PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY OF LONDON.

Series A

CONTAINING PAPERS OF A MATHEMATICAL OR PHYSICAL CHARACTER.

FOR THE YEAR 1896.

VOL. 187.

LONDON:

PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE. W.C.,

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ADVERTISEMENT.

The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

List of Institutions entitled to receive the Philosophical Transactions or Proceedings of the Royal Society.

Institutions marked A are entitled to receive Philosophical Transactions, Series A, and Proceedings.
„ „ B „ „ „ „ Series B, and Proceedings.
„ „ AB „ „ „ „ Series A and B, and Proceedings.
„ „ p „ „ Proceedings only.

America (Central).

Mexico.

p. Sociedad Científica "Antonio Alzate."

America (North). (See United States and Canada.)

America (South).

Buenos Ayres.

AB. Museo Nacional.

Caracas.

B. University Library.

Cordova.

AB. Academia Nacional de Ciencias.

Demerara.

p. Royal Agricultural and Commercial Society, British Guiana.

La Plata.

B. Museo de La Plata.

Rio de Janeiro.

p. Observatorio.

Australia.

Adelaide.

p. Royal Society of South Australia.

Brisbane.

p. Royal Society of Queensland.

Melbourne.

p. Observatory.
p. Royal Society of Victoria.
AB. University Library.

Sydney.

p. Australian Museum.
p. Geological Survey.
p. Linnean Society of New South Wales.
AB. Royal Society of New South Wales.
AB. University Library.

Austria.

Agram.

p. Jugoslavenska Akademija Znanosti i Umjetnosti.
p. Societas Historico-Naturalis Croatica.

Austria (continued).

Brünn.

AB. Naturforschender Verein.

Gratz.

AB. Naturwissenschaftlicher Verein für Steiermark.

Innsbruck.

AB. Das Ferdinandeum.
p. Naturwissenschaftlich - Medicinischer Verein.

Prague.

AB. Königliche Böhmische Gesellschaft der Wissenschaften.

Trieste.

B. Museo di Storia Naturale.
p. Società Adriatica di Scienze Naturali.

Vienna.

p. Anthropologische Gesellschaft.
AB. Kaiserliche Akademie der Wissenschaften
p. K.K. Geographische Gesellschaft.
AB. K.K. Geologische Reichsanstalt.
B. K.K. Naturhistorisches Hof-Museum.
B. K.K. Zoologisch-Botanische Gesellschaft.
p. Oesterreichische Gesellschaft für Meteorologie.
A. Von Kuffner'sche Sternwarte.

Belgium.

Brussels.

B. Académie Royale de Médecine.
AB. Académie Royale des Sciences.
B. Musée Royal d'Histoire Naturelle de Belgique.
p. Observatoire Royal.
p. Société Belge de Géologie, de Paléontologie, et d'Hydrologie.
p. Société Malacologique de Belgique.

Ghent.

AB. University.

Belgium (continued).

Liége.

AB. Société des Sciences.

p. Société Géologique de Belgique.

Louvain.

B. Laboratoire de Microscopie et de Biologie Cellulaire

AB. Université.

Canada.

Hamilton.

p. Hamilton Association.

Montreal.

AB. McGill University.

p. Natural History Society.

Ottawa.

AB. Geological Survey of Canada.

AB. Royal Society of Canada.

Toronto.

p. Astronomical and Physical Society.

p. Canadian Institute.

AB. University.

Cape of Good Hope.

A. Observatory.

AB. South African Library.

Ceylon.

Colombo.

B. Museum.

China.

Shanghai.

p. China Branch of the Royal Asiatic Society.

Denmark.

Copenhagen.

AB. Kongelige Danske Videnskabernes Selskab.

Egypt.

Alexandria.

AB. Bibliothèque Municipale.

England and Wales.

Aberystwith.

AB. University College.

Bangor.

AB. University College of North Wales.

Birmingham.

AB. Free Central Library.

AB. Mason College.

p. Philosophical Society.

Bolton.

p. Public Library.

Bristol.

p. Merchant Venturers' School.

AB. University College.

Cambridge.

AB Philosophical Society.

p. Union Society.

England and Wales (continued).

Cooper's Hill.

AB. Royal Indian Engineering College.

Dudley.

p. Dudley and Midland Geological and Scientific Society.

Essex.

p. Essex Field Club.

Falmouth.

p. Royal Cornwall Polytechnic Society.

Greenwich.

A. Royal Observatory.

Kew.

B. Royal Gardens.

Leeds.

p. Philosophical Society.

AB. Yorkshire College.

Liverpool.

AB. Free Public Library.

p. Literary and Philosophical Society.

A. Observatory.

AB. University College.

London.

AB. Admiralty.

p. Anthropological Institute.

AB. British Museum (Nat. Hist.).

AB. Chemical Society.

A. City and Guilds of London Institute.

p. "Electrician," Editor of the.

B. Entomological Society.

AB. Geological Society.

AB. Geological Survey of Great Britain.

p. Geologists' Association.

AB. Guildhall Library.

A. Institution of Civil Engineers.

p. Institution of Electrical Engineers.

A. Institution of Mechanical Engineers.

A. Institution of Naval Architects.

p. Iron and Steel Institute.

AB. King's College.

B. Linnean Society.

AB. London Institution.

p. London Library.

A. Mathematical Society.

p. Meteorological Office.

p. Odontological Society.

p. Pharmaceutical Society.

p. Physical Society.

p. Quekett Microscopical Club.

p. Royal Agricultural Society.

A. Royal Astronomical Society.

B. Royal College of Physicians.

B. Royal College of Surgeons.

England and Wales (continued).

London (continued).

- *p*. Royal Engineers (for Libraries abroad, six copies).
- AB. Royal Engineers. Head Quarters Library.
- *p*. Royal Geographical Society.
- *p*. Royal Horticultural Society.
- AB. Royal Institution of Great Britain.
- B. Royal Medical and Chirurgical Society.
- *p*. Royal Meteorological Society.
- *p*. Royal Microscopical Society.
- *p*. Royal Statistical Society.
- AB. Royal United Service Institution.
- AB. Society of Arts.
- *p*. Society of Biblical Archæology.
- *p*. Society of Chemical Industry (London Section).
- *p*. Standard Weights and Measures Department.
- AB. The Queen's Library.
- AB. The War Office.
- AB. University College.
- *p*. Victoria Institute.
- B. Zoological Society.

Manchester.

- AB. Free Library.
- AB. Literary and Philosophical Society.
- *p*. Geological Society.
- AB. Owens College.

Netley.

- *p*. Royal Victoria Hospital.

Newcastle.

- AB. Free Library.
- *p*. North of England Institute of Mining and Mechanical Engineers.
- *p*. Society of Chemical Industry (Newcastle Section).

Norwich.

- *p*. Norfolk and Norwich Literary Institution.

Nottingham.

- AB. Free Public Library.

Oxford.

- *p*. Ashmolean Society.
- AB. Radcliffe Library.
- A. Radcliffe Observatory.

Penzance.

- *p*. Geological Society of Cornwall.

Plymouth.

- B. Marine Biological Association.
- *p*. Plymouth Institution.

Richmond.

- A. "Kew" Observatory.

Salford.

- *p*. Royal Museum and Library.

England and Wales (continued).

Stonyhurst.

- *p*. The College.

Swansea.

- AB. Royal Institution.

Woolwich.

- AB. Royal Artillery Library.

Finland.

Helsingfors.

- *p*. Societas pro Fauna et Flora Fennica.
- AB. Société des Sciences.

France.

Bordeaux.

- *p*. Académie des Sciences.
- *p*. Faculté des Sciences.
- *p*. Société de Médecine et de Chirurgie.
- *p*. Société des Sciences Physiques et Naturelles.

Caen.

- *p*. Société Linnéenne de Normandie.

Cherbourg.

- *p*. Société des Sciences Naturelles.

Dijon.

- *p*. Académie des Sciences.

Lille.

- *p*. Faculté des Sciences.

Lyons.

- AB. Académie des Sciences, Belles-Lettres et Arts.
- AB. Université.

Marseilles.

- AB. Faculté des Sciences.

Montpellier.

- AB. Académie des Sciences et Lettres.
- B. Faculté de Médecine.

Nantes.

- *p*. Société des Sciences Naturelles de l'Ouest de la France.

Paris.

- AB. Académie des Sciences de l'Institut.
- *p*. Association Française pour l'Avancement des Sciences.
- *p*. Bureau des Longitudes.
- A. Bureau International des Poids et Mesures.
- *p*. Commission des Annales des Ponts et Chaussées.
- *p*. Conservatoire des Arts et Métiers.
- *p*. Cosmos (M. L'ABBÉ VALETTE).
- AB. Dépôt de la Marine.
- AB. École des Mines.
- AB. École Normale Supérieure.
- AB. École Polytechnique.
- AB. Faculté des Sciences de la Sorbonne.
- AB. Jardin des Plantes.
- *p*. L'Électricien.

France (continued).
Paris (continued).
A. L'Observatoire.
p. Revue Scientifique (Mons. H. DE VARIGNY).
p. Société de Biologie.
AB. Société d'Encouragement pour l'Industrie Nationale.
AB. Société de Géographie.
p. Société de Physique.
B. Société Entomologique.
AB. Société Géologique.
p. Société Mathématique.
p. Société Météorologique de France.
Toulouse.
AB. Académie des Sciences.
A. Faculté des Sciences.

Germany.
Berlin.
A. Deutsche Chemische Gesellschaft.
A. Die Sternwarte.
p. Gesellschaft für Erdkunde.
AB. Königliche Preussische Akademie der Wissenschaften.
A. Physikalische Gesellschaft.
Bonn.
AB. Universität.
Bremen.
p. Naturwissenschaftlicher Verein.
Breslau.
p. Schlesische Gesellschaft für Vaterländische Kultur.
Brunswick.
p. Verein für Naturwissenschaft.
Carlsruhe. See Karlsruhe.
Charlottenburg.
A. Physikalisch-Technische Reichsanstalt.
Danzig.
AB. Naturforschende Gesellschaft.
Dresden.
p. Verein für Erdkunde.
Emden.
p. Naturforschende Gesellschaft.
Erlangen.
AB. Physikalisch-Medicinische Societät.
Frankfurt-am-Main.
AB. Senckenbergische Naturforschende Gesellschaft.
p. Zoologische Gesellschaft.
Frankfurt-am-Oder.
p. Naturwissenschaftlicher Verein.
Freiburg-im-Breisgau.
AB. Universität.
Giessen.
AB. Grossherzogliche Universität.

Germany (continued).
Görlitz.
p. Naturforschende Gesellschaft.
Göttingen.
AB. Königliche Gesellschaft der Wissenschaften.
Halle.
AB. Kaiserliche Leopoldino - Carolinische Deutsche Akademie der Naturforscher.
p. Naturwissenschaftlicher Verein für Sachsen und Thüringen.
Hamburg.
p. Naturhistorisches Museum.
AB. Naturwissenschaftlicher Verein.
Heidelberg.
p. Naturhistorisch-Medizinischer Verein.
AB. Universität.
Jena.
AB. Medicinisch-Naturwissenschaftliche Gesellschaft.
Karlsruhe.
A. Grossherzogliche Sternwarte.
p. Technische Hochschule.
Kiel.
p. Naturwissenschaftlicher Verein für Schleswig-Holstein.
A. Sternwarte.
AB. Universität.
Königsberg.
AB. Königliche Physikalisch - Ökonomische Gesellschaft.
Leipsic.
p. Annalen der Physik und Chemie.
AB. Königliche Sächsische Gesellschaft der Wissenschaften.
Magdeburg.
p. Naturwissenschaftlicher Verein.
Marburg.
AB. Universität.
Munich.
AB. Königliche Bayerische Akademie der Wissenschaften.
p. Zeitschrift für Biologie.
Münster.
AB. Königliche Theologische und Philosophische Akademie.
Potsdam.
A. Astrophysikalisches Observatorium.
Rostock.
AB. Universität.
Strasburg.
AB. Universität.
Tübingen.
AB. Universität.

Germany (continued).

Würzburg.

AB. Physikalisch-Medicinische Gesellschaft.

Greece.

Athens.

A. National Observatory.

Holland. (See NETHERLANDS.)

Hungary.

Buda-pest.

p. Königl. Ungarische Geologische Anstalt.

AB. Á Magyar Tudós Társaság. Die Ungarische Akademie der Wissenschaften.

Hermannstadt.

p. Siebenbürgischer Verein für die Naturwissenschaften.

Klausenburg.

AB. Az Erdélyi Muzeum. Das Siebenbürgische Museum.

Schemnitz.

p. K. Ungarische Berg- und Forst-Akademie.

India.

Bombay.

AB. Elphinstone College.

p. Royal Asiatic Society (Bombay Branch).

Calcutta.

AB. Asiatic Society of Bengal.

AB. Geological Museum.

p. Great Trigonometrical Survey of India.

AB. Indian Museum.

p. The Meteorological Reporter to the Government of India.

Madras.

B. Central Museum.

A. Observatory.

Roorkee.

p. Roorkee College.

Ireland.

Armagh.

A. Observatory.

Belfast.

AB. Queen's College.

Cork.

p. Philosophical Society.

AB. Queen's College.

Dublin.

A. Observatory.

AB. National Library of Ireland.

B. Royal College of Surgeons in Ireland.

AB. Royal Dublin Society.

AB. Royal Irish Academy.

Galway.

AB. Queen's College.

Italy.

Acireale.

p. Accademia di Scienze, Lettere ed Arti.

Bologna.

AB. Accademia delle Scienze dell' Istituto.

Catania.

AB. Accademia Gioenia di Scienze Naturali.

Florence.

p. Biblioteca Nazionale Centrale.

AB. Museo Botanico.

p. Reale Istituto di Studi Superiori.

Genoa.

p. Società Ligustica di Scienze Naturali e Geografiche.

Milan.

AB. Reale Istituto Lombardo di Scienze, Lettere ed Arti.

AB. Società Italiana di Scienze Naturali.

Modena.

p. Le Stazioni Sperimentali Agrarie Italiane.

Naples.

p. Società di Naturalisti.

AB. Società Reale, Accademia delle Scienze.

B. Stazione Zoologica (Dr. DOHRN).

Padua.

p. University.

Palermo.

A. Circolo Matematico.

AB. Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).

A. Reale Osservatorio.

Pisa.

p. Il Nuovo Cimento.

p. Società Toscana di Scienze Naturali.

Rome.

p. Accademia Pontificia de' Nuovi Lincei.

p. Rassegna delle Scienze Geologiche in Italia.

A. Reale Ufficio Centrale di Meteorologia e di Geodinamica, Collegio Romano.

AB. Reale Accademia dei Lincei.

p. R. Comitato Geologico d' Italia.

A. Specola Vaticana.

AB. Società Italiana delle Scienze.

Siena.

p. Reale Accademia dei Fisiocritici.

Turin.

p. Laboratorio di Fisiologia.

AB. Reale Accademia delle Scienze.

Venice.

p. Ateneo Veneto.

AB. Reale Istituto Veneto di Scienze, Lettere ed Arti.

Japan.
Tokiô.
AB. Imperial University.
p. Asiatic Society of Japan.

Java.
Buitenzorg.
p. Jardin Botanique.

Luxembourg.
Luxembourg.
p. Société des Sciences Naturelles.

Malta.
p. Public Library.

Mauritius.
p. Royal Society of Arts and Sciences.

Netherlands.
Amsterdam.
AB. Koninklijke Akademie van Wetenschappen.
p. K. Zoologisch Genootschap 'Natura Artis Magistra.'
Delft.
p. École Polytechnique.
Haarlem.
AB. Hollandsche Maatschappij der Wetenschappen.
p. Musée Teyler.
Leyden.
AB. University.
Rotterdam.
AB. Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.
Utrecht.
AB. Provinciaal Genootschap van Kunsten en Wetenschappen.

New Brunswick.
St. John.
p. Natural History Society.

New Zealand.
Wellington.
AB. New Zealand Institute.

Norway.
Bergen.
AB. Bergenske Museum.
Christiania.
AB. Kongelige Norske Frederiks Universitet.
Tromsoe.
p. Museum.
Trondhjem.
AB. Kongelige Norske Videnskabers Selskab.

Nova Scotia.
Halifax.
p. Nova Scotian Institute of Science.
Windsor.
p. King's College Library.

Portugal.
Coimbra.
AB. Universidade.
Lisbon.
AB. Academia Real das Sciencias.
p. Secção dos Trabalhos Geologicos de Portugal
Oporto.
p. Annaes de Sciencias Naturaes.

Russia.
Dorpat.
AB. Université.
Irkutsk.
p. Société Impériale Russe de Géographie (Section de la Sibérie Orientale).
Kazan.
AB. Imperatorsky Kazansky Universitet.
Kharkoff.
p. Section Médicale de la Société des Sciences Expérimentales, Université de Kharkow.
Kieff.
p. Société des Naturalistes.
Moscow.
AB. Le Musée Public.
B. Société Impériale des Naturalistes.
Odessa.
p. Société des Naturalistes de la Nouvelle-Russie.
Pulkowa.
A. Nikolai Haupt-Sternwarte.
St. Petersburg.
AB. Académie Impériale des Sciences.
B. Archives des Sciences Biologiques.
AB. Comité Géologique.
p. Compass Observatory.
A. Observatoire Physique Central.

Scotland.
Aberdeen.
AB. University.
Edinburgh.
p. Geological Society.
p. Royal College of Physicians (Research Laboratory).
p. Royal Medical Society.
A. Royal Observatory.
p. Royal Physical Society.
p. Royal Scottish Society of Arts.
AB. Royal Society.
Glasgow.
AB. Mitchell Free Library.
p. Natural History Society.
p. Philosophical Society.

Servia.
Belgrade.
p. Académie Royale de Serbie.

Sicily. (See ITALY.)

Spain.

Cadiz.

A. Instituto y Observatorio de Marina de San Fernando.

Madrid.

p. Comisión del Mapa Geológico de Espãna.

AB. Real Academia de Ciencias.

Sweden.

Gottenburg.

AB. Kongl. Vetenskaps och Vitterhets Samhälle.

Lund.

AB. Universitet.

Stockholm.

A. Acta Mathematica.

AB. Kongliga Svenska Vetenskaps-Akademie.

AB. Sveriges Geologiska Undersökning.

Upsala.

AB. Universitet.

Switzerland.

Basel.

p. Naturforschende Gesellschaft.

Bern.

AB. Allg. Schweizerische Gesellschaft.

p. Naturforschende Gesellschaft.

Geneva.

AB. Société de Physique et d'Histoire Naturelle.

AB. Institut National Genevois.

Lausanne.

p. Société Vaudoise des Sciences Naturelles.

Neuchâtel.

p. Société des Sciences Naturelles.

Zürich.

AB. Das Schweizerische Polytechnikum.

p. Naturforschende Gesellschaft.

p. Sternwarte.

Tasmania.

Hobart.

p. Royal Society of Tasmania.

United States.

Albany.

AB. New York State Library.

Annapolis.

AB. Naval Academy.

Austin.

p. Texas Academy of Sciences.

Baltimore.

AB. Johns Hopkins University.

Berkeley.

p. University of California.

United States (continued).

Boston.

AB. American Academy of Sciences

B. Boston Society of Natural History.

A. Technological Institute.

Brooklyn.

AB. Brooklyn Library.

Cambridge.

AB. Harvard University.

B. Museum of Comparative Zoology.

Chapel Hill (N.C.).

p. Elisha Mitchell Scientific Society.

Charleston.

p. Elliott Society of Science and Art of South Carolina.

Chicago.

AB. Academy of Sciences.

p. Field Columbian Museum.

p. Journal of Comparative Neurology.

Davenport (Iowa).

p. Academy of Natural Sciences.

Ithaca (N.Y.).

p. Physical Review (Cornell University).

Madison.

p. Wisconsin Academy of Sciences.

Mount Hamilton (California).

A. Lick Observatory.

New Haven (Conn.).

AB. American Journal of Science.

AB. Connecticut Academy of Arts and Sciences.

New York.

p. American Geographical Society.

p. American Museum of Natural History.

p. New York Academy of Sciences.

p. New York Medical Journal.

p. School of Mines, Columbia College.

Philadelphia.

AB. Academy of Natural Sciences.

AB. American Philosophical Society.

p. Franklin Institute.

p. Wagner Free Institute of Science.

Rochester (N.Y.).

p. Academy of Science.

St. Louis.

p. Academy of Science.

Salem (Mass.).

p. American Association for the Advancement of Science.

AB. Essex Institute.

San Francisco.

AB. California Academy of Sciences.

United States (continued).

Washington.

AB. Patent Office.

AB. Smithsonian Institution.

AB. United States Coast Survey.

B. United States Commission of Fish and Fisheries.

AB. United States Geological Survey.

AB. United States Naval Observatory.

p. United States Department of Agriculture.

A. United States Department of Agriculture (Weather Bureau).

West Point (N.Y.)

AB. United States Military Academy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1896, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to CARL GEGENBAUR, For.Mem.R.S., for his life-long Researches in Comparative Anatomy in all branches of the Animal Kingdom, but chiefly in the History of the Vertebrate Skeleton, as also on account of his Teaching and Influence in reference to a large proportion of Contemporary Anatomists.

The RUMFORD MEDAL to PHILIPP LENARD and to WILHELM CONRAD RÖNTGEN for their Investigations of the Phenomena produced outside a Highly Exhausted Tube, through which an Electrical Discharge is taking place.

A ROYAL MEDAL to CHARLES VERNON BOYS, F.R.S., for his Invention of Quartz Fibres and Investigation of their Properties, his Improvement of the Radio-Micrometer and Investigations with it, for Developments in the Art of Instantaneous Photography, and for his Determination of the Value of the Constant of Attraction.

A ROYAL MEDAL to Sir ARCHIBALD GEIKIE, F.R.S., for his many Original Contributions to Geology, especially those upon the Old Red Sandstone of Western Europe, and the Carboniferous and Tertiary Volcanic Rocks of the British Isles.

The DAVY MEDAL to HENRI MOISSAN for the Isolation of Fluorine, and the Use of the Electric Furnace in the Preparation of Refractory Metals and their Compounds.

The DARWIN MEDAL to GIOVANNI BATTISTA GRASSI for his Researches on the Life-history and Societies of the Termitidæ, and on the Developmental Relationship between Leptocephalus and the Common Eel and other Muraenidæ.

The Bakerian Lecture, "On the Diffusion of Metals," was delivered by W. C. ROBERTS-AUSTEN, C.B., F.R.S.

The Croonian Lecture, "Observations on Isolated Nerve," was delivered by AUGUSTUS D. WALLER, M.D., F.R.S.

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LIST OF ILLUSTRATIONS.

I. *On the Application of the Kinetic Theory to Dense Gases.*

By S. H. Burbury, *F.R.S.*

Received January 12,—Read February 7, 1895.

The motion of a great number of elastic spheres, when their aggregate volume does *not* bear an evanescent ratio to the containing space, has received little attention from writers on the kinetic theory. In what respect, beyond the shortening of the mean free path, will it differ from that of the rare medium usually discussed? I think that the answer to this question is that there exists in all systems, dense or rare, a tendency for the spheres to move together in masses or streams, and so to diminish the mean pressure per unit of area, and the number of collisions per unit of volume and time. And this tendency has an appreciable influence on the form of the motion as soon as the ratio of the aggregate volume of the spheres to the containing space becomes appreciable.

If a part of the system, say n spheres, be at any instant contained in a volume V, they have energy, T_s, of the motion of their common centre of gravity. And they have energy, T_r, of relative motion. As the spheres increase in diameter, the ratio T_r/T_s will be found to diminish on average. But the number of collisions per unit of volume and time, given T, or $T_r + T_s$, depends on T_r, and therefore diminishes by the diminution of T_r.

1. Let M be the mass, c the diameter of a sphere, ρ the number of spheres per unit of volume, p the pressure per unit of surface. Also let us now denote by ρT_r the energy of the motion of the ρ spheres relative to their common centre of gravity, so that T_r is now the mean value per sphere of this energy.

We have then, as is well known,

$$p = \tfrac{2}{3}\rho T_r + \tfrac{1}{3}\Sigma\Sigma Rr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

in which R is the repulsive force, r the distance between a pair of spheres, and the summation includes all pairs in unit of volume. We must first evaluate $\Sigma\Sigma Rr$ on the assumption that no forces act except during collisions.

2. Let q be the relative velocity of two spheres. Let θ be the angle made with q by the line of centres at collision, if a collision takes place. The angle θ may have any value from zero to $\frac{1}{2}\pi$. As the effect of collision the velocity, $\frac{1}{2}q\cos\theta$, in the line of centres is reversed for each sphere. We may assume this reversal to be effected

by the constant finite force $\mathrm{M}q\cos\theta/2dt$ acting on each sphere in the line of centres during the small time $2dt$.

Let us define this short but finite time $2dt$ as the time during which the two spheres *are in collision*, or the *duration of a collision*. Then, during collision, if dt be small enough, the virial of the supposed force is sensibly constant, and we calculate its value as follows. Let λ, μ, ν, be the direction cosines of the line of centres referred to any axes. The coordinates of the point of contact shall be x, y, z. Then those of the centres of the two spheres are, c being the diameter of either sphere,

$$x + \tfrac{1}{2}\lambda c, \qquad y + \tfrac{1}{2}\mu c, \qquad z + \tfrac{1}{2}\nu c \text{ for one sphere,}$$
$$\text{and } x - \tfrac{1}{2}\lambda c, \qquad y - \tfrac{1}{2}\mu c, \qquad z - \tfrac{1}{2}\nu c \text{ for the other.}$$

The component forces acting at the centre of the first sphere are

$$\lambda\mathrm{M}q\cos\theta/2dt, \qquad \mu\mathrm{M}q\cos\theta/2dt, \qquad \nu\mathrm{M}q\cos\theta/2dt.$$

Those acting at the centre of the second sphere are the same with reversed signs.

For two spheres colliding with relative velocity q we find that the virial is $c\mathrm{M}q\cos\theta/2dt$ at each instant during collision. We have to multiply this by the chance that, given two spheres A and B with relative velocity q, they shall be in collision at any given instant.

It is assumed that we are dealing with a space throughout which T_r is constant, and therefore the fact that the relative velocity is q, affords no presumption with regard to the relative position of the two spheres. About the centre of sphere A suppose a spherical surface described with radius c. An element of that surface is $2\pi c^2\cos\theta\sin\theta\,d\theta$. Upon that element of surface form the element of volume $2\pi c^2\cos\theta\sin\theta\,d\theta\,qdt$. And form a similar element of volume on the other side of the sphere A, that is, using $\pi-\theta$ for θ. Then the two spheres are at this instant in collision if the centre of the sphere B is within either of those elements.

Let V be the volume in which n spheres are moving with T_r constant. Then by our assumption B is as likely to be in any part of V as in any other. Therefore the chance that A and B, having relative velocity q, are in collision, is

$$\frac{4\pi c^2\cos\theta\sin\theta\,d\theta\,qdt}{\mathrm{V}}.$$

The average virial for two spheres with relative velocity q is then at each instant

$$\int_0^{\pi/2}\frac{4\pi c^2\cos\theta\sin\theta\,qdt}{\mathrm{V}}\,.\,c\,.\,\frac{\mathrm{M}q\cos\theta}{2dt}\,d\theta$$
$$= \mathrm{M}\,\frac{2}{3\mathrm{V}}\pi c^3q^2 = \mathrm{M}\frac{2}{3\mathrm{V}}\pi c^3\{(u-u')^2+(v-v')^2+(w-w')^2\},$$

if u, v, w, u', v', w', be the component velocities of the two spheres. Now since nT_r is the energy of relative motion of n spheres in volume V,

$$nT_r = M\Sigma \frac{(u-u')^2 + (v-v')^2 + (w-w')^2}{2n},$$

the summation including every pair, and therefore

$$\Sigma\Sigma Rr = \tfrac{2}{3}\pi c^3 \frac{n}{V} 2nT_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

$$= \tfrac{2}{3}\pi c^3 \rho \,.\, 2nT_r, \text{ since } \frac{n}{V} = \rho.$$

Let $\frac{2}{3}\pi c^3 \rho = \kappa$. Then

$$\Sigma\Sigma Rr = \kappa \,.\, 2nT_r \text{ for } n \text{ spheres in V},$$

$$\Sigma\Sigma Rr = \kappa \,.\, 2\rho T_r \text{ for } \rho \text{ spheres in unit of volume.}$$

Substituting this value of $\Sigma\Sigma Rr$ in (1) we obtain

$$p = \tfrac{2}{3}(1 + \kappa)\rho T_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

It is assumed in these results that we are dealing with a space throughout which T_r is sensibly constant.

3. We see then that p is proportional to $T_r + \kappa T_r$. The analogy between this expression and BOLTZMANN'S $T + \chi$, in which χ denotes potential energy, suggests that the law of distribution of velocities among our spheres should be, instead of ϵ^{-hT} as in the rare medium, $\epsilon^{-h(T_r + \kappa T_r)}$, or rather, since there may be stream motion as well as relative motion, $\epsilon^{-h(T + \kappa T_r)}$.

Let us further develop this analogy. In the Clausian equation

$$\tfrac{3}{2}pV = nT_r + \tfrac{1}{2}\Sigma\Sigma Rr,$$

$$\tfrac{1}{2}\Sigma\Sigma Rr = \tfrac{2}{3}\pi c^3 \frac{n^2}{V^2} T_r,$$

where nT_r is the kinetic energy of the motion of n spheres in volume V relative to their common centre of inertia. Hence

$$p = \tfrac{2}{3}\left\{\frac{n}{V} + \tfrac{2}{3}\pi c^3 \frac{n^2}{V^2}\right\} T_r.$$

If the n spheres, being initially contained in volume V_0, be compressed into volume V, and T_r be maintained constant during the process, the work done in compression is

$$W = \int_V^{V_0} p\, dV = \left\{\tfrac{2}{3} n \log \frac{V_0}{V} + \tfrac{4}{9}\pi c^3 n^2 \left(\frac{1}{V} - \frac{1}{V_0}\right)\right\} T_r.$$

The first term on the right-hand side expresses the work which would be done during the process if the spheres were material points, and no collisions took place between them. The second term expresses the amount by which this work is increased by the spheres having finite diameter c and undergoing collisions. And if V_0 be infinite, this additional work is

$$\tfrac{4}{9}\,\pi c^3 \frac{n^2}{V}\, T_r, \quad \text{or} \quad \tfrac{2}{3}\,\kappa n T_r.$$

We see then that the term $n\kappa T_r$ in the index of $\epsilon^{-hn(T+\kappa T_r)}$ represents work done against collisions in compressing the system from an infinite volume to its actual volume with constant T_r. It is analogous to the potential χ in the usual expression $\epsilon^{-h(T+\chi)}$. It might not be inappropriate to call κT_r the *potential of collisions.*

In order to confirm or otherwise the above suggestion, I proceed to consider—

4. *The distribution of energy in a vertical column of gas when in equilibrium in a field of uniform force*, the molecules being equal elastic spheres of diameter c.

Let the column be an infinite cylinder, f, the force, being parallel to its axis.

Take a plane perpendicular to the axis as base, and let s be the height of a point above that plane. Then we have, with the same notation as before,

$$dp/ds = -Mf\rho \qquad (4),$$

and, as before,

$$p = \tfrac{2}{3}(1+\kappa)\,\rho T_r.$$

Here T_r is the average per sphere of the energy of relative motion, which alone is concerned in p. But, in our vertical column, assume for the moment that there is no stream motion that need be taken into account, and, therefore, we may write T instead of T_r, and (3) becomes

$$p = \tfrac{2}{3}(1+\kappa)\,\rho T \qquad (3\text{A}).$$

Now κ contains ρ as a factor by (2). But I will now assume that, not $\overline{T}$, but $\overline{(1+\kappa)T}$, is independent of s, so that we may write $\overline{(1+\kappa)T} = 3/2h$ with h constant. On that assumption (3) and (4) give

$$\rho = \rho_0 \epsilon^{-hMfs} \qquad (5),$$

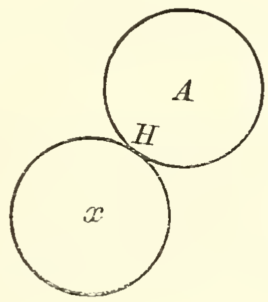
where ρ_0 is the value of ρ when $s = 0$.*

5. Now consider N spheres crossing the plane $s = 0$ with u for vertical component of velocity. Of these, say $N - N'$ reach the plane $s = ds$ without undergoing collision, N' will undergo collision before reaching ds. But for every collision by which one of the N is struck out, there will be another collision by which another sphere is

* See WATSON's 'Kinetic Theory of Gases,' 2nd edition, pp. 56–66.

substituted for it, the substituted sphere having the same vertical velocity, *but not quite the same position on average*, as the original one had at the instant of its collision.

6. If the collision be direct, *i.e.*, the line of centres coincide with the relative velocity, the substituted sphere is advanced in position through a distance equal to c, in the direction of the line of centres, and that without any loss of kinetic energy by the action of the force f. If the collision be not direct, there is an advance, and we proceed to calculate its value. Let A be the centre of a sphere which comes out of collision with u for vertical component of velocity, x the centre of the sphere in collision with it, H the point of contact. Let l denote the vector line of centres xA,



and cos (ul) the cosine of the angle between l and s. Then $\frac{1}{2} c \cos (ul)$ is the projection of HA on the vertical, and $\overline{\frac{1}{2} c \cos (ul)}$ is its average value for all the collisions in question.

7. Again, let A′ be the centre of a sphere which enters collision with vertical velocity u, x' that of the sphere colliding with it, H the point of contact, and let l' denote the vector A′x'. Then, evidently $\overline{\frac{1}{2} c \cos (ul')} = \overline{\frac{1}{2} c \cos (ul)}$.

There will be as many collisions per unit of volume and time of the one class as of the other, and the height of the point of contact H, above the base, is on average the same for one class as for the other. Therefore, taking the collisions in pairs, one from each class, each pair substitutes A for A′ as the sphere with vertical velocity u; and, on average, the substituted sphere A is at the instant of its collision above the original sphere A′ at the instant of its collision by the distance $\overline{c \cos (ul)}$. We have next to show that on average of all collisions of N spheres taking place between $s = 0$ and $s = ds$, $c \cos (ul) = \kappa\, ds$, where $\kappa = \frac{2}{3} \pi c^3 \rho$.

8. Let ω be the actual velocity of the sphere A as it issues from collision, so that u is the vertical component of ω, and $\cos (u\omega) = u/\omega$. Let ψ be the velocity of the other sphere as it issues from collision with A, and E the angle between ω and ψ. Also let q be their relative velocity, so that $\cos (\omega q) = \dfrac{\omega - \psi \cos \mathrm{E}}{q}$.

Whatever be the values of ω, ψ, and E,

$$\overline{\cos (ul)} = \overline{\cos (uq)}\ \overline{\cos (ql)}$$

and
$$\overline{\cos(ql)} = \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \cos\theta \, d\theta \div \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta = \frac{2}{3},$$

therefore
$$\overline{c\cos(ul)} = \frac{2c}{3}\overline{\cos(uq)}.$$

Again whatever be the values of ω, ψ, E,
$$\overline{\cos(uq)} = \overline{\cos(u\omega)}\;\overline{\cos(\omega q)}$$
$$= \frac{u}{\omega}\overline{\cos(\omega q)},$$

therefore
$$\overline{c\cos(ul)} = \frac{2c}{3}\frac{u}{\omega}\overline{\cos(\omega q)} = \frac{2c}{3}\frac{u}{\omega}\frac{\omega - \psi\cos E}{q}.$$

We have to multiply this expression by the number of collisions which N spheres, each having absolute velocity ω, undergo in time dt with other spheres having absolute velocity $\psi \ldots \psi + d\psi$; the angle between ω and ψ being E ... E + dE, and then integrate according to ψ and E. That number is
$$N\rho\pi c^2 q \, dt \, f(\psi) \, d\psi \, \tfrac{1}{2} \sin E \, dE,$$

if $\rho f(\psi) \, d\psi$ be the number per unit of volume of spheres whose velocity is $\psi \ldots \psi + d\psi$, so that $\int_0^\infty f(\psi) \, d\psi = 1$.

Therefore the complete average value of $c \cos(ul)$, that is the average vertical displacement of the substituted spheres, is
$$\tfrac{2}{3} N\rho\pi c^3 \, dt \int_0^\infty d\psi . f(\psi) \int_0^\pi \tfrac{1}{2} \sin E \, dE \, q \frac{u}{\omega} \cos(\omega q),$$

or, since $dt = ds/u$,
$$\tfrac{2}{3} N\rho\pi c^3 \, ds \int_0^\infty d\psi . f(\psi) \int_0^\pi \tfrac{1}{2} \sin E \, dE \, \frac{q\cos(\omega q)}{\omega}.$$

But
$$\int_0^\pi \tfrac{1}{2} \sin E \, dE \, \frac{q\cos(\omega q)}{\omega} = \int_0^\pi \tfrac{1}{2} \sin E \, dE \, \frac{\omega - \psi \cos E}{\omega} = 1,$$

also
$$\int_0^\infty f(\psi) \, d\psi = 1,$$

so our result is
$$N \, \tfrac{2}{3}\pi c^3 \rho \, ds = \kappa N ds.$$

We know (WATSON's 'Kinetic Theory of Gases,' 2nd edition, p. 56) that the quantity of vertical momentum transferred across the plane $s = 0$ per unit of area and time is proportional to p, and, therefore, increases in the ratio $1 : 1 + \kappa$, when the spheres, from being material points, acquire diameter c. We now see that this is true for each separate class of the vertical momenta.

9. Corresponding to N spheres with vertical velocity u at the plane $s = 0$ at the beginning of dt, we have at the end of dt N spheres, the same or substituted, whose average height is $(1 + \kappa)\, ds$ or $(1 + \kappa)\, u\, dt$. But their loss of kinetic energy due to the ascent is on average $\mathrm{M}f ds$ for each sphere. It follows that the average loss due to an ascent ds is, allowing for substitutions, $\dfrac{\mathrm{M}f ds}{1 + \kappa}$.

10. Let then the number per unit of volume of spheres whose energy of vertical velocity is $\frac{1}{2} u^2 \ldots \frac{1}{2}(u^2 + du^2)$ be at the base, where $s = 0$, $\rho_0 \epsilon^{-h\mathrm{Q}}\, du^2$ in which $\mathrm{Q} = \mathrm{M}\,(1 + \kappa)\, \frac{1}{2} u^2$ (A). Then the number per unit of volume which at height s have

$$\mathrm{M}\left(\frac{u^2}{2} - \frac{fs}{1 + \kappa}\right)$$

for energy of their vertical velocity is (remembering (5))

$$\rho_0 \epsilon^{-h\mathrm{M}fs}\, du^2 \epsilon^{-h(\mathrm{Q} - \mathrm{M}fs)},$$

that is $\rho_0 \epsilon^{-h\mathrm{Q}}\, du^2$ (B). The two classes A and B are equally numerous per unit of volume, and since, allowing for collisions and substitutions, the loss or gain of energy due to the force f in passing from the base to ds or *vice versâ* is $\mathrm{M}f ds/\overline{1 + \kappa}$, either class can by ascending or descending (the proper number of substitutions taking place) replace the other. And the assumed law of distribution of velocities is not disturbed by the force f, as spheres pass up and down the column. Now, make $\kappa = 0$ in the above reasoning, and we find it is exactly the reasoning from which, in the ordinary case of a rare medium, we conclude that T is constant throughout the column. The same reasoning leads in the general case to the conclusion that $(1 + \kappa)\,\mathrm{T}$ is constant throughout.

11. The above results are obtained on the hypothesis that no account need be taken of stream motion among our spheres. If, however, there be such stream motion, we have to suppose that the N spheres crossing the plane $s = 0$ were members of a large group having a vertical velocity U of their common centre of gravity, and that the N spheres have vertical velocity u relative to this common centre of gravity. Then in time dt the N spheres will have risen on average the distance ds by virtue of their relative velocity u, where $u = ds/dt$, and a distance $\mathrm{U}\, dt$ by virtue of the common velocity U. Their loss of kinetic energy by the action of the force f in this ascent is

$$\mathrm{M}f\mathrm{U}\, dt + \mathrm{M}f ds/\overline{1 + \kappa},$$

or if $\mathrm{U}\, dt = ds'$

$$\mathrm{M}f\,(ds' + ds/\overline{1 + \kappa}).$$

And now writing T_r for T, and T_s for the energy of the stream motion whose vertical component is U, we find that $T + \kappa T_r$ is constant throughout the column, and we may now write $T + \kappa T_r = 3/2h$.

12. It follows that we cannot express the law of distribution of velocities among the spheres in the form $C\epsilon^{-hT}$, or $C\epsilon^{-h\Sigma\cdot\frac{u^2+v^2+w^2}{2}}$, as in the rare medium, with $T = 3/2h$. The law must be $C\epsilon^{-hQ}$, in which Q is some quadratic function of the velocities. Suppose that for n spheres it is

$$Q_n = a_1 u_1^2 + b_{12} u_1 u_2 + a_2 u_2^2 + \&c.,$$

with corresponding expressions for the components v and w.

If we find the mean value of Q_n by integrating ϵ^{-hQ_n} for all values of u_1, u_2, &c., between the limits $\pm\infty$, we find, there being $3n$ variables or 3 component velocities for each sphere,

$$\overline{Q_n} = 3n/2h.$$

But

$$3n/2h = n\,(\overline{T + \kappa T_r}),$$

therefore,

$$\overline{Q_n} = n\,(\overline{T + \kappa T_r}).$$

This result might be considered to justify the assumption that the law of distribution of velocities is in all cases, including the field of no forces formed by making $f = 0$, accurately expressed as follows. The chance that the velocities of n spheres, forming a group together, shall be $u_1 \ldots u_1 + du_1$, &c., is proportional to $\epsilon^{-hn(T+\kappa T_r)}$, in which

$$nT = \tfrac{1}{2} M\Sigma\,(u^2 + v^2 + w^2)$$
$$nT_r = \tfrac{1}{2} M\Sigma\,\{(u - u')^2 + (v - v')^2 + (w - w')^2\}/n.$$

But we must remember that the whole treatment is based on the consideration of a great number of spheres, so that we cannot safely assume the law to hold when n is small. Let us, then, consider the subject from yet another point of view.

13. I have shown elsewhere ('Science Progress,' November, 1894), that in a dense medium the velocities of contiguous spheres cannot be independent of one another, because there is a presumption that recent collisions of two spheres near to one another have been with the same third sphere, and they have, so to speak, inherited some common velocity from it. In other words contiguous spheres have been exposed to the same environment, and, in the dense medium, environment does not change rapidly; therefore their velocities are not independent.

If, therefore, the spneres contained in a volume V be n in number, and their positions known, the chance that their velocities shall be

$$u_1 \ldots u_1 + du_1, \text{ \&c.,} \qquad v_1 \ldots v_1 + dv_1, \text{ \&c.,}$$

must be of the form

$$C\epsilon^{-h(a_1u_1^2 + b_{12}u_1u_2 + a_2u_2^2 + \&c.)},$$

in which the index is a quadratic function of u_1, u_2, &c., v_1, v_2, &c., and w_1, w_2, &c., but contains no products of the form uv, uw, or vw, and the coefficients b_{12}, &c., are functions of the positions of the spheres.

14. With regard to the forms of these coefficients, we observe that the quadratic function in the index must always be positive, because the chance cannot become infinite for infinite values of the variables, u. The condition for this is that the determinant

$$D = \begin{vmatrix} 2a_1, & b_{12}, & b_{13} \ldots & b_{1n} \\ b_{12}, & 2a_2, & b_{23}, \ldots & \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

and all its coaxial minors, must be positive, and, therefore, every a positive.

15. Again b_{12}, not being zero, expresses the fact that u_1 and u_2 are not independent. But it is also a fact that they are more likely to be of the same than of opposite signs. Therefore, b_{12}, and similarly every b, must be negative or zero.

16. Evidently also the coefficients b must generally diminish in absolute magnitude as the distance between the spheres to which they relate increases, and must become inappreciable at some distance, small compared with the dimensions of our system, but possibly large compared with the diameter of a sphere. The b's must be functions of the positions of the n spheres within V having this property.

Again we may consider either a volume V containing n spheres, or a smaller volume $\frac{n-1}{n}$ V containing $n - 1$ out of the n spheres, that is, all the n spheres except one, which one belongs to the outer layer. If the velocities of that one be denoted by u_n, v_n, w_n, and if Q_n be the quadratic function for the n spheres, Q_{n-1} for the $n - 1$ spheres, we must have

$$\epsilon^{-hQ_{n-1}} = \iiint_{-\infty}^{+\infty} \epsilon^{-hQ_n}\, du_n,\, dv_n,\, dw_n.$$

If we actually perform the integrations indicated for one variable, we get with

$$Q_n = a_1u_1^2 + b_{12}u_1u_2 + a_2u_2^2 + \text{\&c.}$$
$$Q_{n-1} = a'_1u_1^2 + b'_{12}u_1u_2 + a'_2u_2^2 + \text{\&c.,}$$

in which

$$2a'_1 = 2a_1 - b_{1n}^2/2a_n, \qquad 2a'_2 = 2a_2 - b_{2n}^2/2a_n,$$
$$b'_{12} = b_{12} - b_{1n}b_{2n}/2a_n, \text{ \&c.}$$

As the number of spheres included in our group diminishes, the coefficients a diminish, and, since every b is negative, the b's increase in absolute magnitude..

On the other hand, as n increases the a's increase, and every b^2, if changed at all, diminishes. Therefore, as u increases the function Q_n tends to a limiting form. But that limiting form must be nT if $\kappa = 0$, because in that case we know the law to be $\epsilon^{-h\mathrm{T}}$. We must then have ultimately $Q_n = n(\mathrm{T} + \kappa\theta)$, where θ is a quadratic function of the velocities. And we may now assume that *for a sufficiently great* number of spheres comprised in a group, throughout which κ is sensibly constant, the law is expressed by the function $C\epsilon^{-hn(\mathrm{T}+\kappa\mathrm{T}_r)}$, in which $C = \frac{1}{\sqrt{\pi}}\sqrt{D}$, D being the determinant of the coefficients of the quadratic function $\mathrm{T} + \kappa\mathrm{T}_r$.

We have thus obtained certain conditions which the coefficients a_1, b_{12}, &c., must satisfy. Another condition is that the assumed law of distribution of velocities expressed by the function

$$\epsilon^{-h(a_1u_1^2 + b_{12}u_1u_2 + \&c.)}$$

shall not be disturbed by a collision taking place between any two of the n spheres, which collision changes the velocities, but not the positions, of the two spheres in question. To find the values of a_1, b_{12}, &c., to satisfy this condition presents considerable difficulty. It is not, however, at present necessary to solve the problem in that form, as will be seen later. For we have only to consider the positions of the n spheres as unknown, and we obtain a solution sufficient for our purpose. If, namely, it be given that there are at any instant n spheres within a spherical space S, but nothing is known of their positions within S, we have only to assume that the chance of their having at that instant velocities $u_1 \ldots u_1 + du_1$, &c., is

$$\epsilon^{-hQ}\,du_1 \ldots dw_n,$$

with

$$Q = a\Sigma(u^2 + v^2 + w^2) + b\Sigma\Sigma(uu' + vv' + ww')$$

containing only one coefficient a and one b, and we shall find that all necessary conditions are satisfied, including the condition that the assumed distribution shall be unaffected by collisions.

For let $x_1, y_1, z_1, x_2, y_2, z_2$ be the component velocities of two spheres before collision. A collision between the two converts these components into $x'_1, y'_1, z'_1, x'_2, y'_2, z'_2$ in the following manner. Let λ, μ, ν be the direction cosines of the line of centres at collision.

The velocities of the two spheres resolved in the line of centres are, before collision,

$$\lambda x_1 + \mu y_1 + \nu z_1 \quad \text{and} \quad \lambda x_2 + \mu y_2 + \nu z_2$$

respectively. And we have

$$x'_1 = x_1 - \lambda(\lambda x_1 + \mu y_1 + \nu z_1) + \lambda(\lambda x_2 + \mu y_2 + \nu z_2),$$
$$y'_1 = y_1 - \mu(\lambda x_1 + \mu y_1 + \nu z_1) + \mu(\lambda x_2 + \mu y_2 + \nu z_2),$$
&c.

That is,

$$\begin{aligned}
x'_1 &= (1-\lambda^2)x_1 - \lambda\mu y_1 - \lambda\nu z_1 + \lambda^2 x_2 + \lambda\mu y_2 + \lambda\nu z_2 \\
y'_1 &= -\lambda\mu x_1 + (1-\mu^2)y_1 - \mu\nu z_1 + \lambda\mu x_2 + \mu^2 y_2 + \mu\nu z_2 \\
z'_1 &= -\lambda\nu x_1 - \mu\nu y_1 + (1-\nu^2)z_1 + \lambda\nu x_2 + \mu\nu y_2 + \nu^2 z_2 \\
x'_2 &= \lambda^2 x_1 + \lambda\mu y_1 + \lambda\nu z_1 + (1-\lambda^2)x_2 - \lambda\mu y_2 - \lambda\nu z_2 \\
y'_2 &= \lambda\mu x_1 + \mu^2 y_1 + \mu\nu z_1 - \lambda\mu x_2 + (1-\mu^2)y_2 - \mu\nu z_2 \\
z'_2 &= \lambda\nu x_1 + \mu\nu y_1 + \nu^2 z_1 - \lambda\nu x_2 - \mu\nu y_2 + (1-\nu^2)z_2.
\end{aligned}$$

Call these equations A.

But inasmuch as the motion might take place the reverse way with the same values of λ, μ, ν, it must by the same reasoning be true that

$$x_1 = (1-\lambda^2)x'_1 - \lambda\mu y'_1 - \lambda\nu z'_1 + \lambda^2 x'_2 + \lambda\mu y'_2 + \lambda\nu z'_2, \text{ \&c.} \quad . \quad . \quad . \quad . \quad \text{A}',$$

which are the same as equations A with the accents interchanged between the right and left-hand members. If we solve either set of equations we get the other set.

17. Now it is given that at this instant the chance of the spheres having velocities $x_1 \ldots x_1 + dx_1$, &c., is $C\epsilon^{-hQ}\,dx_1 dx_2 \ldots$ in which

$$\begin{aligned}
Q = {} & ax_1^2 + bx_1x_2 + ax_2^2 + ay_1^2 + by_1y_2 + ay_2^2 + az_1^2 + bz_1z_2 + az_2^2 \\
& + bx_1(x_3 + x_4 + \text{\&c.}) + \text{\&c.} \\
& + ax_3^2 + \text{\&c.} + \text{terms containing squares and products of the velocities } x_3 \ldots z_n.
\end{aligned}$$

The chance that after this collision the velocities of the n spheres shall have the values x'_1, y'_1, z'_1, x'_2, y'_2, z'_2, and x_3, y_3, &c., is found by substituting for x_1, y_1, z_1, x_2, y_2, z_2, in the index hQ, their values in terms of x'_1, y'_1, z'_1, x'_2, y'_2, z'_2, as given by equations A′. Effecting this substitution we find that the coefficient of x'^2_1 is

$$\begin{aligned}
& a \cdot \{(1-\lambda^2)^2 + \lambda^2\mu^2 + \lambda^2\nu^2 + \lambda^4 + \lambda^2\mu^2 + \lambda^2\nu^2\} \\
& \quad + b \cdot \{\lambda^2(1-\lambda^2) - \lambda^2\mu^2 - \lambda^2\nu^2\}
\end{aligned}$$

that is a. Similarly, we find that the coefficient of $x'_1x'_2$ in the new index is b, and the coefficient of every product x'_1x_r in the new index is the same as that of x_1x_r in the original index. The new index is then the same function of x'_1, y'_1, z'_1, x'_2, y'_2, z'_2, that the original one was of x_1, y_1, z_1, x_2, y_2, z_2. The assumed distribution of velocities is therefore not affected by any one, and therefore not by any number of collisions.

This, it will be remembered, is on the assumption that all the coefficients a are the same, and all the coefficients b are the same, so far as they have to do with the colliding spheres.

18. Now

$$\frac{1}{M} n (T + \kappa T_r)$$

$$= \left(1 + \frac{n-1}{n}\kappa\right)\frac{u_1^2}{2} - \frac{\kappa}{n} u_1 u_2 - \frac{\kappa}{n} u_1 u_3 + \left(1 + \frac{n-1}{n}\kappa\right)\frac{u_2^2}{2} + \&c.$$

$$= a\Sigma (u^2 + v^2 + w^2) + b\Sigma\Sigma (uu' + vv' + ww'), \text{ if } 2a = 1 + \frac{n-1}{n}\kappa \text{ and } b = -\frac{\kappa}{n},$$

and if we use this for Q_n we find that all necessary conditions are satisfied, including the condition for permanence notwithstanding collisions.

19. If, however, the coefficients a, b were not the same for both colliding spheres, but the form were $a_1 x_1^2 + b_{12} x_1 x_2 + a_2 x_2^2 + b_{13} x_1 x_3 + b_{23} x_2 x_3 + \&c.$, then we should find that the coefficient of x'^2_1 in the new index is $(1 - \lambda^2) a_1 + \lambda^2 a_2$; the coefficient of x'^2_2 is $(1 - \lambda^2) a_2 + \lambda^2 a_1$; the coefficient of $x'_1 x'_2$ is as before, b_{12}; but that of $x'_1 x_3$ is $(1 - \lambda^2) b_{13} + \lambda^2 b_{23}$; and that of $x'_2 x_3$ is $(1 - \lambda^2) b_{23} + \lambda^2 b_{13}$.

The assumed law of distribution cannot in this case be unaffected by collisions, unless (1) all the a coefficients are the same; (2) if the velocities of the two colliding spheres be u_1, v_1, w_1 and u_2, v_2, w_2, and those of any third sphere be u_3, v_3, w_3, then $b_{13} = b_{23}$, that is, the b coefficients must be such functions of the position, that if spheres 1 and 2 are close together $b_{13} = b_{23}$, &c.

To return to the case of the positions of the spheres being unknown.

20. If we form the determinant of the system

$$D = \begin{vmatrix} \left(1 + \frac{n-1}{n}\kappa\right) & -\frac{\kappa}{n} & -\frac{\kappa}{n} & \cdots \\ -\frac{\kappa}{n} & \left(1 + \frac{n-1}{n}\kappa\right) & -\frac{\kappa}{n} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

we find

$$D = (1 + \kappa)^n - n\frac{\kappa}{n}(1 + \kappa)^{n-1}$$

$$D_{11} = (1 + \kappa)^{n-1} - \overline{n-1}\,\frac{\kappa}{n}(1 + \kappa)^{n-2}.$$

Hence we find

$$\overline{u_1^2} = \overline{u_2^2} = \&c. = \frac{D_{11}}{D} = \frac{1}{hM}\cdot\frac{1}{n}\cdot\frac{n + \kappa}{1 + \kappa},$$

therefore

$$n\text{T} = \frac{3}{2h} \cdot \frac{n+\kappa}{1+\kappa}.$$

Also

$$\overline{u_1 u_2} = \frac{\text{D}_{12}}{\text{D}}, \text{ \&c.}$$

[These results are easily obtained by considering the general determinant of n^2 constituents

$$\text{D} = \begin{vmatrix} 2a & b & b & b & . \\ b & 2a & b & . & . \\ . & . & . & . & . \end{vmatrix},$$

in which all the axial constituents are $2a$, and all the non-axial constituents are b. It will be found for $n = 2$, $n = 3$, and thence by induction, that

$$\text{D} = (2a - b)^n + nb\,(2a - b)^{n-1}.$$

Replacing $2a$ by $1 + \overline{n-1}\kappa/n$, and b by $-\kappa/n$, we get the result above stated.]

Also $\overline{v^2}$ and $\overline{w^2}$ have the same value as $\overline{u^2}$. And the whole kinetic energy of the n spheres, or $n\text{T}$, is, on average, $\text{M}\dfrac{3n\overline{v^2}}{2}$, or $n\text{T} = \dfrac{3}{2h}\dfrac{n+\kappa}{1+\kappa}$.

Again,

$$\frac{\kappa}{h\text{M}} n\text{T}_r = \kappa\frac{n-1}{2n}u_1^2 - \frac{\kappa}{n}u_1u_2 - \frac{\kappa}{n}u_1u_3 - \text{\&c},$$

$$+ \kappa\frac{n-1}{2n}u_2^2 - \frac{\kappa}{n}u_2u_3 - \text{\&c.},$$

$$- \text{\&c.},$$

with similar expressions for the v's and w's. But

$$\tfrac{1}{2}\left(1 + \kappa\frac{n-1}{n}\right)\overline{u^2} - \frac{\kappa}{n}\overline{u_1u_2} - \text{\&c.} = \tfrac{1}{2}\left(1 + \kappa\frac{n-1}{n}\right)\frac{\text{D}_{11}}{\text{D}} - \frac{\kappa}{n}\frac{\text{D}_{12}}{\text{D}} - \text{\&c.} = 3n/2h\text{M}$$

by the properties of determinants.

Therefore

$$\kappa n\text{T}_r = \frac{3n}{2h} - \frac{3n}{2h}\frac{\text{D}_{11}}{\text{D}} = \frac{3n}{2h} - \frac{3}{2h}\frac{n+\kappa}{1+\kappa} = \frac{3}{2h}\frac{\overline{n-1}\kappa}{1+\kappa}$$

and

$$n\text{T}_r = \frac{3}{2h}\frac{n-1}{1+\kappa},$$

and therefore

$$n\text{T}_s = n\text{T} - n\text{T}_r = \frac{3}{2h}\left(\frac{n+\kappa}{1+\kappa} - \frac{n-1}{1+\kappa}\right) = 3n/2h,$$

and therefore $T_s/T = (1 + \kappa)/(n + \kappa)$, which increases as κ increases, that is, *cæteris paribus*, as the diameter c increases.

Again, the pressure per unit of area on a plane moving with the stream, and therefore the mean pressure, is $p = \frac{2}{3}(1 + \kappa)\rho T_r$, which is independent of the diameter c.

Now, the spheres being material points with $\overline{T_r} = 3/2h$, the mean pressure is $\frac{2}{3}\rho T_r$; that is, as we may write it, $\frac{2}{3}(1 + \kappa)\rho T_r$, because in this case $\kappa = 0$. As the spheres increase in diameter with $(1 + \kappa) T_r$ unaltered, the mean pressure per unit of area remains unaltered. In other words, it is exactly as much diminished by the conversion of part of the energy of relative motion of contiguous spheres into energy of stream motion as it is increased by the introduction of the term $\Sigma\Sigma Rr$ as the spheres acquire diameter c.

Comparing the actual value of p with what it would be if, with the same total kinetic energy, the spheres were material points, we see that it is diminished in the proportion $1 + \kappa : 1$.

The number of collisions per unit of volume and time is proportional to $c^2 T_r$; that is, to $\kappa^{\frac{2}{3}} / (1 + \kappa)$. It is less in the proportion $1 / (1 + \kappa)$ than it would be if, with the same total kinetic energy, the spheres had velocities independent of one another.

21. From the fact that p is independent of κ, it follows that local variations of density, *i.e.*, of κ, do not involve the expenditure of any work on the whole, and therefore such variations may and will come into being.

22. BOLTZMANN's minimum function continues to diminish by collisions, finally attaining its minimum constant value when the distribution of velocities defined by our assumption in 13 is established, but its actual value when minimum differs by $\frac{1}{2}\log D$ from what it would be if the spheres were material points.

INDEX SLIP.

II. *The Kinematics of Machines.*

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Whatever machine be examined, it will be found to consist of a number of pieces, each of which is connected with one or more other pieces, in such a way as to be capable of some kind of motion relatively to those pieces.

The nature of the relative motion will depend largely on the form of the surfaces of mutual contact, but also on other influences. In some cases it will be of a simple character, in others, more complex.

It is proposed to refer to these as the *elementary mechanical motions,* and to regard a machine as an embodiment of a number of such motions which, together, provide that particular kind of, more or less complicated, movement which is required to serve the purpose of the machine.

The principal object of this paper is to indicate certain geometrical laws which govern the association of these elementary mechanical motions in the composition of machines, and it will be shown that in the examination of the influence of these laws one is led to a systematic classification of all machines, and to the enumeration of an exhaustive list of some, if not all, classes of simple machines.

Further, the idea that a machine is an embodiment of a combination of *elementary motions* (of which it will be found that the number of kinds is comparatively limited), enables one, by adopting a suggestive symbol for each, to indicate the composition of every machine movement by means of a simple formula.

If the machine consists of only two separately moving pieces then only one elementary motion will be involved in the construction.

If there are three pieces, V, X, and Z, for example, V and X may be joined to and in contact with Z, but not connected with one another. In this case the motion of V relatively to Z will be quite independent of the motion of X, and the apparatus should be regarded as a compound mechanism having one link, Z, common to two simple mechanisms.

If V and X are connected together, then, under some circumstances, relative motion of the three pieces will be possible, and there will be three motions included in the mechanism, which will be related to one another.

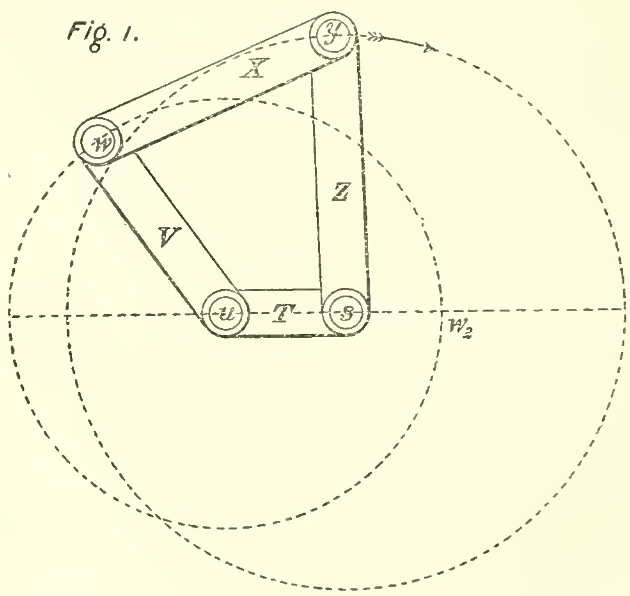
If, in the case in which the pieces move in or parallel to one plane, the links are connected together in one continuous chain, the last being joined to the first, the number of links and motions will often be required to be not less than four to permit of any movement, and, if the number so joined exceeds four, the movement will be indeterminate and the linkage will not be suitable to serve the purpose of a machine in which a definite movement is required.

The general case of a simple plane mechanism is represented in fig. 1, in which four links are united together in one continuous chain by four pins, the axes of which are parallel to one another. If three links only were joined together in this way, they would form a rigid triangle, whereas, if five were so joined the relative movement would be indeterminate.

It will be seen further on that, by imagining the lengths of the links to be suitably changed up to the limits of zero and infinity, this mechanism may be made to move in all the ways which are possible for a simple machine, the parts of which move in or parallel to one plane, also by supposing other changes to occur the same mechanism will be seen to be representative of other classes of mechanisms in which the parts do not move in or parallel to one plane. P. 31.

It will be shown later, p. 36, that machines which consist of more than four links are compound mechanisms, which may be analysed into two or more simple ones, in which one or more links are common to two or more of the simple mechanisms which together compose the compound machine.

Fig. 1.



Simple Plane Mechanisms.

If, in the mechanism of fig. 1, the links are all of finite length, each link will be capable of moving relatively to an adjacent one, either by turning completely and continuously around, which will be referred to as a turning motion and represented by he letter O, or by swinging through a part of a revolution and reciprocating to and fro. Such a motion will be referred to as a swinging motion, and represented by U. Which of these two elementary motions occur at any joint will depend on the proportions which the lengths of the links bear to one another.

There are two geometrical laws which govern the association of these two motions in this mechanism, and which also apply to other derived mechanisms.

Law I. *The sum of the four angles of the plane quadrilateral, s u w y of fig.* 1, *is constant.*

Law II. has to do with the proportions necessary to admit of the complete rotation of one link relatively to the next in sequence. It rests on the established fact that one side of a triangle cannot be greater than the sum of the other two. The expression of this law will be given later.

In fig. 1 suppose the links V and Z to be capable of turning continuously around relatively to T in the direction of the arrow, then, with every revolution, the angle at s may be regarded as having been increased by the amount of 2π, and when, in the movement, the angle between the two links V and T is reduced to zero by w coinciding with w_2, further movement in the same direction may be regarded as causing the angle to become negative.

Regarded in this way Law I. will still hold good, the sum of the four angles estimated from any initial position will be constant.

It will follow from this that it is impossible to have one O motion only in the mechanism, and the other three U motions, or otherwise the sum of the four angles would be increased or diminished by 2π each revolution.

The law permits of there being four O motions. At two of the joints the angles will have to increase, and at the other two to diminish at the same rate, unless it be conceived possible for the rate of turning at one joint to be equal to that of the sum of the other three. The latter imagined movement will be found not to satisfy the conditions imposed by Law II.

Also Law I. permits the combination of three O's and one U if the rate of turning at one of the O's is on the average equal to the sum of the rates at the other two. Such a movement will be found to be not precluded by Law II. Further there may be two O's and two U's.

The capability of complete rotation at any joint will depend on the possibility of the adjacent links getting into the critical positions to lie along the same straight line.

1st. Away from one another.

2nd. Overlying one another.

Suppose Z* to be capable of turning completely around relatively to T, then for Z

* Frequent reference to fig. 1 may be avoided if it is noticed that the small letters for the joints, and the capital letters which denote the intervening links, are in alphabetical order. Small letters $t\ v\ x\ z$ printed in italics are used to represent the length of the corresponding links.

and T to lie in the same straight line away from one another, the four links then forming a triangle, will require that

$$t + z = < v + x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha).$$

Also in order that they may be able to overlie one another, again forming a triangle with the two other links, the difference of t and z must not be less than the difference of v and x,

$$t \backsim z = > v \backsim x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta).$$

Therefore Law II. may be expressed as follows :—

In order that two consecutive links may be capable of turning relatively to one another, it is necessary that the sum of the lengths of those two links should not be greater, nor the difference less, than that of the other two links.

If these two criteria of complete turning motion be applied to each of two joints which are opposite one another, for example, the joint s between T and Z, and the joint w between V and X, it will follow that for complete turning at those joints,

$$\text{either} \quad t = x \quad \text{and} \quad z = v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$\text{or} \quad t = v \quad \text{and} \quad z = x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The equalities (1) are also those which will permit of turning motions at the joints u and y, and if t does not exceed z, the equalities (2) will permit of turning motion at the joint u.

Thus, for four O motions, opposite links must be of equal length.

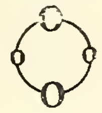
For three O motions and one U, the two links adjacent to the U must be equal to one another, and longer than the other two links, which must also be equal to one another.

It thus happens that in order that there may be O motions at two joints opposite one another, it will be necessary to have an O motion at at least one other joint, so that a combination consisting of two O motions alternating with two U's is not possible.

There being O motions at the two consecutive joints s and u, then, if t added to the length of one of the other links equals the sum of the other two, and, if added to each of the others the sum is less, or if t added to each one of the three gives a sum less than the sum of the two remaining, then, in either case, there can be U motions only at both w and y, but in the former case it will be possible for two complete revolutions to occur at s and u contemporaneously with one complete to and fro swing at each of w and y.

If the proportions are such as to satisfy neither set of conditions previously mentioned, then it will be possible for the motions at all the four joints to be U motions only.

For the motion at the joint of a quadrilateral, which, in the movement, causes a continually diminishing angle, it will be convenient to use a small o, to distinguish it from the motion which causes an increasing angle, which will be represented by a large O.

In the combination of four O's, the two large O's may alternate with the two small o's, in which case the machine movement may be represented by a formula written thus . For convenience in printing the letters may be put in line. The links may be inserted thus —O—o—O—o—, and the end links imagined to be joined up, or the joining up may actually be exhibited thus O—o—O—o; or, in many cases, it will be sufficient to leave the links themselves to the imagination, and show only the sequence of motions thus OoOo. This movement is that which is obtained by the mechanism known as "parallel cranks," which is employed in locomotive engines to connect together two pairs of driving wheels by means of an outside coupling rod.

If the two angles which increase are in sequence, as would be expressed by OOoo or OooO, or ooOO, this movement is that belonging to so-called "anti-parallel cranks."

If two complete rotations take place at one joint in the time occupied by one at another, or in the time taken by a complete swing to and fro, it will be convenient to express that fact by a doubled O, thus, or by O^2.

Thus the combination of three O's and one U may be written thus, oO^2oU, or when reversed thus, Oo^2OU. Considerations of symmetry indicate that the angular velocity at the two single O's must be equal, and therefore, on account of the changing magnitude of the angle at the U joint, it will be alternately greater and less than half the angular velocity at the fourth joint.

As previously described, we may have the movement O^2o^2UU, and also OoUU.

Where a U motion occurs, as for example between X and Z, fig. 1, the construction of the links may be modified to the form shown in fig. 2, and yet precisely the same relative motion of all the links will be retained.

For the link X, of fig. 1, a block X, in fig. 2, is substituted, which swings about a centre y, which is the centre of curvature of the circular slotway formed in the link Z. X may be regarded as being guided by a portion of the surface of a large pin, instead of by the whole of the surface of a smaller one, the centre of the pins occupying the same relative position in each construction.

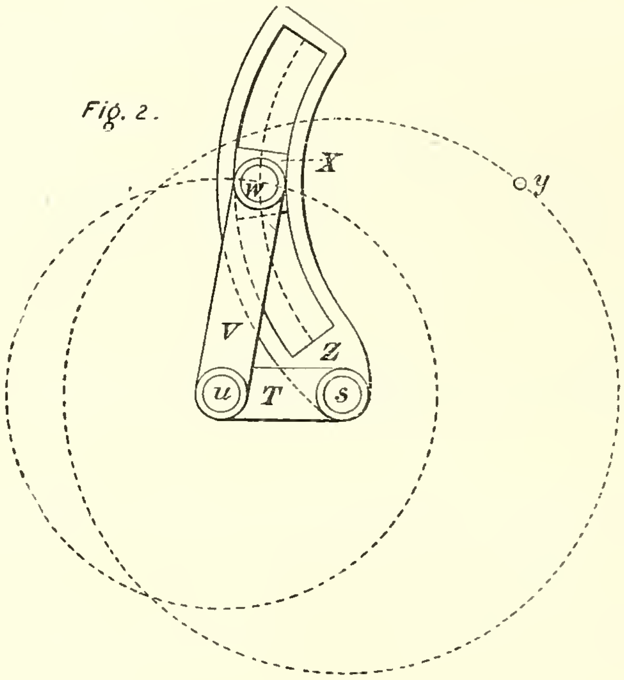
It is easy to conceive the centre of curvature y to be removed to a distance and the slotway to become less curved, until, in the limit when y is at an infinite distance, the slotway will be straight, and a sliding motion between X and Z will be substituted for a swinging motion. The resulting movement may be conceived to be that due to a four-bar mechanism, in which X and Z are infinitely long.

A sliding motion will be represented by the letter I.

This modification of the construction may be repeated where there is another U motion, but not where there is an O motion. The corresponding counterpart of an O motion would be a continuous slide in one direction, to provide for which a really indefinitely long link would be requisite.

The foregoing method of showing the derivation of an I motion from a partial turning motion is due to REULEAUX. It is here introduced for the purpose of showing how the two laws previously enunciated may be applied to mechanisms containing I motions.

Fig. 2.



In the application, a slide is to be regarded as a swing through a zero angle about an infinitely distant centre, and the adjacent links are to be imagined infinitely long, but having, possibly, a finite difference between them.

If in the combination oO^2oU an I is substituted for the U, it will become oO^2oI, in which the proportions between the length of the links previously ascertained must still be retained, that is, the two links joining the O's must be equal, and the other two links joined up by the I are to be conceived infinite and equal, which amounts to the construction being such that the line of slide is parallel to the line joining the axes of the two adjacent o motions. There being no angular velocity at the slide, the angular velocity of the O^2 motion must be exactly twice that of the two o motions.

This property, together with that of symmetry of motion, has caused this combination to be adopted in the design of a high-speed engine.

If a slide is substituted for a swing in the combination OoUU, it will become OoUI, many important examples of which are in every day use, as further described. P. 26.

In the most important examples the line of slide is parallel to the line joining the axes of the adjacent U and O motions, which is equivalent to requiring the two links joined by the I to be imagined equal as well as infinite. But the line of slide may be oblique to the line joining the axes of U and O, short of an amount which would cause the distance between those axes measured pependicularly to the line of slide to be equal to the difference in the lengths of the two finite links which join O and o and o and U. When the distance so measured is equal to the difference, a movement represented by O^2o^2UI may take place, and if the distance exceeds this difference, then all O motions are precluded. All this follows from Law II., and is consistent with the statement, p. 18, of the conditions requisite for OoUU and O^2o^2UU.

Next, if, for the other swing, we substitute a second slide we obtain the movement OoII. In the most important examples of this the two lines of slide are at right angles to one another ; but this is not necessary. It would be possible to increase the expressiveness of the formula by inclining the I's to the actual angle employed.

It is interesting to notice that the movement O^2o^2II is not possible, for with I's instead of U's the conditions requisite for O^2o^2UU could not be satisfied, since the sum of a finite link and an infinite link cannot be equal to the sum of two infinites. It may also be noticed that inasmuch as we cannot have OUoU (p. 18), then, for the same reason, we cannot have OUoI, or OIoI.

From UUUU we may derive by substitution UUUI, and UUII, and UIUI.

In the last mentioned all the links are to be conceived of infinite length, but the axes of the two U's are at a finite though variable distance apart.

The combination UIII is not possible, for a slide is to be regarded as a swing through a zero angle about an infinitely distant centre, and there being no change in the angles of the imaginary quadrilateral at three of the joints, Law I. will preclude a change at the fourth joint.

The movement IIII will be possible, but not only are all the links to be conceived of infinite length, as in UIUI, but there is no finite length whatever in the mechanism. This will permit of the movement being as indefinite as if there were five links of finite length joined by pins.

One of the links and slides may be suppressed, and the mechanism will then provide a definite relative movement represented by III, which may otherwise be shown with the I's inclined to one another, as the lines of slides are.

The fourteen combinations of the OUI motions which have been mentioned exhaust the list of the different possible ways in which they can be associated together in one simple plane mechanism.

The OUI motions permit of the possibility of the two pieces, which so move relatively to one another, being in contact with one another over an area which may be extended to any amount which may be deemed desirable.

From this property great durability against wear is provided, and this advantage, combined with that due to the facility with which the necessary surfaces of contact can be formed, causes these motions to be so very largely employed in the composition of the machines found in ordinary use.

If the foregoing analysis be compared with that instituted by REULEAUX, to which it bears a close resemblance, it will be seen that REULEAUX conceives that the elementary essential components of *machines* are the *pairs of consecutive links* which are in mutual contact, whereas it is here proposed that the *relative motions* of consecutive links should be regarded as the essential elements or components of a *machine movement.*

Whilst the pairs of surfaces of contact of consecutive links should be formed to suit the kind of relative motion which those pieces are required to undergo, yet the forms of those surfaces do not themselves entirely govern the character of the motion.

REULEAUX assumes that the turning and sliding motions are entirely governed by the forms of the surfaces of mutual contact of the consecutive links, but shows that, to insure a more complex relative motion, a restraint is often imposed by means which are external to the two links which so move. Those additional means of constraint have to be included with that due to the forms of the surfaces of mutual contact, in the full conception of a complete pair. The apprehension of what exactly constitutes a pair is often extremely difficult.

REULEAUX, acting on this assumption, does not directly discriminate between a turning and a swinging pair, for there is no difference between the forms of the pairs of surfaces which are suitable for the O and U motions. Yet the difference in the two motions is most apparent, and very important, both kinematically and also from the practical engineer's point of view. So also, whilst the difference between the motions represented by a large O and a small o is easy to understand, and is of much practical importance, yet the necessary difference in the construction of the corresponding pairs is very intangible, and may even be non-existent.

Although REULEAUX supposes a mechanism to be made up of a number of pairs, yet, in some cases, he has to admit that the entire mechanism is needful to complete only one of the pairs. This he refers to as "chain closure." It will be seen that, if what REULEAUX calls a turning pair is differentiated into the pairs requisite for the three motions denoted by OoU, then, whenever either of those motions occurs, the whole mechanism will be needful for the completion of one pair contained in it. Thus, whilst saying that the whole is made up of a number of parts, yet each part complete will include the whole, and so the analysis fails.

In offering the proposed modification of the analysis of machines, the author desires to pay to REULEAUX a tribute of admiration for his work, and to gratefully acknowledge it as the source of his inspiration.

Although exception is taken to some of the premises of REULEAUX, yet the greater portion of his work will remain of undiminished value.

In designing or constructing a machine so that it may embody any one of the previously-mentioned combination of motions, the provision of suitable forms for the surfaces of mutual contact and means of constraint against other motions will be found to exercise a judgment which is required to be extensively informed, the variety of possible forms and means being often so great. What REULEAUX describes under the headings "Complete and Incomplete Closure," "Force Closure," "Inversion of the Pair," "Chamber Pair," &c., will be found of great value in machine design ; but, in the first study of the kinematics of machines in general, it will be found advantageous to avoid the consideration of these details, and to study the conditions necessary to permit of the contemplated motions, without being concerned with what is necessary to preclude other motions.

The great feature of REULEAUX'S theory is the enunciation, for the first time, of the principle known as "the inversion of a mechanism." By the aid of this, the family relationship between machines, which previously were thought to have little or nothing in common, is most clearly established; and all the knowledge which has been acquired relative to one becomes applicable to the consideration of the others.

The belief that the method of analysis here proposed is adapted to more perfectly and consistently express the effects of inversion, has been one of the chief encouragements to the author, in hoping that the adoption of his proposals will be advantageous.

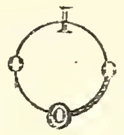
It may be stated that the object aimed at in the design of a machine is to obtain a special kind of movement of one or more pieces, of which the machine is composed, relatively to one's self as a fixed observer or user of the machine.

Any one of the four links (if there are four) may be fixed relatively to the user—may be made, as it is called, the *frame link*, about which all the other pieces move ; and we shall, in general, obtain a different kind of movement of those links with each change of the fixed link, the difference being such as to constitute practically a new machine movement, and yet all the four will be intimately related to each other, for they all have the same relative motions of the pieces composing them.

This idea of the change of the fixed or frame link is what is called "the inversion of a mechanism."

In order to assist in the description of the effect of inversion in causing a change in the machine movement, it will be convenient to adopt the term *primary piece*, originally proposed by RANKINE, for those pieces or links which are in sequence with the frame link and move in contact with it. (RANKINE must have been very near the discovery of the principle of inversion.)

If we invert any one of the previously mentioned mechanisms, and, by making some other link the frame link, find one or both of the new primary pieces to have a different motion to that which the previous primary pieces had, a new machine movement will have been derived ; but if, as often happens, we find after inversion that both primary pieces move just like the previous primary pieces, no new movement will result, but only a repetition of a previous one. Thus there are not necessarily four different machine movements derivable by inversion from every four-linked mechanism.

The link which is selected for the frame may be represented as follows:—If the links and motions are shown in circuit the frame link may be shown by a thickened line, thus,  or if printed in line, thus —O^2—o—I—o—, or thus. O^2—o—I—o (with a bracket joining the end letters). Or, if the links themselves are left to the imagination, the link which unites the end letters of the line, may be understood to be the frame link,* so that O^2oIo means the same thing as either of the above three formulæ, and IoO^2o is an inversion which gives a new machine movement.

oIoO^2 and oO^2oI are inversions which give repetitions only of the previous movements.

An exhaustive list follows of all the possible different machine movements which can be derived by inversion from the combinations of the OUI motions previously enumerated.

They are divided first into four groups. These are sub-divided into fourteen combinations, and the fourteen combinations are still further divided into thirty-two so-called "inversions."

When four different movements are derivable from a combination, each of the four may be regarded as an inversion of either of the others, and it will not be inappropriate in this case to say there are four inversions, or when only three or two different movements are obtainable, to say there are three or two inversions respectively. When only one movement can be obtained it will still be desirable to speak of it as one inversion.

* The author is indebted to one of the Referees, to whom this paper was submitted, for this suggestion and for some others.

Group O^4, Containing 4 Turning Motions.

Combination.	Inversions.	Notes and examples.
OoOo	oOoO	Opposite links equal and parallel to one another. Gives one machine movement only, there being no change by "inversion." Mechanism, known as "parallel-cranks," employed for coupling together two pairs of driving wheels of locomotive engines.
OOoo	OOoo	Opposite links equal but not parallel. Both cranks revolve in the same direction just as in the movement above.
	OooO	Two cranks revolve in opposite directions called anti-parallel cranks.

Group O^3, Containing 3 Turning Motions.

Combination.	Inversions.	Notes and examples.
oO^2oU	oO^2oU O^2oUo	Mechanism consists of two equal short links in sequence and two equal longer ones. It is known as the Kite mechanism.
oO^2oI		There are two finite links of equal length and the line of slide is parallel to the line joining the axes of the two o's.
	oO^2oI	This movement has been used by Bernay in the design of a steam-pump. It may be compared with the common crank-and-connecting-rod engine, in which the length of the connecting rod is equal to the crank. The length of stroke is four times that of the crank. This is supposed to be the advantage due to the use of this mechanism for this purpose.
	O^2oIo	This, used in duplicate, has been employed by Parsons in the design of a high-speed engine. One, primary piece, the crank shaft, revolves at twice the speed of the other primary piece which is the cylinder.

GROUP O^2, Containing 2 Turning Motions.

Combination.	Inversions.	Notes and examples.
OoUU		The link between Oo must be such that when added to either one of the other three the sum must be less than the sum of the two remaining links.
	OoUU	In a beam-engine the portion consisting of the half-beam, connecting-rod, crank shaft, and frame of engine, affords an example.
	oUUO	Example: The drag-link coupling used for connecting two shafts which are parallel, but not in the same straight line.
	UOoU	In this case both primary pieces swing. The 4th inversion will give a repetition of the movement of the 1st.
OoUl		The line of slide must be at a less distance from the axis of the O motion than the difference in the length of the two finite links. In most practical examples of the use of this movement the deviation of the line of slide from the axis of O is zero, as in the following examples.
	OoUl	Example: The much used crank-and-connecting-rod engine.
	oUlO	Example: RIGG's high-speed engine.
	UlOo	Example: The oscillating engine much used in paddle wheel steamers.
	lOoU	Example: STANNAH's pendulum pump.
		Four different machine movements are derivable by inversion from this combination, and each has been adopted in the design of a steam engine.
O^2o^2UU		The link between Oo added to one other link must equal the sum of the two remaining links; but when added to the others the sum must be less.
	O^2o^2UU o^2UUO^2 UO^2o^2U	Two complete rotations occur at two joints, whilst one complete to-and-fro swing occurs at the other two. The 4th fixing gives a movement which is a repetition of the 1st.

Group O^2, Containing 2 Turning Motions—(continued).

Combination.	Inversions.	Notes and examples.
O^2o^2UI	O^2o^2UI o^2UIO^2 UIO^2o^2 IO^2o^2U	In this combination the deviation of the line of slide is equal to the difference in the length of the two finite links. This is the limiting deviation possible compatible with O motions.
OoII.		Only one link is of finite length. The lines of the two slides may be inclined at any angle one to the other > zero. In general, the angle is a right angle as in the following examples.
	OoII	Example: The Yoke pump, a compact engine, much used for feeding boilers. The relative motion between crank and piston is like that due to an infinitely long rod in a crank-and-connecting-rod engine. The motion is said to be "Harmonic."
	oIIO	This movement occurs in Oldham's coupling employed to connect two shafts which are parallel but not in the same straight line. In this case the two shafts have a constant angular velocity ratio which is not the case with the drag-link coupling oUUO previously quoted.
	IOoI	Every point in the link joining Oo moves in an ellipse. The mechanism has been used in an instrument for drawing ellipses. The 4th inversion would produce a movement which would be a repetition of the 1st.

Group U, Containing no Turning Motions.

Combination.	Inversions.	Notes and examples.
UUUU	UUUU	Whichever link is fixed both primary pieces will swing, there will therefore be no change in the movement by inversion. Watt's rectilinear motion is an example of the use of this mechanism.
UUUI	UUUI UUIU	Only two movements by inversion, the other two inversions produce repetitions of the previous movements.
UUII	UUII UIIU IUUI	Three movements by inversion, the 4th being a repetition. There is no difference between the four links required for this combination and those required for OoII.
UIUI	UIUI	Whichever link is fixed, one primary piece will swing and the other slide, so that no new movement will be obtainable by inversion. Two of the infinite links adjacent to an I may be conceived to be equal, causing the line of that slide to be parallel to the line joining the axes of the two U's, as employed in Rapson's slide for steering large vessels, also in the compensating mechanism used in the Worthington steam pump, as pointed out, page 111, 'Cotterill's Applied Mechanics,' edition 1892. Both lines of slide may deviate from the line joining the axes of the U's. If the deviation of each is the same the mechanism has the remarkable property that in one position it may be locked. It appears to be quite new.
III	III	No change in the movement will occur on inversion. This is known as the wedge mechanism. It is employed in the cotter method of adjusting the brasses at the ends of a locomotive connecting rod when worn.

Although the OUI motions constitute the large majority of the relative motions of consecutive directly connected parts of machines, yet there are others, frequently found in use, in which the relative motion is of a more complicated character, for example the relative motion of the teeth of a pair of spur wheels. The relationship of machines in which there are such motions to the foregoing can be best explained by a reference to the series of machines shown in the adjoining five figures.

It has been previously shown that in the mechanism of fig. 2, p. 20, the same relative motions occur as in fig. 1, p. 16, though the construction is different. Suppose now the pin w to be enlarged sufficiently to fit the slotway in the link Z, as in fig. 3, p. 29. The block X may then be suppressed, and the same relative motion of the links T, V, and Z will be retained exactly as before.

The block X being omitted, it will no longer be necessary for the slotway to be curved to the arc of a circle. The radius of curvature may vary from point to point of its length; but, for definiteness of motion, it will be necessary for the width of the slotway to remain constantly equal to the diameter of the pin, as shown in fig. 4. The centre of curvature of any portion of the slotway where the pin w is momentarily situated being y, the instantaneous relative motion of the parts will be the same as would be due to a four-linked mechanism of which the centres of the pins are at s, u, w, y. When the pin w moves to a place where the curvature is different, the relative motion will be the same as that due to a four-linked mechanism in which the lengths of X and Z are different from what they were previously.

If, in figs. 1, 2, 3, 4, the lines joining the centres y, w and s, u are produced to meet in the point r, it may be readily shown that

$$\frac{\text{The angular velocity of V}}{\text{The angular velocity of Z}} \text{ relatively to T} = \frac{sr}{ur}.$$

Fig. 1.

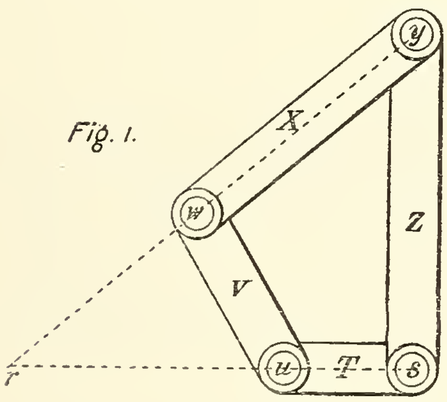


Fig. 2.

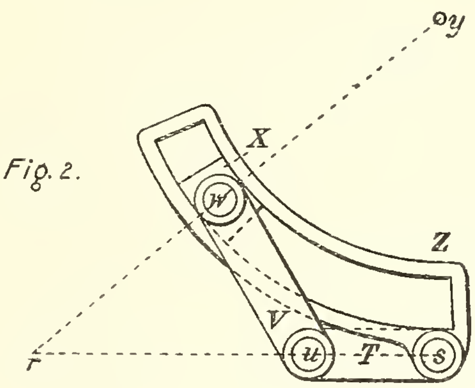


Fig. 3.

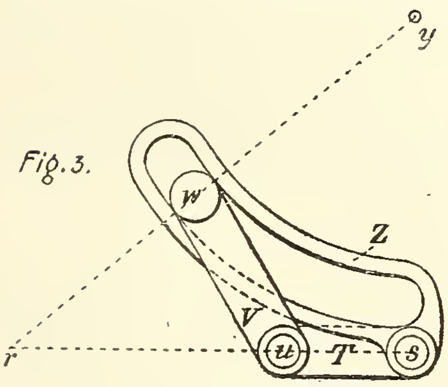


Fig. 4.

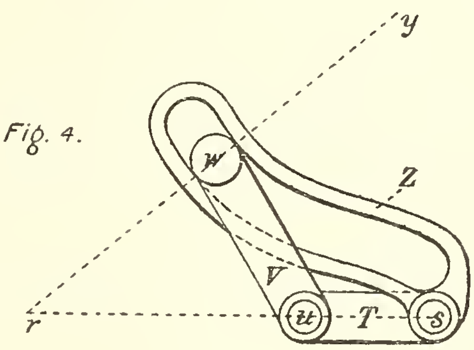
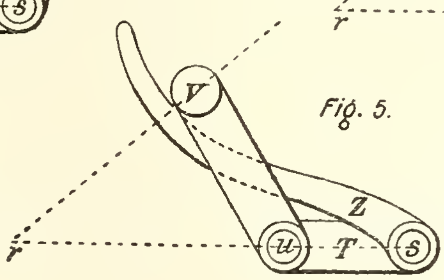


Fig. 5.



In the next consecutive position of the mechanisms 1, 2, and 3, the position of the point r will have changed, and with it the angular velocity ratio. In mechanism fig. 4, it will be possible, by a suitable curve for the slotway, for the intersection of the two lines to remain at a fixed point for a limited amount of motion, when, during that period, the angular velocity ratio will remain constant, or the curvature of the slotway may be so designed as to cause a specially desired change in the angular velocity ratio.

An exceedingly important advantage is thus obtainable from the mechanism of fig 4 as compared with 1 and 2 ; moreover, it is obtained by the use of one less link or moving piece. But these advantages are purchased at the cost of requiring that the consecutive links V and Z be in contact at a point, or along a line of points only instead of over an area. so that, if considerable pressure is transmitted from one link to the other, there is liability to undue abrasion and rapid wear.

If the action of the forces is such as to keep the link V always in contact with one side of the slotway, the other side may be omitted, as shown in fig. 5. In this case the pin *w* need not be cylindrical, but any shape.

The relative motion of V and Z in fig. 5 is comparable to that of two teeth of a pair of spur wheels or two cam surfaces. It will thus be seen to be an amalgamation of two adjacent OU or I motions.

The mechanism may be regarded as a mechanical artifice, whereby we get the movements which would be due to a four-linked mechanism in which the length of the links were capable, during the movement, of being varied in a desired way.

According to this point of view the same two geometrical laws will be applicable to mechanisms containing these motions. For example, from Law I. it will follow that the angular velocity of Z relatively to V must equal the algebraical difference of the angular velocities of Z and V relatively to T.

A convenient and expressive notation for such movements may be derived by suppressing the absent link and joining up the two simple adjacent OUI motions into one more complex. The letter W will naturally stand for an amalgamation of two U's, and the figure eight on its side, or the sign of infinity ∞ for two O's, or would be useful to indicate rolling motion.

The Greek letters Φ and Ψ are also available.

A mechanism consisting of two spur wheels, mounted in a link and in outside gear with one another, may be represented by OWO, and an inversion of it WOO. If the outside of one wheel gears with the inside of the other (an annular wheel), the formula should be OWo.

By the use of pulleys with ropes, belts, or other flexible links, another mechanical device is provided, whereby the angular velocity ratio in consecutive positions of the mechanism may be maintained constant, or caused to vary in a desired way. In this case the result may be considered to be achieved by a perpetual substitution of new links, exactly like the previous ones, and in the previous positions, when those have moved out of place, instead of regarding the movement to be continued by links of altered length, adapted to the new positions as we conceived to be the case with spur-wheels.

The relative motion of a belt to a pulley, which it wraps, is such that every point describes an involute, so the letter C would, in a fairly graphical manner, represent this motion.

A mechanism consisting of two pulleys mounted in a frame joined by a crossed belt, may be represented by OcCO, or oCco. If an open belt is used the corresponding formula would be OcCo, a small c being used for wrapping, and a large C for unwrapping.

It is not easy, if possible, to make an exhaustive list of these mechanisms. What has been said will serve to indicate the method of the application of the previously mentioned principles in considering them.

It will be convenient to class them separately from the plane mechanisms made up of the OUI motions.

Spherical Mechanisms.

The next division of machines to be described will be those in which the simple motions OU only are employed, but in which the axes, instead of being parallel to one another, are so inclined that they all meet in one point.

Referring to the original mechanism, fig. 1, p. 16, suppose the links to be bent so that they lie on or parallel to the surface of a sphere, the axes of the pins will all be radii of the sphere, and if produced will meet at the centre, and the centre lines of the links will be great circles of the sphere.

Reuleaux has shown that the previously described mechanisms, or some of them, have their counterpart in spherical movements ; but he omitted to notice an important exception which forms the foundation for a distinct division of machines, in which helical motions are employed.

There are other points of interest and importance, not observed by Reuleaux, which follow from the application of the geometrical laws which govern the association of the OU motions in a spherical mechanism. For such movements Law I. must be modified as follows :—

The sum of the four angles of the spherical quadrilateral varies, having a value of 3π for a maximum, and 2π for a minimum.

Law II. will remain as before.

It will follow that each one of the combinations of the OU motions in plane mechanisms will have its counterpart in spherical mechanisms.

Reuleaux has further pointed out that if a mechanism containing a sliding motion is adapted to a sphere, then, instead of a sliding motion along a curved link, exactly the same relative motion could be produced by the swinging of one link about one of the poles of the sphere, of which the curved line of slide is the equator. One of the two links joined by the slide would then be required to have a length equal to that of a quadrant of a great circle. The other may be equal to it or not equal, being greater or less at pleasure, for either pole may be selected for the axis of swing.

Also all the combinations of the I with OU motions previously detailed have their counterpart in spherical mechanisms.

REULEAUX has compared one of the best known spherical mechanisms, known as HOOKE's joint, with its plane counterpart, OLDHAM's Coupling, p. 27. In the latter there are three links which are supposed to be equal and infinite. In the former there are three links, the lengths of which are each equal to a quadrant, the fourth being shorter. In HOOKE's joint it is well known that the angular velocity ratio of the connected shafts is not constant, whereas in OLDHAM's Coupling it is.

It is interesting to notice that these two facts immediately follow from Law I., the sum of the four angles of the spherical quadrilateral not being constant.

In spherical mechanisms the relative lengths of the links may be represented by the magnitude of the angles at the centre of the sphere which they subtend. This suggests that a spherical movement may be represented and distinguished from a plane movement by inserting the value of the angles between the letters O and U, which represent the motions.* Thus for examples :—

The spherical counterpart of OoOo is $\beta O \alpha o \beta O \alpha o \beta$, expressing the fact that opposite links are equal to one another.

Of oO^2oU, the spherical counterpart is $\beta O \alpha O^2 \alpha o \beta U \beta$ where $\beta > \alpha$.

Of oO^2oI it is $\frac{1}{2}\pi O \alpha O^2 \alpha o \frac{1}{2}\pi U \frac{1}{2}\pi$. $\alpha < \frac{1}{2}\pi$; and so on for all the other plane mechanisms previously enumerated, with one exception.

The movement III has no spherical counterpart capable of movement, as it will be a spherical triangle, though there is a spherical counterpart to the movement IIII.

It is interesting to notice that the reason which precluded the existence of the combination UIII in plane mechanisms, does not hold in the case of its spherical counterpart, for the sum of the four angles of the spherical quadrilateral is capable of variation. With these exceptions the combinations not possible in plane mechanisms are also impossible in their spherical counterparts.

An enumeration will show that there are only six different ways of combining the OU motions in a spherical mechanism, and out of these we can get only twelve different movements by inversion.

Besides these there are the spherical counterparts of those plane mechanisms which contain in their composition other motions than OUI. The most notable of them is the mechanism consisting of a pair of bevil wheels mounted in a frame, the formula for this would be $O \alpha W \beta O (\alpha + \beta)$.

* The formulæ for plane mechanisms may be made to give information about the length of the links by inserting figures between the letters which represent the motions. Thus O2′o8′UI may be considered to represent a crank and connecting-rod engine of 4′ stroke, the length of the connecting-rod being twice the stroke, and the line of stroke passing through the axis of the rotating crank. If the line of stroke were to deviate by the amount of, say 1′, the fact could be indicated in the formula by O2′o8′U1′I, one of the infinite links being 1′ greater than the other.

Cylindrical Mechanisms.

Whilst the plane mechanism III cannot be adapted to the surface of a sphere, it can be fitted to the surface of a cylinder, and it is the only one of the OUI mechanisms which can.

If such a triangle of slides be imagined to be wrapped around the surface of a cylinder, one at least of the slides will have to become helical. Two or even three may be.

In this way we are led to the very important Third Division of machines containing screw motions.

The projected view of a helix has a shape, much like the letter V, so this letter will suggestively stand for helical motion.

In this division there are the following combinations :—

UVI with three inversions. The common screw press is one example.
VVI with two inversions. The differential screw press and micrometer are examples.
VVU with two inversions, and
VVV one inversion only.

In total, four combinations, containing eight inversions.

But not only is the plane mechanism III capable of being adapted to a cylinder, but also IWI. If for one of the slides of the former mechanism a curved slotway of uniform width fitted by a cylindrical pin were substituted, a mechanism represented by the latter formula would be obtained in which the relative velocities of sliding would not be constant, but vary from point to point. If, further, the resulting mechanism be conceived to be wrapped around a cylinder, we should have one helix of varying pitch. In order that the pin may fit such a helical groove, it will be necessary to restrict the contact between the two to that of one or a pair of points only. Such a mechanism is known as a cylindrical cam, and there are many applications of it in practice.

Adopting the letter H for the motion with the varying-pitch helix, the following four combinations are possible, and they have eleven different inversions, viz. :—

UHI with three inversions.
VHI ,, ,, ,,
VHU ,, ,, ,,
VHV ,, two ,,

No other combinations belonging to this division or sub-division appear to be possible.

Skew Mechanisms.

It is proposed to group all the remaining simple mechanisms in a fourth division. They may be described generally as consisting of those in which the axes of the elementary motions neither meet nor are parallel.

These may be referred to under the name of skew mechanisms.

The general case may be described as consisting of a link T secured to a shaft, which turns or swings in bearings provided in a piece V, which also provides bearings for another shaft which is neither parallel to nor meets the former shaft T. This second shaft has a link X secured to it so that it also turns or swings relatively to V. The two links T and X, which are either cranks or levers, are joined together by a link Z, which has ball and socket connections with T and X.

With such a construction the movement will be of a definite character, and the mechanism will be capable of serving the purpose of a machine. By imagining either or both links T and X to be infinitely long, a slide or slides may be substituted for one or both of the turns or swings.

Thus a variety of combinations are possible, and each one will be capable of a certain number of inversions.

In the formulæ for these it is proposed to use the Greek letter Θ for the movement at the ball and socket connection.

A certain limited number of OUIΘ combinations will be possible of which an exhaustive list could be prepared.

Besides these another set of skew mechanisms will result from suppressing one of the links and amalgamating the two adjacent simple motions. The best known examples of such a combination is seen in the use of a pair of skew-bevil-wheels and in the worm-and-worm-wheel. The relative motion between two such pieces may be represented by the Greek letter Ω.

Summary of Simple Mechanisms.

The method of classification, according to the proposed scheme, may be summarized as follows :—

All simple machine movements may be ranged in four divisions, viz. :

1. Consisting of plane mechanisms in which the pieces move in or parallel to one plane.

2. Spherical mechanisms in which the pieces move in or parallel to the surface of a sphere.

3. Cylindrical mechanisms in which the pieces move in or parallel to the surface of a cylinder ; and

4. Skew mechanisms, being those in which the axes of the turning or swinging motions neither meet nor are parallel.

The mechanisms in each of these divisions may be classed in two sub-divisions :

S being the sub-division in which each pair of consecutive links is in contact one with the other over a surface.

P being those in which one or more pairs of consecutive links are in contact with one another at a point or along a line of points only.

The mechanisms of Sub-division S of Divisions 1 and 2 will consist of those in which the OUI motions only are used.

Those of Division 3 will include the V or helical motion ; and

Those of Division 4 will include the motion Θ, requiring the use of a ball-and-socket joint.

To the pairs of links which have the relative motions denoted by OUIV, REULEAUX has given the name "Lower Pairs."

REULEAUX claimed two characteristics for lower pairs, viz. :

1. Definiteness of motion derived from the forms of the surfaces of mutual contact and depending on nought else.

2. The possibility of distributing the contact of consecutive links over an area which may be extended as much as desired, the contact not being confined to a point or a line of points as in "Higher Pairs."

It has been shown that if the motions represented by Oo and U are required to be differentiated from one another, REULEAUX's so-called turning pair cannot possess the first characteristic. (There is scarcely a mechanism in which the nature of the motion between two consecutive links does not depend on the other links.)

The second characteristic is of considerable value in relation to the liability to abrasion and wear ; but the advantage of greater immunity against wear has to be purchased at the cost of a more complicated construction, and a more restricted character of movement.

As the first characteristic cannot be secured, the author proposes to adopt the second only for the criterion as to whether a mechanism should be regarded as belonging to Sub-division S or P.

Therefore, in Division 4, the motion Θ requiring the ball-and-socket joint should be included in the motions of Sub-division S. REULEAUX considered the ball-and-socket joint a "higher pair."

If the motion of any two consecutive pieces of a mechanism differs from the motions OUIV and Θ, the mechanism will belong to Sub-division P.

Next, the mechanisms included in either of the eight sub-divisions may be separated into sections as numerous as the combinations of the various elementary motions, which will satisfy the governing geometrical laws.

For example, in Sub-division 1*s*, it has been shown that there are fourteen, and only fourteen, possible combinations. (These have been placed in four groups, chiefly to aid the memory in enumerating them.) In the Sub-divisions 2*s* there are only six distinct combinations, and in 3*s* there are four. It is probable that in the other sub-divisions the number of possible combinations will be finite.

The next step in the discrimination of one machine movement from another will consist in distinguishing the various different movements derived from any one combination by inversion.

Compound Mechanisms.

In most of the important machines it will be found that there are more than four, and often many more than four, separately moving pieces and yet the motions are of a perfectly determinate character. It is left to explain how this is so, and how the previous considerations apply to such machines.

In practically every one of these machines it will be found, on examination, that there is a mechanism consisting of not more than four pieces associated together in one continuous chain. The pieces or links of this chain, which for reference may be called A, will therefore have a perfectly definite motion, either relatively to the user of the machine, if one of the links is the frame, or only relatively to one another if neither of those links is the frame.

The compound machine will be formed by one or more of the links of A being combined with other pieces to form a second mechanism B.

Suppose one link only of A to be joined up in chain with three fresh pieces to form the second continuous or closed chain B, then the movement of B will be independent of A, for we may imagine the link which is common to the two to be the frame link, in which case A may be in movement whilst B is at rest, or *vice versâ*, or both may be in motion at the same time.

But more often it will be found that two of the links of A are adopted to compose with two new links the second mechanism B. In this case, a movement of A cannot take place without causing a definite movement in the two new links which, together with two of A, form the second mechanism B. Of the six links, which in this way have definite relative motions, any two may be adopted to form, in conjunction with two additional links, a third mechanism C producing a machine consisting of eight pieces, having a perfectly definite motion relatively to one another, and so on.

Compound mechanisms may be conceived and constructed in which there is a definite relative motion, but in which there is not a four-linked continuous chain of pieces.

The most simple construction of this character may be described as consisting of a continuous chain of six links, of which two pairs of links, which are opposite to one another, are linked together by the addition of two more links. In this case the shortest continuous chain will contain five links. Of the eight links which make up the complete mechanism, four will form a part of three five-linked chains, and four a part of two.

The simultaneous control due to the two or three partial restraints are sufficient to determine a definite motion.

In this mechanism there will be U motions at all the ten joints, and there will be four bars, each containing three joints. Each of these bars will be continuous through a joint, and in the formula may be represented, coupled to three other links, thus —U—U—U— , in which case the formula for the whole mechanism may be shown so

U—U—U—Ω—U—U—U—U—Ω—U

The author has been unable to discover any actual machine in which determinate motion has been derived from such a method of accumulation of partial constraints.

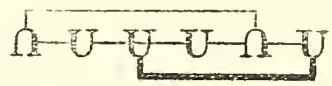
Passing to the compound mechanisms previously referred to, four degrees of complexity in the conjunction of two interdependent simple mechanisms may be specified.

Conjunction I, when two adjacent links, T and V for example, of a mechanism A are adopted, in their exact length, to form with two new links, X_2 and Z_2, the second mechanism B. In this case the new link X_2 is attached to V at the same joint, w, at which X_1 is coupled, and Z_2 as well as Z_1 are united to T at the joint s. In this we have a third mechanism, B′, consisting of the four links, $X_1Z_1Z_2X_2$, united at the joints y_1sy_2w. Supposing U motions at all the joints, the formula for a compound mechanism so conjoined may be written thus, Ω—U—U—Ω—U. An example of this will be found in what is called a double Kite mechanism (see p. 25) consisting of four equal links forming a parallelogram, and two links longer but equal to one another, joined together at one end, and at the other ends united to two opposite corners of the parallelogram. Such a construction forms a part of the mechanism of PEAUCELLIER'S straight-line motion. If the additional links of PEAUCELLIER'S mechanism are omitted, it will be possible, with each Kite, to have the movement UOo^2O as before explained (p. 19). Now, if three links unite at a joint and each of two of them have an O motion relatively to the other then, relatively to one another, the motion must be either U or O^2. A new symbol is therefore wanted to stand for a motion which may be either O, U, or O^2, which it is, to be determined by the context. The letter Q suggests itself, and accordingly the formula for the double Kite would be

Ω—Q—o^2—Q—o^2.

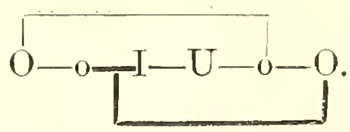
All the previously-mentioned conditions as to the associations of the OUI motions in one chain, and the proportions of links requisite for any combination, will hold good of mechanism B, as well as A. Thus the letter Q in the mechanism above stands for O in each Kite mechanism, and for O^2 in the parallelogram mechanism. It will frequently happen that whilst the proportions of A will permit of O motions at two or more joints, those of B will preclude it, in which case the motions will be restricted to U motions.

In Conjunction II., two adjacent links of A are adopted, but only one in its full length. Thus the two new links which are added to form B are connected to the two chosen ones of A at, say, the joints s and w_2. In this case the third mechanism B′ will consist of five links coupled at the joints w_2, w_1, y_1, s, y_2. Supposing, as before, U motions only, the formula for such a conjunction may be written thus,



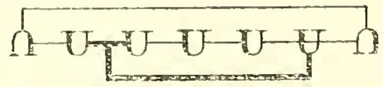
The link shown by the thick line is as explained (p. 24), intended to represent the frame link. The preceding formula will represent a portion of the mechanism of a beam engine consisting of the frame of the engine, the half beam, and the parallelogram and radius rod of WATT's parallel motion.

Another example of Conjunction II. may be quoted for the purpose of showing how the formula may be written when there is an I motion in the mechanism. The portion of a locomotive engine consisting of frame, piston, connecting-rod, crank, outside coupling-rod, and the crank of a second driving wheel, will have a movement which may be written thus,

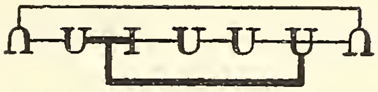


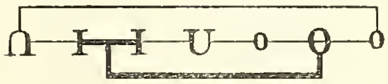
The letter I does not lend itself to showing whether the two adjacent links form one rigid piece or not; but if it be remembered that a slide is to be regarded as a swing about an infinitely distant joint, it will be seen that two lines meeting at an I cannot represent one link continuous through the joint, but must always be understood to indicate two separate links. Thus, no difficulty will occur in the use of the letter I.

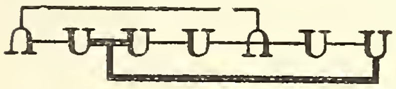
In the next, Conjunction III., neither of the two adjacent links of A, which are adopted, are used with their original length in mechanism B, so that each of these two links possess three joints. Thus, the mechanism B′ will consist of six links connected at the joints w_2, w_1, y_1, s_1, s_2, y_2. Supposing U motions only, the formula for this conjunction may be written thus,

, or so ,

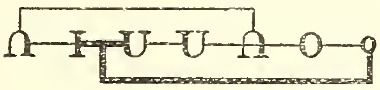
when one of the three-joint links is the frame link.

Of this, two examples may be quoted with which the reader will be sufficiently familiar to dispense with illustrations. One is the mechanism of RICHARD's indicator for taking diagrams of steam pressure in engines, the formula for which may be written thus  and the other is the common crank and connecting-rod engine with eccentric and slide-valve, the formula for which is



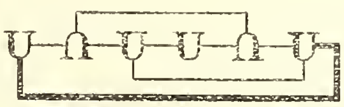
In Conjunction IV. it is the two opposite or alternate links of A to which the two additional links are coupled in the formation of mechanism B. The two selected alternate links of A must each be provided with three joints. Thus, mechanism B, as well as B′, will consist of five links. The formula may be written thus  when one of the two three-jointed links is the frame link. The mechanism of the CROSBY steam indicator affords an example of this conjunction. The same formula will represent the movement if an I be substituted for the third U from the left.

The ATKINSON gas engine may be quoted as another example, the formula being

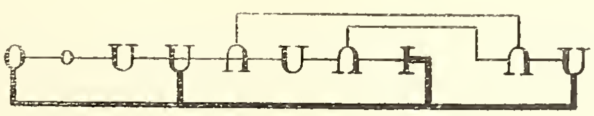


If three or more simple mechanisms are compounded, the varieties in the ways in which they may be conjoined are much more numerous. It is not desirable to set them forth here. Enough has been said to indicate the lines which it is proposed should be followed in systematizing them. The formulæ of some well-known machines, consisting of three or more simple mechanisms, are added.

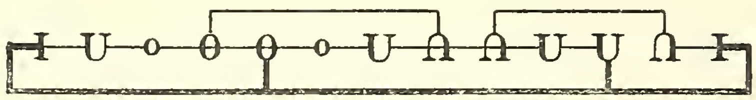
PEAUCELLIER'S straight-line mechanism is an example of a three-fold conjunction, and may be written thus,



The Beam engine, with WATT'S parallel motion, and engine piston driving a rotating shaft is an example of a four-fold conjunction, it may be written thus,

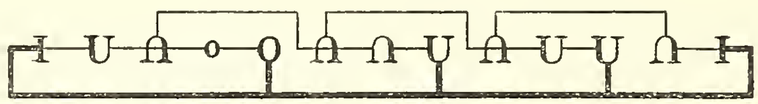


The formula for a reversing engine, with STEPHENSON'S link motion, is



The object of the mechanism is to get a correct relative movement of the two sliding pieces, piston and valve, which are represented by the links adjacent to the I's at the two ends of the formula.

With this may be compared the formula



for an engine with JOY'S valve gear, which is very differently constructed for the same purpose.

INDEX SLIP.

III. *Experiments on Fluid Viscosity.*

By A. MALLOCK.

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THESE experiments consisted of the measurement of the moment transmitted, by fluid viscosity, across the annular space between two concentric cylinders, one of which revolves while the other is stationary.

The fluid used in these experiments was water. Three distinct sets of conditions were tried, viz.:—

(1) Outer cylinder revolving, inner cylinder stationary.
(2) Inner ,, ,, outer ,, ,,
(3) Repetition of series (1) with an annulus of different width.

It had been intended to have had a repetition of series (2) with a larger annulus, but the motion in the fluid in series (2) was so thoroughly unstable that this was not done.

The object of the experiments was chiefly to examine the limits between which the motion of the fluid in the annulus was stable, and the manner in which the stability broke down. For obtaining the actual value of the coefficient of viscosity, other methods, such as the flow through capillary tubes, would be more suitable.

Fig. 1 is a section of the cylinders as arranged for the experiment of series (1). The inner cylinder A is suspended by a torsion wire, attached by a gymbal ring to the top of the stem B. This stem is guided by two rings of balls, held by the gun-metal casting C, the rings being adjustable in their own planes by means of four set screws at each ring, thus allowing the axes of the suspended and revolving cylinders to be made coincident.* The stem B carries a divided circle which is read by the telescope T. E is the outer cylinder, carried on the axis F. This cylinder was driven at constant speeds by a small electromotor connected with a governor which cuts off the current when, and not before, the desired speed is attained. Surrounding

* Though this adjustment was made with ease in the main experiments, preliminary experiments with the axes of the cylinders parallel, but separated by known intervals, had shown that an error in centreing the cylinders, if small, produced a difference of the second order only in the moment transmitted by the fluid.

the cylinder E is another, G, and the space between E and G, and the interior of A, are kept filled with water in which thermometers are placed, and the temperature of the fluid in the annulus between E and A is taken as the mean of the temperatures in A and the outer annulus.

The cylinder A has its floor about half-an-inch up from the lower edge, and during the experiments the space under this floor is filled with air. Thus the fluid in the annulus only touches the cylindrical surface of A. The axis F of the outer cylinder is hollow, and a rod, H, passing through a watertight joint at its upper end, carries a short piece of cylinder K, of the same diameter as A. The arm L at the lower end of H, where it projects beyond F, is held fixed, thus keeping the short cylinder K stationary whilst E revolves. The space between E and K is filled with mercury. The object of this arrangement is to cause the lower surface of the water in the annulus to be in contact with a surface whose velocity at any point is nearly the same as its own. Of course, in the mercury, in consequence of its being in contact with the floor of the cylinder E, there is a certain amount of circulation set up, which causes an outward radial drift in the mercury in contact with the floor and a return current flowing inwards in the mercury surface in contact with the water, but the radial velocity is slow compared with the circumferential, and the water in the annulus between E and A is very nearly in the same condition as it would be if E and A were infinitely long.

Figs. (2) and (3) show the arrangement used in the second and third series of experiments respectively.

Attached to reading telescope T was a small camera-lucida, by means of which the image of the divided circle could be seen on the surface of a cylinder, driven by clock-work and carrying continuous paper, P. On this paper an electric clock recorded seconds, and another pen and electromagnet in connection with a contact-maker on the axis F marked every revolution of the cylinder E. A third pen, worked by a hand lever, was used to follow the motion of the image of the divided circle seen on the paper through the camera-lucida.

In making an experiment the usual course was to bring the wire of the telescope to the zero of the divided circle when everything was at rest, then to start the paper cylinder and mark the position of the zero on the paper. The experimental cylinder was then set in motion, and when the suspended cylinder had reached its approximately stationary position (about four minutes after the revolving cylinder was started) the mark on the divided circle, which was nearest to the centre of the paper, was followed by the third pen for a minute or more, after which the zero mark was again made on the paper, and the number of the division (on the circle) followed by the pen was written on. In some experiments, however, the motions of the suspended cylinder were followed from the moment the revolving cylinder was started. The diagrams so made gave a permanent record of the result of each experiment, which could be analyzed at leisure.

Table I. (p. 47) gives the dimensions and constants of the apparatus.

Fig. 1.

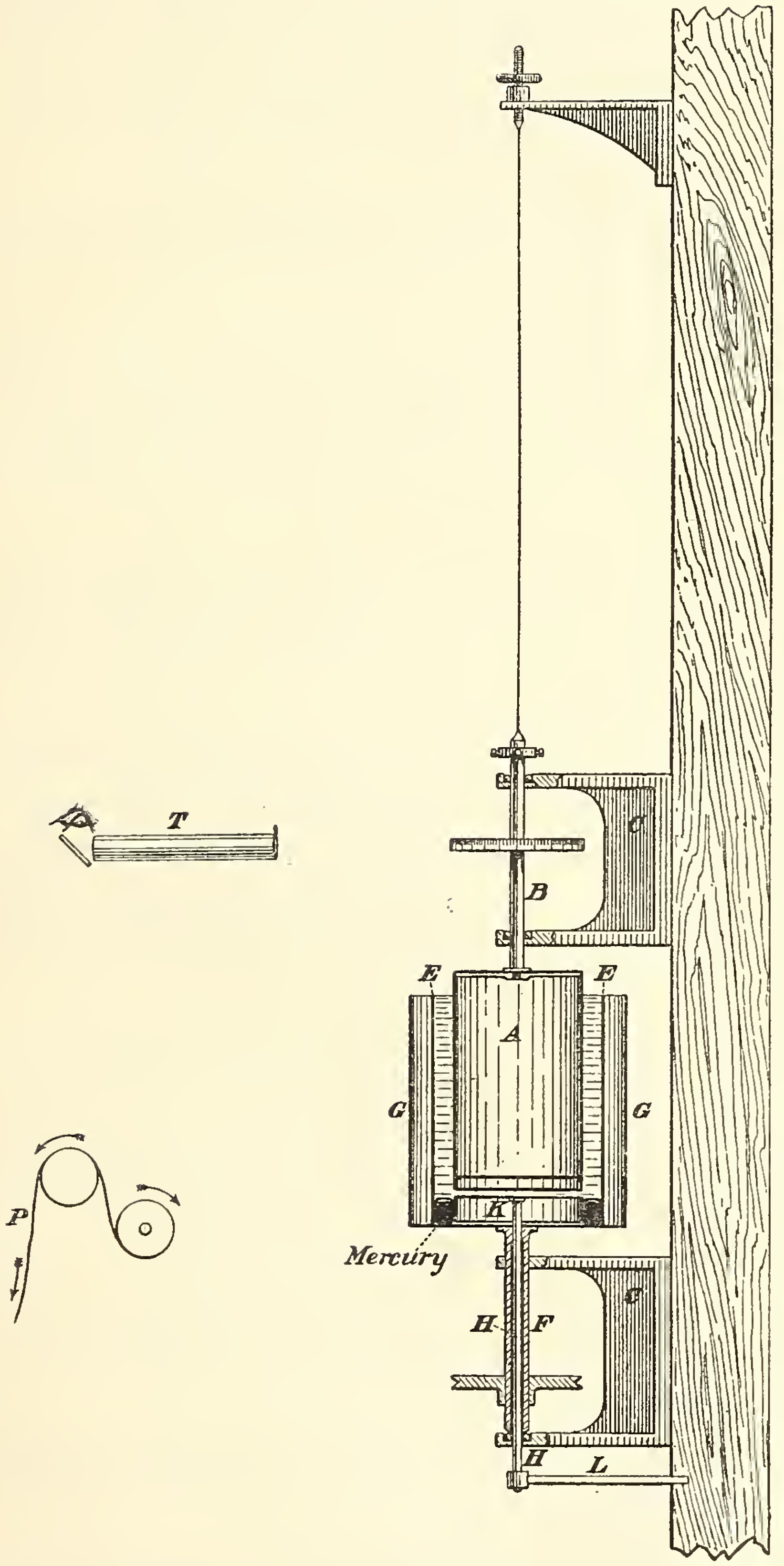


Fig. 2.

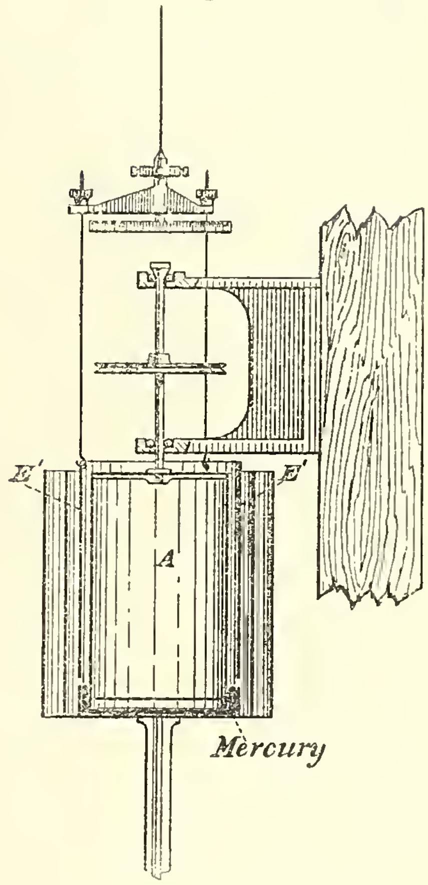
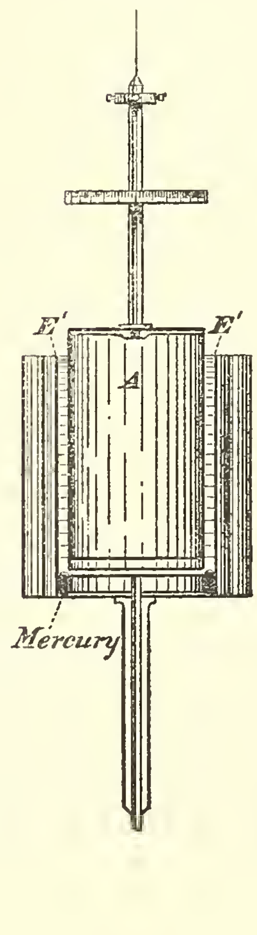


Fig. 3.



The Table II. (pp. 48–52) contains the results of all the experiments (except a few made with the medium cylinder E′) as obtained directly from the analysis of the diagrams taken with the camera-lucida.

These results were afterwards corrected for the variation of the depth of the fluid in contact with the suspended cylinder, due to the curvature of the free surface, and further corrected for the difference, if any, of this depth from 8·5 inches (a value which the dimensions of the cylinder rendered convenient). Most of the quantitative results are summarized in diagrams 5, 6, and 10.

In Diagrams 5 and 6, and also in Diagrams 4, 8, and 9, the abscissa is the number of revolutions per minute of the revolving cylinder and the ordinates are the readings of the divided circle on A.

In Diagrams 5 and 6, in addition to the number of revolutions per minute, the velocity (in feet per second) is given, which two parallel planes must have with reference to one another, if their distance apart is equal to the width of the annulus, and the fluid between them is distorted at the same rate as the fluid at the surface of the cylinder A.

In Diagrams 4, 5, and 6, it will be seen that up to a certain velocity of the cylinders E and E′, the ordinates increase almost exactly as the velocity. Above this velocity, and until a considerably higher velocity is reached, it appears that the ordinates may have any value between certain limits.

At the higher velocities the readings obtained for the ordinates again become very precise and constant, but lie on a curve instead of a straight line. This curve, if continued in the direction of decreasing velocities, will be found to form the upper limit of the value for the ordinates in that part of the diagram where the readings are fluctuating, the lower limit being the continuation of the straight line which precedes it.

The curves of Diagram 7 record the motion of the cylinder A, from the moment of the starting of E until the fluid in the annulus has assumed the steady motion appropriate to the velocity at which E is being driven; and they show that, whether the velocity of E is large or small, provided it does not exceed 150 feet per minute about, the steady motion state is reached in approximately the same time.

The curves of Diagram 7, as immediately obtained from the camera-lucida tracings, are partly obscured by the natural oscillation of the cylinder on its torsion wire, but these oscillations have been eliminated, and the curves show the zero about which the oscillations are taking place, at each instant.

The curves on Diagram 8 are given to show that both the mercury and the short cylinder K have a great effect in increasing the range through which the transmitted moment varies as the first power of the velocity.

Diagram 9 contains the results of the experiments of series (2), in which the outer cylinder (E′) was suspended by the torsion wire while the inner cylinder (A) revolved.

Only a few experiments were made with this arrangement, as the motion of the fluid was eddying and unstable, even at very low velocities.

Diagram 10 gives the coefficient of viscosity in terms of temperature, as deduced from the experiment of series (1) and (3). With these results, for the sake of comparison, I have drawn the curve representing the true value of the coefficient (taken from Professor EVERETT's C.G.S. units).

What the origin of the high values found for the coefficient by my experiments is, I am not at present in a position to explain, but from the fact of the moment transmitted by the fluid being directly proportional to the velocity, I do not think it can be put down to eddy-making in the ordinary sense of the word, or, if due to formation of isolated eddies, the magnitude of and rate at which such eddies are formed must be such as to keep the total transmitted moment (at any rate very nearly) proportional to the velocity.

As bearing on this point the curves of Diagram 11 have been introduced. These curves are actual tracings by the camera-lucida of the angular motion of the divided circle on cylinder A during portions of four experiments.

It will be seen that, for low velocities (curves *a*, *b*) of the revolving cylinder, the suspended cylinder A remains nearly stationary, the chief movement being a slight harmonic oscillation of about fifteen seconds' period (the period of the suspended cylinder on the torsion wire), but that this movement is more marked and more irregular in (*b*) than in (*a*).

When the speed approaches the limits of stability (curve *c*) the disturbance becomes very large, showing that at these speeds the motion of the fluid in the annulus is at times, but not always, irregular. At much higher speed (curve *d*), when the motion is thoroughly unstable, and the fluid, as it were, saturated with eddies, the curve, on the whole, is again fairly straight, but the nature of the irregularities indicates that the forces at work are large.

When the velocity approached that at which instability was liable to occur, it was interesting to notice how small a disturbance of the system was sufficient to change the entire character of the motion. A slight blow on the support which carried the apparatus, or a retardation for a few moments of the rotation of the outer cylinder, was almost sure to produce the effect. (Note that reducing the velocity of E had the effect of increasing for the time the moment acting on A.)

The unstable motion so produced, however, was not necessarily permanent, and I have seen the stable form of motion change to the eddying one and back again many times in succession at irregular intervals, and for no apparent reason, when the speed neared the limit of stability.

The appearance of the surface of the fluid in the annulus was as certain a criterion of the character of the motion as the torsion produced on the inner cylinder. As long as the motion was stable, the surface remained as smooth as glass; the beginning of the instability being marked by the appearance of small dimples and elevations which, when the high velocities were reached, covered the whole surface.

The experiments of series (3), where the width of the annulus was about half-an-inch, give a nearer approach to the true value of the coefficient of viscosity than series (1), where the annulus had a width of an inch. In some former experiments of mine, described in 'Proc. Roy. Soc.,' December, 1888, where the annulus was little more than $\frac{1}{8}$ inch wide, the approximation to the true value was very close.*

The effect of temperature in altering the critical velocity was not as marked as I had expected it to be.

From Professor OSBORNE REYNOLDS' experiments I had supposed that the critical velocity would be proportional directly to the viscosity, but Diagrams 6 and 7 show that in this form of experiment, at any rate, this is not the case. At a temperature of 50° C. the viscosity of water is only about a third of what it is at 0° C., but, at the former temperature, instability begins at a speed only 11 or 12 per cent. less than at the latter.

If we deduce the coefficient of fluid friction from the experiments at the higher speeds of series (1) and (3) it will be found that the formula which best represents the curves is $F = av^{2 \cdot 42}$, and that coefficient of friction is ·058 lb. per square foot of area at 10 feet per second instead of ·23 lb. (FROUDE) and ·22 lb. (UNWIN). And both Mr. FROUDE and Professor UNWIN found the frictional resistance increase as the $1 \cdot 8^{\text{th}}$ power of the velocity for smooth metal surfaces such as those used in my experiments.

It would seem from this that, even when the water in the annulus is in the completely eddying condition, the character of the motion cannot be the same as that in the neighbourhood of Mr. FROUDE's plane or Professor UNWIN's disc.

The case is quite different, however, in the experiments of series (2). Here the motion seems essentially unstable at all speeds, and from such experiments no value of the coefficient of viscosity can be deduced, but the coefficient of friction which they

* Since the above was written it has been pointed out to me, by Sir G. STOKES, that the formula which I used for computing the coefficient of viscosity, from these experiments and also from those of the present series, was incorrect.

The values of μ have, therefore, been recomputed for all the experiments (including those of 1888) from the formula

$$\mu = \frac{F}{A_i V} \frac{r_e^2 - r_i^2}{2r_e}$$

where r_e and r_i are the radii of the external and internal cylinders respectively, F the tangential force acting on the surface of the internal cylinder, A_i the area of the surface of the internal cylinder in contact with the water, and V_e the velocity of the surface of the external cylinder.

The results of the 1888 experiments are indicated by the spots p, q, r, near the curve c, on diagram (10), but they are too close an approximation to the true value of μ to allow of a separate curve being drawn through them.

indicate is a little more than ·2 lb. per square foot at ten feet per second (practically the same as Mr. FROUDE's and Professor UNWIN's).

The exponent also of the velocity is not far from 1·8, but the velocities used in this series were so low that I should not attach much importance to the numerical determinations except as showing that the motion is really different in character from that which takes place when the inner cylinder is stationary and the outer one revolves.

[Note added July 10th, 1895.—Since writing the above, I have, at Sir G. G. STOKES' suggestion, added the set of experiments shown in Diagram 12. These experiments show that the moment transmitted is directly proportional to the depth of the fluid in the annulus measured from the lower edge of the suspended cylinder. They give therefore further and independent evidence that the mercury floor and short cylinder K do really supply boundary conditions such as must exist at any cross-section of an infinitely long annulus, if particles having the same radii on either side of the cross-section are to be without relative motion.]

TABLE I.—Dimension and Constants of Apparatus.

The letters refer to figs. (1), (2), *and* (3).

	Inches.	Centims.
Radius of Cylinder E	3·915	9·943
„ „ E′	3·42	8·687
„ „ A′	3·005	7·632
Height of cylinders	10·0	24·5

Number of divisions on circle, 400.

TANGENTIAL Force at Surface of Cylinder A required to turn it through one division of the circle.

	grain.	gram.
With torsion wire I.	·278	·0181
„ „ II.	·523	·0338

TABLE II.—Results of Individual Experiments.

Explanation of Symbols used in the Table.

θ = Reading of divided circle, during experiment.
θ_0 = Zero reading of divided circle.
H = Height of wet surface of stationary cylinder.
V = Revolutions per minute of revolving cylinder.
T = Temperature, in degrees Centigrade.

Up to Experiment 97 torsion-wire No. (1) was in use. Experiments 98 to 208 were made with torsion-wire No. (2).

No. (1) was a brass wire. No. (2) was a nickel wire.

			θ.	θ_0.	H.	V.	T.	No.
Turn-table moving with cylinder	Water in diving-bell	1893. Nov. 29	28	−0·6	8·6	30·6	11·8	1
			50	..	..	45·6	..	2
			72	..	8·52	59·75	..	3
			95	+0·3	8·5	72·8	..	4
	Air in diving-bell	Nov. 30	93	−0·5	8·5	72·8	13·3	5
			87	−0·5	8·6	73·2	..	6
			86·8	−0·8	8·6	73·6	13·5	7
		”	90	+0·2	8·4	73·5	13·8	8
			62	+2·5	8·45	60·35	..	9
			43·5	+2·6	8·5	46·15	..	10
			25·5	+1·7	8·5	30·9	..	11
			18·4	+1·7	8·5	24·55	..	12
			14·4	+1·8	8·5	20·0	..	13
			10·5	+1·7	8·5	15·06	..	14
			7·2	+1·7	8·5	10·17	14	15
After this experiment H refers to height when water is stationary	Turn-table fixed	Dec. 1	52	+0·2	8·4	73·8	10·5	16
			40	+0·5	8·45	60·1	..	17
			30·2	+0·5	8·5	45·6	..	18
			18	+0·5	..	30·8	..	19
			13·6	+0·9	..	24·2	..	20
			11·5	+0·1	..	19·6	..	21
			8·7	+0·5	..	15·0	..	22
			5·8	+0·2	..	10·13	..	23
Mercury in and turn-table fixed		Dec. 2	41	+0·4	8·55	71·6	8·6	24
			34	+0·3	8·52	58·6	..	25
			25·1	−0·2	8·53	44·9	..	26
			14·8	−0·2	8·55	30·2	8·4	27
Henceforward readings of θ taken by camera-lucida		Dec. 5	14	+0·9	8·5	30	11·6	28
			24·5	+0·9	..	44·6	..	29
			34·8	+1·0	..	59	..	30
Unstable			..	..	..	70·2	..	31
		Dec. 8	33·4	+0·2	8·5	72·1	13·0	32
			33·5	+0·2	8·5	72·0	..	33
			28·5	−0·4	8·55	60	13·1	34
			20·6	+0·5	8·6	45	..	35
			14·5	0	8·6	31	..	36
			5·6	+0·9	..	10·9	..	37
			8·4	+0·7	..	14·9	..	38
			10·5	+0·5	..	19·7	..	39
			13·6	+0·8	..	23·85	13·2	40

TABLE II.—Results of Individual Experiments—(continued).

		θ.	θ_0.	H.	V.	T.	No.
Water spilling	1893. Dec. 10	29·87	+0·9	8·5	56·9	10·9	41
		54·25	..	8·3	85	11·1	42
		101·7	+0·8	8·1	111	11·3	43
		270·6	+0·9	8·25	135·7	11·5	44
		39·8	+1·0	8·55	71·5	11·7	45
		32·6	..	8·5	58·7	11·8	46
		24·6	..	..	44·5	11·8	47
		15·6	+1·2	..	30·5	11·9	48
	Dec. 13	117·7	+0·4	8·6	117	13·5	49
		97·2	..	..	108·4	..	50
		41·8	..	..	85	..	51
		27·5	..	..	56·3	..	52
	Dec. 14	29	0	8·5	59	12·8	53
		40·5	..	..	84	12·9	54
		100·3	..	..	111	13·0	55
		161·8	..	..	135	13·1	56
	Dec. 15	125	+0·4	8·25	130	12·5	57
		100	..	..	108·8	12·75	58
		42·5	..	..	85·5	12·95	59
		28·3	..	..	59·5	13·2	60
More mercury added.							
	Dec. 15	28·1	+0·8	8·65	60	13·2	61
		40·9	..	..	85·5	13·5	62
		95	..	..	111	13·9	63
		135	0	..	136·5	14·3	64
	Dec. 18	48·5	0	8·5	60·1	12	65
Cylinder adjusted for centreing . .	Dec. 20	168·5	+0·4	8·4	134·5	12·1	66
	Dec. 18	74	+0·4	8·2	111	12	67
		72·5	..	..	110·5	..	68
		51·3	..	..	15	..	69
		34	..	..	60·5	12·7	70
	Dec 20	171	+0·4	8·4	135	12·7	71
		110	..	..	112	13	72
		50·8	..	..	88	13·4	73
		32·2	..	..	61·3	13·8	74
	Dec. 21	7·25	+0·4	8·3	9·7	9·5	75
		8·25	..	8·3	14·6	9·7	76
		11	..	..	19·1	9·9	77
		13·4	..	..	23·5	10·1	78
		19·5	..	..	32·4	10·3	79
		27·8	..	..	44·8	10·5	80
		37	..	..	58·3	10·7	81
		46	..	..	70·7	10·9	82
		54·6	..	..	85·5	11·1	83
Unstable and oscillating		110	..	..	111·5	11·3	84
Unstable		162	..	..	135·3	11·5	85
Experiments on the rate at which cylinders approach stationary position (See Diagram 7)	Dec. 27	..	..	8·1	..	12·8	86
		..	..	..	..	..	87
		..	..	..	..	..	88
	Dec. 31	..	..	8·2	..	..	89
		..	..	8·2	..	13·2	90
		..	..	..	..	..	91
		..	..	..	..	..	92
		..	..	..	..	..	93
		..	..	..	..	..	94

TABLE II.—Results of Individual Experiments—(continued).

		θ.	θ_0.	H.	V.	T.	No.
Experiments on the rate at which cylinders approach stationary position (See Diagram 7)	1893. Dec. 31	..	..	..	..	..	95
		..	..	..	..	..	96
		..	..	..	..	..	97
	1894. July 5 .	20·9	+0·1	8	66	19·5	98
		10·2	..	8	44·6	35·5	99
		6·1	+0·1	8	32·8	34	100
	July 7 .	4·6	+0·3	7·9	16·3	13·5	101
		5·5	..	..	20·3	..	102
		8·2	..	..	25·8	..	103
		9·8	..	..	32·5	..	104
		16·1	..	..	52·1	..	105
		19·5	..	..	65·7	14	106
	July 9 .	2·6	0	8	16·4	40	107
		3·4	..	..	23	42	108
		3·9	..	..	26	55	109
		4·6	..	..	33·3	55·5	110
		5·8	..	..	43·3	54·5	111
		8·1	..	..	57·5	53	112
		9·5	..	..	74·2	53	113
		8·7	..	..	60	53	114
		6·8	..	..	45	53	115
		5·5	..	..	31·8	53·5	116
	Date not given, probably July 10	28·4	−0·8	8·3	60·4	2·5	117
		14·1	..	8·4	32·1	..	118
		? 26·21	..	..	55	..	119
		28	..	..	62	..	120
		33·5	..	..	74	..	121
		28·6	..	..	64·6	5·5	122
		33	..	..	73·5	..	123
		32·8	..	..	71·4	4·8	124
		82	..	..	22	..	125
		2·6	..	..	8·8	4·5	126
Extinction diagrams for zero		..	..	..	..	..	127
		..	..	..	..	..	128
More mercury		7·6	−1	8	27·7	16·5	129
		12·3	−1	..	41·3	..	130
		17·1	−1	..	55·0	..	131
		21·1	−1	..	67·5	..	132
Turn-table free — No mercury . .	Date not given, July probably	8·1	0	8	28·1	16·5	133
		12·6	..	..	42·2	..	134
Turn-table free — Eddies begin .		18·8	..	..	55·4	..	135
		22·2	..	..	67·5	..	136
		12·1	−0·5	8	28·4	..	137
		24	..	..	43·0	..	138
		39·4	..	..	56	..	139
		43·8	..	..	68	..	140
		11·6	0	8	28·8	16	141
		17·5	..	..	43	..	142
		23·5	..	..	55·2	..	143
		38·1	+0·6	..	67·8	..	144
	1895. Feb. 4	3·9	0	8·4	13·2	7·5	145
		6·15	..	..	19·4	..	146
		9·6	..	..	26·5	..	147
		12·3	..	..	32·25	..	148
		28·5	..	..	67	..	149

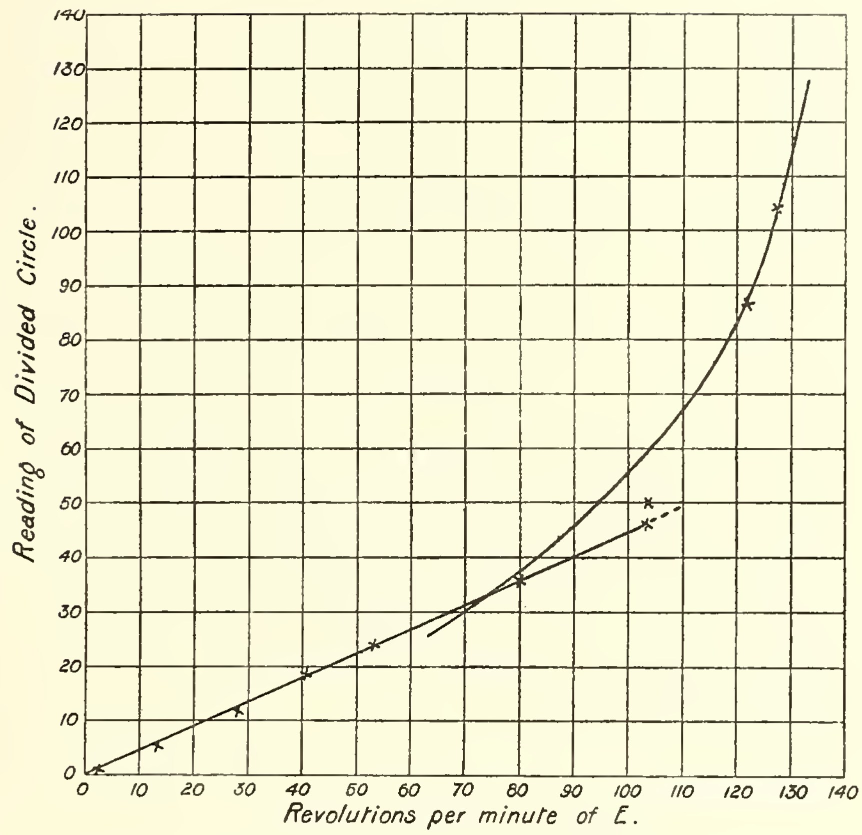
TABLE II.—Results of Individual Experiments—(continued).

		θ.	θ_0.	H.	V.	T.	No.
	1895. Feb. 4	20	..	..	50·1	..	150
		36·2	..	..	78	..	151
		70	..	..	104·2	..	152
	Feb. 5	10·2	0	8·4	27·6	4·9	153
		18	..	..	41·2	..	154
		23·8	..	..	54	..	155
		29·5	..	..	64·7	..	156
		23·5	..	..	53·7	..	157
		35·5	..	..	79·9	..	158
		47·9	..	..	103·8	..	159
		96·3	..	..	122·6	..	160
	Feb. 11	104	0	8·4	126	2	161
		46·8	..	..	103	..	162
		35·5	..	..	80·4	..	163
		24·1	..	..	53	..	164
		18·6	..	..	41·2	..	165
		12	..	8·4	28	..	166
		5·7	..	..	13·4	..	167
		3·5 1·05	+0·9	.. ..	9 2	..	168
7-inch cylinder, 1st time	Feb. 15	11	0	8·35	18	2·5	169
		5·2	..	..	9	..	170
		8·1	..	..	13·25	..	171
		10·7	..	..	18	..	172
		13·6	..	..	22·1	..	173
		16·6	..	..	? 26·6	..	174
		25·3	..	..	42·1	..	175
		33·7	..	..	55	..	176
		30·7	..	..	51·3	..	177
		47·25	..	..	78·1	..	178
		60–100	..	8·1	105	..	179
		135	..	8·0	130	..	180
	Feb. 17	56·1	0	8·3	78·8	2·5	181
		33·5	..	..	51·8	..	182
		32	..	..	54·2	..	183
		26·1	..	..	41·4	..	184
		17·2	..	..	27·2	..	185
		8·1	..	..	13·2	..	186
		6·3	..	..	9	..	187
		2·6	..	..	4·1	..	188
		1·25	..	..	2	..	189
Outer cylinder (7″) suspended	Feb. 24	41·1	+0·2	8·7	17·6	10·5	190
		22·2	..	..	13·1	..	191
		11·8	..	..	8·55	..	192
	Feb. 25	11·4	0	8·7	8·57	4·8	193
		22·5	..	..	13·0	..	194
		32·1	.	..	17·4	..	195
		16·6	..	8·7	10·2	6·3	196
		10·4	..	..	7·6	..	197
		5·5	..	..	5·05	..	198
		3·5	0	8·7	3·4	6·3	199
		3·3	..	8·7	3·43	5·2	200

TABLE II.—Results of Individual Experiments—(continued).

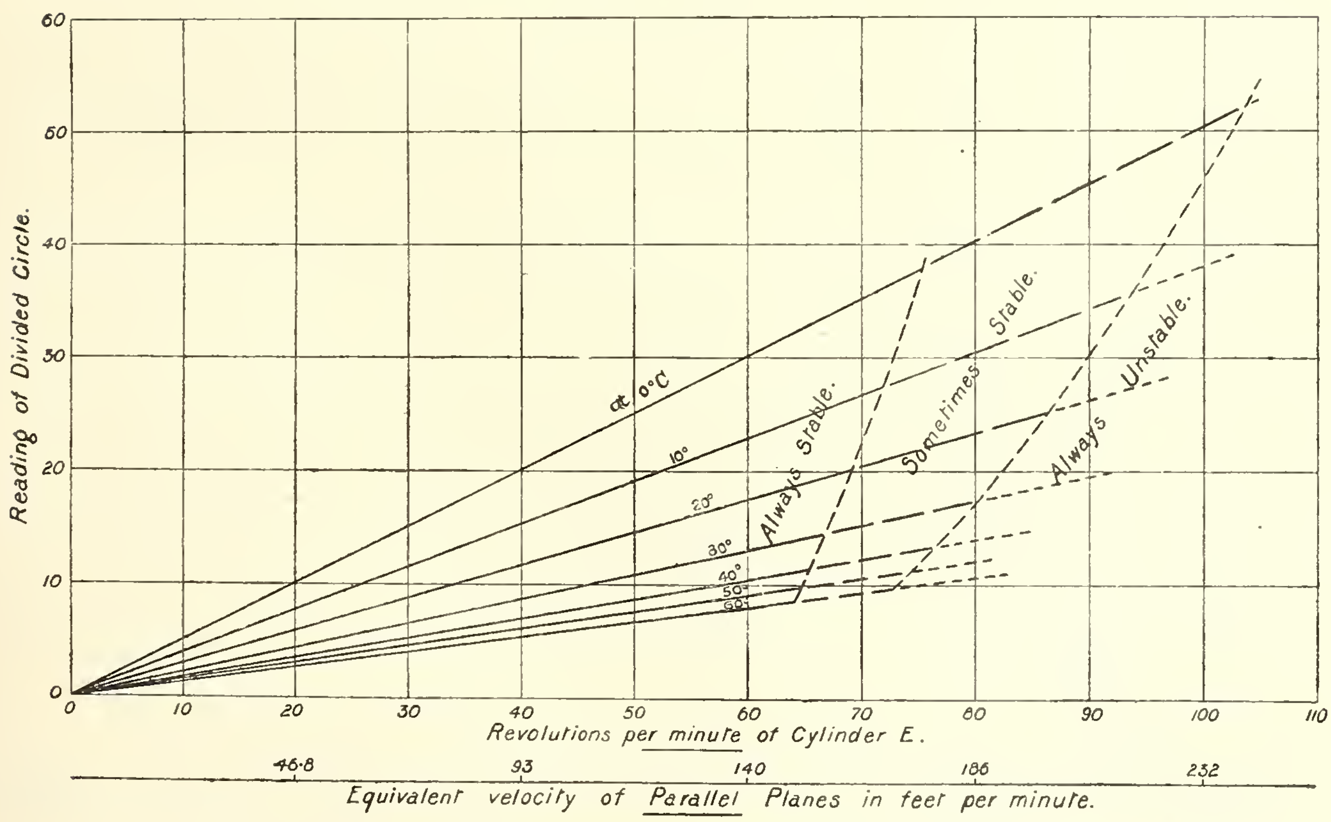
	θ.	θ_0.	H.	V.	T.	No.
	5·5	..	..	5·1	..	201
	15·9	..	..	10·2	..	202
	24·5	..	..	15·1	..	203
	35·9	8·7	..	17·3	5·0	204
	24·0	..	..	13·2	..	205
	13·7	..	..	8·75	..	206
	5·7	..	..	4·9	..	207
	3·2	..	..	3·42	..	208

Diagram 4.



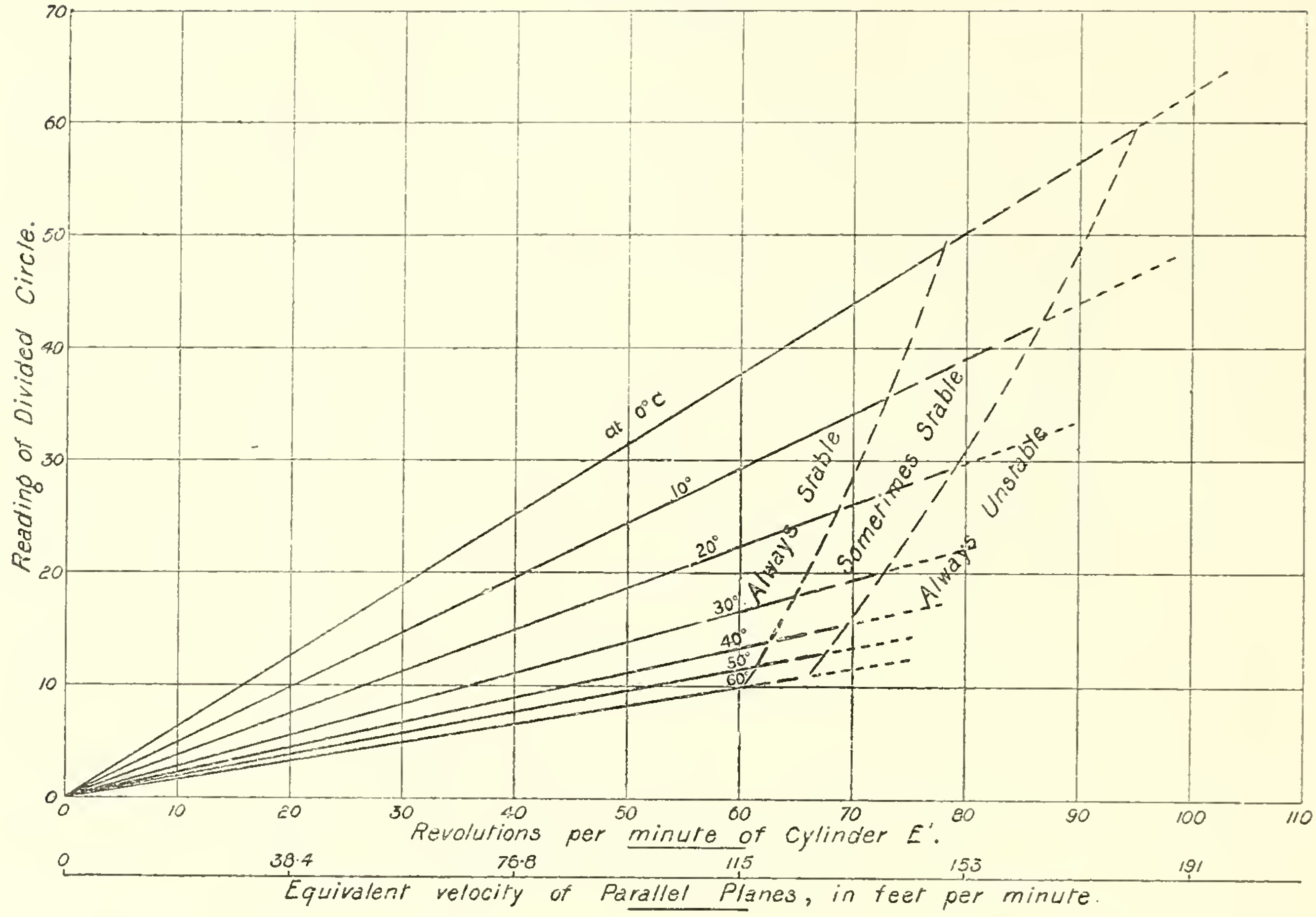
Experiments 160–168. Corrected to H = 8·5, T = 0.

Diagram 5.



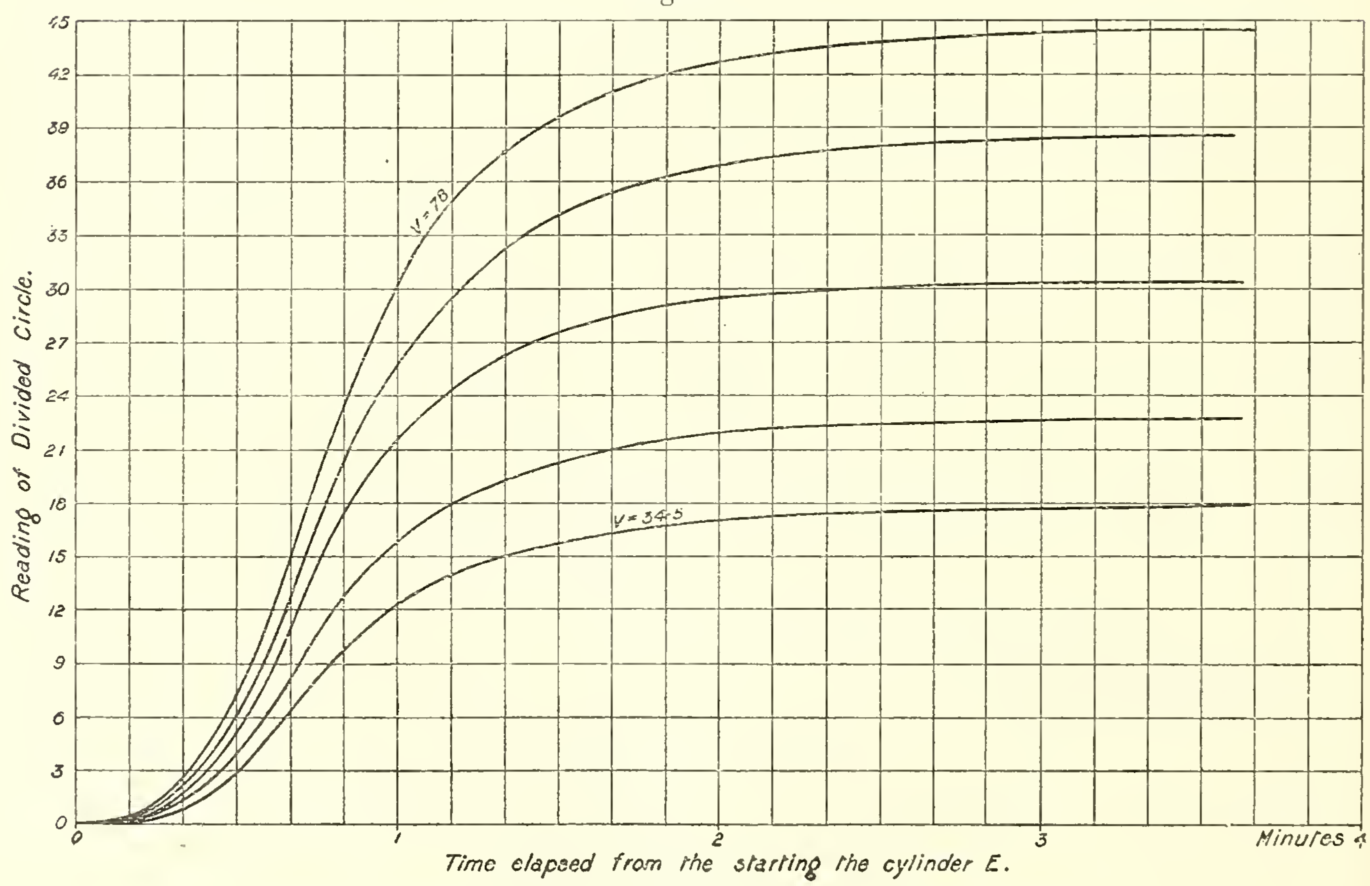
Corrected results of Series 1.

Diagram 6.



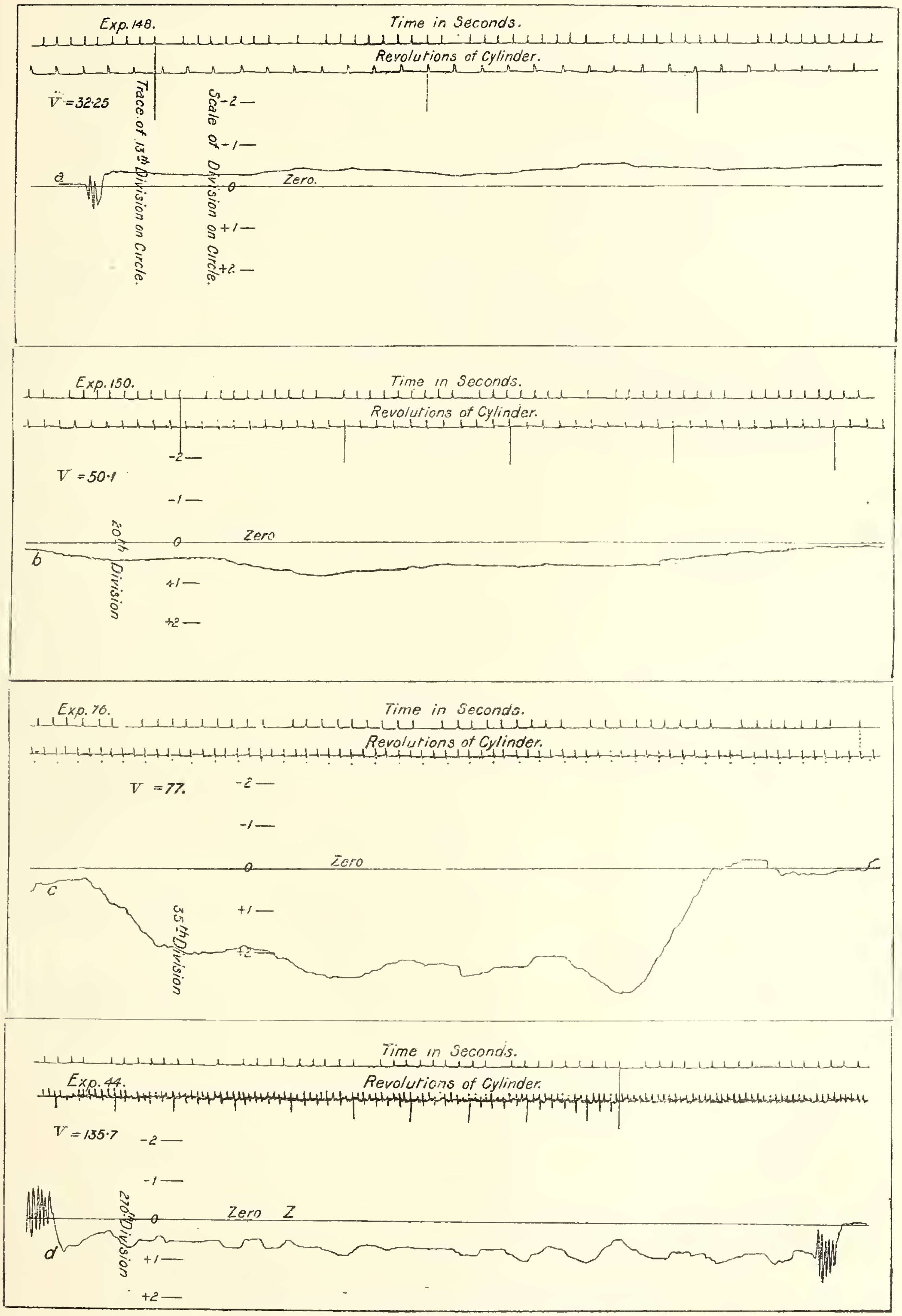
Corrected results of Series 3.

Diagram 7.



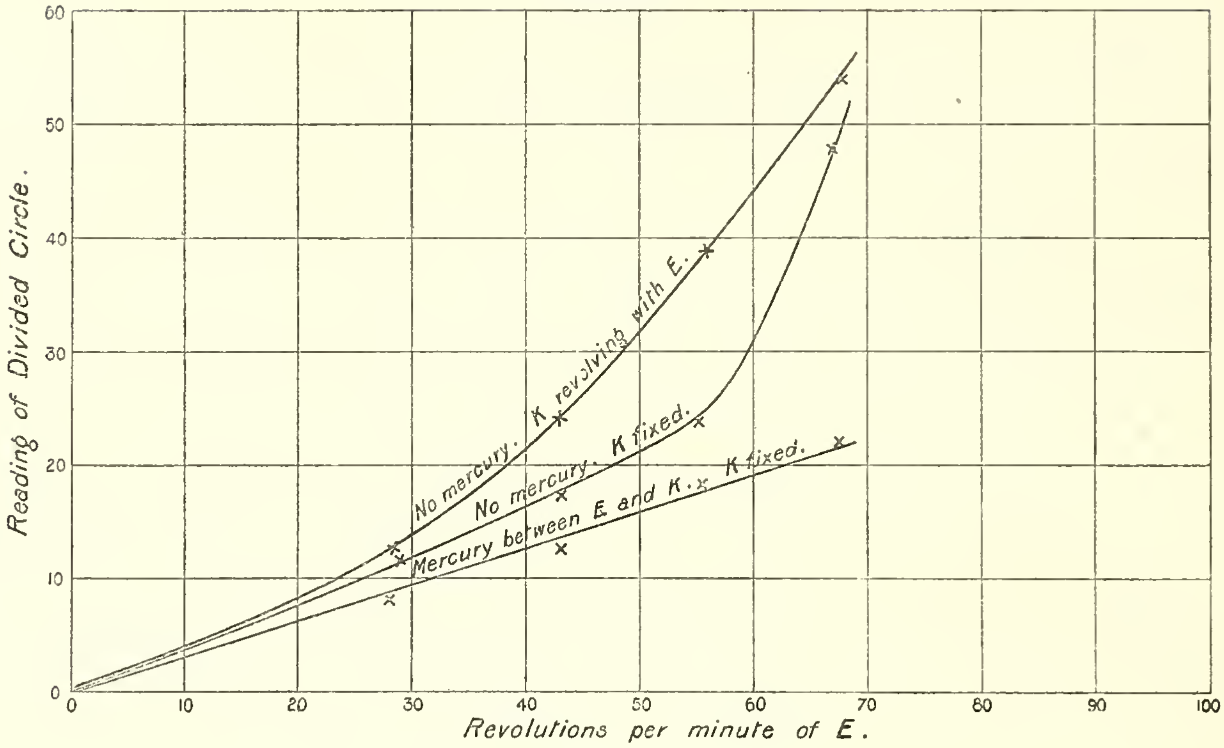
Rate at which Cylinder A approaches its stationary position.

Diagram 11.



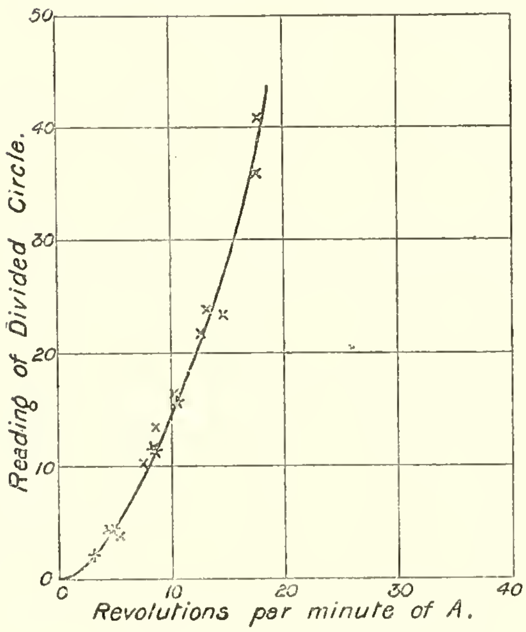
Specimens of diagrams taken with camera-lucida.

Diagram 8.



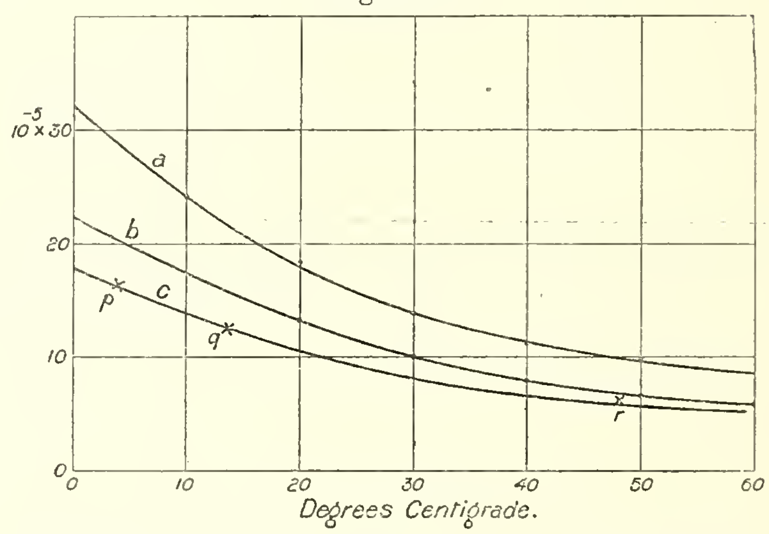
Experiments 133–144.

Diagram 9.



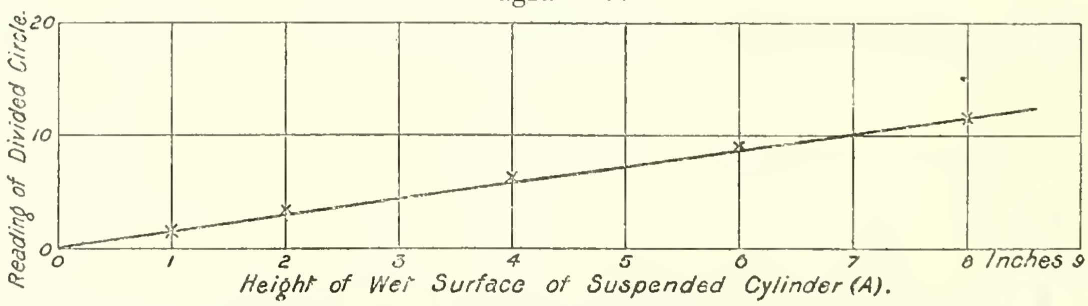
Results of Series 2.

Diagram 10.



Coefficient of viscosity of water in terms of temperature.
Units—centimetre-gramme-second.
a, as deduced from Series 1.
b, ,, ,, ,, 3.
c, as taken from EVERETT's C.G.S. units.
The spots p, q, r are from the results of similar experiments described in 'Proc. Roy. Soc.,' Dec., 1888.

Diagram 12.



To show moment transmitted with different depth of water in the annulus.
42 revolutions per minute. Temperature, 19°·5 C.

INDEX SLIP.

IV. *Researches on the Electric Properties of Pure Substances.*—No. I. *The Electrical Properties of Pure Sulphur.*

By Richard Threlfall, *M.A.*, *Professor of Physics in the University of Sydney, and* Joseph Henry Drapier Brearley, *Deas-Thomson Scholar in the University of Sydney.*

Communicated by Professor J. J. Thomson, *F.R.S.*

Received April 19,—Read May 24, 1894.

[Plates 1–5.]

Preliminary Statement by Professor R. Threlfall.

In 1886 I began to make some experiments on the specific resistance of certain Australian gums, with a view to finding new insulating material. In connection with this I constructed and tested various galvanometers. In 1887 I was joined by Mr. J. A. Pollock, and in 1889 we published conjointly in the 'Philosophical Magazine' a paper on the "Specific Resistance of Imperfectly Purified Sulphur," and on the "Clark Cell, as a source of Small Constant Currents." I also published in the same volume a paper "On the Measurements of High Specific Resistances." All these papers may be considered as preliminary to the present investigation of the behaviour of pure sulphur under electric stress, which has gone on steadily since June, 1889. The remainder of that year, the whole of 1890, and most of 1891, were occupied in preliminary experiments on galvanometers and in making and testing various arrangements of electrodes between which the sulphur to be examined was placed. This work was carried out entirely by Mr. Pollock. Early in 1891 I was joined in the work by Mr. J. H. D. Brearley, who has worked continuously at the subject with me up to the present time. I began the study of sulphur originally with the object of discovering the exact electrical properties of a pure, non-metallic substance, and chose sulphur because it appeared to offer advantages in melting at a moderate temperature, in being capable of existing in several allotropic forms, and, above all, in being capable of being brought to a high degree of purity with comparatively little trouble. With respect to the galvanometer, which has enabled us to perform the experiments described, so far as its general features are concerned, it was designed by me, in 1890, on the lines laid down in the 'Philosophical Magazine,' series 5, vol. 28, p. 473; but the detailed design, as well as the actual construction, I owe to Mr. Cook, who made the instrument in the Laboratory workshop. Most of the work in connection with mounting the moving parts and in installing the instrument was done by Mr. Brearley.

A great, if not the chief, difficulty, in attempting to form a theory of electrochemistry lies in the uncertainty surrounding the data. The memorable researches of Kohlrausch on the conductivity of water, as well as those of Dixon on the effect of a small impurity on the mode of combination of gases, together with the quite allied and equally peculiarly large variations produced in thermoelectric power and contact force phenomena by minute differences of chemical composition, have had a most destructive effect on the theories of twenty years ago. It is not too much to say that the electrical action of most bodies in a pure state is entirely unknown at present, although the fact of the continuity of the electrical properties of certain substances—*e.g.* mercury, gold, and copper—as they are more and more purified, raises a hope that the properties of substances as at present investigated may only differ quantitatively from their properties when pure; but whether this be so or not can only be decided by experiment. The question as to the effect of purification is one which must be discussed in each separate case, and probably over and over again, for the word "pure" has no significance except with respect to a definite state of the art of chemistry. It is to be noted that the final stages in a process of purification must always depend more or less upon hypothesis—for a point is soon reached where the impurities become too small in amount to be amenable to ordinary chemical examination. Theoretically this can be got over by operating on large quantities of substances, but in practice such is not found to be the case, for the greater the bulk of substance to be prepared, the more time is required for its preparation* and the greater is the chance of small and unnoticed accidents occurring. In the case of sulphur this is particularly well marked, for the chief source of impurity is in dust from the air—which leads to a contamination almost proportional to the time required in making the preparations. We look forward to the time when the exclusion of dust during chemical operations of precision will be regarded as of as much importance as its exclusion in bacteriological research. The sulphur we have used is probably as pure as it can be got without taking special precautions to avoid all contact with dusty air. The absence of such precautions vitiates to a great extent many observations of discharge in air, and led Quincke† to explain away the diminution of the resistance of benzene and other liquids of small conductivity with increased voltage per unit length.

In the case we have studied, the physical action of dust particles is not to be feared, and appropriate means were adopted for minimising the possible chemical action between dust and sulphur.

One of the results we have attained with sulphur in a certain condition—is a

* Stas, "Recherches."

† Quincke, "Wied. Ann.," 1886, vol. 28, p. 546.

similar diminution of resistance, and in this case QUINCKE's explanation is impossible. The suggestion is obvious that the phenomena observed by QUINCKE may possibly be real phenomena—not depending on dust particles—and if this be the case, then it is clear that a valuable discovery has been rendered unavailable by the source of uncertainty we are dealing with.

In the following statement we shall explain with some minuteness the methods adopted for purifying our material, and give a summary of the conclusions to which previous observers have been led as to its physical properties. We shall then explain in a general way the methods we have used for determining the electrical properties. This will be followed by a short treatise on the construction of sensitive galvanometers of high resistance, together with an account of the experiments which have led us to the construction of our instrument, of which full details and drawings will be furnished. We shall then proceed to deal with our various experiments. At the present time the experimental work is still in full progress, and is likely to continue, for in investigating properties of substances the end is only reached when the observer's interest falls off. Our chief reason for publishing is that we consider the main features are now pretty well established.

SECTION I.

On the Preparation and Properties of Pure Sulphur.

§ 1. *General Discussion of Causes, Procedure, &c.*—Our information on the transformations of sulphur is chiefly derived from the following authors:—

1856. C. SAINTE-CLAIRE DEVILLE, 'Ann. de Chim. et de Phys.,' Sér. 3, vol. 47, p. 94.
1854. MAGNUS, 'Pogg. Ann.,' 1854, vol. 91, p. 308.
1854. B. C. BRODIE, 'Proc. Roy. Soc.,' vol. 7, p. 24 (March 30th, 1854).
1856. MITSCHERLICH, 'Jour. für prakt. Chem.,' vol. 67, p. 369.
1857. BERTHELOT, 'Ann. de Chim. et de Phys.' (3), vol. 49 (two papers).
O. PETERSEN, 'Zeitschr. für Phys. Chem.,' vol. 8, p. 608.
1884. REICHER, 'Chem. Centr. Blatt,' 1884, p. 450.
1889. J. MONCKMAN, 'Proc. Roy. Soc.' for 1889, vol. 46, p. 136.
1874. D. GERNEZ, 'C.R.,' vols. 79, p. 219, 82, p. 1151, 83, p. 217.

There are a large number of other observers whose work does not bear so precisely on the subject of the present investigation, and whose names are recorded in the "Sachregister zu den Annalen der Physik und Chemie," published in 1888, p. 563.

Amongst the authors mentioned, we have found the papers of BRODIE and BERTHELOT of most service. BRODIE gives an account of the transformations of sulphur, but unfortunately omits many details, particularly as to the mode of purification adopted. BERTHELOT attempts to classify the varieties of sulphur on electro-chemical principles. According to him, sulphur exists in two principal modifications, one, electro-positive sulphur, liberated at the positive-electrode during the electrolysis of chloride of sulphur; the other electro-negative sulphur liberated at the negative electrode during the electrolysis of metallic sulphides.

The electro-positive sulphur comprises all those varieties which are amorphous; the electro-negative all those which are crystalline. The distinction between crystalline and amorphous sulphur appears to us to be much more fundamental than between those varieties which are soluble and those which are insoluble in carbon bisulphide. Indeed, as is well known, mere crystallization from carbon bisulphide often involves the formation of insoluble sulphur. The distinction between the main varieties is considerable from the dynamical point of view—for the thermal changes involved in passing from one to the other are very great—as was shown by BRODIE in a simple experiment and by PETERSEN in an elaborate thermo-chemical research.

We have chiefly examined those varieties of sulphur which are sufficiently definite and permanent to enable us to be sure that we are dealing with pure substances. Thus we have examined the monoclinic variety of sulphur pretty thoroughly, together with what we believe we are the first to recognize as a new variety of monoclinic sulphur, as well as several mixtures of this variety with amorphous sulphur obtained by cooling rapidly from a high temperature. The permanence of monoclinic sulphur we find to be much greater than is generally supposed, at least, if the melting point may be taken as a criterion,* while as for octahedral sulphur, the only possible means at our disposal for obtaining it pure—viz., by the method of GERNEZ—has hitherto failed in our hands. This method is based on the obtaining of sulphur in a state of "surfusion." The sulphur, which is perfectly crystalline to begin with, is melted below 130° C., cooled to, say, 101° C., and then caused to crystallize in the octahedral form by sprinkling the sulphur with a trace of the dust of octahedral crystals. This appears to be a matter of some delicacy, and in the course of a good many trials we have not succeeded to our satisfaction, even when following M. GERNEZ's instructions most minutely. The fact is, that melted sulphur has a very strong tendency to crystallize in the monoclinic form, and this tendency can only be overcome under the most favourable circumstances (if at all) and by the exercise of great care. For most electrical experiments, however, it is necessary to obtain a film of the material between two conducting plates which serve as electrodes, and the conditions attending the production of such films are experimentally unfavourable to forced octahedral crystallization. Again, both FOUSSEREAU ('C.R.' 97, p. 996) and BOLTZMANN, ROMICH and FAJDIGA, and NOWAK ('Sitzungsber. der Wiener Akad.,' vols. 68 and 70) found no appreciable conductivity in crystals of octahedral sulphur, though their methods were not very searching.

* The phenomena attending the transformation of monoclinic into octahedral sulphur will be dealt with during the discussion of special cases.

Section II.

The Preparation and Testing of Samples of Sulphur.

The earliest note we have found on the influence of minute traces of impurity on the properties of sulphur is by Mitscherlich in the 'Journal für Praktische Chemie' for 1856, vol. 67, p. 369. This chemist states that the discoloration of sulphur which has been heated to the boiling-point is to be traced to the presence of organic compounds. One three-thousandth part of "grease" will produce a very deep red coloration in sulphur which has been heated to the boiling-point, and even the greasiness produced by handling is said to notably affect the colour of sulphur. The strongly coloured organic compound does not come over when the sulphur is distilled, and its presence may be detected on the bottom of the retort, where it forms black spots. Mitscherlich considers the supposed coloured varieties of sulphur discovered by Magnus ('Ann. de Chim. et de Phys.,' [3], vol. 47, p. 194) to be merely sulphur contaminated by organic matter; he gives, however, no account of his experiments.

Monckman ('Proc. Roy. Soc.,' 1889, vol. 46, p. 136) noted the black spots which are left in a retort from which sulphur has been distilled, and considers they are produced only when the retort is heated by the bare flame—a process leading, of course, to the heating of the glass much above the boiling-point of the sulphur. Berthelot (*loc. cit.*) is inclined to agree with Mitscherlich as to the influence of small organic impurities on the colour of sulphur.

Stas ('Recherches.—Bull. de l'Académie Royale de Belgique,' 2ème série, vol. 10, p. 253) made use of the vapour of "pure" sulphur, but gives no details as to how it was obtained. We know of no discussion as to the minute purification of sulphur except the one contained in Monckman's paper—in which he refers to a method proposed by one of us, and carries out that method and several others in a very careful manner. We must therefore consider the purification of sulphur from the commencement, since there is practically no previous work to help us—we have, in fact, to solve the problem of purifying sulphur and of recognizing it in the state of purity. Our experiments lead to the following result.

Pure sulphur can be recognized chiefly by its negative characteristics. It evaporates from a platinum basin without residue, it has little or no smell, it has a clear yellow colour even when suddenly cooled from above 180° C. by water; it gives off no gas when heated *in vacuo*, it leaves no black spots when distilled from a retort with or without the bare flame; it has a specific resistance of above 10^{28} C.G.S. units when perfectly crystallized in monoclinic prisms; it exhibits, to all intents and purposes, perfect freedom from "electric absorption" when in this state, and it is perfectly soluble in carbon bisulphide, forming a solution which is absolutely clear. We will now give a short account of how these characteristics were determined.

The question of the colour of sulphur occupied one of us previous to 1888, and in that year, and as a result of a good many trials, it was found that when sulphur is distilled over and over again in glass retorts, black spots always make their appearance in the residue unless the operation is carefully shielded from the action of dusty air. If sulphur is distilled and then exposed to air so that dust can settle on it, then, on repeating the distillation, black spots make their reappearance. We have only once succeeded in absolutely preventing the presence of black spots in a retort from which sulphur was distilled, but we have noted over and over again that just in proportion to the precautions that are taken to exclude dust, so do the black spots become smaller and fewer, and the colour of sulphur suddenly cooled from a high temperature clearer and clearer.

That pure sulphur has no smell we know, because we have prepared a sample by precipitation of hyposulphite, and after five distillations no smell could be detected from it. This observation is important, for we have large quantities of sulphur twice distilled from Sicilian roll sulphur which leaves no residue when burned away in a platinum dish, which remains bright yellow on cooling quickly, which will yield no gas when heated *in vacuo* up to near its boiling-point, and which generally might be considered to be satisfactory had it not the well-known sulphury smell, which, by the way, we consider to be possibly due to traces of chloride of sulphur. The smell is not to be removed by exposure *in vacuo* at any temperature and during many hours, and samples of sulphur which possess it had better be left on one side.

Pure sulphur, of course, leaves no residue on burning in a platinum basin, but we have never succeeded in reaching this stage with sulphur which is burned in the ordinary way. We have, however, got as far as this—that the combustion of 200 grams of sulphur in a platinum dish will leave a stain so small that it cannot be observed unless the platinum is brightly polished. The stain disappears at a red heat, and may be safely attributed to the dust which gets into the dish during the time the sulphur is being introduced, and while the process of burning is being observed. In order to obtain the maximum residue from any sample of sulphur, the last cubic centimetre or so must be boiled off very carefully and without overheating. The residue which, with our purest sulphur, such as we have used in electrical experiments—is after all a mere stain, is far beyond the powers of the balance to detect. It is dark in colour and burns off at a red heat, indicating that it is composed of the same organic compound which was discovered by MITSCHERLICH. We have observed that after a sample of sulphur has been exposed to the air during electrical experiments, it will give a comparatively large residue, although when first prepared the residue could only be observed by close attention. Those samples of sulphur which when suddenly cooled appeared of a buff colour, always gave comparatively large residues, and these were the samples which had been most exposed to the action of air. Of course ordinary sulphur leaves a large quantity of earthy residue besides, which can be got rid of by about two distillations—but we are not discussing such preliminary purifications. Impurities which can be detected by the means above described may be considered to be the more obvious impurities, and do not give rise to any great concern. Above this point, unhappily, we are obliged to proceed by way of hypothesis, and so conduct the process of purification as to tend to eliminate impurities, which our general knowledge of the chemistry of sulphur leads us to suppose may be present in quantities too small for detection by chemical means. Only those impurities are to be feared which will distil with sulphur, or be mechanically carried over during distillation, and dissolve in carbon bisulphide. This practically narrows the investigation down to the consideration of selenium as an impurity.

In order to cause any residual impurity to vary in different samples, we have tried a large number of different sources for our sulphur. Of these we retain three. The first is by decomposing commercial sodium-thiosulphate by pure hydrochloric acid, washing the deposited sulphur with platinum distilled water, and distilling till the residue on combustion is inappreciable. This method has the advantage of involving no contact with carbon compounds, and of giving rise to no impurities except those that are soluble in water, or left on distillation. As to the impurities which may be suspected to be present in the thiosulphate originally, these will be dealt with when we consider soda waste as a source of sulphur or thiosulphate.

The second kind of sulphur which we have employed is that derived from the poly-sulphides of calcium. Some roll sulphur, in powder, was digested with marble-lime in the usual manner. The yellow liquor obtained was decomposed by pure hydro-chloric acid, and the precipitate washed and distilled.

The third kind of sulphur is some recovered sulphur that came into our hands by the kindness of Mr. F. Wright, of Elliott Bros., importers of drugs in this city. We could get no information at all as to where it came from, or what the process of recovery was; but we were told that the sample (amounting to about 5 lbs.) was recovered and very pure. A very few trials showed us that, as far as the negative characteristics go, the claim of purity was well founded, for the sulphur was at least at good as our precipitated sulphur after about three distillations. To find out where it came from, with a view to investigating its history, we wrote to Mr. J. F. Chance (who was known to us as an authority on recovered sulphur) asking him whether he knew of any ordinary process by which sulphur of such a high degree of purity was produced commercially, and whether he could obtain a fresh supply for us. In reply Mr. Chance gave us a general account of the method of sulphur recovery invented by himself, and most kindly presented us with a large quantity of sulphur recovered by his process. On comparing this sulphur with that which we had received from Mr. Wright, it became obvious at once that the latter had been produced by the Chance process: the appearance alone is almost unmistakable. According to Mr. Chance, the moisture and non-volatile impurities amount together to not more than ·01 per cent.—and the non-volatile impurities to perhaps only ·001 per cent.—with regard to the latter number we can say that our experience roughly confirms this estimate, for we never obtained a weighable residue even from the evaporation of several ounces of the fused and filtered sulphur. Traces of selenium and arsenic, however, might occur as volatile impurity, and it was therefore necessary for us to ascertain their presence (or absence) by means of a direct experiment.

After a considerable amount of preliminary work we fell back on the combustion of quantities of about twelve grammes of the sulphur to be tested, in a platinum boat in a combustion tube. The products of combustion being mixed with excess of oxygen and passed over spongy platinum, the resulting sulphur trioxide was received in a large globe cooled by ice and salt. The trioxide was then converted into sulphuric acid, and tested for arsenic and selenium. We made two successful combustions, using in one case the residue from a distillation of Chance sulphur, and in the other the sulphur in the state (twice distilled) in which it was used in our experiments. The combustion requires some precaution. In our arrangements, the tube employed was of hard glass, and was heated for a length of 80 centims., it was 1·4 centim. in internal diameter. In this tube 15 centims. were occupied by the platinum boat containing the sulphur, this was followed by a "mixing" space of red-hot asbestos for 25 centims., and 35 centims. were occupied by spongy platinum and platinized asbestos. Oxygen was admitted by two tubes, one only just entered the combustion tube and provided the oxygen for the burning of the sulphur in the boat; the other ran over the boat and delivered a large excess of oxygen into the "mixing" space. Oxygen was observed to be in excess during the whole process. By regulating the oxygen supply and the temperature of the sulphur in the boat, the combustion can be got to proceed regularly. The combustion tube was bent at the delivery end, and dipped into the receiver. At the close of the operation, a stream of oxygen was passed for some time, the tube being now red-hot throughout—up to the bend—by this means the whole of the contents of the boat were burned and all residues chased out of the combustion tube. Finally the part of the tube which projected into the globe was washed out with platinum-distilled water and the solution added to the solution in the same water of the sulphur trioxide contained in the globe. In the first combustion the oxygen supply was not in sufficient excess during a few moments and the trioxide was coloured blue by combination with a trace of free sulphur. In the second combustion, in which the oxygen was in great excess throughout, the resulting sulphuric acid contained sulphurous acid, and the excess of oxygen also carried off sulphur dioxide during the process. The conclusion seems to be that Sulphur Trioxide is dissociable by spongy platinum at a red heat, even in presence of oxygen—this we did not know, it is not mentioned by LEMOINE ('Etudes sur les Equilibres Chimiques'). The solution of trioxide from the first combustion required to be filtered; the second did not, and was absolutely clear and colourless. We remark on this, for there is a chance of selenium separating during the solution of the trioxide, in the presence of sulphurous acid. In both cases the solution was next boiled with a large excess of sulphurous acid and hydrochloric acid. In neither case was there the slightest precipitate. We are satisfied that, under the conditions of the experiments, this means that there could not have been more than

·03 milligram of selenic acid present, or say ·014 milligram of selenium, or about one part in a million. As at first we under estimated the delicacy of the selenium test, it may be worth while to indicate how the test should be performed in order to give the best results. The points are: (1) a large quantity of hydrochloric acid should be used; (2) the liquid after some minutes' boiling should be cooled before the sulphurous acid is added; (3) the mixture should then be well shaken; (4) on heating to boiling, a precipitate will appear in two or three minutes if selenium is present—at the limit a yellowish rose-colour will be produced, but no actual precipitate. The essential point is to have the volume of liquid as small as possible. For instance, with one cubic centimetre of liquid ·01 milligram of selenic acid may be detected with certainty, and still smaller quantities in proportion as the volume of liquid is smaller.

After testing for selenium, and taking the most refined precautions as to re-agents, the MARSH's test for arsenic was applied. In no case was arsenic discovered. We found by direct experiment that the minimum amount of arsenious oxide we could detect by our appliances was ·02 milligram, and that by the smell of the gas only. The test, however, as made by the nose is just as certain as that made by the eye when a mirror is produced; the latter, however, is less delicate. With respect to precautions, we will only say that the instructions of FRESENIUS were carefully followed, and that this was not the first occasion on which we had made the test.*

We may conclude that if arsenic is present in the sulphur, it is only so to an extent of about one part in a million. The selenium test will be observed to be more delicate than the arsenic test—a fact not generally recognized—and this when the test is made by noting the smell of the evolved hydrogen. If a recognizable mirror be required, the arsenic test is a good deal less delicate.

A test for tellurium and selenium was made for one of us in 1888 by Dr. HELMS, on a sample of sulphur recovered from thiosulphate. These results were negative.

As regards arsenic and selenium, we wish to go further, and to show that it is very improbable that they could have been present even in quantities much less than we could have detected. The results of the enquiry will apply alike to sulphur recovered by the Chance process and to sulphur recovered in the laboratory from commercial sodium thiosulphate (assuming the latter to have come from soda waste). For this purpose it is necessary to consider the LEBLANC process in some detail.

The sulphuric acid employed in the salt-cake process is the source from which arsenic and possibly selenium may be derived. The greater part of the arsenic is got rid of in the salt-cake furnace in the form of chlorides. With respect to selenium, we believe that it will be almost all precipitated in the sulphuric acid chamber, but as the acid is not filtered, as a rule, this cannot be regarded as a purification.

* We may note here that the decomposition of arsine by strong sulphuric acid is a test which, though not quite so delicate as the mirror test, is not by any means a bad one. It is not usually given as a test at all. (The sulphuric acid must be kept concentrated.)

Dr. George Elliott informs us that when burning some samples of sulphur from Japan, containing about 1 per cent. of selenium, for the purpose of making sulphuric acid, the selenium separated in the denitrating tower and in the chamber, in the manner we suggest.* In order to find whether selenium is got rid of in the salt-cake process, we burned a quantity of selenium in a properly bent tube, and passed all the products of combustion into water, burning the selenium part of the time in a brisk current of oxygen, and part of the time with only just enough oxygen to keep the selenium alight. Part of the resulting selenious acid was converted into selenic acid, and a mixture of these acids and selenium, together with any other of the products of combustion of selenium which might have been formed, were added to some sulphuric acid in an experiment imitating the salt-cake process. We found that all the selenium remaining in the salt-cake did not amount to 1 per cent. of that mixed with the sulphuric acid at the commencement of the process, so that the salt-cake process gets rid of most of the selenium as well as the arsenic.

It is possible for some small trace of selenium to get into the black-ash furnace. If it does, it is presumably in the condition of free selenium, sodium selenate, or selenite. Now the trace of free selenium will probably be volatilized to a great extent, and pass off with the other gaseous products. As for the selenite and selenate an experiment to be described indicates that they will be decomposed, yielding free selenium, with which some of the lime will form calcium selenide. We find, however, calcium selenide is decomposed when suspended in water by carbonic acid gas yielding free selenium, but not a trace of selenium hydride. These facts were established as follows:—

About 4 grams of precipitated chalk were converted into calcium selenate, and this was reduced by hydrogen (pure) to calcium selenide. During the operation much of the selenide decomposed into free selenium and lime (?), and there was a considerable evolution of selenium hydride. The process was stopped while the evolution was in full progress. On washing the contents out of the tube we probably had a mixture of free selenium, calcium selenide, calcium selenate, and perhaps other products. A stream of carbon dioxide gas (pure) was passed through this mixture, including the water, contained in a set of potash bulbs, and as a result a large quantity of selenium was deposited, but not the slightest trace of selenium hydride could be discovered either by the smell of the emergent gas or by heating the delivery tube red-hot.

It will be noticed that the great instability of calcium selenide (for we presume that the hydrogen selenide came from the reduction of this salt), when heated to redness, gives us another reason for suspecting that very little calcium selenide is formed in the black-ash furnace—or rather that very little can survive, unless reduction

* A subsequent note from Dr. Elliott throws some doubt on this, for a red precipitate being noticed at the foot of the denitrating tower—when sulphur nearly, if not quite, free from selenium was being burned as a source of sulphur dioxide—an analysis of the precipitate was made. It then turned out to be ferric oxide.

by carbon and carbon gases differs very unexpectedly from reduction by hydrogen. Any alkaline selenide that might be formed is unimportant, because, firstly, it would be soluble in water, and, secondly, it is not stable in solution but deposits selenium, forming, perhaps, some oxyacid of selenium. If this be the case, carbonic acid gas might liberate an oxide of selenium, but this would not volatilize at ordinary temperatures. It is possible, finally, that free selenium in the waste might gradually combine with the excess of lime, if there be an excess, and re-form selenide of calcium. This was tried by boiling finely powdered selenium with excess of lime—but no combination was produced during an hour, and operating on about half a gram of selenium.

The case for the probable absence of selenium from Chance sulphur seems therefore to be pretty well made out, as well as for its absence from commercial thiosulphate of sodium.

Other Methods of Purification.

We have purposely omitted to use any method based on the crystallization of sulphur from organic liquids or sulphur dichloride. Our reasons are very simple—anyone who has crystallized sulphur from carbon disulphide must have noticed that on reducing such crystals to powder, even long after they have appeared perfectly dry, a strong and horrible smell, characterized by all the worst features of the odour of impure carbon bisulphide, becomes immediately perceptible. Since powdering the crystals reduces them only to smaller crystalline fragments, there is no reason why volatile compounds should not continue to linger in them. If this be the case, what are we to expect on distilling? Surely it is gratuitous to assume that organic compounds could not be formed and distil with the sulphur. Supposing these compounds to be without smell, how could they be detected? Again, carbon bisulphide is a liquid almost impossible to purify, it begins to decompose immediately under the action of light, and forms unknown compounds. Judging by the smell still clinging to some carbon disulphide very carefully prepared by one of us, we may hazard a guess that some selenium compound still lingered. As to toluene, turpentine, &c., considered as solvents of sulphur, we have never seriously entertained the idea of using any of them, for fear of introducing volatile carbon compounds—and also because we have an idea that, owing to the comparatively small solubility of sulphur in these substances and the necessarily large amount of the solvent, traces of impurity might possibly accumulate in the sulphur. It surprises us that anyone conversant with, say, the curiously complex contents of rock crystals, should rely on crystallization to remove impurity.

Worst of all possible solvents, in our opinion, is chloride of sulphur. Chlorine is most difficult to purify, and the process of preparation of chloride of sulphur involves uncertainties to which exception may be taken. Even supposing this solvent to be obtained in a pure state, there is no more difficult substance to remove from the sulphur* (compare MONCKMAN, *loc. cit.*). It is altogether a most unsatisfactory body to deal with.

The Gaseous Contents of Sulphur.—On heating sulphur *in vacuo*, there is no doubt that gases are given off, even from the purest samples. The following is an account of some experiments made on this subject in connection with what will later on be described as the "Film of November 18th, 1892." It may be premised that this was a film which had been rapidly cooled from a high temperature, after having been part of some sulphur which had caught fire in an atmosphere containing much carbonic acid gas, and which looked rather dirty. The sulphur composing this film was found to give off a good deal of gas on heating; so, on November 22nd, 1892, a quantity of recovered sulphur, twice distilled, was selected for an experiment having for its object the determination of the conditions under which gas is given off and absorbed by sulphur. The sulphur was melted under an air-pressure of about 5 centims. of mercury; the results were—

Hardly any gas was given off till the thermometer rose to 178° C., and then the evolution of gas increased up to 190° C. At this point the increased viscosity of the sulphur prevented the bubbles coming off, but they were still being formed. At about 230° C. the bubbles began to separate from the sulphur, and continued to do so up to the boiling-point. There was nothing like a sudden evolution of gas at any temperature. The next day the sulphur was re-heated to 240° C. under similar conditions, but not a single bubble of gas could be detected.

A question now arose as to the nature of the gas. Is it air? If so, shaking up the sulphur with air while it cools from the boiling-point might be considered to be likely to induce more air to get absorbed. This was tried, but no bubbles were given off on re-heating under an air-pressure of 5 centims. of mercury only. Is the gas carbon dioxide? The last experiment was repeated in an atmosphere of carbon dioxide, but again no gas was driven off on re-heating. Has the gas anything to do with the sulphur being on fire?—*i.e.*, is the gas an oxide of sulphur dissolved from an atmosphere mainly composed of this oxide? To test this, the sulphur was heated to boiling-point and set on fire in a platinum dish, a bell jar was placed over it, and the sulphur was agitated freely. It was then re-transferred to the vacuum-bath and tested as before, but little or no gas came off.

These experiments give no clue as to what the gas given off really is, but a valuable result is that, when once the gas has been caused to come off, remelting will not cause more gas to be absorbed. In view of the more important questions awaiting solution, we judged it unadvisable to spend any time in further determining the nature of the gas, as we possibly might have done in a vacuum-tube by the help of spectroscopy. We conclude, however, that the purest distilled sulphur ought to be

* 1893.—It can be done however by continued extraction with carbon bisulphide—at all events fairly well (see end of paper).

remelted *in vacuo* before being used—but this we only began to do in 1893, after the above facts were discovered. We should like to hazard the mere guess that the gas is water vapour, and that it is taken up during the formation of the sulphur snow, which is always present when sulphur vapour condenses in a retort and receiver.

General Physical Properties of Purified Sulphur.

We have not any very serious corrections to make to the observations of previous workers—in other words, very pure sulphur does not differ appreciably in ordinary characteristics from sulphur which is only moderately pure. The most obvious physical datum of a fusible solid is the melting-point, and this is, in the case of sulphur, still rather a matter of uncertainty. The most important contribution to our knowledge of this constant was supplied by B. C. BRODIE ('Proc. Roy. Soc.,' vol. 7, p. 24) in the year 1854. The main results, as far as melting-point is concerned, are as follows :—

1. Pure (?) octahedral sulphur crystallized from carbon bisulphide melts at 114°·5 C.
2. But octahedral sulphur begins to change into monoclinic sulphur below this temperature ; and
3. The melting-point of pure monoclinic sulphur is 120° C.
4. Mixtures of octahedral and monoclinic sulphur melt at temperatures between 114°·5 and 120° C.

No data are given as to how the sulphur was purified, nor as to the method adopted for finding the melting-point. We will add to this statement the following account, which may be regarded as summarising a large number of experiments.

Method.—We have always employed the method of heating small particles in thin glass tubes alongside a thermometer. We are aware (though we cannot find the reference) that a recent observer who carefully examined the relative accuracy of different methods, decided that the best method to employ in determining melting-points is that of using a large quantity of the substance and plotting a cooling curve. In the case of sulphur, however, this method is rather difficult for many reasons—some of which will be evident on referring to BRODIE's paper—and with any method the very small conductivity of sulphur constitutes a grave difficulty.

Our thermometers, which we had hoped to have standardized at Kew, were, by an oversight, only standardized up to 100° C., and to send them to Kew for the oversight to be remedied would have required too much time. They were, however, very good. One was six years old at least, and its B.P. was too high by 1°·34 C. We have confidence that the observations are correct, as far as the thermometers go, within about ·1° C., within which limit the thermometers were in accordance at the temperatures dealt with, the boiling-point corrections being made.

Octahedral Sulphur.—The observations were as follow for octahedral sulphur crystallized (and well dried) from carbon bisulphide :—

Sulphur heated in a glycerine bath at rate of about ·5° per minute.

TABLE I.—Melting-point of Octahedral Sulphur.

Number of experiment.	Sulphur began to melt at	Sulphur completely melted at
	° C.	° C.
1	114·4	116·86
2	119·86	120·86
3	116·06	116·06
4	118·46	120·66
Mean . .	117·19	118·61

As BRODIE has shown, octahedral sulphur heated beyond about 100° C. begins to turn into monoclinic sulphur. If the sulphur is in the state of fine powder, the change cannot be avoided even if the sulphur be heated "for the shortest time between 100° and 114°·5 C." As our sulphur was in small fragments, it was possible that the above discrepancies could be attributed to a more or less partial change having occurred. In order to test this, the glycerine bath was brought up to different temperatures near the melting-point before the sulphur was introduced. The results are as follow :—

TABLE II.—Melting-point of pure Octahedral Sulphur.

Number of experiment.	Initial temperature of bath.	Sulphur began to melt at	Sulphur entirely melted at	Remarks.
	° C.	° C.	° C.	
5	116	116	116	Went at once
6	115·86	115·86	115·86	Ditto
7	115·06	115·86	115·86	
8	108	115·46	116	
9	106	115·06	116·06	
Mean (rejecting No. 8)	..	115·69	116·00	

These observations were made by Miss FLORENCE MARTIN, a most careful observer, to whom our thanks are due. She recommends that Experiment No. 8 should be discarded, on the ground of the point of commencement not having been so sharply marked.

It will be noticed that the above tables indicate two things: firstly, that on the whole the melting-point is lower when the sulphur is raised to it suddenly; and secondly, that all the melting-points are considerably above those given by BRODIE. The reason probably is to be found either in the fact that BRODIE used imperfectly purified sulphur, or that there is a constant error in our thermometers, which we have no reason to believe. The fact probably is that BRODIE did not sufficiently purify his crystals from the evil-smelling compounds which they carry with them from the bisulphide.

Monoclinic Sulphur.—The melting-point of perfectly soluble monoclinic sulphur was taken by the same observer in connection with a film which had been experimented upon.

