to another revolving arm $CB$, a rotation having a given ratio of velocity to that with which the latter arm itself revolves, it being understood that the point of attachment of the center $B$ is to be capable of adjustment to a greater or less distance from $C$; a condition which excludes the use of toothed wheel work. The following is the simplest construction which has occurred to me for accomplishing this purpose.

(23.) The horizontal index axis $C$ is surrounded by a grooved wheel $g\,h\,i$, (figs. 4, 5,) which lies behind the index plate (not represented in those figures) and the cross-piece and slider, and is not attached to the axis so as to turn with it, but on the contrary is fixed to the index plate by screws so as to prevent all rotation. The end $f$ of the cross-piece is penetrated by an axis $f$ seen in projection in fig. 5, but lengthwise in fig. 6, whose extremities (to avoid shake and loosening) are pivoted in a bifurcated and recurved prolongation of the cross-piece, which is seen in fig. 6 at $e\,f$. This axis carries on it two wheels also grooved, $c\,d$, $c'd'$, both firmly united to the axis, and therefore incapable of moving unless together as one wheel. A string or band passes round the groove in $g\,h$ and $c\,d$, so that when the axis $C$ is made to revolve, and therefore the wheel $c\,d$ is carried round $g\,h$, the latter remaining immoveable, the *relative* rotation of the one wheel about the other will wrap and unwrap the string round the groove $g\,h$, and thus produce a rotation of the wheel $c\,d$ on its axis $f$, just it would do if all the rest of the apparatus stood still, and the wheel $g\,h$ alone was turned the other way round the axis $C$. Thus a rotation equal and contrary to that of $C$, or, if the wheels $g\,h$, $c\,d$, be of unequal size, in any constant ratio to the latter rotation, is communicated to the wheels $c\,d$, $c'\,d'$. From the latter of these let a string be led round a groove in the wheel $a\,b$, whose axis is the pin $B$ on which the second slider revolves, and which slider is firmly screwed on the face of the wheel, so as to revolve with it. The distance between the axes $f$ and $B$ of these wheels being changeable a re-entering string cannot be used, but it must, after making a complete circumvolution of both, be fastened off at each of its extremities to pegs, and brought by them into the requisite state of tension. The rotation of $c\,d$ will thus be transferred to $a\,b$, increased or diminished in any ratio according to the ratio of the diameters of the wheels $c'\,d'$ and $a\,b$, which must be adjusted accordingly. It is evident that this construction accomplishes the end in question, which might indeed be accomplished without the use of the wheels $c\,d$, $c'\,d'$, were it not necessary to allow the center $B$ completely to attain and even pass across $C$. It is very likely that other and still simpler modes of accomplishing the same object will suggest themselves to others.

To accompany. Sir J Herschels paper




J.W. Lowry Sculp.