TABLE III.—Melting-point of Aged Monoclinic Sulphur.*

Number of experiment.	Sulphur began to melt at	Melting complete.
	° C.	° C.
1	119·46	119·96
2	120·06	120·46
3	119·06	119·96
Mean . .	119·53	120·13

This agrees as well as can be expected with the observations of other observers on monoclinic sulphur. The fact that the melting-point of monoclinic sulphur, even when it has lost its crystallographic properties, is about 120°, may be taken as established by these observations. The want of sharpness in melting-point, which is to be noticed even in the case of monoclinic sulphur, is marked in all the samples we have examined, and is on the whole more pronounced when we deal with mixtures, but not markedly so. We think it is established that sulphur, like selenium, has not a perfectly sharp melting-point—another point of resemblance between these substances which we have not seen noticed. Of course, the effect is much more marked in the case of selenium than in the case of sulphur; but even with the latter the corners of fragments become less sharp and assume a greasy appearance at temperatures a good deal below those at which the fragment is definitely liquid.

The freezing points of sulphur have not been investigated by us, as they have been dealt with most elaborately by GERNEZ ('C.R.,' *loc. cit.*). One disadvantage of the method of publication adopted in those famous "Comptes" is that it effectually prevents any estimate being formed of the reliability of experimental work, the scarcity of space not allowing sufficient details to be given.

The melting points of mixtures of monoclinic and electro-positive sulphur are

* This sulphur was afterwards found by Professor DAVID to have lost its crystallo-optic properties, and could not be distinguished by the polariscope from octahedral sulphur. We have ventured to call this variety "aged monoclinic" sulphur. It is probably generally formed as a preliminary step in the formation of octahedral sulphur from monoclinic at ordinary temperatures.

rather below 120° C., and will be dealt with when the films forming the subject of experiment are considered. The detail of some other physical properties will be dealt with as occasion arises.

Section III.

Methods of Determining the Electrical Constants of Pure Sulphur.

The chief constants determined by us are the Specific Resistance and Specific Inductive Capacity. Observations have also been made on the residual charge of sulphur regarded as a dielectric. No observations on the conduction of electricity through liquid sulphur have as yet been made, this having been provisionally accomplished by previous workers. On the other hand, the phenomena attending conduction through solid sulphur have been made out by us, at least, in their more important features.

It will be convenient here to give a sketch of the information obtained by previous observers.

Resistance of Sulphur.

Wiedemann, 'Lehrbuch,' vol. 1, p. 498. Sulphur is mentioned as a non-conductor.

Boltzmann, 'Wien. Sitzb.,' vol. 70, p. 342. In this paper, which deals more particularly with dielectric constants, it is stated incidentally that sulphur is a perfect insulator, and is free from "dielectrische Nachwirkung."

G. Foussereau, 'C.R.,' vol. 95, 1882. This paper contains an account of a method of determining high specific resistances, together with results as to the resistance of glass, which do not concern us. The method employed is that of placing the substance to be examined between two cylindrical electrodes. These electrodes are in series with a battery and condenser, the latter having its armatures connected to the poles of a Lippmann electrometer. The observation consisted in noting the time required for the electrometer to register a known difference of potential. From this, and the voltage acting on the resistance cell, and the capacity of the condenser, the specific resistance can be calculated. In 'C.R.,' vol. 97, p. 996, 1883, this method is applied to liquid sulphur, the electrodes being concentric zinc cylinders. We did not believe in the freedom of zinc from action on sulphur, but after a good many experiments on bright plates, in no case observing any effect even up to the boiling-point, we must admit that zinc is probably a safe material to use. No details are given by Foussereau as to the methods (if any) used to purify the sulphur. The method adopted for examining the conductivity is free from error so far as the action of the containing vessel goes, but not with respect to the surface conductivity. The sulphur experimented upon is stated as having existed in various states which are named, but the paper is not sufficiently detailed to admit of any judgment being formed as to whether these states were actually attained. The results are in brief:—

Specific resistance of prismatic sulphur at 112°·1 C. = $7·39 \times 10^{12}$ ohms.
Specific resistance of prismatic sulphur at 69° C. = $3·93 \times 10^{15}$ ohms.

Below 69° C., the conductivity ceases to be measurable. With regard to the observation at 69° C., we show that this is not the value for pure sulphur perfectly prismatic. With respect to the value at 112·1° C., all depends on whether the sulphur is heating or cooling; but, in any case, we consider it is probably too low. It is further stated that the same prismatic sulphur left to itself at ordinary temperatures, devitrifies and assumes an appreciable conductivity. Thus :—

After one day, specific resistance at 17° C. = $1·170 \times 10^{15}$ ohms.
After two days, specific resistance at 17° C. = $7·05 \times 10^{14}$ ohms.

This is contrary to our results. Our impression is that probably the sulphur was a mixture of monoclinic and insoluble sulphur, that surface effects were not sufficiently guarded against, and that the sulphur was probably impure. But here again the brevity of the account disarms criticism; we can only say that the results do not agree with ours in any way. There are other remarks on the conductivity of octahedral sulphur, which is said to be zero, and on the conductivity of sulphur in the liquid state, which we have not yet investigated.

E. Duter, 'C.R.,' vol. 106, p. 836. Some experiments with an induction coil on melted sulphur. The author found traces of action both on gold and platinum electrodes.

J. Monckman, 'Proc. Roy. Soc.,' 1889, vol. 46, p. 136. The most complete record of the electrical properties of sulphur hitherto published. He used a voltage of about 60 volts, and a high resistance galvanometer. The electrodes were of gold or carbon, and the paper is more concerned in establishing qualitative properties than absolute quantitative results. The results are more valuable than any of those hitherto mentioned, because the purification of the sulphur is more fully dealt with. All the numerical results refer to melted sulphur, and therefore will not be dealt with here. The most novel observation is that the resistance of "soluble" sulphur depends on the light falling upon it, a result which is inconsistent with that of a number of experiments, designed to test the point by one of the authors, with the assistance of Mr. J. B. Allen.

Specific Inductive Capacity of Sulphur.

An exhaustive list of results up to 1883 is given by Gordon ("Electricity and Magnetism," vol. 1, p. 134 (Table)). The lowest results are those of Faraday and Gordon, viz., 2·24 and 2·58; and the highest recorded values are those of Boltzmann, from 3·84 to 4·773, along a particular axis in octahedral sulphur. Values given by Wüllner and Boltzmann (by condenser method) lie between the above limits. It is clear that the value of the specific inductive capacity of sulphur requires further investigation, if only from the great discrepancies which appear above. Other values which have been obtained for the dielectric constant of sulphur will be found in WIEDEMANN'S 'Lehrbuch,' vol. 2, p. 25, and onwards. The only determination of the constant for sulphur in a special state is that of BOLTZMANN, but the purity of the natural crystal he used was not established.

Methods of Determining High Specific Resistances.

(1.) The most direct method is that by means of a galvanometer. This method was used by one of us in 1888 ('Phil. Mag.,' 1889, vol. 28). The advantages of the method (which will be fully discussed presently) are that it involves a direct determination of voltage and current, and can be made practically independent of any leakage other than the one under investigation. It is also fairly rapid and very flexible. Its drawback is that large voltages are required, and, as generally practised, the method is wanting in sensibility on the one hand, and is more or less dependent on some assumed galvanometric formula on the other. These difficulties we have, however, overcome. The galvanometric method was used with success by the Brothers GRAY, 'Proc. Roy. Soc.,' 1884, vol. 36, p. 287; MONCKMAN, *loc. cit.*; BROOKS, 'Journ. Soc. Tel. Eng.,' vol. 9, p. 5 (1881); AYRTON and PERRY, 'Proc. Roy. Soc.,' vol. 27, p. 219, 1878, and many others.

(2.) *Leakage Methods.*—These are of two classes, and are so well dealt with in Professor GRAY'S book on 'Electric Measurements' that very little remains to be said here, and that little is only by way of criticism. In the first place, the advantage of the method is that it is very sensitive and convenient, and can be carried out without special training by anyone possessing a good electrometer. Also, it works well over a large range of voltages. The disadvantages for very high resistances are—(1) the insulation of the instrument is only made sufficient with great difficulty, (2) the time required for an observation is very long, and this is apt to mask very important effects. Again, if the calculation of the results be made from the rate of fall of potential with time (as was done by J. J. THOMSON and NEWALL, 'Proc. Roy. Soc.,' vol. 42, p. 410, 1887), a certain want of sensitiveness is to be observed. If, on the other hand, two potentials and two times sufficiently far apart for accuracy are observed, an assumption of a law of resistance (generally OHM'S) requires to be made in order to integrate the equation (GRAY, 'Absolute Measurements,' &c., p. 404), since, in the case of a very high resistance, the fall of potential is a small fraction of the total voltage only. This assumption, although philosophically repugnant, probably does not produce any very disastrous effects on the subsequent statement of results. An objection of greater gravity lies in the fact that very often the capacity of the quadrants of the electrometer cannot be neglected, and as these are affected by electric absorption (with electrometers constructed with glass insulation), a source of uncertainty is introduced. If the potential-time-curve-slope be used to give a value of the equivalent resistance for a definite potential, when neither the capacity of the quadrants nor the leakage of the electrometer can be neglected, then the formula of calculation becomes arithmetically very objectionable. This is by no means a slight disadvantage if many results have to be reduced. The capacity of the resistance cell also has to be known with fair accuracy, which, at all events, involves more work. The gravest disadvantage, in our experience, lies in the variability of the rate of leakage of the quadrants. If the experiment last for an hour, as is often necessary, then undiscoverable changes may have been taking place in the insulation resistances of the electrometer. If a large condenser be employed in order to reduce the magnitude of the uncertainties arising from want of accurate knowledge of the other capacities involved, we have to face the difficulty of making a proper allowance for its probably very variable insulation resistance. With our high resistances this difficulty proved insuperable. Taking everything into account, we do not consider this method favourable for the determination of absolute resistances, particularly when the word resistance has no meaning except with respect to an instantaneous voltage.

In the second class of experiments by the leakage method, the charge which leaks is taken up by an auxiliary condenser. This method has been used by BOUTY ('C. R.,' vol. 110, p. 1362) with fair success, on standard condensers, for the purpose of investigating the phenomena of residual charge. For this purpose, and when the capacities of the different parts of the apparatus are known, the method is suitable; but, as a method for measuring absolute resistances, it merely adds to the uncertainties mentioned above—those due to the uncertainty of capacity and insulation of the receiving condenser. In our case these methods were out of the question, as we had to deal with resistances greater than the insulation resistance of our mica condensers. In the application of this method made by FOUSSEREAU ('C. R.,' vol. 95, p. 216) matters were complicated by the use of a LIPPMANN electrometer, the poles of which can surely only insulate fairly well for small voltages, and which presumably has a not necessarily vanishing equivalent capacity. However, we have not had much experience with the LIPPMANN instrument, so our criticism must not be taken for more than it is worth.

A quite different method has been employed by H. KOLLER ('Sitzungsber. Wien. Akad.,' vol. 98, p. 201). This method consists in placing the substance undergoing investigation, in series with a high resistance constructed either of a tube of sulphate of zinc solution, with zinc electrodes, or of a quantity of saturated solution of iodine in carbon bisulphide with platinum electrodes. The latter resistances may be constructed (with proper precautions) as high as 10^{11} SIEMENS' units. A divided circuit is used to give any desired voltage to the circuit containing the artificial resistance in series with the resistance to be measured. The fall of potential between one terminal of the cell or derivation circuit and the resistance cell, and between the same terminal and the artificial resistance, allow (proper connections being made) of a comparison of the resistance in question with that of the artificial resistance. KOLLER was looking for qualitative rather than absolute quantitative results, and for this purpose his method has advantages in its quickness and simplicity. If absolute values are required, the method has many drawbacks. The artificial resistances are subject to considerable variation, and are themselves calibrated one from another in such a manner as to give every opportunity for the accumulation of errors. There is reason to suppose that the resistance to feeble currents, such as would be produced by a few cells, will depend on the previous history, and will not be by any means the same when the current is reversed. In KOLLER's experiments the current was, in general, not reversed; in fact, the peculiar behaviour of the liquids investigated did not permit of the method of reversal being employed in all cases. Since the resistance comparison depends on the comparison of two potential differences, it is necessary to eliminate the electrode effects without taking advantage of the method of reversals. KOLLER allowed for the electrode effects by observing them when no current was flowing, which was, perhaps, as much as the circumstances permitted, but could hardly be considered sufficient when absolute measurements are in question. KOLLER does not appear to have reversed his electrometer (*loc. cit.*, p. 204) in taking his readings; an oversight which, we are persuaded, must damage the authority of his work. We are, however, more concerned to explain why KOLLER's apparently simple and advantageous method was not suitable for the ends we had in view, than to criticise his work. We may add that the power of the method is finally limited by the want of insulation of the electrometer, but it remains probably the most sensitive method known for the examination of high resistances.

Method Adopted for Measuring High Specific Resistances and Specific Inductive Capacity.

The method employed by us* is substantially the same as that explained by one of us and J. A. POLLOCK, in a paper in the 'Philosophical Magazine,' vol. 28, 1889, p. 469. The sulphur forms a film between two aluminium plates. A voltage up to about 300 volts is supplied by a set of small test-tube accumulators. The sulphur, storage cells, and galvanometer, are placed in series, keys being provided for cutting the galvanometer out of the circuit, and for reversing the current through the galvanometer without reversing it in the rest of the circuit. Readings are always taken by the method of reversing the current through the galvanometer, and observing the first elongation of the needle. Two galvanometers are always in commission—one at a very high grade of sensitiveness known as the "new galvanometer" and the other at a lower grade, read by a spot of light on a scale, known as the "old galvanometer." In order to evaluate the readings of the galvanometers, a second set of apparatus is prepared, consisting of a standard Clark cell, a megohm, and various resistance boxes, whereby any voltage, from that of the cell down to one hundred-

* See Plates 2–5.

thousandth of the same, may be included in the galvanometer circuit in series with a megohm. A reversing key for the galvanometer and for the cell is provided. The resistances are adjusted till the first deflection of the galvanometer needle on reversal is, as nearly as may be, the same as the deflection when the sulphur is in circuit. The keys are arranged in such a way that the change from the sulphur circuit to the megohm circuit can be made almost instantaneously. In this way a direct value can be assigned to the current causing any deflection of the galvanometer needle, without relying on any particular galvanometer law ; both circuits are practically inductionless. It remains to measure the voltage of the storage cells. In our earlier experiments the cells were permanently arranged so as to be tested in sections against a set of 50 Clark cells, or a smaller number by the method of differences employed by Lord RAYLEIGH in his experiments on the absolute electromotive force of the Clark cell ('Phil. Trans.,' 1885). The resistance of sulphur, however, depends so much on previous history, voltage, &c., that a refinement of this kind was soon seen to be unnecessary, and in all the later experiments the voltage was obtained simply by placing the cells in series with a megohm and low resistance galvanometer, and observing the double steady deflection. This was compared with the double steady deflection produced by the 100 volts of the electric light circuit, derived from 31 plate E.P.S. accumulators, and measured by a FLEMING and GIMINGHAM voltmeter, which had been calibrated against Clark cells, but which was found to require no correction at or near 100 volts. The proportionality of the readings of this galvanometer to the currents passing, within the limits employed, was established by a special series of experiments. The galvanometer was a low resistance reflecting galvanometer, very strongly controlled, and was read with a lamp and scale.

It very soon became obvious—especially when the sulphur was heated—that some check must be devised, capable of detecting any breaking away of the film from the electrodes, owing to unequal expansion, &c. An arrangement of resistances for comparing capacities by DE SAUTY's method (see GLAZEBROOK, 'B.A. Reports,' Leeds, 1890, p. 102, or 'Electrician,' October 3, 1890) was accordingly set up in a permanent manner, and provided with keys, &c., so that the capacity of the sulphur cell could be obtained within a few minutes of the determination of its resistance. A certain ·2 microfarad division of a condenser by ELLIOTT BROS. was used as the standard, and this was afterwards compared with a condenser by MUIRHEAD, whose corrections were obtained and furnished by Dr. A. MUIRHEAD. The method used by Dr. MUIRHEAD is described by GLAZEBROOK (*loc. cit.*). The uncertainties still remaining in the value of the condenser capacity are probably much too small to require further discussion for the following reasons : When we used the capacity test merely to check the adherence of the film, the absolute value of the standard did not matter, and when we used it to determine the capacity with a view to obtaining the specific inductive capacity of the sulphur, the small thickness of the film (about ·25 millim.) prevented its actual thickness being measured with sufficient accuracy to make it worth while to endeavour to go behind Dr. MUIRHEAD's corrections. It is of course, easy to criticise the DE SAUTY method of comparing capacities as applied to two condensers, one showing absorption and the other not (for sulphur is almost free from this phenomenon), but the largeness of our available voltage and the extreme perfection of our galvanometer permitted us to experiment under very advantageous conditions. For instance, with 100,000 ohms out in one branch, and about 7000 in the other, an appreciable kick was produced when the balance was upset by 1 ohm. Of course, it is necessary to use a thin film of sulphur in order to examine the resistance, and this is not the most satisfactory arrangement for arriving at the best value of the specific inductive capacity. There are two incidental advantages, however, which are of importance in practice. The first is that the correction for the effect of the edges of the plates is small. Thus, the actual area of the opposing plates was in one case found to be 156·445 square centims.; and with a film thickness of ·0241886 centim., the correction for the edges (the plates being nearly square) amounted to ·26 square centim. only, or about ·14 per cent. As it was impossible to insure the plates being *exactly* over one another, and as any deviation from exactitude would diminish the capacity, and as, moreover, it was difficult to avoid the presence of small bubbles (whose area was approximately allowed for), it will be understood that the correction for the edges may be in general left on one side. It was, however, usually applied. Again, the capacity being tolerably large—between 2×10^{-20} and 3×10^{-20} C.G.S. electromagnetic unit—the capacity of the connections and keys could also be neglected without prejudicing the results; they were, however, included. The disturbance produced in this case will be understood by the statement that the capacity of the keys and leads was measured by the ballistic method, as well as by the DE SAUTY method. The throw was 7·8 divisions for a voltage which gave with the sulphur condenser in as well, 388 divisions. This corresponded to a capacity for the sulphur of ·00226 microfarad. Consequently the capacity of the leads and keys, &c., was about 2 per cent. of the capacity of the sulphur condenser. Now we shall show that the specific inductive capacity of prismatic sulphur is about 3·7, and of the same sulphur when mixed with about ·35 per cent. of insoluble sulphur is as much as 5·64 in one set of observations.* Now, whether our results are right or wrong, this shows that the correction of 2 per cent. is not of great importance. It can, however, be supplied if desired, as the key, leads, &c., remained the same throughout, and is supplied with the other corrections in the tables.

The reader may possibly and naturally think that we ought to have observed the specific inductive capacity of sulphur in thick plates in order to get over the uncertainties mentioned, but it was one of our principal objects to discuss the specific inductive capacity as related to the conduction phenomena, and this could only be done with security by making both sets of observations on the same film. That this was a necessity, will be understood after an examination of the tables, which show

* October, 1893.—We believe this high value is due to some error which, however, we cannot trace. The values of K are elaborately dealt with by the method of weighing later on.

that an admixture of less than 1 per cent. of insoluble sulphur with monoclinic sulphur reduces the specific resistance of the latter (in one case) to at least the millionth part. There is still room, however, for a careful study of the specific inductive capacity of sulphur in its several allotropic states, and with varying duration of charge, in order to discover whether the great differences between the values obtained can be accounted for by what KOLLER calls the "Schliessungs Strom," for, from our experiments, residual effect exists only to a very small extent, even in tolerably conducting mixtures of soluble and insoluble sulphur. This is one of the matters which requires further investigation and is dealt with later.

The drawings which accompany this paper will show the exact arrangements of the experiments, as well as the dimensions of the various parts of the apparatus, for the drawings are to scale with the exception of the resistance boxes in the ground plan. We will now mention one or two points in which we have found it requisite to exercise particular caution; the first of these is the insulation of the apparatus. We have found that if the apparatus is to be left in position for any great length of time there appears to be a distinct gain in making use of a combination of different insulating substances. Thus the "old galvanometer," which is very well insulated itself, stands on three bits of ebonite placed on a clean sheet of glass. This sheet of glass is supported by three combined flint-glass and paraffin insulators, made by placing a small cylinder of paraffin on the top of a flint-glass bottle, and inverting over it a cylindrical glass dish. The outer dish serves to protect the paraffin against dust; the whole stands on a large slab of ebonite 1 inch thick. All the apparatus is insulated with extraordinary care, including the galvanometer and set of test-tube storage cells. Then, as a precaution against electrostatic effects, one end of the galvanometer is put definitely to earth by means of a wire soldered to the water supply. The galvanometer is enclosed in an iron case which is kept in connection with the earth, and affords a screen against external electrostatic effects. Referring to the diagram (see Plates), it will be seen that the effect is to put the lower plate of the sulphur condenser and one pole of the battery to earth as well, which insures that except for the small P.D. corresponding to the small current flowing, the greater part of the circuit and all the coils of the galvanometer are at zero potential. This has the effect of throwing the whole stress of insulation on the sulphur film, and on one insulating pole which supports the wire coming from the upper plate, and on two terminals of the reversing key.

The effect of accidental leakage, if any, is always observed by making all the connections except that leading to the top plate of the sulphur condenser, the wire simply hanging from the pole. Though this pole is of splendid flint-glass most carefully varnished and having a resisting length of about one metre, and is provided with an ebonite cross-arm about 20 centims. long, to the ends of which the wire is fastened, the insulation is seldom sufficient till the glass has been dried and the long, thin cross-arm of ebonite scraped. The connecting and reversing keys (see Plate 3, figs. 3 and 2) are formed of pillars of ebonite of from 8 to 20 centims. long, and the two points of the reversing key requiring special insulation are made by placing ebonite cups on the ends of two pillars of fused quartz, each about 10 centims. long. The advantage of fused quartz as an insulator—where it is exposed to dusty air—lies not so much in its great freedom from surface effect, as observed by BOYS ('Phil. Mag.' [5], vol. 28, p. 14), as in the fact that a small oxy-hydrogen flame can be applied to it without fear of breakage, and thus the insulation be made practically perfect for the day, without any rubbing or deranging of contacts. The advantages of the general arrangements are—

(1.) The points at which the insulation requires to be perfect are reduced to the fewest possible.

(2.) The system being definitely earthed at the galvanometer, gives a complete protection against electrostatic effects, which are very noticeable when this precaution is not taken.

(3.) The general insulation of the system is necessary when the evaluation of the galvanometer sensitiveness, or the capacity of the sulphur condenser, has to be determined, and it is a great convenience to secure this by disconnecting a single wire. The insulation of the "new galvanometer," by means of fused quartz rods, will be explained in its proper place (see p. 88).

The necessity for insulating the lower sulphur plate was well shown on one occasion, when it was omitted, to give rise to some trouble. Though this non-insulation produces no effect in so far as the applied voltage goes, yet it has practically the effect of short-circuiting the galvanometer through earth in one position of the reversing key and insulating one terminal of the galvanometer in the other. There are always minute voltages in the circuit, and these will cause a current through the galvanometer when it is short circuited. Hence, on reversing the key a throw will be observed which might be mistaken for a conduction throw due to conduction by the sulphur. As these voltages are very small, however, a sufficient insulation is easily obtained by a system of ebonite, paraffin and glass.

SECTION IV.

On the Construction of Sensitive High Resistance Galvanometers.

In a paper, "On the Measurement of High Specific Resistances" ('Phil. Mag.' [5], vol. 28, p. 466), one of us laid down what he considered to be the most important direction in which to look for improvement in galvanometers. Since then work at the subject has been going on, though with many interruptions, both in the workshop and laboratory. During the time that has elapsed, the matter has received a good deal of discussion, notably from Professor AYRTON, and Messrs. MATHER and SUMPNER ('Phil. Mag.' [5], vol. 30, p. 58).

As to the corrugations on paraffined ebonite pillars, our experience is that they are not to be recommended with bare ebonite, unless the pillar be arranged (as should always be the case) so that they can easily be dismounted and cleaned up with a cutting tool on a lathe; mere washing or sand-papering does little good when very perfect insulation is required.

Theory, which is well understood, indicates the following conditions for maximum sensitiveness :—

(1.) The magnetic force per unit current should be a maximum.

(2.) The magnetic moment of each member of the suspended system should be a maximum.

(3.) The mirror should, if possible, be so good that its defining power is only limited by its size; it should be optically perfect.

(4.) The optical magnification should be a maximum.

(5.) The astaticism of the suspended system should be a maximum.

(6.) The field in which the suspended system moves should be zero when the astaticism is perfect, and the directional force entirely due to the unavoidable torsion of the suspending fibre. Since the astaticism is never perfect, and the field can never be zero, the torsion of the suspended fibre can only operate so as to reduce the sensitiveness, and must therefore be made as nearly zero as possible.

It is obvious that the above conditions are partially incompatible, and, as a matter of fact, it is the last condition which provides a starting-point.

The line of argument is this. In order to prevent changes of zero, we are compelled to use quartz threads. These threads are, however, very stiff. In order to prevent the stiffness being such as to seriously affect the sensitiveness, the threads must be very fine, and, consequently, the suspended apparatus very light. In order to go beyond this, it becomes necessary to determine to what point the optical sensitiveness can profitably be carried—or, what comes to the same thing, the limit of the angular value of the fluctuations of the zero. Under fixed conditions, the weight of the mirror will simply depend on the defining power required. The maker must take into account (1) the optical limit as to smallness, depending on the wave-length of light; and (2) the fact that the mirror must be thick enough neither to deform under its own weight, nor to deform owing to the freeing of internal stresses during its manufacture. Before exact theory can be applied to the construction of a galvanometer it is, therefore, necessary to fix on the weight of the mirror, and though, with the experience we have now acquired with magnetically and thermally screened instruments, we can form an estimate of the minimum weight of the mirror, we could not do so before the instrument was constructed. It was necessary, therefore, to proceed along the lines indicated by the general principles involved, and to make such alterations from time to time in the design as increased knowledge suggested.

An alteration which is now in progress is the application of stronger material than glass to the preparation of the mirrors. Even hard glass, considered as a mineral is very weak, and is far surpassed in strength by many substances whose specific gravity is about the same. Fused quartz, for instance, is much stronger, and several discs have been cut from this substance, and are awaiting manufacture. We are also trying with slices of bloodstone, &c., and expect to obtain a considerable reduction in weight. It must never be lost sight of that the weight of the mirror is an unmixed evil. The mirror in use at present is of glass silvered on the back, with a diameter of 1·1 centim. and a weight of ·0485 gram—it is slightly concave, having a radius of curvature of one metre, which was measured approximately. When we discuss the lines along which further improvements can be made it will be shown that owing to a peculiar mode of scale illumination which we have discovered, these dimensions can probably be a good deal reduced, but as our coils are wound at present, no advantage would be gained by effecting this reduction. The mirror is supported above the magnet system, but we are not sure that this is a good arrangement with such a large mirror—with a smaller one it would not matter. The definition of the mirror* is so good that the image of a millimetre scale in it is read at a distance of 2·67 metres, by an improvised telescope and micrometer eyepiece, whose scale is graduated to ·2 millim. Each division of this scale can be divided by eye into about 5 parts; and by means of the diffraction fringes surrounding the scale images, the latter can be located to this degree of accuracy in the micrometer.

With regard to the designing of the instrument we will take the conditions for maximum sensitiveness as laid down above, in order. We have experienced so much inconvenience ourselves when reading papers, in having to refer from lettering in the text to letters on the drawings, that we have prepared notes explanatory of the drawings to be placed beside them, and will endeavour here to give what explanations are required without making it necessary for the reader to refer to the drawings at every moment.

(1.) *The magnetic force for unit current should be a maximum*—resistance of the coil, or trouble and expense of winding, not to be considered. We have endeavoured to make an improvement in this direction, by using four pairs of coils, one above the other, instead of two, as has hitherto been the case. Preliminary trials showed us that we could hope to make the suspended system about 20 centims. long, and consequently the maximum diameter of the coils was approximately fixed. It will be noticed that the coils are much smaller than those ordinarily employed. This may be taken as a realization of a well-known principle, that when it is a question of resistance or quantity of wire, it is better to have many small coils than few large ones. We began by making a drawing of an instrument with eight tiers of coils, but abandoned it on account of the difficulty of making a suitable magnetic system of the necessary length. It is necessary to wind the wire regularly at the commencement of the winding, in order to get most turns in the most important part. We decided to wind the coils with cylindrical holes. The dimensions of the coils are shown in the

* This mirror was made for us by our assistant, Mr. J. COOK.

drawing (Plate 1). It might possibly be worth while to use fine silver wire, and would certainly be advantageous to have the wire specially lightly insulated, but we are too far from manufacturing countries to be able to take advantage of such expedients. The wire employed by us has a diameter of ·05 millim. It was frequently necessary to make joints, owing to irregularities in the insulation. The best material for reinsulating the joints is a film of collodion, as it takes up less room than anything else we know of.

(2.) *The magnetic moment of each member of the suspended system should be a maximum.* This again is not purely a matter of design.

Our magnets were hardened circular discs stamped out of thin sheet steel. Each magnet consisted of two discs, one on each side of the support. Their dimensions were:

Diameter	·876 centim.
Weight	·0309 gram each.
Thickness	·008 centim.

So that the weight of the suspended part is made up thus:

	gram.
8 discs weigh	·2470
1 mirror weighs	·0485
Wire and cement	·0660
Total weight	·3615

So that the weight of the mirror is only about one-eighth of the whole weight, and the magnets form about five-sevenths of the whole weight.

At first we tried mica as a supporting material, but found it very much inferior to flattened aluminium wire, which can be twisted. A strip of mica, moreover, makes the combination more dead-beat; which, with our method of observation, is equivalent to a reduction of sensitiveness.

We also rolled some sheet steel, ·004 centim. thick, into small cylinders as described by Professor LANGLEY, and compared the intensity of magnetization obtained with that of the small discs. The result is given in the accompanying table; it will be noted the advantage is on the side of the disc. The experiments were made by Mr. POLLOCK. We dare say that better steel can be got; but, unless Mr. ELLERY, F.R.S. (of Melbourne), had kindly come to our assistance, we should have had none at all.

TABLE IV.—Comparison of Magnetic Moments of Discs and Cylinders.

	Disc.	Cylinder.
Mass	·0052 gram	·0045 gram
Diameter	·5 centim.	·13 centim.
Length	. .	·46 „
Time of complete vibration	·333 second	·385 second
Moment of inertia	·00008	·000088
Intensity of field	·26	·26
Magnetic moment	·10957	·090137
Intensity of magnetization	21·07	20·03

Condition (3) has been already treated.

(4.) *The optical magnification should be a maximum.* No part of our work has given us more trouble than the production of good scale images, and, in the end, in no part have we had more success. We began by using the telescope and scale set up in the usual manner, and, as usual, were troubled by want of illumination over some parts of the scale, and excess of illumination over others. The sources of light were also sources of heat, and caused the galvanometer to be unsteady, though it was well boxed in. We will not describe the many ineffectual experiments we made. Finally, the following plan was hit upon. A minute point of light (a kerosene lamp turned down very low) was placed edge-on about 2·9 metres away from the galvanometer. An image of this source was focussed on the centre of the galvanometer mirror by a large lens, 19·5 centims. in diameter, placed at a distance of 55 centims. from the mirror. Directly behind the lens, and between it and the mirror, was placed a scale, which, in our experiments, and after many trials, was made by coating a slip of glass with a film of Canada balsam stained with "nigrosine," and cutting divisions on this film on the dividing engine. The divisions were transparent, the general field being dark, and carefully kept so by placing the engraved scale in a proper recess cut in a black wooden screen. The advantage of the Canada balsam film is not obvious in this case, but the scale had been used in other experiments where double images gave trouble, and then the Canada balsam was of the greatest advantage in helping to prevent their formation, much in the same manner as M. CORNU's varnish cures photographic plates of giving a halo round the image of a bright star ('Journ. de Phys.,' vol. 9, p. 275).

The film was, of course, turned towards the mirror, and the light fell upon it normally. It will be understood that by this we mean that the scale was a tangent to a circle described about the centre of the mirror. The object was, of course, to let the light pass through each division of the scale normally ; and, since the scale was only about 10 centims. long, this condition was sufficiently fulfilled, and the scale did not require to be bent. Real images of the scale divisions were formed at about

260 centims. from the mirror. The reflected beam was, however, intercepted by the telescope, whose object-glass, consisting of a spectacle lens of 10 centims. focal length, was 267 centims. from the mirror. The image was observed in the micrometer eye-piece. When the light was not too bright, and everything well adjusted, the images were exceedingly bright on a dark field, and each was the centre of a system of the most perfectly defined and fine interference fringes. By moving the lamp slightly, these fringes can be got to arrange themselves somewhat unsymmetrically about the luminous image, and thus enable an observer to distinguish a particular fringe with certainty; one crossing the brightest part of the luminous image was always selected; and if the fringes were unsymmetrical one was always much the most sharply marked. The similarity of the tracings of the dividing engine was such that each division had a set of fringes, as nearly as we could see identical with the fringes of the images on each side of it, so that a particular fringe could be selected as the fiducial mark of the image, or the distance between two similar fringes could be taken as the distance between the millimetre divisions on the scale. At first the micrometer eye-piece was provided with a toothed or notched scale, but this not being sufficiently capable of sub-division, it was replaced by a scale divided on thin glass by a diamond, on the dividing engine, to ·2 millim. A candle was then arranged so as to throw a suitable diffused light into the telescope, and, the micrometer scale presenting a diamond-cut edge, the coincidence of its divisions or otherwise with the interference fringes could be made out clearly and with great accuracy. The spider-line of the micrometer was not nearly so fine or so well defined as the interference fringes. By properly adjusting the fringes and illumination, everything became so distinct that it was much easier to read to micrometer scale divisions than it usually is to read with a lamp and scale to one division. We are quite sure that this method of illumination and scale reading will be found valuable by any one requiring to read small deflexions. The only points about which we would suggest caution are the following:—The light falling on the mirror must come from a source as small as possible, and a person setting up the arrangement for the first time would probably make the light much too bright and large for the interference fringes; the source must be a mere spark. The lens must be adjusted so as to be normal to the line joining the mirror and lamp, and this line must pass through its centre. The lens we used was an ordinary one, but it had good adjustments. The divisions on the Canada balsam or other varnish must be fine (we tried photographic transparencies, &c., instead, but they always looked woolly). The eye-piece of the telescope must be capable of very good adjustment. Ours was mounted on a fine tuning-fork stand by König; it is no use unless the stand be very steady, and the adjustments, both vertically, horizontally, and in the altitude must be smooth and good. We made ours out of good brass tube without any difficulty. The source of light should be within reach of the observer at the telescope, in order that the final adjustment may be made by slightly moving the source, and by turning the lamp up and down till the best effect is obtained: this involves having

the reflected ray separated from the incident ray by an angle which is not too large. The method is suitable only for the observation of small angular deflexions, for we have not been able to engrave our scale with numbers finely enough to permit us to see the figures and divisions in the field at the same time. Consequently, a deflexion can only be observed by counting the images of the millimetre divisions as they flash past. With our present arrangements we can only count the divisions when they pass below a certain rate; when the whole deflexion amounts to more than about 20 millims., the divisions pass too quickly to be counted. However finely we might succeed in engraving numbers on the scale, we fear that the interference images would be too blurred to be read. The best method would be to omit every tenth division, say. We did not require to trouble about this, however, for the "old" galvanometer was always set up at hand, and, as soon as the deflexions became too large for the method of observation mentioned, recourse was had to it.

The diagram (Plate 4) will make the arrangement clear. A deflexion of one millimetre division is in angular measure 38·6″, and since each millimetre division of the scale covered about 8 micrometer scale divisions, each divisible into 5 parts, we may consider that the smallest deflexion which could be read with certainty was say one-fortieth of this, or 0·96″. Since the magnification depended on the adjustments, it varied slightly from day to day, but the general magnification was, as we say, such that the distance between two consecutive scale images covered from 8 to 9 micrometer scale divisions. We ought to add that the definition of the mirror was a good deal better than we have ever seen before in a mirror of about the same weight and dimensions. It was worked for us by Mr. COOK, together with a large number of similar mirrors, all about equally good. The best unworked glass we have tried was incomparably worse. The difficulty in making such thin mirrors out of glass arises from the fact that they always change their shape when they come off the polishing tool; but in our experience the change of shape is very regular, thus a flat mirror almost always becomes concave or convex; but it does not do so irregularly, and consequently the definition remains satisfactory. The glass, of course, is ground and polished on both sides. The window through which the light passes to and from the mirror, is a strip of patent plate-glass, selected by the method of observing the reflected images from the front and back, and cutting the glass so that it forms a prism of small angle along the top or bottom edge. Patent plate-glass can be got with fairly flat surfaces, which are, however, generally more or less inclined, and so in cases of this kind the glass must be cut in such a way that the inclination of the surfaces does not disturb the accuracy of the observations. For this method of selecting glass, we are indebted to Professor WRIGHT, of Yale.

(5.) *The astaticism of the suspended system should be a maximum.*

This is a most tiresome condition to satisfy, and we have nothing to add to what was said about it by one of us ('Phil. Mag.' [5], vol. 28, p. 458). In order to magnetize the system astatically, four electro-magnets had to be used; an attempt that was made to screen half the magnetic system, while the other half was being magnetized, failed, even when the screening was attained by pushing half the system into a cylindrical hole bored in a thick soft iron bar. During the process of mounting the discs on aluminium wire, they generally got magnetized, and hence required to be demagnetized before the final magnetizing process was carried out. The demagnetizing was carried out by means of the arrangement of four similar electromagnets, which were supplied with an alternating current. If we had foreseen the necessity, we should have provided these magnets with laminated cores ; as it was, care had to be exercised to prevent them becoming so hot as to melt the paraffin used as a cement for the disc magnets. The final adjustments for astaticism were made by twisting the aluminium wire. This wire was only ·35 millim. in diameter, and so the difficulty of obtaining a perfectly straight system of magnets of about 20 centims. long, and so astatic that the natural period of vibration was from 1·5″ to 2″, will be understood by anyone engaged in similar work. The natural period of vibration of two magnetized discs was a fraction of a second. We had hoped that it would turn out more easy to secure good astaticism with four-magnet systems than with two, but this hope was not justified by the event. The adjusting and mounting of the astatic system is the most difficult part of the manipulation ; the final adjustment must be made by hand twisting, as described in the 'Phil. Mag.' (*loc. cit.*).

(6.) *The fibre must have a minimum torsional coefficient.* In plain English, the fibre must be as thin and long as possible ; and it is, of course, much more important to have the fibre thin than long. The proper thickness was calculated from the results obtained by one of us in a paper on "Quartz Threads" ('Phil. Mag.' [5], vol. 30, p. 99, 1890), and the thread was then picked out by measurement with a microscope ; a factor of safety of about 2 being generally allowed. As a matter of fact, the thread required is very fine, but not unmanageably so : pieces about 25 centims. long were generally used. We find that the thread can be fastened more securely by a little melted shellac than by hard paraffin ; we have had several accidents from the thread slipping out of the paraffin. Of course shellac that has not been overheated or dissolved in alcohol, must be used. Care was always taken to use the thread in an untwisted state.

On the General Design of the Galvanometer for High Degrees of Sensitiveness.

The general principles had been arrived at in 1889 from experience with the old galvanometer, from which fair results had been obtained. The following is the result of our general experience.

(1.) It is of the first importance that the coils be adjustable to the suspended part, as well as the suspended parts to the coils. Unless provision be made for this it is practically beyond the powers of even the most skilful manipulator to make use of magnets properly filling up the coil space, and yet free to turn. This condition determines a large part of the mechanical design. The coils must be carried separately, insulated on brass holders, for the nicety required is not to be obtained by the use of any softer or less rigid material. The framework of the instrument which holds the coils is made in two parts, which are scraped to fit ; no screws or clamps are necessary to hold the two halves of the frame together ; one half, containing half the coils, can be lifted up and carried away from the other half, the plane of separation being the plane containing the magnetic system. In order that the adjustment of the magnet system to the coils may be perfectly made, it is necessary to be able both to look through the axial holes of the coils and also into the narrow space (1·5 millim.) separating the two systems of coils. The method of mounting the coils will be clear from the drawings and description (Plate 1), but it is necessary here to indicate the general procedure. The coils, when wound, were embedded concentrically in rings of ebonite, a little paraffin being poured in and fastened both to the coil and ebonite by means of a hot wire. Four bars of fused quartz had been previously fastened as spokes into the ebonite—regarded as a hub—and by means of a centering apparatus devised by Mr. COOK, each coil with its ebonite and quartz was laid in a proper position in its coil carriage. This was a matter requiring much ingenuity, and was accomplished by means of the adjustable catches into which the ends of the quartz rods drop, and which are screwed to the carriage. The coil being in the centering apparatus, the catches of the carriage are adjusted to the quartz spokes ; the catches are then made fast to the carriage, and then the quartz is made permanently fast to the catches by means of a little plaster of Paris. The quartz spokes are like nails, *i.e.*, they have heads, and are fastened to the ebonite rings by being pushed through them from inside before the coil is centred and cemented by paraffin, which also, of course, cements the head of the quartz spoke. Each coil carriage is screwed to the frame by separate screws, which allow a little latitude for final adjustment to the magnetic system when this is suspended. The detail of the whole of the devices for adjusting coils insulated by quartz rods, we owe to Mr. COOK, and we are not sure that the system would succeed with less perfect workmanship than he put into the construction. The arrangement proved in practice to be everything that could be desired.

(2.) The apparatus carrying the coils and magnet system must be absolutely separate and independent of the parts carrying the controlling magnets. This is a condition of the first importance—however good all the other arrangements may be, they will fail in practice unless this condition is fulfilled. We have tried over and over again with the controlling system more or less connected to the suspended system, but have never had any success with such an arrangement, however massive the brass work. This condition may be regarded as the result of all our experience. At one time we tried supporting the coil system on rubber bungs mounted with more or less elaboration, and on discs of rubber mounted in different ways ; but observations with a mercury trough convinced us that with our concrete and ashphalt block floors, the rubber always did more harm than good. At present the instrument is mounted on a hard smooth slab of concrete, which reposes on a sandstone pillar resting on the concrete on which the floors are laid. The coil system is mounted on a sheet of ebonite supported by insulators on the concrete slab, and the controlling magnet system is supported by leaden pillars passing through the ebonite without touching it, and also resting on the concrete slab. The natural stiffness of the slab, or rather its inertia, is much increased by the fact that it carries the magnetic shield of cast iron weighing about three hundred pounds. This materially lessens any tendency there might be to produce disturbance in the suspended system by adjusting the controlling magnets.

The Magnetic Control.—The whole secret of success lies in this. Instead of regarding the control as a subsidiary part of the apparatus, it must be regarded as the most important part, and must be capable of the finest adjustment. We have found it advisable to use the controlling magnets at such a distance that the residual field is due to them, and not to the earth. In this case, small changes in the direction of the earth's field produce less effect than when the residual field is that due to the magnetic action of the earth. We have a simple but important improvement to report in the matter of control. It turns out that when the sensitiveness is very high, the uniformity of the magnetic field becomes very important. When the field is not sufficiently uniform, the following fatal effect is observed. We will suppose that the earth's field is in excess, and that the magnets are in the magnetic meridian. The control magnet, we will suppose, is lowered gradually; finally, a point is reached at which the period is fairly long, and the suspended part, if nearly at rest, behaves apparently well; now let it be caused to vibrate (in practice, the control can never be lowered without causing the suspended magnets to vibrate more or less). It is always noticed that if the excursion rises beyond a certain very small value, then the suspended magnet swings round perhaps 70° or 80°, and takes up a new position of greater stability than the old one. In order to bring it back the control magnets have to be rotated, with the invariable effect of making the suspended magnet swing past the north and south position, and take up a more or less symmetrical position on the other side. Attempts to bring the needles back by moving external magnets, or working damping arrangements, seldom succeed, and in the end the controlling magnet has usually to be raised, and the tiresome business begun afresh. We have wasted many days over this untoward phenomenon. Suppose, however, that the difficulty is apparently surmounted and the image got on the scale and fairly steady, and observations begun. It is always found that after a time the needle begins to drift, and finally goes over to one of the side positions. This was first noted by Mr. POLLOCK with the old galvanometer in 1889. The cause of these effects is not far to seek. The field produced by a simple control magnet is never uniform. In such a long system as ours, and with the controlling magnet at the top, the lower suspended magnet is directed by the earth's field, and the upper one by the control, the centre ones being probably in a nearly neutral field. The diagram (in MAXWELL, 'Elec. and Mag.,' vol. 2, fig. 15), shows how very far from uniform the field is, at or near the neutral point. If the magnetic system be rotated through a very small distance, it comes to a place where the resultant field is differently directed, and then sails away till a new position of equilibrium is attained. The usual practice of considering the field as only influenced by the earth and controlling magnet, and of regarding the suspended magnet as without influence, is fallacious. The direction of the lines of force depends on three things, the earth, the controlling magnet, and the suspending magnet. MAXWELL'S diagram. fig. 15, represents the case of the discovery well, but the matter is complicated by the fact that the resultant field of earth and control magnet is far from uniform even before it is distorted by the suspended magnet.

It seemed likely that much of the difficulty would be removed by making the residual field more uniform, and with this object two additional and similar control magnets were introduced below the suspended system, and adjusted by trial till they were as nearly as possible symmetrically placed with respect to it and the upper adjustable system. The magnets were of the dimensions and at the distances shown in the drawing (figs. 1 and 2), and it was expected that they would produce a field of surface uniformity sufficient for the purpose, after the manner that the HELMHOLTZ coil arrangement produces a solid uniformity (MAXWELL, vol. 2, fig. 19). The arrangement figured was set up as a trial, but as its success was immediate and complete, we never went on to the selection of magnets of proper length—a step we expected to have to take. Since using the arrangement all the wearisome instability and trouble has been diminished so much that we now adjust our sensitiveness and take observations with no more concern than if we were operating with a wooden galvanometer for a rough resistance balance.*

We must not forget to mention the prime importance of having the fine movements of the adjustable part of the control—the upper magnets in our case—as good as workmanship can make them. The rough adjustment for distance is made by sliding the magnets, screw system and all, up and down. The fine adjustment is by means of an excellent nut and fine screw thread—a device described in 1889. The adjustment in azimuth is by means of the lever and screw device, the rough adjustment by releasing a clamp and twisting the magnet carriage. As the magnets and nut together weigh several pounds, we should have done better had we arranged a counterpoise, so that the freedom of rotation of the screw should not be affected by the great pressure on it, as is the case at present; for, though the bearing surface of the nut was well ground, and was kept well oiled, still, on screwing the magnets up, the friction was enough to slightly twist the wide and thick brass tube, and so to displace the magnets in azimuth.

* Oct., 1893.—After more than a year's experience we have never had the slightest difficulty in obtaining the sensitiveness desired since we made the field uniform.

We believe that it is not only feasible but advisable to further increase the uniformity of the controlled field.

Insulation of the Coils from the Frame.—If the quartz is protected from dust it insulates very well, better than anything else we know of in places which can not be got at constantly for treatment. After being in use for some months, the total insulation of the coils and terminals from the frame was $1 \cdot 2 \times 10^9$ ohms, the measurement being made by means of an electrometer. The method adopted for insulating the terminals and connecting wires does not differ much from ordinary practice. The terminals are of corrugated ebonite, of slender section ; the wire passing into the terminal from the coils passes up the ebonite column without touching it, till it reaches the binding screw. The ebonite terminals themselves are screwed into the glass forming the front and back of the galvanometer, and the insulation is improved by drying and varnishing this glass. Where the wires pass from coil to coil inside, they are supported as little as possible ; but where supports can not be avoided, these are made of needles of fused quartz. All the connections are permanently soldered, and it is to be noted that we have carried out in insulating the coils what we consider to be an essential condition of success, viz., the use of a variety of insulating substances.

There are 12,280 turns in each coil.

Total turns, 98,240.
Total resistance, 43,985 ohms, at 23° C.

Screening from Electrostatic Effects.—We expected to have to interpose a film of mica silvered on one side, between the coils and magnetic system, but this has turned out to be unnecessary. The insulation would certainly suffer if the magnets have to be screened.

Capacity of Coils.—The total capacity of the coils and terminals, the inner casing being put to earth, is 8×10^{-5} microfarad, by measurement.

Avoidance of Thermo-electric Effects.—When the instrument was first set up, some small pieces of stiff German-silver wire were used to pass down inside the ebonite terminals. On short circuiting enormous thermo-electric effects were always noted even when all precautions to secure a constant temperature had been taken. These effects could only be removed by replacing the German-silver by copper, and by electroplating the brass binding screws with copper, and by soldering all joints (except the contact at the binding screws) right up to the galvanometer key.

Screening from Air Currents and Convection Currents due to difference of Temperature.—It was found necessary to seal the coil cases with soft wax, so that the interior of the galvanometer becomes almost air-tight. The sealing is easily accomplished, because, as has been mentioned, there are four glass sides, and the wax was run round the edges of the glass. Thermal effects were guarded against by enclosing the whole apparatus in a mill-board box, coated with tin-foil inside and out, and very conveniently built up around the heavy gunmetal frame carrying the controlling system. The iron slabs of the magnetic screen were also tin-foil coated and a roof was erected on them of mill-board covered with tin-foil. This roof had holes cut in it for the connecting wires to come through, and for the stem of the controlling system. All these precautions were the result of attempts to remedy faults as they made their appearance.

Magnetic Screen.—Four iron slabs were prepared from cast iron and arranged so as to form a symmetrical box, without top or bottom, round the instrument. The screening was not very perfect, and if we could have got soft iron we would have used it. However, the improvement made by this screening was very great, and it was no use trying to improve it by cast iron, for we used as much as we could trust the concrete slab to carry, and put it as near to the instrument as possible. It was supported on a wooden frame. When further improvements are made, the most important of them will be in increasing the magnetic screening, even if it takes half a ton of soft iron. The approximate reduction of the earth's horizontal field due to our screen is from ·26 C.G.S. to about one-twentieth of this. The screen weighed 300 lbs.

Residual Fluctuations of Zero.—These always make their appearance when the single steady deflexion is above 1 micrometer division for 5×10^{-11} ampère. We have endeavoured to trace the cause of this fluctuation. We made a number of experiments by heating the instrument unequally by means of gas furnaces placed in different positions with respect to it, with the result that we discovered that the temperature differences had a considerable effect, but not enough to account for the fluctuations. We habitually use the instrument in a room of fairly constant temperature, and the method of illumination cannot be charged with giving rise to the variations; for the lamp may be turned up so as to radiate, say, a hundredfold as much energy without producing any increased effect. The only radiation reaching the interior from the lamp is that which can pass through the fine scale divisions, and this all impinges on the surface of the mirror. The fact that the instrument is practically air-tight, precludes the theory that external currents of air cause the effect. Besides, there is the mill-board box coated with tin-foil, and outside that the iron and mill-board box.

It remains to consider variations in external magnetic force, and it is to these variations that we attribute the fluctuations observed. Our reasons are: (1) The magnetic screen made the needle much steadier. (2) The residual unsteadiness varies very much from day to day; on some days it is impossible to use the instrument with a period of more than 10 sec., and on others the zero is absolutely still and constant during, say half-an-hour, when the period reaches 25 sec. or 30 sec., and, of course, it always happens that these periods of magnetic peace only occur when we want to measure comparatively low resistances. The fluctuations have, moreover, all the appearance of being due to general causes, for they appear equally at night-time when nothing in the Laboratory is being moved, and when there is no traffic in the road. They are always bad on a windy day; but this may possibly be attributed to small changes of barometric pressure occurring from wind pressure. We are placed at about 150 yards from a road along which a steam tram runs, and every time a tram comes past, the needle moves about violently; in fact, we often obtain the first indication of the approach of a tram from the motion of the needle. All these draw-backs can only be eliminated by increasing the efficiency of the magnetic screen, and this is therefore the next step to take.

Method of Observation.—In order to obtain the maximum reading for a given current, we always obtain a double deflexion, and read the first elongation of the needle. Whether this elongation be produced by the current through the sulphur or by the derived circuit from the Clark cell, there is practically no induction in the circuit, except that in the galvanometer itself. If the derived circuit current be adjusted to give the same throw of the galvanometer as the leakage current, we consider that the currents must be the same.

As an example of the behaviour of the instrument, we will give one complete set of experiments, at a normal sensitiveness, taken on October 17, 1892, a very unsteady day.

TABLE V.—Throws of Galvanometer in Micrometer Scale Divisions.

Date.	Voltage.	Resistance.	Battery position.	Double deflection.	Amplitudes from special experiments.	
					1st–2nd. elongation.	2nd–3rd elongation.
Oct. 17, 1892	2×10^{-5}, Clark	$1{\cdot}044 \times 10^6$ ohms	+ to upper terminal			
,,	,,	,,	,,	A—B, 8 divisions	7	5
,,	,,	,,	,,	B—A, 7·5 ,,	8	6
,,	,,	,,	,,	A—B, 8 ,,	8	6
,,	,,	,,	,,	B—A, 8 ,,	7·5	5·5
,,	,,	,,	— to upper terminal	A—B, 7·5 ,,	Not	taken
,,	,,	,,		B—A, 8·0 ,,		
,,	,,	,,		A—B, 8·5 ,,		
,,	,,	,,		B—A, 7·5 ,,		

Mean deflection, $7{\cdot}9 \pm {\cdot}3$ micrometer divisions.

Period of a double vibration, 14·5 sec.

Values of ρ.—The ratio of the 2nd elongation to the 1st.

$$(1)\ \rho = 7/5 = 1{\cdot}40,$$
$$(2)\ \rho = 8/6 = 1{\cdot}333,$$

(3) $\rho = 7/5{\cdot}5 = 1{\cdot}273$,

(4) $\rho = 7{\cdot}5/5{\cdot}5 = 1{\cdot}364$.

Mean, $\rho = 1{\cdot}3425 \pm {\cdot}042$.

If ϕ be the steady deflection corresponding to the throw θ, then (MAXWELL, II., § 745)

$$\phi = \frac{\rho\theta}{1 + \rho}.$$

Now $\rho = 1{\cdot}3425$. $\theta = 7{\cdot}9$, and therefore $\phi = 4{\cdot}53$. But ϕ is the deflection on reversal, hence deflection from the zero for the current employed is 2·26 divisions. The current is C = E/R.

$$= \frac{2}{10^5} \times \frac{1{\cdot}43}{1{\cdot}044 \times 10^6} \text{ ampères,}$$

$$= 2{\cdot}74 \times 10^{-11} \text{ ampère.}$$

Hence a current of $2{\cdot}74 \times 10^{-11}$ ampère produces on reversal a throw of 7·9 micrometer divisions. Leaving the damping out of consideration, this is the same as a current of $3{\cdot}5 \times 10^{-12}$ ampère, producing a throw of 1 micrometer division on reversal. Taking damping into consideration, this would be slightly increased, since the damping is less the smaller the deflection. Now on several occasions (though not on this particular one) in times of exceptional magnetic quiet, readings could be taken to one-fifth of a micrometer division. If, therefore, we consider that a throw of one-fifth of a micrometer division is the least observable, it follows that we could detect a current of one-fifth of the above, viz., 7×10^{-13} ampère, and this with a time of swing of only 14·5 sec. In practice we do not think that a current so small as 7×10^{-13} ampère could be detected with certainty, even when the galvanometer is steady, because of the small thermo-electric effects that are always exhibited by the contact keys, and which would probably give rise to much greater currents than this.* One of the most important matters awaiting solution is the construction of a reversing key, which shall be free from contact effects.

If we estimate the current required to produce a steady single deflection of 1 micrometer division, we find it is

$$C = \frac{2{\cdot}74 \times 10^{-11}}{\frac{1}{2}\phi} = \frac{2{\cdot}74 \times 10^{-11}}{2{\cdot}26} = 1{\cdot}21 \times 10^{-11},$$

and the current which would produce the smallest perceptible deflection is one-fifth of this, or $2{\cdot}4 \times 10^{-12}$ ampère.

On the same day a similar series of experiments were made with currents corresponding to 5×10^{-5} and 10×10^{-5} part of the voltage of a Clark cell.

* Oct., 1893.—Since then we have habitually measured currents of this order.

The following table is a summary of the results of these and other observations, and shows how necessary it is to calibrate the galvanometer for each particular throw, when these are not large :—

TABLE VI., giving Summary of Experiments made on October 17, 1892; October 25, 1892; and December 1, 1892.

Galvanometer period, 14·5 sec. on October 17, 21 sec. on October 25, and 26 sec. on December 1. 8 Micrometer divisions cover 1 millim. in the scale image.

Date.	Voltage in terms of Clark cell.	Current, in ampères.	Mean throw on reversal, in micrometer divisions.	Sensitiveness for one division on reversal, in ampères.	Values of ϕ for double deflexions. $\phi = \frac{\rho\theta}{1+\rho}$.	Sensitiveness for one division steady deflexion, in ampères (*i.e.*, current required to produce one division steady deflexion).
Oct. 17, 1892	2×10^{-5}	$2{\cdot}739 \times 10^{-11}$	7·9	$3{\cdot}5 \times 10^{-12}$	4·53	$1{\cdot}21 \times 10^{-11}$
" "	5×10^{-5}	$6{\cdot}85 \times 10^{-11}$	18·6	$3{\cdot}68 \times 10^{-12}$	10·32	$1{\cdot}32 \times 10^{-11}$
" "	1×10^{-4}	$1{\cdot}36 \times 10^{-10}$	35·3	$3{\cdot}85 \times 10^{-12}$	18·6	$1{\cdot}46 \times 10^{-11}$
Oct. 25, 1892	1×10^{-5}	$1{\cdot}37 \times 10^{-11}$	10·5 (mean of 14 observations)	$1{\cdot}31 \times 10^{-12}$	6·6	4×10^{-12}
Dec. 1, 1892	5×10^{-5}	..	51·8	$1{\cdot}32 \times 10^{-12}$		
" "	3×10^{-5}	..	28·7			

This table requires little comment. If the sensitiveness were reckoned in terms of the least visible deflexion, it would appear to be five times greater. The deflexions are obviously rather greater for smaller currents than for larger. If we consider what has been said about the state of the resultant field, even under such a fairly strong control as this, it appears that any simple galvanometric law is not to be expected. It is not worth while exhibiting any other tables of sensitiveness, for, though all the sulphur observations might be used for the purpose, in general the sensitiveness for steady deflexion was not required, and was not calculated.* In the tables giving the results by the measurements of the sulphur resistance, the sensitiveness per double deflexion (throw) is given, corresponding to each set of observations. For the sake of rapidity of observation, the sensitiveness was adjusted to suit the work in hand. We do not wish to attempt any comparison between this instrument and those employed by other experimenters: our object was to attempt to combine a high degree of sen-

* Note added September 23, 1893 :—

We have lately had occasion to use the galvanometer at the highest degree of sensitiveness conveniently attainable, in order to study the conduction in films which could not be made very thin. For this purpose the period was raised to 25 sec. The terrestrial magnetic conditions were only fairly steady.

sitiveness with ease and certainty in use, and in this we consider we have succeeded to an extent commensurate with our immediate wants. We do not see our way to the suggestion of any material simplification in design, without sacrificing convenience

The voltage was supplied from the terminals by a two-tenths ohm coil, in series with 100,000 ohms and the Clark cell. The following are the readings :—

Galvanometer reversed from	A – B.	Throw,	18 divisions.
,, ,,	B – A.	,,	16 ,,
,, ,,	A – B.	,,	17 ,,
,, ,,	B – A.	,,	20 ,,
,, ,,	A – B.	,,	19 ,,
,, ,,	B – A.	,,	20 ,,
Mean			18·3 micrometer divisions.

Battery reversed.

Galvanometer reversed . .	A – B.	Throw,	27 divisions.
,, ,, . .	B – A.	,,	24 ,,
,, ,, . .	A – B.	,,	24 ,,
,, ,, . .	B – A.	,,	26 ,,
,, ,, . .	A – B.	,,	23 ,,
,, ,, . .	B – A.	,,	26 ,,
Mean			25 divisions.

Key in circuit only . . .	A – B.	Throw,	0 division.
,, ,, . . .	B – A.	,,	1 ,,
,, ,, . . .	A – B.	,,	0 ,,
,, ,, . . .	B – A.	,,	0. ,,

Coil plugs inserted, so as to allow for resistance of terminals, &c.

Galvanometer reversed . .	A – B.	Throw,	2 divisions.
,, ,, . .	B – A.	,,	4 ,,
,, ,, . .	A – B.	,,	3·1 ,,
,, ,, . .	B – A.	,,	2·5 ,,
Battery reversed.			
,, ,, . .	A – B.	,,	0 ,,
,, ,, . .	B – A.	,,	2 ,,
,, ,, . .	A – B.	,,	2·1 ,,
,, ,, . .	B – A.	,,	3 ,,
Mean			2·3 divisions, say.

Hence, mean deflexion for a resistance of ·2 ohm between the points of derivation is 19·3 divisions. The current is less than $\frac{2 \times 1{\cdot}435}{10^{12}}$ ampères, since the megohm was in series with the galvanometer. The sensitiveness, therefore in ampères per scale division, is

$$\frac{1{\cdot}435 \times 2}{1{\cdot}93 \times 10^{13}} = 1{\cdot}48 \times 10^{-13}.$$

On a really steady day, ·2 micrometer division can be read with absolute certainty; hence, if this condition had obtained on this occasion, the sensitiveness for least observable deflexion would have been 3×10^{-14} ampère.

of adjustment. The improvements which can still be made are (1) greater strength in the support of the controlling system; (2) increased fineness of adjustment of the controlling system; (3) better arrangements for twisting the suspended fibre when the controlling system is mounted; (4) increased uniformity of residual field; (5) more perfect magnetic screening; (6) numbering the divisions of the millimetre scale; (7) observing in some place not affected by traffic along streets or roads. Of these, (5) and (7) are much the most important.

Perhaps we ought to add that we are quite aware that at least as high a degree of sensitiveness as we have attained, may be got out of a much less elaborate instrument —on occasion. We have obtained such results ourselves with the "old" galvanometer. The point we wish to make is not that the sensitiveness is extraordinary, but that, such as it is, it is daily and hourly available with all the ease and certainty of observation generally associated with instruments of, say, a thousand times less sensitiveness.

Section V.

Account of Experiments on Sulphur in Chronological Order.

The first experiments were made with the arrangement of plates figured in the 'Philosophical Magazine' [5], vol. 28, Plate 14, and the diagram of connections was the same as is given in the same volume, p. 470. The plates were of brass truly faced by scraping to a surface plate: they were first platinated by Roseleur's method, and afterwards gilt and burnished many times. The object of the continual gilding and burnishing was to make the gold film as solid as possible; for it was found that if the film was at all spongy, the sulphur penetrated it at a high temperature and acted on the brass below. There were also some few spots that seemed disinclined to gild. These spots were finally drilled out and plugs of gold were inserted by a dentist's apparatus. The result was never perfectly satisfactory. The flatness of the plates, however, was preserved to such a degree that the mere contact of the upper clean plate was strong enough to lift the lower one, which weighed about 2·75 kilograms. These plates were prepared by Mr. Pollock. A film of sulphur was got between the plates by placing the lower one in the gold dish, and pouring in sufficient melted sulphur to cover it to a depth of about half a centimetre. The sulphur used was in this—as in every subsequent experiment unless the contrary be specifically stated—the sulphur recovered by the Chance process, distilled three times in hard-glass retorts. A subsequent examination of the film showed that it was almost entirely monoclinic containing less than 1 per cent. of insoluble sulphur. The screws used for separating the plates had undergone a slight modification, so that the effective area as deduced from a series of measurements by a fine pair of vernier callipers was 193·3 sq. centims. From this it was estimated that the actual area was 190·3 sq. centims. —3 sq. centims. being taken up by bubbles, and by a corner of the film which was broken away in cutting down into the sulphur round the plates in order to increase the surface resistance. We may call the area 190 sq. centims. The micrometer screws were adjusted so as to leave a film space of ·05 centim. This was therefore the thickness of the film. So much difficulty was experienced in screwing the screws back that it was determined for the future to use some other method of separating the gilded plates. During the attempts that were made to get a value of the resistance of the film, several facts came to light on which our subsequent practice was founded.

The first was with respect to the necessity for insulating every part of the apparatus. Our connecting keys proving unfit for such high resistance work, new ones were made out of rods of ebonite about 20 centims. long, and were so arranged that it was easy to get at the ebonite in every part for the purpose of scraping it with bits of broken glass—a very effective way of cleaning it. It was also found that there was a considerable amount of surface conductivity over the sulphur, and we finally were obliged to use phosphorus pentoxide as a drying agent, anything less active failing to give satisfactory results. The battery of test-tube storage cells was reinforced by a water battery* consisting of zinc and copper plates, as described by Professor ROWLAND ('Phil. Mag.' [5], vol. 23, p. 303).

In the final experiment with the film under consideration, a voltage of 551 was used. It was measured—not very accurately but sufficiently so—by the simple means of charging a fraction of a microfarad condenser with it, and observing throws with a galvanometer. Another division of the microfarad was afterwards charged by 40 Clark cells, and the throws similarly observed.

The galvanometer employed was the one described by one of us in the 'Philosophical Magazine,' vol. 28, which we now call the "old" galvanometer. It had a sensitiveness of 1 scale division for $1{\cdot}44 \times 10^{-11}$ ampère, with a deflexion of about 23 scale

* This battery is very convenient when it is once got into good order, but it is not suitable for work of this kind, because many days often elapse between consecutive measurements, and then, as a rule, a good deal has to be done to the battery to get it into good order again. For this reason we no longer use it. The little storage cells are also a great source of trouble and annoyance. A battery of storage cells, to be reliable, requires much attention and ought not to be too small. However, as this battery was first set up in 1887, and is still (October, 1893) in use, perhaps we ought not to complain. The result of our experience is that if we had to make such a battery again, we would attend to the following points: (1) The test-tubes might be advantageously replaced by strong glass cells—of square section, and not less than 3 centims. on each side—inside. (2) The lead plates should be at least 3 millims. thick. (3) They should be formed plates—not pasted in any way. (4) The tops of the cells and the places where the plates bend over should be dipped in marine glue. (5) The cells should not be crowded together, so that they can only be examined by being taken out. It is a mistake to save space at the expense of the satisfaction of this condition. (6) The plates should be separated in each cell by a substantial partition of celluloid or other suitable substance, and it is better to put up with an increased resistance than to cut away most of the partition for the sake of reducing it. (7) The terminal wires should be thick and well coated with marine glue or gutta-percha. (8) The cells should be enclosed in a space which can be shut up, so as to prevent excessive evaporation. (9) The lead plates should not approach the bottom of the cells nearer than about 6 centims.

divisions. Since it was described in the 'Philosophical Magazine,' and before it made its appearance in connection with this work, one of the coils developed a fault. It was therefore replaced by a coil of much fewer turns and only a few hundred ohms resistance, which we happened to have by us. The diminished sensitiveness is due to this cause. It was rather unstable and consequently difficult to observe; owing to this, for the sake of security we have considered 10 scale divisions, instead of one, as the limit of effective sensitiveness. We could detect no current whatever through the sulphur with this instrument and the voltage mentioned.

The resistance is therefore certainly greater than

$$\frac{551}{1{\cdot}44 \times 10^{-10}} = 3{\cdot}83 \times 10^{12} \text{ ohms at (say) } 18^\circ \text{ C.}$$

And the specific resistance is greater than

$$\frac{190}{{\cdot}05} \times 3{\cdot}83 \times 10^{12} = 1{\cdot}5 \times 10^{16} \text{ ohms (say)} = 1{\cdot}5 \times 10^{25} \text{ C.G.S.}$$

This will be discussed in connection with other results.

The capacity of the condenser formed by the sulphur plates was measured ballistically, or rather it was compared with the capacity of the same condenser, when the film had been removed and the plates separated to the same distance by bits of quartz needles. As this was an operation requiring time, one of the divisions of the microfarad condenser was used as a step. The result was that the specific inductive capacity arrived at was 3·5, which is about the mean of the values obtained by BOLTZMANN and WÜLLNER, who probably used sulphur of this kind, *i.e.*, melted at a temperature not much above the melting-point, without any special precautions to ensure solubility. At this time we were not examining sulphur with respect to the resistance of its modifications, and the specific inductive capacity was taken as a check in order to be sure that the plates had not come apart. We were also anxious to see whether we could detect any sort of excessive current on charging, but the value obtained showed us that the closing current was quite normal. Of course the sulphur had been frequently stressed by the repeated application of a large voltage to insure its having reached a steady state. These experiments were partly preliminary; and we took no notice of the change of resistance on reversal, or of the action of time, for the very good reason that we got no measurable deflexion even with the large voltage applied.

On taking the plates apart the film was almost perfect, and showed no sign of staining or of any action at all. The gilding of the plates, however, showed a few black specks where the copper had been acted on through the (spongy ?) gold. This is the only objection we have to this series of experiments. The results were afterwards confirmed in so far as resistance goes by the new galvanometer.

The next two series of experiments failed, owing to the spots on the plates becoming more and more marked, and though some of the results are probably unvitiated by this source of error, we have decided to leave them on one side. So far as they went they were very variable, the variability probably depending on the action of the current itself and on the time. To remove the source of uncertainty, due to the perforation of the gold film, we reluctantly decided to prepare two new plates, and after some difficulty obtained two brass castings, which appeared to be perfect after they were shaped. These were then gilt very heavily—several ounces of gold must have been deposited on them—and the gilding, burnishing, and regilding were carried out as before. A preliminary experiment was then made to find if the sulphur had any action, when it turned out that though the spots were fewer and smaller than before, they were still present. This induced us to put the plates on one side, and some thick aluminium having by this time come to hand, two new plates were made of aluminium. These plates have remained in use ever since. They were got flat on the surface plate, and we trusted to keeping them apart by bits of quartz thread instead of screws, a process that turned out quite satisfactory. A large number of experiments were made on bright and scraped pieces of aluminium, with the object of finding out whether sulphur had any action on the plates during the time when and at the temperature at which they were exposed to its action. No trace of action having been discovered, even when the sulphur was burned on the plates, it was decided that they could be used with confidence. The upper plate was provided with an aluminium handle attached to it by aluminium screws; the handle was always removed during the electrical measurements. The dimensions of the plates were obtained by a series of measurements with the vernier callipers, which will be set down here, as the other absolute measurements depend upon them. These callipers had been found on a previous occasion to agree with the standard metre within very narrow limits.

Top Plate.

MEASUREMENT in terms of the scale of brass callipers by ELLIOTT BROS. Corners named A, B, C, D, in order. Temperature, 24·8° C.

Side A–B.	centims.	*Side B–C.*	centims.
At one end	12·361	At one end	12·653
At middle	12·364	At middle	12·652
At other end	12·363	At other end	12·651
Mean	12·3626	Mean	12·652

Side A–D.	centims.	*Side D–C.*	centims.
At one end . . .	12·651	At one end . . .	12·367
At middle . . .	12·652	At middle . . .	12·367
At other end . .	12·651	At other end . .	12·367
Mean . .	12·6513	Mean . .	12·367

Mean breadth of top plate, $\frac{12·3626 + 12·3670}{2}$ = 12·3648 centims.

Mean length of top plate, $\frac{12·6520 + 12·6513}{2}$ = 12·6516 centims.

Area of top plate, 156·43 sq. centims.

Dimensions of Bottom Plate.

Side A–B.	centims.	*Side B–C.*	centims.
At one end . . .	12·360	At one end . . .	12·646
At middle . . .	12·361	At middle . . .	12·645
At other end . .	12·362	At other end . .	12·644
Mean . .	12·361	Mean . .	12·645

Side D–C.	centims.	*Side D–A.*	centims.
At one end . . .	12·365	At one end . . .	12·657
At middle . . .	12·367	At middle . . .	12·656
At other end . .	12·365	At other end . .	12·656
Mean . .	12·366	Mean . .	12·656

Mean breadth of bottom plate, $\frac{12·361 + 12·366}{2}$ = 12·3635 centims.

Mean length of bottom plate, $\frac{12·645 + 12·6563}{2}$ = 12·65065 centims.

Area of bottom plate	156·40	sq. centims.
Mean area, both plates . . .	156·415	,,
Thickness of top plate	·58	centim.
Thickness of bottom plate . .	·32	,,

The preparation and testing of these different plates lasted from April till October,

1891. On October 6, 1891, a film of recovered sulphur, four times distilled and ·022 centim. thick, was got between the plates, and a long series of experiments commenced with it. Amongst other experiments the amount of residual charge was measured, and, on November 12, 1891, the film was heated up to near the melting point. All the results obtained were so irregular that we could not draw any conclusions from them, and several circumstances led us to suspect that the film had been pierced by electric stress on October 15, with a voltage of 712. The plates were taken apart on January 6, 1892, when it was found that the film no longer adhered to the aluminium plates, and that a discharge had passed at some time or other, forming a lump of dark-coloured substance, and burning a hole in both aluminium plates. All the results obtained between October, 1891, and January, 1892, have, therefore, been rejected, with the exception of the following, all of which were afterwards confirmed, and which we had reason to think were not affected by the piercing of the film.

(1.) Before the film broke down with the voltage of 712 it had been exposed to voltages of about from 648 to 684 volts, giving rise to a small conduction, which was intermittent. In the first experiment a deflexion of 58 divisions was obtained with the battery in one direction, and of only 20 divisions when it was reversed; on reversing again, a deflexion of 68 divisions was obtained. These results are similar to others afterwards obtained with much smaller voltages, and are of importance as showing that the properties of sulphur, considered as a conductor, do not undergo any great qualitative change when the voltage per unit length is increased nearly to the breaking-down point.

The specific resistance of the sulphur, if we take 40 divisions as the average deflexion, is about $6{\cdot}6 \times 10^{22}$ C.G.S.

The film contained an unknown amount of insoluble sulphur, but this resistance is less than any other observed by us at the same temperature (18° C.), while it is known that the amount of insoluble sulphur present could only have been very small as the film was cooled slowly. It will be shown that in general the resistance appears to diminish with an increased voltage, so that very possibly the small resistance observed in this experiment may be due to the high voltage employed, or, rather, to the great electric force in sulphur.

(2.) The sulphur was not cut away from the plates in this instance, but extended from the upper plate to the sides of the gold dish. After the film had been broken down, or, rather, after the time at which we consider it had been broken down, the resistance, as we have said, became very irregular. In seeking for the cause of this we examined several possible explanations, and, amongst them, the influence of surface action. The surface leakage, in this case, must have occurred between the upper plate and the sides of the gold dish, and it was thought that the surface resistance could be increased temporarily by scraping the intermediate sulphur surface. This was done without producing any change in the apparent resistance. The following experiment was then made in order to reduce the surface conductivity (if it existed) to zero. A number of thick rods of fused quartz were prepared, carefully cleaned and heated to redness; when they were sufficiently cool, *i.e.*, when they would just melt the sulphur, they were arranged to lie in the surface of the sulphur between the upper plate and the gold dish, and were caused to lie half embedded in the sulphur. Now, if any surface conductivity exists in the sulphur, it will be checked where the surface of the sulphur is interrupted by the quartz—in other words, the current would have to climb over the exposed part of the quartz rods. Now, the surface conductivity of the quartz is certainly very small—indefinitely less than anything we could detect by means of the old galvanometer. The quartz rods were intended to form a complete fence round the upper plate, and practically formed a fence which was very nearly perfect, being only interrupted by gaps a few millimetres wide at the corners. The result was a slight apparent increase of resistance, but the irregularities were so great that no certain conclusion could be drawn except the following:—The conductivity observed was almost entirely, if not entirely, in the sulphur itself, and not on the surface. This was further proved by removing the drying substance (phosphorus pentoxide) and blowing in a current of warm air saturated with water vapour, when it was again noticed that no appreciable change of resistance took place. We shall see later on that the explanation of this result lies in the fact that sulphur does not appear to condense moisture so as to reduce its apparent resistance, as glass does. With the arrangement described the conductivity of the condensed layer was small compared with that of the sulphur itself. This result was true whether the quartz rods were in or out. The facts as to the conductivity of the surface of annealed sulphur were confirmed and completed at a later time.

With respect to the tests of specific inductive capacity and residual charge, we will only state that by a comparison of this film with a bit of specially-prepared mica, we observed, for the first time, that striking absence of residual effect which will be described when we deal with a film free from the objections we have raised to this one.

It now became clear that we must reduce the electric stress on the sulphur if we desired to be free from the risks of breaking the film. Consequently a number of films were prepared during February, 1891, some soluble, some a mixture of soluble and insoluble sulphur, and these were tested with lower voltages. In one case the film was cooled by carbon dioxide snow. All these films went wrong. In a very thin one the plates were found to touch at one corner. One insoluble film was dirty from having caught fire. The film cooled by carbon dioxide snow was got so cold that it became wet round the edges with condensed moisture, and some of the snow had been in contact with the sulphur, and traces of grease were suspected in the snow. One good soluble film gave no conductivity at all, but our galvanometer was not sensitive enough with the reduced voltage to make the negative result worth anything. A mixed film showed no conductivity till it was heated nearly to melting, when it began to conduct at 110° C., and the conductivity then diminished to about 116° C., when it increased rapidly as the sulphur melted. Some films separated from the plates on heating.

The result of all this was to show us the necessity for a good many small precautions, and induce us to concentrate our efforts on the new galvanometer. This work, which has already been described, occupied us until September, 1892, when the new instrument was got into good working order and the investigation of the sulphur proceeded with. Preliminary experiments, therefore, took nearly two years, though the work was not continuous, and we blundered a good deal. With increased galvanometer sensitiveness came increased difficulty from want of insulation, thermo-electric effects, &c. These took some time and trouble to overcome, and, as our experience and the sensitiveness of the galvanometer increased, we were continually obliged to repeat our observations. The film was the subject of experiment until October 28, 1892, at which time, the plates having come apart, the thickness of the film was measured by a large spherometer. The threads having been measured before they were put between the plates, and previous work having shown that the plates rested exactly on the threads, it was imagined that the thickness of the film would be the same as the diameter of the threads. This, however, turned out to be an erroneous assumption, for the spherometer showed that the thickness varied slightly from point to point, and was more than 1 per cent. greater than the diameter of the threads. In the progress of the work it was necessary to reduce most of the results with this wrong value (*i.e.*, the supposed thickness of the film), but they have been recalculated for this paper. The exact statement of the preparation of the film follows.

Preparation and Properties of Film of September 14, 1892.

The plates having been scrupulously cleaned, as well as the pure gold dish, some twice distilled sulphur (Chance) was melted at a temperature of 145° in the large oil bath. This oil bath was really a large oil oven, heated by two long perforated Bunsen gas tubes. It took about four hours to heat to 120°, and was a proportionally long time in cooling; it was not the same oil bath that was used for heating the sulphur when undergoing examination. Eight threads, each broken off to a diameter of ·24 millim., were placed on the lower plate. These threads were about 1 to 2 centims. long, and their diameter was gauged by a BROWN and SHARP vernier callipers. The upper plate was adjusted to the sulphur, care being taken to avoid the presence of air bubbles by lowering the plate from one end like a microscope cover-slip, and at the same time the plate was lowered gently enough not to wash out the threads. The sulphur stood in the gold dish nearly level with the top of the upper plate. After the plates were adjusted as perfectly as possible (examination afterwards showed that the adjustment was practically perfect), the whole arrangement was put back into the oil bath, and again heated to 145°, after which the temperature might be considered uniform. We desired to avoid distorting the plates by having any want of uniformity in the temperature. The bath was then allowed to cool slowly to 95° C., when the sulphur crystallized, but was still soft. The temperature was then raised to 118° C., kept at this for about an hour, and the gas then turned off, and the bath allowed to cool slowly. It takes six or seven hours to reach the temperature of, say, 30° C. In order to have a check on the nature of the sulphur employed, two auxiliary aluminium plates were prepared at the same time, and furnished with a film of sulphur, under identical conditions with the actual experimental plates. The next day these plates were forced apart, and about 1 gram weight of the sulphur tested by carbon bisulphide. It was found that the sulphur was all but absolutely soluble. We would not like to say that it was not perfectly soluble, for the minute particles left could only be seen when the test-tube was in bright sunlight, and they might have been merely dust motes. However, our impression was that a trace of insoluble sulphur was present; it could not have amounted to more than about ·01 milligram, so that the film may be considered to contain, say, 99,999 parts of soluble sulphur out of 100,000.

After the observations were complete, and the plates taken apart on October 25, it was found that 3 sq. millims. ought to be allowed for the space taken up by the bubbles. The film appeared to be perfectly soluble, nothing in the way of residue could be detected, but only about 1 decigram was available for testing. The melting point was taken five times, and found to be from 120·8° C. to 121·4° C., consequently the sulphur was prismatic. A microscopic investigation showed that there were larger crystals present in the film cemented together by smaller ones, or, possibly, by uncrystalline matter. We were so fortunate as to secure the assistance of an expert mineralogist, Professor DAVID, in the microscopic examination of the film. The investigation of the film at the end of the experiments, so kindly undertaken by Professor DAVID, involved grinding pieces of the film down to very fine slices, in the same manner as rock sections are ground down, and subsequent examination by the polarizing microscope. The results were as follows:—

(1.) The greater portion of the film, probably the whole, is crystalline and anisotropic.

(2.) Most of the crystals, but not all, have their principal axes lying in the plane of the film.

(3.) So far as could be seen, most of the crystals appear to have their principal axes parallel.

(4.) The question as to whether all the crystals were monoclinic was investigated with great care by observing the extinction angles, and comparing the action of the film with that of fresh and aged monoclinic crystals. The result was, that the crystals had certainly ceased to be monoclinic, and within the limit of observation were entirely octahedral. The "extinction angle" of monoclinic sulphur is at 45° to the long axis of the crystal, in the case of the film sulphur the extinction angle had changed to about 8° with the original monoclinic axis. As the extinction orientation of the supposed octahedra is unknown, and the crystals are too small to observe singly, no further conclusion can be obtained.

We desire to express our thanks to Professor DAVID for the foregoing valuable addition to our knowledge of the structure of this film. It clearly consists either of a form of sulphur intermediate between monoclinic and rhombic sulphur with extinction angles of 8° or 9°, and which still preserves the melting-point of prismatic sulphur, or, on the other hand, it may perhaps turn out that the octahedra formed in this way are merely very unstable and return to the monoclinic state so rapidly that in melting-point experiments the reconversion always occurs before the melting can take place. Aged monoclinic crystals of sulphur from various sources were found to behave in the same way. Against the instability theory we have the fact that the melting-point was the same, whatever the initial temperature of the glycerine bath. During the electrical observations the film was probably partly normal monoclinic and partly "aged" monoclinic.

No quartz rods were used to reduce the surface conduction. The film was dried for some days in position in the oil bath by sulphuric acid and phosphorus pentoxide.

The colour of the sulphur was a pure lemon-yellow.

Most probable area of film, 156·415 − ·03 sq. centims.

$$A = 156{\cdot}385 \text{ sq. centims.}$$

Thickness of Film.—The threads were slightly conical and broken, so that the thickest parts were ·024 centim. in diameter, as has been stated. When the plates were taken apart, a series of measurements of the film by the spherometer were made. Bits of the film were taken from each of the sides near the edge, and a bit from the middle. The bit to be measured was placed on the spherometer plate in a marked position—with careful dusting—and then covered by a small square of plane parallel glass, not larger than the fragment of film employed. The spherometer was well oiled and got into a steady state of temperature by handling before the zero was obtained—precautions very necessary, but generally omitted. Ten settings were taken on each portion of the film. The five means of the readings were, in inches,

	·337670
	·336990
	·337840
	·336470
	·336979
Grand mean . .	·3371898

Zero setting on bit of glass. 10 observations, ·346773.

Difference, ·0095832 inch = ·024341 centim. thickness of film.

The differences are probably due to imperfect dusting of the bits of film. The film was very crumbly and the broken bits were strongly electrified and difficult to remove. This measurement of film thickness is the most unsatisfactory part of the work. In calculating specific resistance and specific inductive capacity, the ratio of thickness to area has to be known.

The area has to be corrected for the effect of the edges. In the case of the electrostatic measurement, *i.e.*, the determination of the specific inductive capacity, a correction to the area can be calculated as already mentioned, and may be taken at + ·26 centim. In view of the specific resistance being dependent on so many different factors, it was not worth while to try to calculate a separate correction, so that the corrected area has been used both for the calculation of specific inductive capacity and specific resistance.

This area is 156·385 + ·26 or 156·645 sq. centims.

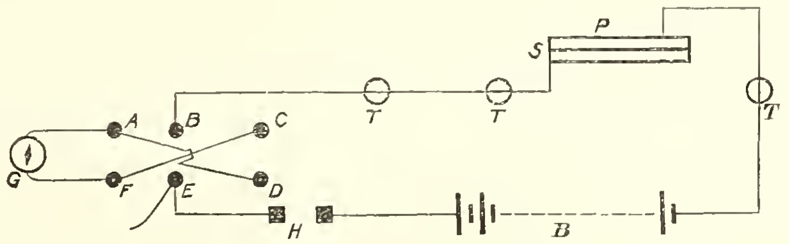
The ratio of corrected area to thickness is, therefore,

$$\frac{156{\cdot}645}{{\cdot}024341} = 6427{\cdot}4.$$

The arrangement of apparatus adopted for making the measurement of the resistance of this film will be understood from the diagram.

Two days experimenting were absorbed in curing leaks, contact effects, &c. The galvanometer was simply used with a lamp and scale.

Fig. 1.



A B C D E F (fig. 1) is a reversing key, put to earth at E.

H is a contact key.

T, T are insulating supports of glass and ebonite.

The terminals of the battery are reversed when necessary by interchanging the wires.

A key, not shown, is provided for connecting A and F or C and D, so as to cut out the galvanometer.

Before and after making an observation the wire is disconnected from the top plate at P, and hung on the support at T. Nothing touches the upper plate except the sulphur and connecting wire. Consequently, if no deflexion occurs when the wire is disconnected from P and hung on T, even when the reversing key is put over, it shows that the insulation is sufficient.

In order to be sure that everything was in order when no deflexion was obtained on the galvanometer, the wire at P was touched with a piece of paper, which, of course, gave a deflexion, and, as additional evidence, the contact at H was broken and the sulphur plates regarded as a condenser discharged and the spark noted.

Observation (1).—Galvanometer sensitiveness, 1 division double deflexion for $5 \cdot 5 \times 10^{-11}$ ampère. Voltage = 313 volts.

No deflexion could be observed.

Therefore resistance of sulphur is greater than

$$\frac{313}{5 \cdot 5 \times 10^{-11}} = 5 \cdot 7 \times 10^{12} \text{ ohms.}$$

And the specific resistance is greater than $3 \cdot 66 \times 10^{25}$ C.G.S.

The sulphur had been exposed during two days to the action of the drying material and to the continued application of voltages of from 150 to 300 volts in both directions.

The next point was to measure the specific inductive capacity. We were delayed in this by finding, during a series of preliminary experiments, that the smaller subdivisions of a standard condenser by ELLIOTT BROS. agreed very badly, and consequently we had to make a number of comparisons in order to form an opinion as to the most reliable condenser division to use—at that time we had no condenser with standardized divisions.

Finally, we decided that the fault lay, as was to be expected, in the smaller divisions of our microfarad; we were therefore obliged to make use of the larger divisions. This necessitated the finding of a way to conveniently bridge the great difference in the capacity of the sulphur condenser and that of ·2 microfarad, the smallest division available. We finally arrived at what we found to be a very satisfactory method. A preliminary experiment showed that, using a certain number of Clark cells, we obtained, by simple charge and discharge of the condenser through the galvanometer, a certain convenient deflexion. The comparison was then made by subdividing the voltage of one Clark cell and charging and discharging the selected division of the microfarad with it until a nearly equal deflexion was observed. This avoids the necessity for determining the logarithmic decrement. The voltage was divided by using points of derivation in a circuit of 100,000 ohms, including a Clark cell.

The capacity of the key and connections was found to be very nearly two hundredths (·02) of the combined capacity of the key, condenser, and connections. This correction is amply accurate enough for the results got by the ballistic method, and is rather too large—uncertainty due to large absorption in the standard prevents really good comparisons being made.

There being a difference between the divisions of the microfarad, one of them was selected as a provisional standard pending calibration when our standard shall arrive.

Let θ be the first elongation of the galvanometer needle.

Let C be the capacity of the condenser.

Let V be the potential difference of the coatings in terms of the Clark cell.

Let K be the ballistic constant of the galvanometer.

Then when the deflexion is small and the logarithmic decrement can be neglected, or eliminated,

$$CV = K\theta; \; K = CV/\theta.$$

We considered that the best way of obtaining a mean value in terms of the divisions of the microfarad would be to take several sets of observations, calculate K from each and then use this mean value to enable us to find the capacity of the condenser. The results are as below.

TABLE VII.—Values of K (Galvanometer constant).

Fraction of microfarad.	Voltage.	Elongation.	K.
·2	$\frac{12}{52}$ Clark	205·0	·0002251
·2	,,	198·4	·0002326
Another 1/3 standard	1/6 Clark	204·5	·0002427

Mean value of K = ·0002335.

The observations on the sulphur were repeated with 21 Clark cells, as it was found that the galvanometer sensitiveness had changed since the day before, when the numbers given were obtained. In this way the mean deflexion was found to be 195·2 divisions.

Hence

$$C = \frac{K\theta}{V} = \frac{\cdot 0002335 \times 195 \cdot 2}{21} = \cdot 00217 \text{ microfarad.}$$

Subtracting ·00004 for key and leads,

$$C = 213 \times 10^{-20} \text{ C.G.S.}$$

Corrected capacity, therefore, is 213×10^{-20}.

If

A be the "effective area" of the film;
d ,, thickness of the film;
μ ,, specific inductive capacity;
V ,, ratio of the units;

$$\mu = \frac{4\pi . d . C . V^2}{A};$$

$$d = \cdot 024341\ ;$$
$$A = 156\cdot 645\ ;$$
$$C = 213 \times 10^{-20}\ ;$$
$$V = 3 \times 10^{10}.$$

Hence

$$\mu = 3\cdot 748,$$

a number very like that previously obtained for the same kind of film (3·5).

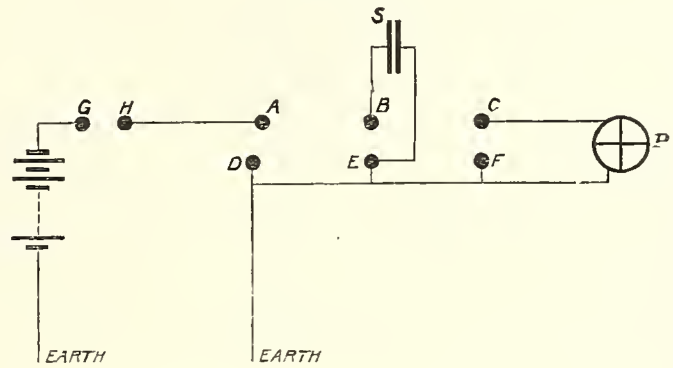
Having obtained a provisional value for the specific inductive capacity of the sulphur, we considered that a suitable time had come to repeat some experiments made a year before on the residual effect, and endeavour to obtain a comparison between sulphur and mica, the latter having been much recommended by M. BOUTY ('C. R.,' vol. 110, p. 1362 ; 'C. R.,' vol. 112, p. 931 ; 'Journ. de Phys.,' vol. 9, p. 288, June, 1890 ; 'Ann. de Chim. et de Phys.' [6], vol. 24, p. 394). We propose to describe our experiments first, and then to discuss this very important matter (the literature of which is overwhelming) when we have described a number of cognate phenomena.

A preliminary experiment was made to find the capacity of the quadrants of our electrometer and of the key and connections used in the investigation of the residual charge. The capacity of the quadrants was rather less than ·00005 microfarad, and of the key and connections about ·000004 microfarad. The sulphur plates have, as has been shown, a capacity of about ·002 microfarad. The smallness of the capacity of the quadrants and key and leads, in comparison with the capacity of the condenser, enables us to disregard them in estimating the residual effect. The mica condenser was prepared from a specially selected bit of mica, such as is used for covering photographs ; it was silvered on both sides, and cleaned along the edges as described by BOUTY, the edges being lightly varnished by dipping in a dilute filtered solution of shellac. This condenser plate was dried at the temperature of the water bath for twelve hours, and was then mounted on quartz insulators in a desiccator over sulphuric acid. It had been in this for a year when our experiments began. Connection was made to the insulated plate by means of an aluminium wire on a fused quartz support. The dimensions of the mica condenser were :—

Length of rectangular silver films . .	7·2	centims.
Breadth	3·6	,,
Thickness of mica	·005	centim.

The capacity was of the order of that of the sulphur condenser. We will not describe any preliminary experiments on the residual effect, because there was nothing novel either in our method or in the difficulties we encountered. The investigation of residual effect is not difficult if the effect be large, but if it be vanishingly small, as it was with the sulphur, and if it be required to assign a limit to this small effect, then numerous difficulties arise which are to be got over by electrostatic screening, avoidance of contact forces, insulation, &c. The following diagram shows our arrangements :—

Fig. 2.



A, B, C, D, E, F (fig. 2), are the six points of a reversing key. The points B and C are insulated by supports of fused quartz. The condenser is inserted at S. GH is a pillar make-and-break key with mercury cups (see Plate 3, fig. 3). Between B and E is placed a wire of aluminium, so arranged that when the rocking key is thrown over from its charging position A B E D to its discharging position B C F E, this wire momentarily touches part of the rocker connected with B, the other end being permanently connected at E. In this way the discharge can be made to occupy a very short time, say ·01 sec., and includes the key. A short-circuiting key is sometimes also kept in C and F till the discharge has taken place. During the charging G and H are connected. HA is a long wire enabling the battery, &c., to be placed far enough away not to complicate matters by direct electrostatic induction. As soon as the rocker is moved over far enough to be insulated from A, this point is earthed and the rocker then moved over to CF, the short-circuiting key being removed after discharge has taken place, and, of course, before the rocker makes contact at C or F. At one time an additional key was used for the short circuiting of the electrometer, but it was given up as merely increasing the trouble of insulation. The electrometer was adjusted to a sensitiveness of about 40 scale divisions per Clark cell, and this did not vary by more than 2 divisions over the scale. Thermo-electric effects, &c., prevented the sulphur from ever giving absolutely no deflexion after the removal of the charge. About two hundred experiments were made on the sulphur and mica, and we know nothing whatever against the summary, which we consider to be a sufficient statement.

TABLE VIII.—Residual Effect in Sulphur and Mica.

Mica. Temperature 18° to 20° C.

1. Date.	2. Number of experiments.	3. Voltage.	4. Charge in scale divisions.	5. Duration of charge.	6. Residual in scale divisions after 30 sec.	7. Residual in scale divisions after 1 min.	8. Percentage of charge coming out after 30 sec.	9. Percentage of charge coming out after 1 min.	10. Remarks.
Sept. 20, 1892	1	40 Clarks	1600	2 min.	12	16	·75	1	
,, ,,	1	,,	,,	3 ,,	12	16	·75	1	
,, ,,	1	,,	,,	1 ,,	12	17	·75	$1\frac{1}{16}$	
,, ,,	1	,,	,,	2 ,,	12	17	·75	$1\frac{1}{16}$	
,, ,,	1	,,	,,	10 sec.	11	..	·67	..	
,, ,,	2	264 volts	264 × 28	1 min.	60	..	·79	..	
Sept. 21, ,,	6	318	318 × 28	1 ,,	40	45	·45	·5	
,, ,,	4	318	318 × 28	1 ,,	Off scale	Off scale	..	..	Mica heated for 38 mins. at 98° C.
Sept. 22, ,,	2	> 298	> 298 × 28	1 ,,	300	..	< 4	..	Not quite recovered, though it had been over P_2O_5 all night.

Sulphur. Temperature 18° to 20° C.

1. Date.	2. Number of experiments.	3. Voltage.	4. Charge in scale divisions.	5. Duration of charge.	6. Residual in scale divisions after 30 sec.	7. Residual in scale divisions after 1 min.	8. Percentage of charge coming out after 30 sec.	9. Percentage of charge coming out after 1 min.	10. Remarks.
Sept. 19, 1892	6	40	1600	1 min.	< 4	< 4	< ·25	< ·25	
Sept. 20, ,,	4	,,	1600	2 ,,	< 1	< 1	< ·06	< ·06	
,, ,,	2	264	264 × 28	1 ,,	< 4	< 5	< ·054	< ·065	
,, ,,	,,	,,	,,	2 sec.	< 4	< 4	< ·054	< ·054	
Sept. 21, ,,	6	> 318	> 318 × 28	1 min.	< 3	< 3	< ·033	< ·033	
Sept. 22, ,,	4	> 298	> 298 × 28	1 ,,	< 3	..	< ·036	..	
,, ,,	2	> 298	> 298 × 28	10 ,,	4	< 6	< ·048	< ·072	
,, ,,	,,	,,	,,	,,	..	After 4 mins. was < 9, or less than ·108 per cent.			

It will be noticed from the above that the residual charge in the case of mica—properly treated and very dry, is less by far than is generally supposed—but by exposure to air, even when heated, the effect is much increased. BOUTY's best results went to show that the residual charge was only about 1 per cent. of the initial charge; these experiments show that mica can be prepared so as to give still less. It is difficult to account for the experiment on September 21st, when the mica had been heated. BOUTY attributes the residual charge almost entirely to the action of the edges, which are varnished, and we are inclined to agree with this, but we do not see how the polarisation can be galvanic in the ordinary sense, for in our case it must have amounted to more than 10 volts. We think it is due to a creeping of the charge (see ROWLAND and NICOLL, 'Phil. Mag.' [5], vol. 2, p. 414, 1881). The sulphur is really phenomenal. We are inclined to attribute most of the small effects observed to the residual sources of uncertainty, contact action, imperfect shielding, imperfect insulation, imperfect prevention of the creeping of the charge, in spite of these effects having been eliminated apparently before the sulphur condenser was inserted. Because these uncertainties are suggested, it must not be thought that they were overlooked at the time. We believe that everything was done that could be done, and that we are, in fact, at about the limit of the applicability of these methods.

However, taking the numbers as they stand, they are sufficiently remarkable, and show that the residual charge, if it exists, is less than, say, four parts in ten thousand of the original charge, even when the duration of charge amounts to ten minutes. In our previous experiments using the film between the old gilded plates, we made use of enormously greater voltages derived from an electrophorus, but some uncertainty arose from the possible parting of the plates, and from the burning of the film where the spark had jumped across. However, we never detected any residual effect at all with the film in question, under circumstances where the residual charge from the mica amounted to more than 50 divisions; this was with a voltage estimated from the spark length at about 5000 volts.

The electric strength of sulphur from the broken film is in the neighbourhood of and probably greater than 730 volts per ·22 millim., or at the rate of about 3300 volts per millim., or 33,000 volts per centimetre, but too much weight must not be attached to this, for the influence of a small air bubble in disturbing the field might be very considerable, and would reduce the strength very materially.

We defer the discussion of these results till the similar data for other films have been dealt with.

The residual charge having been found to be so small, rather greater interest attached to the specific inductive capacity of the film, which had to be taken again in any case in order to give us security as to the permanence of the contact between the plates and the sulphur. As the results of a complete series of very careful experiments by the ballistic method, using one division of the microfarad only, the one chosen as a provisional standard, the capacity of the sulphur was found to be—

$$C = 208 \times 10^{-20} \text{ (uncorrected).}$$
$$\mu = 3{\cdot}697 \text{ (corrected for standard).}$$

A comparison between our condenser and one supplied by Messrs. CLARK, MUIRHEAD, and Co., may now be given; it was made on Jan. 24th, 1893, at a temperature of 23·8 C., by means first of the old, and afterwards of the new galvanometer, with 40 Clark cells by DE SAUTY's method. The leakage through the two condensers was measured, and found to be so small as only to affect the result in the fifth place; it was therefore neglected. We used two 100000 ohm boxes, and a dial box of 10000 ohms, all by ELLIOTT BROS., but the divisions were not compared with our standards. Dr. MUIRHEAD's value for the ·2 microfarad was ·1996 microfarad at 15° C.

As we do not know whether the mica plates are shellacked or only paraffined, we cannot apply any temperature correction—in any case the two condensers probably have about the same temperature coefficients. The result of the comparison was that with 100000 ohms out in one branch, the balance lay between 98815 and 98800 in the other. Taking 98808 as the mean, the capacity of our ·2 division for instantaneous charge is ·20186 microfarad, or say 1 per cent. too large. For the future this value will be employed.

The difference between this value and that formerly obtained is to be ascribed chiefly to the difference between the standard selected and the mean of the three divisions taken before as a standard.

It was thought desirable to check this result by another method, both for the purpose of obtaining independent evidence, and because the ballistic method takes so long to carry out that it is unsuited for measuring the capacity at any given temperature. The well-known method of DE SAUTY was selected after consideration, the sensitiveness of our galvanometers giving us great advantages in carrying it out. By observing the initial kick we also hoped to get the instantaneous capacity comparison (see GLAZEBROOK, 'B.A. Report,' Leeds, 1890, p. 102).

By using 40 Clark cells, a balance for instantaneous charge was given by 100000 and 1065 ohms out in the two branches, one from a dial box, the other from an ordinary box, both by ELLIOTT, but not compared with our standards.

The final corrected result is

$$C = {\cdot}00211,$$
$$\mu = 3{\cdot}708.$$

The sensitiveness of the galvanometer having been increased, as well as our experience in using it, it was thought worth while to determine the resistance again for the purpose of obtaining a higher limit.

Date, October 1st, 1892—Voltage, 304 volts.

Sensitiveness of galvanometer, $3{\cdot}6 \times 10^{-12}$ ampère per "half" tooth of the micrometer.

Limit of certain discrimination, owing to thermal effects, was about ·25 tooth only, but it is put at ·5 tooth for certainty.

Specific resistance is greater than 5×10^{26} C.G.S. (The capacity was taken again just after the test, and found to be the same as before.)

In order to check this, it was made the subject of experiment by the electrometer method. At first we intended to experiment with different voltages in order to test the sulphur with respect to its law of conduction, and made a number of experiments on the lines of the experiments of THOMSON and NEWALL ('Proc. R.S.,' vol. 42, p. 410, 1887). For reasons formerly given, however, we ended by observing the leak during a given time, and then assuming Ohm's law, we will not enlarge on our troubles in this work, for they must have been such as would come to anyone trying to get at a very high resistance by this method. We will, however, ask the reader to believe that all such sources of error as are introduced by incorrect knowledge of capacities and electrometer law—imperfect insulation, &c., were attended to by us with great care, and the results we offer are, we believe, free from objection. Mr. POLLOCK was kind enough to check the algebraical and arithmetical work for us, and as he used no approximations, we give his results rather than ours—with which they agreed within the limit of approximation adopted by us.

From the results, the specific resistance of the sulphur is as follows, neglecting fractions:—

1. Oct. 6, 1892 . *Specific Resistance*, 6×10^{28} C.G.S.
2. Oct. 7, 1892 . *Specific Resistance*, 1×10^{28} C.G.S.
3. Oct. 7, 1892 . *Specific Resistance*, 8×10^{27} C.G.S.

These results were obtained with a voltage of about six Clark cells.

All these observations were taken on sulphur after it had been in its silvered brass box with phosphorus pentoxide for at least twelve hours.

The effect of exposing the sulphur to air was tried on one occasion, when the resistance fell at once (*i.e.*, in ten minutes) about one thousand-fold, and seemed to stay there. After replacing the lid of the box and leaving for about twenty minutes, the resistance had again increased to six times its value when exposed freely to air. This explains partly how it was that we got only a small or negligible effect by exposing to moist air when we used the galvanometer. At the time the plates were exposed to air for ten minutes, as just recorded, the thermometer wet bulb was at 61°·6 F., and the dry bulb at 71°·6 F.—a rather damp day. Nearly all the reduction in resistance took place during the first few minutes of exposure, but the method does not admit of this matter being treated satisfactorily, for it takes too long to get a good measurement. The influence of moisture seems to be to reduce the resistance of the sulphur condenser, but not to such a point as would enable the conductivity to be detected in any measurements hitherto described by means of the galvanometer. It is possible that the small conductivity observed by the electrometer may be wholly due to residual surface action, though later experiments lead us to think that this is not the case. The capacity test was again applied, giving the same result as before, and showing that the plates are still in position. It was decided to heat the sulphur condenser, and observe change of resistance and capacity as the temperature rose. The galvanometer was now more sensitive than before, giving one micrometer division of double elongation for $1{\cdot}31 \times 10^{-12}$ ampère. With 291 volts no effect could be detected. The result is to place the specific resistance above $1{\cdot}4 \times 10^{27}$ C.G.S. The galvanometer now approaches the electrometer in its power of discriminating small conductivities. The temperature was 17°·2 C.

The temperature of the bath was then raised to 50° C., at a rate of about ·2° per minute, and was kept within a degree on either side for about an hour while the resistance and capacity tests were made. The capacity appeared to be exactly the same as before—*i.e.*, it did not change by as much as ·3 per cent., or we should have detected its variation. It will be shown later that this probably indicates an appreciable, but small, positive temperature coefficient of the specific inductive capacity. The resistance test also failed to show any signs of conduction, and the result may be considered to be the same as at 17°·2 C.

The temperature was then raised to 75° C., and kept there for several hours. A slight conductivity showed itself, corresponding to a deflexion of 21·3 divisions, with a voltage of about 285 volts. The specific resistance is now $6{\cdot}8 \times 10^{25}$ C.G.S. After the battery had been charging the sulphur for about an hour and a half at the temperature of 75° the deflexion fell from 21·3 to about 5 divisions, showing that the resistance had increased, say, four times by the continued application of electric stress. This being the first clear case of conductivity we had encountered, we were anxious to note any phenomenon which might present itself. The most striking effect was the discontinuity of the conduction. At first we thought that the sudden sharp kicks given every now and then by the galvanometer were due to accidental causes of some kind, but it soon became clear that the phenomenon was a real one. When the voltage is first applied the discontinuity is more marked than when it has been on for some time. In the case we have before us the galvanometer would take up a definite position of deflexion, and every two or three minutes this would suddenly increase or diminish by a large amount. The only possible way of estimating the conductivity is, therefore, to take a long series of observations and obtain a mean value, and that is what was done above. The impression forced on one while observing at the galvanometer is that a succession of irregular changes are taking place; for instance, that groups of molecules, one after the other, succumb to the electric stress, and signalise their destruction by allowing a current to pass. The conductivity in these cases, however, never falls to zero. It is, perhaps, not too much to say that all the observations are in accordance with a theory that there is a small regular and continuous conduction, superimposed on which there is something very like a disruptive conduction. We have since discovered that the discontinuous conduction occurs also when the conduction is superficial, but that it is a real phenomenon of the conduction through sulphur as well. The temperature was constant between 75°·5 and 74°·3 C. The capacity appears to be absolutely the same as before, *i.e.*, within ·3 per cent.

The temperature raised to 99°·8–100° C. There now appeared to be a slight fall in capacity, say ·3 per cent., but it was so small as to be quite uncertain.

The following resistance results were obtained :—

Double deflexion	of	14·7	divisions at	108° C.	*Six experiments.*
,,	,,	22·5	,,	111° C.	*Four experiments.*
,,	,,	4·5	,,	113°·7 C.	*Six experiments.*
,,	,,	7·2	,,	110°·5 C.	*Five experiments.*

The sensitiveness of the galvanometer had been reduced to one division for $2·2 \times 10^{-11}$ ampère for the sake of rapidity of observation. The specific resistance (mean of above) is about 7×10^{25} C.G.S., but we lay no stress on it, for on applying the capacity test it was found that the plates had come apart.

The results with this film may now be summarized as follows :—

Film of September 14, 1892.

Thickness, ·02431 centim.

Area (effective), 156·645 square centims.

Kind of sulphur, either perfectly soluble, or containing less than 1 part in 100,000 insoluble.

Crystals, monoclinic (aged monoclinic) and octahedral. Optic axis in plane of film.

Melting point, 119°·5 C.–120°·1 C.

Specific inductive capacity at 18°·8 C., corrected in terms of Dr. MUIRHEAD'S standards.

By ballistic method, 3·697.

By instantaneous DE SAUTY *method*, 3·708. The latter value to be preferred ; capacity constant up to 100° C., indicating small temperature increase of specific inductive capacity.

These values were obtained with a voltage of 40 Clark cells.

Residual charge with 300 volts, after charging one minute and discharging one minute, ·03 per cent., say.

Specific resistance at 20° C. by electrometer method, six Clark's, say 1×10^{28} C.G.S. Possibly entirely surface action. Exposed to damp air, resistance diminished in our apparatus 1000 fold.

Specific resistance by galvanometer method with a voltage of 291 volts greater than $1·4 \times 10^{27}$ C.G.S. from 18° to 50° C. At a temperature of 75° C. the specific resistance, after the current was on for, say, ten minutes, is $6 \cdot 8 \times 10^{25}$ C. It was probably a case of *discontinuous* conduction, and the resistance increased about four-fold during an hour's application of the voltage. The voltage was not reversed on the sulphur during these experiments, as such a course complicates matters enormously.

Film of October 28, 1892.

As the experiments just described broke down at a temperature of between 70° and 100° C., it was necessary to make them over again, in order to find how soluble sulphur behaves when heated up to the melting-point. For this purpose the same sulphur was re-melted at a maximum temperature of 140° C., and as there was, in our experience, absolutely no probability of our being able to anneal the film and then re-heat it to the melting-point without disturbing its contact with the plates, we decided to measure its resistance as the temperature fell. The film was prepared on October 28 ; it was annealed during the afternoon, and the following observations taken during the night when tram traffic was suspended and external disturbance less to be feared. The resistances were such that the old galvanometer was most conveniently employed; it was read by a lamp and scale. The temperature fell from 140° C. to 134°·5 in forty minutes. The resistance was so low that only one Clark cell (the large master cell) was used. During an hour the temperature was kept between 134°·5 and 137°·5, and the resistance gradually increased, so that the deflexion fell off from 396 to 354 divisions in half-an-hour. On reversing the battery at a rather lower temperature, 132°·85, the deflexion at once increased to 548, and in the course of about twenty minutes fell to 423. These results are typical. The film probably contained some insoluble sulphur at this temperature ; when it was examined afterwards, several analyses showed a mean content of ·35 per cent. of insoluble sulphur and a melting-point of from 119°·06 to 119°·66 C., but this is no criterion of the actual composition of the film at this stage. The corrected area was 156·445 square centims., and the thickness was ·026188 centim. We will not trouble the reader with such a full account as in the previous case, for the procedure was exactly the same ; the table will, therefore, be rather fuller. At a mean temperature of 135°·6 the mean specific resistance with the battery on both ways was $1 \cdot 5 \times 10^{21}$ C.G.S., which is not much less than the resistance after an infinitely long application of voltage. The capacity at this temperature could only be taken very approximately, owing to the leak ; the result was a specific inductive capacity of 5·7. We believe this is not far from the truth, but lay no stress on it.

We will give the full set of observations of resistance at the next temperature examined, 124°·4 C., partly to illustrate the effect of time and reversal, and partly to enable anyone to form an opinion as to the actual effects obtained. The differences are really greater than they appear, for the ratio of two successive swings with this galvanometer was 2·507, as deduced from twelve experiments with elongations of the same amplitude as occur in the table. The corresponding logarithmic decrement is about ·4. The galvanometer sensitiveness is $1 \cdot 366 \times 10^{-11}$ ampère per scale division (ELLIOTTS, at about 1 metre) of double elongation. We have to thank Mr. POLLOCK for assistance in making all the observations with this film.

Time, 8 hours 55 minutes. Charged for three minutes, + terminal of battery to the upper plate. Voltage 1 Clark.

Temperature	123·5° C.	Galvanometer	reversed	A–B.	D. deflection	338
,,	,,	,,	,,	B–A.	,,	330
,,	,,	,,	,,	A–B.	,,	318
,,	,,	,,	,,	B–A.	,,	318
,,	125° C.	,,	,,	A–B.	,,	306
,,	126° C.	,,	,,	B–A.	,,	310

Time, 9 hours 14 minutes.
Battery reversed.

Temperature	126° C.			Galvanometer reversed	A–B.	D. deflection	414
,,	,,		,,	,,	B–A.	,,	392
,,	126° C.	Time 9 hrs.	20 mins.	Galv. rev.	A–B.	,,	392
,,	,,	,,	,,	,,	B–A.	,,	372
,,	,,	,,	,,	,,	A–B.	,,	363
,,	125°	9	25 ,,	,,	B–A.	,,	348

Mean of all deflections 350·1.

Mean Specific Resistance $1 \cdot 8 \times 10^{21}$ at say 124°·5.

The capacity could not be taken.

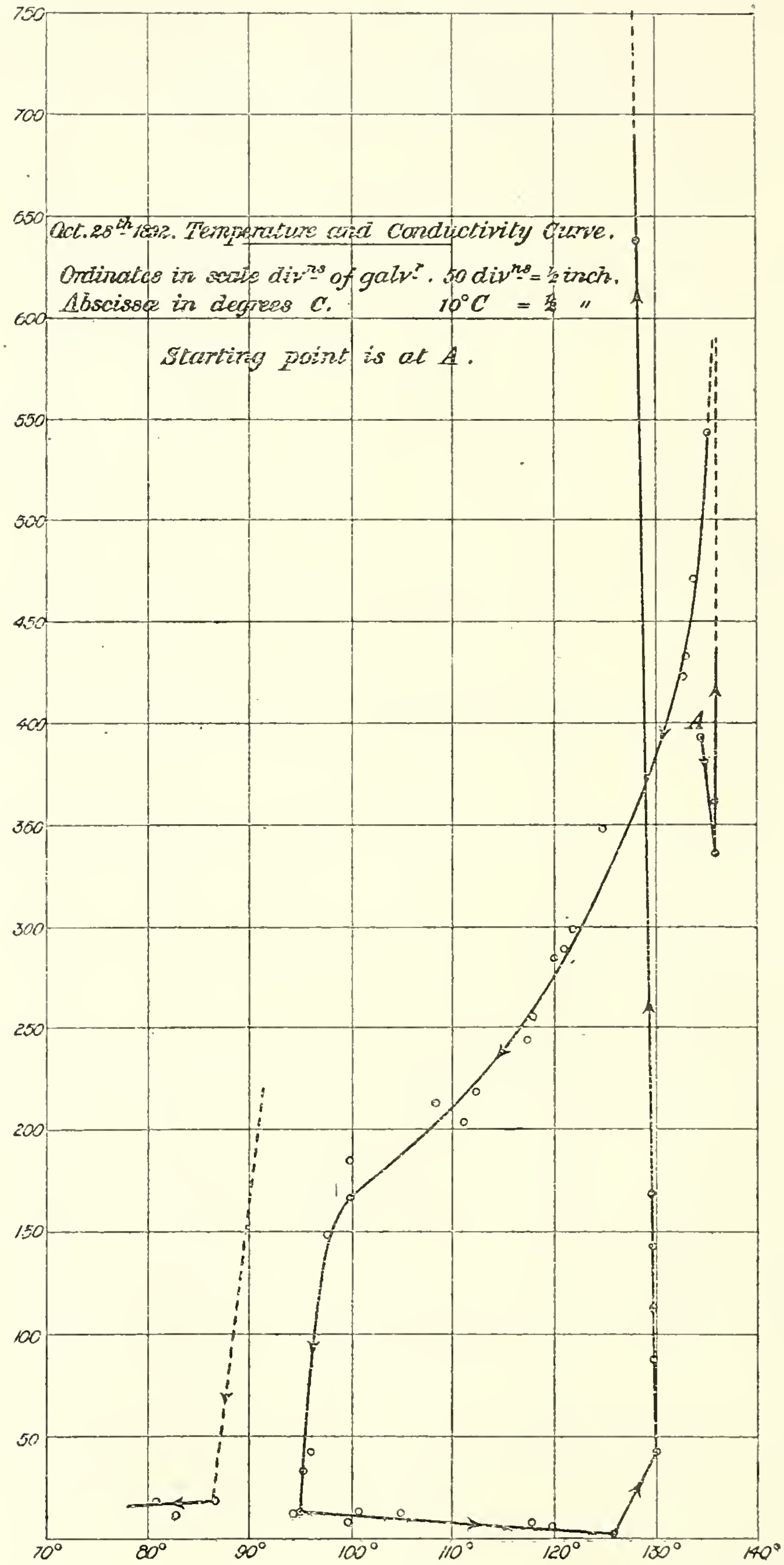
An attempt was made to detect residual polarization by the galvanometer—but without effect—the voltage to be expected being so small as to compare with the thermoelectric effects unavoidable with one part of the circuit so much hotter than the rest. We shall return to this. Observations showing an increasing resistance were made at various points down to 97° C., at which point the sulphur evidently began to crystallize, for the resistance increased so rapidly that we observed best by noting the rate at which the light spot moved over the scale towards the zero. The resistance increased meanwhile about twelvefold. Nothing is to be got by noting the rate of change of resistance with change of temperature, unless we had actually had a rapid thermometer between the plates. Our thermometer was merely in the inner box with the sulphur, and did not indicate the temperature-variation of the sulphur to be expected while it was freezing. The summary of the changes that occur is as follows (see table for numerical details).

(1.) At 100° the resistance is considerably increased.

(2.) Between 100° and 96° the rapid change begins.

(3.) Between 96° and 93° the velocity of the change increases.

Fig. 3.



When the deflection had fallen to 12 divisions, it remained fairly constant. These results probably chiefly refer to the time of crystallization of the sulphur, which, as everyone knows (Gernez, *loc. cit.*), is a process requiring a good deal of time. The sulphur was again heated and the resistance diminished steadily, till at 129·5 the spot began to move at a rate of 10 divisions per second, and went on faster and faster, till after three minutes it had gone off the scale. On cooling, the resistance was undoubtedly permanently reduced—say tenfold; it increased, as before, down to a temperature of 90° C., when the sulphur evidently crystallized and the resistance suddenly increased. A curve has been drawn (fig. 3) showing the variation of resistance with temperature, the observations being for the most part made with the voltage each way; at the points of rapid change this was not possible. The hysteresis of the diagram is partly due to the time required during crystallization. The rate of variation of temperature at important points was never more than one degree in five minutes. The permanent reduction of specific resistance by heating was observed again with another film at a later date, and appears to be a natural phenomenon.

TABLE IX.—Showing Resistance of the Film of October 28th, 1892, at Different Temperatures.

Temperatures.	Specific resistance in C.G.S. units.
135·6° C. falling	$1·5 \times 10^{21}$
124·4 " "	$1·8 \times 10^{21}$
108·5 " "	$2·9 \times 10^{21}$
100 " "	$3·733 \times 10^{21}$
96 " "	$3·3 \times 10^{22}$
95 " "	$3·5 \times 10^{22}$
101·5 " rising	$3·9 \times 10^{22}$
Above this the resistance fell as shown on the curve till about 127°, when the sulphur melted. The two following relations are taken from the curve:—	
At 128° C. rising	$9·9 \times 10^{20}$ (uncertain) change very rapid
On cooling again at 83° C. .	3×10^{22}

* We provisionally attribute the fall of resistance to a change in the amount of insoluble sulphur present, probably, almost certainly, an increase. It may be objected that we ought to have melted our film at a lower temperature, and thus prevented the formation of insoluble sulphur at all. The reply to this is, that unhappily one cannot adjust the plates properly unless one has some time for the operation, and the cooling surface of the plates is so great that sufficient time is not afforded if the sulphur is nearer to its crystallizing point to begin with.

Film of November 9th, 1892.

It was now necessary to examine the behaviour of a film rapidly cooled so as to form a mixture of soluble and insoluble sulphur. Our first attempt failed, owing to

* Further research shows that this provisional conclusion must be modified.

water getting on the film, the second attempt succeeded The sulphur was heated to a temperature of about to 250° C., and kept there for fifteen minutes ; the cooling was effected as rapidly as possible by water, consistently with not allowing any water to touch the sulphur. Some portions of the sulphur were cooled more rapidly than others, and consequently the percentage of insoluble sulphur present varied in samples taken from different parts of the dish. The film itself probably contained about 5·2 per cent. of insoluble sulphur, but this estimate is based on analysis of sulphur cut away from the sides of the plates close to the film. The film itself was heated in the course of the experimental work, which rendered any conclusion from the analysis in its final state inapplicable.

The melting-point of the sulphur, after it had been between the plates as the subject of experiment, was from 119·31° to 119·86° C.

The area of the film (corrected for bubbles, &c.) was 155·845 sq. centims. ; the thickness of the film, by spherometer, was ·02718 centim.

As the specific inductive capacity appeared very different in this case from the values found previously for soluble sulphur, and as the thickness of the film is the most critical measurement in determining the value, we will give the means of four groups of ten settings, each on pieces taken from different parts of the film from which the thickness was deduced. They are

Setting.	Thickness in inches.
(1) ·336764	·009947
(2) ·334774	·011937
(3) ·336211	·010500
(4) ·336190	·010521

Mean, ·335985 inch.

Ten settings on glass. Mean, ·346711.

Difference, ·010726 inch = ·02718 centim.

We consider that the greatest possible error of any group of settings cannot amount to more than 4 per cent., or the film is clearly not quite regular.

A measurement made by focussing with a Zeiss microscope gave about ·03 centim., but it is only of value as a rough check. It will be safe to consider the mean thickness as correct, within, say, 10 per cent., at least.

The sulphur (as soon as the plates were sufficiently cooled by placing the outer dish in water and putting wet filter paper on the upper plate) was set up in the oil bath with drying material. A rough test at a temperature of 22·8° C., voltage 265·3, made with the old galvanometer, gave—

Specific resistance, $4{\cdot}2 \times 10^{24}$ C.G.S.

No marked difference was observed between the conductivity when the battery was first on, and five minutes afterwards, but the resistance slowly increased as usual.

The capacity was measured by the De Sauty method, using the small storage cells with a voltage of 265·3 volts. The instantaneous balance was reached with between 1070 and 1072 out in one arm, and 100,000 out in the other. The resulting capacity was $\cdot 002142 \times 10^{-15}$ C.G.S. Applying a correction of -2 per cent. for the key and $+1$ per cent. for the value of the standard, or, in all, -1 per cent., this becomes ·00212, say. The corrected value of the specific inductive capacity is 4·1814. We can only explain this large value by supposing that the film measurement is at fault, or that the specific inductive capacity of mixed sulphur is really high. The film measurements are given in order to allow the reader to form his own opinion on this point.

The sulphur was kept in the drying box for 24 hours at a temperature of about 22° C. The specific resistance $= 5 \cdot 05 \times 10^{24}$ C.G.S.

So it has not changed much by continued drying; the battery was not reversed, and the mean deflection was 57 divisions. After the battery had been applied for about one hour the double deflection fell to 17 divisions. The voltage was 264 volts. The current having become fairly constant, it was thought advisable to investigate whether the deflection was due to surface conductivity. From previous experiments it is known that the sulphur surface conductivity, after a day's drying, will not make the apparent specific resistance less than of the order of 10^{27} C.G.S., but this was with highly soluble sulphur. It is not likely that the admixture of insoluble sulphur will increase the surface conductivity, but it is possible that it may do so, and in that case we should expect that exposure to damp air would reduce the apparent resistance. The lid was therefore taken off the thermostat, and the plates freely exposed to the air for ten minutes. Dry bulb, 72° F.; wet bulb, 65·5° F. No change having been detected, a current of air which had been passed through a reversed wash-bottle with the water at 50° C., was allowed to play for about five minutes on the sulphur between the upper plate and the dish. The deflexion or reversal of the galvanometer remained precisely the same as before. The conclusion is that the conductivity is probably almost wholly due to the sulphur, and that the latter is not affected by damp air to anything like the same extent as glass is. This is confirmed by some observations to be described, in which, when the sulphur plates are heated in a closed box with drying material to 50° C., the conductivity is increased, whereas, if it were due to surface action, the opposite effect would be expected. The sensitiveness of the galvanometer (new), when these experiments were made, was about $5 \cdot 2 \times 10^{-12}$ ampère per micrometer scale division, and the period was eleven seconds. This (November 10, 1892) was the steadiest day we ever had, the galvanometer could have been read to ·2 division perfectly, had it been necessary. The next day was almost equally steady, and we had a good example of the effect of variable conductivity, which was further shown by the galvanometer needle kicking slightly from time to time. The following table will indicate the effect we mean; we often observed it, but on this occasion the magnetic conditions were so steady that we are able to be quite certain that the effect is not to be explained by accidental irregularity. When taking the sensitiveness of the instrument, within a few minutes, not a trace of such an effect was to be observed, and the insulation of the wire, &c., was found to be perfect, as far as our methods would show.

TABLE showing Irregular Conductivity of Sulphur Film containing a Mixture of Soluble and Insoluble Sulphur.

Date, Nov. 11, 1892. *Voltage*, 268·4.

Sensitiveness, 5·43 × 10^{12} ampères.

Temperature of air, 19°·9 C.,—*of sulphur*, 20°·3 C.

After the voltage has been applied long enough for the sulphur to take up a steady state (say 20 minutes)

A–B,	double	elongation,	6	divisions.
B–A,	,,	,,	18	,,
A–B,	,,	,,	6	,,
B–A,	,,	,,	6	,,
A–B,	,,	,,	6	,,
B–A,	,,	,,	16	,,
A–B,	,,	,,	33	,,
B–A,	,,	,,	29	,,

The capacity was found to be exactly the same as before, and the plates regarded as a condenser gave a bright spark on discharge.

This irregular conduction (though not always so well-marked) was generally observed, and may be considered as a natural phenomenon.

The sulphur was then tested for residual charge in the manner formerly described. Sensitiveness of electrometer was 41 divisions per Clark cell: the insulation of all the apparatus was so good that no corrections had to be applied for leakage. After charging for five minutes with a voltage of 238·6 volts, the residual charge gave a deflection of 9·5 divisions in one minute, and 11 divisions after two or three minutes more. If we take 10 divisions as a basis of calculation, the residual charge is ·147 per cent. of the initial charge, or between three and four times as much as we got before. But the conductivity is, say, a thousand times greater, which seems to show that there is no accounting for the residual effect by considering the conductivity alone. If we adopt MAXWELL'S calculation as a basis, then the variation $\frac{\text{Specific Resistance}}{\text{Specific Inductive Resistance}}$ from layer to layer is greater in the case under consideration than in that formerly dealt with. The conductivity of the film was not great enough as a whole to account for any serious diminution of the residual charge, as was ascertained by observing how the residual charge was held. The voltage of the residual charge was only about ·3 volt in this case, and less in the former case—so that it is conceivably a case of galvanic polarization only. Against this, it may be urged that the residual charge increases with the time—strong evidence that the polarization is dielectric, not galvanic. In order further to test this conclusion 103 volts, instead of 238·6, was applied for the same time (five minutes), and the residual charge, after one minute, was found to be ·187 per cent. of the initial charge. A good many very concordant observations were used in obtaining these values, and there is no doubt about the residue being larger in proportion with the smaller voltage. This result is not in accordance with the results obtained by BOUTY for mica condensers, which was, at all events, partly owing to the charge creeping over the mica edges. In our case, we cannot assert that the small residual charge observed by us was not due to the charge creeping over the sulphur surface. Since the residual is so small that we are unable to discover its origin by experiment, there is not much to be gained by speculation. We may, however, say that this film was probably a great deal more wanting in homogeneity than the perfectly soluble film, and that we expected a much larger residual charge in consequence. That such was not obtained, may either indicate that the possible explanation suggested by MAXWELL does not apply in this case, and that we are actually in presence of the new kind of polarization he suggests, or that though the want of homogeneity is considerable, the variation of the ratio of specific inductive capacity to specific resistance, from point to point, is only small.

As there is every reason to expect that the constitution of a mixed film will continually change, owing to instability of the insoluble sulphur and the gradual tendency of monoclinic to pass into octahedral sulphur, the capacity was taken again in order to detect any change that might have taken place. This was seven days after the film was first made.

The capacity of the condenser was found to have changed somewhat, more than the change of temperature would be likely to account for, temperature 21°·5 for both condensers, corresponding to a change of only ·3° C. The capacity now was such that the balance changed from 1071 to between 1089 and 1090 with 40 Clark cells. This leads to a value of K of 4·247 as against 4·1814 obtained before. There is no doubt about the film having undergone some change. The resistance was, therefore, measured (battery both ways) and found to be $5·018 \times 10^{25}$ C.G.S., with a voltage of 262·8 volts and a galvanometer sensitiveness of $4·9 \times 10^{-12}$ ampère per micrometer division. The resistance has, therefore, increased also. There is no doubt that a real change has taken place, for the increase of specific inductive capacity precludes the possibility of the plates having separated. We attribute this to the conversion of the prismatic sulphur into octahedral, the only change to be expected, so that the specific inductive capacity is higher for the latter than the former. The film was then heated and observations made at from 60° to 64° C. The resistance was found to have diminished, the conduction was unsteady and the usual effects were observed. With battery both ways after the steady state was reached the conduction was found to have increased between 60° and 64°·8 in the ratio of 44·2 to 31·2 (the respective elongations deduced from 24 observations). The value of the resistance at 64°·8 C. is $6\cdot9 \times 10^{24}$ C.G.S. The value of the specific inductive capacity at 67° C. was apparently 4·25, and at 98·5 could not be distinguished from this; the temperature of the standard did not change meanwhile. At 100° C. the resistance was much decreased and the conduction unsteady, the effect on reversal was not so strongly marked. Thus:

Temperature, 100°·5 C.

Battery A.

Double elongation	472
Ditto, reversed	465

Battery B.

Double elongation (at once)	444
Ditto, reversed	465

The results of these experiments are collected in the following tables:—

TABLE X.—Showing Variation of Specific Inductive Capacity of Film of November 9, 1892.

Date.	K.	Temperature of sulphur.	Temperature of standard and resistances.	Voltage.
1892.		° C.	° C.	
Nov. 9	4·18	22·8	22·8	40 Clark's
„ 11	4·18	19·9	20·3	„
„ 16	4·247	21·5	21·5	„
„ 17	4·247	67·5	21·4	40 Clark's and 262 volts.
„ 17	4·247	98·5	21·15	„ „ „

In the above table no corrections are applied for increase in the linear dimensions of the electrodes. If this be done it will be found that the constancy of the capacity as the temperature rises indicates a small positive temperature coefficient in the specific inductive capacity which may, however, be due to a partial annealing. This is dealt with later when we had command of a better standard.

TABLE XI.—Resistance of Film of Mixed Sulphur of November 9th, 1892.

Date.	Voltage.	Resistance.	Specific resistance.	Temperature of sulphur.	Temperature of air.	Remarks.
1892.	volts.	ohms.	C.G.S.	° C.	° C.	
Nov. 9	265·3	$7·358 \times 10^{11}$	$4·178 \times 10^{24}$	22·8	22·8	Directly after cooling
„ 9	„	Half of above	Half of above	„	„	In fifteen minutes
„ 10	264·1	$8·9 \times 10^{11}$	$5·5 \times 10^{24}$	22·6	„	After voltage had been applied for half-an-hour. The resistance was not affected by blowing in damp air
„ 11	268·4	$3·296 \times 10^{12}$	$1·87 \times 10^{25}$	19·9	20·3	Discontinuous conduction. Battery only one way
„ 17	262·8	$8·75 \times 10^{12}$	$5·02 \times 10^{25}$	21·4	20·5	Battery both ways
„ 17	„	$1·2115 \times 10^{12}$	$6·88 \times 10^{24}$	64·8	„	Ten observations. Battery both ways
„ 17	„	$4·945 \times 10^{10}$	$2·836 \times 10^{23}$	100·5	„	Battery both ways

The resistance results may be summed up as follows :—

(1.) The mixed sulphur film is at least 1000 times more conducting than the purely crystalline film.

(2.) The resistance changes from day to day, independently of the surface action, and is always less (*a*) when the voltage is first applied, (*b*) when it is reversed (at ordinary temperature).

(3.) The resistance is discontinuous, particularly when the battery is first applied.

(4.) The resistance decreases nearly two hundredfold between 28° and 100° C.

(5.) Dampness cannot increase the surface action beyond a certain point; thus with our arrangement it could not bring the apparent specific resistance down below 10^{23} C.G.S.

Film of November 18th, 1892.

An attempt was made to increase the amount of insoluble sulphur present by cooling the plates more rapidly. With this object in view the sulphur and plates were heated up to 260° C., and kept at that temperature for half-an-hour. During the heating (which was carried out in an improvised air bath) a stream of pure carbon dioxide, dried by calcium chloride only, was allowed to flow into the air bath. In spite of this some of the sulphur caught fire. The plates were cooled by placing the dish on a block of ice, and covering the upper plate with a zinc tray filled with ice. By this means the metal plates, and with them the sulphur, were reduced to a temperature of nearly 0° C. in less than five minutes. The cooling at the commencement was very rapid, and no water got on to the film or into any part of the apparatus. Several analyses were made of the material from the dish, but they varied between 5 per cent. and 7 per cent. only; rather to our surprise. We believe, however, that the film was more quickly cooled than the sulphur round its edges, in which case the film probably contained upwards of 7 per cent. of insoluble sulphur. The colour of the sulphur in the dish was anything but satisfactory—it was appreciably darker than it ought to have been—especially immediately after cooling. Some samples were cut out with difficulty from round the plates, and on evaporation from platinum left an appreciable stain.

It was thought that possibly the carbon dioxide atmosphere in which the sulphur was heated had led to contamination under the action of the slightly burning sulphur. A number of experiments were therefore made by setting fire to sulphur in partial carbonic acid atmospheres, but no trace of contamination could thus be brought about. We consider the dark colour as probably due to the action of a certain amount of dust which must have collected on the sulphur during the manipulations—some of the same sulphur having been used in a previous experiment.

The results obtained with this film were similar to those with the last film, but with the peculiarities exaggerated. The plates were ultimately caused to come apart by raising the temperature, and it was then found that though the film was not as strongly coloured as the sulphur round the edges, it was distinctly darker than it ought to have been. It was much broken up by air bubbles, and not at all regular in thickness. The upper plate was slightly displaced. The mean corrected area may be taken at 155 sq. centims. The mean thickness at ·021405 centim.*

The melting-point of the sulphur was taken as usual in several experiments in which the temperature of the bath before the sulphur was introduced was gradually raised to within a fraction of a degree of the melting-point, so as to avoid annealing. The constancy of the results showed that this was satisfactorily attained. The sulphur used to find the melting-point was cut from the sides of the plates as close as possible to them. The temperature at which melting commenced was 116·86 ; it was complete at 118·193. The initial temperature of the bath varied from 111·46° to 117·26°. The capacity taken when the film was first dried in the thermostat, gave a corrected value for the specific inductive capacity of the sulphur of about 4·404. The resistance was tested after drying with phosphorus pentoxide for 20 minutes, and after charging with 258 volts for five minutes. The deflection of the galvanometer was about 75 divisions; on reversing the battery and placing the galvanometer in circuit, after about two minutes, the deflection on reversal was too great to observe, but was estimated at about 270 divisions. After keeping the battery on this way for 15 minutes, and again reversing the galvanometer, the deflection amounted to 126 divisions. After

* A subsequent examination of the measurements showed great differences between the thickness in different parts of the film, and the plates were afterwards discovered to have lost their perfect flatness. The absolute values, both of the specific resistance and specific inductive capacity of this film, cannot, therefore, be completely relied upon. It is probable that the value of K is too small, and ρ too large, from the uncertainty mentioned.

10 minutes the battery was again reversed, and the deflection could now be read. Successive elongations on reversal of the galvanometer were

252, 153, 153, 135, 108, 135,

and as before the conduction was discontinuous, the galvanometer needle being thrown violently in the direction of increased conduction every few minutes, and then coming quickly back. The mean effect both ways reduced to a specific resistance of 2×10^{24}. The film was then left for six days in the box with the phosphorus pentoxide, so as to get thoroughly dry, in case it was not so when the above results were obtained.

The effect of time on the resistance was noted as follows :—

Temperature of sulphur, 23·8° C. *Voltage*, 312 volts.

On first applying the voltage + of battery to insulated plate the deflection corresponded to a resistance of 2×10^{12} ohms.

After three minutes this increased to $2{\cdot}6 \times 10^{12}$ ohms ;

After twenty-five minutes to 3×10^{12} ohms.

The battery was then reversed, the initial effect died out in about three minutes, and during ten minutes' observations were made so as to fix a mean value for the resistance, with the previous observations.

Up to three minutes the resistance was $2{\cdot}8 \times 10^{12}$ ohms.

From three to ten minutes mean resistance was $4{\cdot}41 \times 10^{12}$ ohms.

It is not easy to get very accurate values, because the change is always going on, and the discontinuous conduction makes it necessary to have several observations at each point.

To test whether OHM's law was obeyed, we applied a voltage of 155 volts, and took the resistance both ways, up to three minutes, reversing several times first. The mean value was $R = 4{\cdot}69 \times 10^{12}$, instead of $2{\cdot}6 \times 10^{12}$, when we used the higher voltage. This result was probably affected both by the fact that the large voltage was applied first, and by the influence of the time action, though the latter was eliminated so far as possible. We do not think that the deviation from OHM's law observed can be explained away by any of these effects. A substance, whose conductivity may be a function of the quantity of electricity which has passed through it, can hardly be expected to obey OHM's law.

On November 29, after the sulphur had been allowed to rest, the resistance was found, by reversing the battery (312 volts), and allowing the voltage to remain on for three minutes each way. The two sets of deflections were nearly equal, showing that a steady state had been reached. The resulting value was

$$\text{Resistance} = 3{\cdot}6 \times 10^{12} \text{ ohms.}$$

After one hour and twenty minutes this increased (with battery on one way only) to

$$R. = 1{\cdot}06 \times 10^{13}.$$

Half the battery (158 volts) was then applied the same way, and the resulting value was

$$R. = 1{\cdot}5 \times 10^{13},$$

or OHM's law is not obeyed, though the steady state has been reached.

The battery having been on one way for about two hours, it was thought a good opportunity for us to measure the effect of reversal, but, on this being done, the conduction was too great to be observed for several minutes ; after, say, five minutes, it was still six times greater than before the battery was reversed. The full voltage was employed, and the resistance went on increasing, the current falling towards its former value for ten minutes, when observation had to be suspended.

The effect of keeping the sulphur exposed to the action of the battery seemed so definite, and at the same time so curious, that it was thought worth while repeating the observations. This was done on December 1, 1892, at a temperature of 23·6° C., and a voltage of 288·6 volts. After a preliminary test, to see that all was right, the positive end of the battery was put to the top sulphur plate and left, thus connected for one hour; there was then a mean throw on reversal through the galvanometer of 28 divisions ; the current had become practically constant ; the observations were :—

Throw	A — B	30 divisions
,,	B — A	27 ,,
,,	A — B	30 ,,
,,	B — A	25 ,,

The telegraph pole in circuit only gave an elongation of 2 divisions. The mean elongation due to current through the sulphur is therefore 26 divisions. On reversing the battery, and observing, as quickly as possible—within about two minutes—the first two elongations were in the mean 92 divisions ; after five minutes (about) this fell to 71 divisions ; after thirty minutes to 27 divisions. On being again reversed, the first elongations were 81 divisions ; after six minutes 75 ; and after thirty minutes 26 divisions. On again reversing the first effect was only 43 divisions, so that the sulphur behaves as if it got fatigued, and refused after a time to show the effect. If we suppose that the conduction is the expression of the breaking-up and re-arrangement of molecular groupings, such change being from more complex to more simple, this is the effect we should expect when the groups capable of giving way under the applied electric stress became weeded out.

In these experiments of December 1 the smallest specific resistance observed was $2{\cdot}1075 \times 10^{25}$ C.G.S., and the greatest $6{\cdot}079 \times 10^{25}$.

The galvanometer sensitiveness was $1{\cdot}32 \times 10^{-12}$ ampère per micrometer division of double elongation.

Effect of rise of Temperature, December 2nd, 1892.—Temperature of sulphur, 21·1° C. Normal resistance (*i.e.*, after three minutes charging each way) was

$4 \cdot 942 \times 10^{12}$ ohms, and the corresponding specific resistance was $3 \cdot 578 \times 10^{25}$ C.G.S. Voltage, 305 volts. The temperature was raised slowly to 96° C. and kept constant at this temperature. The resistance decreased very rapidly, and was observed to be $R = 3 \cdot 198 \times 10^{10}$ ($\rho = 2 \cdot 315 \times 10^{23}$) before the spot went off the scale and the voltage had to be reduced. This is a reduction of, say, one hundred and sixty fold. The behaviour of the film was now curious. The temperature being allowed to rise from 96° C. to 100° C. in thirty minutes, the conductivity went on increasing so fast that the battery voltage required to be continually reduced at 100°, and after thirty minutes the conductivity had increased to one hundred times (say) its value at 96°. This increase of conductivity did not take place gradually, but by jumps—the galvanometer needle was thrown further and further out by sharp jerks which continually tailed off only to recommence. This effect went on till the temperature of 102° C. was reached, in fifteen minutes, after this the temperature was allowed to fall in order to discover whether the change in the sulphur would reverse. This it did immediately—the light spot coming back as markedly as it had previously moved off. An attempt was then made to reverse the change again, and the temperature was slowly raised to 105°, but the resistance still went on increasing. This suggested that the plates were coming apart, and on examination when all was cold this was found to be the case, for, though no sign of cracking was to be seen, the capacity was found to vary when a tinfoil coated jar placed on the upper plate was filled with about 50 lbs. of mercury. The plates were therefore dismounted, and the upper one was found to be only held in position by the sulphur round its edges. We were pleased to find the capacity test so sensitive and positive. We looked carefully too see if there was any indication of a spark having passed, but could find no reason for supposing that this had taken place ; we do not think we could have missed it if such had been the case.

The following table will sum up the resistance results.

TABLE XII.—Showing Resistance of a Film containing 7 per cent. (?) of Insoluble Sulphur.

Date.	Temperature of sulphur.	Temperature of air.	Voltage in volts.	Specific resistance.	Remarks.
1892.	° C.	° C.			
Nov. 18	22·8	22·8	258	$1·905 \times 10^{24}$	Taken immediately film was supposed to be dry
,, 25	22·7	23·8	312	$1·47 \times 10^{25}$	First application of voltage. Battery not reversed
,, 25	,,	,,	,,	$1·85 \times 10^{25}$	After about 3 minutes. Battery not reversed
,, 25	,,	,,	,,	$2·18 \times 10^{25}$	After 25 minutes. Battery not reversed
,, 25	,,	,,	,,	$2·05 \times 10^{25}$	Mean resistance battery both ways—after 3 minutes
,, 25	,,	,,	155	$3·4 \times 10^{25}$	Battery both ways—after say 10 minutes
,, 29	23·5	24·5	311·7	$2·61 \times 10^{25}$	Mean after 4 minutes. Battery both ways
,, 29	,,	,,	,,	$7·7 \times 10^{25}$	After charging one way for 1 hour 20 minutes
,, 29	,,	,,	154	$1·068 \times 10^{26}$	After charging one way to test for Ohm's law in steady state
Dec. 1	23·6	22	288·6	$2·1 \times 10^{25}$	Least resistance obtainable on reversal after 30 minutes charge
,, 1	,,	,,	,,	$6·079 \times 10^{25}$	Greatest resistance obtainable, battery one way for 30 minute charge
,, 2	21·1	20·6	304·9	$3·578 \times 10^{25}$	Mean resistance both ways after 3 minutes
,, 2	97	,,	,,	$2·3 \times 10^{23}$	After very slow heating—battery only applied at this temperature
,, 2	100	,,	..	..	Resistance decreasing so fast as to be immeasurable, at most specific resistance about 2×10^{21}

Film of April 17th, 1893.

It has been already stated that some uncertainty attached to the constitution of the film just considered, and this was increased when, on examining the electrodes by the true plane, it was found that one of them had not preserved its perfect flatness. Another circumstance conspired to induce us to re-examine mixed films. For this purpose the plates were again adjusted for flatness, and a proper zinc dish constructed to carry ice on the upper plate. In order to obtain further security against surface action we reverted to our old plan of erecting a quartz rod fence along the sulphur surface between the dish and the upper plate. The sulphur employed was some of the Chance sulphur given to us by Mr. CHANCE. This was melted and strained* to get rid of particles of dirt, and was then twice distilled.

With this sulphur we made a film on April 17th, 1893, and put it in the box while still warm with phosphorus pentoxide and sulphuric acid.

The data are—corrected area, 156·64 sq. centims. (including edge correction); thickness, ·022686 centim.

It consisted of soluble prismatic sulphur (probably without admixture of octahedral sulphur), with 5 per cent. of insoluble sulphur. Colour, clear lemon-yellow. When the plates were separated the greater part of the film was still transparent, but it became opaque on rubbing or attempting to detach it from the plates. We think it probable that since the film was much broken and torn up in forcing the plates apart, that it was in the transparent form during the experiments, at all events before it was heated. In this case, it must be considered to have consisted entirely of monoclinic and amorphous sulphur.

An elaborate investigation of the properties was made but, as nothing new turned up, the table will afford sufficient information. The resistance is uniformly higher than in the case of the film of similar composition of November 9th, 1892. We attribute this to the fact that, having now a great deal more material, we probably succeeded with the purification rather better than before. Of course the test of burning two hundred grammes weight was carried out, and resulted in such an almost inappreciable residue that we must have succeeded better than was to be expected in keeping the dust off during the burning, or rather boiling.

* Sulphur is conveniently filtered by means of glass wool in a zinc funnel. A bit of platinum wire gauze, folded several times, should be placed below the funnel to catch any shreds of glass that may come through. This precaution is very necessary. Chance sulphur treated like this has a horrible smell of gas-lime when it is broken up after cooling, which shows that it requires to be distilled if sure results are to be obtained.

TABLE XIII.—Film of April 18, 1893.

The Film was composed of a mixture of 95 per cent. soluble prismatic sulphur with 5 per cent. Insoluble Amorphous Sulphur. Area of film, 156·64 sq. centims. Thickness of film, ·02268 centim.

Date.	Voltage.	Galvanometer sensitiveness. In ampères per scale division of a double deflexion.	Direction of voltage.	Duration of voltage.	Resistance.	Specific resistance.	Temperature in centigrade.	Double deflexion.	Remarks.
1893. Apr. 19	280	$4·95 \times 10^{-12}$	+ to top	23 hours	Greater than $1·12 \times 10^{23}$	Greater than $7·7 \times 10^{26}$	19°·5	·5	The quartz-fence was in position. The plates had been drying for 24 hours by P_2O_5
" 19	"	"	− "	5 minutes	7×10^{21} C.G.S.	$4·8 \times 10^{25}$	"	4	Taken as soon after reversal as convenient
" 24	"	"	+ and −	3 "	$5·6 \times 10^{22}$	$3·9 \times 10^{26}$	18·5	1	After drying for four days. Not changed by removing quartz, exposing to air for 10 minutes, and breathing on. Wet bulb thermometer stood at 60° F.; dry bulb thermometer at 65° F.
" 24	6 Clark's	By electrometer method for leakage in 30 minutes			$7·36 \times 10^{22}$	$5·08 \times 10^{26}$	"	..	The quartz was not replaced. This is to be compared with the other experiment of the same date
" 26	280	$4·95 \times 10^{-12}$	+ and −	5 minutes	$5·6 \times 10^{22}$	$3·9 \times 10^{26}$	31	1	Conduction steady up to 33° C. Quartz not replaced
" 26	"	"	"	"	"	"	45	"	Conductivity remains about the same up about 45°
" 26	"	"	"	"	$1·9 \times 10^{22}$	$1·3 \times 10^{26}$	53	3	Considerable discontinuous conduction. Influence of time well marked
" 26	"	"	"	"	$1·1 \times 10^{22}$	$7·8 \times 10^{25}$	63	5	Ditto ditto
" 26	"	"	"	"	$5·6 \times 10^{21}$	$3·9 \times 10^{25}$	68	10	Ditto ditto
" 26	"	"	"	"	$1·1 \times 10^{21}$	$7·8 \times 10^{24}$	71	About 50	Considerable conduction increases rapidly. Influence of time well marked
" 26	"	"	"	"	7×10^{20}	5×10^{24}	80	" 80	Quartz replaced. Has no effect. Conduction increases rapidly

TABLE XIV.—Film of April 18, 1893 (continued).

Date.	Voltage.	Galvanometer sensitiveness. In ampères per scale division of a double deflexion.	Direction of voltage.	Duration of voltage.	Resistance.	Specific resistance.	Temperature in centigrade.	Double deflexion.	Remarks.
1893. Apr. 26	280	$4·95 \times 10^{-12}$	+ and —	5 minutes	$5·6 \times 10^{20}$	$3·9 \times 10^{24}$	81°	— 100	See special table for details of the rapid rise in conduction and the lag of resistance on cooling
„ 27	315	„	„	„	Say $7·7 \times 10^{21}$	Say $5·3 \times 10^{25}$	18·8	Say 4	So that in spite of the quartz the sulphur has had its conductivity slightly increased by the heating. See tables

DETERMINATION of K.

Comparison of Sulphur Condenser with ·1 microfarad division of a mica standard. All corrections made.

Date.	Voltage.	Galvanometer sensitiveness.	Resistance out in one arm.	Resistance out in other arm.	Capacity.	Temperature.	K.	Temperature coefficient.	Remarks.
1893. Apr. 20	40 Clark's	$4·95 \times 10^{-12}$	10^5	2463 ohms	$2\,463 \times 10^{-21}$	19·5 °C. (both)	4·482	. .	Balance correct to less than 1 ohm in smaller arm
„ 26	„	„	„	„	„	71·3 C.	4·4824	$1·9 \times 10^{-6}$	Ditto

The question of surface conductivity is finally settled by this film ; we got just the same results whether the quartz rods were in or not, or whether we kept the film in the box or exposed it to damp air, or even breathed on the surface of the sulphur between the dish and the upper plate. Using the excellent condenser of Messrs. CLARK, MUIRHEAD, and Co., we were enabled to make the capacity comparisons with greater accuracy, and are thus in a position to assign an approximate value to the temperature coefficient of the specific inductive capacity—which turns out to be positive, and of the approximate value $1 \cdot 9 \times 10^{-6}$ per degree Centigrade.

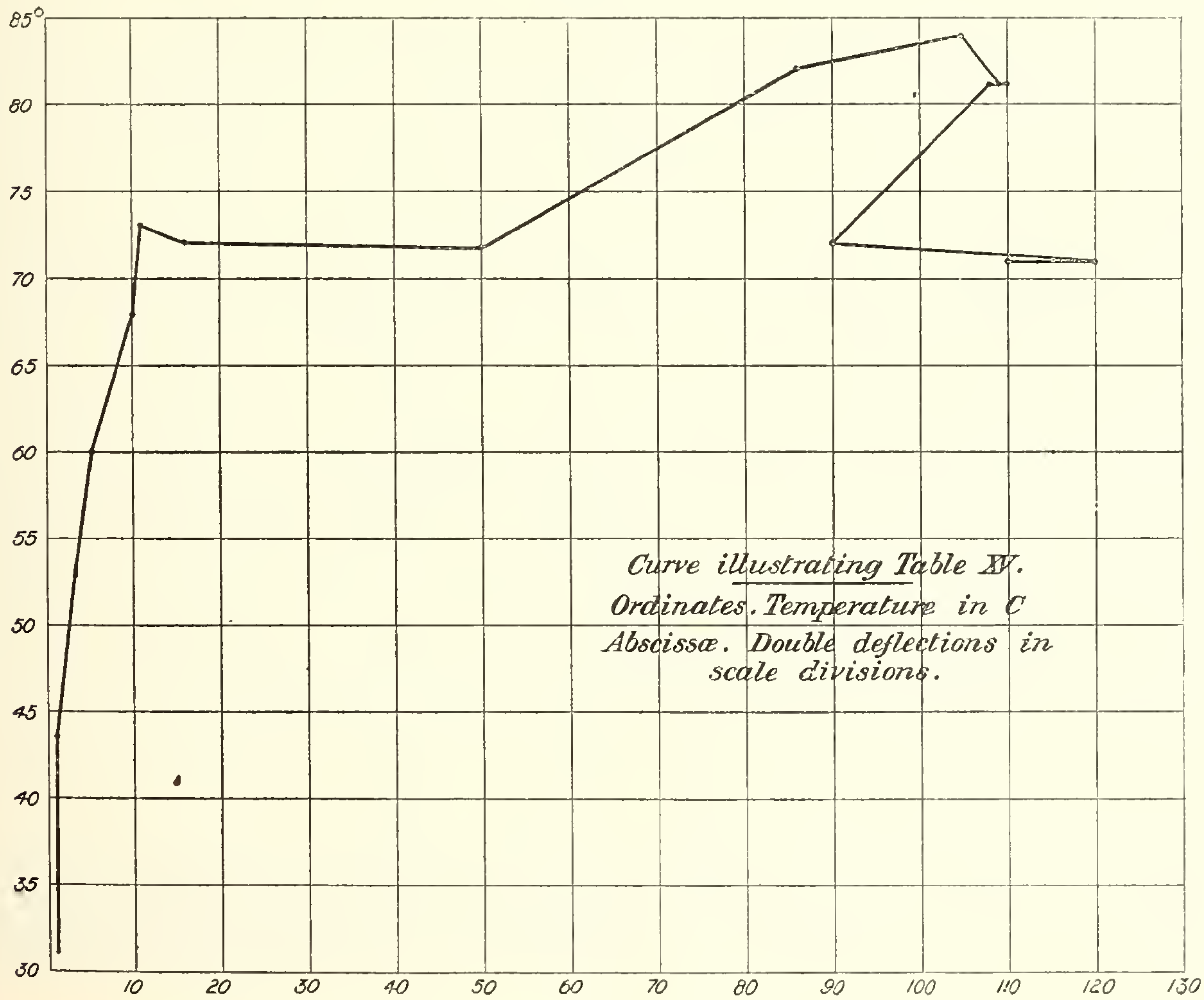
The increase in specific resistance suggests that perhaps even now we are not using a material of sufficient purity. In favour of this view we may state that the film was not quite evenly coloured, some portions appearing of a browner shade than others. On the other hand we know of no reason to suspect any impurity, even the action of dust having been carefully guarded against, and the sulphur itself being admirably pure. It was perhaps, however, insufficiently exhausted of gas. On the other hand, we know that when annealed the resistance of sulphur as we used it is much higher, and it is difficult to credit a supposed impurity with the property of itself undergoing a change of constitution at the same temperature as that at which sulphur anneals. The observations show a very definite reduction of resistance as following a rise of temperature to 80° C., and this persists to some extent on cooling. We noticed this before. We know from a special experiment, in which some finely-powdered sulphur was heated to 80° C. for an hour, that the percentage content of insoluble sulphur is thereby diminished. For instance, in a very careful experiment made on pure, finely-powdered sulphur in quantities of about ten grams, it appeared that the percentage content of insoluble sulphur was 4·01 before annealing, and 3·89 after annealing for several hours. A critical examination showed that this was entirely outside the limits of any possible experimental error. The annealing at 90° C. had been previously observed. We provisionally attribute the decrease of resistance to this partial annealing, which therefore tends to show that the conductivity of mixed sulphur has a maximum value somewhere between the 0 per cent. and 5 per cent. content of insoluble sulphur. The explanation of the temperature curve is, however, not by any means obvious, although it seems safe to draw the following conclusion, assuming that the temperature lag is inappreciable. The conclusion is that there must be at least two causes in operation influencing the resistance. One of these is probably the temperature *per se*, and the other the rate of transformation of insoluble into soluble sulphur, which is probably a complex phenomenon. The complexity may possibly arise from the change itself not being direct, the amorphous sulphur being first converted into a less stable form. There is also evidence, which will be given later, that at least two varieties of insoluble sulphur coexist in a rapidly cooled film.

On referring to the table it will be evident that all the phenomena are similar to those formerly observed, but on a smaller scale, owing to the conduction being less. All the effects of time, of reversal of voltage, &c., were observed as before. We append a short table giving the relation between temperature and resistance, by way of supplement to the general table of results.

TABLE XV.—Voltage, 280 volts.

Time.	Temperature.	Double deflection in scale division.	Remarks.
Not taken	31°	1 division = $4 \cdot 9 \times 10^{-12}$ ampère	
"	44	1 " " "	
"	53	3 " (after 5 minutes)	All effects marked
"	60	5 " " "	" "
"	68·2	10 " " "	" "
"	73·4	11 " " "	" "
"	72	16 " " "	Difference due to temperature ?
"	72	50 " " "	After keeping at 72° for 10 minutes
3 hrs. 17 mins.	82	86 " " "	
4 " 6 "	84	105 " " "	
4 " 35 "	81·8	109 " " "	
5 " 4 "	81·2	110 " " "	
5 " 13 "	81·8	108 " " "	
	72	90 " " "	Cooling slowly
	71	120 " " "	
7 hrs.	71·8	100 " " "	Re-heated slowly (not slow enough ?)

Fig. 4.



Specific Inductive Capacity.—Five pieces of film, measured as usual, gave the following values for the thickness in inches :—

(1) ·009022
(2) ·009917
(3) ·008689
(4) ·008058
Centre (5) ·008978
Mean ·0089316 inch = ·0226863 centim.

These measurements were made with very great care and represent the greatest accuracy the method is capable of; individual measurements of the same bit of film not differing by more than ·00008 inch. The differences are to be partly attributed to partial crushing or crumbling of the film during measurement, though there is clearly also a want of uniformity in the thickness of the film itself.

In obtaining the capacity by the DE SAUTY method, a balance was first obtained when the film was in the box, the lid was then taken off and a new balance obtained; and finally the plates were taken out and supported on their ebonite legs on the laboratory table at some distance from anything else. Removing the lid of the box changed the resistance required for balance from 2517·5 to 2510·5, and placing on the table reduced this to 2510. The connections and keys by themselves require 47 ohms in the adjustable arm. The other was always 100,000 ohms. We see from the small effect produced by lifting the plates out of the box which was to earth, that the free capacity may be neglected compared to the part of the capacity which is due to the dielectric action of the sulphur. We have used the factor $2463/10^5$ in calculating the capacity, allowing for the capacity of the keys, &c. The standard required no correction at all, it was ·1 microfarad. The resulting capacity is therefore 2463×10^{-6} microfarad as stated in the summary. The sulphur condenser did not leak sufficiently to affect the galvanometer by more than one division with 300 volts, and hence with 40 Clarks may be considered to insulate perfectly. The mica standard, however, leaked quite appreciably, so that its resistance was only about $2·3 \times 10^{11}$ ohms. The capacity was measured again at 71°, and the balance was found to be unchanged. In this case the sulphur condenser also leaked, *i.e.*, its resistance was about 10^{12} ohms, as against 7×10^{13}, say when it was cold. In order to find whether these leaks (which are of course exceedingly small) produced any measurable result, we made use of the formula given by GLAZEBROOK, 'Phil. Mag.' [5], vol. 1, p. 376–377, 1881. This is

$$\rho = \frac{R'\rho'}{R}\left(1 - 2\left\{\frac{R'}{\rho'} - \frac{R}{\rho}\right\}\right)$$

where R and R′ are resistances out in the arms, and ρ and ρ' the corresponding insulation resistances of the condensers. We found that the correcting factor was about

$1 + 8 \times 10^{-8}$ in one case,
and $1 + 2 \times 10^{-7}$ in the other.

Consequently our results are quite independent of leakage. Absorption, unless it be instantaneous, is guarded against by observing the point at which the kick vanishes.

Some doubt having arisen as to the correctness of the formula obtained by GLAZEBROOK, Mr. A. POLLOCK was kind enough to look into the matter for us. His conclusion is that instead of the formula quoted we ought to have

$$\rho = \frac{\rho_1 R_1}{R}\left\{1 - \left(2 + \frac{R}{G} + \frac{R'}{G}\right)\left(\frac{R'}{\rho'} - \frac{R}{\rho}\right)\right\},$$

G being the galvanometer resistance, the other letters retaining their meanings. This does not lead to any appreciable correction of our values.

Now, when the sulphur condenser is heated, the difference in the coefficients of expansion of sulphur and aluminium must lead either to the film separating from the aluminium or to both it and the aluminium being thrown into a state of strain. We know that the plates remained firmly attached to the film, from a subsequent examition, and, consequently, we are driven to admit that heating strains both the film and the plates. Now the coefficients of linear expansion as obtained from Sir W. THOMSON's 'Collected Papers' (vol. 3, p. 209), which numbers were taken from "CLARK's Constants of Nature," are for sulphur, say, ·000063, and for aluminium, ·000022. To be quite satisfactory, of course, the linear expansion of this particular kind of sulphur ought to be observed, but the values given (mean of those of FIZEAU and KOPP) are probably not very far wrong.

It is probable that when the aluminium plates are heated they are quite free to expand laterally and also vertically; in fact, the gold dish is so weak and soft, it may be set on one side, and the sulphur round the aluminium plates will, if anything, tend to drag them outwards. We may assume, therefore, that when the condenser is heated the aluminium plates increase in area, and are subject to some little stress, while the sulphur film increases freely in thickness, but is prevented, by sticking to the plates, from expanding beyond their limits. The film being thin and weak, compared to the plates, nearly the whole of the resulting strain is borne by it.

Consequently, any change in condenser capacity observed on heating will be due either to the direct effect of temperature change in altering dimensions or dielectric constant, or to the stresses thereby set up. We will assume the former alternative and see to what results it leads.

The facts are, that when the sulphur is heated, say from 20° C. to 70° C., the mica standard and resistance boxes being at constant temperature, no change in the balance is observed; for instance, it remains steady at 2463×10^{-6} microfarad within one figure in the last place.

The temperature coefficient of specific inductive capacity, which would be required to compensate for the alteration of capacity, computed from the variation of the various dimensions with temperature, is

$$\gamma = + 1{\cdot}89 \times 10^{-6}.$$

This, of course, must be further investigated before we can regard it as a physical constant, but we may take it as pretty certain that there is either a temperature coefficient, or a strain coefficient, or both, or that the specific inductive capacity depends on the composition of the film, for the sulphur is partially annealed at 80° C., and, therefore, possibly at 70° C. It was a distinct oversight not to have tested this at the time by observing the capacity again when the sulphur was cold, and ultimately led to another experiment being undertaken for the special purpose of setting this question at rest. The result of this experiment was to prove that the specific inductive capacity of films, containing rather less insoluble sulphur than this, is distinctly greater the more insoluble sulphur they contain. Hence the temperature coefficient must be greater than the above if it is at all affected by the annealing, for the annealing would almost certainly have reduced the capacity.

While we used the old standard condenser the absorption was so great as to make the observations for capacity insufficiently exact for the present purpose.

Section VI.

The Determination of Specific Inductive Capacity of Varieties of Sulphur by the Method of Weighing.

In the prosecution of these studies by one of us and Mr. J. B. Allen, some experiments were made on the specific inductive capacity of sulphur by the method of weighing. In these experiments a good many sources of electrification suitable for the purpose were investigated, and a good many causes of error detected and rectified. The results, however, only referred to one plate, *i.e.*, to one containing 6·7 per cent. of insoluble sulphur, and even these results were not in such good agreement as could be wished. The best way of preparing plates of sulphur had, however, been discovered, and we felt sure that by gradually improving the apparatus, we should ultimately succeed in obtaining reliable results. The great merit of the weighing method, however, from our point of view, is that it makes the resulting value of the specific inductive capacity absolutely independent of any electrical measurements whatever. In the experiments previously described, we relied on the constancy of the voltage of Clark cells, on standardized resistance coils, and on certificated condensers. It was now felt that an absolute determination of one of the constants ought to be made by a perfectly independent method, in order to supply as complete a check as possible on the previous work. No method could have met our ends more perfectly than the method of weighing, for the result remains entirely independent of the absolute value of anything (as a measurement of K, of course, should do); all that is required is a fairly accurate knowledge of the ratios of the weights employed, and of two distances which can be measured with fair accuracy. The drawback to the method is that it is not as susceptible of being carried to a high degree of accuracy as some more directly electrical methods.

Both measurements of length were reduced to the same standard, by fine callipers of ELLIOTT BROS., whose absolute indications are immaterial, but which we believe from other comparisons to be practically correct.

The comparison of the gilded brass and platinum weights with each other was carefully carried out on a fine balance, and resulted in showing that the weights were so well adjusted that none of the corrections observed can affect our results—the differences being much less than the error of weighing against the electric forces.

As we spent several months in perfecting our apparatus, after the experiments of Mr. ALLEN and one of us were finished, we decided to illustrate our description with a small drawing to scale (Plate 5), which we hope may be of assistance to other people using the method, and which will enable us to shorten our description considerably.

A is the zinc suspended disc. The lower surface was scraped true to a surface plate after the three pillars V, V, V had been soldered to its upper surface. The diameter of this plate was 12·65 centims., and it weighed (uncorrected) 154·3002 grams.

Each of the pillars V, V, V were bored and tapped so as to carry three screws—lying in a plane parallel to the plate. These screws are necessary to allow of the adjustment of the three fine German-silver wires, by which the plate is supported. It is necessary to have these screws well made if accurate and convenient adjustment is sought after. The point of junction of the three wires is attached to a single fine German-silver wire, which passes to below the balance case, where it is hung from a frame originally made for the purpose of weighing quartz rods in water, in order to determine their coefficient of expansion. This frame could be arrested just like the pan of an ordinary balance. The rough adjustment of the length of the wire was made by a minute box-wood shackle. The balance stood on a state shelf, and the rest of the apparatus on a slate table below it, and we may here remark that one of the first desiderata is steadiness and solidity in all the parts of the apparatus. A fine spiral of wire connects the plate to the guard-ring.

B is a guard-ring of zinc, the top and bottom surfaces being turned parallel to each other in a lathe. The inner diameter is 13 centims. and the outer diameter 23·6 centims., leaving the ring with a radial breadth of 5·3 centims. The clearance on each side of the suspended plate is, therefore, ·175 centim. We could have done with less clearance than this, but an examination of the theory showed that no error at all comparable with those of measuring and weighing would be introduced by an annular space of these dimensions.

The guard ring is supported by three strong legs, C, C, made of zinc, and provided with good levelling screws and jamb nuts, D, D. The ring is put to earth by a soldered connection, which ultimately is soldered to a water-pipe. The high potential plate E is a large glass disc, 30·5 centims. in diameter, and optically flat. It was originally silvered, but the silvering having gradually tarnished by contact with sulphur, and its conductivity having become unreliable, it was replaced by some good thin tin-foil, fastened down with thin starch paste, and put on so well that no appreciable deviation from flatness could be detected. It was attached by a clamp and wire, H, to the high potential end of the transformer. The thickness of the plate, 3·3 centims., gets rid of any fear as to flexure producing an appreciable effect.

This plate was supported by three ebonite cones, J, J, 2 centims. in diameter at the thick part, and 5·4 centims. long. These cones were got to insulate well, they were supported by a wooden stand, K, triangular in shape, and carried in turn on three levelling screws, working through brass fittings, and bearing on brass caps supported by three of the double glass and paraffin insulators, M, M, M, we have had occasion to refer to so often before. All this part of the apparatus requires to be well and substantially made, or it tends to tilt over a little. The top of the glass disc was 22·5 centims. above the slate bench. The wire connecting the tin-foil to the transformer was short and thick, but it was found that no alteration was made by twisting it into a spiral, so that no "resonance" effects are to be feared.

Screens of tin-foil pasted on to cardboard completely surrounded all parts of the apparatus. The first screen stood on the guard ring and enclosed the back of the suspended plate, allowing the wire to pass through a small hole. Outside the whole affair was another and larger screen. Both these screens were cylindrical and properly shaped, and were made in two parts joined by tin-foil flaps. The balance was also completely encased in a screen of tin-foil and gauze, allowing the pointer to be observed with ease. The panels of the balance case were separately screened by gauze to allow of the adjusting of weights, &c. The elementary mistake of having bad screen connections was not made. Screens are very necessary to prevent air currents, as well as to hinder electrostatic action. The balance was in a room with a south aspect at the opposite end of our building to the room occupied by machinery.

Balance.—This was a rather short arm balance, which was formerly used for weighing quartz, &c. It has a high range of sensitiveness, though this was diminished to a convenient amount in order to secure other advantages. It was furnished with an adjustable screw stop at R, by Mr. ALLEN. The balance will carry 1 kilogram, so that there is no danger of warping.

Electro-Dynamic Action.—This was tested by insulating the glass plate, the other plate being connected to earth on one side, and to the terminal of the transformer on the other. No trace of any action could be detected, *i.e.*, either nothing at all or else something incomparably smaller than we could approach in the experiments on weighing.

Auxiliary Condenser.—This is necessary to swamp the effect of the changes in capacity produced by putting in or taking out the sulphur. We used ten "gallon" Leyden jars in parallel with the guard-ring condenser.

Transformer.—It appeared from the experiments of one of us and Mr. ALLEN, that of all the methods of excitation tried, nothing was so steady as a large coil used as a transformer, the alternator being excited by a small current only, and driven by rather loose belting from a well loaded gas engine which has a short period of irregularity. The alternator has a very massive field-magnet system which rotates, and this acts sufficiently well as a fly-wheel to smooth most of the irregularities of the gas engine, when these are short in period. Some irregularities, however, remain and impose the limit to the accuracy attainable. It was found best to have a weak field on the alternator, and use all the armature coils, instead of strongly exciting it and using a few armature coils in series with resistances.

This was contrary to expectation. The induction coil could also be used with its break, or with a clock-work break. Mr. ALLEN found, however, that these were not so steady, and the same remark applies to a number of attempts made to use a WIMSHURST machine driven at a constant speed, and having one terminal permanently to earth, and the other earthed by a high resistance vacuum tube, with a bit of caustic potash in it. An irregular excitation will very quickly spoil the knife edges of the balance. The frequency of the alternator was about 60, and we got better results with this than when the frequency was less.

In order to keep a check on the voltage, the latter was measured continuously by a KELVIN static voltmeter reading to 12,000 volts. About 5000 to 6000 was the voltage generally employed,

Adjustments.—The apparatus having been provisionally set up, and all rough adjustments made, the following procedure was adopted. The glass plate was levelled as accurately as possible by an ordinary laboratory spirit-level, reading perhaps to something like 30″ of arc. The guard-ring was then similarly levelled at a suitable distance, from 1·5 to 2 centims. in our experiments, from the lower plate. This was then checked by measuring the distance between the top of the tin-foil and the bottom of the guard-ring, in several places by a wedge reading to ·1 millim. directly. The adjustment being found correct, the hanging plate was adjusted to the guard-ring by eye, the under surface of the ring being flat and level, and the suspended plate also being flat. This can be done with great accuracy, just as in sighting a barometer. Of course the preliminary adjustment was good enough to let the plate hang concentrically in the guard-ring. The adjustment must be made with the plate counterpoised, and is rather tedious, but presents no difficulties that may not be overcome by patience. The lower plate was then excited in order to find if the suspension stretched appreciably, but this was found not to be the case. The screens were then placed in position, and the experiment was ready for observation.

Sulphur Plates.—These were made in two ways, and we consider that too much care cannot be exercised in this part of the work.

"*Insoluble*" *Plates.*—A light sheet zinc mould was prepared, capable of containing a plate about two inches greater in diameter than the one to be employed (which was made usually of rather larger diameter than the guard-ring), and say four or five times as thick. The sulphur, previously filtered as described, through glass wool and platinum gauze, was melted and raised to a sufficient temperature by placing the zinc mould on a hot flat plate, It was noticed that after the filtered sulphur was cold, when it was being broken up for the final melting, it still possessed a strong sulphury smell, which in this case suggested gas lime. This shows that there is an advantage in distilling the sulphur, although the amount of impurity present may be imperceptible on boiling away the filtered sulphur in a polished platinum dish. However, the impurity seems to be got rid of by re-melting in the dish, for the smell was not detected on breaking up the plates at the close of the experiments.

The mould having been filled with sulphur at the proper temperature, *i.e.*, after it had become very viscous, above 170° C., the whole affair was plunged into cold water and agitated so as to cool it as quickly as possible. When the plate was sufficiently stiff, *i.e.*, after some hours, the mould and sulphur were mounted on a lathe, and the bottom of the mould cut completely away, till only a flat surface of sulphur was left. This was then reversed on the face plate, and the free surface turned down till we reached a homogeneous part of the sulphur, or beyond that if desirable. A round cake of the desired size was then cut out. The first plates were then scraped to a surface plate ; but it soon appeared that the natural warping of the plates rendered this an unfruitful labour. Consequently our later practice was to grind the plates down on a sheet of glass-paper, glued to a flat (really flat) iron plate, till they were true to a straight edge. It was necessary to experiment at once before the plates warped too much.

Soluble Plates.—These cannot be handled so freely as plates consisting partly of soluble and partly of insoluble sulphur, however we found the following method gave just as perfect results. The mould consisted of a flat plate of "opal" glass, which is very smooth and generally better than ordinary sheet glass. A ring of plaster of Paris was cast on such a plate and dried, then previously annealed sulphur was carefully heated to just above the melting-point and poured into the mould levelled by the glass plate. The sulphur was caused to cool as fast as convenient, in order to prevent the formation of internal cavities. The plate, when cold, left the glass and plaster easily, and it was, of course, much larger than requisite. In order to prepare the upper surface, the plate was held down flat on the bed of a Brown and Sharpe milling machine, and milled at a good speed till it was homogeneous and of the proper thickness. This gives just as good a flat surface, or rather better, than any of our lathes, and leaves a very fine appearance. Owing to the rapidity of cooling, any impurity from the plaster of Paris is localized to the edge of the plate, *i.e.*, is about half an inch outside the guard-ring, where it can do no harm during the weighing.

Warped Plates.—One of our plates warped in a very regular manner, so as to become concave on one side and convex on the other; the curvature looked spherical and was assumed to be so.

As we did not know what error this might lead to, and still did not wish to lose the plate—on which a good deal of time had been spent—we measured the curvature by a spherometer, and Mr. POLLOCK kindly investigated for us the proper formula. As, however, it involved a difficult integration, it was considered advisable to calculate the effect on the value of K of some extreme assumption as to the thickness of the plate. For this purpose it was supposed that the plate increased in thickness by the same amount all over, as at the boundary of the suspended plate. The radius of curvature was found to be for the concave side of the plate, 133·234 centims.; and for the convex side, 134·177 centims.; and the thickness of the plate was ·94 centim. The diameter of the plate was taken at 5 inches = 12·70 centims. In a particular experiment it was found that the value of K, when the plate was considered as having the maximum effective thickness of ·94407 centim. instead of ·943 centim., would be 3·6836, instead of 3·6878 when the measured thickness was taken. Since this difference is outside the experimental limits of weighing or measuring, and is itself probably so far in excess of the actual correction, even if the lines of force are appreciably "refracted," we decided to ignore the effect of the curvature.

Method of Experimenting.—The experiment consists in observing the attraction in grams weight on the suspended disc, both when the sulphur plate is in and out. To begin with, everything must be carefully dusted, to get rid of shreds of all kinds, and the hanging plate counterpoised—it weighs, as has been stated, about 154·3 grams. The current is then turned on and observed to be steady by the voltmeter—it never really gets steady, the voltmeter oscillating through about one division, corresponding to, say, a "permanent" variation of half a division or 100 volts in 5000, or say, 2 per cent.; this would cause the force to vary by 4 per cent.—this is an extreme limit to the fluctuations. Weights are added until the pointer of the balance is observed just to incline to leave the zero—of course, as soon as it does leave it the force diminishes and the balance tilts over. The point having been ascertained as nearly as possible, the rider is run down to increase the weight by ·005 gram, and if this produces a decided tilt, the observation is taken as exact. This means that the force can not be measured to less than the weight of 5 milligrams owing to fluctuations of potential difference. The forces to be observed vary from about ·5 gram weight without the sulphur to 2 grams with it, and these forces appear in the final result as a ratio which enters to the power of one half. The uncertainties of weighing, therefore, amount to about ·005 gram in one gram, or, say, ·5 per cent., and affect the value of K to about ·25 per cent.

The attraction, without the sulphur, having been observed, the attraction is again measured with the coil commutator reversed, in case there is any outstanding electrification; in general this produces no effect. There was no difference in any of the experiments leading to our values.

The next step is to earth the high potential plate and carefully dust every part of the apparatus, including the sulphur plate; the connecting wire is dusted by running a flame along it. The sulphur plate is then inserted, and an observation made of the value of the counterpoise, in order to find if there is any free electrification. If there is, the plate is taken out and the electrification removed by a flame—a risky proceeding, and one we never indulged in unless it was absolutely necessary. The process of finding the balance was then repeated as before. When a sufficient number of observations, both with the plate in and out, had been made, the screens were removed, and the distance between the guard-ring and the glass plate remeasured. There was never a difference of more than ·1 millim. At the end of the series of experiments all the plates available were experimented on in this manner one after the other, so as to get good comparative results. The apparatus was then dismounted, and the distance between the fixed and suspended plate changed, after which the adjustments were re-made and the whole series gone through again at the new distance, with a view to eliminating accidental errors of setting, weighing, and measuring.

The sulphur plate must not be less than five or six millims. from the suspended plate, or the free electrification at its upper surface has a sufficient P.D. from that of the earth to cause brush discharges. Consequently we always examined some plate or other in the dark to see if this did occur. If it does, of course, the result is that we get too low readings and a too low value of K. On the other hand, the nearer the top of the sulphur plate is to the suspended plate the better, for the greater the forces to be observed. The "art" of the experiment, we should say, lies in adjusting the voltage till brushing just does not occur. This is also provided for by placing the coil terminals so near together that a discharge occurs there before it occurs between the sulphur and the earthed part of the apparatus. It is also exceedingly necessary to remove all dust and sulphur crumbs between each set of weighings.

On thinking the matter over in the light of our present experience, we have an idea that there might be advantages in using a liquid of high electric strength instead of air—in fact, to proceed as QUINCKE did in determining the dielectric constants of liquids by weighing, in order to get the advantage of the large forces he was able to measure.

Analyses of the sulphur plates were made by breaking them up, and taking samples from the active part. This was also necessary in order to find out whether there were any cavities in the plates. In only one plate was a cavity discovered, but happily it was under the guard-ring almost entirely, and was very insignificant.

The partly insoluble plates were annealed in some cases in order that the difference of specific inductive capacity might be observed in one and the same plate. The annealing produced both buckling and cracking; the latter does not matter, and the former has been dealt with. In cases where the plates were annealed the analysis was made on samples collected during the turning or milling of the final surfaces. The formula of calculation is best given in the following form :—

$$\frac{F}{F_1} = \left(\frac{\rho_1 + \rho_2}{\rho_1/K + \rho_2}\right)^2,$$

where F and F_1 are the forces observed, $(\rho_1 + \rho_2)$ is the distance between the fixed and suspended plates, ρ_1 is the thickness of the sulphur, and K is the quantity we are in search of. The results will be sufficiently evident from the following table, which includes unsatisfactory values as well as satisfactory ones. The former are included with a view of showing how we gradually improved our results as we eliminated various sources of error—bad conductivity of silver coating, irregularity of driving, effect of brushes due to imperfect dusting, or too high voltage, &c., &c.

TABLE XVI.—Summary giving Details of Measurements for K by Method of Weighing.

Date.	Temperature.	Thickness of cake, ρ_1.	Distance between plates, $\rho_1 + \rho_2$.	Mean ratio of tensions.	Maximum error in ratio as per cent. of mean ratio.	Values of K.	Mean K.	Per cent. of insoluble sulphur.	No of set.	Remarks.
1893.	° C.	centims.	centims.		per cent.			per cent.		
May 26 . .	17·0	0·943	1·45	3·597	5·01	3·66	* 3·67 Rejected	1·069	(1)	This cake was made from C—S distilled twice. It was made in a zinc dish, May 23, the S being poured in when thick and black, and the whole cooled by immersion in water. The water was not allowed to cover top of cake. It was turned up in the lathe. Diameter about 20 centims. Analysis made of last cut from top and bottom mixed. On breaking up cake (July 6) it was found to be solid all over. Analysed June 30. Rejected from doubts regarding silvered glass plate
„ 26 . .	17·0	0·943	1·73	2·751	3·0	3·68				
„ 30 . .	19·1	0·945	1·73	2·551	6·31	3·169	* Rejected	0·0	(2)	The above cake (1) annealed for 5 hours at 105° C. in thermostat. Cake became buckled and cracked along about three-quarters of a diameter, otherwise quite sound. Analysed and broken up on July 6. Results rejected owing to large per cent. of error, due to defects in silvered glass plate, &c.
„ 30 . .	19·1	0·945	1·42	3·5716	5·74	3·419				
June 2 . .	15·0	1·2476	1·50	6·7192	1·81	3·820	* Rejected	1·437	(3)	Cake of C—S filtered through platinum gauze and glass wool. Kept just molten for 2 to 3 hours and then cast on an opal plate inside a plaster of Paris ring. Not immersed in water. The bottom surface being flat the top one only was ground flat on glass paper. The sample for analysis was taken on June 28. Cake made on June 2. It was broken up, after annealing, on July 6, and only one cavern found in the cake near one edge, and certainly in that part which was under the guard ring
„ 2 . .	15·0	1·2476	1·90	3·7014	4·65	3·727				
„ 5 . .	15·1	1·2476	1·90	3·401	4·30	3·2756	* Rejected	1·437	(4)	(3), (4), and (5) are sets made with this cake. All have been rejected owing to large error, owing, amongst other things, to glass plate not being properly silvered
„ 5 . .	15·1	1·2476	1·68	4·5787	0·53	3·536				
„ 5 . .	15·1	1·2476	1·50	5·958	3·12	3·44				
„ 13 . .	17·3	1·2476	1·88	3·731	2·92	3·65	* Rejected	1·437	(5)	The value K = 3·536 in set (4) is the only reliable one, as per cent. error in ratio is ·53 per cent. only. Any value in which per cent. error in ratio is > 1 per cent. of mean ratio has been rejected
„ 13 . .	17·3	1·2476	1·68	4·6736	1·65	3·58				
„ 23 . .	13·5	1·195	1·68	4·470	1·32	3·86	* Rejected	3·049	(6)	Cake of C.S. filtered through platinum and glass wool. Made as (1), not immersed overhead in water. Turned up in lathe. Made June 20. Seems very solid. Has not been broken up. Analysis made of last cuts front and back. This set rejected owing to large per cent. error in ratio. Analysed June 30. This set was in a measure preliminary
„ 23 . .	13·5	1·195	1·88	3·5094	1·31	3·75				
„ 25 & 26	13·5	1·195	1·88	3·5035	0·01	3·74	3·75	3·049	(7)	Same cake as last (6). The following sets are to be taken as final, and having been made with all the precautions suggested by experience. The glass plate used for H.P. plate has been covered with tinfoil instead of silver
„ 25 & 26	14·0	1·195	1·71	4·216	0·31	3·76				
„ 25 & 26	13·5	1·2476	1·88	3·5935	0·54	3·46	3·51	1·437	(8)	Same cake as (3)
„ 25 & 26	14·0	1·2476	1·71	4·436	0·45	3·56				
„ 25 & 26	13·5	0·945	1·88	2·32	0·00	3·16	3·18 (Mean, 3·162)	0·0	(9)	Same cake as (2), *i.e.*, C—S twice distilled (originally insoluble) and annealed at 105° C. for 5 hours. See remarks on (2). Analysed July 6
„ 25 & 26	14·0	0·945	1·71	2·6	0·00	3·20				
July 5 . .	13·0	1·223	1·70	3·9303	0·33	3·214	3·144 (Mean, 3·162)	0·0	(10)	Cake (3), (8) made by casting C.S. (filtered). This cake (3) was annealed at 105° for $4\frac{1}{2}$ hours in thermostat. It did not buckle in any way, but split all along a diameter and in a few other places. With care a set of readings was taken as shown. Analysed July 6. Annealed June 28
„ 5 . .	13·0	1·223	1·91	3·1	0·00	3·074				

NOTE.—Rejected values marked *.

The net result of the measurements detailed in Table XVI. is as follows :—

Temperature, ·14° C.

For mixed sulphur containing 3 per cent. of insoluble sulphur .	K = 3·75
,, ,, ,, 1·43 ,, ,, . .	K = 3·51
Completely soluble sulphur (aged monoclinic ?)	K = 3·162

The value of K for the soluble sulphur is the mean of values obtained on two different samples.

It would appear from these values that the specific inductive capacity rises as the percentage of insoluble sulphur increases.

Experiments have yet to be made on a cake of sulphur containing a very large amount of insoluble.

The results of the experiments described in the foregoing investigation may be summarized as follows :—

(1.) The specific resistance of perfectly soluble sulphur, either simply monoclinic, or aged monoclinic, *i.e.*, monoclinic sulphur which has lost its crystallographic properties and become opaque, but which practically preserves its old melting-point, is very high, certainly above 10^{28} C.G.S. units.

(2.) An admixture of say 5 per cent. of amorphous sulphur reduces this great specific resistance to say 10^{25} C.G.S. units.

(3.) The conductivity, such as it is, is marked by some peculiarities; it tends to be discontinuous, resembling in this the behaviour of surface films and of bad conductors (glass rods and ebonite) generally.

(4.) The conductivity is always much greater for some time after the battery is reversed, say for three or four minutes.

(5.) The "residual" charge, when the sulphur is regarded as a dielectric, is small.

(6.) The conduction does not obey OHM's law. The specific conductivity is greater, the greater the voltage.

(7.) The specific inductive capacity of mixed sulphur is greater than of pure soluble sulphur.

(8.) The conductivity of mixed films increases enormously while they are annealing, and even perfectly crystalline sulphur shows traces of conductivity at temperatures near the melting-point.

(9.) Melted sulphur conducts comparatively well, say 1000 times as well as a cold mixed film.

Note added April 1, 1895.

Experiments made March and April, 1894.

A possible explanation of the very divergent values of K, as exhibited by different modifications of sulphur, may be sought in the corresponding variations of density.

To test this point, an examination was made of the density of the sulphur employed in the experiments for K by weighing. The accuracy of the determination of density extends to the third figure at least.

The following table will show that the variation of specific inductive capacity is not to be explained by supposing that it is merely proportional to density.

Table showing Relation between the Specific Inductive Capacity and the Density of the Forms of Sulphur.

Description of plate.	Specific inductive capacity of plate.	Density of plate.	Specific inductive capacity ÷ density.
Pressed cake, containing 90 per cent. of insoluble amorphous sulphur . .	2·194	1·11	1·976
Cast cake, containing 3 per cent. of insoluble sulphur	3·75	2·001	1·874
Cast cake, annealed, containing no insoluble sulphur	3·18	2·0103	1·582

DESCRIPTION OF PLATE 1.

New High Grade Galvanometer.

Fig. 1. *Elevation.*

A. Fine adjustment in azimuth for control magnets.

B. Coarse adjustment for raising or lowering same.

C. Fine ,, ,, ,,

D. Brass frame or bridge carrying control system.

E. Iron slabs forming magnetic screen.

F. Wooden stand carrying iron slabs.

G. Brass ring carrying D, supported by three levelling screws on lead pillars, H.

H. Lead pillars carrying G.

I. Ebonite slab insulating and supporting coil system (*e*, *e*, &c., fig. 2).

J. Glass and paraffin insulators supporting ebonite slab I.

K. Concrete slab supporting the whole apparatus.

L. Sandstone pillar supporting K.

M_1, M_2. Top control magnets.

N_1, N_2. Bottom control magnets.

O. Wooden carriage for N_1, N_2.

Fig. 2. *Section.*

A—O. Same as in fig. 1.

Q. Position of millboard case enclosing the coils, &c.

Details of coils and mounting (small letters).

a, *a*. Ebonite terminals.

b, *b*. Brass frames carrying coils.

c and *d*. Brass base.

e. Brass cell, in which the whole of the coils and magnet system are free to turn.

f, *f*. Levelling screws.

Fig. 3. Coil support in plan and section.

a, *a*. Ebonite terminals.

b, *b*. Brass frame.

g, *g*. Quartz rods supporting coil.

h, *h*. Brackets carrying quartz rods.

i, *i*. Carriage fastened to *b*, *b*, but adjustable to magnet system by leaving a play in the holes.

j. Magnet forming part of system.

k. Wire forming coil.

l. Paraffin mounting.

n. Ebonite mounting.

m. Glass front supporting a pair of ebonite pillars, *a*, *a*.

Fig. 1.

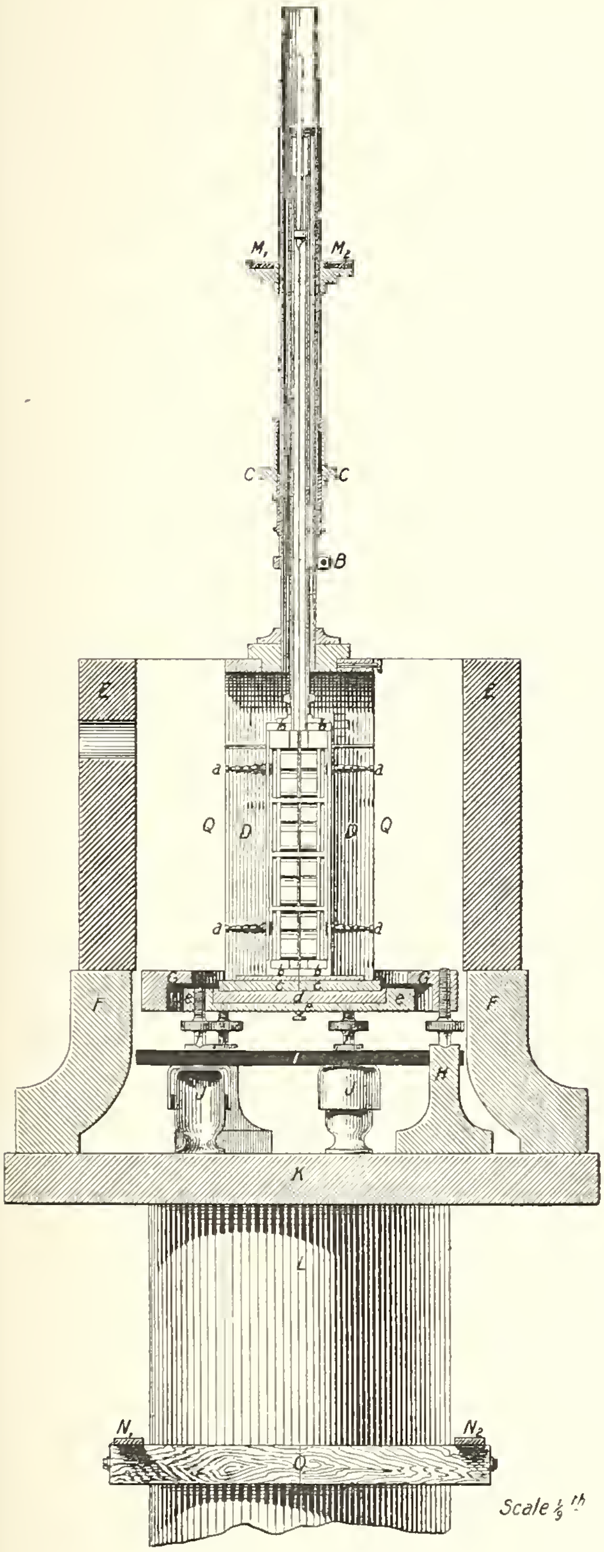


Fig. 2.

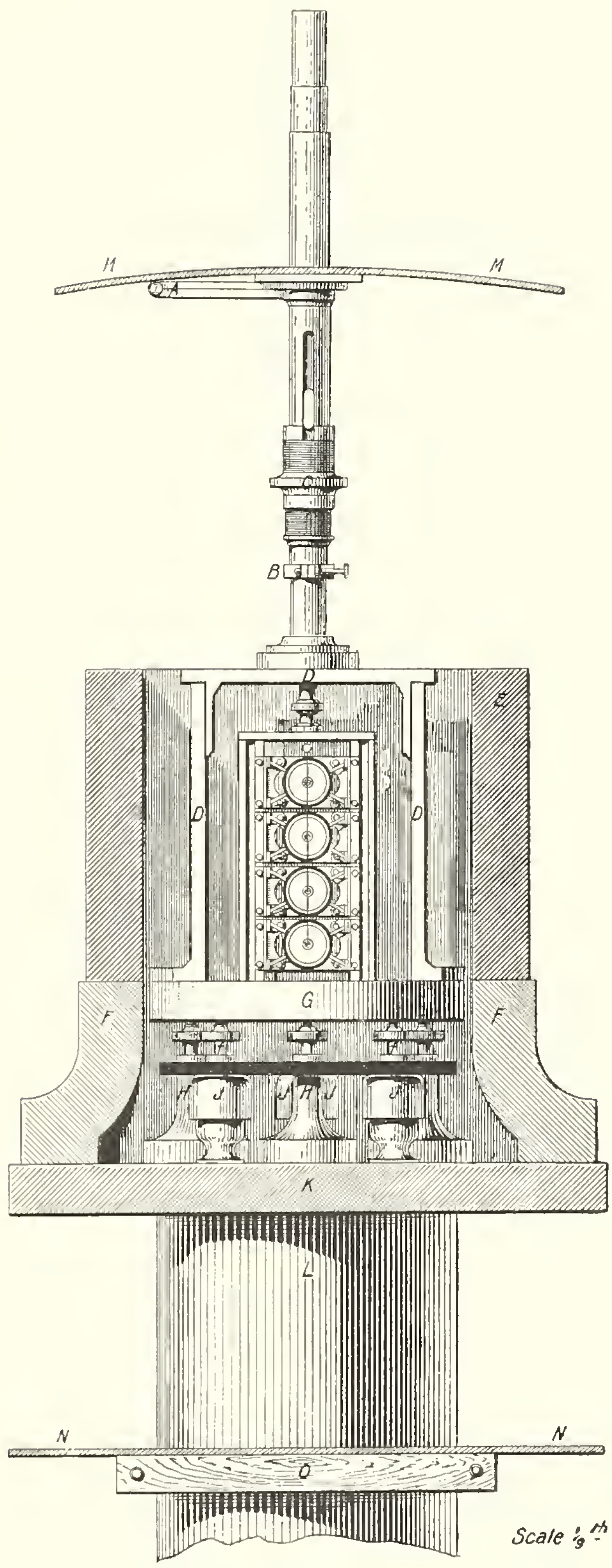
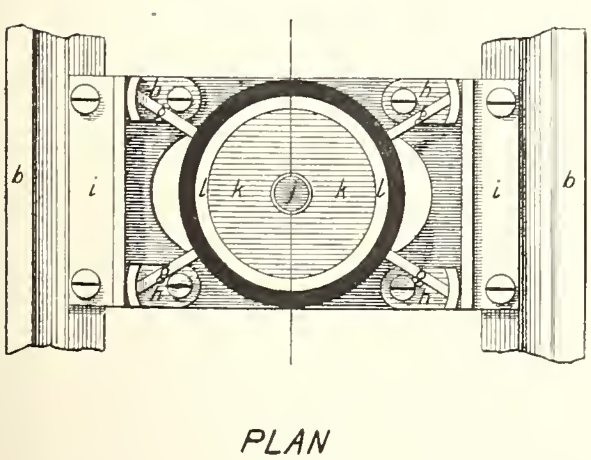
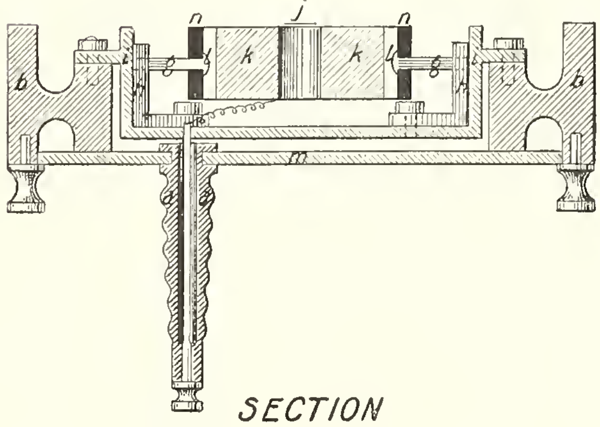


Fig. 3.

COIL SUPPORT
Half Size.



PLAN



SECTION
Shewing Insulation of Terminals.

DESCRIPTION OF PLATE 2.

Fig. 1. Thermostat. A. Section. B. Plan.

a. Outer case of brass.

c. Inner case of brass silvered inside.

b. Heavy mineral oil filling annular space between *a* and *c*.

d. Lead weight to keep *c* in position.

e, *e*. Ebonite blocks to support *h*.

f, *f*. Dish with drying material.

g, *g*. Ebonite cones supporting and insulating glass plate *j*.

h. Innermost brass box, with lid, silvered inside and outside. Holes in lid for thermometers and electrodes. Space between *h* and *b* filled with air.

i. Glass cover.

j. Glass plate supporting gold dish and sulphur.

k_1, k_2. Aluminium plates forming armatures of condenser (see fig. 5, Plate 3).

l. Zinc dish supporting an inner gold dish (see fig. 5).

m_1, m_2. Electrodes from upper and lower aluminium plates.

p_1. Standard thermometer.

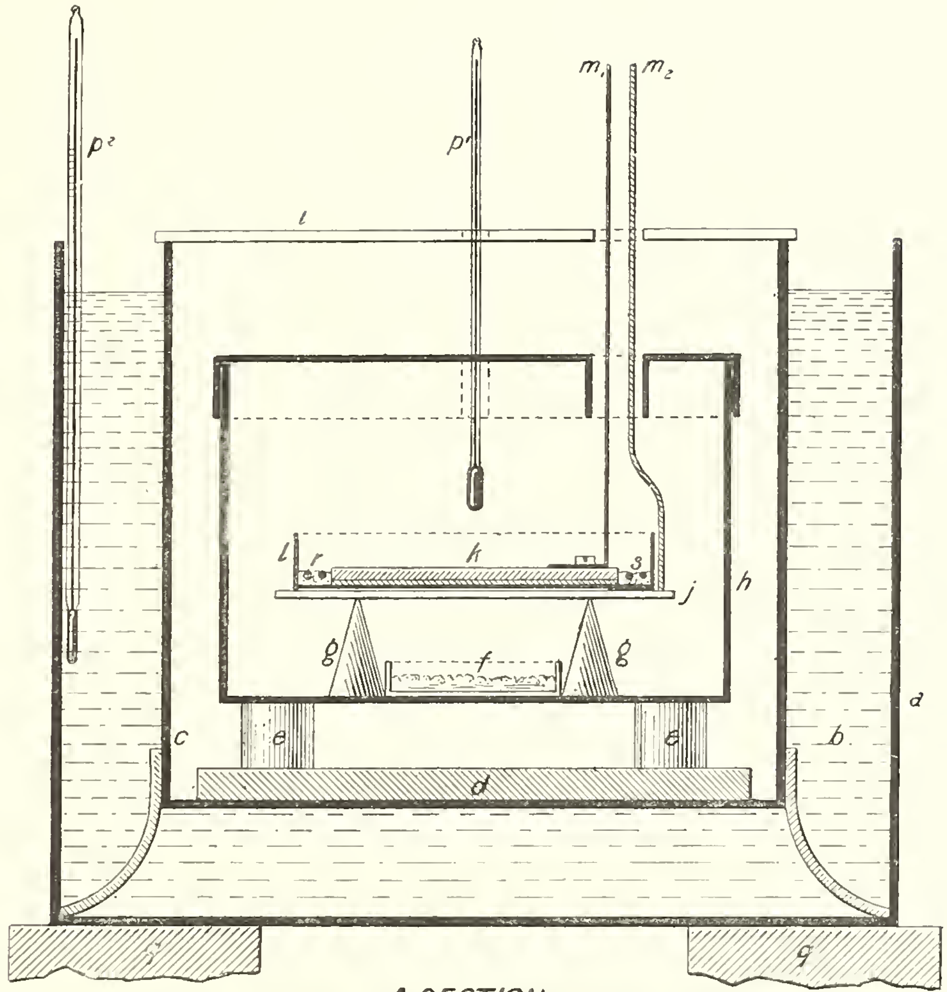
p_2. Ordinary high range thermometer.

q, *q*. Brick supports.

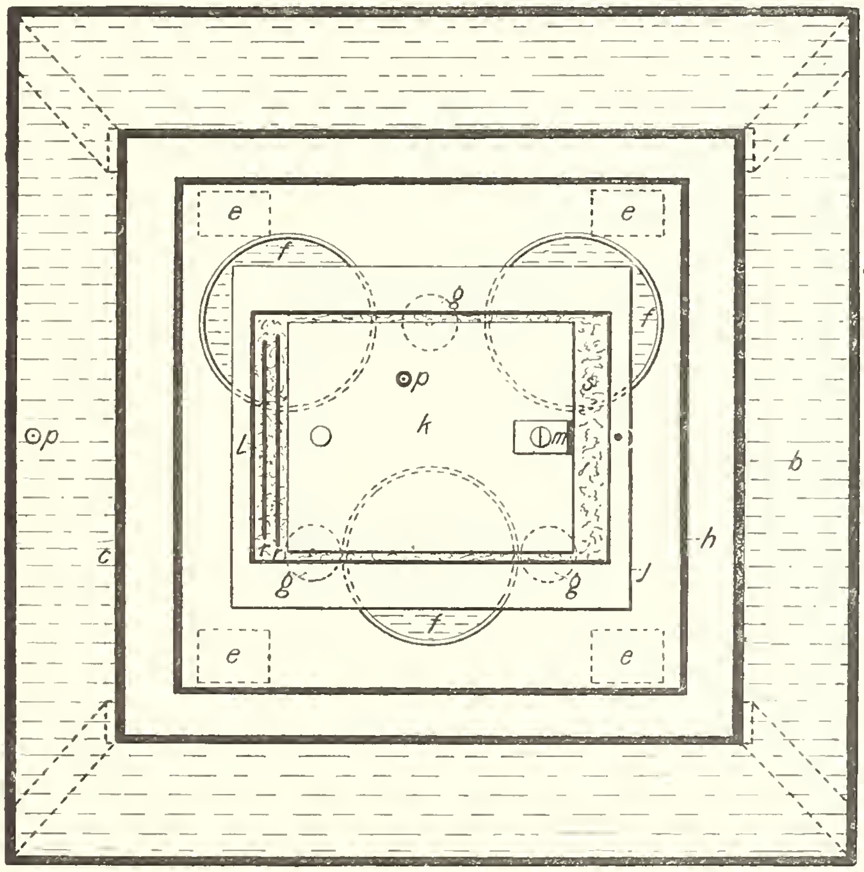
r, *r*. Quartz sticks in sulphur.

s. Surplus sulphur.

Fig. 1.



A. SECTION.



B. PLAN.

0 2 4 6 8 10 20 30 Cms

Scale

DESCRIPTION OF PLATE 3.

Fig. 2. Four-pole ebonite key.

a. Wooden base.

b, *b*, *b*, *b*. Ebonite rods with mercury cups in upper ends.

Fig. 3. Reversing key, &c.

a. Wooden base.

b, *b*, *b*, *b*. Ebonite rods and mercury cup in upper ends.

c, *c*. Rods of fused quartz mounted in ebonite ends, and with mercury cups.

f. Tin-foil wrapped on ebonite handle.

d. Discharging point, only used when determining residual charge to connect g', g^2, thus instantaneously discharging condenser.

For resistance measurements the legs at opposite corners were metallically connected as usual in a commutator.

Fig. 4. Micrometer telescope used with new galvanometer.

a. Object glass.

b. Graduated micrometer head.

c. Scale divided on glass, replacing usual comb.

d. Movable spider line.

e. Eye-piece.

f. Brass mounting, carries *a* and *e*, and is capable of a universal motion (not shown).

Fig. 5. Gold dish and aluminium plates.

a. Aluminium handle used to lift K_1 when making film. This is removed and replaced by electrode *m* when film is in position.

K_1, K_2. Aluminium plates between which the sulphur film is formed.

l. Outer zinc dish.

c. Inner gold dish supported by *l*.

s. Surplus sulphur.

r, *r*. Quartz rods in sulphur.

Note.—Since the sketch was made, a zinc dish has been fixed to the upper aluminium plate K_1. It is scraped true on the bottom and joint sealed with sulphur. The dish can then be filled with pounded ice, when making mixed sulphur films.

Fig. 2.

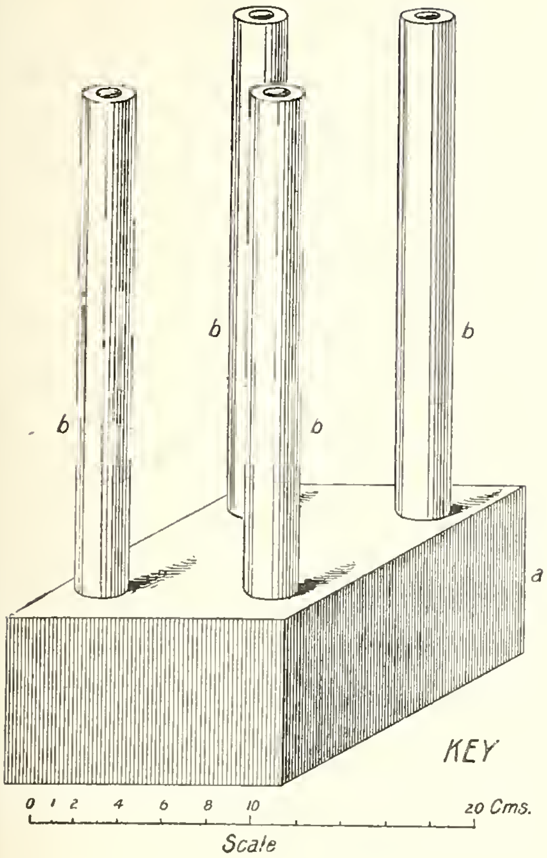


Fig. 3.

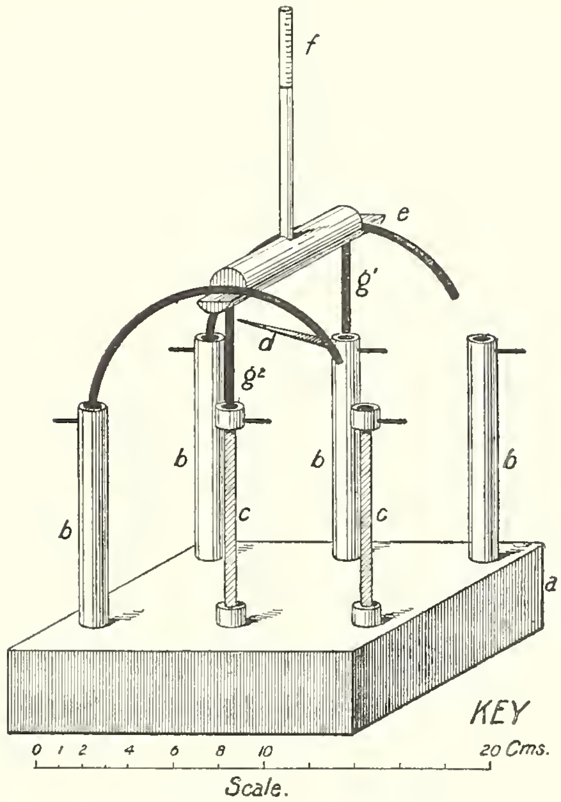


Fig. 4.

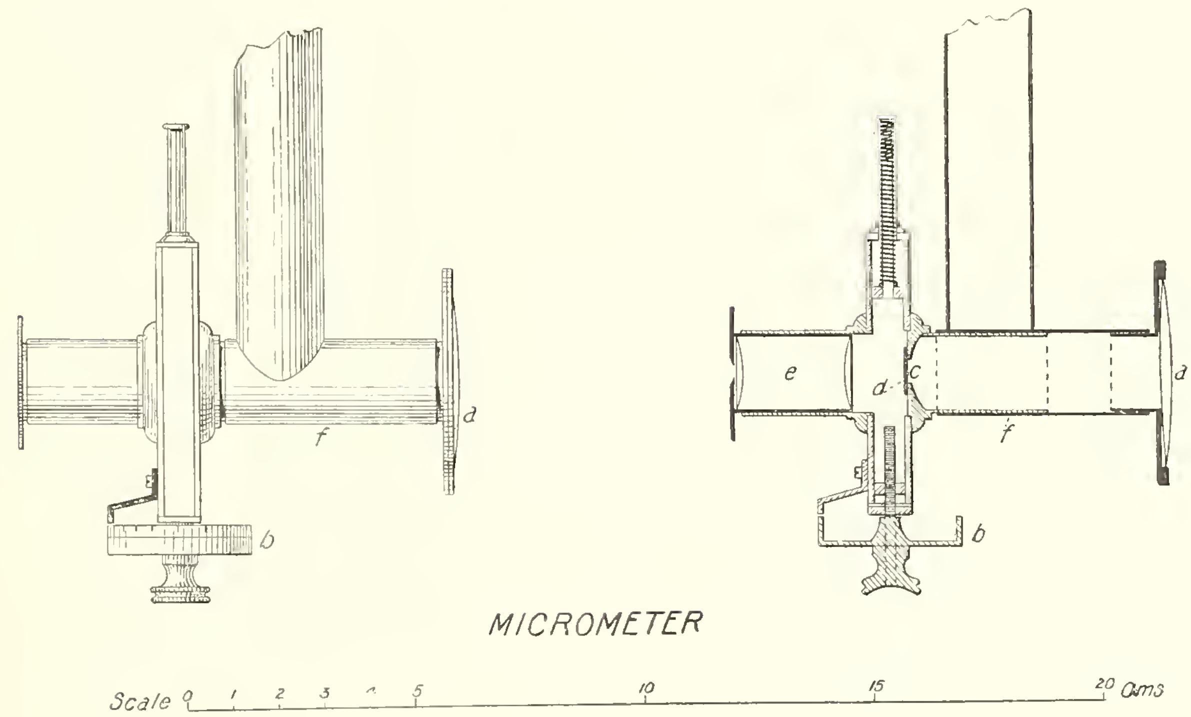
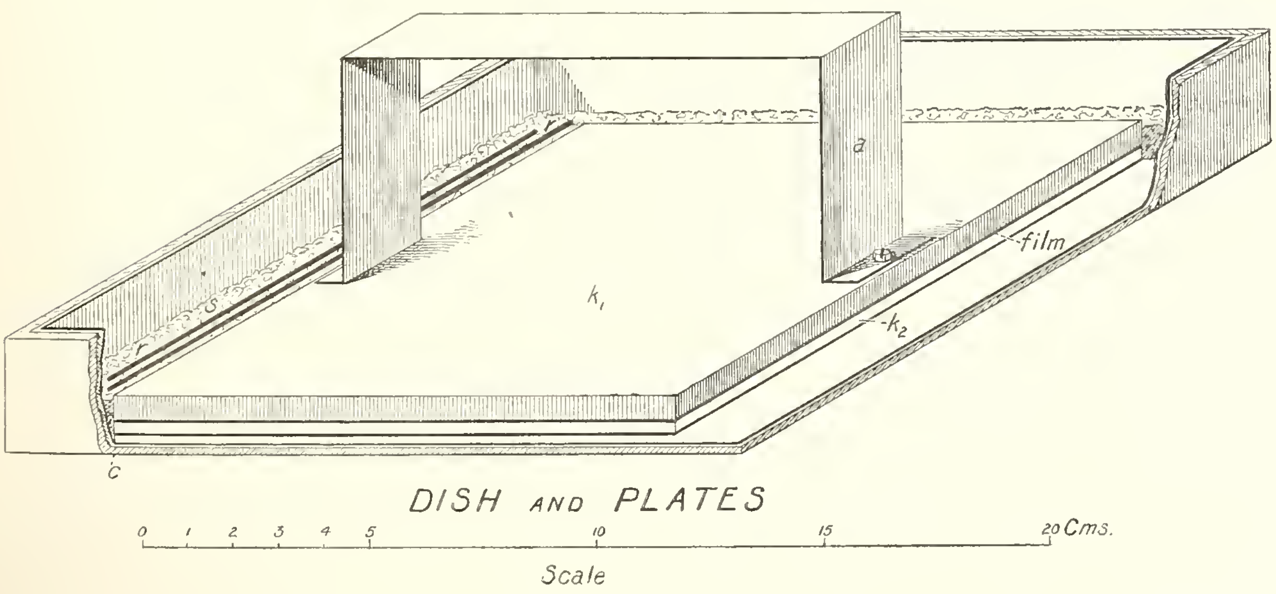
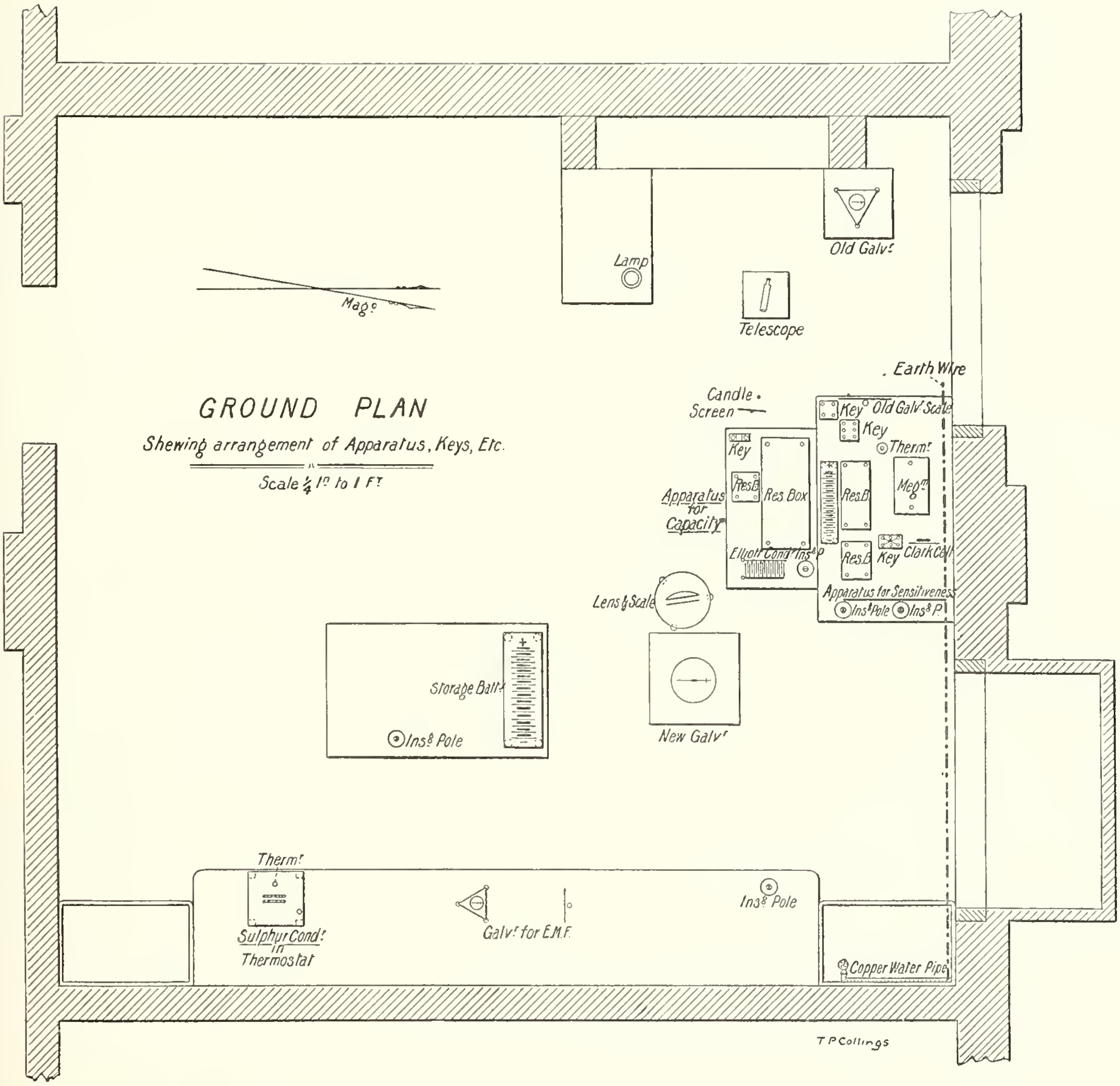


Fig. 5.




GROUND PLAN
Shewing arrangement of Apparatus, Keys, Etc.
Scale ¼ In to 1 Ft
Lamp
Old Galv
Mag
Telescope
Earth Wire
Candle
Screen
Key
Old Galv Scale
Key
Therm
Res B
Res Box
Res B
Meg
Apparatus for Capacity
Elliott Cond
Res B
Key
Clark Cell
Apparatus for Sensitiveness
Ins Pole
Ins P
Lens & Scale
Storage Batt
Ins Pole
New Galv
Therm
Sulphur Cond in Thermostat
Galv for E.M.F.
Ins Pole
Copper Water Pipe
T P Collings

DESCRIPTION OF PLATE 5.

A. Suspended zinc plate attached to balance arm. Connected to earth by a flexible spiral of No. 40 wire, B.W.G.

B. Guard-ring of zinc with parallel faces.

C, C. Zinc legs attached to B by brass screws.

D, D. Brass levelling screws with lock-nuts, for levelling guard-ring.

E. Glass plate covered with a sheet of tin-foil F, which forms the high potential armature of the condenser.

F. Tin-foil surface attached to flat glass plate E.

G. Cake of sulphur.

H. Connection from F to the induction coil.

I. Metallic connection from legs of guard-ring to outer tin-foil case and *earth*.

J, J. Ebonite pillars supporting glass plate.

K. Wooden base supporting J, J, and E.

L, L. Levelling screws by which the tin-foil surface may be brought level and parallel to A.

M, M. Glass and paraffin insulating stands supporting the three levelling screws, L, L, L.

N. Millboard box coated inside and out with tin-foil, and protecting the back of the suspended disc from induction.

P. Large millboard box coated with tin-foil, and surrounding all the apparatus. It is put to earth, as is N.

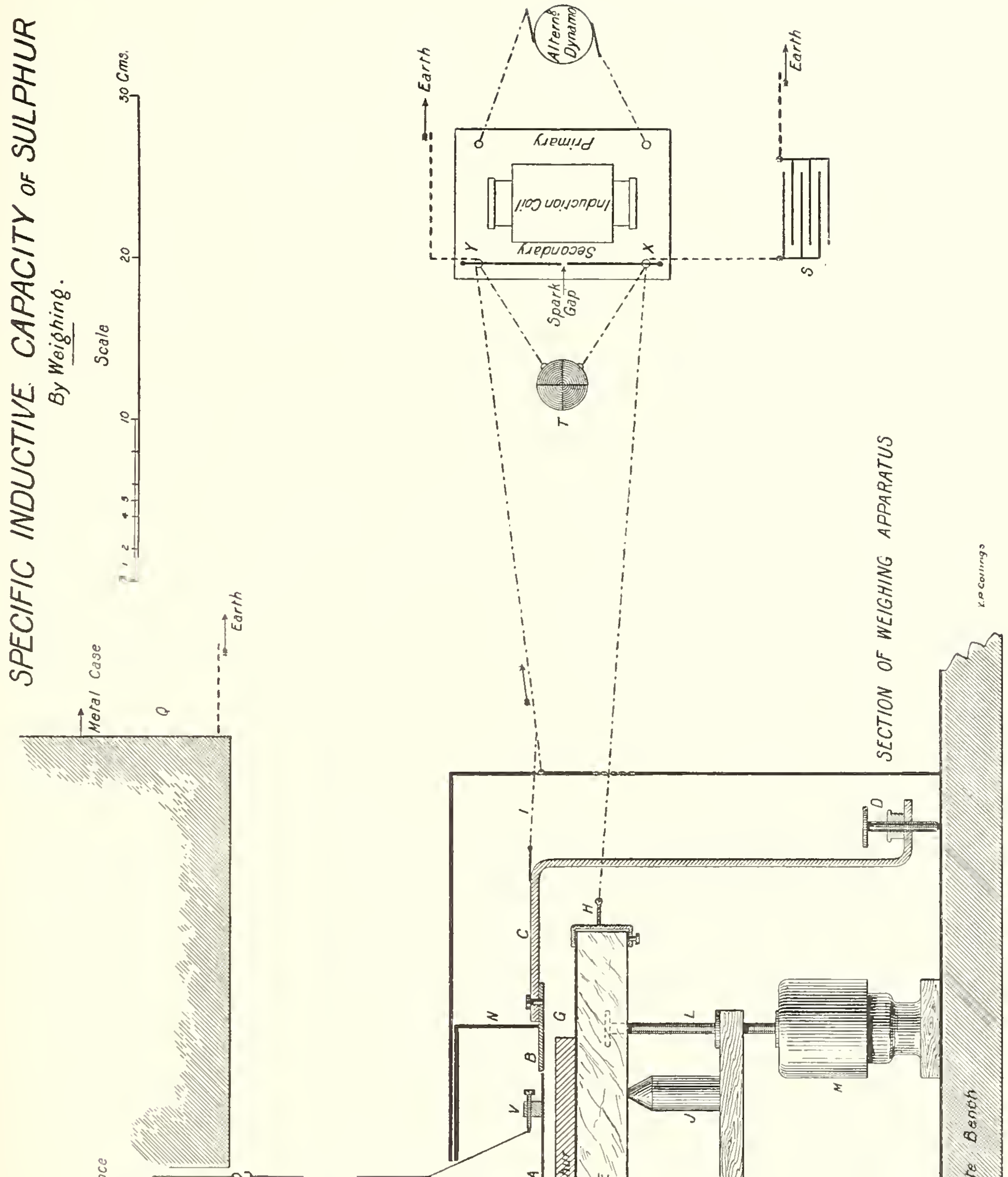
Q. Metallic casing surrounding the balance.

S. Condenser in parallel with secondary terminals to swamp the capacity of the sulphur condenser.

T. High potential Kelvin voltmeter.

X. High potential terminal of secondary.

Y. Low potential terminal of secondary, put to earth on water main. The same earth connection is made throughout.


SPECIFIC INDUCTIVE CAPACITY OF SULPHUR
By Weighing.
Scale
30 Cms.
Earth
Primary
Induction Coil
Secondary
Altern. Dynamo
Spark Gap
Metal Case
Balance
To Balance
Sulphur
SECTION OF WEIGHING APPARATUS
Slate Bench

V. *The Rubies of Burma and Associated Minerals: their Mode of Occurrence, Origin, and Metamorphoses. A Contribution to the History of Corundum.*

By C. Barrington Brown, *Esq., Assoc. R.S.M., F.G.S., and Professor* John W. Judd, *C.B., LL.D., F.R.S., F.G.S.*

Received February 6,—Read March 7, 1895.

[Plate 6.]

Contents.

I. Introduction.

In the year 1798 Greville established and named the mineral species "corundum," the crystallized oxide of aluminium; and in an appendix to Greville's paper, read before this Society,* the Count DE BOURNON correctly defined the crystallographic characters of the species. Four years later the last-mentioned author laid before the Royal Society his very valuable memoir, bearing the title, "Description of the Corundum Stone and its Varieties, commonly known by the names of Oriental Ruby, Sapphire, &c., with Observations on some other Mineral Substances."† In this work the mode of occurrence of corundum is discussed, and an admirable account is given of the minerals with which it is associated in the famous gem-yielding localities of Ceylon, China, and Southern India. In all of these districts DE BOURNON showed that the corundum occurs in crystalline schists; being associated in the Salem district of the Madras Presidency with moonstone, anorthite, fibrolite, diaspore, hornblende, quartz, mica, talc, garnet, zircon, and magnetite; while in Ceylon its chief associates are spinel, pyrrhotite, tourmaline, ceylanite (pleonast), and zircon. DE BOURNON'S memoir is especially noteworthy as containing the first descriptions of two very important rock-forming minerals—anorthite ("indianite") and sillimanite ("fibrolite").

Nearly twenty years later, LESCHENAULT DE LA TOUR was sent by the authorities of the Natural-History Museum of Paris on a scientific mission to the Salem district in Southern India, and an account of his observations—botanical, zoological, and geological—appeared in the official publications of the Museum.‡ LESCHENAULT'S collections are preserved in the Muséum d'Histoire Naturelle, and those of the Count DE BOURNON in the Collège de France, and both these collections have been made the object of a series of careful and exact studies by the able mineralogist and petrographer M. ALFRED LACROIX.§

Since the time of DE BOURNON and LESCHENAULT DE LA TOUR, some further accounts of the gem-bearing rocks of the Salem district in Madras have been published by NEWBOLD,‖ J. CAMPBELL,¶ and E. BALFOUR,** while Dr. KING and Mr. W. BRUCE FOOT have described the general geological features of the whole district in their memoir "On the Geological Structure of Portions of the Districts of Trichinopoly, Salem, and South Arcot, Madras,"†† and a summary of the results of the work of the Geological Survey of India in this district is given in the first edition of the 'Manual of the Geology of India,' by Messrs. H. B. MEDLICOTT and

* "On the Corundum Stone from Asia," by the Rt. Hon. CHARLES GREVILLE, 'Phil. Trans.,' vol. 88 (1798), pp. 403–448.

† *Ibid.*, vol. 92 (1802), pp. 233–326.

‡ 'Mém. du Muséum d'Hist. Natur. de Paris,' vol. 6 (1820), pp. 329–348; vol. 8 (1822), pp. 245–278.

§ Bull. de la Soc. Fr. Min.' (1889), pp. 282–348; see also 'Rec. Geol. Surv. of India,' vol. 24, Part III., p. 155.

‖ 'Asiat. Soc. Journ.,' vol. 7 (1843), pp. 150–171 and 203, 204.

¶ 'Calcutta Journ. Nat. Hist.,' vol. 2 (1842), p. 281.

** 'Select. Rec. Govt. Madras,' vol. 39 (1857), p. 91.

†† Memoirs of the Geological Survey of India,' vol. 4 (1864).

W. T. BLANFORD,* and the second edition of the same work by Mr. R. D. OLDHAM.† Interesting notes on this district are also to be found in Dr. V. BALL's volume on 'The Economic Geology of India' (1881), and in Mr. F. R. MALLET's work on 'The Mineralogy of India' (1887).

Concerning the remarkably similar rocks of Ceylon, there seems to have been very little published in the interval between the appearance of the memoir of DE BOURNON (1802) and that of LACROIX (1889).

Since the period, however, when DE BOURNON's valuable memoir first made geologists and mineralogists familiar with the corundum localities of Salem and Ceylon, a number of works have been published, dealing with the general question of the mode of occurrence of corundum and its associated minerals in other areas. In the Indian peninsula various authors have described the occurrence of corundum in Mysore, North Arcot, Travancore, Coimbatore, Hyderabad, the Central Provinces, Singbuhm, Monghyr, South Rewah, and the Khasi Hills. In South Rewah Mr. F. R. MALLET has given an account of a bed of corundum rock, in places at least thirty yards in thickness, intercalated with the gneiss series of the district.‡ The purplish, granular corundum rock of this locality is described as being associated with diaspore, fibrolite (?), paragonite (euphyllite), and tourmaline (schorl); and the mass is said to be interfoliated with tremolite-schist, amphibolite, and white and green jade—the latter coloured with chromium compounds.

Next in importance to the corundum deposits of the Indian peninsula are the great belts of corundiferous rocks of the eastern United States. These extend along the line of the Appalachian Mountains from Chester in Massachusets to Northern Georgia and have been well described by many authors, among whom may be especially mentioned JOHN DICKSON,§ Dr. CHARLES JACKSON,|| Professor SHEPARD,¶ and Colonel JENKS;** while Mr. T. M. CHATARD, of the U.S. Geological Survey, has given an admirable account of the chief localities and geological relations of the gem-bearing rocks.†† Along this line of country the corundum, occurring either alone or mixed with magnetite (emery), is found in veins of chloritic and vermiculite minerals (ripidolite, jefferisite, &c.), traversing dykes of chromiferous serpentine, which cut through the granites and crystalline schists of the mountain axis. The chief minerals associated with the corundum along this line of country are sillimanite (fibrolite), hercynite, cyanite, smaragdite, zircon, lazurite, rutile, pyrophyllite, and damourite.

* *Loc. cit.*, p. 26.

† *Loc. cit.*, pp. 38, 39.

‡ 'Records Geol. Surv. of India,' vol. 5, p. 20; *ibid.*, vol. 6, p. 43.

§ 'Am. Journ. Sc.,' vol. 3 (1819), p. 4.

|| *Ibid.*, vol. 39 (1865), p. 65.

¶ *Ibid.*, vol. 40 (1865), p. 112; vol. 42 (1866), p. 42; vol. 64 (1868), p. 256.

** 'Quart. Journ. Geol. Soc.,' vol. 30 (1874), p. 303.

†† 'Mineral Resources of the United States' (1883-4), p. 714, &c.

In the district of Mramorsk, near Ekaterinburg, in the Ural Mountains, corundum and emery are described by GUSTAV ROSE as occurring in serpentine and chlorite-schist in association with diaspore and zoisite.* In the Ilmen Mountains, not far from Miask, and in the gold-washings north-east of Zlatousk, KOKSCHAROW has described corundum as being found embedded in anorthite ("barsowite").†

A very valuable contribution to our knowledge of the mode of occurrence of, and the minerals associated with, corundum is contained in the series of papers by Dr. J. LAWRENCE SMITH on the emery formation of Asia Minor.‡ This author shows that the emery of the district in question is a blue corundum, mixed with magnetite, and that it is found, often in masses of considerable size, distributed through a crystalline limestone, which is interfoliated with schists and gneisses. The minerals which accompany the emery, or impure corundum, are diaspore (common), hydrargillite (rare), zinc-spinel (gahnite), pholerite, margarite, muscovite, chloritoid, schorl, chlorite, magnetite, hematite, limonite, pyrite, rutile, ilmenite, and titanoferrite. The localities at which the emery occurs are Gumuch-dagh, Kulah, Adula, and Mauser in Asia Minor, and the neighbouring islands of Naxos, Samos, and Nicaria, in the Grecian Archipelago. More recently, TSCHERMAK has given a fuller account of the emery of Asia Minor, and has shown what an important constituent of this rock is the mineral tourmaline.§

Corundum, in smaller quantities, is known to occur as a constituent of granite and gneiss in the Riesengebirge, Silesia, Auvergne, &c.; in a compact felspar rock, at Mozzo, in Piedmont; in dolomite, with tourmaline, at St. Gothard; while at Orange County, N.Y., and Sussex County, N.J., and other places, corundum has been found with a great variety of minerals in a crystalline limestone.

Corundum has also been detected in masses of gneiss, &c., ejected from volcanoes, as at Königswinter and Niedermendig, &c., and more rarely in the zones formed by contact metamorphism. It has also been detected in the metallic iron of terrestrial origin from Ovifak, Greenland.

It is not necessary to discuss in this place the numerous occurrences of the mineral in its various forms, in river gravels and alluvia, and the various washings from which gold, platinum, and diamonds are obtained.

Upper Burma has long been known to be the source of the magnificent red corundum ("pigeon's blood ruby"), and also of the red spinels (Balas ruby), and of the pink tourmaline (rubellite), a gem which, by the Chinese, is prized even more highly than the true ruby. By Europeans the true, or oriental ruby, is regarded as not only more

* 'Mineralogisch-geognostische Reise nach dem Ural, dem Altai und dem Kaspischen Meere' (1837 to 1842).

† 'Materialien zur Mineralogie Russlands,' vol. 1, p. 30; vol. 2, p. 80.

‡ 'Am. Journ. Sc.,' 2nd series, vol. 7 (1849), p. 283; vol. 9 (1850), p. 289; vol. 10 (1850), p. 354; and vol. 11 (1851), p. 53.

§ "Ueber den Smirgel von Naxos," von G. TSCHERMAK, 'Min. u. Pet. Mitth.,' Bd. 14 (1894), p. 311.

precious than any of the accompanying minerals, but as the most valuable of all gems, and the best coloured varieties fetch far higher prices per carat than diamonds of the finest water.

As might have been expected, however, but very little was known concerning the mode of occurrence of the corundum, spinel, and tourmaline in Burma before that country became a part of the British Empire in 1886. It is probable that all the fine red corundums which found their way into the markets of India originally came from this district, for the Burma mines appear to have been worked from very early times. It is said that the Burmese acquired the mines from the Shans about 1630, but they were regarded as royal property, and very jealously guarded from Europeans.

The existence of the ruby mines of Burma is referred to by many old writers like VINCENT LE BLANC and TAVERNIER. The "Capelan Mountains," mentioned by TAVERNIER* and others as the locality from which the Burmese rubies were derived, appear to be the high grounds around Kyatpyen and Mogok. Some interesting details about the district were collected by JOHN CRAWFORD,† and later by Dr. T. OLDHAM,‡ and further information was given by the Rev. F. MASON (as the result of enquiries made by Captain G. A. STROVER, of Mr. BREDEMEYER§), while Dr. R. ROMANIS and Major HOBDAY (who made the map of the district) were also able to supply Mr. MALLET with some interesting particulars before 1886.||

Very few Europeans are known to have actually visited the ruby mines before the country was annexed by the Indian Government. A runaway English sailor was, in 1830, sent up to blast the rocks by King PHAGYIDORA, but he seems never to have returned. Some time before 1833, the Père GIUSEPPE D'AMATO visited the mines, and published an account of the native methods of working.¶ It is said, too, that in the year 1881, a party of Frenchmen were working at the mines under an engineer in the king's service. About the year 1870 a German mining engineer, named BREDEMEYER, was actually in charge of the ruby mines near Sagyin, twenty-four miles north of Mandalay; but there is no evidence that he was ever permitted to visit the principal mines about Mogok.

When the country was conquered, a map on the scale of four inches to the mile was made under Major J. R. HOBDAY, a first edition of it being published in November, 1886; and in the following month a military expedition to the district was accompanied by Mr. G. S. STREETER, Mr. BILL, and Mr. BEECH, acting on behalf

* TAVERNIER, 'Travels in India,' 1684, p. 143.

† 'Geol. Soc. Trans.,' 2nd series, vol. 1, 1824, pp. 406–408; 'Edinb. New Phil. Journ.,' 1827, p. 366; and 'Journal of an Embassy to the Court of Ava,' 1834.

‡ Appendix to YULE's 'Mission to the Court of Ava,' 1858, p. 347.

§ 'Natural Productions of Burma,' 1850, p. 27; and 'Notes on British Burma,' 1852; see also 'Indian Economist,' vol. 5, p. 14.

|| 'A Manual of the Geology of India,' Part IV., "Mineralogy," 1887, pp. 42–44.

¶ 'Journ. As. Soc. Bengal,' vol. 2, 1833, p. 75.

of parties who desired to obtain a concession of the ruby mines from the British Government.*

In the following year the Secretary of State for India determined to send out an agent to make independent enquiries concerning the value of the mines, and the conditions under which it would be advisable to permit of their being worked. Mr. C. BARRINGTON BROWN was selected for this task, and every facility was given to him by the civil and military authorities of the country for carrying on his researches. The Secretary of State for India also directed that the specimens collected during this expedition should be sent to the Royal College of Science, with the understanding that, after being studied and described, they should be deposited in the British Museum, and in the Museum of Practical Geology, Jermyn Street.

Mr. C. BARRINGTON BROWN's report on the Ruby Mines of Burma was forwarded to the Indian Government, June 15th, 1888; the fuller account of the geology of the country being deferred till the large and interesting collection of rocks and minerals brought from Burma could be examined and described.

Subsequently to Mr. BARRINGTON BROWN's return two interesting notes have been published by Dr. FRITZ NOETLING, Palæontologist to the Geological Survey of India; one a "Report on the Namseka Ruby Mine in the Mainglôn State (Northern Shan States)," and the other a "Report on the Tourmaline Mines near Mainglôn." These mines are both situated in the district closely adjoining the ruby district of Burma.†

In 1889, also, Mr. T. LA TOUCHE, of the Geological Survey of India, gave an account of the sapphire mines in the Zanskar Valley, in Kashmir, where the blue corundum occurs in gneiss, apparently very similar to that of Burma, the minerals associated with it being anthophyllite (kupfferite), tourmaline, and its alteration product cookeite, spodumene, and lazurite.‡

In the following memoir each section is initialled by the author who is responsible for its contents. J. W. J.

II. GEOGRAPHICAL DISTRIBUTION OF THE RUBY-BEARING ROCKS IN UPPER BURMA.

Extending from Wapudoung village, 11 miles east of the military post of Thebayetkin, to the Shan town of Momeit, in an east-north-east direction, is a wide belt of mountainous country, composed of gneissic rocks, containing along its central portion massive beds of crystalline limestone. The breadth of this tract is about 12 miles, in its widest part; and it has a length of 26 miles. It is situated at a distance of 90 miles to the north-north-east of Mandalay. The rocks forming this mountainous district, especially in the central part of its eastern extension, are ruby

* 'Journ. Soc. of Arts,' Feb. 22, 1889; "Precious Stones and Gems," STREETER, 5th edition, 1892, p. 165.

† Dr. NOETLING's reports are dated 13th November, 1890.

‡ 'Records of the Geological Survey of India,' vol. 23, Part II., pp. 59–69.

bearing; for along this line are situated the principal mines where that gem is found—either in the hill-wash on the mountain sides and gullies, in the cavities of the crystalline limestone, and in the limestone itself, and in the alluvia of the rivers and streams. In all likelihood the ruby-bearing rocks extend further to the eastward; but this region was not examined, owing to its lying outside of British territory, and being consequently a dangerous one to traverse. Although the distance from Thebayetkin to Mogok—the principal mining centre—in a straight line is not more than 34 miles, the distance by road, owing to its tortuous course up and down the mountain sides, is 58 miles. Part of this distance, as far as Kabein, is along a graded government road, while beyond to Mogok the remainder is over a straighter but more precipitous Burman pathway.

From Mogok to the southward for some 15 miles the mountains of gneissic rocks gradually become less in height down to the valley of a large stream called Mobaychoung, which has deposited a wide spread of alluvial clays and gravels. The latter are extensively worked for rubellite by the natives, who say that no rubies are ever found in them. Near this the rocks are of a different character, and are either, so far as seen, mica schists, or a passage rock between schist and gneiss.

The area of the ruby-bearing tract proper, as computed by Mr. Penrose, of the Government Survey of India, under the direction of Major Hobday, in 1888, is estimated at 45 square miles in extent; but, taking in the outlying districts, where old excavations were observed, he was of the opinion that this estimate should be increased to 66 square miles. In this latter area there is little doubt that he has included the rubellite mines of Nyoungouk.

There is a small outlying tract of ruby-bearing rocks at Sagyin, 24 miles north of Mandalay and 8 miles from Maddeya, composed of crystalline limestone, forming low hills, rising from the alluvial plain of the Irrawaddy.

Some 15 miles to the northward of Sagyin are two isolated limestone hills, named Nyoungwun and Bodaw, where it is reported by the natives that rubies have been found.

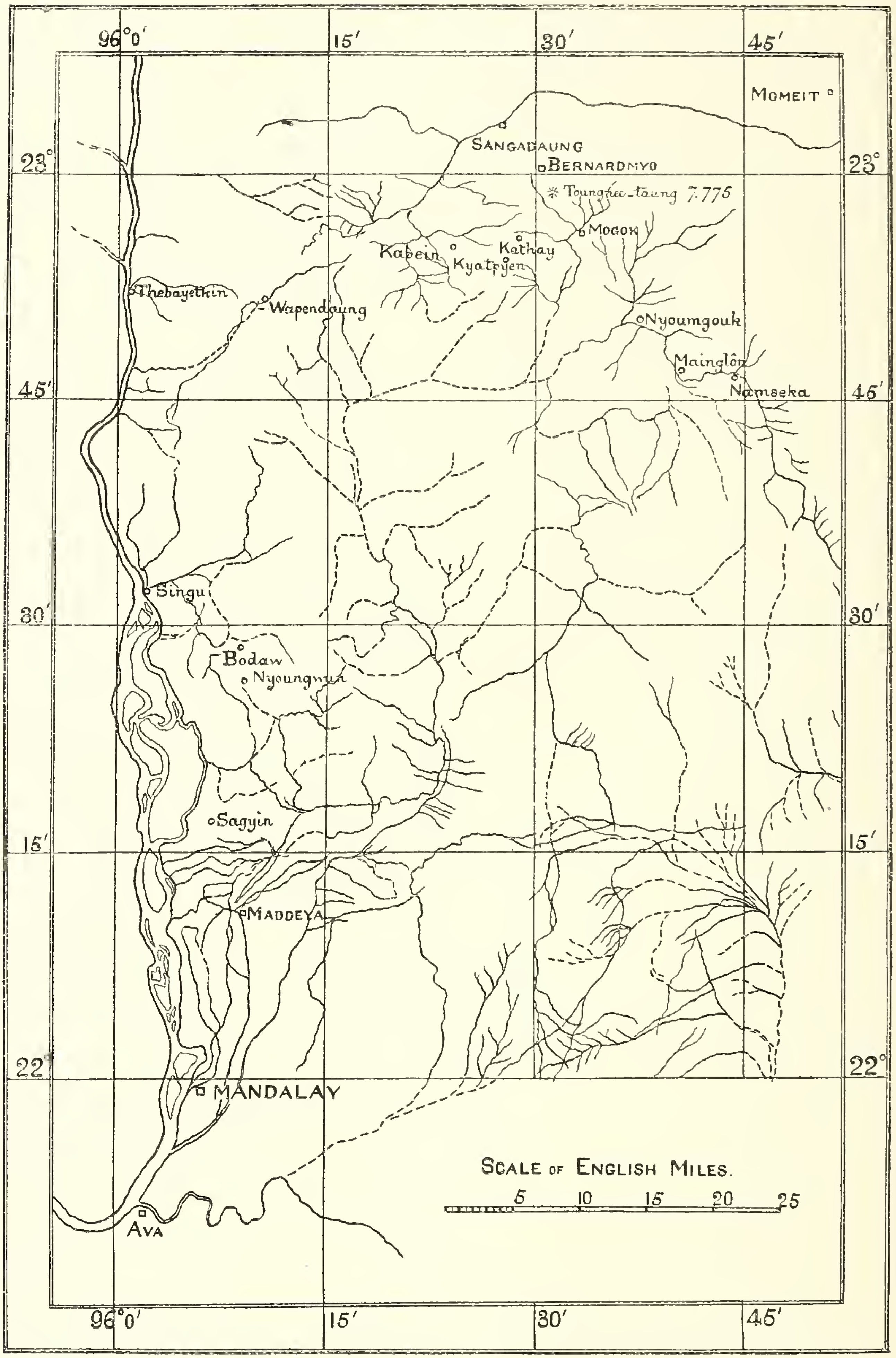
It has been stated that, in making the railway from Rangoon to Mandalay, at Kyoukse, 30 miles from the latter place, an old working in limestone for rubies was discovered.

The ruby-bearing rocks evidently extend over a large portion of Upper Burma on the eastern side of the Irrawaddy, and from thence into the Shan States.

The sketch-map (fig. 1), illustrating the distribution of the gem-bearing localities of this part of Burma, has been compiled from the official map by Major Hobday and his assistants on the Topographical Survey of India.

Before proceeding with the main portions of this paper, it is both a pleasure to me, and my duty also, to state that my investigations of the Burma Ruby Mines, described herein, could not have been completed in the time they were, had it not been for the invaluable assistance I received from the following gentlemen, viz.:—

Fig. 1.



Sketch-map of the country north of Mandalay and east of the River Irrawaddy, showing the relative positions of the gem-yielding localities of Burma.

Mr. G. D. BURGESS, Commissioner of the Northern Division of Burma, whom I accompanied from Mandalay to the mines; the late Mr. FFORD, Deputy Commissioner of the Mines Tract; Mr. J. P. CAREY; Mr. E. BUCHANAN, of the Forest Department; Mr. PENROSE, of the Survey of India, who provided me with working maps of the district, &c., and Major HOBDAY, of the same service, who had me supplied subsequently with a complete set of maps; Mr. BENNETT, the Inspector of Post Offices of Northern Burma; MOUNG KYAW KHINE, the Treasury Officer at Mogok; and Lieutenant ANDERSON, who commanded the Military Police; Mr. FOWLE, the Sub-divisional Officer of the Sagyn District; and also Mr. THIRKELL WHITE, who, on my return to Mandalay, was Acting Commissioner for the time being.

Subsequently, at Simla, when I reported to His Excellency, the Viceroy of India, the Earl of AVA, I was received with great kindness, and was much indebted to Sir EDWARD BUCK and Mr. W. N. LAWRENCE, of the Revenue Department, for assistance and kind attention. C. B. B.

III. PHYSICAL FEATURES OF THE RUBY MINES DISTRICT.

From Thebayetkin on the Irrawaddy river, 600 feet above the sea, the starting point for the ruby mines, the country is composed of a hilly tract which increases gradually in altitude until Wapudoung is reached at a height of 1700 feet. Then commences the great group of mountain ranges of that region, trending irregularly in an easterly direction to Mogok, and extending for a considerable distance in the Shan State of Momeit beyond. From the great main ranges, long spurs descend more or less steeply to the plain of the Irrawaddy on the north, and to the hilly grounds and valley of the Mobaychoung on the south. Amongst these mountains are numerous streams which have cut out gorges and valleys in various directions.

The greatest elevation of the mass to the westward of the Kin-choung river, which flows past Kinua (Kinyua)* at 2200 feet above sea level, is about 3500 feet, while, to the eastward, in the neighbourhood of Mogok, the highest points it attains are at Toungnee-taung (Taungmé) and Cheni-taung, which are respectively 7775 feet and 7362 feet above sea level.

From the Irrawaddy to the head of the Kyatpyen (Kyaukpin-meu-ma) all the country is covered with forest, but eastward of that point the extensive valleys of Kyatpyen, Kathay (Kathé), Mogok, and Injauk (Ingyauk), and a great portion of the mountain sides bordering them, are clothed with grass, dotted here and there with small trees and shrubs, with groves in places.

The principal rivers and streams traversing these mountains are the Mogok-choung, with its affluents the Yense and Yaboo; the Avoo; Kyoukwa; Nammi; Injauk; and the Kinchoung.

The Mogok-choung flows in a southerly direction for a considerable distance and

* Thus on map of India Survey.

empties itself into the Mobay-choung; while the Avoo, flowing past Kathay, is joined by the Naghu, from Kyatpyen, further on, and runs into the Mogok-choung some four miles south of Mogok. Rising some seven miles south of Kinua, the Kin-choung flows northward, being joined by the Injauk near Sagadaung, and probably falls into the Shwali river. The Nammi, rising near Chenitaung, runs a south-westerly course to the Mobay-choung.

At Kauklabin there is a fine stream bordered by a level alluvial patch; while in the valley of the Kin-choung, at Kinua, there is a large stretch of alluvial land. The most picturesque and interesting of the valleys, where wide stretches of river deposits form flat level tracts, bounded on all sides by mountains, are those of Mogok, Kathay, and Kyatpyen. Mogok-valley bottom is at a level of 4100 feet above the sea, and lies in a north-east and south-west direction from the village at the lower end to the foot of the mountains below Chenitaung at the upper, having a length of two miles. It is bounded on the north by spurs from the mountains on that side, and on the south by long slopes of the mountain range from Chenitaung through Zelatneetaung to Panma-ywa; while on the west are irregular masses of hills, continuing westward to Kathay, through which runs the Yeboo river in a narrow valley. Amongst these hills are a number of small alluvial flats, through which course small streams. Kyatpyen valley is 4700 feet above the sea, and that of Kathay is 4800.

Northward of Yeboo village lie the mountainous slopes of the Toungneetaung range, which forms the northern watershed of the mass. This great range sweeps from Toungnee peak in a south-westerly direction and, passing through Sagiwa peak, decreases considerably in altitude at Bolongyi; then curving through Welloo it forms the western termination of Kyatpyen valley. The southern side of this latter valley, and of Kathay, consists of low rounded ridges. From Welloo the mountains slope precipitously for 1500 feet to the valley of Kabein. Injauk valley, on the northern side of Toungnee range, is a comparatively small tract of open rolling land, traversed by a fine stream, and bounded on the south and west by the above range, and by low hills on the north and east. From the crests of these the slopes north and west fall steeply down to the plain of the Shwali river, which is continuous with that of the Irrawaddy. The upper portions of the Taungnee range are clothed with forests, which extend down its northern slopes between Injauk valley and Welloo. To the southward of Mogok, to within a few miles of Lauzee and Nyoungouk, the country is likewise well wooded.

There are many fine views and bold scenic effects amongst these mountains and valleys. Perhaps the most curious of all is the view from Mogok village looking up the valley, where the flat extent of land bordering the river narrows inwards to the foot of the mountains in a triangular form, with the picturesque village of Petswe and its quaint pagodas in the distance, and the lofty Chenitaung as a background. On the mountain sides bounding it the colouring lent by the patches of dark green scattered trees and small groves, the light yellowish tints of the dry grass, and the red, pink, and white patches of soil exposed by landslips, is very pleasing. The numerous *twinlone* mines (see p. 184) in operation on the flat with their tall yellow bamboo balance poles, form a striking feature and give an air of activity to the scene.

C. B. B.

IV. Geological Structure of the District (see Map, p. 183.)

1. *Hill Wash.*

As a result of the action of rain and atmosphere, continued through an immense period of time, portions of the rocks of the country, composed mainly of gneiss, have been completely decomposed to greater or lesser depths, and the interbedded crystalline limestone dissolved to a considerable extent, setting free the insoluble constituents. The resulting materials have been washed down the mountain sides, where they now form thick, flanking masses of loamy clays of various colours, from dark red through light red to pinkish, brown, yellow, and white, together with the harder constituents of the disintegrated rocks. During this process a rude sorting of their component parts has been effected, the sands and gravels being collected together and deposited in irregular layers near the base of the clays. The minerals these rocks contained have thus been liberated and deposited amongst the sands and gravels, where old mountain streams have placed them, amongst huge hard water-worn blocks of rock which have been detached from their original positions and rolled down. Owing to the sorting process, the ruby-bearing earthy clays and sands are found in leads at various levels up the beds of steep gullies and in the red clay hill spurs bordering them. They are invariably in irregular sloping patches, one overlapping the other. Where the sands and gravels are mixed with a dark brownish earthy clay, resulting from the disintegration of the crystalline limestone, they are richer in gems such as the ruby and spinel; and these are more frequently found on the eroded surface of the limestone beds themselves.

This superficial deposit covers large portions of the mountainous tract described; but it is only in certain localities where it is worked for the gems it contains. Owing to the mines in this deposit being large open cuttings, called by the natives Hmyaudwins, they afford fair sections disclosing its nature. In order to show this more clearly, a few of these sections will now be described.

The face of the cutting of No. 11 mine, some 20 feet in depth, is composed of reddish loamy clay, completely covering the pinnacled and eroded surface of layers of crystalline limestone. In the lowest portion of this, amongst the large cavities, is a brown earthy clay, mixed with very slightly waterworn pebbles and sand, which contains minerals, including rubies. Embedded in the deposit amongst rounded gneiss boulders is one block of hornblendic gneiss. This section is an interesting one, as showing how completely the limestone outcrop is hidden from view by the hill wash covering.

At No. 10 mine, near Dattau, in the ruby-bearing clay of a yellowish-brown colour, some of the pebbles are rounded, while others are angular. They consist of quartz and pegmatite, and lie at the base of the red hill wash. Amongst the minerals are flat plates of graphite, and in the clays above are large, clear mica crystals scattered through it.

The length of the cutting at No. 1 mine, which is close to Mogok Stockade, is some 400 yards. The height and width of the face is 60 feet, composed of red loamy clay in its upper part passing downward into a brownish clay containing sand and gravel, in which are rubies, and a few sapphires, and other minerals. This rests on the serrated surface of the white and bluish highly crystalline limestone. In the brownish clay are small blocks of semi-decomposed pegmatite.

On the south side of Mogok valley near Petswe is mine No. 9, the only one in operation on that side of the valley in 1888. It is a very extensive working in red loamy clay, having a depth of 28 feet at the face, containing mingled masses of waterworn blocks of pegmatite, ranging from small sizes up to 3 feet in diameter. These rocks contain garnets, and greatly resemble some portions of the crystalline limestone in outward appearance. The ruby-bearing clay at the base of the cutting is of a yellowish and brownish colour, and contains small blocks of semi-decomposed pegmatite.

No. 13 mine is situated on the north side of the Yeboo (Yebu) at the base of the red hill wash, which is there in its lowest part mingled with great blocks and boulders of gneiss and granulite, the latter being of a coarsely crystalline structure, and of great size, some measuring as much as 20 feet in diameter. The gem-bearing portion consists of white waterworn quartz, gravel, and sand, and is deposited amongst and beneath the boulders on the uneven surface of coarsely crystalline limestone. Opposite Kyatpyen, in mine No. 25, this deposit is composed of dark and light red-coloured loamy clay, amongst which are large waterworn blocks of gneiss and pegmatite. The face of the cutting is 30 feet high, and a pit has been sunk to a depth of 40 feet, in order to cut other layers of gem-bearing material, without apparently reaching the bed rock. This shows the great thickness of the deposit in this spot. Amongst the boulders, at the bottom of the cutting, is the ruby-bearing earthy clay of a brownish colour, which is mixed with a large percentage of exceedingly waterworn gravel composed of quartz and black tourmaline, amongst which is a quantity of mica.

In the red clay face of No. 26 mine, which is close to but at a higher elevation than the last-mentioned mine, there are quantities of small angular chips of quartz; whilst in the gem-bearing clay beneath are numerous flat crystals of graphite, similar to those so frequently seen in the crystalline limestone. In neither of these mines is any limestone to be seen, although the narrow belt of that rock which passes through No. 1 mine should cross near their position.

West of Pingu Hill, in No. 27 mine, which is a large cutting 25 yards long by

20 yards wide, and 10 feet deep, the brown loamy gem-bearing clay encloses large round blocks of decomposed pegmatite, which are reduced to the consistency and appearance of flour, with also some rusty black decomposed crystals. Some pits have been sunk in the floor of the mine to a depth of 18 feet, without reaching bed rock. The upper portion of one pit discloses layers of stream sand and pebbles, some 10 feet in thickness, mixed with brown loam. Beneath this is brown clay, in which are embedded large boulders of gneiss and pegmatite.

At No. 22 mine, near Sinkwa (Shinkwa) village, the section disclosed is some 150 feet wide, with a depth of 20 feet. The nature of the deposit shows a more bedded character than usual. The face of the mine on the west exposes 10 feet of brown clay, of which the upper portion is unproductive.

The clay in another part contains much water-worn pebbles, with blocks of gneiss and quartz which are not rounded. There is a band of blue clay lying horizontally in it, and at the base of the section are angular blocks of grey gneiss.

From a study of these sections it would appear that the brownish loamy clay, containing gems, has been made up of materials derived from the decomposition of gneissic rocks, mixed with a preponderating proportion of clayey matter and minerals derived from the disintegration of the crystalline limestone.

Amongst the ruby clays of this deposit are numerous pebbles of quartz and other rocks, which are completely waterworn, but the majority are not so, whilst in some instances scarcely any of them show any signs of attrition. For instance, the numerous bluish, opaque spinels are in their crystalline form of twin octahedrons, whilst perfect crystals of quartz are not uncommon, occurring with a small amount of water-worn pebbles of the same mineral. Also the rubies and red corundum, though sometimes found in a waterworn condition along with these, have usually, though broken, scarcely suffered from abrasion in any way. From these facts I conclude that none of the minerals in the ruby clays of this deposit have been transported to any great distance from the source of their origin, and that the rounding of some is rather to be attributed to attrition in pot holes, on rock surfaces, in the beds of small mountain streams. The fact of the minerals being intimately mixed with the brown clays, shows also that the whole deposit has been moved down the mountain sides without being submitted to forces sufficient to produce a thorough re-sorting of the component materials.

On washing the ruby-bearing materials of this deposit, and eliminating the clay and fine sand, the remaining portion is found to be made up of quartz, gneiss, pegmatite, black tourmaline, garnet, rock crystal, spinel, and ruby. In some Hmyaudwins, notably No. 1 and No. 2, a few sapphires have been found, but very sparingly. In some of these mines it is said that the upper red loamy clay contains rubies in infinitesimal quantity. In No. 2 mine large pieces of pale red, opaque corundum are found, some of which, though broken, retain a part of their crystalline form.

2. *Alluvium.*

In the larger valleys of the district there are extensive deposits of alluvial matter, consisting of clay, gravel, and sand, which have been laid down by the streams flowing through them. The materials of these vary in different portions of the same valley, being in the upper part of Mogok composed of a brown sandy loam resting upon coarse gravel, beneath which is an admixture of clayey materials containing gravel and sand, together with many rounded blocks of gneiss. In the lowest portion of the gravel and sand, rubies and quantities of garnets are found. This rests upon an under-clay which, in places, is a white floury kaolin containing white mica, the result of the decomposition of the bed rock. The thickness of the whole is from 10 to 12 feet.

Lower down the valley, in front of Mogok, the thickness of the top clay is from 15 to 22 feet, and the ruby-bearing sand and gravel beneath varies from 5 to 7 feet. Beneath this comes a stiff yellowish under-clay containing a few water-worn pebbles. The ruby-bearing material is composed of yellowish sand, in which are coarse pebbles, and rounded blocks of gneiss. It is difficult to say of what the remainder of the deposit is composed after the under-clay is reached, for the miners cannot be induced to dig deeper than the base of the ruby-bearing sand, the under-clay being soft and dangerous to sink through, its weight breaking their light timbering.

Near Taungwee the alluvium is composed of yellow and red loamy clay having a thickness of 24 feet, beneath which is from 3 to 4 feet of yellowish sand with large water-worn blocks of gneiss and pegmatite, resting upon an under-clay of a micaceous character, derived from the decomposition of pegmatite.

Between Taungnee and Mintada, away from the river, the alluvium is of a red and yellow loam amongst large blocks of pegmatite, at the base of which is a thin irregular layer of ruby-bearing gravel. The whole has a thickness of 15 feet, and rests upon the white decomposed bed-rock. Nearer Mintada the deposit is all of blackish clay and sand.

In the Yeboo valley, near the village of that name, it is formed of a brown loam passing into gray clay, of 12 feet in thickness, with dark grey ruby-bearing sand and gravel. I was unable to find any sections of the alluvium of the Kathay and Kyatpyen valleys, owing to there being no twinlones at work in that district; and the numerous remains of small round pits, long since abandoned, did not show the nature of the deposit passed through.

In the Injauk valley near Bernardmyo are old alluvial workings in the form of pits, some of which are 12 feet in depth, sunk through clay of bluish-grey and yellowish colours, with much slightly water-worn quartz gravel which came from the gem-bearing layer at the bottom. The greater portion of the excavated material has been washed away by rain. Some very perfect specimens of rock crystal are seen scattered over the surface of the ground. These mines, it is said, produced sapphires of good quality in former years.

After washing the sand for the rubies they contained, the gravel remaining is chiefly of fragments of quartz, gneiss and pegmatite, amongst which are spinel, garnet, tourmaline, and rock-crystal.

The following is a more detailed account of the deposit as examined in the pits at work. Between No. 8 and No. 9 mine, in a small section on the river bank, is a dark sandy loam resting upon a heavy brown clayey loam, with gravel and blocks of gneiss.

In Twinlone 6, near by, is 6 feet of dark brown loam on gravel in which is gneiss and pegmatite rubble, containing numerous garnets; and in the stream near by are huge rounded blocks of coarse pegmatite, some of which are 30 feet long and 15 feet high.

		ft.	in.
Twinlone 5.	Grey sand and gravel amongst large gneiss boulders	10	0
Twinlone 3.	1. Brown sandy loam	4	0
	2. Bluish-grey sand and gravel with garnetiferous gneiss rubble	3	6
	3. Ruby-bearing gravel and sand	2	6
		10	0
	The bedrock is a decomposed pegmatite.		
Twinlone 1.	1. Brown sandy loam	8	0
	2. Grey finely foliated gneiss and pegmatite rubble, embedded in sand and gravel	3	0
		11	0
Twinlone C.	1. Blackish mud and clay	15	0
	2. Dark greenish-grey ruby-bearing sand, with gneiss pebbles.	2	6
		17	6
Twinlone E.	1. Yellow loam at surface	2	0
	2. Dark loam	5	0
	3. Brownish clay, with gneiss rubble	13	0
	4. Yellowish ruby-bearing sand and gravel	6	0
	5. Stiff yellow under-clay—depth undetermined.		
		27	0+

1st Twinlone I had sunk near to Twinlone E.

	ft.	in.
1. Brownish-yellow clay	3	9
2. Dark brown loam	2	3
3. Drab-coloured clay	6	0
4. Yellowish ruby sand and gravel, with gneiss rubble and a few large blocks of pegmatite	6	0
	18	0

Yellow under-clay.

In No. 4 bed was a sloping parting of grey clay, 2 feet thick at the lower end, and 6 inches at the upper.

		ft.	in.
2nd Twinlone I had sunk.			
	1. Yellow clay	18	0
	2. Yellowish ruby-bearing sand and gravel	12	0
		30	0

This pit did not reach the bottom of the ruby-bearing sand, but the miners refused to sink deeper.

		ft.	in.
Twinlone G.	1. Yellow clay and sand, with black sandy loam	13	6
	2. Grey sand with gneiss pebbles forming the ruby-bearing portion	1	6
		15	0
Twinlone H.	1. Yellow and red clay and loam	22	0
	2. Yellowish ruby-bearing sand, with gneiss, pegmatite, and quartz gravel	3	6
		25	6
	3. Yellow micaceous under-clay.		
Twinlone K.	1. Red and yellow stiff clayey loam	13	0
	2. Yellow ruby-bearing sand, with large blocks of pegmatite and pebbles of gneiss	2	0
		15	0
	3. White floury micaceous under-clay.		
Twinlone M.	1. Brown and grey loam and clay	12	0
	2. Ruby-bearing sand, with gneiss pebbles of variable thickness.		

The alluvial deposits at Nyoungouk, and along the northern side of the Mobay-choung from Nayo, are of two ages; one being the recent alluvium forming the flat valley of the river, and the other an old river gravel deposited at a higher level, which is seen at the base of the hills. This latter is composed almost entirely of quartz and quartzite gravel, which, in some places, is semi-consolidated by the percolation of water, charged with oxide of iron, derived from the red loam resting upon it. This covering is in part a hill wash derived from the mountain sides. The base of the old gravel is generally on a level with, or above the level of, the surface of the present river alluvium.

The latter is composed of a dark loam on grey clays and sands, as seen in the river banks.

In the extensive Hmyaudwin cuttings, on the north side of the river, fair sections of the old gravel are seen, where it has been worked for rubellite. One of these, at the foot of the hills, exposes the following section :—

	ft.	in.
1. Red homogeneous loam	50	0
2. Coarse quartz gravel and rubble in which are blocks of partially decomposed gneiss and water-worn bluish-black clay slate	10	0
	60	0

The section seen in a hmyaudwin face, close to Nyoungouk village, is as follows :—

	ft.	in.
1. Red loam	20	0
2. Yellowish sand and clay, with pebbles	10	0
3. White quartz gravel, mixed with a small amount of light grey and yellow sand, the whole being more or less iron stained. Amongst this are some blocks of pegmatite containing black crystals of tourmaline and small elongated water-worn blocks of hard dark-coloured clay slate in which are thin bands of quartz	10	0
	40	0

This gravel (No. 3) contains the rubellite and some garnet.

3. *Sandstone.*

About one quarter of a mile from Thebayetkin the crystalline limestone is covered, between that and the Irrawaddy river, by a deposit of grey, friable sandstone, probably of Tertiary age. The junction of these rocks is obscure. The sandstone beds are seen on the back of the Thebayetkin ridge, and in its face on the river bank, where a cliff exposes grey, friable sandstone, containing some small quartz pebbles, while water-worn blocks of a somewhat similar sandstone form layers in it. These beds dip south at an angle of 50°.

4. *Mica Schist.*

About one mile north of Lazee, a village situated on the north bank of the Mobaychoung, this rock is met with succeeding the gneissic rocks, which are there chiefly represented by pegmatites. The first exposure is of a light greenish colour containing a large percentage of silvery mica. This rock is again seen in the bed of the Mobaychoung, not far eastward of Lazee, inclining to the south-east at an angle of 38°. On the hills at Nayo loose blocks of mica schist are met with. From Nayo there is a fine view of the surrounding hill country, which is all deeply covered with red loam, presenting a sterile appearance.

It would appear that the pegmatite passes into mica schist, which probably forms a large tract of country to the southward.

Higher up the Mobaychoung, beyond Nyoungouk, there must be a further change in the nature of the rocks, probably from mica schist to clay slate, for waterworn pebbles of the latter are met with in the old gravel beds of the Mobaychoung, in the vicinity of Nyoungouk.

5. *Gneissic Rocks.*

As before stated, the rocks composing the earth's surface over the large extent of country embracing the ruby mines tract are composed chiefly of gneiss. These are covered in places by hill-wash on the mountain sides, and alluvial deposits in the valleys, but every here and there they appear at the surface in large exposures, enabling their varieties and structure to be studied. They contain bands of crystalline limestone of various thicknesses, which form a subordinate, though most important, part of the whole, insomuch that all along their outcrops are situated the gem mines which form the chief industry of the district.

The chief variety of gneissic rocks met with is a hard, coarse-grained, grey compact gneiss of somewhat massive character, which shows its true foliated structure by the weathering of its surface. This passes into a fine-grained gneiss of lighter colour and finer foliation, which in parts shows a slight contortion of its foliation. In a few places some of these show a very contorted foliation, and resemble the form once known in Germany as Stangel gneiss. Amidst the ordinary gneisses there occur massive and thin interfoliated layers of a rock composed almost entirely of quartz and felspar, the latter often in very large crystals, which sometimes assumes the character of an ordinary graphic granite. For this rock I employ the term pegmatite. It does not, as far as I was able to observe, constitute veins or intrusive masses, but forms a member of the gneissic series with which it is interfoliated, and possibly interbedded. This bedded character, both in this rock and in the gneiss, may be in appearance only, and due to tabular jointing, coinciding with the planes of foliation; but when examining such exposures as those of Chenitaung mountain and on Toungnee pass (to be afterwards described), one cannot but be struck with the resemblance of this to true bedding. The last variety to be mentioned, though not extensively developed, is seen on the range north of Kathay, near Sagiwa mountain, and also near Bernardmyo. It is of a coarse texture, of a greenish to greenish-yellow colour, and of a somewhat waxy appearance, containing in places sparsely scattered crystals of black mica, similar to that sometimes seen in the coarser pegmatites. A layer of dark hornblendic rock is met with in a few places, which may be a hornblende gneiss, but it occupies only a very subordinate position.

The main range on the south side of Mogok valley, from Chenitaung to Panma, is composed of gneiss of the massive, coarse variety; but to the southward it passes into the lighter and finer textured sorts, with interfoliated pegmatite, almost as far as the Mobaychoung.

The mountains north of Mogok, and onwards to Bernardmyo, are principally formed of the hard, finely-foliated variety; and a similar gneiss is also well developed south of Kyatpyen and Kathay.

The gneiss from the valley of Kyatpyen westward to Wapudoung is of the ordinary finely-foliated, grey variety, which near Shwaynambin shows a contorted foliation. There was no opportunity of examining the rocks more than superficially in passing over the road from Kabein to Thebayetkin, as the journey was performed on horseback in long stages, which left little time for stoppage on the way.

The strike of the foliation, and apparent bedding, varies from north-east and south-west on the south side of Mogok valley, to east and west on the southern slopes of Toungnee; then from south-west and north-east from Sagiwa to Pingu hill, and in a general east and west direction to Kabein, and, as far as could be made out, continuing in that direction to a point halfway between Nampan mountain and Kauklabin, where it curves round to the northward. Its dip is in a southerly direction at right angles to the strikes above enumerated, generally at angles of from 20° to 80°, while in a few places it is vertical, as seen in the Chenitaung range near Zelatnetaung. On the mountain side near Wapudoung it dips west at an angle of 45°, and evidently in the vicinity of that place it becomes horizontal, or nearly so, for beyond to within one quarter of a mile of Thebayetkin one of the great inclosed layers of crystalline limestone—the outcrop of which is seen in the gneiss on the mountain side above mentioned—assumes an almost horizontal position, forming the rock surface onwards. In two places only on this portion of the road are the gneissic rocks seen; one appearing about two miles west of Wapudoung, and the other some three miles from Thebayetkin. The first is where gneiss and pegmatite form a band for about half-a-mile; and the second where thin beds of gneiss crop out at very low angles in the crystalline limestone.

Crystals of garnet are more or less disseminated through the gneissic mass, and it is possible it contains rubies and spinels as well, but no evidence of this has yet been obtained.

On the path leading from Mogok to Ongain the foliation of some gneiss beds is slightly contorted, and dips at an angle of 80°.

Between No. 4 and No. 2 mines there is a large exposure of fine-grained, grey gneiss 40 feet in height, the foliation of which dips southwardly at an angle of 75°.

On the mountain side on the road from Mogok to Howet there is an exposure of solid grey gneiss 80 feet in height, of a fine-grained variety, containing garnets. Lower down the hill in the same mass are interfoliated layers of coarse, white, granular gneiss containing garnets of rounded and lenticular forms, very impure and oxidized in parts. Amongst them are flat, black crystals of a micaceous character. This section is an exceptional one as regards the inclination of the foliation, which inclines to the northward.

The dividing ridge between Mogok and Momeit valleys is very precipitous between the pathway in the pass and Chenitaung peak, where an extensive section is exposed of bands of gneiss dipping evenly to the south-east at an angle of 35°. So regular are these bands, that from a distance they greatly resemble beds of sandstone.

Here they are of finely foliated grey-and-whitish gneiss somewhat decomposed, in bands of from 10 to 20 feet in thickness. The last one, where further ascent was rendered difficult by the precipitous nature of the ridge, was a coarse white granular gneiss, containing little mica, but with large nodules of garnet. Near the top of the pass is crystalline limestone in the gneiss, occupying a width of 50 feet, and containing large crystals of white felspar and small crystals of graphite. Half-way down the mountain below this are some loose blocks of "Stangel Gneiss," outwardly bearing a resemblance to stems of fossil trees.

Towards the point at the head of Mogok valley, where the Yenee river emerges from the limestone, the red loam rests upon the semi-decomposed gneiss, with huge rounded blocks of the latter rock on the surface ; the hills thus formed close in on both sides of the river, constituting a barrier, which separates the level bottom of the valley from a second extensive flat alluvial area, extending up to the head of the river.

Two-thirds of a mile southward of Mogok, in the narrow gorge of the Mogokchoung, is an exposure 300 yards in width, of dark-coloured, fine-textured gneiss in vertical bands, varying from a foot or two in width to 10 feet. Interfoliated with these are two layers of crystalline limestone, the first being 30 feet wide, in which is a band of gneiss one foot in width ; and in the second, which is 20 feet wide, are several narrow vertical bands of dark-coloured, fine-textured gneiss. There are also bands of pegmatite in the gneiss, one of which is five feet in thickness. From this section, laid bare by the river, showing the limestone, when it is not visible along the range either east or west, owing to the covering of hill-wash, it must be inferred that in many other places there are small bands of crystalline limestone in the gneiss, which are not seen at the surface on the hill-sides. It is only the large bands of limestone of great thickness, whose outcrops stand out in jagged masses above the surrounding surface, that prove the existence of this rock and form an important feature in the geology of the district. In the limestone beds above mentioned, are violet-coloured crystals of garnet, and the same are also seen as accessories in the adjoining gneiss.

From Mogok valley, on the road to Momeit *viâ* Kyaukwa, up to the pass, where rock outcrops are visible, they are all of gneiss, thin-bedded and grey, with interfoliated coarse, whitish gneiss, and pegmatite containing garnets. At the pass itself all the rocks are whitish pegmatite. In a line with the dip of the gneiss on the peak close to Letnytaung (Lennu Taung), a flat-lying, grey gneiss is seen on this road, which probably is the same as that of 40 feet in thickness which caps the crystalline limestone of the peak, and does not coincide with the usual dip, but slants irregularly in a south-easterly direction at an angle of 35°. It may be that here there has been some disturbance, in the form of a more or less horizontal fault, or a "lateral thrust." To the east of the pass the mountains of gneiss are extremely decomposed to a great depth, and contain a band of pegmatite in which are bluish crystals of apatite. Near by to the south, this rock contains large crystals of dark-coloured mica. From the pass down to the river in the bottom of the narrow valley and along to Kyaukwa, all the surface rocks immediately along the pathway are of finely-laminated gneiss, apparently inclining south.

South of the Yenee, near the village of Taungwa (Taungywa), not far from the bridge, on the hillside there is an enormous block of dark pegmatite, which contains a reddish felspar, and much quartz. This curious rock is rudely foliated, and resembles a block of gneiss, but differs in not containing any mica. Below this rock in the gneiss are three small veins of quartz, two horizontal, and one vertical. The only other places where veins of quartz were observed was on the road to Howet, where a small one occurs, at a short distance south of Kyatpyen; and near Kyauktaing village, where one of 18 inches thick could be traced for a short distance.

The first four miles along the new military road from Bernardmyo to Kathay, the cuttings disclose semi-decomposed gneiss containing coarse, light-coloured pegmatites, in some of which are small, round, decomposed crystals of a brown, iron-stained mineral, probably garnet. Near the dividing ridge on the same road, is a small section of decomposed gneiss which is foliated in various directions, and incloses rounded eyes of pegmatite around which its foliation curves.

The hill to the north-east of the cantonments of Bernardmyo is composed of evenly foliated grey gneiss, in parts of which are nests of white, finely crystalline quartz. In the stream at the foot of the descent from Taungnee to the Injauk valley are some blocks of a very dark-coloured coarse gneiss, hornblendic gneiss, and pegmatite. One quarter of a mile from the bridge across the Injaukchoung is a white quartz-schist, apparently interfoliated in the gneiss.

Near Thaungla the gneiss is seen in alternating bands of fine and coarse varieties, the former varying in width from a few inches to one foot, whilst the latter are of much greater thickness.

In ascending Sagiwa from the Kathay-Bernardmyo road, a set of massive, greenish-yellow beds of gneiss, containing in parts large scattered crystals of black mica, are crossed with ordinary gneiss near the top, on which is a coarse pegmatite. Resting on this is fine-textured gneiss with contorted foliation, enclosing white quartzite containing crystals of graphite. The peculiar greenish gneiss having a waxy appearance, is well-developed on the dividing ridge beyond Sagiwa to the north, and can be traced south-westward for over a mile towards Bolongui. This rock when weathered is on the surface of a light yellow colour.

From Kyatpyen for a mile south, the gneiss is thinly foliated and wavy in parts; and onwards to Nayaw (Nounghwai), some six miles to the south-south-west, it contains interfoliated pegmatite. At two miles from Kyatpyen the rocks are much decomposed, and landslips disclose the red, pink, and white colouring of the resulting clays and loams, similar to those seen on the south-east of Mogok valley. Close to the village of Nayaw is a section of coarse-jointed gneiss, dipping south-west at an angle of 50°.

Crossing the Yevoo, a quarter of a mile below Kathay, is a hard set of gneiss rocks showing a slightly contorted foliation, which forms a small fall.

In the gneiss from Kyatpyen valley head to Bolong there is an extensive exposure of pegmatite, of a coarsely crystalline texture, the greater portion of which is made up of large felspar crystals which have a satiny lustre. This rock differs from its other developments in having a large amount of whitish mica.

Not far from Pingu hill, on the west, there is a small section disclosing the junction of the gneiss and a limestone band. Here, at the plane of contact, the rock is of a nearly black colour, resembling a hornblendic gneiss ; and is in a band 4 feet in thickness, inclining south-west at an angle of 35°. This rock also occurs near Kyatpyen valley head, near Welloo, at No. 13 mine, and near Dattau ; and is evidently a continuous band adjoining one of the great limestone masses.

On the north side of Pingu hill, which is very steep and joined by a ridge from Kyauktaing, the rocks are of gneiss. Then for 50 feet of the ascent these, in large blocks, are mingled with masses of the limestone. They have evidently slipped down the uneven surfaces of the limestone beds, and in some instances have fallen into the open cavities in that rock. On the top of the hill the gneiss forms a thin layer, dipping south. From Panma, on the south, the winding path leading to the summit (5,660 feet) is on a comparatively easy slope over gneiss, until near the top, where there are thin limestone outcrops. The first portion of the ascent is of white pegmatite, succeeded by grey gneiss higher up. Near the top are some curious gneiss bands of 2 feet in thickness, which are very friable, somewhat resembling an altered sandstone. A block of limestone near the summit has a band of greenish rock in it, $1\frac{1}{2}$ inch in thickness, of a gneissose character.

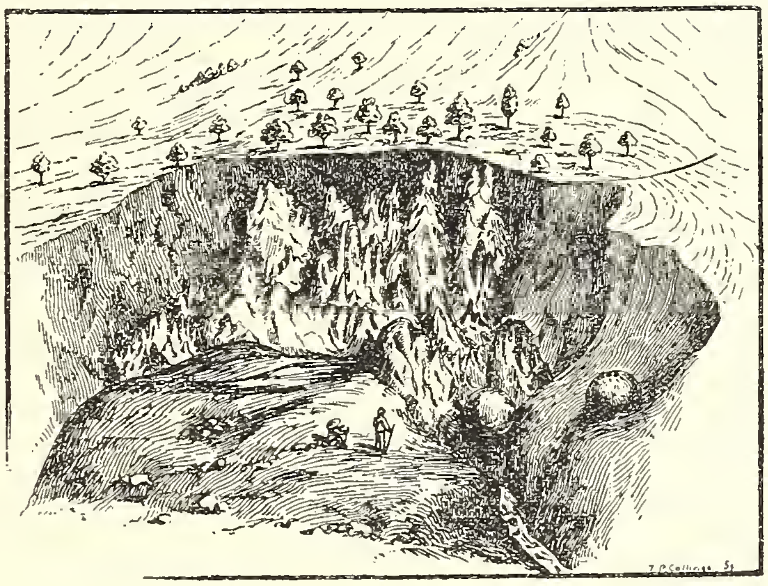
At the foot of Mandalay hill, not far from the late King's Palace, are bands of pegmatite of white, greenish, and reddish colours, in which is an interfoliated band of crystalline limestone, containing brownish mica crystals.

6. *Crystalline Limestone.*

I have deemed it advisable to give a description of this rock, and its mode of occurrence, separately from that of the gneiss in which it forms great bands, owing to its very interesting nature. As far as the limited investigations of the geology of the district went, it was only possible to partly trace out the extension of some of the larger bands, from the head of Mogok valley to Kabein ; and even the result of this work, to a great extent, must be looked upon as provisional only.

The outcrops of this rock are easily discerned crossing the mountain sides and spurs of the open portions of the country, in the form of dark grey masses rising above the surface of the enclosing gneiss. Their true white colour is completely disguised by a dark greyish *lichen*, which coats and clings tenaciously to their surface. On the other hand, on many parts of mountain slopes their continuity is completely hidden by the thick covering of hill-wash; while, in the valley bottoms, they are rendered invisible by the deposits of alluvial matters. But in many cases, in these positions, their existence is made known through the removal of these superficial deposits by the operations of the native miners, in extracting the clays containing rubies (see fig. 2).

Fig. 2.



Hmyaudwin No. 11.
Showing pinnacled surface of limestone where hill-wash has been removed.

It is of the usual composition and character of ordinary crystalline limestones, being made up of finely crystalline or granular limestone in layers, together with irregularly shaped bands of very coarsely crystalline limestone of white and bluish colours, which are interfoliated with the gneissose rocks. The surface of the limestone is always serrated and pinnacled by atmospheric action, and contains multitudes of sinuous caves leading down in the direction of the bedding planes to greater or lesser depths. One of those in mine *a* is said to have been followed by the miners to a depth of 390 feet. In some instances there are also horizontal caves traversed by small streams for considerable distances. At mine *a*, near Kyauksan, which is situated in a small depression on the limestone, the water led by a trench down the mountain to the mine, for the purpose of disintegrating and washing the ruby clay brought out of the tortuous natural pits, is led down an abandoned one in a somewhat turbid state, and finds its way to a cave, the mouth of which is at the head of the Kathay valley, some distance off. This cave, which I explored for a considerable distance, has a good-sized stream flowing in it, which, though rendered turbid by the mine's gutter, contains far more water than is artificially let into it. It has a winding course, sometimes along the planes of foliation but also frequently across them, and is from 20 to 30 feet in height in places.

There are eleven distinct limestone exposures between the falls on the Mogok-choung and the peak next below Toungnee on the southern slope of that mountain.

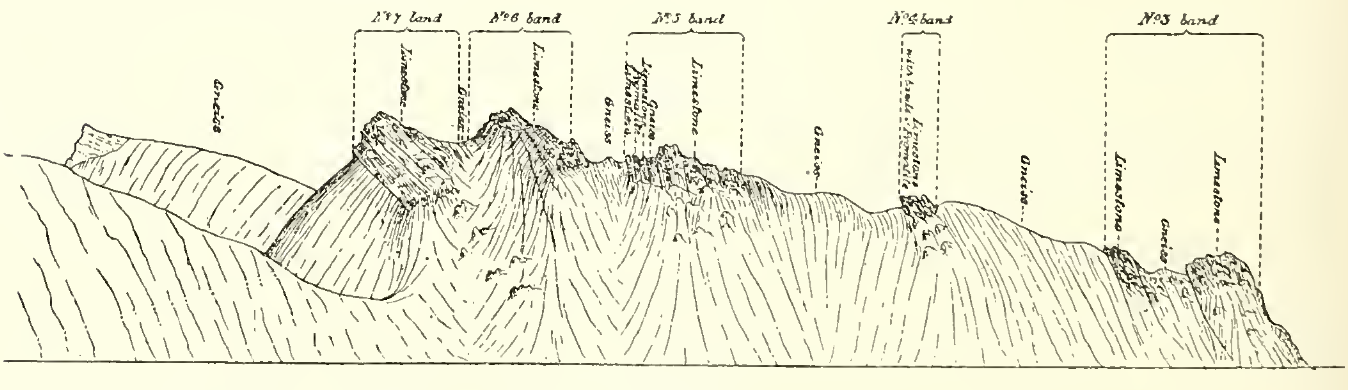
The long spur from this peak down to Mogok valley discloses the existence of nine of these; one more is seen in the valley, and one in the above-mentioned falls. These vary in width, across their outcrops, from 20 feet to 700 feet.

The principal band No. 3 (see fig. 3), along which the most important mines are situated, is some 300 feet in width and, as far as it can be traced out, extends in a sinuous line from Dattau through the end of Letnytaung spur, on the edge of Mogok valley, onwards in a westerly direction through Bobadaung. There it curves in a south-westerly direction past Pyanbin to Pingu hill, where its course is altered to the north-westward as far as Bolong. From thence it passes through Welloo and Kyaukmyo to Kabein.

Beyond this westwards there was no attempt made to fully investigate the limestone band further; in fact, had time allowed, the work would have been almost rendered impossible, owing to the trackless nature of the great forest-covered mountains of this portion of the district. In places, however, where some of the bands crossed the roadway onwards, they were noted; and I found that (as before-mentioned) one band, after curving to the northward in the neighbourhood of Wapudoung, became almost horizontal, forming, with but one exception, the surface rock along the road nearly to Thebayetkin (see map, fig. 11).

Although these bands are somewhat parallel to each other in parts, they are not so in others, and it would be impossible to expect much regularity in their relations to each other, or much conformity in the width of each for any distance, associated as they are with a class of rock which must have been subjected to considerable disturbance and displacement. In all cases their dips are conformable to the foliation of the gneissose rocks (see fig. 3).

Fig. 3.



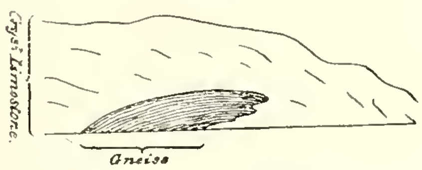
Sketch of part of Letnytaung spur, showing the limestone bands in the gneiss.

Commencing with the small bands at Mogokchoung Falls, and ending with that on the peak next below Taungnee—a distance of four miles—as far as I was able to roughly estimate them, their widths were as follows;—

No. 1. 50 feet; 20, and 30 feet, in two bands, in Mogokchoung Falls.
,, 2. 150 ,, on the north side of Mogok valley.
,, 3. 300 ,, in two bands of 200 and 100 feet, with 100 feet of gneiss between, on the north edge of Mogok valley, at end of Letnytaung spur.
,, 4. 50 ,, on Letnytaung spur.
,, 5. 400 ,, in two bands of 300 and 100 feet, with 100 feet of gneiss between, on Letnytaung spur.
,, 6. 250 ,, close to Letnytaung peak.
,, 7. 400 ,, at Letnytaung peak.
,, 8. 50 ,, between Letnytaung peak and that below Taungnee.
,, 9. 300 ,, ,, ,, ,, ,,
,, 10. 700 ,, ,, ,, ,, ,,
,, 11. 70 ,, in peak next below Taungnee.

The bands from No. 2 to No. 11, inclusive, occupy a distance of $2\frac{1}{2}$ miles, and their collective thickness, across their outcrops on the surface, is about 2670 feet.

Fig. 4.



Gneiss in crystalline limestone.

Small bands of gneiss are frequently seen in the crystalline limestone (see fig. 4), as well as those of pegmatite in a few instances. As a general rule the former are evenly foliated, and apparently undisturbed, as far as the small sections exposed allow them to be seen, but in one or two instances this is otherwise. For instance, a small band, 2 feet 6 inches wide, in the limestone (No. 7), on the side of Letnytaung spur, is contorted and twisted in a remarkable manner, both the gneiss and limestone having the appearance of being at one time subjected to a vertical, and also lateral pressure, by which they were rendered, as it were, of a plastic state (see fig. 5). Near by, on the mountain top, a band of gneiss, 1 foot 6 inches wide, seems to have been ruptured by the limestone being forced through it, when the whole appears as if it were subjected to an intense pressure when in a plastic condition (see fig. 6).

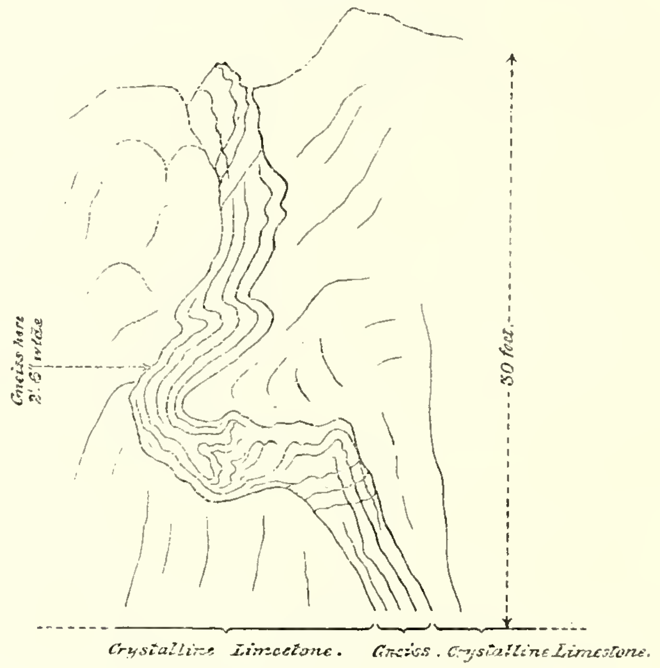
A small section of these bands is seen on Taungnee Pass, where they are undisturbed, their planes of foliation dipping south at an angle of 55°.

This is as follows (see fig. 7):—

1. White crystalline limestone, containing spinel.

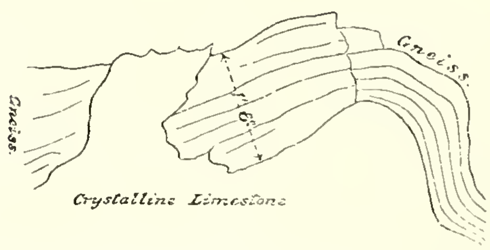
2. Finely foliated grey gneiss, slightly decomposed, 4 feet in width.
3. White crystalline limestone, 2 feet.
4. Thinly foliated gneiss, 6 feet.
5. Coarsely crystalline white limestone, containing violet-coloured spinel and green crystals of augite, together with graphite, 16 feet.
6. Finely foliated gneiss, 15 feet.
7. Coarsely crystalline limestone.

Fig. 5.



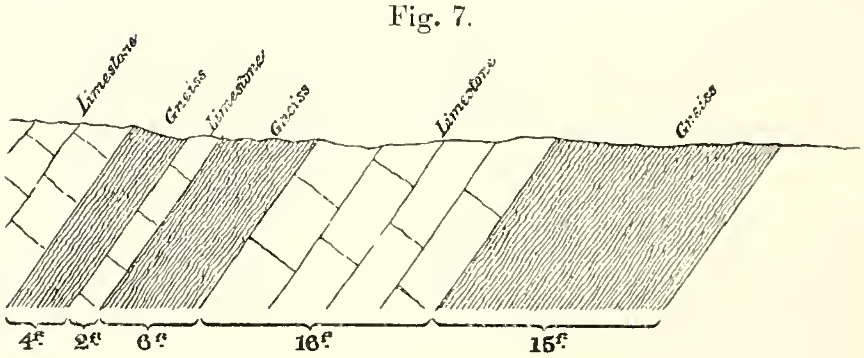
Highly contorted gneiss and crystalline limestone.

Fig. 6.



Crystalline limestone forced through gneiss.

Fig. 7.



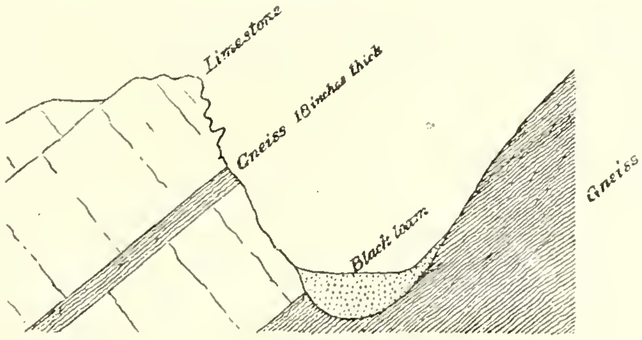
Section on Taungnee Pass.

Another section, occurring near mine No. 15, is as follows (see fig. 8):—

1. Coarsely crystalline white limestone, containing brownish mica crystals, in parallel planes giving it a schistose structure, which are most numerous at the plane of contact with the next band.
2. Hard grey compactly crystalline grey gneiss, 18 inches.
3. Coarsely crystalline limestone, 8 feet.
4. Finely foliated grey gneiss.

These dip to the south-south-east at an angle of 40°.

Fig. 8.



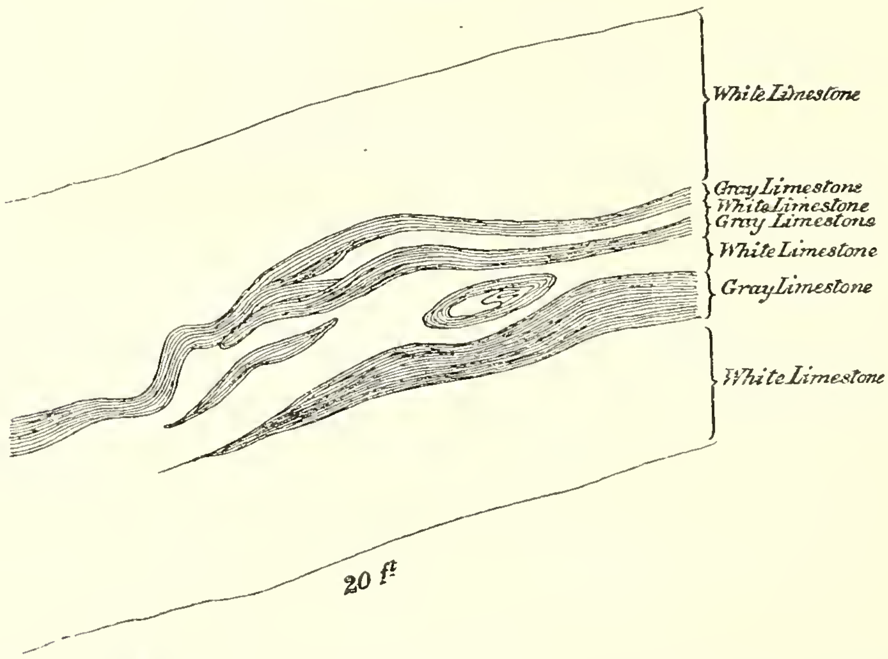
Section near No. 15 Loodwin.

It is difficult to find good sections showing the junction of the large limestone bands with the main masses of gneiss, where they could be easily observed. In those few places where an examination was rendered comparatively easy, it was seen that near the plane of contact the limestone has a schistose appearance, owing to the contained crystals of whitish and brownish mica being arranged in planes parallel to the foliation of gneiss. In that part also appeared to be a greater variety of accessory minerals, such as graphite, white opaque felspar, and violet-coloured spinel. Further away the limestone became more coarsely crystalline, passing into a sort of white opaque, or bluish and greyish semi-transparent calc-spar, in rhombohedrons which sometimes attain a size of over six inches across.

The best section of the junction, as well as of one of the main limestone bands, is seen near Pyagone, where the latter (probably No. 3 band) crosses through a ridge from Bobedaung. Even here it is difficult to accurately examine this, owing to the somewhat broken surface of the outcrop, and to the covering of hill-wash upon it in places. The junction is very clearly defined in a small cliff-like section at No. 16 mine. Here the grey gneiss, of an evenly and finely laminated variety, rests on the limestone, its foliation dipping to the south, like that of the latter rock, at an angle of 45°. Although the plane of contact is clearly defined, yet the gneiss firmly adheres to the limestone. Tracing the section northward it is as follows:—

1. Finely laminated gray gneiss.
2. Crystalline limestone, containing spinel, gneiss, felspar, &c., 18 inches.
3. Finely laminated gneiss, 6 inches.
4. Coarsely crystalline limestone, containing graphite, light-coloured mica, and green crystals of augite, 50 feet.
5. Very coarsely crystalline limestone in irregular shaped bands, alternating with ordinary crystalline limestone, probably 150 feet.
6. Finely crystalline limestone, containing graphite, mica, and spinel, probably 100 feet. At mine *e*, in this, there is a band of grey gneiss, the foliation of which is inclined at an angle of 75°, corresponding with the inclination of the limestone in this part.
7. Gneiss underlying the limestone just beyond mine *e*, probably 600 feet.
8. Band of crystalline limestone, probably 150 feet.
9. Gneiss.

Fig. 9.



Section showing bands of grey limestone in white limestone at mine *d*.

This band is again seen cropping out in spurs near Pyanbin on both the east and west sides of the basin-shaped depression at *b* mine. On the eastern side it has been exposed by the detrital covering, having been removed from its jagged surface, and slopes south-easterly at an angle of 75°. On the western side it has the same dip, and rests upon the gneiss at the same angle. It again appears on the eastern side of Kathay valley, but its greater portion onwards is hidden by alluvium as far as Panma village. From thence onwards it passes through the north-eastern side of Pingu hill, and is seen outcropping here and there through Bolong, Welloo, and Kyaukmyo, to Kabein (see Map, fig. 11, p. 183).

Some of the large bands north of Letnytaung pass eastwardly into the Kyaukwa valley towards Momeit, and eastwardly up the slopes of the Taungnee range, and over it into its descent northwards, where they are crossed on the road from Kathay to Bernardmyo.

Besides the band above mentioned, another one is seen on the western side of Kathay valley, which is probably the western extension of that at No. 1 mine. Where the new road crosses it the cuttings disclose 60 feet of crystalline limestone, then 30 feet of gneiss, and then 120 feet of limestone. To the east of this section there is a deep depression, in which this band is seen, but beyond, all the way to Mogok, its outcrop is not again visible at the surface. It is hidden by the alluvium of Kathay valley, but at mine A, at Kyatpyen valley-head, at Ya-a, and at Kyaukpyasa, it is seen again. At the latter place one portion of it constitutes a peak composed of coarsely crystalline limestone 40 feet in width, rising 50 feet above the gneiss on its north side, and forming a precipice on the south side of between 300 and 400 feet in height. A large solitary crystal of light-coloured mica of 4 inches across is seen in one part of it.

This variety of rock is met with in other places, viz.: on the mountain slope on each side of Taungnee pass and elsewhere, rising from the surface of the limestone in narrow, uneven developments, which greatly resemble the ruined walls of an old fortress, some being 15 feet in thickness, and as much as 30 to 40 feet in height.

Halfway between Mogok and Kathay are the outcrops of two small bands of limestone, which are probably an extension westward of the two seen in the Mogok-choung falls.

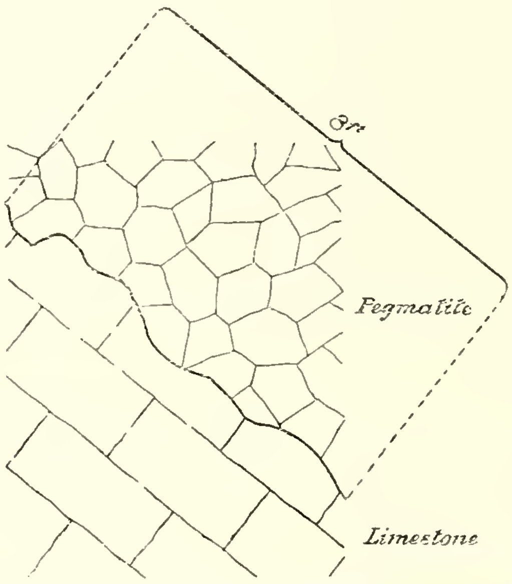
In the limestone bands No. 4 and No. 5, on Letnytaung spur, there are bands of coarse pegmatite containing large black mica and tourmaline crystals. One of these in No. 5 is 15 feet in thickness, and it is difficult to determine whether it is intrusive or not. It appears to be more vertical than the enclosing rock.

On the pass at the head of Kyatpyen valley there is a section showing a junction of whitish porphyritic pegmatite (in which are large crystals of felspar) and coarsely crystalline limestone. The former rests upon the latter, adhering firmly to it, and seems to fill the inequalities in the limestone surface. These dip south at an angle of 40° (see fig. 10). In places we find an interfoliation of white and grey limestones (see fig. 9).

From Kabein, the new government road runs in west-north-west direction past Kinua to Shwaynambin, leaving the main limestone outcrops to the south; but not far on from Kinua a high peak of gneiss on the right is seen, capped by a band of limestone dipping north-west at an angle of 20°. It would appear that a great boss of granite, which I had no opportunity of examining, has forced up the gneiss and altered the dips from southwardly to northwardly. Before arriving at Shwaynambin a band of limestone is seen in the gneiss, and another small one a little past that village. From this the general direction of the road to Nampan mountain is south-west, so that it again crosses limestone bands, which are undoubtedly some of those extending westwards from the ruby mines. The first of these, which is crossed between

Nampan and Kyauklabin, at about two miles from the latter place, appears to be only 90 feet in width. Another similar outcrop is seen before coming to Kyauklabin, and the next is crossed beyond that place, dipping south at an angle of 45°. These are followed by a few very small bands.

Fig. 10.



Section on pass between Kyatpyen and Kabein.

Descending the mountain to Wapudoung, the tortuous-graded road crosses and recrosses a thick limestone band all down the hill, interbanded with gneiss. Where this is first met with on the mountain top it appears to dip southwardly at an angle of 45°, while half way down it seems to incline to the west at a much lower angle. It would appear that this band of limestone, and the gneiss as well, curves sharply round from west to north, and becomes almost horizontal in the vicinity of Wapudoung, further on continuing with slight undulating dips* in that position all over the country to the vicinity of Thebayetkin. On approaching the latter place, its surface is covered in places with white marl, derived from its disintegration ; while the beds of streams are coated with calcareous tuff. Beyond this, for a quarter of a mile to the Irrawaddy river, the rocks are of sandstone.

In the Sagyin district, 24 miles northward of Mandalay, the crystalline limestone forms a line of hills, surrounded by the alluvium of the Irrawaddy river, trending north and south. Of these, two are ridge-like, while the third, about one mile to the north of them, is formed of a low group. The most southern hill, rising to a height of 500 feet, is one mile in length, and has a width of a quarter of a mile at its base. This, and the next, slope gently towards the east with the dip of the limestone, but very steeply to the west, owing to the jointing of the limestone being at right angles to the bedding.

* The general inclination is at an angle of about 1° to the west.

On the southern hill about its centre, and halfway up its eastern slope, are some large quarries, where a granular limestone has been obtained, from which most of the multitude of images of Budda, with which the pagodas of the surrounding country are supplied, have been sculptured.* From these quarries a mass of limestone was excavated, from which was sculptured a huge Godda (Gautama) for the Arakan pagoda in Mandalay. The mass was so bulky that a canal was constructed from the foot of the hill to a side channel of the Irrawaddy, in order to convey it to Mandalay. The largest quarry of the six to be seen there, is 100 feet long, 50 feet wide, and 30 feet deep. The limestone bed quarried is from six to eight feet in thickness, of a pure white to pale bluish-grey tints, and apparently free from foreign ingredients. Its dip varies from an angle of 27° to 40°.

Filling the interstices and clefts in the limestone of the southern hill, is a semi-consolidated red earth, somewhat resembling laterite, which has been derived from the disintegration of the limestone and gneissic rocks, and which is said to sparingly contain light-coloured rubies. This material in the second hill is of a more consolidated nature, and of a greyish-red to reddish colour, forming irregular layers of from 2 inches to 18 inches in width. Near the southern end these have been extensively excavated over a tract 200 yards in length on the slope from the base up, and of 100 yards in width, in order to obtain the embedded rubies. With these are associated crystals of spinel and water-worn pebbles of brown hydrated oxide of iron, the latter showing that they, and the accompanying minerals, have been deposited along with the red earth in their present position. Some of the clefts have been filled with a dark-coloured earth. A few small pieces of gneiss are seen in the red earth, which may have been derived from thin bands of that rock in the limestone.

The enclosing bluish-tinted, coarsely crystalline limestone beds are as seen immediately underlying the granular rock of the quarries in the southern hill.

A specimen of limestone, which I obtained at the old workings, contained pale pink crystals of ruby, with iron pyrites, and purplish to blue crystals of sapphire.

The rock forming the third hill, on its eastern side, is a coarsely crystalline bluish and white limestone, one layer of which contains rubies. Here the rock has been blasted along a line, some 5 feet wide, up the slope of the hill, for the purpose of obtaining them.

On the north-west side of this group is an old cave working, together with a pit in the alluvium close by, which were formerly called the Royal Loo of King Mindoon.

In the main set of limestone bands passing through Letnytaung spur, not far from the Momeit road, and one mile north-east of Mogok, is situated the quarry which was formerly worked for the extraction of rubies from the matrix. The short limestone outcrop there exposed, traced from its southern end northwards, shows some 40 feet

* This rock takes a fine polish, and in Mandalay is known by the name of Alabaster, from being mistaken for that substance.

of white crystalline limestone, then a wide bed of white granular limestone, which abuts against a crystalline limestone of a few feet in width, in which are bands of reddish-brown mica crystals, graphite, small crystals of felspar, reddish spinels, and light green augite. This passes into a coarsely crystalline semi-opaque limestone (the matrix of the ruby) of about 20 feet in width; beyond which comes white crystalline limestone extending for some 90 feet, to where it is seen in some caves, of a coarsely crystalline variety.

Although the bedding of these is somewhat obscure, yet they evidently dip slightly to the southward.

The quarry, which is in the coarsely crystalline semi-opaque limestone above mentioned, is 20 feet wide at the face, and has been cut in for 15 feet, to where a small drift in its bottom has been advanced a short distance, making the entire length of the cutting at that part some 30 feet.

The rubies are found in the rock over a space of 6 feet in width, extending almost vertically from the bottom of the quarry to the surface of the ground; while the direction of the productive portion slopes to the south at an angle of 70°. On either side of this the rock changes from a bluish-grey to pure white. Along the centre line, where the rubies are most numerous, are small developments of a greyish mineral (diaspore), enclosing small crystals of iron pyrites, and where these occur the miners assert that the rock is the most productive. There is also a small irregular vein-like structure of the same mineral traversing this part for a short distance. Besides these there are small dull-greenish crystals of disaspore, and light green crystals of augite. In some of the specimens I obtained by blasting, magnetic pyrites was found in contact with the rubies.

At the mines close to Bobedaung village there are some fine beds of statuary marble, similar to those seen at the ruby quarry at Mogok; and here the adjoining coarsely crystalline limestone has been quarried for the rubies it contains.

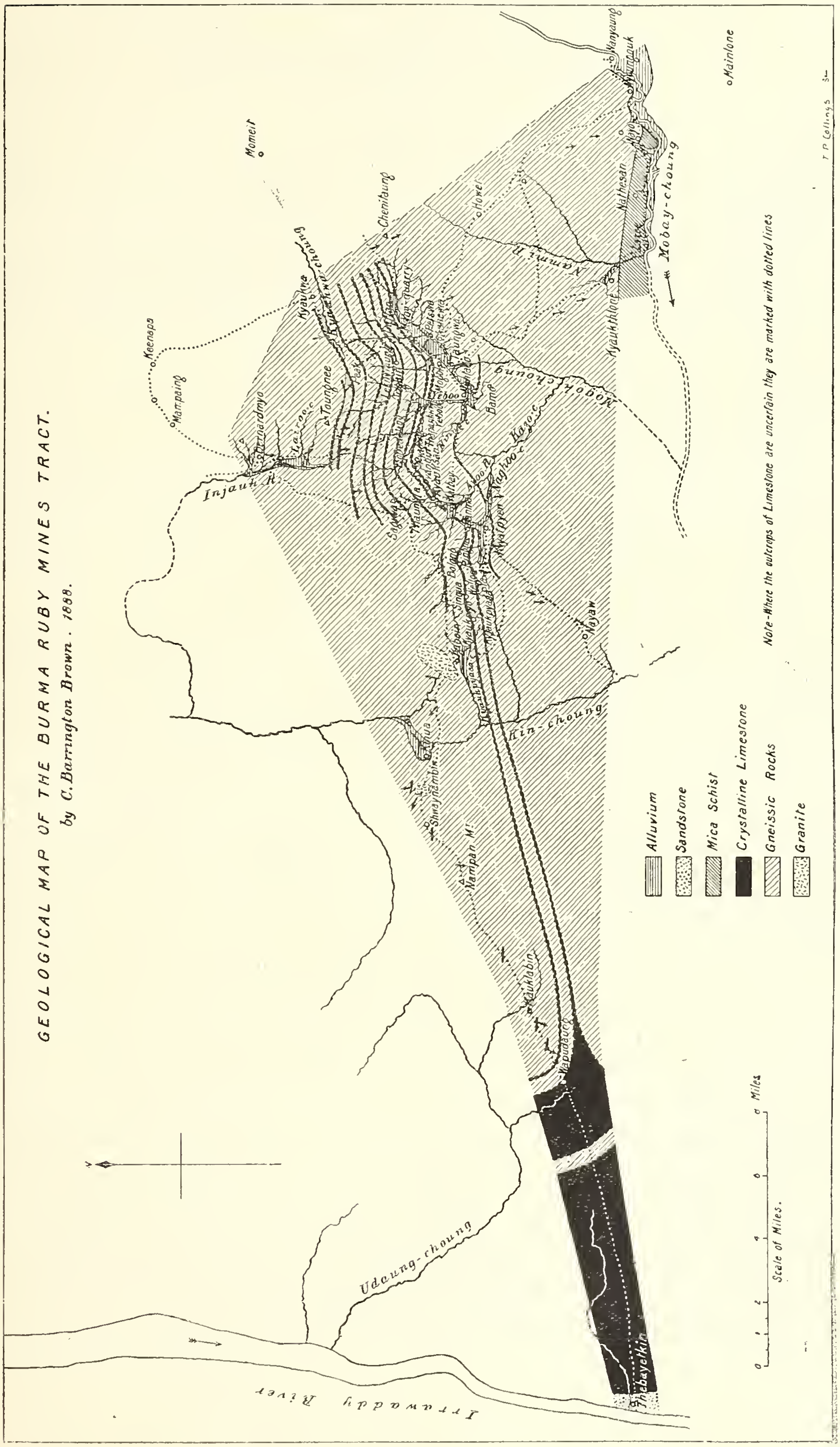
This band is evidently the western extension of the one in which the Mogok quarry is situated. A cave working near by, called the Royal Loo, was formerly considered to be a very rich one. Moung Guy, a Mogok mine owner, informed me that when this mine was opened many years ago, the bones of a large animal were discovered in the ruby clay and loam in the cave's entrance, amongst which a number of fine rubies were found. He had seen one of these bones some ten years ago, and from his description it was evidently a rib of four inches in width.

Between Thaungla and Pyanbin, in the continuation westward of the same band, rubies have been found in the limestone matrix.

7. *Granite.*

There is a large exposure of grey granite near Kabein, which there was no opportunity of examining, except a very small portion of it crossed on the new road. This is probably a pegmatite. C. B. B.

Fig. 11.


GEOLOGICAL MAP OF THE BURMA RUBY MINES TRACT.
by C. Barrington Brown. 1888.
Alluvium
Sandstone
Mica Schist
Crystalline Limestone
Gneissic Rocks
Granite
Note-Where the outcrops of Limestone are uncertain they are marked with dotted lines
Scale of Miles.
Miles
Irrawaddy River
Udaung-choung
Kin-choung
Mobay-choung
Injauk R.
Kyaukwa-choung
Nagoh-choung
Thabayetkin
Momeit
Chenitaung
Keenapa
Mampaing
Nayaw
Mainlone
Nampan Mt.
Kyauktaung
Howei
Nathesan
Shwaynambin
T. P. Collings

V. Economics. (See Map, fig. 11, p. 183.)

1. *Mines.*

Mining for rubies is carried on in three ways, according to the different positions of the ruby-bearing deposits.

In the case of the alluvial deposits, where the sands and gravels are in a layer below the level of the beds of the rivers, they are reached by sinking pits called by the miners Twinlones, which, in the Shan language, means Round pits. Formerly these were made round and of small diameters, but now they are sunk square and of large size.

Where the ruby-bearing material is in hill-wash it is reached by open cuttings called Hmyaudwins, which means Water-mines.

And where the ruby earth is dug out of natural vertical tunnels and caves in the limestone, the excavations are known by the name of Loodwins—that is, Crooked or Twisted mines. To these may be added a fourth class of mine where the rubies occur in the coarse crystalline limestone (their matrix), from which they are obtained by blasting the rock; and these may be termed Quarry mines.

(*a.*) *Twinlones.*—These pits vary in size from 2 to 9 feet square. The mode of sinking and timbering these is as follows:—After a few feet have been excavated, strong posts, 12 feet in length, are driven down in each corner, and in the case of one 9 feet square, three more posts are driven at equal distances apart along each side. Short timbers are wedged across between each, and at intervals of about two feet light flat timbers are placed across each way. Wattles and dry grass, or twigs with leaves, are forced in the spaces between the upright posts to support the clay walls. This method of timbering is continued with the sinking to the bottom of the pit. When the excavation reaches the ends of the uprights, another set of posts is driven down inside the first, and these generally pass through the ruby-bearing sands and gravels. If necessary a third set is put in. Upon completing the pit and having sent all the ruby sand to the surface, the timbering is taken out for use in a new pit. Round pits at the present day are few in number, and are sunk merely to test the presence of the ruby sand, but, as seen in old workings in the Kyatpyen valley, and elsewhere, they must have been formerly extensively used instead of those of square shape now in vogue. The balance poles used both for hoisting the materials excavated, and the water accumulated in the pits, are made of strong bamboos. A large basket, filled with stones, is used as a balance weight at the butt, and to the long end overhanging the pit is attached a rope, or thin pole, carrying a basket. Some pits have five of these balance poles. Water-tight bamboo baskets are used for hoisting out the water. Some shallow twinlones are kept free from water by rude but ingenious bamboo pumps, placed in a slanting position.

The ruby-bearing sand is placed in heaps, and washed in small flat baskets made of the outer portion of the bamboo, somewhat similar in shape to the wooden batea. Their mesh is such that small particles of mineral matter will not pass through, while the water is all drained off. As soon as one pit is finished, generally within eight to ten days for a large one, and four to five for a small one; another pit is commenced near by.

(*b*.) *Hmyaudwins.*—These, the most numerous of all the classes of mines, are open cuttings of an elongated form, the lower ends of which are open to a hill or gully side. On commencing this sort of mine some outlay of capital is required, in order to bring a supply of water from some adjacent or distant stream, along the mountain sides to the head of the working. In one instance one of these gutters is over two miles in length. Where a ravine intervenes, the water is conducted across in a number of bamboo troughs laid side by side, supported on a bamboo structure, held together by strong cross pieces and stays. The water is delivered at the head of the cutting through bamboo poles, the ends of which are closed, the water passing through an opening in their top which causes it to fall in a more or less scattered manner. It flows away through a trench in the bottom of the working, which forms a ground sluice.

The operation of excavating the ruby-bearing portion is performed with long and short-handled spuds, and the stuff is thrown in heaps, upon which the falling water is directed. As the face of the cutting becomes undermined, the superincumbent clay and loam slips down and is washed away. The large stones in it are thrown to one side in heaps, or formed into walls to support the refuse, as well as the sides of the ground sluice, which is advanced towards the face of the working as the process of excavation proceeds.

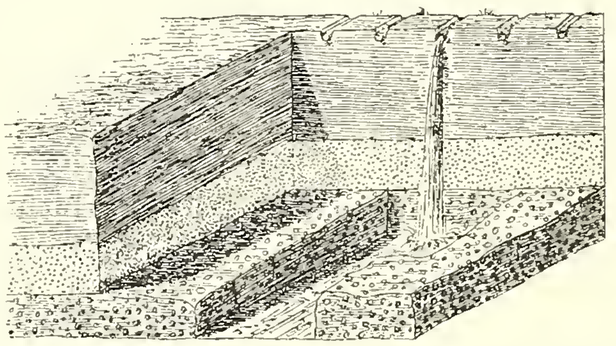
When the ruby-bearing material has been sufficiently softened by the falling water, the larger stones are picked out and thrown to one side, the remainder being hauled with hoes into the upper end of the sluice, where it is puddled. Two or three wooden riffles, of two feet or more in height, arrest the sand and gravel containing all the minerals. This is taken out, placed in basket bateas and washed clear of the remaining clayey particles when it is carefully searched for rubies, spinels, &c. The ruby sand which may escape from the upper portion of sluice is collected behind riffles placed along its entire length, and washed from time to time; but the principal and valuable portions are procured from the first 12 feet of the sluice head.

The extensive Rubellite Mines at Nyoungouk (see fig. 12) are worked somewhat on the Hmyaudwin principle, the water being delivered by a number of short bamboo spouts placed along the top of the face of the cutting, for the purpose of removing the top clay and for softening the binding material of the gravel, which contains the stones sought for. The work of excavation is very neatly executed, the sides as well as the face being made vertical. By dashing water against the exposed face of the layer of gravel and sand, with shovel-shaped baskets, the miners are enabled to see the rubellites and pick them out by hand. All the produce of these mines is sent to

China, where pink rubellite meets with a ready sale, but there are few purchasers of it in the European market. The transparent pink variety having an internal feathery structure, is most highly prized.

(*c.*) *Loodwins.*—Although at the time of my investigations only eight of these mines were in operation, the immense number of old cave mines to be seen in the crystalline limestone attested to the importance of these in former times. From there, it is said, have been procured the finest and largest rubies discovered in the whole district. Without doubt the rubies in them have been liberated from the rock itself, by the action of water and deposited amongst detrital matters in the sinuous channels, caves, and interstices in it, which extend in every direction and to great depths. The brownish clayey loam which fills these interstices and contains the rubies, is derived chiefly from the disintegration of the limestone.

Fig. 12.



Rubellite Mine at Nyoungouk.

Provided with small short-handled spuds, baskets, and small oil-lamps, the miners descend the pits, dig out the loam, which in instances where these are very tortuous they bring to the surface themselves ; but usually the proceeds of their work are hoisted up perpendicular pits sunk in wide fissures by means of balance poles. By these pits, and tunnels connected with the workings which form passages from one to the other, ventilation is procured.

Blasting with gunpowder manufactured by the natives has sometimes to be resorted to.

In these mines no ladders are used, the miners descending and ascending the narrow tortuous passages with apparent ease.

Owing to the fissured nature of the limestone, there is usually no water to contend with in the workings, and none is used for softening the loamy clay, which is taken directly to the nearest water-supply and there washed in baskets as in the cases before described.

The depths of some of these earth-filled natural shafts is very considerable.

(*d.*) *Quarry Mines.*—These were formerly worked in four places, viz :—

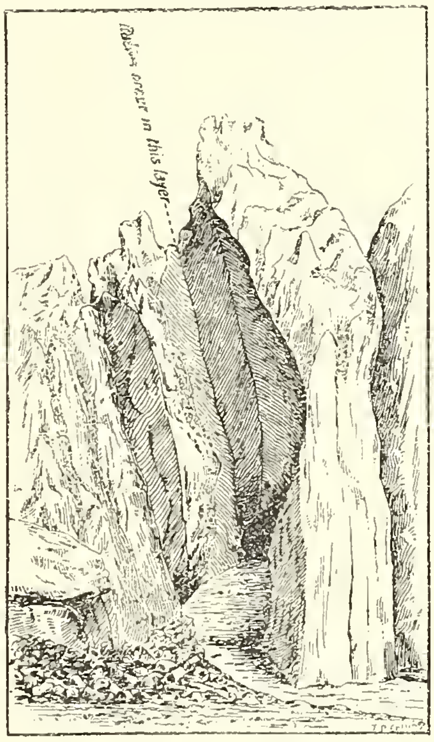
1. At the Ruby Quarry near Mogok.

2. At Bobedaung in the Royal Loo.
3. At Taungla.
4. At the northern hill of Sagyin.

Previous to my examination of the mines no true rubies had ever been seen by Europeans in their matrix, though crystals of opaque reddish corundum had been discovered in limestone in other parts of the world. It therefore became an interesting matter to discover whether the ruby existed in this celebrated ruby-bearing tract in the matrix.

Failing to discover anything but red garnets in the gneissose rocks, and red spinels in the crystalline limestone, I endeavoured, by questioning the mine-owners through

Fig. 13.



Ruby Quarry and Cave.

my interpreter, as to whether they had met with the ruby enclosed in rock, but at first without success. They must have known that it had been found, but were evidently loth to acknowledge it, or offer any information on the subject. My interpreter, upon whom the importance of making the fullest inquiries was constantly impressed, at last learned from a miner the locality of the Ruby Quarry in limestone at Mogok, from which he had extracted rubies by blasting.

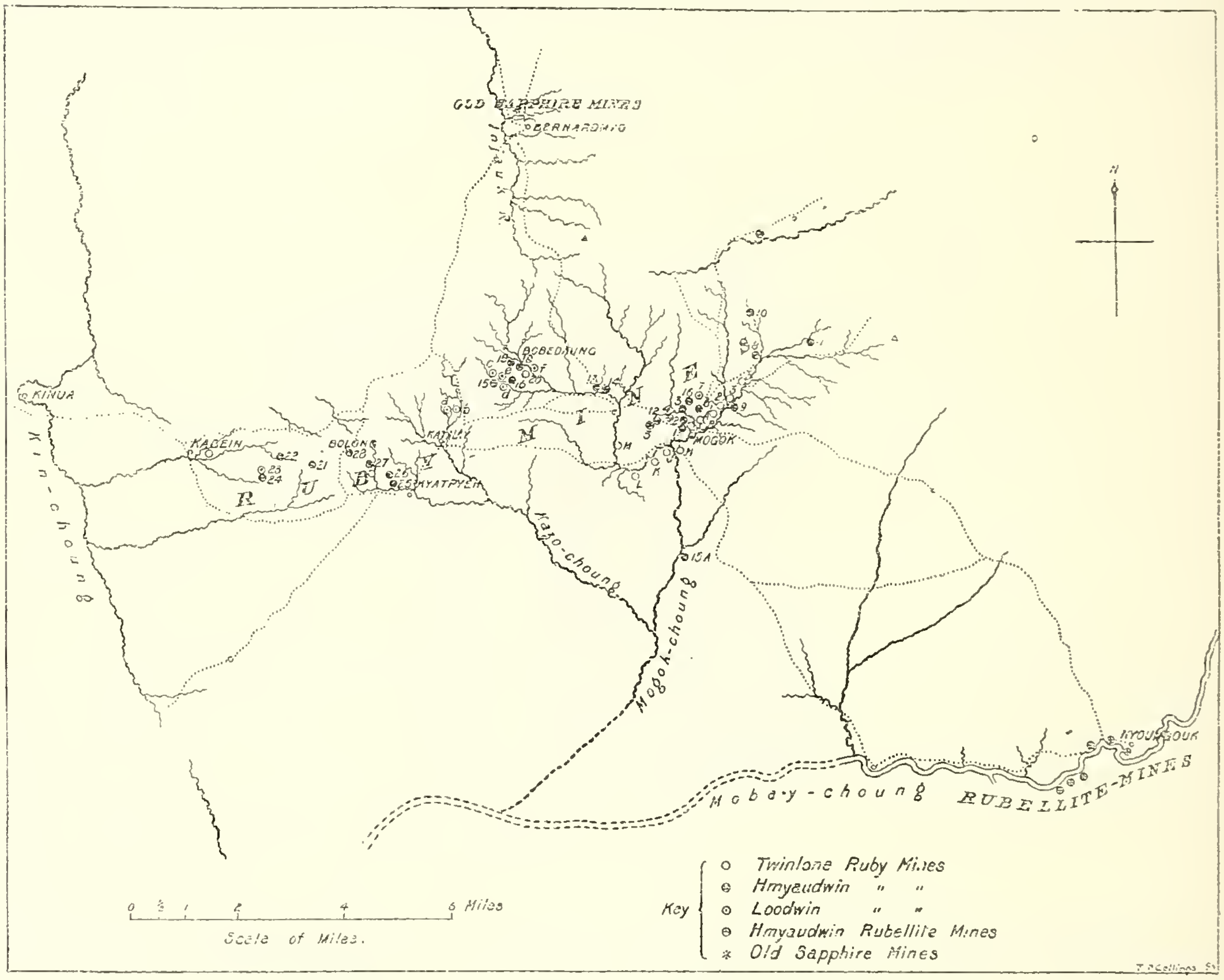
Upon examining this, no rubies could be seen in the face of the rock, but on getting two men who had worked there previously to put in a few shots and blow down a block, I had the gratification of seeing rubies of fine colour in the pure white limestone for the first time.

This quarry had been worked for fifteen years up to the time of the British occupation, when the authorities prohibited the use of gunpowder by the miners; but the fact was unknown to the officials or anyone visiting the mines previously to the date of my obtaining the specimens.

After this, one of the leading mine owners communicated to me the localities where the matrix had been worked for rubies.

The men whom I employed in a period of ten days procured fourteen good-sized rubies, besides numerous smaller ones, from $1\frac{1}{2}$ cubic foot of rock. After drilling

Fig. 14.



Map showing the position of the gem-mines of Upper Burma.

shallow holes and blowing down small blocks of limestone, they proceeded to break them up with hammers, and so obtained the specimens; but, owing to the jarring of the rock by powder and hammer, the rubies disclosed were all more or less injured, so that their commercial value was greatly reduced. These miners formerly obtained in this way rubies to the value of Rs. 200 per month, and on one occasion extracted a ruby which they sold for Rs. 300. No doubt there are other places besides those mentioned where rubies exist in the limestone, and when a process is found by which the rock may be cut out and burned in a kiln the contained precious stones will be procured uninjured, and in sufficient numbers to make this mode of obtaining them a source of profit.

I had some blasting done at Sagyin, which disclosed a few small off-coloured rubies, of no value, in the rock. The rubies obtained in the second hill in the consolidated red earth have hitherto been looked upon as being in the matrix, but this is not the case.

The foregoing description of the manner in which these mines are worked by the natives tends to show how primitive were the modes adopted at the time of my visit in 1888. In the hands of Europeans, with capital at command, the quantity of precious stones produced should be greatly increased were proper methods of working adopted. The great drawback to Twinlone mining is the quantity of water in the ruby-bearing gravel, and it would be a costly matter to drain such valleys as Mogok, Kathay, Kyatpyen, and Yeboo. But, could this be effected in the Mogok Valley there is a large quantity of ruby sand and gravel still remaining to be dealt with.

In some places adits could be driven at the foot of the hills into the limestone, thereby tapping the deposits in its caves and natural channels, which, from experience gained by native miners, is known to be more or less rich in rubies.

In February, 1888, the number of mines, consisting of Twinlones, Hmyaudwins, and Loodwins, was 77, employing 771 native miners. No quarry mines were then in operation, owing to the prohibition placed on the use of gunpowder by the Government. For the same reason, many of the Loodwins were lying idle. C. B. B.

VI. Petrology.

The large and interesting collections brought home by Mr. C. Barrington Brown, which have been submitted to me for study by the Secretary of State for India, consist of the following classes of materials :—

(*a*) Specimens of the rocks of the district—gneisses, granulites, pegmatites, crystalline limestones, &c.—collected *in situ*.

(*b*) Selections of rock-fragments and minerals found in the ruby-bearing gravels. The rock-fragments are for the most part similar to those collected *in situ*; but they are often water-worn and are sometimes in a very disintegrated condition, owing to the action of atmospheric denudation.

(*c*) Specimens of the reddish-brown earth in which the rubies and other gem-minerals occur.

(*d*) Samples of the materials obtained by the rough washing of the ruby-earths before picking.

In dealing with these materials I have studied the rock-specimens, whether collected *in situ* or as derived fragments in the gravel, by the aid of thin sections under the microscope, where necessary isolating and submitting to different physical and chemical tests their several constituent minerals.

Specimens of the ruby-earths have been washed in the laboratory, and the minute fragments of minerals in them isolated by the use of heavy liquids. The particles thus obtained have been studied microscopically, various physical and chemical tests being applied for their identification.

The collections of materials from the washings at the mines have proved to be of great value as a means of estimating the nature of the minerals present, and (in some cases) of the relative proportions of the different minerals occurring in these alluvial deposits.

The limestones have been studied, not only by thin sections, but also by dissolving considerable quantities of them in acid, and separating the minerals left behind by the aid of heavy liquids. In this way it has been possible to determine not only the several minerals present in the calcareous masses, but also the proportions which the carbonates bear to the silicates and other minerals scattered through them.

In studying this series of rocks I have received the greatest assistance from the admirable work of LACROIX on the rocks of Ceylon and Southern India, and his comparisons of these with the pyroxene- and scapolite-rocks of other areas, such as Brittany, Central Europe, Spain, Algeria, Western Africa, Eastern Africa, Norway, Sweden, and the United States.* My studies of the rocks has also been facilitated by the receipt of numerous specimens from former students and correspondents; in particular, I may mention valuable collections obtained by me from Mr. C. BARRINGTON BROWN and Mr. P. BOSWORTH SMITH, from Ceylon and Southern India, and from the latter gentleman and from Mr. T. H. HOLLAND, of the Geological Survey of India, from the Salem district; from Mr. NASON and Professor F. D. ADAMS I have received interesting examples of the pyroxene- and scapolite-rocks of the North American Continent; while to Mr. F. R. MALLET I am indebted for specimens from South Rewah, and for much valuable information concerning several Indian localities.

In the task of working out these rocks I have been aided by several of my assistants in the Geological Laboratories of the Royal College of Science, especially by Mr. T. W. HOLLAND, F.G.S. (now of the Geological Survey of India, and the Presidency College, Calcutta); by Mr. G. W. CARD, F.G.S. (now curator of the Geological Survey Museum, Sydney, N.S.W.); and by Mr. T. BARRON. To another assistant, Mr. F. CHAPMAN, I am also especially indebted for the great skill he has shown in making microscopic sections, often from very troublesome materials.

The corundum-bearing rocks of Southern Asia appear to have a very wide distribution, and, so far as they are known, to exhibit remarkably uniform characters. The

* "Contributions à l'étude des Gneiss à Pyroxène et des Roches à Wernerite," par ALFRED LACROIX. 'Bull. de la Soc. Fr. de Mineralogie.' April, 1889. A translation of the portions of this memoir relating to Ceylon and the Salem district has been made by Mr. F. R. MALLET, and published in the 'Records of the Geological Survey of India,' vol. 24, Plate 3, pp. 155–200.

chief areas in which they have been described are Ceylon, the Salem district of the Madras presidency, the district of South Rewah in the Bengal presidency (where, in the midst of the gneiss series, beds of corundum rock are found, with others of limestone, dolomite, magnetite, and quartzite), the North-Western Himalayas, Bokara, Burma, and the Shan States of Siam.

In all these districts the rocks containing corundum appear to be highly crystalline gneisses, sometimes passing into schists, and frequently including masses of limestone and dolomite, the latter rocks being, as a rule, highly crystalline, and containing many silicates and other minerals ("calciphyres and cipollinos"). Intrusive masses of granite or pegmatite also occur. There is no evidence, so far as I am aware, opposed to the view that the whole of these rocks must be regarded as of Archæan age.

Now, as has been so well shown by LACROIX, one very remarkable and striking feature of the gneisses of Ceylon and Salem, in which districts corundum also occurs, is the prevalence of bands of granulite and gneiss of more basic composition than the rest of the surrounding metamorphic masses. These basic gneisses and granulites contain anorthite* and various forms of pyroxene, and not unfrequently scapolite, wollastonite, sillimanite (fibrolite), kyanite, andalusite, and calcite, as well as spinel, corundum, tourmaline, and other accessory minerals. As we shall have to show in the sequel, the gneiss series of Burma which yields the famous rubies contains a number of subordinate bands of these remarkable basic gneisses, composed of pyroxene, anorthite, and scapolite, with many accessory and secondary minerals; and in all their essential features these present the closest resemblances to the corundiferous rocks of Ceylon and the Salem district.

It is only in comparatively recent years that the importance and wide distribution of the pyroxene-bearing gneisses and granulites has come to be fully recognised; and there is still, unfortunately, much diversity in opinion and practice in respect to the names by which they are to be distinguished. The older writers designated them by the name of "trap-granulite," while BARROIS, following the usage of DANA and STERRY HUNT, employs the term "pyroxenite" for them. This latter term, however, as LACROIX insists, is more properly applied to rocks, either "massive" or schistose, which are almost wholly made up of pyroxenes; and the last-mentioned author describes the rocks with plagioclase felspar and wernerite as "pyroxene-gneisses" ("gneiss à pyroxène)." The Canadian geologists have called the same rocks after the predominating felspar, "anorthite-gneiss," and sometimes "augite-syenite-gneiss," or "anorthosites."

The "trap-granulites" of Saxony, which were long ago carefully studied by NAUMANN,† SCHEERER,‡ and STELZNER,§ have in recent years been made the subject

* As long ago as 1802 Count DE BOURNON pointed out the association of corundum with anorthite ("indianite") and fibrolite in the Salem district; and in 1839 G. ROSE gave the name of "barsowite" to the Ural variety of Anorthite which contains corundum.

† 'Jahrb. d. k.k. geol. Reichsanst.' (1856), vol. 7, pp. 766–771.

‡ 'Neues Jahrb. f. Min., &c.,' 1873, p. 673.

§ *Ibid.*, 1871, p. 244.

of valuable monographs by DATHE* and J. LEHMANN.† Their peculiar characters and their relations to the "flaser-gabbros" and other igneous masses which have been subjected to dynamo-metamorphic action have been rendered familiar to all geologists by the writings of these authors.

In Brittany, a compact rock of this class has long been known as furnishing materials for pre-historic weapons, and the characters and distribution of the pyroxene-bearing rocks in this area have in recent years been worked out by WHITMAN CROSS,‡ BARROIS,§ and LACROIX.||

In 1882 Professor F. BECKE gave a very careful description of the peculiar characters of the pyroxene-gneisses of the Waldviertel district in Lower Austria,¶ and pointed out the association of these with rocks containing scapolite and crystalline limestone.

Rocks of the same kind have been known for some time in Scandinavia, and have been especially studied at Oedegarden and other localities near Bamle by BRÖGGER and REUSCH,** TORNEBOHM,†† SJÖGREN,‡‡ and SVEDMARK.§§

On the North American continent, ADAMS and LAWSON|||| have called attention to the rocks of the same mineralogical composition in Canada; while their presence and wide range in the United States have been pointed out by LACROIX,¶¶ NASON,*** and others.

Areas composed of similar rocks have been described in Bavaria by GUMBEL,††† in the Pyrenees and Central France by GOUNARD and LACROIX,‡‡‡ in Spain by CALDERON,§§§ in Algeria by DELAGE,|||||| off the West Coast of Africa, opposite to the Azores, by LACROIX,¶¶¶ in South Africa (Hereroland) by WULF**** in Eastern Africa

* 'Zeitsch. d. deutsch. geol. Gesellsch.,' vol. 29 (1877), p. 274.

† 'Entstehung der altkrystallinischen Schiefergesteine' (1884).

‡ TSCHERMAK'S 'Min. und Petr. Mittheil.,' vol. 6, 1884, p. 369.

§ 'Bull. Soc. Géol. du Nord,' vol. 15, 1888, p. 69.

|| 'Compt. Rend.,' 1880, p. 1,011; and 'Bull. Soc. Min. Fr.,' vol. 12, 1889, pp. 83-361.

¶ TSCHERMAK'S 'Min. und Petr. Mittheil.,' vol. 4, 1882, p. 365.

** 'Zeitsch. d. deutsch. Geol. Gesellsch,' vol. 27, 18 5, p. 676.

†† 'Sverige Geol. Fören. i Stockholm Forh.,' vol. 5, 1882, p. 10.

‡‡ 'Sverige Geol. Fören. i Stockholm Forh.,' vol. 6, 1883, p. 447.

§§ 'Sverige Geol. Fören. i. Stockholm Forh.,' vol. 7 (1884), p. 293.

|||| 'Canadian Record of Science.'

¶¶ 'Bull. Soc. Min. Fr.,' vol. 12, 1889, pp. 267–281.

*** 'Ann. Rep. State Geologist for New Jersey for 1889,' p. 30.

††† 'Geol. Beschr. d. Ostbayer. Grenzgebirge,' vol. 2, 1868, p. 355.

‡‡‡ 'Compt. Rend.,' vol. 97, 1882, p. 1,447, and 'Bull. Soc. Min. Fr.,' vol. 6, 1883, p. 3; vol. 9, 1886, p. 46.

§§§ 'Mem. de la Com. del Mapa Geol.,' vol. 2, 1887, p. 297.

|||||| 'Géol. du Sahel d'Alger. Montpellier,' 1888, p. 149.

¶¶¶ 'Bull. Soc. Min. Fr.,' vol. 12, 1889, p. 171.

**** TSCHERMAK'S 'Min. und Pet. Mittheil.,' vol. 8, 1887, p. 193.

(Kilimanjaro) by MÜGGE,* in Brazil by O. DERBY and LACROIX,† and in New Caledonia by GARNIER and HEURTEAU.‡

In studying the series of gneissic rocks of Burma by the aid of the general descriptions and the specimens supplied to me by Mr. BARRINGTON BROWN, the circumstance which appears most striking is the marked difference in chemical composition between the various bed-like masses which compose it. From highly acid rock, made up of orthoclase and quartz, with a little muscovite, but no ferromagnesian silicate, we find every gradation—through normal biotite-gneisses and schists—into rocks made up of quartz, basic felspars, and pyroxenes (pyroxene-granulites, "trap-granulites," and pyroxene-gneisses); and these, by the disappearance of the felspar and quartz, pass into pyroxenites and amphibolites. The tendency of the basic felspars to break up and give rise to minerals of the scapolite family, by a process which I have described in an earlier memoir,§ and which LACROIX proposes to call "Werneritization,"|| side by side with the change of both enstatites and augites into amphiboles, gives rise also to many interesting varieties of rocks. How far the associated limestones are to be regarded as resulting from still further alteration of the scapolite-rocks will be discussed in the sequel.

For the purpose of convenient description, we may classify the various rocks of the Burma area according to the following table:—

A. Rocks of intermediate composition forming the great bulk of the foliated masses of the district.
   1. Biotite gneisses.
   2. ,, granulites.
   3. ,, schists.

   These rocks often contain much garnet.

B. Rocks of acid composition, intercalated with the common biotite gneisses, &c.
   1. Pegmatites and graphic granites.
   2. Aplites and granulites ("Weissstein").
   3. Granular quartzites.
   4. Orthoclase-epidote rocks.

C. Rocks of basic composition intercalated with the common biotite gneisses, &c.
   1. Augite (Sahlite, green diopside, &c.) gneiss.
   2. Augite-granulite (with garnet, &c.).
   3. Enstatite- (hypersthene) gneiss.
   4. Enstatite-granulite (with garnet, &c.).

* 'Neues. Jahrb. f. Min.,' etc., vol. 4, 1886, p. 577.

† 'Bull. Soc. Min. Fr.,' vol. 12, 1889, p. 266.

‡ 'Ann. des Mines,' 6th Series, vol. 12, 1867, p. 1; *ibid.*, 7th Series, vol. 9, 1876, p. 232.

§ 'Mineralogical Magazine,' vol. 8, 1889, p. 186.

|| 'Bull. Soc. Min. Fr.,' vol. 13, 1890, p. 35.

5. Scapolite-gneiss.
6 ,, granulite.
7. Pyroxenites.
8. Amphibolites.
9. Lapis-lazuli (Lazurite-scapolite-diopside-epidote rock).

D. Crystalline limestones intercalated in the gneisses, and usually associated with the augite- and scapolite-rocks.
Cipollinos.
Calciphyres.

E. In addition to the rocks *in situ*, we have the interesting gravels and earthy rocks formed by the atmospheric disintegration of all the other rocks.

1. *Gneiss, Granulite, and Schist.*

These rocks which constitute the great mass of the Burma area now under consideration are closely related to one another, the dominant rock is a biotite-gneiss, which passes into the other two types, granulitic and schistose, by the most insensible gradations. The minerals of which these rocks are constituted are as follows :—

1. *Quartz.*—This mineral, which is usually abundant, is often darkened, as seen in thin sections, by the very numerous bands of inclusions, containing liquids with moving bubbles, that traverse the grains in all directions. Under polarized light, too, the quartz grains often show undulatory extinction and other indications of having been subjected to strain. With the quartz exhibiting these phenomena we, however, find grains—usually of smaller size—which are perfectly clear and are probably of later date than the great mass of the quartz-grains in the rock.

Felspar.—While both orthoclastic and plagioclastic felspars are present in all these rocks, the relative proportions of the two varieties varies greatly in different specimens. In the great majority of cases the felspars are somewhat cloudy, showing distinct signs of chemical alteration (kaolinization). The phenomenon that especially characterises the felspars of these rocks, however, is that known to the French petrographers as the presence of "quartz of corrosion." The grains of felspar are riddled in all directions by irregular, and often ramifying, veins of a mineral of higher refractive index than the felspar itself. Sometimes these inclusions show no trace of relationship in their position to the planes of symmetry in the crystal of felspar ; but, in other cases, they clearly conform to definite directions within the crystal in which they are developed, and thus resemble "planes of schillerization." The crystals with those parallel inclusions often show very beautifully the sheen of murchisonite and moonstone. The nature and origin of some of these changes will be discussed in the chapter on Mineralogy.

Biotite.—The mica, which is approximately uniaxial, is black by reflected light and brown in thin sections by transmitted light. It occurs in irregular scales which, in some varieties of the rock, are somewhat rare, but in others very abundant. The pleochroism is intense.

$\mathfrak{c}$ and $\mathfrak{b}$, greyish and reddish-brown.
$\mathfrak{a}$, yellow to red-brown.

Absorption $\mathfrak{c} > \mathfrak{b} > \mathfrak{a}$. This is so great parallel to $\mathfrak{c}$ and $\mathfrak{b}$ that the mineral appears black when traversed by rays vibrating in these directions.

While the mass of the rocks is made up of these three minerals others appear in smaller quantities.

Zircon is abundant, enclosed in all the minerals of the rock. When found in the biotite, pleochroic halos are exhibited by the latter mineral. *Apatite* is somewhat rare, as are also *magnetite* and *titanoferrite*. The latter sometimes exhibits the transformation to *sphene*, and the former to *limonite*, and occasionally into *hematite*.

Garnet (almandine) is often a very abundant accessory constituent of the rocks, and not infrequently is found undergoing change along the cracks which traverse it. In some cases the garnet is so abundant that it must be regarded as an essential constituent.

Cordierite, *sillimanite* (intergrown with biotite), *graphite*, with primary *sphene*, also occur in these rocks.

These gneissic rocks vary greatly in the coarseness of their grain. The most abundant type is a moderately coarse-grained gneiss which occasionally exhibits large "eyes," composed of quartz and felspar, and with the foliated structure fairly well-marked. In other cases, the rock is seen to be made up of more or less rounded grains of quartz and felspar, with some biotite and abundant garnets; and it may then be called a "mica granulite." On the other hand, the biotite occasionally increases in amount, while the felspar diminishes, and, the foliated structure of the rock becoming more pronounced, it approximates to a true mica-schist. Hornblende appears to occur but rarely in this series of rocks.

The greatest interest attaches however not to these gneisses themselves, but to the remarkable rocks which form subordinate members of the series, and are found interfoliated with them. Some of these are of more acid composition than the gneisses, and may be classed as pegmatites and aplites, passing into quartzites; others are of more basic composition, and include scapolite gneiss with various forms of pyroxene-gneiss, passing into pyroxenites and amphibolites. With these latter occur lazurite-diopside rocks (lapis-lazuli) and various forms of impure limestones (cipolines and calciphyres). We shall proceed to consider, firstly, the acid rocks intercalated with the gneiss series of Burma, then the basic rocks, and finally the remarkable limestones which Mr. Barrington Brown has shown to be the parent rock, both of the red corundum (rubies) and the red spinel (rubicelle or balas ruby).

2. *Acid Rocks.—"Pegmatites" and "Aplites."*

These rocks, which occur as bands in the gneiss, consist mainly of an alkali-felspar with more or less quartz and certain accessory minerals. Mr. BARRINGTON BROWN's study of these Burma rocks in the field has led him to the same conclusion as that arrived at by Mr. F. R. MALLET in mapping the gneissic series in South Rewah *—namely, that these acid rocks are not granitic intrusions, but that they constitute an integral part of the mass of crystalline schists. Very similar rocks occur in Ceylon and in the Salem district, and have been admirably described by M. ALF. LACROIX.† We shall, however, in the sequel, have to point out some very interesting points of distinction between the Burma pegmatites and aplites, and those of Ceylon and Southern India.

The pegmatites of Burma vary very greatly in their texture. Sometimes the individual crystals of felspar are as large as a man's fist, and occasionally they attain even greater dimensions; but this coarse rock passes by insensible gradations into one of a more finely granular character, to which the name of aplite or granitell may be given. When, as is frequently the case, garnets are present, the rock passes into a true granulite or leptynite ("Weissstein" of German authors). Examples of the largely crystalline types (pegmatites) are found at several points about Mogok, at the pass leading from that place to Momeit, and many other points in the gneiss area; and fragments of them are very widely distributed in the ruby earths. The more finely granular varieties appear to have an equally wide distribution; very interesting examples of this type are found at Mandalay hill—some of them being stained of reddish and greenish tints by alteration products.

The "pegmatites" often, but not invariably, exhibit traces of the graphic structure, and sometimes pass into true graphic granites. The finer grained rocks ("aplites") usually exhibit a more or less distinct granulitic structure.

The proportions of the several mineral constituents in these acid rocks are not less variable than their texture and structure. As a rule, the felspar greatly predominates, and the quartz and mica are quite subordinate, the latter mineral being not unfrequently altogether absent. When the quartz is present in small quantities, it is usually found only as scattered grains enclosed in the felspar; when the quartz is present in larger quantities, it occurs intergrown with the felspar, to form a true graphic granite. Occasionally both quartz and mica are present in considerable quantities, and the rock becomes a typical muscovite granite, or rather granitic gneisss.

The characters presented by the constituent minerals of these rocks are as follows:—

The felspar is an *Orthoclase*, which is nearly always of a white colour. Though it frequently exhibits, in its undulatory extinction, evidence of having been subjected to mechanical stresses, yet it seldom or never exhibits the microcline structure. In

* 'Manual of the Geology of India,' vol. 1, p. 21.

† 'Bull. de la Soc. Fr. de Mineralogie.' 1889.

this respect the Burma pegmatites appear to offer a striking contrast to those of Ceylon and Salem, in which microcline is the predominant felspar while plagioclase (oligoclase) is often present also. The last-named mineral is but seldom found in the Burma pegmatites.

These orthoclases in the Burma pegmatites are not only remarkable for exhibiting the structure described by French petrographers as "quartz of corrosion," but for other peculiarities some of which will be noticed in the chapter on Mineralogy. The illustrations of the production of various kinds of schiller structure and of the alterations resulting in the play of colours characteristic of moonstone are of a very interesting character.

All these structures in the orthoclase felspars are doubtless due to incipient alteration; and in the fragments derived from these rocks, and included in the "ruby-earth," we find these changes carried out to their fullest extent, the orthoclase being sometimes converted into a kaolin, not unfrequently into a hydrous potash-mica (damourite or gilbertite) and at other times into an epidote.

Epidote is often found developed along the cracks of these orthoclases, and in one instance (viz., in a specimen collected on the road between Mogok and Momeit), the epidote has the peculiar colour and pleochroism of withamite, due, no doubt, to the fact that it contains some manganese. In another case the epidote and felspar are found so curiously intergrown as to give rise to a structure, which, when seen upon the weathered surface of the rock, was thought to be of organic origin.

The phenomena exhibited by these orthoclases in the various stages of their decomposition are of a very interesting kind and are worthy of the most careful study. In the ruby-earth, the orthoclase exhibiting the last stages of the alteration of the mineral is found, the masses crumbling between the fingers into a powder of kaolinite, muscovite, &c.

The *quartz* in these pegmatites is of the same kind as that seen in the accompanying gneisses, and is usually full of bands of inclusions; perfectly clear quartz, probably of secondary origin, also occurs in them however.

The *mica* is almost always a biaxial one—muscovite or damourite. Occasionally, however, biotite occurs, and these biotite rocks form a link between the pegmatites and the ordinary gneisses of the district.

Among the accessory minerals, fibrolite (which is often enclosed in the orthoclase), garnet (almandine), zircon, and cordierite most frequently occur. Plagioclase felspar, as has been already pointed out, is comparatively rare in them.

By the reduction in size of the grain of these pegmatites and a replacement of the graphic by the granulitic habit, the pegmatites pass into the aplites. These usually consist almost entirely of quartz and orthoclase in more or less rounded granules. Occasionally, as at Ingouk, near Bernardmyo, a rock of this class is found in which the quartz predominates over the orthoclase to such an extent that it passes in places into a granular quartzite. The only mineral which it contains, besides quartz, and traces of orthoclase, being a little muscovite and grains of zircon. In an old Hmyaudwin, near Momeit-road pass, fragments of a similar rock are found, passing into a quartz-schist by the increase of muscovite, and containing also grains of graphite.

At Nyoungouk an aplite is found, containing a felspar with a structure like that of anorthoclase (cryptoperthite), some quartz, a little plagioclase and a blue tourmaline (indicolite), showing a beautiful zoned structure, and in places passing into pink tourmaline (rubellite). See Plate 6, fig. 8.

At Mandalay Hill the aplite, which contains both orthoclase and plagioclase felspar with quartz, is sometimes changed to a greenish colour by the development of ferromagnesian silicates (epidote and chlorite ?) and at other times acquires a pinkish tint by the development of scales of hematite from the included magnetite. This rock also contains corundum (sapphire).

When these finer-grained (aplite) rocks contain, as they frequently do, garnets (almandine) they assume all the characters of ordinary granulites or leptynites ("Weissstein" of the Germans). A rock of this class is found at Yenee River Falls, near Mogok, in which the pale-red garnets exhibit anomalous double refraction; in this rock too, the beautiful green chrome-mica (fuchsite) is found. A few grains of brown biotite are not rare in some of these aplites and granulites, which thus graduate into the ordinary biotite-gneisses of the district. A very good example of one of these rocks constituting a transition between the aplites and the ordinary biotite gneisses is found between Mogok and Momeit.

In their general characters these foliated pegmatites and aplites, intercalated among the biotite-gneisses, present a very close analogy with the massive pegmatites and aplites that occur as the so-called "contemporaneous" or "segregation" veins in so many of the eruptive granites.

There are numerous examples of rocks, transitional in character, between these acid types and the normal gneisses of the district. Thus, at Letnytaung Mountain, we find a rock rich in plagioclase,—which forms large crystals, just as in the pegmatites,—with much brown mica and a dark brown hornblende. This hornblende is remarkable for its intense pleochroism, the scheme of pleochroism and absorption being as follows:—

$\mathfrak{a}$. Yellowish-brown.
$\mathfrak{b}$. Very dark greenish-brown.
$\mathfrak{c}$. Intensely dark brownish-green.

$\mathfrak{c} = \mathfrak{b} > \mathfrak{a}$.

There is a very striking resemblance between the pleochroism absorption, extinction angle, and double refraction between this variety of hornblende found in the basic gneisses and the well-known "basaltin" or basaltic hornblende found in basic lavas. This rock also contains a considerable amount of titanoferrite, passing by alteration into leucoxene (secondary sphene).

On a ridge near this same mountain there occurs a very similar rock, containing orthoclase and some plagioclase felspar, the brown hornblende, and a little biotite. The great point of interest about this rock, however, is the circumstance that both scapolite and a large amount of calcite have been developed in it, apparently at the expense of a lime-felspar (see fig. 17, p. 203).

Similar rocks, intermediate in character between the acid rocks and the normal gneisses, are found as fragments in the ruby-gravels. Thus, in the Hmyaudwin No. 15*a*, at Mogok, a specimen of a pegmatite or very coarse gneiss was found which contains orthoclase crystals, sometimes including patches of plagioclase, and tufts of fibrolite needles; plagioclase felspar sometimes passing into scapolite; an abundance of a dark brown biotite, which is sensibly biaxial but with a small angle between the optic axes; and a few grains of a pale-green augite (sahlite) exhibiting twin lamellæ.

3. *Basic Rocks.—Pyroxene- and Scapolite-Gneisses and Granulites, Pyroxenites, &c.*

It is in the more basic rocks associated with the gneissic series in Burma, with their closely associated crystalline limestones, that the chief interest of the geologist is centred; for it is undoubtedly in connexion with these rocks that the rubies, spinels, and other gem-minerals are found. These rocks contain, in addition to a lime-felspar (anorthite or labradorite), several varieties of pyroxene (white and green diopside, sahlite, and hypersthene), which may be replaced by hornblende or biotite, and in many cases a considerable quantity of scapolite, wollastonite, and calcite.

Rocks of this class occur in bands which are subordinate to the gneiss of the district, and are common both in the Mogok valley and in the Injouk valley. Thicker masses of the same rock are found intercalated in the gneisses on the ridge between Mogok and Bernardmyo. The frequency of fragments of these rocks in the ruby-bearing gravels testifies to their very wide distribution.

The rocks present a characteristic dark greenish-grey colour, often accompanied by a greasy lustre, and are thus easily distinguished from the ordinary gneisses with which they are associated.

The coarser grained rocks of this class may be distinguished as *pyroxene-gneisses*, or as *pyroxene-scapolite-gneisses*, when the latter mineral is largely developed in them. These rocks appear to the eye as coarse aggregates of a grey felspar (anorthite, or a variety like bytownite) and a black or very dark-green augite, and various accessory minerals, among which sphene is the most abundant and conspicuous.

Very coarse varieties of these pyroxene gneisses occur as fragments in the Hmyaudwins No. 3, at Mogok, and No. 27, in the Kyatpyen districts; the rocks have suffered somewhat from atmospheric disintegration, but their essential characteristics can be made out in thin sections under the microscope.

The felspars exhibit lamellar twinning in a very local manner, the lamellæ dying out in certain portions of the crystals, and large areas of the latter being altogether free from lamellar structure. The extinctions of these lamellæ point to their being a variety rich in lime, and this conclusion is confirmed by micro-chemical examination of fragments. The felspars often betray traces of much alteration, and not unfrequently exhibit both opalescence and the rich play of colours so common in labradorite. Schillerization is frequently displayed, and the so-called "quartz of corrosion" is very common in them.

Some of the felspars in these pyroxene gneisses are converted wholly or partially into scapolite, and numerous beautiful examples of the manner in which this change takes place may be found. As I have shown to be the case at Bamle, the conversion of the lime-felspar into scapolite (dipyre) appears to be a deep-seated one and to be one of the results of the dynamo-metamorphic action to which these rocks have been subjected. The scapolite itself often shows the first signs of chemical alteration and becomes cloudy in ordinary light, while with the polariscope the colours due to double refraction are patchy and variable.

The pyroxene in these rocks is an augite very variable in colour, as seen by transmitted light, and forms crystals up to two centimetres in length. Sometimes it is of a rich dark green, at other times of a very pale green, becoming occasionally quite colourless. Whenever any colour can be distinguished, a *very faint* pleochroism can also be detected, the colours being a bluer green for the rays traversing the crystal parallel to the $\mathfrak{a}$ axis, and a yellower green for the rays parallel to $\mathfrak{b}$ and $\mathfrak{c}$. The absorption is so slight that I could not satisfy myself as to the direction in which it is greatest. This pyroxene often shows the beginning of alteration, ferruginous staining appearing along the cracks of the crystal. Schiller enclosures also appear in it parallel to both the orthopinacoid (100) and the basal plane (001). The augite also shows the beginning of uralitization, the development of the hornblende being accompanied by a separation of magnetite, which renders the mineral almost opaque.

Sphene is present in some of these rocks to almost as great an extent as the augite, and in this respect the Burma rock strongly resembles the well-known augite-felspar-sphene rock of Bamle. The sphene occurs in two forms. In the first place, we have the large wedge-shaped crystals from 5 to 10 millims. in length, of a yellowish-brown colour with slight but distinct pleochroism :—

$\mathfrak{a}$, very pale yellow,
$\mathfrak{b}$, greenish-yellow,
$\mathfrak{c}$, reddish-yellow.

Not unfrequently this brown sphene shows signs of alteration. Along the cleavage-cracks, especially on the outside of the crystals, black opaque deposits of ilmenite are seen, and this is sometimes accompanied by the formation of the white opaque material, so commonly observed when titanoferrite is undergoing decomposition. On the other hand, we find a number of smaller grains of the perfectly colourless sphene (leucoxene or titanomorphite) which is so frequently found resulting from the alteration of titanoferrite. It seems probable that in this rock we have examples of both kinds of paramorphism between ilmenite and sphene. The original coloured sphene is seen passing through a white opaque compound into titanoferrite ; and on the other hand a colourless sphene occurs which is probably of secondary origin and results from an alteration of the titanoferrite which originally formed a constituent of the rock.

The other accessory constituents of these interesting pyroxene gneisses are apatite, which is found enclosed in all the other minerals of the rock, zircon, which is abundant, and occasionally corundum. The existence of this mineral in these peculiar basic gneisses is a very interesting and, as we shall hereafter show, a very significant fact.

Some of the chief types of these pyroxene gneisses are illustrated in figs. 15, 16, and 17, and in Plate 6, fig. 1.

Rocks of similar mineralogical constitution to these coarse pyroxene-gneisses, but of finer texture and assuming a more or less perfect granulitic habit, appear to be by no means rare in Burma. They exhibit a great diversity in the characters of their constituent minerals, but nearly all consist of a lime- or lime-soda-felspar (more or less altered to scapolite and sometimes to calcite and quartz) and a pyroxene which may be some form either of enstatite or augite, or may be partially or wholly converted into a hornblende or a biotite. These rocks may be designated pyroxene-granulites and pyroxene-scapolite granulites, and I will proceed to describe some of their most interesting varieties.*

On the ridge below Tounguee a rock was found *in situ*, which proves on microscopic study to be a *sahlite-scapolite-granulite* (see Plate 6, fig. 3). In this rock a white augite (sahlite) is very abundant and constitutes not less than one half of the mass. The mineral seen in thin sections shows no trace of colour or pleochroism, it has the usual augite cleavages, but no trace of schiller planes, and its extinction-angle is low—apparently not exceeding 40°. With the augite a few grains of an enstatite (bronzite) also occur. The remainder of the rock is almost entirely made up of a scapolite often exhibiting traces of the commencement of decomposition. There is no distinct felspar found in the rock, but the scapolite not unfrequently shows, by the relics of a lamellar structure in it, that it has been produced by the alteration of a plagioclase. Zircons are found enclosed in all the other minerals, and a few garnets and crystals of sphene with corundum (?) are among the accessory minerals of this very interesting rock.

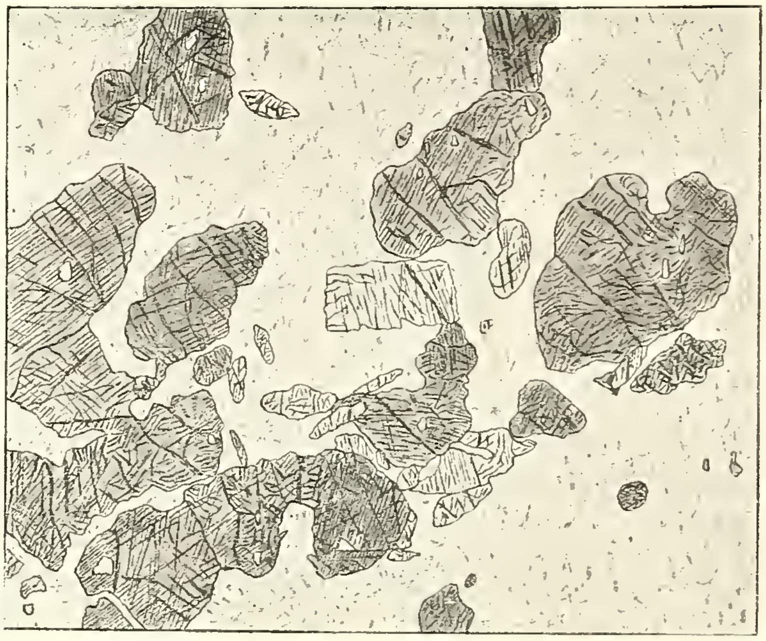
A rock closely related to the last is found at Letnytaung mountain. It is an *augite-scapolite-granulite* (see Plate 6, fig. 2) and consists of the following minerals :—A dark-coloured augite, exhibiting in thin sections the purplish tints of the varieties of this mineral containing titanium, but with scarcely a trace of pleochroism, constitutes more than one-third of the rock ; the remainder of the rock is made up of

* The interesting rock recently described by Mr. T. H. HOLLAND as "charnockite," or hypersthene granite, is a massive rock of similar composition, but rich in orthoclase.

scapolite, which has clearly been derived from a plagioclase felspar, quartz, some of which is quite free from bands of inclusions and is probably secondary, and calcite, grains of which are by no means rare; the accessory minerals are garnet, sphene, and zircon.

On the road from Mogok to Momeit a rock occurs *in situ* which contains a white augite (sahlite or malacolite), a ferriferous enstatite (hypersthene), and a little biotite with some magnetite or titanoferrite in large grains (see Plate 6, fig. 4). The white augite is similar to that found at Toungnee, and the hypersthene shows the usual pleochroism of an enstatite moderately ferriferous. The remainder of the rock is almost wholly made up of grains of scapolite, some of which show, however, in the

Fig. 15.



Pyroxene-gneiss, from Mogok. × 3 diameters. The dark crystals are a pale-green augite. The paler crystals a yellowish-brown sphene, and the rest of the slide is made up of a basic felspar near anorthite, which, in this instance, has undergone but little change.

presence of faint traces of lamellar structure that they have been produced by the alteration of a plagioclase. The brown biotite of this rock appears to have been formed by the alteration of the sahlite.

A variety of the sahlite-granulite rock from Toungnee hill is interesting as exhibiting the alteration of the white augite into brown biotite and the plagioclase felspar into scapolite. Every stage of these interesting changes may be traced in this particular rock, which also contains a considerable quantity of hypersthene.

Near the cantonments at Bernardmyo there occurs a coarser grained variety of these rocks, distinguished by the limited amount of ferro-magnesian silicates and the presence of orthoclase. The augite of this rock is largely altered to hornblende and biotite. In No. 13 Hmyaudwin at Mogok a very similar rock (see Plate 6, fig. 1)

Fig. 16.

Pyroxene-scapolite-gneiss from Mogok. × 3 diameters. Very dark-green pleochroic crystals of augite, are seen scattered through a mass of felspar and scapolite crystals, the latter being so full of inclusions as to show an opalescence like that of "moonstone." In ordinary light the scapolite is distinguished from the felspar by its slightly darker tint.

Fig. 17.

Hornblende-biotite gneiss passing into a caciphyre, from the ridge near Letnytaung × 6 diameters. The dark crystals are deep brown hornblende, with intergrowths of biotite. On the left we have somewhat decomposed plagioclase felspar, while over the rest of the slide calcite prevails.

occurs in fragments in the ruby gravels, but this rock contains hypersthene as well as augite.

Pyroxene-granulites, with both augite and enstatite (hypersthene) and the plagioclase felspar more or less completely altered to scapolite, occur, by no means rarely, in the ruby gravels, as at the Hmyaudwins Nos. 9 (see Plate 6, fig. 6) and 13 at Mogok. In many cases, however, the pyroxenes are more or less completely altered into hornblende and biotite. In one case I have found the augite changed to the schillerized forms—diallage and pseudo-hypersthene.

While some of the pyroxene-gneisses are rich in felspar and contain some orthoclase, others are remarkable for the abundance of the ferro-magnesian silicates, and when all the alumino-alkaline silicates disappear, these rocks pass into the pyroxenites, or by alteration into amphibolites. These rocks consist of an augite, sometimes nearly colourless, at other times green, and occasionally purplish in tint, and more or less hornblende, evidently formed by the alteration of the pyroxene. Enstatite (hypersthene or bronzite) is also sometimes present. The colourless minerals—plagioclase in various stages of Werneritization—and quartz and calcite are so rare that they can only be regarded as accessory. In addition we find sphene, both original and secondary (titanomorphite), apatite and titanoferrite.

At the mine No. 14 at Mogok there occurred a rock of this kind, consisting of nearly equal parts of a purplish-brown augite and a deep-brown hornblende, very similar to basaltine (basaltic hornblende). There are clear indications that the hornblende is paramorphic after the augite. In this case only minute and inconsiderable quantities of the white minerals (felspar, quartz, calcite, and scapolite) are present.

In a Loodwin, near the Momeit-Road Pass, a rock consisting of a dark-green augite, occasionally changed to a brown hornblende, is found. The rock also contains some wollastonite, and is interesting, as it is seen to be interfoliated with a crystalline limestone, the one rock passing insensibly into the other.

At the pass from Kyatpyen into the next valley to the west, there is a rock which seems to have originally consisted of a colourless augite with some brown sphene. The greater part of the augite has now been converted into a brown hornblende, and the pyroxenite has been converted into an amphibolite.

The connection between these rocks, so rich in ferro-magnesian silicates ("ultrabasic gneisses") and the calcareous rocks is often of the most intimate kind.

The last of these basic rocks which we have to notice are those which contain the beautiful blue minerals—coloured by a sulphur compound—hauyn and lazurite (lasurit of BRÖGGER). The blue crystals are isotropic, and vary greatly in the intensity of their tints, consisting in all probability of haüyne, and perhaps sodalite, as well as lazurite. The blue mineral is accompanied by a colourless pyroxene (diopside), with calcite, scapolite, epidote, and other colourless minerals, the whole forming a very beautiful lapis-lazuli rock. (See Plate 6, fig. 9.)

There are two varieties of this rock. In one the blue isotropic minerals do not form more than one-third of the mass, the remainder consisting of granules of white diopside, with a very fine ground mass, consisting largely of alteration products like scapolite and calcite.

In the other variety, the blue minerals (which vary in tint from an intense violet-blue to a pale lavender-blue), make up more than two-thirds of the whole, but enclose a number of granules of the white diopside, with some patches of decomposition products.

Although these beautiful lapis-lazuli rocks have not been detected *in situ*, their presence in the ruby-gravels with rocks that have all clearly been derived from the disintegration of the gneiss of the Burma area, shows that they must occur as members of this gneiss series. The specimens examined and described were found near Thabanpin. Different varieties of the rock are shown, by determinations made by Mr. T. BARRON, to vary in specific gravity from 2·86 to 2·94.

4. *Crystalline Limestones.*

As has been already pointed out, the connection of these calcareous rocks with the gneissic rocks among which they are intercalated, is of the most intimate character. With the basic pyroxene and scapolite-rocks, and especially with the pyroxenites and amphibolites into which these graduate, the calcareous rocks appear to be always closely associated.

The idea that great beds of crystalline limestone, intercalated in a series of foliated rocks, must necessarily have resulted from the metamorphism of calcareous strata of organic origin, which have alternated with other aqueous or igneous masses, finds no support from the characters presented by the rocks of Burma.

Between gneissic rocks containing numerous crystals of calcite and dolomite, and rocks consisting mainly of those minerals, with the various constituents of the basic gneisses—augites, enstatites, scapolites, phlogopites, &c.—scattered through them, we find every gradation, and the limestones and gneisses are also found interfoliated in the most intimate manner.

That some impure crystalline limestones (hemithrènes, kalkaphanites, &c.) have been formed from basic igneous rocks, there is not the smallest doubt,* and in the same manner the study of the Burma limestones leads to the conclusion that the basic gneisses may undergo a corresponding metamorphosis, the breaking up of the lime-felspars leading to the formation of an unstable scapolite, from which and from other unstable minerals originate, as a consequence of further change, the calcite and dolomite; while the more stable silicates become enclosed in the crystallizing mass of carbonates, and new minerals are formed during the progress of these complicated chemical actions.

The capricious manner in which the gneisses and calcareous rocks of Burma are

* See Liebrich, 'Neues Jahrb. für Min.,' &c., for 1893, Bd. 2, p. 75.

associated, and the unexpected manner in which the one type of rock seems to replace the other, has been illustrated by Mr. BARRINGTON BROWN in his account of their relations as seen in the field.

In their general appearance and texture these metamorphic limestones of Burma exhibit the widest diversity. Sometimes, as in the bands that occur at Bernardmyo, they are finely granular (saccharoid) in character ; at other times, as is well exemplified in the ruby-cave at Mogok, the individual crystals of calcite may grow to the size of a man's fist. There are few more beautiful limestones than this rock of Mogok, with its delicate blue tint and the large and brilliant cleavage-faces of its twinned calcite crystals.

The crystalline limestones of Burma also differ greatly from one another in the proportions and the nature of the minerals enclosed in them. Some of the limestones are almost free from foreign minerals, while others are made up to the extent of more than half their mass of various silicates, oxides, and sulphides.

Many of the limestones contain a notable proportion of dolomite. Sometimes the dolomite grains are of the same size as those of calcite, with which they are intergrown, and only differ from them by the comparative rarity of twins following the $-\frac{1}{2}$R (110) planes of the crystal and their insensibility to the action of cold dilute acids. In other cases, however, the dolomite in these limestones make up aggregates of minute rhombs, which are left behind when the rock is digested with acetic acid or with dilute hydrochloric acid in the cold, but are rapidly dissolved in boiling acid.

We may group these crystalline limestones into two classes according to the nature of the foreign minerals that are present in them, as follows :—

(1) *Cipollinos*, in which the predominant foreign constituent is a micaceous mineral, and the rock usually exhibits a more or less distinctly foliated structure, thus passing into a calc-schist (kalkschiefer). Rocks of this class occur at Mandalay Hill and on the Mogok side of the pass on the Bernardmyo road, near Toungnee mountain, and many other localities too numerous to specify. Similar limestones are very common as fragments in the various pits opened in the alluvial deposits for obtaining rubies.

(2.) *Calciphyres*.—This old name of BRONGNIART'S may be conveniently applied to the rocks, consisting of a ground mass of calcite crystals, through which various pyroxenes and other minerals are distributed like porphyritic constituents. The well-known red limestone of Ballyphetrich (Tiree marble) was made by BRONGNIART the type of his Calciphyres, and there are many interesting points of resemblance between the Hebridean limestone and the very beautiful rocks of the same class in Burma.

The minerals that have been found in these Burma limestones are very numerous. They may be conveniently divided into two classes as follows :—

(*a*.) Minerals which are identical with those found in the associated basic gneisses, pyroxenites, and amphibolites.

(*b.*) Minerals which occur in the limestones, but have not been found in the masses with which the limestones are interfoliated.

There are a few minerals (which must be relegated to a third group, as they occur rarely in the pyroxene-gneisses and pyroxenites, but are much more abundant in the limestones.

Belonging to the class (*a*) are the following species :—

Diopside (white and pale-green).
Bright-green augite.
Purplish-brown augite, passing into diallage and pseudo-hypersthene.
Sahlite (white, aluminous augite.)
Enstatite.
Bronzite.
Hypersthene.
Quartz.
Orthoclase (Murchisonite and Moonstone).
Oligoclase.
Anorthite.
Hornblende (a variety near Basaltic Hornblende).
Biotite and other micas, including Fuchsite.
Scapolite.
Zircon.
Magnetite.
Titanoferrite.
Sphene (both original and secondary).
Rutile (in inclusions).
Garnet (almandine).

(*b.*) Among the minerals which especially characterize the limestones are the following :—

Phlogopite (changing to vermiculites).
Wollastonite.
Corundum (ruby).
Spinel.
Pyrrhotite.
Diaspore.
Hematite.
Limonite.
Apatite (Moroxite.)
Graphite.

The minerals which occur in the gneisses but have not certainly been detected in the limestones are :—

Tourmaline (Rubellite, Indicolite, &c.).

The results obtained by separating the silicates from the carbonates by the action of dilute hydrochloric acid in the case of the beautiful blue limestone from the ruby cave of Burma, were as follows :—The proportion of insoluble residue was found to vary in different cases from 2 per cent. to 3·34 per cent. The amount of insoluble minerals in a number of average fragments, weighing together 688·46 grammes, was 21·23 grammes. This gives a percentage of 3·17, which is probably near a true average. The limestone of the Burma ruby cave is a pure calcite, almost wholly free from dolomite.

5. *Gravels and Earthy Deposits.*

These deposits are made up entirely of fragments of the rocks found *in situ*. The fragments are sometimes all angular or sub-angular ; but well-rounded water-worn pebbles are by no means unfrequent in them. The fine materials of the earths consist of scales of kaolinite, mica, &c., stained to a yellowish-brown tint by ochreaceous material. J. W. J.

VII. Mineralogy.

In studying the series of minerals obtained from the interesting corundiferous localities of Burma, there are three classes of facts which appear to have a special significance and to be worthy of study for their important scientific bearings.

First. The association of the various species of minerals in the rocks of Burma, which contain the rubies and other gems ; and the comparison of this association with the facts observed in other areas, where the same gems occur. This study of paragenesis constitutes a line of enquiry of great promise to the mineralogist.

Secondly. This association of minerals in particular rocks of different areas is suggestive of the conditions under which the Burma gems—the corundum (ruby), spinel and tourmaline (rubellite)—may have been formed, and the chemical reactions to which they owe their origin.

Thirdly. Not less interesting than the question of the origin of the ruby and its associated gems, are the problems concerning their alteration and destruction, which, as we shall show, have been continually going on, as the result of both deep-seated and surface action.

1. *Association of Minerals in the Gem-localities of Burma.*

The investigations of Mr. C. Barrington Brown have shown that the parent rock of the famous rubies of Burma (the "pigeon's-blood rubies") is undoubtedly a highly crystalline limestone which encloses crystals of corundum, spinel, pyrrhotite, graphite, phogopite, and many other minerals. Equally decisive is the evidence that he has brought forward to show that the fine red tourmalines (rubellites) of the same district in Burma do not occur in these limestones in association with the rubies and spinels, but come from another locality, their parent rock being a gneiss or schist. The significance of these facts will become apparent when we consider the association of minerals at other localities where corundum is found in limestone, as at St. Gothard, in Switzerland, and in the limestone belt of Orange Co., N.Y., and Sussex Co., N.J., in the United States.

There are two kinds of evidence which are available in determining the nature of the minerals associated with the ruby and spinel in Burma. The first and most satisfactory is derived from examination of the limestones which contain the gems. This may be done either by means of thin sections, or by dissolving the limestone in acid and separating and determining the various insoluble minerals left behind. Nature, however, has done this latter kind of work for us on a grand scale, and in the alluvial deposits we find great accumulations of the insoluble minerals which were originally embedded in the limestone, and here we naturally obtain larger and finer crystals than we can hope to procure by carrying on the work of solution in the laboratory. It must not be forgotten, however, that there is an element of uncertainty in the evidence afforded by these alluvial deposits, as some of the minerals found in them may be derived, not from the corundiferous limestones, but from other associated rocks in the district; hence it will not be wise to accept the evidence afforded by the alluvial deposits, unless it is corroborated by the fact of the same minerals being actually found in the limestone or in the residues obtained by its solution.

In the following list are included all the species of minerals which I have been able to determine in the crystalline limestone of Mogok and the surrounding districts and in the granular limestone of Sagyin. In the remarks about the several minerals I have, however, often availed myself of information derived from the study of the finer and larger specimens yielded by the alluvial deposits.

1. *Corundum.*—This mineral rarely presents crystalline outlines, for, as we shall presently see, chemical disintegration and alteration into diaspore and various silicates has gone on from the exterior, and the core of unaltered aluminium oxide left behind is of irregular form. Occasionally, however, well defined crystals of corundum are obtained, and it is a noteworthy circumstance that, as has been already pointed out by Mr. F. R. MALLET, the Burma rubies generally assume peculiar and characteristic combinations of forms.* Crystals made up of the rhombohedron and basal plane (R and 0R : 111, 100) are very common, though the prism (∞ P2, $10\bar{1}$) occasionally enters into the combination. Forms in which the prism and pyramids (so constantly seen in the corundums from Ceylon, Southern India, United States, &c.) are found predominating, are, so far as my observation goes, very rare; though several such combinations are figured by Mr. MALLET.

The forms present in the ruby-crystals of Burma are thus seen to be the same as those found in the artificial rubies made by MM. FRÉMY and FEIL; but in the arti-

* 'A Manual of the Geology of India,' Part IV., Mineralogy (1887), p. 43, Plate 1, figs. 6, 7, 8.

ficial crystals the degree of development of the basal plane (0R) is usually much greater, and the crystals thus come to assume a tabular character.

In order to obtain the specific gravity of the Burma rubies, a selection of clear finely coloured gems was made from the washings brought from Burma by Mr. BARRINGTON BROWN, and their density was determined by the pycnometer. The average density of the Burmese gem was found to be 4·03. DE BOURNON found the average density of twenty rubies, of which the locality is not stated, to be 3·98. Most of the associated red corundums of Burma have a considerably lower density, and in the case of one crystal, showing marked indications of chemical change, it was found to be as low as 3·74.

The colours of the Burma corundums are very variable. Every shade from white through various shades of pink up to the deep crimson (pigeon's-blood colour), which is so highly prized, may be found. The stones are not unfrequently parti-coloured, and variable in tint in different portions. Mr. BARRINGTON BROWN brought home a remarkable specimen, which was colourless at one end and graduated at the other end into the most beautiful and intense red. All the coloured varieties of corundum occur, but in much smaller quantities than the red ones; blue of every depth of tint (sapphires), yellow (oriental topaz), purple (oriental amethyst) are not rare; but the green varieties (oriental emerald, oriental chrysolite, and oriental aquamarines) with the colourless ones (white sapphires) appear to be of very exceptional occurrence.

The pleochroism of the Burma rubies of course varies with their colour, but there seem to be constant peculiarities in the pleochroism of gems of different districts which can be detected by the aid of the dichroiscope. Thus the sapphires of Burma, like those of Ceylon, show tints of blue and *straw-colour* with the dichroiscope, while the Siam sapphires give blue and a decided *green* colour. In the same way, while the fine rubies of Burma give crimson and *aurora-red* tints for the ordinary and extraordinary ray respectively, the less finely coloured stones from Siam give crimson and a *brownish-red* tint for the two rays.

The Burma rubies not unfrequently exhibit cavities, sometimes of considerable size, and often of very irregular form. These cavities sometimes contain a liquid and bubble (carbon dioxide), but in other cases appear to be filled with solid matter—a reddish-brown glass in some instances. Not unfrequently we find the rubies filled with acicular or platy inclusions arranged along the rhombohedral planes and intersecting one another at angles of 60°. These give rise to the beautiful phenomenon of asterism. In some cases, a well-marked zoned structure is exhibited by crystals, in which layers filled with inclusions, and giving a striking "schiller," alternate with clear bands without inclusions. A remarkable example of this kind is in the possession of Mr. STREETER.

In the great majority of cases the Burma rubies exhibit a perfectly irregular or conchoidal fracture, like that of quartz. Chemical alteration sometimes leads to the formation of a secondary parting along the basal plane, while still more rarely the familiar gliding planes following the primitive rhombohedron have been developed by pressure and intensified by subsequent chemical change.

The clear and brilliantly coloured rubies seldom, if ever, attain to great dimensions, either in the Burma limestones or in the alluvia; but masses of red corundum are occasionally found weighing several pounds. [Occasionally specimens are found showing a perfect transition from the dull red crystalline corundum into the transparent and lustrous ruby. I am indebted to the Burma-Ruby-Mining Company for a beautiful illustration of this transition.—November 30, 1895.]

2. *Spinels.*—In the beauty of their colour and lustre the spinels found in the Burma limestones are scarcely inferior to the corundums, but they are of course wanting in that play of colour which results from the possession of the property of pleochroism. They more frequently exhibit their proper crystalline form—the octahedron without modifications—than do the corundums; but twinned forms appear to be rare. Various tints of red are the most common colours, specimens from the palest pink to the deepest crimson being found. The average specific gravity of these red spinels, determined on a number of carefully selected examples, was 3·52. With the red spinels, however, there occur more rarely various shades of blue, violet, brownish, and occasionally black (ceylanite). The faces usually exhibit etched figures, and, as we shall see in the sequel, alteration forms of the mineral are very abundant. The fracture is almost always conchoidal, and only rarely do we find traces of the parting (probably of secondary origin) parallel to the octahedral faces.

The proportion of spinel to corundum, at different localities in Burma, appears to vary to a very marked extent. At Sagyin, spinels are very abundant, and rubies comparatively rare, while in the limestone of Mogok, the red corundum abounds and the spinels are in nothing like such great excess. In a mass of gems brought home from a twinlone at Mogok, by Mr. BARRINGTON BROWN, I found that the spinels weighed 54·39 grams, or 64 per cent. of the whole, and the corundums 30·33 grams, or 36 per cent. All the spinels from this pit were various shades of red, with the exception of 0·39 gram of shades of blue spinel. Among the corundums, 25·87 grams were various shades of red, 1·20 grams were blue (sapphire), 2·57 were yellow (oriental topaz), and 0·69 purple (oriental amethyst).

A prospecting pit, sunk by Mr. BARRINGTON BROWN at Bernardmyo, yielded, however, very different results. The washing of about two cubic yards of earth yielded 76·71 grams gems, in which I found 63·89 grams, or 83 per cent., of spinels, mostly red, and 12·82 grams, or 17 per cent., of corundum. The latter comprised 5·65 grams of colour and clearness, entitling them to be called "rubies," 1·26 grams of sapphire, and 6·91 grams of red corundum.

3. *Zircon* occurs but rarely in association with the corundums and spinels of Burma. Well-developed crystals are found in the washings, consisting of the combination of prism and pyramid (∞ P and P : 110, 111). These zircons are nearly colourless, have a remarkable lustre, and may easily be mistaken for diamonds. In a mass of lime-

stone brought from the ruby-cave at Mogok by Mr. BARRINGTON BROWN, there occurs a colourless crystal of high lustre with curved faces, which was naturally taken, at first sight, for a diamond. A careful measurement of the angles by Mr. H. A. MIERS, of the British Museum, however, suggested that the mineral must be zircon. Strange to say, however, when this specimen was sealed up in a vacuum-tube by Mr. CROOKES, and submitted to an electric discharge, it gave a phosphorescence very similar to that of many diamonds, and very different from that of most zircons. But an undoubted zircon from washings at Burma, when sealed up and exposed to the electric discharge, exhibited the same remarkable phosphorescence. For the application of these important tests, and much other valuable aid in the study of these minerals, I am greatly indebted to Mr. W. CROOKES, F.R.S.

4. *Garnets*, which are so abundant in the gneiss rocks that are interfoliated with the limestones, occur also in the latter rocks themselves. Almandine, andradite, and more rarely grossular garnets occur, crystals over a pound in weight being sometimes found. Like the accompanying minerals, the garnets are often found undergoing various stages of alteration; and in the gravels there are frequently found pseudomorphs of garnet crystals, which crumble to pieces when handled.

5. *Apatite* of a beautiful and unusual blue tint is found in some of the washings, mingled with the other gems. It forms well-defined crystals.

6. *Felspars* of several species, both orthoclastic and plagioclastic, abound in the washings. The orthoclase often exhibits the Murchisonite parting, with the peculiar schiller due to reflection from air films characteristic of that variety. Not unfrequently the varieties with brilliant reflections from included scales (sunstone), and those with opalescence and chatoyance (moonstones), are found. In many cases the felspar-crystals, as they occur in the washings, are completely converted into kaolin, and sometimes they are, to a greater or less extent, changed into epidote. In one case, a large felspar-crystal was found in which alteration into epidote had gone on along the two principal planes of cleavage, the result being a mass which weathered, with surfaces that suggested organic structure.

7. *Quartz* of different varieties is found in nearly all the ruby-earths. Milk-quartz, rose-quartz, and amethyst occur, but are rare, while citrine and smoky quartz are common. Many beautiful examples of parallel growths are seen, and, in the case of a fine example from a mine at Mogok, this parallel growth is accompanied by a spiral twisting of the principal axis of the crystal, resulting in forms like those of Trabecsthal, which have been lately so admirably investigated by TSCHERMAK.* Even the quartz has not escaped the alteration so universally exhibited by the minerals of these ruby-bearing rocks. We find the quartz-crystals to be, in nearly all cases, pitted all over with naturally etched figures (*Verwitterungs-figuren*), often of a very interesting character.

8. *Micas.*—Phlogopite is one of the most common constituents of the corundiferous

* 'Denks. k.k. Akad. Wissensch., Wien Math.-Naturwissensch. Cl.,' Bd. 61 (1891), p. 365.

limestones of Burma; and in the ruby-earths hexagonal plates of mica with a diameter of from 20 to 30 centims. are frequently found. These phlogopites are some clear and almost colourless, but more usually show tints from smoke-brown to black. The angle between the optic axes of these phlogopites is shown by the interference figures to be small, never exceeding 20 or 30 degrees, and is often much less. In the ruby-earth muscovite, probably an alteration product from corundum, orthoclase and other minerals are frequently found.

Fuchsite, or the beautiful green chrome-mica, certainly occurs in some of the samples of ruby-earth from Burma, and is also sometimes seen in the limestone containing rubies from Mogok (see Plate 6, fig. 10).

9. *Amphiboles* are common in the ruby-earths and the limestone, and include several varieties of both green and brown hornblende with a variety near to arfvedsonite.

10. *Pyroxenes* are particularly common, and include a nearly colourless sahlite, green diopside, and a green soda-pyroxene (ægerine), with a jade-like variety among the augites, and both bronzite and hypersthene among the enstatites; *wollastonite* is by no means rare in some of the limestones and associated rocks.

11. *Fibrolite* (sillimanite) is also a constituent of the limestones.

12. *Scapolite* is found in the limestone, but has not been detected in the ruby-earths; it has probably disappeared through alteration processes, giving rise to the formation of calcite.

13. Besides *Muscovite* and *Gilbertite*, other silicates found in these limestones, which probably represent decomposition products, are *Margarite*, and several other clintonites, and various chlorites and vermiculites.

14. *Kaolinite* (*nacrite*) is very abundant, probably derived from the felspars; and, indeed, this and similar hydrated silicates of aluminium appear to make up the bulk of the ruby-earths.

15. *Lapis-lazuli.*—Among the interesting minerals brought from the ruby-earths of Burma, are great blocks of lapis-lazuli. These are of two varieties. In one, the quantity of blue mineral is so great that the rock-masses have a deep indigo tint, the quantity of white minerals being, as seen in thin sections under the microscope, comparatively small. In the other variety we have a white mass speckled with blue. Microscopic study of these rocks shows that we have several isotropic, blue minerals present (lazurite, haüyne, sodalite) and various nearly colourless minerals, including white-diopside, wollastonite, scapolite, epidote, and calcite.

16. *Graphite*, often well crystallized, occurs in plates up to 8 or 10 centims. in diameter in the ruby-earths. It is also found to be present in almost all the limestones.

17. *Pyrrhotite*, like graphite, is among the most widely distributed of the minerals in the limestones. It appears, like graphite, to be a constant associate of the rubies and spinels.

In addition to these undoubted constituents of the Burma limestones, there are several other minerals which have been brought from Burma, but have not yet been detected in the limestones. Among these are beryl (aquamarine) and danburite.

The absence of certain minerals from the Burma limestone is very noteworthy, especially if we compare the association of minerals there with that found in other corundiferous limestones—like those of Orange County, N.Y., and Sussex County, N.J., and the well-known rock of St. Gothard. So far as my observations go, I can find no trace of the silicates with fluorine and boric acid. Neither topaz, the tourmalines, the axinites, or the chondrodites have been found, though most of these minerals are found abundantly in the corundiferous limestones of the Eastern United States and the Alps. Chondrodite has been reported as occurring in Burma limestones; but, I think, the yellow decomposition products of corundum may in some instances have been mistaken for the mineral in question.

[Since this was written, my friend, Mr. T. H. HOLLAND, F.G.S., of the Geological Survey of India, has sent me some remarkable specimens of a limestone from Sagyin, in which very fine spinels are found associated with much chondrodite. My friend, Dr. W. T. BLANFORD, informs me that a limestone of this kind also occurs just outside Mandalay. But of the occurrence of chondrodite in the Mogok limestones, I have hitherto been unable to obtain any evidence.—October 7th, 1895.]

2. *Origin of the Corundum, Spinels, and other Minerals occurring in the Limestone of Burma.*

The investigations carried on with respect to the mode of origin of the Burmese gems, which have been detailed in the preceding pages, show that the parent rock of the corundum and spinel is a limestone—either highly crystalline or saccharoidal—which contains a great variety of included minerals, silicates, and oxides, with graphite and pyrrhotite. The limestone which the rock of Burma most closely resembles is undoubtedly that of Orange County, N.Y., and Sussex County, N.J., which are associated with the remarkable deposits of zinc ore at Franklin Furnace and Ogdensburg, N.J. In comparing the minerals found in these limestones of North America, as given by Mr. J. F. KEMP,* and those of Burma, we shall be struck by the large number of species in common. But there is one striking and significant difference that must be borne in mind. The tourmalines and chondrodites so abundant in and characteristic of the North American limestone, appear to be nearly, if not altogether, absent from the Mogok limestone.

At St. Gothard, corundum is also found embedded in a dolomitic limestone, and here, too, it is associated with tourmaline.

The researches of LAWRENCE SMITH† and TSCHERMAK‡ have shown that the emery

* J. F. KEMP, "The Ore Deposits at Franklin Furnace and Ogdensburg, N.J." ('Trans. N.Y. Acad. Sci.,' vol. 13 (1893), pp. 76–98).

† 'Am. J. Sc.,' vol. 7 (1849), p. 283; *ibid.*, vol. 10 (1850), p. 354; and vol. 10 (1851), p. 53.

‡ 'Min. u. Pet. Mitth.,' Bd. 14 (1894), p. 311.

variety of corundum of Asia Minor also occurs in limestone. The last-mentioned author has proved that the following minerals enter into the constitution of the emery itself—corundum, magnetite, tourmaline, chloritoid, muscovite, margarite, and calcite—the last three being probably products of the alteration of the corundum itself. The list of minerals found in the corundum-bearing limestone, according to these two authors, includes diaspore, gibbsite, (hydrargillite), spinels (including gahnite or zinc spinel), muscovite, phlogopite, biotite, vesuvian, margarite, chloritoid, and other clintonites, rutile, ilmenite, magnetite, hematite, staurolite, kyanite, pholerite, kaolinite, and tourmaline.

With respect to the origin of the remarkable limestone in which the Burmese gems occur, I have been unable to find the slightest evidence that it has been formed by the alteration and recrystallization of an organic deposit; on the contrary, all the facts point to a totally different conclusion.

The limestones of Burma are, as we have seen, most closely associated and intimately interfoliated with granitic and gneissic rocks, and especially with the unstable pyroxene and scapolite-bearing gneisses. Between these pyroxene-gneisses and the limestone we find every intermediate type of rock. Gneisses in which calcite is an important constituent (see fig. 17, p. 203) pass quite insensibly into rocks in which the same varieties of pyroxenes, amphiboles, and mica occur, but in which, by the increase of calcite, the rock becomes a true "calciphyre," like that of Tiree.

Now all modern researches point to the conclusion that the pyroxene- and other gneisses resembling those of Burma were of igneous origin, and the limestones have certainly not the appearance of having become associated with the gneisses by the intrusion of the latter among them. On the contrary, when we come to study the metamorphism of the minerals in these gneisses, calcite is found to be constantly one of the final products of the changes which have gone on in the rocks. The abundant lime-soda felspars (labradorites) of the pyroxene-gneisses have been everywhere converted into scapolites (dipyre, &c.), the change having being brought about (as in the case of the analogous rock of Bamle, which I have investigated) by the action of hydrochloric acid under pressure. For this change LACROIX has proposed to use the term "Werneritization." But the scapolite is itself a very unstable mineral, and calcite is the constant product of its decomposition. It appears to me probable, therefore, that the calcite so frequent in these highly crystalline rocks, whether occurring as disseminated crystals through the gneiss or as great interfoliated masses, is really neither altered organic limestones nor of ordinary chemical (tufaceous) origin, but have resulted from the metamorphism of the lime-bearing felspars.

[If this be the case, it is, of course, necessary to suppose that the calcium carbonate has been often transported to new localities in solution, while the basic aluminium and other silicates have in some cases been broken up, so as to give rise to the formation of corundum, spinel, and the various other minerals occurring in the limestone or in the rocks so closely associated with it.—October 7th, 1895.]

If this view of the origin of the gem-bearing limestones be accepted, we may possibly find means of accounting for the formation of the gems, corundum, and spinel themselves.

It may be remarked at the outset of our enquiry that the association of corundum and spinel, with such varied minerals in different cases, points to the conclusion that these gems have not always owed their origin to the same set of causes.

In the great belt of corundiferous rocks in the eastern part of the United States, corundum has been shown to be constantly associated with ultra-basic rocks (peridotites), more or less completely converted into serpentine, and included in a series of highly metamorphic masses.

But corundum is no less frequently found in association with more acid rocks. Near Mozzo, in Piedmont, corundum occurs in a felspar-rock, in the Riesengebirge in granite, and in the Zanskir mountains of Cashmere and many other localities in gneissic rocks.

At Pipra, in S. Rewah, and at other points in Central India, masses of corundum-rock of enormous thickness and extent have been found interfoliated with the crystalline gneisses and schists.*

Corundums are found, probably produced by contact metamorphism, in basaltic lavas (Unkel-on-Rhine, and especially near Le Puy, &c.), and in blocks ejected from volcanoes (Laacher See, Niedermendig, Königswinter, &c.).

In limestones, corundum occurs usually in association with tourmaline and chondrodite, as at Orange Co., N.Y., and Sussex Co., Jersey, St. Gothard, Naxos, and other localities; and at other times, as in Burma, in limestones without the minerals containing boric acid and fluorine.

There are two other modes of occurrence of corundum which are of especial interest. HENRI STE.-CLAIRE DEVILLE showed† that when the natural hydrated aluminium oxide of Baux (bauxite) is fused with caustic soda, digested with water, and then treated with nitric acid, a few grains of considerable hardness are left behind. These hard grains resist the action of acids, but dissolve in fused potassium bisulphate. On analysis they were found to be aluminium oxide containing traces only of iron, titanium, and vanadium. The hard grains were by these tests proved to be corundum.

M. MOISSAN has shown that the remarkable iron-masses of Ovifak, Disco Island, Greenland, contain disseminated grains of corundum sapphire.‡

Spinel probably occurs in all, or nearly all, the different associations in which we find corundum, and, indeed, the association of the two minerals with one another is of frequent occurrence.

* See 'Mining Mag.,' vol. 11 (1895), p. 57; and 'Records of Geol. Survey of India,' vol. 5, p. 20, and vol. 6, p. 43.

† 'Ann. de Chim. et de Phys.,' vol. 61, p. 309; 'Chemical News,' vol. 4 (1861), p. 341.

‡ 'Compt. Rend.,' 116, p. 269.

If we now turn our attention to the processes which have been devised for causing aluminium oxide to assume the crystallized form, and thus produce corundum, we shall find that no less than twenty methods are now known by which this end may be attained. They are as follows :—

1. Fusing aluminium oxide in oxyhydrogen flame (GAUDIN, 1869).
2. Heating alum alone or with potassium sulphate (GAUDIN, 1839).
3. Heating aluminium chloride in closed tube (suggested by GAY LUSSAC, accomplished by MEUNIER, 1880).
4. Heating aluminium oxide with borax (EBELMEN, 1851).
5. Heating aluminium fluoride with boric acid (DEVILLE and CARON, 1857).
6. Acting on aluminium oxide, at a red heat, with hydrofluoric acid (HAUTEFEUILLE, 1865).
7. Heating sodium aluminate with hydrochloric acid (DEBRAY, 1865).
8. Heating aluminium phosphate with potassium sulphate (DEBRAY, 1865).
9. Heating aluminium oxide with minium in a crucible (FRÉMY and FEIL, 1877).
10. Heating aluminium oxide with a fluoride in the presence of an alkali (FRÉMY and VERNEUIL, 1887).
11. Melting together microcline and fluorspar (FOUQUÉ and M. LEVY, 1884).
12. Heating to redness for one hour cryolite and a silicate (LACROIX, 1887).
13. Heating aluminium oxide with silica and cryolite (MEUNIER, 1880).
14. Melting nepheline (P. HAUTEFEUILLE and A. PERREY, 1890).
15. Acting on aluminium oxide by water at a red heat (MEUNIER, 1880).
16. Heating aluminium oxide with soda to 530° to 535° C. for 20 hours (FRIEDEL, 1891).
17. Heating solution of aluminium chloride to 350° C. in closed vessel (SENARMONT, 1850).
18. Heating solution of aluminium nitrate to 350° C. in closed vessel (SENARMONT, 1850).
19. Heating aluminium oxide with water and trace of ammonium fluoride to 300° C. for 10 hours (BRUHNS, 1889).
20. Heating solution of aluminium sulphate in closed tube to 160° to 180° C. (WEINSCHENK, 1890).

It would be rash to suggest that any one of the methods enumerated above was the one by which corundum has been naturally produced in any particular rock. But it is evident that by these, or similar, reactions, crystallized aluminium oxide may have been formed in deep-seated rocks, under enormous pressure, at even moderate temperatures, and that, with sufficient time, the crystals may have grown to considerable dimensions.

The general conclusions concerning the origin of the rubies of Burma, to which we have been led by these studies then are as follows : Pyroxene-gneisses abound, with an unstable basic felspar (labradorite or anorthite), which is easily converted by the action of minute quantities of hydrochloric acid under pressure into a scapolite ; the scapolite in turn breaking up into various hydrated aluminium silicates and calcite. In some cases, however, the basic silicate may be converted directly, by carbonic or other acids, into a mass of hydrated silicates, quartz, and calcite. Examples of such a change are found in the cavities of many amygdaloidal basic rocks. LIEBRICH has recently shown that among the products of decomposition in a basalt of Rudigheim, near Hanau, is a special form of calcium carbonate occurring in nodular concretions with spheroidal and concentric structure, in masses up to and more than a foot in diameter. It is noteworthy that this limestone is associated with clay, *bauxite*, and hyalite.* A similar case, occurring in South Africa, has been communicated to me by Mr. D. DRAPER, F.G.S. In this connection, it must be remembered how frequently calcareous materials make their appearance in masses of altered basic rocks—the "kalk-diabases," "kalk-aphanites," &c., of many authors. The rocks known to French geologists as "hemithrènes," are probably of similar origin.

While the limestones are being formed from basic felspars, the aluminium silicates taking up water may also be attacked by sulphuric, hydrochloric, boric, or hydrofluoric acid acting at moderate temperatures, and the salts of aluminium thus formed are easily decomposed ; the aluminium oxide, either hydrated (diaspore, gibbsite, bauxite, &c.) being set free, or under certain conditions of temperature and pressure the anhydrous oxide itself being formed. The slowly liberated oxide may assume the crystalline form, and thus give rise to corundum. That the crystallization of the aluminium oxide took place under great pressure, and probably at moderate temperatures, is indicated by the circumstance that the crystals include not only cavities containing supersaturated solutions of chlorides, sulphates, &c., but also, in some cases, liquid carbon dioxide, which remains liquid at all ordinary temperatures, below the critical temperature for that gas.

3. *Metamorphoses of the Rubies and Associated Minerals of Burma.*

At the earth's surface, as is well known, corundum, or the crystallized oxide of aluminium, is one of the most unalterable of substances. Fragments found in river gravels and sands, though perfectly water-worn, show no trace of chemical alteration in their surfaces. On the other hand, there can be no doubt that conditions must exist in the earth's crust, under which chemical change of this mineral does take place ; this is abundantly proved by the frequency with which undoubted

* 'Neues Jahrb.,' 1893, vol. 2, p. 75.

pseudomorphs of corundum occur. Among the minerals found replacing corundum as pseudomorphs are muscovite (damourite), various forms of spinel, andalusite, fibrolite, cyanite, margarite, chloritoid, zoisite, ripidolite, and other chlorites, various vermiculites, kaolin, and other substances. The hydrates of alumina, diaspore and gibbsite, are seldom, if ever, found as pseudomorphs after corundum because (as is so well seen in the case of the diaspore of Dilln, near Schemnitz) the hydrated oxides of aluminium very readily enter into combination with silica, forming various silicates. The spinels, so commonly associated with corundum, are also frequently altered; pseudomorphs after spinel in hydrotalcite, serpentine, talc, and hydrous biotites being well known to mineralogists.

Between the years 1849 and 1851, the late Professor J. LAWRENCE SMITH, published several important memoirs on corundum and emery.* One very striking fact established by the experiments and analyses of this able observer, was that all forms of corundum and emery (excluding the fine gem-varieties known as ruby, sapphire, &c.) contain water up to about 3 per cent., with varying proportions of silica, lime and iron oxide. LAWRENCE SMITH also showed that the abrasive power of corundum steadily diminishes as the proportion of water present in it increases; and he was clearly of the opinion that the water in these specimens of corundum is combined with a portion of the alumina forming a hydrate disseminated through the mass. In connection with this subject he remarks: "Of all the specimens that I have collected, none offer so much interest as those composed of diaspore embedded in corundum; here we see the two minerals passing one into the other, without being able, in many places, to distinguish the line of separation, so imperceptible is the gradation. After what has been said in respect to corundum, it is not astonishing to see the connection of alumina, more or less hydrated, with a hydrate of alumina of definite composition."†

To another distinguished American mineralogist, the late Dr. F. A. GENTH, we are also indebted for many valuable observations which illustrate the ease with which corundum becomes hydrated, and then, by combination with silica, is converted into a great variety of crystallized minerals. In his valuable memoir on "Corundum, its Alterations and Associated Minerals," published in 1873, Dr. GENTH showed by a series of careful analyses, how remarkable have been the series of metamorphoses which this mineral has undergone, in the great corundiferous belt of the Eastern United States.‡ RAMMELSBERG, it is true, in the second edition of his 'Handbuch der Mineralchemie,' published in 1873, threw some doubts on the results announced by GENTH; § but in a later memoir, published in 1882, the latter author fully established

* 'Am. Journ. Sci.,' series 2, vol. 7, (1849), p. 283. *Ibid.*, vol. 9 (1850), pp. 289, 354. *Ibid.*, vol. 11 (1851), p. 53.

† *Ibid.*, p. 58.

‡ 'Proc. Am. Phil. Soc.,' vol. 13 (1873), pp. 361–406.

§ 'Handbuch der Mineralchemie,' 2nd edition, Spec. Theil, p. 147.

the accuracy of his conclusions.* As the result of his studies, GENTH wrote, "May it not be that the diaspore is so very finely distributed through the corundum, that even the best microscopic or other examination could not detect it, as I have just shown with regard to the admixture of corundum and spinel."†

In many cases, alteration of corundum into diaspore can be seen, in thin sections under the microscope, to have taken place along the planes of secondary parting following the rhombohedron. The reduction in specific gravity of the mineral serves as a measure of the hydration that has taken place. Thus, in an altered corundum from the United States, I found the specific gravity to be 3·88, while in a still more altered specimen from Rekwanna, Ceylon, given to me by Mr. C. BARRINGTON BROWN, the density was as low as 3·79. As the density of unaltered corundum is 4·02, and of diaspore is 3·32, we may calculate that the American specimen consists of four parts of corundum united with one part of diaspore; while the Ceylon specimen is made up of two parts of corundum with one of diaspore. In the last-mentioned specimens the presence of diaspore can be detected by the naked eye. The corundum crystals break easily along the rhombohedral parting-planes; and these are seen to be covered with films of diaspore, exhibiting its characteristic colour and lustre. Common corundum has an average specific gravity of 3·93, and may, therefore, be regarded as a combination of six parts of corundum with one part of diaspore. It must be remembered, however, that in many cases the change has proceeded one step farther, and various silicates have been formed from the hydrated alumina.

The hydration of corundum, and the union of the diaspore thus produced with silica and certain bases to form various silicates, is admirably illustrated in the case of the Burma examples. Scattered through the highly crystalline limestone we find a number of rubies, most of these being enveloped by a mass which clearly consists of the products of alteration of the crystallized aluminium oxide. In some cases the rubies are only surrounded by a thin shell of these alteration products, but in other instances every trace of the original ruby has disappeared, and the products of its decomposition fill the space once occupied by it. Under the microscope thin sections enable us to follow out the several stages of the metamorphism that has taken place. Immediately around the unaltered ruby there is always a layer, varying in thickness, of pure diaspore (hydrated alumina); but as we pass outwards we find this replaced by mixtures of various hydrated silicates, such as margarite, damourite, kaolin, &c., into which the diaspore insensibly passes (see Plate 6, figs. 10, 11, 12). The latter change is quite similar to that observed in the well-known case of the diaspore of Dilln, near Schemnitz, which has been investigated by A. HUTZELMANN‡ and other authors.

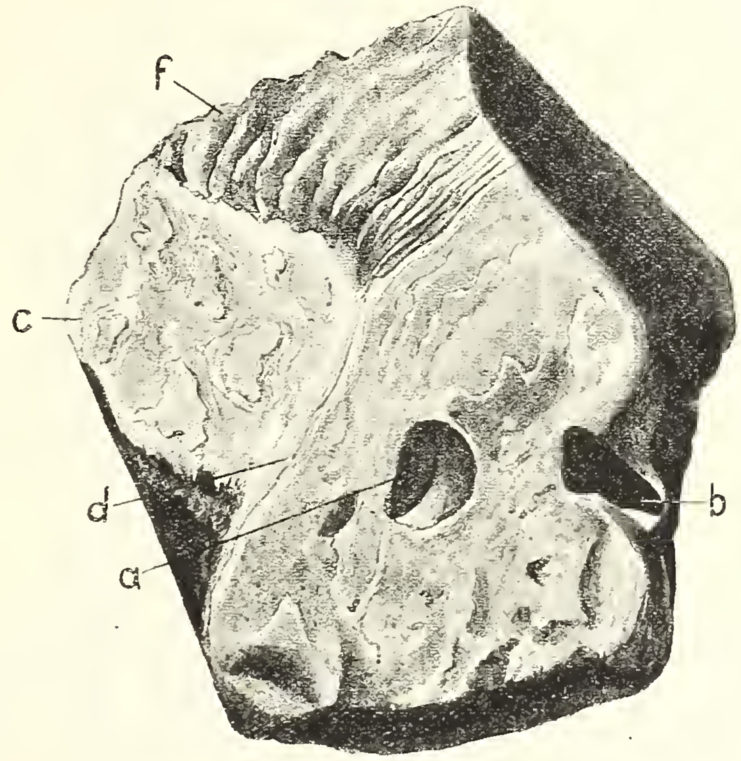
* 'Proc. Am. Phil. Soc.,' vol. 20 (1882–3), pp. 381–392. See also, 'Am. Journ. Sci.,' 3rd series, vol. 39 (1890), pp. 47–50.

† 'Proc. Am. Phil. Soc.,' vol. 13 (1873), p. 372.

‡ 'Bull. Freund. der Naturwiss.,' Wien; 'POGG. Ann.,' vol. 28, p. 575; 'Am. Journ. Sci.,' 2nd ser., vol. 10 (1850), p. 247; also Berg u. Hütten. Zeitschr.,' vol. 10 (1851), 11–12.

In a recent communication to the Mineralogical Society* I have shown that in the corundum, which has not been subjected to pressure and thus had gliding planes developed in it, the fracture is conchoidal like that of quartz. In such untwinned

Fig. 18.



crystals there is a plane of chemical weakness parallel to the basal plane (0R., 100). This is shown by the frequency of a pearly lustre on that face, due to the development of films of diaspore within the crystal, and sometimes also by a tendency of the crystal to break up parallel to this plane. Now the presence of this plane of chemical weakness is very strikingly exhibited by some Burmese rubies.

Fig. 18 represents a much-altered ruby crystal, magnified four diameters. The basal plane has been attacked irregularly, and the deep holes *a* and *b* show how capriciously such erosive action sometimes goes on ; at *c* we have a rhombohedral face undergoing the kind of exfoliation, of which we are about to speak as characteristic of those planes of the crystal, but at its upper part, *d*, we see that the crystal is made up near the much-weathered basal plane of alternate layers of corundum and diaspore. The same fact is still better shown on the fractured surface *f*. It is this alternation of diaspore layers which gives the pearly lustre so often exhibited on the basal plane of corundum crystals.

When, in consequence of pressure, gliding planes (similar to those produced in calcite) have been formed in corundum, chemical action tends to take place along these gliding or secondary twinning planes.

In rubies embedded in the limestone, the faces parallel to the rhombohedron are

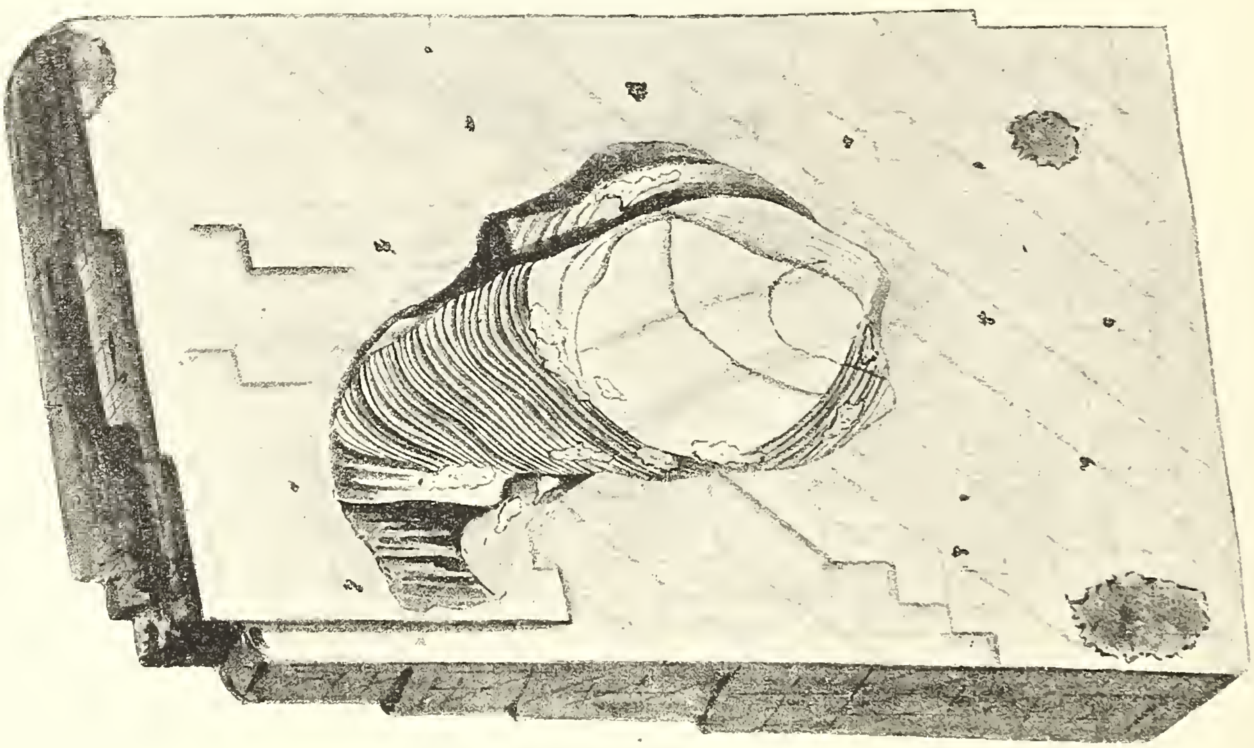
* 'Mineralogical Magazine,' vol. 11 (1895), p. 49.

seen to display a series of step-like ridges like those of a contoured model, or those exhibited by the well-known Babel quartz (see fig. 18, *c*.).* This character is admirably illustrated by the fine ruby (evidently from Burma) which has been presented by Mr. RUSKIN to the British Museum collection, under the name of the "Edwardes Ruby." It is evident that the hydration and conversion of the corundum into diaspore has not gone on with perfect uniformity, but has been controlled by the existence of planes of chemical weakness parallel to the faces of the primitive rhombohedron.

In some cases, however, the eating into the corundum crystal by the formation of the hydrate (diaspore) appears to go on in a very capricious manner indeed; irregular depressions are formed extending quite into the interior of the crystal (see fig. 18, *a* and *b*), and thus it is gradually reduced to a shapeless mass.

The most characteristic of all the methods of breaking up of the Burmese rubies by hydration and chemical action is that exhibited when the rhombohedral faces of the crystals are attacked along the twinning planes that become solution-planes.

Fig. 19.



In the specimen figured above (fig. 19), which is enlarged 5 diameters, a piece of the Mogok limestone, showing both cleavage and twinning planes, is seen having embedded

* [It might, of course, be argued that the peculiar forms of these corundums and spinels are due to irregularities in the growth of the crystals, as in the Babel quartz and certain varieties of fluor and many other minerals. But that, in the case of the corundums and spinels of Burma, the appearances presented are actually due to corrosion seems to be proved by the fact that in so many instances the products of alteration can be seen still surrounding the unaltered gems.—October 7, 1895.]

in it a fine ruby of excellent colour. The rhombohedral faces of the crystal are found to exhibit the characteristic step-like surfaces, and upon these white patches of diaspore can still be seen. This mass of limestone exhibits several pseudomorphs after smaller rubies, and some grains of graphite and pyrrhotite, all enclosed in the calcite.

Fig. 20.

[There is a remarkable analogy between the way in which the rubies of Burma break up along their rhombohedral planes, during hydration, and the disintegration of the diamond during its oxidation (combustion). Mr. J. JOLY, F.R.S., in his account of experiments made to determine the thermal expansion of the diamond ('Nature,' March 22, 1894) writes as follows :—

"At a temperature of 850°, and indeed below this, observations were stopped by the 'efflorescence' upon the surface of the diamond of flaky particles which wriggled and twisted in a peculiar manner, finally disappearing. Once started, the 'combustion' continued till the temperature of the oven was lowered to 712°. Cooling the oven, I subsequently photographed one face of the diamond. The picture obtained shows the face with a lamellar appearance, which was produced entirely by the heating, as at starting the faces were smoothly curved. Such an appearance is occasionally observed upon specimens of diamond. This photograph, as well as the curve of expansion, were shown at the *soirée* of the Royal Society in June, 1892."

I am greatly indebted to Mr. JOLY for copies of this very interesting photograph, and for permission to insert a reproduction of it (fig. 20) in illustration of this paper.] —December 8, 1895.

But there is yet another method of the breaking up of corundum and spinel crystals, which is of the greatest interest to crystallographers and mineralogists.

The hydrates and silicates have in some cases been produced along more or less irregular depressions, but these have shown such a general tendency to follow definite directions within the crystal that the result has been a mass built up of granules, each having an approximately crystalline form. Fig. 21 shows a crystal of black spinel (ceylanite), the summit magnified 2 diameters, the faces of which show etched figures, but at one angle this etching has gone so far as to leave a number of octahedral figures standing out in relief. In fig. 22 (also magnified 2 diameters) of a purple spinel from Burma the whole crystal is found broken up in the same manner, into an aggregate of small rhombohedral polysynthetic crystal of some authors. These resultant masses resemble in a striking way the models built up of "fundamental forms" by which Haüy and other crystallographers have sought to illustrate their ideas of crystals architecture. In the case of the spinel, the form of the granules into which the large crystals break up, is the regular octahedron. A very large and remarkable specimen of spinel from Burma, in the possession of Mr. Streeter, exhibits this peculiarity in the same striking manner as the smaller ones figured. Beginning with natural etched figures (Verwitterungsfiguren), we find in these cases that the depressions become deeper and deeper till the whole crystal is reduced to a skeleton-like mass.

[A very interesting problem which suggests itself to the student of these remarkable changes in the corundum and spinel, is that of the time and place at which they must have taken place. That, at ordinary temperatures and pressures, both

Fig. 21.

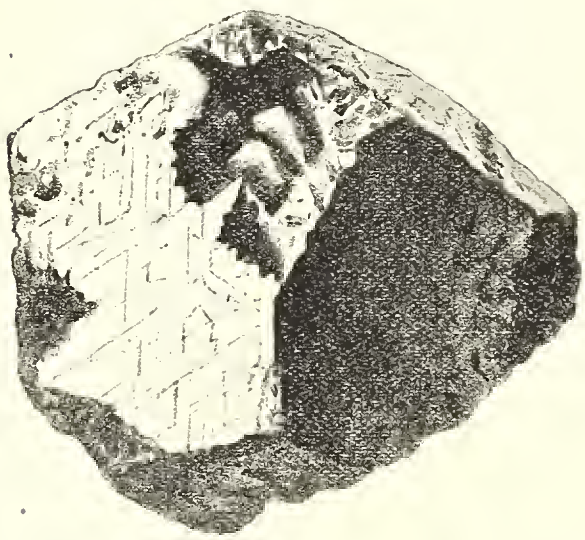
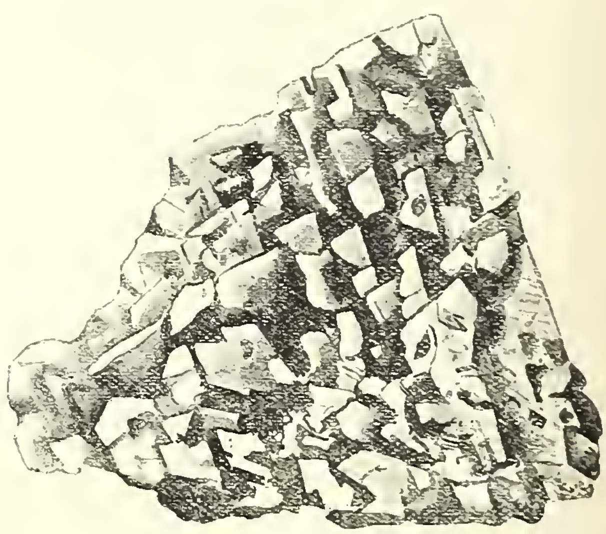


Fig. 22.



corundum and spinel are very slowly, if at all, acted upon by atmospheric agencies, is shown by the condition of the specimens found in alluvial deposits. In spite of the extreme hardness of the gems, they may be found to have suffered greatly from mechanical attrition while exhibiting little if any evidence of chemical change. The hydration of the oxides and their union with silica must have been brought about, therefore, while they still constituted portions of deep-seated masses, and were acted upon by solvents under pressure. The very interesting specimens from Sagyin, recently sent me by Mr. HOLLAND, seem to prove that, in certain cases, the corrosion of the spinel crystals must have taken place *before* their enclosure in the limestone. The fine spinels, nearly an inch in diameter, are found with their surfaces showing the incipient alteration to which I have referred. In these cases, it would seem that the growth of the spinel crystals and their partial alteration by solvent action going on irregularly over their surfaces, must both have been accomplished before the deposition of the calcite in which they are now found to be completely embedded. In the case of the Mogok limestones, however, the fact that the unaltered rubies are seen lying in the midst of a mass of their alteration products, the whole being enclosed in the limestone, seems to prove conclusively that the changes took place *after* the formation of the latter rock.—October 7, 1895].

VIII.—SUMMARY OF RESULTS.

The chief scientific results to which we have been conducted by these studies, are as follows :—

1. The famous rubies and spinels of Burma have now been found *in situ* in a highly crystalline limestone, containing various silicates and oxides, with pyrrhotite (magnetic pyrites) and graphite.

2. The equally famous rubellite (red tourmaline), so highly prized by the Chinese, though found in the same district, does not occur in actual association with the corundum and spinel in the limestones, but is found in certain acid rocks (aplites) associated with the gneisses and schists.

3. The limestone containing rubies and spinels is very intimately associated with certain highly basic foliated rocks :—pyroxene - gneisses and granulites, with pyroxenites and amphibolites.

4. Unlike most crystalline limestones which have yielded corundum and spinel, the Mogok rock does not appear to contain silicates combined with fluorine and boron.

5. The source of the crystallized aluminium oxide (corundum), the aluminate of magnesium (spinel), and the calcite in which these minerals are embedded, appears to be the basic lime-felspar (anorthite) and associated minerals of the pyroxene-gneisses. The anorthite is often converted into scapolite, from which calcite and various hydrated aluminous silicates have been formed by further alteration.

6. The hydrous aluminous silicates have been shown, under certain conditions, to break up and give rise to silica (opal) and hydrated aluminium oxide (diaspore, gibbsite, bauxite, &c.).

7. The hydrated aluminium oxide, under other conditions of temperature and pressure, becomes dehydrated and may crystallize as corundum.

8. While the crystallized aluminium oxide at the earth's surface is one of the

most unalterable of substances, this is not the case with the mineral as it exists at considerable depths in the earth's crust.

9. The anhydrous oxide easily takes up water and recrystallizes as diaspore, and many examples of corundum, as suggested by LAWRENCE SMITH and GENTH, really consist of intimate admixtures of the anhydrous and hydrous oxides of aluminium.

10. Unaltered corundum is, like quartz, a mineral without any true cleavage and breaks with a conchoidal fracture.

11. The common partings parallel to the primitive rhombohedron which are so often found in corundum, are gliding planes produced by pressure.

12. In addition to these partings, produced by mechanical means, there are others which are developed by chemical means, namely, hydration and union with silica and other oxides. This action tends to take place along definite crystalline planes—solution planes—which are parallel to the base and hexagonal prism.

13. If, however, the crystal has been subjected to pressure, and had the rhombohedral gliding planes developed in it, such planes become secondary-solution planes, along which chemical changes proceed very rapidly.

14. By the formation of diaspore and the union of diaspore with silica and other oxides, the corundum gradually diminishes in hardness, lustre, and density; the changes sometimes appearing to go on in a perfectly capricious manner from the surface of the crystal, but more usually following either the primary solution planes, or, if the crystals have been subjected to mechanical change, the secondary solution planes. This change took place in some cases before, and in others after, the minerals were embedded in the limestone.

15. The final result of these processes is to convert the corundum crystal into one of those pseudomorphs in hydrated silicates, damourite, margarite, chlorites, vermiculites, &c., which are so familiar to all mineralogists.

EXPLANATION OF PLATE 6.

NOTE.—In describing the actual dimensions of the objects represented on this plate, the magnifying power of the objective is given as the numerator, and the reduction for the purposes of drawing as the denominator. The quotient, therefore, gives the actual linear enlargement.

Fig. 1. Fine-grained pyroxene-gneiss, resembling a gabbro in its characters. It consists of plagioclase felspar, with a little quartz and orthoclase, the latter showing much of the "quartz of corrosion" of French authors. The crystals thus attacked are indicated by faint shading. The pyroxenes, which are present in moderate quantities, belong to both augite and

enstatite; the former showing the two sets of secondary parting planes, characteristic of "pseudo-hypersthene," the latter being highly ferriferous and strongly pleochroic (hypersthene). The rock was obtained from Hmyaudwin, No. 13, Mogok. $\times \frac{10}{3}$.

Fig. 2. Pyroxene-granulite, with scapolite and calcite. This rock consists largely of an untwinned felspar in rounded grains, among which are many grains of scapolite and a considerable number of grains of calcite. The rock is very distinctly foliated, and streams of liquid cavities can be traced passing continuously through contiguous grains. The ferro-magnesian silicate is a purplish, non-pleochroic augite, probably titaniferous. The rock was obtained, *in situ*, at Letnytoung mountain. $\times \frac{25}{3}$.

Fig. 3. Sahlite-scapolite granulite. In this rock the whole of the felspar appears to be converted into scapolite, with some calcite. The pyroxene is a white augite (sahlite) which, on the outer margin of its granules, sometimes shows tendency towards an alteration to a brown pleochroic material (biotite). Granules of sphene are by no means rare. The rock was collected, *in situ*, at Toungnee mountain. $\times \frac{25}{3}$.

Fig. 4. Sahlite-hypersthene-biotite-granulite. This rock contains not only a monoclinic pyroxene, like the last, but also a considerable amount of a highly coloured and pleochroic rhombic pyroxene (hypersthene), as well as a number of grains of biotite and magnetite, or titanoferrite. There is much quartz, a little plagioclase felspar, and possible orthoclase also. The rock was collected, *in situ*, on the road leading across the pass between Mogok and Momeit. $\times \frac{25}{3}$.

Fig. 5. Pyroxene-biotite-hornblende-hypersthene-granulite with scapolite. This rock differs from the last in having a brown hornblende in addition to the colourless pyroxene, in the relative abundance of the several constituents, and in the large amount of plagioclastic felspar in places largely converted into scapolite. The quartz is present in much smaller quantities A part, at least, of the colourless pyroxenic constituent appears to be wollastonite. This rock was collected in *a* Loodwin, Kathay.

Fig. 6. Hornblende-sahlite-granulite. The pyroxene in this rock sometimes shows the first trace of a paramorphic change into hornblende by very faint pleochroism. The plagioclase felspar is almost entirely unaltered, and there is little or no quartz or calcite in the rock. The rock was collected in No. 9, Hmyaudwin, Mogok. $\times \frac{25}{3}$. (N.B.—The part of the slide figured shows but little sahlite, this mineral being somewhat locally distributed.)

Fig. 7. Pyroxenite associated with the limestones of Burma. This rock consists almost wholly of a green pyroxene with a little quartz and plagioclase felspar and some sphene. In other cases, however, the pyroxene is

largely replaced by a green or brown hornblende, the latter resembling basaltine in its scheme of pleochroism. At the upper part is seen an interesting intergrowth of pyroxene with plagioclase felspar and quartz resembling the "centric structure" of BECKE. $\times \frac{10}{3}$. (These pyroxenites and amphibolites alternate with limestones, and frequently graduate insensibly into the calciphyres.)

Fig. 8. Tourmaline-felspar-quartz rock of Nyoungouk, which yields the rubellite of Burma, the tourmaline which is often beautifully zoned, is an indicolite. The felspar is plagioclastic and sometimes partially changed to scapolite, and the quartz is small in quantity.

Fig. 9. Lapis-lazuli rock—consisting of white diopside, scapolite, and two blue constituents, one deeply tinted, the other pale-tinted (haüyn and lazurite (?)), both perfectly isotropic. In different examples of this rock, the proportions of the white and blue minerals to one another vary greatly. The rock was not found *in situ*, but in blocks of considerable size in ruby workings at Thabanpin. $\times \frac{25}{3}$.

The last three figures illustrate the general nature of the Mogok limestone, and of the alterations which take place in the spinels and rubies enclosed in it.

Fig. 10. Section of limestone from ruby-cave, Mogok—a crystal of corundum is seen lying in this limestone, partially converted into diaspore and various secondary silicates. A bent crystal of the beautiful green chrome-mica (fuchsite) appears in the lower part of the mass and a granule of pyrrhotite at the right. $\times \frac{40}{3}$.

Fig. 11. Spinel and ruby in the midst of the limestone. Ruby-cave, Mogok, 11*a*, shows a crystal of spinel much eaten into and undergoing peripheral alteration, shown by change of colour. Masses of secondary silicates, pseudomorphous after spinel, and scales of graphite are also seen. $\times \frac{25}{3}$. 11*b*, from another part of the same slide, shows two granules of corundum (ruby) broken up and enveloped in diaspore, the whole being surrounded by a zone of mixed silicates. These are evidently examples of ruby undergoing change in the midst of the limestone rock. $\times \frac{75}{3}$.

Fig. 12. Large ruby in the midst of the limestone of Mogok, partially changed into various silicates. Smaller crystals, two intact, and two partially altered, are seen near it. $\times \frac{10}{3}$.

J. W. J.

M.P. Parker del. et lith.

West, Newman imp.

INDEX SLIP.

VI. *Alternate Current Dynamo-Electric Machines.*

By J. Hopkinson, *F.R.S.*, *and* E. Wilson.*

Received April 4,—Read May 2, 1895.

The paper deals experimentally with the current induced in the coils and in the cores of the magnets of alternate current machines by the varying currents and by the varying positions of the armature. It is shown that such currents exist and that they have the effect of diminishing to a certain extent the electromotive force of the machine when it is working on resistances as a generator without having a corresponding effect upon the phase of the armature current. It is also shown that preventing variations in the coils of the electromagnet does not, in the machine experimented upon, greatly affect the result, and that the effect of introducing copper plates between the magnets and the armature has not a very great effect upon the electromotive force of the armature, the conclusion being that the conductivity of the iron cores is sufficient to produce the main part of the effect. A method of determining the efficiency of alternate current machines is illustrated and the results of the experiments for this determination are utilised to show that in certain cases of relation of phase of current to phase of electromotive force the effect of the local currents in the iron cores is to increase instead of to diminish the electromotive force of the machine.

I.

In algebraic discussions of the theory of alternate current machines, it has usually been assumed that the electromotive force due to the magnets is a periodic function, the same whether there is a current in the armature or not, and that the effect of the current in the armature can be represented by regarding the armature as having self-induction. It has been pointed out, too, that the coefficient of self-induction will generally vary with the position of the armature in the field. To state exactly the same thing in another way it has been assumed that the electromotive force of the magnets is a periodic function independent of the current in the armature, and that the effect of the armature current on the induction through the armature can be

* The large majority of the experiments herein described were made in the summer of 1893 and a considerable part of the paper was then written. We have to thank Mr. F. Lydall, one of the student demonstrators at King's College at that time, for much assistance.

represented as an armature reaction which vanishes at the moment when the current in the armature vanishes.

To state the matter in the form of an equation, let E be the electromotive force of the machine on open circuit, R the resistance of the armature circuit, x the current in the armature, T the periodic time, then it is assumed that

$$Rx = E - (Lx)^{\cdot};$$

E being independent of x, L being, if you please, a coefficient of self-induction constant or variable, or, if you prefer it, Lx representing the change in the induction through the armature due to the current in the armature, vanishing with x.

It is easy to see that this statement is true in some cases. For example, it is very nearly true in the older machines with permanent magnets. Or imagine a machine without iron in the magnets or armature, consisting merely of two circuits—one the magnet circuit, the other the armature circuit—movable in relation to each other. If the current in the magnet circuit is kept precisely constant, either by inserting a great self-induction in its circuit external to the machine, or by inserting such a resistance and using so high an electromotive force that any disturbing electromotive forces are inappreciable compared with it, the preceding statement is strictly accurate. But if the magnet current is not forced to be constant the problem is more complicated.

Stating the matter in the language of self- and mutual-induction, let x and y be the currents in armature and magnet circuits, R and r their resistances, L and N their self-induction, M sin $2\pi t/T$ their mutual-induction, F the constant electromotive force applied to the magnet, the equations for the system are :—

$$\left.\begin{aligned} Rx &= -M\,(y \sin 2\pi t/T)^{\cdot} - Lx^{\cdot} \\ ry &= F - M\,(x \sin 2\pi t/T)^{\cdot} - Ny^{\cdot} \end{aligned}\right\}$$

These equations can be solved by approximations if the variations in the value of y are small.

First,

$$y = \frac{F}{r}; \qquad x = -\frac{MF}{r}\frac{2\pi}{T}\frac{R\cos 2\pi t/T + 2\pi L/T \sin 2\pi t/T}{R^2 + (2\pi L/T)^2}.$$

Second, substituting this value of x, we obtain

$$ry = F + \frac{M^2F}{r}\cdot\frac{2\pi/T}{R^2 + (2\pi L/T)^2}\left\{\frac{R}{2}\sin\frac{4\pi t}{T} + \frac{\pi L}{T}\left(1 - \cos\frac{4\pi t}{T}\right)\right\}^{\cdot} - Ny^{\cdot}.$$

This gives periodic terms in y, the period being one-half the period of the mutual induction.

Introducing these terms into the first equation, we see that the term in $2\pi t/T$ in the electromotive force of the machine is modified, and that terms in $6\pi t/T$ are introduced. The former may have real practical importance, and it is one of the objects of the present paper to ascertain how far it exists and is of importance in actual machines.

Returning to machines as ordinarily constructed, in these the current in the magnet coils is not compelled to be constant, and any rapid variation of the induction in the magnet cores will induce currents in those cores. The variations in the current in the magnet coils and the currents in the cores both tend to annul the variations in the induction in the core arising from the current in the armature, but they will not tend to alter the *average* effect of the currents in the armature on the induction through the magnets. That there will be such an average effect is not difficult to see. Suppose the armature coils to be fixed in line with the magnets of the machine, any current in the armature will have its full effect in increasing or diminishing the field through the magnets. Suppose the armature coils to be fixed midway between the magnets, any current in the armature will then have practically little or no effect in increasing or diminishing the field through the magnets. If the armature be connected through resistance, inductive or non-inductive, and the machine run in the ordinary way, the current in the armature will lag in phase behind the electromotive force E. The result is that when the armature is opposite to the magnets there is a current in the armature tending to demagnetize the magnets, and adding together the effects of the armature in all positions, there is an average effect tending to demagnetize the magnets. If the machine had a constant current round the magnets and divided iron in the magnets, this average effect, as well as its variations, would be fully accounted for by the term $(Lx)^{\cdot}$, call it self-induction or call it armature reaction, as you please. But inasmuch as the variations are in part annulled by the variations of current in the magnet-winding and the local currents in the magnet cores, we have a part of the diminution of the electromotive force E of the machine unaccompanied by retardation of phase of the current in the armature.

Before giving any experimental results, it will be well to describe the machines used and the general method of experiment adopted.

The two dynamos experimented upon were constructed by Messrs. SIEMENS BROS., and are of the same pattern and size but are of an old type. They are mounted upon a common base-plate, their pulleys being provided with flanges and bolts, so that any desired phase difference can be given to the armatures, for the accurate setting of which a graduated circle is provided, or so that the armatures can be run independently of each other. The pulleys have each a diameter of 12 inches, and are suited for a 6-inch belt—the shaft is prolonged for the purpose of carrying a revolving contact-maker and a small pulley for driving a Buss tachometer.

Each dynamo has a series of 24 electromagnets (see fig. 1), there being 12 on either side of the armature. The core of each magnet (A) is of wrought-iron $2\frac{1}{16}$ inches diameter, and $6\frac{1}{4}$ inches long: and is wound with 5 layers, 35 convolutions per layer, of copper wire 3·5 millims. diameter. The electromagnets are bolted to circular cast-iron frames (B), which serve also for supporting the bearings of the armature shaft. The centres of the 12 electromagnets on each frame are equally spaced out on a circle $8\frac{7}{8}$ inches radius concentric with the axle of the machine. Each cast-iron frame has a cross-sectional area of 4·8 sq. inches. The opposing pole pieces of the electromagnets have an air space of $1\frac{1}{8}$ inch between them, through which the armature coils rotate. The 24 electromagnet windings are coupled in series, and have a total resistance of 1·8 ohms, the normal exciting current being about 22 amperes.

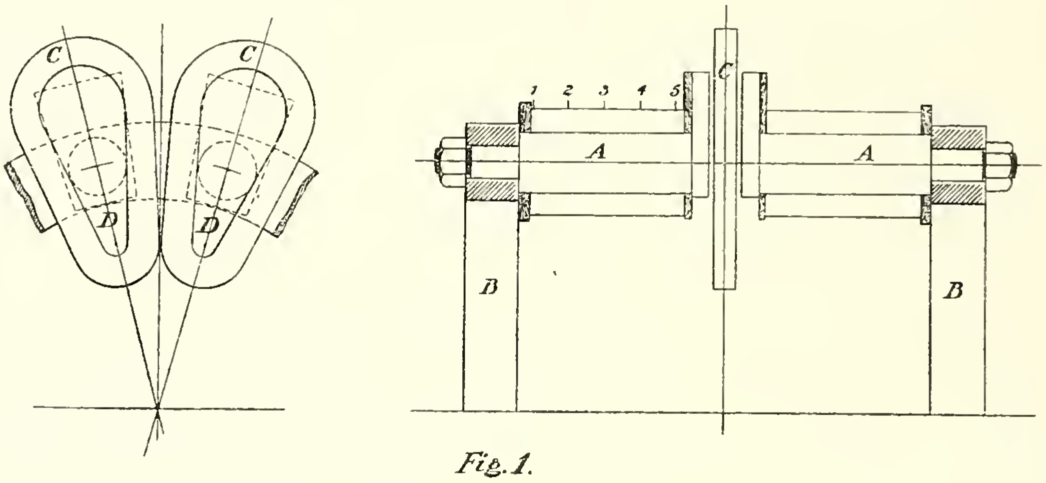


Fig. 1.

The armature of each dynamo consists of 12 coils or bobbins (C) with wooden cores (D) $\frac{7}{8}$ inch thick. Each core is 7 inches long (radially), with rounded ends—the outer being struck to a circle $3\frac{1}{8}$ inches and the inner $1\frac{1}{8}$ inches diameter. The ends of the respective coils are brought to a screw-plug commutator board fixed to the shaft, by means of which a series of combinations can be made. Each coil consists of 10 layers, 8 convolutions per layer, 2·2 millims. copper wire, having a resistance (cold) of ·187 ohm, and at a speed of 1000 revolutions per minute, with normal excitation and fully loaded, is intended to give 50 volts at its extremities. The full load current for each coil is 16·6 amperes, so that at 1000 revolutions per minute, or a frequency of 100 complete periods per second, the machine should give, with its 12 armature coils in parallel, an output of 200 amperes 50 volts. The two terminal rings of the screw-plug commutator are connected by conductors to two gun-metal collector rings, insulated from one another and from the shaft by means of ebonite. Each collector ring is provided with two 1-inch copper wire brushes, carried by adjustable bar-holders fixed to the terminal blocks of the dynamo.

The potential difference between any two points at any epoch is determined by means of a Kelvin quadrant electrometer and a revolving contact-maker fixed to the shaft of the dynamo. The contact-maker consists of a disc of gun-metal which carries two rings, one of gun-metal insulated from the disc, the other of ebonite. Into the latter is inserted a strip of metal $\frac{5}{64}$ inch wide, which is in permanent contact with the gun-metal ring. Two insulated brushes are attached to a movable brush-holder, so that one presses on each ring. The circuit between the two brushes is completed once in each revolution, and the position of the contact can be read off by a pointer attached to the holder on a circle $13\frac{1}{2}$ inches diameter, divided into 360 degrees. The two points between which it is desired to measure the potential difference are connected through the contact-maker to a condenser and the quadrant electrometer, as shown in fig. 2, in which A and B are the points, C the revolving

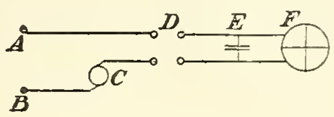


Fig. 2.

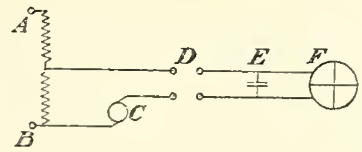
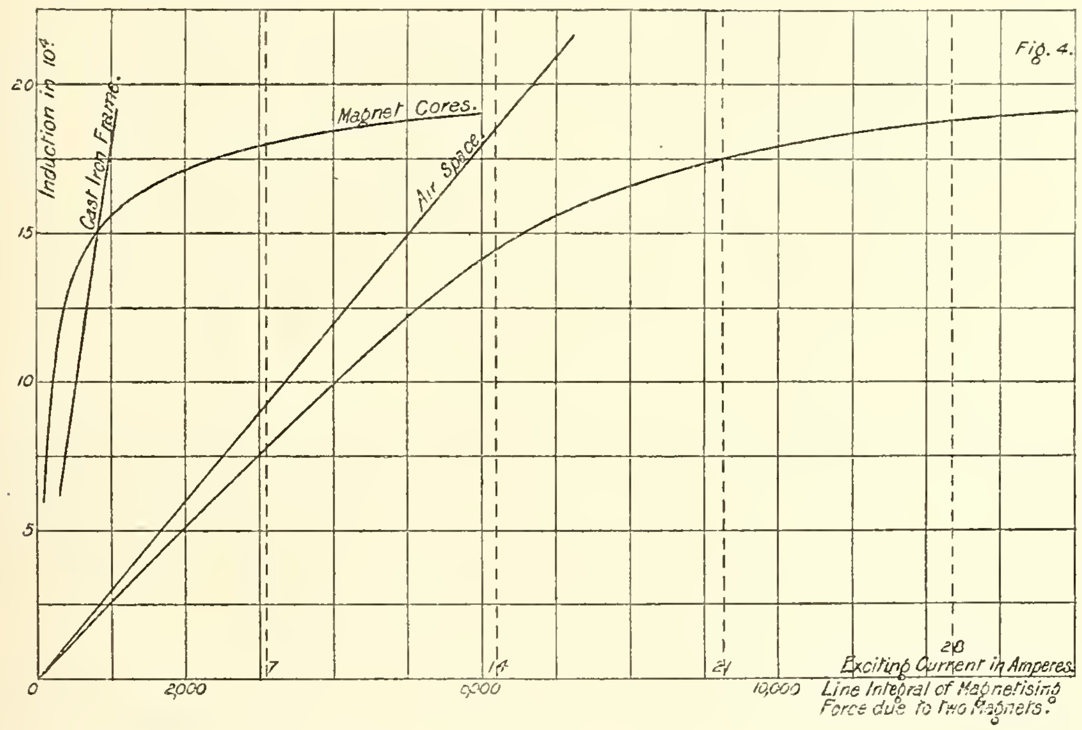


Fig. 3.

contact-maker, D the reversing switch of the electrometer, E the condenser, and F the quadrant electrometer. By plotting as ordinates the volts measured at any epoch, and as abscissæ the position of the contact-maker as representing time, the curve of potential is obtained. The electrometer is standardised by means of a Clark cell, so that the deflections on the scale can be reduced to volts: when the potential difference between A and B was too great for the electrometer, it was reduced in any desired ratio by two considerable non-inductive resistances introduced between A and B, as shown in fig. 3.

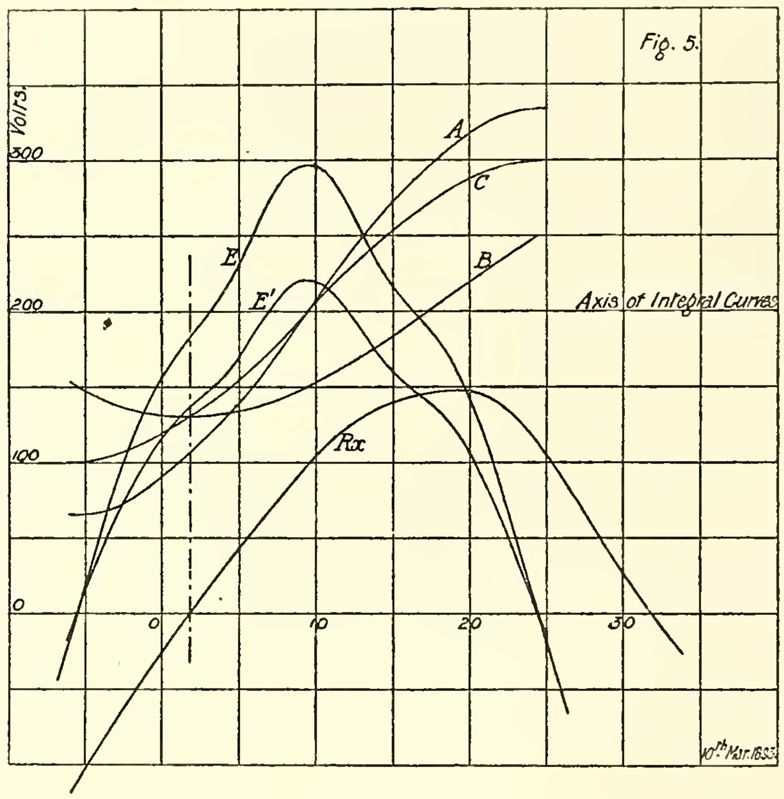
Fig. 4.



The characteristic curve of the alternator is given in fig. 4, and shows the relation between the total induction I, between one pole piece and the opposing one, in terms of the line integral of magnetising force due to the two windings in series on the two respective magnet cores; the scale of amperes in the magnet winding is also given horizontally.

In fig. 5 the speed of the alternator experimented upon was 923 revolutions per minute, or a frequency of 92·3 complete periods per second. For the purpose of obtaining a marked effect from the current in the armature a large current was taken out of the armature, and the current in the magnet winding was only 8 amperes. A KELVIN multicellular voltmeter placed across the terminals of the machine read 190 volts on open circuit, and 81 volts when loaded, and a KELVIN ampere balance in the external non-inductive circuit read 40 amperes. The armature bobbins were coupled 6 in series 2 parallel between the brushes, the total resistance of the circuit (R) was 2·52 ohms, the armature resistance alone being ·55 ohm.

Fig. 5.



Curve E is the electromotive force curve of the machine when there is no current in the armature. Rx is the curve of electromotive force deduced from the potential difference taken between the terminals of the alternator when supplying current through non-inductive resistances. The curves E and Rx have been integrated, the corresponding integral curves being A and B respectively.

That the ordinary theory does not fully account for the facts is easily shown. We have $\text{R}x = \text{E} - (\text{L}x)^{\cdot}$. Integrate both sides from any fixed epoch 0 to any time t and we have

$$\int_0^t \text{R}x\,dt = \int_0^t \text{E}\,dt - \text{L}\,(x_t - x_0).$$

Each term of this equation consists of a constant part and of a periodic part. The constant parts must be equal and also the periodic parts. We have to deal only with the periodic parts. The curves A and B, fig. 5, represent the periodic parts of each of the first two terms; the difference of ordinates of these curves should at all times be equal to Lx. In particular, and this is the only point of moment, as we do not know how L may vary, when Lx vanishes $\int Rx\,dt = \int E\,dt$.

If the effect of the current in the armature on the induction through the armature could be represented by a term which vanishes with the current in the armature, it is obvious that the curves A and B would cross at the epoch when $x = 0$. They do not. The difference of inductions as given by these curves at this epoch is 25 per cent. of the induction which would *then* traverse the armature coil if the machine were running on open circuit, that is to say, 25 per cent. of the ordinate at this epoch given by the A curve. If it is assumed that the *average* effect of the armature current upon the induction in the magnets is such as to lower this induction by 25 per cent., the ordinates of the E curve would be decreased in like proportion, giving the curve E′ of which C is the integral. The difference at any epoch between the curve C and the Rx curve is to be fully accounted for by the term $(Lx)^{\cdot}$.

We may put it in this way. In this machine the armature current at the times when it has a value affects the field at the instant when the armature current is zero. The effect is produced by variations induced in the current in the field magnet-winding and in the solid iron of the magnets by the varying current in and the varying position of the armature. That these induced currents must exist is obvious, and it is easy enough to measure them in the copper coils. They have the effect of causing the armature reaction to produce an average effect upon the magnetism of the fields by partially annulling its periodic effect. If the current in the armature did not lag behind the electromotive force of the magnets E, there would be little or no diminution of the average magnetism of the magnets. We may correctly say that this diminution of the magnetism of the magnets is due to the self-induction of the armature causing a lag of current. The effect arises from the self-induction of the armature modified by currents induced in the magnets.

The effect on the magnets of any current in the armature is greatest when the armature bobbins are opposite the pole-faces, it is *nil* or small when half-way between the pole-faces. We may therefore represent its effect at any instant approximately as proportional to the expression

$$\frac{2\pi L/T \sin 2\pi t/T + R \cos 2\pi t/T}{(2\pi L/T)^2 + R^2} \sin 2\pi t/T$$

or

$$\frac{\pi L/T}{(2\pi L/T)^2 + R^2} - \frac{\pi L/T \cos 4\pi t/T - R \sin 4\pi t/T}{(2\pi L/T)^2 + R^2}.$$

The constant term causes the fall in average magnetism, the periodic term causes currents in the magnets, and its effect on the magnetism is partly annulled thereby. The effect will vary as the square of the current if this is small. The effect of self-induction in diminishing the apparent electromotive force of the machine varies as

Fig. 6.

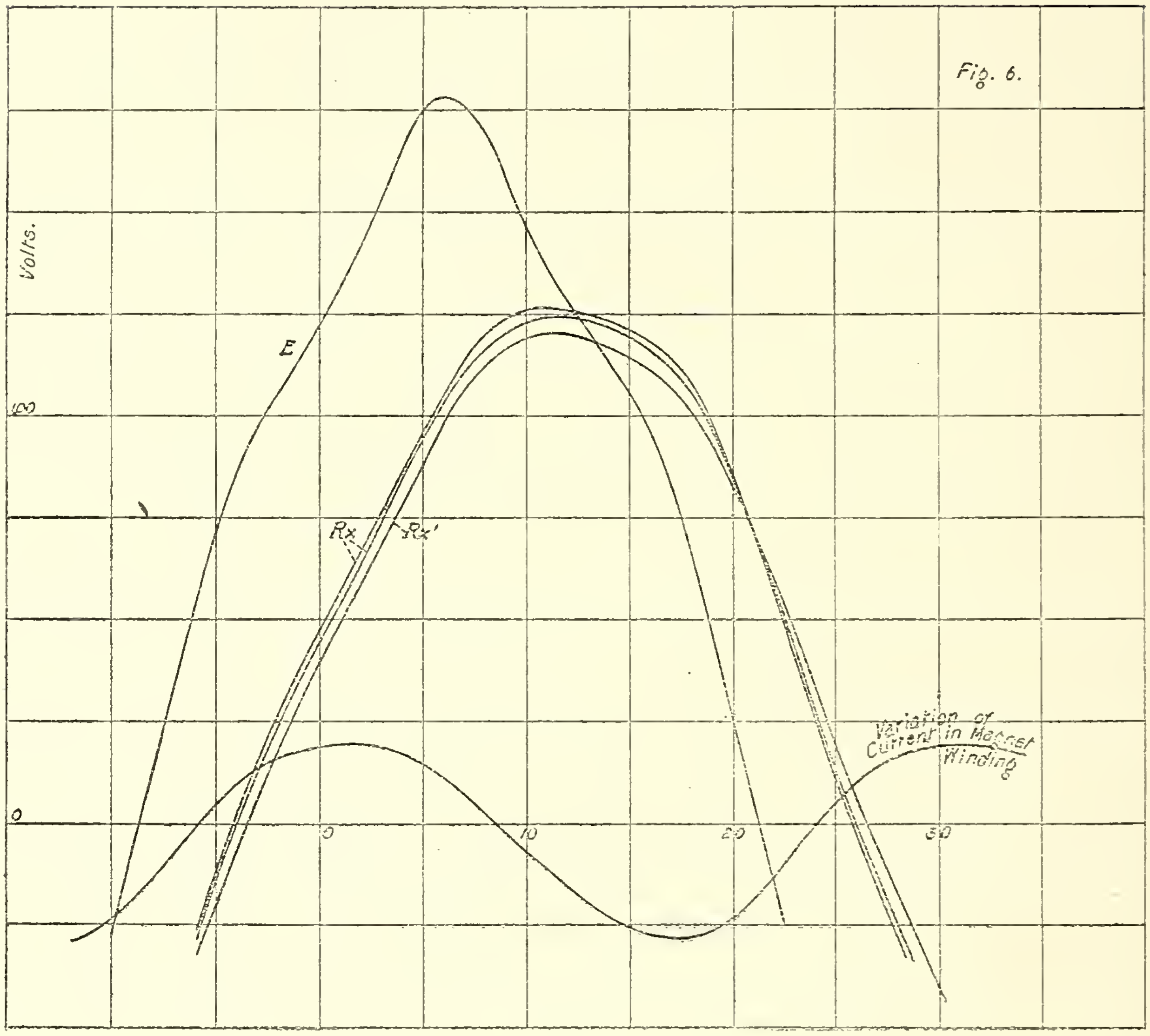
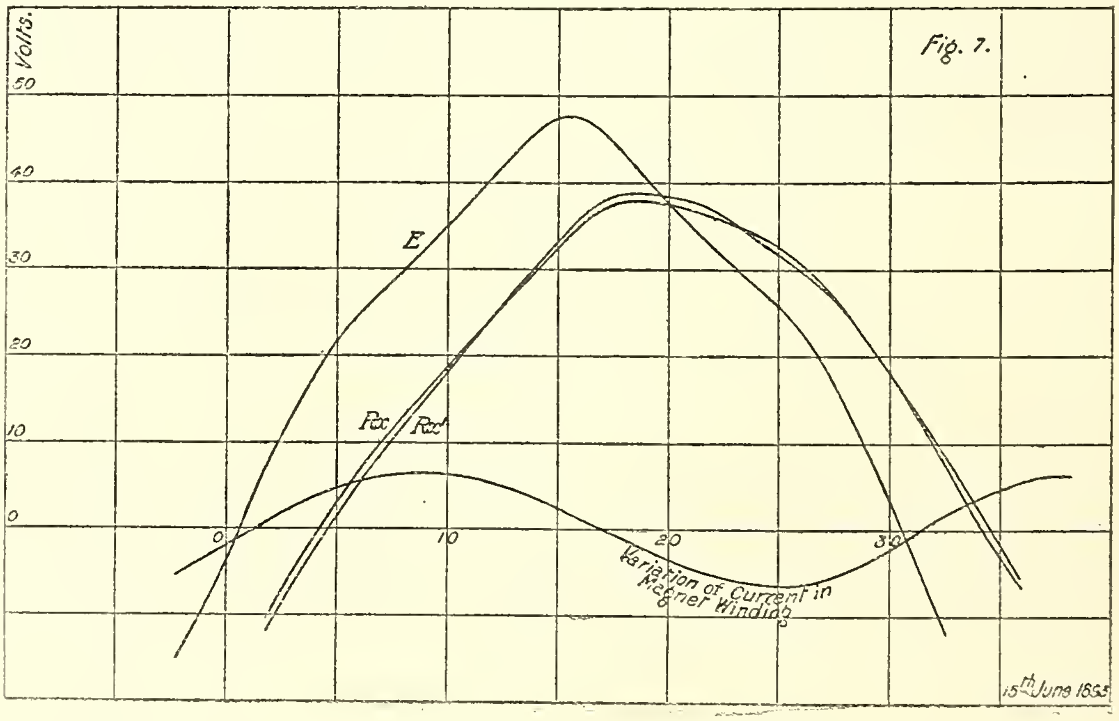


Fig. 6.
Volts.
100
E
Rx
Rx'
0
0
10
20
30
Variation of Current in Magnet Winding

Fig. 7.


Volts.
Fig. 7.
50
40
30
20
10
0
E
Rx
Rx'
0
10
20
30
Variation of Current in Magnet Winding

the square of the current, so that we may expect the two effects, that due to the reaction of the armature on itself and that brought about by the armature inducing currents in the magnets, to vary together.

The lag of phase is less than we should expect from the diminution of electromotive force, or the electromotive force suffers greater diminution than we should expect from the angle of lag of phase.

We have tried a number of experiments for the purpose of tracing the variations of current in the magnets, and also with the intention of increasing or diminishing the effect we have observed. It is easy enough to trace the variations in the current round the magnets by measuring at points during the period the potential difference between the ends of a non-inductive resistance in series with the magnets. These variations are shown in figs. 6 and 7, in which the armature bobbins were coupled, 4 series, 3 parallel, and 12 series, respectively. We see, as we should expect, that the variations have a periodic time one-half the periodic time of the machine. But the current round the magnet could be made constant by exciting the two machines with the same current, loading each to the same degree, and placing their armatures one-fourth part of a period apart in phase.

The machines were run under conditions set forth in Table I., and the curves are given in figs. 5, 6, 7, 8. An electromotive force curve E, was observed with no current in the armature, and a curve of potential difference was taken between the brushes with the alternator loaded on a non-inductive resistance. Rx is this curve *with* variations in the exciting current, and Rx' *without* variations, in each case corrected for the resistance of the armature. Figs. 6 and 7 show the curves of actual electromotive force, Rx and Rx', when the current in the magnet winding is allowed to vary and when its variations are stopped. It will be seen that they do not differ a great deal. What currents are stopped in the magnet winding no doubt turn up in the substance of the cores themselves and have an effect not differing greatly. Curve E represents electromotive forces observed. Rx represents the potential difference taken between the brushes and corrected for the resistance of the armature, with the alternator working on a non-inductive resistance *with* variations in the exciting current. Rx' *without* such variations in each case. The induced currents are in either case adequate to nearly stop the variation of induction. Fig. 8 shows the same thing.

An exploring coil was wound and placed on one of the magnet limbs and the electromotive force in it was observed in terms of the time for the various positions of the exploring coil, marked 1, 2, 3, 4, 5, in fig. 1. Both the amplitude and the epoch varied with the position of the coil, but, in all cases, the periodic time was half the periodic time of the machine. It does not seem worth while to publish the curves connecting electromotive force and time.

Lastly, we tried to exaggerate the effects; for this purpose we introduced plates of copper, $\frac{1}{8}$ inch thick, in the form of two flat rings between the pole faces and the armature. Curves 9, 10, 11, 12, give the results for two different currents round

Fig. 8.

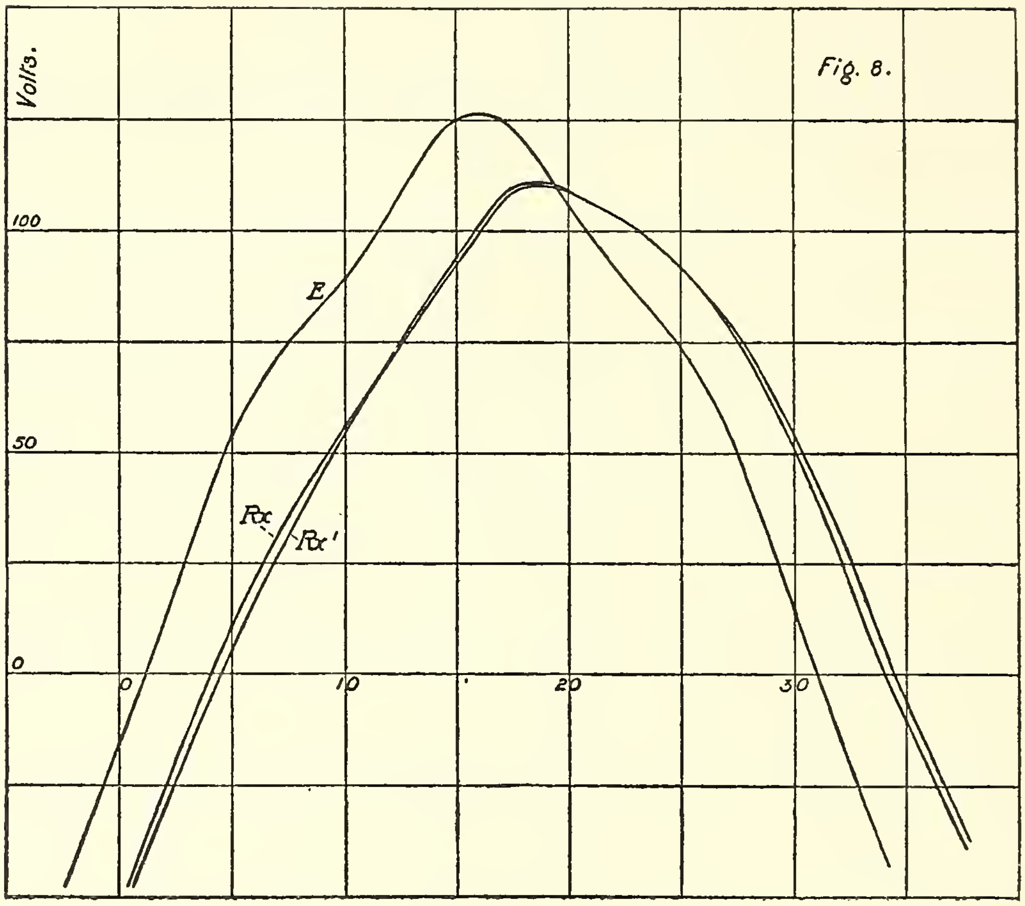

Volts.
Fig. 8.
100
50
0
E
Rx
Rx'
0
10
20
30

Fig. 9.

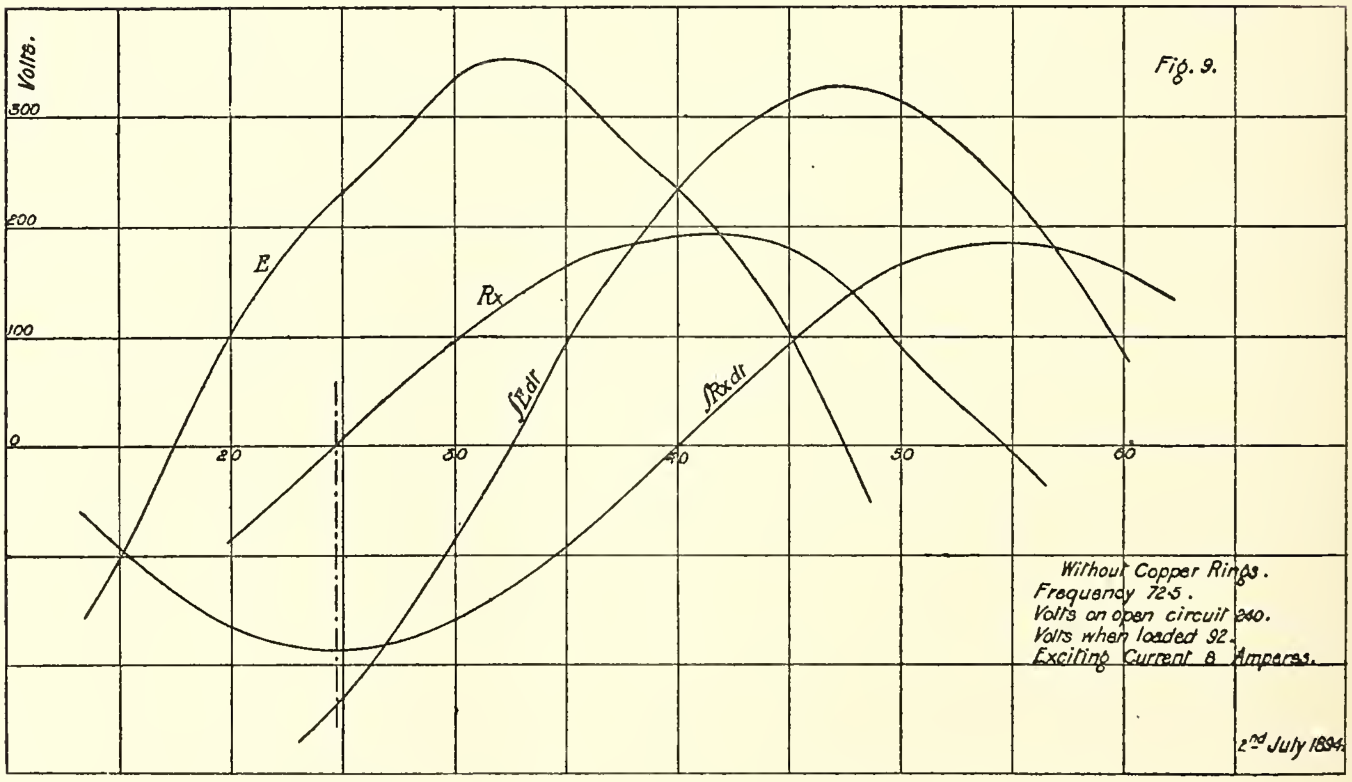

Volts.
Fig. 9.
300
200
100
0
E
Rx
∫Edt
∫Rxdt
20
30
40
50
60
Without Copper Rings.
Frequency 72·5.
Volts on open circuit 240.
Volts when loaded 92.
Exciting Current 8 Amperes.
2nd July 1894.

Fig. 10.

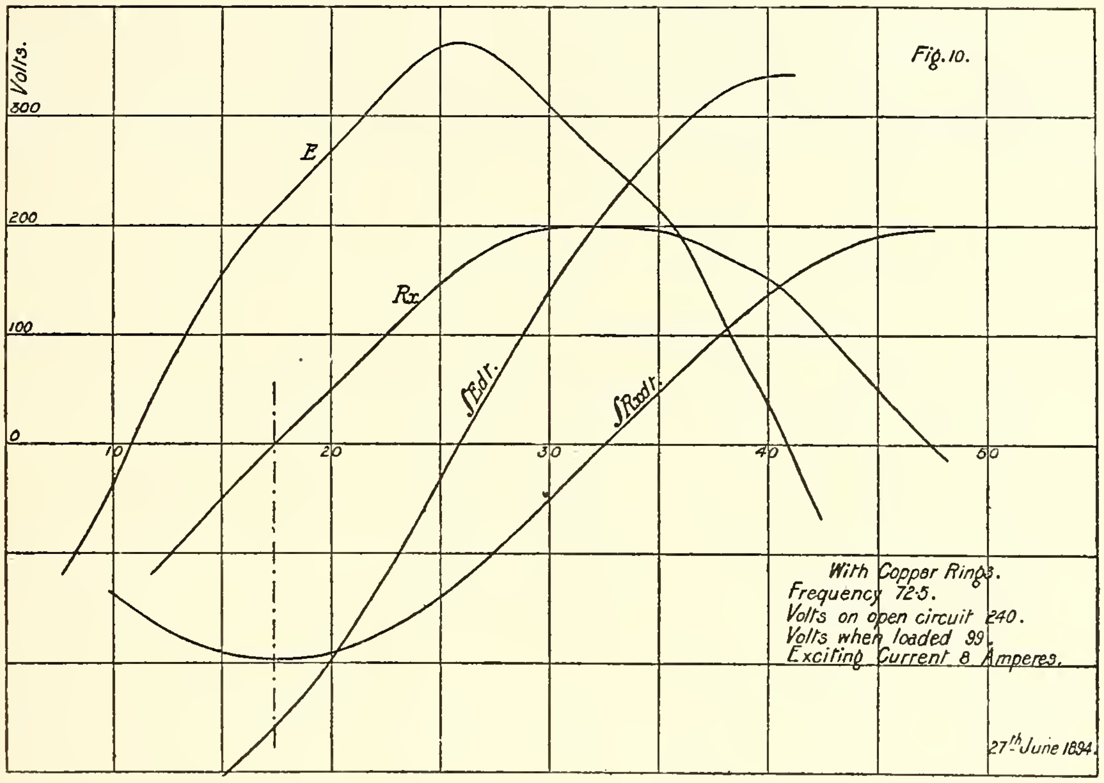
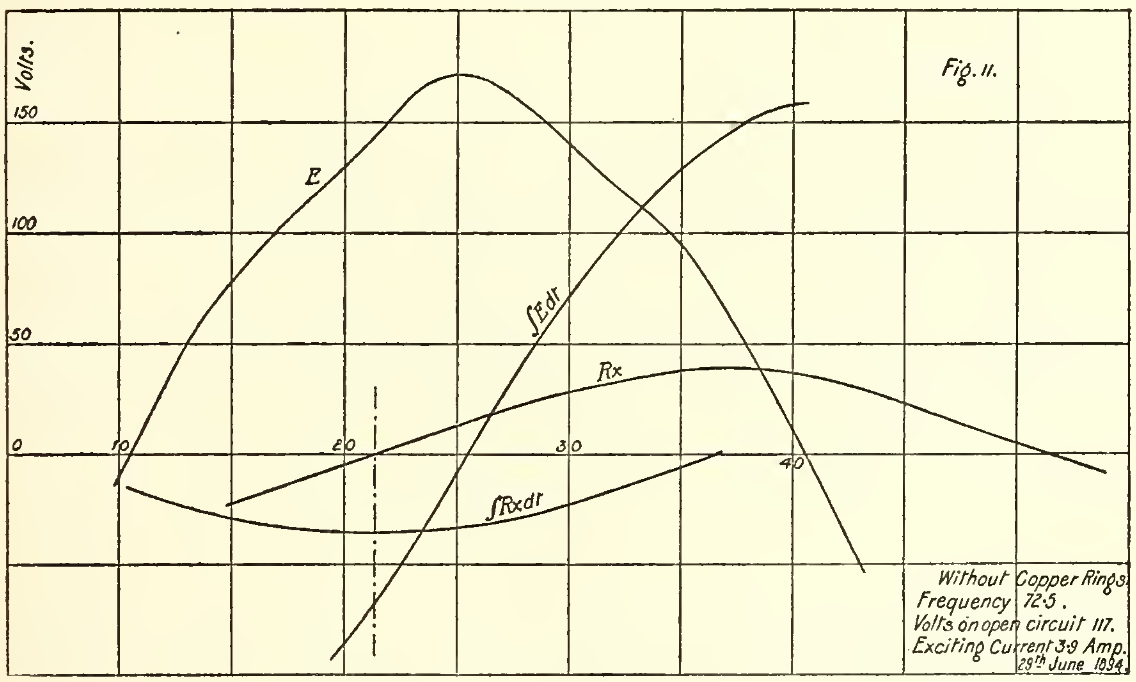


Fig. 11.

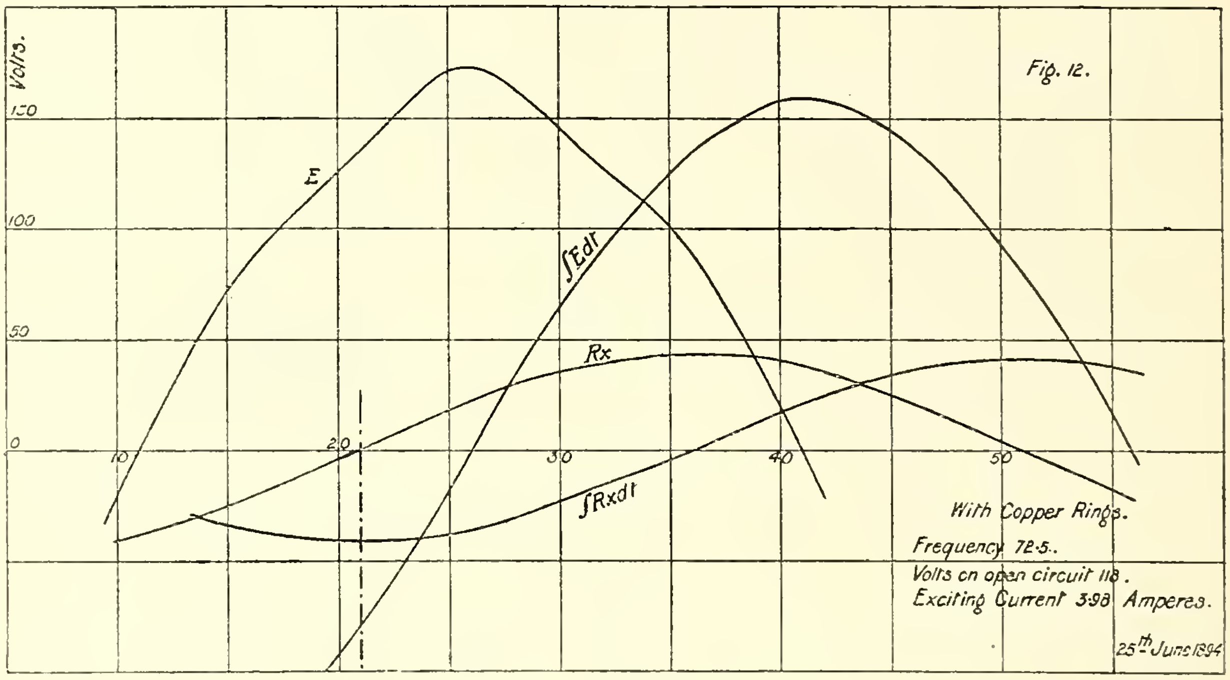


the magnets with the copper plates in and the copper plates absent. A comparison shows that the copper plates do not make a great deal of difference. The principal effect is to diminish the current induced in a coil on the magnet placed at position 5, fig. 1, close behind the copper plate.

It may be inferred that in this machine there is conductivity enough in the magnet cores to have in large measure the effect indicated, and that the effect cannot be greatly diminished by compelling the magnetizing current to be constant in the magnet coils, nor can it be greatly increased by exaggerating the currents induced about the magnets by intentionally introducing additional conductivities around them. The effect of each is merely to alter the place where the currents occur.

Recently machines have been built, with finely-divided pole pieces to the magnets by Messrs. MATHER and PLATT and by the BRITISH THOMSON-HOUSTON COMPANY.

Fig. 12.



It was obviously desirable to obtain a verification with a machine of totally different construction. For this purpose we had available the first model made of the alternating machines manufactured by Messrs. MATHER and PLATT. It has an iron core in the armature which projects and extends beyond the armature coils. It was treated in exactly the same way as the SIEMENS' machines but with a fairly full load. The results are shown in fig. 21, from which it will be observed that the total induction actually observed when the machine is loaded is about 11 per cent. below the induction inferred from the electromotive force on open circuit.

II.

The following experiments were primarily made for the purpose of determining the efficiency of the machine, but they will be seen in Section III. to have an important bearing upon the principal subject of this paper.

For the purpose of finding the efficiency of the alternators, when running as generator and motor,* the two armatures were rigidly mechanically coupled together,

* This is the same method of test which has been applied to direct current machines (see 'Phil. Trans.,' R.S., 1886, p. 331).

the leading machine being generator, and were connected in series with a non-inductive resistance r, and a KELVIN ampere balance C, as shown in fig. 13. The potential difference of the generator was measured at different epochs by means of the KELVIN quadrant electrometer Q, and the contact maker K, the potential applied to the electrometer being reduced by the non-inductive resistances r_1, r_2. For corresponding epochs a curve of potential was taken across r, this gives the current passing between the machines and also the difference of potential difference between motor and generator.

The power difference, or loss in the combination, was supplied by a shunt-motor through shafting and belts, and was determined by observing the watts supplied to the motor when driving the shafting, the alternator belt being removed, and then observing the watts taken to drive the alternator when loaded—the speed being the same in each case. The difference gives the power absorbed by the combination.

It was found that the watts required to drive the shafting alone were 1681; the watts required to drive the shafting and alternators when excited, but not loaded, were 2479, the difference being in part due to currents induced in the metal frames of the armature.

Tables II. and III. give for about half and full load (with regard to current only) the data for getting at the watts given out by generator and received by motor, and have been obtained from the potential and current curves. The phase difference between the armatures was $\frac{1}{20}$th and $\frac{1}{10}$th period respectively.

Table IV. shows how the efficiencies of generator, motor, and combination have been obtained; and also gives the allocation of losses in the system.

The frequency was about 70 periods per second, and, since the machines are built for 100 periods per second, the figures must be taken only as illustrative of the method of test.

The alternators being connected, as shown in fig. 13, we were able to vary the exciting currents of the two machines by means of the adjustable resistances r_3, r_4. The armatures were coupled $\frac{1}{10}$th of a period apart in phase, and the following experiments were made.

1. The alternators were equally excited with a current of 17·5 amperes and run at a speed of 716 revolutions per minute, corresponding with a frequency of 71·6 periods per second, and the following curves obtained (see fig. 14).

E_G, E_M are the E.M.F.'s of generator and motor when running on open circuit.

PD_G, PD_M are the potential difference of generator and motor respectively when a current of 42·2 amperes ($\sqrt{\text{mean}^2}$) was passing through the armatures.

e_G, e_M are the E.M.F.'s of the respective machines when loaded, that is to say, they are the curves PD_G, PD_M corrected for current into armature resistance.

E is the E.M.F. of the combination when not loaded, that is, it is the difference of the curves E_G, E_M.

x is the curve of current passing between the machines, and is proportional to the

Fig. 13.

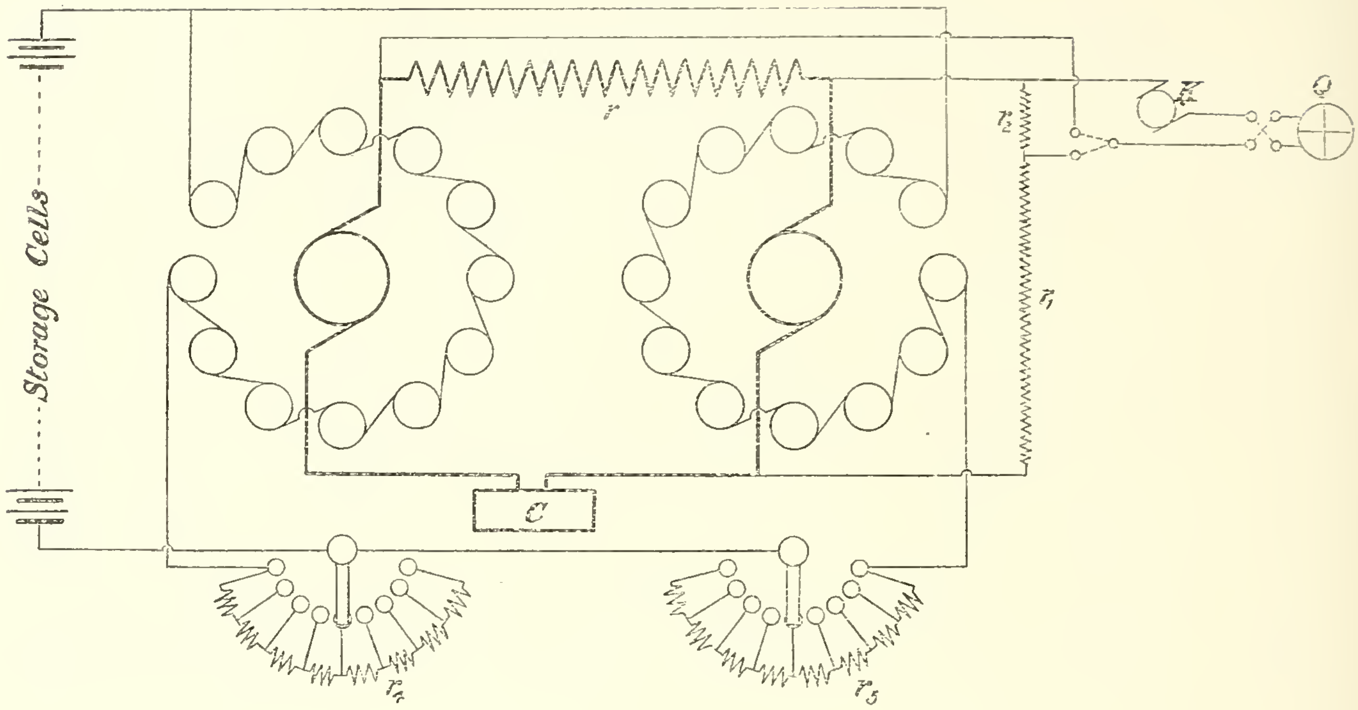

Storage Cells
r
C

Fig. 14.

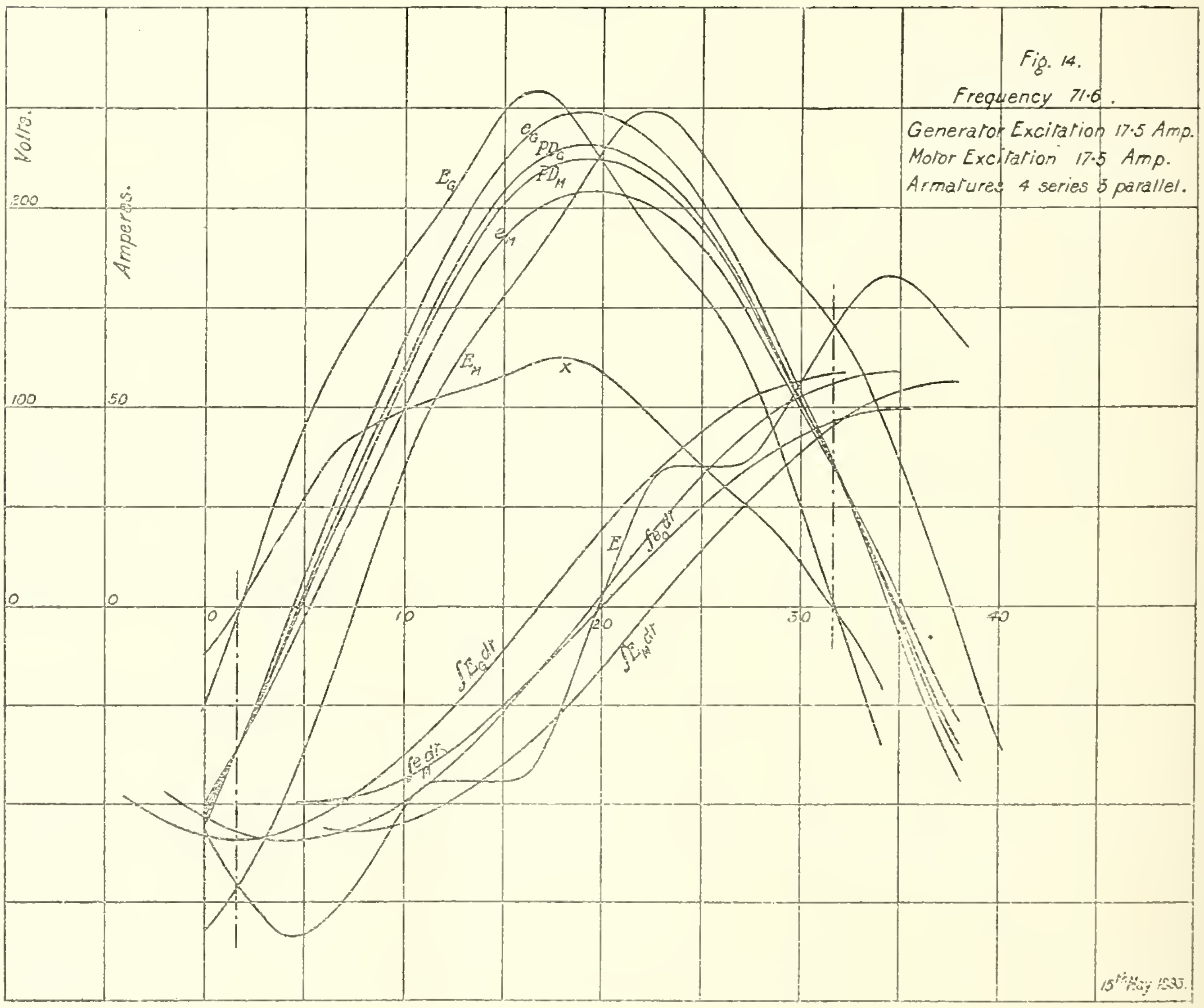


Fig. 14.
Frequency 71·6.
Generator Excitation 17·5 Amp.
Motor Excitation 17·5 Amp.
Armatures 4 series 5 parallel.
Volts.
Amperes.
200
100
50
0
0
10
20
30
40

E.M.F (e) of the combination when loaded, the connecting leads being non-inductive. This electromotive force is the difference of the curves e_G, e_M.

2. In fig. 15* the frequency is 71, the motor is excited with 18·6 amperes and the

Fig. 15.

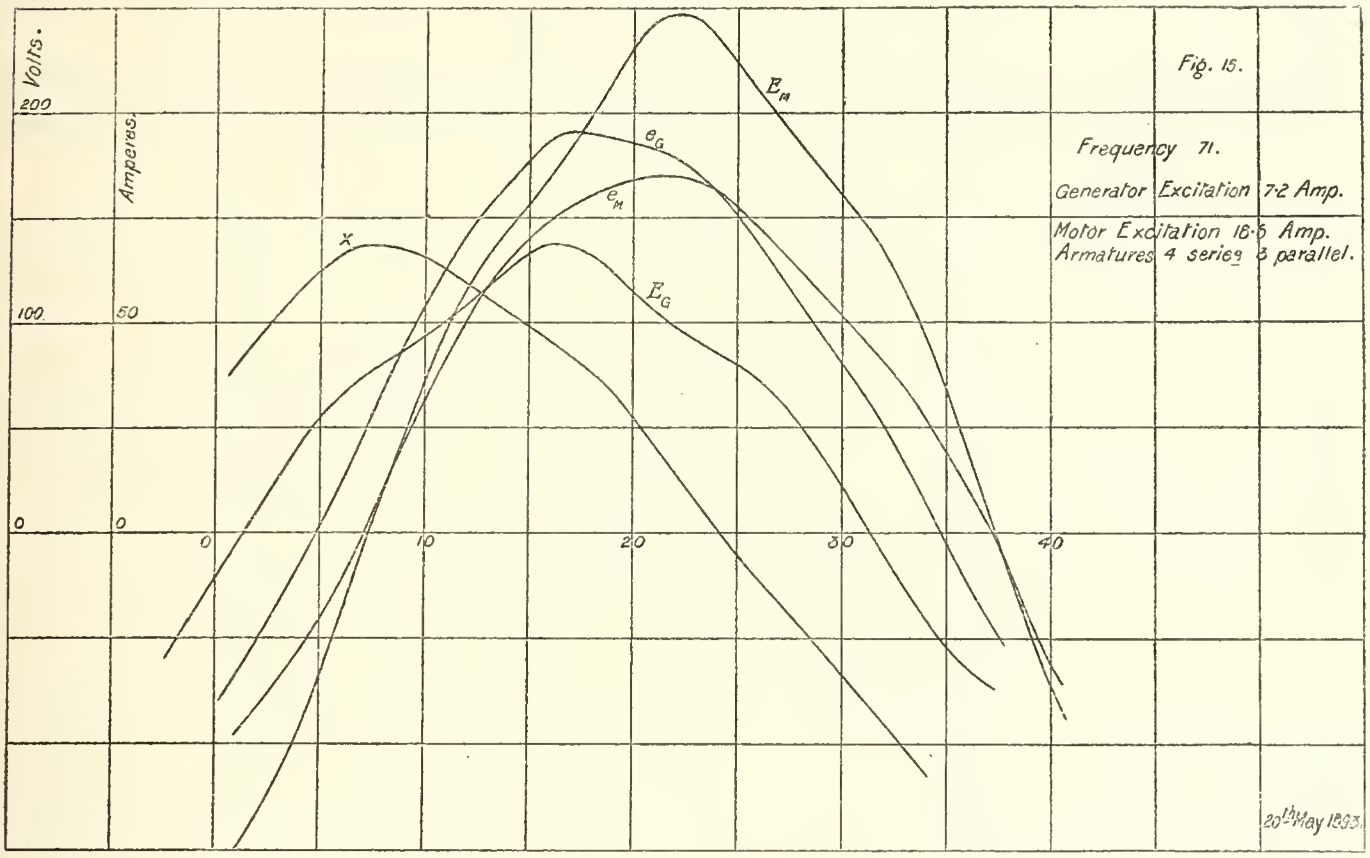
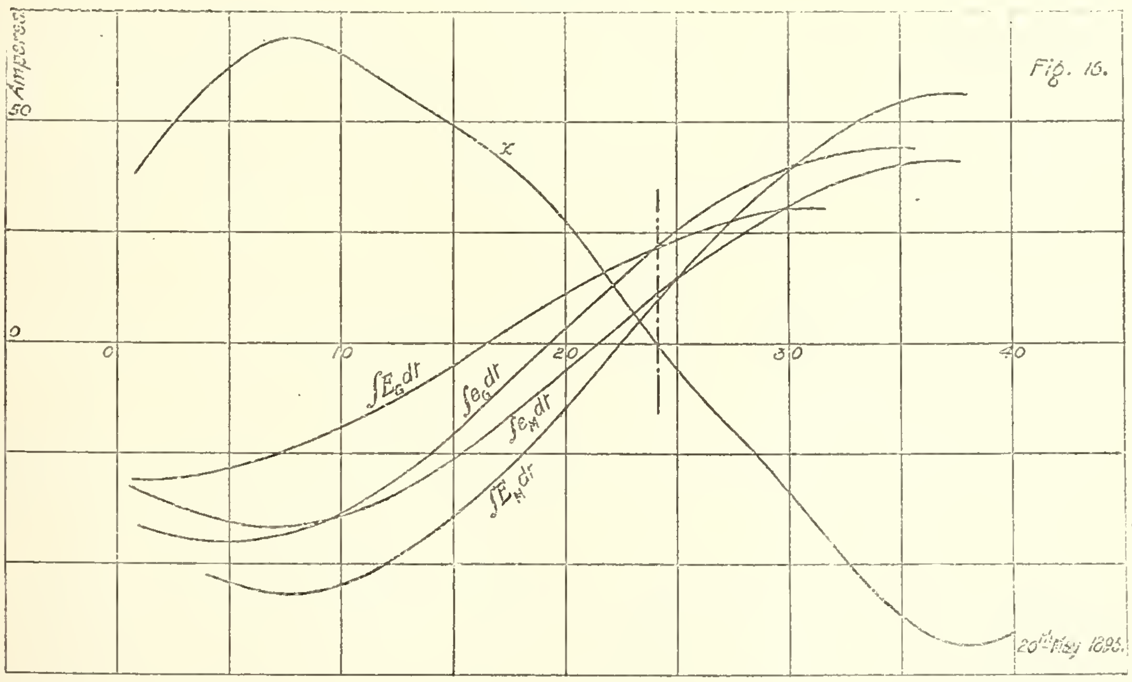


Fig. 16.



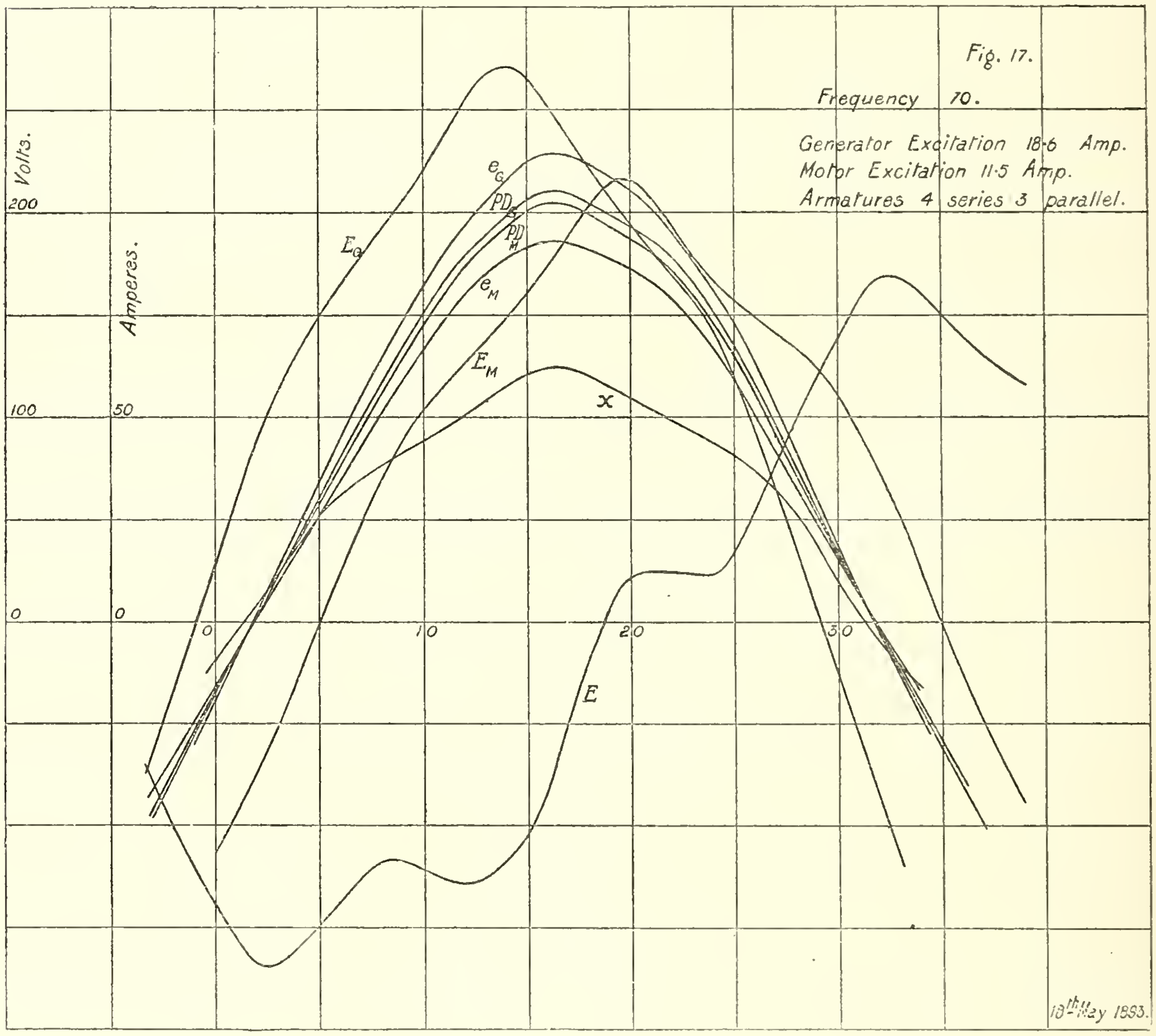
* *Vide* HOPKINSON, Institute of Civil Engineers Lecture delivered 1883; Institute of Civil Engineers, November, 1884; or "Original Papers on Dynamo Machinery," pp. 58 and 148.

generator with 7·2 amperes. In this case the motor is working at higher E.M.F. on open circuit than the generator. Curves corresponding to those in the Experiment 1 were obtained and are marked in a similar manner.

The potential curves in fig. 15 have been integrated, and the integral curves so obtained are given in fig. 16. These give therefore the inductions in terms of the time.

3. In fig. 17 the frequency is 70, the generator is excited with 18·6 and the motor with 11·5 amperes. The generator is working with a higher E.M.F. on open circuit

Fig. 17.



than the motor. The potential curves have been integrated, and the integral curves are given in fig. 18.

It was observed that when the exciting current of the motor was decreased, that of the generator being kept fairly constant (the two machines being equally excited with about 18 amperes each to begin with), the current between the machines gradually decreased until a critical point was reached, when a further diminution of the motor exciting current had the effect of increasing the current between the machines.* It was also observed that the watts given out by the generator did not vary in the proportion of the currents between the machines.

Fig. 18.

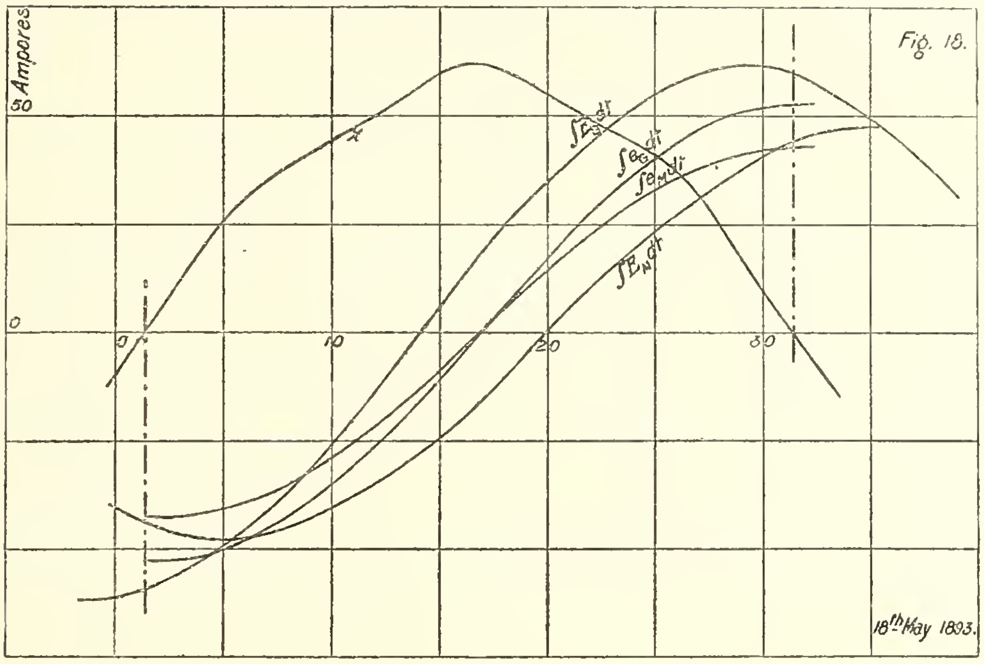


Table V. gives the watts given out by generator for three values of its exciting currents 18·2, 18·6, and 19 amperes, the corresponding exciting currents for the motor being 18, 11·5, and 8·4 amperes. The current between the machines is a minimum for exciting currents of 18·6 and 11·5 in the two machines, and the curves in fig. 17 have been taken under these conditions.

This is a point of practical importance in transmission of power by alternate currents, since the size of the conductor between motor and generator is mainly determined by the current transmitted. The cause is readily explained from the curves.

Starting with the conditions as in Experiment 2, where the motor is more highly excited than the generator, we see that the current (x) is accelerated in phase with regard to the potential difference of the machines. On increasing the exciting current of the generator until the machines are equally excited as in Experiment 1, the current (x) is still accelerated with regard to the potential difference, but not to such an extent. On diminishing the motor exciting current until the machines are excited as in Experiment 3, the current (x) is in phase with the potential difference. For a given power transmitted this will be the point of maximum efficiency with regard to the intermediate conductors. Any further diminution of the motor exciting

* This effect has been independently observed by Mr. Mordey and Mr. Kapp (see 'Journal of Institute of Electrical Engineers,' vol. 22, pp. 128, 173).

current has the effect of retarding the current (x) with regard to the potential difference, and consequently for the same watts transmitted and the same potential difference the current must be increased. The case in which the conductors between motor and generator have considerable induction or capacity has not been worked out.

The losses in the system can be supplied electrically (instead of by belt as in these experiments) as in the case of direct current machines.*

III.

The results of the last section are valuable in relation to the effects of induced currents in the magnets, the subject of Section I. With a machine working as a simple generator the current lags behind the electromotive force on open circuit by any amount from 0° to 90°. But when a generator and motor are run rigidly coupled together, the current may lead the generator electromotive force, or may lag and the motor may lag by any amount from 90° to 270°. Regarding the relative phases of electromotive force of machines and of current, the machine is a generator when the current is from 0° to 90° behind the machine; it is a motor from 90° to 270°, and again a generator from 270° to 360°. We have already stated that we should naturally expect that the induced currents in the magnets would have little or no effect when E.M.F. and current were in the same phase, and that they would have a maximum effect when the two were 90° apart, or at quarter centres. We should expect further that, as a generator can be made into a motor by reversing the current in the armature, wherever local currents diminish E.M.F. of a generator, they would increase E.M.F. of a motor. That is, we should expect that local currents would diminish E.M.F. from lead 0° to 180°, and increase E.M.F. from 180° to 360°. As a fact, we find this to be partially verified; it seems that local currents diminish E.M.F. from a negative angle of comparative small amount, perhaps 30°, to considerably more than 90°, and that they increase E.M.F. from 180° to over 270°.

Referring to the curves in fig. 14, x and E_G are in phase, and $\int e_G \, dt$ would need increasing 3 per cent. to meet $\int E_G \, dt$, when x vanishes E_M lags 216°, and $\int e_M \, dt$ needs diminishing, that is the currents have increased the E.M.F. In fig. 15 a very small change in the observations would change the character of the results.

We have taken another set of curves shown in figs. 19 and 20. In fig. 19, we have a very small current 3·3 amperes in the generator magnets, and the current is in phase with the generator, the motor being 264° behind the current. The generator is about 12 per cent. low owing to local currents, and the motor is 25 per cent. of its actual value high. To obtain a better standard we excited a machine with 3·3 amperes and passed the same current as before through the armature and an ordinary non-inductive resistance, and found the current lagged 72°, and the E.M.F.

* See 'Engineering,' 24th March, 1893.

of the machine was diminished 50 per cent. instead of 12 per cent. The results are given in fig. 20.

Fig. 19.

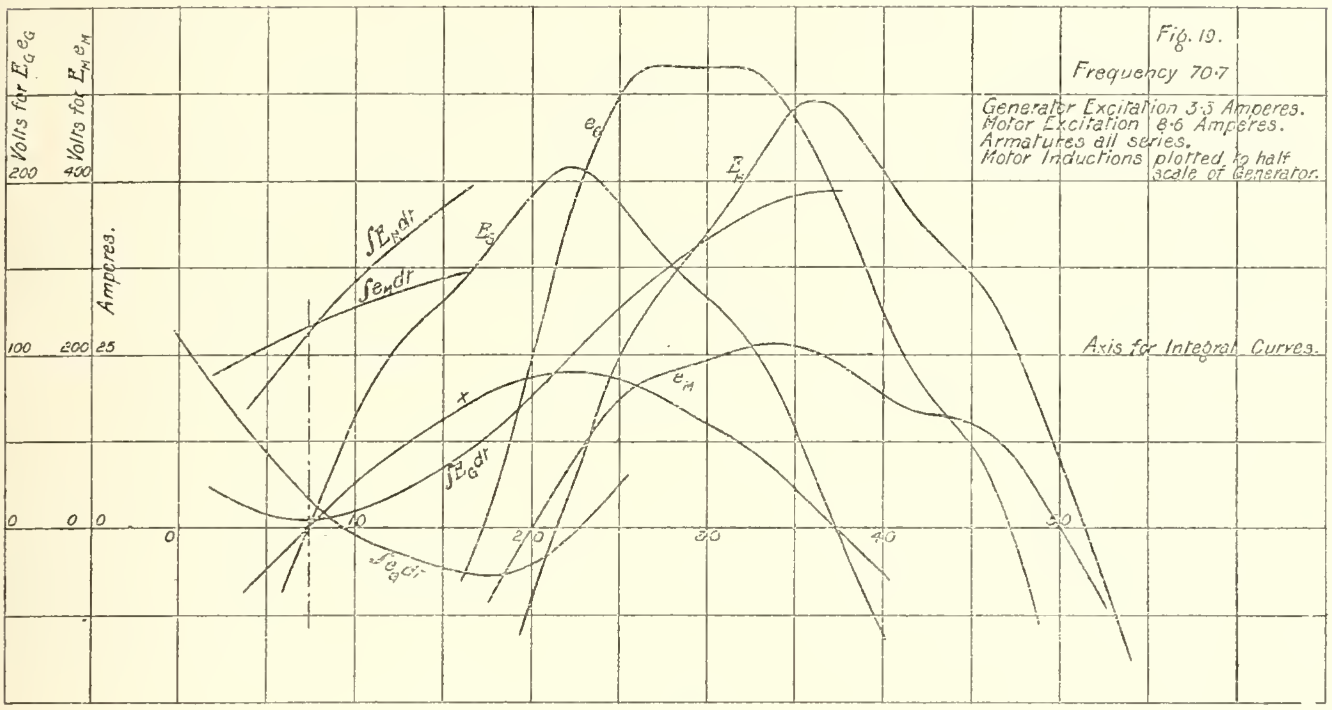
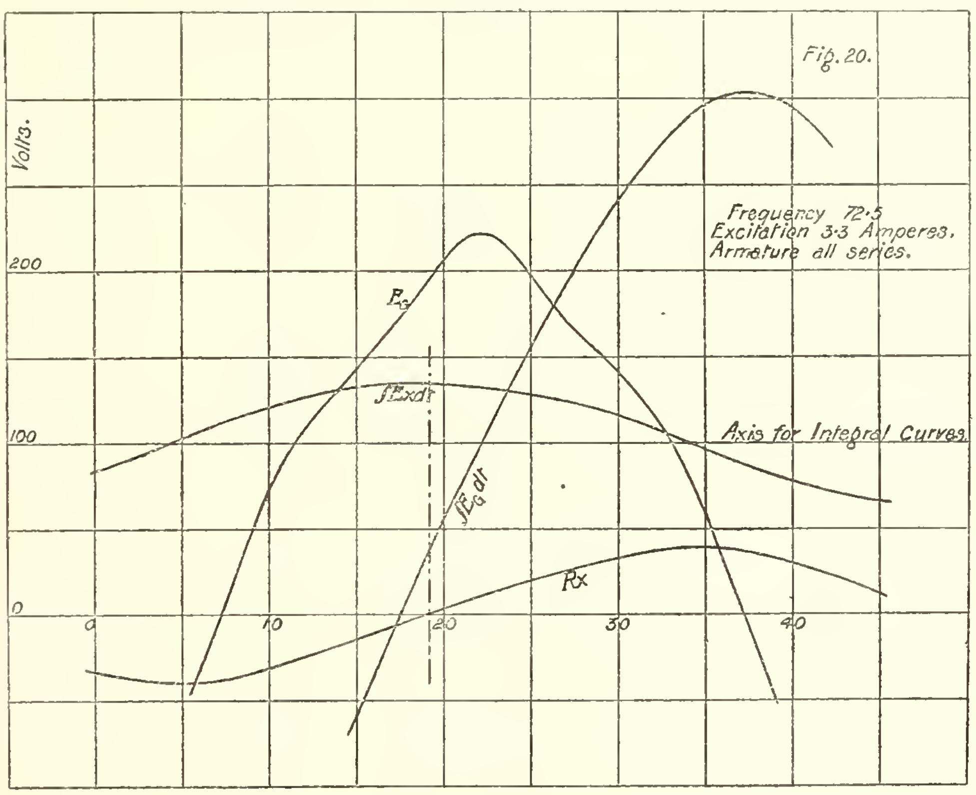


Fig. 20.



In considering the applications of these results, it must be remembered that the machines have been worked far outside the limits of practice for the purpose of accentuating the effect. If we confined ourselves to these limits we should still find these effects, but smaller in amount.

Fig. 21.

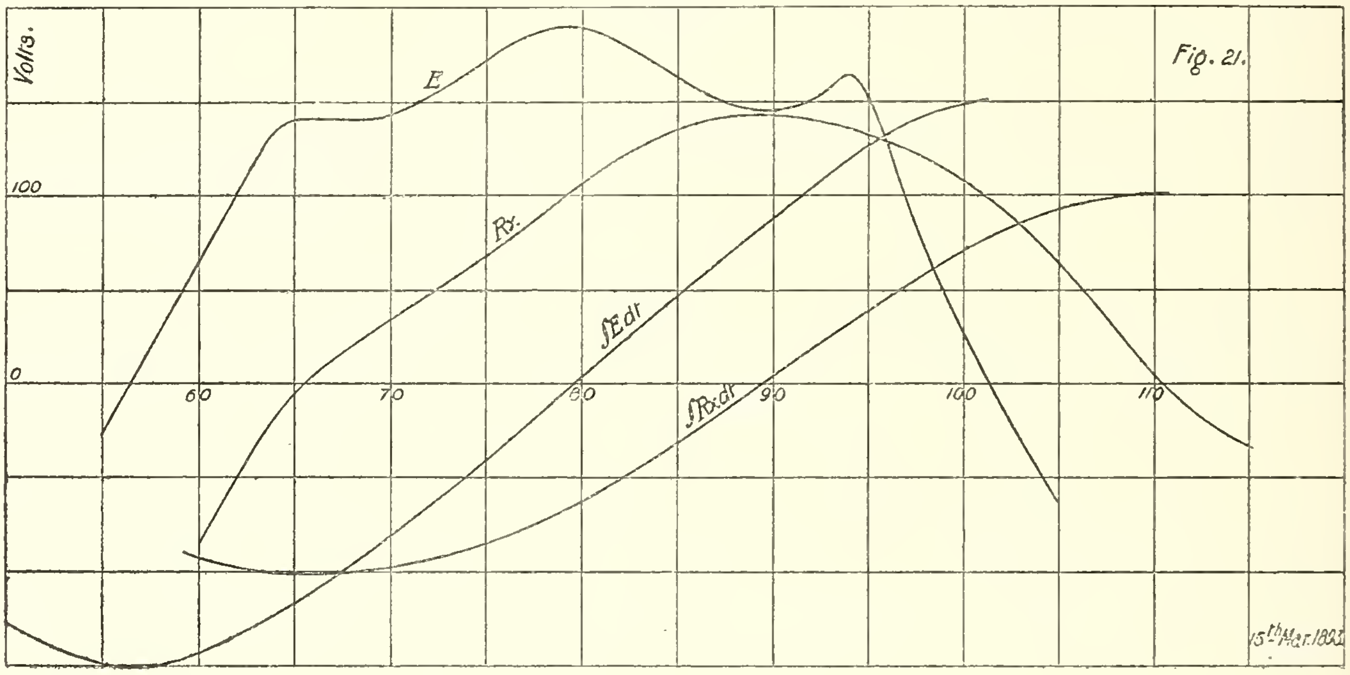


TABLE I.

Fig.	Frequency.	* $\frac{b}{a}$. With variations in current in magnet winding.	* $\frac{b}{a}$. Without variations in current in magnet winding.	Amperes ($\sqrt{\text{mean}^2}$) per armature bobbin. With variations in current in magnet winding.	Amperes ($\sqrt{\text{mean}^2}$) per armature bobbin. Without variations in current in magnet winding.	Exciting current in winding on magnets. Amperes. Amperes when normal.	Exciting current in winding on magnets. Amperes. Maximum variation both sides from normal (per cent.)	Remarks.
5	92·3	·255	. .	20	. .	8	. .	
	97·7	·149	. .	15	. .	7	. .	
6	95	·13	·147	13·6	13·6	7	22	External resistance altered so as to give same current
6	95	·104	·147	14·1	13·6	7	22	External resistance same in each experiment
7	9·2	·093	·085	8·9	9·1	6	21	External resistance same in each experiment
8	11·7	·05	·035	17·6	17·6	16	20	External resistance same in each experiment

* $a = \frac{1}{2}\int_0^{\frac{T}{2}} \mathrm{E}\, dt - \int_{\mathrm{E}=0}^{x=0} \mathrm{E}\, dt$ = ordinate of curve A when $x = 0$.

$b = a - \frac{1}{2}\int_0^{\frac{T}{2}} \mathrm{R}x\, dt$ = difference of ordinates of curves A and B when $x = 0$, which should be zero on the usual theory.

TABLE II.—Efficiency Test of SIEMENS' W. 12 Alternators. Frequency 70·3 periods per second. Half Load.

Angle in 60ths of a period.	Potential at terminals of generator.		Current between machines.			Potential at terminals of motor.		Watts given out by generator.	Watts given to motor.
	Electrometer deflection.	Volts.	Electrometer deflection.	Amperes.	Sq. of amperes $\sqrt{\text{mean}^2} = 21·57$ THOMSON balance } 21·63	Difference of potential difference.	Volts.		
0	−159	−224	+ 96	+23·1	533·6	+2·57	−221·4	−5174	−5114
2	−138	−194·5	+ 76	+18·3	334·9	+2·03	−192·5	−3559	−3523
4	−119	−167·7	+ 62	+14·9	222·0	+1·66	−166·0	−2499	−2473
6	− 99	−139·5	+ 44	+10·6	112·3	+1·18	−138·3	−1479	−1568
8	− 71	−100·0	+ 17	+ 4·1	16·8	+0·45	− 99·5	− 410	− 408
10	− 34	− 47·9	− 18	− 4·3	18·5	−0·48	− 48·4	+ 206	+ 208
2	+ 6	+ 8·4	− 50	−12·0	144·0	−1·34	+ 7·1	− 101	− 85
4	+ 46	+ 64·8	− 79	−19·0	361·0	−2·11	+ 62·7	−1231	−1191
6	+ 79	+111·3	− 95	−22·9	524·4	−2·54	+108·8	−2548	−2491
8	+104	+146·5	−101	−24·3	590·5	−2·7	+143·8	−3560	−3494
20	+124	+174·7	−107	−25·8	665·7	−2·86	+171·8	−4507	−4432
2	+145	+204·3	−116	−27·9	778·4	−3·1	+201·2	−5700	−5614
4	+166	+233·9	−127	−30·6	936·4	−3·4	+230·5	−7158	−7053
6	+179	+252·2	−129	−31·1	967·2	−3·45	+248·7	−7842	−7734
8	+174	+245·2	−116	−27·9	778·4	−3·1	+242·1	−6840	−6754
					6984·1			52402	51726
					465·6			3493	3448

TABLE III.—Efficiency Test of SIEMENS' W. 12 Alternators. Frequency 69·2 periods per second. Full Load.

Angle in 60ths of a period.	Potential at terminals of generator.		Current between machines.			Potential at terminals of motor.		Watts given out by generator.	Watts given to motor.
	Electrometer deflection.	Volts.	Electrometer deflection.	Amperes.	Sq. of amperes $\sqrt{\text{mean}^2} = 42·8$ THOMSON balance } 42·0	Difference of potential difference.	Volts.		
0	− 64	− 90·2	+ 25	+ 6·02	36·24	+0·67	− 89·53	− 543	− 539
2	− 35	− 49·3	− 34	− 8·2	67·24	−0·91	− 50·21	+ 404	+ 412
4	− 1	− 1·4	−100	−24·1	580·7	−2·67	− 4·07	+ 34	+ 98
6	+ 32	+ 45·1	−152·4	−36·7	1347·0	−4·07	+ 41·03	− 1655	− 1506
8	+ 63	+ 88·8	−189·5	−45·6	2079·4	−5·06	+ 83·74	− 4049	− 3819
10	+ 94	+132·5	−211	−50·8	2580·8	−5·64	+126·86	− 6731	− 6445
2	+121	+170·5	−222·4	−53·6	2873·0	−5·95	+164·55	− 9140	− 8819
4	+142	+200·1	−236	−56·8	3226·0	−6·31	+193·79	− 11367	−11009
6	+155·3	+218·8	−251·5	−60·6	3672·0	−6·73	+212·07	− 13260	−12850
8	+161·2	+227·1	−251·5	−60·6	3672·0	−6·73	+220·37	− 13763	−13354
20	+160·2	+225·7	−228·2	−55	3025·0	−6·11	+219·59	− 12413	−12080
2	+152·4	+214·7	−188·5	−45·4	2061·0	−5·04	+209·66	− 9746	− 9518
4	+138	+194·4	−147·4	−35·5	1260·3	−3·94	+190·46	− 6902	− 6761
6	+118	+166·3	−113	−27·2	739·8	−3·02	+163·28	− 4523	− 4441
8	+ 92	+129·6	− 75	−18·1	327·6	−2·01	+127·59	− 2346	− 2309
					27548·08			96000	92940
					1836·54			6400	6196

TABLE IV.

No.	Description of Magnitude.	Half load.	Full load.
	Frequency in complete periods per second	70·3	69·2
	Phase difference between armatures in fractions of a complete period	$\frac{1}{20}$	$\frac{1}{10}$
1	Watts given out by generator (see Tables II. and III.) . . .	3493	6400
2	Watts given to motor (see Tables II. and III.)	3448	6196
3	Watts dissipated in generator armature = $(\sqrt{\text{mean}^2} \text{ C.})^2$ ·275 ohm	128·5	481·2
4	Watts dissipated in motor armature = $(\sqrt{\text{mean}^2} \text{ C.})^2$ ·275 ohm	128·5	481·2
5	Watts dissipated in generator magnet winding	537	537
6	Watts dissipated in motor magnet winding	537	537
7	Watts dissipated in connections between machines	45	204
8	Watts absorbed by combination through belt	1848	2941
9	Total electrical power developed in generator = No. 1 + No. 3	3621	6881
10	Half Watts absorbed by system *minus* half known Watts = $\frac{1}{2}$ {No. 8 − (No. 3 + No. 4 + No. 7)}	746	889
11	Total power given to generator = No. 5 + No. 9 + No. 10 . .	4904	8307
12	Percentage efficiency of generator = $\frac{\text{No. 1}}{\text{No. 11}} \times \frac{1}{100}$	*71·2*	*77·0*
13	Percentage loss in generator armature	2·61	5·79
14	Percentage loss in generator magnet winding	10·9	6·46
15	Percentage sum of all other losses in generator	15·29	9·75
16	Percentage efficiency of motor = $\left(\frac{\text{No. 9}+\text{No. 10}-\text{No. 8}}{\text{No. 2}+\text{No. 6}}\right)\frac{1}{100}$	*63·2*	*71·7*
17	Percentage loss in motor armature	3·22	7·14
18	Percentage loss in motor magnet winding	13·5	7·98
19	Percentage sum of all other losses in motor	20·1	13·18
20	Percentage efficiency of combination = (No. 12 × No. 16) $\frac{1}{100}$	*45·0*	*55·2*

Table V.

Angle in 60ths of a period.	Current in genr. mags. 18·2, motor 18, $\sqrt{\text{mean}^2}$ volts on motor 155. Frequency 70.				Current in genr. mags. 18·6, motor 11·5, $\sqrt{\text{mean}^2}$ volts on motor 143. Frequency 70.				Current in genr. mags. 19, motor 8·4, $\sqrt{\text{mean}^2}$ volts. on motor 132·5. Frequency 70.			
		Current.				Current.				Current.		
	Volts $\sqrt{\text{mean}^2} = 158$.	Electrometer deflection.	Amperes $\sqrt{\text{mean}^2} = 43·3$.	Watts given out by generator.	Volts $\sqrt{\text{mean}^2} = 146$.	Electrometer deflection.	Amperes $\sqrt{\text{mean}^2} = 40·5$.	Watts given out by generator.	Volts $\sqrt{\text{mean}^2} = 135·5$.	Electrometer deflection.	Amperes $\sqrt{\text{mean}^2} = 43·6$.	Watts given out by generator.
0	− 53·3	44	− 9·9	+ 528	− 33·8	42	+ 9·45	− 319	− 23·8	124	+27·9	− 664
3	+ 13·6	155·3	−35·0	− 476	+ 23·3	51	−11·48	− 267	+ 34·4	4	+ 0·9	+ 310
6	+ 75·9	219·4	−49·4	− 3,750	+ 80·3	138	−31·06	− 2,494	+ 81·2	88	−19·8	− 1,608
9	+146·8	249·5	−56·2	− 8,250	+134·2	182·5	−41·08	− 5,513	+127·5	143·5	−32·3	− 4,118
12	+192·6	263·5	−59·3	−11,420	+179·6	222·3	−50·05	− 8,988	+167·5	202·2	−45·5	− 7,622
15	+219·7	277	−62·4	−13,710	+206·2	265·5	−59·76	−12,320	+188·4	263·5	−59·3	−11,170
18	+224·2	251	−56·5	−12,670	+205·2	267·5	−60·22	−12,360	+184·2	282·8	−63·7	−11,730
21	+203·2	188·4	−42·4	− 8,616	+186·6	231·2	−52·04	− 9,710	+165·4	263·5	−59·3	− 9,808
24	+166·6	128	−28·8	− 4,798	+150·6	194·2	−43·72	− 6,584	+132·3	245·5	−55·3	− 7,316
27	+108·6	53	−11·9	− 1,292	+ 96·2	141·5	−31·85	− 3,064	+ 80·4	205	−46·1	− 3,706
				64,454				61,919				57,432

VII. *Mathematical Contributions to the Theory of Evolution.—III. Regression, Heredity, and Panmixia.*

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Contents.

(1.) *Introductory.*

There are few branches of the Theory of Evolution which appear to the mathematical statistician so much in need of exact treatment as those of Regression, Heredity, and Panmixia. Round the notion of panmixia much obscurity has accumulated, owing to the want of precise definition and quantitative measurement. The problems of regression and heredity have been dealt with by Mr. FRANCIS GALTON in his epoch-making work on 'Natural Inheritance,' but, although he has shown exact methods of dealing, both experimentally and mathematically, with the problems of inheritance, it does not appear that mathematicians have hitherto developed his treatment, or that biologists and medical men have yet fully appreciated that he has really shown how many of the problems which perplex them may receive at any rate a partial answer. A considerable portion of the present memoir will be devoted to the expansion and fuller development of Mr. GALTON's ideas, particularly their application to the problem of *bi-parental inheritance.* At the same time I shall endeavour to point out how the results apply to some current biological and medical problems. In the first place, we must definitely free our minds, in the present state of our knowledge of the mechanism of inheritance and reproduction, of any hope of reaching a mathematical relation expressing the degree of correlation between individual parent and individual offspring.* The causes in any individual case of inheritance are far too complex to admit of exact treatment; and up to the present the classification of the circumstances under which greater or less degrees of correlation between special groups of parents and offspring may be expected has made but little progress. This is largely owing to a certain prevalence of almost metaphysical speculation as to the causes of heredity, which has usurped the place of that careful collection and elaborate experiment by which alone sufficient data might have been accumulated, with a view to ultimately narrowing and specialising the circumstances under which correlation was measured. We must proceed from inheritance in the mass to inheritance in narrower and narrwoer classes, rather than attempt to build up general rules on the observation of individual instances. Shortly, we must proceed by the method of statistics, rather than by the consideration of typical cases. It may seem discouraging to the medical practitioner, with the problem before him of inheritance in a particular family, to be told that nothing but averages, means, and probabilities with regard to large classes can as yet be scientifically dealt with; but the very nature of the distribution of variation, whether healthy or morbid, seems to indicate that we are dealing with that sphere of indefinitely numerous small causes, which in so many other instances has shown itself only amenable to the calculus of chance, and not to any analysis of the individual instance. On the other hand, the mathematical theory will be of assistance to the medical man by answering, *inter alia,* in its discussion of regression the problem as to the average effect upon the offspring of given degrees of morbid variation in the parents. It may enable the physician, in many cases, to state a belief based on a high degree of probability, if it offers no ground for dogma in individual cases.

One of the most noteworthy results of Mr. FRANCIS GALTON's researches is his discovery of the mode in which a population actually reproduces itself by regression and fraternal variation. It is with some expansion and fuller mathematical treatment of these ideas that this memoir commences.

* The physical and arithmetical statements of WEISMANN's "Theory of Germ Plasm" offer, so far as I have been able to interpret them, no sound basis for a quantitative theory of heredity in the mathematician's sense.

(2.) *Definitions.*

It is necessary to give definitions to several current biological conceptions, in order to introduce them into our mathematical analysis.

(*a.*) *Variation.*—If a curve be constructed, of which the ordinate y is such that $y\,\delta x$ measures the frequency with which an organ lying in size between x and $x + \delta x$, occurs in a considerable population (500 to 1000 or more), the constants which, for any particular organ for any particular animal determine the form of this curve, are termed the *constants of variation*, or more briefly, the variation of the given organ.

The assumption is made that the frequency is continuous, or that we really reach a curve. In the great majority of cases, where real statistical methods have been used, continuous curves (or, practically, polygons) have been found, and we shall assume this continuity to hold in all cases to which our formulæ are applied.

The size of the organ (x) which corresponds to the ordinate (y) through the centroid of the frequency curve, is termed the *mean*; the size of the organ, which corresponds to the ordinate bisecting the area of the frequency curve, is termed the *median*; the size of the organ corresponding to maximum frequency is termed the *mode.*

We assume, what may be considered as fairly established, that variation curves in zoometry, and more especially anthropometry, approximate closely to probability curves. When the variation curve has more than one mode, it may, as a rule, be resolved into simple probability curves, each with a single mode, and it may be even heterogeneous and require resolution, when only one mode is apparent.* These probability curves may be skew, and in this case the treatment of the problem of heredity involves a discussion of skew-correlation,† but in a very great range of cases the frequency is sufficiently closely given by the normal probability curve. Here the variation is defined by a single constant,‡ the standard deviation σ, and the equation to the curve is given by

$$y = \frac{\mathrm{N}}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)},$$

and we shall confine our attention to such variation in the present memoir. The following assumption, therefore, lies at the basis of our present treatment of heredity. The variation of any organ in a sufficiently large population—which may be selected in any manner other than by this organ itself from a still larger population—is closely defined by a normal probability curve.

(*b.*) *Correlation.*—Two organs in the same individual, or in a connected pair of

* On resolution and skew variation, see 'Contributions to the Mathematical Theory of Evolution,' Memoirs I. and II., 'Phil. Trans.,' vols. 185 and 186.

† Dealt with in a memoir not yet published.

‡ Inheritance can be treated by single-constant variation in the case of most organs in human adults, but it could not be dealt with in like manner in the of case pedigree buttercups, see DE VRIES: 'Berichte der Deutschen Botanischen Gesellschaft,' 1894 and 1895.

individuals, are said to be correlated, when a series of the first organ of a definite size being selected, the mean of the sizes of the corresponding second organs is found to be a function of the size of the selected first organ. If the mean is independent of this size, the organs are said to be non-correlated. Correlation is defined mathematically by any constant, or series of constants, which determine the above function.

The word "organ" in the above definitions of variation and correlation must be understood to cover any measurable characteristic of an organism, and the word "size," its quantitative value.

(*c.*) *Natural Selection.*—This is of two kinds: *Secular Natural Selection* is measured by the changes due solely to mortality, in the mean and standard deviation of the variation-curve as we pass from one adult generation to the next. In statistical observations on man it is by no means easy—as we shall indicate later—to differentiate it from the effects of sexual selection, and of altered sanitary conditions.

Periodic Natural Selection may leave no trace of itself in the adult variation-curves of successive generations; it is measured by the changes due solely to mortality in the mean and standard deviation of the variation curves at successive stages of the *same* generation—*due allowance being made for the changes of the variation-constants due to growth.* In other words, if we watched a generation from birth to the adult stage, carefully preserving it from any form of selective mortality, such as arises from the struggle for existence, we should still find changes in the variation-constants due to the law of growth. If now the same generation be subjected to the struggle for existence, *i.e.*, placed in its natural surroundings, the variation-constants will differ from their values at the corresponding stages of the unselected growth. This difference is due to the selective mortality, *i.e.*, to natural selection. But this selective mortality may go on and still leave the variation-constants of the adult stage of each generation the same. In this case we speak of it as periodic natural selection. It repeats itself in each generation, but produces no secular change. It maintains an adult standard, but is not a factor of progressive evolution.

No estimate of periodic natural selection can be formed until the law of growth has been accurately ascertained by a series of observations on individuals. The influence of secular natural selection will be allowed for in our investigations by supposing the means and standard deviations of successive adult variation-curves to be not necessarily the same.*

(*d.*) *Sexual Selection.*—Sexual Selection† is of two kinds, due respectively to what

* Variation-curves for non-adult populations appear to be frequently skew. I propose in another paper to discuss the general law of selection on the basis of skew curves and with any arbitrary law of growth.

† I think DARWIN'S view would be of the following kind. Let A be the most attractive female, *a* the most efficient male, Z the least attractive female and *z* the least efficient male. Then supposing only these four, *a* and *z* would both desire A with the result that (1) *a* would drive away *z* or (2) kill him. In the first case *z* would be free to mate with Z, but if he did so they would tend to produce a miserable

may be spoken of as individual and tribal taste. Tribal taste manifests itself in the preference of one sex as a whole for mating with members of the other sex having special characteristics, or to the rejection as mates by one sex of members of the other having special characteristics. The preference and rejection being in neither case absolute, but relative. This type of sexual selection, which may be spoken of as *preferential mating*, is measured by the differences in mean and standard deviation between the variation-curves for the whole adult population of one sex, and for the mated portion of it. For example, the mean height and mean variation in height of women generally are not identical, or are not necessarily identical with the mean height and mean variation in height of wives. Preferential mating may have reference to any organ or measurable characteristic of either sex.

Individual taste on the other hand does not denote the exclusion from mating of any section of the population of either sex. It is due to the preference of individuals with an organ or characteristic of given size for mates with the same or another organ or characteristic of a size, the average of which differs from the whole population average. This type of sexual selection which may be spoken of as *assortative mating* is measured mathematically by the coefficient of correlation between the two organs or characteristics in mated pairs.

It will be obvious that preferential mating and assortative mating are fundamental ideas to be quantitatively allowed for in any theory of heredity. Their action may often be in entirely opposite directions.*

(*e*.) *Reproductive Selection*.—One pair may produce more offspring than another, and in this manner give through heredity greater weight to their own characteristics. For example, the mean height of mothers is not identical, or is not necessarily identical with the mean height of wives, nor is the standard-deviation of fathers identical or necessarily identical with the standard-deviation of husbands. Further, the means and standard-deviations of mothers or fathers of sons may be different from those of mothers or fathers of daughters. The quantitative measure of reproductive selection is the correlation between the size of any selected organ in either male and female and their reproductivity, the reproductivity being measured by the number of their offspring in either sex or both sexes.

offspring fated to die out. Of course a might in certain cases after (1) mate with both A and Z. None of these possibilities corresponds exactly to what is described in this section as assortative mating, which in no way necessitates the exclusion from mating of z. a and z are not indeed competitors, but seeking different qualities in their mates. Thus, in man for example, the intellectual and non-intellectual might, and possibly do, sort themselves out in pairs, *i.e.*, there is a correlation between intellectual capacity of husband and wife.

* For example, preferential mating might lead in a highly social community to the rejection of consumptive mates, while assortative mating might, through localisation or community of habit, lead to considerable consumptive correlation. Thus sexual selection as a whole may influence in diverse ways the inheritance of the consumptive taint.

The importance of determining whether there is any correlation between reproductivity and a given organ of either parent appears to be great. For, if there be, it is not easy to understand how, even in the absence of both natural and sexual selection, a population can remain in a stable state. For example, suppose the mean father or the mean mother or both to be taller than the mean man or the mean woman or both, then this reproductive selection would appear to involve a gradual increase of height in the population in the same manner as selective breeding of animals by man might do. It is probable, therefore, that if reproductive selection be demonstrated by a finite value of the correlation constants, the instability of the population which results is partially or completely screened by natural selection.*

(*f.*) *Heredity.*—Given any organ in a parent and the same or any other organ in its offspring, the mathematical measure of heredity is the correlation of these organs for pairs of parent and offspring. If the organs be the same for parent and offspring, the heredity may be spoken of as *direct*, if they be different as *cross*. The word organ here must be taken to include any characteristic which can be quantitatively measured.

If the organs are not those of parent and offspring, but of any two individuals with a given degree of blood relationship, the correlation of the two organs will still be the proper measure of the strength of heredity for the given degree of relationship. *Cf.* § 6.

(*g.*) *Regression.*—Regression is a term which has been hitherto used to mark the amount of abnormality which falls on the average to the lot of offspring of parents of a given degree of abnormality. The mathematical measure of this special regression is the ratio of the mean deviation of offspring of selected parents from the mean of all offspring to the deviation of the selected parents from the mean of all parents. This may be further elucidated as follows:—Let parents, having an organ or characteristic of given deviation from the average or normal, be termed a "parentage," let the offspring of a parentage be termed a "fraternity." Then the *coefficient of regression* may be defined as the ratio of the mean deviation of the fraternity from the mean offspring to the deviation of the parentage from the mean parent. Both parentage and fraternity may be either male or female. It will be noted that we have so framed our definition of regression, that it marks the deviation of the fraternity from the filial and not the parental mean. We are thus able to allow for secular natural selection and reproductive selection. We shall see in the sequel that the coefficient of regression is a function of the variations in parents and offspring, and further of the correlations which define parental heredity and assortative mating. Further, as in heredity, the deviation or abnormality in parentage and fraternity may be measured with respect to the same or different organs; we have thus *direct* and *cross* regression.

From this special definition of regression in relation to parents and offspring, we

* I hope shortly to publish a paper on "Reproductive Selection in Man," and show how completely it appears to screen Natural Selection in the case of *civilised* man.

may pass to a general conception of regression. Let A and B be two correlated organs (variables or measurable characteristics) in the same or different individuals, and let the sub-group of organs B, corresponding to a sub-group of A with a definite value a, be extracted. Let the first of these sub-groups be termed an *array*, and the second a *type*. Then we define the coefficient of regression of the array on the type to be the ratio of the mean-deviation of the array from the mean B-organ to the deviation of the type a from the mean A-organ. The following are illustrations of types and arrays :—

Type.	Array.
Organ of given magnitude in—	Distribution of the correlated organs in—
Parent	Fraternity.
Offspring	Parentage.
Wife.	Male matage.
Husband	Female matage.
Given value of—	Distribution of correlated—
Height	Spans.
Cephalic index	Alveolar indices.
Barometric height	Heights at second station.
Local wages	Local pauper percentages.
Etc.	Etc.

It will be seen in the sequel that for the same pair of correlated organs or characteristics, the coefficient of regression is, if the law of frequency be the normal law, the same for the arrays corresponding to all types. But the coefficient is not the same when the type and array organs are interchanged, *e.g.*, the regression of husbands (male matage) on wives is not the same as the regression of wives (female matage) on husbands.

(*h.*) *Panmixia.*—Suppose that starting from a population of given mean and variation for any particular organ, secular natural selection of definite amount takes place for p generations and produces a population of another definite mean and variation for this same organ. Now suppose natural selection, whether periodic or secular, to be suspended for q generations, and sexual selection to be non-extant or negligible, then those members of the general population which were formerly weeded out, will now mix with all the other members of the population, and the results of interbreeding are spoken of as *panmixia*. The mathematical measure of the result on the given organ of panmixia acting for q generations is the change in mean and variation of the population with regard to that organ during these q generations. Should the mean and variation of the population tend with increase of q to approach the mean and variation of the population $p + q$ generations previously, panmixia may be said to reverse natural selection.

We have now defined the chief factors which will be dealt with in the present memoir, and shown how they are to be quantitatively measured. We shall now proceed to their mathematical analysis on the fundamental assumption that the variations with which we are about to deal obey the normal law of frequency.

(3.) *Correlation with special reference to the Problem of Heredity.*

(*a.*) *Historical.*—The fundamental theorems of correlation were for the first time and almost exhaustively discussed by BRAVAIS ('Analyse mathématique sur les probabilités des erreurs de situation d'un point.' Mémoires par divers Savans, T. IX., Paris, 1846, pp. 255-332) nearly half a century ago. He deals completely with the correlation of two and three variables. Forty years later Mr. J. D. HAMILTON DICKSON ('Proc. Roy. Soc.,' 1886, p. 63) dealt with a special problem proposed to him by Mr. GALTON, and reached on a somewhat narrow basis* some of BRAVAIS' results for correlation of two variables. Mr. GALTON at the same time introduced an improved notation which may be summed up in the 'GALTON function' or coefficient of correlation. This indeed appears in BRAVAIS' work, but a single symbol is not used for it. It will be found of great value in the present discussion. In 1892 Professor EDGEWORTH, also unconscious of BRAVAIS' memoir, dealt in a paper on 'Correlated Averages' with correlation for three variables ('Phil. Mag.' vol. 34, 1892, pp. 194-204.) He obtained results identical with BRAVAIS', although expressed in terms of 'GALTON's functions.' He indicates also how the method may be extended to higher degrees of correlation. He starts by assuming a general form for the frequency of any complex of n organs each of given size. This form has been deduced on more or less legitimate assumptions by various writers. Several other authors, notably SCHOLS, DE FOREST and CZUBER, have dealt with the same topic, although little of first-class importance has been added to the researches of BRAVAIS. To Mr. GALTON alone is due the idea of applying these results—usually spoken of as "the laws of error in the position of a point in space"—to the problem of correlation in the theory of evolution.

The investigation of correlation which will now be given does not profess, except at certain stated points, to reach novel results. It endeavours, however, to reach the necessary fundamental formulæ with a clear statement of *what assumptions are really made*, and with special reference to what seems legitimate in the case of heredity.

(*b.*) *Theory of Correlation.*—Let $\eta_1, \eta_2, \eta_3 \ldots \eta_n$ be the deviations from their respective means of a complex of organs or measurable characteristics. These organs may be in the same or in different individuals, or partly belong to one and partly to another individual. The complex may be constituted by a natural or artificial tie

* The coefficient of correlation was assumed to be the same for the arrays of all types, a result which really flows from the normal law of frequency.

of any kind, but the tie is to remain the same for every complex, whether it be the result of mating or parentage, or flow from any physiological or social relation, &c.

We shall now assume that the sizes of this complex of organs are determined by a great variety of *independent* contributory causes, for example, magnitudes of other organs not in the complex, variations in environment, climate, nourishment, physical training, various ancestral influences, and innumerable other causes, which cannot be individually observed or their effects measured. Let these causes be m in number, m being generally much greater than n, and let their deviations from their mean intensities be $\epsilon_1, \epsilon_2, \epsilon_3, \ldots \epsilon_m$, then $\eta_1, \eta_2, \eta_3, \ldots \eta_n$ will be functions of $\epsilon_1, \epsilon_2, \epsilon_3, \ldots \epsilon_m$. Further, certain of the ϵ's will appear only in certain of the η's, and the ϵ's will not be fully determined for a given η complex.

We shall in the next place assume that the variations in intensity of the contributory causes are small as compared with their absolute intensity, and that these variations follow the normal law of distribution.* The mean complex being reached with the mean intensities of contributory causes, we have by the principle of the superposition of small quantities :

$$\left.\begin{array}{l} \eta_1 = \alpha_{11}\epsilon_1 + \alpha_{12}\epsilon_2 + \alpha_{13}\epsilon_3 + \ldots + \alpha_{1m}\epsilon_m, \\ \eta_2 = \alpha_{21}\epsilon_1 + \alpha_{22}\epsilon_2 + \alpha_{23}\epsilon_3 + \ldots + \alpha_{2m}\epsilon_m, \\ \ldots\ldots \\ \eta_n = \alpha_{n1}\epsilon_1 + \alpha_{n2}\epsilon_2 + \alpha_{n2}\epsilon_3 + \ldots + \alpha_{nm}\epsilon_m, \end{array}\right\} \quad \ldots\ldots \quad \text{(i).}$$

Here any of the system of α's may be zero.

Further, the chance that we have a conjunction of contributory causes lying between ϵ_1 and $\epsilon_1 + \delta\epsilon_1$, ϵ_2 and $\epsilon_2 + \delta\epsilon_2 \ldots e_m$ and $\epsilon_m + \delta\epsilon_m$ will be given by

$$P = Ce^{-\left(\frac{\epsilon_1^2}{2\kappa_1^2} + \frac{\epsilon_2^2}{2\kappa_2^2} + \frac{\epsilon_3^2}{2\kappa_3^2} + \ldots + \frac{\epsilon_m^2}{2\kappa_m^2}\right)} \times \delta\epsilon_1\delta\epsilon_2\delta\epsilon_3, \ldots \delta_m, \quad \ldots\ldots \quad \text{(ii)}$$

where the standard deviations of the variation distributions for $\epsilon_1, \epsilon_2, \epsilon_3, \ldots \epsilon_n$ are respectively $\kappa_1, \kappa_2, \kappa_3, \ldots \kappa_n$, and C is a constant.

Now by aid of the equations (i.) let n of the variables ϵ, say, the first n, be replaced by the variables η, then the probability that we have a complex with organs lying between η_1 and $\eta_1 + \delta\eta_1$, η_2 and $\eta_2 + \delta\eta_2 \ldots \eta_n$ and $\eta_n + \delta\eta_n$, together with a series of contributory causes lying between ϵ_{n+1} and $\epsilon_{n+1} + \delta\epsilon_{n+1}$, ϵ_{n+2} and $\epsilon_{n+2} + \delta\epsilon_{n+2} \ldots \epsilon_m$ and $\epsilon_m + \delta\epsilon_m$ will be

$$P' = C'e^{-\frac{1}{2}\phi^2}\,\delta\eta_1\,\delta\eta_2 \ldots \delta\eta_n\,\delta\epsilon_{n+1}\,\delta\epsilon_{n+2} \ldots \delta\epsilon_m$$

* This may be taken at any rate as a first approximation. It is at this point that the theory of skew-correlation diverges from our present treatment.

where C′ is a constant, a function of C and the a's, and ϕ^2 consists of the following parts:—

(i.) A quadratic function of the η's from η_1 to η_n,
(ii.) A quadratic function of the ϵ's from ϵ_{n+1} to ϵ_m,
(iii.) A series of functions of the type:

$$\begin{array}{l}\epsilon_{n+1}\,(b_{1,\,n+1}\eta_1 + b_{2,\,n+1}\eta_2 + \ldots + b_{n,\,n+1}\eta_n),\\ \epsilon_{n+2}\,(b_{1,\,n+2}\eta_1 + b_{2,\,n+2}\eta_2 + \ldots + b_{n,\,n+2}\eta_n),\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ \epsilon_m\,(b_{1,\,m}\eta_1 + b_{2,\,m}\eta_2 + \ldots + b_{n,\,m}\eta_n),\end{array}$$

where some of the b's may be zero.

Now if P′ be integrated for the values from $-\infty$ to $+\infty$ of all the contributory causes $\epsilon_{n+1}, \epsilon_{n+2} \ldots \epsilon_m$, we shall have the whole chance of a complex with organs falling between η_1 and $\eta_1 + \delta\eta_1$, η_2 and $\eta_2 + \delta\eta_2 \ldots \eta_n$ and $\eta_n + \delta\eta_n$. But every time we integrate with regard to an ϵ, ϵ_{n+1}, say, we alter the constants of each contributory part of ϕ^2, but do not alter the triple constitution of ϕ^2, except to cause one ϵ to disappear from its (ii.) and (iii.) constituents. At the same time we alter C′ without introducing into it any terms in η. Thus, finally, after $m - n$ integrations, ϕ^2 is reduced to its first constituent, or we conclude that the chance of a complex of organs between η_1 and $\eta_1 + \delta\eta_1$, η_2 and $\eta_2 + \delta\eta_2 \ldots \eta_n$ and $\eta_n + \delta\eta_n$ occurring is given by

$$\mathrm{P} = \mathrm{C}e^{-\frac{1}{2}\chi^2}\,\delta\eta_1\,\delta\eta_2 \ldots \delta\eta_n \quad \ldots\ldots\ldots \quad \text{(iii.)}$$

where χ^2 is a quadratic function of the η's. This is the law of frequency for the complex.

Now our deduction of (iii.) seems to have considerable justification in case of heredity. We allow for an indefinite number of quite inappreciable and unascertainable independent contributory causes. We suppose that some of these causes are common to parent and offspring; how many and in what degree we make no pretence at saying. We assume, however, that the action of these causes does not differ very widely in intensity throughout the special range of organisms from which our complexes are drawn, and further, that the variation in intensity of any contributory cause follows that law of frequency, which we know to be at any rate approximately true, for distributions of physical and organic variation similar in character to those in which we may with a high degree of probability suppose the phenomena of heredity to ultimately have their origin.

Having thus deduced, with special reference to our particular topic, BRAVAIS' law of frequency, I propose to consider its characteristics in two special cases, as it is needful to deduce for our present purposes one or two, I believe, novel results.

(4.) *Special Case of Two Correlated Organs.*

(*a.*) *Theory.*—Let x and y be the deviations of a pair of organs (or measurable characteristics) from their respective means. Let σ_1 and σ_2 be the standard deviations of x and y, treated as independent variations. Let N be the total number of pairs and $z \times \delta x \, \delta y$ the frequency of a pair falling between x and $x + \delta x$, y and $y + \delta y$, then, by BRAVAIS' form,

$$z = C \times e^{-(g_1 x^2 + 2hxy + g_2 y^2)}$$

where g_1, g_2, and h are constants.

Integrate z for all values of y from $-\alpha$ to $+\alpha$, and we must have the normal curve of x-variation, hence

$$\frac{1}{2\sigma_1^2} = g_1 (1 - h^2/g_1 g_2).$$

Similarly integrating z for all values of x, we have

$$\frac{1}{2\sigma_2^2} = g_2 (1 - h^2/g_1 g_2).$$

Now integrate z for all values of x and y to obtain the total frequency, and we have

$$N = C\pi/\sqrt{g_1 g_2 - h^2}.$$

If we now write r for $-h/\sqrt{g_1 g_2}$, we can throw z into the form

$$z = \frac{N}{2\pi\sigma_1\sigma_2} \frac{1}{\sqrt{1 - r^2}} e^{-\frac{1}{2}\left\{\frac{x^2}{\sigma_1^2(1-r^2)} - \frac{2xyr}{\sigma_1\sigma_2(1-r^2)} + \frac{y^2}{\sigma_2^2(1-r^2)}\right\}}.$$

(*b.*) *On the best Value of the Correlation Coefficient.*—This is the well-known Galtonian form of the frequency for two correlated variables, and r is the GALTON function or coefficient of correlation. The question now arises as to what is *practically* the best method of determining r. I do not feel satisfied that the method used by Mr. GALTON and Professor WELDON will give the best results. The problem is similar to that of determining σ for a variation-curve, it may be found from the mean error or the median, but, as we know, the error of mean square gives the theoretically best results.

Let the n pairs of organs be $x_1, y_1, x_2, y_2, x_3, y_3$, &c. . . . then the chance of the observed series for a given value of r varies as

$$\frac{1}{(1-r^2)^{\frac{1}{2}n}} e^{-\frac{1}{2}\left\{\frac{x_1^2}{\sigma_1^2(1-r^2)} - \frac{2x_1y_1r}{\sigma_1\sigma_2(1-r^2)} + \frac{y_1^2}{\sigma_2^2(1-r^2)}\right\}}$$

$$\times e^{-\frac{1}{2}\left\{\frac{x_2^2}{\sigma_1^2(1-r^2)} - \frac{2x_2y_2r}{\sigma_1\sigma_2(1-r^2)} + \frac{y_2^2}{\sigma_2^2(1-r^2)}\right\}}$$

$$\times e^{-\frac{1}{2}\left\{\frac{x_3^2}{\sigma_1^2(1-r^2)} - \frac{2x_3y_3r}{\sigma_1\sigma_2(1-r^2)} + \frac{y_3^2}{\sigma_2^2(1-r^2)}\right\}}$$

$$\times \ldots\ldots,$$

or, S denoting summation, since $\sigma_1^2 = \mathrm{S}(x^2)/n$, $\sigma_2^2 = \mathrm{S}(y^2)/n$, the chance varies as

$$\frac{1}{(1-r^2)^{\frac{1}{2}n}} e^{-n\left\{\frac{1-\lambda r}{1-r^2}\right\}},$$

where λ is written for $\mathrm{S}(xy)/(n\sigma_1\sigma_2)$, and $\mathrm{S}(xy)$ corresponds to the product-moment of dynamics, as $\mathrm{S}(x^2)$ to the moment of inertia.

Now, assume r to differ by ρ from the value previously selected, and expand by TAYLOR'S theorem, after expressing the function, in the following manner :—

$$u_r = \frac{1}{(1-r^2)^{\frac{1}{2}n}} e^{-n\left\{\frac{1-\lambda r}{1-r^2}\right\}} = e^{n\left\{\frac{1}{2}\log(1-r^2) - \frac{1-\lambda r}{1-r^2}\right\}}.$$

We have

$$\frac{1}{n}\log u_{r+\rho} = \frac{1}{n}\log u_r + \frac{(1+r^2)(\lambda-r)}{(1-r^2)^2}\rho + \tfrac{1}{2}\frac{\lambda(2r^3+6r) - 1 - 6r^2 - r^4}{(1-r^2)^3}\rho^2$$

$$+ \tfrac{1}{6}\frac{\lambda(6+36r^2+6r^4) + 4r^5 - 6r^4 - 28r^3 - 18r}{(1-r^2)^4}\rho^3 + \&c.$$

Hence $\log u_r$ and therefore u_r is a maximum when $r = \lambda$, for the coefficient of ρ^2 is then negative. Thus, it appears that the observed result is the most probable, when r is given the value $\mathrm{S}(xy)/(n\sigma_1\sigma_2)$. This value presents no practical difficulty in calculation, and therefore we shall adopt it. It is the value given by BRAVAIS, but he does not show that it is the best.*

(*c.*) *Probable Error of the Correlation Coefficients.*—Assuming that r has this value, we may put $\lambda = r$ in the above result, and we find

$$u_{r+\rho} = u_r e^{-\frac{n(1+r^2)}{(1-r^2)^2}\frac{\rho^2}{2} - \frac{2nr(r^2+3)}{(1-r^2)^3}\frac{\rho^3}{3} - \&c.}$$

Now $u_{r+\rho}$ is the chance of the observed series on the assumption that the coefficient

* It seems desirable to draw special attention to this best value of the correlation coefficient, as it has hitherto been frequently calculated by methods of somewhat arbitrary character, involving only a portion of the observations.

of correlation r is $r + \rho$ instead of r. Hence the above is the law of distribution of variation in the coefficient of correlation. If the second term be negligible as compared with the first, we see that ρ follows the normal law of distribution. Thus we may say that with sufficient accuracy for most cases the standard deviation of a coefficient of correlation is

$$\frac{1 - r^2}{\sqrt{n\,(1 + r^2)}},$$

or its probable error $= \cdot 674506 \dfrac{1 - r^2}{\sqrt{n\,(1 + r^2)}}$.

The ratio of the first term neglected to the term retained

$$= \tfrac{4}{3} \frac{r\,(r^2 + 3)}{(r^2 + 1)\,(1 - r^2)}\,\rho,$$

or to determine the order, giving ρ its probable value on a first approximation, we have

$$\text{ratio} = \tfrac{4}{3} \frac{1}{\sqrt{n}} \frac{r\,(r^2 + 3)}{(r^2 + 1)^{\frac{3}{2}}} \times \cdot 674506.$$

This may be shown to be a maximum for $r^2 = 1$, and the ratio then takes the value $\dfrac{1 \cdot 272}{\sqrt{n}}$, or the second term in this most unfavourable case will only be about 4 per cent. of the first when $n = 1000$. For $r = \cdot 5$, the ratio takes the value $1 \cdot 046/\sqrt{n}$ or for $n = 1000$ is about 3·3 per cent.

It will be sufficient, therefore, for most practical purposes to assume that the probable error of a coefficient of correlation

$$= \cdot 674506 \frac{1 - r^2}{\sqrt{n\,(1 + r^2)}}.$$

(*d.*) *Constancy of Correlation Coefficients for Local Races.*—This result is not only of importance in dealing with the problem of heredity, it is crucial for determining whether constancy of correlation is characteristic of all races of the same species. Mr. GALTON has suggested that the coefficient of correlation might be found to be constant for any pair of organs in different families of the same race. Professor WELDON has determined a series of coefficients of correlation for shrimps and crabs, which he thinks justify him in assuming "as at least an empirical working rule that GALTON'S function has the same value in all local races. The question whether the empirical rule is rigidly true will have to be determined by fuller investigation, based on larger samples."*

* 'Roy. Soc. Proc.,' vol. 54, p. 329, 1893.

Now whether the sample be large enough or not seems to depend on the just determined value of the probable error, and in Professor WELDON'S case the probable error is so small, as compared with the value determined for GALTON'S function, that I think we may safely draw conclusions from his results.

Taking the case of shrimps, we have for the most reliable determination of r, that for total length of carapace and length of post-spinous portion* :—

	n.	r.	$p.e.$ of r.
Plymouth	1000	·81	·0057
Southport	800	·85	·0050

Thus the difference between the r's is not very large, but still between five and six times the probable error (·0075) of their difference.

Taking two cases from Professor WELDON'S results for crabs,† with r's of considerably different order, we have :—

		n.	r.	$p.e.$ of r.
Breadth, frontal, and R. antero-lateral margin	Naples	1000	·29	·0187
	Plymouth	1000	·24	·0196
R. antero-lateral margin, and L. dentary margin	Naples	1000	·60	·0117
	Plymouth	1000	·70	·0089

With these probable errors the identity of the first pair of r's is unlikely ; the identity of the second excessively improbable.

The conclusions therefore to be drawn from our results are these :—The samples taken were sufficiently large to determine r with close practical accuracy. Hence, therefore, unless there were large errors of measurement, or in the determination of r, the evidence of these observations is against the constancy of GALTON'S function for local races of the same species. If the differences in the values of r be attributable not to deviation in the sample from the mean, but to experimental error or to methods of calculation, then it would appear that the methods adopted or the measurements are not sufficiently close to supply an answer to the problem proposed, it being an essential condition of the requisite observations that the experimental, or the arithmetic error shall be less than the probable error of the sample. It seems to me extremely improbable that the divergence should be due to errors of measurement, and Professor WELDON'S papers, I venture to think, illustrate not the constancy of

* 'Roy. Soc. Proc.,' vol. 51, p. 2, 1892.

† 'Roy. Soc. Proc.,' vol. 54, p. 327, 1893.

correlation in species, but the equally interesting point of the extent and manner of its variation in local races.

(5.) *Regression, Uniparental Inheritance, and Assortative Mating.*

(*a.*) *General Formulæ.*—On the basis of the above discussion we can obtain the formulæ requisite for calculating scientific measures of uniparental inheritance and assortative mating.

Let male or female parents solely be kept in view, and let male or female parents be considered which have an organ or measurable characteristic differing h from that of the general population of male or female parents. Then the frequency of a variation x in the same or any other organ of the offspring is given by

$$z = \frac{N}{2\pi\sigma_1\sigma_2}\,\frac{1}{\sqrt{(1-r^2)}}\,e^{-\frac{1}{2}\left\{\frac{x^2}{\sigma_1^2(1-r^2)}-\frac{2xhr}{\sigma_1\sigma_2(1-r^2)}+\frac{h^2}{\sigma_2^2(1-r^2)}\right\}}.$$

The offspring, therefore, have variation following a normal distribution about the mean

$$x_0 = r\,\frac{\sigma_1}{\sigma_2}\,h,$$

and with standard deviation $\sigma_1\sqrt{(1-r^2)}$.

Hence, by our definition, the coefficient of regression $= x_0/h = r\sigma_1/\sigma_2$, and the variability of the offspring of the selected parents is reduced from that of the general population of offspring in the ratio of $\sqrt{(1-r^2)}$ to 1. We thus have a measure of the manner in which selection of parents reduces the variability in offspring, *i.e.*, tends to make the latter closer to a definite type. This result is achieved even with promiscuity in the case of one parent, if there be selection in the case of the other. The greater closeness of approach to type when both parents are selected will be dealt with under biparental inheritance.

We note that the coefficient of regression and the restriction of variability are the same whatever type of parent be adopted, or the closeness with which selection leads to a given type of offspring is independent of the parent adopted and the type of offspring which results from this parent.*

* This is, of course, true of the regression and variability of the array corresponding to any type whatever, when frequency follows the normal law. Mr. G. U. YULE points out to me that if the coefficient of regression be constant for the arrays of all types, then it follows that *whatever be the law of frequency*, the coefficient of regression must $= r\sigma_1/\sigma_2$, where $r = S(xy)/(n\sigma_1\sigma_2)$. This much generalises the formula. At the same time, in the case of skew-correlation, the coefficient of regression usually varies with the type, and the fundamental problem is to determine what function it is of the type. Let bridegrooms of age differing by p years from the mean age of all bridegrooms have an array of brides with a mean age differing q years from the mean age of all brides; then p/q is *not* constant for all values of p.

These results have been reached by Mr. GALTON in his work on 'Natural Inheritance.' He, however, supposes the population to be stable, and makes the mean and variation of successive generations the same, *i.e.*, x_0 is measured from the mean of the general population of parents, and σ_1 taken equal to σ_2. It seems better to keep our formulæ perfectly general, and allow for possible natural selection of the secular kind as well as for possible reproductive selection.

(*b.*) *Special case of Stature in Man.*—In order to get some idea of the nature of direct and cross inheritance, of assortative mating, &c., in man, I have, in conjunction with Professor W. F. R. WELDON, issued a circular and card appealing for help in collecting family measurements. We hope eventually to procure 1000–2000 families with data of height, span, and arm-length, but it may be many months, or even years, before sufficient material has been accumulated to allow of fairly definite statements being made. Meanwhile, Mr. GALTON, with his accustomed generosity, has placed at my disposal the family data on which his work on 'Natural Inheritance' was based. These data contain statistics with regard to one organ, *height*, for about 200 families. The number is not sufficiently great to make the probable error of quite small enough dimensions in several cases, and so allow of definite conclusions. The data do not offer, as those we are collecting, material for the treatment of cross as well as direct inheritance. Nevertheless, the drift of Mr. GALTON's statistics is in many cases obvious enough, and even in other cases, where the weight of the numerical results is not great, the conversion of our formulæ into numbers will still assist the reader to understand their significance, and serve to some extent for comparison when wider series of statistics are forthcoming.* Hence, in the numerical results of this paper, I wish more to draw attention to method than emphasise general laws. Mr. GALTON's families appear to have been drawn from the upper middle classes, and therefore any conclusions formed must not be hastily extended to the whole community.

* Only those who have attempted to get the measurements of, say 20 families, will appreciate the difficulty of the task of completing even 200 for one organ. Parents and children must be alive and fall within suitable limits of age; and what is more, their interest must be aroused.

The following tables give the chief results :—

TABLE I.—Variation.

Class.	Number.	Mean Height in inches.	Probable Error of M.H.	S.D. in inches.	Probable Error of S.D.
Males	683	69·215	·066	2·592	·047
Husbands	200	69·136	·126	2·628	·089
Sons	483	69·247	·081	2·617	·057
Fathers in general . . .	935	69·175	·055	2·501	·039
Fathers of sons	483	69·106	·071	2·325	·050
Fathers of daughters . .	452	69·248	·086	2·731	·061
Females	652	64·043	·061	2·325	·043
Wives	200	63·859	·110	2·303	·078
Daughters	452	64·118	·075	2·347	·053
Mothers in general . . .	935	64·099	·051	2·308	·036
Mothers of sons	483	64·054	·072	2·334	·051
Mothers of daughters . .	452	64·147	·072	2·274	·051

TABLE II.—Correlation.

Class.	Coefficient r.	Probable Error of r.
Husbands and wives . . .	·0931	·0473
Fathers and sons	·3959	·0241
Fathers and daughters . . .	·3603	·0260
Mothers and sons	·3018	·0267
Mothers and daughters . . .	·2841	·0281

TABLE III.—Regression.

Class.	Coefficient of Regression.
Assortative Mating :—	
Husbands on wives	·1062
Wives on husbands	·0816
Inheritance :—	
Fathers on sons	·3517
Sons on fathers	·4456
Fathers on daughters	·4192
Daughters on fathers	·3096
Mothers on sons	·2692
Sons on mothers	·3384
Mothers on Daughters	·2753
Daughters on mothers	·2932

TABLE IV.—Variation in Selected Groups.

Class of Selected.	S.D. in inches.	S.D. in inches.	Unselected.
Matage:—			
Wives of selected husbands	2·293	2·303	All wives
Husbands of selected wives	2·617	2·628	All husbands
Parentage:—			
Fathers of selected sons	2·135	2·325	All fathers of sons
Fathers of selected daughters . . .	2·548	2·731	All fathers of daughters
Mothers of selected sons	2·225	2·334	All mothers of sons
Mothers of selected daughters . . .	2·180	2·274	All mothers of daughters
Fraternity:—			
Sons of selected fathers	2·403	2·617	All sons
Daughters of selected fathers . . .	2·189	2·347	All daughters
Sons of selected mothers	2·495	2·617	All sons
Daughters of selected mothers . . .	2·250	2·347	All daughters

TABLE V.—Sexual Ratio.

Class.	Ratio of Means.	Ratio of S. D.'s.	Ratio of V.'s.*
Husbands to wives	1·082	1·141	1·055
Males to females	1·081	1·115	1·032
Sons to daughters	1·080	1·115	1·032
Fathers to mothers	1·079	1·084	1·005

* V = the "coefficient of variation" or percentage of variation in organ = 100 S. D. ÷ (mean). See below.

N.B.—Mr. GALTON excluded from his calculations the larger families, but it seems to me that large families form an essential feature of the community. Two brothers are more likely to be two brothers of a large than of a small family, and, accordingly, large families ought to be given their proportionate weight. The whole problem, indeed, of reproductive selection turns upon the inclusion of large families.

Explanation of the Tables.—These tables were calculated in the following manner: Table I. A father or mother appears once for each child in this Table. The mean heights of each group were then calculated, as well as their standard deviations (S.D.) or deviations of mean square. The probable errors of the means and standard deviations were then found by means of the formulæ

$$p.e. \text{ of M.H.} = \cdot 674506 \times \text{S.D.}/\sqrt{n},$$
$$p.e. \text{ of S.D.} = \cdot 674506 \times \text{S.D.}/\sqrt{2n},$$

where n is the number of cases recorded in the second column of the Table.

To obtain Table II., tables of double entry* were formed for the class enumerated in the first column, *e.g.*, height of husband and height of wife as the variables x and y, and frequency of each pair of heights as z. From this table S (xy) was calculated by very laborious but straightforward arithmetic. This product moment was reduced to parallel axes (x', y') through the centroid of the system and r determined from the formula $r = \text{S}(x'y')/n\sigma_1\sigma_2$ (see p. 265). The *p.e.* of r was then found from the formula on p. 266.

The coefficients of regression, in Table III., have the value $r\sigma_1/\sigma_2$, given on p. 267, where, if σ_1 be the standard deviation of A, and σ_2 of B, $r\sigma_1/\sigma_2$ is the regression of an A array on a B type, and $r\sigma_2/\sigma_1$, the regression of a B array on an A type.

In Table IV., the array is first stated and then the type; *e.g.*, in the first line the type is the husband of given height, the array the distribution of all wives of husbands of this height. The first S.D. is that of the array obtained from the formula S.D. $= \sigma_1\sqrt{1-r^2}$, of p. 267, σ_1 being the second S.D. of Table IV., or the S.D. of the whole group from which the array has been extracted by selecting a particular value of the correlated group.

Table V. gives the ratio for corresponding groups of the two sexes of the constants given in Table I.

Now, a consideration of the probable errors recorded in Tables I. and II. shows us that, in several cases, definite conclusions may be drawn, and in certain other cases very probable conclusions. In particular, the probable errors of the correlation coefficients of inheritance are sufficiently small to show that these coefficients give the chief features of heredity in the group and for the characteristic we are dealing with. We may note one or two special features.

(i.) *Natural Selection.*—We are dealing with two adult populations, and, therefore, should only expect to find traces of secular natural selection. The data, however, are not suited, either by their nature or number, to illustrate this point. There is a slight increase in height of sons over height of husbands, and a larger increase in height of daughters over height of mothers. Neither can be definitely asserted to be significant. Even if they were significant they might be accounted for by (a) shrinkage due to old age,† and (b) increased physical activity and exercise in the middle classes of the younger generation, especially daughters. If we turn from means to S.D.'s we see again an insignificant change in the range of variation of husbands and sons, the sons being slightly less variable than fathers. This result, were it necessary to account for it, would be more likely due to our having taken sons from a less general population than husbands—a point to be borne in mind

* It did not seem necessary to publish these tables, but the corresponding tables will be published when the fuller data for heredity in man, which I am at present collecting, are complete.

† In my own collection of data, several parents state that they are now shorter than they used to be. The shrinkage in the case of fathers of sons cannot be great in Mr. GALTON's statistics, to judge by the means, unless we suppose a sensible regression in sons' stature.

when statistics of this kind are collected, and more than one son in a family is included. There is a more significant difference in the variation of wives and daughters. It is, however, in the opposite sense to what we may suppose would be produced by natural selection, or by the fact that we have drawn daughters from a less general population than wives. There is no definite evidence as to natural selection to be drawn from these results accordingly.

(ii.) *Sexual Selection.*—(*a.*) *Preferential Mating.*—We have no general populations to compare with those of husbands and wives. If we suppose the population stable, and treat sons and daughters as characteristic of the general unmarried population, husbands are not a significant selection from sons. Possibly the difference between the variation in daughters and wives might be accounted for by a distaste for very tall or very short wives in the middle classes. The difference is, however, not very significant, but it should be borne in mind in dealing with a larger range of statistics.

(*b.*) *Assortative Mating.*—Although the probable error (Table II.) is about half the coefficient of correlation, it is unlikely that the latter can be really zero, and although we must not lay very great stress on the actual value of r, still we are justified in considering that there is a definite amount of assortative mating with regard to height going on in the middle classes. It may be expressed by saying that wives 1″ taller than the mean will have on an average husbands ·11″ taller than the mean, and husbands 1″ taller than the mean, wives on an average ·08″ taller than the mean (Table III.). Table IV. shows us that the variation in matages would hardly be discoverable directly from our present range of statistics.*

(iii.) *Reproductive Selection.*—Although in the matter of means we cannot assert significance between the heights of males in general and fathers in particular, it is quite possible that such will reveal itself in more ample data. On the other hand, we see at once that fathers are definitely less variable than husbands, and fathers of sons remarkably less variable than fathers of daughters. Thus, while the height of a father is less closely related to his chances of having a daughter, any tendency to normality is of service in the chances of having a son. Reproductivity in males seems to be thus essentially correlated to height, and again, height to be potential in the question of male or female offspring.

An endeavour to directly calculate the correlation of reproductivity and height is

* Of course 200 couples give graphically nothing like a surface of correlation, nor can any section of it be taken as a fair normal curve. We assume *à priori* that 1000 couples would give a fair surface. This is practically what I have found for skull-measurements, 900 give an excellent curve, 50 a doubly, or even trebly, peaked polygon. None the less, sets of 50 skulls give means and S.D.'s in close accord. For example, in Professor FLINDERS PETRIE's newly discovered race, 50 male crania from T. and Q. graves give for cephalic index: Mean, 72·96, S.D., 2·82; while 53 male crania from General and B. graves give: Mean, 72·92, S.D., 2·95. The 103 crania together give: Mean, 72·938, S.D., 2·885, with a probable error of S.D. = ·29. The variation *curves* would not suggest any such close agreement at all. The constants, however, suffice to show the homogeneous character of the two sets of excavations.

frustrated by the obvious fact that size of adult families does not follow any approach to a normal distribution. Thus, I find in 205 adult families the following frequency:—

Title.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(1) Sons and daughters	..	32	22	23	31	23	19	18	18	8	5	5	..	..	..	1
(2) Sons only . .	25	43	46	42	30	10	5	3	..	..	1					
(3) Daughters only	25	56	44	34	21	13	5	3	3	1						

This Table shows the number of those families in which (1) the number of sons and daughters, (2) the number of sons only, (3) the number of daughters only correspond, with the title in the top line.

Now, although, as I propose to show later, the quantity, $r = \mathrm{S}\,(xy)/(n\sigma_1\sigma_2)$, is really a significant characteristic of correlation, just as σ_1 and σ_2 are significant for variation even in the case of skew frequency, still there is little to be gained by working it out in this particular case, where, the statistics being insufficient to accurately determine the skew law of frequency, we shall not be able to find what we want—the law of regression.*

But several points as to paternal reproductivity may be learnt from these families. In the first place, of the 25 families with no sons, the father in 5 cases only was below the mean, in 20 cases above the mean height. The mean height of fathers in general is 5′ 9″·17, but of sonless fathers is 5′ 11″·03. Of the 25 daughterless fathers, 14 are below and 11 above the mean height; the mean height of the daughterless father being 5′ 8″·71. Or, the same point may be emphasised in this way : If short fathers be taken as those below 5′ 6″·5, and tall fathers as those above 5′ 11″·5, short fathers have 65 sons and tall fathers 67 sons. We should accordingly, with our proportion of sons and daughters, expect 61 daughters to short fathers and 63 to tall fathers, but we find short fathers with 73 and tall fathers with 81. This point in reproductive selection, that mediocre fathers have more tendency to sons and exceptional fathers to daughters, seems of considerable importance in relation to the prepotency of paternal inheritance. A similar point, but less emphatically significant, may be noted in the case of mothers. Mothers of daughters are less variable than mothers of sons. Without laying too great stress on statistics of so small a range and of one characteristic only, we may still suggest that it might be worth while to investigate whether the offspring of a mediocre parent and an abnormal parent do not tend to follow the sex of the mediocre parent.

* Much more complete statistics of size in families have recently been sent to me by Mr. F. Howard Collins. They give a remarkably smooth skew frequency distribution, thus demonstrating the need of the theory of skew correlation when we are dealing with reproductive selection. I propose to illustrate this in a memoir on skew correlation.

Finally, it is impossible to more than *hazard* suggestions as to reproductive selection in relation to mothers' height. It will be noticed that both mothers of sons and mothers of daughters are taller than wives, and, further, daughters, while taller than wives, are not so tall as mothers of daughters. Hence, while the difference in height of daughters and wives might be due to natural selection or improved physical training, it might also be accounted for by greater reproductivity as to daughters in tall women, *i.e.*, mothers of daughters taller than wives, and this tallness being transmitted in a lesser extent to daughters. This would be a case of secular change due to reproductive selection. The statistics are, however, too few to make the differences in the mean heights of wives, daughters and mothers, very definitely significant.

(iv.) *Inheritance.*—Mr. GALTON has concluded from his data that the coefficient of regression is ·3333 from father to son or from son to father, and by the assumption of the "midparent" has practically given the mother an equal prepotency with the father in heredity. The fuller theory developed in this paper does not seem in entire agreement with these conclusions. In the first place, the theory of uni-parental inheritance shows us that it is not the constancy of variation in two successive generations with which we have to deal, but the question whether sons have the same degree of variability as the "fathers of sons," and this must be definitely answered in the negative. Table II. shows us that there are undoubtedly significant differences in the coefficients of correlation, which may be summed up in the words *prepotency in heredity of the father.* It must be remembered that this is only for one characteristic, *height*, but in this characteristic both sons and daughters, on the average, take very considerably more after their father than after their mother. Turning to Table V., we see that the ratio of the mean heights of the two sexes, considered in three different classes, is practically the same, *i.e.*, 1·08, or 13 to 12, as Mr. GALTON has expressed it. Now, in Table III. we see that the coefficients of regression in paternal inheritance are not only sensibly greater than those of maternal inheritance, but, as these coefficients have to be multiplied by the *absolute* deviations of father or mother from their means to obtain the absolute deviations of offspring, and as these absolute deviations will be in the ratio of 13 to 12, there is a considerable further reduction to be made in comparing the strength of maternal with that of paternal heredity.

Thus it may be said that paternal heredity is to maternal heredity, in the case of sons, as ·4456 to $\cdot 3384 \times \frac{12}{13}$ or to ·3124, and in the case of daughters, $\cdot 3096 \times \frac{13}{12}$ or ·3354 to ·2932. Thus, while daughters inherit less from both their parents on the average than sons, both—and sons especially—take more after their father than their mother. The inferior inheritance of daughters may, to some extent, be counterbalanced by the law already noticed, that exceptional fathers have more often daughters than sons.

We may illustrate this by two examples—the regression of grandson on grandfather, and of great-grandson on great-grandfather when the inheritance is respectively through the male and female lines.

	Male line.	Female line.
Grandson on grandfather	·1986	·0885
Great-grandson on great-grandfather	·1048	·0307

In the first case, the strength of inheritance is more than double through the male; in the second, more than triple through the male what it is through the female line. Were this law of inheritance true, not only of stature, but of other physical, and especially of mental characteristics, some justification might be found for confining hereditary peerages initially given for merit to the male line. Meanwhile, it cannot be too strongly emphasized that the present results apply only to one organ, are based on comparatively few families drawn from a special class of the community, and thus stand in need of careful criticism in the light of ampler statistical material.

Another point already briefly referred to, which seems of significance, is the inequality of regression in the case of ascent and descent in the direct line. It may seem paradoxical to assert that sons are more like fathers than fathers are like sons, but the solution is bound up in the statement that fathers of sons are less variable than sons, or, in another form, that every son is not to the same degree a potential father. Similarly, the opposite paradox that fathers are, on the average, more like their daughters than daughters are like their fathers, finds its solution in the relatively great variability of fathers of daughters.

In Table IV. are tabulated alongside, in each case, the standard deviation for the corresponding general population, the standard deviations for inheritance from selected classes. Here again we see a general law for height, which deserves to be investigated for other organs, and for a variety of animals, namely, we breed "truer to the type," have less variability in offspring, if we breed from selected males rather than from selected females. We shall see later the effect of selecting both parents.

(*c*.) *On Further Relations between Correlation, Regression, and Variability.* (i.) *The Coefficient of Variation V.*—In dealing with the comparative variation of men and women (or, indeed, very often of the two sexes of any animal), we have constantly to bear in mind that relative size influences not only the means but the deviations from the means. When dealing with absolute measurements, it is, of course, idle to compare the variation of the larger male organ directly with the variation of the smaller female organ. The same remark applies also to the comparison of large and small built races.

If the absolute measurements* have in the case of man to be on the average altered in the ratio of 13 to 12 to obtain those of the woman, if Mr. GALTON has gone so far as to replace any woman by an equivalent man on this basis, then, clearly, to compare

* The ratio 13 to 12 is not only true of stature, but approximately of several other organs, weight, brain-capacity, &c., &c.

deviations in man and woman, we must alter the deviations in the same ratio. Freeing ourselves from this particular ratio, we may take as a measure of variation the ratio of standard deviation to mean, or what is more convenient, this quantity multiplied by 100. We shall, accordingly, define V, the coefficient of variation, as the percentage variation in the mean, the standard deviation being treated as the total variation in the mean; since the p.e. = ·674,506 × S.D., V multiplied by ·674,506 may be called the "probable percentage variation." Of course, it does not follow because we have defined in this manner our "coefficient of variation," that this coefficient is really a significant quantity in the comparison of various races; it may be only a convenient mathematical expression, but I believe there is evidence to show that it is a more reliable test of "efficiency" in a race* than absolute variation. At present, however, we will merely adopt it as a convenient expression for a certain function, and proceed to examine its relation to correlation.

Let m_1, m_2 be the means of two correlated organs; σ_1, σ_2 their standard deviations; r their coefficient of correlation; V_1, V_2 their coefficients of variation; and R_1, R_2 the respective regressions for deviations d_2 and d_1 of the two organs.

Now

$$R_1 = r\frac{\sigma_1}{\sigma_2}d_2 = r\frac{V_1}{V_2} \times \frac{m_1 d_2}{m_2},$$

or

$$\frac{R_1}{m_1} = r\frac{V_1}{V_2} \times \frac{d_2}{m_2},$$

and similarly

$$\frac{R_2}{m_2} = r\frac{V_2}{V_1} \times \frac{d_1}{m_1}.$$

But we see that the amounts d_2/m_2 and d_1/m_1 are equally significant deviations in the case of the second and first organ, while the amounts R_1/m_1 and R_2/m_2 are equally significant regressions in the case of the first and second organ.†

It follows, therefore, that *the significances of the mutual regressions of the two organs are as the squares of their coefficients of variation.*

Hence inequality of coefficients of variation marks inequality of mutual regressions. Now coefficients of variation are rarely, if ever, equal for the same organ in corresponding classes of men and women. In dealing with male and female skull measurements for a great variety of races, this inequality is often very marked, and, therefore, differences of variation tell, especially in mutual regression in the case of sexual selection and inheritance from the opposite sex. They are sufficient, I think, to preclude Mr. GALTON's theory of the mid-parent from being considered as more than a

* By "race efficiency," I would denote stability, combined with capacity to play a part in the history of civilization. I hope later to publish details of variation, especially in skull measurements of different races of man, the data of which I have been for some years reducing.

† For example, 1'' and $\frac{12}{13}$'' I term equally significant deviations or regressions in the stature of man and woman, and 1'' and $\frac{13}{12}$'' in the stature of woman and man.

first approximation. Turning to Table V., we see that variation in height is greater for males than females; but while very sensible for husbands and wives, and sons and daughters, it is insignificant for fathers and mothers. This superiority of male to female variation, as measured by the coefficient of variation, is in accordance with the usual belief that the male is more variable than the female, but it is entirely out of accordance with the great bulk of the statistics I have so far reduced. The belief seems to have arisen from a very loose notion of how variation is to be estimated. These stature statistics of the English middle classes seem to some extent anomalous. For example, I find from statistics of stature in the German working classes:—

Male coefficient of variation . . . = 4·0245,
Female ,, ,, . . . = 4·2582.

Ratio of female to male coefficient = 1·058, thus more than reversing the highest English ratio, that of husbands and wives. It is noteworthy that, while the variation is thus reversed, the ratio of the mean heights equals 1·078, and remains practically the same. These remarks are introduced in order to prevent any too hasty generalisation as to the nature of male and female correlation based on a current belief in the greater intensity of male variation.

(ii.) *Coefficient of Correlation and Coefficients of Variation.*—Let x and y be two correlated organs, and let ξ and η be corresponding deviations from the mean values m_1 and m_2. We shall suppose that ζ and η are so small that the squares of the ratios ξ/m_1 and η/m_2 are negligible as compared with the first powers. Let r be the coefficient of correlation of x and y, σ_1, σ_2 their standard deviations, v_1, v_2 their coefficients of variation, and let z be any function $f(x, y)$ of x and y with a deviation ζ, corresponding to ξ and η, and a standard deviation, mean and coefficient of variation respectively Σ, M, and V.

Differentiating $z = f(x, y)$ and remembering our hypothesis as to the smallness of the variations, we have:

$$\zeta = f_x \xi + f_y \eta.$$

Squaring:

$$\zeta^2 = f_x^2 \xi^2 + f_y^2 \eta^2 + 2 f_x f_y \xi\eta.$$

Summing for every possible value of ξ and η, and dividing by n the total number of correlated pairs:

$$\frac{S(\zeta^2)}{n} = f_x^2 \frac{S(\xi^2)}{n} + f_y^2 \frac{S(\eta^2)}{n} + 2 f_x f_y \frac{S(\xi\eta)}{n},$$

or,

$$\Sigma^2 = f_x^2 \sigma_1^2 + f_y^2 \sigma_2^2 + 2 f_x f_y \sigma_1 \sigma_2 \times \frac{S(\xi\eta)}{n\sigma_1\sigma_2}.$$

Now, if there were no correlation, we should have: $\Sigma^2 = f_x^2 \sigma_1^2 + f_y^2 \sigma_2^2$; hence any law of frequency whatever which causes $S(\xi\eta) = 0$,—for example, if it be equally likely that η occurs with an equal negative or positive value of ξ,—will show that x and y are independent variations. Hence, if we define $r = S(\xi\eta)/(n\sigma_1\sigma_2)$ as the coefficient of correlation, we see that it has a significance extending much further than the normal law of error. Just as σ_1, σ_2 are radii of gyration (and independent of any special law of error), so $S(\xi\eta)$ is a product moment, and its vanishing marks the absence of correlation, or directions of independent variation.

We see, then, that the coefficient of correlation may be found from

$$r = \frac{\Sigma^2 - f_x^2\sigma_1^2 - f_y^2\sigma_2^2}{2f_xf_y\sigma_1\sigma_2},$$

or by calculating standard deviations.

The question naturally arises as to what is the best value of $f(x, y)$. This will often be already answered by the data themselves. A common case is that in which the variations in x and y are given, and the variation in their ratio or the index x/y is calculated. In this particular instance $f_x = M/m_1$ and $f_y = -M/m_2$. Hence

$$r = \frac{v_1^2 + v_2^2 - V^2}{2v_1v_2}$$

We thus throw back the determination of correlation on ascertaining three coefficients of variation.

This formula, while less general than the one previously given, in that we have neglected squares of small quantities, is more general in that we have not limited ourselves to any special law of frequency.

(iii.) *Example.*—The formula may be illustrated by the following statistics taken from a not yet published paper on variation in man. r = coefficient of correlation between length and breadth.

Adult Male Crania.

Professor Flinders Petrie's newly discovered race.*

Length of skull	$m_1 = 185{\cdot}2777$,	$\sigma_1 = 5{\cdot}7783$,	$v_1 = 3{\cdot}1187$
Breadth of skull	$m_2 = 135{\cdot}0194$,	$\sigma_2 = 4{\cdot}4076$,	$v_2 = 3{\cdot}4183$
Cephalic index, B/L .	$M = 72{\cdot}9379$,	$\Sigma = 2{\cdot}8848$,	$V = 3{\cdot}9551$

$r = {\cdot}2705.$

* Professor Flinders Petrie kindly replied to my request for 100 skulls of a homogeneous race, 3,000 to 4,000 years old, by bringing back to England the finest anthropological collection—skeletons as well as crania—known to me. The collection was packed and brought to England at the charge of Mr. A. B. Pearson-Gee. Mr. Herbert Thompson has made a series of measurements on 301 skulls, ♂ and ♀, details of which will be published later, and the above constants are calculated from his measurements. The date of the new race is about 3000 B.C.

MODERN German (Bavarian Peasants*).

Length of skull . . .	$m_1 = 180{\cdot}58$,	$\sigma_1 = 5{\cdot}8441$,	$v_1 = 3{\cdot}2363$
Breadth of skull . . .	$m_2 = 150{\cdot}47$,	$\sigma_2 = 5{\cdot}8488$,	$v_2 = 3{\cdot}8871$
Cephalic index, B/L .	$M = 83{\cdot}41$,	$\Sigma = 3{\cdot}5794$,	$V = 4{\cdot}2913$

$r = {\cdot}2849$.

MODERN French (Parisians†).

Length of skull . . .	$m_1 = 181{\cdot}85$,	$\sigma_1 = 5{\cdot}9420$,	$v_1 = 3{\cdot}2675$
Breadth of skull . . .	$m_2 = 144{\cdot}93$,	$\sigma_2 = 5{\cdot}2139$,	$v_2 = 3{\cdot}5975$
Cephalic index, B/L .	$M = 79{\cdot}82$,	$\Sigma = 3{\cdot}7865$,	$V = 4{\cdot}7438$

$r = {\cdot}0474$.

The probable error of r in all three cases lies between ·06 and ·07. Now it is clear that had we only dealt with the race from Egypt and the Bavarians, we might easily have concluded that the coefficient of correlation was constant for local races of man, and had remained so for nearly 5,000 years. The French numbers completely upset this view. In order to test my French results I give another series from the Anthropological Collection at Munich; the skulls are those of French soldiers who died at Munich during the Franco-German war.

MODERN French (Peasants).

Length of skull . . .	$m_1 = 179{\cdot}93$,	$\sigma_1 = 6{\cdot}2987$,	$v_1 = 3{\cdot}5006$
Breadth of skull . . .	$m_2 = 143{\cdot}51$,	$\sigma_2 = 5{\cdot}4208$,	$v_2 = 3{\cdot}7772$
Cephalic index, B/L .	$M = 79{\cdot}7857$;	$\Sigma = 3{\cdot}8410$,	$V = 4{\cdot}8141$

$r = {\cdot}1265$.

This collection numbers only 57 crania, and the probable error of r is about ·09, but clearly we have the same general features as in the previous French series. In particular the closeness in the line for the cephalic index constants is remarkable. The value of r might possibly be the same as for the Parisians; it is highly improbable that it should agree with the value of r for the Germans or the race from Egypt. We are compelled to conclude, therefore, that it is very unlikely that "GALTON'S function" is constant for all local races of man.

* Calculated from measurements given by Professor J. RANKE: 'Beiträge zur physischen Anthropologie der Bayern,' Bd. I, S. 88, Kapitel VI., I may take this opportunity of acknowledging the extreme kindness of Professor RANKE in helping me in a variety of ways.

† Calculated from measurements extracted from the manuscripts of M. PAUL BROCA, which I owe to the courtesy of M. MANOUVRIER. He has responded to my request by forwarding to me copies of a great variety of measurements, which will be largely used in a paper on variation in man.

An examination of the above numbers brings out a fact which I am not sure has been noted before: namely, the alteration from dolicocephaly to brachycephaly appears to chiefly depend upon an alteration in the breadth and not in the length of the skull. We see too that, if variation be judged, not by standard-deviations, but by the coefficients of variation advocated in this paper, the breadth of skull is in all cases a sensibly more variable quantity than the length, and, further—a point to which I shall return on another occasion—that the more civilised races are the more variable. Both of these results have, I believe, very important bearing on the mathematico-statistical theory of evolution. On the present occasion the above example is only given to illustrate the relation of variation to correlation.

(6.) *Collateral Heredity.*

(*a.*) *Stature in Man.*—The whole theory of correlation as applied to uniparental inheritance may be at once applied to correlation between brothers, sisters and brothers and sisters. To illustrate the theory I give the following tables, again based on Mr. GALTON'S statistics.

In the pairs sister-sister and brother-brother the elder sister or the elder brother has been taken first in order to ascertain the effect of earlier birth on correlation. In the pairs sister-brother, I had no data as to relative age.

TABLE VI.—Variation.

Class.	Number.	Mean Height in inches.	Probable error of M.H.	S.D. in inches.	Probable error of S.D.
Elder sisters of sisters	595	63·869	·0617	2·2303	·0436
Younger sisters of sisters . . .	595	64·199	·0695	2·5119	·0491
Elder brothers of brothers . .	605	69 0174	·0715	2·6080	·0506
Younger brothers of brothers .	605	69·0814	·0725	2·6434	·0513
Sisters of brothers	1181	63·9274	·0440	2·2430	·0311
Brothers of sisters	1181	69·0963	·0533	2·7164	·0377

TABLE VII.—Correlation.

Class.	Coefficient r.	Probable error of r.
Sister-sister	·4436	·0203
Brother-brother	·3913	·0216
Sister-brother	·3754	·0158

TABLE VIII.—Regression.

Class.	Coefficient of Regression.
Younger sister on elder sister	·4996
Elder sister on younger sister	·3939
Younger brother on elder brother . . .	·3966
Elder brother on younger brother . . .	·3860
Sister on brother.	·3100
Brother on brother	·4547

TABLE IX.—Variation in Selected Groups.

Selected fraternity.	S.D. in inches.	S.D. in inches.	Unselected.
Younger sisters of selected elder .	2·2512	2·5119	All younger sisters of sisters
Elder sisters of selected younger .	1·9989	2·2303	All elder sisters of sisters
Younger brothers of selected elder .	2·4327	2 6434	All younger brothers of brothers
Elder brothers of selected younger .	2·4000	2·6080	All elder brothers of brothers
Sisters of selected brother . . .	2·0789	2·2430	All sisters of brothers
Brothers of selected sister	2·5178	2·7164	All brothers of sisters

These tables have been calculated in precisely the same manner as the previous series.

(*b*.) *Conclusions.*—Now these results seem at several points of very great suggestiveness. In the first place, with regard to variation, we see that elder sisters are significantly more mediocre than younger sisters ; younger sisters are taller and more variable. The same difference appears in the case of elder and younger brothers, but the probable errors do not allow us in this case to assert that the difference is certainly significant. To illustrate this conclusion we give the constants for pairs of sisters, no respect being paid to relative age.

TABLE X.—Sisters of Sisters.

Mean height	64·0454
Probable error of mean height	·0655
S.D.	2·3668
Probable error of S.D.	·0463
Coefficient of correlation r	·4386
Probable error of r	·0205
Coefficient of regression	·4386
S.D. of array of sisters of selected sister	2·1270

It will be seen from this table that elder and younger sisters of sisters are respectively less and more variable than sisters of sisters in general. It will be noted also that sisters of brothers are, both in stature and variation, nearer akin to elder sisters of sisters than to younger sisters. It deserves accordingly to be investigated whether or not sisters are not on the average older than brothers—on this point I have no data. As sisters of brothers approximate to elder sisters of sisters, so brothers of sisters correspond more closely to younger than to elder brothers of brothers. These are points which require fuller investigation, when ampler statistics are forthcoming. Turning to correlation we note that the coefficients in the case of collateral inheritance are slightly greater than in the case of direct inheritance. It will be remarked at once that the values are much less than those given by Mr. GALTON, "Natural Inheritance," p. 133, who has himself drawn attention to the considerable difference between the constants for collateral inheritance given by his R.F.F. Data and by his Special Data. Mr. GALTON having kindly allowed me to use his data, I have recalculated from the formula $r = \mathrm{S}(xy)/(n\sigma_1\sigma_2)$ the value of r for the Special Data, taking my pairs of brothers precisely as I had done for the Records of Family Faculties. I find $r = {\cdot}5990$ with a probable error of $\pm{\cdot}0124$. This value is not as high as Mr. GALTON'S, but differs very widely from the value ·3913 given above.

In making the calculations, however, I was much struck by the peculiarities presented by a certain portion of the data, which I will speak of as the Essex contribution. The brothers therein were very short and remarkably close together. I therefore went through the calculations again, separating the Essex contribution, and with the following results:—

MR. GALTON'S Special Data.

	Whole population.	Essex contribution.	Remainder.
Mean height	68·544	67·797	68·797
Probable error	·0402	·1013	·0457
r for brothers	·5990	·7175	·5574
Probable error of r	·0124	·0200	·0152

Now the probable error of the difference of the Essex contribution and the remainder is ·1111″ for height and ·0251 for correlation. Thus difference in height is nine times, and the difference in correlation more than six times the corresponding probable error. It is absolutely necessary therefore to conclude that the Essex contribution differs significantly from the remainder of the data. Now the Essex contribution appears to be drawn from brothers in a volunteer regiment, and I am inclined to think there may be two sources accounting for its peculiarities, (*a*) unconscious selection as to height by those who join the volunteers, (*b*) a greater correlation among the agricultural and working classes than among the middle classes. At any rate the great variation within the family to be found in the R.F.F. data does not repeat itself either in the Essex contribution or in other portions of the special data, which appear also to be drawn from military and working class sources.

I would accordingly suggest that the R.F.F. data and the Special data give different results, because the latter are largely drawn from a different class of the population from the former (and possibly in the case of volunteer regiments by a method which itself tends to emphasise correlation). I should expect that the influence of natural selection is far greater—witness the greater infantile mortality—in the working classes, and that accordingly we should find the variation in a fraternity sensibly less, or the correlation much greater. I believe, then, that difference of variation in different classes of the community will ultimately be found to account for part, if not all, of the difference between the two values given for the correlation of brothers by the Special data and by the R.F.F. data.

Considering the amount by which the elimination of a portion only of the heterogeneity of the Special data reduces r, it does not seem likely that the R.F.F. data are so wide of the mark in the correlation values as might at first be thought. I doubt whether the correlation coefficients for collateral inheritance—at any rate in the middle classes—can be greater than ·5. I have not at present sufficient data of my own to make a trustworthy determination of brother-brother correlation, but I was able to find the correlation of 237 brother-sister pairs from about 160 families. The measurements were taken without boots, and give values for the mean heights of brothers and sisters sensibly over 69″ and 64″ respectively. The families were all middle-class families—mostly those of male and female college students. They thus approximate to Mr. GALTON'S R.F.F. series. The result was

$$r = \cdot 4703 \pm \cdot 0308$$

The previous result was

$$r = \cdot 3754 \pm \cdot 0158$$

The probable error of the difference therefore $= \cdot 0346$ and the difference ·095, between two and three times the probable error. The two differ, of course, considerably,* but they are nearer together than to Mr. GALTON'S ·67, and being entirely independent series, may be taken to justify the statement made above that the coefficient for the middle classes can hardly exceed ·5. Thus there is not, I think, sufficient ground at present for forming any definite conclusion as to the manner in which lineal is related to collateral heredity. It does not seem to me necessary that the coefficient for the former should be half that for the latter, as supposed by Mr. GALTON.

In some respects, indeed, the Special data verify the conclusions we may draw from the R.F.F. data. Thus R.F.F., Special data, and the two components into which I have divided the latter, all four agree in making the younger brother taller than the elder brother. The variability of both brothers is practically equal in the Special data and slightly greater than that of the R.F.F. data—2·656 as compared with 2·626 —a difference not significant, and which, if it were, might be put down to the mixture of classes in the Special data.

Assuming that the regression coefficients in Table VIII. give the relative if not the absolute values for collateral inheritance, we draw from them a few suggestions for further inquiry when the statistics are forthcoming. In the first place, sisters are more like each other than brothers. At any rate, the younger sister is more like the elder sister than brother is like brother. If this appears to contradict the principle that sons are more like their parents than daughters, a solution of the paradox must be sought in the relative variabilities of daughters, elder sisters, and younger sisters. To compare the strength of inheritance in brothers and sisters, we have to consider not ·3100 and ·4547, but these coefficients of regression multiplied and divided respectively by 13/12, or ·3358 and ·4197, whence we see that the brother takes more after the sister than the sister after the brother.

It will be wise, however, to lay no great stress on these results, until a wider series of statistics has been collected.

The following example must be taken only as the roughest approximation, but so far as it goes as confirming the above results.

An exceptional grandmother in Baden† had a length-breadth head index of 90, her 20 grandchildren had a mean head index of 83·55, with a S.D. = 3·025. The mean head index of the general population‡ was 83·15 with S.D. = 3·63. Thus, if r_1 be the regression of offspring on parent, and r_2 of offspring on each other, $r_1^2 \times 6·85 = ·4$, and $\sqrt{(1 - r_2^2)} = 3·025/3·63$.

Hence, $r_1 = ·24$ and $r_2 = ·55$. Considering the large probable error of the S.D. of the fraternity (·32), these results indicate inheritance in head indices of the same order as in stature.

* The difference is to be expected. Mr. GALTON'S R.F.F. series allows for due weight being given to the variability in large families. My statistics take only four members at a maximum, and frequently only two out of each family.

† O. AMMON, 'Die natürliche Auslese beim Menschen,' p. 13. Three children were unmeasured, and I have accordingly had to disregard this generation.

‡ Calculated from results for 6748, BADENSER, given by AMMON, p. 67.

(7.) *Special Case of Three Correlated Organs.*

We need not stay long over the general theory as it has been fully treated by BRAVAIS. We indicate its general outline in a modified form. By p. 263 we have, if x, y, z be the deviations from the means of the three organs, and σ_1, σ_2, σ_3 their standard deviations,

$$P = Ce^{-\frac{1}{2}\left(\lambda_1 \frac{x^2}{\sigma_1^2} + \lambda_2 \frac{y^2}{\sigma_2^2} + \lambda_3 \frac{z^2}{\sigma_3^2} - \frac{2yz}{\sigma_2\sigma_3}\nu_1 - \frac{2zx}{\sigma_3\sigma_1}\nu_2 - \frac{2xy}{\sigma_1\sigma_3}\nu_3\right)} dx\, dy\, dz.$$

This may be written in either the form,

$$P = Ce^{-\frac{1}{2}\lambda_1\left(\frac{x}{\sigma_1} - \frac{\nu_3 y}{\lambda_1\sigma_2} - \frac{\nu_2}{\lambda_1}\frac{z}{\sigma_3}\right)^2} \times e^{-\frac{\lambda_2\lambda_1 - \nu_3^2}{2\lambda_1}\left(\frac{y}{\sigma_2} - \frac{z}{\sigma_3}\frac{\nu_1\lambda_1 + \nu_2\nu_3}{\lambda_2\lambda_1 - \nu_3^2}\right)^2}$$

$$\times e^{-\frac{\lambda_1\lambda_2\lambda_3 - 2\nu_1\nu_2\nu_3 - \lambda_1\nu_1^2 - \lambda_2\nu_2^2 - \lambda_3\nu_3^2}{2(\lambda_1\lambda_2 - \nu_3^2)}\frac{z^2}{\sigma_3^2}} dx\, dy\, dz \quad . \qquad . \quad \text{(A)},$$

or,

$$P = Ce^{-\frac{1}{2}\lambda_1\left(\frac{x}{\sigma_1} - \frac{\nu_3}{\lambda_1}\frac{y}{\sigma_2} - \frac{\nu_2}{\lambda_1}\frac{z}{\sigma_3}\right)^2}$$

$$\times e^{-\frac{1}{2}\left\{\left(\frac{y}{\sigma_2}\right)^2\frac{\lambda_2\lambda_1 - \nu_3^2}{\lambda_1} + \left(\frac{z}{\sigma_3}\right)^2\frac{\lambda_3\lambda_1 - \nu_2^2}{\lambda_1} - \frac{2yz}{\sigma_2\sigma_3}\frac{\nu_1\lambda_1 + \nu_2\nu_3}{\lambda_1}\right\}} dx\, dy\, dz \quad . \quad . \quad . \quad \text{(B)}.$$

Integrating A for x, y, z successively between $\pm\infty$, we have, if n be the number of correlated triplets, and

$$\chi = \lambda_1\lambda_2\lambda_3 - 2\nu_1\nu_2\nu_3 - \lambda_1\nu_1^2 - \lambda_2\nu_2^2 - \lambda_3\nu_3^2,$$

$$n = C\,.\,(2\pi)^{3/2}\,\sigma_1\sigma_2\sigma_3/\sqrt{\chi},$$

or,

$$C = n\sqrt{\chi}/((2\pi)^{3/2}\,\sigma_1\sigma_2\sigma_3).$$

Integrating B for x between $\pm\infty$, we have

$$P' = C'e^{-\frac{1}{2}\left\{\left(\frac{y}{\sigma_2}\right)^2\frac{\lambda_2\lambda_1 - \nu_3^2}{\lambda_1} + \left(\frac{z}{\sigma_3}\right)^2\frac{\lambda_3\lambda_1 - \nu_2^2}{\nu_1} - \frac{2yz}{\sigma_2\sigma_3}\frac{\nu_1\lambda_1 + \nu_2\nu_3}{\lambda_1}\right\}} dy\, dz.$$

But this must be the correlation distribution for y and z treated independently of x, or, comparing with p. 264, if r_1, r_2, r_3 be the three correlation coefficients for the pairs yz, zx, xy respectively, we have

$$\lambda_1/(\lambda_1\lambda_2 - \nu_3^2) = 1 - r_1^2 = \lambda_1/(\lambda_3\lambda_1 - \nu_2^2).$$

$$(\nu_1\lambda_1 + \nu_2\nu_3)/(\lambda_2\lambda_1 - \nu_3^2) = r_1.$$

Integrating A for x and y from $\pm\infty$, we must have the distribution for z treated independently, or a normal distribution σ_3; this gives at once

$$\lambda_1\lambda_2 - \nu_3^2 = \chi.$$

Hence we have by symmetry the equations,

$$\lambda_1 = \chi(1 - r_1^2), \qquad \lambda_2 = \chi(1 - r_2^2), \qquad \lambda_3 = \chi(1 - r_3^2),$$
$$\nu_1\lambda_1 + \nu_2\nu_3 = \chi r_1, \qquad \nu_2\lambda_2 + \nu_3\nu_1 = \chi r_2, \qquad \nu_3\lambda_3 + \nu_1\nu_2 = \chi r_3.$$

We easily deduce

$$\chi^2(r_1 - r_2r_3) = (\lambda_2\lambda_3 - \nu_1^2)(\nu_1\lambda_1 + \nu_2\nu_3) - (\nu_2\lambda_2 + \nu_3\nu_1)(\nu_3\lambda_3 + \nu_1\nu_2) = \nu_1\chi,$$

or,

$$\nu_1 = \chi(r_1 - r_2r_3),$$

and similarly,

$$\nu_2 = \chi(r_2 - r_3r_1), \quad \nu_3 = \chi(r_3 - r_1r_2).$$

Finally,

$$\chi^2\{(r_1 - r_2r_3)(1 - r_1^2) + (r_2 - r_1r_3)(r_3 - r_1r_2)\} = \chi r_1,$$

or,

$$\chi(1 - r_1^2 - r_2^2 - r_3^2 + 2r_1r_2r_3) = 1.$$

Thus all the constants are determined, and we have,

$$\mathrm{P} = \frac{n\sqrt{\chi}}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} e^{-\frac{1}{2}\chi\left\{\frac{x^2}{\sigma_1^2}(1 - r_1^2) + \frac{y^2}{\sigma_2^2}(1 - r_2^2) + \frac{z^2}{\sigma_3^2}(1 - r_3^2) - 2(r_1 - r_2r_3)\frac{yz}{\sigma_2\sigma_3} - 2(r_2 - r_3r_1)\frac{zx}{\sigma_3\sigma_1} - 2(r_3 - r_1r_2)\frac{xy}{\sigma_1\sigma_2}\right\}} dx\,dy\,dz.$$

This agrees with BRAVAIS' result, except that he writes for r_1, r_2, r_3 the values $\Sigma(yz)/(n\sigma_2\sigma_3)$, etc., which we have shown to be the best values (see *loc. cit.*, p. 267).

Obviously we have the following general results. If Σ_1 be the standard deviation of a group of x-organs selected with regard to values h_2 and h_3 of y and z,

$$\Sigma_1 = \frac{\sigma_1}{\sqrt{\{\chi(1 - r_1^2)\}}} = \sigma_1\sqrt{\frac{1 - r_1^2 - r_2^2 - r_3^2 + 2r_1r_2r_3}{1 - r_1^2}}$$

and if h_1 be the deviation of the mean of the selected x-organs from the x-mean of the whole population

$$h_1 = \frac{r_3 - r_1r_2}{1 - r_1^2}\frac{\sigma_1}{\sigma_2}h_2 + \frac{r_2 - r_1r_3}{1 - r_1^2}\frac{\sigma_1}{\sigma_3}h_3.$$

Expressions of the form $\dfrac{r_3 - r_1r_2}{1 - r_1^2}$ will be spoken of as coefficients of double correlation, and expressions of the form $\dfrac{r_3 - r_1r_2}{1 - r_1^2}\dfrac{\sigma_1}{\sigma_2}$ as coefficients of double regression.*

* [The above values for Σ_1 and h_1 are still true, as Mr. G. U. YULE points out to me, *whatever be the law of frequency*, provided the standard-deviations of all arrays be the same and h_1 be a linear function of h_2 and h_3.]

(8.) *Double Regression and Biparental Inheritance.*

(*a.*) *General Formulæ and Comparison with the Theory of the Midparent.*—If we apply the results of Section (7) to the problem of inheritance, we obtain some interesting results. Let r_1 = coefficient of correlation for the same or different organs in two parents, *i.e.*, be the measure of assortative mating; r_3 = coefficient of correlation of organs of offspring and male parent, *i.e.*, be the measure of paternal inheritance; r_2 = coefficient of correlation of organs of offspring and female parent, *i.e.*, be the measure of maternal inheritance; then the above formulæ express the chief characteristic of biparental inheritance as modified by assortative mating. If r_1, as probably is frequently the case, be small, then we see that the effect of assortative mating is to reduce the deviation of the offspring. Suppose there were no assortative mating, then the mean deviation of the offspring of selected parents would be

$$h_1 = r_3 \frac{\sigma_1}{\sigma_2} h_2 + r_2 \frac{\sigma_1}{\sigma_2} h_3,$$

and the actual value r_1, being small, is clearly less than this. Again, even admitting the insignificance of the assortative mating in some cases, we see that, unless $r_2 = r_3$, and further special relations hold between the variations of parents and offspring, this formula is not reducible to a mid-parent formula.

For example, in the case of stature, consider the male offspring of two pairs of parents. In the first case, let the father be 4″ and the mother ·923″ above the average; in the second, let the father be 1″ and the mother 3″·692 above the average. In both cases the height of the mid-parent is 2″·5 above the average, and the average male offspring will, on the mid-parent theory, exceed the mean by 1″·67. But in the first case, the bi-parental formula gives 1″·95, and in the second, 1″·52. In the case of the female offspring of the same pairs, the mid-parental formula gives 1″·54 for both pairs, and the bi-parental formula 1″·41 and 1″·25 respectively. These differences are due to the prepotency of paternal inheritance, and to the inequality of the variation in different male and female groups.

These results have, of course, no greater validity than the statistics upon which they are based—a validity which Mr. GALTON has been very careful to weigh ('Natural Inheritance,' pp. 73, 131), but, I think, they suffice to show that the mid-parent theory must be looked upon as only an approximation of a rough kind.

It must further be borne in mind, that the variability of a fraternity with given mid-parent is, if assortative mating be neglected, $\Sigma_1 = \sigma_1 \sqrt{1 - r_2^2 - r_3^2}$; or if r_2 be $= r_3$, it is equal to $\sigma_1 \sqrt{1 - 2r_2^2}$, and *not* $\sigma_1 \sqrt{1 - r^2}$.

(*b.*) *Effect of Assortative Mating on Cross Heredity.*—Our formula of course applies to the problems I have classed as those of *cross heredity*. Unfortunately, I have no statistics at present to give any illustration of the intensity of cross-heredity.

Still one or two remarkable general principles may be noticed. Let us suppose, what is not improbable, that there is a first organ, say in the father, which has no sensible correlation with a second organ in the offspring, but that the latter organ in the mother is closely correlated by assortative mating with the first organ in the father. The formula for regression in the offspring of parents having the deviations h_2 and h_3 in the two organs (or characteristics) will now be

$$h_1 = - \frac{r_1 r_2}{1 - r_1^2} \frac{\sigma_1}{\sigma_2} h_2 + \frac{r_2}{1 - r_1^2} \frac{\sigma_1}{\sigma_3} h_3.$$

This shows us that the possession in any exceptional degree of the first organ by the father will actually reduce the amount of the second organ which the offspring inherits from the mother. Let a special example be used to illustrate this. Suppose the problem to be the inheritance of artistic sense from the mother and (h_1, h_3) be measures of the deviations of this sense in son and mother from the normal. Suppose further that h_2 be a measure of the father's physique, say his girth of chest. Now it is conceivable that artistic sense in the mother may be closely correlated with physique in father. If now we deal with artistic sense of the son as related to physique in father and artistic sense in mother, we conclude that exceptional physique in the father will *reduce* the exceptional artistic sense which the son inherits from his mother. Similarly, the exceptional physique which the son would inherit from his father would be reduced by exceptional artistic sense in his mother. It will be noted that these results have no relation whatever to the coexistence or not of artistic sense with physique in the father or the mother. They depend entirely on the influence of assortative mating. It is remarkable that, given mothers of high artistic sense, then this will be handed down in a greater degree to those offspring whose fathers have a physique below the average, than to those of fathers who have a physique above the average.

The above example is not to be taken as a demonstrated truth, but as an illustration of the effect of assortative mating on cross-heredity. Innumerable similar statements can be made, but it seems desirable to await the collection of definite statistics before discussing them at length.

The only statistics which are at present at my disposal for the consideration of bi-parental inheritance are Mr. GALTON'S "Family Records," and to these I now turn.

(c.) *Bi-parental Inheritance of Stature.*

TABLE XII.—Correlation as Influenced by Assortative Mating.

Class.	Correlation coefficients.	
	Modified by mating.	Direct.
Daughters and fathers	·3368	·3603
,, mothers	·2528	·2841
Sons and fathers	·3710	·3959
,, mothers	·2673	·3018

TABLE XIII.—Regression Coefficients as Influenced by Assortative Mating.

Class.	Regression coefficients.	
	Modified by mating.	Direct.
Daughters on fathers	·2895	·3096
,, mothers	·2609	·2932
Sons on fathers	·4176	·4456
,, mothers	·2997	·3384

Thus we see that both in correlation and regression very sensible differences are made by the introduction of bi-parental formulæ.

TABLE XIV.—Variation in Selected Groups.

Class.	Standard deviations.			
	(i.) All offspring.	(ii.) Offspring of selected mother.	(iii.) Offspring of selected father.	(iv.) Offspring of selected mother and father.
Sons	2·617	2·495	2·403	2·300
Daughters	2·347	2·250	2·189	2·108

GENERAL FORMULÆ FOR REGRESSION IN STATURE:—

h_2 = deviation of father; h_3 = deviation of mother.

Sons:—The mean height of array of sons corresponding to fathers of height h_2 and mothers of height h_3 is

$$h_1 = \cdot 4176\, h_2 + \cdot 2997\, h_3,$$

or,

$$= \cdot 4176\, h_2 + \cdot 2766\, (1\cdot 08\, h_3).$$

Daughters:—The mean height of array of daughters corresponding to fathers of height h_2 and mothers of height h_3 is

$$h'_1 = \cdot 2895\, h_2 + \cdot 2609\, h_3.$$
$$= \cdot 3136\, (h_2/1\cdot 08) + \cdot 2609\, h_3.$$

In the second expressions given with both formulæ, the parental heights are exhibited in terms of the equivalent heights of the sex of the offspring.

Explanation of the Tables.—Table XII. gives the value of the correlation coefficient as influenced by assortative mating, *e.g.*, $\frac{r_3 - r_1 r_2}{1 - r_1^2}$. The values of the simple correlation coefficients (r_1, r_2, r_3) are taken from Table II. Against each coefficient is placed the value of the "direct" coefficient, on the supposition that $r_1 = 0$—*e.g.*, r_3—in order to exhibit immediately the influence of assortative mating.

Table XIII. gives the regression coefficients as influenced by assortative mating, *e.g.*, $\frac{r_3 - r_1 r_2}{1 - r_1^2} \frac{\sigma_1}{\sigma_2}$ (see p. 286), and "direct" or uninfluenced by such mating, *e.g.*, $r_3 \frac{\sigma_1}{\sigma_2}$; the former are calculated from the values given in Tables XII. and I., and the latter are reproduced from Table III.

Table XIV. exhibits the decreasing variation in arrays of sons and daughters, when we select (i.) neither father nor mother, (ii.) a mother of given type, (iii.) a father of given type, and (iv.) both mother and father of given types; (i.) is taken from Table I., (ii.) and (iii.) from Table IV., and (iv.) is calculated from the formula for Σ_1 deduced on p. 287.

(*d.*) *Conclusions. Prepotency of Father.*—These tables bring out the essential prepotency of the father in the case of both sons and daughters, the ratio of the contributions being 42 to 28 in the first case and 31 to 26 in the second case. A prepotency of the father* in other characteristics has been noted by Mr. GALTON in his "Hereditary Genius," but it is there attributed to the greater ease with which the male characteristic (genius) makes itself apparent. It deserves, however, to be

* Prepotency of either parent might, I think, be easily tested statistically in the case of morbid inheritance, particularly in tubercular disease. Dr. R. E. THOMPSON ('Family Phthisis,' pp. 89 and 95), indicates a prepotency of the mother in both male and female inheritance of this disease.

considered whether there is not, at any rate in many characteristics, an actual and not apparent male prepotency. It is, perhaps, needless to point out the sensible, if small, modifications introduced into inheritance by assortative mating.

Lastly, we note in Table XIV. the increasing tendency to "breed truer" as we select (i.) mother, (ii.) father, and (iii.) both mother and father.

(9.) *On Some Points connected with Morbid Inheritance.*

(*a.*) *On the Skipping of Generations.*—It must be carefully borne in mind that the formulæ we have discussed make not the least pretence to explain the mechanism of inheritance. All they attempt is to provide a basis for the quantitative measure of inheritance—a schedule, as it were, for tabulating and appreciating statistics. At the same time we may reasonably ask whether our formulæ are wide enough to embrace certain of the more isolated and remarkable features of heredity. Let the subscripts 1, 2, 3, 4 refer respectively to father, mother, son, daughter. Thus, σ_3 would be the S.D. of the son population, h_2 a deviation of a mother from the mean of mothers, r_{14} the correlation coefficient of fathers and daughters, and so on. Now if we consider the general form for single correlation :

$$z = z_0 e^{-\frac{1}{2}\left(\frac{x^2}{\sigma'^2} - \frac{2rxy}{\sigma'\sigma''} + \frac{y^2}{\sigma''^2}\right)\frac{1}{1-r^2}},$$

we may give any values whatever to σ' and σ'', and any value to r, which is less than unity, and deduce the theoretical results. Let us suppose r to be of finite value, but that σ'' is very small as compared with σ'. Then the regression of y on $x = h'r\sigma''/\sigma'$ will be very small, while the regression of x on $y = h''r\sigma'/\sigma''$ will be large. On the other hand, the deviation in y will never be very remote from its mean. All this is perfectly true whatever be the value of r.

Now let us apply this to some secondary sexual characteristic, say hair on the face. A very small amount of hair on the woman's face, with a very large amount of hair on the man's face, is compatible with a large value of r; a small amount of hair on the woman's face may be accounted for by a low mean and very small standard deviation. The regression from father to daughter will be expressed by

$$h_4 = r_{14}\frac{\sigma_4}{\sigma_1}h,$$

or, since σ_4 is extremely small, the daughter will hardly differ sensibly from the mean small hairiness of women. The regression from daughter to daughter's son will be

$$r_{23}\frac{\sigma_3}{\sigma_2}h_4 = r_{23}r_{14}\frac{\sigma_3}{\sigma_2}\frac{\sigma_4}{\sigma_1}h_1$$

or, since σ_2 and σ_4 are nearly, if not practically, equal, and σ_3 and σ_1 also, we have—regression from grandfather to grandson through the female line $= r_{23}r_{14}h_1$.

This may be a very sensible quantity, if the correlation coefficients are of considerable magnitude. What we have here, then, is the *skipping of a generation*, the inheritance of an especially male characteristic through the female line. The same reasoning would apply to the inheritance of an especially female characteristic through the male line. The formula, of course, gives no *explanation* of why σ_4 is small and r_{14} finite. It is only suggested that these outlying facts of heredity are not necessarily inconsistent with the formula. It may be argued that this account of skipping a generation would only apply to a characteristic which actually exists in both sexes, even if only in a small degree in one of them, and further, it assumes the distribution of this small degree to be of a normal character. This argument would certainly touch characteristics functionally necessary and peculiar to one sex; it may be doubted how far it would affect the question of secondary sexual characteristics, which may have rudimentary values in the sex of which they are not characteristic. It must further be remembered, however, that our correlation formulæ are perfectly true for *cross* heredity, and accordingly the idea of rudimentary value may be pushed a good way, even to the idea of latency in a second closely-allied organ. The idea of latency here is not to be pressed into any theory of panmixia or of germ plasm. Given that certain bulls get good milkers, we have the problem, what organ or characteristic, rudimentary or not, in bulls has the highest numerical coefficient of correlation with the milk-giving capacity of the cows they beget? We may not be able to ascertain this organ or characteristic, but the problem is really a statistical one, and does not assert anything as to the mechanism of heredity. The skipping of a generation in secondary, or even in primary, sexual characteristics, does not seem accordingly to present anything of a character which our formula fails to cover. In particular, in the case of morbid inheritances, such as gout and colour-blindness, which, while peculiarly male diseases, are yet handed down through the female line, our formula seems to be of considerable suggestiveness. This suggestiveness essentially depends on the independence of the two factors—correlation and variation—which are components of the formula. Thus, while there appears to be no necessary relation between power of transmitting and capacity for developing a disease, the independence of correlation and variation will probably allow us to account for most special cases. The reader must be careful to note that we are not compelled to give r or σ meanings relating directly to the intensity of the disease; they may refer to the size of organs or intensity of characteristics on which the liability to the disease or its intensity directly or indirectly depends. Bearing this in mind, we have only to put r_{13} finite, or vanishingly small, while both σ_1 and σ_3 are finite, to grasp (i.) how gout may be transmitted from grandfather through either son or daughter to grandson, and yet (ii.) how colour-blindness and hæmophilia are transmitted, as a rule, through daughter only to grandson—in both cases the daughter generally herself escaping (r_{14} finite and σ_4 very small). The protection of the transmitting sex is due, not to smallness of correlation, but to relative smallness of variation in that sex.

(*b.*) *General Formulæ for Four Correlated Organs.*—Another point—especially important for the problem of morbid inheritance—is the relative ages at which a characteristic appears in parent and offspring. DARWIN has noted how a characteristic appearing at a given age in the parent will reappear at the same age—sometimes indeed earlier—in the offspring. In particular, inherited diseases tend to develop themselves at an earlier date in the offspring than in the parent in proportion to the intensity of the inheritance. This appears to be especially the case in gout, rheumatic fever, diabetes, and phthisis.*

Now, the quantities with which we have to deal here are *four* in number, ages of parent and offspring on appearance of disease and intensities of the disease in the parent and offspring. We require, accordingly, the formulae for triple correlation. Proceeding, as in the earlier discussions, we find, if x_1, x_2, x_3, x_4 be the deviation of the four quantities from their respective means, σ_1, σ_2, σ_3, σ_4 their standard deviations, r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}, the six correlation coefficients pair and pair of organs or characteristic $z\,\delta x_1\,\delta x_2\,\delta x_3\,\delta x_4$, the frequency out of a total of n quadruplets of the quadruplets with organs or characteristics between x_1, x_2, x_3, x_4 and $x_1+\delta x_1$, $x_2+\delta x_2$, $x_3+\delta x_3$, $x_4+\delta x_4$:

$$z = \frac{n\sqrt{\chi}}{4\pi^2\sigma_1\sigma_2\sigma_3\sigma_4}\, e^{-\frac{\chi}{2}\left\{\lambda_1\left(\frac{x_1}{\sigma_1}\right)^2+\lambda_2\left(\frac{x_2}{\sigma_2}\right)^2+\lambda_3\left(\frac{x_3}{\sigma_3}\right)^2+\lambda_4\left(\frac{x_4}{\sigma_4}\right)^2-2\nu_{12}\frac{x_1x_2}{\sigma_1\sigma_2}-2\nu_{13}\frac{x_1x_3}{\sigma_1\sigma_3}-2\nu_{14}\frac{x_1x_4}{\sigma_1\sigma_4}-2\nu_{23}\frac{x_2x_3}{\sigma_2\sigma_3}-2\nu_{24}\frac{x_2x_4}{\sigma_2\sigma_4}-2\nu_{34}\frac{x_3x_4}{\sigma_3\sigma_4}\right\}}$$

where

$$\lambda_1 = 1 - r_{23}^2 - r_{34}^2 - r_{42}^2 + 2r_{23}r_{34}r_{42},$$

$$\nu_{12} = r_{12}(1 - r_{34}^2) - r_{13}r_{23} - r_{14}r_{24} + r_{34}(r_{14}r_{23} + r_{13}r_{24}),$$

and

$$\begin{aligned}1/\chi = 1 &- r_{12}^2 - r_{13}^2 - r_{14}^2 - r_{23}^2 - r_{24}^2 - r_{34}^2 + r_{12}^2r_{34}^2 + r_{23}^2r_{14}^2 + r_{13}^2r_{24}^2 \\ &+ 2(r_{23}r_{24}r_{34} + r_{34}r_{14}r_{13} + r_{12}r_{14}r_{24} + r_{12}r_{13}r_{23}) \\ &- 2(r_{12}r_{14}r_{23}r_{34} + r_{14}r_{13}r_{23}r_{24} + r_{12}r_{13}r_{24}r_{34}),\end{aligned}$$

while the remaining λ's and ν's may be written down by symmetry from λ_1 and ν_{12}.

Accordingly we have for regression the formula

$$h_1 = \frac{\nu_{12}}{\lambda_1}\frac{\sigma_1}{\sigma_2}h_2 + \frac{\nu_{13}}{\lambda_1}\frac{\sigma_1}{\sigma_3}h_3 + \frac{\nu_{14}}{\lambda_1}\frac{\sigma_1}{\sigma_4}h_4,$$

and for the standard deviation of a group of organs x, corresponding to selected organs h_2, h_3, h_4 (*i.e.*, an array)

$$\Sigma_1 = \sigma_1/\sqrt{\chi\lambda_1}.$$

* Here, as elsewhere, I have to thank my friend, Dr. R. T. RYLE, for the kindness with which he has allowed me to examine the material he has collected with regard to morbid inheritance.

(*c.*) *Antedating of Family Diseases.*—We may now apply these results to the case of morbid inheritance, making the following assumptions :—

(a.) The distribution of the disease with regard to both age and intensity will be taken to be the same for any two successive generations, and to be normal.

(b.) The age at which the disease appears and its intensity are both directly inherited, but the age of appearance and intensity of the disease in the parent are not directly correlated with the intensity of the disease and the age of its reappearance in the offspring.

Let ϵ be the coefficient of correlation between the age of appearance of disease in the parent and the age of the offspring at its reappearance ; let σ be the standard-deviation for the frequency of the disease at different ages, and M the mean age at which the disease appears in the population.

Let η be the coefficient of correlation between the intensity of the disease in the parent and the intensity of the disease in the offspring ; let σ' the standard-deviation of the intensity-frequency and M′ be the mean intensity.*

Let $M + A_1$, $M' + I_1$, be the mean age of the appearance of the disease and its mean intensity for an array of offspring, whose parents exhibited the disease when $M + A_2$ years old with an intensity $M' + I_2$.

Let the subscripts 1, 2, 3, 4 refer respectively to age of offspring, age of parent, and intensity in offspring and intensity in parent. Then, in the formula for triple correlation, we must put :

$$r_{12} = \epsilon, \quad r_{34} = \eta, \quad r_{13} = r_{24} = \kappa, \quad r_{14} = r_{23} = 0.$$

Hence :

$$\lambda_1 = \lambda_2 = 1 - \eta^2 - \kappa^2, \quad \nu_{12} = \epsilon + \eta(\kappa^2 - \epsilon\eta),$$
$$\lambda_3 = \lambda_4 = 1 - \epsilon^2 - \kappa^2, \quad \nu_{34} = \eta + \epsilon(\kappa^2 - \epsilon\eta),$$
$$\nu_{13} = \nu_{24} = \kappa - \kappa(\kappa^2 - \epsilon\eta),$$
$$\nu_{14} = \nu_{23} = -\kappa(\epsilon + \eta),$$
$$1/\chi = 1 - \epsilon^2 - \eta^2 - 2\kappa^2 + (\kappa^2 - \epsilon\eta)^2.$$

Substituting these values in the regression formula, we find :

$$A_1 = \frac{\epsilon + \eta(\kappa^2 - \epsilon\eta)}{1 - \eta^2 - \kappa^2} A_2 + \frac{\kappa - \kappa(\kappa^2 - \epsilon\eta)}{1 - \eta^2 - \kappa^2} \frac{\sigma}{\sigma'} I_1 - \frac{\kappa(\epsilon + \eta)}{1 - \eta^2 - \kappa^2} \frac{\sigma}{\sigma'} I_2.$$

Now as the parents in the group $M + A_2$, $M' + I_2$ are in no way selected by

* It might be difficult to get a mathematical measure of the intensity of a disease. For simple theory as apart from statistical measurements, such is, however, unnecessary. The terms used in medical works, acute, subacute, chronic, &c., sufficiently indicate that the relative intensity of various cases is a fact duly recognized by the trained medical mind, if it cannot always be quantitatively expressed.

parentage, or influenced by heredity, being general statistics, we shall assume that, on the average, $A_2 = \kappa \frac{\sigma}{\sigma'} I_2$, and hence:

$$A_1 = \frac{\kappa - \kappa(\kappa^2 - \epsilon\eta)}{1 - \eta^2 - \kappa^2} \frac{\sigma}{\sigma'} (I_1 - \eta I_2) = \left\{\kappa + \frac{\kappa\eta(\eta + \epsilon)}{1 - \eta^2 - \kappa^2}\right\} \frac{\sigma}{\sigma'} (I_1 - \eta I_2).$$

Similarly:

$$I_1 = \left\{\kappa + \frac{\kappa\epsilon(\eta + \epsilon)}{1 - \epsilon^2 - \kappa^2}\right\} \frac{\sigma'}{\sigma} (A_1 - \epsilon A_2).$$

These formulæ give the chief influence of age of appearance and intensity of disease in parent upon intensity and age of appearance in the offspring. If we suppose κ *positive*, *i.e.*, if increased age of appearance means for the diseased population as a whole increased intensity, then intensity of disease in parents tends to lower the age at which the disease appears in the offspring, and this tendency to antedate is the greater, the greater the correlation (η) between intensity of the disease in parent and child, *i.e.*, the stronger the hereditability of the disease. If κ be *negative*, *i.e.*, increased age of appearance means for the diseased population as a whole decreased intensity, then the opposite result will follow, for A_1 will have a less negative value than if $\eta = 0$, *i.e.*, the age of offspring be raised towards the mean.* It would thus seem possible that the antedating of inheritance in the case of gout and diabetes might correspond to a post-dating in the cases of diseases intenser in youthful incidence.

Our second formula shows that for diseases with increased intensity at increased age of appearance, a late age of appearance in the parent decreases the intensity of appearance in the offspring, while the reverse holds if the disease is intenser for youthful than for senile incidence.

It must be noted that the correlation between intensity and age without regard to heredity is given by:

$$I_1 = \kappa \frac{\sigma'}{\sigma} A_1,$$

so that heredity affects the constant of correlation κ by multiplying it by the quantity:

$$1 + \frac{\epsilon(\eta + \epsilon)}{1 - \epsilon^2 - \kappa^2}.$$

The second part of this expression is by no means necessarily negligible as compared with the first part, if heredity be strong. For example, with the order of correlation we have found between parent and offspring, in the case of stature the

* Generally but not absolutely, for $\eta^2 + \kappa^2$ for some diseases may be > 1, and, if not *very* different, then the second term is the important term

second term might be $\frac{1}{3}$ to $\frac{1}{4}$, while, for values of the order ·7 in the correlation coefficients, it would be a much more important term than the first, *i.e.*, heredity would completely obscure the general correlation between intensity and age.

Similar remarks apply, of course, to the formula

$$A_1 = \kappa \frac{\sigma}{\sigma'} I_1,$$

and the modification of its κ by the factor

$$1 + \frac{\eta(\eta + \epsilon)}{1 - \eta^2 - \kappa^2}.$$

While the above discussion has been adapted particularly to the problem of morbid inheritance, it should be noted that the general formulæ for triple correlation apply to a number of interesting problems on the inheritance of two faculties by the offspring from the parent. In particular, the above special formulæ in η, ϵ, and κ apply without modification to any case when (*a*) the two faculties are correlated in like manner (κ) in parent and offspring, (*b*) the two faculties are each directly inherited (η and ϵ), (*c*) there is an insensible or zero amount of cross heredity. I do not stay to develop the formulæ at present, because I hope to return to them when I have more ample statistics to illustrate the properties of cross heredity from.

(*d*.) *On the Skewness of Disease Curves.*—There is one qualifying remark which must, however, be made before we leave the topic of morbid inheritance. We have assumed that the frequency surface for intensity and age of appearance of disease is a normal correlation surface. This, however, is only an approximation. If we add together all the intensities for each age, we shall have a frequency with age curve for the disease, and if the correlation surface were a true normal surface, this would be a true normal curve. In many diseases, possibly in all, it is however, a distinctly skew curve, and this whether we take the case-frequency or the mortality-frequency. This has been illustrated in "Contributions to Mathematical Theory of Evolution, II." ('Phil. Trans.,' vol. 186, A.), Plate 12, for enteric fever.* The following statistics illustrate the same skewness for a disease more distinctly associated with heredity† :—

PHTHISIS : 2000 cases with History of Parental Phthisis.

Age . . .	1	10	15	20	25	30	35	40	45	50	55	60
Frequency	26	100	436	549	392	217	149	65	27	6	9	4

* It is, I think, true for all fevers, some of which, however, have κ positive and others κ negative.

† R. E. THOMSON, 'Family Phthisis,' p. 22, London, 1884.

It is clear that we have here to deal with a skew curve of the kind discussed in my second memoir, and the intensity-age distribution must be a skew correlation surface to give rise to such a curve. The full treatment, accordingly, of morbid inheritance requires a discussion of skew correlation. I hope to be able to return to it again when dealing with the general theory of disease distributions. Meanwhile, the considerations of this section are based on an approximate theory, which, however, can hardly fail to give the main outlines of the subject, if a more accurate development might be requisite when actual statistics were forthcoming to be dealt with.

(10.) *Natural Selection and Panmixia.*

(*a.*) *Fundamental Theorem in Selection.*—The general theory of correlation shows us that taking $p+1$ correlated organs, if we select p of them of definite dimensions, the remaining organ will follow a normal law of distribution, of which the standard-deviation and mean can be determined. Now, in the problem of natural selection, we do not select absolutely definite dimensions, and the p organs selected may be specially correlated together in selection, in a manner totally different from their "natural" correlation or correlation of birth. We, therefore, require a generalised investigation of the following kind: Given $p+1$ normally correlated organs, p out of these organs are selected in the following manner: each organ is selected normally *round a given mean*, and the p selected organs, pair and pair, are correlated in any arbitrary manner. What will be the nature of the distribution of the remaining $(p+1)^{\text{th}}$ organ?

Geometrically in p-dimensional space we have a correlation surface of the p^{th} order among the p organs, and out of this, with any origin we please, we cut an arbitrary correlation surface of the p^{th} order—of course, of smaller dimensions—the problem is to find the distribution of the $(p+1)^{\text{th}}$ organ related to this arbitrary surface cut out of what we may term the natural surface.

If the p organs are organs of ancestry—as many as we please—and the $(p+1)^{\text{th}}$ organ that of a descendant, we have here the general problem of natural selection modified by inheritance.

We will distinguish the two correlation surfaces as the unselected and the selected. Let $\beta_1, \beta_2, \beta_3, \ldots$ be the regression coefficients of the $(p+1)^{\text{th}}$ organ on the p organs for unselected correlation, then for values of the p organs $h_1, h_2, h_3, \ldots$ from their respective means, the $(p+1)^{\text{th}}$ organ will have a distribution centering round $\beta_1 h_1 + \beta_2 h_2 + \beta_3 h_3 + \ldots$, and a standard deviation σ given by the general theory of correlation (*i.e.*, the S.D. of the array). Similarly, for values $h_1 + x_1$, $h_2 + x_2$, $h_3 + x_3 \ldots$ of the p organs, the $(p+1)^{\text{th}}$ will have a distribution with standard-deviation σ and centre

$$\beta_1 (h_1 + x_1) + \beta_2 (h_2 + x_2) + \beta_3 (h_3 + x_3) + \ldots = \zeta + \mathrm{S}\,(\beta_1 x_1), \text{ say.}$$

Thus a deviation of the p^{th} organ lying between v and $v + dv$ from the mean of these organs will occur with a frequency varying as

$$dv\, e^{-\frac{\{v-\zeta+S(\beta_1 x_1)\}^2}{2\sigma^2}}.$$

Now let the selected correlation surface centering round $h_1, h_2, h_3, \ldots$ be given by

$$z = \text{constant} \times e^{-\frac{1}{2}(a_{11}x_1^2 + a_{22}x_2^2 + \ldots + 2a_{12}x_1x_2 + \ldots)}.$$

Then the total frequency of the p^{th} organ lying between v and $v + dv =$

$$\text{Constant} \times dv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots e^{-\frac{\{v-\zeta-S(\beta_1 x_1)\}^2}{2\sigma^2} - \frac{1}{2}\{S(a_{11}x_1^2)+2S(a_{12}x_1x_2)\}} dx_1 dx_2 dx_3 \ldots$$

To carry out the integrations, let us first transfer the expression in the exponential power to its "centre," writing $v - \zeta = u$, and $x_1', x_2', x_3', \ldots$ as the coordinates of the centre.

To find the "centre" we have the equations:

$$\beta_1 (u - S(\beta_1 x_1'))/\sigma^2 = a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \ldots,$$
$$\beta_2 (u - S(\beta_1 x_1'))/\sigma^2 = a_{21}x_1' + a_{22}x_2' + a_{23}x_3' + \ldots,$$
$$\beta_3 (u - S(\beta_1 x_1'))/\sigma^2 = a_{31}x_1' + a_{32}x_2' + a_{33}x_3' + \ldots,$$
$$\ldots\ldots\ldots\ldots$$

hence

$$\Delta x_1' = (\beta_1 A_{11} + \beta_2 A_{12} + \beta_3 A_{13} + \ldots)(u - S(\beta_1 x_1'))/\sigma^2,$$
$$\Delta x_2' = (\beta_1 A_{21} + \beta_2 A_{22} + \beta_3 A_{23} + \ldots)(u - S(\beta_1 x_1'))/\sigma^2,$$
$$\Delta x_3' = (\beta_1 A_{31} + \beta_2 A_{32} + \beta_3 A_{33} + \ldots)(u - S(\beta_1 x_1'))/\sigma^2,$$
$$\ldots\ldots\ldots\ldots$$

where Δ is the determinant of the a's, and the A's are its minors, clearly $a_{ij} = a_{ji}$ and $A_{ij} = A_{ji}$. Multiplying these equations by $\beta_1, \beta_2, \beta_3 \ldots$ respectively, and adding we find

$$\begin{aligned}\sigma^2\Delta S(\beta_1 x_1') &= \{\beta_1^2 A_{11} + \beta_2^2 A_{22} + \beta_3^2 A_{33} + \ldots \\ &\qquad + 2A_{12}\beta_1\beta_2 + 2A_{13}\beta_1\beta_3 + \ldots\}(u - S(\beta_1 x'_1)) \\ &= \{S(\beta_1^2 A_{11}) + 2S(A_{12}\beta_1\beta_2)\}(u - S(\beta_1 x'_1)),\end{aligned}$$

hence

$$S(\beta_1 x_1') = \frac{\chi u}{\sigma^2\Delta + \chi}$$

where

$$\chi = S(\beta_1^2 A_{11}) + 2S(A_{12}\beta_1\beta_2).$$

We can now transfer the exponential expression to its centre and we find for the frequency

$$\text{Constant} \times du\, e^{-\frac{u^2}{2}\left(\frac{1}{\sigma^2} - \frac{\chi}{\sigma^2\Delta + \chi}\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots e^{-\frac{1}{2}\left\{S\left(x^2_1\left(a_{11} + \frac{\beta_1^2}{\sigma^2}\right)\right) + S\left(2x_1x_2\left(a_{12} - \frac{\beta_1\beta_2}{\sigma^2}\right)\right)\right\}} dx_1 dx_2 dx_3.$$

Where x_1, x_2, $x_3 \dots$, now denote the coordinates transferred to the new origin. The integrations can then be performed without changing the u factor, and finally the frequency

$$= \text{constant} \times du\, e^{-\frac{1}{2}u^2/\left(\sigma^2 + \frac{\chi}{\Delta}\right)}.$$

Hence we notice the following important results :

(a.) The $p + 1^{\text{th}}$ organ follows a normal distribution.

(b.) Its standard deviation Σ is given by

$$\Sigma^2 = \sigma^2 + \beta_1^2 \frac{A_{11}}{\Delta} + \beta_2^2 \frac{A_{22}}{\Delta} + \dots + 2\beta_1\beta_2 \frac{A_{12}}{\Delta} + 2\beta_1\beta_3 \frac{A_{13}}{\Delta} + \dots$$

(c.) Its mean (since $u = v - \zeta$) $= \beta_1 h_1 + \beta_2 h_2 + \beta_3 h_3 + \dots$

We conclude that

(i.) so long as selection is normal, however complex may be the system of organs selected, and however complex their correlation, the distribution of any single organ remains normal. This possibly accounts for the persistency with which normal grouping reappears in nature.

(ii.) If we select organs *varying* about any means whatever, the mean of the correlated organ resulting from this selection will be identical with the mean we should have obtained by selecting organs actually at the means of selection.

(iii.) The standard deviation of the organ which results from the selection is *not* that of an array (σ) arising from selection of the organs actually at the means, but is (as we might expect) greater. This greater variability is due to the expression

$$\beta_1^2 \frac{A_{11}}{\Delta} + \beta_2^2 \frac{A_{22}}{\Delta} + \dots + 2\beta_1\beta_2 \frac{A_{12}}{\Delta} + \dots$$

which admits of the following interpretation.

Consider the selection correlation surface

$$z = \text{constant} \times e^{-\frac{1}{2}(a_{11}x_1^2 + a_{22}x_2^2 + \dots + 2a_{12}x_1x_2 + \dots)}$$

and give x_1 and x_2 chosen values η_1 and η_2.

Transfer the remaining variables to the "centre." The equations to do this are

$$
\begin{aligned}
f + \eta_1 a_{11} + \eta_2 a_{12} &= \eta_1 a_{11} + \eta_2 a_{12} + a_{13} x_3' + a_{14} x_4' + \ldots \\
g + \eta_1 a_{12} + \eta_2 a_{22} &= \eta_1 a_{21} + \eta_2 a_{22} + a_{23} x_3' + a_{24} x_4' + \ldots \\
0 &= \eta_1 a_{31} + \eta_2 a_{32} + a_{33} x_3' + a_{34} x_4' + \ldots \\
0 &= \eta_1 a_{41} + \eta_{42} a_{42} + a_{43} x_3' + a_{44} x_4' + \ldots \\
&\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

where f and g are written for $a_{13}x'_3 + a_{14}x'_4 + \ldots$ and $a_{23}x'_3 + a_{24}x'_4 + \ldots$ respectively. Solving, we find

$$
\begin{aligned}
A_{11}(f + \eta_1 a_{11} + \eta_2 a_{12}) + A_{12}(g + \eta_1 a_{12} + \eta_2 a_{22}) &= \eta_1 \Delta, \\
A_{21}(f + \eta_1 a_{11} + \eta_2 a_{12}) + A_{22}(g + \eta_1 a_{12} + \eta_2 a_{22}) &= \eta_2 \Delta.
\end{aligned}
$$

Hence

$$
f = \frac{\eta_1 A_{22} - \eta_2 A_{12}}{A_{11}A_{22} - A_{12}^2}\Delta - \eta_1 a_{11} - \eta_2 a_{12},
$$

$$
g = \frac{\eta_2 A_{11} - \eta_1 A_{12}}{A_{11}A_{22} - A_{12}^2}\Delta - \eta_1 a_{12} - \eta_2 a_{22}.
$$

But the exponential expression with its origin changed is given by

$$
\begin{aligned}
z = \text{constant} &\times e^{-\frac{1}{2}(a_{11}\eta_1^2 + 2a_{12}\eta_1\eta_2 + a_{22}\eta_2^2 + f\eta_1 + g\eta_2)} \\
&\times e^{-\frac{1}{2}(a_{33}x_3^2 + a_{44}x_4^2 + \ldots + 2a_{34}x_3x_4 + \ldots)}.
\end{aligned}
$$

Integrating between the limits $\pm\infty$ for all the variables $x_3, x_4, x_5 \ldots$, we shall have the correlation surface for η_1, η_2, or substituting for f and g

$$
z' = \text{constant} \times e^{-\frac{1}{2}\frac{\Delta}{1-(A_{12}^2/A_{11}A_{22})}\left\{\frac{\eta_1^2}{A_{11}} + \frac{\eta_2^2}{A_{22}} - 2\eta_1\eta_2\frac{A_{12}}{A_{11}A_{22}}\right\}}.
$$

Comparing this with the formula on p. 264, we see that if ρ_{12} be the correlation coefficient of x_1, x_2 and s_1, s_2 their standard deviations

$$
\rho_{12}^2 = A_{12}^2/A_{11}A_{22} \quad s_1^2 = A_{11}/\Delta \quad s_2^2 = A_{22}/\Delta \quad \text{or} \quad \rho_{12}s_1s_2 = A_{12}/\Delta \quad . \quad . \quad (\epsilon).
$$

Thus we conclude that the standard deviation for the organ resulting from the selection is given by

$$
\Sigma^2 = \sigma^2 + \beta_1^2 s_1^2 + \beta_2 s_2^2 + \ldots + 2\beta_1\beta_2\rho_{12}s_1s_2 + \ldots
$$

Here σ, β_1, β_2 . . . refer to the natural or unselected correlation surface, and $s_1, s_2, \ldots \rho_{12} \ldots$ to the selection correlation surface.

(*b*.) EDGEWORTH'S *Theorem*.—We may stay for a moment over the results (ϵ) above in order to deduce Professor EDGEWORTH'S Theorem,* which we shall shortly require to use. By the theory of minors (SALMON'S 'Higher Algebra,' 1866, p. 24) we have

* Briefly stated with some rather disturbing printer's errors in the 'Phil. Mag.,' vol. 34, p. 201, 1892.

$$\Delta^{p-1} = \begin{vmatrix} A_{11}, & A_{12}, & A_{13} \dots \\ A_{21}, & A_{22}, & A_{23} \dots \\ A_{31}, & A_{32}, & A_{33} \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

$$= \Delta^p s_1^2 s_2^2 s_3^2 \dots \begin{vmatrix} 1, & \rho_{12}, & \rho_{13} \dots \\ \rho_{21}, & 1, & \rho_{23} \dots \\ \rho_{31}, & \rho_{32}, & 1 \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix}$$

Hence $1/\Delta = s_1^2 s_2^2 s_3^2 \dots R$, where R is the determinant formed by the correlation coefficients with a diagonal of units.

Further, if B_{11}, $B_{22} \dots B_{12} \dots$ be the minors of the A-determinant, and R_{11}, $R_{22}, \dots R_{12}, \dots$ of the R-determinant, we have (SALMON, *loc. cit.*):

$$a_{11} = B_{11}/\Delta^{p-2} = \Delta R_{11} s_1^2 s_2^2 s_3^2 \dots / s_1^2 = R_{11}/(R s_1^2),$$
$$a_{22} = B_{22}/\Delta^{p-2} = \Delta R_{22} s_1^2 s_2^2 s_3^2 \dots / s_2^2 = R_{22}/(R s_2^2),$$
$$a_{12} = B_{12}/\Delta^{p-2} = \Delta R_{12} s_1^2 s_2^2 s_3^2 \dots / s_1 s_2 = R_{12}/(R s_1 s_2).$$
$$\dots \dots \dots$$

Thus, the correlation surface may be written

$$z = \frac{\mu n}{(2\pi)^{\frac{1}{2}p} s_1 s_2 s_3 \dots \sqrt{R}} e^{-\frac{1}{2R}\left(R_{11}\frac{x_1^2}{s_1^2} + R_{22}\frac{x_2^2}{s_2^2} + \dots + 2R_{12}\frac{x_1 x_2}{s_1 s_2} + \dots\right)}$$

where n is the total number of sets of p organs and μ is a numerical factor denoting the number of $(p+1)^{\text{th}}$ organs corresponding to each set—in inheritance what may be termed a factor of reproductivity*—which is assumed to be practically constant, if not over the whole unselected correlation surface, at least over the selected portion of it.

(*c*.) *Selection of Parentages. Correlation Coefficients for Ancestry.*—The results on p. 300 and p. 301 for the regression ζ and the standard-deviation Σ when p correlated organs are arbitrarily selected about p means will, I think, be found to express the chief features of natural selection. A few special corollaries may follow here.

Cor. 1.—If a single parentage be selected with mean h_1 above the mean of the general population and standard deviation s_1, then $\beta_1 = r_{01}\frac{\sigma_0}{\sigma_1}$, where r_{01} is the

* The variation of this factor is, however, the essential feature of reproductive selection, as I shall show on another occasion.

correlation coefficient of parent and offspring, and σ_1, σ_0 their standard-deviations in the unselected state. Thus we have

$$\xi = r_{01} \frac{\sigma_0}{\sigma_1} h_1, \quad \Sigma^2 = \sigma_0^2 (1 - r_{01}^2) + r_{01}^2 \frac{\sigma_0^2}{\sigma_1^2} s_1^2.$$

If the parent and offspring are of the same sex and there be no reproductive selection, $\sigma_0 = \sigma_1$, and we have

$$\xi = r_{01} h, \quad \Sigma^2 = \sigma_0^2 (1 - r_{01}^2) + r_{01}^2 s_1^2.$$

Cor. 2.—If a bi-parentage be selected with parental means h_1, h_2, standard-deviations s_1, s_2, and coefficient of assortative mating ρ_{12}, then

$$\begin{aligned} \xi &= \frac{r_{01} - r_{12} r_{02}}{1 - r_{12}^2} \frac{\sigma_0}{\sigma_1} h_1 + \frac{r_{02} - r_{12} r_{01}}{1 - r_{12}^2} \frac{\sigma_0}{\sigma_2} h_2, \\ &= \beta_1 h_1 + \beta_2 h_2, \end{aligned}$$

$$\Sigma^2 = \sigma_0^2 (1 - r_{01}^2 - r_{02}^2 - r_{12}^2 + 2 r_{01} r_{02} r_{12}) + \beta_1^2 s_1^2 + \beta_2^2 s_2^2 + 2 \beta_1 \beta_2 \rho_{12} s_1 s_2.$$

Let us use these results to investigate how the offspring of a selected parentage or bi-parentage degenerate. At first sight, it would appear that with our general proposition the discussion of the effect of p selections would be perfectly straightforward. So it is, but the conclusion which follows, although it might have been foreseen, is remarkable in its consequences. We have only to calculate out the β's for p selected ancestors, and we obtain the regression ζ in the descendant by putting in the values h_1, h_2, h_3, . . . of the means of the selected ancestors. For example, suppose now a parent, a grandparent, and a great-grandparent to have been selected. We can find the β's at once from the results on p. 294. If 1, 2, 3, 4 denote the successive generations, and r the correlation coefficient of parent and offspring, we find

$$r_{12} = r; \; r_{13} = r^2, \; r_{14} = r^3, \; r_{23} = r, \; r_{24} = r^2, \; r_{43} = r,$$

whence we deduce at once

$$\begin{aligned} \lambda_1 &= 1 - 2r^2 + r^4, & \nu_{12} &= r(1 - 2r^2 + r^5), \\ \nu_{13} &= \nu_{14} = 0, & 1/\chi &= (1 - r^2)^3, \end{aligned}$$

or

$$\beta_1 = r \frac{\sigma_1}{\sigma_2}, \quad \beta_2 = \beta_3 = 0. \quad \Sigma_1 = \sigma_1 \sqrt{1 - r^2}.$$

Similarly, if we take offspring (1), parent (2), and maternal and paternal of the same sex, grandparents (3 and 4), we have :

$$r_{12} = r, \; r_{13} = r^2, \; r_{14} = r^2, \; r_{23} = r, \; r_{24} = r, \; r_{34} = 0,$$

whence

$$\lambda_1 = 1 - 2r^2, \quad \nu_{12} = r(1 - 2r^2), \quad \nu_{13} = \nu_{14} = 0, \quad 1/\chi = 1 - 3r^2 + 2r^4,$$

or,

$$\beta_1 = r\frac{\sigma_1}{\sigma_2}, \quad \beta_2 = \beta_3 = 0, \quad \Sigma_1 = \sigma_1\sqrt{1 - r^2}.$$

Thus we see that in both cases the grandparents *are quite indifferent*, when the immediate parent has been selected.

These theorems can be at once generalised by means of EDGEWORTH's theorem. Suppose we select a complete parentage for p generations in the case of partheno-genetic reproduction, or a parentage of one sex, say males, in the case of sexual reproduction, then in either case our scheme of subscripts of the correlation-coefficients, $\longrightarrow$ marking a generation, is

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \longrightarrow . \quad . \quad . \quad . \longrightarrow p$$

and

$$R = \begin{vmatrix} 1 & r & r^2 & r^3 \ldots r^{p-1} \\ r & 1 & r & r^2 \ldots r^{p-2} \\ r^2 & r & 1 & r \ldots r^{p-3} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r^{p-1} & r^{p-2} & r^{p-3} & \ldots \ldots 1 \end{vmatrix}.$$

Multiply the second line by r, and subtract from the first, and we have

$$R = (1 - r^2)R_{11}.$$

Take R_{1q} $(q < p)$, and we have

$$R_{1q} = \begin{vmatrix} r & 1 & r & r^2 \ldots r^{q-3},\ r^{q-1} \ldots r^{p-2} \\ r^2 & r & 1 & r \ldots r^{q-4},\ r^{q-2} \ldots r^{p-3} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r^{p-1} & r^{p-2} & \ldots & \ldots r^{p-q+2},\ r^{p-q} \ldots 1 \end{vmatrix}.$$

Multiply the second column by r, and subtract from the first, and we have $R_{1q} = 0$ if $q > 2$.

If $q = 2$, we have

$$R_{12} = \begin{vmatrix} r & r & r^2 \ldots r^{p-2} \\ r^2 & 1 & r \ldots r^{p-3} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ r^{p-1} & r^{p-3} & \ldots \ldots 1 \end{vmatrix},$$

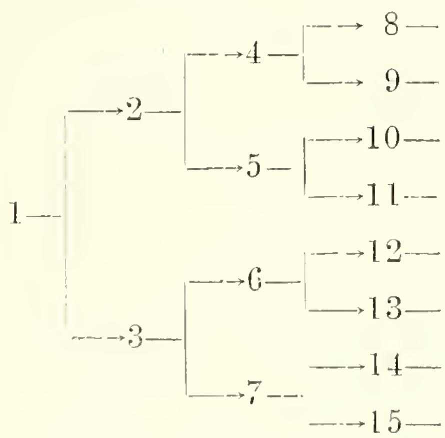
or, dividing the first column by r, $\quad R_{12} = rR_{11}$.

Hence $\zeta = r\dfrac{\sigma_1}{\sigma_2}h_2$, and $\sigma'^2 = \sigma_1^{\,2}(1 - r^2)$,

or precisely the results we should have obtained by selecting only the immediate parent.

To simplify the analysis for biparental selection, assume that the correlation coefficients of both parents are equal, and that there is no assortative mating.

We have the scheme for the correlation-coefficients subscripts, $\longrightarrow$ marking a generation :



and so on.

Thus r_{mn} is at once expressible as zero, or a power of r, the simple coefficient of correlation for parent and offspring, according as m and n do not or do lie in the direct descent.

Hence we find

$$
R = \left|\begin{array}{cccccccccccccccc}
1 & r & r & r^2 & r^2 & r^2 & r^2 & r^3 & r^3 & r^3 & r^3 & r^3 & r^3 & r^3 & r^3 & \ldots \\
r & 1 & 0 & r & r & 0 & 0 & r^2 & r^2 & r^2 & r^2 & 0 & 0 & 0 & 0 & \ldots \\
r & 0 & 1 & 0 & 0 & r & r & 0 & 0 & 0 & 0 & r^2 & r^2 & r^2 & r^2 & \ldots \\
r^2 & r & 0 & 1 & 0 & 0 & 0 & r & r & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
r^2 & r & 0 & 0 & 1 & 0 & 0 & 0 & 0 & r & r & 0 & 0 & 0 & 0 & \ldots \\
r^2 & 0 & r & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & r & r & 0 & 0 & \ldots \\
r^2 & 0 & r & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & r & r & \ldots \\
r^3 & r^2 & 0 & r & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
r^3 & r^2 & 0 & r & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
r^3 & r^2 & 0 & 0 & r & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
r^3 & r^2 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
r^3 & 0 & r^2 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
r^3 & 0 & r^2 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
r^3 & 0 & r^2 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
r^3 & 0 & r^2 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|
$$

Add the second and third rows, multiply them by r and subtract from the first, and we find :

$$R = (1 - 2r^2)\,R_{11}.$$

If $q > 3$ and $< p$ we have

$$R_{1q} = \begin{vmatrix} r & 1 & 0 & r & r & 0 & 0 & r^2 & r^2 & r^2 & r^2 & 0 & 0 & 0 & 0 & \dots \\ r & 0 & 1 & 0 & 0 & r & r & 0 & 0 & 0 & 0 & r^2 & r^2 & r^2 & r^2 & \dots \\ r^2 & r & 0 & 1 & 0 & 0 & 0 & r & r & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ r^2 & r & 0 & 0 & 1 & 0 & 0 & 0 & 0 & r & r & 0 & 0 & 0 & 0 & \dots \\ r^2 & 0 & r & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & r & r & 0 & 0 & \dots \\ r^2 & 0 & r & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & r & r & \dots \\ r^3 & r^2 & 0 & r & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ r^3 & r^2 & 0 & r & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Whence, adding the second and third columns, multiplying by r and subtracting from the first, we have $R_{1q} = 0$.

Lastly, $R_{12} = rR_{11}$ and $R_{13} = rR_{11}$; for R_{12} is of the form of R_{1q} above without the second column. Divide the first column by r, and subtract the second column (the *third* of R_{1q} above) and it becomes the first of R_{11}, the remainder is identical in R_{12} and R_{11}. Hence, $R_{12} = rR_{11}$. Similarly we find R_{13}. Thus we conclude that

$$\zeta = r\,\frac{\sigma_1}{\sigma_2}\,h_2 + r\,\frac{\sigma_1}{\sigma_3}\,h_3, \qquad \sigma'^2 = \sigma_1^2\,(1 - r^2 - r^2).$$

the formulæ for biparental inheritance with equal parental correlation, and no assortative mating. The analysis for unequal parental correlation and assortative mating follows the same lines, is far more lengthy, but leads to the same result, *i.e.*, no gain by selection of the same amount, oft repeated.

(*d.*) *Secular Natural Selection and Steady Selection. Focus of Regression.*—We thus see that, on the theory with which we are concerned, a knowledge of the ancestry beyond the parents in no way alters our judgment as to the size of organ or degree of characteristic probable in the offspring, nor its variability.* *An exceptional father is as likely to have exceptional children if he comes of a mediocre stock as if he comes of an exceptional stock.* The value of ζ will be no greater nor the value of Σ_1 less if the parents have been selected for p generations than if they have been selected for one only. This result seems to me somewhat surprising, but I cannot see how it is to be escaped so long as we assume the normal distribution of frequency, which appears in so many cases to be a close approximation to fact. It is of course possible

* This seems specially noteworthy; it would seem natural to suppose that the offspring of a long selected stock would be less variable than those of one just started—that the offspring of a gradually created variety would be more stable than those so to speak of a sport. It appears not.

that in some manner repeated selection causes a progression of the "focus of regression," by which term I would understand the mean of the general population from which selection has originally taken place. I have been very careful so far not to hazard any statement with regard to this focus of regression. I have measured only the amount by which the offspring of exceptional parents diverge, not from the mean of the parental population but from the mean of the offspring population. In this manner our formulæ allowed for the play of secular natural selection. It is quite true that the word "regression" thus loses its accustomary meaning, which it can only bear if the population be stable and the means of two generations sensibly identical; this is the case for which, I think, the word was introduced by Mr. GALTON.* The sense given to it in the present paper is accordingly a technical one; as already defined, it is the ratio of the mean deviation of the offspring of a selected parentage to the deviation in the parent which characterises the selection, the deviations in offspring and parents being respectively measured from the means of the corresponding general populations. Now here, at the very outset of our consideration of panmixia arises a very real difficulty, which is vital for the whole theory of evolution by natural selection. According to Mr. GALTON the population being stable, or no secular natural selection or reproductive selection taking place, there is a regression of the offspring of selected parents towards the mean of a certain general population, and the "grandchildren" also regress to the same mean. We shall see then that unless correlation is perfect $\left(r\frac{\sigma_2}{\sigma_1} = 1\right)$ no amount of continued selection would suffice to prevent a race from regressing to an original general population when that selection was suspended. Panmixia in the sense of its most ardent supporters would be demonstrated. But the difficulty is not the establishment of panmixia, but as to what is to be considered the "original general population." On the theory of evolution by natural selection that general population has itself been produced by a series of selections, and selections probably affecting its mean as well as its standard-deviation, hence how is it possible to pick out any particular stage of general population as the "focus of regression," and assert that regression of the offspring of parents now selected takes place towards that stage of evolution? Where is the focus of regression to be placed for the profile angle of man? About 80° to 90° or nearer the 40° to 70° of the anthropoid apes? The further back the better for those who believe that suspension and reversal of natural selection are identical, but no manipulating whatever of the human mortality tables would allow for a "focus of regression" very considerably below that of the current general population. Hence it would seem essential that successive selections must connote some progression of the focus of regression. This progression may be continuous with continuous natural selection, or it may take place by starts and leaps, as

* We can at once restore the true notion of regression, as Mr. GALTON points out to me, by measuring each organ or characteristic in terms of its own standard-deviation. It will then be a coefficient of correlation and a proper fraction.

indicated in Mr. GALTON's idea of organic stability. In either case panmixia would only carry back the mean to the current focus of regression, and so be a very minute reversal of natural selection.

What our theory really shows is a regression of the offspring of selected parents towards the mean of general offspring. This latter mean, supposing no secular natural selection, can, it seems to me, only be determined by experiment. It can hardly agree with the general parental mean, if the parents themselves are the product of natural selection. On the other hand, the statistics actually obtained for stable, or sensibly stable, populations seem to mark a focus of regression close to the mean of the current population, and, therefore, a progression of the focus due to past selection.* Meanwhile, till experiment has settled how continuous selection affects the focus of regression, we may see whither extreme hypotheses lead us. Such are:

(1.) The focus of regression remains stable during selection.

(2.) The focus of regression is the mean of the population from which parents have been selected.

(*e.*) *Focus of Regression Stable during Selection.*

(i.) *Steady Selection cannot be Secular or Produce Truer Breeding.*—We have seen that on this hypothesis ancestry, as distinct from immediate parentage, is indifferent. Thus, in the case of parthenogenetic reproduction, or of sexual reproduction with one parent selected, we have seen that one selection leads to the distribution (Cor. 1, p. 301):

$$\zeta = r_{01}\frac{\sigma_0}{\sigma_1}h_1, \quad \Sigma^2 = \sigma_0^2(1 - r_{01}^2) + r_{01}^2\frac{\sigma_0^2}{\sigma_1^2}s_1^2,$$

and if out of this we again select a parentage, defined by h_1 and s_1, we shall obtain the same distribution of offspring, and this however often the process be repeated. We must increase the divergence (h_1) of the selected from the general population or its concentration ($1/s_1$) or both, if we require any progressive effect from continual selection. The same remarks apply to bi-parental selection (m_1, m_2, s_2, s_3, ρ_{12}). Persistent selection only suffices to keep the mean and variation at a definite distance from those of the general population. Or, on the hypothesis of a stationary focus of regression, we conclude that *steady selection, however long it persists, can only be periodic and not secular.*

This point seems of such importance that it may be best to illustrate it by an example drawn from our Table I. and Table III. The mean height of fathers being about 69″·2, the regression of the average sons of fathers of 6′ in height is about 1″·25, or the average height of sons of a 6′ fatherhood = 70″·45. Hence, if we select fathers forming a normal distribution of any standard round 6′, we shall have a normal distribution of sons round 70″·45. If we select a second parentage from

* The determination of the focus of regression for some organ in selected domestic ducks for several generations and comparison with the means for wild and general domestic ducks would seem a possibility.

those taller sons averaging 6′, their sons will still only average 70″·45, and, however long we persist in this process of selection, we shall produce no secular change; the population will remain after the p^{th} selection just where it was both as to mean and variation after the first. The only way to produce a secular change is to continually increase the standard of the selected (or to alter the focus of regression). No steady selection would appear to produce "truer breeding."

(ii.) *Panmixia and Uni-parental Regression.*—Continual selection of the same magnitude for p generations, merely giving us the same mean and variation, we may now ask what would be the effect of suspending natural selection for q generations.

Take first the case of parthenogenetic reproduction, or that of uni-parental regression. The first parentage after suspension of natural selection will have $m_2 r_3 \sigma_1/\sigma_2$ for its mean, and $\sqrt{\{\sigma_1^2 (1 - r_3^2) + s_2^2 r_3^2 \sigma_1^2/\sigma_2^2\}}$ for its standard-deviation. Successive parentages can be found by substituting these values successively in themselves for the quantities m_2 and s_2. We find at once that after q generations of suspended selection the mean of the population will differ from the focus of regression by

$$m_2 (r_3 \sigma_1/\sigma_2)^q,$$

and the standard-deviation will be given by

$$\Sigma_q^2 = \sigma_1^2 (1 - r_3^2) \frac{1 - (r_3 \sigma_1/\sigma_2)^{2q-2}}{1 - (r_3 \sigma_1/\sigma_2)^2} + (r_3 \sigma_1/\sigma_2)^{2q-2} s_1^2.$$

Now if the population simply repeat itself without any natural selection (if there be no reproductive selection at work) $\sigma_1 = \sigma_2$, and in most cases I have come across $r_3 \sigma_1/\sigma_2$ is a fraction. Hence, as q is indefinitely increased $m_2 (r_3 \sigma_1/\sigma_2)^q$ becomes indefinitely small, and $\Sigma_q^2 = \sigma_1^2$, or $= \sigma_1^2 \dfrac{1 - r_3^2}{1 - (r_3 \sigma_1/\sigma_2)^2}$, if σ_1 be not equal to σ_2.

We see, therefore, that both as to mean and variation the population with suspended natural selection tends to rapidly regress to the general population from which it was selected. This is still true if there has been a continuous secular, as distinguished from a periodic natural selection, for we have only to suppose m_2 and s_2 to be the final result of such selection. If then the focus of regression does not progress with continuous selection, all that has been asserted as to the effect of suspended natural selection holds, at least so far as concerns a return to the condition of things which prevailed when the focus of regression was the mean of the general population. But unfortunately the advocates of panmixia want more than this, namely, either an indefinite regression of the focus of regression itself, or to place it, if steady, at an indefinitely distant point. The first result would be perfectly parallel with our second hypothesis—a progression of the focus of regression,—but would demand rather a reversal than a suspension of natural selection. The second result seems quite inconsistent with any statistics of successive generations yet taken; it demands a mortality due to natural selection, which its propounders have hardly appreciated.

(iii.) *Panmixia and Bi-parental Regression.*—The process by which corresponding results may be deduced for bi-parental selection may now be briefly indicated.

We suppose both natural selection and assortative mating to have gone on in any manner for any number of generations, the final effect, however, if the focus of regression be not changed, will be:

$$\begin{aligned}
\text{Mean of males} &= \beta_2 m_2 + \beta_3 m_3 = \mu_1, \text{ say,} \\
\text{Mean of females} &= \beta'_2 m_2 + \beta'_3 m_3 = \mu'_1, \text{ say,} \\
(\text{S.D. of males})^2 &= \sigma^2 + \beta_2^2 s_2^2 + \beta_3^2 s_3^2 + 2\beta_2\beta_3 s_2 s_3 \rho = \epsilon_1^2, \\
(\text{S.D. of females})^2 &= \sigma'^2 + \beta'^2_2 s_2^2 + \beta'^2_3 s_3^2 + 2\beta'_2\beta'_3 s_2 s_3 \rho = \eta_1^2,
\end{aligned}$$

where m_2, m_3, s_2, s_3, ρ defines the last step of the natural and sexual selections, and β'_2, β'_3 are the regression-coefficients for females.

Now, selection of all sorts ceasing, we must use for the regression-coefficients no longer their values modified by sexual selection, but simply:

$$\begin{aligned}
\beta_2 &= r_3\,\sigma_1/\sigma_2, & \beta_3 &= r_2\,\sigma_1/\sigma_3, \\
\beta'_2 &= r'_5\,\sigma'_1/\sigma'_2, & \beta'_3 &= r'_2\,\sigma'_1/\sigma'_3, \\
\sigma^2 &= \sigma_1^2\,(1 - r_2^2 - r_3^2) & \sigma'^2 &= \sigma'^2_1\,(1 - r'^2_2 - r'^2_3),
\end{aligned}$$

obtained from the general values, p. 287, by putting $r_1 = 0$. Here r'_2 and r'_3 are respectively the maternal and paternal correlation coefficients for inheritance in the female line. Further, we have very closely $\sigma_1 = \sigma_2 = \sigma'_2$ and $\sigma'_1 = \sigma'_3 = \sigma_3$. If μ_p, μ'_p give the male and female means, ϵ_p, η_p the male and female standard-deviations, after p generations in which natural and sexual selection have both been suspended, we have:

$$\begin{aligned}
\mu_p &= \beta_2\mu_{p-1} + \beta_3\mu'_{p-1}, \\
\mu'_p &= \beta'_2\mu_{p-1} + \beta'_3\mu'_{p-1}, \\
\epsilon_p^2 &= \sigma^2 + \beta_2^2\epsilon_{p-1}^2 + \beta_3^2\eta_{p-1}^2, \\
\eta_p^2 &= \sigma'^2 + \beta'_2\epsilon_{p-1}^2 + \beta'^2_3\eta_{p-1}^2.
\end{aligned}$$

Solving the equations for the means first, we have:

$$\mu_p = \mathrm{A}_1\gamma^{p-1}{}_1 + \mathrm{A}_2\gamma^{p-1}{}_2,$$

$$\mu'_p = \mathrm{A}_1\,\frac{(\gamma_1 - \beta_2)}{\beta_3}\,\gamma_1^{p-1} + \mathrm{A}_2\,\frac{\gamma_2 - \beta_2}{\beta_3}\,\gamma_2^{p-1},$$

where

$$A_1 = \frac{\mu'_1\beta_3 + \mu_1(\beta_2 - \gamma_2)}{\gamma_1 - \gamma_2},$$

$$A_2 = -\frac{\mu'_1\beta_3 + \mu_1(\beta_2 - \gamma_1)}{\gamma_1 - \gamma_2},$$

and

$$\left.\begin{matrix}\gamma_1\\ \gamma_2\end{matrix}\right\} = \tfrac{1}{2}\{\beta_2 + \beta'_3 \pm \sqrt{(\beta_2 - \beta'_3)^2 + 4\beta_3\beta'_2}\}.$$

Since the β's for parental inheritance will be < ·5, it follows that γ_1 and γ_2 are proper fractions, hence by taking p sufficiently large, we can make μ_p and μ'_p as small as we please.

This result is equally true whether the β's be those for assortative mating or not. Thus we conclude that suspended natural selection, whether accompanied by sexual selection or not, would ultimately result in a regression of means to the foci of regression of the two sexes.

(iv.) *Panmixia for Human Stature.*—It is worth while illustrating this by an example. Let us suppose that owing to natural selection, the *mean* of the male human population were pushed up to 4″ above its present level, and the *mean* of the female population were pushed up 3″ above its present level, and then let us inquire how they would regress in p generations of suspended natural selection with and without that factor of sexual selection we have termed assortative mating.

(a.) *Without Assortative Mating.*—We must take the values of the β's from Table III.:

$$\beta_2 = \cdot 4456, \quad \beta_3 = \cdot 3384, \quad \beta'_2 = \cdot 3096, \quad \beta'_3 = \cdot 2932.$$

Further

$$\mu_1 = 4'', \qquad \mu'_1 = 3''.$$

We find

$$\gamma_1 = \cdot 7069, \qquad \gamma_2 = \cdot 0419,$$
$$A_1 = 3\cdot 9549, \qquad A_2 = \cdot 0451,$$

whence

$$\mu_p = 3\cdot 9549\,(\cdot 7067)^{p-1} + \cdot 0451\,(\cdot 0419)^{p-1},$$
$$\mu'_p = 3\cdot 0538\,(\cdot 7067)^{p-1} - \cdot 0538\,(\cdot 0419)^{p-1}.$$

Thus, in four generations ($p = 5$) the males will have sunk to ·9876″ and the females to ·7626″ from the old means* before natural selection started, while in nine generations ($p = 10$), the mean of the males will have sunk to ·2036″, and the mean of the females to ·1816″ from the old means; thus the means of the general populations of both sexes have been sensibly carried back by panmixia to the focus of regression.

* The smallness of the contributions given by the second terms in the values of μ_p, μ'_p is to be noted.

(b.) *With Assortative Mating.*—We must take the values of the β's from Table XIII.:

$$\beta_2 = \cdot 4176, \quad \beta_3 = \cdot 2997, \quad B'_2 = \cdot 2895, \quad B'_3 = \cdot 2609.$$

We find

$$\gamma_1 = \cdot 6440 \qquad \gamma_2 = \cdot 0344,$$
$$A_1 = 3\cdot 9893 \qquad A_2 = \cdot 0107,$$

whence

$$\mu_p = 3\cdot 9893\ (\cdot 6440)^{p-1} + \cdot 0107\ (\cdot 0344)^{p-1},$$
$$\mu'_p = 3\cdot 0136\ (\cdot 6440)^{p-1} - \cdot 0136\ (\cdot 0344)^{p-1}.$$

As before, we note the small importance of the second terms. After four generations ($p = 5$), we have $\mu_p = \cdot 6862$ and $\mu'_p = \cdot 5184$; while after nine generations we have $\mu_p = \cdot 1180$ and $\mu'_p = \cdot 0892$.

Now the effect of assortative mating here, even so little of it as may be detected in regard to stature in human mating, is of the exactly opposite character to what some of the current language on panmixia would have led us to believe. *The more assortative mating the more rapid is the regression.* The maximum of regression would be reached, if this factor of sexual selection exhibited perfect correlation.* Hence, assortative mating, if unaccompanied by a stringent natural selection, appears rather to emphasize than retard the action of panmixia.

(v.) *Effect of Panmixia on Variation.*—We now turn to the second part of our problem, the determination of the standard deviations after p generations of suspended natural selection and assortative mating. This involves the solution of the equations ϵ_p and η_p on p. 310.

We find

$$\epsilon_p^2 = \epsilon_1^2 + C_1 \frac{g_1^{p-1} - 1}{g_1 - 1} + C_2 \frac{g_2^{p-1} - 1}{g_2 - 1},$$

$$\eta^2_p = \eta_1^2 + C_1 \frac{g_1 - \beta_2^2}{\beta_3^2} \frac{g_1^{p-1} - 1}{g_1 - 1} + C_2 \frac{g_2 - \beta_2^2}{\beta_3^2} \frac{g_2^{p-1} - 1}{g_2 - 1},$$

where

$$\left.\begin{matrix} g_1 \\ g_2 \end{matrix}\right\} = \tfrac{1}{2} \{\beta_2^2 + \beta_3'^2 \pm \sqrt{(\beta_2^2 - \beta_3'^2)^2 + 4\beta_3^2\beta_2'^2},$$

and C_1 and C_2 are to be found from

* This is not absolutely accurate, for r_2 and r_3 are not equal, so that all the β's do not take their smallest value for $r_1 = 1$. But assuming r_2 and r_3, r'_2 and r'_3 not very sensibly different, the result stated would practically follow. The whole reasoning in the text is, indeed, subject to another limitation, it is supposed that the constants of parental inheritance and of assortative mating are independent and characteristic of the race. The former, however, may really depend upon the latter. The dependence is very improbably so close as to reverse the principle stated.

$$C_1 + C_2 = \sigma^2 + (\beta_2^2 - 1)\,\epsilon_1^2 + \beta_3^2\eta_1^2,$$
$$C_1 g_1 + C_2 g_2 = \beta_2^2\,(\sigma^2 + (\beta_2^2 - 1)\,\epsilon_1^2 + \beta_3^2\eta_1^2) + \beta_3^2\,(\sigma'^2 + \beta_2'^2\epsilon_1^2 + (\beta_3'^2 - 1)\,\eta_1^2).$$

ϵ_p^2 and η_p^2 can then be found at once, if the values of the constants are known.

Remembering that g_1 and g_2 will be proper fractions, we can easily find the effect of continued panmixia by putting $p = \infty$.

We have

$$\epsilon_\infty^2 = \epsilon_1^2 + \frac{C_1 + C_2 - C_1 g_2 - C_2 g_1}{g_1 g_2 - (g_1 + g_2) + 1} = \frac{\sigma^2\,(1 - \beta_3'^2) + \sigma'^2\beta_3^2}{\beta_2^2\beta_3'^2 - \beta_3^2\beta_2'^2 - \beta_2^2 - \beta_3'^2 + 1},$$

after some rather lengthy reductions. Similarly

$$\eta_\infty^2 = \frac{\sigma^2\beta_2'^2 + \sigma'^2\,(1 - \beta_2^2)}{\beta_2^2\beta_3'^2 - \beta_3^2\beta_2'^2 - \beta_2^2 - \beta_3'^2 + 1}.$$

If we substitute in these the values of the β's, and of σ and σ' given on p. 310, we find :

$$\epsilon_\infty = \sigma_1, \qquad \eta_\infty = \sigma'_1.$$

Thus we see that indefinitely prolonged panmixia carries back not only the means of both sexes, but their distributions about the means to the state of affairs when the foci of regression were themselves the means of the population.*

The all-important question concerning panmixia is, as we have seen, that of the position and stability of the focus of regression, and it seems to me that this is a question which it is only possible to settle by experiments. Nor do the experiments, at least from the theoretical standpoint, seem attended by difficulties which are insuperable. It is not necessary to select a parthenogenetically reproductive race, it is not necessary even to select both parents, it would be sufficient to deal with the regression from one selected parent, if this were most convenient.† The simple test is this :—If M_1 be the mean of selected parents, m_1 the mean of their offspring, and M_2 be the mean of another group of selected parents (*e.g.*, selected out of the

* In order to ascertain whether the standard deviations would return to their old values, supposing natural selection to be suspended, but assortative mating maintained, we should have to solve a series of equations of the type :

$$\epsilon_p^2 = \sigma^2 + \beta_2^2\epsilon_{p-1}^2 + \beta_3^2\eta_{p-1}^2 + 2\beta_2\beta_3 r_1\epsilon_{p-1}\eta_{p-1},$$
$$\eta_p^2 = \sigma'^2 + \beta'^2_2\epsilon_p r + \beta'^2_3\eta_{p-1}^2 + 2\beta'_2\beta'_3 r_1\epsilon_{p-1}\eta_{p-1},$$

and then substitute the values of σ_1, σ_1' and the β's from p. 286 in ϵ_∞ and η_∞. I have not yet solved these equations. In turning the above formulae into numbers, the caution given in the footnote, p. 312, must be borne in mind, *i.e.*, the correlation coefficients for inheritance during assortative mating may differ somewhat from those holding when it is suspended.

† Perhaps a common father and series of selected mothers would give the best results.

group m_1 by any series of selections and breedings) and m_2 their offspring-mean, is $(M_2m_1 - M_1m_2)/(M_2 - m_2 - M_1 + m_1)$ constant for all stages of selection? If it be, it is the stable focus of regression, and $1 - \frac{m_1 - m_2}{M_1 - M_2}$ is the coefficient of regression.

(f.) *Progression of the Focus of Regression with Natural Selection.*

(i.) *General Remarks on Regression and Fixedness of Character.*—Our first hypothesis certainly favours the general views of those who support the doctrine of panmixia, although to be quite consistent they must:

(i.) Place the focus of regression back at the zero size of an organ or the zero degree of intensity of a characteristic.

(ii.) Assume much nearer approximation to unity in their coefficients of regression than any measurement as yet suggests, or

(iii.) Demand a far higher mortality of *periodic* natural selection than has anywhere as yet been demonstrated.

Professor WEISMANN has no difficulty, apparently, about (i.): "As soon as natural selection ceases to operate upon any character, structural or functional, it begins to *disappear*." ("Essays on Heredity," 1889, p. 90.) He talks of functionless organs losing in size with the suspension of natural selection "until the last remnant finally *disappears*" (*ibid.*, p. 292), while "the disposition of the tail to become rudimentary, in cats and dogs, may be explained in the simplest way, by the process which I have formerly called panmixia," *i.e.*, suspension of natural selection (*ibid.*, p. 430). This explanation "in the simplest way" fails entirely to say whether (ii.) or (iii.) is to be accepted after assuming the truth of (i.). What is quite clear is that in the only case where either the coefficients of regression or the mortality can at present be even approximately stated neither (ii.) nor (iii.) hold. Fox-terriers and domestic ducks may be bred with a comparatively small mortality, but how great must be the coefficients of regression if their foci of regression are to be placed only as far back, say, as at general populations of jackals and wild ducks.* Apart from cases of atavism, which may be looked upon as improbable variations amply allowed for by theory, we do note, even in dogs, a regression towards a distant ancestry (DARWIN: "Animals and Plants under Domestication," vol. 1, pp. 37, *et seq.*). In these cases, however, change of environment seems in some way more important than the suspension of natural selection. We have, so far, evidence in favour of Mr. GALTON's view of positions of stability for the focus of regression. It seems, indeed, to be a general opinion among breeders that a character can be fixed, a stock made to breed truer by repeated selection.

Thus DARWIN writes on "Fixedness of Character:" "It is a general belief amongst breeders, that the longer any character has been transmitted by a breed, the more fully it will continue to be transmitted. I do not wish to dispute the truth of

* Professor WEISMANN would place the focus of regression for domestic ducks much further back, presumably in a wingless stage. ("Essays on Heredity," p. 90.)

the proposition that inheritance gains strength simply through long continuance, but I doubt whether it can be proved. In one sense the proposition is little better than a truism; if any character has remained constant during many generations, it will be likely to continue so, if the conditions of life remain the same. So again in improving the breed, if care be taken for a length of time to exclude all inferior individuals, the breed will obviously tend to become truer, as it will not have been crossed during many generations by an inferior animal." ("Animals and Plants under Domestication," vol. 2, p. 37.)

Down to the words "if the conditions of life remain the same," all is consistent with the extreme theory of panmixia, but making a breed *truer* by selection for many generations is only consistent with belief in a progression of the focus of regression, or in a change towards unity in the coefficient of regression with continued selection. The latter alternative would, I think, be quite inconsistent with our whole theory of heredity as applied to a practically stable population. As we cannot mathematically deal with a theory of progression of the focus of regression without some hypothesis of the nature of progression with continued selection, we will assume an extreme case, and suppose the focus to progress very rapidly, *i.e.*, that offspring regress to the mean of the population from which their parents have been immediately selected. This will at least offer some explanation of animals breeding truer with persistent selection, if at the same time it leads to results inconsistent with the extreme theory of panmixia.

(ii.) *Panmixia and Bi-parental Selection.*—Let h_1, s_1 be the paternal, h_2, s_2 the maternal distribution at each selection. Then with assortative mating after p generations, the standard-deviations of the male and female populations will be of the same form as after one generation and be given by the ϵ_1, η_1 of p. 310. Now this result is not like the stable focus of regression out of accord, I think, with experience. It is noteworthy how comparatively little difference there is in the variation constants of the different races of man, although in many cases pretty severe selection may have been supposed to have been in progress for many generations. For example, the mean cephalic index varies from 70 to 83, but the probable deviation from this mean only varies from about 2 to 2·7, so that even very primitive races (where the variation is small and we may suppose the selection has been severe, or the strain is very pure), do not "breed much truer" than highly civilised races with a far less mortality. The difference between the variation of the most and least variable races is probably not more than the β-terms in the values of ϵ_1 and η_1 (p. 310) may be able to account for.

Turning now to the alteration of the male and female means in p-generations of selection, let as before β_2, β_3, β'_2, β'_3, be the regression coefficients and u_n, v_n, the distances from m_2, m_3, of the means of the male and female populations out of which the nth bi-parentage (m_2, m_3, s_2, s_3, ρ) is selected.

Hence : $u_n - (\beta_2 u_n + \beta_3 v_n)$ and $v_n - (\beta'_2 u_n + \beta'_3 v_n)$ are the distances from m_2, m_3

of the means of the male and female populations from which the $\overline{n+1}^{\text{th}}$ bi-parentage is selected.

Thus we have the finite difference equations:

$$u_{n+1} = u_n - (\beta_2 u_n + \beta_3 v_n)$$
$$v_{n+1} = v_n - (\beta'_2 u_n + \beta'_3 v^n)$$

Assume:

$$u_n = A\chi^{n-1}, \quad v_n = \beta\chi^{n-1}.$$

Hence:

$$A(\chi - 1 + \beta_2) = -B\beta_3, \quad B(\chi - 1 + \beta'_3) = -A\beta'_2,$$

or,

$$(\chi - 1)^2 + (\beta_2 + \beta'_3)(\chi - 1) + \beta_2\beta'_3 - \beta_3\beta'_2 = 0,$$

or,

$$\chi_1 = 1 - \gamma_1 \text{ and } \chi_2 = 1 - \gamma_2,$$

where γ_1 and γ_2 have the same values as on p. 311. Thus:

$$u_p = A_1(1-\gamma_1)^{p-1} + A_2(1-\gamma_2)^{p-1},$$

$$v_p = A_1\frac{\gamma_1 - \beta_2}{\beta_3}(1-\gamma_1)^{p-1} + A_2\frac{\gamma_2 - \beta_2}{\beta_3}(1-\gamma_1)^{p-1};$$

where

$$A_1 = \frac{(\beta_2 - \gamma_2)m_2 + \beta_3 m_3}{\gamma_1 - \gamma_2}$$

$$A_2 = -\frac{(\beta_2 - \gamma_1)m_2 + \beta_3 m_3}{\gamma_1 - \gamma_2}.$$

Now this solution* is the same as that on p. 311, except (i.) that u_p and v_p, unlike μ_p and μ'_p, are measured from the selected means, *i.e.*, the mean heights of the male and female populations are respectively $m_2 - u_p$ and $m_3 - v_p$ after p-generations; (ii.) that in the values u_p and v_p $(1-\gamma_1)^{p-1}$ and $(1-\gamma_2)^{p-1}$ replace γ_1^{p-1} and γ_2^{p-1}. We conclude, accordingly, since γ_1, γ_2, and, therefore, $1-\gamma_1$, $1-\gamma_2$ are proper fractions, that u_p and v_p grow smaller and smaller, or, if selection be long enough continued, the means of the male and female populations will ultimately pass to the selection means.

Of course, if selection be suspended at the n^{th} generation, regression will take place as on p. 310, but only to the nearest focus of regression, *i.e.*, $m_2 - u_n$, $m_3 - v_n$. Thus the effect of n selections has been to raise the general means permanently by these amounts.

* The uniparental or parthenogenitic results for progression of the focus follow at once by simply putting $\beta_3 = \beta'_2 = \beta'_3 = 0$ in the above formulæ.

(iii.) *Panmixia for Human Stature.*—It is instructive to note the value of these expressions for the case of stature in man. We have at once from the numbers on p. 311, supposing p-selections of male and female populations averaging 4″ and 3″ above the present mean, the following results :

(a.) *Without Assortative Mating.*

$$u_p = 3\cdot9549\,(\cdot2933)^{p-1} + \cdot0451\,(\cdot9581)^{p-1},$$
$$v_p = 3\cdot0538\,(\cdot2933)^{p-1} - \cdot0538\,(\cdot9581)^{p-1}.$$

Thus, in five generations ($p = 5$) $u_5 = \cdot0673$ and $v_5 = -\cdot0227$, or the male and female means have been raised 3″·9327 and 3″·0227 respectively. Thus, we see that the males have been raised by selection very near to the selection average, while the females have actually been raised beyond it.* Thus, continued selection would now keep down, and not raise, the female mean, panmixia corresponding to a rise in the mean.

(b.) *With Assortative Mating.*

$$u_p = 3\cdot9893\,(\cdot3560)^{p-1} + \cdot0107\,(\cdot9656)^{p-1}$$
$$v_p' = 3\cdot0136\,(\cdot3560)^{p-1} - \cdot0136\,(\cdot9656)^{p-1}.$$

Thus, in five generations, $u_5 = \cdot0744$ and $v_5 = \cdot0366$, or the male and female means have been raised 3″·9256 and 2″·9634 respectively. The means are accordingly raised less rapidly with this form of sexual relation, the female mean, indeed, having in the five generations not yet overshot the selection mean.

(iv.) *Concluding Remarks on Regression and Fixedness of Character.*—Accordingly on this hypothesis, with the correlation coefficients of inheritance anything like their value in man, five generations of selections of the type required in *both* parents would suffice to establish a breed. This seems more or less consonant with breeders' opinions, which, in part at any rate, may be presumed to represent their experience. If, however, anything like this hypothesis be true, then the suspension of natural selection would not be followed by a rapid regression, or even a slow persistent regression, that would require a reversal of natural selection, *i.e.*, a selection of those previously destroyed and a destruction of those previously selected. On this hypothesis, indeed, it would be probably best to keep the term panmixia for that suspension of assortative mating which we have seen assists, rather than retards, the processes of natural selection.

Several fairly sound reasons could be given why the focus of regression should be taken as the mean of the population from which the parents have been selected, but the sole safe argument appears to be *experiment*.

* This results, of course, from breeding from an average father very much taller relatively than the average mother selected.

The two hypotheses with which we have dealt give practically the two extremes; observation and experiment are perfectly able to determine between them, or to settle whether an intermediate theory is necessary which will give a progression, but a slower progression, to the focus of regression. There are many ways in which analysis can put on the brake, if it be really needful.

At present, all this memoir proposes is to show that such subjects as inheritance, regression, assortative mating and panmixia, are capable of perfectly direct quantitative treatment, and that such treatment, and not somewhat vague discussion of individual instances or of metaphysical possibilities, is what alone can settle the chief problems of evolution. What is wanted is a wide extension of the experimental and statistical work of Mr. FRANCIS GALTON and Professor WELDON. Such numbers as appear in this memoir must be looked upon as illustrative and tentative only. I hope later to publish, for a very limited field, namely, skull measurements in man, a more complete numerical study with mathematical discussion of variation and correlation.

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VIII. *The Rotation of an Elastic Spheroid.*

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Communicated by Professor G. H. Darwin, *F.R.S.*

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Introduction.

It is well known that, if a rigid body whose principal moments of inertia are $\mathfrak{A}$, $\mathfrak{A}$, $\mathfrak{C}$ be set rotating about its axis of symmetry, and then be subjected to a slight disturbance, it will execute oscillations about its mean position, in consequence of which the axis of rotation will undergo periodic displacements relatively to the body in a period which bears to the period of rotation the ratio $\mathfrak{A} : \mathfrak{C} - \mathfrak{A}$. The object of the present investigation is to determine to what extent this period will be modified if the body, instead of being perfectly rigid, is capable of elastic deformations.

The problem has important bearings in connection with the theory of the Earth's rotation. The remarkable researches of Dr. S. C. Chandler, published in a series of papers in the 'Astronomical Journal,'* have placed it almost beyond a doubt that the axis of rotation of the Earth is subject to a series of displacements, the most important of which consists of a periodic motion in all respects similar in character to the oscillation mentioned above, but having a period considerably in excess of that which theory would require if the Earth could be regarded as perfectly rigid. It is natural to suppose that this motion has its origin in the same cause, but that the theory by which the period has previously been assigned is in some respects defective. The most plausible attempt which has yet been made to correct this theory is that given by Newcomb,† who shows, by an elegant geometrical method, that the elasticity of the solid portions of the Earth and the mobility of the ocean will each have the effect of prolonging the period. He then proceeds to obtain a numerical estimate of this extension, basing his calculations on certain results given by Thomson and Tait‡ with reference to the deformation of an elastic sphere. In order to make these results applicable, several assumptions have to be made which do not appear to me

* Vol. 11, *et seq.* For a summary of Chandler's results, *vide* 'Science,' May 3, 1895.

† 'Monthly Notices of the Royal Astronomical Society,' March, 1892.

‡ 'Natural Philosophy,' Part II., § 837.

8.5.96

to be well founded; and it is with a view to examining these assumptions that I have attempted to exhibit the solution of the problem in an analytical form.

The analysis in the present paper is confined to the case of a homogeneous spheroid of revolution, composed of isotropic, incompressible, gravitating material, while no account is taken of the surface waters. For further simplification we have also supposed that the figure conforms to that required for hydrostatic equilibrium, so that when the body is undisturbed we may suppose it free from strain in its interior. We have reason to suppose that this condition is approximately realized in the case of the Earth.

In §§ 1–2, I have obtained the rigorous dynamical equations for the oscillations of such a system. The form of these equations is, however, such as to render an approximate solution necessary. Hence it has been assumed in the subsequent sections that the ellipticity of the spheroid and, consequently, the angular velocity of rotation, are small quantities. This assumption leads to a considerable reduction in the differential equations which express the elastic displacements, and, in fact, reduces them to the familiar equations for the equilibrium of a strained elastic body The boundary conditions are also similar in form to the surface equations which are obtained in treating of the problem of the deformations of an elastic sphere, and as the solution of this problem is well known, we are in a position to obtain a solution of our differential equations applicable to the problem in hand. §§ 3–4 are devoted to the transformation of the equations into a form convenient for solution, while the actual solution is given in § 5.

The method of approximation followed up to this point fails to lead to a determination of the period. When, however, the body is supposed perfectly rigid, we are able to determine the period accurately by means of the equations of angular momentum for the whole system. This suggests the use of an analogous process, which is employed in § 6, to determine the period when elastic deformations are taken into account.

The principal results of this paper will be found in § 8, where they are compared with the hypotheses made by NEWCOMB. It is found that the general character of the motion agrees with that assumed by NEWCOMB, but that his quantitative law as to the displacement of the pole, due to elastic distortion, is slightly in error. The bearing of these results on the theory of the Earth's constitution is discussed in the final section (§ 9).

§ 1. *Differential Equations of Motion of Isotropic, Incompressible, Elastic Solid, referred to axes rotating uniformly.*

Take as axis of z the axis of rotation, and let the system be rotating with angular velocity ω about this axis. Let $x_0 + u$, $y_0 + v$, $z_0 + w$ be the coordinates at the time

t of the particle, which, when the body is unstrained, is at the point x_0, y_0, z_0. The velocity components of this particle will be

$$\begin{aligned}
\mathrm{U} &= \frac{d}{dt}(x_0 + u) - \omega(y_0 + v) = \dot{u} - \omega(y_0 + v),\\
\mathrm{V} &= \frac{d}{dt}(y_0 + v) + \omega(x_0 + u) = \dot{v} + \omega(x_0 + u),\\
\mathrm{W} &= \frac{d}{dt}(z_0 + w) \qquad\qquad\qquad\quad = \dot{w}.
\end{aligned}$$

The components of acceleration will be

$$\begin{aligned}
\dot{\mathrm{U}} - \mathrm{V}\omega &= \ddot{u} - \dot{v}\omega - \{\dot{v} + \omega(x_0 + u)\}\,\omega = \ddot{u} - 2\omega\dot{v} - \omega^2 u - \omega^2 x_0,\\
\dot{\mathrm{V}} + \mathrm{U}\omega &= \ddot{v} + \dot{u}\omega + \{\dot{u} - \omega(y_0 + v)\}\,\omega = \ddot{v} + 2\omega\dot{u} - \omega^2 v - \omega^2 y_0,\\
\dot{\mathrm{W}} &\qquad\qquad\qquad\qquad\qquad\qquad\quad = \ddot{w}.
\end{aligned}$$

If P, Q, R, S, T, U denote the six components of stress at the point x, y, z (where $x = x_0 + u$, &c.), X, Y, Z the components of bodily force at this point, and ρ the density of the material, the equations of motion may be written down in the same manner as if the axes were fixed, provided that we replace the accelerations d^2u/dt^2, d^2v/dt^2, d^2w/dt^2 by the values we have found above. Thus we have*

$$\begin{aligned}
\frac{\partial \mathrm{P}}{\partial x} + \frac{\partial \mathrm{U}}{\partial y} + \frac{\partial \mathrm{T}}{\partial z} + \rho\mathrm{X} &= \rho(\ddot{u} - 2\omega\dot{v} - \omega^2 u - \omega^2 x_0),\\
\frac{\partial \mathrm{U}}{\partial x} + \frac{\partial \mathrm{Q}}{\partial y} + \frac{\partial \mathrm{S}}{\partial z} + \rho\mathrm{Y} &= \rho(\ddot{v} + 2\omega\dot{u} - \omega^2 v - \omega^2 y_0),\\
\frac{\partial \mathrm{T}}{\partial x} + \frac{\partial \mathrm{S}}{\partial y} + \frac{\partial \mathrm{R}}{\partial z} + \rho\mathrm{Z} &= \rho\ddot{w}.
\end{aligned}$$

We have here employed two different sets of independent variables. The quantities involved on the left have been supposed to be expressed as functions of the variables x, y, z, t, that is to say, the coordinates of a definite point on which we fix our attention and the time ; these variables are analogous to the Eulerian system in Hydrodynamics. On the other hand, the quantities u, v, w on the right have been regarded as functions of x_0, y_0, z_0, t, which correspond to the Lagrangian system of independent variables in Hydrodynamics. It is desirable for us to retain only one set of variables ; we propose to select the former set and proceed to examine the modified form of the equations of small motion.

If the symbol d/dt be used as above to denote partial differentiation with respect

* LOVE, 'Elasticity,' vol. 1, p. 60.

to the time, on the supposition that x_0, y_0, z_0, remain constant, and $\partial/\partial t$ be used when x, y, z, are the other independent variables, we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{du}{dt}\frac{\partial}{\partial x} + \frac{dv}{dt}\frac{\partial}{\partial y} + \frac{dw}{dt}\frac{\partial}{\partial z}.$$

Hence, if we neglect squares and products of the small quantities u, v, w, we obtain

$$\dot{u} = du/dt = \partial u/\partial t, \qquad \ddot{u} = d^2u/dt^2 = \partial^2 u/\partial t^2, \text{ \&c.}$$

Thus, the only modification necessary on the right will be to replace $x_0 + u$ by x, and $y_0 + v$ by y, and the equations of motion become

$$\left.\begin{aligned} \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} + \rho X &= \rho\,(\ddot{u} - 2\omega\dot{v} - \omega^2 x) \\ \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} + \rho Y &= \rho\,(\ddot{v} + 2\omega\dot{u} - \omega^2 y) \\ \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} + \rho Z &= \rho\ddot{w} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

We have retained the fluxional notation, it being understood that the dots now denote differentiation with regard to the time on the supposition that x, y, z remain constant.

Since the stress-strain relations do not involve differentiation with regard to the time, they may be written down in the same manner as when the axes are fixed. To the degree of approximation to which we are going, we may replace $\partial u/\partial x_0$, &c., by $\partial u/\partial x$, &c., and therefore when the material is isotropic and incompressible, the components of stress are given by

$$\left.\begin{aligned} P &= -p + 2\mathfrak{n}\frac{\partial u}{\partial x_0} = -p + 2\mathfrak{n}\frac{\partial u}{\partial x} \\ Q &= -p + 2\mathfrak{n}\frac{\partial v}{\partial y_0} = -p + 2\mathfrak{n}\frac{\partial v}{\partial y} \\ R &= -p + 2\mathfrak{n}\frac{\partial w}{\partial z_0} = -p + 2\mathfrak{n}\frac{\partial w}{\partial z} \\ S &= \mathfrak{n}\left(\frac{\partial w}{\partial y_0} + \frac{\partial v}{\partial z_0}\right) = \mathfrak{n}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ T &= \mathfrak{n}\left(\frac{\partial u}{\partial z_0} + \frac{\partial w}{dx_0}\right) = \mathfrak{n}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ U &= \mathfrak{n}\left(\frac{\partial v}{\partial x_0} + \frac{\partial u}{\partial y_0}\right) = \mathfrak{n}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where p denotes the hydrostatic pressure at x, y, z, and $\mathfrak{n}$ the rigidity.

If further, the bodily forces be derivable from a potential function V, so that

$$X = \partial V/\partial x, \quad Y = \partial V/\partial y, \quad Z = \partial V/\partial z \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

on substituting the values (2), (3), in (1), we obtain

$$\frac{\mathfrak{n}}{\rho}\nabla^2 u + \frac{\partial}{\partial x}\left\{V + \tfrac{1}{2}\omega^2(x^2+y^2) - \frac{p}{\rho}\right\} = \ddot{u} - 2\omega\dot{v},$$

$$\frac{\mathfrak{n}}{\rho}\nabla^2 v + \frac{\partial}{\partial y}\left\{V + \tfrac{1}{2}\omega^2(x^2+y^2) - \frac{p}{\rho}\right\} = \ddot{v} + 2\omega\dot{u},$$

$$\frac{\mathfrak{n}}{\rho}\nabla^2 w + \frac{\partial}{\partial z}\left\{V + \tfrac{1}{2}\omega^2(x^2+y^2) - \frac{p}{\rho}\right\} = \ddot{w}.$$

Putting $\mathfrak{n}/\rho = n$, $V + \frac{1}{2}\omega^2(x^2+y^2) - p/\rho = \psi$, these equations take the form

$$\left.\begin{aligned} n\nabla^2 u + \partial\psi/\partial x &= \ddot{u} - 2\omega\dot{v}, \\ n\nabla^2 v + \partial\psi/\partial y &= \ddot{v} + 2\omega\dot{u}, \\ n\nabla^2 w + \partial\psi/\partial z &= \ddot{w}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

We must in addition express the fact that the material is incompressible; this is done by the equation

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The equations (4), (5), which are the rigorous equations for the vibrations of a rotating, incompressible, elastic solid, are theoretically sufficient to determine u, v, w, ψ, subject to certain boundary conditions. Eliminating v, u in turn from the first two of equations (4), we obtain

$$\left[\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^2}{\partial t^2}\right]u = -\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\frac{\partial\psi}{\partial x} + 2\omega\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial y}\right),$$

$$\left[\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^2}{\partial t^2}\right]v = -\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\frac{\partial\psi}{\partial y} - 2\omega\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial x}\right).$$

Applying the operators $\partial/\partial x$, $\partial/\partial y$, and making use of (5),

$$\left[\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^2}{\partial t^2}\right]\frac{\partial w}{\partial z} = \left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right).$$

Hence, on eliminating w by means of the third of (4), we obtain the following equation for ψ:—

$$\left[\nabla^2\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)^2 + 4\omega^2\frac{\partial^4}{\partial t^2 . \partial z^2}\right]\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

In the case where n is zero, it will be noticed that the last equation reduces to POINCARÉ's differential equation for the oscillations of a rotating mass of liquid.* It may easily be shown by retaining any one of the four quantities u, v, w, ψ, and eliminating the other three, that each one of these quantities is a solution of the equation (6).

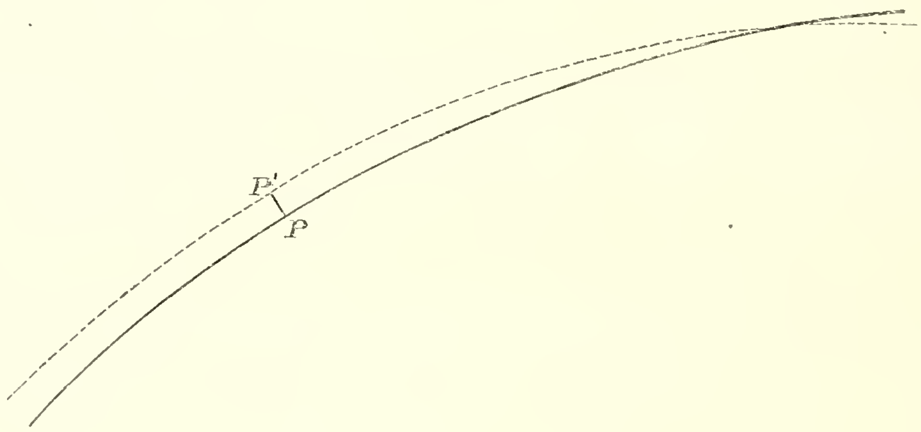
If the motion relatively to the moving axes consists of a simple harmonic vibration in period $2\pi/\lambda$, we may suppose u, v, w, ψ each proportional to $e^{i\lambda t}$, and the equations (4), (5), (6) will then become

$$\left.\begin{aligned} (n\,\nabla^2+\lambda^2)\,u+2\omega i\lambda v &= -\,\partial\psi/\partial x,\\ (n\,\nabla^2+\lambda^2)\,v-2\omega i\lambda u &= -\,\partial\psi/\partial y,\\ (n\,\nabla^2+\lambda^2)\,w \qquad\qquad &= -\,\partial\psi/\partial z,\\ \partial u/\partial x+\partial v/\partial y+\partial w/\partial z &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (7),$$

$$[\nabla^2(n\,\nabla^2+\lambda^2)^2-4\omega^2\lambda^2\,\partial^2/\partial z^2]\,\psi=0 \quad \ldots\ldots\ldots \quad (8).$$

§ 2. *Boundary Conditions.*

The conditions to be satisfied at the boundary are that the components of surface-traction should vanish at all points of the displaced surface. We proceed to replace these conditions by certain analytical conditions at the mean surface.



Take a point P on the mean surface and let the normal at P meet the surface of the distorted body in P'.

Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction-cosines of the normal PP' and $\cos\alpha+l_1$, $\cos\beta+m_1$, $\cos\gamma+n_1$ the direction-cosines of the normal at P' to the displaced surface; also let $PP'=\zeta$. Then l_1, m_1, n_1, ζ will be small quantities of the order of the displacements u, v, w.

Let P, Q, R, S, T, U denote the components of stress at P and the same letters

* 'Acta Mathematica,' vol. 7, p. 356.

accented the components of stress at P′, and let dn' denote an element of the normal to the mean surface. Then to our order of approximation we have

$$\mathrm{P}' = \mathrm{P} + \zeta\, \partial\mathrm{P}/\partial n', \quad \mathrm{Q}' = \mathrm{Q} + \zeta\, \partial\mathrm{Q}/\partial n', \text{ \&c.}$$

Thus the component of surface-traction at P′ parallel to the axis of x is

$$\mathrm{P}'(\cos\alpha + l_1) + \mathrm{U}'(\cos\beta + m_1) + \mathrm{T}'(\cos\gamma + n_1),$$

$$= \left(\mathrm{P} + \zeta\frac{\partial\mathrm{P}}{\partial n'}\right)(\cos\alpha + l_1) + \left(\mathrm{U} + \zeta\frac{\partial\mathrm{U}}{\partial n'}\right)(\cos\beta + m_1) + \left(\mathrm{T} + \zeta\frac{\partial\mathrm{T}}{\partial n'}\right)(\cos\gamma + n_1),$$

$$= \mathrm{P}\cos\alpha + \mathrm{U}\cos\beta + \mathrm{T}\cos\gamma + \zeta\left(\frac{\partial\mathrm{P}}{\partial n'}\cos\alpha + \frac{\partial\mathrm{U}}{\partial n'}\cos\beta + \frac{\partial\mathrm{T}}{\partial n'}\cos\gamma\right)$$
$$+ l_1\mathrm{P} + m_1\mathrm{U} + n_1\mathrm{T},$$

if we neglect small quantities of the second order.

Now in the small terms we may replace P, Q, R, &c., by their values in the steady motion. Since we have supposed the body when undisturbed to be free from tangential stress in its interior, we have in this case

$$\mathrm{P} = \mathrm{Q} = \mathrm{R} = -p, \quad \mathrm{S} = \mathrm{T} = \mathrm{U} = 0$$

throughout, while at the surface also $p = 0$.

Therefore $l_1\mathrm{P} + m_1\mathrm{U} + n_1\mathrm{T}$, $\partial\mathrm{U}/\partial n'$, $\partial\mathrm{T}/\partial n'$ may be put equal to zero, and $\frac{\partial\mathrm{P}}{\partial n'} = -\frac{\partial p}{\partial n'} = -\rho\frac{\partial}{\partial n'}\{\mathrm{V} + \tfrac{1}{2}\omega^2(x^2 + y^2)\} = +\rho g$ say, where g denotes the value of gravity (inclusive of centrifugal force) at the surface. Thus the x-component of surface-traction at P′ is

$$\mathrm{P}\cos\alpha + \mathrm{U}\cos\beta + \mathrm{T}\cos\gamma + \rho g\zeta\cos\alpha.$$

Introducing the values of P, U, T in terms of the displacements and equating this expression to zero, we obtain

$$-p\cos\alpha + \mathfrak{n}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma\right) + \mathfrak{n}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial v}{\partial x}\cos\beta + \frac{\partial w}{\partial x}\cos\gamma\right)$$
$$= -\rho g\zeta\cos\alpha;$$

and, in like manner, by considering the components of surface-traction in the directions of the axes of y and z,

$$-p\cos\beta + \mathfrak{n}\left(\frac{\partial v}{\partial x}\cos\alpha + \frac{\partial v}{\partial y}\cos\beta + \frac{\partial v}{\partial z}\cos\gamma\right) + \mathfrak{n}\left(\frac{\partial u}{\partial y}\cos\alpha + \frac{\partial v}{\partial y}\cos\beta + \frac{\partial w}{\partial y}\cos\gamma\right)$$
$$= -\rho g\zeta\cos\beta,$$
$$-p\cos\gamma + \mathfrak{n}\left(\frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma\right) + \mathfrak{n}\left(\frac{\partial u}{\partial z}\cos\alpha + \frac{\partial v}{\partial z}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma\right)$$
$$= -\rho g\zeta\cos\gamma.$$

These equations express the fact that the surface-tractions at the mean surface are equivalent to a normal stress equal to the weight of the harmonic inequality, and they might have been written down at once from this consideration. It has, however, been thought preferable to verify them at length; it will be seen that the shorter procedure is only justifiable in the case where the material is initially in a state of hydrostatic equilibrium. If in the zero configuration this condition is not satisfied the form of the boundary equations will be much more complicated.

Let us now replace p by the function ψ of the previous section. By the definition of ψ we have

$$p/\rho = \mathrm{V} + \tfrac{1}{2}\omega^2(x^2+y^2) - \psi$$
$$= \text{non-periodic terms} + v' - \psi,$$

where v' denotes the potential due to the harmonic inequalities.

The non-periodic terms vanish at the surface in virtue of the conditions for steady motion, and therefore the boundary equations may be written

$$\left.\begin{aligned}
&\psi\cos\alpha + n\left\{\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma + \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial v}{\partial x}\cos\beta + \frac{\partial w}{\partial x}\cos\gamma\right\} \\
&\qquad = (v' - g\zeta)\cos\alpha \\
&\psi\cos\beta + n\left\{\frac{\partial v}{\partial x}\cos\alpha + \frac{\partial v}{\partial y}\cos\beta + \frac{\partial v}{\partial z}\cos\gamma + \frac{\partial u}{\partial y}\cos\alpha + \frac{\partial v}{\partial y}\cos\beta + \frac{\partial w}{\partial y}\cos\gamma\right\} \\
&\qquad = (v' - g\zeta)\cos\beta \\
&\psi\cos\gamma + n\left\{\frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma + \frac{\partial u}{\partial z}\cos\alpha + \frac{\partial v}{\partial z}\cos\beta + \frac{\partial w}{\partial z}\cos\gamma\right\} \\
&\qquad = (v' - g\zeta)\cos\gamma
\end{aligned}\right\} \quad (9),$$

where

$$\zeta = u\cos\alpha + v\cos\beta + w\cos\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

The rigorous method of procedure would be first to solve the equation (8) for ψ. Having found ψ we could introduce its value into the right-hand members of (7) and proceed to find solutions of these equations consistent with the boundary conditions (9). Unfortunately the form of equation (8) is such as to make it appear hopeless to carry out this process, and we must have recourse to some method of approximation before we can advance further.

§ 3. *Change of Variables.*

An elastic body, such as that with which we are dealing, will, of course, be capable of an infinite number of independent normal types of vibration. The equations of motion we have found in § 1 are applicable to any one of these types, while the substitution of the solutions in the boundary equations should lead to an equation for the determination of the frequencies. In general, the values of u, v, w will become very small when n is large in such a manner that nu, nv, nw approach finite limits, while the admissible values of λ become large of the order $n^{\frac{1}{2}}$. That type of oscillation with which we are concerned is, however, unique in character in that it continues to exist even when the rigidity is perfect. If, then, in the expressions for u, v, w in terms of n we suppose n to be made infinite, u, v, w should approach finite limits which we will denote by u_0, v_0, w_0.

The quantities u_0, v_0, w_0 denote the displacements of a body which is supposed perfectly rigid. Now the most general small displacement of such a body consists of a translation whose components we may denote by ξ, η, ζ, and a rotation whose components we denote by θ_1, θ_2, θ_3. Thus, the most general values of u_0, v_0, w_0 geometrically possible are

$$\xi - y\theta_3 + z\theta_2,\ \eta - z\theta_1 + x\theta_3,\ \zeta - x\theta_2 + y\theta_1.$$

When, however, we are dealing with the rotation of a rigid body not subject to external disturbing force the quantities ξ, η, ζ, θ_3 will not appear, and we may take

$$u_0 = z\theta_2,\ v_0 = -z\theta_1,\ w_0 = -x\theta_2 + y\theta_1 \quad \ldots \ldots \quad (11).$$

Let us now suppose that

$$u = u_0 + u_1,\ v = v_0 + v_1,\ w = w_0 + w_1,$$

where u_0, v_0, w_0 have the values (11). This is equivalent to supposing that the body is first displaced by rotation as a whole through small angles θ_1, θ_2 about Ox, Oy and is then subjected to elastic distortion, the displacements due to the distortions being u_1, v_1, w_1.

Further, let $\zeta = \zeta_0 + \zeta_1$, $v' = v'_0 + v'_1$, where

$$\left.\begin{aligned} \zeta_0 &= u_0 \cos\alpha + v_0 \cos\beta + w_0 \cos\gamma \\ \zeta_1 &= u_1 \cos\alpha + v_1 \cos\beta + w_1 \cos\gamma \end{aligned}\right\} \quad \ldots \ldots \quad (12),$$

and where v'_0, v'_1 denote the parts of v' due to the harmonic inequalities ζ_0, ζ_1 respectively.

If V_0 denote the non-periodic part of V, $\mathrm{V}_0 + v'_0$ will be the potential at the point

x, y, z due to the attraction of the body when it is rotated without distortion through small angles θ_1, θ_2. If then x_1, y_1, z_1 denote the coordinates of this same point referred to axes obtained by rotating the old axes with the body, on putting in evidence the arguments of the function V_0, we have

$$V_0(x, y, z) + v'_0 = V_0(x_1, y_1, z_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

But the direction cosines of the two sets of axes are given by the scheme

	x	y	z
x_1	1	0	$-\theta_2$
y_1	0	1	θ_1
z_1	θ_2	$-\theta_1$	1

whence $x_1 = x - z\theta_2$, $y_1 = y + z\theta_1$, $z_1 = z + x\theta_2 - y\theta_1$.

Introducing these values in (12) and expanding by TAYLOR's theorem, we find

$$\begin{aligned} V_0(x, y, z) + v'_0 &= V_0(x - z\theta_2,\ y + z\theta_1,\ z + x\theta_2 - y\theta_1) \\ &= V_0(x, y, z) - z\theta_2 \frac{\partial V_0}{\partial x} + z\theta_1 \frac{\partial V_0}{\partial y} + (x\theta_2 - y\theta_1)\frac{\partial V_0}{\partial z} \end{aligned}$$

But from the definition of g we have, at the surface,

$$g \cos \alpha = -\frac{\partial V_0}{\partial x} - \omega^2 x, \ g \cos \beta = -\frac{\partial V_0}{\partial y} - \omega^2 y, \ g \cos \gamma = -\frac{\partial V_0}{\partial z}$$

and, therefore,

$$\begin{aligned} v'_0 &= z\theta_2 (g \cos \alpha + \omega^2 x) - z\theta_1 (g \cos \beta + \omega^2 y) - (x\theta_2 - y\theta_1) g \cos \gamma \\ &= g (u_0 \cos \alpha + v_0 \cos \beta + w_0 \cos \gamma) + \omega^2 (\theta_2 xz - \theta_1 yz), \end{aligned}$$

or

$$v'_0 - g\zeta_0 = \omega^2 (\theta_2 xz - \theta_1 yz),$$

whence, finally,

$$v' - g\zeta = \omega^2 (\theta_2 xz - \theta_1 yz) + v'_1 - g\zeta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If now we change our variables from u, v, w to u_1, v_1, w_1, the equations of motion become

$$\left.\begin{aligned} n\nabla^2 u_1 + \lambda^2 u_1 + 2\omega i\lambda v_1 &= -\partial\psi/\partial x - \lambda^2 z\theta_2 + 2\omega i\lambda z\theta_1 \\ n\nabla^2 v_1 + \lambda^2 v_1 - 2\omega i\lambda u_1 &= -\partial\psi/\partial y + \lambda^2 z\theta_1 + 2\omega i\lambda z\theta_2 \\ n\nabla^2 w_1 + \lambda^2 w_1 &= -\partial\psi/\partial z + \lambda^2 (x\theta_2 - y\theta_1) \\ \partial u_1/\partial x + \partial v_1/\partial y + \partial w_1/\partial z &= 0 \end{aligned}\right\} \quad . \quad . \quad (15),$$

while in virtue of (14) the boundary equations may be written

$$\left.\begin{aligned} &\psi\cos\alpha + n\left\{\frac{\partial u_1}{\partial x}\cos\alpha + \frac{\partial u_1}{\partial y}\cos\beta + \frac{\partial u_1}{\partial z}\cos\gamma + \frac{\partial u_1}{\partial x}\cos\alpha + \frac{\partial v_1}{\partial x}\cos\beta + \frac{\partial w_1}{\partial x}\cos\gamma\right\} \\ &\qquad - (v'_1 - g\zeta_1)\cos\alpha = \omega^2(\theta_2 xz - \theta_1 yz)\cos\alpha, \\ &\psi\cos\beta + n\left\{\frac{\partial v_1}{\partial x}\cos\alpha + \frac{\partial v_1}{\partial y}\cos\beta + \frac{\partial v_1}{\partial z}\cos\gamma + \frac{\partial u_1}{\partial y}\cos\alpha + \frac{\partial v_1}{\partial y}\cos\beta + \frac{\partial w_1}{\partial y}\cos\gamma\right\} \\ &\qquad - (v'_1 - g\zeta_1)\cos\beta = \omega^2(\theta_2 xz - \theta_1 yz)\cos\beta, \\ &\psi\cos\gamma + n\left\{\frac{\partial w_1}{\partial x}\cos\alpha + \frac{\partial w_1}{\partial y}\cos\beta + \frac{\partial w_1}{\partial z}\cos\gamma + \frac{\partial u_1}{\partial z}\cos\alpha + \frac{\partial v_1}{\partial z}\cos\beta + \frac{\partial w_1}{\partial z}\cos\gamma\right\} \\ &\qquad - (v'_1 - g\zeta_1)\cos\gamma = \omega^2(\theta_2 xz - \theta_1 yz)\cos\gamma. \end{aligned}\right\} (16).$$

§ 4. *Reduction of the Equations when the Body is of the form of a Spheroid of Small Ellipticity.*

The only assumption we have made as yet as to the form of the free surface is that it is a possible figure of equilibrium for a rotating mass of liquid, so that the body may be free from strain when rotating uniformly. The simplest form which can occur and that which presents the greatest interest is the case of a spheroid of revolution of small ellipticity ϵ. We propose for the future to confine ourselves to this case. If we neglect the square of ϵ the angular velocity of rotation is related to ϵ by the equation

$$\epsilon = 15\omega^2/16\pi\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where the density ρ is expressed in gravitational units. If then the units of length and time be so chosen that ρ is finite, ω will be a small quantity of the order $\epsilon^{\frac{1}{2}}$.

Take as the equation to the free surface $r = a\{1 + \epsilon T_2\}$ where

$$T_2 = (x^2 + y^2 - 2z^2)/3a^2.$$

Then the direction cosines of the normal to the surface $r = a\ \{1 + \epsilon Q_n\}$, where Q_n is a solid harmonic of order n, are

$$\frac{x}{r} + a\epsilon\left\{\frac{nx}{r^2}Q_n - \frac{\partial Q_n}{\partial x}\right\}, \text{ \&c.,}$$

and thus we have

$$\cos\alpha = \frac{x}{r} + a\epsilon\left\{\frac{2x}{r^2}T_2 - \frac{\partial T_2}{\partial x}\right\},$$

$$\cos\beta = \frac{y}{r} + a\epsilon\left\{\frac{2y}{r^2}T_2 - \frac{\partial T_2}{\partial y}\right\},$$

$$\cos\gamma = \frac{z}{r} + a\epsilon\left\{\frac{2z}{r^2}T_2 - \frac{\partial T_2}{\partial z}\right\},$$

whence

$$\begin{aligned}\zeta_0 &= z\theta_2\cos\alpha - z\theta_1\cos\beta - (\theta_2 x - \theta_1 y)\cos\gamma \\ &= a\epsilon\left[\theta_2\left(x\frac{\partial T_2}{\partial z} - z\frac{\partial T_2}{\partial x}\right) + \theta_1\left(z\frac{\partial T_2}{\partial y} - y\frac{\partial T_2}{\partial z}\right)\right] \\ &= a\epsilon\left[-\theta_2 \, . \, \frac{2xz}{a^2} + \theta_1 \, . \, \frac{2yz}{a^2}\right],\end{aligned}$$

or

$$\zeta_0 = -\frac{2\epsilon}{a}[\theta_2 xz - \theta_1 yz] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Now observation indicates that in the case of the Earth the oscillation in question differs but slightly in type from the motion of a rigid body, whence we conclude that u_1, v_1, w_1 must be small compared with u, v, w. We propose therefore to make the assumptions, leaving the verification thereof to our subsequent work, that u_1, v_1, w_1, ψ contain the small factor ω^2, while λ is a small quantity of the order ω^3. The latter assumption is justifiable when the body is perfectly rigid, since in this case we have

$$\frac{\lambda}{\omega} = \frac{\mathfrak{C} - \mathfrak{A}}{\mathfrak{A}} = \epsilon = \frac{15\omega^2}{16\pi\rho}.$$

If then, we neglect small quantities of the order ω^4 or ϵ^2, the equations of motion (15) reduce to

$$\left.\begin{aligned} n\nabla^2 u_1 = -\frac{\partial\psi}{\partial x}, \quad n\nabla^2 v_1 = -\frac{\partial\psi}{\partial y}, \quad n\nabla^2 w_1 = -\frac{\partial\psi}{\partial z}, \\ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (19).$$

Since all the terms in the boundary equations involve the small factor ω^2, we may replace $\cos\alpha$, $\cos\beta$, $\cos\gamma$, by x/r, y/r, z/r respectively, with errors only of the same order ω^4, and thus the approximate form of the boundary equations is

$$\left.\begin{aligned}\psi x + n\left(r\frac{\partial}{\partial r} - 1\right)u_1 + n\frac{\partial}{\partial x}(u_1x + v_1y + w_1z) - (v_1' - g\zeta_1)\,x \\ = x\,.\,\omega^2(\theta_2xz - \theta_1yz), \\ \psi y + n\left(r\frac{\partial}{\partial r} - 1\right)v_1 + n\frac{\partial}{\partial y}(u_1x + v_1y + w_1z) - (v_1' - g\zeta_1)\,y \\ = y\,.\,\omega^2(\theta_2xz - \theta_1yz), \\ \psi z + n\left(r\frac{\partial}{\partial r} - 1\right)w_1 + n\frac{\partial}{\partial z}(u_1x + v_1y + w_1z) - (v_1' - g\zeta_1)\,z \\ = z\,.\,\omega^2(\theta_2xz - \theta_1yz).\end{aligned}\right\} \quad (20).^*$$

To the same order we may suppose these equations to hold good at the surface $r = a$, instead of at the surface $r = a\{1 + \epsilon T_2\}$.

§ 5. *Determination of the Elastic Distortions.*

The modified forms (19) of the equations of motion are simply the equations to which we are led in determining the displacements in a strained elastic solid and the solutions of them, when either the displacements or the surface-tractions at the surface of a sphere are given, are well known.† We may readily adapt these solutions so as to satisfy the boundary conditions (20).

Denote for brevity the function $\omega^2(\theta_2xz - \theta_1yz)$ by S_2.

From (19) we obtain at once $\nabla^2\psi = 0$. This equation replaces the more complicated form (8); a particular solution of it is $\psi = AS_2$.

Introducing this value of ψ into the left-hand members of (19), we find

$$\nabla^2u_1 = -\frac{A}{n}\frac{\partial S_2}{\partial x}, \quad \nabla^2v_1 = -\frac{A}{n}\frac{\partial S_2}{\partial y}, \quad \nabla^2w_1 = -\frac{A}{n}\frac{\partial S_2}{\partial z},$$

particular solutions of which are

$$u_1 = -\tfrac{1}{10}\frac{A}{n}r^2\frac{\partial S_2}{\partial x}, \quad v_1 = -\tfrac{1}{10}\frac{A}{n}r^2\frac{\partial S_2}{\partial y}, \quad w_1 = -\tfrac{1}{10}\frac{A}{n}r^2\frac{\partial S_2}{\partial z} \quad . \quad (21).$$

These do not satisfy the last of equations (19), but they make

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = -\tfrac{2}{5}\frac{A}{n}S_2.$$

* For the physical significance of these equations, *vide* § 8 *infra*.

† *Vide* LOVE, 'Elasticity,' vol. 1, chap. 10.

We must therefore add to the particular integrals just found complementary functions which satisfy the equations

$$\left.\begin{aligned} &\nabla^2 u_1 = 0, \quad \nabla^2 v_1 = 0, \quad \nabla^2 w_1 = 0 \\ &\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = + \tfrac{2}{5}\frac{A}{n} S_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Now the functions $r^7 \frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)$, $r^7 \frac{\partial}{\partial y}\left(\frac{S_2}{r^5}\right)$, $r^7 \frac{\partial}{\partial z}\left(\frac{S_2}{r^5}\right)$ are spherical harmonic functions which remain finite at the origin, and hence, if we take

$$\left.\begin{aligned} u_1 &= B\frac{\partial S_2}{\partial x} + Cr^7 \frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right), \\ v_1 &= B\frac{\partial S_2}{\partial y} + Cr^7 \frac{\partial}{\partial y}\left(\frac{S_2}{r^5}\right), \\ w_1 &= B\frac{\partial S_2}{\partial z} + Cr^7 \frac{\partial}{\partial z}\left(\frac{S_2}{r^5}\right), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

the first three of equations (22) will be satisfied identically, and the fourth will also be satisfied if

$$-3.7.C = \tfrac{2}{5}\frac{A}{n}, \quad \text{or} \quad C = -\tfrac{2}{105}\frac{A}{n}.$$

Therefore, adding together the particular integrals (21) and the complementary functions (23), we obtain as a set of solutions of the equations of motion which remain finite at the origin

$$\left.\begin{aligned} \psi &= AS_2, \\ u_1 &= -\tfrac{1}{10}\frac{A}{n} r^2 \frac{\partial S_2}{\partial x} + B\frac{\partial S_2}{\partial x} - \tfrac{2}{105}\frac{A}{n} r^7 \frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right), \\ v_1 &= -\tfrac{1}{10}\frac{A}{n} r^2 \frac{\partial S_2}{\partial y} + B\frac{\partial S_2}{\partial y} - \tfrac{2}{105}\frac{A}{n} r^7 \frac{\partial}{\partial y}\left(\frac{S_2}{r^5}\right), \\ w_1 &= -\tfrac{1}{10}\frac{A}{n} r^2 \frac{\partial S_2}{\partial z} + B\frac{\partial S_2}{\partial z} - \tfrac{2}{105}\frac{A}{n} r^7 \frac{\partial}{\partial z}\left(\frac{S_2}{r^5}\right), \end{aligned}\right\} \quad . \quad . \quad (24).$$

We must now prove that these solutions are sufficiently general to satisfy the boundary conditions (20) at the surface of the sphere $r = a$.

From (24) we obtain

$$u_1 x + v_1 y + w_1 z = -\tfrac{1}{5}\frac{A}{n} r^2 S_2 + 2BS_2 + \tfrac{2}{35}\frac{A}{n} r^2 S_2 = -\tfrac{1}{7}\frac{A}{n} r^2 S_2 + 2BS_2 \quad . \quad (25).$$

Since in the small terms which already contain S_2 as a factor we may treat the spheroid as a sphere of radius a, we have

$$\zeta_1 = u_1 \frac{x}{a} + v_1 \frac{y}{a} + w_1 \frac{z}{a} = \frac{1}{a}\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right] S_2 \quad . \quad . \quad . \quad . \quad . \quad (26),$$

and therefore

$$v_1' = \tfrac{4}{5}\pi\rho a \zeta_1 = \tfrac{4}{5}\pi\rho\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right] S_2,$$

whence

$$v_1' - g\zeta_1 = (\tfrac{4}{5}\pi\rho a - \tfrac{4}{3}\pi\rho a)\,\zeta_1 = -\tfrac{8}{15}\pi\rho\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right] S_2 \quad . \quad . \quad (27).$$

Consider now the different terms of the left-hand members of the boundary equations (20). By means of the formula

$$xS_2 = \frac{r^2}{5}\frac{\partial S_2}{\partial x} - \frac{r^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)$$

we obtain at the boundary

$$x\psi = A\left\{\frac{a^2}{5}\frac{\partial S_2}{\partial x} - \frac{a^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\right\}.$$

Also we have

$$\left(r\frac{\partial}{\partial r} - 1\right)u_1 = -\tfrac{1}{5}\frac{A}{n}r^2\frac{\partial S_2}{\partial x} - \tfrac{4}{105}\frac{A}{n}r^7\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right),$$

and from (25)

$$\frac{\partial}{\partial x}(u_1x + v_1y + w_1z) = -\tfrac{2}{7}\frac{A}{n}xS_2 - \tfrac{1}{7}\frac{A}{n}r^2\frac{\partial S_2}{\partial x} + 2B\frac{\partial S_2}{\partial x}$$

$$= -\tfrac{2}{7}\frac{A}{n}\left\{\frac{r^2}{5}\frac{\partial S_2}{\partial x} - \frac{r^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\right\} - \tfrac{1}{7}\frac{A}{n}r^2\frac{\partial S_2}{\partial x} + 2B\frac{\partial S_2}{\partial x};$$

therefore, at the surface,

$$\frac{\partial}{\partial x}(u_1x + v_1y + w_1z) = -\tfrac{1}{5}\frac{Aa^2}{n}\frac{\partial S_2}{\partial x} + 2B\frac{\partial S_2}{\partial x} + \tfrac{2}{35}\frac{A}{n}a^7\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right).$$

Lastly, from (27),

$$x(v_1' - g\zeta_1) = -\tfrac{8}{75}\pi\rho\left[2B - \tfrac{1}{7}\frac{Aa^2}{n}\right]\left[a^2\frac{\partial S_2}{\partial x} - a^7\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\right]$$

and therefore the first boundary equation becomes

$$\frac{\partial S_2}{\partial x}\left\{A\frac{a^2}{5} - \tfrac{1}{5}Aa^2 - \tfrac{1}{5}Aa^2 + 2nB + \tfrac{8}{75}\pi\rho a^2\left(2B - \tfrac{1}{7}\frac{Aa^2}{n}\right)\right\}$$
$$+ a^5\cdot\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right)\left\{-A\frac{a^2}{5} - \tfrac{4}{105}Aa^2 + \tfrac{2}{35}Aa^2 - \tfrac{8}{75}\pi\rho a^2\left(2B - \tfrac{1}{7}\frac{Aa^2}{n}\right)\right\}$$
$$= \frac{a^2}{5}\frac{\partial S_2}{\partial x} - \frac{a^7}{5}\frac{\partial}{\partial x}\left(\frac{S_2}{r^5}\right).$$

The remaining boundary equations can be written down by replacing x by y, z respectively. Hence, all three boundary equations will be satisfied provided A, B are subject to the relations

$$-\mathrm{A}a^2 + 10n\mathrm{B} + \tfrac{8}{15}\pi\rho a^2\left(2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n}\right) = a^2,$$

$$\mathrm{A}a^2 - \tfrac{2}{21}\mathrm{A}a^2 + \tfrac{8}{15}\pi\rho a^2\left(2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n}\right) = a^2.$$

Solving these equations, we obtain

$$\mathrm{A} = \frac{21}{19 + \frac{8}{3}\dfrac{\pi\rho a^2}{n}}, \quad \mathrm{B}n = \frac{4a^2}{19 + \frac{8}{3}\cdot\dfrac{\pi\rho a^2}{n}},$$

$$2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n} = \frac{5a^2/n}{19 + \frac{8}{3}\dfrac{\pi\rho a^2}{n}}.$$

Now, let $\epsilon' = 5\omega^2 a^2/38n$, so that ϵ' will denote the ellipticity which would be induced in a sphere of radius a by centrifugal force when distortion is resisted by elasticity alone.* Then the above values may be written

$$\mathrm{A} = \frac{21}{19\{1 + \epsilon'/\epsilon\}}, \quad \mathrm{B}\omega^2 = \frac{8\epsilon'/5}{1 + \epsilon'/\epsilon},$$

$$\left(2\mathrm{B} - \tfrac{1}{7}\frac{\mathrm{A}a^2}{n}\right)\omega^2 = \frac{2\epsilon'}{1 + \epsilon'/\epsilon}.$$

Finally from (24)

$$\left.\begin{aligned}
u = u_0 + u_1 &= z\theta_2 + \mathrm{B}\omega^2 z\theta_2 - \tfrac{5}{42}\frac{\mathrm{A}a^2\omega^2}{n}\frac{r^2}{a^2}\theta_2 z + \tfrac{2}{21}\frac{\mathrm{A}a^2\omega^2}{n}\frac{x(\theta_2 xz - \theta_1 yz)}{a^2} \\
&= z\theta_2\left\{1 + \frac{\frac{8}{5}\epsilon' - \frac{r^2}{a^2}\epsilon'}{1 + \epsilon'/\epsilon}\right\} + \tfrac{4}{5}\frac{\epsilon'}{1 + \epsilon'/\epsilon}\frac{x(\theta_2 xz - \theta_1 yz)}{a^2} \\
v = v_0 + v_1 &= -z\theta_1\left\{1 + \frac{\frac{8}{5}\epsilon' - \frac{r^2}{a^2}\epsilon'}{1 + \epsilon'/\epsilon}\right\} + \tfrac{4}{5}\frac{\epsilon'}{1 + \epsilon'/\epsilon}\frac{y(\theta_2 xz - \theta_1 yz)}{a^2} \\
w = w_0 + w_1 &= (-x\theta_2 + y\theta_1)\left\{1 - \frac{\frac{8}{5}\epsilon' - \frac{r^2}{a^2}\epsilon'}{1 + \epsilon'/\epsilon}\right\} + \tfrac{4}{5}\frac{\epsilon'}{1 + \epsilon'/\epsilon}\frac{z(\theta_2 xz - \theta_1 yz)}{a^2}
\end{aligned}\right\} \quad (28),$$

* *Cf.* THOMSON and TAIT, "Natural Philosophy," Part II., § 837.

and from (18), (26)

$$\zeta = \zeta_0 + \zeta_1 = -\frac{2\epsilon}{a}(\theta_2 xz - \theta_1 yz) + \frac{2}{a}\frac{\epsilon'}{(1+\epsilon'/\epsilon)}(\theta_2 xz - \theta_1 yz)$$
$$= -\frac{2}{a}\left\{\epsilon - \frac{\epsilon'}{1+\epsilon'/\epsilon}\right\}(\theta_2 xz - \theta_1 yz),$$
$$= -\frac{2}{a}\left\{\frac{\epsilon}{1+\epsilon'/\epsilon}\right\}(\theta_2 xz - \theta_1 yz) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

§ 6. *Determination of the Period.*

The method we have followed hitherto has enabled us to express the displacements at any point of the body by means of two arbitrary constants θ_1, θ_2, but the quantity λ, whose value it is our chief object to determine, has entirely disappeared. We have, in fact, verified that the equations (7), (8), and the boundary conditions (9) are approximately satisfied by the forms (28), while to the same order zero is the approximate value of λ. We require now to have recourse to a method which will enable us to carry our approximations to the value of λ further.

By using the well-known equations of motion of a body of changing form Professor WOODWARD* has shown that the determination of the period may be reduced to the evaluation of the disturbing angular velocities due to the flow of the material relatively to the principal axes of the body. As we have now expressed the displacements, and consequently the velocities, at any point in terms of θ_1, θ_2, which may be taken as the displacements of the body as a whole, we are in a position to calculate these disturbing angular velocities. We might then introduce their values in Professor WOODWARD's equations and proceed to the determination of the period by his method. We propose, however, to make use of equivalent equations which express that the rates of change of angular momentum for the system as a whole about the axes Ox, Oy are zero. It seems somewhat preferable, on account of the additional simplicity of the motion of the axes themselves, to refer to these axes rather than to the moving axes used by WOODWARD, which correspond with our axes Ox_1, Oy_1.

If h_1, h_2 denote the components of angular momentum about Ox, Oy, we have

$$h_1 = \iiint \{\dot{w}y - (\dot{v} + x\omega)z\}\, dm,$$

where dm denotes an element of mass, and the integral is taken throughout the volume contained by the displaced surface. Replacing the integral by an integral taken throughout the mean volume and a surface integral, we have

$$h_1 = \iiint \{(\dot{w}y - \dot{v}z) - \omega xz\}\, dm - \rho \iint \zeta\omega xz dS,$$

* 'Astronomical Journal,' xv., No. 345.

or since $\iiint xz\,dm \iint xz\,dm = 0$ and v, w, ζ are each proportional to $e^{i\lambda t}$,

$$h_1 = i\lambda \iiint (wy - vz)\,dm - \rho\omega \iint \zeta xz\,dS,$$

$$\dot{h}_1 = -\lambda^2 \iiint (wy - vz)\,dm - \rho\omega i\lambda \iint \zeta xz\,dS.$$

Similarly

$$h_2 = i\lambda \iiint (uz - wx)\,dm - \rho\omega \iint \zeta yz\,dS,$$

$$\dot{h}_2 = -\lambda^2 \iiint (wy - vz)\,dm - \rho\omega i\lambda \iint \zeta yz\,dS.$$

The equations of angular momentum are

$$\dot{h}_1 - h_2\omega = 0, \qquad \dot{h}_2 + h_1\omega = 0.$$

Replacing h_1, h_2, $\dot{h}_1$, $\dot{h}_2$ by the values we have just found we obtain the following rigorous equations

$$\left.\begin{aligned} -\lambda^2 \iiint (wy - vz)\,dm - \rho\omega i\lambda \iint \zeta xz\,dS \qquad\qquad & \\ - i\lambda\omega \iiint (uz - wx)\,dm + \rho\omega^2 \iint \zeta xz\,dS = 0, & \\ -\lambda^2 \iiint (uz - wx)\,dm - \rho\omega i\lambda \iint \zeta yz\,dS \qquad\qquad & \\ + i\lambda\omega \iiint (wy - vz)\,dm - \rho\omega^2 \iint \zeta xz\,dS = 0, & \end{aligned}\right\} \quad . \quad . \quad (30).$$

Now by using the approximate forms (28) and denoting by M the mass of the spheroid, we have

$$\iiint (wy - vz)\,dm = \iiint \{ -xy\theta_2 + (y^2 + z^2)\,\theta_1\}\,dm + \iiint (w_1 y - v_1 z)\,dm$$
$$= \tfrac{2}{5}\mathrm{M}a^2\theta_1 + \text{terms of order } \epsilon,$$

$$\iiint (uz - wx)\,dm = \iiint \{(z^2 + x^2)\,\theta_2 - xy\theta_1\}\,dm + \iiint (u_1 z - w_1 x)\,dm$$
$$= \tfrac{2}{5}\mathrm{M}a^2\theta_2 + \text{terms of order } \epsilon.$$

Also from (29),

$$\iint \zeta yz\,dS = \frac{2\theta_1}{a}\left(\frac{\epsilon}{1 + \epsilon'/\epsilon}\right) \iint y^2z^2\,dS = \tfrac{8}{15}\pi a^5\theta_1 \left(\frac{\epsilon}{1 + \epsilon'/\epsilon}\right),$$

$$\iint \zeta xz\,dS = -\tfrac{8}{15}\pi a^5\theta_2 \frac{\epsilon}{1 + \epsilon'/\epsilon},$$

the errors being of the order ϵ^2.

Hence neglecting terms of order ω^6 only (λ being regarded as of order ω^3) the equations (30) reduce to

$$-i\lambda\omega \,.\, \tfrac{2}{5}\mathrm{M}a^2\theta_2 + \rho\omega^2\theta_1 \,.\, \tfrac{8}{15}\pi a^5 \frac{\epsilon}{1 + \epsilon'/\epsilon} = 0,$$

$$+ i\lambda\omega \,.\, \tfrac{2}{5}\mathrm{M}a^2\theta_1 + \rho\omega^2\theta_2 \,.\, \tfrac{8}{15}\pi a^5 \frac{\epsilon}{1 + \epsilon'/\epsilon} = 0.$$

These equations will be consistent if $\theta_1 = i\theta_2$

$$\lambda = \frac{\tfrac{8}{15}\pi\rho a^5\omega^2 \dfrac{\epsilon}{1 + \epsilon'/\epsilon}}{\tfrac{2}{5}\mathrm{M}a^2\omega} = \omega \frac{\epsilon}{1 + \epsilon'/\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

This gives the value of λ with errors of the order ω^5.

§ 7. *Numerical Values.*

If we suppose the rigidity of our body to become perfect we should obtain $\epsilon' = 0$, and therefore $\lambda/\omega = \epsilon$.

This is the value we should arrive at if we started by neglecting the elastic distortions. We see now that it is too large and that consequently the effect of elastic deformation is to diminish the frequency or to prolong the period.

The expressions for ϵ, ϵ' are

$$\epsilon = \frac{15\omega^2}{16\pi\rho}, \qquad \epsilon' = \frac{5\omega^2 a^2}{38n}.$$

Taking the sidereal day as 86164 mean solar seconds and using BOYS's values* for the mean density of the Earth and the constant of gravitation, viz.: 5·5270 and $6{\cdot}6576 \times 10^{-8}$ we find from the above formula that for a spheroid of the same mean density as the Earth, rotating in a sidereal day,

$$\epsilon = \frac{1}{232}.$$

Again taking the Earth's mean radius as $6{\cdot}371 \times 10^8$ centimetres and $\mathfrak{n} = 8{\cdot}19 \times 10^{11}$,† which is the rigidity of steel, we find

$$n = \mathfrak{n}/\rho = 1{\cdot}482 \times 10^{11},$$

* 'Proc. Roy. Soc.,' 1894, p. 132.

† EVERETT, 'Units and Physical Constants,' pp. 61, 65.

whence

$$\epsilon' = \frac{1}{522},$$

and finally

$$\lambda/\omega = \frac{\epsilon}{1 + \epsilon'/\epsilon} = \frac{1}{335}.$$

We conclude that for a homogeneous spheroid of the same size and mean density as the Earth, the period would be extended from 232 days to 335 days in consequence of elastic distortions, if we suppose the rigidity to be that of steel.

If we take into account the variations in the density of the Earth's strata the problem presented becomes much more complicated. We must replace ϵ by the Precessional Constant, which will no longer be equal to the surface ellipticity. Its value may however be accurately determined by means of data furnished by the Theory of Precession ; this value is known to be 1/305. Hence the effective value of ϵ is diminished in the ratio 232 : 305.

As regards the effect of heterogeneity on the value of ϵ', we are, at present, only in a position to make speculations. Professor NEWCOMB points out that in calculating the mean density, greater weight should be given to the density of the superficial layers on account of their greater effective inertia, and hence ϵ' should also be diminished. A reasonable hypothesis seems to be that it is diminished in the same ratio as ϵ. If we make this hypothesis, we find that if the effective rigidity of the Earth were as great as that of steel, the period of the Eulerian nutation would become

$$\frac{305 \times 335}{232} \text{ days} = 440 \text{ days}.$$

This period is slightly in excess of CHANDLER'S observed period of 427 days. We therefore conclude that the effective rigidity of the Earth is slightly greater than that of steel.

If we make the same hypothesis as above, with regard to the effects of the variations of density, we may easily calculate what degree of rigidity would be consistent with CHANDLER'S observed period. We find for the period of a homogeneous spheroid of the same degree of rigidity

$$\frac{427 \times 232}{305} \text{ days} = 326 \text{ days}.$$

Putting $\lambda/\omega = \frac{1}{326}$, $\epsilon = \frac{1}{232}$ in (31) we obtain

$$1 + \epsilon'/\epsilon = \tfrac{326}{232}$$

$$\epsilon' = \tfrac{94}{232}\epsilon = \tfrac{1}{572},$$

and therefore

$$\mathfrak{n} = \frac{8 \cdot 19 \times 572}{522} \times 10^{11} = 8 \cdot 98 \times 10^{11}.$$

§ 8. *Physical Characteristics of the Motion.*

The height of the waves at the free surface is given by the formula (29), viz. :—

$$\zeta = -\frac{2}{a}\frac{\epsilon}{1 + \epsilon'/\epsilon}(\theta_2 xz - \theta_1 yz).$$

Comparing this with equation (18) we see that ζ may be obtained from ζ_0 by changing θ_1, θ_2 into $\theta_1 \frac{\epsilon}{\epsilon + \epsilon'}$, $\theta_2 \frac{\epsilon}{\epsilon + \epsilon'}$. Thus, to our degree of approximation, the free surface will remain a spheroid of revolution of ellipticity ϵ. The position of the axis of this spheroid may be found by rotating the axis Oz through small angles $\theta_1 \frac{\epsilon}{\epsilon + \epsilon'}$, $\theta_2 \frac{\epsilon}{\epsilon + \epsilon'}$, about O$x$, O$y$ respectively.

Again, from (28) we see that with a high degree of approximation the displacements at the point x, y, z will be given by

$$u = z\theta_2, \; v = -z\theta_1, \; w = -x\theta_2 + y\theta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

in other words, the displacements due to elastic deformation will be negligible compared with the displacements due to the rotation of the body as a whole. We have seen, however, that the elastic distortions will produce an appreciable effect in displacing the axis of figure.

The modified forms (32) indicate that θ_1, θ_2 are the displacements of an axis sensibly fixed in the earth, which we may call the mean axis of figure, while the motion, at any instant, will consist very approximately of a rotation, as a rigid body, with angular velocity ω about an axis whose direction-cosines are $\dot{\theta}_1/\omega$, $\dot{\theta}_2/\omega$, 1. This axis we shall hereafter refer to as the instantaneous axis of rotation. It is, of course, only when we neglect the displacements due to elastic distortion that such an axis exists.

We have found that when $\lambda = \omega\left(\frac{\epsilon^2}{\epsilon + \epsilon'}\right)$, $\theta_1 = i\theta_2$. Taking

$$\theta_1 = \phi e^{i\lambda(t-\tau)}$$

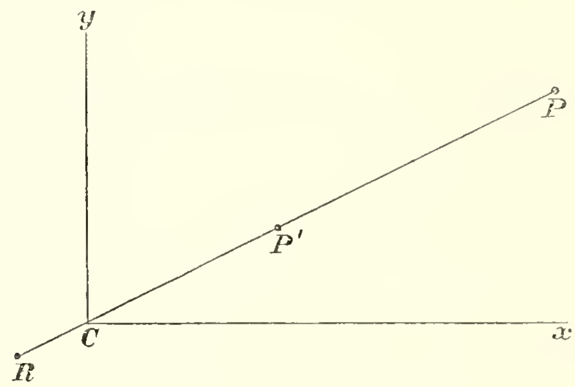
where ϕ, τ are real, we have

$$\theta_2 = -i\phi e^{i\lambda(t-\tau)}.$$

Adding to these the solutions obtained by changing the sign of i, wherever it occurs, we find as the real solution corresponding to the frequencies $\pm\,\omega\,\frac{\epsilon^2}{\epsilon+\epsilon'}$

$$\theta_1 = 2\phi\cos\lambda\,(t-\tau), \qquad \theta_2 = 2\phi\sin\lambda\,(t-\tau),$$

$$\dot\theta_1/\omega = -\,2\phi\,\frac{\lambda}{\omega}\sin\lambda\,(t-\tau), \qquad \dot\theta_2/\omega = +\,2\phi\,\frac{\lambda}{\omega}\cos\lambda\,(t-\tau).$$



Let C be the point where the axis Oz cuts the surface of the spheroid, and let us take a pair of rectangular axes Cx, Cy, having C as origin and parallel to the original axes Ox, Oy. Let P, P′, R be the points in which the mean axis of figure, the axis of the deformed figure and the instantaneous axis meet the plane Cxy. Taking the radius of the spheroid as unity, the coordinates of P are θ_2, $-\,\theta_1$, or

$$2\phi\sin\lambda\,(t-\tau), \qquad -\,2\phi\cos\lambda\,(t-\tau).$$

The coordinates of P′ are $\theta_2\frac{\epsilon}{\epsilon+\epsilon'}$, $-\,\theta_1\frac{\epsilon}{\epsilon+\epsilon'}$, or

$$2\phi\,\frac{\epsilon}{\epsilon+\epsilon'}\sin\lambda\,(t-\tau), \qquad -\,2\phi\,\frac{\epsilon}{\epsilon+\epsilon'}\cos\lambda\,(t-\tau),$$

and the coordinates of R are $\dot\theta_1/\omega$, $\dot\theta_2/\omega$, or

$$-\,2\phi\,\frac{\epsilon^2}{\epsilon+\epsilon'}\sin\lambda\,(t-\tau), \qquad +\,2\phi\,\frac{\epsilon^2}{\epsilon+\epsilon'}\cos\lambda\,(t-\tau).$$

We thus see that the points P, P′, R all lie on the same straight line through C, and that this line revolves about C with uniform angular velocity λ relatively to the moving axes Cx, Cy. These axes are themselves rotating with angular velocity ω about C in the same direction, and, hence, the actual angular velocity of the line PP′R about C is $\omega\,\{1+\epsilon^2/(\epsilon+\epsilon')\}$. The distances of R, P, P′ from C remain constant, while since CR is of the order ϵ compared with CP or CP′, which are themselves small quantities, we may suppose the point R to coincide with C. We conclude that the axis of rotation remains sensibly fixed in direction while the mean axis of figure and the axis of the distorted figure describe cones of revolution about it in a period approximately equal to the period of rotation, these three axes always remaining in a plane.

Let us next consider the motion of the instantaneous axis of rotation relatively to the Earth. Take a new set of axes Ox_1, Oy_1, Oz_1 found by rotating the old set through small angles θ_1, θ_2 about Ox, Oy; these new axes will be sensibly fixed in the Earth. The motion at any instant consists of a rotation whose components about the old axes are $\dot{\theta}_1$, $\dot{\theta}_2$, ω; resolving these rotations about the new axes, we find for the components

$$\dot{\theta}_1 - \omega\theta_2, \quad \dot{\theta}_2 + \omega\theta_1, \quad \omega.$$

Hence the direction-cosines of the axis of rotation relatively to the axes Ox_1, Oy_1, Oz_1 are

$$(\dot{\theta}_1 - \omega\theta_2)/\omega, \quad (\dot{\theta}_2 + \omega\theta_1)/\omega, \quad 1,$$

or

$$-2\phi\left(1 + \frac{\epsilon^2}{\epsilon + \epsilon'}\right)\sin\lambda(t - \tau), \quad +2\phi\left(1 + \frac{\epsilon^2}{\epsilon + \epsilon'}\right)\cos\lambda(t - \tau), \quad 1.$$

Thus, relatively to the Earth, the axis of rotation will describe a cone of revolution about the mean axis of figure with angular velocity λ, the direction of motion in this cone being the same as the direction of the Earth's rotation. This motion would manifest itself by a periodic change in the latitude of places on the Earth's surface, as found by astronomical observations.

The circumstances we have here described are the well-known characteristics of the Eulerian nutation,* with the additional feature that the axis of figure is displaced by centrifugal force towards the axis of rotation. This latter fact has been assumed by NEWCOMB† and other writers‡ as the basis of their work. The law of displacement of the pole of figure assumed by NEWCOMB is, however, not verified. We find

$$\mathrm{PR} : \mathrm{PP'R} = \epsilon + \epsilon' : \epsilon,$$

or

$$\mathrm{PP'} : \mathrm{P'R} = \epsilon' : \epsilon \quad \ldots\ldots\ldots\ldots \quad (a),$$

whereas NEWCOMB has taken

$$\mathrm{PP'} : \mathrm{P'R} = \mathrm{E'} : \epsilon \quad \ldots\ldots\ldots\ldots \quad (b),$$

where E′ denotes the ellipticity which would be induced in a sphere by centrifugal

* TISSERAND, 'Mécanique Céleste,' vol. 2, p. 494.

† 'Monthly Notices,' March, 1892.

‡ For an account of previous work, *vide* TISSERAND, 'Mécanique Céleste,' vol. 2, chap. XXX.

force, if distortion were resisted by gravitation as well as by elasticity. The quantity ϵ' is the ellipticity which would be induced if elasticity were the only resisting factor.

The reason Professor NEWCOMB'S hypothesis is at fault is that the ellipticity ϵ, called by him the "natural" ellipticity of the spheroid, is itself partly maintained by centrifugal force, and that if the rotation were annulled, the spheroid would be distorted so that its ellipticity would no longer be ϵ, but E, say.

The ellipticity induced by rotation about a displaced axis has to be superposed on the ellipticity E about the original axis of rotation, and not on the ellipticity ϵ. Thus, in (*b*), ϵ should be replaced by E. Now it is obvious that $\epsilon = \mathrm{E} + \mathrm{E}'$, and by a formula given by THOMSON and TAIT*

$$\frac{1}{\mathrm{E}'} = \frac{1}{\epsilon} + \frac{1}{\epsilon'}.$$

Thus

$$\mathrm{E}' = \frac{\epsilon\epsilon'}{\epsilon + \epsilon'}, \text{ and } \mathrm{E} = \frac{\epsilon^2}{\epsilon + \epsilon'},$$

and therefore

$$\mathrm{E}' : \mathrm{E} = \epsilon' : \epsilon.$$

With the above correction, then, the laws (*a*) and (*b*) become identical.

The spheroid will be distorted not only by centrifugal force about a displaced axis but by the relaxation of centrifugal force about the original axis. The second disturbing factor has been neglected by Professor NEWCOMB.

The disturbing potential will be the difference of the rotation-potentials due to rotation with angular velocity ω about Oz and about Oz_1, that is to

$$\begin{aligned} &\tfrac{1}{2}\omega^2 (x^2 + y^2) - \tfrac{1}{2}\omega^2 (x_1^2 + y_1^2) \\ &\quad = \tfrac{1}{2}\omega^2 (x^2 + y^2) - \tfrac{1}{2}\omega^2 \{(x - z\theta_2)^2 + (y + z\theta_1)^2\} \\ &\quad = \omega^2 (\theta_2 xz - \theta_1 yz). \end{aligned}$$

It is obvious that the equations (19), (20) are the equations for the distortion of an elastic sphere when distorted by forces throughout its mass derivable from this potential function.

Finally the angular velocity of R about P is λ, and therefore the angular velocity of R as viewed from P′ is

$$\lambda \cdot \frac{\mathrm{RP}}{\mathrm{RP}'} = \lambda \frac{\epsilon + \epsilon'}{\epsilon} = \omega\epsilon,$$

which is the third hypothesis made by NEWCOMB.

* 'Natural Philosophy,' Part II., § 840.

§ 9. *Conclusions.*

The existence of the Eulerian nutation, and the fact that it would give rise to a variation in latitude, was first predicted theoretically on the assumption that the Earth could be regarded as a rigid body. Our present work, however, shows that the hypothesis of perfect rigidity, though affording a sufficiently close approximation to the circumstances presented by nature to specify the character of the oscillation, is totally inadequate to lead to a correct determination of the period unless the Earth possesses a very much higher degree of rigidity than is met with in substances which have been subjected to experiment. The only knowledge we have of the amount of the Earth's rigidity arises from the very vague indications furnished by Tidal Theory, and we must therefore have recourse to observation to determine the period with accuracy. This has been effected by Dr. CHANDLER, who, as the result of a discussion of a very large number of observations, has assigned 427 days as the true period. This period is, as the present theory requires, considerably in excess of the Eulerian period of 305 days.

In § 7 we have endeavoured to obtain a numerical estimate of the effective rigidity of the Earth which would be consistent with this observed period, and we have found it to be slightly greater than that of steel, a result which agrees sufficiently closely with the requirements of Tidal Theory. Various causes however combine to render this result liable to a considerable amount of uncertainty. In the first place, our present analysis applies only to a homogeneous spheroid composed of isotropic material, neither of which conditions are fully realized in the case of the Earth. In the second place, there are probably other causes in addition to the elastic deformations of the solid parts of the Earth, which tend to modify the period. In particular we have completely neglected the effects of the mobility of the ocean. According to NEWCOMB these effects will be small compared with the effects of elastic deformation, but different writers have expressed widely different opinions on the subject. Thus WOODWARD* is of opinion that they alone would be sufficient to fully account for the observed extension of the period. NEWCOMB'S view is to some extent confirmed by the smallness of the tide having a 427-day period which has been made the subject of observation by BAKHUYSEN† and A. S. CHRISTIE,‡ and as it has appeared to me to be quite open to question in what manner the results would be affected, I have thought it best not to apply any correction on this account; the results, then, must only be regarded as provisional, pending more complete mathematical investigations on the subject.

In a previous paper§ I have investigated the effects of an internal fluid nucleus,

* 'Astron. Jour.,' No. 345, vol. 15.

† 'Astron. Nach.,' No. 3261.

‡ 'Astron. Jour.,' No. 351.

§ 'Phil. Trans.,' A, 1895.

and found that if the central solid portions of a rigid spheroid were replaced by liquid, the theoretical estimate of the period of oscillation would be diminished. It might appear then that our estimate of the Earth's rigidity would be diminished if we suppose there to be a central fluid nucleus. That this is not so, I think the following considerations will show.

In accordance with the results of the above-mentioned paper the effect of internal fluidity would be to increase the effective value of the quantity we have denoted by ϵ. When the external crust is of considerable thickness, the increase in this quantity is, however, very slight; thus for a crust of about 2,000 miles in thickness we find ϵ is increased in the ratio 305 : 300. Now it seems that if the central solid portions of the Earth were replaced by fluid the increase in the value of ϵ', which denotes the ellipticity due to rotation, would be much more rapid, and that consequently λ/ω would rapidly diminish. We conclude then that the increased effects of the elastic deformations would more than counteract the influence of the reduced effective inertia due to internal fluidity, and that with a given degree of rigidity the period of oscillation would be still further prolonged. The degree of rigidity of the crust necessary to account for a given period would also have to be increased, and as the estimate we have already found is high, the evidence furnished by the latitude-variation still seems opposed to the existence of an internal fluid nucleus.

Finally we may consider the effects of the Earth's viscosity. Unless this be so great that the present work is inapplicable, a circumstance which seems to be quite precluded from the close agreement of our results with observation, the chief effect of internal friction will be to cause the oscillation in question to gradually die out without producing any material change in the period. The dissipative forces arise entirely from the distortion of the parts of the system, and consequently no such forces will occur if the system be absolutely rigid throughout. Now we have seen that the motion consists very approximately of a rotation as a rigid body, and that the elastic distortions are exceedingly minute. Hence we conclude that a very small amount of dissipative force will be called into play, and thus if the motion is once set up, there appears to be no difficulty in accounting for its continuance for a very considerable period, possibly extending over several centuries.

INDEX SLIP.

IX. *On the Magnetical Results of the Voyage of H.M.S. "Penguin,"* 1890–93.

By Captain E. W. Creak, *R.N., F.R.S.*

Received May 8,—Read June 20, 1895.

In the recent magnetic surveys conducted in different countries, the details of which have been published, one point stands out prominently from the rest, that the more minute the survey, the more surely do the observations show that the needle is subject to "local" and "regional" magnetic disturbances, varying in amount from the normal values of the magnetic elements, as deduced from extended observations made over the whole country.

A reference alone to that recent and most valuable contribution to terrestrial magnetism, "A Magnetic Survey of the British Isles," by Professors A. W. Rücker, F.R.S., and T. E. Thorpe, F.R.S., is quite sufficient to show the certainty of these disturbances.

Our knowledge of the magnetic elements on land and their disturbances is constantly being added to, but there is a much larger area for exploration, which, whilst leaving the dry land to the observers on land, seems specially to belong to those whom we may term the seagoing magneticians, namely, the broad sea, the coasts washed by the sea, and what is equally important to science and navigation, the land under the sea. It is a fact that as yet we have not obtained anything like an exact knowledge of the form which the "isomagnetics" may take on going from the assumed normal lines, passing from over the deep sea to cross depths of water under 100 fathoms, until the dry land with its known disturbances is reached.

Although along those parts of the coasts of great continents more commonly visited, several series of observations of the magnetic elements have been made by the war-vessels of various nationalities, for the coasts of Australia, from Adelaide westward round north to Cape York, there were, previous to 1885, only some three or four stations at which either Dip or Force had been observed.

To remedy this defect as far as possible, Admiral Wharton, F.R.S., Hydrographer to the Admiralty, caused H.M. surveying vessel "Meda" to be furnished with the necessary magnetic instruments, with which the elements were observed at twelve stations, distributed between King George's Sound and Cossack, in N.W. Australia, by Navigating-Lieutenant Dockrell, of that ship.

Unfortunately the continuance of this series was cut short by the close of the survey, but not until a remarkable disturbance of the compass observed on board the "Meda," when approaching Cossack (Port Walcott), in N.W. Australia, the disturbance being evidently caused by magnetic bodies in the land under the sea. Time only sufficed to approximately localize the position of greatest disturbance.

Towards the close of 1889 H.M.S. "Penguin" was appropriated for surveying service in Australia, under the command of Captain W. U. MOORE, R.N. Once more the Hydrographer decided that the magnetic survey of the western coasts of Australia should be proceeded with as far as the requirements of the hydrographic survey would admit. Lieutenant J. W. COMBE, R.N., of the "Penguin," was selected to make the observations, and was therefore specially instructed at the Admiralty and at Kew Observatory in the use of the several magnetic instruments supplied to the "Penguin."

The "Penguin" is a composite-built screw steam vessel of 1130 tons displacement and 700 indicated horse-power, and consequently a suitable vessel as regards size for magnetic observations at sea. The amount of iron, however, used in her construction, made her practically an iron ship, and the magnetic observations were confined to those of the Declination or Variation when the ship could be swung, or, in other words, when her head could be placed on eight or more points equally distributed round the compass, and for Dip and Force when the Relative instruments used on board could be compared with the Absolute instruments on land under proper conditions :—

The following is a list of the instruments supplied :—

For Absolute observations on land	(1) Unifilar Magnetometer, No. 25, by ELLIOT, with two magnets. (2) BARROW'S Dip Circle.
For Relative observations on board ship	A Fox Dip and Intensity Apparatus, No. C. 10.

Also an Admiralty Standard Compass for observations on board and on land.

Base Station.

Kew Observatory was the adopted base station, where the Unifilar Magnetometer and Dip Circle were verified and Constants obtained. On return from abroad Lieutenant COMBE repeated the observations to test the condition of the instruments after their three years' work, during which they were subjected to great change of climate and the chances inseparable from frequent transit from ship to shore.

In order to show how far the absolute instruments remained in good order under such circumstances, the following final observations were made at Kew and other fixed observatories. The symbols δ for the Declination, H for the Horizontal Force expressed in C.G.S. units, and θ for the Dip have been adopted.

		δ.	H.	θ.	Needle.
Kew Observatory . . .	Mean for Sept., 1893 .	17° 27'·3 W.	·18243	+67° 26'·1	
H.M.S. "Penguin" . .	5th and 6th Sept., 1893	17 29·3	25 a., ·18262 25 d., ·18257	+67 25·2 +67 22·7 +67 27·5 +67 26·5	1 2 3 4

		δ.	H.	θ.	Needle.
Melbourne Observatory .	Mean for 1891 . . .	7° 58'·5 E.	·23479	−67° 12'·9	
H.M.S. "Penguin" . .	March, 1891 . . .	*8 5·7	25 a., ·23475 25 d., ·23453	−67 15·6 −67 14·3 −67 15·2 −67 17·3	1 2 3 4

		δ.	H.	θ.	Needle.
Hong Kong Observatory	Mean for 1892 . . .	0° 33'·6 E.	·36352	+32° 3'·5	
H.M.S. "Penguin" . .	. . . .	0 35·4	25 a., ·36397 25 d., ·36388	+32 3·0 +32 2·5 +32 3·1 +32 4·3	1 2 3 4

The final results obtained at Kew, in September, 1893, have been corrected for the diurnal range obtained from the Report of the Kew Committee, and the mean results obtained at Kew, for the month of September, have been adopted as a standard of comparison.

* Mean of six observations at different hours on three days, corrected for diurnal variation from Observatory curves.

DIFFERENCES between "Penguin" and Observatories.

	δ.	H.	θ.	Needle.
At Kew	+2′0	25 a., +·00019 25 d., +·00014	−0′9 −3·4 +1·4 +0·4	1 2 3 4
At Melbourne . . .	+7·2	25 a., −·00004 25 d., −·00026	+2·7 +1·4 +2·3 +4·4	1 2 3 4
At Hong Kong . . .	+1·8	25 a., +·00045 25 d., +·00036	−0·5 −1·0 −0·4 +0·8	1 2 3 4

Observing how nearly the "Penguin's" results agree with those of Kew and Hong Kong, it seems fair to assume that, in spite of the increased discordance observed at Melbourne, the instruments remained in a satisfactory condition throughout the period of observation.

It is not proposed to record here in full the moments of the two magnets observed in the Force observations, but it may be explained that the several values were treated graphically, with the following results:—

	Magnet 25, *a*.	Magnet 25, *d*.
March, 1890	m = ·029172	·027568
December, 1890	= ·029094	·027494
Febraary, 1892	= ·029050	·027469
September, 1893	= ·028995	·027328

Thus, during the first nine months, the moments of both magnets declined somewhat rapidly, after which they slowly diminished in value until the close of the series at Kew. It may be noted that, throughout the series, no single value of m differed more than $\pm$ ·00001 from those of the curve obtained from the whole set employed for obtaining H.

The value of P has been calculated by the formula

$$(\text{Log } A_1 - \log A_2) \times 5{\cdot}64 = P$$

where A_1 and A_2 are the values of m/H at the two distances. The mean value for the whole series is

$$P = -\ {\cdot}00080228.$$

Considering the importance of a uniform system of exhibiting the value of the observer's work in observations of similar nature, it was originally intended to adopt that of Professors RÜCKER and THORPE, as explained in their "Magnetic Survey of Great Britain" (see 'Phil. Trans.,' 1890). The difficulty then arose of there being no means for ascertaining the solar diurnal variations and effects of disturbances on the elements observed at places distributed over so large an area of the world. Thus, defects of observation and movements of the needle from magnetical causes could not be separated, and RÜCKER and THORPE's method was reluctantly abandoned.

With regard to the use of the Fox Dip and Intensity apparatus, it should be understood that, being a relative instrument, it was constantly compared with the absolute instruments. The index errors affecting the Dip observations were known to the nearest minute, and the change of magnetic force in the deflectors used in the force observations were ascertained by obtaining the values of the "weight equivalents" at four stations, so that the magnetic condition of the instrument was known whenever it was used. The temperature corrections were too small to be applied to force observations taken under conditions of small change of temperature.

The results obtained on land with the absolute instruments are given in Table I.; those with relative instruments in Table II.

Local Magnetic Disturbances.

Although the amount of local magnetic disturbance over given areas, and the causes thereof, have for some years past been a subject of close enquiry among magneticians, it does not appear that anything like close attention has been paid to those local magnetic disturbances experienced on board ship which are independent of any direct action from iron or steel used in the construction of the ship. It is certain, however, that conclusions have been drawn and promulgated that are absolutely unfounded. Amongst them may be mentioned the erroneous impression that visible land, when miles distant, affects a ship's compasses, to the danger of the ship.

In view of placing the local magnetic disturbances observed in depths of water under 100 fathoms of water on a proper basis, instructions were given to H.M.S. "Penguin" to devote as much time to such an enquiry as her special surveying duties would permit. H.M.S. "Meda," having reported on the remarkable disturbance of the compass experienced on board in the neighbourhood of Cossack, Port Walcott, in N.W. Australia, the results of a magnetic survey were of great importance to navigation, and consequently some days were devoted to the examination of the port. A discussion of the observations made, with diagrams, forms the concluding and it is presumed the most important part of this paper. The results of observations made on islands and at Cossack for local magnetic disturbance will now be given in order of time.

Perim Island.

Two stations two miles apart were selected on this island for the absolute observations: (1) at Signal Hill, near the west end of the island and about 100 feet above the sea; (2) the high lighthouse near the east end, about 200 feet above the sea, the instruments being set up on lava rock. There were also three auxiliary stations where the Fox apparatus was employed, and relative observations made: Nos. 1 and 2 on the north coast, on broken coral; and No. 3 in a sandy bay at the S.E. end of the island. The declination was only observed at the high stations with a value at both $= 3° 16'$ W. Comparing this with the value obtained on board by swinging in 19 fathoms of water $= 3° 25'$ W., the difference of 9′ shows but small disturbance.

Inclination—With this element there is a difficulty in obtaining an exact normal value, but as the observation at Auxiliary Station No. 3 was made on a sandy beach, and the results agree with the Chart of Normal Values at the Admiralty, it has been adopted.

Station.	Observed Dip.	Normal.	Disturbance.
	° ′	° ′	° ′
Signal Hill	5 27 N.	3 30	+1 57
High Lighthouse . . .	5 3	..	+1 33
Auxiliary Station No. 1 .	5 4	..	+1 34
Auxiliary Station No. 2 .	5 4	..	+1 34

Horizontal Force.—Assuming the value of H at Auxiliary Station No. 3 as normal, we have the following:—

Station.	Observed H.	Normal.	Disturbance.
Signal Hill	·359	·355	+·004
High Lighthouse . . .	·341	..	−·014
Auxiliary Station No. 1 .	·355	..	− 0
Auxiliary Station No. 2 .	·352	..	−·003

Baudin Island.

On this island the three elements were observed at two stations: (A) on the south slope of the island, on hard compact gray sandstone, with fragments of ironstone, situated 100 feet above the sea; (B) on the foreshore, where the beach consisted of coralline sand and coralline limestone. Two specimens of rock from this island were found to be magnetic, with values of k = ·000217 and ·000529 in C.G.S. units (see Appendix B).

Station B values have been adopted as the normal for the following reasons. The ship was swung in 19 fathoms of water near the island in two successive years with the same results as regards Declination = 2° 11′ E. The Dip and Force agree nearly with the best normal values on the charts.

DISTURBANCES.

	δ.	θ.	H.
	° ′		
Station A	3 16 W.	− 42·55	·3572
Station B, adopted normal . . .	2 14 E.	− 40·7	·3574
Disturbance	5 30 W.	− 2·48	·0002

Port Walcott (Cossack).

Although the name of the township of Cossack may be more familiar than the name of the port in which it is situated, still, as the observations about to be considered refer chiefly to the port approaches to Cossack, and extend over an area of some miles, the name Port Walcott has been adopted.

The accompanying map of Port Walcott shows the relative positions of the points where land observations were made and the region of magnetic disturbance in land under the sea. The latter region we may hereafter refer to as the "Magnetic Shoal," over which the "magnetic soundings"* were taken (see *post*).

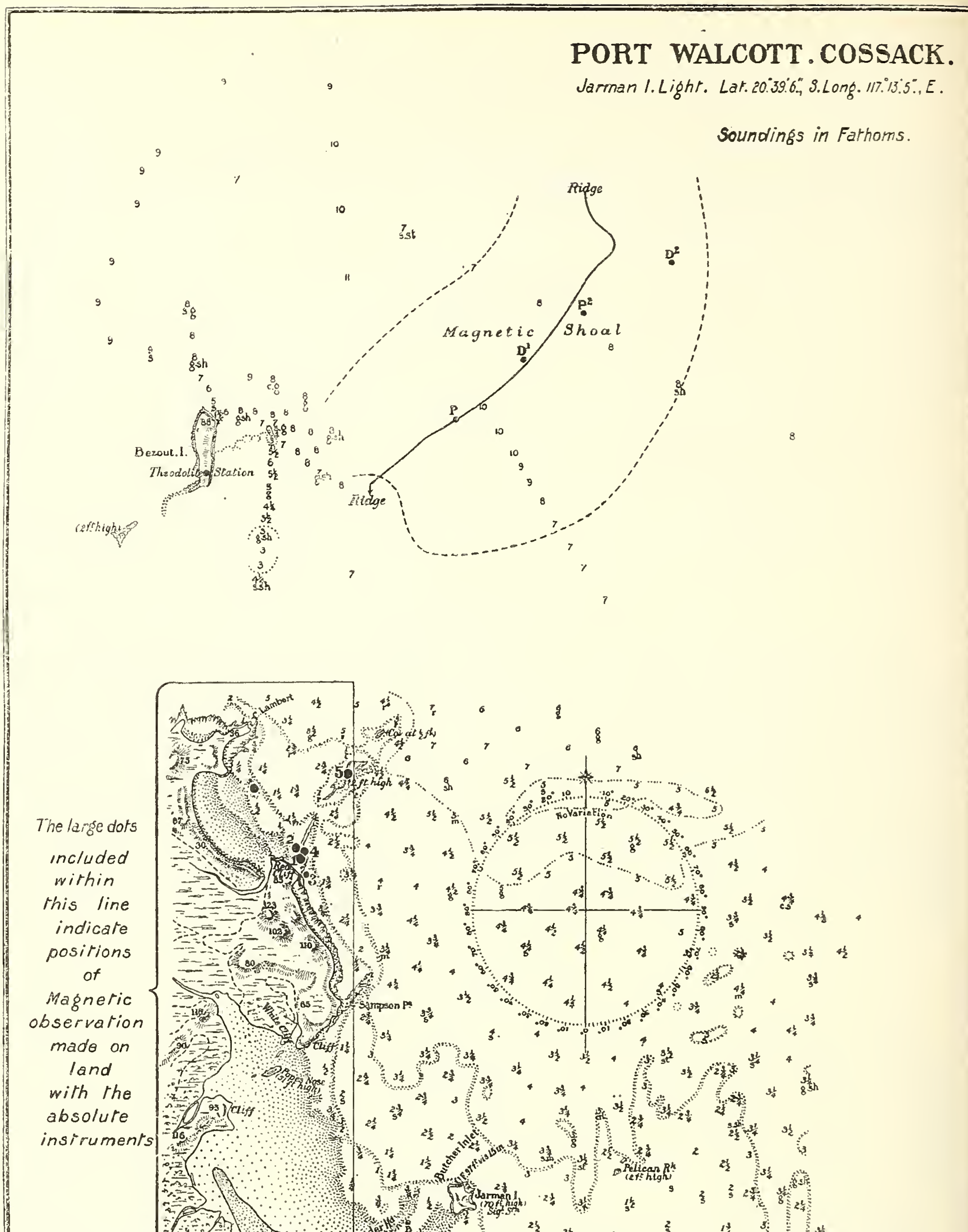
Land Observations.

There is the usual difficulty in this place of determining the normal values of the magnetic elements, but an observation was made with the Fox apparatus on board the ship in 19 fathoms when approaching the port, giving a corrected value of 51°. This reduced for difference of Dip, due to difference of the latitude, makes the Dip at Reader Head (see map) = 51° 16′, the observation on land at the same place being = 51° 20′.

Again, by swinging the ship 4 miles east of Reader Head station, the Declination was observed to be = 0° 15′ E., as compared with 0° 4′ W. on the head.

The station at Reader Head being also in a sandy neighbourhood, was therefore adopted as the position of normal values.

* The term "magnetic soundings" is here meant to apply to the several magnetic disturbances in analogy with the soundings taken to determine the position and extent of a shoal of sand, for example. —E. W. CREAK.



PORT WALCOTT. COSSACK.
Jarman I. Light. Lat. 20°39'6" S. Long. 117°13'5" E.
Soundings in Fathoms.
Ridge
Magnetic Shoal
Bezout. I.
Theodolite Station
Ridge
The large dots included within this line indicate positions of Magnetic observation made on land with the absolute instruments
Lambert
No Variation
Sampson Pt
White Cliff
Cliff
Butcher Inlet
Pelican Rk (2 ft high)
Jarman I. (70 ft high)
Reader Hd
Perseverance Rk (Beacon)
Cossack
Red Hill
Causeway
Tisard I.
Cables 10 5 0 1 Scale. 2 3 4 Sea Miles
T.P. Collings Sc.

Land Disturbances.

Declination.

Normal.	Bezout I.	Cape Lambert.	Red Cliff stations.			
			2.	3.	4.	5.
0° 4′ W.	0° 22′ W.	0° 4′ W.	7° 56′ W.	1° 11′ E.	2° 53′ E.	1° 4′ E.
Disturbance .	0 18 W.	..	7 52 W.	1 15 E.	2 57 E.	1 8 E.

Inclination.

Normal.	Bezout I.	Cape Lambert.	Red Cliff stations.				
			1.	2.	3.	4.	5.
51° 20′ S.	50° 2′ S.	50° 13′ S.	52° 18′	54° 12′	50° 41′	52° 30′	51° 36′
Disturbance .	+ 1 18	+ 1 7	− 0 58	− 2 52	+ 0 39	− 1 10	− 0 16

Horizontal Force (Metric Units).

This was only observed at No. 1 Red Cliff station besides the normal, with a difference of 0·0878 as a disturbance.

There is nothing in the amount of the above disturbances to call for special remark, as they have often been largely exceeded in other countries.

There is, however, one point which requires notice, and that is the different signs shown by the Dip disturbances, the north-seeking pole of the needle being repelled at four Dip stations, and attracted at three stations out of the seven. Attention is called to this as bearing upon subsequent results obtained on board the ship.

Disturbances Caused by Land under the Sea.

The instruments employed for observing on board the ship :—

1. Standard compass on poop, 75 feet above the sea-bottom.
2. Fox apparatus (C. 10) for Dip and Force 82 feet above the sea-bottom.
3. A compass occupying the place of the Fox apparatus when removed, and called the "Fox compass."

Before proceeding to show the remarkable amount of the disturbances of the magnetic elements observed at Port Walcott, it must be remembered that the only means available for obtaining the observations was on board a composite-built* vessel with steam machinery; in fact, that the observers had *nolens volens* placed the ship, herself a disturbing magnet, between the instruments and the source of disturbance.

It was therefore necessary, first of all, to determine to what extent this interposed magnet, the ship, disturbed the needles on board. For this purpose the ship was swung off Baudin Island in 19 fathoms of water on May 6, 1891: (*a*) for values of Dip and Intensity on the eight principal points of the compass; (*b*) simultaneous observations of the Standard and Fox compasses for deviation on all points of the compass.

Adopting the methods and formulæ described in the Admiralty 'Manual of Scientific Enquiry' (Art. Terrestrial Magnetism) for 1886, the following table of Disturbance of the Dip and Total Force, caused by the Horizontal Forces of the ship, was computed:—

Ship's head.	Correction for Dip.	Correction for Total Force.
	° ′	met. units.
North . . .	−0 17	+·074
N.E.	+1 27	+·192
East	+2 26	+·305
S.E.	+1 29	+·177
South . . .	+0 17	+·014
S.W.	+1 48	+·118
West	+2 26	+·217
N.W.	+0 49	+·157

Next adopting the notation of the Admiralty 'Manual for Deviations of the Compass' (1893), the following coefficients, representing the horizontal disturbances of the ship, were computed from the observed deviations:—

	B.	C.	D.
	° ′	° ′	° ′
Standard compass . .	+0 45	−0 11	+3 55
Fox compass	+0 35	−0 4	+2 22

Having corrected the Dips for effects of the ship's horizontal forces, the resulting mean Dip was compared with the normal values at Baudin Island, and found to differ only 2′. The Vertical Force of the ship was therefore considered zero.

Now at Port Walcott the coefficients B, C, D were found to be:—

* Composite-built ships have iron frames and wood planking.

	B.	C.	D.
Standard compass . .	−0° 26′	−0° 32′	+4° 2′
Fox compass	+0 50	0 0	+2 20

or nearly the same as off Baudin Island, and as the Dip corrected for horizontal forces in the ship differed only 10′ from the normal, it was considered that the tables of correction for the three elements obtained off Baudin Island might for the purposes of this discussion be used for the observations over the "magnetic shoal" at Port Walcott, and this has been done.

Although no direct observations of the mean Horizontal Force to north (or λ of the Admiralty 'Manual') were obtained at the compass position where the Declination observations were made, still on comparing the best values of the Horizontal Force obtained on board with those on land, a small diminution was found, giving a mean force to north = ·98 (considering the land force = 1·0) at the Fox position.

At the Standard compass position the mean force to north may be assumed as ·97.

Having so far defined the magnetic condition of the ship, we are in a position to review the order of observation at the magnetic shoal and consider the results.

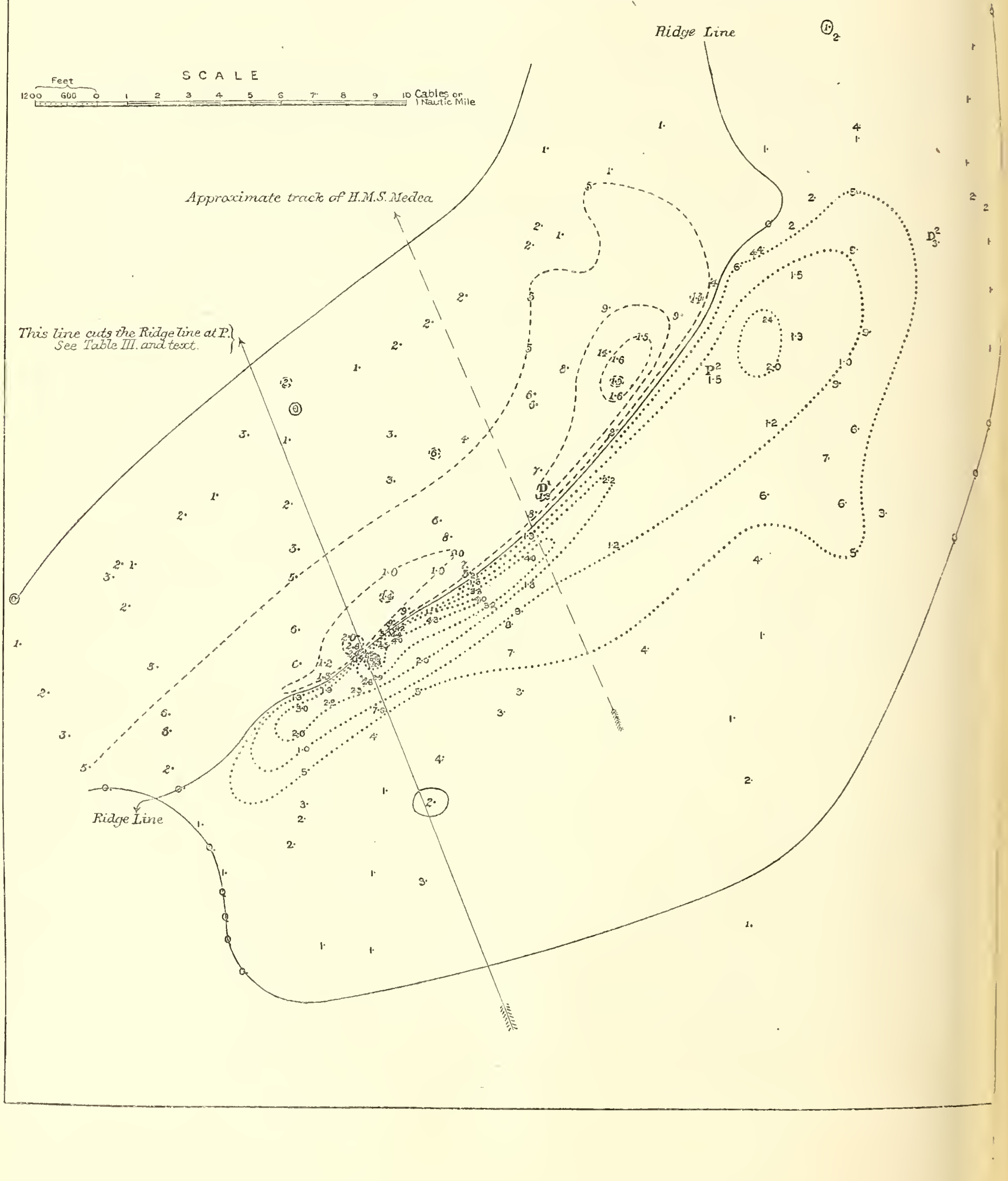
A preliminary examination of the general limits of the area of Disturbance at Port Walcott was made in November, 1890. The 22nd to the 25th of the following April were devoted to completing the magnetic survey of the shoal, the distribution of the observers being as follows :—Lieutenant TANCRED, with a theodolite, was placed on Bezout Island, to take the true bearings of the Standard compass as the ship changed her position. Lieutenant PARRY, on board the ship, took compass bearings of the theodolite station on Bezout with the Standard compass, the direction of the ship's head being noted by Sub-Lieutenant OLIVER. The Dip and Force observations were made by Lieutenant COMBE, the position of the ship being fixed and depth of water taken at every magnetic observation.

On the first day the ship was run across the area of disturbance in the north and south directions, whilst the observations of Declination were made, buoys being placed at positions of greatest disturbance. On the second day the Declination observations were continued over the eastern and western extremities of the shoal, special attention being given to the spaces between the buoys.

The third and fourth days were occupied with observations of the Dip and Force with such additional observations of the Declination as could be made, the method of ensuring reliable results being as follows. The position of maximum disturbance had already been pointed out by the Declination observations ; the ship was therefore moored in its immediate neighbourhood, one anchor being let go on the west side of the point of greatest westerly disturbance ; the other anchor on the east side of the greatest easterly disturbance. Thus, by working the cables as requisite, the ship

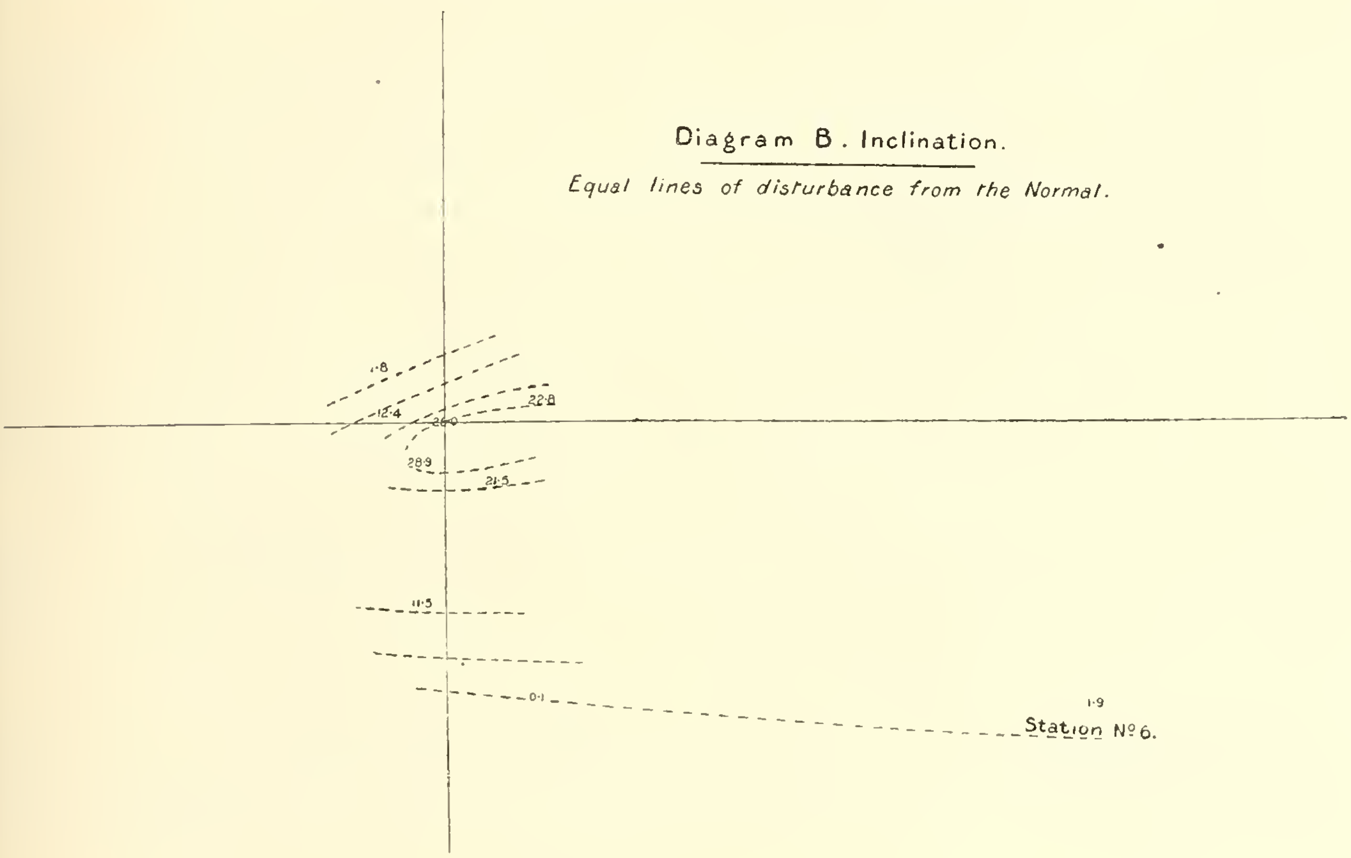
Diagram A. Declination disturbances given in Curves of equal value 5° 10° 20° 30° 40° and 50°

The figures represent the amount of disturbances in degrees and the dots show the points of observation.
.... denote equal values of disturbance of the needle Eastward / ------ " " " " Westward } of the true Meridian.
——— are lines of no disturbance
The Declination at Reader's Head near Cossack being taken as zero.
The letters D^1, D^2 and P^2 show the positions of disturbance of Vertical force – see Table III.



was hauled over the area of maximum disturbance and fixed in any desired position. The direction of the stream also favoured the retention of the ship's head in a given direction, an important factor when ship observations are concerned.

With the exception of one observation, marked No. 6, the data for Diagrams B, C, D were obtained in this manner, but the remaining observations at Stations 11, 12, and 13 were made with the ship under way. The observations of the four days are recorded in Tables III. and IV., with their various corrections and final results.



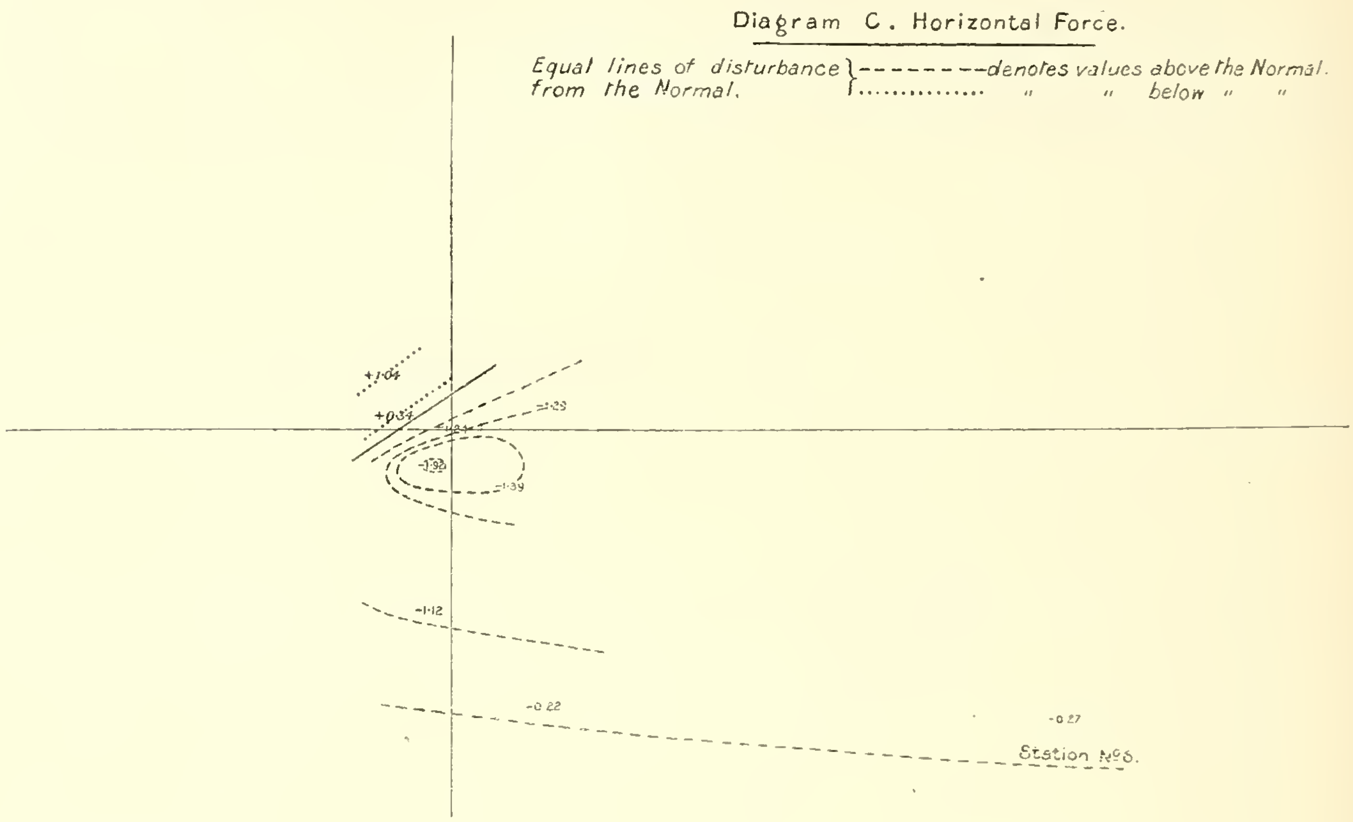
Graphic Representation of the Observed Disturbances.

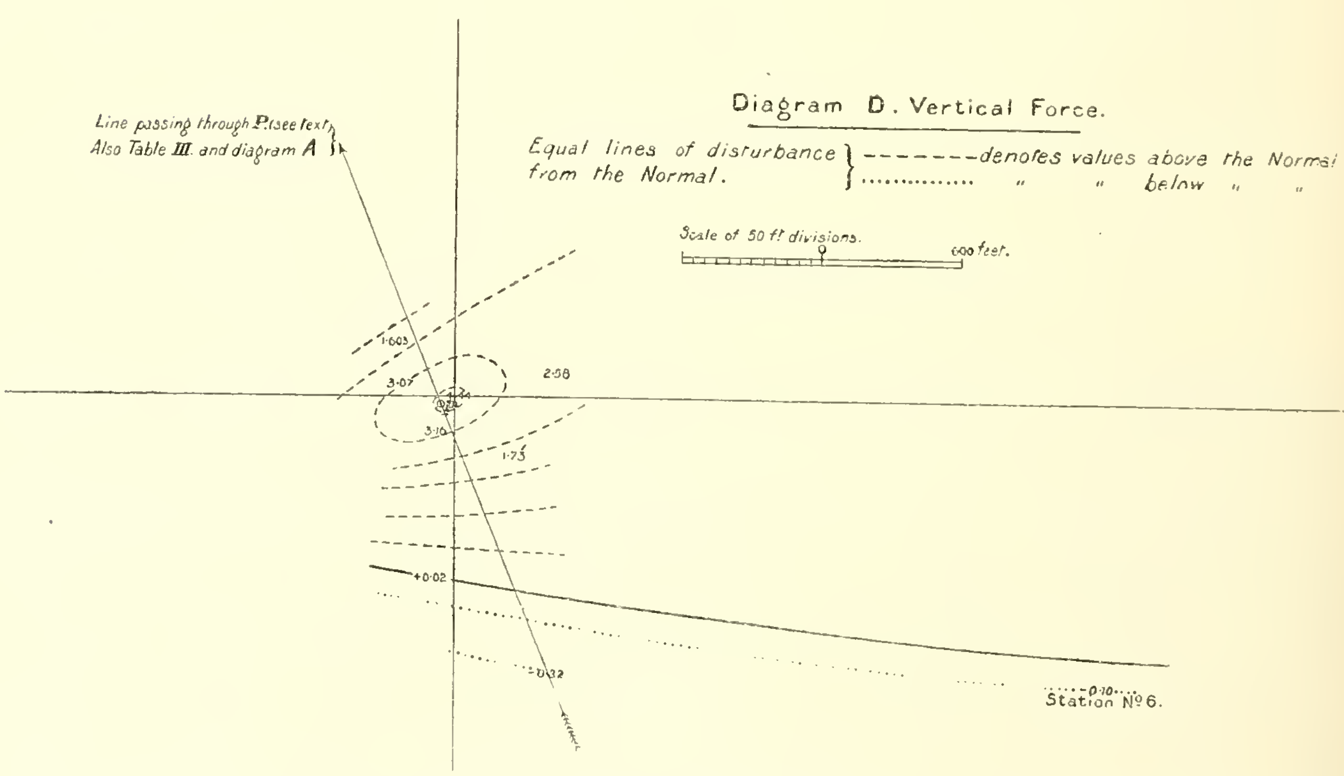
On Diagram A are shown lines of equal values of the disturbances in the Declination taken from Table IV. The figures represent the values of disturbances expressed in degrees, the dots denoting the position of the ship's Standard compass at each observation. Lines of no disturbance are drawn plain, easterly disturbances dotted, and westerly pecked. The values of the lines are 5°, 10°, 20°, 30°, 40°, and 50°. Whilst the general direction of this magnetic shoal is N. 50° E. (true), the approximate dimensions of it are 3 miles long by $1\frac{1}{4}$ miles at its widest part.

On Diagram B the observed disturbances of the Dip are shown, and on C and D those of the Horizontal and Vertical Forces respectively. The figures are taken from Table III., the dots representing the position of the ship as well as the decimal point, and the curves of equal value are drawn.

It may possibly be considered that the observed values of the Declination and

Horizontal Force should have been corrected for reduction of the mean Horizontal Force due to the iron of the ship previously mentioned. This correction was abandoned in view of the fact that, with such large disturbances in which only the





nearest degree of Declination and the Horizontal Force to the second place of decimals were considered, such correction was unnecessary. Moreover, it is highly probable that the ship with her iron frames and her keel between 30 and 40 feet nearer the source of disturbance than the observing instruments, placed in different parts of so powerful a magnetic field, was subject to a measure of induction which there was no means of gauging.

It was suggested that observations to test the effects of this induction might be made on a wooden raft, but neither the state of the sea nor time available permitted this.

On the possible Cause of the Disturbances.

Having in possession the amount of the several disturbances observed, the question arises what is the disturbing cause ? It is certain, by the sea soundings, that the source of the disturbance was some 82 feet below the Force instruments.

Referring to Diagram A, the central line of no disturbance is clearly defined by the change of sign in the observed values of the disturbances, and this line may be termed a "ridge line" extending with its sinuosities for a distance of 3·5 miles, and passing through the point of greatest upward Vertical Force disturbance. Points of decreasing value of upward Vertical Force are shown at P^2 and D^1, whilst at D^2 the disturbance is comparatively small and downwards.

Also in Diagram D a transverse sectional line will be seen passing across the curves. This line passes through the area of greatest disturbance in the Vertical Force and of the Declination, and its direction has been selected as also passing through points in the curves which are best authenticated by observation.

The principal results, therefore, may be expressed shortly as follows :—At a point situated N. 78° E. (true), distant 2·155 miles from the Station on Bezout Island, there exists in the land below the sea a source of magnetic disturbance, causing disturbances of the following magnitude :—

Declination.		Dip.	Horizontal Force.		Vertical Force.
26° W. to 56° E.		−29°	+1·04 to −1·92		−4·44 to +0·32
on N. side	on S. side.		on N. side	on S. side.	

Of the nature of the land under the sea causing such abnormal magnetic disturbances, there is scant information upon which to form any decided opinion. A geological survey of the most disturbed part of the coast at Red Cliff was made by Mr. WALKER, Chief Engineer of the "Penguin," which is given in full with diagram in Appendix B.

A number of specimens of rock and sand were collected by the "Penguin" in several parts of Australia, which Professor RÜCKER has been kind enough to examine. Among these was a specimen of sand taken from the bottom where the magnetic shoal lies, but none of those brought from Port Walcott showed any signs of magnetic susceptibility (see Appendix B).

Professor JUDD, F.R.S., has been also kind enough to look over the geological paper, and he writes: "The only possible chance I can see for other conclusions" (he had previously considered from Appendix A that there was no geological formation to account for the magnetic phenomena) "is, that the term quartzite is used for volcanic rock, or that ironstone dykes are really a decomposed igneous rock (basaltic diabase) which at a slight depth would be found to show their normal magnetic character."

In conclusion, I would remark that the highest credit is due to Lieutenant COMBE, R.N., who, under Captain MOORE, had charge of the magnetic observations. There is no link missing in the chain of evidence as to their completeness in every detail. The management of the ship during the survey of the magnetic shoal was fully fitted to a successful issue.

I have to thank Professors RÜCKER and JUDD for valuable assistance with reference to Appendices A and B.

It is evident, from the scant but well authenticated reports from different parts of the world, that there is much to be done in the direction of ascertaining the position and dimensions of local magnetic disturbances of land under the sea, and it is hoped that what has been done by the "Penguin" will be a source of emulation to others whose lot is cast in a seafaring life.

TABLE I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93.

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ. Date.	Needle.	Time.	θ.	Absolute Horizontal Force = H. Date.	Magnet.	Time.	H.
(1) MALTA.	1890.		° ′	1890.			° ′	1890.			
Site of Spencer's Monument N. 45° W. (true) 159 yards	8 March	8.50 A.M.	9 42·1 W.	4 March	3	10 A.M. to	51 12·1 N.	4 March	25 *a*	2 P.M.	·26509
Lat. 35° 52′ 57″ N.					4	noon	51 4·3 N.	7 ,,	25 *d*	1.45 P.M.	·26509
Long. 14° 30′ 42″ E.				7 ,,	4	10 A.M. to	51 6·4 N.	7 ,,	25 *a*	2 P.M.	·26441
					4	noon	51 16·0 N.				
(2) SUEZ.											
West end of Canal Breakwater	15 March	3.30 P.M.	4 30·1 W.	15 March	3	9 A.M.	40 33·6 N.	15 March	25 *a*	1.15 P.M.	·30296
Lat. 29° 55′ 52″ N.					4	10 A.M.	40 35·4 N.		25 *d*	1.45 P.M.	·30388
Long. 32° 33′ 33″ E.					4	noon	40 43·4 N.				
					3	2 P.M.	40 39·3 N.				
					4	4 P.M.	40 43·7 N.				
(3) PERIM.											
(1) Signal Hill Station	27 March	2 P.M.	3 15·4 W.	27 March	3	3.30 P.M.	5 27·2 N.	29 March	25 *a*	12.30 P.M.	·35903
Lat. 12° 38′ 45″ N.	28 ,,	5.30 P.M.	3 17·3 W.	28 ,,	3	11 A.M.	5 27·3 N.		25 *d*	1.30 P.M.	·35904
Long. 43° 23′ 41″ E.				28 ,,	4	11 A.M.	5 25·4 N.				
					3	3 P.M.	5 27·5 N.				
					4	3 P.M.	5 30·5 N.				
(2) High Lighthouse Station	1 April	2.30 P.M.	3 18·9 W.	1 April	By two needles of Fox instrument corrected for I.E.	11.30 A.M.	5 3·5 N.	1 April	25 *d*	noon	·34108
Lat. 12° 39′ 0″ N.	2 ,,	5.15 P.M.	3 13·1 W.						25 *a*	1.30 P.M.	·34102
Long. 43° 25′ 41″ E.											

TABLE I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93 (continued).

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ. Date.	Needle.	Time.	θ.	Absolute Horizontal Force = H. Date.	Magnet.	Time.	H.
(4) BAUDIN ISLAND.	1890.		° ′	1890.			° ′	1890.			
(1) South Slope Station	19 July	8.40 A.M.	3 15·9 W.	19 July	4	10.30 A.M.	42 54·2 S.	19 July	25 *a*	2 P.M.	·35720
Lat. 14° 7′ 50″ S.		5.10 P.M.	3 16·3 W.		4	3 P.M.	42 56·1 S.		25 *d*	1.30 P.M.	·35720
Long. 125° 36′ 13″ E.											
(2) Sandy Point Station	21 July	4.40 P.M.	2 19·5 E.	21 "	3	10 A.M.	40 7·3 S.	21 "	25 *a*	1.20 P.M.	·35742
Lat. 14° 7′ 50″ S.	22 "	8.20 A.M.	2 9·7 E.		3	2 P.M.	40 5·8 S.		25 *d*	1 P.M.	·35740
Long. 125° 36′ 20″ E.					4	10 A.M.	40 9·8 S.				
	1891.										
Same station as No. (2) above	7 May	8.15 A.M.	2 13·8 E.		4	2 P.M.	40 5·0 S.				
		4 A.M.	2 15·0 E.								
(5) PORT DARWIN.	1890.										
Fort Hill Station	30 June	4.30 P.M.	2 39·6 E.	28 June	3	10.30 A.M.	36 46·6 S.	30 June	25 *a*	2.15 P.M.	·36588
Lat. 12° 28′ 28″ S.	1 July	7.30 A.M.	2 38·7 E.		4	10.30 A.M.	36 53·1 S.	30 "	25 *d*	1.30 P.M.	·36589
Long. 130° 50′ 37″ E.	1 "	4.40 P.M.	2 39·4 E.		3	2.30 P.M.	36 51·8 S.	1 July	25 *a*	1.20 P.M.	·36622
	2 "	7.30 A.M.	2 37·9 E.		4	2.30 P.M.	36 52·0 S.				
				1 July	3	11.30 A.M.	36 49·4 S.				
					4	11.30 A.M.	36 52·8 S.				
(6) ADMIRALTY GULF.											
Low rocks	9 July	8.15 A.M.	2 2·5 E.	9 July	3	10 A.M.	39 55·3 S.	9 July	25 *a*	1.40 P.M.	·35855
Lat. 14° 3′ 50″ S.	9 "	5 P.M.	2 6·9 E.		4	10 A.M.	39 57·0 S.		25 *d*	1.6 P.M.	·35875
Long. 125° 52′ 34″ E.	10 "	8.20 A.M.	2 0·9 E.		3	2 P.M.	39 55·6 S.	10 "	25 *a*	2 P.M.	·35844
		3.40 P.M.	2 5·5 E.		4	2 P.M.	39 58·0 S.		25 *d*	1.30 P.M.	·35845
				11 "	3	10.30 A.M.	39 53·7 S.	11 "	25 *a*	1.40 P.M.	·35871
					4	10.30 A.M.	39 56·7 S.				
					3	2.30 P.M.	39 54·2 S.				
					4	2.30 P.M.	39 56·3 S.				

TABLE I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93 (continued).

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ.				Absolute Horizontal Force = H.			
				Date.	Needle.	Time.	θ.	Date.	Magnet.	Time.	H.
(7) FANNY BAY.	1890.		° ′	1890.			° ′	1890.			
Lat. 12° 26′ 10″ S.	7 August	4.45 P.M.	2 37·6 E.	7 August	4	10 A.M.	36 49·7 S.	7 August	25 *a*	noon	·36580
Long. 130° 50′ 8″ E.	8 ,,	8.15 A.M.	2 41·1 E.		4	3 P.M.	36 48·6 S.		25 *d*	0.30 P.M.	·36554
				8 ,,	4	10 A.M.	36 47·8 S.				
					3	10 A.M.	36 49·4 S.				
					3	2 P.M.	36 48·5 S.				
					3	3 P.M.	36 48·2 S.				
(8) COSSACK.											
Station S. 81° W., 2,280 feet from Reader Head	5 Nov.	6.20 A.M.	0 2·6 W.	4 Nov.	3	10.30 A.M.	51 22·1 S.	5 Nov.	25 *d*	1.0 P.M.	·31917
	4 ,,	4.20 P.M.	0 5·7 W.		4	10.30 A.M.	51 18·4 S.		25 *a*	1.30 P.M.	·31909
		7.20 A.M.	0 3·1 W.		3	2 P.M.	51 20·2 S.				
Lat. 20° 39′ 42″ S. Long. 117° 11′ 51″ E.					4	3 P.M.	51 19·7 S.				
(9) BEZOUT ISLAND (near Cossack).											
South part of Island, 88 feet high	6 Nov.	7.30 A.M.	0 22·0 W.	6 Nov.	3	9.30 A.M.	50 2·1 S.				
Lat. 20° 32′ 45 S. Long. 117° 11′ 15″ E.					4	10.30 A.M.	50 1·8 S.				
(10) ROEBUCK BAY.											
Broome Station	29 Oct.	5.30 P.M.	1 19·8 E.	29 Oct.	4	11 A.M.	46 17·0 S.	29 Oct.	25 *a*	noon	·34058
Lat 17° 57′ 36″ S.		8 A.M.	1 12·3 E.	29 ,,	4	4 P.M.	46 15·9 S.		25 *d*	,,	·34028
Long. 122° 14′ 32″ E.				30 ,,	3	11 A.M.	46 16·4 S.				
					3	4 P.M.	46 16·0 S.				
					4	11 A.M.	46 15·1 S.				
					4	3 P.M.	46 14·7 S.				

Table I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93 (continued).

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ. Date.	Needle.	Time.	θ.	Absolute Horizontal Force = H. Date.	Magnet.	Time.	H.
(11) Gascogne River.	1890.		° ′	1890.			° ′	1890.			
Mouth of River, on sand hills.	11 Nov.	4.30 P.M.	1 50·1 W.	11 Nov.	4	10 A.M.	56 19·0 S.	11 Nov.	25 *d*	10 A.M.	·29389
Lat. 24° 53′ 45″ S.					3	11 A.M.	56 21·4 S.		25 *a*	10 A.M.	·29377
Long. 113° 39′ 40″ E.					3	2 P.M.	56 22·2 S.				
					4	3 P.M.	56 19·2 S.				
(12) Fremantle.											
Arthur Head Lighthouse, S. 15° E.	25 Nov.	5.30 P.M.	3 46·3 W.	25 Nov.	3	10 A.M.	63 45·4 S.	25 Nov.	25 *a*	1 P.M.	·25357
(true) 170 feet.	26 ,,	7 A.M.	3 48·6 W.		4	11 A.M.	63 45·0 S.	26 ,,	25 *a*	1 P.M.	·25364
Lat. 32° 3′ 14″ S.	26 ,,	5.30 P.M.	3 48·2 W.		4	2 P.M.	63 45·3 S.		25 *d*	1 P.M.	·25340
Long. 115° 44′ 25″ E.					3	3 P.M.	63 45·8 S.				
				26 ,,	3	10 A.M.	63 46·3 S.				
					3	3 P.M.	63 46·8 S.				
(13) Hobart.	1891.			1891.				1891.			
Kangaroo Point Station.	23 Feb.	5 P.M.	9 55·3 E.	23 Feb.	3	10 A.M.	71 8·9 S.	23 Feb.	25 *a*	noon	·20169
Lat. 42° 52′ 55″ S.	25 ,,	8.30 A.M.	9 43·8 E.		3	3 P.M.	71 9·4 S.	25 ,,	25 *a*	,,	·20172
Long. 147° 22′ 15″ E.		5.20 P.M.	9 53·4 E.		4	10 A.M.	71 8·2 S.		25 *d*	,,	·20155
	26 ,,	7.40 A.M.	9 45·9 E.		4	3 P.M.	71 8·3 S.	26 ,,	25 *d*	,,	·20191
				26 ,,	3	11 A.M.	71 9·1 S.	27 ,,	25 *a*	,,	·20172
					3	3 P.M.	71 9·6 S.				
(14) Melbourne.											
At the Observatory.	17 March	5.20 P.M.	8 10·4 E.	17 March	3	10 A.M.	67 15·7 S.	17 March	25 *a*	noon	·23475
Lat. 37° 49′ 53″ S.		5.30 P.M.	8 9·3 E.		4	11 A.M.	67 17·1 S.		25 *d*	1 P.M.	·23453
Long. 144° 58′ 42″ E.	19 ,,	8.40 A.M.	8 0·5 E.		3	3 P.M.	67 14·7 S.				
		4.45 P.M.	8 8·0 E.		4	3 P.M.	67 17·6 S.				
	20 ,,	9 A.M.	8 3·4 E.	18 ,,	2	11 A.M.	67 14·2 S.				
		3·45 P.M.	8 8·8 E.		1	noon	67 15·4 S.				
					1	2.40 P.M.	67 15·8 S.				
					2	3.30 P.M.	67 14·4 S.				

TABLE I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93 (continued).

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ.				Absolute Horizontal Force = H.			
				Date.	Needle.	Time.	θ.	Date.	Magnet.	Time.	H.
(15) ADELAIDE.	1891.		° ′	1891.			° ′	1891.			
Observatory.	30 March	7.30 A.M.	5 35·2 E.	30 Mar.	3	10.30 A.M.	65 27·8 S.	30 March	25 *a*	noon.	·24775
Lat. 34° 55′ 32″ S.		5.20 P.M.	5 40·4 E.		1	11.30 A.M.	65 26·1 S.		25 *d*	1.20 P.M.	·24776
Long. 138° 35′ 5″ E.		8.20 A.M.	5 34·8 E.		1	3.0 P.M.	65 26·7 S.	31 "	25 *a*	1.30 P.M.	·24742
					3	3.30 P.M.	65 27·8 S.		25 *d*	0.45 P.M.	·24750
(16) ALBANY.											
Wakefield Point, ruin of old Commissariat	8 April	4 P.M.	4 0·5 W.	7 April	3	10 A.M.	66 18·0 S.	7 April	25 *a*	2 P.M.	·23965
Store, S.W. (true),		8 A.M.	4 5·6 W.		3	3 P.M.	66 19·5 S.		25 *d*	1.30 P.M.	·23966
110 yards.	7 "	4 P.M.	4 2·4 W.	8 "	4	10 A.M.	66 17·2 S.	8 "	25 *a*	2.10 P.M.	·23888
Lat. 35° 2′ 0″ S.		8 A.M.	4 8·1 W.		4	3 P.M.	66 17·2 S.		25 *d*	1.50 P.M.	·23881
Long. 117° 54′ 3″ E.				7 "	1	3 P.M.	66 17·6 S.				
					1	11 A.M.	66 16·6 S.				
(17) NEAR COSSACK.											
Cape Lambert. Lat. 20° 35′ 45″ S. Long. 117° 11′ 20″ E.	22 April	8 A.M.	0 4·5 W.	22 April	3	10 A.M.	50 13·3 S.				
Red Cliff Station, No. 1. Lat. 20° 36′ 20″ S. Long. 117° 11′ 36″ E.				22 "	3	noon	52 18·5 S.	22 April	25 *a*		·31039
Red Cliff Point, No. 2 Station. Lat. 20° 36′ 16″ S. Long. 117° 11′ 33″ E.	22 "	4.50 P.M.	7 56·1 W.	22 "	3	4.0 P.M.	54 11·9 S.				—

TABLE I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93 (continued).

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ.				Absolute Horizontal Force.			
				Date.	Needle.	Time.	θ.	Date.	Magnet.	Time.	H.
(17) NEAR COSSACK (continued).	1891.		° ′	1891.			° ′	1891.			
Red Cliff Point, No. 3 Station. Lat. 20° 36′ 30″ S. Long. 117° 11′ 43″ E.	23 April	8 A.M.	1 11·4 E.	23 April	3	9.30 A.M.	50 41·1 S.				
Red Cliff Point, No. 4 Station. Lat. 20° 36′ 18″ S. Long. 117° 11′ 41′ E.	23 "	11 A.M.	2 53·3 E.	23 "	3	10.30 A.M.	52 30·5 S.				
Near Red Cliff Point, No. 5 Station. Station on 2-feet rock. Lat. 20° 35′ 36″ S. Long. 117° 12′ 11″ E.	23 "	3.30 P.M.	1 4·0 E.	23 "	3	2 P.M.	51 35·6 S.				
(10) ROEBUCK BAY.											
Broome Station Lat. 17° 57′ 36″ S. Long. 122° 14′ 32″ E. B.A.T. flagstaff N. 16½ W. distant 200 yards	28 April	8 A.M. 5 P.M.	1 17·0 E. 1 17·3 E.	28 April	3 4 4 3	10 A.M. 11 A.M. 2 P.M. 3 P.M.	46 22·0 S. 46 19·0 S. 46 20·0 S. 46 21·5 S.	28 April 29 "	25 *a* 25 *d*	noon "	·34044 ·34046
(4) BAUDIN ISLAND.											
East Sandy Point Lat. 14° 7′ 50″ S. Long. 125° 36′ 20″ E.	7 May	8.15 A.M. 4 P.M.	2 13·8 E. 2 15·0 E.								

TABLE I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93 (continued).

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ.				Absolute Horizontal Force.			
				Date.	Needle.	Time.	θ.	Date.	Magnet.	Time.	H.
(18) DAMMA.	1891.		° ′	1891.			° ′	1891.			
N. Side of Koelewatte	7 Nov.	7.40 A.M.	2 14·8 E.	6 Nov.	3	11.30 A.M.	27 40·4 S.	7 Nov.	25 *a*	2 P.M.	·37574
Harbour. Ruined		4.40 P.M.	2 20·7 E.		3	2.30 P.M.	27 43·6 S.		25 *d*	1.30 P.M.	·37572
watch-house bore				9 „	3	11 A.M.	27 40·4 S.	9 „	25 *d*	1.15 P.M.	·37594
N. 85° W. 100 yards					3	2 P.M.	27 39·6 S.				
Lat. 7° 8′ 45″ S.				7 „	4	10 A.M.	27 43·5 S.				
Long. 128° 40′ 18″ E.					4	3 P.M.	27 43·8 S.				
(19) AMBOINA.											
(Same station as	14 Nov.	8 A.M.	2 24·6 E.	14 Nov.	3	10.30 A.M.	21 0·8 S.	14 Nov.	25 *a*	1.40 P.M.	·38381
H.M.S. "Challen-		4.30 P.M.	2 30·6 E.		3	3 P.M.	21 2·7 S.		25 *d*	1 P.M.	·38334
ger" in 1874)	16 „	4.40 P.M.	2 32·3 E.	16 „	3	10 A.M.	21 0·3 S.	16 „	25 *d*	1 P.M.	·38345
Lat. 3° 41′ 40″ S.					4	11 A.M.	21 4·8 S.				
Long. 128° 10′ 7″ E.					4	2 P.M.	21 4·6 S.				
					3	3 P.M.	21 1·0 S.				
(20) TERNATE.											
(Same station as	23 Nov.	7.40 A.M.	2 2·1 E.	23 Nov.	3	10.30 A.M.	11 25·4 S.	23 Nov.	25 *a*	1.30 P.M.	·38413
H.M.S. "Challen-		4.15 P.M.	2 4·1 E.		3	2.30 P.M.	11 22·3 S.		25 *d*	1 P.M.	·38398
ger" in 1874)				24 „	4	10 A.M.	11 25·7 S.	24 „	25 *a*	1.30 P.M.	·38434
Lat. 0° 47′ 25″ N.					3	11 A.M.	11 24·4 S.				
Long. 127° 23′ 15″ E.					3	2 P.M.	11 21·8 S.				
					4	3 P.M.	11 26·6 S.				
(21) SAMBOANGAN.											
(Same position as	30 Nov.	7.20 A.M.	2 4·4 E.	30 Nov.	4	10.20 A.M.	1 36·0 N.	30 Nov.	25 *a*	1.50 P.M.	·38349
H.M.S. "Challen-		4.50 P.M.	2 5·6 E.		3	11.20 A.M.	1 37·3 N.	1 Dec.	25 *a*	1.10 P.M.	·38365
ger," 1874)	1 Dec.	7.20 A.M.	2 2·8 E.		3	2.50 P.M.	1 35·8 N.		25 *d*	0·40 P.M.	·38368
Lat. 6° 54′ 20″ N.					4	3.30 P.M.	1 33·0 N.				
Long. 122° 3′ 51″ E.				1 Dec.	4	10.30 A.M.	1 36·7 N.				
					4	2.20 P.M.	1 33·7 N.				

TABLE I.—H.M.S. "Penguin." Table of Magnetic Elements Observed during 1890–93 (continued).

Place of observation.	Date.	Time.	Declination = δ.	Inclination or Dip = θ.				Absolute Horizontal Force = H.			
				Date.	Needle.	Time.	θ.	Date.	Magnet.	Time.	H.
(22) MANILA.	1891.			1891.				1891.			
(Same position as	8 Dec.	8.50 A.M.	0 46·8 E.	8 Dec.	4	10.20 A.M.	17 14·7 N.	8 Dec.	25 *a*	2.10 P.M.	·37699
H.M.S. "Challen-		4.40 P.M.	0 47·8 E.		3	11 A.M.	17 16·2 N.		25 *d*	1.40 P.M.	·37709
ger," 1874)	9 ,,	7.50 A.M.	0 47·3 E.		3	3 P.M.	17 15·7 N.	9 ,,	25 *d*	0.30 P.M.	·37717
Lat. 14° 35′ 25″ N.				9 ,,	4	3.30 P.M.	17 13·9 N.				
Long. 120° 58′ 15″ E.					4	10·40 A.M.	17 13·2 N.				
Water line station, 123 yards from principal position	9 ,,	9 A.M.	0 47·9 E.		4	1.20 P.M.	17 13·2 N.				
Inner station, 190 yards inland from principal position		9.20 A.M.	0 45·3 E.								
(23) HONG KONG.	1892.			1892.				1892.			
Observatory.	28 Jan.	5 P.M.	0 35·4 E.	27 Jan.	4	noon	32 3·3 N.	27 Jan.	25 *a*	2 P.M.	·36393
Lat. 22° 18′ 12″ N.	29 ,,	8.40 A.M.	0 34·5 E.		4	3 P.M.	32 5·2 N.	28 ,,	25 *d*	0.40 P.M.	·36384
Long. 114° 10′ 30″ E.		4.20 P.M.	0 35·7 E.	28 ,,	3	10 A.M.	32 2·4 N.	29 ,,	25 *a*	0.30 P.M.	·36401
	4 Feb.	10.30 A.M.	0 37·7 E.		3	3.30 P.M.	32 4·9 N.	4 Feb.	25 *d*	0.50 P.M.	·36393
		3.20 P.M.	0 36·0 E.		2	11 A.M.	32 1·8 N.				
		5 P.M.	0 33·2 E.		2	2.40 P.M.	32 3·3 N.				
				29 ,,	1	10.20 A.M.	32 1·8 N.				
					1	4 P.M.	32 4·2 N.				
				29 ,,	3	11.20 A.M.	32 2·5 N.				
					3	3 P.M.	32 2·8 N.				
KEW OBSERVATORY.	1893.			1893.				1893.			
(Base station.)	5 Sept.	0.50 P.M.	17 38·15 W.	5 Sept.	1	11·40 A.M.	67 26·4 N.	5 Sept.	25 *a*	4.40 P.M.	·18260
	6 ,,	2.20 P.M.	17 34·03 W.	4 ,,	2	10.40 A.M.	67 24·3 N.	6 ,,	25 *a*	0.20 P.M.	·18235
				4 ,,	3	3.20 P.M.	67 27·8 N.		25 *d*	11.50 A.M.	·18257
					4	0.20 P.M.	67 27·1 N.				
					4	0.40 P.M.	67 27·8 N.				

TABLE II.—H.M.S. "Penguin." Table of Magnetic Elements Observed with Relative Instruments during 1890–93.

Place of observation.	Date.	δ.	Date.	θ.	Date.	H.	Z.	Remarks.
SUEZ BAY.	1890.	° ′	1890.	° ′	1890.			
Lat. 29° 50′ 0″ N. Long. 32° 32′ 30″ E.	15 March	4·31 W.						δ obtained by swinging ship in 19 fathoms of water
STRAIT OF BABELMANDEB.								
Lat. 12° 35′ 0″ N. Long. 42° 25′ 0″ E.	27 "	3·25 W.						δ obtained by swinging ship in 19 fathoms of water
PERIM ISLAND.								
Auxiliary Station, No. 1. Lat. 12° 39′ 47″ N. Long. 43° 23′ 35″ E.			31 March	5 4·3 N.	31 March	·35541	·03154	Observed with Fox circle C. 10, compared with two stations at which absolute observations were made
Auxiliary Station, No. 2. Lat. 12° 39′ 47″ N. Long. 43° 24′ 46″ E.			31 "	5 3·7 N.	31 "	·35222	·03120	
Auxiliary Station, No. 3. Lat. 12° 38′ 0″ N. Long. 43° 25′ 40″ E.			2 April	3 30·0 N.	2 April	·35568	·02175	
BAUDIN ISLAND.								
Lat. 14° 0′ 0″ S. Long. 125° 36′ 5″ E.	19 July	2·11 E.						δ obtained by swinging ship in 21 fathoms of water
Lat. 14° 2′ 0″ S. Long. 125° 36′ 0″ E.	1891. 6 May	2·11 E.						δ obtained by swinging ship in 21 fathoms of water
PORT WALCOTT (Cossack.)	1890.							
Lat. 20° 35′ 0″ S. Long. 117° 16′ 0″ E.	5 Nov.	0·15 E.						δ obtained by swinging ship in 8 to 9 fathoms
LAUNCESTON (Tasmania.)	1891.		1891.		1891.			
Lat. 41° 26′ 0″ S. Long. 147° 8′ 35″ E.	5 April		5 April	69 58·6 S.	5 April	·021120	·57958	

TABLE III.—Magnetic Shoal, Port

Abstract of Observations for Declination, Inclination,

Date.	Time.	Number of Obser-vation.	Position of ship. Bezout △. Bearing.	Position of ship. Bezout △. Dis-tance.	Dis-tance of Fox circle from bottom of the sea.	Magnetic direction of ship's head.	Observed Declination.	North seeking end of Needle repelled to	Magnetic direction ship's head by Fox compass.	Observed Inclina-tion or Dip.
			° ′	Miles.	Feet.	°	° ′ ″		° ′	° ′
24th April, 1891.	A.M. 9.10 – 9 40	1	S. 84 14 W.	2·200	83½	N. 45 E.	5 40 0	E.	N. 45 38 E.	52 31·3
	10.20 – 10.32	2	S. 82 15 W.	2·128	83½	N. 53 E.	22 44 0	E.	N. 54 3 E.	64 13
	P.M. 12.40	3	S. 79 45 W.	2·188	80½	N. 50 E.	53 25 0	E.	..	74 8
	12.50 – 1.10	3	S. 79 45 W.	2·188	80½	N. 53 E.	55 56 0	E.	N. 54 3 E.	74 11
	2.20 – 2.37	4	S. 78 49 W.	2·170	80½	S. 66 E.	1 12 0	E.	N. 115 45 E.	78 53
	3.10 – 3.29	5	S. 78 28 W.	2·131	83½	S. 20 E.	{26 12 0 / 25 45 0}	W.	N. 160 35 E.	62 30
	5.30 – 5.50	6	S. 85 12 W.	2·579	83½	N. 12 E.	2 41 0	E.	N. 11 52 E.	52 27
25th April, 1891.	A.M. 6.0 – 6.10	6*a*	S. 85 12 W.	2·579	83	S. 34 E.	..	..	N. 145 51 E.	55 5
	6.45 – 7.5	6*b*	S. 85 12 W.	2·579	83	N. 22 E.	..	..	N. 21 19 E.	52 47
	9.13 – 9.26	7	S. 79 27 W.	2·148	82	N. 22 E.	14 17 0	E	N. 21 19 E.	80 21
	10.6 – 10.20	8	S. 78 38 W.	2·240	83½	N. 7 E.	44 34 0	E.	N. 6 32 E.	73 50
	11.0 – 11.15	9	S. 77 27 W.	2·126	83½	S. 86 E.	19 42 0	W.	N. 94 48 E.	55 5
	11.50 – 12.2	10	S. 78 27 W.	2·132	83½	S. 78 E.	{25 12 0 / 24 39 0}	W.	N. 102 45 E.	67 5
	P.M. 2.45	11	S. 71 0 W.	2·851	82	S. 60 E.	..	..	N. 121 13 E.	53 56
	2.53	11	S. 71 0 W.	2·851	82	S. 44 E.	12 50 0	W.	N. 137 17 E.	..
	3.7 – 3.40	12	S. 68 4 W.	3·501	80½	S. 71 E.	15 8 0	E.	N. 110 20 E.	68 39
	4.40	13	S. 66 6 W.	4·330	..	S. 71 E.	..	..	N. 110 20 E.	52 21
	5.45	13*a*	S. 66 6 W.	4·330	..	S. 81 E.	..	..	N. 100 18 E.	53 31
	6.0	13	S. 66 6 W.	4·330	79	North.	2 42 0	E.	..	..
	6.3	13*b*	S. 66 6 W.	4·330	..	N. 9 W.	..	..	N. 10 25 W.	49 19

NOTE.—The letters P, D^1, P^2, D^2, point to the values of the Vertical

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and Total Force with Fox Apparatus C. 10.

Deviation in Inclination for ship's head.	Corrected Dip.	Observed Total Force.	Deviation in Total Force for ship's head.	Corrected Total Force.	Horizontal Force.	Vertical Force.	Disturbances from the Normal.				
							Declination.	Horizontal Force.	Vertical Force.	Number of Observation.	Dip.
° ′	° ′						° ′				° ′
−1 24	51 7	4·5294	+·1775	4.7069	2·9547	3·6640	+ 5 40	−0·218	−0·319	1	+ 0 7
−1 42	62 31	4·2947	+·2038	4·4985	2·0760	3·9908	+ 22 44	−1·116	+0·024	2	+11 31
−1 42	72 26	..	..	..	..	..	+ 53 25	..	..	3	+21 26
−1 42	72 29	5·7713	+·2038	5·9751	1·7984	5·6980	+ 55 56	−1·394	+1·731	3	+21 29
−1 56	76 57	8·3723	+·2619	8·6342	1·9495	8·4114	+ 1 12	−1·243	+4·444 P	4	+25 57
−0 30	62 0	8·0099	+·0484	8·0583	3·7831	7·1154	−{26 12 / 25 45}	+0·591	+3·148	5	+11 0
−0 12	52 15	4·7556	+·0931	4·8487	2·9683	3·8338	+ 2 41	−0·224	−0·133	6	+ 1 15
−0 55	54 10	4·7542	+·1037	4·8579	2·8440	3·9384	..	−0·348	−0·029	6*a*	+ 3 10
−0 28	52 19	4·7385	+·1116	4·8501	2·9648	3·8384	..	−0·227	−0·129	6*b*	+ 1 19
−0 28	79 53	7·1250	+·1176	7·2426	1·2721	7·1301	+ 14 17	−1·920	+3·163	7	+28 53
0	73 50	6·7405	+·0807	6·8212	1·8992	6·5515	+ 44 34	−1·293	+2·584	8	+22 50
−2 20	52 45	6·6958	+·3020	6·9978	4·2357	5·5702	− 19 42	+1·044	+1·603	9	+ 1 45
−2 18	64 47	7·4109	+·2960	7·7069	3·2835	6·9724	−{25 12 / 24 39}	+0·091	+3·005	10	+13 47
−1 50	52 6	..	..	..	..	..	..	..	..	11	+ 1 6
..	..	6·5676	+·1531	6·7207	4·1284	5·3029	− 12 50	+0·936	+1·336 D[1]	11	..
−2 04	66 35	7·0213	+·2776	7·2989	2·9007	6·6976	+ 15 8	−0·291	+2·731 P[2]	12	+15 35
−2 04	50 17	..	..	..	..	..	..	..	..	13	− 0 43
−2 18	51 13	..	..	..	..	..	..	..	..	13*a*	+ 0 13
..	..	4·8856	+·0738	4·9594	3·1690	3·8148	+ 2 42	−0·023	−0·152 D[2]	13	..
+0 40	49 59	..	..	..	..	..	..	..	..	13*b*	− 1 01

Force at the position of the corresponding letters on Diagram A.

TABLE IV.—Observations of Declination. Magnetic Shoal, Port Walcott.

Date.	Time.	Position of ship, Bezout Island △.		Bearing of Bezout △ by Standard.	Ship's head by Standard.	Deviation for apparent position ship's head.	Apparent magnetic bearing of Bezout △.	True bearing of Bezout Island △.	Deflection of Standard compass.	North-seeking end of needle repelled to	Remarks.
		Bearing.	Distance.								
	A.M.		miles.								
April 22, 1891.	10 5	N. 83° 41′ W.	1·903	N. 82° 0′ W.	N. 22° W.	3° 44′ W.	N. 85° 44′ W.	N. 83° 41′ W.	2° 17′	E.	A correction of 14′ W. index correction has been applied to each of the deflections recorded, and the normal Declination at Bezout Island has been taken =0
	10 7	N. 85 52 W.	1·924	N. 88 40 W.	N. 15 E.	0 55 E.	N. 87 45 W.	N. 85 52 W.	2 7	E.	
	10 10	N. 87 45 W.	1·940	S. 87 30 W.	N. 27 E.	2 12 E.	S. 89 42 W.	N. 87 45 W.	2 47	E.	
	10 13	S. 89 43 W.	1·932	S. 88 20 W.	N. 20 W.	3 22 W.	S. 84 58 W.	S. 89 43 W.	4 59	E.	
	10 16	S. 86 55 W.	1·929	S. 81 20 W.	N. 32 W.	4 10 W.	S. 77 10 W.	S. 86 55 W.	9 59	E.	
	10 18	S. 86 5 W.	1·923	S. 69 20 W.	N. 19 W.	3 22 W.	S. 65 58 W.	S. 86 5 W.	20 21	E.	
	10 20	S. 83 18 W.	1·917	S. 57 0 W.	N. 23 W.	3 44 W.	S. 53 16 W.	S. 83 18 W.	30 16	E.	
	10 22	S. 81 57 W.	1·917	S. 70 40 W.	N. 4 W.	1 26 W.	S. 69 14 W.	S. 81 57 W.	12 57	E.	
	10 25	S. 79 9 W.	1·934	S. 84 20 W.	N. 11 E.	0 40 E.	S. 85 0 W.	S. 79 9 W.	5 37	W.	
	10 28	S. 75 7 W.	1·963	S. 81 20 W.	N. 7 E.	0 5 W.	S. 81 15 W.	S. 75 7 W.	5 54	W.	
	10 32	S. 71 30 W.	2·001	S. 77 0 W.	N. 2 E.	0 42 W.	S. 76 18 W.	S. 71 30 W.	4 34	W.	
	10 36	S. 68 11 W.	2·035	S. 73 20 W.	N. 10 W.	1 26 W.	S. 71 54 W.	S. 68 11 W.	3 29	W.	
	10 41	S. 63 52 W.	2·079	S. 66 0 W.	N. 8 E.	0 5 W.	S. 65 55 W.	S. 63 52 W.	1 49	W.	
	10 45	S. 58 49 W.	2·169	S. 61 20 W.	N. 3 W.	1 26 W.	S. 59 54 W.	S. 58 49 W.	0 51	W.	
	10 49	S. 54 3 W.	2·285	S. 57 40 W.	N. 4 W.	1 26 W.	S. 56 14 W.	S. 54 3 W.	1 57	W.	
	10 57	S. 57 21 W.	2·640	S. 60 40 W.	S. 9 E.	1 4 W.	S. 59 36 W.	S. 57 21 W.	2 1	W.	
	10 59	S. 58 20 W.	2·640	S. 57 30 W.	S. 23 W.	3 41 E.	S. 61 11 W.	S. 58 20 W.	2 37	W.	
	11 4	S. 62 41 W.	2·476	S. 66 0 W.	S. 7 E.	0 20 W.	S. 65 40 W.	S. 62 41 W.	2 45	W.	
	11 7	S. 65 52 W.	2·404	S. 68 30 W.	S. 6 W.	1 43 E.	S. 69 13 W.	S. 65 52 W.	3 7	W.	
	11 12	S. 72 34 W.	2·299	S. 81 0 W.	S. 9 W.	2 9 E.	S. 83 9 W.	S. 72 34 W.	10 21	W.	
	11 14	S. 75 7 W.	2·271	S. 86 10 W.	S. 13 W.	2 51 E.	S. 89 1 W.	S. 75 7 W.	13 40	W.	
	11 16	S. 77 47 W.	2·237	S. 72 0 W.	South	0 52 E.	S. 72 52 W.	S. 77 47 W.	5 9	E.	
	11 18	S. 79 36 W.	2·196	S. 30 30 W.	S. 44 E.	3 18 W.	S. 27 12 W.	S. 79 36 W.	52 38	E.	
	11 20	S. 81 24 W.	2·189	S. 56 0 W.	S. 33 E.	3 12 W.	S. 52 48 W.	S. 81 24 W.	28 50	E.	
	11 22	S. 84 5 W.	2·178	S. 78 40 W.	S. 8 E.	1 4 W.	S. 77 36 W.	S. 84 5 W.	6 43	E.	
	11 24	S. 86 19 W.	2·171	S. 84 20 W.	S. 16 E.	2 0 W.	S. 82 20 W.	S. 86 19 W.	4 13	E.	
	11 29	N. 89 3 W.	2·192	S. 89 0 W.	S. 1 E.	0 52 E.	S. 89 52 W.	N. 89 3 W.	1 19	E.	
	11 33	N. 82 24 W.	2·187	N. 84 0 W.	South	0 52 E.	N. 83 8 W.	N. 82 24 W.	0 58	E.	
	11 39	N. 74 59 W.	2·214	N. 76 10 W.	South	0 52 E.	N. 75 18 W.	N. 74 59 W.	0 35	E.	
	11 47	N. 82 24 W.	2·339	N. 83 50 W.	North	1 10 W.	N. 85 0 W.	N. 82 24 W.	2 50	E.	
	11 52	N. 88 26 W.	2·354	N. 87 20 W.	N. 18 E.	1 28 E.	N. 85 52 W.	N. 88 26 W.	2 20	W.	
	11 54	S. 88 6 W.	2·377	S. 86 40 W.	N. 10 W.	2 16 W.	S. 84 24 W.	S. 88 6 W.	3 56	E.	
	11 57	S. 82 44 W.	2·328	S. 79 30 W.	North	1 10 W.	S. 78 20 W.	S. 82 44 W.	4 38	E.	

TABLE IV.—Observations of Declination. Magnetic Shoal, Port Walcott (continued).

Date.	Time.	Position of ship, Bezout Island △.		Bearing of Bezout △ by Standard.	Ship's head by Standard.	Deviation for apparent position ship's head.	Apparent magnetic bearing of Bezout △.	True bearing of Bezout Island △.	Deflection of Standard compass.	North-seeking end of needle repelled to	Remarks.
		Bearing.	Distance.								
	A.M.		miles.								
April 22, 1891.	12 0	S. 80° 20′ W.	2·344	S. 61° 0′ W.	N. 3° E.	0° 42′ W.	S. 60° 18′ W.	S. 80° 20′ W.	20° 16′	E.	
	12 3	S. 77 9 W.	2·403	S. 37 0 W.	N. 17 W.	3 0 W.	S. 34 0 W.	S. 77 9 W.	43 23	E.	
	12 5	S. 74 1 W.	2·447	S. 83 30 W.	N. 8 E.	0 12 E.	S. 83 42 W.	S. 74 1 W.	9 41	W.	
	12 7	S. 70 27 W.	2·501	S. 74 30 W.	N. 22 E.	1 44 E.	S. 76 14 W.	S. 70 27 W.	5 33	W.	
	12 11	S. 65 19 W.	2·585	S. 69 40 W.	N. 7 W.	1 43 W.	S. 71 23 W.	S. 65 19 W.	5 50	W.	
	P.M.										
	1 37	S. 52 45 W.	3·386	S. 55 0 W.	S. 10 E.	1 4 W.	S. 53 56 W.	S. 52 45 W.	0 57	W.	
	1 43	S. 56 0 W.	3·225	S. 58 30 W.	S. 4 E.	0 13 E.	S. 58 43 W.	S. 56 0 W.	2 29	W.	
	1 47	S. 59 26 W.	3·081	S. 63 20 W.	S. 4 W.	1 18 E.	S. 64 38 W.	S. 59 26 W.	4 58	W.	
	1 52	S. 62 14 W.	2·995	S. 67 40 W.	S. 5 E.	0 14 E.	S. 67 54 W.	S. 62 14 W.	5 26	W.	
	1 55	S. 65 28 W.	2·929	S. 69 40 W.	S. 5 W.	1 43 E.	S. 71 23 W.	S. 65 28 W.	5 51	W.	
	2 0	S. 68 57 W.	2·866	S. 75 30 W.	S. 1 E.	0 52 E.	S. 76 22 W.	S. 68 57 W.	7 11	W.	
	2 4	S. 71 31 W.	2·797	S. 77 30 W.	S. 10 W.	2 35 E.	S. 80 5 W.	S. 71 31 W.	8 20	W.	
	2 7	S. 73 8 W.	2·769	S. 63 30 W.	S. 26 E.	2 55 W.	S. 60 35 W.	S. 73 8 W.	12 47	E.	
	2 10	S. 74 30 W.	2·748	S. 37 50 W.	S. 54 E.	2 43 W.	S. 35 7 W.	S. 74 30 W.	39 37	E.	
	2 14	S. 76 26 W.	2·717	S. 61 20 W.	S. 22 E.	2 49 W.	S. 58 31 W.	S. 76 26 W.	18 9	E.	
	2 18	S. 78 25 W.	2·683	S. 68 30 W.	S. 7 W.	1 13 E.	S. 69 43 W.	S. 78 25 W.	8 56	E.	
	2 22	S. 80 54 W.	2·626	S. 77 30 W.	S. 30 E.	3 6 W.	S. 74 24 W.	S. 80 54 W.	6 44	E.	
	2 26	S. 83 53 W.	2·639	S. 79 10 W.	S. 2 W.	1 28 E.	S. 80 38 W.	S. 83 53 W.	3 29	E.	
	2 35	S. 82 10 W.	3·058	S. 79 40 W.	N. 5 W.	1 43 W.	S. 77 57 W.	S. 82 10 W.	4 27	E.	
	2 40	S. 75 29 W.	3·021	S. 67 0 W.	N. 16 W.	3 0 W.	S. 64 0 W.	S. 75 29 W.	11 43	E.	
	2 43	S. 71 20 W.	3·060	S. 51 0 W.	N. 7 W.	1 43 W.	S. 49 17 W.	S. 71 20 W.	22 17	E.	
	2 46	S. 68 52 W.	3·128	S. 69 20 W.	N. 17 E.	1 28 E.	S. 70 48 W.	S. 68 52 W.	1 42	W.	
	2 49	S. 66 56 W.	3·185	S. 80 30 W.	N. 30 E.	2 12 E.	S. 82 42 W.	S. 66 56 W.	15 32	W.	
	2 50	S. 65 56 W.	3·206	S. 84 40 W.	N. 17 E.	0 45 E.	S. 85 25 W.	S. 65 56 W.	19 15	W.	
	2 51	S. 64 57 W.	3·228	S. 81 30 W.	N. 4 E.	0 42 W.	S. 80 48 W.	S. 64 57 W.	15 37	W.	
	2 55	S. 61 56 W.	3·273	S. 70 40 W.	N. 11 E.	0 40 E.	S. 71 20 W.	S. 61 56 W.	9 10	W.	Correction for I. E. of Standard compass, 14′ W. applies in each case
	3 5	S. 63 13 W.	3·786	S. 62 0 W.	S. 23 E.	2 49 W.	S. 59 11 W.	S. 63 13 W.	4 16	E.	
	3 10	S. 66 22 W.	3·726	S. 46 0 W.	S. 43 E.	3 17 W.	S. 42 43 W.	S. 66 22 W.	23 53	E.	
	3 15	S. 68 36 W.	3·689	S. 50 40 W.	S. 13 E.	1 43 W.	S. 48 57 W.	S. 68 36 W.	19 53	E.	
	3 18	S. 71 12 W.	3·613	S. 59 30 W.	S. 7 E.	0 22 W.	S. 59 8 W.	S. 71 12 W.	12 18	E.	
	3 22	S. 74 50 W.	3·537	S. 68 30 W.	S. 1 W.	0 52 E.	S. 69 22 W.	S. 74 50 W.	5 42		

TABLE IV.—Observations of Declination. Magnetic Shoal, Port Walcott (continued).

Date.	Time.	Position of ship, Bezout Island △. Bearing.	Position of ship, Bezout Island △. Distance.	Bearing of Bezout △ by Standard.	Ship's head by Standard.	Deviation for apparent position ship's head.	Apparent magnetic bearing of Bezout △.	True bearing of Bezout Island △.	Deflection of Standard compass.	North-seeking end of needle repelled to	Remarks.
April 22, 1891.	P.M. 3 26	S. 78° 2′ W.	miles. 3·462	S. 76° 20′ W.	S. 14° E.	2° 0′ W.	S. 74° 20′ W.	S. 78° 2′ W.	3° 56′	E.	
	3 29	S. 82 8 W.	3·443	S. 80 0 W.	S. 1 W.	0 52 E.	S. 80 52 W.	S. 82 8 W.	1 30	E.	
	3 33	S. 86 29 W.	3·314	S. 84 40 W.	S. 4 W.	1 18 E.	S. 85 58 W.	S. 86 29 W.	0 45	E.	
	3 36	S. 89 44 W.	3·373	N. 89 40 W.	S. 14 E.	2 0 W.	S. 88 20 W.	S. 89 44 W.	1 38	E.	
	3 48	N. 82 20 W.	3·395	N. 83 20 W.	S. 10 W.	2 35 E.	N. 80 45 W.	N. 82 20 W.	1 21	W.	
April 23, 1891.	A.M. 9 24	N. 70 8 W.	1·858	N. 66 40 W.	N. 22 W.	3 44 W.	N. 70 24 W.	N. 70 8 W.	0 30	E.	
	9 28	N. 72 47 W.	1·772	N. 70 20 W.	N. 10 W.	2 16 W.	N. 72 36 W.	N. 72 47 W.	0 3	E.	
	9 31	N. 75 24 W.	1·741	N. 75 0 W.	N. 6 E.	0 5 W.	N. 75 5 W.	N. 75 24 W.	0 5	W.	
	9 34	N. 78 5 W.	1·709	N. 80 0 W.	N. 26 E.	2 0 E.	N. 78 0 W.	N. 78 5 W.	0 9	E.	
	9 38	N. 78 38 W.	1·705	N. 79 40 W.	N. 11 E.	0 40 E.	N. 79 0 W.	N. 78 38 W.	0 36	E.	
	9 40	N. 79 34 W.	1·698	N. 78 0 W.	N. 8 W.	2 0 W.	N. 80 0 W.	N. 79 34 W.	0 40	E.	
	9 44	N. 82 15 W.	1·653	N. 82 10 W.	N. 5 E.	0 5 W.	N. 82 15 W.	N. 82 15 W.	0 14	E.	
	9 49	N. 84 32 W.	1·602	N. 86 40 W.	N. 19 E.	1 28 E.	N. 85 12 W.	N. 84 32 W.	0 54	E.	
	9 53	N. 88 14 W.	1·536	N. 84 20 W.	N. 19 W.	3 22 W.	N. 87 42 W.	N. 88 14 W.	0 18	W.	
	9 57	S. 88 17 W.	1·504	N. 88 0 W.	N. 1 W.	1 10 W.	N. 89 10 W.	S. 88 17 W.	2 19	W.	
	10 1	S. 83 59 W.	1·497	S. 88 40 W.	N. 19 E.	1 28 E.	N. 89 52 W.	S. 83 59 W.	5 55	W.	
	10 6	S. 81 43 W.	1·500	S. 85 10 W.	N. 36 E.	2 57 E.	S. 88 7 W.	S. 81 43 W.	6 10	W.	
	10 10	S. 75 27 W.	1·491	S. 82 50 W.	N. 21 W.	3 44 W.	S. 79 6 W.	S. 75 27 W.	3 25	W.	
	10 16	S. 67 10 W.	1·461	S. 70 40 W.	N. 1 W.	1 10 W.	S. 69 30 W.	S. 67 10 W.	2 6	W.	
	10 21	S. 61 41 W.	1·497	S. 62 30 W.	N. 18 E.	1 28 E.	S. 63 58 W.	S. 61 41 W.	2 3	W.	
	10 29	S. 58 30 W.	1·147	S. 61 20 W.	S. 19 E.	2 33 W.	S. 58 47 W.	S. 58 30 W.	0 3	W.	
	10 33	S. 66 29 W.	1·095	S. 70 50 W.	S. 30 E.	3 6 W.	S. 67 44 W.	S. 66 29 W.	1 0	W.	
	10 35	S. 75 47 W.	1·133	S. 80 40 W.	S. 28 E.	3 0 W.	S. 77 40 W.	S. 75 47 W.	1 39	W.	
	10 38	S. 82 7 W.	1·174	S. 88 30 W.	S. 28 E.	3 0 W.	S. 85 30 W.	S. 82 7 W.	3 9	W.	
	10 42	S. 89 28 W.	1·228	N. 82 10 W.	S. 30 E.	3 6 W.	N. 85 16 W.	S. 89 28 W.	5 2	W.	
	10 45	N. 86 9 W.	1·321	N. 82 0 W.	S. 37 E.	2 47 W.	N. 84 47 W.	N. 86 9 W.	0 18	W.	
	10 48	N. 82 15 W.	1·352	N. 80 0 W.	S. 31 E.	3 6 W.	N. 83 16 W.	N. 82 15 W.	1 15	E.	
	11 7	S. 78 52 W.	2·025	N. 86 50 W.	S. 17 E.	2 15 W.	N. 89 5 W.	S. 78 52 W.	11 49	W.	
	11 9	S. 80 43 W.	2·019	S. 83 30 W.	S. 25 E.	2 54 W.	N. 86 24 W.	S. 80 43 W.	12 39	W.	
	11 10	S. 81 43 W.	2·023	S. 66 10 W.	S. 44 E.	3 16 W.	S. 62 54 W.	S. 81 43 W.	19 3	E.	
	11 12	S. 82 44 W.	2·037	S. 64 10 W.	S. 47 E.	3 3 W.	S. 61 7 W.	S. 82 44 W.	21 51	E.	

TABLE IV.—Observations of Declination. Magnetic Shoal, Port Walcott (continued).

Date.	Time.	Position of ship, Bezout Island △.		Bearing of Bezout △ by Standard.	Ship's head by Standard.	Deviation for apparent position of ship's head.	Apparent magnetic bearing of Bezout △.	True bearing of Bezout Island △.	Deflection of Standard compass.	North-seeking end of needle repelled to	Remarks.
		Bearing.	Distance.								
	A.M.		miles.								
April 23, 1891.	11 17	S. 81° 24′ W.	2·164	S. 57° 30′ W.	N. 31° W.	4° 10′ W.	S. 53° 20′ W.	S. 81° 24′ W.	28° 18′	E.	
	11 18	S. 80 26 W.	Slowly crossing the trough of vanishing repulsion near the focus, but not fixing	S. 70 0 W.	N. 42 W.	4 22 W.	N. 65 38 W.	S. 80 26 W.	15 2	E.	
	11 18½	S. 79 31 W.		S. 80 0 W.	N. 9 W.	2 0 W.	S. 78 0 W.	S. 79 31 W.	1 8	E.	
	11 19	S. 78 54 W.		N. 82 0 W.	N. 11 E.	0 40 E.	N. 81 20 W.	S. 78 54 W.	19 32	W.	
	11 25	S. 76 55 W.	2·275	S. 88 0 W.	S. 11 E.	1 43 W.	S. 86 17 W.	S. 76 55 W.	9 8	W.	
	11 26	S. 78 29 W.	..	S. 39 0 W.	S. 57 E.	2 32 W.	S. 36 28 W.	S. 78 29 W.	42 15	E.	
	11 27	S. 79 10 W.	..	S. 37 20 W.	S. 59 E.	2 25 W.	S. 34 55 W.	S. 79 10 W.	44 29	E.	
	11 40	S. 78 30 W.	2·270	S. 41 0 W.	N. 77 W.	2 6 W.	S. 38 54 W.	S. 78 30 W.	40 16	E.	
	11 41	S. 77 43 W.	2·290	S. 37 50 W.	N. 80 W.	2 6 W.	S. 35 44 W.	S. 77 43 W.	42 13	E.	
	11 42	S. 76 27 W.	2·321	West	N. 27 W.	4 1 W.	S. 85 59 W.	S. 76 27 W.	9 18	W.	
	11 47	S. 71 52 W.	2·520	S. 83 20 W.	S. 27 E.	3 0 W.	S. 80 20 W.	S. 71 52 W.	8 14	W.	
	11 49	S. 73 0 W.	2·529	S. 85 40 W.	S. 21 E.	2 49 W.	S. 82 51 W.	S. 73 0 W.	9 37	W.	
	11 50	S. 74 23 W.	2·541	S. 84 20 W.	S. 24 E.	2 54 W.	S. 81 26 W.	S. 74 23 W.	6 49	W.	
	11 51	S. 74 48 W.	2·546	S. 81 20 W.	S. 28 E.	3 0 W.	S. 78 20 W.	S. 74 48 W.	3 18	W.	
	11 52	S. 75 13 W.	2·550	S. 76 40 W.	S. 33 E.	3 12 W.	S. 73 28 W.	S. 75 13 W.	1 59	E.	
	11 53	S. 75 38 W.	2·553	S. 63 0 W.	S. 49 E.	3 7 W.	S. 59 53 W.	S. 75 38 W.	15 59	E.	
	11 54	S. 76 3 W.	2·558	S. 45 0 W.	S. 69 E.	2 4 W.	S. 42 56 W.	S. 76 3 W.	33 21	E.	
	11 55	S. 76 40 W.	2·569	S. 38 0 W.	S. 77 E.	1 32 W.	S. 36 28 W.	S. 76 40 W.	40 6	E.	
	11 56	S. 77 17 W.	2·579	S. 47 50 W.	S. 71 E.	1 56 W.	S. 45 54 W.	S. 77 17 W.	31 37	E.	
	11 59	S. 79 0 W.	2·637	S. 73 20 W.	S. 57 E.	2 32 W.	S. 70 48 W.	S. 79 0 W.	8 26	E.	
	P.M.										
	1 35	S. 78 49 W.	3·771	S. 73 20 W.	N. 13 E.	0 56 E.	S. 74 16 W.	S. 78 49 W.	4 47	E.	
	1 40	S. 76 13 W.	3·771	S. 74 40 W.	N. 24 W.	3 52 W.	S. 70 48 W.	S. 76 13 W.	5 39	E.	
	1 42	S. 73 44 W.	3·764	S. 66 0 W.	N. 10 E.	0 40 E.	S. 66 40 W.	S. 73 44 W.	7 18	E.	
	1 48	S. 69 36 W.	3·909	S. 60 0 W.	N. 8 E.	0 12 E.	S. 60 12 W.	S. 69 36 W.	9 38	E.	
	1 51	S. 68 39 W.	4·009	S. 59 40 W.	N. 6 E.	0 5 W.	S. 59 35 W.	S. 68 39 W.	9 18	E.	

TABLE IV.—Observations of Declination. Magnetic Shoal, Port Walcott (continued).

Date.	Time.	Position of ship, Bezout Island △. Bearing.	Position of ship, Bezout Island △. Distance.	Bearing of Bezout △ by Standard.	Ship's head by Standard.	Deviation for apparent position of ship's head.	Apparent magnetic bearing of Bezout △.	True bearing of Bezout Island △.	Deflection of Standard compass.	North-seeking end of needle repelled to	Remarks.
	P.M.	° ′	miles.	° ′	°	° ′	° ′	° ′	° ′		
April 23, 1891.	1 55	S. 64 53 W.	4·062	S. 57 40 W.	N. 2 W.	1 26 W.	S. 56 14 W.	S. 64 53 W.	8 53	E.	
	1 57	S. 62 43 W.	4·149	S. 58 50 W.	N. 1 W.	1 10 W.	S. 57 40 W.	S. 62 43 W.	5 17	E.	
	2 0	S. 60 9 W.	4·261	S. 57 30 W.	N. 1 E.	1 10 W.	S. 56 20 W.	S. 60 9 W.	4 3	E.	
	2 4	S. 56 20 W.	4·363	S. 55 0 W.	N. 4 E.	0 42 W.	S. 54 18 W.	S. 56 20 W.	2 16	E.	
	2 8	S. 52 41 W.	4·523	S. 53 20 W.	N. 4 W.	1 43 W.	S. 51 37 W.	S. 52 41 W.	1 18	E.	
	2 16	S. 56 13 W.	4·966	S. 56 0 W.	S. 6 E.	0 20 W.	S. 55 40 W.	S. 56 13 W.	0 47	E.	
	2 20	S. 57 49 W.	4·908	S. 60 0 W.	S. 37 E.	2 47 W.	S. 56 13 W.	S. 57 49 W.	1 50	E.	
	2 23	S. 59 3 W.	4·872	S. 56 40 W.	S. 8 W.	2 10 E.	S. 58 50 W.	S. 59 3 W.	0 27	E.	
	2 26	S. 59 55 W.	4·756	S. 58 10 W.	S. 4 W.	1 18 E.	S. 59 28 W.	S. 59 55 W.	0 41	E.	
	2 30	S. 61 27 W.	4·647	S. 60 50 W.	S. 5 E.	0 20 W.	S. 60 30 W.	S. 61 27 W.	1 11	E.	
	2 33	S. 63 44 W.	4·548	S. 64 30 W.	S. 16 E.	2 0 W.	S. 62 30 W.	S. 63 44 W.	1 30	E.	
	2 36	S. 65 7 W.	4·524	S. 66 40 W.	S. 30 E.	3 6 W.	S. 63 34 W.	S. 65 7 W.	1 47	E.	
	2 38	S. 65 48 W.	4·562	S. 67 30 W.	S. 35 E.	3 12 W.	S. 64 18 W.	S. 65 48 W.	1 44	E.	
	2 41	S. 67 5 W.	4·534	S. 66 0 W.	S. 3 E.	0 16 E.	S. 66 16 W.	S. 67 5 W.	1 3	E.	
	2 44	S. 68 52 W.	4·472	S. 68 20 W.	S. 8 E.	0 20 W.	S. 68 0 W.	S. 68 52 W.	1 6	E.	
	2 48	S. 71 16 W.	4·420	S. 72 0 W.	S. 15 E.	2 0 W.	S. 70 0 W.	S. 71 16 W.	1 30	E.	
	2 52	S. 74 17 W.	4·348	S. 72 50 W.	S. 6 W.	1 43 E.	S. 74 33 W.	S. 74 17 W.	0 2	W.	
	2 55	S. 76 20 W.	4·254	S. 74 50 W.	S. 6 W.	1 43 E.	S. 76 33 W.	S. 76 20 W.	0 1	E.	
	3 1	S. 78 57 W.	4·131	S. 77 50 W.	S. 2 W.	1 18 E.	S. 79 8 W.	S. 78 57 W.	0 3	E.	
	3 7	S. 77 3 W.	3·898	S. 78 0 W.	N. 22 W.	3 44 W.	S. 74 16 W.	S. 77 3 W.	3 1	E.	
	3 11	S. 72 54 W.	3·878	S. 70 20 W.	N. 18 W.	3 0 W.	S. 67 20 W.	S. 72 54 W.	5 48	E.	
	3 13	S. 70 22 W.	3·864	S. 65 30 W.	N. 36 W.	4 0 W.	S. 61 30 W.	S. 70 22 W.	9 6	E.	
	3 16	S. 67 39 W.	3·802	S. 58 50 W.	N. 29 W.	4 0 W.	S. 54 50 W.	S. 67 39 W.	13 3	E.	
	3 18	S. 66 38 W.	3·861	N. 54 30 W.	N. 14 W.	2 38 W.	S. 51 52 W.	S. 66 38 W.	15 0	E.	
	3 25	S. 62 46 W.	3·935	S. 65 0 W.	N. 22 W.	3 44 W.	S. 61 16 W.	S. 62 46 W.	1 44	E.	
	3 29	S. 58 54 W.	3·971	S. 60 20 W.	N. 10 W.	2 16 W.	S. 58 4 W.	S. 58 54 W.	1 4	E.	
	3 33	S. 54 49 W.	4·170	S. 56 50 W.	N. 13 W.	2 38 W.	S. 54 12 W.	S. 54 49 W.	0 51	E.	
	3 37	S. 50 38 W.	4·275	S. 53 10 W.	N. 19 W.	3 22 W.	S. 49 48 W.	S. 50 38 W.	1 4	E.	
	3 49	S. 55 51 W.	4·354	S. 52 30 W.	S. 30 W.	4 12 E.	S. 56 42 W.	S. 55 51 W.	0 37	W.	
	3 55	S. 60 31 W.	4·252	S. 55 40 W.	S. 22 W.	3 41 E.	S. 59 21 W.	S. 60 31 W.	1 24	E.	
	3 59	S. 61 59 W.	4·033	S. 57 30 W.	S. 19 W.	3 8 E.	S. 60 38 W.	S. 61 59 W.	1 35	E.	
	4 2	S. 62 4 W.	3·876	S. 58 20 W.	S. 44 W.	4 27 E.	S. 62 47 W.	S. 62 4 W.	0 29	W.	
	4 3	S. 63 9 W.	3·807	S. 55 20 W.	S. 44 W.	4 27 E.	S. 59 47 W.	S. 63 9 W.	3 36	E.	

TABLE IV.—Observations of Declination. Magnetic Shoal, Port Walcott (continued).

Date.	Time.	Position of ship, Bezout Island △. Bearing.	Position of ship, Bezout Island △. Distance.	Bearing of Bezout △ by Standard.	Ship's head by Standard.	Deviation for apparent position of ship's head.	Apparent magnetic bearing of Bezout △.	True bearing of Bezout Island △.	Deflection of Standard compass.	North-seeking end of needle repelled to	Remarks.
	P.M.	° ′	miles.	° ′	°	° ′	° ′	° ′	° ′		
April 23, 1891.	4 5	S. 63 13 W.	3·711	S. 53 20 W.	S. 45 W.	4 27 E.	S. 57 47 W.	S. 63 13 W.	5 39	E.	
	4 6	S. 63 28 W.	3·630	S. 63 20 W.	S. 35 W.	4 22 E.	S. 67 42 W.	S. 63 28 W.	4 0	W.	
	4 8	S. 63 46 W.	3·546	S. 69 40 W.	S. 54 W.	4 21 E.	S. 74 21 W.	S. 63 46 W.	10 21	W.	
	4 10	S. 64 4 W.	3·479	S. 69 20 W.	S. 59 W.	4 0 E.	S. 73 20 W.	S. 64 4 W.	9 2	W.	
	4 13	S. 64 21 W.	3·350	S. 75 20 W.	S. 45 W.	4 27 E.	S. 79 47 W.	S. 64 21 W.	15 12	W.	
	4 16	S. 64 35 W.	3·218	S. 80 20 W.	N. 86 W.	1 19 W.	S. 79 1 W.	S. 64 35 W.	14 12	W.	
	4 19	S. 64 18 W.	3·073	S. 67 40 W.	S. 46 W.	4 27 E.	S. 72 7 W.	S. 64 18 W.	7 35	W.	
	4 22	S. 64 46 W.	2·942	S. 65 50 W.	S. 55 W.	4 19 E.	S. 70 9 W.	S. 64 46 W.	5 9	W.	
	4 25	S. 65 47 W.	2·680	S. 65 50 W.	S. 38 W.	4 25 E.	S. 70 15 W.	S. 65 47 W.	4 14	W.	
	A.M.										
April 24, 1891.	7 21	S. 62 2 W.	1·039	S. 65 50 W.	N. 9 W.	2 0 W.	S. 63 50 W.	S. 62 2 W.	1 34	W.	
	7 26	S. 63 46 W.	1·455	S. 69 10 W.	S. 65 E.	2 11 W.	S. 66 59 W.	S. 63 46 W.	2 59	W.	
	7 30	S. 63 13 W.	1·540	S. 68 30 W.	N. 23 W.	3 44 W.	S. 64 46 W.	S. 63 13 W.	1 19	W.	
	7 34	S. 60 44 W.	1·761	S. 65 0 W.	S. 70 E.	2 11 W.	S. 62 49 W.	S. 60 44 W.	1 51	W.	
	7 38	S. 60 36 W.	1·880	S. 62 20 W.	N. 2 E.	0 52 W.	S. 61 28 W.	S. 60 36 W.	0 38	W.	
	7 40	S. 57 5 W.	2·055	S. 58 0 W.	N. 67 E.	2 48 E.	S. 60 48 W.	S. 57 5 W.	3 29	W.	
	7 44	S. 57 22 W.	2·250	S. 57 30 W.	N. 7 E.	0 15 W.	S. 57 15 W.	S. 57 22 W.	0 21	E.	
	7 50	S. 57 36 W.	2·482	S. 60 50 W.	S. 73 E.	1 48 W.	S. 59 2 W.	S. 57 36 W.	1 12	W.	
	7 52	S. 57 51 W.	2·591	S. 59 30 W.	N. 2 W.	1 26 W.	S. 58 4 W.	S. 57 51 W.	0 1	E.	
	7 57	S. 57 15 W.	2·763	S. 61 30 W.	S. 75 E.	1 40 W.	S. 59 50 W.	S. 57 15 W.	2 21	W.	
	8 1	S. 57 7 W.	2·903	S. 63 0 W.	N. 23 W.	3 44 W.	S. 59 16 W.	S. 57 7 W.	1 55	W.	
	8 6	S. 56 39 W.	3·153	S. 62 30 W.	S. 62 E.	2 18 W.	S. 60 12 W.	S. 56 39 W.	2 19	W.	
	8 9	S. 57 14 W.	3·276	S. 60 30 W.	N. 10 W.	2 16 W.	S. 58 14 W.	S. 57 14 W.	0 46	W.	
	8 12	S. 55 52 W.	3·430	S. 63 0 W.	S. 71 E.	1 56 W.	S. 61 4 W.	S. 55 52 W.	4 58	W.	
	8 16	S. 55 34 W.	3·516	S. 59 10 W.	N. 14 W.	2 38 W.	S. 56 42 W.	S. 55 34 W.	0 54	W.	
	8 20	S. 55 17 W.	3·738	S. 59 50 W.	S. 51 E.	2 55 W.	S. 56 55 W.	S. 55 17 W.	1 24	W.	
	9 26	S. 84 14 W.	2·200	S. 75 34 W.	N. 42 E.	3 13 E.	S. 78 47 W.	S. 84 14 W.	5 40	E.	
	10 30	S. 82 15 W.	2·128	S. 56 57 W.	N. 58 E.	2 48 E.	S. 59 45 W.	S. 82 15 W.	22 44	E.	

TABLE IV.—Observations of Declination. Magnetic Shoal, Port Walcott (continued).

Date.	Time.	Position of ship, Bezout Island △. Bearing.	Position of ship, Bezout Island △. Distance.	Bearing of Bezout by Standard.	Ship's head by Standard.	Deviation for apparent position of ship's head.	Apparent magnetic bearing of Bezout △.	True bearing of Bezout Island △.	Deflection of Standard compass.	North-seeking end of needle repelled to	Remarks.
April 24, 1891.	P.M.		miles.								
	1 9	S. 79° 45′ W.	2·188	S. 23° 28′ W.	N. 51° E.	3° 6′ E.	S. 26° 34′ W.	S. 79° 45′ W.	53° 25′	E.	
	2 2	S. 79 36 W.	Heaving in the port cable	S. 22 20 W.	N. 56 E.	2 48 E.	S. 25 8 W.	S. 79 36 W.	54 42	E.	
	2 5	S. 79 28 W.		S. 24 8 W.	N. 69 E.	1 36 E.	S. 25 44 W.	S. 79 28 W.	53 58	E.	
	2 8	S. 79 13 W.		S. 35 35 W.	N. 79 E.	0 30 E.	S. 36 5 W.	S. 79 13 W.	43 22	E.	
	2 10	S. 79 11 W.		S. 32 42 W.	N. 74 E.	1 14 E.	S. 33 56 W.	S. 79 11 W.	45 29	E.	
	2 13	S. 79 10 W.	Veering the starboard	S. 34 20 W.	N. 79 E.	0 30 E.	S. 34 50 W.	S. 79 10 W.	44 34	E.	
	2 14	S. 79 9 W.		S. 54 0 W.	S. 81 E.	1 21 W.	S. 52 39 W.	S. 79 9 W.	26 44	E.	
	2 16	S. 79 7 W.	2·169	S. 53 22 W.	S. 88 E.	1 0 W.	S. 52 22 W.	S. 79 7 W.	26 59	E.	
	2 17	S. 79 5 W.	Heaving in port	S. 62 10 W.	S. 73 E.	1 48 W.	S. 60 22 W.	S. 79 5 W.	18 59	E.	
	2 19	S. 79 4 W.		S. 74 28 W.	S. 65 E.	2 4 W.	S. 72 24 W.	S. 79 4 W.	6 54	E.	
	2 21	S. 78 49 W.	2·170	S. 80 5 W.	S. 67 E.	2 4 W.	S. 78 1 W.	S. 78 49 W.	1 12	E.	
	2 49	S. 78 47 W.	Heaving in port cable	S. 89 10 W.	S. 40 E.	2 55 W.	S. 86 15 W.	S. 78 47 W.	7 14	W.	
	2 53	S. 78 47 W.		N. 80 30 W.	S. 29 E.	3 0 W.	N. 83 30 W.	S. 78 47 W.	17 29	W.	
	2 55	S. 78 43 W.		N. 77 40 W.	S. 27 E.	3 0 W.	N. 80 40 W.	S. 78 43 W.	20 23	W.	
	2 58	S. 78 39 W.		N. 74 50 W.	S. 24 E.	2 54 W.	N. 77 44 W.	S. 78 39 W.	23 23	W.	
	2 59	S. 78 37 W.	Veering the starboard cable	N. 73 30 W.	S. 22 E.	2 49 W.	N. 76 19 W.	S. 78 37 W.	25 10	W.	
	3 4	S. 78 35 W.		N. 72 50 W.	S. 21 E.	2 49 W.	N. 75 39 W.	S. 78 35 W.	25 32	W.	
	3 6	S. 78 31 W.		N. 72 30 W.	S. 19 E.	2 33 W.	N. 75 3 W.	S. 78 31 W.	26 12	W.	
	3 11	S. 78 28 W.	2·131	N. 73 0 W.	S. 20 E.	2 33 W.	N. 75 33 W.	S. 78 28 W.	25 45	W.	
	4 12	S. 79 30 W.	2·188	S. 21 50 W.	N. 67 E.	1 58 E.	S. 23 48 W.	S. 79 30 W.	55 56	E.	
	4 42	S. 85 12 W.	2·579	S. 85 40 W.	S. 40 E.	2 55 W.	S. 82 45 W.	S. 85 12 W.	2 41	E.	
	A.M.										
	9 14	S. 79 27 W.	2·148	S. 63 40 W.	N. 22 E.	1 44 E.	S. 65 24 W.	S. 79 27 W.	14 17	E.	
	10 6	S. 78 38 W.	2·240	S. 35 0 W.	N. 4 E.	0 42 W.	S. 34 18 W.	S. 78 38 W.	44 34	E.	
	11 1	S. 77 27 W.	2·126	N. 81 28 W.	S. 83 E.	1 9 W.	N. 82 37 W.	S. 77 27 W.	19 42	W.	
	11 44	S. 78 27 W.	2·132	N. 73 0 W.	S. 48 E.	3 7 W.	N. 76 7 W.	S. 78 27 W.	25 12	W.	
	11 53	S. 78 27 W.		N. 75 0 W.	S. 76 E.	1 40 W.	N. 76 40 W.	S. 78 27 W.	24 39	W.	
	P.M.										
	2 51	S. 71 0 W.	2·851	S. 87 20 W.	S. 42 E.	3 16 W.	S. 84 4 W.	S. 71 0 W.	12 50	W.	
	3 24	S. 68 4 W.	3·501	S. 54 50 W.	S. 76 E.	1 40 W.	S. 53 10 W.	S. 68 4 W.	15 8	E.	
	4 41	S. 66 6 W.	4·330	S. 66 10 W.	S. 58 E.	2 32 W.	S. 63 38 W.	S. 66 6 W.	2 42	E.	

Appendix A.

Geological Observations at Red Cliffs, near Cossack, N. W. Australia.

On April 25, 1891, I landed at the "Red Cliffs," between Cape Lambert and Reader Head, near Cossack, N.W. Australia, and made the following notes on their geological structure :—

(1.) The cliff runs in a general N.W. and S.E. direction for about a quarter of a mile, and is about 30 feet in average height ; the face is very rugged, but it being almost entirely free from vegetation, some very excellent sections are presented.

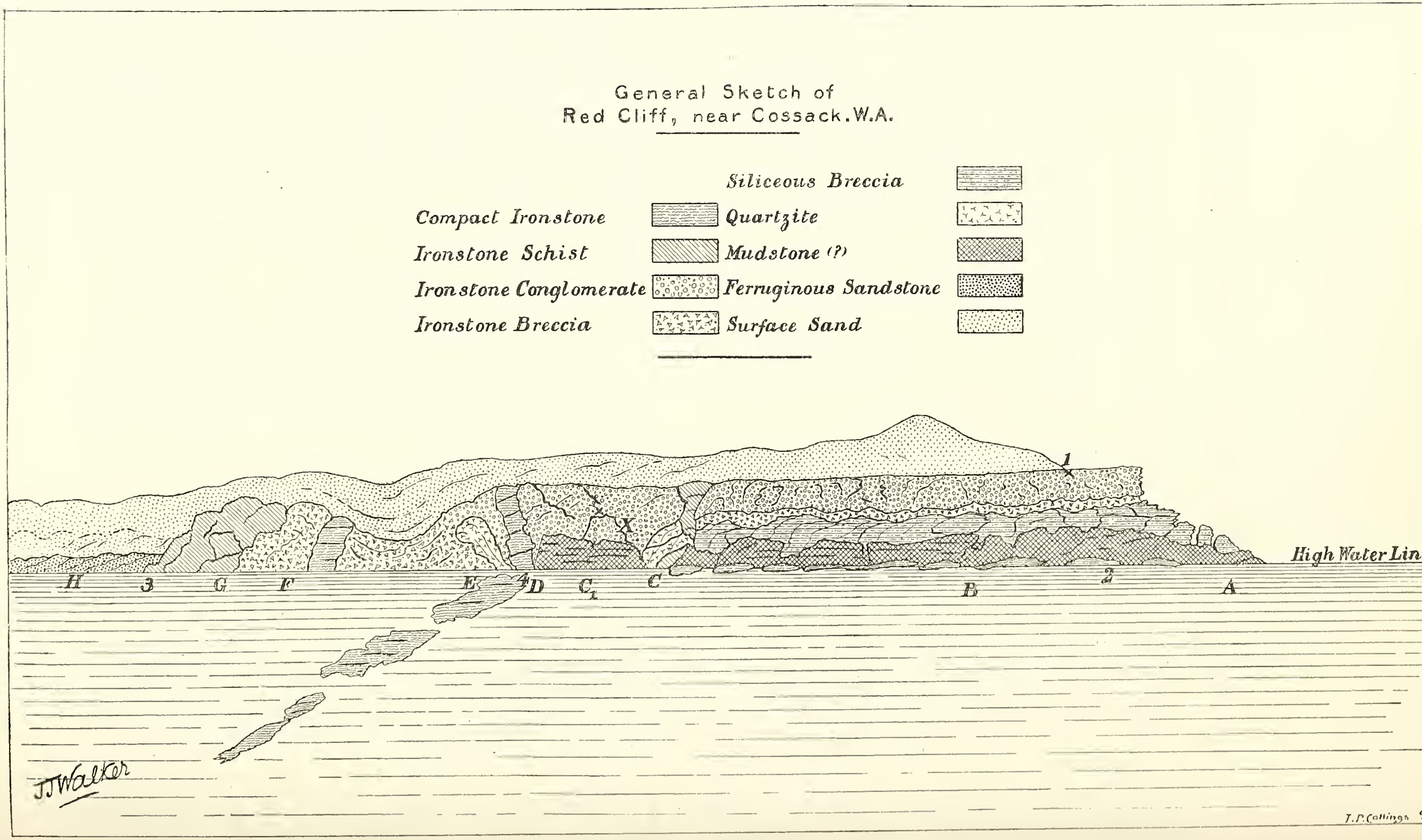
(2.) At the N.W. end (A in the general sketch), the beach is strewn with huge blocks of breccia and ironstone conglomerate, fallen from the cliffs above. Commencing from high-water-mark (it being unfortunately nearly high tide when I landed), there is first a layer of yellowish sandstone (?), about 10 feet thick, the bedding of which is nearly horizontal. On this rests another bed of the same thickness of a siliceous breccia, crowded with angular fragments of various forms of quartzite. At the top of this layer, the paste in which the fragments are embedded becomes highly ferruginous, and passes into a cap of "ironstone conglomerate," consisting of rounded nodules of (apparently) brown hæmatite embedded in a reddish-brown paste, which effervesces strongly when treated with hydrochloric acid.

(3.) Proceeding along the cliff to (B), large masses of compact ironstone (? brown hæmatite) *in situ* are seen, cropping out of the beach at high-water-mark.

(4.) At (C) is a small gap in the cliff, just beyond which is a well-marked dyke-like mass of quartz, about 4 feet wide, in layers alternately light and dark coloured, and bedded almost vertically. The "strike" of these layers is approximately N.E. and S.W. On the north side of the dyke-like mass is a narrow platform of ferruginous sandstone, having the same dip and strike. At (C) is a small mass or vein (marked X in general sketch) of a curious whitish soft mineral, which I am not able to name.

(5.) At (D) the cliffs are intersected by a very remarkable dyke-like mass, or vein, of compact ironstone, about 20 feet in width, nearly vertical, strike about N.N.E. and S.S.W. It appears to be continued out to sea in the same direction in a series of reefs, awash at low water. Just beyond it, at (E), is another dyke-like mass of quartzite, similar to that at (C), but the layers are better defined, and in places much more contorted ; the width of this dyke-like mass is about 10 feet. Here the cap of ironstone conglomerate ceases.

(6.) A little further on, in a small hollow or glen in the cliff, another massive dyke-like mass of ironstone, similar to that at (D), crops out. Then comes, at (F), a mass of "ironstone breccia" (fragments of quartzite embedded in a highly ferruginous paste), which passes at (G) into what I venture to call "ironstone schist," consisting of alternate layers of whitish quartzose stone and ironstone, about an inch in average


General Sketch of
Red Cliff, near Cossack. W.A.
Siliceous Breccia
Compact Ironstone
Quartzite
Ironstone Schist
Mudstone (?)
Ironstone Conglomerate
Ferruginous Sandstone
Ironstone Breccia
Surface Sand
1
X
High Water Line
H
3
G
F
E
4
D
C_1
C
B
2
A
J.J.Walker
J.P.Collings Sc

thickness, dipping at an angle of 85°, and striking approximately N.E. and S.W. In many places, however, the beds are much contorted.

(7.) The cliff ceases at (H) and is succeeded by a sandy beach resting on a platform of coarse ferruginous sandstone.

(8.) The summit of the cliff from (A) to (D) is nearly level and strewn with fragments and nodules of ironstone. Further inland the country is very barren and sandy, with scanty grass and a few low shrubs. A quartz dyke-like mass, probably a continuation of that at (C), was noticed about a quarter of a mile inland from the edge of the cliff.

(9.) Magnetical observations were taken at positions (1), (2), (3), and (4) on the general sketch. A series of the most characteristic rocks and minerals were collected by me and are forwarded with these notes.

(Signed) J. J. WALKER,
Chief Engineer,
H.M.S. "Penguin."

APPENDIX B.

The whole of the geological specimens forwarded from H.M.S. "Penguin" have been examined by Professor RÜCKER, F.R.S. Of these, the undermentioned list specially refers to positions in the vicinity of Bezout and Baudin Islands of special magnetic interest.

1*a*, 2*b*, 3*c*, 4*d*. Bezout Island, near Cossack.

5. Reader Hill, Cossack.

6. Sand, Cossack—Reader Hill bearing (true) N. 81° E., 760 yards.

18. Broome, Roebuck Bay.

19. Sand, 3 feet below surface. Magnetic observation spot, Broome.

21. Sand, from surface of place of magnetic observation, Broome.

22. Sand, east side of Baudin Island, off Cape Voltaire.

24. Sand, from magnetic shoal off Port Walcott, in 8 fathoms ; lat. 20° 31′ 35″ S. long. 117° 13′ 2″ E.

25. Baudin Island, off Cape Voltaire.

26. Baudin Island, off Cape Voltaire (basement).

Of the above list of specimens, Nos. 25 and 26 only are magnetic, their respective values of k in C.G.S. units being 0·000217 and 0·000529.

INDEX SLIP.

X. Bakerian Lecture.—*On the Diffusion of Metals.*

By W. C. Roberts-Austen, *C.B., F.R.S., Professor of Metallurgy, Royal College of Science; Chemist of the Mint.*

Received and Read February 20, 1896.

[Plates 7, 8.]

PART I.—DIFFUSION OF MOLTEN METALS.

Gold, Platinum, and Rhodium, diffusing in Molten Lead and in Molten Bismuth.

The diffusion of molten and solid metals has long demanded investigation, their molecular mobility being of great interest in relation to the constitution of matter, and its results of much industrial importance.

The analogy of alloys to ordinary saline solutions has often been pointed out, and many experiments have recently been devoted to comparing the action of osmotic pressure in saline solutions and in alloys, as measured by the lowering of the freezing point which is caused by the addition to the solvent of a small quantity of another body.* The general effect is the same whether the solvent is a liquid like water or a molten metal. Very little attention has, however, been given to the consideration of the molecular movements which enable two or more molten metals to mix spontaneously and form a truly homogeneous fluid mass, although it is by such an investigation that the analogy of an alloy to a saline solution may reasonably be expected to be more clearly revealed than by any other method of research. A single example of the spontaneous mixing of two metals may be useful. In preparing the alloy of gold and copper used for coinage, some 33 kilos. of gold and 3 kilos. of copper are melted together in a single crucible, and the results of assays on the first and the last portions of metal poured from the crucible, seldom differ by more than one ten-thousandth part. Such a fluid mass of standard gold owes this remarkable uniformity in composition not only to the mechanical stirring by which the blending of

* Heycock and Neville, 'Trans. Chem. Soc.,' vol. 55, 1889, p. 666; vol. 57, 1890, pp. 376 and 656; vol. 59, 1891, p. 936; vol. 61, 1892, pp. 914 and 888; vol. 65, 1894, pp. 31 and 65; vol. 67, 1895, p. 1024; and Roberts-Austen, 'Proc. Roy. Soc.,' vol. 49, 1891, p. 347.

the gold and copper was roughly effected. The molecular mobility of the metals has influenced the result, and the metals dissolved in each other, become, by a spontaneous process, spread or diffused uniformly; in this case the uniformity is not materially disturbed when the solidification of the mass is effected.

In view of the great interest connected with such action, the absence of direct experiments is remarkable, but this may perhaps be explained by the difficulty of conducting them. The sources of these difficulties are many. Such metals as are suitable for study require a more or less elevated temperature to melt them, and, where diffusion is concerned, small variations in temperature may be of much importance, for, as GRAHAM showed, the rate of diffusion of salts in water is greatly increased by a small rise in temperature, the diffusibility of chloride of sodium, for instance, being more than doubled by a rise of 33°. It is now well-known that the osmotic pressure of a salt in solution is measured by the diffusion which takes place. A rise of temperature, therefore, which augments the osmotic pressure, must also increase the rate of diffusion. GRAHAM further pointed out that the inequality of diffusion which various saline substances exhibit at a low temperature, becomes less at a high temperature, and he therefore concluded that "it would appear to be the effect of a high temperature to assimilate diffusibilities" of different salts.*

In the case of molten metals, the necessity for working at high temperatures, which until quite recently could not be readily measured even with approximate accuracy, and the fear that the value of the results would be impaired by the action of convection currents, must have deterred physicists from undertaking experiments on the diffusion of molten metals. OSTWALD'S statement† with reference to the diffusion of salts, that "to make accurate experiments on diffusion is one of the most difficult problems in practical physics," may well have given rise to doubts whether any method which seemed to be available for conducting such investigations with molten metals would afford trustworthy results. The difficulties are obvious, but my long connection with GRAHAM'S researches made it almost a duty to attempt to extend his work on liquid diffusion to metals, and, therefore, fourteen years ago the present investigation was undertaken, but it was abandoned because I was unable to measure with sufficient accuracy the temperature at which diffusion took place, and it has only been resumed during the past two years.

As regards the history of the subject, I believe that a brief communication of my own on the mobility of gold and silver in molten lead, to the Chemical Section of the British Association at the meeting at Southport in September, 1883, embodied the results of the first experiments which were ever made with the direct object of investigating the diffusion of molten metals and alloys, other than those of mercury which are fluid at the ordinary temperature. I stated that "while molten copper and antimony interpenetrate but slowly, the mobility of gold and silver in molten lead is

* GRAHAM, 'Collected Papers,' p. 570, or 'Phil. Trans. Roy. Soc.,' 1861, p. 183–224.

† OSTWALD, 'Solutions,' English Edition, 1891, p. 122.

comparatively rapid."* As regards mercury and its fluid amalgams, the history is more extended, for in 1713 HOMBERG † may be said to have at least foreshadowed the diffusion of metals, both solid and liquid, in his paper "On Substances which Penetrate and which pass through Metals without their being Melted." He incidentally showed, by experiment, the extreme rapidity with which mercury will penetrate a bar of tin.

In November, 1883, GUTHRIE ‡ published a remarkable paper "On Certain Molecular Constants," in which the diffusion of zinc, lead, tin, sodium, and potassium in mercury was studied, and he stated that these metals, which, of course, are much lighter than mercury, "appear after a month's interval in appreciable quantity at a depth of a foot beneath the surface, when the temperature is about 16° or 17°." He concludes his paper by offering "a general curve of amalgamation," which he thought would represent the rate at which the metals examined by him alloy with mercury, and this curve may also, he says, "represent the relative rates of elementary atomic and molecular diffusion generally."

GUTHRIE held that as the mercury he employed was a good conductor of heat, there was not much fear of the disturbing influence of convection currents. The existence of such currents, nevertheless, gave me much anxiety in the earlier experiments with molten metals which were begun in the year 1881, and will now be described.

From the outset of this research both molten lead and bismuth were chosen as suitable fluids in which the diffusion of other metals could be studied. Advantage was also taken of the fact that at temperatures well above the melting point, neither of these metals unite with iron. The precious metals, also, when alloyed with lead or bismuth, do not show any tendency to unite with iron unless it is very clean and bright. Tubes of wrought iron, therefore, proved to be most useful in conducting the inquiry. In the first instance single tubes filled with lead were arranged vertically in a bath of lead which was kept well above its melting point. Weighed quantities of heated, but still solid, gold or platinum were then rapidly lowered through the lead in little covered receptacles of iron to the bottom of the tube, and when by the aid of a rigid steel wire the removal of the covers was gently effected, the gold was exposed to the lead; it became rapidly dissolved, and diffusion began. The tubes filled with molten lead, in which diffusion took place, were about 200 millims. long, and many such tubes were arranged in a single bath, which was carefully kept hotter at the top than at the bottom so as to avoid as much as possible the carrying of the precious metal upwards by any streams of lead which might rise as convection currents from the bottom of the tube. The main result of

* ROBERTS-AUSTEN, 'British Association Report,' 1883, p. 402.

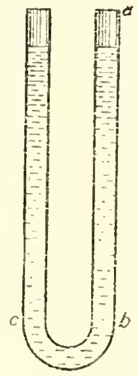
† HOMBERG, 'Mém. de l'Acad. Royale des Sciences,' 1713 (vol. published 1739), p. 306.

‡ GUTHRIE, 'Phil. Mag.,' vol. 16, 1883, p. 321. W. S. HUMPHREYS has recently made experiments on the diffusion of zinc, lead, tin, bismuth, silver and copper in mercury. 'Trans. Chem. Soc.,' vol. 69, 1896, p. 243.

these earlier experiments was to show that samples of lead (which were removed by sucking them from the upper part of the tubes into stems of tobacco pipes) always revealed the presence of weighable quantities of gold after a lapse of the first three hours, while a sample withdrawn at the end of a second period of three hours did not indicate the presence of a commensurate amount of the precious metal. This fact either pointed to defects in the method, to the transmission of gold by convection currents, or to the very rapid diffusion of gold when minute quantities of this metal are present in lead. I believe that the latter will ultimately prove to afford the true explanation of the facts observed.

The vertical tubes were then replaced by U-tubes of wrought iron, each limb of which was 230 millims. long and 10 millims. internal diameter. The tubes were filled with lead, and heated externally in a bath of molten lead, in an appliance which will be described immediately, and the precious metal, in the form of a rich alloy with lead, was inserted into one limb of the tube. Experiments proved that the gold falling by gravity became rapidly and uniformly distributed through the column of lead in that side of the tube, *a.b.*, Fig. 1, into which it was introduced, and

Fig. 1.



in the rounded part of the tube, *b*, *c*, at the base of the U. After a given number of days the tubes were cautiously withdrawn from the bath and cooled from the bottom so as to solidify the alloy. The tube was then carefully divided by transverse saw cuts into measured sections, which were numbered consecutively, and the alloy could then be readily melted from each section and weighed, after which the amount of precious metal it contained was determined by analysis.

It was found, however, that the use of U-tubes greatly increases the difficulties of calculation, as there is great uncertainty in any assumption as to the distribution of the diffusing metal during the experiment at points near the bend. It was found that in all these experiments, the gold or platinum was very evenly distributed by gravity through the limb into which the metal was first introduced.

The calculations were finally made on the assumption that the concentration at the bend was, throughout the experiment, equal to the mean of its initial and final concentration. I, nevertheless, determined to sacrifice a long series of results, as they are less trustworthy than those obtained later by the use of single tubes.

The continuance of these experiments is mainly due to the interest which Lord KELVIN has always taken in them, and in a letter dated 4th of November, 1883, he reminded me of the necessity for maintaining a graduated temperature from the top to the bottom of the diffusion tubes, and he has from time to time given me, on points of detail, suggestions, which I need hardly say have been adopted.

Description of the Apparatus employed in the later Experiments.

The tubes that contain the lead in which the diffusion takes place are arranged in a little furnace of special construction. In the earlier experiments they were, as has been already stated, placed in a bath of molten lead, but this was abandoned in favour of an air bath with double walls which can be heated above the melting point of lead and readily maintained at definite temperatures. The drawing, fig. 2, Plate 7, shows the general arrangement. The diffusion tubes are closed at the base, and two of such tubes are shown at TT′, fig. 2, Plate 7, placed in a cylinder of iron, I, 3 inches in diameter and 7 inches high. The sectional plan (fig. 2, Plate 7) shows six of the diffusion tubes arranged symmetrically in this iron cylinder, which is enclosed in a second cylinder of thick copper, C, $4\frac{1}{4}$ inches* in diameter and $8\frac{1}{4}$ inches high. The lower half of this copper vessel is surrounded with a layer of asbestos cloth, A. There is a lid composed of two discs of copper, DD′, with asbestos between them. If, for any special reason, U-tubes should be employed, one open end of each tube might communicate with a hole in the double lid, and the metals, the diffusion of which is to be studied, are introduced through this hole. They fall by gravity to the base of the U-tube, and then rise by diffusion up its opposite limb.

The heating is effected by a series of clay gas-burners BB′, mounted on a ring RR′; the burners surrounding the upper portion of the copper cylinder. Investing cases of fireclay GG′G″G‴, and a lid of clay H, completes the construction of the furnace.

It is, of course, a matter of great importance to obtain a gas supply of constant volume, and this has been effected by means of a regulator, shown in fig. 3, Plate 7, and a delicate gauge not shown in the plate is also provided, and by its aid any variation in the pressure of the gas is indicated.

The absence of a trustworthy method for the measurement of the temperature would have rendered it impossible to conduct these experiments, but by the aid of thermo-junctions such measurement can readily be effected. These thermo-junctions JJ′J″, are shown in fig. 2, Plate 7, and the method of using them will now be indicated. I have, however, already fully described the method elsewhere,† and the general

* It is convenient to give these dimensions in inches, but a metric scale is also provided with the drawing.

† 'Roy. Soc. Proc.,' vol. 49, 1891, p. 347. 'Inst. Civil Engineers Proc.,' vol. 110, 1891–2, Part iv.

distribution of the apparatus which may be used for such measurement of temperature up to the melting-point of platinum, and a brief enumeration of the several parts of the appliance is, therefore, all that need be given here. As shown in Plate 7, fig. 1, it consists of a camera about five feet long, in which the galvanometer is placed. This camera has three doors, and is made separate from the portion of the apparatus which contains the moving photographic plate. The two parts are connected at J, by flexible leather, the object being to enable the plane of the sensitized plate to be adjusted at right angles to the ray of light from the stationary galvanometer mirror F. Inside the camera is a focussing tube T, containing a lens L, which receives external light from the mirror H, and transmits it to the galvanometer mirrors F and M. Of these mirrors, M is movable, and is carried by the coil of the galvanometer, while F is stationary on an adjustable arm fixed to the supports of that instrument, its function being to send a ray of light from the mirror H to the slit B, and thus to trace a datum line as the photographic plate travels upwards. The temperature is recorded by the variations in the position of the spot of light derived from the movable mirror M. There is a screen S to cut off light reflected from the brasswork of the galvanometer. The end of the tube T is provided with an adjustable brass slit, by means of which the width of the photographic lines traced on the plate can be varied. The mirror H is mounted on a block, which can be adjusted so that external light may be brought from either side. The focussing of the lens L may be effected from outside the camera. Plug connections are provided at the back of the instrument, and the wires *a*, *b*, connect the galvanometer with the thermo-junctions at the source of heat; cold junctions being interposed, as shown in fig. 4, Plate 7. The photographic plate is secured to its carrying slide C by means of little cams, and the carrier C is enclosed in a case K provided with a light-tight door, N. The case K is held in position by a pin, P. The connection of the photographic plate with the driving clock is shown at D. The sensitized plate moves in front of the slit B, and is lifted by a weight actuating a fine clock, constructed by the well-known horologist, Mr. DAVID GLASGOW. It may be mentioned that the galvanometer stands on three plates, which ensure steadiness by providing the well-known combination of the "hole," "slot," and "plane."

It will be seen on reference to Plate 7, fig. 2, that in the centre of the inner cylinder of iron which contains the diffusion tubes there is a central tube of metal L. Into this tube several thermo-junctions, JJ′J″, usually three in number, are inserted to various depths. Each thermo-junction is, as fig. 2 shows, suitably protected and insulated by clay tubes. These clay tubes nearly block up the inner space and prevent the free circulation of air, and the four cold junctions with which they communicate are shown in fig. 4, Plate 7. These junctions are, by the aid of a switch S, fig. 4, actuated by clockwork, placed in turn in communication with the galvanometer, and by the deflection of its mirror M (fig. 1) the temperature of any given junction at any moment can either be observed on a transparent scale, or recorded on a photographic plate. By this means a continuous record of the temperature at three positions in the air bath may be readily obtained, but as regards each of the three positions this record is intermittent, and the indications afforded by each individual thermo-junction form a dotted not a continuous line. The intermittence is, however, very rapid, and the result is three clear and distinct time-temperature records which enable any variation in temperature to be readily detected and measured. The records, which are very numerous, show that while there was no rapid variation of temperature, there was, however, a gradual fall in temperature from the beginning to the end of the experiment, and this is due to the fact that the burners became slightly obstructed by the products of the combustion of the gas, and in experiments which are now in progress a thermostat is employed. The occasional slight adjustment of the gas-taps by hand in accordance with the indications of the thermo-junctions rendered it possible to maintain a very constant temperature. Much care was taken to prove that the method of inserting the thermo-junctions down a central tube really indicated the mean differences of temperature between the upper, middle and lower portions of the bath, and it was shown that there was no object in continuing the use of a lead bath which was first adopted, as it greatly complicated the manipulation.

Now that the possibility of making accurate measurements on the diffusion of molten metals has been demonstrated, and as it has been shown that convection currents are not established, even when the temperatures at the top and the bottom of the diffusion tubes only differ by 35° C., it will be well to attempt to do without this graduated temperature altogether, trusting to the difference in densities of the solvent and the diffusing metal. It will also be necessary to set up apparatus which will automatically maintain for lengthy periods any given temperature between the ordinary temperature and 600° C.

At the end of a given period varying from six hours to seven days, the diffusion tubes were removed, cooled from below, carefully measured, and then cut into transverse numbered sections, the metallic contents of each of which were weighed and analysed. When gold is the diffusing metal, determination of its amount in the various sections of lead can be effected with remarkable precision by the ordinary method of cupellation assay, provided that check assays on samples synthetically prepared are made simultaneously, and the rigorous accuracy of such assays is so well known to the chemists who have had actual experience in the matter, that it is unnecessary to submit evidence on this point. The estimation of platinum or of rhodium in lead is a matter of greater difficulty, but details of the analysis need not be given.

If the bath is unprotected below, and heated from the top, the top will be very much hotter than the bottom, and thus the rate of diffusion would be considerably greater near the top than near the bottom of each tube. The value of the results would also be much impaired if the bath were inadequately protected on the top, for if the upper surface of the bath is cooled, convection currents will be established. The lead, moreover, at the surface oxidises, and forms a detachable slag, and so relatively increases the percentage of gold or platinum in the upper layer of lead. It was frequently noticed that the top layer in the tubes was slightly richer than the one next below it.

In the experiments, the results of which are plotted in figs. 3 and 4 (p. 399), to which reference will subsequently be made, the temperature, as indicated by the top thermo-junction, was gradually raised to 550°, at which point it was maintained practically without variation during the greater part of the experiment, which lasted nearly seven days. The temperature, however, slowly fell during the last two days to 500°, owing to the choking up of the burners. The middle thermo-junction showed temperature 25°, and the lowest one 35°, below this; the mean temperature shown by the middle junction being given as the temperature of the experiment. It should be pointed out that these temperatures are based on measurements which assume the melting point of gold to be 1045°, and lead 325°. Recent work seems to show that 1045° may be some 15° too low.*

It may be well to offer here a few general considerations respecting the phenomena to be observed.

It is now held that liquid diffusion is the result of osmotic pressure.† A movement of the particles (molecules or atoms) of the dissolved substance takes place, and a molecular force drives them from the place where they are more closely packed and, therefore, exert greater pressure, and impels them to positions in which they are more widely distributed. This movement continues until the concentration, and, therefore, the pressure of the diffusing metal, is constant throughout the liquid. GRAHAM'S method of studying liquid diffusion consisted in filling wide-mouth phials of glass with the solutions of salts, which were allowed to diffuse outwards into water contained in capacious cylinders. This method could not well be imitated in the present experiments, as the manipulation, and the calculation of the results obtained by such a method present great difficulty. Hence the adoption in the present research of vertical tubes, as has already been described. In the earliest experiments made by me, in 1883, the little spheres of precious metal obtained from each measured section of lead were arranged on a card scale at measured distances;‡ each of the little spheres, therefore, represented the amount of gold in the section of the tube from which it had been derived, and their general appearance, when arranged as has just been described, suggested that a trustworthy method had been secured. It appeared probable that the law of diffusion of salts, framed by FICK, would also apply

* See HEYCOCK and NEVILLE, 'Trans. Chem. Soc.,' vol. 67, 1895, p. 160. ROBERTS-AUSTEN, 'Nature,' May 9, 1895, p. 40. LE CHATELIER, 'Comptes Rendus,' vol. 121, 1895, p. 323.

† NERNST, 'Zeitsch. für Physikal. Chemie,' vol. 2, 1888, p. 613.

‡ Specimens of these records were exhibited to Section B. of the British Association, at the Montreal Meeting, 1884.

to the diffusion of one metal in another. FICK's law states that "the quantity of salt, which diffuses through a given area, is proportional to the difference between the concentrations of two areas infinitely near each other."

FOURIER's theory of thermal conduction was applied by FICK to the phenomena of material diffusion generally. The law of diffusion is thus stated by Lord KELVIN :* The rate of augmentation of the "*quality*," per unit of time, is equal to the diffusivity multiplied into the rate of augmentation per unit of space of the rate of augmentation per unit of space of the "*quality*." In the case of diffusion of salts or metals, the "*quality*" is concentration of the matter diffused, or deviation of concentration from some mean or standard considered.

The movement in linear diffusion may therefore be expressed by the differential equation

$$\frac{dv}{dt} = k\frac{d^2v}{dx^2}.$$

In this equation, x represents the distance in the direction in which the diffusion takes place; v is the degree of concentration of the diffusing metal, and t the time; k is the diffusion constant, that is, the number which expresses the quantity of the metal, in grammes, diffusing through unit area (1 sq. centim.) in unit time (one day) when unit difference of concentration (in grammes, per cubic centim.) is maintained between the two sides of a layer one centim. thick. The unit of diffusivity has the dimensions $[L^2T^{-1}]$; so that diffusion constants may be expressed in square centimetres per day. The constant has a definite value for each pair of metals (that is for the diffusing metal and its solvent) at a particular temperature, and the object of the experiments on diffusion is to determine this value.

In the equation, dv/dt denotes the increase in the degree of concentration which takes place at any point during unit time, dv/dx represents the difference between the degrees of concentration at the two sides of a layer of unit thickness, and d^2v/dx^2 in the equation represents the change which takes place in dv/dx as the position of the point under consideration is moved unit distance along the tube.

In Plate 8, and in the figures in the text, x is represented by abscissæ, v by ordinates, dv/dx is the tangent of the inclination of the curve to the horizontal, and d^2v/dx^2 is the change in this value occurring for unit change in x; that is, the curvature of the diffusion curve.

It was not, however, until the experiments and calculations of the results were far advanced that evidence was obtained as to the applicability to the present method of investigation of the tables calculated by STEFAN† for the diffusion of salts. By the help of these tables the diffusion constant can be determined, in the case of the single-tube experiments, if the distribution of the dissolved diffusing metal is known.

* 'Mathematical and Physical Papers,' vol. 3, 1890, p. 428.

† STEFAN, 'Wien. Akad. Ber.,' vol. 79, 1879, p. 161.

Experimental Results.

The following tables, A, B, C, D, embody the results of four diffusion experiments, made simultaneously, two with gold and two with platinum, in straight vertical tubes. They serve to indicate the method of calculating the diffusivity of the respective metals from the data afforded by the experiments. In the case given in Table A, for example, the alloy of gold and lead was allowed to diffuse upwards into pure lead in the way already described. This comparatively rich alloy of gold and lead containing 30 per cent. of gold, occupied 2·108 centims. of the length of the tube. The tube (the whole length of which was about 16 centims.) was subsequently cut into sections, each of which was 1·054 centims. long, or half the length occupied by the rich alloy of gold. The reason for the adoption of this length was as follows :— In GRAHAM's experiments on the diffusion of salts, which form the basis of STEFAN's tables, the concentrated solution from which diffusion started, occupied two of the sections into which the contents of the diffusion cylinders were divided for analysis. The length of one of these sections, or half of the length of the portion of the tube occupied by the alloy from which diffusion takes place, is denoted in the calculation by the letter h. In Table A there are fourteen such sections, and these are numbered consecutively in column 1, while column 2 gives the weight in grammes of the lead-gold alloy obtained from each of the respective sections and of the pure gold extracted from it. Column 3 gives the percentage of gold present in each section calculated from the numbers given in column 2. In column 4 these percentages of gold have been corrected so as to give the amounts of gold in equal volumes of the " solution." It must be remembered that the density of the gold-lead alloys increases with the percentage of gold, so that the concentration of the gold, that is, the weight present in equal volumes of the alloy, is not truly represented by the percentage given in column 3. A correction of sufficient accuracy may, however, be introduced by assuming that the fluid densities of the alloys are proportional to their calculated densities, and the numbers so corrected are given in column 4.

EXTRACT of four pages from the note book, showing method of calculating the results of two gold and two platinum diffusions in lead. The experiments were begun on October 9, 1894, and occupied 6·96 days. The mean temperature was 492° C.

TABLE A.—Tube 1. Gold in Lead.

1	2		3	4	5	6		7
Number of section of diffusion tube.	Weight in grammes of metal from each section—		Percentage of gold in each section.	Corrected for density to equal volumes.	Divided by 0·005719.	Theoretical numbers for—		$\frac{h}{2\sqrt{kt}}$ (by interpolation).
	lead-gold alloy.	gold.				$\frac{h}{2\sqrt{kt}}=0·11.$	$\frac{h}{2\sqrt{kt}}=0·12.$	
1	2·63	0·1876	7·13	7·38	1291	1217	1322	0·117
2	3·20	0·2247	7·02	7·26	1270	1188	1286	0·118
3	3·15	0·2096	6·65	6·87	1201	1135	1217	0·118
4	3·02	0·1846	6·11	6·29	1100	1058	1120	0·117
5	3·57	0·1950	5·46	5·60	979	965	1004	0·113
6	2·52	0·1212	4·81	4·92	860	860	874	0·110
7	3·16	0·1317	4·17	4·25	743	749	742	0·119
8	2·93	0·1030	3·52	3·58	626	640	613	0·115
9	3·19	0·0903	2·83	2·87	502	538	496	0·119
10	2·67	0·0593	2·22	2·24	392	447	393	0·120
11	2·74	0·0477	1·74	1·75	306	370	308	0·120
12	2·61	0·0354	1·35	1·36	238	310	244	0·121
13	3·16	0·0308	0·97	0·97	170	269	201	0·125
14	4·37	0·0374	0·85	0·85	149	249	180	0·125
			Sum. .	57·19			Mean . .	0·1184

h = 1·054 centims. when cold = 1·082 centims. at 492° C., therefore kt = 20·88.
t = 6·96 days, therefore k = 3·00 sq. centims. per diem.

The tables given by STEFAN for calculating absolute diffusivities from the results of GRAHAM'S experiments with salts, give for special values of the factor $\frac{h}{2\sqrt{kt}}$ the concentrations in each section of a diffusion cylinder, on the assumption that the original *two* volumes of diffusing solution of a salt, taken by GRAHAM, contained 10,000 units of salt.

It follows from this that the sum of the numbers representing the concentration of the total number of sections will always equal 10,000. The numbers given in column 4 must therefore be divided by such a number as will make the sum of their quotients 10,000. This common divisor is found by adding up the numbers in column 4 and dividing the result by 10,000, and column 5 gives the quotients of the numbers in column 4 divided by the common divisor 0·005719.

Column 6 gives the theoretical concentration in each section in cases where the factor $\frac{h}{2\sqrt{kt}}$ has the values 0·11 and 0·12, because the actual results of this experiment (on the diffusion of gold in lead) were seen by inspection to lie between these values.

STEFAN* applied the formal analogy between diffusion-movements and wave-motions, to the calculation of GRAHAM'S results ; both motions are expressed by the differential equation already quoted, $\frac{dv}{dt} = k\frac{d^2v}{dx^2}$. STEFAN calculated his results for the case of a cylinder of infinite length. In order to apply his results to a cylinder of limited length, which is the case in these lead-gold experiments, use is made of the principle of reflection and superposition, in accordance with which the quantity of a substance that would have passed beyond the end of this limited cylinder is considered to be totally reflected, distributed by diffusion, and retained in the several sections of the tube.

The figures in column 6 have been adjusted in this manner to suit the diffusion tube, which was divided (as column 1 of Table A shows) into fourteen sections.

The final column, 7, gives for each section the value of $\frac{h}{2\sqrt{kt}}$ equivalent to the concentration actually observed, calculating by interpolation the figures which lie between those actually given in the double column 6. If the diffusion were in accordance with FICK'S law and the experiments free from error, all the numbers in column 7 should be identical. It will be seen that they do agree closely, and that the differences which occur may be attributed to experimental errors. It will be evident that the value to be attached to an experiment may be gathered from the degree of uniformity exhibited by the numbers in column 7.

* STEFAN, *loc. cit.*, and OSTWALD, 'Solutions,' p. 130.

TABLE B.—Tube 2. Gold in Lead.

1	2		3	4	5	6		7
Number of section of diffusion tube.	Weight in grammes of metal from each section—		Percentage of gold in each section.	Corrected for density to equal volumes.	Divided by 0·0054785.	Theoretical numbers for—		$\frac{h}{2\sqrt{kt}}$ (by interpolation).
	lead-gold alloy.	gold.				$\frac{h}{2\sqrt{kt}}=0·13.$	$\frac{h}{2\sqrt{kt}}=0·14.$	
1	2·44	0·1855	7·603	7·887	1440	1427	1530	0·1313
2	3·72	0·2768	7·442	7·717	1408	1381	1474	0·1329
3	3·64	0·2529	6·948	7·186	1312	1295	1369	0·1321
4	3·35	0·2111	6·302	6·500	1187	1176	1225	0·1322
5	3·50	0·1931	5·518	5·666	1034	1035	1056	
6	3·54	0·1688	4·769	4·879	891	882	878	
7	3·20	0·1242	3·882	3·954	722	728	706	0·1327
8	3·36	0·1050	3·125	3·170	579	585	544	0·1315
9	3·85	0·0928	2·411	2·439	445	475	407	0·1324
10	3·79	0·0691	1·823	1·839	336	352	298	0·1330
11	3·29	0·0451	1·371	1·379	252	272	217	0·1338
12	3·17	0·0349	1·101	1·107	202	218	163	0·1330
13	2·99	0·0313	1·047	1·062	194	191	136	
			Sum. .	54·785			Mean . .	0·1325

h = 1·194 centims. cold = 1·225 centims. at 492° C., therefore kt = 21·37.

t = 6·96 days, therefore k = 3·07 sq. centims. per diem.

TABLE C.—Tube 3. Platinum in Lead.

1	2		3	4	5	6		7
Number of section of diffusion tube.	Weight in grammes of metal from each section—		Percentage of platinum in each section.	Corrected for density to equal volumes.	Divided by 0·0048654.	Theoretical numbers for—		$\frac{h}{2\sqrt{kt}}$ (by interpolation).
	lead platinum alloy.	platinum.				$\frac{h}{2\sqrt{kt}}=0·18$.	$\frac{h}{2\sqrt{kt}}=0·19$.	
1	3·11	0·2805	9·020	9·744	2003	1927	2023	0·188
2	3·37	0·2845	8·442	9·076	1865	1817	1895	0·186
3	3·40	0·2527	7·434	7·925	1629	1614	1664	0·183
4	3·68	0·2265	6·155	6·492	1334	1352	1368	
5	3·67	0·1850	5·041	5·268	1083	1067	1054	
6	3·94	0·1383	3·510	3·619	744	793	760	0·194
7	3·84	0·0933	2·430	2·482	510	556	513	0·190
8	3·32	0·0530	1·596	1·618	332	368	324	0·188
9	3·98	0·0404	1·015	1·024	210	230	193	0·185
10	3·65	0·0216	0·592	0·595	122	137	108	0·185
11	3·16	0·0132	0·418	0·420	86	83	60	0·179
12	4·33	0·0169	0·390	0·391	80	57	39	
			Sum. .	48·654			Mean . .	0·186

$h = 1·245$ centims. cold $= 1·277$ centims. at 492°, therefore $kt = 11·79$.

$t = 6·96$ days, therefore $k = 1·69$ sq. centims. per diem.

TABLE D.—Tube 4. Platinum in Lead.

1	2		3	4	5	6		7
Number of section of diffusion tube.	Weight in grammes of metal from each section—		Percentage of platinum in each section.	Corrected for density to equal volumes.	Divided by 0·0048556.	Theoretical numbers for—		$\frac{h}{2\sqrt{kt}}$ (by interpolation).
	lead-platinum alloy.	platinum.				$\frac{h}{2\sqrt{kt}}=0·18.$	$\frac{h}{2\sqrt{kt}}=0·19.$	
1	3·17	0·2795	8·817	9·509	1958	1927	2023	0·183
2	3·85	0·3435	8·922	9·613	1980	1817	1895	0·201
3	3·49	0·2690	7·707	8·235	1696	1614	1664	0·196
4	4·47	0·2761	6·177	6·517	1344	1352	1368	0·175
5	2·62	0·1280	4·885	5·093	1049	1067	1054	0·194
6	4·12	0·1471	3·571	3·682	758	793	760	0·191
7	3·62	0·0891	2·462	2·514	517	556	513	0·189
8	3·40	0·0491	1·444	1·462	301	368	324	0·195
9	3·73	0·0342	·917	0·925	190	230	193	0·191
10	3·57	0·0169	·473	0·475	102	137	108	0·191
11	3·26	0·0099	·304	0·305	62	83	60	0·189
12	3·28	0·0074	·226	0·226	46	57	39	0·186
			Sum .	48·556			Mean . .	0·190

h = 1·27 centims. cold = 1·303 centims. at 492°, therefore kt = 11·76.
t = 6·96 days, therefore k = 1·69 sq. centims. per diem.

In figs. 3, 4, the diffusions which are given in Tables B and C are plotted in thick lines, with distance and concentration as coordinates. The curves in dotted lines give theoretical distributions for two values of $\frac{h}{2\sqrt{kt}}$, which nearly agree with the experimental results, and between which the true value of $\frac{h}{2\sqrt{kt}}$ appears to lie.

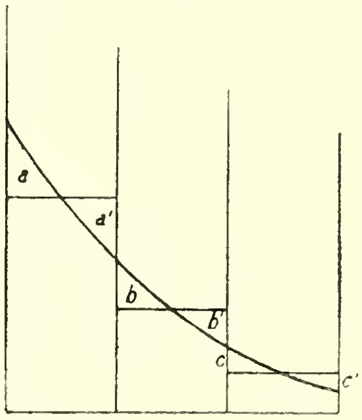
In these curves the horizontal length of the figure, which represents the height of the diffusion-tube, is divided by the sectional lines into as many equal parts as there are sections of the tube. The concentration in each section was marked by a horizontal pencil line, and the continuous curve shown in fig. 2, and in figs. 3, 4, was drawn through these lines in such a way that the area included in each section still represented the average concentration in it. This method was adopted to avoid the slight error which would have been introduced if the average concentration had been plotted at the mid-point of each section. This mode of plotting, fig. 2, in which the area $a = a'$, $b = b'$, and $c = c'$, renders it impossible to specify the *points* through which the curve has been drawn, but it may be remembered that the average position of the curved line in each section represents the result of one analysis.

The dotted lines are plotted from figures calculated from STEFAN'S tables for two values of the factor $\frac{h}{2\sqrt{kt}}$, between which the experimental values are found to fall in most of the sections. They are plotted in the same manner as the thick continuous lines, and the value of k corresponding to each value of $\frac{h}{2\sqrt{kt}}$ is calculated and placed opposite the dotted line to which it belongs.

In fig. 4 it was necessary to plot these dotted lines for more widely divergent values of k than was the case in fig. 3.

Fig. 3, representing the diffusion of gold in lead, is seen to agree very well with theory, while fig. 4, representing the diffusion of platinum in lead, shows considerable irregularities, which are probably due to errors in the platinum assay. The mean of the figures in column 7 is taken as the correct value for $\frac{h}{2\sqrt{kt}}$, and from this, knowing the values of h and t, that of k may be obtained.

Fig. 2.



It will be remembered that the length of each section (Table A) was 1·054 centims., which is equal to half the length of the part of the tube occupied by the rich gold alloy. As, however, this measurement was made on cold solidified metal, it must be corrected to the temperature at which the experiment was actually conducted. This correction is based on experimental determinations of the linear expansion by heat of the molten metals contained in the iron tubes. The corrected value of h, 1·082 centims., and the value of t, 6·96 days, are substituted in the equation $\frac{h}{2\sqrt{kt}} = 0·1184$, giving 3·00 as the value of k, the units being the *centimetre* and *day*. The experiments, the results of which are embodied in Tables B, C, and D, were conducted simultaneously with those in Table A, which have just been described. All the experiments in this series were, therefore, subjected to exactly the same conditions as regards time and temperature.

Experiments A and B were diffusions of gold, and C and D of platinum, in lead. The results of these gave k for gold in lead as 3·00 and 3·07, and for platinum in lead as 1·69 and 1·69 respectively, the temperature in each case being 492° C. It may

Fig. 3 (plotted from Table B).

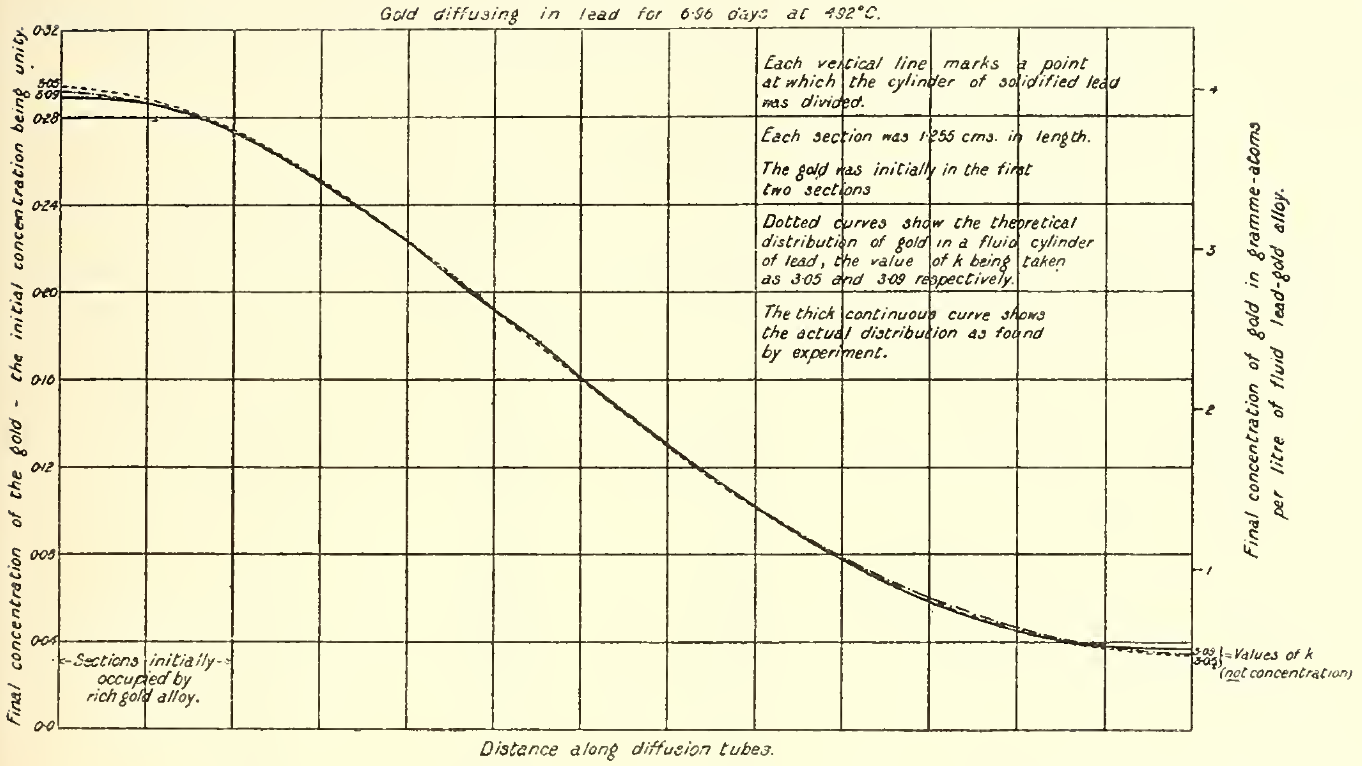
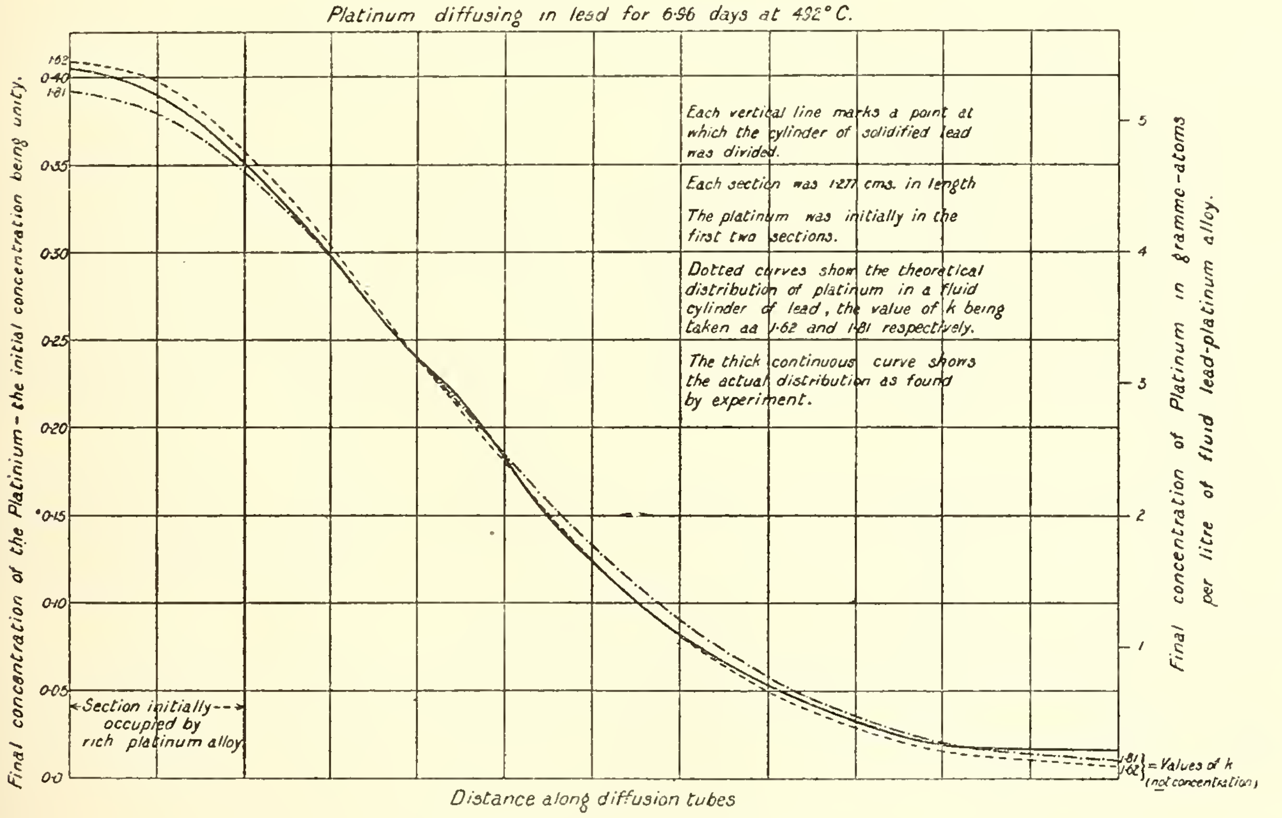


Fig. 4 (plotted from Table C).



be thought that this is an unnecessarily high temperature, but as some experiments are in progress in which lead alloys rich in gold are used, it was considered advisable to work at a temperature at which the entire series of alloys would melt.

Table E contains the results of the more recent experiments which have been made; k is given in sq. centims. per diem, and also in sq. centims. per second. The result for the diffusion of lead in tin is, however, less trustworthy than the other data.

TABLE E.

Diffusing metal.	Solvent.	Temperature.	k in square centims.	
			Per diem.	Per second.
		°		
Gold	Lead . . .	492	3·00	$3·47 \times 10^{-5}$
,, ...	,, . . .	492	3·07	3·55 ,,
Platinum . .	,, . . .	492	1·69	1·96 ,,
,, . .	,, . . .	492	1·69	1·96 ,,
Gold	,, . . .	555	3·19	3·69 ,,
,, ...	Bismuth . .	555	4·52	5·23 ,,
,, ...	Tin	555	4·65	5·38 ,,
Silver . . .	,, ...	555	4·14	4·79 ,,
Lead . . .	,, ...	555	3·18	3·68 ,,
Gold	Lead . . .	550	3·185	3·69 ,,
Rhodium . .	,, . . .	550	3·035	3·51 ,,

These results are presented graphically in Plate 8, which represents the theoretical distribution of the several metals after diffusion has proceeded for seven days, the temperature being close to 500° C.

The horizontal ordinate represents distance in the direction in which diffusion takes place, the actual length of the plate being the same as that of the tube. The vertical ordinate represents concentration. It will be evident that if the plate be turned so as to make the horizontal line vertical, the actual distance the metals diffused upwards will at once be apparent. Each of the diffusing metals occupied at the beginning of the experiment the length a, b of the tube, and they all had the same initial concentration.

If the time had been infinitely extended, the final condition of each experiment would be represented by the horizontal line d, e, and the relative diffusivity of each pair of metals is shown by the degree of approximation which the distribution in that experiment has made to the final condition of things.

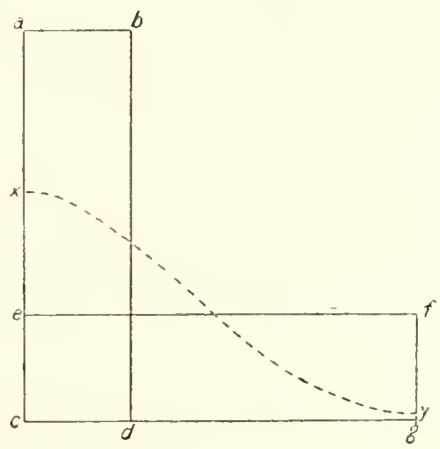
What this final condition would be may be made clear by the aid of fig. 5. The length c, d represents that part of the tube which is occupied by the gold alloy from which diffusion takes place, the alloy having an original concentration denoted by $c\ a$, so that the area $a\ b\ d\ c$, represents the whole quantity of gold in the experiment; all the gold being initially to the left of the line d, b. The final state of complete diffusion is represented by the area $c\ e\ f\ g$, which is the same as $a\ b\ d\ c$, since the quantity of gold remains unaltered.

In the same manner the area *c x y g*, would represent the distribution of gold at one particular stage of the diffusion.

The greater the diffusive power of the metal represented by the area *c x y g*, the sooner will it become identified with the area *c e f g*, and consequently in experiments which have lasted for equal times, the flatter the line *x*, *y*, the greater is the diffusivity of the metal it represents.

The diffusion of chloride of sodium in water at 18° C. has been plotted in Plate 8 (as a dotted line) for comparison. It will be seen that (at 18° C.) common salt is much less diffusive in water than the metals as yet examined are at 500° in the solvents lead or bismuth.

Fig. 5.



The conditions under which the experiments on the diffusion of metals were conducted differed so much, that it would have been meaningless to have plotted the actual distributions in a single plate. The time and temperature were different in each set of experiments, and the value of *h* was also variable. Plate 8 accordingly represents the theoretical distribution calculated from the experimental value of *k* corrected to 500°, and for the assumed time, 7 days, and the value of *h*, shown in this figure, 1·27 centims.; the tube being divided into 14 sections. The small extent to which these theoretical distributions differ from those actually measured, may be seen in figs. 3 and 4, and in tables A, B, C, and D. The curves were drawn, for convenience, by plotting the mean concentration of each section at its mid point. The error so introduced is almost invisible on the curves and does not affect their *relative* position. No scale of the initial concentration is attached because the distribution of the diffusing metal is but slightly affected by small differences in concentration.

The concentration of the diffusing metal, and as affecting this the duration of the experiment are, nevertheless, of much importance. The more rapid diffusivity in the alloys, poor in precious metal, is extremely marked. As this was especially the case in experiments which only lasted for a short period, where many of the segments contained very dilute solutions, it may be in part an effect requiring careful study, since it points to a possible simplification of the gold molecule when the osmotic pressure is small. This effect appears to be connected with the dilution of the gold below a concentration of about 0·005 gramme-atom per litre of lead, corresponding, at 450°, to an osmotic pressure of about one-third of an atmosphere.

The results given in Table E show that the diffusion of metals is not increased nearly so rapidly by a rise in temperature of 50° as the ordinary aqueous diffusion of salts is. This is probably due to the fact that the resistance presented by metals to diffusing metallic molecules is not much reduced by heat, and it may be that the molecules of the diffusing substance are not so liable to disruption by a rise in temperature.

It may be well to wait until more results have been obtained before attempting to deduce evidence as to the molecular condition of the metals composing these alloys, though, in this respect, metallic diffusion presents several advantages over the diffusion of salt solutions, the latter being very limited as regards the choice of a solvent and range of temperature at which experiments could be made, and, moreover, if a salt is dissociated into its ions its diffusion rates will be modified. The fact that when water is used, both the salt and the solvent are chemical compounds, renders their diffusivities less directly significant than those of metals, because, with the exception of gaseous elements, molten metals present the simplest possible case which can occur, as they represent the diffusion of one element into another.

A few general deductions may, however, be drawn.

It will be seen that gold diffuses more rapidly in bismuth and in tin than it does in the heavier metal lead. It has also been observed (though Table E does not embody the results) that platinum diffuses faster in bismuth than in lead. The diffusion of platinum and of gold is increased in about equal ratio by the substitution of bismuth for lead as a solvent. On the other hand, platinum diffuses much more slowly in lead than gold does, although their atomic weights and their densities do not greatly differ. Rhodium, another metal of the platinum group, diffuses in lead nearly as fast as gold does, but if allowance be made for the smaller atomic weight, it will be found to agree fairly well with platinum. This would point to the conclusion that the platinum metals are molecularly more complex than either gold or silver, as a complex molecule exerts less osmotic pressure and diffuses more slowly than a comparatively simple one.

The early workers on diffusion of salts used water as a solvent; KAWALKI,* however, has recently given tables of diffusions of salts in both water and alcohol. He found that there is a fairly constant ratio between the diffusivities of salts in the two solvents. Experiments seem to show that this is also true of two metallic solvents such as lead and bismuth.

Calculations of the Older Results.—The results obtained from the earlier experiments (which were conducted in U-tubes) have, as already stated, been omitted, because the great difference of temperature between the top and bottom of the tubes made

* 'Wied. Ann.,' vol. 52, 1894, p. 300.

the meaning of the final distribution rather uncertain. The results of these and some of the very early experiments have been dealt with by the aid of tables of theoretical distributions, similar to those of STEFAN, but calculated for the case of an infinite tube having constant concentration at one end, which nearly corresponded to the actual conditions under which my earlier experiments were conducted. The values given by these experiments agree very fairly with those obtained later, but the inequalities of temperature made the value of k vary from point to point of the tube, so that it was impossible to determine it with any degree of precision.

Diffusions of Amalgams in Mercury.—A number of experiments were made on the diffusion, at ordinary temperatures, of gold and other metals in mercury, which will be included in a subsequent communication. I may, however, mention, for the sake of comparison, that the diffusivity of gold in mercury at 11° C. is 0·72 sq. centim. per day, the diffusivity of gold in lead being 3·0 sq. centims. per day at 500°. As already stated, Dr. GUTHRIE* published in 1883 particulars of some experiments of this kind, the metals he selected being zinc, tin, lead, sodium, and potassium, diffusing in mercury. He did not make any calculations with a view to obtain either the absolute or the relative diffusivities of these metals in mercury; indeed, after giving the percentage of the diffusing metal in successive quantities of mercury, he observes: "It is scarcely worth while dividing these diffusion *percentages* by the so-called atomic weights of the metals." His experiments were complicated by the fact that in some cases he employed solid metals instead of fluid amalgams, as the source of the diffusing metal. Approximate results have, however, been obtained from his data, and from certain measurements of the original apparatus, now deposited in the South Kensington Museum. These show that his inference "that potassium and sodium have a far greater diffusive energy than the heavier metals examined" is not supported by the actual result of his experiments. His results, calculated by the method given in the present paper, give the values of k, in square centimetres per day, as follows:—

Tin	in mercury	at about 15°	1·22
Lead	,,	,,	1·0
Zinc	,,	,,	1·0
Sodium	,,	,,	0·45
Potassium	,,	,,	0·40.

* *Loc. cit.*

PART II.—DIFFUSION OF SOLID METALS.

GOLD, DIFFUSING INTO SOLID LEAD.

The experiments described in the first part of this paper naturally suggested the enquiry whether gold would still permeate lead if the temperature were maintained at a point far below the melting point of lead. Would diffusion take place through solid lead at the ordinary temperature, or must a certain amount of viscosity be given to it by the application of a moderate heat? These were questions which demanded attention.

Historical.

The history of the diffusion of solids is full of interest, and it may be convenient, as far as possible, to group the facts which are known, rather than to deal with them in strict chronological order.

"*Kernel Roasting.*"—There has long been a prevalent belief that diffusion can take place in solids, and the practice in conducting certain important industrial processes supports this view. One of these processes, which is of comparatively ancient date, has certainly been employed since 1692 at Agordo, and its results are as follows: When lumps of cupriferous iron pyrites are subjected to very gradual roasting with access of air, the copper becomes concentrated as a "kernel" of nearly pure sulphide of copper in the centre of a mass of ferric oxide, while, at the same time, the silver originally present in the ore travels outwards and forms a glistening shell on the exterior. These complicated changes must be effected in the solid by a movement allied to diffusion.

Cementation Processes.—Of all the processes which depend on the diffusion of solids probably the most interesting is the truly venerable one by which silver may be recovered from either plates or globules of solid gold by "cementation," the name being derived from the "cement" or compound in which the plates were heated. Its nature was indicated by PLINY, and the manipulation it involved was minutely described by GEBER in the 8th century, as well as by many of the early metallurgists; SAVOT,* for instance, pointed out in the early part of the 17th century that "cementation" will deprive gold of the silver it contains, "however small" the amount of the latter metal may be, so that it will be evident that the elimination of the silver from the centre of a mass of solid gold must also be effected by an inter-molecular movement allied to diffusion. The evidence, however, is not conclusive, because gaseous chlorine intervenes, and may even play an important part in the penetration of the solid metal.

In another ancient "cementation" process, the conversion of strongly heated but

* 'Discours sur les Médailles antiques,' 1627, p. 76.

still solid iron into steel is effected by the passage of solid carbon into the interior of the mass of iron, and the explanations which have from time to time been given of the process form a voluminous literature. LE PLAY considered cementation, which is really a slow creeping action of one solid into another, to be "an unexplained and mysterious operation," and he attributed the transmission of the carbon to the centre of the iron solely to the action of gaseous carbonic oxide. GAY LUSSAC* confessed that a study of the process shook his faith "in the belief generally attributed to the ancient chemists that *corpora non agunt nisi soluta*," for it is certain, he adds, "that all bodies, solid, liquid, or aëriform, act upon each other, but, of the three states of bodies, the solid state is the least favourable to the exercise of chemical affinity."

In 1881, M. A. COLSON† communicated a paper to the Académie des Sciences, in which he showed that when iron is heated in carbon there is a mutual interpenetration of carbon and iron at so low a temperature as 250°. The interpenetration of solids, as distinguished from the diffusion of two metals in each other, has received attention from many experimenters, of whose work brief mention will only be given, as the subject of this part of the paper is the diffusion of solid metals. COLSON pointed out that pure silver diffuses as chloride in dry chloride of sodium, and he states that calcium passes into platinum when the latter is heated in lime, and that silica diffuses through carbon and yields its silicon to platinum. The permeation of strongly heated porcelain by carbon has been demonstrated by MARSDEN, VIOLLE, and other experimenters. SPRING,‡ in 1885, showed that solid barium sulphate and sodium carbonate react on each other until an equilibrium is established.

Fig. 6.

Any lingering doubt as to whether gas need necessarily intervene in the cementation of iron was, I may point out, removed by an experiment of my own,§ in 1889, which showed that pure iron may be carburized by diamond *in vacuo*, at a temperature far below the melting point of iron and under conditions which absolutely preclude the presence or influence of occluded gas. I am indebted to my friend, M. OSMOND, for a photograph (from which fig. 6 is prepared) of a section, magnified 100 diameters, through a piece of electro-iron, which had been heated to 1500° and carburized from the upper end by contact with the diamond form of carbon, and this section clearly shows the gradual penetration of the iron by carbon. The white grains at the base are unchanged iron. The beautiful work of OSMOND on the transforma-

* 'Ann. de Chim. et de Phys.,' vol. 17, 1846, p. 221.

† 'Comptes Rendus,' vol. 93, 1881, p. 1074; vol. 94, 1882, p. 26.

‡ 'Bull. de l'Acad. Roy. de Belgique,' vol. 10, 1885, p. 204.

§ 'Nature,' vol. 41, 1889, p. 14.

tions of iron, and of iron and carbon, affords, moreover, a striking proof of the molecular mobility of solid iron at a temperature which is at least 600° below its melting point.

The history of the formation of alloys by cementation will be traced subsequently.

The Penetration of Solid Metals by Gases.—This subject was, as is well known, investigated by GRAHAM, "the leading atomist of his generation," but before his attention was specially directed to it, a mass of experimental evidence led him in 1863 to express views of singular interest in the beautiful paper* which embodied his "speculative ideas respecting the constitution of matter." In this paper he pointed out that in solids, some of the molecules may still be in the liquid or even the gaseous condition, and his words are very definite. He says, "the three conditions [solid, liquid, and gaseous] probably always co-exist in every liquid or solid substance, but one predominates over the others. . . . Liquefaction or solidification may not, therefore, involve the suppression of either the atomic or the molecular movement, but only the restriction of its range." He subsequently, in 1866,† gave singular point to these speculations by his discovery of the penetration of solid metals by gases.

After GRAHAM'S death, evidence as to the molecular mobility of metals came slowly. E. WIEDEMANN‡ showed, in 1878, that solid metals were not necessarily inert, as changes which involve atomic movement take place in bismuth-lead alloys, and the clear evidence thus afforded of allotropic change in the solid, recently formed the subject of some experiments of my own.§

Confirmation of the accuracy of GRAHAM'S views as to the co-existence of liquid and gaseous molecules in a solid, was afforded twenty years later by Professor W. SPRING.|| In 1886 his admirable investigation on the solidification of alloys of lead and tin, afforded him experimental evidence that in these alloys active molecular movement is continued after the alloys have become solid. He says, and it is well to quote his interesting words, "on serait porté à penser qu'entre deux molécules de deux corps solides il y a un va-et-vient perpétuel d'atomes," and he adds, "if the two molecules are of the same kind, chemical equilibrium will not be disturbed, but if they are different this movement will be revealed by the formation of new substances."

Formation of Alloys by Cementation.—The fact that alloys can be formed by the union of two metals at a temperature below the melting point of the more fusible of the two has long been known to metallurgists, and perhaps the most striking fact in the more modern history of the subject was recorded in 1820 by FARADAY and

* 'Phil. Mag.,' February, 1864; 'Collected Papers,' p. 299.

† 'Phil. Trans. Roy. Soc.,' 1866, pp. 399–439.

‡ 'WIED., Ann.,' 3, 1878, p. 237.

§ Second Report Alloys Research Committee, 'Proc. Inst. Mech. Engineers,' 1893, p. 127.

|| 'Bull. de l'Acad. Royale de Belgique,' vol. 11, 1886, p. 355.

STODART,* who, in the course of an investigation on the alloys of iron with other metals, noted their failure to produce certain alloys by "cementation," but consider it "remarkable," in the case of platinum, that it will unite with steel at a temperature at which the steel itself is not affected. They also show that solid steel and platinum, in the form of bundles of wires, may be welded together "with the same facility as could be done with iron or steel," and they observe that on etching the surface of the welded mass by an acid "the iron appeared to be alloyed with the platinum." Their interest in this singular fact led them to promise some direct experiments on "this apparent alloy by cementation," that is, by the interpenetration of solids. Since this time there have been many more or less isolated observations bearing on the subject, and brief reference may be made to the more important of them in chronological order. In 1877 CHERNOFF† showed that if two surfaces of iron of the same nature be placed in intimate contact and heated to about 650° they will unite. In 1882 SPRING‡ made his remarkable experiments on the formation of alloys by strongly compressing their constituent metals at the ordinary temperature, while in 1885, O. LEHMANN§ suggested, and in 1888 HALLOCK|| demonstrated that compression is not necessary, as alloys might be formed by placing carefully-cleaned pieces of two constituent metals in juxtaposition and heating them to the melting point of the alloy to be formed, which was, in some cases, 150° below the melting point of the more fusible of the two metals. In 1889 COFFIN showed, and I have repeatedly verified the accuracy of his experiment, that if the freshly-fractured surfaces of a steel rod, 9·5 millims. square, be placed together and heated to below redness, they will unite so firmly that it is difficult to pull them apart by hand. The steel is highly carburized and the diffusion of a carbide of iron probably plays an important part in effecting the union. There must have been molecular interpenetration in this case, though the steel was at least 1000° below its melting point. In 1894 SPRING¶ proved that if the carefully surfaced ends of cylinders of two metals were strongly pressed together and maintained for eight hours at temperatures which varied from 180° to 400°, interpenetration would take place, true alloys being formed at the junction of the two metals.

In these experiments, which are of great interest, the temperatures at which the cylinders were maintained were below the melting point of the more fusible of the two metals. Care appears to have been taken to avoid heating them up to the melting point of the *eutectic* alloy, though it was in some cases close to it. The necessity for

* 'Quarterly Journal of Science,' vol. 9, 1820, p. 319.

† 'Revue Universelle des Mines,' vol. 1, 1877, p. 411.

‡ 'Ber. der Deutsch. Chem. Gesell.,' 15, 1882, p. 595.

§ 'Wied. Ann.,' 24, 1885, p. 1.

|| Communicated to Phil. Soc. of Washington, Feb. 18, 1888; 'Zeitschr. Phys. Chem.,' 2, 1888, p. 6, or 'Chem. News,' vol. 63, 1891, p. 17.

¶ 'Bull. de l'Acad. Royale de Belgique,' vol. 28, 1894, p. 23.

this precaution will be obvious as the union of the two compressed cylinders might easily be effected by the fusion of an eutectic alloy with a relatively low melting point.

I observed in 1887 that an electro-deposit of iron on a clean copper plate will adhere so firmly to it that when they are severed by force a copper film is actually stripped from the copper plate and remains on the iron, thus affording clear evidence of the interpenetration of metals at the ordinary temperature. I found that this interpenetration of copper and iron will take place through films of electro-deposited nickel.* MYLIUS and FROMM have shown that metals interpenetrate and form alloys when they are precipitated by electrolysis from their aqueous solutions.†

The diffusion of metals in each other must be closely connected with the evaporation of solid metals or alloys at temperatures far below their melting points, and it will be well before describing the new experiments on diffusion in solid metals to briefly recall the facts which are already known. It is not necessary to go further back for definite views on the subject than to the time of BOYLE,‡ who thought that "even such (bodies) as are solid may respectively have their little atmospheres," . . . "for" he adds, "no man, I think, has yet tried whether glass, and even gold, may not in length of time lose their weight."

BOYLE'S opinion was correct, for mercury which has been *frozen* by extreme cold does, as MERGET§ showed two centuries later, evaporate into the atmosphere surrounding it; a fact which is of much interest in connection with GAY LUSSAC'S well-known observation that the vapours emitted by ice and by water both at 0°C. are of equal tension. DEMARÇAY‖ has proved that *in vacuo* metals evaporate sensibly at lower temperatures than they do at the ordinary atmospheric pressure, and he suggests that even metals of the platinum group will be found to be volatile at comparatively low temperatures.

Thus he finds that cadmium volatilizes at 160°, zinc at 184°, and lead and tin at 360°, and subsequently SPRING¶ (1894) demonstrated that zinc is volatile at atmospheric pressure at about 300°, cadmium at about 400°, while even copper is slightly volatile at the latter temperature.

MOISSAN** has stated quite recently that the vapour tension of solid silicon enables it to unite with iron and chromium by true "cementation" at a temperature of 1200°, which is much below the fusing point of these metals.

It must be borne in mind that the interesting facts recorded by the various experimenters whose names I have cited hardly come within the prevailing conditions in

* 'Journ. Iron and Steel Inst.,' Part I., 1887, p. 73.

† 'Ber. der D. Chem. Gesell.,' vol. 27, 1894, p. 630.

‡ 'Collected Works.' Shaw's edition, 1738, vol. 1, p. 400.

§ 'Ann. de Chim. et de Phys.,' vol. 25, 1872, p. 121.

‖ 'Comptes Rendus,' vol. 95, 1882, p. 183.

¶ *Loc. cit.*, 1894, p. 42.

** 'Comptes Rendus,' vol. 121, 1895, p. 621.

the ordinary diffusion of liquids, in which the diffusing substance is usually in the presence of a large excess of the solvent which is supposed to exert but little chemical action on it. This condition has been fully maintained in the experiments on the diffusion of liquid metals which are described in the first part of the present paper. It must also be remembered that VAN'T HOFF* has made it highly probable that the osmotic pressure of substances existing in a *solid* solution is analogous to that in liquid solutions and obeys the same laws, and it is also probable that the behaviour of a solid mixture, like that of a liquid mixture, would be greatly simplified if the solid solution were very dilute.

NERNST expresses the hope that it may be possible to measure by indirect methods the osmotic pressure of substances existing in solid solutions. I trust that the following experiments will sustain this hope by affording measurements of the results of osmotic pressure in masses of *solid metals* at the ordinary atmospheric pressure, and at a temperature at which it has hitherto been scarcely possible to detect diffusion in them.

The following experiments constitute, so far as I am aware, the first attempt to actually measure the diffusivity of one solid metal in another. It must be borne in mind that the union of two clean surfaces of metal, and even the interpenetration of two metals to a slight depth below the surfaces does not necessarily depend on diffusion alone, as the metals become united in a great measure by viscous flow. The nature of welding demands investigation, but the union of metals by welding is effected most energetically when the metals are in the colloidal condition, in which true diffusion is least marked. It may be observed that discs of gold and lead, pressed together at the ordinary temperature for three months, were found to have welded together more perfectly than two similar discs kept in contact at 100° for six weeks, although at least ten times more metal had interdiffused in the latter case than in the former.

Diffusion of Gold in Solid Lead.

The attempt was first made to ascertain whether diffusion of gold in solid lead could be measured at a temperature of 250°, that is 75° below the melting point of lead. With this object in view, thin plates of gold were fused on to the end of cylindrical rods of lead, 14 millims. in diameter and 7 centims. long. This could readily be effected by the point of a blow-pipe flame, and, when the cylinder of lead was kept cool by immersion in water to within a few millimetres of its end, the gold rapidly alloyed with the metal, but, as many analyses showed, did not penetrate the cylinder of lead more than a millimetre. Such cylinders were maintained for thirty-one days in a little iron chamber lined with asbestos, the temperature of which only varied by a degree or two from 250° C. The cylinders were then measured and

* 'Zeitschr. Phys. Chem.,' vol. 5, 1890, p. 322.

cut up into sections, and the amount of precious metal in the respective sections was determined by analysis. It should be observed that gold and lead are singularly well adapted for such an investigation, as the amount of precious metal diffusing can, by the aid of a delicate balance, be determined with wonderful precision. The balance used was a short-beam one, with a pointer at each end of the beam, moving over a graduated ivory scale, the divisions of which were read by means of fixed lenses. This balance was specially designed for the verification of the weights used in gold assaying at the Mint. Its maximum load is 0·5 gramme, and it will readily indicate the five hundred thousandth part of a gramme.

Gold in Lead at 251°.

The results of two experiments are shown in the following tables.

The total length of the cylinder was 6·5 centims., in the case of Experiment I, and 7·0 centims. in No. II; fragments of gold were fused on their lower ends. Time, thirty-one days.

Number of section (counting from base).	Weight of alloy in grammes.	Weight of gold in grammes.	Gold per cent.	Diffusivity in sq. centims. per diem.
I. 1	26·9	0·0824	0·3064	
2	22·2	0·0053	0·0239	
3	19·6	0·0015	0·0076	0·023
4	22·3	0·00013	0·0006	
5	24·6	0·00002	0·0001	
II. 1	2·83	0·1803	0·3700	
2	2·55	0·0020	0·0784	
3	3·12	0·0013	0·0417	
4	2·84	0·0006	0·0211	0·03
5	2·99	0·00026	0·0087	
6	5·92	0·00013	0·0022	
7	6·03	0·00001	0·0002	
8	4·40			

In calculating the diffusivity in this and in the following experiments, the initial concentration of the solid lead-gold alloy from which diffusion starts was deduced from the general trend of the concentration curve as plotted from the above figures. The first section in each case, which of course contained pieces of partially alloyed gold, was neglected.

Gold in Lead at 200°.

The next experiments were made at 200° and only lasted ten days. The results of two of these are given below. In No. I the cylinder of lead was 2·5 centims. long, and had a plate of gold fused on to its end. In No. II the cylinder was 1·0 centim. long, and the plate of gold was merely held against the carefully surfaced end by means of a binding screw.

Number of section (counting from base).	Weight of alloy in grammes.	Weight of gold in grammes.	Gold per cent.	Diffusivity in sq. centims. per diem.
I. 1	1·50	0·2147	14·3	
2	1·05	lost		
3	2·00	0·00055	0·0275	
4	2·14	0·000185	0·0086	0·007
5	2·18	0·000022	0·0010	
6	3·95	trace		
7	4·60	none		
II. 1	0·39	0·00082	0·210	
2	0·49	0·000140	0·029	0·008
3	0·82	0·000187	0·023	
4	2·12	0·000220	0·010	
5	2·50	0·000040	0·002	

There are many other results of similar experiments at 200°, which closely agree with those given. There is, however, one point that must not be lost sight of. The lead cylinder was solid, but the experiments just described were conducted at a higher temperature than the melting-point of the *eutectic* alloy of gold and lead, which melts at nearly 200°. It may be that directly gold diffuses into lead it forms an alloy which is fluid at the temperature of the experiment and, therefore, though the lead itself is solid, it may, nevertheless, contain the gold in the form of a fluid alloy which diffused as such. Great care was consequently taken to ascertain whether gold would still diffuse in solid lead at a temperature well below the melting-point of the eutectic alloy (200°), and for this purpose the temperature chosen was 165°.

Gold in Lead at 165°.

The length of the cylinder in Experiment 1 was 0·64 centim., and it was maintained at this temperature for thirty days. The length of the cylinder in Experiment 2 was 0·60 centim., and it was heated for twenty-nine days. In each case diffusion started from a 5 per cent. alloy of gold and lead, pressed against the end of the lead cylinder,* and there was no need, therefore, to reject the first section of the cylinder.

* This precaution was necessary, as the diffusion must be very slow, and it was important to avoid

Number of section (counting from base).	Weight of alloy in grammes.	Weight of gold in grammes.	Gold per cent.	Diffusivity in sq. centims. per diem.
I. 1	0·64	0·00025	0·039	
2	2·33	0·00069	0·030	0·005
3	2·02	0·00030	0·015	
II. 1	0·80	0·00031	0·039	
2	2·06	0·00060	0·029	0·004
3	1·40	0·00027	0·019	

Gold in Lead at 100°.

In the following, and concluding experiment of the series, the lead cylinders, 0·45 centim., were maintained in a water oven at a temperature of 100° for forty-one days. Each cylinder had a plate of very pure gold pressed against its surfaced end.

Number of section (counting from base).	Weight of alloy in grammes.	Weight of gold in grammes.	Gold per cent.	Diffusivity in sq. centims. per diem.
I. 1	0·22	0·00013	0·059	
2	1·19	0·00006	0·005*	0·00002
3	1·98			
II. 1	0·22	0·00034	0·155	
2	1·52	0·00010	0·007	0·00002
3	1·43			

It is remarkable that gold placed at the bottom of a cylinder of *solid* lead, 7 centims. long, should, at 250°, appear in notable quantity at the top of it in less than a month. The diffusivity of gold in *solid* lead is, however, slow when compared with diffusion in the *fluid* metal. The diffusivity of gold in lead at 500° was shown in the first part of this paper to be 3·0 In *solid* lead, on the other hand, the diffusivity at 250° is 0·03, or $\frac{1}{100}$th of the diffusivity at 500°. It is also clear that gold will diffuse into solid lead at the very moderate temperature of 100°, which is 225° below its melting point, a fact which must be considered to be remarkable, and one the existence of which has hitherto been unsuspected. The diffusivity is, however, only $\frac{1}{100,000}$th of that which occurs in fluid lead.

any complication which might arise if the cylinder were heated by fusing gold on to the lead. At the conclusion of the experiment the gold alloy was simply detached by breaking it off the lead cylinder.

* It may be thought that 0·005 per cent. of gold is rather a small quantity to measure with accuracy, but it represents no less than $1\frac{1}{2}$ oz. of gold per ton of lead, an amount of precious metal far in excess of that which assayers are accustomed to deal with in the valuation of auriferous commercial lead.

Welding of Gold and Lead, and Diffusion of Gold in Lead at the Ordinary Temperature.

The fact that two clean surfaces of lead will weld together at the ordinary temperature is well known. It may be well, however, to state in connection with the experiments of DEMARÇAY (see p. 408) on the volatilization of metals *in vacuo* at comparatively low temperatures that if the ends of small bars of gold or silver be surfaced, pressed against lead, and maintained for four days *in vacuo* at a temperature of only 40°, the interpenetration of the two metals will be so complete that their separation can only be effected by the application of a load of 70 lbs. per square inch of the sectional area of the bars, or no less than $\frac{1}{3}$ of the breaking strain of lead.

It remains to be seen whether diffusion can be measured in solid lead at the ordinary temperature, and, with this object in view, cylinders have been prepared and set aside for future examination.

In searching for evidence of diffusion in solid metals at the ordinary temperature, it will be well to examine certain alloys used in art metal-work by the Japanese, who often employ an alloy of copper containing a small proportion of gold (called *Shakudo*), which is soldered or welded in alternate layers with pure copper. The gold in the copper enables it to assume a beautiful purple patina when it is treated with suitable pickling solutions, which leave the pure copper of a red colour. In this way very singular banded effects are produced. Many of the specimens are centuries old, and I have attempted, by grinding away the existing patina and re-pickling the surface, to ascertain whether the widening of the coloured bands would show that, in the course of time, gold had diffused from the *Shakudo* layers and had passed into the copper. I believe that there is evidence that it does do so, but the enquiry is full of difficulty, and needs training in micrography, of which my friend, M. OSMOND, is a master. We propose to study this part of the subject together, and I only allude to it here because, if diffusion occurs in copper, silver, and gold at the ordinary temperature, its results should be revealed in the products of this ancient oriental art.

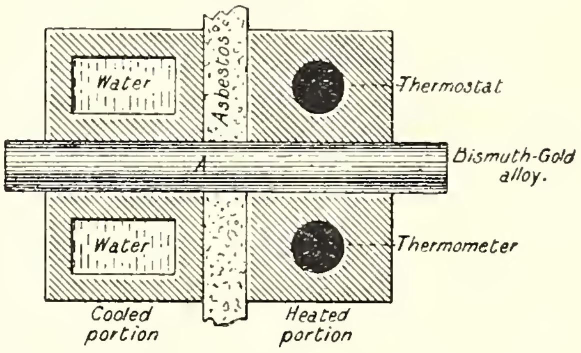
Diffusion of Gold in Solid Silver.

A short cylinder of silver, containing 20 per cent. of gold, was pressed against the carefully-surfaced end of a cylinder of pure silver, 1 centim. in diameter and 2·5 centims. long. These cylinders were kept in an annealing furnace for ten days, at a temperature which never exceeded 800°, and was, therefore, 160° below the melting point of silver (960°). The lowest melting point of the gold-silver series of alloys is 850°. The cylinder was then cut into sections in the usual way, and the amount of diffused gold determined by analysis. As the temperature was intermittent the true diffusivity cannot be taken, but I am satisfied that the diffusivity of solid gold in solid silver at 800° is of the same order as that of gold in lead at 200°. It would appear, therefore, that the melting points of the metals have a dominating influence on the resistance offered to diffusion.

Diffusion from a Hot to a Cold Portion of Solid Metal.

It is well known that, in the liquid diffusion of salt solutions, osmotic pressure will drive molecules from the hot to the cold portions. Professor THORPE suggested that an experiment of this nature should be tried with solid alloys. The principle of "reflection," which would, of course, be involved in such an experiment, has already been alluded to under the liquid diffusion of metals (see Part I., p. 394). The difficulty is to obtain a solid alloy of uniform composition, but, after many experiments, a rod of bismuth was obtained, in which it was believed that 3·75 per cent. of gold was uniformly distributed. This was arranged, as shown in fig. 7, which represents, in sectional plan, a double block of brass, enabling half the rod to be heated

Fig. 7.



to 170°, which is below the melting point of the bismuth-gold *eutectic* alloy, while the other half can be cooled by a stream of water. The heating was maintained for six days. The result proved that there was a distinct concentration of 0·1 per cent. gold at the point A, where the bar entered the cooled chamber. Similar results have been obtained with gold-lead alloys, and, if the bar be maintained at 240°, which is above the melting point of the *eutectic* alloy, the effect is very marked. I offer these statements with some reserve, as they require confirmation.

The data now published form but a small portion of the investigation which has been long in progress. The manipulation it involved was singularly difficult and tedious, and it would probably have been far less advanced than it is, if I had not had the advantage of the aid of one of my former students at the Royal School of Mines, Mr. ALFRED STANSFIELD, B.Sc. He made the calculations which were necessary to extend STEFAN's tables to the range covered by the present experiments, and has shown untiring interest in conducting that part of the series which has been undertaken during the past two years.

The founder of this lectureship directed in 1768 that the subject be selected from "Such part of Natural History or Experimental Philosophy as the Council of the Royal Society shall be pleased to appoint," and in the century which followed the date of Mr. BAKER's bequest, these branches of knowledge seemed to diverge widely. The investigation and measurement of molecular movement has, however, gradually joined them in the closer union which GRAHAM did so much to effect. His work in experimental physics, more than that of any other investigator, taught the physiologist that tracing the relations of the phenomena of life as revealed in diffusion, transpiration, and osmosis will afford Natural History its most precious records.

The evidence gathered by the metallurgist of active atomic movement in fluid and solid metals may sustain the hope of the physiologist that he will ultimately be able to measure the atomic movements upon which vitality and thought depend.

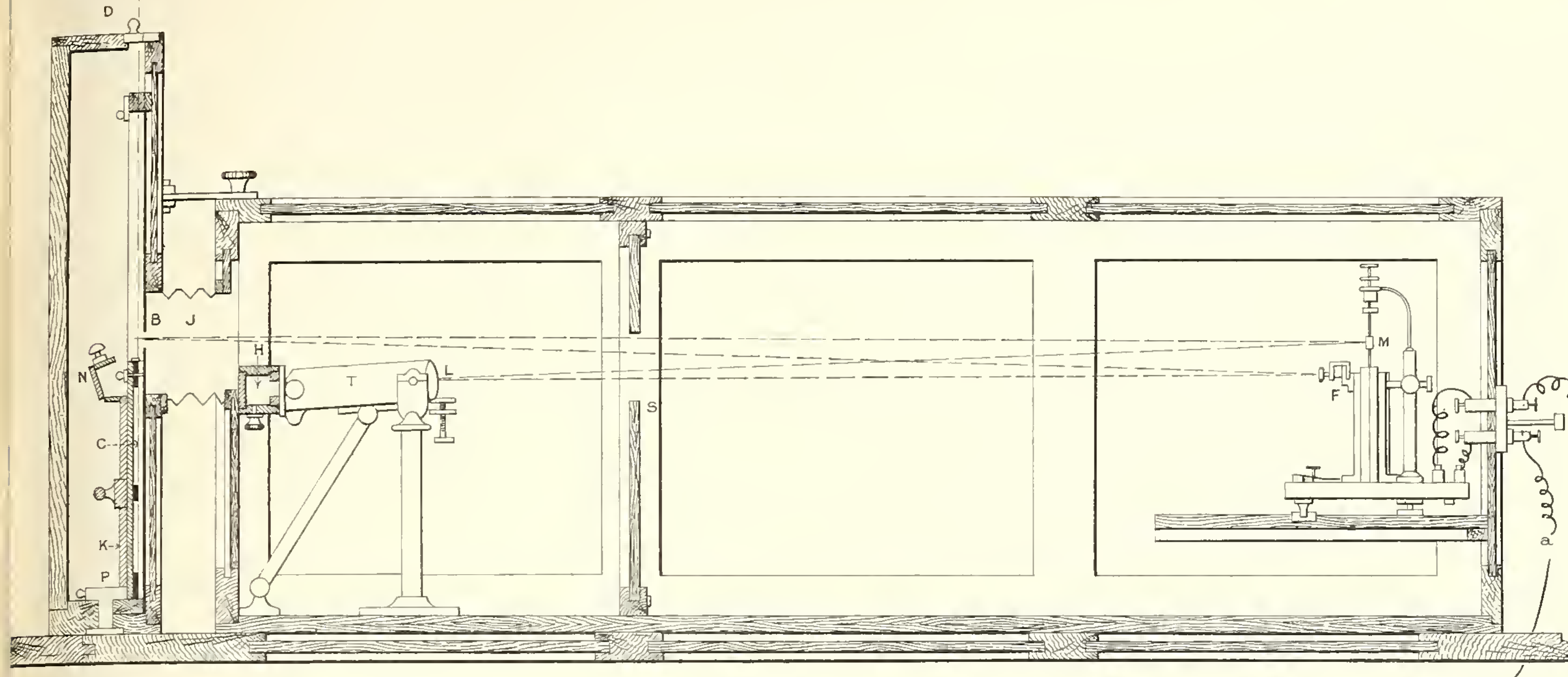


Fig. 1

Scale for Fig. 1.

12 6 0 1 2 3 Feet
10 0 10 20 30 40 50 60 70 80 90 100 Centimetres.

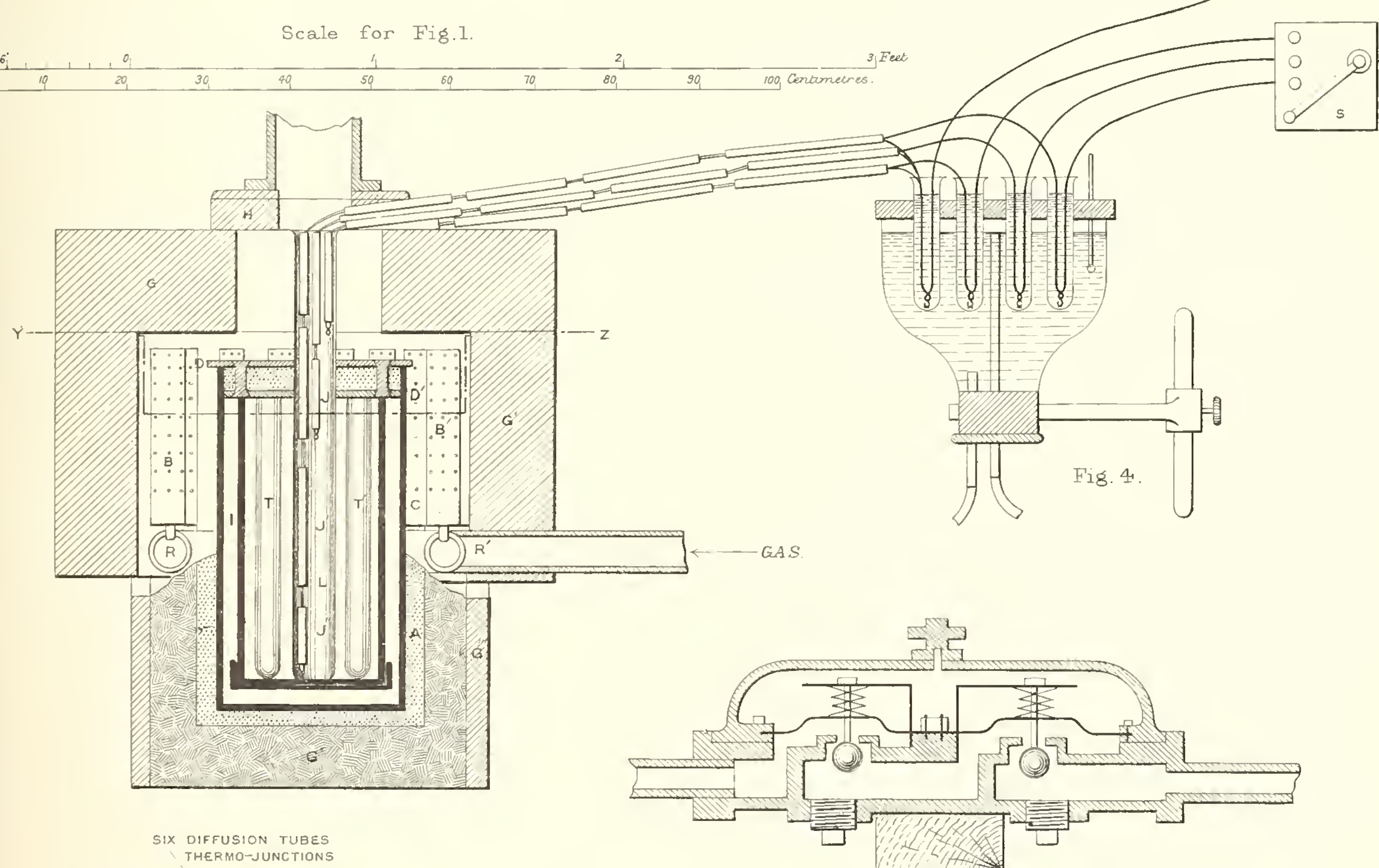
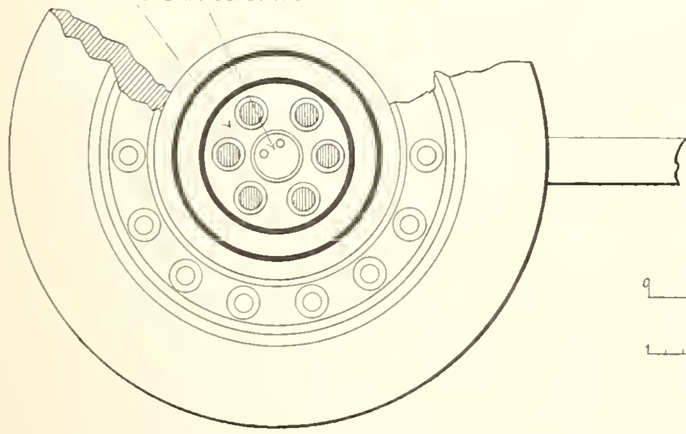


Fig. 4.

Fig. 3.



SECTION ON LINE Y. Z.

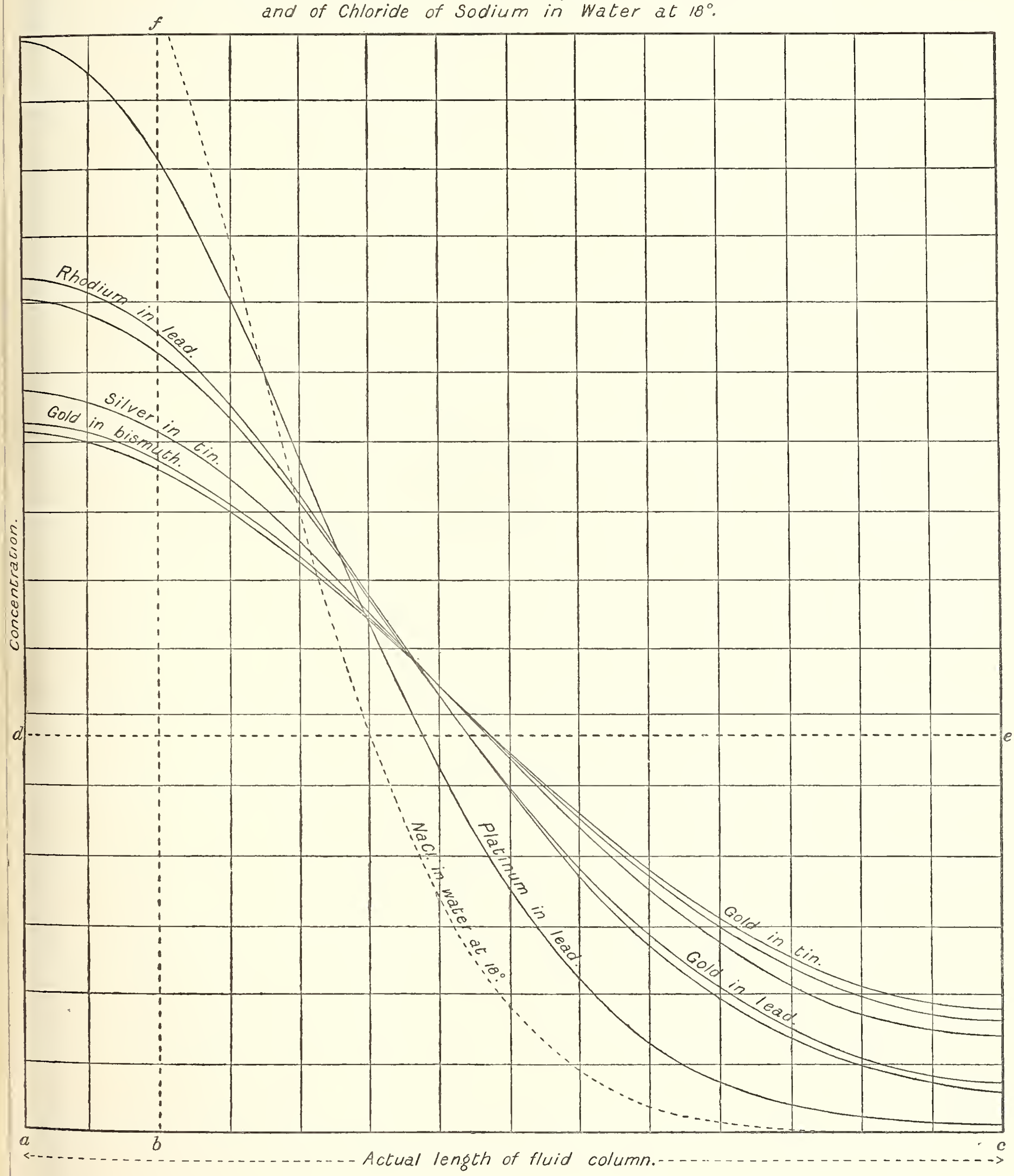
Fig. 2.

Scale for Figs 2.3.4.

0 6 1 Foot
0 10 20 Centimetres

W.C. Roberts-Austen del. West, Newman, lith.

The results of Diffusion for seven days of fluid metals at 500°, and of Chloride of Sodium in Water at 18°.



INDEX SLIP.

XI. *On the Structure of Metals, its Origin and Changes.*

By M. F. Osmond *and Professor* Roberts-Austen, *C.B., F.R.S.*

Received June 10,—Read June 18, 1896.

[Plates 9, 10.]

It has been shown by Herbert Tomlinson that the atomic volume of metals is intimately connected with their thermal capacity* and with Young's modulus.† He considers in view of the work of Wertheim,‡ of Maxwell,§ and of Heen,‖ and as the result of his own experiments, that the value of the product of the elasticity E, when multiplied by a fractional power of the atomic volume, A/D, is a constant for all metals. $E\ (A/D)^{7/3} = 181 \times 10^4$. The divergencies shown by several metals from this mean value arise from the fact that the presence of small amounts of impurity makes a great difference in their elasticity.

Sutherland¶ finds a close relation between the atomic volume and the rigidity of metals, and considers that this rigidity is "in its essence a kinetic phenomenon almost as simple in character as the elasticity of perfect gases."

Professor Fessenden,** moreover, has urged that the cohesion of metals is proportional to some power of the atomic volume, and he considers that the rigidity varies as the fifth power of the distance of the centre of the atoms, or as (atomic volume)$^{5/3}$. These facts are given merely to show that the atomic volume of the added element is very important.

Some years ago, one of us purified gold with great care and alloyed seventeen separate portions of it with foreign elements, in quantities which were in all cases close to 0·2 per cent., and from each sample of this alloyed gold, bars were cast 88 millims. long by 7·5 millims. wide by 5·2 millims. thick. The metal was in each case poured into a closed iron mould heated to about 500°; the cooling was therefore not very rapidly effected. The tensile strength, elongation, and reduction of sectional

* 'Roy. Soc. Proc.,' vol. 38 (1884–85), p. 488.
† 'Roy. Soc. Phil. Trans.,' 1883, p. 32.
‡ 'Ann. de Chim. et de Phys.,' vol. 12, 1844.
§ 'Roy. Soc., Phil. Trans.,' vol. 126, 1866.
‖ 'Bull. de l'Acad. Royale de Belgique,' vol. 4 (1882).
¶ 'Phil. Mag.,' vol. 32, 1891, p. 41.
** 'Chem. News,' vol. 66, 1892, p. 206.

area (striction) were determined, and the results given in the following table were published* in the Philosophical Transactions in 1888.

Name of added element.	Tensile strength.	Elongation, per cent. (on 3 inches.	Impurity, per cent.	Atomic volume of impurity.	Reduction of area at fracture, per cent.
	Tons per sq. in.				
Potassium . .	Less than 0·5	Not perceptible	Less than 0·2	45·1	Nil.
Bismuth . . .	0·5 (about)	,,	0·210	20·9	,,
Tellurium. . .	3·88	,,	0·186	20·5	,,
Lead	4·17	4·9	0·240	18·0	Very slight.
Thallium . . .	6·21	8·6	0·193	17·2	15
Tin	6·21	12·3	0·196	16·2	Not measured.
Antimony . .	6·0 (about)	qy.	0·203	17·9	54
Cadmium . . .	6·88	44·0	0·202	12·9	See note †
None. Pure gold	7·00	30·8	*none*	—	,,
Silver	7·10	33·3	0·200	10·1	,,
Palladium . .	7·10	32·6	0·205	9·4	75
Zinc	7·54	28·4	0·205	9·1	74
Rhodium . . .	7·76	25·0	0·21 (about)	8·4	See note †
Manganese . .	7·99	29·7	0·207	6·8	60
Indium . . .	7·99	26·5	0·290	15·3	72
Copper . . .	8·22	43·5	0·193	7·0	See note †
Lithium . . .	8·87	21·0	0·201	11·8	60
Aluminium . .	8·87	25·5	0·186	10·45	46

These results indicated, in a general way, that the tenacity and ductility of gold is increased by the presence of 0·2 per cent. of an added element of smaller atomic volume than that of gold itself, while on the other hand, these properties are diminished when the atomic volume of the added element is greater than that of gold. There are, as might be expected, exceptions and irregularities, but it is strange that they are not more numerous and more marked. The weight of the added element is in all cases close to 0·2 per cent., but the atomic percentage differs widely. It will be interesting to ascertain in a future research, what is the effect of adding to gold equal numbers of atoms of foreign elements.

The investigations which have been conducted in later years have revealed the complexities of the question. Even the purest metals are not, from a mechanical point of view, homogeneous. Under the influence of internal forces which tend to make them crystalline, and of external stresses which are set up by contraction during cooling, the invisible molecules become arranged in visible and more or less highly organised groups. These groups are separated from each other either by planes of cleavage or by joints which are often surfaces of least cohesion, and therefore of weakness. This is especially the case when these joints have been accentuated by the evolution of dissolved gas at the moment of the solidification of the metal.

In alloys, chemical homogeneity may, in turn, disappear, and free metals, chemical compounds, or various alloys, may fall out of solution from the liquid mass and finally

* 'Roy. Soc. Phil. Trans.,' vol. 179, 1888, A, p. 339.

† These test-pieces drew out after the manner of pitch, that is, as a viscous solid.

the *eutectic* alloy solidifies, but the presence of a residual fluid facilitates the arrangement of the parts which have previously solidified.

One of us, in collaboration with M. WERTH,* was probably the first to direct attention to the influence which these fusible residues, to which the name of "cements" was given, could exert on the working of steel and on the mechanical properties of the finished products. Since then, M. ANDRÉ LE CHATELIER† has repeatedly insisted on this point, correctly enough as a principle, though perhaps with a tendency to generalise too much from ideas which are in themselves accurate. The Reports of the Alloys Research Committee, organised by the Institution of Mechanical Engineers,‡ have, by the aid of autographic curves of the cooling of alloys, brought into prominence a certain number of instances of liquation in the cases of copper and silver, copper and bismuth, gold and aluminium, copper and tin.

Micrographs also reveal the existence of numerous constituents in a great number of alloys; in an alloy of 78 per cent. of gold with 22 of aluminium for instance, grains of a definite compound $Au\ Al_2$, of a brilliant purple colour, are separated by a fine network of a white alloy of very different composition,§ and numerous other analogous examples are to be found in the work of BEHRENS,|| of GUILLEMIN,¶ of CHARPY,** and others.

In short, we are led to distinguish in metals and alloys both the *visible* structure and the *molecular* structure, and, between these, such methods of investigation as are possible, enable a well-defined line of demarcation to be traced. Attention must therefore be directed to ascertaining to what extent the mechanical properties of a given sample of metal are due to each of these kinds of structure, and how far to such relations as are possible between them. This being the case, we considered that it would be interesting to submit the gold, containing 0·2 per cent. of various elements, to micrographical examination, and, fortunately, the identical specimens which were submitted by one of us to the Royal Society eight years ago, had been preserved intact, and were available for examination.

Gold is a metal which may readily be purified to a high degree. It does not oxidize in air at any temperature. The influence of occluded gases appears to be small; and the quantity of the added element is in each case so small as to favour the view that, at least, most of the impurity remains dissolved throughout the mass without there being liquation either of definite or indefinite compounds. We have, in fact,

* OSMOND and WERTH, "Ann. des Mines," vol. 8, 1885, p. 5.

† "Proc. Inst. Mech. Engineers," April, 1893, p. 191.

‡ *Ibid.*, October, 1891, p. 543; April, 1893, p. 102; April, 1895, p. 238.

§ This observation was not printed, but a diagram of the section was shown at a lecture delivered at the Royal Institution, 1891, and has been continuously used since by one of us in class teaching.

|| "Das Mikroskopische Gefüge der Metalle und Legierungen," Leipzig, Voss, 1894.

¶ "Commission des méthodes d'essai des matériaux de construction," T. II., p. 19.

** "Bull. de la Soc. d'Encouragement," February, 1896.

reason to think that many disturbing causes are, if not eliminated, at least reduced to a minimum in this series of alloys. It will be possible up to a certain point to apportion the effect of these disturbing causes, and eventually to set aside such complications as tend to conceal the effect of atomic volume in the researches to which reference has been made.

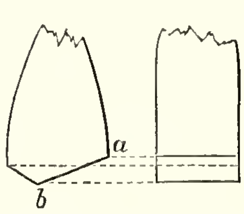
With the exception of some facts stated by BEHRENS, and of some early experiments of our own, on the alloys of gold and aluminium, to which we have just referred, Professor ARNOLD was the first to examine the alloys of gold. He took the experiments of one of us on the influence of impurities on the mechanical properties of gold as the basis of his work,* and made some careful drawings of etched sections of pure gold alloyed with about 0·2 per cent. of various impurities. We consider, however, that photographs of micro-sections are far preferable to drawings, and we agree with M. CHARPY,† in thinking that drawings, however careful, give a very incomplete idea of the appearance of etched metallic surfaces. We have therefore great pleasure in submitting to the Society micro-photographs from the identical specimens of gold described by one of us in 1888, which formed the starting point for the later investigations. The results of the micrographic examination of these specimens of gold forms the first part of the present paper.

PART I.

The alloys examined may be conveniently represented by the chemical symbol of the metal added to the gold, which was in each case in the proportion of 0·2 per cent. They are the twelve following: K, Bi, Zr, Rh, In, Li, Se, Zn, Pd, Te, Sb, Al. Thus "K" would signify gold containing 0·2 per cent. of potassium and so on. It will also be convenient to supplement this series by a sample of pure gold which was cooled under precisely the same conditions as the "test pieces" into which the alloys submitted to mechanical tests were cast.

These test pieces had been divided by two chisel cuts close to the part which had been restricted by traction, so that the portions which were preserved resembled the form shown in fig. 1. The structure would have been examined on a section at right angles to the axis of the bar, but this method of procedure proved to be inconvenient, as it did not make the most of the precious metal, and it would have notably diminished the dimensions of the fragments, which were already too small to be conveniently handled. It was necessary, therefore, to be content with rubbing the face *a b* on emery paper until it became flat, hence the sections are oblique in relation to the axis of the bar. We should have preferred to examine bars which had not been distorted by traction, but it was important to use the old specimens which were available.

Fig. 1.



* 'Engineering,' vol. 61, 1896, p. 176.

† 'Bull. Soc. d'Encourage,' vol. 1, 1896, p. 200.

Polishing.—Even the finest emery paper tears the metal and produces relatively deep scratches. These are effaced by the use of *Brillant Belge*, a commercial product of which the exact composition is unknown, though it is widely used by jewellers, as well as for domestic use in cleaning copper. This *Brillant Belge* is spread on a sheepskin covering a revolving plate; it, however, leaves scratches. The polishing is continued with English rouge of the best quality, prepared by the calcination of pure oxalate of iron, as prescribed by M. Henrivaux. This rouge is spread dry in small quantities on chamois leather covering, as in the case of the *Brillant Belge*, the surface of a revolving disc. In order to polish successfully, a certain lightness of hand is necessary, otherwise the rouge scratches and soils the surface of the gold; with a little practice, however, a polish which appears very beautiful to the naked eye may be obtained, but it is nevertheless imperfect. This may be seen from photographs 15 and 16, where the principal mass of the metal is not etched or hardly so. There are a few large, purely accidental scratches which give no trouble, while the surface remains finely striated. We have not succeeded in doing better by this treatment; gold is very soft and it spreads without receiving a polish when such powders are employed as do not actually scratch it. It is, moreover, very difficult to prepare powders in a sufficiently pure state to exclude all traces of gritty constituents, but this relative imperfection of the polishing is in some respects advantageous and affords a method of investigation which has been successfully employed in other circumstances. If the impurities in the little section under examination have become concentrated in compounds which have liquated from the mass, these are generally harder than the pure gold, and their presence will probably be revealed by a more mirror-like polish. Further, polishing on a soft substance like chamois leather, will cause these liquated portions to appear in relief. Finally their colour in certain cases differs more or less from that of gold, and enables them to be distinguished from the mass.

We have insisted upon the indications furnished by polishing, as they are often very useful in metallography, by enabling a constituent to be defined. The sunk lines, which may be the result of a general attack by etching (or attacking the section with a reagent) are in themselves by no means characteristic of the presence of a free constituent in the mass. In default of the additional evidence afforded by polishing, they probably represent lines of minimum compactness, along which the etching is deeper, or even, according to Commander Hartmann, lines of tension.*

In fact, polishing revealed some cavities which were relatively important in gold alloyed with bismuth, thallium, or potassium. In this last case, and especially in the alloy with indium, we observed grey filaments, which were possibly due to the incorporation of traces of rouge in a network of fissures, although they might also be attributed to the presence of a cement, and in all cases they deserved attention.

Partial Attack.—It is known that certain alloys of gold, especially those with iron,

* 'Distribution des déformations dans les métaux soumis à des efforts;' Paris, Berger-Levrault, 1896.

zinc, copper, lead, silver, &c., are attacked by certain acids which do not attack the gold itself.* This fact led us to hope that the use of suitable reagents would afford useful data. Pure nitric acid (spec. grav. 1·33) was without action in the case of twelve of the specimens which were subjected to its action for five minutes at a temperature close to its boiling point, but it developed, on the section of the alloy with indium, a network of either dotted or continuous lines, to which we shall again refer after we have described the results of a general attack by *aqua regia.* This observation confirms the indications afforded by polishing, and puts beyond question the existence of a cement in the alloy of gold with indium, and this alloy has not been submitted to the action of other reagents which attack it partially (see photo. 18).

If the alloys were kept boiling in pure hydrochloric acid for five minutes, no result was produced. In a preliminary trial, where the specimens had not been previously perfectly cleaned, the rouge adhering to their surfaces was converted into ferric chloride, and all the sections were slightly tarnished. When these conditions prevailed only the alloy of potassium showed a black polygonal network *in intaglio.*

Sulphuric acid (spec. grav. 1·84) is supposed to be without action upon gold. On heating our alloys, however, for five minutes, followed by a treatment for ten minutes, at a temperature at which white fumes were evolved (200–250°), the acid became tinted and the specimens were slightly attacked.† Their polish was, nevertheless not materially diminished, but the greater part of them showed a network of fine lines of nearly uniform width, and without special coloration (see Plate 10, photo. 16).

As these lines presented exactly the same characters in pure gold and in the alloys, it is evident that they do not afford an indication of the presence of a cement. They are joints of contiguous grains. But we will examine this question of joints later in detail. For the moment we would only note the following exception. In the case of the alloy with potassium, instead of a network of sunk joints, the network seemed to be in slight relief corresponding to that which had already been revealed by the action of impure hydrochloric acid. We have, therefore, three indications which taken separately are slender; but they are concordant, and point to the probability of the existence of a cement in the alloy in question.

General Attack by Reagents.—The polished surfaces were immersed three times in *aqua regia* (half hydrochloric, half nitric acid), the temperature being a little below boiling point, until effervescence began, that is to say, after about seven seconds each time.‡ Micrographic examination afforded the following results:—

* 'L'or,' par CUMENGE et FUCHS; Paris, DUNOD, pp. 96–102; 'Encyclopédie chimique de Frémy.' H. LOUIS, 'Trans. of the Amer. Inst. of Min. Engineers,' vol. 24, p. 705 (1894).

† The absence of nitrous derivatives was verified by testing by ferrous sulphate.

‡ Other attempts have been made to attack the alloys attached to the positive pole of a weak battery, by a dilute solution of cyanide of potassium (3·5 of the solid salt in 100 of water), but this reagent taught nothing more than *aqua regia* and was abandoned.

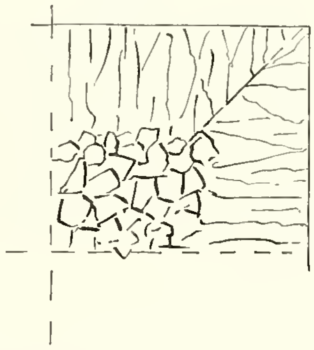
1. First a general structure, visible under small magnification (twenty diameters for instance), and sometimes even to the naked eye. This structure is that of ingots of steel, and, probably, also of most metals which have been poured in ingot moulds.

The mass subdivides itself into jointed groups, the intimate nature of which we will consider later, but the groups may be distinguished from each other either by a different general orientation of elementary crystallization, or by being more or less energetically attacked (see Plate 9, photos. 1, 2, 3, 4, 5, 6). It follows from this that the same group may appear either brilliant or dark, golden, brick-red, or deep purple, according to whether it reflects more or less light into the objective under a definite mode of illumination. The photographs 1 and 6, for instance, represent the same micro-section lighted obliquely, but from opposite directions. The general orientation is, however, subject in a given group to local changes (photo. 12), but the limits of the subdivisions are then ill defined.

If we eliminate the complications introduced into our observations by the obliquity of the sections, and by the deformation of the test pieces caused by traction, it will be seen (photos. 1, 2, 3, 4, 5) that the groups are of two kinds, and resemble those which CHERNOFF found in ingots of steel.

Those at the periphery are prismatic and at right angles to the surface of cooling; while those in the interior are, roughly speaking, equi-axial, so that a transverse section of a quarter of an ingot may be diagrammatically represented by fig. 2. The genesis of this double structure has already been explained by one of us in a paper on the "Cellular Theory as applied to Metals."*

Fig. 2.



The absolute dimensions of the groups, and the relative importance of prisms and equi-axial polyhedra, vary in ingots of steel with the rapidity of cooling, with the temperature of the metal at the moment of pouring, with the amount of occluded gas and with chemical composition. We have detected in our alloys of gold analogous variations. From this point of view, the thirteen specimens may be divided into five classes, of which each is represented by a photograph in which the micro-section is enlarged to seventeen diameters and illuminated obliquely by SORBY's parabolic mirror.

First, the prisms are of considerable size and occupy the whole section, even to the centre (Au, Bi, Zr, Rh, Zn, Pd, Tl; see photos. 1 and 6).

Second, the prisms do not occupy all the mass, their mean size is smaller than that of the metals in the first category, and a central core is occupied by polyhedra, which are roughly equi-axial (Li, Se; see photo. 2 of the alloy with lithium).

Third, the prisms stop about midway between the periphery and the longitudinal axis of the section, while the central core becomes of considerable importance (In, Sb, see photo. 3 of alloy with indium).

* 'Ann. des Mines,' 8th Series, vol. 8, pp. 5–84.

Fourth, the border of prisms become restricted and the polyhedra dominant; both prisms and polyhedra being of small size (K; see photo. 5).

Fifth, the distinction between the groups is very slight (Al; see photo. 4). This is only true for rather more than half of the section, the rest of it would be more fittingly placed in the second category.

The question arises—are the differences which we have signalised caused solely by the difference in the composition of the alloys, or can they to some extent be attributed to variations in the experimental conditions under which the bars were prepared, such variations, for instance, as those which occurred in the temperature of the metal at the moment of pouring? We cannot as yet tell. It is, however, certain that there is no relation between either the structure of the mass, the appearance of the fractures, the melting point of the alloyed elements, and the mechanical properties given in the table on page 418.

Every iron metallurgist who sees our photographs will probably think, at first sight, that the alloy with potassium (photo. 5) will possess the best mechanical properties and that the structure indicated by photograph 1 is deplorable.

The fact is that from a mechanical point of view the alloy of gold with potassium is the worst of all the series, and photograph 1 might equally well represent alloys which vary in tenacity from less than half a ton per square inch to 7·75 tons per square inch, and either are incapable of being stretched or may be elongated 32·6 per cent.

There remains to be considered the groupings (polyhedral or prismatic), from the triple point of view of their external form, their internal structure, and their mutual relations in the neighbourhood of each other.

II. Hitherto we have advisedly employed the vague term "groups" to designate the patches which are differentiated on the etched section by variations of colour and lustre.

This is because we are uncertain as to the true nature of these groups, and it appeared useful to preserve a distinction which is ordinarily ignored between crystals which are defined by natural characteristic inclinations, and the *pseudo* crystals which have been arrested in a more or less haphazard way in consequence of the independent growth of neighbouring groups.* The latter groups have already been called "cells" or "grains," but the word grains appears in this case to be more suitable, in view of the fact that interposed foreign matter is usually absent.

Microscopical examination, with a moderate enlargement of 100 to 300 diameters, enabled us to see the parts respectively played in our alloys by crystals and by grains. Pure gold when alloyed with 0·2 per cent. of bismuth, zirconium, rhodium, zinc, palladium or thallium, is entirely formed of grains. The gold alloyed with lithium or selenium only contains grains in the prismatic envelope, while in the interior the grains have a tendency to pass to crystallites, and the result is mixed jointing, with

* The substance of the mass may, however, be crystalline.

complicated sutures analogous to those in cranial bones. This tendency is still more marked in the alloys with indium and antimony, especially in the interior, but it is visible in the prisms at the periphery (photo. 14). The alloy with potassium is a curious mixture of little grains, of perfectly defined crystallites, and of mixed forms (photo. 11). The appearance of certain portions of the alloy with aluminium approaches that of the alloys with lithium and selenium, but the bulk is rather crystalline.

III. The interior paste, whether it be in grains or crystallites, is certainly crystalline, as might have been anticipated from the fact that the colour of the grains varies with the incidence of the light. When examined, after etching, under an enlargement of 1000 diameters, this paste presents three aspects :—

(*a*) Clear vermicular structures on a dark ground.
(*b*) Parallel grooves.
(*c*) Pointed crystals of the same orientation, in lines nearly parallel.

The photographs 21, 9, and 10 show these different aspects, which, it must be remembered, do not correspond with true differences of structure, but with different sections of a single structure. As far as it is possible to judge, it is a question of files of little crystals penetrating each other, the general orientation of which remains constant in the area of each grain.

In this paste, resulting from primary and rapid crystallization, which constitutes the principal mass of the greater part of our alloys, some small secondary crystals separate themselves with relative distinctness. These are probably small cubes, the diameter of which is about 2·5 μ ; they have a tendency to range themselves in groups and pass progressively into crystallites by a series of intermediate stages, which it is easy to follow through the whole series of the specimens, and thus to connect one with another.

During the course of this secondary crystallization five stages or classes can be distinguished, and each can be designated by a number :

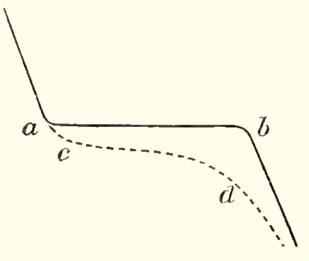
0. Absence of little crystals (photo. 10, pure gold ; 1000 diameters).
1. Small detached crystals isolated indiscriminately.
2. Small crystals in rows at regular intervals (photo. 14, in which the magnification is 150 diameters, and 17, 500 diameters).
3. Small connected crystals, often joined in parallel lines, in which they may lose their individuality, and are illustrated by the same photos. as in Class 2.
4. The parallel lines thus formed become aggregated crystallites (photo. 13, Al, 150 diameters).

None of the specimens belong exclusively to a single class, as one part of a section represents a certain class and the rest of it another, but in a given section the secondary crystallization attains certain limits which it does not exceed, and among these five classes our thirteen specimens are distributed as follows :—

Between Classes 0 and 1 : pure gold, alloyed with Bi, Zr, Rh, Pd, Zn.
" " 0 " 2 : alloys with Tl, Sb, Se.
" " 0 " 3 : " " K, In.
" " 1 " 4 : " " Li, Al. A large portion of the alloy with aluminium belongs entirely to type 4.

This more or less accentuated development of a secondary crystallization is, in itself, interesting in connection with the phenomena observed by one of us* in the curves which represent the solidification of gold (fig. 3). In the presence of small quantities of lead, or of certain other bodies, the horizontal portion of the curve, which represents the actual solidification of the metal, is not sharp at its angles as shown in the line *a*, *b*, but is rounded as at *c*, *d*, and is often inclined at an angle to the datum line. This points to the existence of a transitory pasty state which may facilitate the secondary crystallization.†

Fig. 3.



It may be asked whether the small crystals above-mentioned, and their aggregates do not represent an allotropic form of gold.‡ Some explanation may thus be afforded of the mechanical properties of the alloys of gold with 0·2 per cent. of aluminium, and of gold with 0·2 per cent. of lithium, which are, at the same time, the most tenacious and the most perfectly-crystallized of the entire series. But, on the other hand, this same kind of crystallization connects the alloy of gold and bismuth, for instance, with the alloy of gold and rhodium, these being the members of our series which are very wide apart from the point of view of tenacity and extensibility.

It must be admitted, moreover, that the influence of the secondary crystallization on the mechanical properties of the alloys, supposing it exists, is at least greatly complicated by some other more powerful influence.

IV. It only remains for us to consider the relations which result from the association of contiguous grains or crystallites, that is to say, from their surfaces in contact, or joints. The general crystalline orientation of the paste on both borders of the joint is not the same, and the elementary crystals belonging to two different groups may not be as firmly united as they would be in the interior of a homogeneous group; a joint, therefore, becomes a surface of weakness (sometimes passing progressively into a fissure), which etching by a reagent enlarges and renders visible. In fact, all our etched surfaces are divided by a network of joints, which correspond with the outlines of the grains and crystallites.

The joints revealed by the action of sulphuric acid are simple furrows, which occur

* 'Proc. Inst. Mech. Engineers,' Oct. 1891, p. 558–564.

† Secondary crystallization is at a minimum in the case of pure gold.

‡ Compare the notes of H. Louis on "The Allotropism of Gold," and "Further Experiments on Amorphous Gold," 'Trans. Amer. Inst. of Mining Engineers,' vol. 24, pp. 182 and 705, 1894.

in all our specimens, except in the alloy with potassium.* When revealed by the action of *aqua regia*, these joints usually appear under the microscope as being of a brick-red or deep purple colour, and are of variable thickness. In certain cases they are only mathematical lines, that is to say, without thickness, and their existence is only revealed by the different coloration of the grains of which they form the limit (see photos. 12, Rh, 150 diameters; and 9, Rh, 1000 diameters). In other cases, the joints revealed by etching have a definite thickness (see photos. 15, Tl; 14, In; 11, K; 150 diameters). These joints, which are more or less thick, when carefully examined under high magnification, may be represented by two types. One of these is formed of a sunk line, generally very fine, on the edges of which crystals, apparently little cubes, are ranged, which are secondary crystallizations. These are sometimes isolated, but more usually run together, and are joined in continuous files (see photo. 17, In, 500 diameters). Other joints are simply constituted by the separation for a short distance of primary crystals, between which the acid can penetrate more easily and etch away a deeper channel. We will call the latter, as distinguished from the former, *non-crystalline.*†

Considered from the point of view of the nature and the thickness of their joints,‡ our thirteen specimens may be divided into five groups.

1st. Pure gold; alloys with Zr, Al, Rh, Pd, Zn; joints which are usually non-crystalline, of a thickness which equals 0 to 2 μ ($\mu = 0{\cdot}001$ mm.), the mathematical joints or very fine ones are dominant.

2nd. Alloy with lithium; the thickness of the joints equals 0 to 3·5 μ; the thick joints are crystalline but are rare and disconnected, the mathematical joints, or very fine joints are dominant.

3rd. Alloys with Sb, Se; crystalline joints which are often broken; the thickness of these equals 1 μ to 3·5 μ, or a mean thickness of 2 μ.

4th. The alloys with In, K; crystalline continuous joints, with a thickness of 1 to 4 μ, or a mean thickness of 2 to 2·5 μ.

5th. Alloys with Tl, Bi; crystalline joints, which are continuous, often passing into

* The alloy of gold and indium has not been etched by sulphuric acid.

† It may be well to record an appearance which is sometimes met with, and may lead to error. Certain thick non-crystalline joints simulate the presence of an interposed cement, for, on account of the difference of level between the depth of the joint and the surface, shadows are cast. This is most clearly shown by the photograph 10 (1000 diameters). The plane etched by *aqua regia* had not been repolished after the previous etching by sulphuric acid. It looks as if a film of foreign substance was interposed between two neighbouring grains, but, on looking at it closer, it is evident that the traces of this film join those of one of the grains, and it is not possible to admit a doubt that we have merely to deal with the structure of pure gold.

‡ By the expression "thickness of a joint," we understand, in the case of the crystalline joints, the sum of the joint properly so-called, and of the crystalline border, which it is usually impossible to separate from it. We have, in fact, measured in all cases the thickness of the dark line. It must not be forgotten that it is a question of the apparent dimensions, which are much exaggerated by the etching.

fissures, the thickness equals 1 μ to 5 μ, or a mean thickness of 2·5 μ. Generally speaking the condition of these joints appears to be closely related to the mechanical properties of the alloy (table, p. 418.).

Conclusions.—We do not contest in any way, as our previous publications abundantly prove, the importance of the part which may be played in the mechanical properties of the alloys by the residues which remain liquid after the main mass of the alloy has solidified, the alloys being tested either at the ordinary temperature or when heated. But, in order that it may be possible for such cements to intervene and affect the mechanical properties of alloys, the cements must at least have a real existence. Nothing indicates that they do exist in ten out of twelve of our alloys, but we would not even express ourselves too positively on this point, for some new method of etching may reveal new facts. The impurities which are sought for may happen to concentrate themselves beyond the particular region which has been sectioned. These are, however, for the present gratuitous suppositions. Polishing only indicates the presence of cement in two cases. The little secondary crystals which we have already described might readily be mistaken for cements of definite or indefinite composition if they were found only in certain specimens and then in such proportions as could be accepted. But we meet with them everywhere and in all cases their appearance is constant in forms and dimensions, and moreover we see them collect into crystallites which pervade the whole mass. These crystals are therefore usually and indubitably due to the crystallization of gold itself, although the alloying substances sometimes (indium and probably potassium) join up the crystals in question. For the same reason the dark lines of the joints, traced as furrows by the etching, are very rarely the empty tracks of cement which has been dissolved away by *aqua regia*; their formation, which it is easy to follow in all its phases, directly connects them with secondary crystallization. We are led to the belief that in the case of ten of our alloys of gold with about 0·2 per cent. of various impurities, solidification of the whole mass, although prolonged and less rapid than in the case of pure gold, has been directly accomplished without interruption, and that the foreign bodies have remained partly or wholly as *solidified* solutions, the impurities being dissociated into their atoms in both solid and liquid. We can at least say, without going beyond the actual evidence before us, that the dissemination of the foreign bodies eludes the power of the methods of investigation which we have employed. Under these conditions it is difficult to invoke, as explaining the mechanical properties of the alloy, the intervention of hypothetical cements with relatively low fusing points.

The absolute dimensions of the grains or crystallites cannot, as we have already seen, account for the mechanical properties. The micrographic examination of the copper-zinc alloys (brasses) has recently led M. CHARPY* to a similar conclusion, and it is now certain that the large size of the grains does not, in itself, constitute a

* *Loc. cit.* He also shows that ordinary brasses may be effectively annealed at 500°.

defective or an undesirable form of structure. If the metallurgy of steel points to a different conclusion, it is probably owing to the presence of gases which are often abundant and are apt to accumulate between the grains; these gases are the less divided, and consequently are the more dangerous, when the grains are large and the total surface of the joints is small for a unit of volume. Our attention has been specially directed to this point of M. Werth, who justly attaches great importance to the question of the presence of gas in steel.

The more or less advanced state of the crystallization compared with the results of the tests for tensile strength does not show any direct relation.

Finally, micrographical examination only leaves us the joints themselves to account for the observed mechanical properties. Here the concordance is fairly good.* Without wishing to attribute a degree of precision which does not belong to them to the dimensions of joints which have been widened by etching, and perhaps modified to some extent by the longitudinal stress to which the bars were subjected, we can readily see, in a general way, that the thick and crystalline joints correspond to the alloys of low tenacity, while the converse is also true. But if the mechanical properties are in direct relation to the thickness of the joints and to the atomic volume of the alloyed elements, we may fairly conclude that a relation of cause to effect should exist between these two last variables.

This conclusion appears rather unforeseen. If we seek its interpretation, we are face to face with a very important but very complex question, that of the genesis of the joints in the metal.

A joint is often a surface of weakness, as we have already stated, in that it alone marks a sharp change of organization and constitutes the artificial border of two natural groups. The more it diverges from a mathematical surface and acquires sensible thickness the weaker it becomes. How, then, is a joint formed? Evidently by an internal stress, if occluded gas, which appears negligible in the case of gold, be left out of the question. Such tractional stress is produced during solidification, cooling and consequent shrinkage (and sometimes experimental proof of it is found in the presence of scraps of metal attached to the edges of a fissure). In order that a joint may be opened it is necessary (and it is enough) that the joint should be subjected, at a given moment, to a greater weight than its breaking strain, and that this breaking strain is less than the elastic limit of the metal in the interior of the grains.

We have then four factors at work:

1. The stresses established by the change of volume of the metal when it solidifies.

2. The stresses established by shrinkage, which themselves depend (the conditions of cooling remaining constant) on the coefficient of expansion of the metal.

3. The elastic limit and the "*deformability*" of the metal which forms the body of the grains.

* The alloy with indium is an exception, but we have verified the presence of cement which has had, in this case, a favourable influence on the mechanical qualities.

4. The strength of the joint, which is *nil* towards the end of the solidification, and remains insignificant for a longer time if the solidification is prolonged by the presence of impurity in the way indicated in fig. 3. The presence of a crystalline envelop covering the joints, which appears to be an unfavourable condition, as it is specially marked in the alloys of low tenacity, is probably also connected with the prolongation or want of sharpness in solidification occasioned by the presence of certain foreign bodies. These envelopes without doubt represent the parts which have solidified last.

But all the factors we have just reviewed co-operate directly in determining the condition of the joints, and indirectly the mechanical properties of the cooled metal. The co-efficient of expansion, the elastic limit and plasticity of the bodies of the grains and the delayed solidification) are directly dependent on the molecules and atoms. It is therefore not surprising that in a case which is relatively so simple as that of gold the influence of the atomic volume of the added elements can be directly observed. Furthermore, the relation between the mechanical properties of the metals and the atomic volume of the dissolved impurities is an experimental fact. We can now explain the greater part of the apparent exceptions. The exceptional behaviour of indium is explained by the presence of a cement, and that of lithium and of aluminium by the development of secondary crystallization.* But as the conversion of heat into work is effected in the steam engine by a series of intermediate stages, so here also a more or less complex mechanism intervenes between the extreme terms which are in relation to each other. The foregoing research is a small contribution to the study of one of the elements of this mechanism.

Part II.—On some Phenomena of Annealing.

Etching by sulphuric acid at a temperature of between 200° to 250° revealed a network of joints on the gold alloyed with bismuth and thallium, which the previous attack with *aqua regia* had not rendered apparent. All the specimens (except the alloy with indium which had not been subjected to the action of sulphuric acid) were then repolished and again etched with *aqua regia* under the same conditions as before.

The pure gold and the alloys with K, Pd, Zn, Rh, Li, Se, and Zr did not undergo any change, but the appearance of the four alloys with Bi, Tl, Sb, and Al were more or less altered by the treatment.

The bismuth alloy, which was initially adequately represented by the zirconium-gold, Photograph No. 6, for a magnification of 17 diameters, and by the thallium-gold, Photograph No. 15, which shows details (magnified 150 diameters), is now represented by Photographs No. 7 (17 diameters) and No. 20 (150 diameters). The large grains of the melted metal become sub-divided, after heating for five minutes to between 200° to 250°, into a number of small polyhedral grains. The effect is just

* Zinconium still presents a very striking exception.

the same as is caused by annealing steel castings at a temperature of about 800°. Nothing remains of the original structures. In the new grains the paste, the secondary crystallization, and the joints present the same characteristics as in the old grains.

The alloy with thallium undergoes a strictly analogous transformation, but the dominant lines of the initial structure have been preserved in several places, notwithstanding the internal rearrangement (photo. 19, 150 diameters). The alloy with antimony behaved in a very different fashion; this alloy, which was adequately represented before annealing at 200° by the gold-indium, photograph 3 (17 diameters) and by 14 for the details (150 diameters), is now represented by the photographs 8 (17 diameters) and 22 (150 diameters). The original structure has disappeared, but the new organization shows neither the polyhedral grains nor the continuous network of joints; the incipient grains, indicated by the crystalline orientations, have not clean faces, and the sulphuric acid only traces fragments of broken joints. This structure recalls that of hardened steel of medium hardness.

The alloy with aluminium undergoes the same transformation, but only locally and partially. Generally speaking, it seems that bismuth, thallium, antimony, and aluminium, when present in the proportion of about 0·2 per cent., behave in respect to gold in the same way as carbon does with regard to steel, but at a much lower temperature.

It also appears to be evident that the bodies in question must have been present in the solid metal in a state closely resembling the fluid. None of these bodies possess a very high melting point, and, as is natural, this circumstance favours the maintenance of fluid molecules at a low temperature. The temperature, however, of annealing remains much below that of melting aluminium or antimony, and even below the melting points of the *eutectic* alloy of aluminium and gold (about 600°) or of antimony and gold (440°), and it at most reaches that of the fusion of bismuth. On the other hand, lithium, with a point of fusion below that of bismuth, and zinc with a fusing point below that of antimony, have not exerted a similar effect in lowering the temperature of annealing.* The melting point of the impurities, although it is not without influence, is not the sole factor to be considered. It should be observed that lead, judging from what is known of its action on gold, probably behaves like bismuth and thallium. Amalgamation also appears to be a phenomenon of the same kind, possibly occurring, owing to the liquidity of mercury, at a still lower temperature. But gold, mercury, thallium, lead, and bismuth follow each other in the classification of elements based on increasing atomic weight, and are grouped on the same horizontal line of MENDELEEF'S table. This coincidence is curious. Whatever it may signify, this transformation of the structure of a metal, at a temperature so far below its melting point, in the presence of only two-tenths per cent. of a foreign body, is probably not an isolated fact, and appears to open a new field for research.

* We are not speaking of potassium, which appears to be concentrated in a cement.

Description of Plates 9 and 10.

Number of micro-section.	Description of specimen.		Reagent by which the section was etched.	Method of illumination.	Enlargement, linear.
	Gold alloyed with 0·2 per cent. of	Treatment of the alloy.			
1	Zirconium	Poured in a mould .	Aqua regia . .	Oblique .	17
2	Lithium.	,, ,,	,, ,, . .	,, .	17
3	Indium	,, ,,	,, ,, . .	,, .	17
4	Aluminium	,, ,,	,, ,, . .	,, .	17
5	Potassium	,, ,,	,, ,, . .	,, .	17
6	Zirconium	,, ,,	,, ,, . .	,, .	17
7	Bismuth	Annealed at about 200° to 250°	,, ,, . .	,, .	17
8	Antimony	,, ,,	,, ,, . .	,, .	17
9	Rhodium	Poured in a mould .	,, ,, . .	Normal .	1000
10	Nothing (pure gold) .	,, ,,	,, ,, . .	,, .	1000
11	Potassium	,, ,,	,, ,, . .	,, .	150
12	Rhodium	,, ,,	,, ,, . .	,, .	150
13	Aluminium	,, ,,	,, ,, . .	,, .	150
14	Indium	,, ,,	,, ,, . .	,, .	150
15	Thallium	,, ,,	,, ,, . .	,, .	150
16	Nothing (pure gold) .	,, ,,	Sulphuric acid .	,, .	150
17	Indium	,, ,,	Aqua regia . .	,, .	500
18	Indium	,, ,,	Nitric acid . .	,, .	150
19	Thallium	Annealed at about 200° to 250°	Aqua regia . .	,, .	150
20	Bismuth	,, ,,	,, ,, . .	,, .	150
21	Bismuth	,, ,,	,, ,, . .	,, .	1000
22	Antimony	,, ,,	,, ,, . .	,, .	150

Notes.

Photograph No. 1 would represent equally well either gold alloyed with Bi, Rh, Zn, Pd, Tl, or pure gold prepared under the same conditions.

Photograph No. 2 would also represent gold alloyed with Se, and poured in a mould; while No. 3 would represent gold alloyed with antimony.

Photograph No. 11 shows a mixture of grains and crystallites, a secondary crystallization belonging to classes 0 to 3 with thick crystalline joints.

Photograph No. 12 presents a characteristic case in which crystallites are absent; there is no secondary crystallization, and there are very fine non-crystalline joints.

Photograph No. 13 shows the complete development of secondary crystallization, with variable and disconnected joints.

Photographs Nos. 14 and 17 show the intermediate stages of secondary crystallization, the joints being thick and crystalline.

Photograph No. 15 shows a very thick joint crossing a paste of primary crystallization.

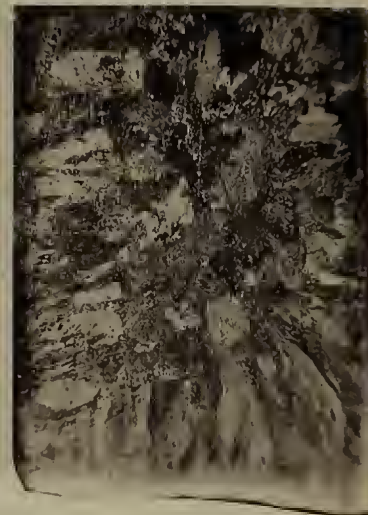
Photograph No. 16 shows the general aspect of joints, which would be traced by sulphuric acid on most of the specimens.

Neither these photographs (11 to 17) nor photographs 1 to 6, 9, 10, and 21 are characteristic only of the alloys from which they were taken; they have been chosen as typical illustrations of the organization of primary crystallization, of the variable development of joints, and, in fact, of the general phenomena which have been described in the text.

Photographs 7, 8, 19, 20, and 22 are of a more special nature; they show the effects of annealing at a given temperature on a given alloy.

1

2



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6

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Phototypie Berthaud, Paris.

NOTE.—The sections, 1 to 22, are all from bars which were cast, in heated moulds, and fractured by longitudinal stress. In all cases, except Nos. 16, 18, they were etched with *aqua regia*. In some cases (Nos. 7, 8, 19–22) the sections were also annealed by heating them to a temperature between 200° and 250° C.

No. 1. Gold alloyed with 0·2 °/$_{\circ}$ of Zirconium. × 17 diam. Illuminated obliquely.

No. 2. Gold alloyed with 0·2 °/$_{\circ}$ of Lithium. × 17 diam. Illuminated obliquely.

No. 3. Gold alloyed with 0·2 °/$_{\circ}$ of Indium. × 17 diam. Illuminated obliquely.

No. 4. Gold alloyed with 0·2 °/$_{\circ}$ of Aluminium. × 17 diam. Illuminated obliquely.

No. 5. Gold alloyed with 0·2 °/$_{\circ}$ of Potassium. × 17 diam. Illuminated obliquely.

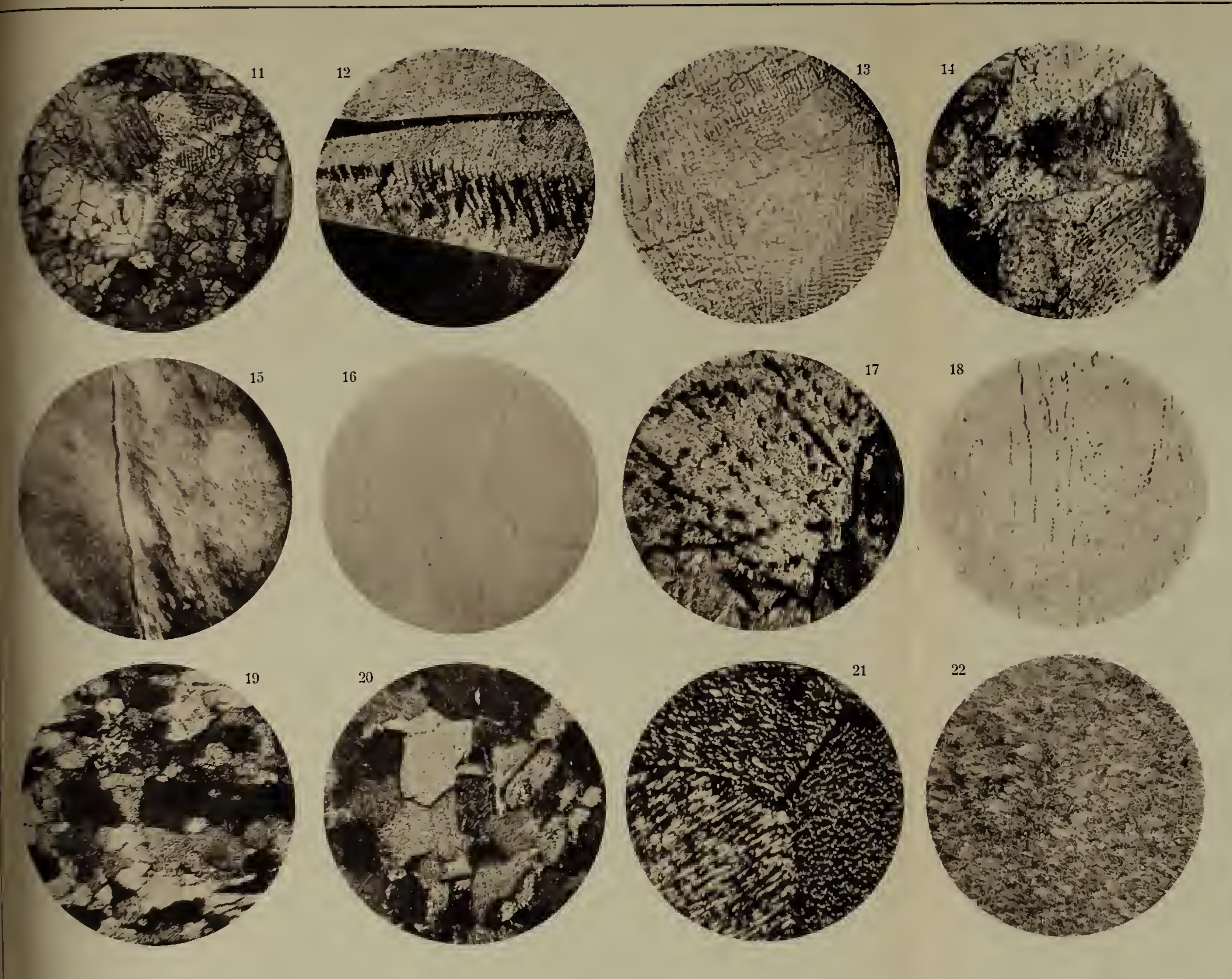
No. 6. Same as No. 1, but illuminated from the opposite direction.

No. 7. Gold alloyed with 0·2 °/$_{\circ}$ of Bismuth. Annealed. × 17 diam. Illuminated obliquely.

No. 8. Gold alloyed with 0·2 °/$_{\circ}$ of Antimony. Annealed. × 17 diam. Illuminated obliquely.

No. 9. Gold alloyed with 0·2 °/$_{\circ}$ of Rhodium. × 1000 diam. Illuminated perpendicularly.

No. 10. Pure Gold. × 1000 diam. Illuminated perpendicularly.



Phototypie Berthaud, Paris.

No. 11. Gold alloyed with 0·2 °/ₒ of Potassium. × 150 diam. Illuminated perpendicularly.

No. 12. Gold alloyed with 0·2 °/ₒ of Rhodium. × 150 diam. Illuminated perpendicularly.

No. 13. Gold alloyed with 0·2 °/ₒ of Aluminium. × 150 diam. Illuminated perpendicularly.

No. 14. Gold alloyed with 0·2 °/ₒ of Indium. × 150 diam. Illuminated perpendicularly.

No. 15. Gold alloyed with 0·2 °/ₒ of Thallium. × 150 diam. Illuminated perpendicularly.

No. 16. Pure Gold. Etched with sulphuric acid. × 150 diam. Illuminated perpendicularly.

No. 17. Gold alloyed with 0·2 °/ₒ of Indium. × 500 diam. Illuminated perpendicularly.

No. 18. Gold alloyed with 0·2 °/ₒ of Indium. Etched with nitric acid. × 150 diam. Illuminated perpendicularly.

No. 19. Gold alloyed with 0·2 °/ₒ of Thallium. Annealed. × 150 diam. Illuminated perpendicularly.

No. 20. Gold alloyed with 0·2 °/ₒ of Bismuth. Annealed. × 150 diam. Illuminated perpendicularly.

No. 21. Gold alloyed with 0·2 °/ₒ of Bismuth. Annealed. × 1000 diam. Illuminated perpendicularly.

No. 22. Gold alloyed with 0·2 °/ₒ of Antimony. Annealed. × 150 diam. Illuminated perpendicularly.

INDEX SLIP.

XII. *On the Determination of the Photometric Intensity of the Coronal Light during the Solar Eclipse of April 16th,* 1893.

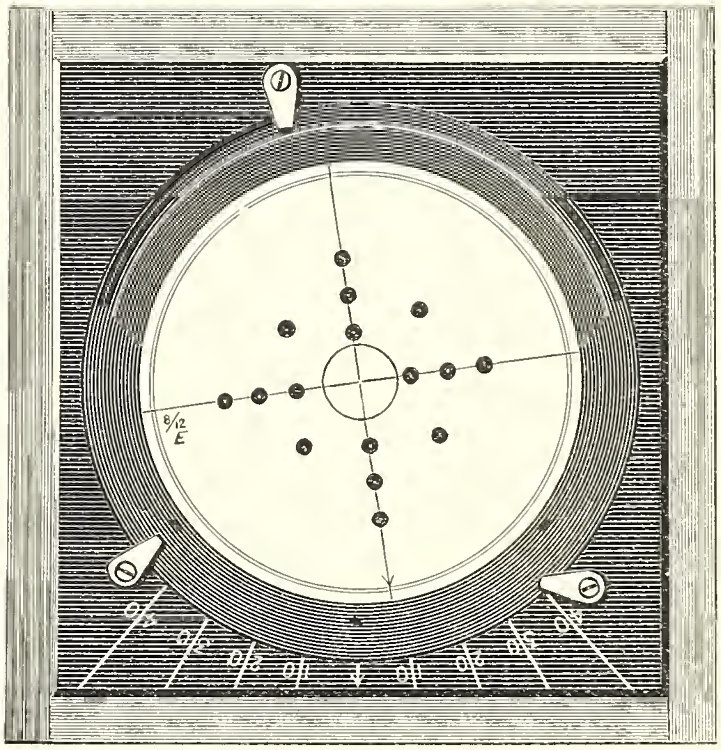
By Captain W. DE W. ABNEY, *C.B., R.E., F.R.S., and* T. E. THORPE, *LL.D., F.R.S.*

Received April 14,—Read April 30, 1896.

IN the Introduction to our paper "On the Determination of the Photometric Intensity of the Coronal Light during the Solar Eclipse of August 28th, 1886," which the Society did us the honour to print in the 'Philosophical Transactions' (A, 1889, p. 363), we gave an account of the attempts which had been made from time to time since the eclipse of December 22, 1870, the first occasion on which such measurements were made, to ascertain the amount of light emitted by the corona. So far as we know no other attempt of the kind has been made since the date of our last paper. We may therefore at once pass to the description of the methods adopted on the present occasion.

The methods, as well as the instruments, used by us for the measurement of the coronal light during the eclipse of April 16th, 1893, were substantially the same as those employed in Grenada during the eclipse of August 28th, 1886, with certain modifications suggested by our experience on that occasion. For an account of the principle of these methods, as well as for the description of the instruments themselves, we may refer to the paper above cited. It will suffice here to say that one instrument was designed to measure the comparative brightness of the corona at different distances from the moon's limb, whilst a second was arranged to measure the total brightness of the corona, excluding as far as possible the sky effect. The first instrument, from the mode in which it was constructed, will be called the equatorial photometer; the second will be termed the integrating photometer. In both cases the principle of photometry adopted was that of BUNSEN, the intensity of the coronal light being compared with that of a glow-lamp, according to the method of ABNEY and FESTING ('Phil. Trans.,' 1886, 'Proc. Roy. Soc.,' 1887, 43). In the case of the equatorial photometer, a telescope by SIMMS, lent by the Astronomer-Royal, was employed. It had an aperture of 6 inches and the object-glass had a focal length of 78 inches, forming an image of the moon 0·76 inch in diameter. The image was received on a circular white screen contained in the photometer-box and placed in the focus of the object-glass. In the centre of the screen was traced a circle of the diameter of the image of the moon, and during the observation the moon's disc was made to fall exactly within the circle. As the telescope was equatorially mounted with clockwork, the image could be kept stationary within the circle. The screen was of RIVES' paper of medium thickness, and round the pencilled circle a series of small grease-spots about $\frac{1}{8}$ of an inch in diameter were made. For the mode of making the grease-spots and testing the screens we may refer to our original paper. Fig. 1, reproduced from this paper, shows the screen mounted in its circular frame. The screen could be rotated so as to bring the spots into any desired angular position, and it could be removed at pleasure by releasing it from the buttons which held it within the frame. The box to hold the screen in the focus of the telescope and the VARLEY carbon resistance-apparatus used to increase or diminish the light of the standard glow-lamp were identical with those employed during the 1886 eclipse, and are described in our previous paper (*loc. cit.*, pp. 367, *et seq.*).

Fig. 1.



The most important variations in the mode of making the observations, as compared with the previous measurements, consisted in the manner of connecting up the lamp, resistance-apparatus, and galvanometer, and in the method of reading the current-strength. On the former occasion the brightness of the lamp was increased by adding resistance to the shunt; consequently in the measures made the highest readings of the galvanometer corresponded with the lowest intensity of light. On the present occasion the better plan of putting the galvanometer and resistance in the main circuit was employed; hence high readings of the galvanometer mean high intensities of light. The lamps used during the present eclipse were EDISON-SWAN'S 8 volts, 5-candle power. The relation between the current-strength and light intensity expressed in terms of SIEMEN'S amyl-acetate lamp, in the case of the two lamps actually employed, is seen from the following tables.

LAMP C 6. Used with the Equatorial Photometer. Mark 184 D.

Amperes.	Light units.	Amperes.	Light units.	Amperes.	Light units.
0·8	·004	1·00	·030	1·15	·102
0·85	·0065	1·01	·034	1·20	·140
0·9	·0100	1·05	·048	1·25	·193
0·95	·017	1·10	·073	1·30	·262

LAMP C 3. Used with the Integrating Photometer. Mark 185 D.

Amperes.	Light units.	Amperes.	Light units.	Amperes.	Light units.
0·8	·002	1·00	·026	1·20	·093
0·85	·005	1·05	·035	1·25	·125
0·90	·010	1·10	·048	1·30	·167
0·95	·017	1·15	·066		

The readings of current strength in the case of the equatorial photometer were made by means of a Weston ammeter, which is especially convenient for the purpose. In the case of the integrating instrument an Evershed ammeter was used. These were procured for us by Professor AYRTON, and were carefully tested by him in order to ensure that their indications were strictly comparable. He was also good enough to select the glow-lamps for us and to have them standardised in his laboratory. On again standardising them after making use of them for the observations they were found to be practically unchanged. Our best thanks are due to Professor AYRTON for the interest he displayed in our work, and for the great amount of time and trouble he spent on our behalf.

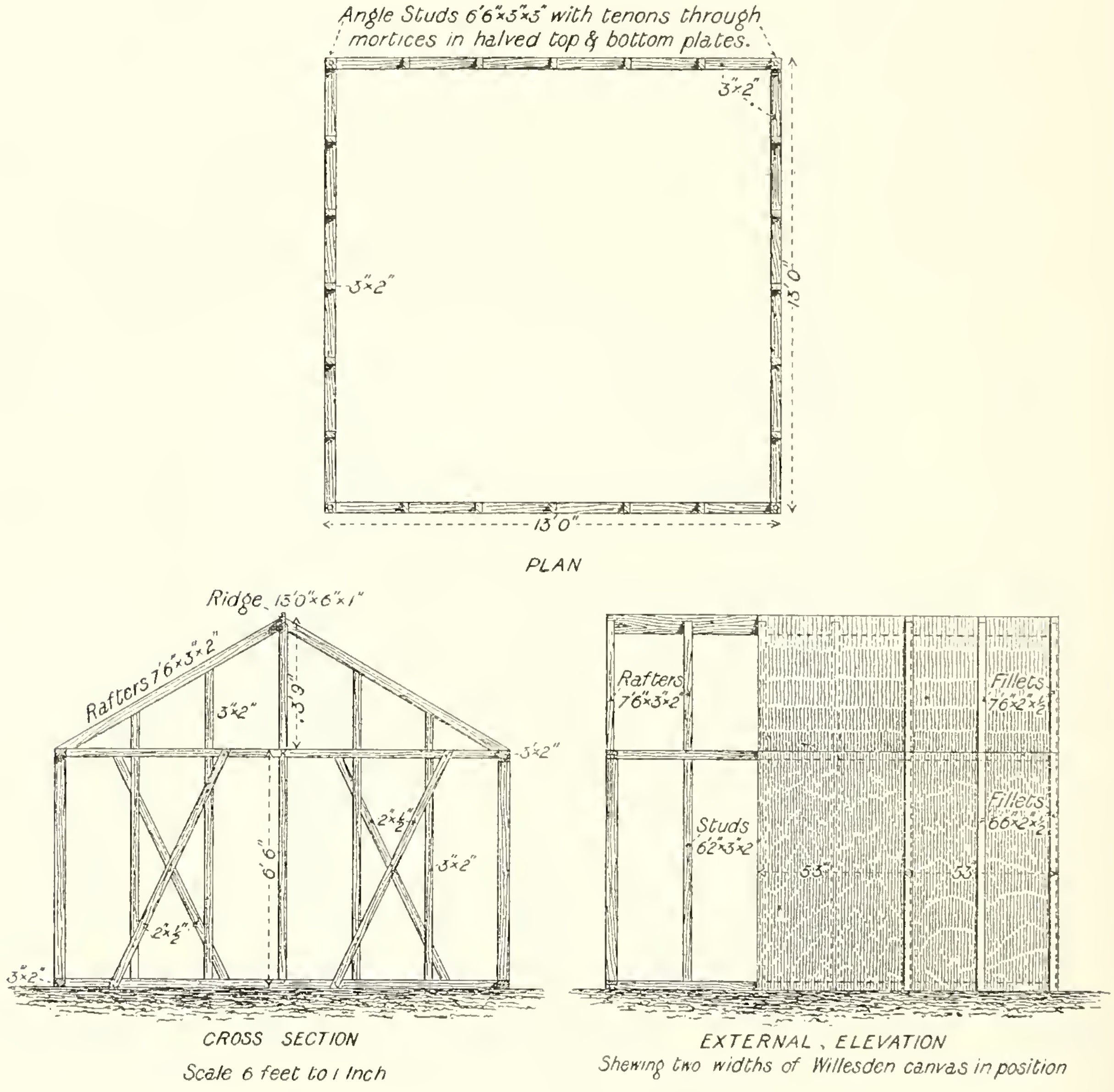
As regards the rest of the work, nature of battery, instructions to observers, &c., the details were identical with those already given in our previous paper.

The Joint Committee representing the Royal Society, the Royal Astronomical Society, and the Solar Physics Committee of the Department of Science and Art, which was charged with the superintendence of the arrangements for observing the Eclipse of April 16, 1893, directed that the photometric measurements of the coronal light should be part of the work of the expedition to be despatched to Senegambia.

Professor THORPE (who was in charge of the expedition) was entrusted with the equatorial photometer measurements, and was assisted by Mr. P. L. GRAY, B.Sc., who undertook the galvanometer readings, whilst the observations with the integrating photometer were to be made by Mr. JAMES FORBES, jun., of the Royal College of Science.

The station actually selected for the African party was Fundium, in Senegal, a small town on the south bank of the Salum river, about thirty miles from the sea and near the central line of the shadow. The observers were sent out from England by mail-steamer to Bathurst, the head-quarters of the British possessions on the Gambia, and thence transferred to H.M.S. *Alecto*, which vessel, by the kindness

Fig. 2.



HUT used by British Eclipse Expedition, 1893.

of the Admiralty, had been told off to assist the expedition. The *Alecto* left Bathurst for the Salum on April 2, and arrived at Fundium on the afternoon of the following day. M. ALLYS, the Administrator of the district, which is in French territory, received the party very courteously, and placed the ground in the rear of his house at their disposal. The best thanks of the expedition are due to the Administrator and to the French authorities for their kindness.

The position offered by M. ALLYS was well adapted to our work. It was enclosed within a stockade, was conveniently near the moorings of the *Alecto*, and sufficiently removed from the huts of the people, who are here mainly Sereres and Wolofs, to secure the necessary privacy. The greater number of the packing cases containing the instruments were landed before nightfall of the day of arrival, the ground being meanwhile measured out, and the positions for the various observing huts of the party assigned. With the help of the men of the *Alecto*, together with a few Sereres who were engaged to dig and to make concrete for the foundations for the equatorials, the erection of the huts and instruments proceeded as rapidly as the excessive heat would permit, and before the end of the week everything was in adjustment, and we had a clear six days before us for the necessary drill and final preparations. The huts used on this occasion were designed, and their construction in England superintended, by Quarter-Master Sergeant KEARNEY, R.E. As they answered their purpose remarkably well, being readily and quickly put together and taken down, and being sufficiently rigid when erected, it may be desirable to give a short account of them here in view of future expeditions. The design is seen in plan, cross-section, and external elevation in fig. 2, which is drawn to a scale of 6 feet to an inch.

The materials required are :—

		feet.
8 pieces white deal scantling	3″ × 2″ × 13′ =	104
2 ,, ,, ,, ,,	3″ × 2″ × 10′ =	20
12 ,, ,, ,, ,,	3″ × 2″ × 7′ 6″ =	90
18 ,, ,, ,, ,,	3″ × 2″ × 6′ 2″ =	111
		325
Allowance for waste in trimming . .		25
		350
4 pieces white deal scantling	3″ × 3″ × 6′ 6″ =	26
Allowance for trimming		2
		28
1 ridge piece 6″ × 1″ × 13′	=	13
Allowance		1
		14
12 pieces for battens 3″ × ½″ × 8′ 6″	=	102
Allowance for trimming		18
		120

Willesden canvas 53″ wide, 46 yards.

The canvas should be previously cut to the sizes required, viz.:—

3 pieces 28″ long.
6 ,, 6′ 6″ long.
2 ,, 6′ 6″ cut diagonally.

The canvas is fastened to the hut by means of fillets (as shown in the drawing). These should be 2″ × $\frac{1}{2}$″, cut into the following lengths: 18 pieces 7′ 6″ and 32 pieces 6′ 6″, making in all 343 feet. The studs and rafters should be fastened by 3-inch French wire nails (say, 3 lbs.), and the fillets over the Willesden canvas by 1-inch French nails (say, 1 lb.); these are easier to drive by unskilled hands than the ordinary kind, and are less liable to split the wood.

The climatic conditions at Fundium were wholly different from those with which we had to contend at Grenada. At the latter place our chief difficulties were due to the frequent rains and constant humidity.

At Fundium, on the other hand, we had excessive dryness; at noon there was often a difference of 20° F. between the wet and dry-bulb thermometers, and with the exception of the evening before the eclipse, there was not even a trace of dew at night. This excessive dryness, combined with the high temperature—it occasionally rose to 110° or 112° in the huts—was very trying to the woodwork. But our main trouble was the dust, which was excessively fine and light, and deposited itself as an impalpable powder over the apparatus, and greatly interfered with the proper running of the clocks. However, by covering the base of the huts with layers of the shells of some variety of *Cardium*, which were found in large numbers near the beach, and by frequently watering the ground, we to some extent kept down the cloud which every foot-fall otherwise raised.

On the afternoon of the 14th the weather changed slightly for the worse. Up to that time the wind had been mainly in the east, and the sky almost unclouded. On that day the wind went round to the west, the temperature fell considerably and there was more cloud and haze in the sky and a certain amount of dew in the evening.

On the 16th, the day of the eclipse, the conditions were slightly better as regards the amount of cloud, but the haze and general opalescence of the air was not less distinct. The morning was bright and clear, but the effects of the westerly winds were to be seen in the milky colour of the sky, and, as the sun rose higher, in the grey appearance of the heavens. During the whole period of the eclipse, however, the sky was cloudless, except for a few thin wispy cirri near the horizon, and although, of course, the temperature fell considerably as the phase of totality approached, there was not the slightest appearance of condensing moisture and there was no increase of cloud although a considerable increase of wind. In this respect the conditions were entirely different from those in the West Indies. At Grenada all the observations show, in the clearest manner, that the results were affected by the precipitation of moisture after the first 60 or 70 seconds of the totality ; dark clouds rapidly gathered in the neighbourhood of the disc, and at about one minute before the calculated end of the total phase the moon and corona were wholly obscured. The air was saturated with moisture ; a slight shower had indeed fallen a few minutes before the beginning of totality, and the lowering of the temperature, consequent on the obscuration of the solar disc, undoubtedly caused the gradual condensation of moisture. Hence, therefore, we could only regard, at most, the first six out of the possible sixteen observations made with the equatorial photometer as valid ; and only three, at the outside, out of the sixteen readings made by the integrating photometer were of any value as measuring the total brightness.

On the present occasion, owing to the much more favourable conditions, the greater number of the measurements made by the equatorial photometer, twenty in all, as well as ten measurements made by Mr. FORBES with the integrating photometer, are available. The results of these measurements are given in the following tables :—Table I. gives the value in light of the readings of the equatorial photometer. The numbers of grease-spots in Column I. correspond with the order in which they were read. Column II. gives the calculated distance of each spot from the sun's centre in terms of the solar semi-diameter. Column III. gives the readings on the ammeter ; and Column IV. the corresponding value of the light in SIEMENS' units at one foot. Column V. gives the approximate time in seconds after the beginning of totality when the readings were made.

TABLE I.—Readings on the Equatorial Photometer reduced to the value of Light-Intensity.

I.	II.	III.	IV.	V.
1	1·66	1·060	·058	10
2	2·70	1·015	·037	16
3	3·80	·954	·018	23
4	1·30	1·070	·061	30·5
5	2·88	1·021	·038	42
6	3·12	·951	·019	48·5
7	1·76	1·186	·137	61
8	2·32	·969	·023	71
9	3·28	·928	·015	79
10	1·70	1·088	·069	92
11	2·82	1·020	·039	101·5
12	3·88	·961	·021	107
13	2·46	·950	·018	121
14	2·46	·920	·013	131
15	2·42	·877	·009	141·5
16	2·64	·950	·018	151
1	1·66	1·110	·084	162
4	1·30	1·060	·058	172
7	1·76	1·170	·123	185
10	1·70	1·030	·042	193·5

Spots 1 and 10 are in the line of the apparent direction of the moon's path across the solar disc ; it will be noticed that on repeating the readings one area has gained as much in intensity as the other has lost, whereas spots 4 and 7 which are at right-angles to the moon's apparent motion, have remained practically unchanged in intensity.

Table II. gives the value of Mr. FORBES's readings with the integrating photometer.

TABLE II.—Readings with the Integrating Photometer reduced to values of Light-Intensity.

No. of readings.	Ammeter reading.	Value of light at 1 foot from screen in SIEMENS' units.	Approximate time of readings from beginning of totality.
			seconds.
1	1·05	0·029	3
2	1·03	0·026	12
3	1·02	0·024	30
4	0·99	0·022	45
5	1·00	0·022	70
6	1·04	0·027	94
7	1·035	0·026	120
8	1·04	0·027	147
9	1·05	0·029	184
10	1·05	0·029	206
Average light = 0·026.			

The measurements with the equatorial photometer clearly show that the visual brightness of the corona of the 1893 eclipse varied within comparatively wide limits, and that, at all events close to the moon's limb, there were marked differences in local intensity. The readings of spots 1, 2, 3, and 10, 11, 12, which are at the opposite sides of the lunar disc are fairly concordant among themselves. Spot 7, which was comparatively close to the limb, was probably affected by the proximity of a prominence; it will be noticed that the readings were repeated at an interval of more than a couple of minutes and are in substantial agreement.

It may here be stated that anything in the nature of personal bias on the part of either observer was impossible from the very manner in which the work had to be done ; Mr. GRAY, of course, could not see the comparisons actually made, nor could Dr. THORPE have any knowledge of the readings of the current strength at the particular moment of comparison; each portion of the joint work was therefore wholly independent. If the several values taken in the direction of the poles and equator are grouped as before, excluding Spot 7, for the reason above given, the averages, when treated as in our former paper, will be found to afford a curve almost identical in character with that already given (*loc. cit.*, p. 380), showing, as formerly stated, that the diminution in intensity from the moon's limb outwards is less rapid than that demanded by the law of inverse squares.

The results are as follows :—

Distances in solar semi-diameters.	Photometric intensity.		
	Observed.		Law of inverse squares.
	1893.	1886.	
1·6	0·060	0·066	0·066
2·0	0·048	0·053	0·042
2·4	0·038	0·043	0·029
2·8	0·030	0·034	0·022
3·2	0·024	0·026	0·016
3·4	0·018	0·021	0·013

These numbers would appear to show that the actual brightness of the corona was probably not very dissimilar at the two eclipses, the slight apparent diminution observed during the 1893 eclipse being, in all probability, due to the haze or opalescence in the air which, as already stated, prevailed at the time. This haze, caused more by suspended and finely-divided solid matter than by precipitated moisture, undoubtedly contributed to the general sky-illumination at the time of totality. The actual gloom during this phase of the eclipse at Fundium was certainly much less than at Grenada in 1886. It must not, however, be forgotten that the altitude of the sun was very different on the two occasions. At Grenada it was only about 19° ; the amount of cloud was from 7 to 8 (overcast = 10) at the time of totality ; and much of it was in the neighbourhood of the sun ; whereas at Fundium the sun's altitude was 52° and the sky was of a bluish-grey colour and practically free from cloud.

The effect of these different conditions in the sky in the neighbourhood of the disc is seen in Mr. FORBES's measurements when compared with those of Lieutenant DOUGLAS at Grenada. The ten fairly concordant observations at Fundium give an average value of 0·026 Siemens unit at one foot from the screen ; the highest and probably the most accurate value observed by Lieutenant DOUGLAS fifteen seconds after totality, with the same photometer, although with a different lamp and galvanometer, was 0·0197.

The observations are not sufficiently precise to enable any valid comparison to be made between the brightness of the corona at the poles and at the equator, nor are they numerous enough to make it worth while to attempt to seek for the law connecting the decrease of brightness of the corona in terms of its distance from the limb.

In conclusion we desire to acknowledge our great indebtedness to Captain LANG and the officers of H.M.S. *Alecto* for their ready assistance and for the zeal and intelligence with which they co-operated in the work of the expedition. The party is also under many obligations to Captain FESTING and the other officers of H.M.S. *Blonde* for their kindness and hospitality on the voyage home from Bathurst to Las Palmas.

[Since this paper was written we have to deplore the death, from fever, at St. Kitts, of Mr. Forbes. He was a young man of great promise, and a most careful and painstaking observer, and his amiable disposition and uniform courtesy endeared him to every member of the expedition.]

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XIII. *On a Type of Spherical Harmonics of Unrestricted Degree, Order, and Argument.*

By E. W. Hobson, *Sc.D., F.R.S.*

Received December 23, 1895,—Read January 16, 1896.

Introduction.

The ordinary system of Spherical Harmonics or Laplace's functions is obtained from Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

by choosing special values of V which satisfy this differential equation, and are of the forms

$$r^n \begin{matrix}\cos \\ \sin\end{matrix} m\phi \,.\, u_n^m(\mu), \quad \text{or} \quad r^{-n-1} \begin{matrix}\cos \\ \sin\end{matrix} m\phi \,.\, u_n^m(\mu),$$

where n and m are real positive integers, x, y, z being expressed in terms of r, μ, ϕ by means of the relations

$$x = r(1-\mu^2)^{\frac{1}{2}} \cos\phi, \quad y = r(1-\mu^2)^{\frac{1}{2}} \sin\phi, \quad z = r\mu\,;$$

the function $u_n^m(\mu)$ is a particular integral of a certain ordinary linear differential equation of the second order, and is known as Legendre's associated function of degree n and order m; these solutions, in which μ is restricted to be real and to lie between the values ± 1, and in which m is restricted to be less than or equal to n, are the solutions of Laplace's equation which are required in the very important class of potential problems in which the boundary of the space considered consists of either one or two complete spheres, or of surfaces which differ only slightly from spheres.

It appears, however, that the functions $\begin{matrix}\cos \\ \sin\end{matrix} m\phi \,.\, u_n^m(\mu)$ are required for the solution of certain potential problems in which the boundaries are of forms other than complete spheres, and in some of these cases the values of n, m, and μ are not subject to the restrictions which hold in the case of the primary potential problems in which the boundaries are complete spheres. In the case in which the boundary is a spheroid or two confocal spheroids, the functions $u_n^m(\mu)$ of both kinds are required, in which, although n and m are still real integers, μ may have values which are real and greater than unity. The functions for which n is fractional or complex are required for the solution of potential problems in which the boundary consists of coaxal circular cones and of spheres with the centre at the vertex of the cones. For potential problems connected with the anchor-ring functions are required for which n is half an odd integer, and μ is greater than unity. For the space bounded by two spherical bowls with a common rim, solutions in which n is complex of the form $-\frac{1}{2}+p\iota$, and μ is greater than unity, have been applied. Solutions in which m is not an integer are sometimes of use, for example, in the potential problem for the portion of an anchor-ring cut off by two planes through the axis of the ring, which are inclined to one another at an angle not a sub-multiple of two right angles.

The expressions

$$r^n \begin{matrix}\cos\\ \sin\end{matrix} m\phi \,.\, u_n^m(\mu), \quad r^{-n-1} \begin{matrix}\cos\\ \sin\end{matrix} m\phi \,.\, u_n^m(\mu),$$

in which $u_n^m(\mu)$ represents any particular integral of the differential equation which it satisfies, and in which the degree n, the order m, and the argument μ may have any real or complex values, are a special type of Spherical Harmonics in the extended sense of the term, which applies to all solutions of LAPLACE'S equation; the investigation of their forms reduces to that of two particular integrals, here denoted by $P_n^m(\mu)$, $Q_n^m(\mu)$, of the differential equation which $u_n^m(\mu)$ satisfies. The forms and properties of the functions required for various potential problems have been investigated by various writers, the investigations resting usually on a more or less independent basis; thus, for example, we possess separate theories of Toroidal functions, Conal functions, &c. It is obviously desirable that all these special functions should be treated as parts of a general theory; thus an investigation of the forms and properties of the two functions $P_n^m(\mu)$, $Q_n^m(\mu)$ for unrestricted values of n, m, μ is required for the consolidation of the various special results which have been obtained in connection with special potential problems. To do this by means of the modern methods applicable to linear differential equations is the object of the present memoir.

In the standard treatise of HEINE, the forms and properties of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$ are investigated for complex values of μ, the degree n and the order m being primarily real and integral; various extensions are made to cases in which n is not so restricted, but in default of a general definition of the functions for unrestricted values of n and m, these extensions are fragmentary, incomplete, and in some cases erroneous. Many of the series which satisfy the differential equation for unrestricted values of the degree and order have been given by THOMSON and TAIT,* and a general treatment of the series has been given by OLBRICHT,† who obtains seventy-two hyper-

* See 'Natural Philosophy,' vol. 1, Part I., Appendix B.

† See OLBRICHT, 'Studien über die Kugel- und Cylinder-functionen,' Halle, 1887.

geometric functions which satisfy the differential equation, at least half of which are convergent at any given point of the μ-plane.

In order that the relations between the various particular integrals in the form of series may be exhibited, it appears to be most convenient to start from integral expressions which satisfy the differential equation; this is the course adopted in the present memoir. A definition of the two functions $P_n^m(\mu)$, $Q_n^m(\mu)$ by means of integrals taken along complex paths, which shall be valid for unrestricted values of the degree and order, has been rendered possible by the introduction independently by JORDAN* and POCHHAMMER† of the use of integrals with double circuits; the use of such integrals has the great advantage over the employment of integrals taken between limits, that the constants have to satisfy no convergency conditions, and thus that the functions may be defined by means of expressions which have a definite meaning for all values of the constants.

In the special case $m = 0$, the zonal functions $P_n(\mu)$, $Q_n(\mu)$ can be completely defined by means of integrals with single circuits; this has been done by SCHLÄFLI,‡ who bases his theory of the series which represent these functions upon such definitions.

In the first part of the present memoir the two functions $P_n^m(\mu)$, $Q_n^m(\mu)$ are defined by means of integrals in such a manner that the functions are uniform over the whole μ-plane, which, however, has a cross-cut extending along the real axis from the point $\mu = 1$ to $\mu = -\infty$; these definitions are so chosen that in the ordinary case of real integral values of n and m, the functions coincide with the well-known functions used in ordinary Spherical Harmonic Analysis; from these definitions various series are obtained which represent the functions in various domains of the μ-plane. Special conventions are made as to the meaning to be attached to the functions at points in the cross-cut. Various other integral expressions are obtained which would serve as alternative definitions of the functions. It is shown that all the known definite integral expressions for the functions in restricted cases due to LAPLACE, DIRICHLET, HEINE, and MEHLER are special cases of the more general formulæ. In the latter part of the memoir various definite integral formulæ are deduced for cases in which the degree and order are subject to special restrictions. In conclusion, the forms of the functions required for the potential problems connected with the ring, the cone, and the bowl are deduced from the general formulæ; in particular, convergent series are obtained for the tesseral toroidal functions.

As much confusion is caused by the variety of notation used by different writers, it is convenient to state here for purposes of comparison the relations between the symbols used in the most important works on the subject; for this purpose the

* See 'Cours d'Analyse,' vol. 3.

† See various papers in volumes 35 and 36 of the 'Mathematische Annalen.'

‡ See a tract "Ueber die beiden Heine'schen Kugelfunctionen." Bern, 1881.

ordinary case of integral values of n and m is the only one which has to be considered.

HEINE uses the symbols $P_m^{(n)}(\mu)$, $\mathfrak{P}_m^{(n)}(\mu)$, $Q_m^{(n)}(\mu)$, $\mathfrak{Q}_m^{(n)}(\mu)$, which are connected with the symbols $P_n^m(\mu)$, $Q_n^m(\mu)$ used in this memoir by the relations

$$P_m^{(n)}(\mu) = P_{-m}^{(n)}(\mu) = (\mu^2 - 1)^{-\frac{1}{2}m}\mathfrak{P}_m^{(n)}(\mu) = \frac{1 . 2 . 3 \ldots n - m}{1 . 3 \ldots 2n - 1} P_n^m(\mu)$$

$$Q_m^{(n)}(\mu) = Q_{-m}^{(n)}(\mu) = (\mu^2 - 1)^{-\frac{1}{2}m}\mathfrak{Q}_m^{(n)}(\mu) = (-1)^m \frac{1 . 3 \ldots 2n + 1}{1 . 2 . 3 \ldots n + m} Q_n^m(\mu).$$

THOMSON and TAIT use the symbols $\Theta_n^{(m)}(\mu)$, $\vartheta_n^{(m)}(\mu)$, which are connected with HEINE'S $P_m^{(n)}(\mu)$ by the relations

$$(-1)^{\frac{1}{2}m} P_m^{(n)}(\mu) = \Theta_n^{(m)}(\mu) = \frac{2^{n-m} . n!\,(n-m)!}{(2n)!\; m!} \vartheta_n^{(m)}(\mu).$$

FERRERS uses $T_n^m(\mu)$ for what is denoted here by $(-1)^{\frac{1}{2}m} P_n^m(\mu)$, except in the case of a real μ lying between ± 1, in which case $T_n^m(\mu)$ and $P_n^m(\mu)$ are identical.

The Gaussian function $\Pi(x)$, which is equivalent to $\Gamma(x+1)$, is used throughout the memoir.

Definition of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$ by means of definite integrals.

1. If, in the differential equation

$$(1-\mu^2)\frac{d^2V}{d\mu^2} - 2\mu\frac{dV}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\} V = 0 \quad . \quad . \quad . \quad . \quad (1),$$

which is satisfied by LEGENDRE'S associated functions, we substitute $V = (\mu^2 - 1)^{\frac{1}{2}m} W$, then W satisfies the differential equation

$$(1-\mu^2)\frac{d^2W}{d\mu^2} - 2(m+1)\mu\frac{dW}{d\mu} + (n-m)(n+m+1)W = 0 \quad . \quad . \quad (2).$$

If, in the expression on the left-hand side of (2), we substitute

$$W = \int (t^2-1)^n (t-\mu)^{-n-m-1}\, dt,$$

we find

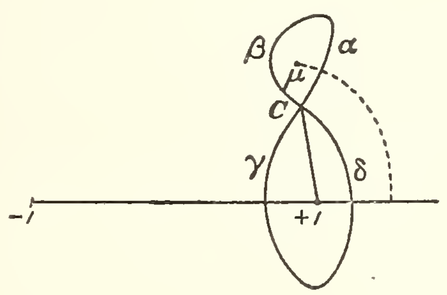
$$\left\{(1-\mu^2)\frac{d^2}{d\mu^2} - 2(m+1)\mu\frac{d}{d\mu} + (n-m)(n+m+1)\right\}\int (t^2-1)^n (t-\mu)^{-n-m-1}\, dt$$
$$= -(n+m+1)\int \frac{d}{dt}\{(t^2-1)^{n+1}(t-\mu)^{-n-m-2}\}\, dt.$$

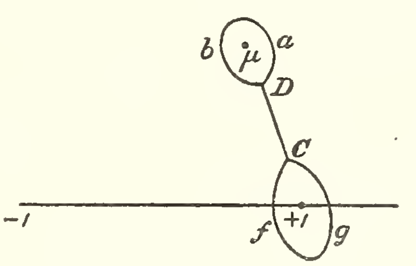
It appears thus that the differential equation (2), is satisfied by

$$W = \int (t^2 - 1)^n (t - \mu)^{-n-m-1} \, dt,$$

for unrestricted values of n and m, provided the integral is taken along a closed path, *i.e.*, one such that the integrand $(t^2 - 1)^n (t - \mu)^{-n-m-1}$ attains the same value when the path has been completely described, as that with which it commenced. The integrand has, in general, the four singular points $t = +1$, $t = -1$, $t = \mu$, $t = \infty$, and we shall see that it is possible to choose two distinct closed paths, defined with reference to these singular points, which will represent the values of W required for the two LEGENDRE'S associated functions.

2.





If the variable t, starting from a point C, describes a path in which a positive (counter-clockwise) turn is made round the point μ, then a positive turn round the point 1, then a negative turn round μ, and lastly a negative turn round 1, such a path will be closed, *i.e.*, the integrand $(t^2 - 1)^n (t - \mu)^{-n-m-1}$ will have the same value at C at the beginning and at the end of the path. In the first figure the path will be (C$\alpha\beta$C, C$\gamma\delta$C, C$\beta\alpha$C, C$\delta\gamma$C); in the second figure it will be (CD, DabD, DC, CfgC, CD, DbaD, DC, CgfC). In POCHHAMMER'S notation, the value of V will be

$$V = (\mu^2 - 1)^{\frac{1}{2}m} \int_c^{(\mu+,\, 1+,\, \mu-,\, 1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1} \, dt,$$

which will satisfy the equation (1); it is necessary to specify precisely the values of the multiple valued functions in the integral, in order that the integral may have a definite value.

First, to define the meaning of $(\mu^2 - 1)^{\frac{1}{2}m}$, let $\mu - 1 = re^{\iota\theta}$, $\mu + 1 = r'e^{\iota\theta'}$, and suppose μ to have moved from a point in the real axis for which $\mu > 1$, along any path up to its actual position; we shall suppose that $\theta = 0$, $\theta' = 0$, when μ is in the real axis and greater than unity, the value of $(\mu^2 - 1)^{\frac{1}{2}m}$ at any point will then be $(rr')^{\frac{1}{2}m} \left\{ \cos \frac{m}{2} (\theta + \theta') + \iota \sin \frac{m}{2} (\theta + \theta') \right\}$, where θ, θ' are the angles the lines joining μ and 1, μ and -1 make with the real axis; θ and θ must be restricted each to lie between $\pm\pi$, in order that a single value may be assigned to $(\mu^2-1)^{\frac{1}{2}m}$; by $(rr')^{\frac{1}{2}m}$ is denoted $e^{\frac{1}{2}m\log(rr')}$ where $\log(rr')$ has its real value; the value of $(\mu^2-1)^{\frac{1}{2}m}$ has thus been uniquely specified for all values of μ, except those which are real and lie between $+1$ and $-\infty$. Next, in $(t^2-1)^n=(t-1)^n(t+1)^n$, we shall suppose the phase of $t-1$ to commence with the value ϕ at C, where ϕ is the angle (between $\pm\pi$) the line joining C to $+1$ makes with the positive direction of the real axis; the phase of $t+1$ at C we shall suppose to be ϕ', where ϕ' is the angle (between $\pm\pi$) the line joining C to -1 makes with the positive direction of the real axis; if at C, $t-1=ke^{\iota\phi}$, $t+1=k'e^{\iota\phi'}$, the value of $(t^2-1)^n$ will be $e^{n\log(kk')}\,.\,e^{n\iota(\phi+\phi')}$ where $\log(kk')$ has its real positive value; after the positive turn round 1, $(t^2-1)^n$ will have become $e^{n\log(kk')}\,.\,e^{n\iota(2\pi+\phi+\phi')}$.

The phase of $t-\mu$ we shall choose to be such that it is zero when t passes through that point of the path for which $t-\mu$ is a positive real quantity, thus the initial value of $t-\mu$ at C is $\rho e^{-(\pi-\psi)\iota}$, where ψ is the angle (between $\pm\pi$) which the line Cμ makes with the positive direction of the real axis, hence $(t-\mu)^{-(n+m+1)}$ changes from $\rho^{-(n+m+1)}e^{(\pi-\psi)(n+m+1)\iota}$ to $\rho^{-(n+m+1)}e^{-(n+m+1)(\pi+\psi)\iota}$, in going from C round the point μ to C again, $\rho^{-(n+m+1)}$ denoting $e^{-(n+m+1)\log_e\rho}$, where $\log_e\rho$ has its real positive value.

3. Let us now consider the value of

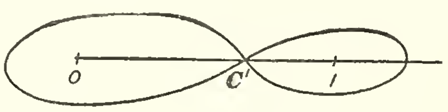
$$c_n^m(\mu^2-1)^{\frac{1}{2}m}\int_c^{(\mu+,1+,\mu-,1-)}\frac{1}{2^n}\frac{(t^2-1)^n}{(t-\mu)^{n+m+1}}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

with the specifications of the phases just given, in the case in which μ is such that mod. $\frac{1}{2}(1-\mu)<1$. We shall make the substitution $t-1=(\mu-1)u$; it will be convenient to place the path so that C is on the straight line joining 1 and μ, so that u has a real value less than unity when t is represented by the point C.

The integral becomes

$$c_n^m(\mu^2-1)^{\frac{1}{2}m}\int_{c'}^{(1+,0+,1-,0-)}(\mu-1)^{-m}u^n(u-1)^{-n-m-1}\left(1+\frac{\mu-1}{2}u\right)^n du$$

where C′ is the point corresponding to C.



In this integral the initial phase of u at C′ is zero, that of $u-1$ is $-\pi$, and $\left(1+\frac{\mu-1}{2}u\right)^n$ has the value given by the Binomial expansion.

On performing the expansion, we obtain

$$c_n^m\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m}\sum_{r=0}^{\infty}\frac{\Pi(n)}{\Pi(r)\,\Pi(n-r)}\left(\frac{\mu-1}{2}\right)^r\int_{c'}^{(1+,0+,1-,0-)}u^{n+r}(u-1)^{-n-m-1}du.$$

The expression

$$e^{-\pi\iota(a+b)}\int_{c'}^{(1+,\,0+,\,1-,\,0-)} u^{a-1}(1-u)^{b-1}\,du$$

has been denoted by POCHHAMMER by $\epsilon\,(a, b)$; it has the advantage over the Eulerian integral $\int_0^1 u^{a-1}(1-u)^{b-1}\,du$ of having a definite finite value for all values of a and b. In $\epsilon\,(a, b)$, the quantity $1-u$ has the phase 0 initially at C′, so that $u-1=(1-u)\,e^{-\pi\iota}$. The principal properties of $\epsilon\,(a, b)$ are the following:—

(1)′ $\epsilon\,(a, b) = \epsilon\,(b, a)$,

(2)′ $\epsilon\,(a+r, b) = (-1)^r \dfrac{a(a+1)\ldots(a+r-1)}{(a+b)(a+b+1)\ldots(a+b+r-1)}\,\epsilon\,(a, b)$,

$$\epsilon\,(a-r, b) = (-1)^r \frac{(a+b-1)\ldots(a+b-r)}{(a-1)(a-2)\ldots(a-r)}\,\epsilon\,(a, b).$$

(3)′ $\epsilon\,(a, b) = -4\sin a\pi\sin b\pi\,.\,\mathrm{E}\,(a, b)$

when the real parts of a, b are positive, $\mathrm{E}\,(a, b)$ denoting the Eulerian integral

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,du,$$

which is equal to

$$\frac{\Pi\,(a-1)\,\Pi\,(b-1)}{\Pi\,(a+b-1)}.$$

By means of (2) this theorem can be extended to the case in which the real parts of a, b are not necessarily positive.

(4)′ $\epsilon\,(a, b) = \epsilon\,(1-a-b, b) = \epsilon\,(a, 1-a-b)$.

We have

$$\int^{(1+,\,0+,\,1-,\,0-)} u^{n+r}(u-1)^{-n-m-1}\,du$$

$$= e^{(n+m+1)\pi\iota}\int^{(1+,\,0+,\,1-,\,0-)} u^{n+r}(1-u)^{-n-m-1}\,du$$

$$= e^{(n+r)\pi\iota}\,\epsilon\,(n+r+1, -n-m),$$

hence, since

$$\epsilon\,(n+r+1, -n-m) = (-1)^r \frac{(n+1)(n+2)\ldots(n+r)}{(1-m)(2-m)\ldots(r-m)}\,\epsilon\,(n+1, -n-m),$$

the expression (3) becomes

$$c_n{}^m e^{n\pi\iota}\,\epsilon\,(n+1,\,-n-m)\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m}\Sigma\,\frac{\Pi\,(n+r)}{\Pi\,(r)\,\Pi\,(n-r)}\,\frac{1}{(1-m)\ldots(r-m)}\left(\frac{\mu-1}{2}\right)^r,$$

or

$$c_n{}^m e^{n\pi\iota}\,\epsilon\,(n+1,\,-n-m)\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\,n+1,\,1-m,\,\frac{1-\mu}{2}\right),$$

where F is used with the ordinary notation, for the hyper-geometric series.

In virtue of the property (4)′, we have $\epsilon\,(n+1,\,-n-m)=\epsilon\,(n+1,\,m)$; and from (3)′ we have

$$\epsilon\,(n+1,\,m)=4\sin n\pi\sin m\pi\,.\,\frac{\Pi\,(n)\,\Pi\,(m-1)}{\Pi\,(n+m)};$$

hence, whatever n and m may be, the expression (3) becomes

$$c_n{}^m.\,e^{n\pi\iota}.\,4\sin n\pi\sin m\pi\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m}\frac{\Pi\,(n)\,\Pi\,(m-1)}{\Pi\,(n+m)}\,\mathrm{F}\left(-n,\,n+1,\,1-m,\,\frac{1-\mu}{2}\right).$$

4. In the case $m=0$, we have, since $\Pi\,(-m)\,\Pi\,(m-1)=\pi\operatorname{cosec} m\,\pi$,

$$c_n{}^0\int^{(\mu+,\,1+,\,\mu-,\,1-)}(t^2-1)^n\,(t-\mu)^{-n-1}\,dt=c_n{}^0.\,e^{n\pi\iota}.\,4\pi\sin n\pi.\,\mathrm{F}\left(-n,\,n+1,\,1,\,\frac{1-\mu}{2}\right),$$

when mod. $\frac{1}{2}\,(1-\mu)<1$; in accordance with usage we take the LEGENDRE's function $\mathrm{P}_n\,(\mu)$ of the first kind to be given by $\mathrm{P}_n\,(\mu)=\mathrm{F}\left(-n,\,n+1,\,1\,\frac{1-\mu}{2}\right)$, hence, if we choose $c_n{}^0$ equal to $\frac{e^{-n\pi\iota}}{4\pi\sin n\pi}$, we have

$$\mathrm{P}_n\,(\mu)=\frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\int^{(\mu+,\,1+,\,\mu-,\,1-)}\frac{1}{2^n}\,(t^2-1)^n\,(t-\mu)^{-n-1}\,dt.$$

The integral on the right-hand side defines $\mathrm{P}_n\,(\mu)$ over the whole plane, the function represented in the domain of the point 1 by the series, being analytically continued over the whole plane.

In order to obtain a definition of $\mathrm{P}_n{}^m\,(\mu)$, we shall first consider the case when m is a real positive integer, and shall then define $\mathrm{P}_n{}^m\,(\mu)$ for general values of m in such a way that the definition agrees with the usual definition for the special case in which m is a real integer.

When m is a positive integer, we may define $\mathrm{P}_n{}^m\,(\mu)$ by means of the formula $\mathrm{P}_n{}^m\,(\mu)=(\mu^2-1)^{\frac{1}{2}m}\,\frac{d^m\mathrm{P}_n\,(\mu)}{d\mu^m}$; thus in this case

$$\mathrm{P}_n{}^m\,(\mu)=\frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\,\frac{\Pi\,(n+m)}{\Pi\,(n)}\,(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,\,1+,\,\mu-,\,1-)}\frac{1}{2^n}\,(t^2-1)^n\,(t-\mu)^{-n-m-1}\,dt$$

so that in this case $c_n^m = \frac{e^{-n\pi\iota}}{4\pi \sin n\pi} \frac{\Pi(n+m)}{\Pi(n)}$. We shall choose this value of c_n^m for all values of n and m, thus obtaining a definition of the function $P_n^m(\mu)$ for all values of n and m real or complex; $P_n^m(\mu)$ is accordingly defined by the expression

$$P_n^m(\mu) = \frac{e^{-n\pi\iota}}{4\pi \sin n\pi} \frac{1}{2^n} \frac{\Pi(n+m)}{\Pi(n)} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(\mu+,\, 1+,\, \mu-,\, 1-,)} (t^2-1)^n (t-\mu)^{-n-m-1}\, dt \quad (4)$$

for unrestricted values of n and m, the phases of the expressions in the integrand being assigned as in Art 2. In order that this function $P_n^m(\mu)$ may be a single-valued function of μ we must suppose that a cross-cut is made along the real axis from the point 1 to $-\infty$, so that the phases of $\mu - 1$, $\mu + 1$ in $(\mu^2 - 1)^{\frac{1}{2}m}$ are restricted to lie between $\pm\pi$, the function is then, when we take into account the remarks which we have to make in the next article, a single-valued function over the whole plane so cut, the values at points indefinitely close to one another on opposite sides of the cross-cut being in general different. It should be observed that the integrand in the integral for a given value of μ varies continuously in crossing the cross-cut which has no reference to the variable t, but applies to μ only.

When μ in such that mod. $(1 - \mu) < 2$, we have

$$P_n^m(\mu) = \frac{\sin m\pi}{\pi} \Pi(m-1) \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)$$

$$= \frac{1}{\Pi(-m)} \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) \quad . \quad . \quad . \quad . \quad (5).$$

The formula (5) represents the function $P_n^m(\mu)$ over that part of the plane which is contained within a circle of radius 2 with its centre at the point $\mu = 1$; this function can be analytically continued over the whole plane and (with the cross-cut) the function so continued is uniform, and is given by the definite integral formula (4) which affords a general definition of the function.

When m is an integer positive or negative, the expression (4) can be simplified; in this case the integrand returns to its initial value after a positive turn round each of the points μ and 1, denoting the parts of the integral taken round $C\alpha\beta C$, $C\gamma\delta C$ (fig. 1, Art. 2) by P and Q respectively, the complete integral is

$$P + Q - Pe^{2n\pi\iota} - Qe^{(n+m+1)2\pi\iota},$$

or

$$(1 - e^{2n\pi\iota})(P + Q);$$

now $P + Q$ is the integral taken along a curve which encloses both the points μ and $+1$, and is described positively, hence, in the case in which m is an integer, the formula (4) becomes

$$P_n^m(\mu) = \frac{1}{2\pi\iota}\frac{\Pi(n+m)}{\Pi(n)}\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^n}\int^{(\mu+,1+)}(t^2-1)^n(t-\mu)^{-n-m-1}dt \qquad (6).$$

When $m = 0$ we have

$$P_n(\mu) = \frac{1}{2\pi\iota}\int^{(\mu+,1+)}\frac{1}{2^n}(t^2-1)^n(t-\mu)^{-n-1}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which agrees with the definition given by SCHLÄFLI.

The only case of failure of the formulæ (4) and (6) is that in which $n + m$ is a negative integer; in that case $\Pi(n+m)$ is infinite and the integral is zero, and the product can be evaluated by the rule for undetermined forms $0 \times \infty$; we have

$$\Pi(n+m) = -\frac{\operatorname{cosec}(m+n)\pi}{\Pi(-m-n-1)},$$

and the limiting value of

$$\frac{1}{\sin(m+n)\pi}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^n(t-\mu)^{-n-m-1}dt$$

is

$$-\frac{1}{\pi\cos(m+n)\pi}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^n(t-\mu)^{-n-m-1}\log_e(t-\mu)\,dt,$$

thus

$$P_n^m(\mu) = \frac{e^{-n\pi\iota}}{4\pi\sin n\pi}$$

$$\cdot\frac{1}{2^n\pi\cos(m+n)\pi}\frac{1}{\Pi(n)\Pi(-m-n-1)}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^n(t-\mu)^{-n-m-1}\log_e(t-\mu)\,dt.$$

If in (5) we change n into $-n-1$, the hypergeometric series is unaltered, thus within the circle of convergence $P_n^m(\mu)$ is equal to $P_{-n-1}^m(\mu)$; it follows that the same relation holds over the whole plane; we accordingly obtain another expression for $P_n^m(\mu)$ by changing n into $-n-1$ in the formula (4), we thus have

$$P_n^m(\mu) = P_{-n-1}^m(\mu)$$

$$= -\frac{e^{-n\pi\iota}}{4\pi\sin n\pi}\cdot 2^{n+1}\frac{\Pi(m-n-1)}{\Pi(-n-1)}(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^{-n-1}(t-\mu)^{n-m}dt,$$

$$= -\frac{e^{-n\pi\iota}}{4\pi\sin(n-m)\pi}\cdot 2^{n+1}\frac{\Pi(n)}{\Pi(n-m)}(\mu^2-1)^{\frac{1}{2}m}\int^{(\mu+,1+,\mu-,1-)}(t^2-1)^{-n-1}(t-\mu)^{n-m}dt \quad (8).$$

The formula (8) will serve equally with (4), as a definition of $P_n^m(\mu)$; it does not appear to be easy to prove directly their equivalence.

As regards the formula (5),

$$P_n^m(\mu) = \frac{1}{\Pi(-m)} \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right),$$

we may remark that

(α) When n is a real integer and m is not so, the series is finite, and therefore $P_n^m(\mu)$ is an algebraical function.

(β) When m is a real positive integer and n is not so, the formula may be written

$$P_n^m(\mu) = \frac{1}{2^m} \frac{\Pi(n+m)}{\Pi(n-m)} \frac{1}{\Pi(m)} (\mu^2-1)^{\frac{1}{2}m} F\left(m-n, n+m+1, m+1, \frac{1-\mu}{2}\right).$$

(γ) When n and m are both positive real integers, and $n > m$, it falls under case (β), the series being however finite since the first element $m-n$ of the hypergeometric series is a negative integer, thus $P_n^m(\mu)$ is an algebraical function.

(δ) When n and m are both positive integers, and $n < m$, case (β) shows that $P_n^m(\mu)$ is zero; in order to obtain an integral of the differential equation we must take $\Pi(n-m) P_n^m(\mu)$ which is finite.

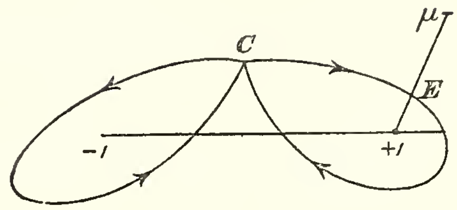
5.

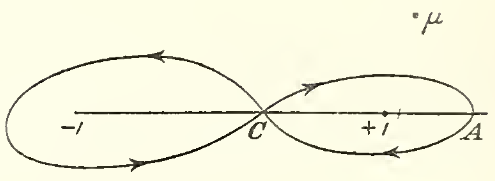
(a.) (b.)

In defining the function $P_n^m(\mu)$ by means of a definite integral taken round a closed path, in which turns are made round the points μ and 1, but none round the point -1, it is necessary to specify the position of the path with reference to the point -1. The figures (a) and (b) represent two distinct paths for the same value of μ, but the integrals obtained from them will be, in general, different in value, as one path cannot be brought by continuous deformation into coincidence with the other without crossing the point -1, which is a singular point for the integrand. We shall consequently specify that the path by means of which $P_n^m(\mu)$ is defined in (4), is one which does not cut the real axis between -1 and $-\infty$, or is, at all events, a path which can by continuous deformation be brought, without crossing the point -1, into a path which does not cut the real axis between -1 and $-\infty$.

6. Another closed path for the integrand $(t^2-1)^n (t-\mu)^{-n-m-1}$ is that in which

a positive turn round the point -1 is followed by a negative turn round the point $+1$.





Consider thus the expression

$$f_n^m (\mu^2 - 1)^{\frac{1}{2}m} \int_C^{(-1+, 1-)} \frac{1}{2^n} (t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt,$$

taken along the path as in either of the figures. The phase of $t - \mu$ will be measured as before; those of $t - 1$, $t + 1$, we shall take to be such that they vanish at the instant when t passes in the integration through the point A of the real axis, for which $t - 1$, $t + 1$ are both real and positive; thus, in the second figure, the initial phrases of $t - 1$, $t + 1$ at C are π and -2π respectively.

Let $t - \mu = (\mu - t)\, e^{-\iota\pi}$, then the phases of $\mu - t$ are such that at the point E, where the line joining μ and 1 cuts the path, the phase of $\mu - t$ is the angle (between $\pm\pi$) the line makes with the positive direction of the real axis; the expression becomes

$$f_n^m (\mu^2 - 1)^{\frac{1}{2}m} \int^{(-1+, +1-)} \frac{1}{2^n} e^{(n+m+1)\iota\pi} (t^2 - 1)^n (\mu - t)^{-n-m-1}\, dt.$$

Suppose now that mod. $\mu > 1$, the path of integration can then always be so placed that mod. t is everywhere less than mod. μ; expanding by the binomial theorem, the expression becomes

$$f_n^m (\mu^2 - 1)^{\frac{1}{2}m} \cdot \frac{1}{2^n} e^{(n+m+1)\iota\pi} \sum_{r=0}^{r=\infty} \int^{(-1+, +1-)} \frac{\Pi(n+m+r)}{\Pi(n+m)\,\Pi(r)} \frac{1}{\mu^{n+m+1+r}} (t^2 - 1)^n t^r\, dt.$$

To evaluate $\int^{(-1, 1-)} (t^2 - 1)^n t^r\, dt$, we may place the path so that the two loops are exactly equal, C being half-way between the points 1 and -1; it is thus seen that the integral vanishes when r is odd, and that when r is even and equal to $2s$ it is equal to

$$-2 \int_0^{(+1+)} (t^2 - 1)^n t^{2s}\, dt;$$

making the substitution $t' = t^2$, we see that $t' - 1$, or $(t - 1)(t + 1)$ is such that its phase increases from $-\pi$ to π during the integration, we thus have

$$-\int_0^{(+1+)} (t'-1)^n t'^{s-\frac{1}{2}}\,dt,$$

which can easily be shown to be equal to $2\iota \sin n\pi \dfrac{\Pi(n)\,\Pi(s-\frac{1}{2})}{\Pi(n+s+\frac{1}{2})}$.

The expression with which we commenced is now reduced to the form

$$f_n^m \cdot \frac{1}{2^n} \cdot 2\iota \sin n\pi \cdot e^{(n+m+1)\iota\pi} (\mu^2-1)^{\frac{1}{2}m} \sum_{s=0}^{\infty} \frac{\Pi(n+m+2s)}{\Pi(n+m)\,\Pi(2s)} \frac{\Pi(n)\,\Pi(s-\frac{1}{2})}{\Pi(n+s+\frac{1}{2})} \frac{1}{\mu^{n+m+2s+1}},$$

which is

$$f_n^m \cdot \frac{1}{2^n} \cdot 2\iota \sin n\pi \cdot e^{(n+m+1)\iota\pi} (\mu^2-1)^{\frac{1}{2}m} \frac{\Pi(n)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}$$

$$\cdot \frac{1}{\mu^{n+m+1}} \mathrm{F}\left(\frac{n+m}{2}+1, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right).$$

When n is a positive integer, we have in accordance with the usual definition of $Q_n(\mu)$,

$$Q_n(\mu) = \frac{1}{2^{n+1}} \cdot \frac{\Pi(-\frac{1}{2})\,\Pi(n)}{\Pi(n+\frac{1}{2})} \frac{1}{\mu^{n+1}} \mathrm{F}\left(\frac{n}{2}+1, \frac{n+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right);$$

hence, in this case, if we take

$$f_n^0 = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi}$$

we have

$$Q_n(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} \int^{(-1+,\,1-)} \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-1}\,dt.$$

Defining $Q_n^m(\mu)$ when m is a positive integer, by means of the equation

$$Q_n^m(\mu) = (\mu^2-1)^{\frac{1}{2}m} \frac{d^m}{d\mu^m} Q_n(\mu),$$

we have

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} (\mu^2-1)^{\frac{1}{2}m} \frac{\Pi(n+m)}{\Pi(n)} \int^{(-1+,\,1-)} \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt,$$

we should consequently, when m and n are positive integers, choose f_n^m equal to

$$\frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} \cdot \frac{\Pi(n+m)}{\Pi(n)}.$$

We shall now assign this value to f_n^m, whatever m and n are; we thus obtain the formula

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} \cdot \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int^{(-1+,\,1-)} \frac{1}{2^n} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt \quad (9),$$

which we shall take as the definition of the function $Q_n^m(\mu)$ for unrestricted values of m and n.

When mod. $(\mu) > 1$, $Q_n^m(\mu)$ is represented by the expression

$$Q_n^m(\mu) = \frac{e^{m\iota\pi}}{2^{n+1}} \frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \frac{1}{\mu^{n+m+1}} \mathrm{F}\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, \frac{1}{\mu^2}\right) \quad (10).$$

The uniform function obtained by continuing the function in (10) over the whole plane, with the exception of the cross-cut along the real axis from $+1$ to $-\infty$, is represented by the expression in (9).

When n is such that the real part of $n+1$ is positive, the definition (9) can be simplified, the integral being then reducible to one along a line joining the points ± 1. The path may be as in the figure; then, since the integrals along the loops round the points 1 and -1 become indefinitely small when the loops are made indefinitely small, we have

$$\int_c^{(-1+,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt = (e^{n\pi\iota} - e^{-n\pi\iota}) \int_{-1}^{1} (1-t^2)^n (t-\mu)^{-n-m-1}\,dt$$
$$= 2\iota \sin n\pi \int_{-1}^{1} (1-t^2)^n (t-\mu)^{-n-m-1}\,dt\,;$$

hence, when $n+1$ has its real part positive, we may substitute for (9) the definition

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{2} \cdot \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int_{-1}^{1} \frac{1}{2^n} (1-t^2)^n (t-\mu)^{-n-m-1}\,dt$$
$$= \frac{e^{m\iota\pi}}{2^{n+1}} \frac{\Pi(n+m)}{\Pi(n)} (\mu^2-1)^{\frac{1}{2}m} \int_{-1}^{1} (1-t^2)^n (\mu-t)^{-n-m-1}\,dt. \quad . \quad . \quad . \quad (11).$$

The integral may be taken along the real axis, $(1-t^2)^n$ denoting $e^{n\log(1-t^2)}$, where the logarithm has its real value.

It will be observed that when n is a positive integer, the form (9) is undetermined $(\infty \times 0)$; we can, however, in this case use the formula (11). When n is a negative integer, the value of $Q_n^m(\mu)$, as given by (9), is in general finite, since

$$\sin n\pi\,.\,\Pi(n) = -\frac{\pi}{\Pi(-n-1)}\,;$$

if, however, $n+m$ is also a negative integer, or if m is zero, the value of $Q_n^m(\mu)$ is infinite, so that the factor $\Pi(n+m)$ must be rejected if we wish to obtain a finite solution of the differential equation.

Proof of a relation between $Q_n^m(\mu)$ and $Q_n^{-m}(\mu)$.

7. If we apply to the formula (10) the known theorem

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha; \gamma-\beta, \gamma, x)$$

we have when mod $\mu > 1$,

$$Q_n^m(\mu) = \frac{e^{m\iota\pi}}{2^{n+1}} \frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})} (\mu^2-1)^{-\frac{1}{2}m} \frac{1}{\mu^{n-m+1}} F\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right).$$

The expression for $Q_n^{-m}(\mu)$ is obtained by writing $-m$ for m, in the formula (10); we have thus the relation

$$\frac{e^{-m\iota\pi}\,Q_n^m(\mu)}{\Pi(n+m)} = \frac{e^{m\iota\pi}\,Q_n^{-m}(\mu)}{\Pi(n-m)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which must hold over the whole plane; it is obvious that $Q_n^{-m}(\mu)$ satisfies the differential equation (1), as that equation is unaltered by changing the sign of m. The result in (12) may also be obtained by transforming the integral in (9) by means of the transformation $(t-\mu)(t'-\mu) = \mu^2 - 1$, which is equivalent to an inversion with respect to the point μ. On making the substitution, we find

$$(\mu^2-1)^{\frac{1}{2}m}\int^{(-1+,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt$$
$$= -(\mu^2-1)^{-\frac{1}{2}m}\int^{(1+,\,-1-)} (t'^2-1)^n (t'-\mu)^{-n+m-1}\,dt'.$$

Corresponding to the phase $-\pi$ of t^2-1, the phase of t'^2-1 is π; also to the phase $-\pi$ of $t-\mu$, in the case in which μ is real and greater than unity, the phase of $t'-\mu$ is π, hence, in order that in the integral on the right-hand side the phases may be measured in the same way as on the left-hand side, the factor $e^{2n\pi\iota-2(n-m+1)\pi\iota}$, or $e^{2m\pi\iota}$, must be introduced; we thus obtain

$$(\mu^2-1)^{\frac{1}{2}m}\int^{(-1+,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt$$
$$= (\mu^2-1)^{-\frac{1}{2}m} e^{2m\pi\iota}\int^{(-1+,\,1-)} (t'^2-1)^n (t'-\mu)^{-n+m-1}\,dt',$$

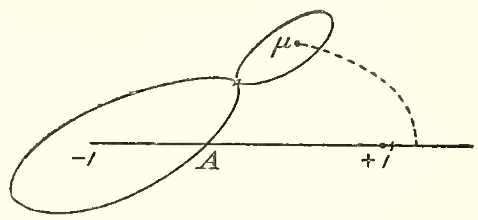
and thus the result (12) is proved.

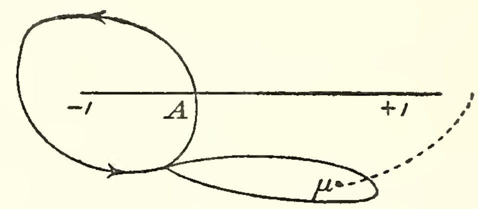
Expression for $Q_n^m(\mu)$ when mod $(\mu+1)$ and mod $(\mu-1)$ are less than 2.

8. It will be necessary to obtain an expression for the integral

$$(\mu^2 - 1)^{\frac{1}{2}m} \int^{(\mu+,\ -1+,\ \mu-,\ -1-)} \frac{1}{2^n} (t^2 - 1)^n (t - \mu)^{-n-m-1} \, dt,$$

analogous to the corresponding integral round the singular points μ, 1, obtained in Art. 3. To define the phases of the integrand we shall distinguish the cases in which the imaginary part of μ is positive, and is negative.





We suppose μ to move from a point in the real axis for which its value is greater than unity, up to its actual position, the path of integration being drawn as in the figures; it will be observed that as μ moves from a position on the positive side of the real axis to one on the negative side, the path cannot be displaced from its first position to the second one without crossing the singular point $+1$, it is therefore necessary to distinguish the two cases.

In the first figure the phase of $t - 1$ at A is $+\pi$, and in the second figure it is $-\pi$, in both cases the phase of $t + 1$ at A is zero, and that of $t - \mu$ is measured as before.

Put $t + 1 = (\mu + 1) u$, the expression then becomes

$$\left(\frac{\mu - 1}{\mu + 1}\right)^{\frac{1}{2}m} \int^{(1+,\ 0+,\ 1-,\ 0-,)} u^n \left(\frac{\mu + 1}{2} u - 1\right)^n (u - 1)^{-n-m-1} \, du,$$

now we put

$$\frac{\mu + 1}{2} u - 1 = e^{\iota\pi}\left(1 - \frac{\mu + 1}{2} u\right)$$

or,

$$\frac{\mu + 1}{2} u - 1 = e^{-\iota\pi}\left(1 - \frac{\mu + 1}{2} u\right)$$

according as the imaginary part of μ is positive or negative, in both cases the phase of $1 - \frac{\mu + 1}{2} u$ is zero at A, and then $\left(1 - \frac{\mu + 1}{2} u\right)^n$ will have that value which is given by the expansion by the Binomial Theorem.

We have for the integral

$$\left(\frac{\mu - 1}{\mu + 1}\right)^{\frac{1}{2}m} e^{\pm n\pi\iota} \int^{(1+,\ 0+,\ 1-,\ 0-)} u^n \left(1 - \frac{\mu + 1}{2} u\right)^n (u - 1)^{-n-m-1} \, du,$$

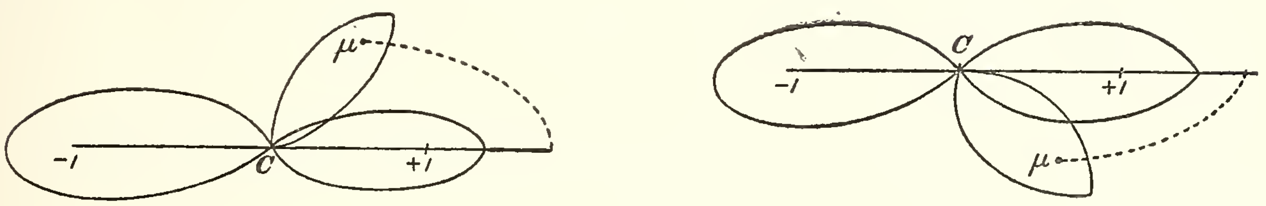
the upper or lower sign being taken in $e^{\pm n\pi\iota}$, according as μ is above or below the real axis. When mod. $(\mu + 1) < 2$, this expression can be evaluated exactly as in Art. 3, the result being obtained by writing $-\mu$ for μ; we thus find at once

$$(\mu^2 - 1)^{\frac{1}{2}m} \int^{(\mu+,\,-1+,\,\mu-,\,-1-)} \frac{1}{2^n} (t^2 - 1)^n (t - \mu)^{-n-m-1} dt$$

$$= e^{2n\pi\iota} \,.\, 4 \sin n\pi \sin m\pi \,\frac{\Pi(n)\,\Pi(m-1)}{\Pi(n+m)} \left(\frac{\mu - 1}{\mu + 1}\right)^{\frac{1}{2}m} \mathrm{F}\left(-n,\, n+1,\, 1-m,\, \frac{1+\mu}{2}\right). \quad (13)$$

when μ is above the real axis, the exponential factor being omitted when μ is below the real axis.

9.



Let L, M, N denote the values of the integral $\int (t^2 - 1)^n (t - \mu)^{-n-m-1} dt$ taken along loops from C round the three points -1, 1, μ respectively, in the positive directions, the phases at C being as follows :

of $t - 1$, π in the first figure, and $-\pi$ in the second,

of $t + 1$, zero,

of $t - \mu$, $-(\pi - \phi)$, where ϕ is the (positive or negative) angle the line joining C to μ makes with the positive direction of the real axis. We have at once

$$\int_c^{(\mu+,\,1+,\,\mu-,\,1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1} dt = \mathrm{N} + \mathrm{M}e^{-2\pi(m+n+1)\iota} - \mathrm{N}e^{2\pi n\iota} - \mathrm{M},$$

$$\int_c^{(\mu+,\,-1+,\,\mu-,\,-1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1} dt = \mathrm{N} + \mathrm{L}e^{-2\pi(m+n+1)\iota} - \mathrm{N}e^{2\pi n\iota} - \mathrm{L},$$

the phases in the integrands being measured as just stated.

To express $\int_c^{(-1+,\,1-)} (t^2 - 1)^n (t - \mu)^{-n-m-1} dt$, in which, as in Art. 6, the phase of $t - 1$ at C is $+\pi$, and that of $t + 1$ is -2π, we have for the value of the integral

$$\mathrm{L}e^{-2n\pi\iota} - \mathrm{M}e^{-2n\pi\iota}, \text{ or } \mathrm{L} - \mathrm{M}$$

according as μ is above or below the real axis.

It follows that

$$\int_c^{(-1+,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt$$

$$= \frac{e^{-2n\pi\iota}}{1-e^{-2\pi(m+n)\iota}}\left\{\int_c^{(\mu+,\,1+,\,\mu-,\,1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt - \int_c^{(\mu+,\,-1+,\,\mu-,\,-1-)} (t^2-1)^n (t-\mu)^{-n-m-1}\,dt\right\},$$

or

$$= \frac{1}{1-e^{-2\pi(m+n)\iota}} \times \text{the same expression} \quad . \quad . \quad . \quad . \quad . \quad (14),$$

according as μ is above or below the real axis.

10. The relation (14) enables us to find the expression for $Q_n^m(\mu)$ in series, for values of μ which are such that mod. $(1+\mu)$ and mod. $(1-\mu) < 2$. Using the formulæ (5), (9), (13), we find at once

$$Q_n^m(\mu) = \frac{\pi e^{m\pi\iota}}{2\sin(m+n)\pi}\,\frac{1}{\Pi(-m)}\left\{e^{\mp n\pi\iota}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\,n+1,\,1-m,\,\frac{1-\mu}{2}\right) - \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n,\,n+1,\,1-m,\,\frac{1+\mu}{2}\right)\right\} \quad (15),$$

the upper or the lower sign being taken in $e^{\mp n\pi\iota}$, according as the imaginary part of μ is positive or negative.

When m is zero, we have

$$Q_n(\mu) = \frac{\pi}{2\sin n\pi}\left\{e^{\mp n\pi\iota} F\left(-n,\,n+1,\,1,\,\frac{1-\mu}{2}\right) - F\left(-n,\,n+1,\,1,\,\frac{1+\mu}{2}\right)\right\} \quad (16).$$

The particular case (16) agrees, when μ is above the real axis, with the result obtained by SCHLÄFLI.

If we use the relation (12) between $Q_n^m(\mu)$ and $Q_n^{-m}(\mu)$, we can write (15) in the form

$$Q_n^m(\mu) = \frac{\pi e^{m\pi\iota}}{2\sin(n-m)\pi}\,\frac{\Pi(n+m)}{\Pi(n-m)}\,\frac{1}{\Pi(m)}\left\{e^{\mp n\pi\iota}\left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n,\,n+1,\,1+m,\,\frac{1-\mu}{2}\right) - \left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\,n+1,\,1+m,\,\frac{1+\mu}{2}\right)\right\} \quad (17).$$

When $n+m$ is a positive integer, the expression (10) shows that $Q_n^m(\mu)$ has in general a finite value, hence we see from (15), that in this case

$$e^{\mp n\pi\iota}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n,\,n+1,\,1-m,\,\frac{1-\mu}{2}\right) = \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n,\,n+1,\,1-m,\,\frac{1+\mu}{2}\right);$$

this result is proved by HEINE* for the special case in which n and m are both integers. We see, therefore, that when $n+m$ is a positive integer, the formula (15) is undetermined, the formula (17) must in that case be used.

When $n-m$ is a positive integer we must use (15), since (17) become in this case undetermined. When $n+m$ is a negative integer, $Q_n^m(\mu)$ is infinite, but we can take $Q_n^m(\mu)\sin(n+m)\pi$ as a finite solution of the differential equation.

When n and m are both real integers, and m is positive and $> n$, the form (17) is finite, but if $m \leqq n$ both the forms (15), (17) are undetermined, and must be modified by applying the rule for the determination of undetermined forms 0/0.

The functions $P_n^m(\mu)$, $Q_n^m(\mu)$ are defined by OLBRICHT for the general case, by means of equations, in our notation,

$$P_n^m(\mu) = \text{constant}\left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1+m, \frac{1-\mu}{2}\right),$$

$$Q_n^m(\mu) = \text{constant}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1+m, \frac{1+\mu}{2}\right),$$

this definition of $Q_n^m(\mu)$ is, however, not consistent with the usual definition as in (10), in the form of a hypergeometric series whose fourth element is $\frac{1}{\mu^2}$.

Relation between the functions Q_n^m, Q_{-n-1}^m, P_n^m.

11. In the formula (15), write $-n-1$ for n, we have then

$$Q_{-n-1}^m(\mu) = \frac{\pi e^{m\pi\iota}}{2\sin(m-n-1)\pi}\cdot\frac{1}{\Pi(-m)}\left\{e^{\pm(n+1)\pi\iota}\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) - \left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1+\mu}{2}\right)\right\}.$$

On eliminating the second hypergeometric series between this equation and (15), we find

$$\begin{aligned} &Q_n^m(\mu)\sin(n+m)\pi - Q_{-n-1}^m(\mu)\sin(n-m)\pi \\ &= \frac{\pi e^{m\pi\iota}}{2\Pi(-m)}\left(e^{-n\pi\iota}+e^{n\pi\iota}\right)\left(\frac{\mu+1}{\mu-1}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) \\ &= \pi e^{m\pi\iota}\cos m\pi \cdot P_n^m(\mu), \qquad \text{by (5).} \end{aligned}$$

We thus obtain the formula

* See 'Kugelfunctionen,' vol. 2, pp. 238, 336.

$$P_n^m(\mu) = \frac{e^{-m\pi\iota}}{\pi \cos n\pi}\{Q_n^m(\mu)\sin(n+m)\pi - Q_{-n-1}^m(\mu)\sin(n-m)\pi\} \quad . \quad (18).$$

This relation which has been proved to hold over the domain of the point $\mu = -1$, must hold over the whole plane.

In the case $m = 0$, we have

$$P_n(\mu) = \frac{\tan n\pi}{\pi}\{Q_n(\mu) - Q_{-n-1}(\mu)\}.$$

If $n + m$ is a positive real integer, we have

$$P_n^m(\mu) = -\frac{2}{\pi}\cdot e^{-m\pi\iota}\cos m\pi \,.\, Q_{-n-1}^m(\mu).$$

If $n - m$ is a negative real integer, the relation (18) becomes

$$P_n^m(\mu) = \frac{2}{\pi}e^{-m\pi\iota}\sin m\pi \,.\, Q_n^m(\mu),$$

we see therefore that in this case the two functions $P_n^m(\mu)$, $Q_n^m(\mu)$ are not distinct.

Changing m into $-m$, in (18), we have

$$P_n^{-m}(\mu)$$
$$= \frac{e^{-m\pi\iota}}{\pi\cos n\pi}\left\{\frac{\Pi(n-m)}{\Pi(n+m)}Q_n^m(\mu)\sin(n-m)\pi - \frac{\Pi(-n-m-1)}{\Pi(-n+m-1)}Q_{-n-1}^m(\mu)\sin(n+m)\pi\right\}$$
$$= \frac{e^{-m\pi\iota}}{\pi\cos n\pi}\cdot\frac{\Pi(n-m)}{\Pi(n+m)}\sin(n-m)\pi\,\{Q_n^m(\mu) - Q_{-n-1}^m(\mu)\},$$

hence on substituting for $Q_{-n-1}^m(\mu)$ its value given by (18), we have

$$P_n^{-m}(\mu) = \frac{\Pi(n-m)}{\Pi(n+m)}\left\{P_n^m(\mu) - \frac{2}{\pi}e^{-m\pi\iota}\sin m\pi \,.\, Q_n^m(\mu)\right\} \quad . \quad . \quad . \quad (19).$$

Remembering the relation between $P_n^m(\mu)$, $P_{-n-1}^m(\mu)$, we see that of the eight solutions $P_n^m(\mu)$, $P_{-n-1}^m(\mu)$, $P_n^{-m}(\mu)$, $P_{-n-1}^{-m}(\mu)$, $Q_n^m(\mu)$, $Q_{-n-1}^m(\mu)$, $Q_n^{-m}(\mu)$, $Q_{-n-1}^{-m}(\mu)$, of the equation (1), six have been expressed in terms of the other two.

Expressions for $P_n^m(-\mu)$, $Q_n^m(-\mu)$ in terms of $P_n^m(\mu)$, $Q_n^m(\mu)$.

12. Since the differential equation (1) is unaltered by substituting $-\mu$ for μ, it follows that $P_n^m(-\mu)$, $Q_n^m(-\mu)$ are particular integrals of the differential equation, and are therefore expressible in terms of $P_n^m(\mu)$, $Q_n^m(\mu)$.

The phases of $\mu+1$, $\mu-1$ in $(\mu+1)^{\frac{1}{2}m}$, $(\mu-1)^{\frac{1}{2}m}$ being restricted to lie between π and $-\pi$, on changing μ into $-\mu$, we must put $-\mu-1=e^{\mp\pi\iota}(\mu+1)$, $-\mu+1=e^{\mp\pi\iota}(\mu-1)$, where the upper or lower sign is taken according as the imaginary part of μ is positive or negative; we have therefore from (5)

$$\mathrm{P}_n^{\,m}(-\mu)=\frac{1}{\Pi(-m)}\left(\frac{\mu-1}{\mu+1}\right)^{\frac{1}{2}m}\mathrm{F}\left(-n,\,n+1,\,1-m,\,\frac{1+\mu}{2}\right);$$

on substituting for the series its value given by (15), we find the relation

$$\mathrm{P}_n^{\,m}(-\mu)=e^{\mp n\pi\iota}\,\mathrm{P}_n^{\,m}(\mu)-\frac{2\sin(n+m)\pi}{\pi}\,.\,e^{-m\pi\iota}\,\mathrm{Q}_n^{\,m}(\mu)\quad\ldots\quad(20).$$

Again from (10), we have since $(-\mu)^{n+m+1}=\mu^{n+m+1}\,.\,e^{\mp(n+m+1)\iota\pi}$, where the sign is chosen as before,

$$\mathrm{Q}_n^{\,m}(-\mu)=-\,e^{\pm n\pi\iota}\,\mathrm{Q}_n^{\,m}(\mu)\quad\ldots\ldots\ldots\quad(21).$$

In the particular case of a real integral value of n, we have

$$\mathrm{P}_n^{\,m}(-\mu)=(-1)^n\,\mathrm{P}_n^{\,m}(\mu)-\frac{2}{\pi}(-1)^n\sin m\pi\,.\,e^{-m\pi\iota}\,.\,\mathrm{Q}_n^{\,m}(\mu)$$

$$\mathrm{Q}_n^{\,m}(-\mu)=(-1)^{n+1}\,\mathrm{Q}_n^{\,m}(\mu).$$

Expression for $\mathrm{P}_n^{\,m}(\mu)$ in powers of $\frac{1}{\mu}$, when mod $\mu>1$.

13. In the formula (10), the expression for $\mathrm{Q}_n^{\,m}(\mu)$ in a series of powers of $\frac{1}{\mu}$ has been obtained for the domain of $\mu=\infty$; we shall now employ the relation (18) to express $\mathrm{P}_n^{\,m}(\mu)$ in a similar manner. We find by changing n into $-n-1$ in (10),

$$\begin{aligned}&\mathrm{Q}_{-n-1}^{\,m}(\mu)\\&=2^n.e^{m\pi\iota}\frac{\Pi(m-n-1)\,\Pi(-\frac{1}{2})}{\Pi(-n-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right)\\&=-\,2^n e^{m\pi\iota}\frac{\Pi(-\frac{1}{2})\,\Pi(n-\frac{1}{2})}{\Pi(n-m)}\,\frac{\cos n\pi}{\sin(n-m)\,\pi}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right).\end{aligned}$$

Hence we find

$$\begin{aligned}&\mathrm{P}_n^{\,m}(\mu)\\&=\frac{\sin(n+m)\,\pi}{2^{n+1}\cos n\pi}\cdot\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\frac{1}{\mu^{n+m+1}}\mathrm{F}\left(\frac{n+m+2}{2},\frac{n+m+1}{2},n+\tfrac{3}{2},\frac{1}{\mu^2}\right)\\&+2^n\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}\mathrm{F}\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right)\quad\ldots\quad(22).\end{aligned}$$

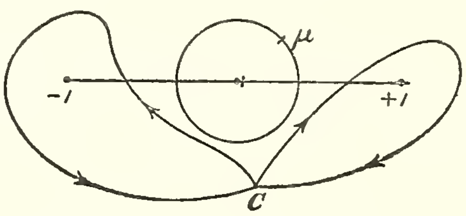
In the particular case $m = 0$, we have

$$P_n(\mu) = \frac{\tan n\pi}{2^{n+1}} \frac{\Pi(n)}{\Pi(n+\frac{1}{2})\Pi(-\frac{1}{2})} \frac{1}{\mu^{n+1}} F\left(\frac{n}{2}+1, \frac{n+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right)$$
$$+ 2^n \frac{\Pi(n-\frac{1}{2})}{\Pi(n)\Pi(-\frac{1}{2})} \mu^n F\left(\frac{1-n}{2}, -\frac{n}{2}, \tfrac{1}{2}-n, \frac{1}{\mu^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (23).$$

It will be observed that when $n + m$ is a positive integer, the expression (20) reduces to its second term, but not so when $n + m$ is a negative integer, since $\sin(n+m)\pi \,.\, \Pi(n+m)$ is then finite. HEINE gives* as the expression for $P_n(\mu)$, when n is unrestricted, a formula which is equivalent to the second term in (23); his formula is, therefore, only correct when n is a real integer.

Expressions for P_n^m, Q_n^m *in series of powers of* μ, *when mod.* $\mu < 1$.

14. It will be convenient to obtain the expansion of $Q_n^m(\mu)$, first in powers of μ, when mod. $\mu < 1$, and afterwards to deduce the corresponding series for $P_n^m(\mu)$.



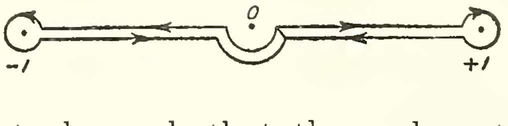
Taking the formula

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} (\mu^2-1)^{\frac{1}{2}m} \frac{\Pi(n+m)}{\Pi(n)} \int_c^{(-1+,1-)} \frac{1}{2^n}(t^2-1)^n (t-\mu)^{-n-m-1}\, dt.$$

Consider first the case in which the imaginary part of μ is positive; the path of integration can be so chosen, as in the figure, that, for every point of it, mod $t >$ mod μ; the term $(t-\mu)^{-n-m-1}$ can then be expanded in ascending powers of μ, and we thus find

$$Q_n^m(\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} (\mu^2-1)^{\frac{1}{2}m} \frac{1}{2^n\Pi(n)} \sum_{r=0}^{\infty} \frac{\Pi(n+m+r)}{\Pi(r)} \mu^r \int_c^{(-1+,1-)} (t^2-1)^n t^{-n-m-r-1}\, dt.$$

Let us now consider the integral, $\int^{(-1+,1-)} (t^2-1)^n t^p\, dt$.



First, suppose n and p to be such that the real parts of $n+1$, $p+1$ are both positive, the path of integration may then be as in the second figure, the loops round

* 'Kugelfunctionen,' vol. 1, p. 38.

the points 1, -1 being indefinitely small, and the semi-circles round the point 0 being so also; the parts of the integral taken along the loops and semi-circles in the limit vanish, and we have only to consider integrals taken along the real axis. The integral consists of four parts, the following :—

(1.) From 0 to -1, phase of t equal to $-\pi$, and the phases of $t-1$, $t+1$ equal to π, -2π respectively.
(2.) From -1 to 0, phase of t equal to $-\pi$, and the phases of $t-1$, $t+1$ equal to π and 0 respectively.
(3.) From 0 to 1, phases of t, $t-1$, $t+1$ equal to 0, π, 0 respectively.
(4.) From 1 to 0, phases of t, $t-1$, $t+1$ equal to 0, $-\pi$, 0 respectively.

Taking v for the modulus of t, we have in the first two parts of the integral $t = ve^{-\iota\pi}$, and in the other two parts $t = v$; hence, the integral is

$$\int_0^1 (1-v^2)^n v^p \left(e^{n\iota\pi} \,.\, e^{-2n\iota\pi} \,.\, e^{-\overline{p+1}\iota\pi} - e^{n\iota\pi} \,.\, 1 \,.\, e^{-\overline{p+1}\iota\pi} + e^{n\iota\pi} \,.\, 1 \,.\, 1 - e^{-n\iota\pi} \,.\, 1 \,.\, 1\right) dv,$$

or

$$(e^{n\pi\iota} - e^{-n\pi\iota})(1 + e^{-p\pi\iota}) \int_0^1 (1-v^2)^n v^p \, dv,$$

which is equal to

$$2\iota \sin n\pi \,.\, (1 + e^{-p\pi\iota}) \,.\, \tfrac{1}{2} \frac{\Pi(n)\, \Pi\left(\frac{p-1}{2}\right)}{\Pi\left(n + \frac{p+1}{2}\right)}.$$

To show that the result is the same, when the real parts of $n+1$, $p+1$ are not both positive, we find by integration by parts

$$\int^{(-1+,\,1-)} (t^2-1)^n t^p \, dt = \frac{2n+p+3}{p+1} \int^{(-1+,\,1-)} (t^2-1)^n t^{p+2} \, dt,$$

and also

$$\int^{(-1+,\,1-)} (t^2-1)^n t^p \, dt = -\frac{2n+p+3}{2n+2} \int^{(-1+,\,1-)} (t^2-1)^{n+1} t^p \, dt.$$

By successive use of these two formulæ, we find

$$\int^{(-1+,\,1-)} (t^2-1)^n t^p \, dt = (-1)^\lambda \frac{\Pi\left(\frac{p+1}{2}+n+s\right) \Pi\left(\frac{p-1}{2}\right)}{\Pi\left(\frac{p+1}{2}+n\right) \Pi\left(\frac{p-1}{2}+s\right)} \cdot \frac{\Pi\left(\frac{p+1}{2}+n+s+\lambda\right)}{\Pi\left(\frac{p+1}{2}+n+s\right)} \cdot \frac{\Pi(n)}{\Pi(n+\lambda)}$$
$$\times \int^{(-1+,\,1-)} (t^2-1)^{n+\lambda}\, t^{p+2s} \, dt,$$

where λ, s are positive integers which we can so choose that the real parts of $n + \lambda + 1$, $p + 2s + 1$ are both positive, in which case

$$\int^{(-1+,\,1-)} (t^2 - 1)^{n+\lambda}\, t^{p+2s}\, dt = \iota \sin (n + \lambda)\pi \,.\, (1 - e^{-\overline{p+1+2s}\iota\pi}) \frac{\Pi (n + \lambda)\, \Pi \left(\frac{p + 2s - 1}{2}\right)}{\Pi \left(n + \lambda + \frac{p + 2s + 1}{2}\right)}$$

whence we find, as before,

$$\int^{(-1+,\,1-)} (t^2 - 1)^n\, t^p\, dt = \iota \sin n\pi \,.\, (1 + e^{-p\pi\iota}) \frac{\Pi (n)\, \Pi \left(\frac{p - 1}{2}\right)}{\Pi \left(n + \frac{p + 1}{2}\right)}.$$

We have now, letting $p = - n - m - r - 1$,

$$Q_n^m (\mu) = \frac{e^{-(n+1)\iota\pi}}{4\iota \sin n\pi} (\mu^2 - 1)^{\frac{1}{2}m} \frac{\iota \sin n\pi}{2^n \Pi (n)} \sum_{r=0}^{\infty} (1 - e^{(n+m+r)\pi\iota}) \frac{\Pi (n + m + r)}{\Pi (r)} \frac{\Pi (n)\, \Pi \left(- \frac{n + m + r}{2} - 1\right)}{\Pi \left(\frac{n - m - r}{2}\right)} \mu^r$$

$$= \frac{e^{-(n+1)\iota\pi}}{2^{n+2}} (1 - e^{(n+m)\pi\iota}) \sum_{s=0}^{\infty} \frac{\Pi \left(\frac{- n - m - 2s}{2} - 1\right) \Pi (n + m + 2s)}{\Pi (2s)\, \Pi \left(\frac{n - m - 2s}{2}\right)} \mu^{2s}$$

$$+ \frac{e^{-(n+1)\iota\pi}}{2^{n+2}} (1 + e^{(n+m)\pi\iota}) \sum_{s=0}^{s=\infty} \frac{\Pi \left(\frac{- n - m - 2s - 1}{2} - 1\right) \Pi (n + m + 2s + 1)}{\Pi (2s + 1)\, \Pi \left(\frac{n - m - 2s - 1}{2}\right)} \mu^{2s+1}.$$

By the known transformation theorem $\Pi (- x)\, \Pi (x - 1) = \pi \operatorname{cosec} x\pi$, we have

$$\frac{\Pi \left(\frac{- n - m - 2s}{2} - 1\right)}{\Pi \left(\frac{n - m - 2s}{2}\right)} = \frac{\Pi \left(\frac{m - n + 2s}{2} - 1\right)}{\Pi \left(\frac{n + m + 2s}{2}\right)} \frac{\operatorname{cosec} \left(\frac{n + m + 2s}{2} + 1\right)\pi}{\operatorname{cosec} \left(\frac{m - n + 2s}{2}\right)\pi}$$

$$= - \frac{\Pi \left(\frac{m - n - 2}{2} + s\right)}{\Pi \left(\frac{m + n}{2} + s\right)} \frac{\sin \frac{m - n}{2}\pi}{\sin \frac{m + n}{2}\pi},$$

also

$$\frac{\Pi \left(\frac{- n - m - 2s - 1}{2} - 1\right)}{\Pi \left(\frac{n - m - 2s - 1}{2}\right)} = \frac{\Pi \left(\frac{m - n + 2s - 1}{2}\right)}{\Pi \left(\frac{n + m + 2s + 1}{2}\right)} \frac{\operatorname{cosec} \left(\frac{m + n + 2s + 1}{2} + 1\right)\pi}{\operatorname{cosec} \left(\frac{m - n + 2s + 1}{2}\right)\pi}$$

$$= - \frac{\Pi \left(\frac{m - n - 1}{2} + s\right)}{\Pi \left(\frac{m + n + 1}{2} + s\right)} \frac{\cos \frac{m - n}{2}\pi}{\cos \frac{m + n}{2}\pi},$$

hence the expression for $Q_n^m(\mu)$ becomes

$$Q_n^m(\mu) = -\frac{\iota e^{(m-n)\frac{\pi\iota}{2}}}{2^{n+1}}(\mu^2-1)^{\frac{1}{2}m}\sin\frac{m-n}{2}\pi \cdot \frac{\Pi(n+m)\,\Pi\left(\frac{m-n-2}{2}\right)}{\Pi\left(\frac{n+m}{2}\right)} F\left(\frac{n+m+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$

$$+\frac{e^{(m-n)\frac{\pi\iota}{2}}}{2^{n+1}}(\mu^2-1)^{\frac{1}{2}m}\cos\frac{m-n}{2}\pi \cdot \frac{\Pi(n+m+1)\,\Pi\left(\frac{m-n-1}{2}\right)}{\Pi\left(\frac{n+m+1}{2}\right)} \mu F\left(\frac{m-n+1}{2}, \frac{m+n+2}{2}, \tfrac{3}{2}, \mu^2\right) \quad (24).$$

The known transformation

$$\frac{\Pi(2x)}{\Pi(x)} = 2^{2x}\frac{\Pi(x-\frac{1}{2})}{\Pi(-\frac{1}{2})}$$

gives us

$$\frac{\Pi(n+m)}{\Pi\left(\frac{n+m}{2}\right)} = 2^{n+m}\frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(-\frac{1}{2}\right)}$$

and

$$\frac{\Pi(n+m+1)}{\Pi\left(\frac{n+m+1}{2}\right)} = 2^{n+m+1}\frac{\Pi\left(\frac{n+m}{2}\right)}{\Pi\left(-\frac{1}{2}\right)},$$

also

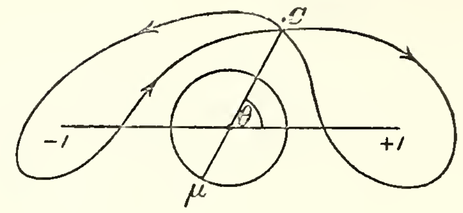
$$\Pi\left(\frac{m-n-2}{2}\right) = \frac{\pi \operatorname{cosec}\left(\frac{n+2-m}{2}\right)\pi}{\Pi\left(\frac{n-m}{2}\right)}, \quad \Pi\left(\frac{m-n-1}{2}\right) = \frac{\pi \operatorname{cosec}\left(\frac{n-m+1}{2}\right)\pi}{\Pi\left(\frac{n-m-1}{2}\right)};$$

hence the formula (24) may be written

$$Q_n^m(\mu) = -\frac{\iota e^{(m-n)\frac{\pi\iota}{2}}}{2}2^m\frac{\Pi\left(\frac{n+m+1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}(\mu^2-1)^{\frac{1}{2}m} F\left(\frac{n+m+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$

$$+e^{(m-n)\frac{\pi\iota}{2}}\,2^m\frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\mu F\left(\frac{m-n+1}{2}, \frac{m+n+2}{2}, \tfrac{3}{2}, \mu^2\right). \quad . \quad (25).$$

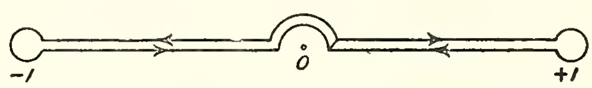
15. Next suppose the real part of μ is negative, the path of integration in the formula for $Q_n^m(\mu)$ may then be placed as in the figure, in which the line joining C and μ passes through the point $t = 0$. At C the phase of $t - \mu$ is $-(2\pi - \theta)$, and



that of t is θ, so that the phase of $1-\frac{\mu}{t}$ is -2π, thus $\left(1-\frac{\mu}{t}\right)^{-n-m-1}$ is equal to $e^{2(n+m)\pi\iota}$ times the value given by the Binomial expansion ; we have therefore

$$Q_n^m(\mu)=\frac{e^{-(n+1)\iota\pi}}{2\iota\sin n\pi}\cdot\frac{1}{2^n\,\Pi(n)}e^{-2n\pi\iota}(\mu^2-1)^{\frac{1}{2}m}\,\Sigma\,\frac{\Pi(n+m+r)}{\Pi(r)}\int_c^{(-1+,\,1-)}(t^2-1)^n\,t^{-n-m-r-1}\,dt.$$

The value of $\int_c^{(-1+,\,1-)}(t^2-1)^n\,t^p\,dt$ may be found as before by first considering the case in which the real parts of n and p are greater than unity ; in the present case,



the phases of t are π in the integral from 0 to -1, and from -1 to 0, and zero in the integrals from 0 to 1, and 1 to 0, hence the integral is equal to

$$\int_0^1(1-v^2)^n\,v^p\left(e^{n\pi\iota}e^{-2n\pi\iota}e^{\overline{p+1}\,\iota\pi}-e^{n\pi\iota}\,.\,1\,.\,e^{\overline{p+1}\,\iota\pi}+e^{n\pi\iota}\,.\,1\,.\,1-e^{-n\pi\iota}\,.\,1\,.\,1\right)dv,$$

or

$$\left(e^{n\pi\iota}-e^{-n\pi\iota}\right)\left(1+e^{p\pi\iota}\right).\,\tfrac{1}{2}\,\frac{\Pi(n)\,\Pi\left(\frac{p-1}{2}\right)}{\Pi\left(n+\frac{p+1}{2}\right)}.$$

The extension of this result to the case in which one or both of the quantities n, p have their real parts greater than -1, can be made as before.

We thus find after reduction, as in the preceding case,

$$Q_n^m(\mu)=\frac{\iota e^{(\frac{3}{2}m+\frac{1}{2}n)\pi\iota}}{2}\,.\,2^m\,.\,\frac{\Pi\left(\frac{n+m-1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\,F\left(\frac{n+m+1}{2},\frac{m-n}{2},\tfrac{1}{2},\mu^2\right)$$

$$+\,e^{(\frac{3}{2}m+\frac{1}{2}n)\pi\iota}\,.\,2^m.\,\frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\,\mu F\left(\frac{m-n+1}{2},\frac{m+n+2}{2},\tfrac{3}{2},\mu^2\right)\quad.\quad(26),$$

which is the formula that corresponds to (25).

16. In (25), change n into $-(n+1)$, we then find, after some transformation of the numerical factors,

$$Q_{-n-1}^{m}(\mu)$$

$$= -\tfrac{1}{2}e^{(m+n)\frac{\pi\iota}{2}}.\,2^m\frac{\cos\frac{n+m}{2}\pi}{\sin\frac{n-m}{2}\pi}\cdot\frac{\Pi\left(\frac{m+n-1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\,F\left(\frac{m+n+1}{2},\frac{m-n}{2},\tfrac{1}{2},\mu^2\right)$$

$$-\iota e^{(m+n)\frac{\pi\iota}{2}}.\,2^m\frac{\sin\frac{n+m}{2}\pi}{\cos\frac{n-m}{2}\pi}\cdot\frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}(\mu^2-1)^{\frac{1}{2}m}\,\mu F\left(\frac{m+n+2}{2},\frac{m-n+1}{2},\tfrac{3}{2},\mu^2\right).$$

On substituting these values of $Q_n^m(\mu)$, $Q_{-n-1}^m(\mu)$ in the formula

$$P_n^m(\mu) = \frac{e^{-m\pi\iota}}{\pi\cos n\pi}\{Q_n^m(\mu)\sin(n+m)\pi - Q_{-n-1}^m(\mu)\sin(n-m)\pi\},$$

we find, after some reduction, for the case in which the imaginary part of μ is positive,

$$P_n^m(\mu) = e^{-m\pi\iota}.\,2^m\cos\frac{n+m}{2}\pi.\frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(\frac{n-m}{2}\right)\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\,F\left(\frac{m+n+1}{2},\frac{m-n}{2},\tfrac{1}{2},\mu^2\right)$$

$$+e^{-m\pi\iota}.\,2^m\sin\frac{n+m}{2}\pi.\frac{\Pi\left(\frac{n+m}{2}\right)}{\Pi\left(\frac{n-m-1}{2}\right)\Pi(\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\,\mu F\left(\frac{m+n+2}{2},\frac{m-n+1}{2},\tfrac{3}{2},\mu^2\right) \quad (27).$$

When the imaginary part of μ is negative, we obtain in a similar manner a formula which differs from (27) only in having the exponential factor $e^{m\pi\iota}$ instead of $e^{-m\pi\iota}$.

From (27) it is seen that when $m+n$ is an integer, only one of the two hyper-geometric series is required to express $P_n^m(\mu)$, the first or the second according as $n+m$ is even or odd.

Definition of the functions P_n^m, Q_n^m for real values of μ which are less than unity.

17. The functions $P_n^m(\mu)$, $Q_n^m(\mu)$ have been defined as uniform functions of μ for all points in the plane of μ in which a cross-cut is made along the real axis from 1 to $-\infty$; at points indefinitely close to one another on opposite sides of the cross-cut the values of $P_n^m(\mu)$ or of $Q_n^m(\mu)$ will in general be different. We shall consider first

the values $P_n^m(\mu + 0 \,.\, \iota)$, $P_n^m(\mu - 0 \,.\, \iota)$ on opposite sides of the cross-cut for real values of μ lying between the values ± 1.

Referring to the expression (5), we see that in this case

$$P_n^m(\mu + 0 \,.\, \iota) = \frac{1}{\Pi(-m)} e^{-\frac{1}{2}m\pi\iota} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)$$

$$P_n^m(\mu - 0 \,.\, \iota) = \frac{1}{\Pi(-m)} e^{\frac{1}{2}m\pi\iota} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)$$

hence we have the relation

$$e^{\frac{1}{2}m\pi\iota} P_n^m(\mu + 0 \,.\, \iota) = e^{-\frac{1}{2}m\pi\iota} P_n^m(\mu - 0 \,.\, \iota)$$
$$= \frac{1}{\Pi(-m)} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) \quad (28).$$

It is convenient to define the function $P_n^m(\mu)$ for real values of μ between $+1$ and -1, in such a way that its value shall be real for real values of m and n; the definition which we give is that for such values of μ,

$$P_n^m(\mu) = e^{\frac{1}{2}m\pi\iota} P_n^m(\mu + 0 \,.\, \iota) = e^{-\frac{1}{2}m\pi\iota} P_n^m(\mu - 0 \,.\, \iota)$$
$$= \frac{1}{\Pi(-m)} \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right) \quad . \quad . \quad (29).$$

From (27), we find in this case

$$P_n^m(\mu) = 2^m \cos\frac{n+m}{2}\pi \,.\, \frac{\Pi\left(\frac{n+m-1}{2}\right)}{\Pi\left(\frac{n-m}{2}\right)\Pi\left(-\frac{1}{2}\right)} (1-\mu^2)^{\frac{1}{2}m} F\left(\frac{m+n+1}{2}, \frac{m-n}{2}, \frac{1}{2}, \mu^2\right)$$

$$+ 2^m \sin\frac{n+m}{2}\pi \,.\, \frac{\Pi\left(\frac{n+m}{2}\right)}{\Pi\left(\frac{n-m-1}{2}\right)\Pi\left(\frac{1}{2}\right)} (1-\mu^2)^{\frac{1}{2}m} F\left(\frac{m+n+2}{2}, \frac{m-n+1}{2}, \frac{3}{2}, \mu^2\right).$$

when $(1-\mu^2)^{\frac{1}{2}m}$ denotes $e^{\frac{1}{2}m \log_e(1-\mu^2)}$ and $\log_e(1-\mu^2)$ has its real value.

We see from (29) that when m is zero, or an even integer, the values of the function on the opposite sides of the cross-cut are equal, so that in this case the cross-cut is necessary, so far as the function $P_n^m(\mu)$ is concerned, only from -1 to $-\infty$.

18. Next, let us consider the values of $Q_n^m(\mu)$ on opposite sides of the cross-cut for values of μ lying between ± 1; from (15) we have

$$Q_n^m(\mu + 0 \,.\, \iota) = \frac{\pi e^{m\pi\iota}}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left\{e^{-(n+\frac{1}{2}m)\pi\iota}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)\right.$$
$$\left. - e^{\frac{1}{2}m\pi\iota}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1+\mu}{2}\right)\right\}$$

and

$$Q_n^m(\mu - 0 \,.\, \iota) = \frac{\pi e^{m\pi\iota}}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left\{e^{(n+\frac{1}{2}m)\pi\iota}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)\right.$$
$$\left. - e^{-\frac{1}{2}m\pi\iota}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1+\mu}{2}\right)\right\},$$

from these equations we find

$$e^{-\frac{1}{2}m\pi\iota} Q_n^m(\mu + 0 \,.\, \iota) - e^{\frac{1}{2}m\pi\iota} Q_n^m(\mu - 0 \,.\, \iota)$$
$$= \frac{\pi e^{m\pi\iota}}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left(e^{-(n+m)\pi\iota} - e^{(n+m)\pi\iota}\right)\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right),$$

hence we have the relation

$$e^{-\frac{1}{2}m\pi\iota} Q_n^m(\mu + 0 \,.\, \iota) - e^{\frac{1}{2}m\pi\iota} Q_n^m(\mu - 0 \,.\, \iota) = -\iota\pi e^{m\pi\iota} P_n^m(\mu) \quad . \quad . \quad (30),$$

where $P_n^m(\mu)$ is defined as in the last Art. In the particular case $m = 0$, (30) reduces to HEINE's relation

$$Q_n(\mu + 0 \,.\, \iota) - Q_n(\mu - 0 \,.\, \iota) = -\iota\pi P_n(\mu).$$

It is convenient to define $Q_n^m(\mu)$ for real values of μ between $+1$ and -1 by means of the equation

$$e^{m\pi\iota} \,.\, Q_n^m(\mu) = \tfrac{1}{2}\{e^{-\frac{1}{2}m\pi\iota} Q_n^m(\mu + 0 \,.\, \iota) + e^{\frac{1}{2}m\pi\iota} Q_n^m(\mu - 0 \,.\, \iota)\} \quad . \quad . \quad (31),$$

which gives us

$$Q_n^m(\mu) = \frac{\pi}{2\sin(n+m)\pi}\,\frac{1}{\Pi(-m)}\left\{\cos(n+m)\pi \,.\, \left(\frac{1+\mu}{1-\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1-\mu}{2}\right)\right.$$
$$\left. - \left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m} F\left(-n, n+1, 1-m, \frac{1+\mu}{2}\right)\right\}.$$

We have also

$$Q_n^m(\mu) = -2^{m-1}\sin\frac{m+n}{2}\pi\,\frac{\Pi\left(\frac{n+m-1}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m}{2}\right)}(1-\mu^2)^{\frac{m}{2}} F\left(\frac{n+m+1}{2}, \frac{m-n}{2}, \tfrac{1}{2}, \mu^2\right)$$

$$+ 2^{m-1}\cos\frac{m+n}{2} \,.\, \frac{\Pi\left(\frac{n+m}{2}\right)\Pi(-\frac{1}{2})}{\Pi\left(\frac{n-m-1}{2}\right)}(1-\mu^2)^{\frac{m}{2}}\mu\, F\left(\frac{m-n+1}{2}, \frac{m+n+2}{2}, \tfrac{3}{2}, \mu^2\right).$$

In the case $m = 0$, (31) agrees with HEINE's definition of the function $Q_n(\mu)$ for real values of μ between ± 1. Objections have been raised by SCHLÄFLI to this definition of $Q_n(\mu)$, on the ground that the function does not satisfy LEGENDRE's equation. There does not, however, appear to be in reality any question of principle involved; it is merely a matter of convenience to give a definition of $Q_n(\mu)$, which shall give real values of the function in the real axis, when n is real. It must, moreover, be remembered that although we have drawn the cross-cut along the real axis, it might have been drawn along any line we please joining the points ± 1, and thus the function $Q_n(\mu)$ may be regarded as satisfying the differential equation of LEGENDRE for points in or near the real axis, the surface over which the function is uniform being a different one from that which we have hitherto postulated, and the function being a linear combination of the two independent integrals of LEGENDRE's equation which we have defined and used.

19. For values of μ near that part of the real axis which is between -1 and $-\infty$, we see from the expression (10), that

$$Q_n^m(\mu + 0\,.\,\iota)$$
$$= \frac{e^{m\pi\iota}}{2^{n+1}}\cdot\frac{\Pi(n+m)\,\Pi(-\tfrac{1}{2})}{\Pi(n+\tfrac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\,.\,e^{-(n+1)\iota\pi}\frac{1}{(-\mu)^{n+m+1}}\,\mathrm{F}\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right),$$

$$Q_n^m(\mu - 0\,.\,\iota)$$
$$= \frac{e^{m\pi\iota}}{2^{n+1}}\cdot\frac{\Pi(n+m)\,\Pi(-\tfrac{1}{2})}{\Pi(n+\tfrac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\,.\,e^{(n+1)\iota\pi}\,.\,\frac{1}{(-\mu)^{n+m+1}}\,\mathrm{F}\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right),$$

where $(\mu^2-1)^{\frac{1}{2}m}$ here denotes $e^{\frac{1}{2}m\log_e(\mu^2-1)}$, the logarithm having its real positive value; we thus have

$$e^{n\pi\iota}\,Q_n^m(\mu + 0\,.\,\iota) = e^{-n\pi\iota}\,Q_n^m(\mu - 0\,.\,\iota) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and we may define $Q_n(\mu)$, for real values of μ between -1 and $-\infty$, to be equal to either of the expressions in (30) with its sign changed, thus

$$Q_n^m(\mu) = \frac{e^{m\pi\iota}}{2^{n+1}}\cdot\frac{\Pi(n+m)\,\Pi(-\tfrac{1}{2})}{\Pi(n+\tfrac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\frac{1}{(-\mu)^{n+m+1}}\,\mathrm{F}\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\tfrac{3}{2}, \frac{1}{\mu^2}\right)$$

where $(\mu^2-1)^{\frac{1}{2}m}$ has the meaning given above.

To express the relation between $P_n^m(\mu)$, $P_n^m(-\mu)$, $Q_n^m(\mu)$, $Q_n^m(-\mu)$ when μ is real and lies between ± 1.

20. We have from (20), if θ lies between 0 and $\frac{1}{2}\pi$,

$$P_n^m(-\cos\theta - 0 \cdot \iota) = e^{-n\pi\iota} P_n^m(\cos\theta + 0 \cdot \iota) - \frac{2\sin(n+m)\pi}{\pi} e^{-m\pi\iota} Q_n^m(\cos\theta + 0 \cdot \iota),$$

hence

$$e^{\frac{1}{2}m\pi\iota} P_n^m(-\cos\theta)$$
$$= e^{-n\pi\iota} \cdot e^{-\frac{1}{2}m\pi\iota} P_n^m(\cos\theta) - \frac{2\sin(n+m)\pi}{\pi} e^{-m\pi\iota} \cdot e^{\frac{1}{2}m\pi\iota}\{Q_n^m(\cos\theta) - \tfrac{1}{2}\iota\pi P_n^m(\cos\theta)\},$$

or

$$P_n^m(-\cos\theta) = P_n^m(\cos\theta)\{e^{-(n+m)\pi\iota} + \iota\sin(n+m)\pi\} - \frac{2\sin(n+m)\pi}{\pi} Q_n^m(\cos\theta),$$

hence we have

$$P_n^m(-\cos\theta) = P_n^m(\cos\theta)\cos(n+m)\pi - \frac{2}{\pi}\sin(n+m)\pi \cdot Q_n^m(\cos\theta) \quad . \quad (33).$$

It is easily verified by means of the formula in Arts. 17 and 18, that when $\theta = \frac{1}{2}\pi$,

$$(1 - \cos\overline{n+m}\,\pi) P_n^m(0) = -\frac{2}{\pi}\sin(n+m)\pi \cdot Q_n^m(0),$$

hence (33) does not involve a discontinuity in the value of $P_n^m(\cos\theta)$, as θ changes from 0 to π.

We have, also when θ is between 0 and $\frac{1}{2}\pi$,

$$Q_n^m(-\cos\theta - 0 \cdot \iota) = -e^{n\pi\iota} Q_n^m(\cos\theta + 0 \cdot \iota),$$

or

$$e^{\frac{m\pi}{2}}\left\{Q_n^m(-\cos\theta) + \frac{\iota\pi}{2} P_n^m(-\cos\theta)\right\} = -e^{n\pi\iota} \cdot e^{\frac{3m\pi}{2}}\left\{Q_n^m(\cos\theta) - \frac{\iota\pi}{2} P_n^m(\cos\theta)\right\},$$

hence, by means of (33), we obtain the relation

$$Q_n^m(-\cos\theta) = -Q_n^m(\cos\theta)\cos(n+m)\pi - \tfrac{1}{2}\pi\sin(n+m)\pi \cdot P_n^m(\cos\theta) \quad (34).$$

When m and n are real integers we have

$$P_n^m(-\cos\theta) = (-1)^{n+m} P_n^m(\cos\theta), \quad Q_n^m(-\cos\theta) = (-1)^{n+m+1} Q_n^m(\cos\theta).$$

Expansion of $P_n^m(\mu)$, $P_n^m(\mu)$ in powers of $\mu - \sqrt{\mu^2 - 1}$.

21. If we make $(\mu - \sqrt{\mu^2 - 1})^2$, for which we shall write ξ, the independent variable in the differential equation (2), we find that the equation takes the form

$$\xi^2(1-\xi)\frac{d^2W}{d\xi^2}+\xi\{\tfrac{1}{2}-m-(m+\tfrac{3}{2})\xi\}\frac{dW}{d\xi}-\tfrac{1}{4}(n-m)(n+m+1)(1-\xi)W=0.$$

Let $W=\xi^{\frac{1}{2}(n+m+1)}W'$; we then find, on substitution, the following differential equation for W':

$$\xi(1-\xi)\frac{d^2W'}{d\xi^2}+\{(n+\tfrac{3}{2})-(n+2m+\tfrac{5}{2})\xi\}\frac{dW'}{d\xi}-(n+m+1)(m+\tfrac{1}{2})W'=0.$$

Comparing this with the equation,

$$\xi(1-\xi)\frac{d^2W'}{d\xi^2}+\{\gamma-(\alpha+\beta+1)\xi\}\frac{dW'}{d\xi}-\alpha\beta W'=0,$$

which is satisfied by $W'=F(\alpha,\beta,\gamma,\xi)$, we see that if $\alpha=n+m+1$, $\beta=m+\frac{1}{2}$, $\gamma=n+\frac{3}{2}$, the equations are identical. It follows that our fundamental equation (1) is satisfied by

$$V_1=z^{-(n+m+1)}(\mu^2-1)^{\frac{1}{2}m}F\left(\tfrac{1}{2}+m,\ n+m+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right),$$

or by

$$V_2=z^{n-m}(\mu^2-1)^{\frac{1}{2}m}F\left(\tfrac{1}{2}+m,\ m-n,\ \tfrac{1}{2}-n\ \frac{1}{z^2}\right),$$

where z denotes $\mu+\sqrt{\mu^2-1}$.

In z we suppose $\sqrt{\mu^2-1}$ to be measured as hitherto, so that it has a single value at every point of the μ-plane in which a cross-cut is made along the real axis from $+1$ to $-\infty$.

It will be seen that mod z is greater than unity over the whole plane, the real part of $\sqrt{\mu^2-1}$ having the same sign as the real part of μ; on the imaginary axis z is purely imaginary.

In order to express the solutions V_1, V_2 in terms of $P_n^m(\mu)$, $Q_n^m(\mu)$, it will be sufficient to compare these solutions for values of μ whose modulus is very large, with the expressions (10), (22).

These latter formulæ show that for such values of μ, the principal parts of $Q_n^m(\mu)$, $P_n^m(\mu)$ are,

$$\frac{e^{m\pi\iota}}{2^{n+1}}\frac{\Pi(n+m)\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-(n+m+1)},$$

$$\frac{\sin(n+m)\pi}{2^{n+1}\cos n\pi}\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-n-m-1}+2^n\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m},$$

respectively; for similar values of μ we have

$$V_1=(2\mu)^{-n-m-1}(\mu^2-1)^{\frac{1}{2}m},\quad V_2=(2\mu)^{n-m}(\mu^2-1)^{\frac{1}{2}m}.$$

It follows that, since V_1, V_2 must both be linear functions of $P_n^m(\mu)$, $Q_n^m(\mu)$,

$$Q_n^m(\mu)=2^m e^{m\pi\iota}\frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}z^{-(n+m+1)}\,F\left(\tfrac{1}{2}+m,\ n+m+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right) \quad (35).$$

$$P_n^m(\mu)=2^m\frac{\sin(n+m)\pi}{\cos n\pi}\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}z^{-(n+m+1)}\,F\left(\tfrac{1}{2}+m,\ n+m+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right)$$
$$+\,2^m\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\,\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}z^{n-m}\,F\left(\tfrac{1}{2}+m,\ m-n,\ \tfrac{1}{2}-n,\ \frac{1}{z^2}\right). \quad . \quad (36).$$

These formulæ, (35), (36), are the expressions for $Q_n^m(\mu)$, $P_n^m(\mu)$ in series of powers of $\frac{1}{z}$; the series are convergent over the whole plane.

In the particular case $m=0$, we have

$$Q_n(\mu)=\frac{\Pi(n)\,\Pi(-\frac{1}{2})}{\Pi(n+\frac{1}{2})}z^{-(n+1)}\,F\left(\tfrac{1}{2},\ n+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right) \quad . \quad . \quad . \quad (37).$$

$$P_n(\mu)=\tan n\pi\,.\,\frac{\Pi(n)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}z^{-(n+1)}\,F\left(\tfrac{1}{2},\ n+1,\ n+\tfrac{3}{2},\ \frac{1}{z^2}\right)$$
$$+\,\frac{\Pi(n-\frac{1}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}z^n\,F\left(\tfrac{1}{2},\ -n,\ \tfrac{1}{2}-n,\ \frac{1}{z^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

The particular cases of (37), (38), in which n is a real integer, are given by HEINE.* It will be observed that the case of a real integral value of n is the only one in which $P_n(\mu)$ is represented by a single hypergeometric series. Exceptional cases of the four formulæ will be considered below.

A Second Class of Definite Integral Expressions for $P_n^m(\mu)$, $Q_n^m(\mu)$.

22. By using the definite integral forms which satisfy the hypergeometric equation, we see that the expressions

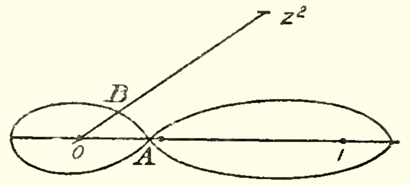
$$(\mu^2-1)^{\frac{1}{2}m}z^{-n-m-1}\int u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m}du \quad . \quad . \quad (A)$$

$$(\mu^2-1)^{\frac{1}{2}m}z^{-n-m-1}\int u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du \quad . \quad . \quad (B)$$

satisfy the differential equation (1), the integrals being taken along closed paths, such that after a complete description of such path the integrand attains its initial value.

* See 'Kugelfunctionen,' vol. 1, p. 129.

In (A) or (B), n may be changed into $-n-1$, and m into $-m$; we thus have eight different forms which satisfy the differential equation, and as in each case two independent closed paths may be chosen, we obtain, on the whole, sixteen definite integrals which satisfy the differential equation (1). We shall proceed to express these definite integrals in terms of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$.



Consider the integral (A), the path of integration consisting of a loop described positively round the point 1, followed by a loop positively round 0, another negatively round 1, and lastly a loop negatively round 0. When the loops are placed as in the figure, we shall suppose that the phases of u, $1-u$ initially at A are zero, and that the phase of $1-\frac{u}{z^2}$ is zero at B; when the loops are displaced into any other position the proper phases will be obtained by the principle of continuity. We have then

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,0+,1-,0-)} u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m}du$$

$$=\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\sum_{r=0}^{r=\infty}\frac{\Pi(m+r-\frac{1}{2})}{\Pi(r)\,\Pi(m-\frac{1}{2})}\frac{1}{z^{2r}}\int^{(1+,0+,1-,0-)} u^{n+m+r}(1-u)^{-m-\frac{1}{2}}du;$$

now

$$\int^{(1+,0+,1-,0-)} u^{n+m+r}(1-u)^{-m-\frac{1}{2}}du = e^{(n+r+\frac{3}{2})\pi\iota}\,\epsilon(n+m+r+1,\,-m+\tfrac{1}{2})$$

$$=e^{(n+r+\frac{3}{2})\pi\iota}\cdot(-1)^r\frac{(n+m+1)\ldots(n+m+r)}{(n+\frac{3}{2})\ldots(n+r+\frac{1}{2})}\,\epsilon(n+m+1,\,-m+\tfrac{1}{2})$$

$$=e^{\pi(n+\frac{3}{2})\iota}\frac{(n+m+1)\ldots(n+m+r)}{(n+\frac{3}{2})\ldots(n+r+\frac{1}{2})}\cdot 4\pi\sin(n+m)\pi\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(m-\frac{1}{2})}.$$

Hence

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,0+,1-,0-)} u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m}du$$

$$=-\iota e^{\pi n\iota}\,.\,4\pi\sin(n+m)\pi\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\,F\left(n+m+1,\,m+\tfrac{1}{2},\,n+\tfrac{3}{2},\,\frac{1}{z^2}\right).$$

Comparing this result with the formula (31), we have

$$Q_n^m(\mu)$$

$$=\iota e^{(m-n)\pi\iota}\,.\,2^m\frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\pi}\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,0+,1-,0-)} u^{n+m}(1-u)^{-\frac{1}{2}-m}\left(1-\frac{u}{z^2}\right)^{-\frac{1}{2}-m}du \quad (39).$$

If in this expression we put $u = hz$, and make h the independent variable, we have

$$Q_n^m(\mu) = \iota e^{(m-n)\pi\iota} \cdot 2^m \cdot \frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n+m)\pi} (\mu^2-1)^{\frac{1}{2}m} \int^{(\frac{1}{z}+,\, 0+,\, \frac{1}{z}-,\, 0-)} \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} dh \quad (40).$$

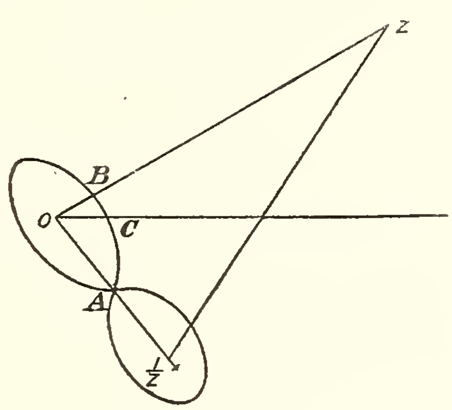
In the particular case $m = 0$, we find

$$Q_n(\mu) = \frac{\iota e^{-n\pi\iota}}{4 \sin n\pi} \int^{(\frac{1}{z}+,\, 0+,\, \frac{1}{z}-,\, 0-)} \frac{h^n}{(1-2\mu h+h^2)^{\frac{1}{2}}} dh \quad . \quad . \quad . \quad . \quad (41).$$

Using the theorem (12), we deduce from (40) the formula

$$Q_n^m(\mu) = \iota e^{(m-n)\pi\iota} \cdot 2^{-m}$$

$$\frac{\Pi(n+m)}{\Pi(n-m)} \cdot \frac{\Pi(-m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n-m)\pi} (\mu^2-1)^{-\frac{1}{2}m} \int^{(\frac{1}{z}+,\, 0+,\, \frac{1}{z}-,\, 0-)} \frac{h^{n-m}}{(1-2\mu h+h^2)^{\frac{1}{2}-m}} dh \quad (42).$$



It will be observed that in the formulæ (40), (41), (42), the phases of the integrand are to be measured as follows:—Draw the figure in the h-plane corresponding to the figure we have drawn in the u-plane; the points z, $\frac{1}{z}$ correspond to the points z^2, 1 respectively; the initial phase of h at A is to be the same as that of $\frac{1}{z}$, and will therefore be zero at the first passage through C; the phase of $1 - hz$ in the product $1 - 2\mu h + h^2$, which equals $(1-hz)\left(1-\frac{h}{z}\right)$, will be initially zero at A, and that of $1 - \frac{h}{z}$ will be zero at B. When the figure is displaced in any manner the phases can be found from the foregoing specifications by means of the principle of continuity.

23. If the real parts of $n + m + 1$ and $\frac{1}{2} - m$ are positive, the integral in (40) can be reduced to the form

$$(1 - e^{(n+m)2\pi\iota})\,(1 - e^{-(m+\frac{1}{2})2\pi\iota}) \int_0^{\frac{1}{z}} \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} dh,$$

thus we have

$$Q_n^m(\mu) = e^{m\pi\iota} 2^m \Pi(m - \tfrac{1}{2}) \Pi(-\tfrac{1}{2}) \frac{\cos m\pi}{\pi} (\mu^2 - 1)^{\frac{1}{2}m} \int_0^{\frac{1}{z}} \frac{h^{n+m}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh \qquad (43)$$

when the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive.

In particular

$$Q_n(\mu) = \int_0^{\frac{1}{z}} \frac{h^n}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

provided the real part of $n + 1$ is positive.

Similarly we find from (42)

$$Q_n^m(\mu) = e^{m\pi\iota} \cdot 2^{-m} \frac{\Pi(n + m)\Pi(-\frac{1}{2})}{\Pi(n - m)\Pi(m - \frac{1}{2})} (\mu^2 - 1)^{-\frac{1}{2}m} \int_0^{\frac{1}{z}} \frac{h^{n-m}}{(1 - 2\mu h + h^2)^{\frac{1}{2}-m}} dh \quad . \quad (45)$$

provided that the real parts of $n - m + 1$, $m + \frac{1}{2}$ are positive.

In the formulæ (43), (44), (45), change h into $\frac{1}{h}$, we then find

$$Q_n^m(\mu) = e^{m\pi\iota} \cdot 2^m \Pi(m - \tfrac{1}{2}) \Pi(-\tfrac{1}{2}) \frac{\cos m\pi}{\pi} (\mu^2 - 1)^{\frac{1}{2}m} \int_z^{\infty} \frac{h^{m-n-1}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh \quad (46)$$

when the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive.

$$Q_n(\mu) = \int_z^{\infty} \frac{h^{-n-1}}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where the real part of $n + 1$ is positive,

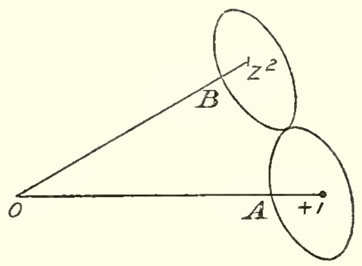
$$Q_n^m(\mu) = e^{m\pi\iota} 2^{-m} \frac{\Pi(n + m)\Pi(-\frac{1}{2})}{\Pi(n - m)\Pi(m - \frac{1}{2})} (\mu^2 - 1)^{-\frac{1}{2}m} \int_z^{\infty} \frac{h^{-n-m-1}}{(1 - 2\mu h + h^2)^{\frac{1}{2}-m}} dh \quad . \quad (48),$$

when the real parts of $n - m + 1$, $m + \frac{1}{2}$, are positive.

24. Next, consider the expression

$$(\mu^2 - 1)^{\frac{1}{2}m} z^{-n-m-1} \int^{(1+,\, z^2-)} u^{n+m} (1 - u)^{-\frac{1}{2}-m} \left(1 - \frac{u}{z^2}\right)^{-\frac{1}{2}-m} du.$$

Suppose that the phases of u, $1 - u$ are zero, when the point A in which the path cuts the real axis between 0 and 1 is reached, and that $1 - \frac{u}{z^2}$ has its phase zero at the point B in which $\frac{u}{z^2}$ is real and less than unity.



Transform the integral by means of the equation $u = z^2 - (z^2 - 1)v$, so that v is the independent variable, we have then,

$$-(\mu^2 - 1)^{\frac{1}{2}m} z^{-n-m-1} \int^{(1+,\,0-)} z^{2(n+m)} \left\{1 - \left(1 - \frac{1}{z^2}\right)v\right\}^{n+m} (z^2 - 1)^{-\frac{1}{2}-m}$$
$$(v-1)^{-\frac{1}{2}-m}\left(1 - \frac{1}{z^2}\right)^{-\frac{1}{2}-m} v^{-\frac{1}{2}-m}(z^2-1)\,dv,$$

or,

$$-(\mu^2 - 1)^{\frac{1}{2}m} z^{n+3m}(z^2 - 1)^{-2m} \int^{(1+,\,0-)} \left\{1 - \left(1 - \frac{1}{z^2}\right)v\right\}^{n+m} (v-1)^{-\frac{1}{2}-m} v^{-\frac{1}{2}-m}\,dv\,;$$

in this integral $v - 1$ has the phase zero at that point of the path in which v is positive and greater than unity; this expression may be written in the form

$$-\frac{1}{2^{2m}}(\mu^2 - 1)^{-\frac{1}{2}m} z^{n+m} \Sigma(-1)^r \left(1 - \frac{1}{z^2}\right)^r \frac{\Pi(n+m)}{\Pi(n+m-r)\,\Pi(r)} \int^{(1+,\,0-)} v^{-\frac{1}{2}-m+r}(v-1)^{-\frac{1}{2}-m}\,dv\,;$$

now

$$\int^{(1+,\,0-)} v^{-\frac{1}{2}-m+r}(v-1)^{-\frac{1}{2}-m}\,dv = 2\iota\cos m\pi\,.\,\frac{\Pi(-\frac{1}{2}-m)\,\Pi(-\frac{1}{2}-m+r)}{\Pi(-2m+r)},$$

hence the expression becomes

$$-\frac{1}{2^{2m}}(\mu^2 - 1)^{-\frac{1}{2}m} z^{n+m} \frac{\Pi(-\frac{1}{2}-m)\,\Pi(-\frac{1}{2}-m)}{\Pi(-2m)}\, 2\iota\cos m\pi\,\mathrm{F}\left(-n-m,\ \tfrac{1}{2}-m,\ 1-2m,\ 1-\frac{1}{z^2}\right),$$

or

$$-\frac{1}{2^{2m}}(\mu^2 - 1)^{-\frac{1}{2}m}\,.\,z^{n+m}\,.\,\frac{\Pi(2m-1)}{\{\Pi(m-\frac{1}{2})\}^2}\,.\,4\pi\iota\sin m\pi\,\mathrm{F}\left(-n-m,\ \tfrac{1}{2}-m,\ 1-2m,\ 1-\frac{1}{z^2}\right),$$

or

$$-(\mu^2 - 1)^{-\frac{1}{2}m} z^{n+m}\, 2\iota\sin m\pi\,.\,\frac{\Pi(m-1)\,\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,\mathrm{F}\left(-n-m,\ \tfrac{1}{2}-m,\ 1-2m,\ 1-\frac{1}{z^2}\right).$$

If we use the known transformation

$$F(\alpha, \beta, \gamma, x) = \frac{\Pi(\gamma - \alpha - \beta - 1)\,\Pi(\gamma - 1)}{\Pi(\gamma - \alpha - 1)\,\Pi(\gamma - \beta - 1)} F(\alpha, \beta, 1 + \alpha + \beta - \gamma, 1 - x)$$
$$+ \frac{\Pi(\alpha + \beta - \gamma - 1)\,\Pi(\gamma - 1)}{\Pi(\alpha - 1)\,\Pi(\beta - 1)} (1 - x)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, x),$$

we find for our expression

$$-2\iota \sin m\pi \,.\, (\mu^2 - 1)^{-\frac{1}{2}m} z^{n+m} \frac{\Pi(m-1)\,\Pi(-\frac{1}{2})}{\Pi(m - \frac{1}{2})} \left\{ \frac{\Pi(n - \frac{1}{2})\,\Pi(-2m)}{\Pi(n - m)\,\Pi(-m - \frac{1}{2})} \right.$$
$$F\left(-n - m, \tfrac{1}{2} - m, \tfrac{1}{2} - n, \frac{1}{z^2}\right) + \frac{\Pi(-n - \frac{3}{2})\,\Pi(-2m)}{\Pi(-n - m - 1)\,\Pi(-\frac{1}{2} - m)} z^{-2n-1}$$
$$\left. F\left(n - m + 1, -m + \tfrac{1}{2}, \tfrac{3}{2} + n, \frac{1}{z^2}\right) \right\},$$

which can be written

$$-\frac{\pi\iota}{2^{2m-1}} (\mu^2 - 1)^{-\frac{1}{2}m} \frac{1}{\Pi(m - \frac{1}{2})} \left\{ \frac{\Pi(n - \frac{1}{2})}{\Pi(n - m)} z^{n+m} F\left(-n - m, \tfrac{1}{2} - m, \tfrac{1}{2} - n, \frac{1}{z^2}\right) \right.$$
$$\left. + \frac{\sin(n + m)\pi}{\cos n\pi} \frac{\Pi(n + m)}{\Pi(n + \frac{1}{2})} z^{m-n-1} F\left(n + 1 - m, \tfrac{1}{2} - m, n + \tfrac{3}{2}, \frac{1}{z^2}\right) \right\}$$

or, if we use the transformation

$$F(\alpha, \beta, \gamma, x) = (1 - x)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, \gamma, x),$$

it can be written

$$-\frac{2\pi\iota}{\Pi(m - \frac{1}{2})} \left\{ \frac{\Pi(n - \frac{1}{2})}{\Pi(n - m)} z^{n-m} (\mu^2 - 1)^{\frac{1}{2}m} F\left(\tfrac{1}{2} + m, m - n, \tfrac{1}{2} - n, \frac{1}{z^2}\right) \right.$$
$$\left. + \frac{\sin(n + m)\pi}{\cos n\pi} \frac{\Pi(n + m)}{\Pi(n + \frac{1}{2})} z^{-m-n-1} (\mu^2 - 1)^{\frac{1}{2}m} F\left(\tfrac{1}{2} + m, n + m + 1, n + \tfrac{3}{2}, \frac{1}{z^2}\right) \right\};$$

on referring to (22), we see that it is equal to

$$-\frac{2\pi\iota\,\Pi(-\frac{1}{2})}{\Pi(m - \frac{1}{2})} \frac{1}{2^m} P_n^m(\mu);$$

we thus obtain the formula

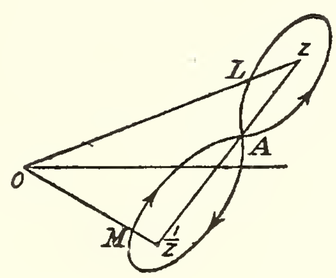
$$P_n^m(\mu)$$
$$= \frac{\iota}{2\pi} 2^m \frac{\Pi(m - \frac{1}{2})}{\Pi(-\frac{1}{2})} z^{-n-m-1} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(1+,\, z^2-)} u^{n+m} (1 - u)^{-\frac{1}{2}-m} \left(1 - \frac{u}{z^2}\right)^{-\frac{1}{2}-m} du \quad (49);$$

on making the substitution $u = hz$, this becomes

$$P_n^m(\mu) = \frac{\iota}{2\pi} 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(\frac{1}{z}+, z-)} \frac{h^{n+m}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh,$$

or

$$P_n^m(\mu) = \frac{1}{2\pi\iota} \cdot 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(z+, \frac{1}{z}-)} \frac{h^{n+m}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh \quad . \quad (50).$$



In this integral the phases of $1 - hz$, $1 - \frac{h}{z}$ are to be zero at the points M, L in which the lines joining the points $\frac{1}{z}$, z to the origin cut the path.

In the particular case $m = 0$, we have

$$P_n(\mu) = \frac{1}{2\pi\iota} \int^{(z+, \frac{1}{z}-)} \frac{h^n}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh,$$

which is reducible to

$$P_n(\mu) = \frac{1}{2\pi\iota} \int \frac{h^n}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51),$$

the integral being taken along a closed path which includes both the points z, $\frac{1}{z}$, and excludes the point 0.

By changing n into $-(n+1)$, we obtain the formulæ

$$P_n^m(\mu) = \frac{1}{2\pi\iota} 2^m \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int^{(z+, \frac{1}{z}-)} \frac{h^{m-n-1}}{(1 - 2\mu h + h^2)^{m+\frac{1}{2}}} dh \quad . \quad (52),$$

$$P_n(\mu) = \frac{1}{2\pi\iota} \int \frac{h^{-n-1}}{(1 - 2\mu h + h^2)^{\frac{1}{2}}} dh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53),$$

the integral (53) being taken as in (51).

25. Next consider the expression (B), the path being the same as in Art. 22; we find

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,\,0+,\,1-,\,0-)} u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du$$

$$=\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\Sigma\frac{\Pi(n+m+r)}{\Pi(r)\Pi(n+m)}\frac{1}{z^{2r}}\int^{(1+,\,0+,\,1-,\,0-)} u^{r+m-\frac{1}{2}}(1-u)^{n-m}du.$$

Now

$$\int^{(1+,\,0+,\,1-,\,0-)} u^{r+m-\frac{1}{2}}(1-u)^{n-m}du = e^{(n+r+\frac{3}{2})\iota\pi}\,\epsilon\left(r+m+\tfrac{1}{2},\,n-m+1\right)$$

$$=e^{(n+r+\frac{3}{2})\iota\pi}\,.\,(-1)^r\frac{(m+\frac{1}{2})\ldots(m+r-\frac{1}{2})}{(n+\frac{3}{2})\ldots(n+r+\frac{1}{2})}\,\epsilon\left(m+\tfrac{1}{2},\,n-m+1\right)$$

$$=e^{(n+\frac{3}{2})\iota\pi}\frac{(m+\frac{1}{2})\ldots(m+r-\frac{1}{2})}{(n+\frac{3}{2})\ldots(n+r+\frac{1}{2})}\,.\,4\cos m\pi\sin(n-m)\pi\frac{\Pi(m-\frac{1}{2})\,\Pi(n-m)}{\Pi(n+\frac{1}{2})},$$

hence the expression becomes

$$e^{(n+\frac{3}{2})\iota\pi}\,.\,4\cos m\pi\sin(n-m)\pi$$

$$.\,\frac{\Pi(m-\frac{1}{2})\,\Pi(n-m)}{\Pi(n+\frac{1}{2})}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\,.\,F\left(n+m+1,\,m+\tfrac{1}{2},\,n+\tfrac{3}{2},\,\frac{1}{z^2}\right).$$

Comparing this with the expression (31), for $Q_n^m(\mu)$, we see that

$$Q_n^m(\mu)=\iota e^{(m-n)\iota\pi}\frac{2^m}{4\cos m\pi\sin(n-m)\pi}\,\frac{\Pi(n+m)}{\Pi(n-m)}\,\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}$$

$$.\int^{(1+,\,0+,\,1-,\,0-)} u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du\;.\qquad(54).$$

26. We shall now consider the expression

$$\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int^{(1+,\,z^2+,\,1-,\,z^2-)} u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1}du.$$

Using the transformation $u=z^2-(z^2-1)v$, the expression becomes

$$-(\mu^2-1)^{-\frac{1}{2}m}\frac{z^{n+m}}{2^{2m}}\int^{(1+,\,z^2+,\,1-,\,z^2-)}\left\{1-\frac{z^2-1}{z^2}v\right\}^{m-\frac{1}{2}}(v-1)^{n-m}v^{-n-m-1}dv,$$

which can be expanded into the form

$$-(\mu^2-1)^{-\frac{1}{2}m}\frac{z^{n+m}}{2^{2m}}\Sigma\frac{\Pi(m-\frac{1}{2})}{\Pi(r)\Pi(m-r-\frac{1}{2})}(-1)^r\left(1-\frac{1}{z^2}\right)^r\int^{(1+,\,z^2+,\,1-,\,z^2-)}(v-1)^{n-m}v^{r-n-m-1}dv.$$

Now

$$\int^{(1+,\, z^2+,\, 1-,\, z^2-)} (v-1)^{n-m}\, v^{r-n-m-1}\, dv = e^{(r-2m+1)\pi\iota} \in (n-m+1\ \ r-n-m)$$

$$= -e^{(r-2m)\pi\iota}\,.\,(-1)^r\,.\,\frac{(-n-m)(1-n-m)\ldots(r-n-m-1)}{(1-2m)(2-2m)\ldots(r-2m)} \in (n-m+1,\ -n-m)$$

$$= e^{-2m\pi\iota}\,\frac{(-n-m)\ldots(r-n-m-1)}{(1-2m)\ldots(r-2m)}\,.\,4\pi\sin(n-m)\pi\sin(n+m)\pi\,\frac{\Pi(n-m)\,\Pi(-n-m-1)}{\Pi(-2m)}$$

$$= -e^{-2m\pi\iota}\,.\,2^{2m}\,.\,2\pi\sin 2m\pi\sin(n-m)\pi$$

$$.\,\frac{\Pi(n-m)}{\Pi(n+m)}\,\frac{\Pi(m-\frac{1}{2})\,\Pi(m-1)}{\Pi(-\frac{1}{2})}\,.\,\frac{(-n-m)\ldots(r-n-m-1)}{(1-2m)\ldots(r-2m)}\,.$$

Thus the expression becomes

$$e^{-2m\pi\iota}\,.\,2\pi\sin 2m\pi\sin(n-m)\pi$$

$$\frac{\Pi(n-m)}{\Pi(n+m)}\,\frac{\Pi(m-\frac{1}{2})\,\Pi(m-1)}{\Pi(-\frac{1}{2})}\, z^{n+m}\,(\mu^2-1)^{-\frac{1}{2}m}\, \mathrm{F}\left(\tfrac{1}{2}-m,\ -n-m,\ 1-2m,\ 1-\frac{1}{z^2}\right).$$

As in Art. 21, this expression can be shown to be equal to

$$e^{-2m\pi\iota}\,.\,4\pi^2\cos m\pi\,.\,\sin(n-m)\pi\,.\,\frac{\Pi(n-m)\,\Pi(m-\frac{1}{2})}{\Pi(n+m)\,\Pi(-\frac{1}{2})}\,\frac{1}{2^m}\,\mathrm{P}_n^m(\mu),$$

hence we have the formula

$$\mathrm{P}_n^m(\mu) = \frac{2^m e^{2m\pi\iota}}{4\pi^2}\,\frac{1}{\cos m\pi\sin(n-m)\pi}\,\frac{\Pi(n+m)\,\Pi(-\frac{1}{2})}{\Pi(n-m)\,\Pi(m-\frac{1}{2})}\,\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}$$

$$\int^{(1+,\, z^2+,\, 1-,\, z^2-)} u^{m-\frac{1}{2}}\,(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1} du \quad . \quad (55),$$

and in particular when $m = 0$

$$\mathrm{P}_n(\mu) = \frac{1}{4\pi^2\sin n\pi}\,\frac{1}{z^{n+1}}\int^{(1+,\, z^2+,\, 1-,\, z^2-)} u^{-\frac{1}{2}}\,(1-u)^n\left(1-\frac{u}{z^2}\right)^{-n-1} du \quad . \quad (56).$$

A third class of definite integrals which represent the function $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_n^m(\mu)$.

27. If we put $\mu^2 = \mu'$, we find that the differential equation (2) becomes, when μ' is made the independent variable,

$$\mu'(1-\mu')\frac{d^2\mathrm{W}}{d\mu'^2} + \left(\tfrac{1}{2} - \frac{2m+3}{2}\mu'\right)\frac{d\mathrm{W}}{d\mu'} + \frac{(n-m)(n+m+1)}{4}\mathrm{W} = 0.$$

We see that this equation is the differential equation satisfied by the hypergeometric series $F(\alpha, \beta, \gamma, \mu')$, where $\alpha = \frac{m-n}{2}$, $\beta = \frac{m+n+1}{2}$, $\gamma = \frac{1}{2}$; we thus see that the differential equation (1) is satisfied by either of the expressions

$$(\mu^2-1)^{\frac{1}{2}m}\int u^{\frac{n+m-1}{2}}(1-u)^{\frac{-m-n+1}{2}}(1-\mu^2 u)^{\frac{n-m}{2}}\,du,$$

$$(\mu^2-1)^{\frac{1}{2}m}\int u^{\frac{m-n-2}{2}}(1-u)^{\frac{n-m}{2}}(1-\mu^2 u)^{-\frac{n+m+1}{2}}\,du,$$

when, as in the other cases, the integrals are taken along closed paths. We thus obtain a third class of definite integrals, by which the functions $P_n^m(\mu)$, $Q_n^m(\mu)$ can be represented. It is unnecessary to obtain the exact expressions for the functions in four of these definite integral expressions, as all the results of interest may be obtained from the two classes which have been already considered.

The existence of these three classes of definite integrals which satisfy the fundamental differential equation (1) is equivalent to the result obtained by OLBRICHT, that the equation is satisfied by three distinct RIEMANN's P-functions,

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ \frac{1}{2}m & -n & \frac{1}{2}m & \frac{1-\mu}{2} \\ -\frac{1}{2}m & n+1 & -\frac{1}{2}m & \end{Bmatrix},$$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ -\frac{n}{2} & m & -\frac{n}{2} & \frac{\mu+\sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}} \\ \frac{n+1}{2} & -m & \frac{n+1}{2} & \end{Bmatrix},$$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ -\frac{n}{2} & \frac{m}{2} & 0 & \frac{1}{1-\mu^2} \\ \frac{n+1}{2} & -\frac{m}{2} & \frac{1}{2} & \end{Bmatrix}.$$

Expansion of $P_n^m(\mu)$, $Q_n^m(\mu)$ in powers of $\frac{\mu\pm\sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}}$.

28. In the formula

$$Q_n^m(\mu) = \iota e^{(m-n)\pi\iota}\,.\,2^m\,.\,\frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\pi}(\mu^2-1)^{\frac{1}{2}m}\int^{(\frac{1}{z}+,\,0+,\,\frac{1}{z}-,\,0-)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh \quad . \quad (40)$$

change from h to w as independent variable in the integral, where $h = \frac{1}{z}(1 - w)$; we then have

$$Q_n^m(\mu) = -\iota e^{(m-n)\pi\iota} \cdot 2^m \cdot \frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n+m)\pi}$$

$$\cdot \frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\int^{(0+,\,1+,\,0-,\,1-)} u^{-m-\frac{1}{2}+r}(1-u)^{n+m}\left(1+\frac{u}{z^2-1}\right)^{-m-\frac{1}{2}} du$$

$$= -\iota e^{(m-n)\pi\iota} \cdot 2^m \cdot \frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi \sin(n+m)\pi}$$

$$\cdot \frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\sum_{r=0}(-1)^r \cdot \frac{\Pi(m+\frac{1}{2}+r)}{\Pi(r)\,\Pi(m-\frac{1}{2})}\frac{1}{(z^2-1)^r}\int^{(0+,\,1+,\,0-,\,1-)} u^{-m-\frac{1}{2}+r}(1-u)^{n+m}\,du.$$

On evaluating the definite integrals we find

$$Q_n^m(\mu)$$
$$= e^{m\pi\iota} \cdot 2^m \cdot \frac{\Pi(-\frac{1}{2})\,\Pi(n+m)}{\Pi(n+\frac{1}{2})}\frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\,F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, n+\tfrac{3}{2}, \frac{1}{1-z^2}\right) \quad (57),$$

which gives an expression for $Q_n^m(\mu)$ in powers of $\frac{\mu - \sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}}$, which is convergent for the part of the plane over which this expression has its modulus less than unity.

Using the formula

$$P_n^m(\mu) = \frac{e^{-m\pi\iota}}{\pi \cos n\pi}\{Q_n^m(\mu)\sin(n+m)\pi - Q_{-n-1}^m(\mu)\sin(n-m)\pi\},$$

we find from (57)

$$P_n^m(\mu)$$
$$= \frac{2^m\Pi(-\frac{1}{2})}{\pi}\left\{\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\frac{\sin(n+m)\pi}{\cos n\pi}\frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\,F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, n+\tfrac{3}{2}, \frac{1}{1-z^2}\right)\right.$$
$$\left. + \frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)}\frac{z^{m+n+1}}{(z^2-1)^{m+\frac{1}{2}}}(\mu^2-1)^{\frac{1}{2}m}\,F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, -n+\tfrac{1}{2}, \frac{1}{1-z^2}\right)\right\}.$$

Now by the known formula for the transformation of a hypergeometric series whose fourth element is $1 - x$, into a linear function of series whose fourth element is x, we find

$$F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, n+\tfrac{3}{2}, -\frac{z^2}{1-z^2}\right)$$
$$= \frac{\Pi(n-\frac{1}{2})\,\Pi(n+\frac{1}{2})}{\Pi(n-m)\,\Pi(n+m)}F\left(m+\tfrac{1}{2}, -m+\tfrac{1}{2}, \tfrac{1}{2}-n, \frac{1}{1-z^2}\right)$$
$$+ \frac{\Pi(-n-\frac{3}{2})\,\Pi(n+\frac{1}{2})}{\Pi(m-\frac{1}{2})\,\Pi(-m-\frac{1}{2})}\frac{1}{(1-z^2)^{n+\frac{1}{2}}}\left(-\frac{z^2}{1-z^2}\right)^{-n-\frac{1}{2}}F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m, n+\tfrac{3}{2}, \frac{1}{1-z^2}\right),$$

and thence, after some reduction

$$P_n^m(\mu) = \frac{2^m \Pi(-\frac{1}{2})}{\pi} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})} \left\{ e^{-(m-\frac{1}{2})\pi\iota} \frac{z^{m-n}}{(z^2-1)^{m+\frac{1}{2}}} (\mu^2-1)^{\frac{1}{2}m} F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m, n+\tfrac{3}{2}, \frac{1}{1-z^2}\right) \right.$$
$$\left. + \frac{z^{m+n+1}}{(z^2-1)^{m+\frac{1}{2}}} (\mu^2-1)^{\frac{1}{2}m} F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m, n+\tfrac{3}{2}, \frac{-z^2}{1-z^2}\right) \right\}. \quad (58).$$

This formula expresses $P_n^m(\mu)$ in powers of $\dfrac{\mu \pm \sqrt{\mu^2-1}}{2\sqrt{\mu^2-1}}$.

29. Let $\mu = \cos\theta$, then remembering that $P_n^m(\cos\theta) = e^{\frac{1}{2}m\pi\iota} P_n^m(\cos\theta + 0 . \iota)$, we have

$$P_n^m(\cos\theta)$$
$$= \frac{2^m \Pi(-\frac{1}{2})}{\pi} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})} e^{m\pi\iota} \sin^m\theta \left\{ e^{-(m-\frac{1}{2})\pi\iota} \frac{e^{-(n+\frac{1}{2})\iota\theta}}{(2e^{\frac{1}{2}\iota\pi}\sin\theta)^{m+\frac{1}{2}}} F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m, n+\tfrac{3}{2}, \frac{-e^{-\iota\theta}}{2e^{\iota\pi/2}\sin\theta}\right) \right.$$
$$\left. + \frac{e^{(n+\frac{1}{2})\iota\theta}}{(2e^{\iota\pi/2}\sin\theta)^{m+\frac{1}{2}}} F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m, n+\tfrac{3}{2}, \frac{e^{\iota\theta}}{2e^{\iota\pi/2}\sin\theta}\right) \right\}.$$

Hence

$$P_n^m(\cos\theta) = \frac{2}{\sqrt{\pi}} \frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})} \left[\frac{\cos\left(\overline{n+\frac{1}{2}}\,\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{1}{2}}} + \frac{1^2-4m^2}{2 . \overline{2n+3}} \frac{\cos\left(\overline{n+\frac{3}{2}}\,\theta - \frac{3\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{3}{2}}} \right.$$
$$\left. + \frac{1^2-4m^2 . 3^2-4m^2}{2 . 4 . \overline{2n+3} . \overline{2n+5}} \frac{\cos\left(\overline{n+\frac{5}{2}}\,\theta - \frac{5\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{5}{2}}} + \ldots \right] . (59);$$

this series represents $P_n^m(\cos\theta)$ for unrestricted values of n and m, provided it is convergent, which is the case when $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$.

To find the corresponding expression for $Q_n^m(\cos\theta)$, we have from (57),

$$Q_n^m(\cos\theta + 0 . \iota)$$
$$= e^{m\pi\iota} . 2^m . \frac{\Pi(-\frac{1}{2})\Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{e^{(m-n)\iota\theta}(e^{\frac{1}{2}\iota\pi}\sin\theta)^m}{e^{(m+\frac{1}{2})\iota\theta}(2e^{\iota\pi/2}\sin\theta)^{m+\frac{1}{2}}} F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m, n+\tfrac{3}{2}, \frac{-e^{-\iota\theta}}{2e^{\iota\pi/2}\sin\theta}\right)$$
$$= e^{m\pi\iota} \frac{\Pi(-\frac{1}{2})\Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{e^{-(n+\frac{1}{2})\theta\iota - \iota\pi/4}}{(2\sin\theta)^{\frac{1}{2}}} \left\{ 1 - \frac{1^2-4m^2}{2 . \overline{2n+3}} \frac{e^{-\iota(\theta+\pi/2)}}{2\sin\theta} \right.$$
$$\left. + \frac{1^2-4m^2 . 3^2-4m^2}{2 . 4 . \overline{2n+3} . \overline{2n+5}} \frac{e^{-2\iota(\theta+\pi/2)}}{2\sin\theta} - \ldots \right\}$$

Similarly we find

$$Q_n^m(\cos\theta - 0 . \iota) = e^{m\pi\iota} . \frac{\Pi(-\frac{1}{2})\Pi(n+m)}{\Pi(n+\frac{1}{2})} . \frac{e^{\iota(n+\frac{1}{2})\theta + \iota\pi/4}}{(2\sin\theta)^{\frac{1}{2}}} \left\{ 1 - \frac{1^2-4m^2}{2 . \overline{2n+3}} . \frac{e^{\iota(\theta+\pi/2)}}{2\sin\theta} + \ldots \right\}$$

thence using the relation

$$e^{m\pi\iota}\,Q_n^m(\cos\theta) = \tfrac{1}{2}\{e^{-\frac{1}{2}m\pi\iota}\,Q_n^m(\mu + 0\,.\,\iota) + e^{\frac{1}{2}m\pi\iota}\,Q_n^m(\mu - 0\,.\,\iota)\} \quad . \quad . \quad (29)$$

we find

$$Q_n^m(\cos\theta) = \sqrt{\pi}\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\left\{\frac{\cos\left(\overline{n+\frac{1}{2}}\,\theta + \frac{\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{1}{2}}} - \frac{1^2 - 4m^2}{2\,.\,2n+3}\,.\,\frac{\cos\left(\overline{n+\frac{3}{2}}\,\theta + \frac{3\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{3}{2}}}\right.$$

$$\left. + \frac{1^2 - 4m^2\,.\,3^2 - 4m^2}{2\,.\,4\,.\,2n+3\,.\,2n+5}\,\frac{\cos\left(\overline{n+\frac{5}{2}}\,\theta + \frac{5\pi}{4} + \frac{m\pi}{2}\right)}{(2\sin\theta)^{\frac{5}{2}}} - \ldots\right\} \quad (60),$$

the convergency condition for this series is the same as for (59).

It may be remarked that the series (57) is convergent if μ is a real positive quantity greater than unity, ($= \cosh\psi$) provided $\psi > \frac{1}{2}\log 2$, or $\cosh\psi > \frac{3}{2\sqrt{2}}$; in that case we have

$$Q_n^m(\cosh\psi) = e^{m\pi\iota}\,\sqrt{\pi}\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\,\frac{e^{-(n+\frac{1}{2})\psi}}{(2\sinh\psi)^{\frac{1}{2}}}\left\{1 - \frac{1^2 - 4m^2}{2\,.\,2n+3}\,\frac{e^{-\psi}}{2\sinh\psi}\right.$$

$$\left. + \frac{1^2 - 4m^2\,.\,3^2 - 4m^2}{2\,.\,4\,.\,2n+3\,.\,2n+5}\,.\,\frac{e^{-2\psi}}{(2\sinh\psi)^2} - \ldots\right\} \quad (61),$$

where $\cosh\psi > \frac{3}{2\sqrt{2}}$.

The corresponding series for $P_n^m(\cosh\psi)$ is not convergent.

30. The series (59), (60) are convergent, provided θ lies between $\frac{\pi}{6}$ and $\frac{5\pi}{6}$; it will now however be shown that in case m and n are real, and $n + m - 1$, $\frac{1}{2} + m$ are positive, a finite number of terms of the series will represent approximately the values of $P_n^m(\cos\theta)$, $Q_n^m(\cos\theta)$ when the restriction as to the value of θ is removed. To prove this, it will be necessary to estimate the remainder after any number of terms in the series (57).

It has been shown by DARBOUX* that if x is a complex quantity, MACLAURIN'S theorem takes the form

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}\,.\,\lambda\,.\,f^r(\theta' x)$$

where θ' is a proper fraction, and λ denotes some quantity whose modulus is not greater than unity. Applying this result to the expansion in Art. 28, by which (57) was obtained, we see that the remainder after r terms of the series for $Q_n^m(\cos\theta + 0\,.\,\iota)$ is

$$-\iota\,.\,e^{m\pi\iota}\,.\,\frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{4\pi\sin(m+n)\pi}\,e^{-(n+\frac{1}{2})\iota\theta - \frac{\iota\pi}{4}}\,.\,\frac{1}{\sqrt{2\sin\theta}}\,.\,\frac{1}{(z^2-1)^r}\,(-1)^r\,\frac{\Pi(m+\frac{1}{2}+r)}{\Pi(r)\,\Pi(m-\frac{1}{2})}$$

$$\int^{(0+,\,1+,\,0-,\,1-)} u^{-m-\frac{1}{2}+r}\,(1-u)^{n+m}\,\lambda\left(1 + \frac{\theta' u}{z^2-1}\right)^{-m-\frac{1}{2}} du,$$

* See LIOUVILLE'S 'Journal,' Series III., vol. 4.

where λ is a quantity whose modulus is less than unity; suppose r so great that $r - m + \frac{1}{2}$ is positive, the integral may then be replaced by

$$- 4 \sin (m - \tfrac{1}{2} + r) \pi \sin (n + m) \pi \int_0^1 u^{-m-\frac{1}{2}+r} (1 - u)^{n+m} \lambda \left(1 + \frac{\theta' u}{z^2 - 1}\right)^{-m-\frac{1}{2}} du$$

where the integral is now taken along the real axis. We have now

$$1 + \frac{\theta' u}{z^2 - 1} = 1 - \frac{\iota \theta' u \,.\, e^{-\iota\theta}}{2 \sin \theta} = 1 - \frac{\theta' u}{2} - \iota \frac{\theta' u \cot \theta}{2},$$

the modulus of this expression is $\left\{\left(1 - \frac{\theta' u}{2}\right)^2 + \frac{1}{4}\theta'^2 u^2 \cot^2 \theta\right\}^{\frac{1}{2}}$, which is always greater than $\left(1 - \frac{\theta' u}{2}\right)$, and therefore always greater than $\frac{1}{2}$; it follows that the modulus of $\left(1 + \frac{\theta' u}{z^2 - 1}\right)^{-m-\frac{1}{2}}$ is always less than $2^{m+\frac{1}{2}}$, hence also the modulus of $\lambda \left(1 + \frac{\theta' u}{z^2 - 1}\right)^{-m-\frac{1}{2}}$ is always less than $2^{m+\frac{1}{2}}$; put $\lambda \left(1 + \frac{\theta' u}{z^2 - 1}\right)^{-m-\frac{1}{2}} = \rho (\cos \chi + \iota \sin \chi)$, where ρ, χ are functions of u, and $\rho < 2^{m+\frac{1}{2}}$ for all values of u; we have then

$$\int_0^1 u^{-m-\frac{1}{2}+r} (1-u)^{n+m} \lambda \left(1+\frac{\theta' u}{z^2 - 1}\right)^{-m-\frac{1}{2}} du = \int_0^1 u^{-m-\frac{1}{2}+r} (1-u)^{n+m} \rho (\cos \chi + \iota \sin \chi)\, du,$$

in this integral the real part and the coefficient of ι in the imaginary part are each less than $2^{m+\frac{1}{2}} \int_0^1 u^{-m-\frac{1}{2}+r} (1 - u)^{n+m} du$, hence the modulus of the expression is less than $2^{m+1} \int_0^1 u^{-m-\frac{1}{2}+r} (1 - u)^{n+m} du$. Now the $r + 1^{\text{th}}$ term of the series (57) is obtained by putting $\theta' = 0$, $\lambda = 1$ in the expression for the remainder after r terms, it has thus been shown that the modulus of the remainder after r terms is less than 2^{m+1} times the modulus of the $r + 1^{\text{th}}$ term, and this is true for all values of θ, not merely for those for which the series is convergent. The two quantities $e^{-m\pi\iota} Q_n^m (\cos \theta + 0 \,.\, \iota)$, $e^{-m\pi\iota} Q_n^m (\cos \theta - 0 \,.\, \iota)$ are conjugate complex quantities, hence the remainders after r terms in the series for $Q_n^m (\cos \theta + 0 \,.\, \iota)$, $Q_n^m (\cos \theta - 0 \,.\, \iota)$ are of the form

$$(X \pm \iota Y)\, e^{m\pi\iota} \frac{\Pi(-\frac{1}{2})\, \Pi(n+m)}{\Pi(n+\frac{1}{2})} \frac{e^{\mp(n+\frac{1}{2})\iota\theta \mp \iota\pi/4}}{(2 \sin \theta)^{\frac{1}{2}}} (-1)^r \frac{1^2 - 4m^2 \ldots \overline{2r-1}|^2 - 4m^2}{2 \,.\, 4 \ldots 2r \,.\, 2n+3 \ldots 2n+2r+1} \frac{e^{\mp r\iota(\theta+\pi/2)}}{(2 \sin \theta)^r},$$

when X and Y are each less than $2^{m+\frac{1}{2}}$; using (29) we now see that the remainder in the series (56) for $Q_n^m (\cos \theta)$, is of the form

$$(X^2 + Y^2)^{\frac{1}{2}} \,.\, \sqrt{\pi} \,.\, \frac{\Pi(n + m)}{\Pi(n + \frac{1}{2})} (-1)^r \frac{1^2 - 4m^2 \,.\, 3^2 - 4m^2 \ldots (2r - 1)^2 - 4m^2}{2 \,.\, 4 \ldots 2r \,.\, (2n + 3) \ldots (2n + 2r + 1)}$$

$$\frac{\cos\left(n + \frac{2r+1}{2}\,\theta + \frac{2r+1}{4}\,\pi + \frac{m\pi}{2} - \beta\right)}{(2 \sin \theta)^{r+\frac{1}{2}}},$$

when β denotes $\tan^{-1}\frac{Y}{X}$; finally this remainder is numerically less than

$$2^{m+1}\sqrt{\pi}\,.\,\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})}\,\frac{1^2-4m^2\,.\,3^2-4m^2\ldots(2r+1)^2-4m^2}{2\,.\,4\ldots 2r\,.\,(2n+3)\ldots(2n+2r+1)}\,\frac{1}{(2\sin\theta)^{r+\frac{1}{2}}},$$

it has thus been shown that for all real values of m and n such that $n+m-1$, $m+\frac{1}{2}$ are positive, the series (60) may be used to obtain an approximate value of $Q_n^m(\cos\theta)$ for all values of θ between 0 and π; if the first r terms of the series are taken, the error is certainly less than 2^{m+1} times what we get by writing unity for the cosine in the $r+1^{\text{th}}$ term, r being any number greater than $m+\frac{1}{2}$. A particular case of this theorem, namely, that in which $m=0$, and n is an integer, has already been obtained otherwise by STIELTJES.*

It has been shown that $\iota\pi e^{m\pi\iota}\,P_n^m(\mu) = e^{\frac{1}{2}m\pi\iota}\,Q_n^m(\mu-0\,.\,\iota) - e^{-\frac{1}{2}m\pi\iota}\,Q_n^m(\mu+0\,.\,\iota)$, it therefore follows that the series (59) for $P_n^m(\cos\theta)$, may, under the same conditions as regards n, m, be used to obtain approximate values of $P_n^m(\cos\theta)$, the error being limited in the same manner as in the case of (60).

Approximate Values of $P_n^m(\mu)$, $Q_n^m(\mu)$ when n is a large real quantity and μ is real.

31. It is well known that when n is a large integer, $\frac{\Pi(n)}{\Pi(n+\frac{1}{2})}$ is approximately equal to $\frac{1}{\sqrt{n}}$, it follows from (59), (60) that the asymptotic values of $\frac{\Pi(n)}{\Pi(n+m)}P_n^m(\cos\theta)$, $\frac{\Pi(n)}{\Pi(n+m)}Q_n^m(\cos\theta)$ for a large real integral value of n are given by

$$\frac{\Pi(n)}{\Pi(n+m)}P_n^m(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}}\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right)$$

$$\frac{\Pi(n)}{\Pi(n+m)}Q_n^m(\cos\theta) = e^{m\pi\iota}\sqrt{\frac{\pi}{2n\sin\theta}}\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}+\frac{m\pi}{2}\right). \quad . \quad (62).$$

These expressions are generalizations of the known asymptotic values

$$P_n(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}}\sin\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}\right)$$

which was given by LAPLACE, and

$$Q_n(\cos\theta) = \sqrt{\frac{\pi}{2n\sin\theta}}\cos\left(\overline{n+\tfrac{1}{2}}\,\theta+\frac{\pi}{4}\right)$$

given by HEINE.†

* 'Annales de la Faculté des Sciences de Toulouse,' vol. 4, in a paper entitled "Sur les Polynômes de Legendre."

† 'Kugelfunctionen,' vol. 1, p. 175.

To obtain a closer approximation for large values of n, we use the theorem

$$\Pi(n) = \sqrt{2\pi n} \,.\, e^{-n} n^n \left(1 + \frac{1}{12n} + \dots\right)$$

we have

$$\frac{\Pi(n)}{\Pi(n+\frac{1}{2})} = \frac{\sqrt{n} \,.\, e^{-n} \,.\, n^n \left(1 + \dfrac{1}{12n}\right)}{\sqrt{n+\frac{1}{2}} \,.\, e^{-(n+\frac{1}{2})} (n+\frac{1}{2})^{n+\frac{1}{2}} \left(1 + \dfrac{1}{12n+6}\right)}, \text{ approximately}$$

$$= \frac{1}{\sqrt{n}} \left(1 + \frac{1}{2n}\right)^{-1} e^{\frac{1}{2}} \,.\, \left(1 + \frac{1}{2n}\right)^{-n}, \text{ neglecting terms in } \frac{1}{n^{\frac{5}{2}}},$$

now

$$\log \left(1 + \frac{1}{2n}\right)^{-n} = -n\left(\frac{1}{2n} - \frac{1}{8n^2}\right) = -\tfrac{1}{2} + \frac{1}{8n},$$

hence

$$\left(1 + \frac{1}{2n}\right)^{-n} = e^{-\frac{1}{2}}\left(1 + \frac{1}{8n}\right), \text{ approximately,}$$

or

$$\frac{\Pi(n)}{\Pi(n+\frac{1}{2})} = \frac{1}{\sqrt{n}}\left(1 - \frac{1}{2n}\right)\left(1 + \frac{1}{8n}\right) = \frac{1}{\sqrt{n}}\left(1 - \frac{3}{8n}\right)$$

when terms in $\frac{1}{n^{\frac{5}{2}}}$ . . . are neglected. We thus find as an approximation to $\frac{\Pi(n)}{\Pi(n+m)} P_n^m(\cos\theta)$, by taking the first two terms in (59),

$$\sqrt{\frac{2}{n\pi \sin\theta}} \left(1 - \frac{3}{8n}\right) \left\{\sin\left(\overline{n+\tfrac{1}{2}}\theta + \frac{\pi}{4} + \frac{m\pi}{2}\right) - \frac{1^2 - 4m^2}{4n} \frac{1}{2\sin\theta} \cos\left(\overline{n+\tfrac{3}{2}}\theta + \frac{\pi}{4} + \frac{m\pi}{2}\right)\right\}$$

or

$$\frac{\Pi(n)}{\Pi(n+m)} P_n^m(\cos\theta) = \sqrt{\frac{2}{n\pi\sin\theta}} \left[\left(1 - \frac{1-2m^2}{4n}\right) \sin\left(\overline{n+\tfrac{1}{2}}\,\theta + \frac{\pi}{4} + \frac{m\pi}{2}\right) - \frac{1-4m^2}{8n} \cot\theta \cos\left(\overline{n+\tfrac{1}{2}}\,\theta + \frac{\pi}{4} + \frac{m\pi}{2}\right)\right] \quad (63).$$

Similarly we find

$$\frac{\Pi(n)}{\Pi(n+m)} Q_n^m(\cos\theta) = \sqrt{\frac{\pi}{2n\sin\theta}} \left\{\left(1 - \frac{1+2m^2}{4n}\right) \cos\left(\overline{n+\tfrac{1}{2}}\theta + \frac{\pi}{4} + \frac{m\pi}{2}\right) + \frac{1-4m^2}{8n} \cot\theta \sin\left(\overline{n+\tfrac{1}{2}}\theta + \frac{m\pi}{2}\right)\right\} \quad . \quad (64).$$

In (63), (64), n is large but not necessarily integral, and m is not necessarily integral.

32. When μ is real and greater than unity, let it be denoted by $\cosh\psi$; in Art. 28, $P_n^m(\mu)$ has been expressed in terms of two hypergeometric series, in both of

which the fourth element is $\frac{1}{1-z^2}$; when $z = e^{\psi}$, this expression for $P_n^m(\mu)$ becomes approximately, when n is large,

$$P_n^m(\cosh\psi) = \frac{1}{\sqrt{\pi}}\left\{\frac{\Pi(n+m)}{\Pi(n)}\cdot\frac{\Pi(n)}{\Pi(n+\frac{1}{2})}\frac{\sin(n+m)\pi}{\cos n\pi}\frac{e^{-n\psi}}{\sqrt{2e^{\psi}\sinh\psi}}\left(1-\frac{1-4m^2}{4n}\frac{e^{-\psi}}{2\sinh\psi}\right)\right.$$
$$\left.+\frac{\Pi(n)}{\Pi(n-m)}\cdot\frac{\Pi(n-\frac{1}{2})}{\Pi(n)}\frac{e^{(n+1)\psi}}{\sqrt{2e^{\psi}\sinh\psi}}\left(1+\frac{1-4m^2}{4n}\cdot\frac{e^{-\psi}}{2\sinh\psi}\right)\right\},$$

except when n (supposed positive) is half an odd integer, the first term is very much less than the second, on account of the factor $e^{-n\psi}$; hence

$$\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\cosh\psi) = \frac{1}{\sqrt{\pi n}}\left(1-\frac{1}{8n}\right)\frac{e^{n\psi}}{\sqrt{1-e^{-2\psi}}}\left(1+\frac{1-4m^2}{4n}\cdot\frac{e^{-2\psi}}{1-e^{-2\psi}}\right),$$

or,

$$\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\cosh\psi) = \frac{1}{\sqrt{\pi n}}\cdot\frac{e^{n\psi}}{\sqrt{1-e^{-2\psi}}}\left\{1-\frac{3}{8n}+\frac{m^2}{n}+\frac{1-4m^2}{4n}\cdot\frac{1}{1-e^{-2\psi}}\right\} \quad (65).$$

The asymptotic value of $\frac{\Pi(n-m)}{\Pi(n)}P_n^m(\cosh\psi)$ is therefore $\frac{1}{\sqrt{\pi n}}\cdot\frac{e^{n\psi}}{\sqrt{1-e^{-2\psi}}}$, except in the case in which n is equal to half an odd integer.

From (61) we see that the approximate value of $\frac{\Pi(n)}{\Pi(n+m)}Q_n^m(\cosh\psi)$, for large values of n is

$$e^{m\pi\iota}\sqrt{\pi}\cdot\frac{1}{\sqrt{n}}\left(1-\frac{3}{8n}\right)\frac{e^{-(n+1)\psi}}{\sqrt{1-e^{-2\psi}}}\left\{1-\frac{1^2-4m^2}{4n}\frac{e^{-2\psi}}{1-e^{-2\psi}}\right\},$$

or

$$\frac{\Pi(n)}{\Pi(n+m)}Q_n^m(\cosh\psi) = e^{m\pi\iota}\sqrt{\frac{\pi}{n}}\cdot\frac{e^{-(n+1)\psi}}{\sqrt{1-e^{-2\psi}}}\left\{1-\frac{1}{8n}+\frac{m^2}{n}-\frac{1-4m^2}{4n}\cdot\frac{1}{1-e^{-2\psi}}\right\} \quad (66),$$

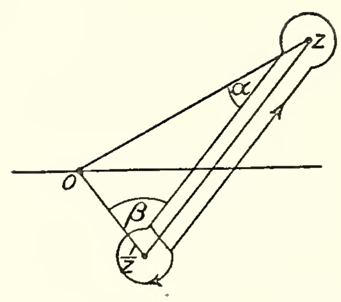
the asymptotic value is $e^{m\pi\iota}\sqrt{\frac{\pi}{n}}\cdot\frac{e^{-(n+1)\psi}}{\sqrt{1-e^{-2\psi}}}$.

It may be remarked that the semi-convergent expressions for BESSEL's functions $J_m(x)$, $Y_m(x)$ may be obtained from the series (59), (60), by putting $\theta = x/n$ and proceeding the limit $n = \infty$.

Expressions for $P_n^m(\mu)$, as definite integrals taken along real paths.

33. In (50) change m into $-m$, we have then

$$P_n^{-m}(\mu) = \frac{1}{2\pi\iota}\cdot\frac{1}{2^m}\cdot\frac{\pi\sec m\pi}{\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}(\mu^2-1)^{-\frac{1}{2}m}\int^{(z+,1/z-)}h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}dh;$$



now suppose the real part of $m+\frac{1}{2}$ is positive, then provided the real part of μ is positive the figure is as in Art. 24 ; we may take the path of integration to be two straight lines on opposite sides of the straight line joining the points $z, \frac{1}{z}$, and two indefinitely small circles round these points ; the integrals round these circles will vanish. If the real part of μ had been negative, so that the line joining $z, \frac{1}{z}$, were on the left of 0, the path of the integral (50) must have been placed so that 0 was on its left hand, and thus we could not have reduced the integral to integrals along the line joining $z, \frac{1}{z}$; it is therefore essential in what follows that the real part of μ be supposed to be positive.

We have now

$$\int^{(z+,\,1/z-)} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh = (e^{-\iota\pi\overline{2m-1}}-1)\int_{1/z}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh,$$

where the phases of $1-hz, 1-\frac{h}{z}$ in the integral on the right-hand side, are $2\pi-\beta$, and α respectively, we thus have

$$\mathrm{P}_n^{-m}(\mu) = \frac{1}{2^m}\cdot\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}(\mu^2-1)^{-\frac{1}{2}m}\int_{1/z}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh.$$

Let $h=\mu+(\mu^2-1)^{\frac{1}{2}}\cos\psi$, then

$$dh = -\sqrt{\mu^2-1}\,\sin\psi\,d\psi$$

$$1-2\mu h+h^2 = -(\mu^2-1)\sin^2\psi = e^{+\iota\pi}(\mu^2-1)\sin^2\psi$$

since the phase of $1-2\mu h+h^2$ is $\alpha-\beta+2\pi$, which is $2\lambda+\pi$, and the phase of μ^2-1 is 2λ; thus

$$\mathrm{P}_n^{-m}(\mu) = \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\int_0^{\pi}(\mu+\sqrt{\mu^2-1}\cos\psi)^{n-m}\sin^{2m}\psi\,d\psi \quad . \quad (67).$$

Or, using equation (19), we have

$$\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \, . \, \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi (\mu + \sqrt{\mu^2-1} \cos\psi)^{n-m} \sin^{2m}\psi \, d\psi \quad (68).$$

This relation holds for all values of n and m, subject to the conditions that the real parts of $m + \frac{1}{2}$, μ are positive; the phase of $\mu + \sqrt{\mu^2-1}\cos\psi$ is the same as that of μ when $\psi = \frac{1}{2}\pi$.

In (68) change n into $-n-1$, we then have, on using the relation (18), after some reduction

$$\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \, . \, \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\mu + \sqrt{\mu^2-1}\cos\psi)^{n+m+1}} d\psi \quad . \quad (69).$$

34. From (68), (69) it is easy to find the corresponding formulæ for the case in which the real part of μ is negative; in this case we have

$$\mathrm{P}_n^m(-\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi . \mathrm{Q}_n^m(-\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{e^{\mp m\pi\iota}}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int_0^\pi e^{\mp(n-m)\pi\iota} (\mu + \sqrt{\mu^2-1}\cos\psi)^{n-m} \sin^{2m}\psi \, d\psi.$$

The expression on the left-hand side is equal to

$$e^{\mp n\pi\iota} \mathrm{P}_n^m(\mu) - \frac{2\sin(n+m)\pi}{\pi} e^{-m\pi\iota} \mathrm{Q}_n^m(\mu) + \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi . e^{\pm n\pi\iota} \mathrm{Q}_n^m(\mu);$$

hence

$$\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin n\pi . e^{\pm(n-m)\pi\iota} \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{1}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int_0^\pi (\mu + \sqrt{\mu^2-1}\cos\psi)^{n-m} \sin^{2m}\psi \, d\psi \quad (70),$$

where the upper or lower sign is to be taken according as the imaginary part of μ is positive or negative; (70) corresponds to (68).

In a similar manner we find, corresponding to (69),

$$-e^{\mp 2n\pi\iota} \mathrm{P}_n^m(\mu) + \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi . e^{\mp(n+m)\pi\iota} \mathrm{Q}_n^m(\mu)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{1}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m} \int_0^\pi \frac{\sin^{2m}\psi}{(\mu + \sqrt{\mu^2-1}\cos\psi)^{n+m+1}} d\psi \quad (71),$$

where, as before, the upper or lower sign is taken according as the imaginary part of μ is positive or negative.

35. When $\mu = \cos\theta$, and θ lies between 0 and $\frac{1}{2}\pi$, the expression on the left-hand side of (68) becomes, on putting $\mu = \cos\theta + 0 \,.\, \iota$,

$$e^{-\frac{1}{2}m\pi\iota} P_n^m(\cos\theta) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, e^{\frac{3}{2}m\pi\iota} \left\{ Q_n^m(\cos\theta) - \frac{\iota\pi}{2} \,.\, P_n^m(\cos\theta) \right\},$$

and on the right-hand side $(\mu^2 - 1)^{\frac{1}{2}m} = e^{\frac{1}{2}m\pi\iota} \sin^m\theta$, hence (68) becomes

$$\cos m\pi \,.\, P_n^m(\cos\theta) - \frac{2}{\pi} \sin m\pi \,.\, Q_n^m(\cos\theta)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{1}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \sin^m\theta \int_0^\pi (\cos\theta + \iota \sin\theta \cos\psi)^{n-m} \sin^{2m}\psi \, d\psi \quad . \quad (72).$$

Again, on putting $\mu = \cos\theta - 0 \,.\, \iota$, we find in a similar manner

$$\cos m\pi \,.\, P_n^m(\cos\theta) - \frac{2}{\pi} \sin m\pi \,.\, Q_n^m(\cos\theta)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{1}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \sin^m\theta \int_0^\pi (\cos\theta - \iota \sin\theta \cos\psi)^{n-m} \sin^{2m}\psi \, d\psi \,.\, (73).$$

Again, putting $\mu = \cos\theta + 0 \,.\, \iota$ in (55), we have

$$\cos m\pi \,.\, P_n^m(\cos\theta) - \frac{2}{\pi} \sin m\pi \,.\, Q_n^m(\cos\theta)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\sin^m\theta}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\mu \pm \sqrt{\mu^2-1}\cos\psi)^{n+m+1}} d\psi \quad . \quad (74).$$

Next let us consider the case in which θ lies between $\frac{1}{2}\pi$ and π; we find from (70), by putting $\mu = \cos\theta \pm 0 \,.\, \iota$,

$$e^{\mp m\pi\iota} \,.\, P_n^m(\cos\theta) \{1 \pm \iota \sin n\pi \,.\, e^{\pm n\pi\iota}\} - \frac{2}{\pi} \sin n\pi \,.\, e^{\pm(n-m)\pi\iota} Q_n^m(\cos\theta)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\sin^m\theta}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi (\cos\theta \pm \iota \sin\theta \cos\psi)^{n-m} \sin^{2m}\psi \, d\psi \quad . \quad (75),$$

this corresponds to (72), (73); the phase of $\cos\theta \pm \iota \sin\theta \cos\psi$, when $\psi = \frac{1}{2}\pi$, is $+\pi$ or $-\pi$ according as the upper or the lower signs are taken in the exponentials.

Again, corresponding to (74), we find that when θ lies between $\frac{1}{2}\pi$ and π,

$$-e^{\mp(n+m)\pi\iota} \,.\, P_n^m(\cos\theta) \{e^{\mp n\pi\iota} + \iota \sin n\pi\} + \frac{2}{\pi} \sin n\pi \,.\, e^{\mp(n+m)\pi\iota} Q_n^m(\cos\theta)$$
$$= \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\sin^m\theta}{2^m \Pi(-\frac{1}{2}) \Pi(m-\frac{1}{2})} \int_0^\pi \frac{\sin^{2m}\psi}{(\cos\theta \pm \iota \sin\theta \cos\psi)^{n+m+1}} d\psi \quad . \quad (76),$$

where, as before, the phase of $\cos\theta \pm \iota\sin\theta\cos\psi$ is $\pm\pi$, when $\psi = \frac{1}{2}\pi$, according as the upper or lower signs are taken in the exponentials.

36. In the important case in which m is a positive integer, we find, from (68) and (70), that

$$\frac{\Pi(n+m)}{\Pi(n-m)}\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi(\mu+\sqrt{\mu^2-1}\cos\psi)^{n-m}\sin^{2m}\psi\,d\psi$$

is equal to

$$P_n^m(\mu), \quad \text{or} \quad P_n^m(\mu) - \frac{2}{\pi}e^{\pm n\pi\iota}\sin n\pi\,.\,Q_n^m(\mu) \quad . \quad . \quad . \quad . \quad (77),$$

according as the real part of μ is positive or negative.

From (69), (71), we find in this case

$$P_n^m(\mu) = \frac{\Pi(n+m)}{\Pi(n-m)}\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi\frac{\sin^{2m}\psi}{(\mu+\sqrt{\mu^2-1}\cos\psi)^{n+m+1}}d\psi,$$

when the real part of μ is positive, and

$$-e^{\mp 2n\pi\iota}P_n^m(\mu) + \frac{2}{\pi}\sin n\pi\,.\,Q_n^m(\mu)$$

$$= \frac{\Pi(n+m)}{\Pi(n-m)}\frac{(\mu^2-1)^{\frac{1}{2}m}}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi\frac{\sin^{2m}\psi}{(\mu+\sqrt{\mu^2-1}\cos\psi)^{n+m+1}}d\psi \quad . \quad . \quad (78),$$

when the real part of μ is negative, the upper or lower sign in the exponential being taken according as the imaginary part of μ is positive or negative.

When $\mu = \cos\theta$, we have in the case in which m is a positive integer,

$$P_n^m(\cos\theta) = (-1)^m\frac{\Pi(n+m)}{\Pi(n-m)}\frac{\sin^m\theta}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi(\cos\theta\pm\iota\sin\theta\cos\psi)^{n-m}\sin^{2m}\psi\,d\psi. \quad (79),$$

when θ may have any value between 0 and π.

Also

$$P_n^m(\cos\theta) = (-1)^m\frac{\Pi(n+m)}{\Pi(n-m)}\frac{\sin^m\theta}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi\frac{\sin^{2m}\psi}{(\cos\theta\pm\iota\sin\theta\cos\psi)^{n+m+1}}d\psi,$$

where θ lies between 0 and $\frac{1}{2}\pi$, and

$$-e^{\mp n\pi\iota}P_n^m(\cos\theta)(e^{\mp n\pi\iota}+\iota\sin n\pi) + \frac{2}{\pi}e^{\mp n\pi\iota}\sin n\pi\,.\,Q_n^m(\cos\theta)$$

$$= (-1)^m\frac{\Pi(n+m)}{\Pi(n-m)}\frac{\sin^m\theta}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\int_0^\pi\frac{\sin^{2m}\psi}{(\cos\theta\pm\iota\sin\theta\cos\psi)^{n+m+1}}d\psi \quad . \quad (80),$$

where θ lies between $\frac{1}{2}\pi$ and π.

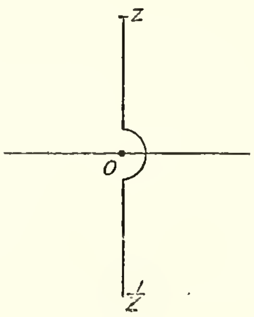
Remarks on HEINE'S *definition of* $P_n(\mu)$.

37. HEINE has proposed* to define the function $P_n(\mu)$ for complex values of n and μ, by means of the expression

$$P_n(\mu) = \frac{1}{\pi}\int_0^\pi (\mu + \sqrt{\mu^2 - 1}\cos\psi)^n \, d\psi.$$

It will appear from what we have shown in Arts. 33–36, that this definition is not a valid one, as the function given by the definite integral for values of μ with a negative real part is not the analytical continuation of the function given by the same definite integral for values of μ with a positive real part; it follows that $P_n(\mu)$ can be defined by the above expression only for values of μ with a positive real part.

The fact that the definite integral is of ambiguous meaning at the imaginary μ axis is clear if we attend to the phases of the integrand $(\mu + \sqrt{\mu^2 - 1}\cos\psi)^n$, or h^n; μ being purely imaginary there is a value of ψ between 0 and π for which h vanishes, and in passing through this value of ψ the phase of the integrand changes by a finite amount. The h integral in Art. 33 is taken along a path joining $z, \frac{1}{z}$ which has the point $h = 0$ on the left hand side, thus for purely imaginary values of μ the path may be placed

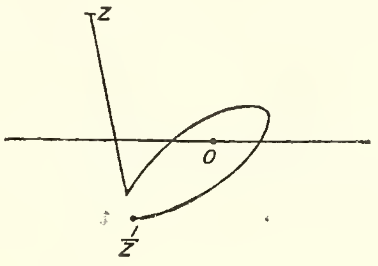


as in the figure, with a semi-circular portion to avoid the point $h = 0$; we thus see that in the above definite integral there must be a sudden diminution of phase $n\pi$ in the integrand as $\cos\psi$ passes through the value $\frac{-\mu}{\sqrt{\mu^2 - 1}}$; if this be taken into account the definite integral will represent the function $P_n(\mu)$ for purely imaginary values of μ; there is however nothing in the definite integral itself which decides apart from convention what the change of phase in the integrand shall be as it passes through its zero value.

Next suppose μ to cross the imaginary axis, the h integral can then be taken from $\frac{1}{z}$ to z along a loop round the point $h = 0$, and then along a straight line to the

* 'Kugelfunctionen.,' Vol. 1, p. 37.

point $h = z$, but cannot be taken directly from $\frac{1}{z}$ to z; it thus appears that the function $P_n(\mu)$ is no longer represented by the definite integral, but that the value



of the definite integrals involves $Q_n(\mu)$ as well as $P_n(\mu)$; in fact, we have shown in (70) that in this case

$$\frac{1}{\pi}\int_0^\pi (\mu + \sqrt{\mu^2 - 1}\cos\psi)^n \, d\psi = P_n(\mu) - \frac{2}{\pi} e^{\pm n\pi\iota} \sin n\pi \, . \, Q_n(\mu),$$

where the upper or lower sign is to be taken in the exponential according as the imaginary part of μ is positive or negative.

The only case in which HEINE's definition is valid for all values of μ is when n is a real integer.

HEINE deduces from his definition that for unrestricted values of n, the function $P_n(\mu)$ is represented when mod $\mu > 1$, by the series

$$\frac{1}{2^n} \cdot \frac{\Pi(2n)}{\Pi(n)\,\Pi(n)} \mu^n \, F\left(-\frac{n}{2}, \frac{1-n}{2}, \tfrac{1}{2} - n, \frac{1}{\mu^2}\right),$$

this result, following from the incorrect definition, is erroneous, the correct expression being given by (23) and involving two hypergeometric series.

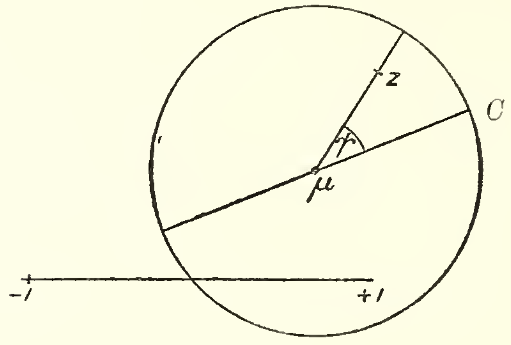
It was to be expected *à priori* that as $P_n(\mu)$ was defined by means of an integral taken along a path containing the singular points μ and $+1$, but excluding -1, the function so defined would not in general possess any kind of symmetry about the imaginary axis.

Definite Integral Expressions for $P_n^m(\mu)$ when m is a Real Integer.

38. When m is a real integer, the formula (4) for $P_n^m(\mu)$ becomes

$$P_n^m(\mu) = \frac{1}{2\pi\iota} \frac{\Pi(n+m)}{\Pi(n)} \frac{1}{2^n} (\mu^2 - 1)^{m/2} \int^{(\mu+,\,1+)} (t^2 - 1)^n (t - \mu)^{-n-m-1} \, dt.$$

Suppose the real part of μ to be positive, and the path of integration to be a circle with centre at the point μ, and of radius greater than mod $(\mu - 1)$ and less than



mod $(\mu + 1)$. On this circle take a point C such that the angle between μz and μC is ψ, and let ϕ be the angular distance of any point of the circle from C. If we put $t = \mu + \sqrt{\mu^2 - 1}\, e^{\iota(\phi-\psi)\mp u}$, the point t represents, for different values of ϕ, points on a circle of centre μ and radius $e^{\mp u}$ mod $(\sqrt{\mu^2 - 1})$; we must thus take u to be such that $e^{\mp u}$ mod $(\sqrt{\mu^2 - 1}) >$ mod $(\mu - 1)$, and $<$ mod $(\mu + 1)$, or $u < \frac{1}{2} \log \text{mod} \dfrac{\mu+1}{\mu-1}$. Take the circle commencing at C to be the path of integration; we have

$$t^2 - 1 = 2\sqrt{\mu^2 - 1}\, .\, e^{\iota(\phi-\psi)\mp u}\left[\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\right].$$

Hence we have

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi}\,\frac{\Pi(n+m)}{\Pi(n)} \int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n e^{-m\iota(\phi-\psi)\pm m u}\, d\phi,$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n (\cos m\phi - \iota \sin m\phi)\, d\phi$$
$$= \mathrm{P}_n^m(\mu)\ \frac{\Pi(n)}{\Pi(n+m)}\, e^{-m\iota(\psi \mp \iota u)}.$$

On changing m into $-m$, and remembering that

$$\mathrm{P}_n^{-m}(\mu) = \frac{\Pi(n-m)}{\Pi(n+m)}\,\mathrm{P}_n^m(\mu)$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n (\cos m\phi + \iota \sin m\phi)\, d\phi$$
$$= \mathrm{P}_n^m(\mu) \cdot \frac{\Pi(n)}{\Pi(n+m)}\, e^{+m\iota(\psi \mp \iota u)},$$

we thus obtain the formulæ

$$\frac{1}{2\pi}\int_0^{2\pi}\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^n \frac{\cos}{\sin} m\phi\, d\phi$$

$$= \mathrm{P}_n^m(\mu)\frac{\Pi(n)}{\Pi(n+m)}\frac{\cos}{\sin} m(\psi\mp\iota u) \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

If we change n into $-(n+1)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\frac{\cos}{\sin} m\phi}{\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^{n+1}}\, d\phi$$

$$= \mathrm{P}_n^m(\mu)\frac{\Pi(n-m)}{\Pi(n)}(-1)^m\frac{\cos}{\sin} m(\psi\mp\iota u) \quad . \quad . \quad . \quad . \quad (82).$$

In these formulæ n is unrestricted, m is a real integer, and u is any real positive quantity less than $\frac{1}{2}\log\mathrm{mod}\frac{\mu+1}{\mu-1}$, and the real part of μ is positive.

Formulæ corresponding to (81), (82) have been given by HEINE in the case in which n is a positive integer.*

If in (81), (82) we put $u=0$, $\psi=0$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}(\mu+\sqrt{\mu^2-1}\cos\phi)^n\frac{\cos}{\sin} m\phi\, d\phi = \frac{\Pi(n)}{\Pi(n+m)}\mathrm{P}_n^m(\mu) \quad . \quad . \quad . \quad (83).$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\frac{\cos}{\sin} m\phi}{(\mu+\sqrt{\mu^2-1}\cos\phi)^{n+1}}\, d\phi = (-1)^m\frac{\Pi(n-m)}{\Pi(n)}\mathrm{P}_n^m(\mu) \quad . \quad . \quad (84).$$

Definite Integral Expressions for $\mathrm{Q}_n^m(\mu)$.

39. When the real parts of $m+\frac{1}{2}$, $n-m+1$ are both positive, the formula (54), for $\mathrm{Q}_n^m(\mu)$ reduces to

$$\mathrm{Q}_n^m(\mu) = e^{m\pi\iota}\,.\,2^m\frac{\Pi(n+m)}{\Pi(n-m)}\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\frac{(\mu^2-1)^{\frac{1}{2}m}}{z^{n+m+1}}\int_0^1 u^{m-\frac{1}{2}}(1-u)^{n-m}\left(1-\frac{u}{z^2}\right)^{-n-m-1} du$$

on changing the independent variable to v, where $u=\frac{v-1}{v+1}$, we then have

$$\mathrm{Q}_n^m(\mu) = e^{m\pi\iota}\,.\,2^m\,.\,\frac{\Pi(n+m)}{\Pi(n-m)}\,.\,\frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})}\,.\,(\mu^2-1)^{\frac{1}{2}m}\,.\,2^{n-m+1}$$

$$\int_1^\infty (v^2-1)^{m-\frac{1}{2}}\left\{z+\frac{1}{z}+v\left(z-\frac{1}{z}\right)\right\}^{-n-m-1} dv,$$

* See 'Kugelfunctionen,' vol. 1, p. 211.

or, on putting $v = \cosh w$, this becomes

$$Q_n^m(\mu) = \frac{1}{2^m} e^{m\pi\iota} \frac{\Pi(n+m)}{\Pi(n-m)} \cdot \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} (\mu^2-1)^{\frac{1}{2}m}$$
$$\int_0^\infty \{\mu + \sqrt{\mu^2-1}\cosh w\}^{-n-m-1} \sinh^{2m} w \, dw \quad . \quad (85),$$

where the real parts of $m + \frac{1}{2}$, $n - m + 1$ are positive.

If $m = 0$, we have

$$Q_n(\mu) = \int_0^\infty \{\mu + \sqrt{\mu^2-1}\cosh w\}^{-n-1} dw \quad . \quad . \quad . \quad . \quad . \quad (86)$$

The particular case of (85), when m and n are real integers, is given by HEINE.*

When μ has a real value less than unity, we have, on using (31),

$$Q_n^m(\cos\theta) = \frac{1}{2^{m+1}} \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} \sin^m\theta \left\{\int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta + \iota\sin\theta\cosh w)^{n+m+1}} dw + \int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta - \iota\sin\theta\cosh w)^{n+m+1}} dw\right\};$$

and from (30),

$$P_n^m(\cos\theta) = \frac{1}{2^m \cdot \iota\pi} \cdot \frac{\Pi(n+m)}{\Pi(n-m)} \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} \sin^m\theta \left\{\int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta - \iota\sin\theta\cosh w)^{n+m+1}} dw - \int_0^\infty \frac{\sinh^{2m} w}{(\cos\theta + \iota\sin\theta\cosh w)^{n+m+1}} dw\right\}.$$

40. In the formula

$$Q_n^m(\mu) = e^{m\iota\pi} . 2^m . \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \cos m\pi \, (\mu^2-1)^{\frac{1}{2}m} \int_0^{\frac{1}{z}} \frac{h^{n+m}}{(1-2\mu h + h^2)^{m+\frac{1}{2}}} dh \quad . \quad (43),$$

which holds, provided the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive; put $h = \mu - \sqrt{\mu^2-1}\cosh w$, then when $h = 0$, we have $w = w_0 = \frac{1}{2}\log_e\frac{\mu+1}{\mu-1}$, and when $h = \frac{1}{z}$, $w = 0$, thus since $1 - 2h\mu + h^2 = (\mu^2-1)\sinh^2 v$, we have

$$Q_n^m(\mu)$$
$$= e^{m\pi\iota} . 2^m . \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \cos m\pi . (\mu^2-1)^{-\frac{1}{2}m} \int_0^{w_0} (\mu - \sqrt{\mu^2-1}\cosh w)^{n+m} \sinh^{-2m} w \, dw \quad (87),$$

where $w_0 = \frac{1}{2}\log \text{mod.} \frac{\mu+1}{\mu-1}$, and the real parts of $n + m + 1$, $\frac{1}{2} - m$ are positive.

* See 'Kugelfunctionen,' vol. 1, p. 222.

If $m = 0$, we have

$$Q_n(\mu) = \int_0^{w_0} (\mu - \sqrt{\mu^2 - 1} \cosh w)^n \, dw \quad . \quad . \quad . \quad . \quad . \quad . \quad (88).$$

It is interesting to compare (87) with the formula obtained by changing m into $-m$, in (85),

$$Q_n^m(\mu)$$
$$= e^{m\pi\iota} \cdot 2^m \cdot \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})} \cos m\pi \cdot (\mu^2 - 1)^{-\frac{1}{2}m} \int_0^\infty \{\mu + \sqrt{\mu^2 - 1} \cosh w\}^{m-n-1} \sinh^{-2m} w \, dw \quad (89),$$

which holds under the same conditions as (87).

In (87) change m into $-m$, we have then

$$Q_n^m(\mu)$$
$$= e^{m\pi\iota} \cdot 2^m \frac{\Pi(n+m)}{\Pi(n-m)} \cdot \frac{\Pi(-\frac{1}{2})}{\Pi(m-\frac{1}{2})} (\mu^2 - 1)^{\frac{1}{2}m} \int_0^{w_0} (\mu - \sqrt{\mu^2 - 1} \cosh w)^{n-m} \sinh^{2m} w \, dw \quad (90),$$

which holds when the real parts of $n - m + 1$, $\frac{1}{2} + m$ are positive.

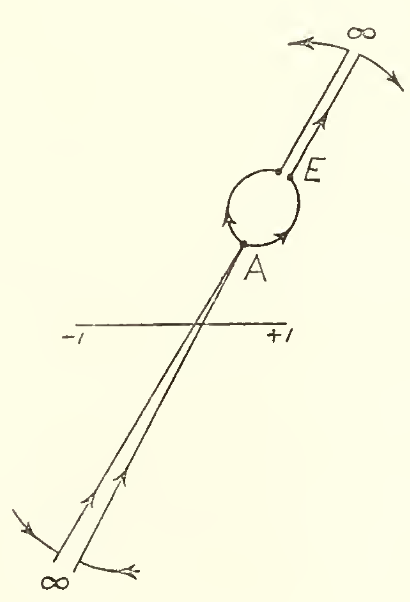
41. In the formula (9), change n into $-n-1$, we then have

$$Q_{-n-1}^m(\mu) = \frac{e^{n\pi\iota}}{4\iota \sin(n-m)\pi} \cdot \frac{\Pi(n)}{\Pi(n-m)} 2^{n+1} \int^{(-1+,\,1-)} X \, dt,$$

where

$$X = (\mu^2 - 1)^{\frac{1}{2}m} (t^2 - 1)^{-n-1} (t - \mu)^{n-m}.$$

Place the path as in the figure; starting from A, a semicircle of centre μ is first described, then a straight line from E to ∞, a semicircle of infinite radius, then a straight path from ∞' to A, followed by a similar path taken negatively round the point $+1$.



If the real part of $n - m + 1$ is positive, the integrals along the semicircles with μ as centre vanish when the radius is made indefinitely small. If the real part of $n + m + 1$ is positive, the integrals along the infinite semicircles vanish. We thus have,

$$Q_{-n-1}^{m}(\mu)$$
$$= \frac{e^{n\pi\iota}}{4\iota\sin(n-m)\pi}\frac{\Pi(n)}{\Pi(n-m)}2^{n+1}\left\{e^{-n\pi\iota}\,.\,2\cos m\pi\int_{\mu}^{\infty}X\,dt - e^{-n\pi\iota}\,.\,2\cos n\pi\int_{\mu}^{\infty'}X\,dt\right\},$$

where in the integrals X commences with the phase it has at A initially. The phase of $t + 1$ at A is $-(2\pi - \gamma)$.

From equation (8), we have

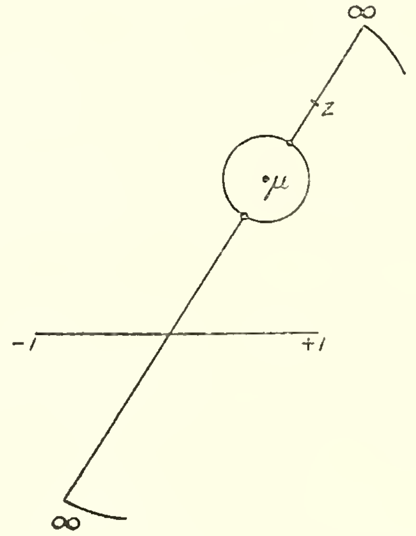
$$P_{n}^{m}(\mu) = \frac{-e^{+n\pi\iota}}{4\pi\sin(n-m)\pi}\cdot\frac{\Pi(n)}{\Pi(n-m)}\cdot 2^{n+1}\int^{(\mu+,\,1+,\,\mu-,\,1-)}(t^2-1)^{-n-1}(t-\mu)^{n-m}\,dt,$$

where the phase of $t + 1$ in the integrand is γ at A, and thus

$$(t^2-1)^{-n-1}(t-\mu)^{n-m} = e^{-2n\pi\iota}X;$$

hence

$$P_{n}^{m}(\mu) = \frac{-e^{-n\pi\iota}}{4\pi\sin(n-m)\pi}\cdot\frac{\Pi(n)}{\Pi(n-m)}2^{n+1}\int^{(\mu+,\,1+,\,\mu-,\,1-)}X\,dt.$$



Taking the path as in the figure, we have, provided the real parts of $n \pm m + 1$ are positive,

$$\int^{(\mu+,\,1+,\,\mu-,\,1-)}X\,dt = e^{2(n-m)\pi\iota}\int_{\mu}^{\infty'}X\,dt - e^{(n-3m)\pi\iota}\int_{\mu}^{\infty}X\,dt + e^{-(n+m)\pi\iota}\int_{\mu}^{\infty}X\,dt - \int_{\mu}^{\infty'}X\,dt$$
$$= -e^{-2m\pi\iota}\,.\,2\iota\sin(n-m)\pi\int_{\mu}^{\infty}X\,dt\,.\,+e^{(n-m)\pi\iota}\,.\,2\iota\sin(n-m)\pi\int_{\mu}^{\infty'}X\,dt,$$

therefore

$$P_n^m(\mu) = \frac{-e^{-n\pi\iota}}{4\pi} \cdot \frac{\Pi(n)}{\Pi(n-m)} \cdot 2^{n+1}\left\{-2\iota\, e^{-2m\pi\iota}\int_\mu^\infty X\,dt + 2\iota \,.\, e^{(n-m)\pi\iota}\int_\mu^{\infty'} X\,dt\right\}.$$

Substituting for $Q_{-n-1}^m(\mu)$ its value in terms of $Q_n^m(\mu)$, $P_n^m(\mu)$ given by (18), we have

$$Q_n^m(\mu)\sin(n+m)\pi - \pi\cos n\pi \,.\, e^{m\pi\iota}P_n^m(\mu)$$
$$= \frac{1}{4\iota}\cdot\frac{\Pi(n)}{\Pi(n-m)}2^{n+1}\left\{2\cos m\pi\int_\mu^\infty X\,dt - 2\cos n\pi\int_\mu^{\infty'} X\,dt\right\}.$$

On substituting the value of $P_n^m(\mu)$ in this equation, we have

$$Q_n^m(\mu) = 2^n \,.\, e^{-n\pi\iota}\frac{\Pi(n)}{\Pi(n-m)}\cdot(\mu^2-1)^{\frac{1}{2}m}\int_\mu^\infty (t^2-1)^{-n-1}(t-\mu)^{n-m}\,dt,$$

which holds for all values of n and m, such that the real parts of $n+m+1$, $n-m+1$ are both positive.

In this formula, when t is just greater than μ, the phase of $t-1$ is the same as that of $\mu-1$, but the phase of $t+1$ is less by 2π than that of $\mu+1$, hence if we wish the phase of t^2-1 to be that of μ^2-1, the result must be multiplied by $e^{2n\pi\iota}$; again the phase of $t-\mu$ is that at A, and this is less by π than the phase of $\sqrt{\mu^2-1}$, hence, in order that the phase of $t-\mu$ may be that of $\sqrt{\mu^2-1}$, we must multiply by $e^{-(n-m)\pi\iota}$; the formula now becomes

$$Q_n^m(\mu) = 2^n \,.\, e^{m\pi\iota}\frac{\Pi(n)}{\Pi(n-m)}(\mu^2-1)^{\frac{1}{2}m}\int_\mu^\infty (t^2-1)^{-n-1}(t-\mu)^{n-m}\,dt;$$

on substituting $t = \mu + \sqrt{\mu^2-1}\,.\,e^u$, which gives us $\dfrac{t^2-1}{2(t-\mu)} = \mu + \sqrt{\mu^2-1}\cosh u$, where u is a real quantity, we have

$$Q_n^m(\mu) = e^{m\pi\iota}\cdot\frac{\Pi(n)}{\Pi(n-m)}\cdot\frac{1}{2}\int_{-\infty}^\infty \frac{e^{-mu}}{(\mu+\sqrt{\mu^2-1}\cosh u)^{n+1}}\,du,$$

or

$$Q_n^m(\mu) = e^{m\pi\iota}\cdot\frac{\Pi(n)}{\Pi(n-m)}\cdot\int_0^\infty \frac{\cosh mu}{(\mu+\sqrt{\mu^2-1}\cosh\mu)^{n+1}}\,du \quad . \quad . \quad . \quad . \quad (91),$$

where the real parts of $n+m+1$, $n-m+1$ are positive.

In (91), the phase of $\mu+\sqrt{\mu^2-1}\cosh u$ is equal to that of $\mu+\sqrt{\mu^2-1}$ when $u=0$. The formula (91) is a generalization of the formula given by HEINE;*

* 'Kugelfunctionen,' vol. 1, p. 223.

his formula he proves, for the case in which m is a real integer, by a method of transformation which cannot be applied to obtain the more general result (91).

42. In the formula

$$Q_n^m(\mu) = \frac{e^{m\pi\iota}}{2^{n+1}} \frac{\Pi(n+m)}{\Pi(n)} (\mu^2 - 1)^{\frac{1}{2}m} \int_{-1}^{1} (1 - t^2)^n (\mu - t)^{-n-m-1} dt \quad . \quad . \quad (11),$$

which holds, provided the real part of $n + 1$ is positive; let

$$t = \mu - \sqrt{\mu^2 - 1} \,.\, e^u, \text{ then } 1 - t^2 = \sqrt{\mu^2 - 1} \,.\, e^u \{2\mu - 2\sqrt{\mu^2 - 1} \cosh u\},$$

hence

$$Q_n^m(\mu)$$

$$= \frac{e^{m\pi\iota}}{2^{n+1}} \cdot \frac{\Pi(n+m)}{\Pi(n)} (\mu^2 - 1)^{\frac{1}{2}m} \int_{\log_e \sqrt{\frac{\mu+1}{\mu-1}}}^{\log_e \sqrt{\frac{\mu-1}{\mu+1}}} \frac{(\mu^2-1)^{\frac{1}{2}n} \,.\, e^{nu} \{2\mu - 2\sqrt{\mu^2-1} \cosh \mu\}^n}{(\mu^2-1)^{\frac{n+m+1}{2}} e^{(n+m+1)u}} (-\sqrt{\mu^2-1} \,.\, e^u) du,$$

or

$$Q_n^m(\mu) = e^{m\pi\iota} \,.\, \frac{\Pi(n+m)}{\Pi(n)} \int_0^{\log_e \sqrt{\frac{\mu+1}{\mu-1}}} \{\mu - \sqrt{\mu^2 - 1} \cosh u\}^n \cosh mu \, du \quad . \quad (92).$$

This formula holds for all values of n and m such that the real part of $n + 1$ is positive.

The Evaluation of a certain Definite Integral.

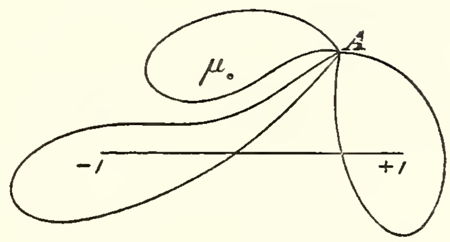
43. Suppose n and m are such that $n - m$ is a real integer, and that they are otherwise unrestricted; in this case the integral

$$\frac{1}{2\pi\iota} (\mu^2 - 1)^{\frac{1}{2}m} \int \frac{1}{2^n} (t^2 - 1)^n (t - \mu)^{-n-m-1} dt$$

taken round a closed path which includes the three singular points $1, -1, \mu$ will satisfy the fundamental equation (2), since the integrand attains its original value after description of the closed path. We shall take the path to be a circle with centre at the point μ; if we put $t = \mu + \sqrt{\mu^2 - 1} \,.\, e^{\iota(\phi-\psi)\mp u}$, as in Art. 38, in this case we must have $u > \frac{1}{2} \log \operatorname{mod} \frac{\mu+1}{\mu-1}$, and the integral becomes

$$\frac{1}{2\pi} \int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n e^{-m\iota(\phi-\psi) \pm mu} d\phi.$$

This integral has been evaluated in Art. 38, when $u < \frac{1}{2} \log \operatorname{mod} \frac{\mu + 1}{\mu - 1}$; we proceed to evaluate it in the present case $u > \frac{1}{2} \log \operatorname{mod} \frac{\mu + 1}{\mu - 1}$. We shall denote the definite integral by $\frac{1}{2\pi\iota} \mathrm{I}(n, m)$.



Denote by L, M, N the integrals of $\frac{1}{2^n}(t^2 - 1)^n (t - \mu)^{-n-m-1}$ taken along loops from the point A, round the points -1, 1, μ respectively, then

$$(\mu^2-1)^{\frac{1}{2}m} \int^{(\mu+,\, 1+,\, \mu-,\, 1-)} \frac{1}{2^n}(t^2-1)^n (t-\mu)^{-n-m-1}\, dt = \mathrm{N} + \mathrm{M}e^{-4n\pi\iota} - \mathrm{N}e^{2n\pi\iota} - \mathrm{M}$$
$$= (1 - e^{2n\pi\iota})\{\mathrm{N} + \mathrm{M}e^{-4n\pi\iota}(1 + e^{2n\pi\iota})\}$$

$$(\mu^2-1)^{\frac{1}{2}m} \int^{(-1+,\, +1-)} \frac{1}{2^n}(t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt = (\mathrm{L} - \mathrm{M})e^{-2n\pi\iota}.$$

Also

$$\mathrm{I}(n, m) = \mathrm{N} + \mathrm{L}e^{-4\pi n\iota} + \mathrm{M}e^{-2\pi n\iota},$$

hence

$$(1 - e^{2\pi n\iota})\,\mathrm{I}(n, m) = (\mu^2 - 1)^{\frac{1}{2}m} \int^{(\mu+,\, 1+,\, \mu-,\, 1-)} \frac{1}{2^n}(t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt$$
$$- (1 - e^{-2\pi n\iota})(\mu^2 - 1)^{\frac{1}{2}m} \int^{(-1+,\, 1-)} \frac{1}{2^n}(t^2 - 1)^n (t - \mu)^{-n-m-1}\, dt,$$

or

$$- e^{n\pi\iota} \,.\, 2\iota \sin n\pi \,.\, \mathrm{I}(n, m) = \frac{\Pi(n)}{\Pi(n + m)} \,.\, 4\pi \sin n\pi \,.\, e^{n\pi\iota}\, \mathrm{P}_n^m(\mu)$$
$$- \frac{\Pi(n)}{\Pi(n + m)} \,.\, 8 \sin^2 n\pi \,.\, \mathrm{Q}_n^m(\mu),$$

we thus have

$$\frac{1}{2\pi} \int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1} \cos(\phi - \psi \pm \iota u)\}^n e^{-m\iota(\phi-\psi) \pm mu}\, d\phi$$
$$= \frac{\Pi(n)}{\Pi(n + m)} \left\{ \mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-n\pi\iota} \sin n\pi\, \mathrm{Q}_n^m(\mu) \right\} \quad . \quad (93),$$

where $n - m$ is a real integer, and $u > \frac{1}{2} \log_e \bmod \frac{\mu + 1}{\mu - 1}$. It has been shown in Art. 11 that the expression in (93) is zero when $n - m$ is a negative integer.

When m and n are both integers

$$\frac{1}{2\pi}\int_0^{2\pi}\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n e^{-m\iota(\phi-\psi)\pm mu}\, d\phi = \frac{\Pi(n)}{\Pi(n+m)} \mathrm{P}_n^m(\mu) \,. \quad (94),$$

the right-hand side is zero when n and m are positive, and $n < m$, since in this case $\mathrm{P}_n^m(\mu) = 0$.

Next change m into $-m$, in the formula (93), the expression on the right-hand side then becomes

$$\frac{\Pi(n)}{\Pi(n-m)}\left\{\mathrm{P}_n^{-m}(\mu) - \frac{2}{\pi} e^{-n\pi\iota} \sin n\pi \,.\, \mathrm{Q}_n^{-m}(\mu)\right\}$$

or

$$\frac{\Pi(n)}{\Pi(n+m)}\left\{\mathrm{P}_n^m(\mu) - \frac{2}{\pi} e^{-m\pi\iota} \sin m\pi \,.\, \mathrm{Q}_n^m(\mu)\right\} - \frac{\Pi(n)}{\Pi(n+m)} \cdot \frac{2}{\pi} \sin n\pi \,.\, e^{-(n+2m)\pi\iota} \mathrm{Q}_n^m(\mu),$$

which reduces to $\frac{\Pi(n)}{\Pi(n+m)} \mathrm{P}_n^m(\mu)$, since $n + m$ is a real integer; we thus have the formula

$$\frac{1}{2\pi}\int_0^{2\pi}\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n e^{m\iota(\phi-\psi)\mp mu}\, d\phi = \frac{\Pi(n)}{\Pi(n+m)} \mathrm{P}_n^m(\mu) \quad (95),$$

which holds for all values of m and n such that $m + n$ is a real integer; when m and n are positive integers such that $m > n$, we have $\mathrm{P}_n^m(\mu) = 0$, and the integral in (95) vanishes.

44. In (93) change n into $-n-1$, we have then

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-m\iota(\phi-\psi)\pm mu}}{\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^{n+1}}\, d\phi$$
$$= \frac{\Pi(-n-1)}{\Pi(m-n-1)}\left\{\mathrm{P}_n^m(\mu) + \frac{2}{\pi} e^{n\pi\iota} \sin n\pi\, \mathrm{Q}_{-n-1}^m(\mu)\right\}$$

where $m - n$ is a real integer; now, suppose m and n are both real integers, it is then necessary to evaluate the undetermined form on the right-hand side; to do this, suppose the modulus of μ is greater than unity, and substitute the series in powers of $\frac{1}{\mu}$ for the functions $\mathrm{P}_n^m(\mu)$, $\mathrm{Q}_{-n-1}^m(\mu)$; the expression then becomes

$$\frac{1}{\Pi(m-n-1)}\left\{\frac{\pi\Pi(-n-1)}{\Pi(m+n-1)\Pi(-m-n)}\frac{1}{2^{n+1}\cos n\pi}\frac{1}{\Pi(n+\frac{1}{2})\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-n-m-1}\right.$$
$$F\left(\frac{n+m+2}{2},\frac{n+m+1}{2},n+\tfrac{3}{2},\frac{1}{\mu^2}\right)$$
$$\left.+2^n\frac{\Pi(-n-1)}{\Pi(n-m)}\frac{\Pi(n-\frac{1}{2})}{\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}F\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right)\right\}$$
$$-\frac{2}{\pi}e^{(n+m)\pi\iota}2^n\frac{\pi\Pi(m-n-1)\Pi(-\frac{1}{2})}{\Pi(n)\Pi(-n-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{n-m}F\left(\frac{m-n+1}{2},\frac{m-n}{2},\tfrac{1}{2}-n,\frac{1}{\mu^2}\right)\Bigg\}.$$

The ratio of the coefficients of the last two terms can easily be shown to be -1, thus the result reduces to

$$\frac{1}{\Pi(m-n-1)}\frac{\pi}{\Pi(m+n-1)}\frac{(-n-1)\ldots(-n-m+1)}{2^{n+1}\cos n\pi}\frac{1}{\Pi(-\frac{1}{2})\Pi(n+\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\mu^{-n-m-1}$$
$$F\left(\frac{n+m+2}{2},\frac{n+m+1}{2},n+\tfrac{3}{2},\frac{1}{\mu^2}\right),$$

or to

$$\frac{(-1)^{n+1}}{\Pi(m-n-1)}\frac{1}{\Pi(n+m)\Pi(n)}Q_n^m(\mu).$$

This result must hold whether mod μ is greater or less than unity; hence when m and n are real integers

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-m\iota(\phi-\psi)\pm mu}}{\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^{n+1}}d\phi=\frac{(-1)^{n+1}}{\Pi(m-n-1)}\frac{1}{\Pi(n+m)\Pi(n)}Q_n^m(\mu)\quad(96),$$

when $m>n$, and is equal to zero when $m \overline{\gtrless} n$.

The case in which m and $m+n$ are negative would require special examination, but the result in that case may be deduced from (96); change ψ, u into $-\psi$, $-u$, and ϕ into $2\pi-\phi$, we thus find

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{m\iota(\phi-\psi)\pm mu}}{\{\mu+\sqrt{\mu^2-1}\cos(\phi-\psi\pm\iota u)\}^{n+1}}d\phi=\frac{(-1)^{n+1}}{\Pi(m-n-1)}\frac{1}{\Pi(n+m)\Pi(n)}Q_n^m(\mu)$$

when $m>n$, and is equal to zero when $m \overline{\gtrless} n$ . . . . (97).

The results in (94), (96), (97) agree with those of HEINE,* the more general formulæ (93), (95) are not given by him.

45. Results such as those in Arts. 38, 43, 44, could be foreseen by a consideration of the fact that $(z+\alpha x+\beta y)^n$ satisfies LAPLACE's equation $\nabla^2V=0$, provided α, β are any constants such that $\alpha^2+\beta^2=-1$; this holds for complex values of n, and when x, y, z are not restricted to be real. Let $\alpha=-\iota\cos(\psi\mp\iota u)$, $\beta=-\iota\sin(\psi\mp\iota u)$, then, since $z=r\mu$, $x=\iota r\sqrt{\mu^2-1}\cos\phi$, $y=\iota r\sqrt{\mu^2-1}\sin\phi$, we have

* 'Kugelfunctionen,' vol. 1, p. 211.

$$(z + \alpha x + \beta y)^n = r^n \{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n;$$

we should, therefore, expect that if $\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n$ is expanded in cosines and sines of multiples of ϕ, say $\Sigma(u_m \cos m\phi + v_m \sin m\phi)$, the coefficients u_m, v_m would be linear functions of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$.

Let $w = \sqrt{\mu^2 - 1}\, e^{\pm\iota(\phi - \psi \pm \iota u)}$, we then find that

$$\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n = (2w)^{-n}\ (\mu + w - 1)^n (\mu + w + 1)^n.$$

If $u <$ log mod $\sqrt{\frac{\mu + 1}{\mu - 1}}$, one of the expressions $(\mu + w - 1)^n$, $(\mu + w + 1)^n$ can be expanded in positive powers, and the other in negative powers of w; if, however, $u >$ log mod $\sqrt{\frac{\mu + 1}{\mu - 1}}$, both expressions can be expanded in positive powers, or both in negative powers, according to the sign taken in $\pm \iota u$.

Case I.—$u <$ log mod $\sqrt{\frac{\mu + 1}{\mu - 1}}$.

In this case all the powers of w in the expansion are of positive or negative integral degree, thus

$$\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n = \sum_{m=0}^{\infty} u_m \cos m\phi + v_m \sin m\phi,$$

where m has all positive integral values.

We have

$$u_m = \frac{1}{\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n \cos m\phi$$

$$= 2P_n^m(\mu)\frac{\Pi(n)}{\Pi(n + m)}\cos m(\psi \mp \iota u), \qquad \text{by (81)},$$

except that

$$u_0 = P_n(\mu),$$

also

$$v_m = \frac{1}{\pi}\int_0^{2\pi} \{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n \sin m\phi$$

$$= 2P_n^m(\mu)\frac{\Pi(n)}{\Pi(n + m)}\sin m(\psi \mp \iota u),$$

hence

$$\{\mu + \sqrt{\mu^2 - 1}\cos(\phi - \psi \pm \iota u)\}^n$$

$$= P_n(\mu) + 2\sum_{m=1}^{\infty}\frac{\Pi(n)}{\Pi(n + m)}P_n^m(\mu)\cos m(\phi - \psi \pm \iota u) \qquad (98),$$

this formula which holds for all real and complex values of n is a generalization of a well-known formula, namely the case in which n is a positive integer, in which case the series is a finite one, since $P_n^m(\mu) = 0$, when m and n are positive integers such that $m > n$.

Case II. $u > \log \text{mod.} \sqrt{\frac{\mu+1}{\mu-1}}$.

In this case the expansion of $\{\mu + \sqrt{\mu^2-1}\cos(\phi - \psi - \iota u\}^n$ in powers of w consists of powers whose indices differ from n by a real integer; thus

$$\{\mu + \sqrt{\mu^2-1}\cos(\phi - \psi - \iota u)\}^n = \Sigma u_m e^{m(\phi-\psi-\iota u)\iota}$$

where m has the values $n, n-1, n-2, \ldots$

To determine u_m, multiply both sides of the equation by $e^{-m(\phi-\psi-\iota u)\iota}$; then, since $\int_0^{2\pi} e^{(m'-m)(\phi-\psi\pm\iota u)\iota}\,d\phi = 0$, when m, m' are different real integers, we have

$$u_m = \frac{1}{2\pi}\int_0^{2\pi}\{\mu + \sqrt{\mu^2-1}\cos(\phi-\psi-\iota u)\}^n e^{-m(\phi-\psi-\iota u)\iota}\,d\phi$$

$$= \frac{\Pi(n)}{\Pi(n+m)}\left\{P_n^m(\mu) - \frac{2}{\pi}\cdot e^{-n\pi\iota}\sin n\pi \,.\, Q_n^m(\mu)\right\}, \quad \ldots \text{ by (93).}$$

We have thus obtained the expansion

$$\{\mu + \sqrt{\mu^2-1}\cos(\phi-\psi-\iota u)\}^n$$
$$= \Sigma\frac{\Pi(n)}{\Pi(n+m)}\left\{P_n^m(\mu) - \frac{2}{\pi}e^{-n\pi\iota}\sin n\pi \,.\, Q_n^m(\mu)\right\}e^{m\iota(\phi-\psi-\iota u)} \quad \ldots \quad (99),$$

when m has the values $n, n-1, n-2, \ldots$ and the expansion holds for all real or complex values of n; in the special case in which n is a positive integer, we have

$$\{\mu + \sqrt{\mu^2-1}\cos(\phi-\psi-\iota u)\}^n = \sum_{m=0}^{m=n}\frac{\Pi(n)}{\Pi(n+m)}P_n^m(\mu)\,.\,e^{\iota m(\phi-\psi-\iota u)} \quad . \quad (100).$$

When n is a negative integer, change it into $-n-1$; we thus find, on using the formula (97)

$$\{\mu + \sqrt{\mu^2-1}\cos(\phi-\psi-\iota u)\}^{-n-1}$$
$$= \frac{(-1)^{n+1}}{\Pi(n)}\Sigma\frac{1}{\Pi(m-n-1)\,\Pi(m+n)}Q_n^m(\mu)\,e^{-\iota m(\phi-\psi-\iota u)} \quad \ldots \quad (101),$$

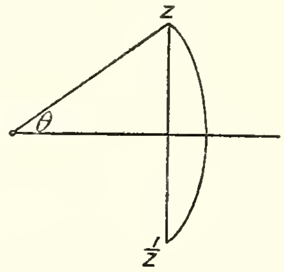
where m has the values $n+1, n+2, n+3, \ldots$

Generalization of DIRICHLET'S *and* MEHLER'S *Expressions for* $P_n(\cos\theta)$ *as a Definite Integral.*

46. It has been shown in Art. 33, that provided the real part of $m+\frac{1}{2}$ is positive,

$$P_n^{-m}(\mu) = \frac{1}{2^m}\cdot\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}(\mu^2-1)^{-\frac{1}{2}m}\int_{\frac{1}{z}}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh,$$

let $\mu=\cos\theta+0\,.\,\iota$, the line joining the points $z, \dfrac{1}{z}$ on the h-plane is perpendicular



to the real axis, and the path of integration may be taken to be a circular arc with centre at the origin; let $h=e^{\iota\phi}$, then remembering that the phase of $1-2\mu h+h^2$ increases from $2\pi-\theta$, at the lower limit to $2\pi+\theta$, at the upper one, we have

$$(1-2\mu h+h^2)^{m-\frac{1}{2}} = e^{2\pi(m-\frac{1}{2})\iota}\,e^{(m-\frac{1}{2})\iota\phi}\,(2\cos\phi-2\cos\theta)^{m-\frac{1}{2}},$$

hence

$$P_n^{-m}(\cos\theta) = e^{-\frac{1}{2}m\pi\iota}\,P_n^{-m}(\cos\theta+0\,.\,\iota)$$

$$= e^{-\frac{1}{2}m\pi\iota}\frac{1}{2^m}\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,.\,e^{-\frac{1}{2}m\pi\iota}\sin^{-m}\theta\,.\,e^{2\pi(m-\frac{1}{2})\iota}$$
$$\int_{-\theta}^{\theta} e^{(n-m)\iota\phi}\,.\,e^{(m-\frac{1}{2})\iota\phi}(2\cos\phi-2\cos\theta)^{m-\frac{1}{2}}\,.\,\iota e^{\iota\phi}\,d\phi,$$

or

$$P_n^{-m}(\cos\theta) = \frac{2}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\int_0^{\theta}\frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi-2\cos\theta)^{\frac{1}{2}-m}}\,d\phi \quad . \quad (102).$$

From Art. 11 we find

$$P_n^{-m}(\cos\theta) = \frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,P_n^m(\cos\theta) - \frac{2}{\pi}\sin m\pi\,.\,Q_n^m(\cos\theta)\right\},$$

hence

$$\frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,P_n^m(\cos\theta) - \frac{2}{\pi}\sin m\pi\,.\,Q_n^m(\cos\theta)\right\}$$
$$= \frac{2}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\,.\int_0^{\theta}\frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi-2\cos\theta)^{\frac{1}{2}-m}}\,d\phi \quad . \quad (103).$$

A particular case of (102), or (103) is MEHLER'S form of one of DIRICHLET'S expressions for $P_n(\cos\theta)$,

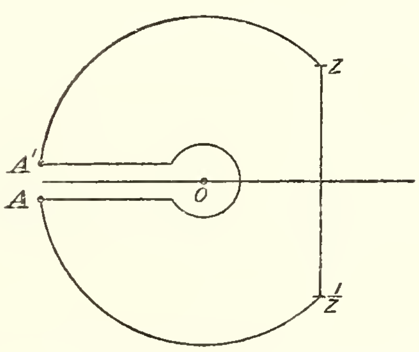
$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{2(\cos\phi - 2\cos\theta)^{\frac{1}{2}}}.$$

The formula (103) holds for all values of n and m, real or complex, provided the real part of $m+\frac{1}{2}$ is positive. When m is a real integer, we have for unrestricted values of n,

$$(-1)^m \frac{\Pi(n-m)}{\Pi(n+m)} P_n^m(\cos\theta) = \frac{2\sin^{-m}\theta}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi}{(2\cos\phi - 2\cos\theta)^{\frac{1}{2}-m}}d\phi \quad . \quad (104).$$

47. Next let us suppose the real parts of $n-m+1$, and of $m+\frac{1}{2}$ to be positive; the path of the integral

$$\int_{1/z}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh$$



can be taken as in the figure to consist of two circular arcs of unit radius, two straight portions along the real axis, and a circle of indefinitely small radius round the point $h=0$; under the above conditions as to m and n, the circle contributes nothing to the value of the integral.

In the integral taken along the arc joining the points $\frac{1}{z}$ and -1, the phase of $1-2\mu h+h^2$ is $3\pi-\phi$, where $h=e^{-\iota\phi}$; in the integral, from -1 to z, it is $\pi+\phi$, where $h=e^{\iota\phi}$; the two integrals together make up

$$\int_\theta^\pi \{e^{-(n-m)\iota\phi+(m-\frac{1}{2})(3\pi-\phi)\iota}(-\iota e^{-\iota\phi}) - e^{(n-m)\iota\phi+(m-\frac{1}{2})(\pi+\phi)\iota}(\iota e^{\iota\phi})\}(2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi,$$

or

$$e^{2\pi(m-\frac{1}{2})\iota}\int_\theta^\pi 2\iota\cos\left[(n+\tfrac{1}{2})\phi-(m+\tfrac{1}{2})\pi\right](2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi.$$

In the integrals from $h=1$ to $h=0$, the phase of $1-2\mu h+h^2$ is 2π; let $h=e^{-\iota\pi}e^{-v}$, for the lower path, and $h=e^{\iota\pi}e^{-v}$, for the upper path; these portions of the integral give us

$$-\int_0^\infty \{e^{-(n-m+1)\iota\pi}.\, e^{-(n+\frac{1}{2})v}.\, e^{2\pi\iota(m-\frac{1}{2})} - e^{(n-m+1)\iota\pi}.\, e^{-(n+\frac{1}{2})v}.\, e^{2\pi\iota(m-\frac{1}{2})}\}\,(2\cosh v + 2\cos\theta)^{m-\frac{1}{2}}\,dv,$$

or

$$-e^{(m-\frac{1}{2})2\pi\iota}\, 2\iota\sin(n-m)\pi\int_0^\infty \frac{e^{-(n+\frac{1}{2})v}}{(2\cosh v + 2\cos\theta)^{\frac{1}{2}-m}}\,dv$$

we thus obtain the formula

$$\begin{aligned} P_n^{-m}(\cos\theta) &= \frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,P_n^m(\cos\theta) - \frac{2}{\pi}\sin m\pi\,.\,Q_n^m(\cos\theta)\right\} \\ &= \frac{2\sin^{-m}\theta}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\left\{\int_\theta^\pi \frac{\cos[(n+\frac{1}{2})\phi-(m+\frac{1}{2})\pi]}{(2\cos\theta-2\cos\phi)^{\frac{1}{2}-m}}\,d\phi\right. \\ &\qquad \left. + \cos(n+\tfrac{1}{2}-m)\pi\int_0^\infty \frac{e^{-(n+\frac{1}{2})v}}{(2\cosh v+2\cos\theta)^{\frac{1}{2}-m}}\right\} \quad . \quad (105), \end{aligned}$$

which holds provided the real parts of $m+\frac{1}{2}$, $n-m+1$ are positive. If $n-m$ is a positive real integer this becomes

$$\begin{aligned} &\frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,P_n^m(\cos\theta) - \frac{2}{\pi}\sin m\pi\,.\,Q_n^m(\cos\theta)\right\} \\ &\qquad = \frac{2\sin^{-m}\theta}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\int_\theta^\pi \frac{\cos[(n+\frac{1}{2})\phi-(m+\frac{1}{2})\pi]}{(2\cos\theta-2\cos\phi)^{\frac{1}{2}-m}}\,d\phi \quad . \quad . \quad (106). \end{aligned}$$

When m and n are both positive integers, and $n \geqq m$, we obtain

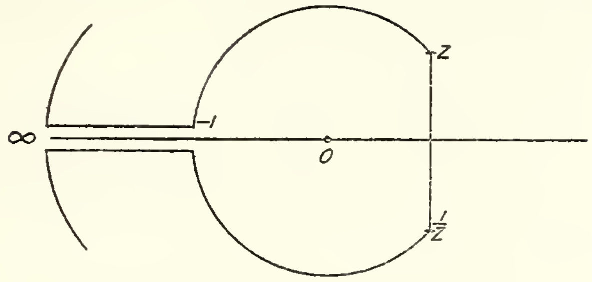
$$\frac{\Pi(n-m)}{\Pi(n+m)}\,P_n^m(\cos\theta) = \frac{2\sin^{-m}\theta}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi}{(2\cos\theta-2\cos\phi)^{\frac{1}{2}-m}}\,d\phi \quad . \quad (107),$$

which becomes, when $m=0$,

$$P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi}{(2\cos\theta-2\cos\phi)^{\frac{1}{2}}}\,d\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (108),$$

which is the second expression given by MEHLER for $P_n(\cos\theta)$. The formulæ (105), (106), (107) are therefore generalizations of the known formulæ of MEHLER and DIRICHLET.

48. Next suppose the condition that the real part of $n-m+1$ is positive does not necessarily hold, but that the real part of $n+m$ is negative, and that of $m+\frac{1}{2}$ is positive; we may replace part of the path of integration in the last Art. by straight paths from -1 to $-\infty$ along the real axis, and a circle of infinite radius. From $\frac{1}{z}$ to -1, the phase of $1-2\mu h+h^2$ is $\pi-\phi$, where ϕ is initially equal to θ; from -1 to z, the phase of $1-2\mu h+h^2$ is $3\pi+\phi$, where ϕ is equal to θ at the



point z. The part of the integral for $P_n^{-m}(\cos\theta)$ which consists of integrations along the two finite circular axes is

$$\frac{1}{2^m}\cdot\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,e^{-m\pi\iota}\sin^{-m}\theta\int_\theta^\pi\Big\{e^{-(n-m)\iota\phi+(m-\frac{1}{2})(\pi-\phi)\iota}\,(-\iota e^{-\iota\phi})$$
$$-\,e^{(n-m)\iota\phi+(m-\frac{1}{2})(3\pi+\phi)\iota}\,(\iota e^{\iota\phi})\Big\}\,(2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi$$

or

$$\frac{1}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\int_\theta^\pi 2\cos\{(m+\tfrac{1}{2})\,\pi+(n+\tfrac{1}{2})\,\phi\}\,(2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi.$$

The part of the integral which is taken along the circle of infinite radius is zero, and the part taken along the real axis is

$$\frac{1}{2^m}\cdot\frac{e^{-\iota\pi(m-\frac{1}{2})}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\,e^{-m\pi\iota}\sin^{-m}\theta\int_0^\infty\Big\{e^{-\iota\pi(n-m+1)+(n+\frac{1}{2})v}$$
$$-\,e^{\iota\pi(n-m+1)+(n+\frac{1}{2})v+(m-\frac{1}{2})4\pi\iota}\Big\}\,(2\cos\theta+2\cosh v)^{m-\frac{1}{2}}\,dv$$

or

$$\frac{1}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sin^{-m}\theta\int_0^\infty 2e^{(n+\frac{1}{2})v}\cos(n+\tfrac{1}{2}+m)\,\pi\,(2\cos\theta+2\cosh v)^{m-\frac{1}{2}}\,dv,$$

we thus obtain the formula

$$P_n^{-m}(\cos\theta)=\frac{\Pi(n-m)}{\Pi(n+m)}\left\{\cos m\pi\,.\,P_n^m(\cos\theta)-\frac{2}{\pi}\sin m\pi\,.\,Q_n^m(\cos\theta)\right\}$$
$$=\frac{1}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\cdot\frac{2}{\sin^m\theta}\left\{\int_0^\pi\cos(\overline{n+\tfrac{1}{2}}\phi+\overline{m+\tfrac{1}{2}}\pi)\,(2\cos\theta-2\cos\phi)^{m-\frac{1}{2}}\,d\phi\right.$$
$$\left.+\int_0^\infty e^{(n+\frac{1}{2})v}\cos(n+\tfrac{1}{2}+m)\,\pi\,.\,(2\cos\theta+2\cosh v)^{m-\frac{1}{2}}\,dv\right\}\qquad(109)$$

which holds, provided the real part of $n+m$ is negative, and that of $m+\frac{1}{2}$ is positive.

When the real part of n is between 0 and -1, and the real part of m is between

$\frac{1}{2}$ and $-\frac{1}{2}$, both the formulæ (109) and (105) hold. In the special case $m=0$, we find by adding the two expressions in (105) and (109),

$$P_n(\cos\theta) = \frac{2}{\pi}\cos(n+\tfrac{1}{2})\pi\int_0^\infty \frac{\cosh(n+\frac{1}{2})v}{(2\cos\theta+2\cosh v)^{\frac{1}{2}}}\,dv \quad . \quad . \quad . \quad (110)$$

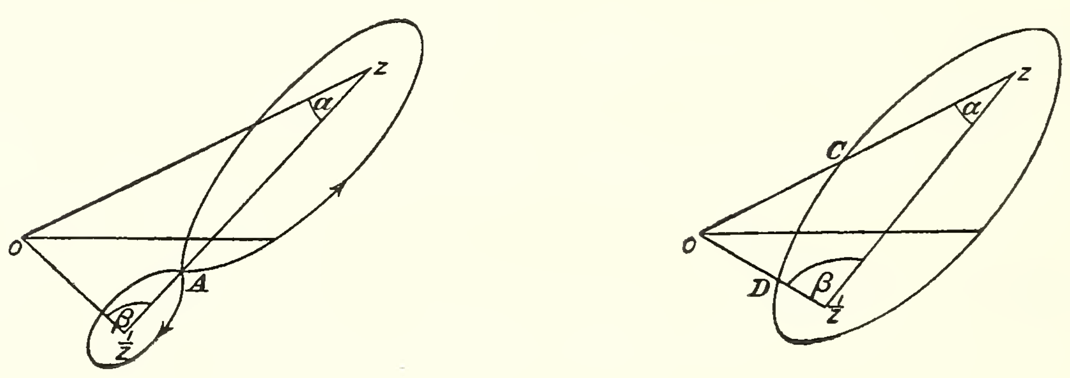
when the real part of n is between 0 and -1.

A definite integral form for $P_n(\mu)$, *when the real part of* n *is between* 0 *and* -1.

49. Taking the formula

$$P_n^m(\mu) = \frac{1}{2\pi\iota}\cdot 2^m\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\int^{(z+,\,1/z-)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh,$$

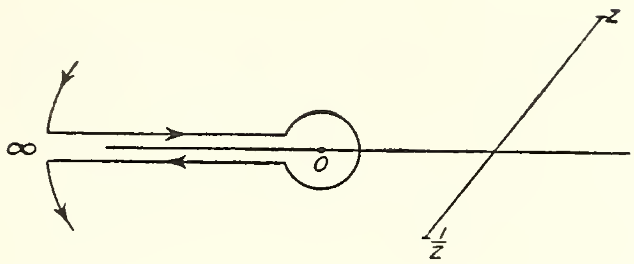
we see that, provided m is half a real integer, and also $\frac{1}{2}-m$ is positive, the path may be replaced by one which consists of a single curve enclosing both the points $z, \frac{1}{z}$.



In the first figure the initial phases at A are $2\pi-\beta$, for $1-hz$, and $-(2\pi-\alpha)$ for $1-\frac{h}{z}$. In the second figure the phase of $1-\frac{h}{z}$ is zero at C, and that of $1-hz$ is 2π at D. The formula becomes

$$P_n^m(\mu) = \frac{1}{2\pi\iota}2^m\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}(\mu^2-1)^{\frac{1}{2}m}\int^{(z+,\,1/z+)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh.$$

Now suppose the real part of $n-m$ is negative, and that of $n+m+1$ is positive; we may replace the path by one round a circle of infinite radius, straight paths along



the real axis, and a circle round the point O; the circular paths contribute nothing to the value of the integral, and we have

$$\mathrm{P}_n^m(\mu) = \frac{1}{2\pi\iota}\, 2^m\, \frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}\,(\mu^2-1)^{\frac{1}{2}m}\int_{-\infty}^{\infty} - \{e^{(n+m+1)\iota\pi-2\iota\pi(m+\frac{1}{2})}$$
$$- e^{-(n+m+1)\iota\pi+2\iota\pi(m+\frac{1}{2})}\} e^{(n+\frac{1}{2})u}(2\cosh u + 2\mu)^{-m-\frac{1}{2}}\,du$$

$$= -\frac{2^{m+1}}{\pi}\cdot\frac{\Pi(m-\frac{1}{2})}{\Pi(-\frac{1}{2})}\sin(n-m)\pi\,.\int_0^{\infty}\cosh(n+\tfrac{1}{2})u\,.\,(2\cosh u + 2\mu)^{-m-\frac{1}{2}}\,du,$$

this holds for all values of μ of which the real part is positive, provided m is half an integer, and is less than $\frac{1}{2}$, also provided the real part of $n - m$ is negative and of $n + m + 1$ is positive; the only value of m which satisfies these conditions is $m = 0$; we thus obtain the formula

$$\mathrm{P}_n(\mu) = \frac{2}{\pi}\cos(n+\tfrac{1}{2})\pi\int_0^{\infty}\frac{\cosh(n+\frac{1}{2})u}{(2\cosh u + 2\mu)^{\frac{1}{2}}}\,du\,. \quad . \quad . \quad . \qquad (111),$$

which holds, provided the real part of n is between 0 and -1, and that of μ is positive.

Definite Integral Expressions for $\mathrm{P}_n^m(\mu)$, *when* μ *is real and greater than unity.*

50. In the formula

$$\mathrm{P}_n^{-m}(\mu) = \frac{1}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\, e^{-\iota\pi(m-\frac{1}{2})}(\mu^2-1)^{-\frac{1}{2}m}\int_{1/z}^{z} h^{n-m}(1-2\mu h+h^2)^{m-\frac{1}{2}}\,dh$$

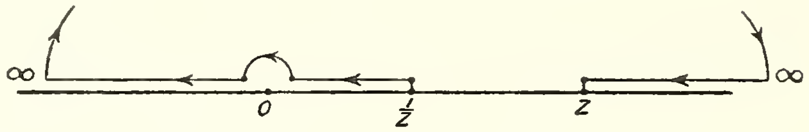
where the real part of $m + \frac{1}{2}$ is positive; when μ is real and greater than unity, put $\mu = \cosh\psi$, then $z = e^{\psi}$, $\dfrac{1}{z} = e^{-\psi}$, thus putting $h = e^{\phi}$, we obtain the formula

$$P_n^{-m}(\cosh\psi)$$

$$= \frac{1}{2^m \Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})} \sinh^{-m}\psi \int_0^\psi 2\cosh(n+\tfrac{1}{2})\phi\,(2\cosh\psi - 2\cosh\phi)^{m-\frac{1}{2}}\,d\phi \quad . \quad (112)$$

where the real part of $m + \frac{1}{2}$ is positive, and in particular

$$P_n(\cosh\psi) = \frac{2}{\pi}\int_0^\psi \frac{\cosh(n+\frac{1}{2})\phi}{\sqrt{2\cosh\psi - 2\cosh\phi}}\,d\phi \quad . \quad . \quad . \quad . \quad (113).$$



The path joining the points z, $1/z$ can be placed as in the figure, along the real axis from $1/z$ to $-\infty$, except for a small semicircle round the point 0, then a semicircle of infinite radius, and lastly a straight path along the real axis from $+\infty$ to z. If the real part of m lies between $\frac{1}{2}$ and $-\frac{1}{2}$, and if the real part of $n - m + 1$ is positive, and if $n + m$ is negative, the straight portions of the path are the only ones which contribute anything to the value of the integral; in this way we find that under the conditions just specified.

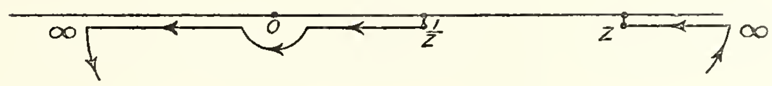
$$P_n^{-m}(\cosh\psi) = \frac{\sinh^{-m}\psi}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\left\{\int_\psi^\infty -2\iota\sinh(\overline{n+\tfrac{1}{2}}\phi - \iota m\pi)\,(2\cosh\phi - 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right.$$

$$\left. + \int_0^\infty e^{(n+\frac{1}{2})\phi + (n+\frac{1}{2})\iota\pi}(2\cosh\phi + 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right\}.$$

In a similar manner, we can prove, by taking the semicircles below the real axis that under the same conditions, $P_n^{-m}(\cosh\psi)$ is given by the formula

$$P_n^{-m}(\cosh\psi) = \frac{\sinh^{-m}\psi}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\left\{\int_\psi^\infty 2\iota\sinh(\overline{n+\tfrac{1}{2}}\phi + \iota m\pi)\,(2\cosh\phi - 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right.$$

$$\left. + \int_0^\infty e^{(n+\frac{1}{2})\phi - (n+\frac{1}{2})\iota\pi}(2\cosh\phi + 2\cosh\psi)^{m-\frac{1}{2}}\,d\phi\right\}.$$



Multiply the two formulæ by $e^{-(n+\frac{1}{2})\iota\pi}$, $e^{(n+\frac{1}{2})\iota\pi}$, and subtract them, we then have the formula

$$\mathrm{P}_n^{-m}(\cosh\psi) = \frac{\sinh^{-m}\psi}{2^m\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\operatorname{cosec}(n+\tfrac{1}{2})\pi\int_\psi^\infty\{e^{(n+\frac{1}{2})\iota\pi}\sinh(\overline{n+\tfrac{1}{2}}\phi+\iota m\pi)$$

$$+\,e^{-(n+\frac{1}{2})\iota\pi}\sinh(\overline{n+\tfrac{1}{2}}\phi-\iota m\pi)\}\,(2\cosh\phi-2\cosh\psi)^{m-\frac{1}{2}}\,d\phi \quad (114),$$

where the real part of n is between 0 and -1; and the real part of m is between $\pm\frac{1}{2}$. If $m=0$, we have

$$\mathrm{P}_n(\cosh\psi) = \frac{2}{\pi}\cot(n+\tfrac{1}{2})\pi\int_\psi^\infty\frac{\sinh(n+\frac{1}{2})\phi}{\sqrt{2\cosh\phi-2\cosh\psi}}\,d\phi \quad . \quad . \quad (115).$$

Definite Integral Formulæ for $\mathrm{Q}_n^m(\cos\theta)$, *under Special Conditions.*

51. When the real parts of $n+m+1$, $\frac{1}{2}-m$ are positive, we have

$$\mathrm{Q}_n^m(\cos\theta+0.\iota) = e^{m\pi\iota}.2^m\Pi(m-\tfrac{1}{2})\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}.e^{\frac{m\pi\iota}{2}}.\sin^m\theta\int_0^{1/z}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh \quad . \quad (43),$$

take the path of integration to be from 0 to 1, along the real axis, then from 1 to $\frac{1}{z}$ along an arc of a circle of unit radius with its centre at the origin; along the straight path, $1-2\mu h+h^2$ has the phase zero, and along the circular arc it has the same phase as h, hence, writing in the first part of the integral $h=e^{-u}$ and in the second part $h=e^{-\iota\phi}$,

$$\mathrm{Q}_n^m(\cos\theta+0.\iota)$$

$$= e^{\frac{3}{2}m\pi\iota}.2^m.\Pi(m-\tfrac{1}{2})\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}.\sin^m\theta\left\{\int_0^\infty\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh u-2\cos\theta)^{m+\frac{1}{2}}}\,du\right.$$

$$\left.-\int_0^\theta\frac{\iota e^{-(n+\frac{1}{2})\iota\phi}}{(2\cos\phi-2\cos\theta)^{m+\frac{1}{2}}}\,d\phi\right\}.$$

Next take the path to be from 0 to $-\infty$ along the real axis, along a semicircle of infinite radius to $+\infty$, from $+\infty$ along the real axis to 1, and from 1 along a circular arc whose centre is the origin to the point $\frac{1}{z}$. If the real part of $n-m$ is negative, the part of the integral taken along the infinite semicircle is zero; we have

then, writing $h = e^{-\iota\pi} . e^u$, in the first integral, $h = e^u$, in the second integral, and $h = e^{-\iota\phi}$, in the third integral,

$$Q_n^m(\cos\theta + 0 . \iota)$$

$$= e^{\frac{3}{2}m\pi\iota} . 2^m . \Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}\sin^m\theta\left\{\int_{-\infty}^{\infty} e^{-(n+m+1)\iota\pi}\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u + 2\cos\theta)^{m+\frac{1}{2}}}\,du\right.$$

$$\left. - \int_0^\infty \frac{1}{e^{(m+\frac{1}{2})2\pi\iota}}\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u - 2\cos\theta)^{m+\frac{1}{2}}}\,du - \int_0^\theta \frac{\iota e^{-(n+\frac{1}{2})\iota\phi}}{e^{(m+\frac{1}{2})2\pi\iota}(2\cos\phi - 2\cos\theta)^{m+\frac{1}{2}}}\,d\phi\right\}.$$

If the real part of m lies between $\pm\frac{1}{2}$, and if the real parts of $n+m+1$, $m-n$ are positive, both the formulæ we have found for $Q_n^m(\cos\theta + 0 . \iota)$ hold. Multiply the first expression by $e^{-m\pi\iota}$, and the second by $e^{m\pi\iota}$, and then add; we find

$$2\cos m\pi . Q_n^m(\cos\theta + 0 . \iota)$$

$$= e^{\frac{1}{2}m\pi\iota} . 2^m . \Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi} . \sin^m\theta\left\{\int_0^\infty \frac{2\cosh(n+\frac{1}{2})u}{(2\cosh u - 2\cos\theta)^{m+\frac{1}{2}}}\,du\right.$$

$$\left. + e^{-(n-m+1)\iota\pi}\int_0^\infty \frac{2\cosh(n+\frac{1}{2})u}{(2\cosh u + 2\cos\theta)^{m+\frac{1}{2}}}\,du\right\} \quad . \quad (116),$$

which holds, provided the real part of m lies between $\pm\frac{1}{2}$, and those of $n+m+1$, $m-n$ are positive.

If $m = 0$, we have

$$Q_n(\cos\theta + 0 . \iota) = \int_0^\infty \frac{\cosh(n+\frac{1}{2})u}{(2\cosh u - 2\cosh\theta)^{\frac{1}{2}}}\,du + e^{-(n+1)\iota\pi}\int_0^\infty \frac{\cosh(n+\frac{1}{2})u}{(2\cosh u + 2\cos\theta)^{\frac{1}{2}}}\,du \quad (117),$$

which holds provided the real part of n is between 0 and -1.

It is to be remembered that $Q_n(\cos\theta + 0 . \iota) = Q_n(\cos\theta) - \frac{\iota\pi}{2}P_n(\cos\theta)$.

Formula for $Q_n^m(\cosh\psi)$ under special conditions.

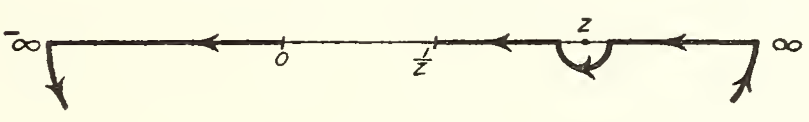
52. When μ is real and greater than unity, let $\mu = \cosh\psi$, we then have, provided the real parts of $n+m+1$, $\frac{1}{2}-m$ are positive,

$$Q_n^m(\cosh\psi) = e^{m\pi\iota} \,.\, 2^m \,.\, \Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}\sinh^m\psi\int_0^{1/z}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh.$$

Let $h = e^{-u}$, we have then, taking the path along the real axis,

$$Q_n^m(\cosh\psi) = e^{m\pi\iota} \,.\, 2^m \,.\, \Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}\sinh^m\psi\int_\psi^\infty\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}\,du, \quad (118).$$

If we take the path to be from 0 to $-\infty$ along the real axis, along an infinite semicircle from $-\infty$ to $+\infty$, along a straight path from ∞ to $\frac{1}{z}$, avoiding the



point z by describing a small semicircle; the integrals along the semicircles vanish provided the real part of $n-m$ is negative, we then have

$$Q_n^m(\cosh\psi) = e^{m\pi\iota}.2^m.\Pi(m-\tfrac{1}{2})\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}\sinh^m\psi\left\{\int_{-\infty}^{\infty}\frac{e^{-(n+m+1)\iota\pi}.e^{(n+\frac{1}{2})u}}{(2\cosh\psi+2\cosh u)^{m+\frac{1}{2}}}\,du\right.$$
$$\left.-\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{e^{(2m+1)\iota\pi}(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}\,du-\int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{e^{(m+\frac{1}{2})\iota\pi}(2\cosh\psi-2\cosh u)^{m+\frac{1}{2}}}\,du\right\}.$$

In a similar manner, by taking the semicircles above the real axis, we can show that

$$Q_n^m(\cosh\psi) = e^{m\pi\iota}.2^m\Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}\sinh^m\psi\left\{\int_{-\infty}^{\infty}\frac{e^{(n+m+1)\iota\pi}.e^{(n+\frac{1}{2})u}}{(2\cosh\psi+2\cosh u)^{m+\frac{1}{2}}}\,du\right.$$
$$\left.-\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{e^{-(2m+1)\iota\pi}(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}\,du-\int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{e^{-(m+\frac{1}{2})\iota\pi}(2\cosh\psi-2\cosh u)^{m+\frac{1}{2}}}\,du\right\}.$$

Multiplying the first expression by $e^{(n+m+1)\iota\pi}$, and the second by $e^{-(n+m+1)\iota\pi}$, and subtracting, we then have

$$Q_n^m(\cosh\psi)\sin(n+m)\pi$$
$$=e^{m\pi\iota}.2^m.\Pi(m-\tfrac{1}{2})\Pi(-\tfrac{1}{2})\frac{\cos m\pi}{\pi}\sinh^m\psi\left\{\sin(n-m)\pi\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du\right.$$
$$\left.-\sin(n+\tfrac{1}{2})\pi\int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh\psi-2\cosh u)^{m+\frac{1}{2}}}du\right\};$$

where the real part of m is less than $\frac{1}{2}$, and the real parts of $n+m+1$, $m-n$ are positive.

Put $m=0$, we have then

$$Q_n(\cosh\psi)=\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{\frac{1}{2}}}du-\cot n\pi\int_0^\psi\frac{e^{-(n+\frac{1}{2})u}}{(2\cosh\psi-2\cosh u)^{\frac{1}{2}}}du,\quad(119)$$

where the real part of n lies between 0 and -1.

Expressions for $Q_n^m(\cosh\psi)$, *when* $n-\frac{1}{2}$ *is a real integer.*

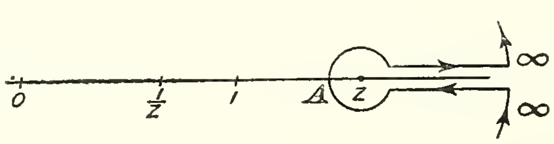
53. When $n-\frac{1}{2}$ is a real integer, the formula

$$Q_n^m(\mu)=\iota e^{(m-n)\pi\iota}.2^m.\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{4\pi\sin(n+m)\pi}(\mu^2-1)^{\frac{1}{2}m}\int^{(1/z,\,0,\,1/z-,\,0-)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}dh$$

may be replaced by

$$e^{2m\pi\iota}.2^m.\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{2\pi}(\mu^2-1)^{\frac{1}{2}m}\int^{(1/z+,\,0+)}\frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}dh,$$

the path may, as in the figure, consist of a circle of infinite radius, straight paths along the real axis from ∞ to z, and a small circle round the point z.



If the real parts of $m-n$, $\frac{1}{2}-m$ are positive, the only effective parts of the integral are those along the real axis. The phase of $1-2\mu h+h^2$, at A is π; we thus find,

$$Q_n^m(\cosh\psi)$$
$$=e^{2m\pi\iota}.2^m.\frac{\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{2\pi}\sinh^m\psi\left\{\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du\right.$$
$$\left.-\int_\psi^\infty\frac{e^{(n-m)2\pi\iota}.e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du\right\},$$
$$=e^{(m+n)\pi\iota}.2\iota\sin(m-n)\pi.\frac{2^m\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{2\pi}\sinh^m\psi\int_\psi^\infty\frac{e^{(n+\frac{1}{2})u}}{(2\cosh u-2\cosh\psi)^{m+\frac{1}{2}}}du,$$

or

$Q_n^m(\cosh\psi)$

$$= \frac{\cos m\pi\, 2^m\, \Pi(m-\tfrac{1}{2})\,\Pi(-\tfrac{1}{2})}{\pi} \sinh^m\psi \int_\psi^\infty \frac{e^{(n+\frac{1}{2})u}}{(2\cosh u - 2\cosh\psi)^{m+\frac{1}{2}}}\, du \quad . \quad . \quad . \quad . \quad (120),$$

where $n-\frac{1}{2}$ is a real integer, and the real parts of $m-n$, $\frac{1}{2}-m$ are positive.

If $m=0$, we have

$$Q_n(\cosh\psi) = \int_\psi^\infty \frac{e^{(n+\frac{1}{2})u}}{(2\cosh u - 2\cosh\psi)^{\frac{1}{2}}}\, du \quad . \quad . \quad . \quad . \quad . \quad (121),$$

where $n-\frac{1}{2}$ is a negative real integer.

For all values of m and n such that $n-\frac{1}{2}$ is a real integer, the path may be taken to be a circle of radius unity with the origin as centre; we obtain on putting $h=e^{\iota\phi}$, since $h^2-2\mu h+1 = he^{\iota\pi}(2\cosh\psi - 2\cos\phi)$,

$$Q_n^m(\mu) = e^{2m\pi\iota}\,.\,2^m\,.\,\frac{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{2\pi}\sinh^m\psi\int_{-\pi}^{\pi}\frac{e^{(n-\frac{1}{2})\iota\phi}\,.\,\iota\,.\,e^{\iota\phi}}{e^{(m+\frac{1}{2})\iota\pi}(2\cosh\psi - 2\cos\phi)^{m+\frac{1}{2}}}\,d\phi,$$

or

$$Q_n^m(\mu) = \iota e^{(m-\frac{1}{2})\iota\pi}\,.\,\frac{2^m\,\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\pi}\sinh^m\psi\int_0^\pi\frac{\cos(n+\frac{1}{2})\phi}{(2\cosh\psi - 2\cos\phi)^{m+\frac{1}{2}}}\,d\phi \quad (122).$$

Recurrent Relations for Successive Values of n, m in $P_n^m(\mu)$, $Q_n^m(\mu)$.

54. Denote the integral $\int \frac{h^{n+m}}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\,dh$, by $U(n, m)$, the integral being taken along any closed path, that is, one in which after completion the integrand returns to its initial value.

We find

$$\frac{dU(n,m)}{d\mu} = (2m+1)\int\frac{h^{n+m+1}}{(1-2\mu h+h^2)^{m+\frac{3}{2}}}\,dh = (2m+1)\,U(n, m+1);$$

also

$$\frac{d}{dh}\cdot\frac{\mu-h}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} = \frac{2m}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} + (2m+1)\frac{\mu^2-1}{(1-2\mu h+h^2)^{m+\frac{3}{2}}}\,.$$

Hence

$$(\mu^2-1)\frac{dU(n,m)}{d\mu} = \int h^{n+m+1}\,.\left\{\frac{-2m}{(1-2\mu h+h^2)^{m+\frac{1}{2}}} + \frac{d}{dh}\cdot\frac{\mu-h}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}\right\}dh$$

$$= -2mU(n+1, m) - \int\frac{\mu-h}{(1-2\mu h+h^2)^{m+\frac{1}{2}}}(n+m+1)\,h^{n+m}\,dh$$

$$= -2mU(n+1, m) - (n+m+1)\{\mu U(n,m) - U(n+1,m)\},$$

or,

$$(\mu^2 - 1)\frac{d\mathrm{U}(n, m)}{d\mu} = (n - m + 1)\,\mathrm{U}(n + 1, m) - (n + m + 1)\,\mu\mathrm{U}(n, m).$$

Referring to the formulæ (40), (50) for $\mathrm{Q}_n^m(\mu)$, $\mathrm{P}_n^m(\mu)$, we see that by choosing specified closed paths for the integration in $\mathrm{U}(n, m)$, each of the functions is of the form $\mathrm{C}_m(\mu^2 - 1)^{\frac{1}{2}m}\mathrm{U}(n, m)$; we thus obtain the formulæ

$$\left.\begin{aligned}(\mu^2 - 1)\frac{d\mathrm{P}_n^m(\mu)}{d\mu} &= (n - m + 1)\,\mathrm{P}_{n+1}^m(\mu) - (n + 1)\,\mu\mathrm{P}_n^m(\mu)\\ (\mu^2 - 1)\frac{d\mathrm{Q}_n^m(\mu)}{d\mu} &= (n - m + 1)\,\mathrm{Q}_{n+1}^m(\mu) - (n + 1)\,\mu\mathrm{Q}_n^m(\mu)\end{aligned}\right\} \quad . \quad (123).$$

Next let $\mathrm{V}(n, m) = \mathrm{U}(-n - 1, m)$, we have then by changing n into $-n - 1$ in the relation which has been found above for U,

$$(\mu^2 - 1)\frac{d\mathrm{V}(n, m)}{d\mu} = -(n + m)\,\mathrm{V}(n - 1, m) + n\mu\,\mathrm{V}(n, m);$$

special cases of this relation are

$$\left.\begin{aligned}(\mu^2 - 1)\frac{d\mathrm{P}_n^m(\mu)}{d\mu} &= n\,\mu\mathrm{P}_n^m(\mu) - (n + m)\,\mathrm{P}_{n-1}^m(\mu)\\ (\mu^2 - 1)\frac{d\mathrm{Q}_n^m(\mu)}{d\mu} &= n\,\mu\mathrm{Q}_n^m(\mu) - (n + m)\,\mathrm{Q}_{n-1}^m(\mu)\end{aligned}\right\} \quad . \quad . \quad . \quad (124),$$

from (123), (124), we have at once

$$\left.\begin{aligned}(2n + 1)\,\mu\mathrm{P}_n^m(\mu) - (n - m + 1)\,\mathrm{P}_{n+1}^m(\mu) - (n + m)\,\mathrm{P}_{n-1}^m(\mu) = 0\\ (2n + 1)\,\mu\mathrm{Q}_n^m(\mu) - (n - m + 1)\,\mathrm{Q}_{n+1}^m(\mu) - (n + m)\,\mathrm{Q}_{n-1}^m(\mu) = 0\end{aligned}\right\} \quad . \quad (125),$$

these recurrent relations between the functions for different values of n hold for general complex values of m and n.

55. It has been remarked in Art. 1, that W, which is equivalent to $\mathrm{U}(n, m)$, satisfies the differential equation

$$(1 - \mu^2)\frac{d^2\mathrm{U}(n, m)}{d\mu^2} - 2(m + 1)\,\mu\frac{d\mathrm{U}(n, m)}{d\mu} + (n - m)(n + m + 1)\,\mathrm{U}(n, m) = 0,$$

now

$$\frac{d\mathrm{U}(n, m)}{d\mu} = (2m + 1)\,\mathrm{U}(n, m + 1), \quad \frac{d^2\mathrm{U}(n, m)}{d\mu^2} = (2m + 1)(2m + 3)\,\mathrm{U}(n, m + 2)$$

thus

$$(1 - \mu^2)(2m + 1)(2m + 3)\,\mathrm{U}(n, m + 2) - 2(m + 1)(2m + 1)\,\mu\,.\,\mathrm{U}(n, m + 1) \\ + (n - m)(n + m + 1)\,\mathrm{U}(n, m) = 0;$$

referring to the formulæ (40), (50), we see that, as special cases of this result,

$$\left.\begin{aligned}P_n^{m+2}(\mu) + 2(m+1)\frac{\mu}{(\mu^2-1)^{\frac{1}{2}}}P_n^{m+1}(\mu) - (n-m)(n+m+1)P_n^m(\mu) = 0\\ Q_n^{m+2}(\mu) + 2(m+1)\frac{\mu}{(\mu^2-1)^{\frac{1}{2}}}Q_n^{m+1}(\mu) - (n-m)(n+m+1)Q_n^m(\mu) = 0\end{aligned}\right\}(126),$$

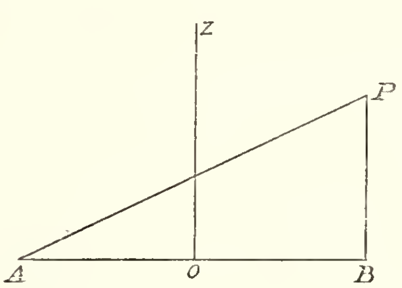
the formulæ (125), (126), are well known for the case in which m and n are real integers.

If $\mu = \cos\theta$, then introducing the modification of Art. 17 into the symbols P_n^m, Q_n^m, we have

$$\left.\begin{aligned}P_n^{m+2}(\cos\theta) - 2(m+1)\cot\theta . P_n^{m+1}(\cos\theta) + (n-m)(n+m+1)P_n^m(\cos\theta) = 0\\ Q_n^{m+2}(\cos\theta) - 2(m+1)\cot\theta . Q_n^{m+1}(\cos\theta) + (n-m)(n+m+1)Q_n^m(\cos\theta) = 0\end{aligned}\right\}(127).$$

Toroidal Functions.

56. If A, B are points at the extremities of a diameter of a fixed circle, and the coordinates of any point P in a plane through AB perpendicular to the plane of the circle, are denoted by σ, θ, ϕ, where $\sigma = \log\frac{AP}{BP}$, $\theta = \angle$ APB, and ϕ is the angle the plane APB makes with a fixed plane through the axis $0z$ which bisects AB and is perpendicular to the plane of the circle, it is known*



that the normal functions requisite for the solution of potential problems connected with the anchor ring are

$$P_{n-\frac{1}{2}}^m(\cosh\sigma)\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\theta \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi,$$

$$Q_{n-\frac{1}{2}}^m(\cosh\sigma)\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\theta \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi.$$

* See C. NEUMANN'S 'Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe,' Halle, 1864. W. M. HICKS, "Toroidal Functions," 'Phil. Trans.,' 1879. A. B. BASSET, "On Toroidal Functions," 'American Journal of Mathematics,' vol. 15. W. D. NIVEN, "On the Ring Functions," 'Proc. Lond. Math. Soc.,' vol. 24.

The functions $P_{n-\frac{1}{2}}^m(\cosh\sigma)$, $Q_{n-\frac{1}{2}}^m(\cosh\sigma)$, where m and n are positive integers, are consequently called toroidal functions. Various expressions for these functions may be found, as particular cases of the various definite integral expressions which have been given above for $P_n^m(\mu)$, $Q_n^m(\mu)$.

We find from (113)

$$P_{n-\frac{1}{2}}(\cosh\sigma) = \frac{2}{\pi}\int_0^\sigma \frac{\cosh n\phi}{\sqrt{(2\cosh\sigma - 2\cosh\phi)}}$$

Also from (81), (82),

$$P_{n-\frac{1}{2}}^m(\cosh\sigma) = \frac{1}{\pi}\cdot\frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-\frac{1}{2})}\int_0^\pi (\cosh\sigma + \sinh\sigma\cos\phi)^{n-\frac{1}{2}}\cos m\phi\, d\phi;$$

$$= \frac{(-1)^m}{\pi}\cdot\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\int_0^\pi \frac{\cos m\phi}{(\cosh\sigma + \sinh\sigma\cos\phi)^{n-\frac{1}{2}}}\, d\phi.$$

From (68), (69)

$$P_{n-\frac{1}{2}}^m(\cosh\sigma)$$

$$= \frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\cdot\frac{1}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\sinh^m\sigma\int_0^\pi (\cosh\sigma + \sinh\sigma\cos\phi)^{n-m-\frac{1}{2}}\sin^{2m}\phi\, d\phi,$$

$$= \frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\,\frac{1}{2^m\Pi(-\frac{1}{2})\Pi(m-\frac{1}{2})}\sinh^m\sigma\int_0^\pi \frac{\sin^{2m}\phi}{(\cosh\sigma + \sinh\sigma\cos\phi)^{n+m+\frac{1}{2}}}\, d\phi.$$

Again from (92), we find

$$Q_{n-\frac{1}{2}}^m(\cosh\sigma) = (-1)^m\frac{\Pi(n+m-\frac{1}{2})}{\Pi(n-\frac{1}{2})}\int_0^{\log\coth\frac{\sigma}{2}} (\cosh\sigma - \sinh\sigma\cosh w)^{n-\frac{1}{2}}\cosh mw\, dw,$$

and from (122),

$$Q_{n-\frac{1}{2}}^m(\cosh\sigma) = (-1)^m.\frac{2^m\Pi(m-\frac{1}{2})\Pi(-\frac{1}{2})}{\pi}\sinh^m\sigma\int_0^\pi \frac{\cos n\phi}{(2\cosh\sigma - 2\cos\phi)^{m+\frac{1}{2}}}\, d\phi.$$

In the case in which the real part of $n - m + \frac{1}{2}$ is positive, we find from (90) and (91),

$$Q_{n-\frac{1}{2}}^m(\cosh\sigma)$$

$$= (-1)^m.2^m.\frac{\Pi(n+m-\frac{1}{2})\Pi(-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})\Pi(m-\frac{1}{2})}\sinh^m\sigma\int_0^{\log\coth\frac{\sigma}{2}} (\cosh\sigma - \sinh\sigma\cosh w)^{n-m-\frac{1}{2}}\sinh^{2m}w\, dw,$$

$$= (-1)^m\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})}\int_0^\infty \frac{\cosh mw}{(\cosh\sigma + \sinh\sigma\cosh w)^{n+\frac{1}{2}}}\, dw.$$

57. From (125), (126), we find, on writing $n - \frac{1}{2}$ for n, the relations

$$2n\cosh\sigma\,.\,\mathrm{P}_{n-\frac{1}{2}}{}^{m}(\cosh\sigma)-(n-m+\tfrac{1}{2})\,\mathrm{P}_{n+\frac{1}{2}}{}^{m}(\cosh\sigma)-(n+m-\tfrac{1}{2})\,\mathrm{P}_{n-\frac{3}{2}}{}^{m}(\cosh\sigma)=0,$$

with a similar relation for the Q functions, and

$$\mathrm{P}_{n-\frac{1}{2}}{}^{m+2}(\cosh\sigma)+2(m+1)\coth\sigma\,.\,\mathrm{P}_{n-\frac{1}{2}}{}^{m+1}(\cosh\sigma)$$
$$-(n-m-\tfrac{1}{2})(n+m+\tfrac{1}{2})\,\mathrm{P}_{n-\frac{1}{2}}{}^{m}(\cosh\sigma)=0,$$

with a similar relation for the Q functions. Formulæ similar to these have been employed by HICKS to calculate the functions successively.

58. It is important to have series for $\mathrm{P}_{n-\frac{1}{2}}{}^{m}(\cosh\sigma)$, $\mathrm{Q}_{n-\frac{1}{2}}{}^{m}(\cosh\sigma)$ in powers of $e^{-\sigma}$, so that the values of the functions may be calculated approximately for considerable values of σ. The required series for $\mathrm{Q}_{n-\frac{1}{2}}{}^{m}(\cosh\sigma)$ is given at once by (35); we thus have

$$\mathrm{Q}_{n-\frac{1}{2}}{}^{m}(\cosh\sigma)$$
$$=(-1)^m\,2^m\,\frac{\Pi(n+m-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\Pi(n)}\sinh^m\sigma\,.\,e^{-(n+m+\frac{1}{2})\sigma}\,\mathrm{F}(m+\tfrac{1}{2},n+m+\tfrac{1}{2},n+1,e^{-2\sigma}),$$

and in particular

$$\mathrm{Q}_{n-\frac{1}{2}}(\cosh\sigma)=\frac{\Pi(n-\frac{1}{2})\,\Pi(-\frac{1}{2})}{\Pi(n)}\,e^{-(n+\frac{1}{2})\sigma}\,\mathrm{F}(\tfrac{1}{2},n+\tfrac{1}{2},n+1,e^{-2\sigma}).$$

This is the expansion in powers of $e^{-\sigma}$, of the elliptic integral to which

$$\int_0^\infty\frac{dw}{\sqrt{(\cosh\sigma+\sinh\sigma\cosh w)}}$$

is reduced by means of the substitution $\cosh\sigma+\sinh\sigma\cosh w=\operatorname{cosec}^2\theta\,.\,e^{\sigma}$.

The corresponding series for $\mathrm{P}_{n-\frac{1}{2}}{}^{m}(\cosh\sigma)$ must be obtained from (36), which requires, however, in this case modification. We observe that in the formula

$$\mathrm{P}_{n}{}^{m}(\cosh\sigma)$$
$$=2^m\frac{\sin(n+m)\pi}{\cos n\pi}\cdot\frac{\Pi(n+m)}{\Pi(n+\frac{1}{2})\,\Pi(-\frac{1}{2})}\sinh^m\sigma\,e^{-(n+m+1)\sigma}\,\mathrm{F}(m+\tfrac{1}{2},n+m+1,n+\tfrac{3}{2},e^{-2\sigma})$$
$$+\,2^m\frac{\Pi(n-\frac{1}{2})}{\Pi(n-m)\,\Pi(-\frac{1}{2})}\sinh^m\sigma\,.\,e^{(n-m)\sigma}\,\mathrm{F}(m+\tfrac{1}{2},m-n,\tfrac{1}{2}-n,e^{-2\sigma})$$

when $n-\frac{1}{2}$ is a positive integer p_0; the second series has after a finite number of terms, infinite coefficients, moreover the coefficient sec $n\pi$ of the first series is infinite.

The expression for $\mathrm{P}_{n}{}^{m}(\cosh\sigma)$, gives us, therefore, first a finite series

$$2^m \frac{\Pi(p_0)}{\Pi(p_0+\frac{1}{2}-m)\,\Pi(-\frac{1}{2})} \sinh^m\sigma \,.\, e^{(p_0+\frac{1}{2}-m)\sigma}\left\{1 + \frac{(\frac{1}{2}+m)(p_0+\frac{1}{2}-m)}{1\,.\,p_0} e^{-2\sigma}\right.$$
$$+ \frac{(\frac{1}{2}+m)(\frac{3}{2}+m)(p_0-m+\frac{1}{2})(p_0-m-\frac{1}{2})}{1\,.\,2\,.\,p_0\,.\,p_0-1} e^{-4\sigma} + \dots$$
$$\left.+ \frac{(\frac{1}{2}+m)\dots(\frac{1}{2}+m+p_0-1)(p_0-m+\frac{1}{2})\dots(-m+\frac{1}{2})}{1\,.\,2\dots p_0\,.\,p_0(p_0-1)\dots 1} \,.\, e^{-2p_0\sigma}\right\},$$

which we shall denote by S_1; and second the undetermined form

$$2^m \frac{\sin(p+\frac{1}{2}+m)\pi}{\cos(p+\frac{1}{2})\pi} \,.\, \frac{\Pi(p+\frac{1}{2}+m)}{\Pi(p+1)\,\Pi(-\frac{1}{2})} \sinh^m\sigma \,.\, e^{-(p+m+\frac{3}{2})\sigma}\, F(m+\tfrac{1}{2}, p+\tfrac{3}{2}+m, p+2, e^{-2\sigma})$$
$$+2^m \frac{\Pi(p)}{\Pi(p+\frac{1}{2}-m)\,\Pi(-\frac{1}{2})} \frac{(\frac{1}{2}+m)\dots(\frac{1}{2}+m+p_0)(p-m+\frac{1}{2})\dots(p-m+\frac{1}{2}-p_0)}{1\,.\,2\dots(p_0+1)\,.\,p(p-1)\dots(p-p_0)} \,.\, \sinh^m\sigma$$
$$e^{(p+\frac{1}{2}-m-2p_0-2)\sigma}\left\{1 + \frac{\frac{1}{2}+m+p_0+1\,.\,p-m+\frac{1}{2}-p_0-1}{p_0+2\,.\,p-p_0-1} e^{-2\sigma} + \dots\right\}$$

where in the limit, $p = p_0$.

The numerical coefficient of the second series is equal to

$$2^m \,.\, \frac{\Pi(p)}{\Pi(p+\frac{1}{2}-m)\,\Pi(-\frac{1}{2})} \,.\, \frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(m-\frac{1}{2})} \,.\, \frac{\Pi(p-m+\frac{1}{2})}{\Pi(p-p_0-m-\frac{1}{2})} \,.\, \frac{\Pi(p-p_0-1)}{\Pi(p)} \,.\, \frac{1}{\Pi(p_0+1)},$$

which is equal to

$$\frac{2^m}{\Pi(p_0+1)} \,.\, \frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(m-\frac{1}{2})\,\Pi(-\frac{1}{2})} \,.\, \frac{\Pi(p_0-p+m-\frac{1}{2})}{\Pi(p_0-p)} \,.\, \frac{\sin(p_0-p+m+\frac{1}{2})\pi}{\sin(p_0-p+1)\pi}\,.$$

Now the limiting value of the ratio

$$\frac{\sin(p+\frac{1}{2}+m)\pi}{\cos(p+\frac{1}{2})\pi} \Big/ \frac{\sin(p_0-p+m+\frac{1}{2})\pi}{\sin(p_0-p+1)\pi}$$

when $p = p_0$, is easily seen to be -1, thus the coefficients of the two series are equal and opposite infinities.

Evaluating the indeterminate form according to the known rule, we obtain first an expression, which we shall denote by S_2; this is

$$S_2 = 2^m \frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})\,\Pi(p_0+1)} \,.\, \sinh^m\sigma \,.\, L_{p=p_0} \frac{p-p_0}{\sin(p-p_0)}$$
$$L_{p=p_0} \frac{d}{dp}\left[\frac{\Pi(p_0-p+m-\frac{1}{2})}{\Pi(p_0-p)} \sin(p_0-p+m+\tfrac{1}{2})\pi \,.\, e^{-(m+\frac{3}{2})\sigma+(p-2p_0)\sigma}\right.$$
$$\left.\left\{1 + \frac{\frac{3}{2}+m+p_0\,.\,p-p_0-m-\frac{1}{2}}{p_0+2\,.\,p-p_0-1} \,.\, e^{-2\sigma} + \dots\right\}\right];$$

we also obtain the expression

$$S_3 = -2^m(-1)^{p_0}\sinh^m\sigma \cdot \frac{d}{dp_0}\left\{\frac{\Pi(p_0+\frac{1}{2}+m)}{\Pi(p_0+1)\,\Pi(-\frac{1}{2})}\sin(p_0+\tfrac{1}{2}+m)\pi\right.$$

$$\left. e^{-(p_0+m+\frac{3}{2})\sigma}\,F(m+\tfrac{1}{2}, p_0+\tfrac{3}{2}+m, p_0+2, e^{-2\sigma})\right\},$$

since $\cos(p+\frac{1}{2})\pi = -(-1)^{p_0}(p-p_0)$, in the limit.

We have now, on the whole, picking out the terms in S_2, S_3, obtained by differentiating the exponential function

(1) The finite series S_1,

(2) $$2^{m+1}\,.\,\sin(m+\tfrac{1}{2})\pi\,.\,\frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(p_0+1)\,\Pi(-\frac{1}{2})}\,\sigma\,.\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}$$
$$F(m+p_0+\tfrac{3}{2}, m+\tfrac{1}{2}, p_0+2, e^{-2\sigma}),$$

(3) $$-2^{m+1}\frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(p_0+1)\,\Pi(-\frac{1}{2})}\cos(m+\tfrac{1}{2})\pi\,.\,\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}$$
$$F(m+\tfrac{1}{2}, m+p_0+\tfrac{3}{2}, p_0+2, e^{-2\sigma}),$$

$$+\frac{2^m\,.\,\Pi(p_0+m+\frac{1}{2})}{\Pi(-\frac{1}{2})\,\Pi(p_0+1)}\sin(m+\tfrac{1}{2})\pi\left\{\frac{\Pi'(0)}{\Pi(0)}-\frac{\Pi'(m-\frac{1}{2})}{\Pi(m-\frac{1}{2})}-\frac{\Pi'(p_0+m+\frac{1}{2})}{\Pi(p_0+m+\frac{1}{2})}\right.$$
$$\left.+\frac{\Pi'(p_0+1)}{\Pi(p^0+1)}\right\}\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}F(m+\tfrac{1}{2}, m+p_0+\tfrac{3}{2}, p_0+2, e^{-2\sigma}),$$

in the case of the ordinary ring functions (m integral), the first term vanishes on account of the factor $\cos(m+\frac{1}{2})\pi$.

(4) $$\frac{2^m\sin(m+\frac{1}{2})\pi}{\Pi(-\frac{1}{2})}\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}.$$

$$\sum_{s=1}^{s=\infty}\frac{\Pi_0(p_0+m+s+\frac{1}{2})}{\Pi(p_0+s+1)}\,\frac{\Pi(m+s-\frac{1}{2})}{\Pi(m-\frac{1}{2})\,\Pi(s)}\left\{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{s}\right.$$
$$-\frac{1}{m+\frac{1}{2}}-\frac{1}{m+\frac{3}{2}}-\ldots-\frac{1}{m+s-\frac{1}{2}}+\frac{1}{p_0+2}+\ldots+\frac{1}{p_0+s+1}$$
$$\left.-\frac{1}{p_0+m+\frac{3}{2}}-\ldots-\frac{1}{p_0+m+s+\frac{1}{2}}\right\}e^{-2s\sigma}.$$

Confining ourselves to the case in which m is integral, we can simplify the expression in (2); we have

$$\frac{\Pi'(p_0+1)}{\Pi(p_0+1)}=\frac{1}{p+1}+\frac{1}{p}+\ldots+\frac{1}{1}+\frac{\Pi'(0)}{\Pi(0)}$$

$$\frac{\Pi'(m-\frac{1}{2})}{\Pi(m-\frac{1}{2})}=\frac{1}{m-\frac{1}{2}}+\frac{1}{m-\frac{3}{2}}+\ldots+\frac{1}{\frac{1}{2}}+\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}$$

$$\frac{\Pi'(m+p_0+\frac{1}{2})}{\Pi(m+p_0+\frac{1}{2})}=\frac{1}{m+p_0+\frac{1}{2}}+\ldots+\frac{1}{\frac{1}{2}}+\frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})},$$

hence

$$\frac{\Pi'(0)}{\Pi(0)} - \frac{\Pi'(m-\frac{1}{2})}{\Pi(m-\frac{1}{2})} - \frac{\Pi'(p_0+m+\frac{1}{2})}{\Pi(p_0+m+\frac{1}{2})} + \frac{\Pi'(p_0+1)}{\Pi(p_0+1)}$$
$$= 2\left\{\frac{\Pi'(0)}{\Pi(0)} - \frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})}\right\} + \left[\frac{1}{1} + \ldots + \frac{1}{p_0+1} - \left(\frac{1}{\frac{1}{2}} + \ldots + \frac{1}{m-\frac{1}{2}}\right)\right.$$
$$\left. - \left(\frac{1}{\frac{1}{2}} + \ldots + \frac{1}{p_0+m+\frac{1}{2}}\right)\right].$$

Now use the known theorem $\Pi(x-1)\,\Pi(x-\frac{1}{2}) = \sqrt{2\pi}\,.\,2^{\frac{1}{2}-2x}\,.\,\Pi(2x-1)$; on taking logarithms and differentiating, and then putting $x = \frac{1}{2}$, we find

$$\frac{\Pi'(0)}{\Pi(0)} - \frac{\Pi'(-\frac{1}{2})}{\Pi(-\frac{1}{2})} = \log_e 4.$$

Taking (2), (3), and (4) together, we now have the expression

$$(-1)^m 2^{m+1}\frac{\Pi(p_0+m+\frac{1}{2})}{\Pi(p_0+1)\Pi(-\frac{1}{2})}\log(4e^\sigma).\sinh^m\sigma.e^{-(p_0+m+\frac{3}{2})\sigma}F(m+p_0+\tfrac{3}{2}, m+\tfrac{1}{2}, p_0+2, e^{-2\sigma})$$
$$+(-1)^m 2^m.\frac{\sinh^m\sigma\,.\,e^{-(p_0+m+\frac{3}{2})\sigma}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sum_{s=0}^{s=\infty}\left(u_{p_0+s+1}+u_s+v_{m+s-\frac{1}{2}}-v_{p_0+m+s+\frac{1}{2}}\right)$$
$$\frac{\Pi(p_0+m+s+\frac{1}{2})\,\Pi(m+s-\frac{1}{2})}{\Pi(s)}e^{-2s\sigma},$$

where

$$u_r \text{ denotes the series } \frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r}$$

and

$$v_{r+\frac{1}{2}} \text{ denotes the series } \frac{1}{\frac{1}{2}}+\frac{1}{\frac{3}{2}}+\ldots+\frac{1}{r+\frac{1}{2}}.$$

On changing p_0 into $n-1$, we now have the complete expression for the ring function $P_{n-\frac{1}{2}}{}^m(\cosh\sigma)$, ($m$ integral),

$$P_{n-\frac{1}{2}}{}^m(\cosh\sigma)$$
$$= \frac{2^m\Pi(n-1)}{\Pi(n-m-\frac{1}{2})\,\Pi(-\frac{1}{2})}\sinh^m\sigma\,.\,e^{-(n-m-\frac{1}{2})\sigma}\left[1+\frac{(\frac{1}{2}+m)(n-\frac{1}{2}-m)}{1\,.\,n-1}e^{-2\sigma}+\ldots\right.$$
$$\left.+\frac{(\frac{1}{2}+m)\ldots(\frac{1}{2}+m+n-2)(n-m-\frac{1}{2})\ldots(-m+\frac{1}{2})}{1\,.\,2\ldots n-1\,.\,n-1\,.\,n-2\ldots 1}e^{-2(n-1)\sigma}\right]$$
$$+(-1)^m 2^{m+1}.\frac{\Pi(m+n-\frac{1}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\log(4e^\sigma)\sinh^m\sigma\,.\,e^{-(n+m+\frac{1}{2})\sigma}F(m+n+\tfrac{1}{2}, m+\tfrac{1}{2}, n+1, e^{-2\sigma})$$
$$+(-1)^m 2^m.\frac{\sinh^m\sigma\,.\,e^{-(n+m+\frac{1}{2})\sigma}}{\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\sum_{s=0}^{s=\infty}\left(u_{s+n}+u_s-v_{m+s-\frac{1}{2}}-v_{m+n+s-\frac{1}{2}}\right)$$
$$\frac{\Pi(m+n+s-\frac{1}{2})\,\Pi(m+s-\frac{1}{2})}{\Pi(s)}e^{-2s\sigma}.$$

The particular case $m = 0$, gives as the expression for the zonal function,

$$P_{n-\frac{1}{2}}(\cosh\sigma)$$

$$= \frac{\Pi(n-1)}{\Pi(n-\frac{1}{2})\Pi(-\frac{1}{2})} e^{-(n-\frac{1}{2})\sigma}\left[1 + \frac{\frac{1}{2}.n-\frac{1}{2}}{1.n-1}e^{-2\sigma} + \ldots + \frac{\frac{1}{2}.\frac{3}{2}\ldots(n-\frac{3}{2})(n-\frac{1}{2})(n-\frac{3}{2})\ldots\frac{1}{2}}{1.2\ldots n-1.n-1\ldots 1} e^{-2(n-1)\sigma}\right]$$

$$+ \frac{2\,\Pi(n-\frac{1}{2})}{\Pi(n)\,\Pi(-\frac{1}{2})}\log(4e^{\sigma})\,e^{-(n+\frac{1}{2})\sigma}\,\mathrm{F}(n+\tfrac{1}{2}, \tfrac{1}{2}, n+1, e^{-2\sigma})$$

$$+ \frac{1}{\pi}e^{-(n+\frac{1}{2})\sigma}\sum_0^\infty (u_{n+s} + u_s - v_{s-\frac{1}{2}} - v_{n+s-\frac{1}{2}})\frac{\Pi(n+s-\frac{1}{2})\,\Pi(s-\frac{1}{2})}{\Pi(s)}e^{-2s\sigma}.$$

This particular case has been obtained by other methods by BASSET, and by W. D. NIVEN.

The case in which m is fractional has really been included in the above investigation; the simplification of the coefficients in the expression (2) does not apply to the general case.

MEHLER'S *Functions for the Cone.*

59. The normal potential functions for problems in which the boundaries are coaxal circular cones* are spherical harmonics of complex degree $-\frac{1}{2} + p\iota$; it is therefore desirable to consider the forms which the functions $P_n(\cos\theta)$, $Q_n(\cos\theta)$ take when n is of this form; $P_{-\frac{1}{2}+p\iota}(\cos\theta)$ will be denoted by $K_p(\cos\theta)$.

We find from (103), (110), (111),

$$K_p(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cosh pu}{\sqrt{2\cos u - 2\cos\theta}}\,du,$$

$$= \frac{2}{\pi}\cosh p\pi \int_0^\infty \frac{\cos pv}{\sqrt{2\cosh v + 2\cos\theta}}\,dv;$$

these formulæ have been proved by other methods by MEHLER and by HEINE.†

From (103), we obtain the new formula

$$K_p^m(\cos\theta) = (-1)^m \frac{2}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\frac{\Pi(n+m)}{\Pi(n-m)}\sin^{-m}\theta\int_0^\theta \frac{\cosh pu}{(2\cos u - 2\cos\theta)^{\frac{1}{2}-m}}$$

where m is any positive quantity.

From the above formulæ, we see that $P_{-\frac{1}{2}+p\iota}(\cos\theta) = P_{-\frac{1}{2}-p\iota}(\cos\theta)$.

From (117), we have,

* See MEHLER'S paper in CRELLE'S 'Journal,' vol. 68.

† See 'Kugelfunctionen,' vol. 2, p. 221.

$$Q_{-\frac{1}{2}+p\iota}(\cos\theta) - \frac{\iota\pi}{2}P_{-\frac{1}{2}+p\iota}(\cos\theta) = Q_{-\frac{1}{2}+p\iota}(\cos\theta + 0\,.\,\iota)$$

$$= \int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u - 2\cos\theta}}\,du - \iota e^{+p\pi}\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u + 2\cos\theta}}\,du,$$

hence, changing p into $-p$, and adding the two equations, we find

$$Q_{-\frac{1}{2}+p\iota}(\cos\theta) + Q_{-\frac{1}{2}-p\iota}(\cos\theta) - \iota\pi P_{-\frac{1}{2}+p\iota}(\cos\theta)$$

$$= 2\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u - 2\cos\theta}}\,du - 2\iota\cosh p\pi\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u + 2\cos\theta}}\,du,$$

hence

$$\frac{\cosh p\pi}{\pi}\{Q_{-\frac{1}{2}+p\iota}(\cos\theta) + Q_{-\frac{1}{2}-p\iota}(\cos\theta)\} = \frac{2\cosh p\pi}{\pi}\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u - 2\cos\theta}}$$

$$= K_p(-\cos\theta).$$

Thus we can use $K_p(\cos\theta)$, $K_p(-\cos\theta)$, as the two independent functions.

It thus appears that the expressions given by MEHLER and HEINE for $K_p(\cos\theta)$, $K_p(-\cos\theta)$, are particular cases of the general formulæ we have obtained above.

Potential Functions for the Bowl.

60. It has been shown by MEHLER that for potential problems in which the boundaries are spherical bowls with a common circular rim, the functions $K_p(\mu)$ can be used, μ being in this case real and greater than unity, say $\mu = \cosh\psi$.

We find from (111), that

$$P_{-\frac{1}{2}+p\iota}(\cosh\psi) = K_p(\cosh\psi) = \frac{2}{\pi}\cosh p\pi\int_0^\infty \frac{\cos pu}{\sqrt{2\cosh u + 2\cosh\psi}}\,du,$$

and from (113), we find

$$K_p(\cosh\psi) = \frac{2}{\pi}\int_0^\psi \frac{\cos pv}{\sqrt{2\cosh\psi - 2\cosh v}}\,dv,$$

also from (115),

$$K_p(\cosh\psi) = \frac{2}{\pi}\coth p\pi\int_\psi^\infty \frac{\sin pw}{\sqrt{2\cosh w - 2\cosh\psi}}\,dw,$$

these formulæ are all proved by HEINE* by other methods.

From (112) we have

$$K_p^m(\cosh\psi) = \frac{2(-1)^m}{2^m\,\Pi(-\frac{1}{2})\,\Pi(m-\frac{1}{2})}\frac{\Pi(n+m)}{\Pi(n-m)}\sinh^{-m}\psi\int_0^\psi \frac{\cos pu}{(2\cosh\psi - 2\cosh u)^{\frac{1}{2}-m}}\,du$$

* See 'Kugelfunctionen,' vol. 2, p. 220.

From (118), we have

$$Q_{-\frac{1}{2}+p\iota}(\cosh\psi) = \int_{\psi}^{\infty} \frac{e^{-p\iota u}}{\sqrt{2\cosh u - 2\cosh\psi}}\,du.$$

Hence

$$Q_{-\frac{1}{2}+p\iota}(\cosh\psi) + Q^{-\frac{1}{2}-p\iota}(\cosh\psi) = 2\int_{\psi}^{\infty} \frac{\cos pu}{\sqrt{2\cosh u - 2\cosh\psi}}\,du,$$

hence defining $K^p(-\cosh\psi)$ by means of the formula

$$K_p(-\cosh\psi) = \frac{2}{\pi}\cosh p\pi \int_0^{\infty} \frac{\cos pu}{\sqrt{2\cosh u - 2\cosh\psi}}\,du,$$

we have

$$\frac{\cos p\pi}{\pi}\left\{Q_{-\frac{1}{2}+p\iota}(\cosh\psi) + Q_{-\frac{1}{2}-p\iota}(\cosh\psi)\right\} = K_p(-\operatorname{ccsh}\psi).$$

It thus appears that the known expressions for these functions are immediately derivable from the general formulæ obtained in the present memoir.

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XIV. *Magnetization of Liquids.*

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The magnetization of liquids is a subject which has been examined by several methods, and although the relative values of the coefficient of magnetization for different bodies agree tolerably accurately, still there are large discrepancies in the absolute values given by different observers. An account of nearly all these methods is given in Wiedemann's 'Electricität,' vol. 3.

The difficulty of finding the magnetizing force is no doubt a serious objection to the accurate determination of the coefficient k in absolute measure by magnetometer methods, which, on the other hand, are very convenient for finding the relative values of k for different bodies when magnetized by large forces.

The following null method of arranging an induction balance to determine k in absolute measure gives very accurate results, and is very sensitive even when the forces are as small as 1 C.G.S. unit.

The apparatus which was used consisted of a primary and a secondary circuit. A quantity of electricity $2Mi/R_2$ is induced in the secondary when the current in the primary is changed from $-i$ to $+i$, M being the mutual induction of the two circuits, and R_2 the resistance of the secondary. A commutator is used to commute the connection of the primary with the battery, and at the same time to alter the connections of the secondary with the galvanometer in such a way as to send the induced currents in the same direction through the galvanometer, so that it will give a deflection corresponding to a current $\frac{2Mi}{R_2}p$, p being the number of times per second the direction of i is changed.

By this means a very small value of M can be detected.

The mutual induction M consisted of three parts :—

1. The induction (α) of a long solenoid (wound on a glass tube) on a secondary wound round its centre.
2. The mutual induction (β) of two large circular coils.
3. An adjustable induction (γ) which could be altered by small known amounts.

$$M = \alpha + \beta + \gamma.$$

21.10.96

The connections were so made that (α) and (β) almost cancelled one another, so that when the commutator revolved, and contact was made in the primary circuit with the battery, the *deflection* of the galvanometer in the secondary circuit could be reduced to zero by adjusting γ, thus making $M = 0$.

Let

A = internal area of the section of the solenoid tube.
N = number of turns in the primary coil.
N' = number of turns in the secondary coil.
$2l$ = length of primary.

The mutual induction $\frac{4\pi NN'A}{2l}$ is changed to $(1 + 4\pi k)\ \frac{4\pi MN'A}{2l}$ by introducing a liquid into the solenoid whose coefficient of magnetization is k. M is no longer zero, as the deflection of the galvanometer indicates, and if m is the amount by which γ must be diminished so as to reduce this deflection to zero we have

$$4\pi k \, . \, \frac{4\pi MN'A}{2l} = m$$

The balance $M = 0$ being again restored.

We have thus a means of measuring k in terms of quantities which are easily found without involving in the formula the rate of rotation of the commutator or the value of the primary current.

The method is therefore very easily adapted to determine whether k varies with the force, since the latter can be changed by altering the resistance of the primary circuit.

The general plan of the arrangements is shown in fig. 1. The continuous line denoting the primary and the dotted line the secondary circuit. The solenoid A is connected with the other parts of the apparatus as shown. The two circular coils B are separated by an ebonite sheet; the upper one belonging to the secondary circuit could be moved about till the mutual induction of the pair cancelled the mutual induction of the two coils A. The adjustable induction γ is represented at C (and also in fig. 2), the primary circuit leading to the mercury cups at E by which the required changes in induction are made. D and G represent the commutator and galvanometer. The sensitiveness is not much diminished by increasing R the resistance of the primary, as it will be shown that in this case the rate at which the commutator revolves can also be increased without any danger of the Foucault current generated in the liquid having any effect on the secondary current, as far as its effect on the galvanometer is concerned.

It is convenient to have a key, K, in the primary circuit by means of which the current can be sent in either direction to the terminals, T, so as to reverse the direction of the current in the secondary circuit. The *deflection* of the galvanometer could thus have been increased by the multiplication method, but in all the experiments which were *done* the apparatus was found sensitive enough to give k correct to the seventh decimal place by the ordinary deflection.

Fig. 1.

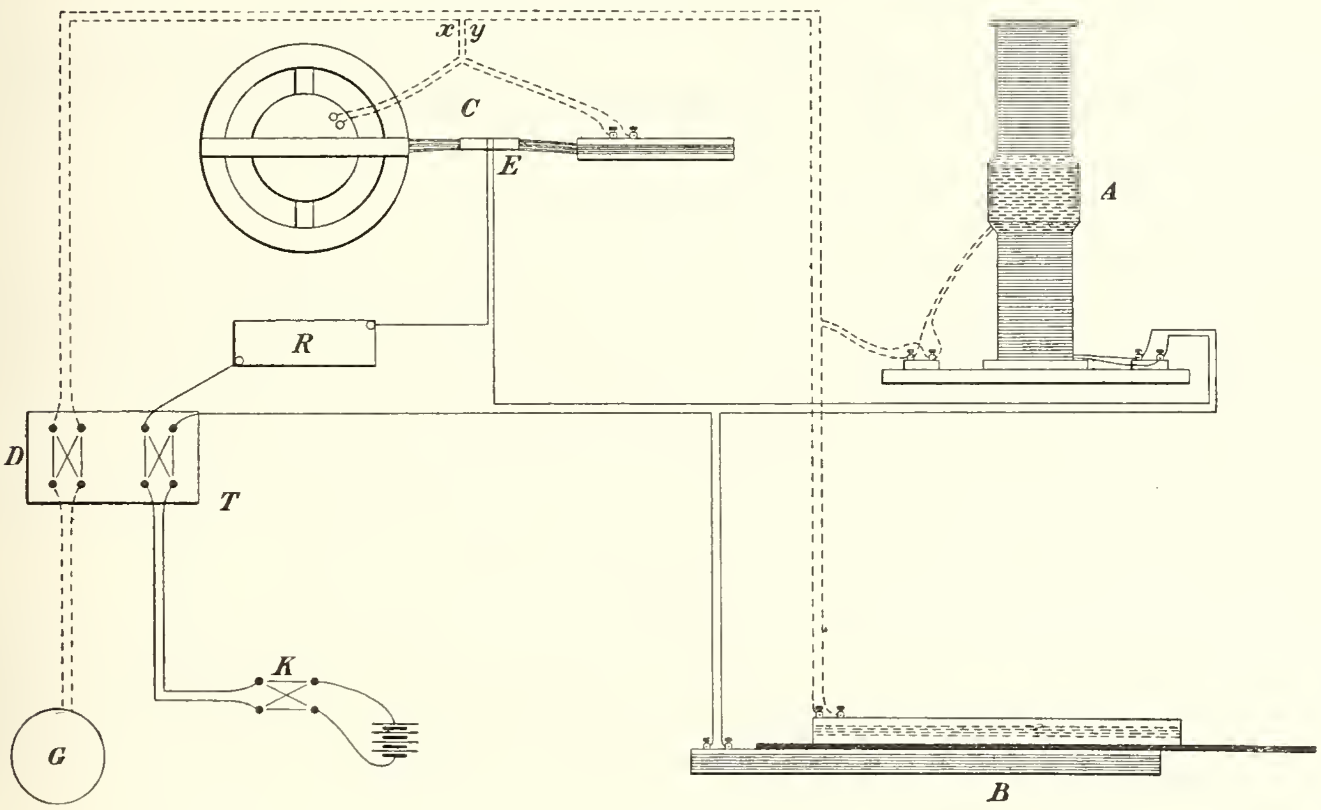
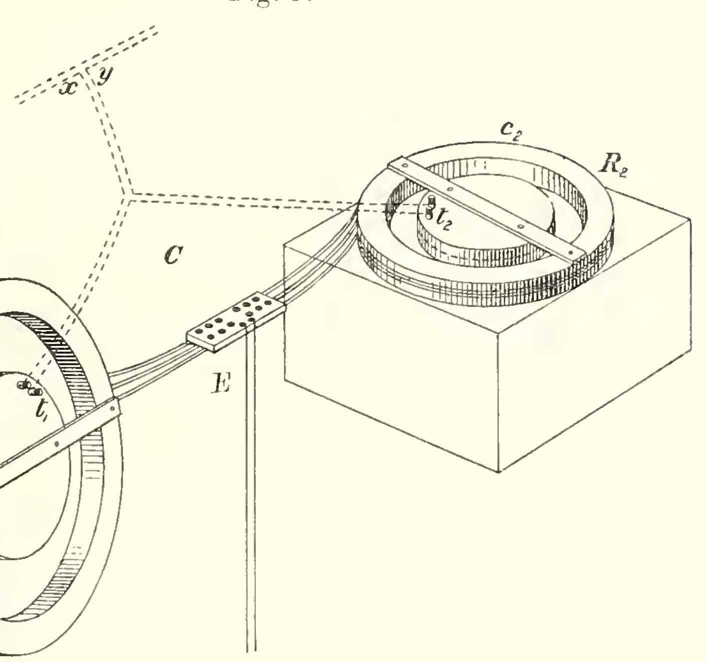


Fig. 2.



The great difficulty in the experiment is to ensure the perfect insulation of the secondary circuit, since the great electromotive force of the primary at break will cause a current to leak to the secondary if good non-conductors are not used to separate the two circuits.

It is necessary that A, B, and C should be at long distances apart, so as to satisfy the two following conditions:—

1. The primary circuit of B must have no lines of force through the secondary of A.
2. Changing the number of coils in the primary of C should have no effect on the galvanometer when C is cut out of the secondary circuit (by joining x and y).

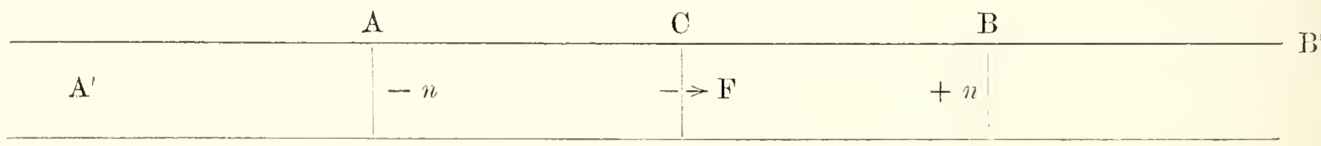
Details of Apparatus.

The primary circuit of the solenoid contained 1509 turns, wound in four layers on a cylindrical glass jar 47 centims. long and about 4 centims. internal radius. The secondary, of 841 turns, was carefully insulated from it by means of several layers of paraffin paper, and occupied a length 4 centims. each side of the centre.

The mean internal sectional area at the centre perpendicular to the axis of the tube was found by observing with a cathetometer the rise in level of water in the tube, when a known additional volume was added to it.

The area thus obtained was 50·6 sq. centims.

Let a = radius and l half the length of the solenoid AB. The force F due to unit current is less than $4\pi n$ by the amount contributed by the two productions AA′ and BB′ of the solenoid, A′ and B′ being theoretically at infinity.



The solenoids, A′A and B′B, can be replaced by uniform surface distributions of magnetism, $-n$ and $+n$, over the plane circular ends at A and B, as far as forces at places external to A′A′ and BB′ are concerned, n being the number of turns per centim.*

Hence the force F at the centre of the solenoid is less than $4\pi n$ by that due to a pole, $-\pi na^2$ at A and $+\pi na^2$ at B, so that

$$F = 4\pi n\left(1 - \frac{a^2}{2l^2}\right).$$

The second correction for the ends of the solenoid is found by considering the induction through the secondary circuit due to the magnetism induced on the surface of the liquid. In cases where k is small this correction is easily made since terms involving k^2 may be neglected.

* MAXWELL, 'Electricity and Magnetism,' vol. 2, sect. 676.

Let ϕ be the potential of the magnetic force; dv the element of the normal to the boundary of the liquid. There is a distribution σ of magnetism on the surface of the liquid, where $\sigma = - k\, d\phi/dv$, which gives rise to a small force f, at C, in the direction opposite to F. So that, if B is equal to the mean area of the secondary coil and N′ its number of turns, there is a diminution in the total flux of induction through the secondary circuit equal to $\mathrm{A}f\mathrm{N}'\mu + (\mathrm{B} - \mathrm{A})f\mathrm{N}'$, due to the magnetism induced on the surface of the liquid.

Since $\mu = 1 + 4\pi k$, and f involves k as a factor, we may put $\mu = 1$, and the correction reduces to $\mathrm{B}f\mathrm{N}'$.

Now, since $\sigma = - k\, d\phi/dv$, we may, in determining the normal component $- d\phi/dv$ of the magnetic force, omit k and other small corrections and consider the case of a long solenoid of length $2l$ and find the force at the ends parallel to the axis, and the normal force at the cylindrical boundary, when the space inside is occupied by air.

It is easy to see that $\phi = - 4\pi nz + \phi_1$, z being measured from A to B along the axis, and ϕ_1 the potential of the distribution $-n$ at A, and $+n_1$ at B, so that at the ends of the solenoid the component L of the magnetic force is

$$- d\phi/dz = 4\pi n - 2\pi n = 2\pi n \text{ at A or B.}$$

The following simple consideration also shows that the force parallel to the axis, at the ends is approximately half its value at the centre. The force at B due to the long solenoid AB′ is $4\pi n$. This must be contributed to equally by the parts AB and BB′, hence, when the latter is removed, the force parallel to the axis is $2\pi n$.

The double integral $\iint \frac{d\phi}{dv}\, ds$ vanishes over any closed surface within which $\Delta^2\phi = 0$ at every point, so that by considering the closed surface made up of the two planes perpendicular to the axis at C and B, and the intermediate part of a cylinder of any section whose generators are parallel to the axis, and integrate over it. We get

$$- \iint \frac{d\phi}{dv}\, ds \text{ over the plane end at C}$$

$$= \iint \frac{d\phi}{dv}\, ds \text{ over the length } l \text{ of the cylindrical surface and the plane end at B.}$$

But $d\phi/dv = 4\pi n$ at C, and $- 2\pi n$ at B.

Hence

$$\iint \frac{d\phi}{dv}\, ds \text{ over the length } l \text{ of the cylinder} = - 4\pi n\,(\mathrm{A}) + 2\pi n\,(\mathrm{A}) = - 2\pi n\,(\mathrm{A}),$$

where (A) = area of the section of the cylinder.

Now let this cylinder be a body whose coefficient of magnetization is small, and let it extend from A to B, since the surface distribution is $- k\, d\phi/dv$, we see that:

When a current flows through a solenoid, and magnetizes a cylindrical body of equal length whose value for k is small, there are uniform distributions $+$, and $-$, $2\pi nk$ induced on the plane ends, and of the total quantities $+$, and $-$, $4\pi nk$ (A) induced, one-half resides on the ends and the other half on the cylindrical surface.

This latter distribution could not be got rid of by producing the ends of the cylinder beyond the magnetizing coil.

The actual value of the normal force N at a point distance y from the end of a solenoid of circular section of radius a is given by the equation

$$N = \left[\frac{2y^2 + 4a^2}{a\sqrt{y^2 + 4a^2}} K - \frac{2\sqrt{y^2 + 4a^2}}{a} E\right] n,$$

K and E being the complete elliptic integrals of the first and second kind, the modulus k being

$$\frac{2a}{\sqrt{y^2 + 4a^2}}$$

(N may also be found by differentiating the expression for V in THOMSON and TAIT'S 'Natural Philosophy,' Ex. II., Section 546).

Making all the corrections, it is found that the change in induction m is

$$4\pi k\, 4\pi n N' \,.\, A\left[1 - \frac{a^2}{2l^2} - \frac{B}{A}\left(\frac{\cdot 0530}{\pi}\right)\right].$$

Substituting the following values,

$$n = \tfrac{1509}{47}; \quad N' = 841; \quad A = 50\cdot 6; \quad a = 4\cdot 5; \quad 2l = 47; \quad B = 93;$$

we get

$$m = k \times 2155\cdot 10^5 (1 - \cdot 049),$$

therefore

$$k = \frac{m}{205\cdot 10^6}.$$

The adjustable inductance was made in two separate parts, c_1 and c_2, shown in fig. 2. The larger changes in induction were made by c_1, the secondary circuit of which consisted of seven turns wound round the inner bobbin, 18·61 centims. in diameter, and had its terminals at t_1.

On the outer surface of the wooden ring R, 37·17 centims. in diameter, were wound 10 turns of wire of small diameter close beside one another. The ends of 1, 3, and 6 of these circles of wire were twisted together and terminated in 6 mercury cups. The ring R and the bobbin were placed in concentric and coplanar positions, and were then fixed rigidly together by means of ebonite bars.

The second part of c_2 was similar in principle, and by means of it smaller changes of induction could be made. It had only 1 turn in the secondary, the 6 ends of 1, 3, and 6 turns on the outer ring terminating in 6 mercury cups, beside those belonging to the larger inductance. The radius of the inner circle of wire was 12·4, and that of the outer circles 25 centims.

The planes of c_1 and c_2 were perpendicular to one another, each going through the centre of the other, so that when the number of coils of c_1 in series with the primary was altered there should be no change in the magnetic force perpendicular to the plane of c_2, and *vice versâ*. The induction of one circle of the primary on the corresponding secondary is, for

$$c_1,\ 707, \text{ and for } c_2,\ 65.^*$$

Foucault Currents.

Objections have often been made to the induction method of finding k by using a commutator, as it was supposed that, in the case of conductors, the induced currents had an appreciable effect on the current through the secondary circuit.

This, however, is not the case when a galvanometer is used in the secondary circuit, but would introduce an error if a telephone were substituted.

In order to prove this, let us consider the case of currents generated in a circular cylinder by the current $i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)$ starting in the primary, E, R, and L being the E.M.F. resistance and self induction of the primary.

We see from symmetry that the induced currents are in circles round the axis, and, if r is the perpendicular distance of any point from the axis, and u the strength of the current per unit section,

$$u = f(rt).$$

Considering the circuit of radius r, and thickness dr, extending through unit length of the cylinder, the current flowing round the circuit is $u\,dr$, the resistance of the circuit is $2\pi\sigma r/dr$, where σ is the specific resistance of the liquid.

Therefore

$$2\pi r\sigma u = -\frac{dN}{dt},$$

where N is the total number of lines of induction going through the circuit.

$$N = 4\pi^2\int_0^r r^2 u\,dr + 4\pi^2 r^2\int_r^a u\,dr + 4\pi^2 r^2\frac{nE}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right).$$

Hence we have

* MAXWELL, 'Electricity and Magnetism,' vol. 2, chap. iv., App.

$$(1.)\quad 4\pi^2\int_0^r r^2\frac{du}{dt}\,dr + 4\pi^2 r^2\int_r^a\frac{du}{dt}\,dr + 2\pi r\sigma u = -\,4\pi^2 r^2\frac{n\mathrm{E}}{\mathrm{L}}\epsilon^{-\frac{\mathrm{R}}{\mathrm{L}}t}.$$

Differentiating with respect to r, and dividing by r and differentiating a second time, we get

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} - \frac{4\pi}{\sigma}\frac{du}{dt} = 0.$$

Let $u = v\epsilon^{-m\sigma t}$, where v is a function of r only and m as yet undetermined.

Hence we have:

$$r^2\frac{d^2v}{dr^2} + r\frac{dv}{dr} + (4\pi m r^2 - 1)\,v = 0.$$

Hence

$$v = \mathrm{A}_m\mathrm{J}_1\sqrt{4\pi m}\,r,$$

where J_1 is BESSEL's internal function of 1st order.

Hence

$$u = \Sigma\mathrm{A}_m\,\epsilon^{-m\sigma t}\,\mathrm{J}_1\sqrt{4\pi m}\,r,$$

substituting in (1) and using the equations

$$\mathrm{J}_1(x) = -\frac{d}{dx}\mathrm{J}_0(x), \qquad \frac{d}{dx}(x^2\mathrm{J}_2) = x^2\mathrm{J}_1,$$

and

$$x^2\,[\mathrm{J}_2(x) + \mathrm{J}_0(x)] = 2x\mathrm{J}_1 x,^*$$

we get

$$\Sigma\mathrm{A}_m\sigma\epsilon^{-m\sigma t}\sqrt{4\pi m}\,r^2\pi\mathrm{J}_0(\sqrt{4\pi m}\,a) = -\frac{4\pi^2 r^2 n\mathrm{E}}{\mathrm{L}}\epsilon^{-\frac{\mathrm{R}}{\mathrm{L}}t}.$$

Hence $\mathrm{J}_0\sqrt{4\pi m}\,a = 0$, except when $m\sigma = \mathrm{R}/\mathrm{L} = p$, in which case

$$\mathrm{A}_p\mathrm{J}_0\sqrt{\frac{4\pi p}{\sigma}}\,a = -\sqrt{\frac{4\pi p}{\sigma}}\,\frac{n\mathrm{E}}{\mathrm{R}}.$$

The other coefficients are determined by the condition that $u = 0$, when $t = 0$. From what follows, however, it will be seen that it is unnecessary to determine them, as they do not appear in the result.

The number of lines of force through the secondary due to the induced currents in the liquid is

$$4\pi^2\int_0^a r^2u\,dr = \Sigma\mathrm{B}_m\epsilon^{-m\sigma t} + \mathrm{B}_p\epsilon^{-pt}.$$

* FORSYTH's 'Differential Equations.'

The equation determining the secondary current i_2 is

$$L_2 \frac{di_2}{at} + R_2 i_2 = -\frac{dN}{at},$$

where dN/at consists of a number of terms of the form $-\lambda B\epsilon^{-\lambda t}$.

The solution of $L_2 \frac{di_2}{at} + R_2 i_2 = \lambda B\epsilon^{-\lambda t}$ is

$$i_2 = \frac{\lambda B\,(\epsilon^{-\lambda t} - \epsilon^{-p't})}{R_2 - \lambda L_2}, \text{ where } p' = \frac{R_{(2)}}{L_{(2)}},$$

and the quantity q_2 is

$$q_2 = \frac{B\left[(1 - \epsilon^{-\lambda t}) - \frac{\lambda}{p'}(1 - \epsilon^{-p't})\right]}{R_2 - \lambda L_2}.$$

Let us consider the combined effect of the increase in permeability, and the currents generated in the liquid, on the quantity q_2.

The first gives rise to the term $4\pi k M i$, and the second to the series

$$\Sigma B_m \epsilon^{-m\sigma t} + Bp\epsilon^{-pt} \text{ in N.}$$

Hence

$$\frac{dN}{dt} = \frac{4\pi k M E}{R} p\epsilon^{-pt} - \Sigma m B_m \epsilon^{-m\sigma t} - pB_p\epsilon^{-pt}$$

Substituting and bearing in mind that $\Sigma B_m + B_p = 0$, we get

$$(2) \quad q_2 = \frac{4\pi k M E}{R} \frac{\left[(1 - \epsilon^{-pt}) - (1 - \epsilon^{-p't})\frac{p}{p'}\right]}{R_2 - pL_2} + \Sigma \frac{B_m\left[\frac{m\sigma}{p'}\epsilon^{-p't} - \epsilon^{-m\sigma t}\right]}{R_2 - m\sigma L_2} + \frac{B_p\left(\frac{p}{p'}\epsilon^{-p't} - \epsilon^{-pt}\right)}{R_2 - pL_2}.$$

In order to select the important terms from this equation we must find the approximate values of m, p, and p'.

The smaller values of the roots of $J_0(x) = 0$ are*

2·404,
5·520,
8·654,
11·792.

So that the smallest value of m is given by the equation

* Lord Rayleigh, 'Theory of Sound,' Vol. 1, section 206.

$$\sqrt{4\pi m a} = 2{\cdot}4,$$

therefore

$$m = {\cdot}03,$$

and taking $\sigma = 10^9$, we get

$$m\sigma = 3.10^7, \qquad R = 37.10^9; \qquad L = 67.10^6,$$

so that $p = 500$, and p' about the same value.

Hence we see that in equation (2), if we seek the values of q_2 for times greater than $\frac{1}{50}$th of a second after the primary current has been made, it is evident that terms having $\epsilon^{-m\sigma t}$ as a multiplier may be neglected in comparison with those multiplied by ϵ^{-pt} and $\epsilon^{-p't}$, so that the terms in the sum Σ, reduce to

$$\Sigma \frac{B_m m\sigma\epsilon^{-p't}}{R_2(p' - m\sigma)} = -\frac{\epsilon^{-p't}}{R_2}\Sigma B_m = \frac{B_p\epsilon^{-p't}}{R_2}.$$

Hence :—

$$(3) \quad q_2 = -\frac{4\pi k ME}{RR_2} - \frac{B_p}{R_2}\frac{\epsilon^{-p't} - \epsilon^{-pt}}{\left(\frac{p}{p'} - 1\right)},$$

and

$$B_p = 4\pi^2\int_0^a r^2 A_p J_1 \sqrt{\frac{4\pi p}{\sigma}}\, r dr$$

$$= -4\pi n N' \frac{\pi a^2 E}{R} \frac{J_2\sqrt{\frac{4\pi p}{\sigma}}\,a}{J_0\sqrt{\frac{4\pi p}{\sigma}}\,a} = -\frac{ME}{R_1R_2}\frac{J_2\sqrt{\frac{4\pi p}{\sigma}}\,a}{J_0\sqrt{\frac{4\pi p}{\sigma}}\,a}.$$

Hence the quantity induced in the secondary due to the presence in the solenoid of a liquid having conductivity σ is

$$\frac{ME}{R_1R_2}\frac{J_2\sqrt{\frac{4\pi p}{\sigma}}\,a}{J_0\sqrt{\frac{4\pi p}{\sigma}}\,a} \cdot \frac{\epsilon^{-p't} - \epsilon^{-pt}}{\left(\frac{p}{p'} - 1\right)},$$

and the ratio of this to the quantity induced owing to a value of $k = 10^{-7}$ is,

$$\frac{1}{4\pi 10^{-7}} \cdot \frac{J_2\sqrt{\frac{4\pi p}{\sigma}}\,a}{J_0\sqrt{\frac{4\pi p}{\sigma}}\,a} \cdot \frac{\epsilon^{-p't} - \epsilon^{-pt}}{\frac{p}{p'} - 1}.$$

p is increased when the resistance of the primary is increased, its smallest value being 500, which is also about the value of p'.

$$\sqrt{\frac{4\pi p}{\sigma}}\, a = \tfrac{1}{100}, \quad J_0(x) = 1, \quad \text{and} \quad J_2(x) = \frac{x^2}{8}$$

when x is small, so that the above ratio reduces to :

$$\frac{1}{32\pi 10^{-3}} \cdot p' t\, \epsilon^{-p't},$$

and letting $t = \frac{1}{30}$, which is about the time the commutator allowed the secondary to be connected in the positive direction, say, with the galvanometer after the process of changing the current in the primary from 0 to $i = \frac{E}{R}\left(1 - \epsilon^{-\frac{L}{R}t}\right)$ was completed, we get a value for this ratio far less than unity $\frac{500}{3}\epsilon^{-50/3}$; in fact, the commutator might have been going twice as fast without the currents induced in the liquid having any effect on the value of k in the eighth decimal place.

Results of the Experiments.

In carrying out an experiment for the determination of k, it is necessary in all cases to syphon the liquid out of he solenoid immediately after m has been found, and test the balance again to see whether it is restored by replacing m. When this condition is fulfilled, it is evident that the mutual induction of the system of coils has not been altered while the experiment has been going on.

It does not always happen that an exact balance can be got by the method of changing the number of turns in the primary of c, for if a certain adjustment gives a deflection of two divisions positive, a reduction of one turn in the primary of c_2 gives a deflection of three divisions negative, so that in this case the correct reduction can be taken as two-fifths of 65, which is sufficiently accurate, since all that is necessary in order to determine k in the 7th decimal place is to be able to detect an induction of 20 C.G.S. units.

In all the following experiments the magnetising force was varied from 1 to 9 C.G.S. units, and in no case was there any change in k.

Ferric Chloride.

The values of k were found for four solutions of ferric chloride, the quantities of the salt per cub. cemtim. being proportional to 1, 2, 3, and 4.

The values of $10^7 k$ were

69, 148, 220, and 298.

These numbers are not proportional to 1, 2, 3, and 4; but when we add 7·7 (the value of $-10^7 k$ for the solvent) to each, we get values for k which are proportional to the weights of iron per cub. centim. This shows that the action of the solvent is to diminish the value of 10^7k by 7·7, which is the value of -10^7k for water.

The actual values found for distilled water in three different experiments were

$$10^7k = -8·1, \quad -7·5, \quad \text{and} \; -7·5,$$

the mean being $-7·7$.

The values for 10^7k for water given by different observers are:—

HENRICHSEN −7·51, QUINCKE −4·278, HOWARD −4·248, and WÄBNER −2·758.*

A solution of ferric chloride, whose value for 10^7k at 10° centigrade, was

$$230·5 - 7·7,$$

was found by gravimetric analysis to contain ·0865 gram of iron per cub. centim. Hence the general formula for k, for ferric chloride dissolved in water, is

$$10^7k = 2660 \text{ W} - 7·7$$

W being the weight of iron per cub. centim. of the solution.

Ferric Sulphate.

The values of k for three different solutions of ferric sulphate were found, the quantity of iron per cub. centim. in each being proportional to 2, 3, and 4, giving the values

$$52, \quad 79·3, \quad \text{and } 110, \text{ for } 10^7k.$$

If to these we add 7·7 we get the values of k for the salt alone which are in the same proportion as the quantity of salt per cub. centim.

A solution, whose value for 10^7k at 7° centigrade was $86·2 - 7·7$, contained ·0320 gram of iron per cub. centim., which gives for ferric sulphate the formula

$$10^7k = 2690 \text{ W} - 7·7.$$

It will be seen from the curves showing the variation of k with temperature that at 10° centigrade the value of k will be 1·5 per cent. less than at 7°, so that we get

$$10^7k = 2660 \text{ W} - 7·7, \text{ at } 10° \text{ centigrade},$$

exactly the same formula as for ferric chloride.

* WIEDEMANN, 'Electricitat,' vol. 3, pp. 1292 and 1322.

Ferric Nitrate.

An experiment was tried with ferric nitrate and estimated gravimetrically, like the chloride and the sulphate, and it also gave the formula

$$10^7 k = 2660 \text{ W} - 7 \cdot 7.$$

thus showing that the coefficient k depends only on the quantity of iron present, and the nature of the solvent.

Ferrous Chloride.

A solution of $FeCl_2$, which contained ·0409 gram of iron per cub. centim., had a coefficient of magnetization k given by the formula

$$10^7 k = 84 \cdot 5 - 7 \cdot 7 \text{ at } 12° \text{ centigrade.}$$

The solution had been boiled for several hours with iron wire, so as to reduce any ferric salt to the ferrous state and then filtered. So that for ferrous chloride we have the formula

$$10^7 k = 2060 \text{ W} - 7 \cdot 7.$$

Ferrous Sulphate.

A more complete examination was made of ferrous sulphate, the values for $10^7 k$ for two solutions, one containing twice the amount of iron per cub. centim. as the other, were found to be

127·3, and 60·5,

so that when they are corrected for the water in which they were dissolved they have the ratio 2 : 1.

A solution was examined similarly to the ferrous chloride and gave a formula $10^7 k = 2090 \text{ W} - 7 \cdot 7$, the quantity of iron having been estimated gravimetrically by boiling a specimen with nitric acid and proceeding in the ordinary way.

As this formula did not coincide with that got for the ferrous chloride a more complete examination was made of a solution which contained a small amount of ferric sulphate. Having found the value of k, 25 cub. centims. of the solution were extracted and the quantity of ferrous salt found by a standard solution of permanganate of potash, another specimen having been treated with nitric acid gave the total quantity of iron present, thus providing a means of finding the amount of ferric iron in the solution.

The numbers thus obtained were :—

Weight of iron in ferrous state, per cub. centim., ·0689,
Weight of iron in ferric state, per cub. centim., ·0064,

$$10^7 k = 158 - 7\cdot 7,$$

subtracting 17, which is the value of $10^7 k$ for ·0064 gram per cub. centim. in ferric sulphate, we get

$$10^7 k = 141 - 7\cdot 7 \text{ for } \cdot 0689 \text{ of iron in the ferrous state.}$$

Hence

$$10^7 k = 2050 \text{ W} - 7\cdot 7,$$

differing from the formula obtained for ferrous chloride by only ·5 per cent.

The same result was obtained for a solution containing ·01 gram of ferric iron and ·0118 of ferrous iron.

The presence of free sulphuric acid in the sulphates was found not to affect the value of k, by testing whether an addition of the acid altered it.

The liquid containing the two sulphates in nearly equal proportions was further examined by gradually neutralizing the free sulphuric acid with caustic potash till an almost neutral solution was obtained.

This did not alter the coefficient of magnetization of the salt, showing that no change in the acid radical had as yet taken place, but when the potash was added till the dark precipitate was formed, the value of k immediately rose.

This experiment shows that a salt corresponding to the magnetic oxide does not exist in solution, and that when the latter is dissolved in an acid it gives a mixture of ferrous and ferric salts.

Salts in the Dry State.

Estimations were also made in the dry state by filling a glass tube with the salt and putting it into the solenoid. These experiments could not with ease be performed with the same accuracy as those made on the solutions, since the density of the salt is variable along the tube, and the amount of moisture and water of crystal lization in the body introduce errors when estimating the weight of iron.

The experiments showed roughly that the magnetic properties of the salts are the same as when they are in solution, the actual numbers got for ferrous sulphate were $10^7 k = 841$, the amount of iron per cub. centim., found by weighing the body as $FeSo_4 + 7H_2O$, was ·241. This value of k differs by only 4 per cent. from the value calculated from the formula $10^7 k = 2050$ W.

Iron in the Acid Radical.

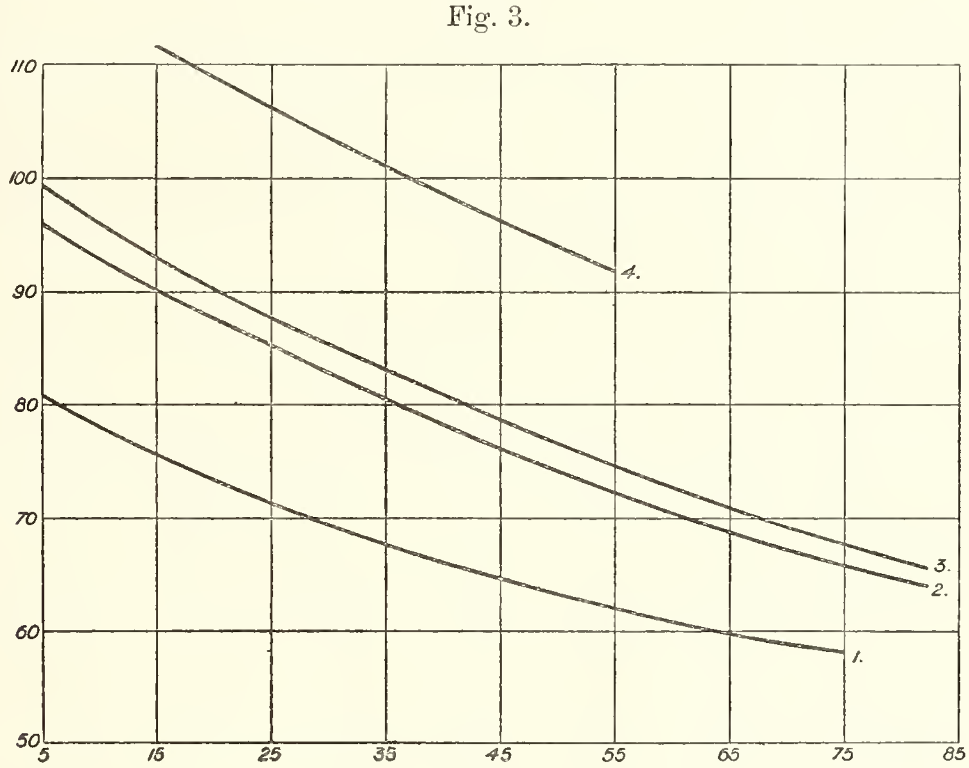
Solutions of potassium ferricyanide and ferrocyanide each containing ·029 gram of iron per cub. centim. gave values of k the same as if water alone were present, so that when iron appears in the acid radical it is at least 100 times less magnetic than when it takes the part of a metal in a salt.

Effect of Temperature.

In estimating the effect of temperature it is necessary to use a tube of about one-third the sectional area of that on which the solenoid is wound, so as to be able to prevent the latter from being heated by radiation and convection.

The time required to keep the hot tube inside the solenoid while adjusting is about half-a-minute, so that it is an easy matter to prevent the coils getting heated. The efficiency of the method employed was proved by the balance returning to its original zero when the liquid was taken out and the reduction in the turns in c restored. Uniformity of temperature along the column of the liquid is secured by inverting the tube containing it before inserting it into the solenoid.

Fig. 3.



The results of these experiments are best shown by drawing curves, fig. 3, the x ordinates denoting temperature centigrade, the y ordinates being proportional to k, a length corresponding to 50 being subtracted from the latter for convenience of representation.

The first curve was got from experiments done on a solution of ferric chloride containing ·086 grm. of iron per cub. centim., the 2nd from ferrous chloride containing ·148 grm. of iron per cub. centim., the 3rd from ferric sulphate containing

·105 grm. of iron per cub. centim., and the 4th from an alcoholic solution of ferric chloride containing ·127 grm. of iron per cub. centim.

The curves show that there is a great diminution in k as the temperature rises, amounting to more than ·5 per cent. per degree centigrade at the lower temperatures.

The curves show that $k = k_0(1 - \alpha t)$, where α is a function of the temperature, and is very approximately independent of the acid radical.

The diminution in density of the salt due to expansion contributes such a small amount to the rapid fall in k that it is not necessary to make any correction for it.

The solvent itself did not show any variation due to temperature which could be detected by this method, and as the alcoholic solution gives the same temperature coefficient as the solutions in water, it is probable that the change due to temperature is a property of the molecules of the salt itself.

The first conclusion that the above results leads us to is that the magnetization is entirely due to the iron, and is accurately the same for all acid radicals connected with it, so long as the iron remains either in the ferrous or the ferric state. The only other electromagnetic phenomenon we know of that also possesses this property is the atomic charge, so that possibly the polarity of the metallic atom may be due to its rapid rotation carrying with it its atomic charge. The ratio of the atomic charge in the ferric to that in the ferrous state is 3 : 2. There is a variety of suppositions as to the nature of the rotations of the iron atom in the molecule which will account for the ratio 266 : 206 of the two values of k.

A simple case presents itself by supposing the intensity of the polarity proportional to $q_1\omega_1$ and $q_2\omega_2$ in the two cases, q_1 and q_2 denoting the charges, and ω_1 and ω_2 the rotations, the axes of which coincide with the axis of polarity. In this case, if H is the applied force, we get the displacements proportional to $q_1\omega_1$H and $q_2\omega_2$H, so that the ratio of the values of k is $q_1^2\omega_1^2 : q_2^2\omega_2^2 = 266 : 206$, so that $\omega_2 = 1{\cdot}32\omega_1$.

The next consideration is the controlling force, which acts in such a way that the induced magnetization is proportional to the applied force. It cannot arise from any action of the solvent, since the magnetic properties of the salts are the same in the dry state as when they are in solution. Also, since k is accurately proportional to the density, we see that each molecular magnet behaves in exactly the same way whether the neighbouring magnets are near it or not.

It is, therefore, highly improbable that the surrounding molecules contribute to the controlling force, as there is no variation in it when the mean distances undergo large changes.

We are, therefore, led to the conclusion that there is no controlling force due to external bodies, and that the magnetization is a phenomena due to the perturbations in the angle of inclination of the axis of polarity to the direction of force H.

Let θ denote this angle, which is a function of the time, $\theta = \theta_0 + \delta\theta$ where θ_0 is the value of θ when H = 0, and $\delta\theta$ the perturbation at any time due to H, and is proportional to H, so that $\delta\theta = \mathrm{H}f(t)$ where f differs from the different atoms according to θ_0 and the initial circumstances.

The intensity I of magnetization is equal to

$$\Sigma \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{M} \cos\theta \, dt,$$

where M is the magnetic moment of the molecular magnet, and T a time which is large compared with the periods.

Therefore,

$$\mathrm{I} = \Sigma \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{M}\left(\cos\theta_0 - \delta\theta \sin\theta_0\right) dt,$$

$$\mathrm{I} = \mathrm{H}\Sigma \frac{-1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{M} \sin\theta_0 f(t) \, dt.$$

This explanation accounts for diamagnetism, for if we consider the very simple case of a magnet rotating rapidly in the horizontal field of the earth, the effect of the applied force H will be to make the north pole move more slowly when it is south of the axis of suspension than when it is north, so that its mean position is south of the axis of suspension, thus causing the magnet to act like a diamagnetic body.

The following table shows the values of 10^7k for salts in solution, w being the weight of salt per cub. centim., the forces ranging from 1 to 9 C.G.S. units.

	10^7k.
Fe_2Cl_6	$916w - 7{\cdot}7$
$Fe_2(SO_4)_3$	$745w - 7{\cdot}7$
$Fe_2(NO_3)_6$	$615w - 7{\cdot}7$
$FeCl_2$	$908w - 7{\cdot}7$
Fe_2SO_4	$749w - 7{\cdot}7$

INDEX SLIP.

XV. *The Total Eclipse of the Sun, April 16th, 1893 —Report and Discussion of the Observations relating to Solar Physics.*

By J. NORMAN LOCKYER, *C.B., F.R.S.*

Received April 17,—Read April 30, 1896.

[PLATES 11–14.]

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25.11.96.

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(2.) *Plates.*

PART I.—THE OBSERVATIONS.

I. Introduction.

The results obtained by Professor Respighi and myself during the eclipse of 1871 in India, in which part of the attack consisted in the employment of slitless spectroscopes—a method of work at which we had arrived independently—indicated the extreme value of such observations.

For my own observations in 1871 I had arranged a train of five prisms without either collimator or observing telescope. "I saw four rings with projections defining the prominences. In brightness, C came first, then F, then G, and last of all 1474K. Further, the rings were nearly all the same thickness, certainly not more than 2′ high, and they were all enveloped in a band of continuous spectrum."*

* "Nature," vol. 5, p. 218, 1872.

RESPIGHI's observations were made with a telescope of $4\frac{1}{2}$ inches aperture with a large prism of small angle in front of the object glass. A negative eyepiece magnifying 40 times and having a field view large enough to include the whole of the spectrum was employed. The principal results obtained by RESPIGHI were as follows* :—

"At the very instant of totality, the field of the telescope exhibited a most astonishing spectacle. The chromosphere at the edge which was the last to be eclipsed . . . . . . was reproduced in the four spectral lines C, D_3, F, and G, with extraordinary intensity of light . . . . . .

"Meanwhile the coloured zones of the corona became continually more strongly marked, one in the red corresponding with the line C, another in the green, probably coinciding with the line 1474 of KIRCHHOFF's scale, and a third in the blue perhaps coinciding with F."

"The green zone surrounding the disc of the moon was the brightest, the most uniform and the best defined."

My observation† was made intermediately between the two observations of Professor RESPIGHI. The observations may be thus compared :—

RESPIGHI	C	D_3	F.G.	Chromosphere and prominences at beginning of totality.
LOCKYER	C	1474 (faint)	F.G.	Corona 80 secs. after beginning of totality.
RESPIGHI	C	1474 (strong)	F.	Later.

I had no object glass to collect light, but I had more prisms to disperse it, so that with me the rings were not so high as those observed by RESPIGHI, because I had not so much light to work with; but such as they were, I saw them better, because the continuous spectrum was more dispersed, and the rings (the images of the corona) therefore did not overlap. Hence, doubtless RESPIGHI missed the violet ring which I saw; but both that and 1474 were very dim, while C shone out with marvellous brilliancy, and D_3 was absent.

In arranging for the eclipse of 1875 in Siam and the Nicobars, the method was further developed by the introduction of photography, and the first results of this extension were given in the report of the Eclipse Expedition of that year. They showed clearly that with the rapid dry plates of to-day a considerable increase of dispersion might be attempted.

The object glass employed on this occasion had an aperture of $3\frac{3}{4}$ inches and a focal length of 5 feet, while the prism had a refracting angle of 8 degrees.

Two photographs were obtained with exposures of one and two minutes respectively. Both are reproduced in the Report,‡ and they show only such

* 'Nature,' vol. 5, p. 237, 1872.

† 'Brit. Assoc. Report,' 1872, p. 331.

‡ 'Phil. Trans.,' 1878, vol. 169, Part 1, p. 139.

differences as can be attributed to difference of phase. The dispersion was very small compared with the size of the sun's image, so that the photographs present the appearance of an ordinary photograph of the eclipsed sun which is slightly distended in the direction of dispersion. The various prominences each show three images, two of which were identified with H_β, H_γ, while the others were found to correspond to a wave length of about 3957.

It was suggested (Report, p. 149) that this represented the H and K radiations of calcium, and this is fully confirmed by the results obtained in 1893, to say nothing of results obtained in other eclipses.

In addition to the protuberances, the photographs show a well defined circular boundary of the moon's limb at a position corresponding to H_γ. This was considered to be an indication that hydrogen was one of the substances existing in an incandescent state in the corona itself, for although coronal rings corresponding to H_β and H_δ were not photographed, their absence may possibly be accounted for by the fact that the plates employed were most sensitive to the H_γ region.

There are also indications of a continuous spectrum from the lower parts of the corona, shown by well defined structure running parallel to the direction of dispersion.

I next proceed to remark very briefly upon the photographic results obtained since 1875.

In 1878, near the sun-spot minimum, the method was employed by several observers, myself among them, but no *bright* rings were recorded. The maximum sun-spot conditions previously observed had entirely changed; indeed with a slit spectroscope the 1474 line was very feeble, and was only seen by a few of the observers, and hydrogen lines were similarly feeble.*

Part of my own equipment for this eclipse consisted of a small grating placed in front of an ordinary portrait camera, and with this I obtained a photograph showing only a very distinct continuous spectrum.†

The method was employed by Dr. SCHUSTER in Egypt in 1882; the camera was of 3 inches aperture and 20 inches focal length, with a prism having a refracting angle of 60°.‡

The single photograph obtained (which was not reproduced) was stated to show two rings, which were considered to be due to the lower parts of the corona, and therefore to correspond to true coronal light. The wave-length of one of these rings was measured to be 5315; it is due to the green corona line (1474K). The second was stated to be coincident with D_3. The ring in the green was particularly strong in the south-western quadrant, and hardly visible at some other points of the sun's

* 'American Journal of Science,' vol. 16, p. 243.

† With a duplicate grating I observed the spectrum of the eclipsed sun, and again in three different orders, saw nothing but continuous spectrum ('Nature,' vol. 18, 1878, p. 459).

‡ 'Phil. Trans.,' vol. 175, 1884, p. 262.

limb. The yellow ring was much fainter on the whole, but more uniform all round the sun.

In 1883 the same instrument used in Egypt in 1882 was employed, as well as a 6-inch achromatic telescope, and a concave ROWLAND grating of 5 feet focus, arranged for taking ring spectra in the first and second orders.

It is stated in the report* that the photographs "possess no features of interest," and neither reproductions, nor drawing, nor measurements are given.

The prismatic camera employed in the eclipses of 1882 and 1883 was again used in the West Indies in 1886. Only the spectra of some prominences seem to have been recorded. There is no mention of rings. The hydrogen lines as well as K and *f* are noted.†

While on the one hand the photographic results, to which reference has been made, certainly did not come up to the expectations raised by my observations of 1871, on the other, subsequent solar investigations confirmed my opinion that this was the best way of studying the lower parts of the sun's atmosphere, provided an instrument of much greater light-grasping power could be employed.

I determined, therefore, when arranging for the observations to be made during the eclipse of 1893, to renew the attack with the largest telescope and the greatest dispersion at my command.

The Solar Physics Committee is now in possession of a prismatic camera of 6 inches aperture. I decided, therefore, to employ it, all the more because the work on stellar spectra at Kensington had given abundant proof of its excellence.

The object glass of this instrument, corrected for the photographic rays, was constructed by the Brothers HENRY. The correction is such that it is unnecessary to incline the back of the camera, and hence some of the objections which have been made to the use of this form of spectroscope are overcome. The large refracting angle of the prism (45°) obviously increases the value of the instrument for eclipse work. This instrument was placed at the disposal of the Eclipse Committee, by the Solar Physics Committee, and was entrusted to Mr. FOWLER, who took the photographs at the African station.

Although no other instrument of this power was available, it seemed important that a series of similar photographs should be attempted at another point on the line of totality. A spectroscope belonging to the Astrophysical Laboratory of the Royal College of Science was lent for the purpose by the Department of Science and Art, and a siderostat used in conjunction with it was lent by the Royal Society. These instruments formed part of the equipment of the Brazilian expedition, and were placed in charge of Mr. SHACKLETON, Computer to the Solar Physics Committee.

The following sections give, first, reports of the operations at the two stations contributed by Messrs. FOWLER and SHACKLETON, in charge of the instruments at

* 'Phil. Trans.,' 1889 A, vol. 180, p. 122.

† 'Phil. Trans.,' 1889 A, vol. 180, p. 319.

the African and Brazilian stations respectively; then a detailed description of the phenomena recorded, followed by a discussion of the method employed in dealing with the photographs.

The spectrum of the corona and its possible variation, the wave-lengths and intensities of the prominence and chromospheric lines are next studied, and finally the loci of absorption in the sun's atmosphere are considered.

Much labour and time have already been spent in dealing with the chemical part of the inquiry, and I have been driven to the conclusion that before the bearing of the eclipse observations on our knowledge of the spectrum of each chemical substance is given, much more inquiry must be undertaken (*a*) into the old observations, (*b*) into the spectrum of stars and nebulæ, and (*c*) into certain questions for which new observations are necessary.

This chemical part of the inquiry will, therefore, be set out in a future Memoir.

II.—The African Station (Mr. Fowler's Report).

Locality.

The station selected in West Africa as offering the greatest facilities, combined with good chances of fine weather, was Fundium (or Foundiougne, as it is called by the French), on the Salum River. According to the Admiralty Chart, the village is in latitude 14° 3′ N. and longitude 16° 30′ W. It was, therefore, only about six miles south of the central line of eclipse.

The expedition left Liverpool on the morning of March 18, 1893, by the British and African Company's s.s. *Teneriffe*, and arrived at Bathurst, on the Gambia, on March 31. Instruments and observers were there transferred to H.M.S. *Alecto*, the special service gunboat on the West Coast of Africa, then in charge of Lieutenant-Commander Lang, R.N. Leaving Bathurst on April 2, the party proceeded to the selected station, arriving there on April 3.

A very suitable site for the instruments was offered by the Administrator, in the grounds surrounding his own house, and it was at once accepted as satisfying all requirements. It was quite close to one of the wharves, and had the advantage of being partially enclosed. Vegetation in the neighbourhood was very sparse, and an almost perfectly clear horizon was obtained.

A neighbouring site was already occupied by M. Deslandres, of the Paris Observatory, when the British expedition arrived at Fundium.

With the assistance of the officers and men of H.M.S *Alecto*, the instruments were taken ashore without delay, the concrete base was laid down, and the hut erected. By April 10 matters were sufficiently advanced to admit of full rehearsals with the remainder of the observers.

Magnificent weather prevailed during the stay of the expedition at Fundium, so that the adjustments of the instruments were made without any difficulty. The sun was not obscured by clouds on any of the thirteen days preceding the eclipse, near the time of day at which the eclipse took place, and only on a few days was there a slight haze.

The temperature in the early afternoon was usually about 90° F.

On the day of the eclipse the sky was a little more hazy than on previous days, but, on the whole, the conditions of observation were excellent.

The dismantling of the instruments was commenced immediately after the eclipse, and by the following evening everything was safely on board H.M.S. *Alecto*. On April 18 the expedition left Fundium and arrived at Bathurst next day. As the return mail steamer did not leave Bathurst for a considerable time, H.M.S. *Blonde* was kindly placed at the disposal of the expedition by the Admiralty and conveyed the party to Grand Canary, whence, after several days' delay, the passage to England was completed by the s.s. *Mequinez*.

The success of the expedition was in great measure due to the generous assistance rendered by the Admiralty in granting the use of H.M.S. *Alecto*. As there were no means of direct communication with Fundium except by a man-of-war, the expedition would otherwise have been almost impossible, and it would be difficult to over-estimate the value of the skilled help which thus became available. The subsequent grant of H.M.S. *Blonde* prevented the necessity of the expedition remaining without adequate accommodation in an unhealthy climate some two or three weeks after the work was done.

Thanks are also due to the French Government for the facilities afforded to the observers for landing in French territory; to the African Steamship Company for a material reduction of passage money; and for help in other ways to R. B. LLEWELLYN, C.M.G., the administrator at Bathurst, and M. VICTOR ALLYS, the Administrator at Fundium.

Personnel.

The general arrangements for the expedition were made by a Joint Committee of the Royal Society, the Royal Astronomical Society, and the Solar Physics Committee.

Quartermaster HALLETT, H.M.S. *Alecto*, acted as general timekeeper for the expedition, and for the detailed time records required for the prismatic camera, Lieutenant SHIPTON, R.N., gave invaluable aid. In the absence of mechanical contrivances for making the exposures, Chief Artificer MILLIGAN performed the duties entrusted to him as perfectly as could have been desired. To these, as well as to the ships' carpenters and others kindly placed at the disposal of the expedition by Lieutenant-Commander LANG, R.N., who himself did everything possible to ensure the success of the work, the best thanks of all friends of science are due.

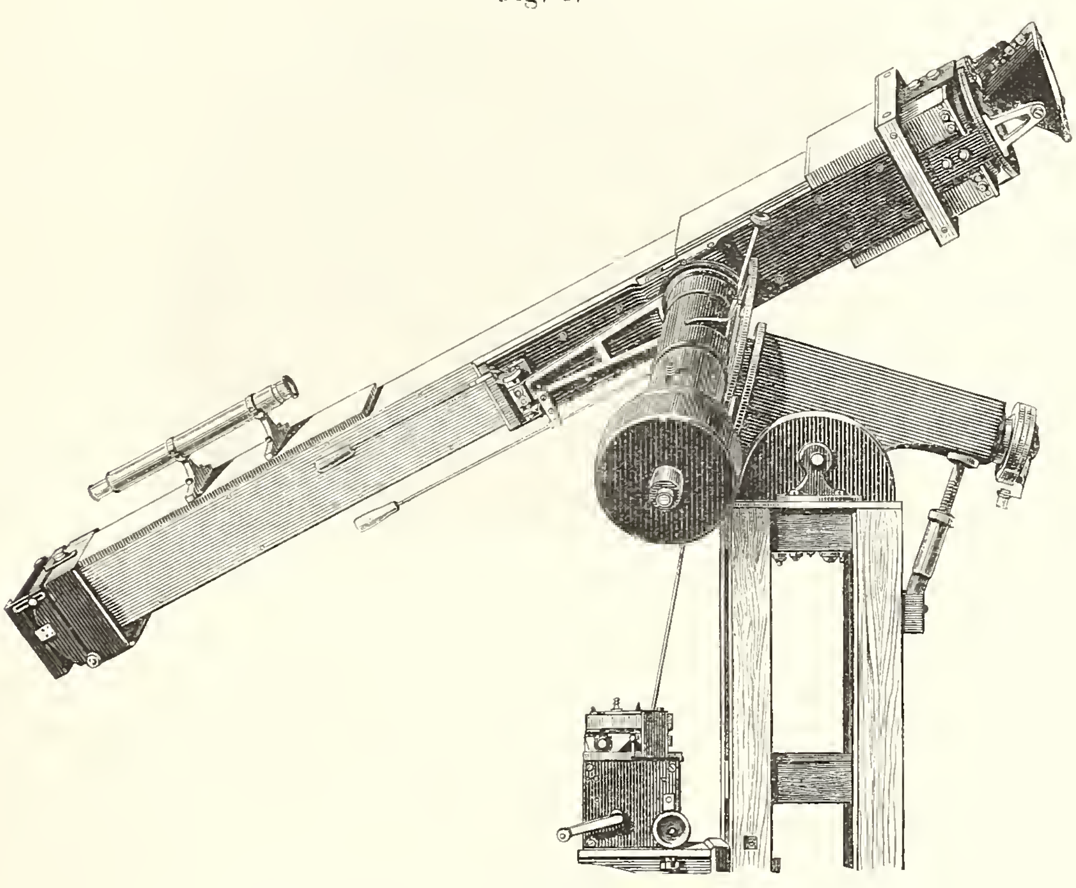
T. B. MACAULAY, of the Bathurst police force, was attached to the expedition as interpreter, and accompanied the observers to Fundium.

The Instrument Employed.

The 6-inch prismatic camera employed in Africa has a focal length of 7 feet 6 inches, and the spectrum obtained is about 2 inches long from F to K. Rings corresponding to the inner corona are about seven-eighths of an inch in diameter.

The object glass and prism, with the square tube to which they were attached, were kindly lent for the occasion by the Department of Science and Art, and the equatorial mounting was that of Professor LOCKYER'S 6-inch Cooke refractor. The tube is a strong mahogany one, square in section, and it was attached to the declination axis by means of a suitable iron plate. In order to reduce the weight of the instrumental equipment, the heavy iron pillar of the equatorial was replaced by a rough wooden stand which was filled up with concrete after being placed in position. Provision was made for the clock bracket and fine adjustments of the polar axis, and the whole arrangement was quite satisfactory.

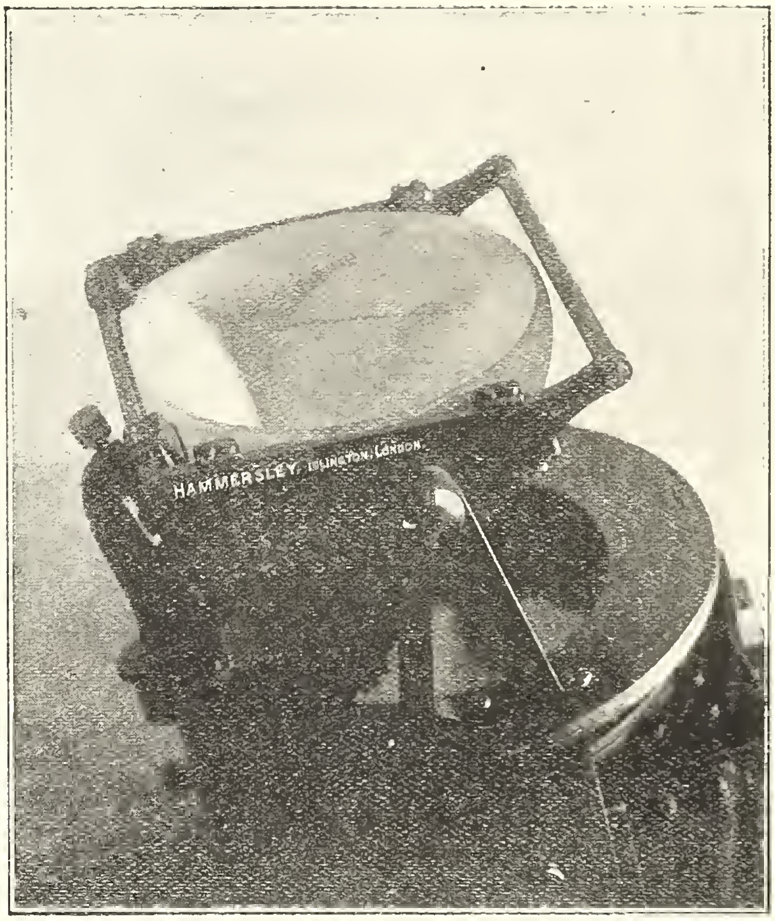
Fig. 1.



Prismatic Camera mounted on equatorial stand.

Fig. 1 represents the instrument as adjusted for use in latitude 14° 3′ N. When actually in use, the camera was steadied by a stiff wooden rod screwed to the end of the tube, and bearing on the end of the declination axis; this did not interfere with the driving gear and materially contributed to the successful results, as on account of the great weight of the prism it was necessary to bring a large part of the tube forward to the eye end. The brass cap which protected the camera from light other than that which passed through the prism and object glass, is not shown in the diagram. The details of the attachment of the prism to the object glass cell is shown in fig. 2; provision is made for adjustment to minimum deviation and for rotating the prism and clamping it in any desired position.

Fig. 2.



Details of prism mounting.

The whole instrument worked very satisfactorily, except for a slight backlash in the driving screw, which could not be corrected. On a few of the plates there is evidence of a trail of the spectrum during the exposure, in consequence of this defect, and this trail seems to be partly answerable for the absence of some of the fainter details from some of the negatives. No trouble was experienced with the driving clock.

The observatory provided for the instrument was 13 feet square, and 6 feet high, with a gable roof 4 feet in height. It consisted of a rough wooden framework, which had been completely prepared and marked before leaving England, covered with Willesden water-proof canvas. A portion of the covering of the roof was arranged so that it could be readily opened to admit of observations. The hut was very easily erected and it satisfied all requirements at a very small cost.

Conditions of the Eclipse in Africa.

The results obtained with the prismatic camera will vary in detail according to the conditions of eclipse, and it is therefore desirable to indicate the conditions obtaining in 1893.

At the African station, the apparent diameter of the sun and moon were respectively 31′ 55″·4 and 33′ 35″·9 ; the altitude at the time of eclipse was about 53°. The first contact, according to the 'Nautical Almanac' Circular, took place at an angle of about 130°, reckoned from the North point towards the West, and the last contact at an angle of 57° East of the North point. Hence, at the commencement of totality the contact would occur about 60° East of the North point, and at the end of totality about 128° W.

A calculation for the relative movement of the moon and sun during totality gives 0″·37 per second. On this relative movement depends the length of time during which the chromosphere will be visible after the beginning and before the end of totality; thus, if the chromosphere were 5″ in depth, it would have been visible for about 13 seconds on this particular occasion. The apparent depth of the chromosphere seen at the commencement of totality will be diminishing at the rate of 0″·37 per second, while that seen near the end of totality, on the opposite limb, will be increasing at the same rate. The spectroscopic appearances in successive photographs vary accordingly.

The duration of totality calculated for a place very near the selected station was 4 minutes 12·4 seconds ('Nautical Almanac' Circular, No. 14). As measured by M. COCULESCO, a member of the French Expedition at Fundium, who was specially occupied with this question, the duration was 4 minutes 11 seconds.*

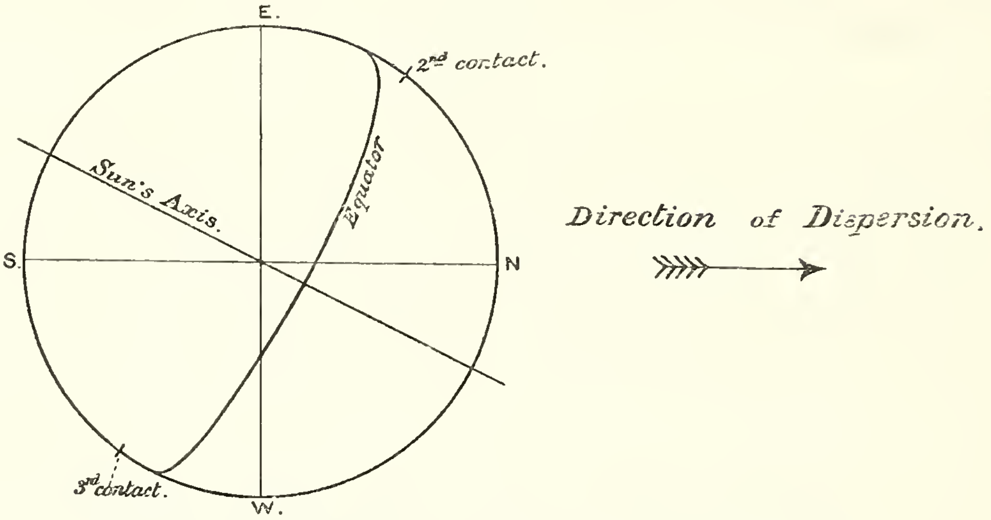
A consideration of the conditions of the eclipse indicated that on the whole the best position for the prism was that giving the dispersion in a North and South line ; so long as the direction of dispersion is not nearly tangential to the sun at the points of contact, it matters little what is its direction as regards the photographs taken during totality. For the photographs out of totality, where the object is to study the phenomena at the cusps, the direction of dispersion must be such that the spectra of the two cusps are not superposed, and, if other circumstances permit, the best position of dispersion would be perpendicular to the line joining the cusps.

With an equatorially mounted prismatic camera, the position of the prism giving dispersion in a North and South line is by far the most convenient for practically working the instrument, as the deviation of the prism can be readily corrected by a movement in declination alone, and photographs of stellar spectra can be taken for focussing purposes. Hence, for the sake of greater simplicity in working, it was decided to work with a North and South dispersion, although this involved a sacrifice

* 'Comptes Rendus,' 1893, vol. 116, p. 1238.

of the phenomena appearing at one of the cusps in the photographs taken shortly after the end of totality.

Fig. 3.



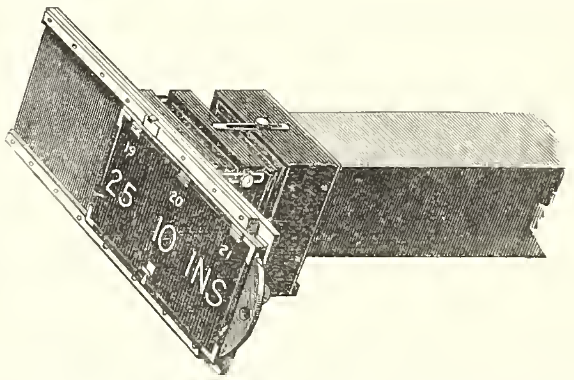
Showing position of sun's axis and equator, April 16, 1893.

Fig. 3 shows the position of the sun's poles and equator in relation to the direction of dispersion.

The Dark Slides Used.

The construction of the camera and dark slides, or plate-holders, was based on the plan devised and adopted by Professor LOCKYER for the large pictures of the corona, which he hoped to obtain in the West Indies in 1886. Fig. 4 will make them readily understood. The slides are about 13 inches in length by 7 inches broad, and have 3 compartments, each taking a plate 6 inches by 4 inches.

Fig. 4.



Details of dark slide employed in West Africa.

The camera at the end of the long wooden tube has an opening 6 inches square, and a rectangular frame 24 inches long, with a central aperture 6 inches by 4 inches, and provided with grooves to take the slides, was symmetrically attached to it. A dark slide being placed in the frame, so that the first compartment was opposite the middle of the telescope tube, the shutter was then opened to its full extent, and an exposure made; the plate in the second compartment was next brought to the middle of the frame, by pushing the slide along, and also exposed; again, by moving the slide along, the third plate was brought into position and exposed, after which the shutter was closed and the slide withdrawn. During the exposure of any one of the three plates in a slide, the other two were protected from light by the rectangular frame.

The upper edge of each dark slide was notched in three places, corresponding to the positions of the three plates which it contained, and, as each plate came to the proper position for exposure, as the slide was pushed along, a spring catch automatically dropped into its place.

Upon the back of each dark slide six numbers were painted in clear white figures. A small series of numbers corresponded to the numbering of the 30 plates to be exposed during the eclipse, and a larger series indicated the exposures to be given to each plate, so that it was unnecessary to refer to any list.

These time-saving devices are of the highest importance in eclipse work, and too much attention cannot be given to them. The arrangements in West Africa worked admirably, and it was possible to change from one plate to another in about a second when a slide was once inserted, and to change the whole slide in 5 seconds. Longer intervals, however, were allowed to elapse between the exposures, in order that the instrument might steady itself, and to correct the backlash of the driving screw.

Method of Focussing.

The instrument was focussed by photographing the spectra of some of the brighter stars. This is the only satisfactory method of focussing the prismatic camera, as rays from a star fall on the prism under exactly the same conditions as those from the eclipsed sun. If a slit and collimator be employed, identical conditions can only be obtained when the collimator is perfectly achromatic and absolutely adjusted for parallel rays.

From C to the extreme ultra-violet the focus is sufficiently constant with the Henry lens, to enable the use of a swing back to be dispensed with, and this is an immense advantage in work with the prismatic camera, for the reason that opposite sides of the rings of light corresponding to the various radiations will be equally well in focus.

The Plates Employed.

There was a little uncertainty with regard to the kind of plates which would be most suitable in the climate of West Africa, and four different commercial brands were therefore taken out, namely :—

MAWSON and SWAN'S "Stellar" plates.
EDWARDS' isochromatic plates.
Ilford isochromatic plates.
Ilford special rapid plates.

These were tested on the spot and all were found very satisfactory. Hence, some of each were employed for the work during the eclipse.

The plates were all placed in the dark slides and the films carefully numbered with pencil on the evening preceding the eclipse.

Since the isochromatic plates appear to answer as well for the blue and violet as the ordinary ones, while giving also information as to the green and yellow, it may be desirable to employ them exclusively in future.

Exposure of the Plates.

As the instrument was not provided with an exposing shutter, the exposures were made by covering and uncovering the prism with a piece of thick card. In this part of the work I was assisted by Lieutenant SHIPTON, R.N., and Chief Artificer MILLIGAN, of H.M.S. *Alecto*, and the following plan was adopted after numerous rehearsals: The dark slide being placed in proper order on a packing case close to the camera, one was inserted and the figures indicating the time of the exposure were noted, the prism being meanwhile covered; my calling out of the time of exposure was the signal for Mr. MILLIGAN to remove the card and for Lieutenant SHIPTON to commence counting the number of seconds announced, at the termination of which he gave the signal "over," the prism was covered and the dark slide moved on for the next plate. In the case of the "instantaneous" exposures, "snap" was called, the card removed for a moment by Mr. MILLIGAN, and the time noted by Lieutenant SHIPTON. After rehearsals the arrangements worked without a hitch.

A pendulum clock had been provided for recording the times, but for some reason, probably from the sand getting among the gearing, it ceased to work satisfactorily on the day of the eclipse, and a navigator's deck-watch was used in its place.

As very little idea could be formed of the length of exposure required to give the best results with this instrument, the exposures were arranged by Professor LOCKYER in a series which was repeated three times during totality. Thus, in case one particular exposure was found better than the others a good record at different stages of the eclipse would be secured. As will be seen by reference to the table on page 566, the actual exposures in the series were instantaneous, 5 seconds, 25 seconds, and 10 seconds.

Near the middle of totality a specially long exposure of 40 seconds was interpolated with the object of photographing the coronal spectrum free from admixture with the spectrum of the chromosphere.

In addition it was very important to attempt to secure records of the phenomena as nearly as possible at the beginning and end of totality. For this reason the series of exposures to which reference has been made were not commenced until an instantaneous exposure had been made, and after they were completed four more plates were taken with short exposures, in the hope that one of them might be exposed within the two seconds preceding the end, the determination of the end of totality being less certain than of the beginning.

A similar arrangement of exposures was made in the case of the photographs taken out of totality, the series in this case being 8 seconds, 2 seconds, and instantaneous. The spectrum was observed on the ground glass screen between the exposures, and seeing the great illumination of the field, I took upon myself at the last moment the responsibility of a small departure from the table drawn up and substituted two instantaneous exposures for the 2 and 8 seconds in Photographs Nos. 5 and 6, fearing that the longer exposures would fog the plates.

Plates Obtained.

A complete list of the photographs taken is given in the appended table. Column 1 contains reference numbers to the photographic plates; column 2 the brand of plate employed; column 3 the times of beginning and ending each exposure, as recorded by a deck-watch; and column 4 the amounts of exposure, "Inst." indicating an exposure given as quickly as possible by hand.

Table of Exposures.

No.	Kind of plate.	Times by deck watch.			Exposure.	Remarks.
		hrs.	mins.	secs.		
1	Edwards' Isoch.	2	17	20	Inst.	About $6\frac{1}{2}$ mins. before totality
2	,, ,,	2	17	50·52	2 secs.	,, 6 ,, ,, ,,
3	,, ,,	2	18	21·29	8 ,,	,, $5\frac{1}{2}$,, ,, ,,
4	,, ,,	2	18	55	Inst.	,, 5 ,, ,, ,,
5	,, ,,	2	20	55	,,	,, 3 ,, ,, ,,
6	,, ,,	2	23	19	,,	,, $\frac{1}{2}$,, ,, ,,
7	Mawson	2	23	58	,,	First photo. during totality.
8	,,	2	24	0	,,	
9	,,	2	24	6·11	5 secs.	
10	,,	2	24	21·46	·25 ,,	
11	,,	2	24	48·58	10 ,,	
12	,,	2	25	2	Inst.	
13	,,	2	25	14·19	5 secs.	
14	,,	2	25	24·49	·25 ,,	
15	,,	2	25	51·61	10 ,,	About mid-eclipse
16	Edwards' Isoch.	2	26	10	Inst.	
17	,, ,,	2	26	12·52	40 secs.	
18	,, ,,	2	26	55·60	5 ,,	
19	Mawson	2	27	10·35	25 ,,	
20	,,	2	27	38·48	10 ,,	
21	,,	2	27	50	Inst.	Last photo. in totality
22	Ilford, Isoch.	2	28	3·8	5 secs.	3 secs. after totality
23	,, ,,	2	28	10	Inst.	10 ,, ,, ,,
24	,, ,,	2	28	11	,,	11 ,, ,, ,,
25	Ilford, Special	2	28	41·49	8 secs.	41 ,, ,, ,,
26	,, ,,	2	29	41·43	2 ,,	1 min. 41 secs. after totality
27	,, ,,	2	30	42	Inst.	2 mins. 42 ,, ,, ,,
28	Edwards' Isoch.	2	31	42·50	8 secs.	3 ,, 42 ,, ,, ,,
29	,, ,,	2	32	42·44	2 ,,	4 ,, 42 ,, ,, ,,
30	,, ,,	2	33	42	Inst.	5 ,, 42 ,, ,, ,,

The recorded times are not to be taken as the true local times at which the exposures were made, as the error of the watch was not precisely known, and the object was simply to note the durations of the exposures and the intervals elapsing between them. No attempt was made to determine the exact moments of commencement and end of totality, but the photographs taken after totality enable the end to be approximately determined from the measured width of photosphere uneclipsed. The end of totality, reckoned by the deck-watch as determined in this way, was as follows :—

	h.	m.	s.
From photograph 22	2	28	1
,, ,, 23	2	28	2
,, ,, 24	2	28	0
Mean.	2	28	1

Taking the duration to be 252 seconds, as calculated, the commencement of totality would occur at 2 hours 23 minutes 49 seconds, watch time. Accepting these times, Photograph No. 7 would thus be taken 9 seconds after the beginning of totality, and No. 21, 11 seconds before the end. The appearances on the Plates, however, seem to indicate that these intervals should be more nearly equal, and it would probably be no great error to suppose that totality commenced at 2 hours 23 minutes 48 seconds, and ended at 2 hours 28 minutes 0 seconds, by the deck-watch. The delay in commencing the photographs at the beginning of totality was occasioned by an error in the signal agreed upon for the whole party, so that Photographs Nos. 22, 23, and 24, some of which should have been taken before the end of totality, according to the table drawn up, were not taken until the sun had re-appeared. Fortunately, these three Plates are perhaps not very much less valuable than if they had been exposed at the times arranged.

III. The Brazilian Station (Mr. Shackleton's Report).

Locality.

The station selected in Brazil as being nearest the central line and point of longest duration, was a small village on the coast called Para Curu; it is situated about 50 miles North of the important town of Ceará (Fortaleza), and as read off from the Admiralty Chart is in latitude 3° 24′ S. and longitude 39° 1′ W., the latter corresponding to a difference of 2 hours 36 minutes 4 seconds W. of Greenwich.

The expedition left Southampton on February 23rd, 1893, by the R.M.S. *Tamar*, and arrived at Pernambuco on March 11th; here everything was more or less in disorder on account of the insurrection at Rio de Janeiro, and after nearly a week's delay with Custom House officialism, a start was made by a small coasting steamer, the s.s. *San Francisco*, which arrived at Ceará in six days. The instruments were again disembarked and lay in a store for about a week, until a steamer was ready to proceed northwards; this was the s.s. *Oriente*, the owners of which kindly consented for her to put into the bay at Parazinho and land the eclipse party. Arriving here on March 30th, the baggage was got on shore by means of surf rafts or catamarans, but no help was forthcoming from the natives until two days later, as they were celebrating Easter festival. During this forced idleness the time was spent in procuring a suitable site at Para Curu, some 1½ miles away, and at length a clearing on an eminence was obtained which had a good sea horizon to the north, the observing station being south of the line. To this place the baggage was removed on April 3rd by the aid of bullock wagons, and just a week before the day of the eclipse, huts had been built and the instruments adjusted. The weather continued very fine until within three days previous to the eclipse, during which rehearsals were gone through to get accustomed to the work, and also to find out the minimum time which could be allowed with safety to move the dark slides from one plate to another, and also to effect a complete change of slides; this was found to be 2 and 8 seconds respectively

After this the sky was continually obscured by clouds and rain fell heavily; this seemed likely to persist, but no rain fell on the morning of the eclipse, and fortunately about half-an-hour before totality commenced, the sky began to clear and remained perfectly so during the whole time of total eclipse.

As soon as the photographs had been taken, the dark slides were removed to a place of safety and the huts were covered over and made fast. This had scarcely been accomplished, when a tropical downpour followed, which continued during the remaining part of the day. Similar weather was experienced by a party of astronomers from Rio de Janeiro, who had their station at a village some miles distant, and who paid a visit to Para Curu, but the compliment could not be returned in consequence of the difficulty and unfamiliarity with bush travelling, besides the fatigue experienced with the thermometer registering 96° in the shade during the day and 88° during the night. The following three days were spent in taking the instruments to pieces and packing up, after which there was nothing to be done but to wait for the steamer which was to call on its way south; this proved to be the s.s. *Colombo*, which put in appearance seven days after the eclipse. The instruments were again got aboard by the rafts, and passage was then taken back to Ceará, where the baggage was shipped on a cargo steamer bound for Liverpool, so as to prevent the many transhipments which would otherwise have been necessary. After a few days' waiting, passage was again taken to Pernambuco by the s.s. *Brazil;* thence the return voyage to England was completed by the R.M.S. *Trent*.

Personnel.

In the absence of such skilled assistance as could be offered by a man-of-war, most of the work was of necessity performed by myself, but thanks are due to Sir J. B. STONE for photographing the instruments; to M. OLSEN (photographer at Ceará), for use of dark room; and to the interpreter, Mr. A FURLEY (Ceará Harbour Works), for assistance in erection of huts and instruments.

The Brazilian Instrument.

The prismatic camera employed in Brazil was simply a large photographic spectroscope deprived of its collimator, mounted on an iron table. The dispersive train consisted of two prisms, each having a refracting angle of 60° and a clear aperture of 3 inches. The object glass was a Dallmeyer portrait lens, 5 D, aperture 3·25 inches, focal length 19 inches.

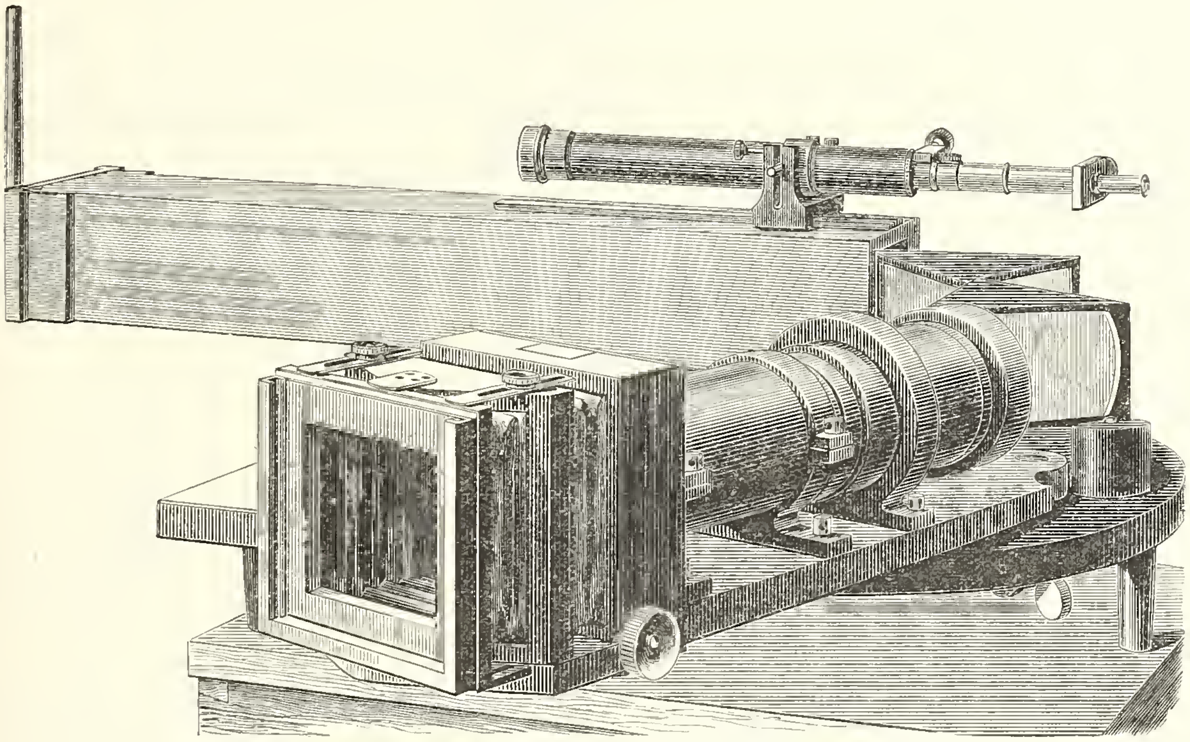
As explained in the report by Mr. FOWLER, the direction of the refracting edges of the prisms has its maximum efficiency defined by the position-angle of the cusps, but with this form of prismatic camera the faces of prisms were of necessity vertical, and therefore the direction of dispersion could not be controlled; fortunately, however, it so happened that the cusps were favourably situated.

The diameters of the moon and sun were such that this camera gave images of ·183 inch and ·176 inch respectively. The length of the spectrum given by the combination was 2·5 inches from D_3 to K or 1·65 inches from F (H_β) to K.

The light from the sun was reflected on to the prisms from the mirror of a 12-inch Cooke siderostat, and to keep any extraneous light from entering the camera, a wooden tube was put in the place ordinarily occupied by the collimator; at the end of the tube was a shutter which could be closed and opened from the camera end with a cord, and by means of this the length of exposures was regulated.

On the top of the wooden tube was placed a small telescope to serve as finder, it being directed to a portion of the mirror not utilized by the prismatic camera; the arrangement is shown in the accompanying figure.

Fig. 5.



Prismatic camera used in Brazil.

A small hut to enclose the siderostat was erected, and, with the exception of one side and the front, which were made of canvas on wooden frames (so as to be easily moved), it was built of wood. The fall for the clock weights was made by erecting a sort of gallows behind the hut, over which the clock cord was directed by pulleys. About ten feet away to the North, the hut to hold the prismatic camera was built, this was also small, being only sufficiently large to hold the instrument, and constructed in the same way as the one described above, the observer taking his place outside.

The siderostat was set down on a firm concrete base and adjusted for latitude by watching the image of a star or the sun with a theodolite set in a meridian line North of the mirror, and seeing that it did not leave the cross wires; then the spectroscope was placed on a large packing case, loaded with gravel for rigidity, with its collimator in a North and South line directed to the centre of the mirror, and the prisms set to minimum deviation for F (Hβ).

Conditions of Eclipse.

At the Brazilian station the diameters of the sun and moon were 31′ 55″·4 and 36′ 6″·6 respectively; this difference of diameters gave a duration of 4 minutes 43 seconds, hence the relative movement was at the rate of ·253″ per second.

The first contact took place at an angle of 136° W. of the North point, and the last contact at an angle of 45° E. of the North point; therefore the direction of the moon's motion cut the North and South line of the sun very nearly at an angle of 45°.

The local mean times of commencement and ending of totality were 11 hours 40 minutes 51 seconds and 11 hours 45 minutes 34 seconds respectively, giving a duration of 4 minutes 43 seconds, and the altitude of the sun was 76°.

The Greenwich mean time of totality was 2 hours 17 minutes, thus preceding the total phase at the African station by 1 hour 20 minutes, and following that of the Chilian station by 1 hour 10 minutes.

Dark Slides Employed.

Three dark slides were provided; they were made so as to travel up and down by means of a rack and pinion, in a frame fitting into the back of the camera. Each slide had eight compartments for plates ($4\frac{1}{4} \times 1\frac{3}{4}''$), and the plates were successively brought into position by raising the slide (fig. 6).

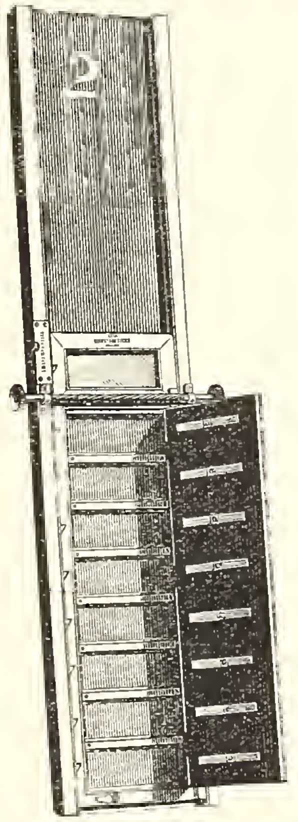
Corresponding to each compartment there was a notch at the side, which indicated the exact place for the plate when a spring detent in the frame fell into the notch, the detent being so arranged as to allow the slide to move only in one direction.

The number of each slide was cut in deep Roman numerals, so that it could be known by touch when it was in the velvet bag, without bringing it into the light, for it was necessary to know which one was being dealt with, as they contained different kinds of plates.

Method of Focussing.

The collimation for parallel rays had been effected before leaving England, but this was again verified, and trial photographs were then taken till an excellent focussed spectrum was obtained from D to beyond K in the ultra-violet. To obtain the same conditions as would hold during the eclipse, the collimator was then removed and an attempt made to photograph the spectrum of a star (Arcturus); as perhaps the best way of ensuring good definition from one end of the spectrum to the other is to get the spectrum of a star in focus, for collimation can only be effected over a small region, unless the collimating lens is perfectly achromatic. To make the image of the star to travel in a direction parallel of the edges of the prisms (in order to give breadth of the spectrum) it was necessary to throw the polar-axis slightly out of adjustment and to keep the clock going to star rate by means of the fine adjustment in right ascension ; after several trials a photograph was obtained which showed the spectrum of the star,

Fig. 6.



Dark slide employed in Brazil.

but only between $H\beta$ and $H\gamma$, and this seemed to be fairly well in focus. From this time up to the day of the eclipse there was no possibility of trying this again, so that the focus had to be taken as final. It so happened, however, that the spectrum was in focus between λλ 58 and 44 and again between λλ 37 and 35. The same thing appears to have occurred, only more seriously, when the instrument was previously used during the eclipse of 1883.

Plates Employed.

As stated in Mr. FOWLER's report, there was some uncertainty as to the kind of plate best suited for this work, and for that reason similar brands to those used by him were employed with this instrument, except that of the isochromatic ; those made by EDWARDS were solely used, the other makes proving (in the preliminary trials) not to be as suitable under the climatic conditions. About twelve of the plates were backed with pitch dissolved in benzene during the night preceding the eclipse; the remaining twelve were left untouched, and do not appear to have suffered in any way from the effects of halation.

Exposure of the Plates.

All the operations for exposing the plates were performed by myself, durations being allowed according to a table drawn up by Professor LOCKYER. The plates were successively brought into position with the rack and pinion of the dark slide, and the length of exposure was regulated by means of the shutter in front of the prisms.

About the beginning and end of totality the exposures were short, whilst at mid-eclipse the longest duration was given, followed by one of instantaneous exposure. Also at successive intervals similar exposures were made, as shown by the appended tables:—

No.	Exposures.	No.
5	Instantaneous	9
6	5 seconds	10
7	30 ,,	11

No.	Exposures.	No.
6	5 seconds	15
7	30 ,,	16
8	15 ,,	17
9	Instantaneous	18
10	5 seconds	19

Thus not only are individual plates comparable, by a reason of like exposures, but also a series of them.

The Plates Obtained.

A list of the photographs taken is given in the accompanying table. The first column gives the numbers of the photographs for use in subsequent references, and indicates also the order in which the photographs were taken. The second column indicates the kind of photographic plate, while the third column indicates the amount of exposure, as reckoned by a watch; "instantaneous" means that the shutter was opened and closed again as quickly as possible. The intervals elapsing between the exposures of successive plates are shown in Column 4. These could not be reckoned directly by the watch, but from previous experience and the summation of times of exposure deducted from the total duration they were indirectly estimated. When taken in conjunction with the lengths of exposure, the intervals enable us to form an estimate of the interval from the beginning or end of totality at which any one of the plates was taken. This method of recording the times was adopted because no time-keeper was available. Photograph No. 2 was probably taken within two seconds after the commencement of totality, while the end of totality took place between the exposure of Photographs Nos. 18 and 19.

No.	Kind of plate.	Exposure.	Interval of change.	Remarks.
1	Mawson	Inst.	$\frac{1}{2}$ min.	
2	,,	2 secs.	2 secs.	Commencement of totality
3	,,	8 ,,	2 ,,	
4	,, (Stellar)	Inst.	2 ,,	
5	,, ,,	,,	2 ,,	
6	,,	5 secs.	2 ,,	
7	,,	30 ,,	2 ,,	
8	,,	15 ,,	2 ,,	
9	,, ,,	Inst.	2 ,,	
10	Isochromatic (Edwards)	5 secs.	2 ,,	
11	,, ,,	30 ,,	2 ,,	
12	,, ,,	60 ,,	2 ,,	Middle of eclipse
13	,, ,,	Inst.	2 ,,	
14	,, ,,	30 secs.	2 ,,	
15	,, ,,	5 ,,	2 ,,	
16	Mawson	30 ,,	10 ,,	
17	,,	15 ,,	2 ,,	
18	,, (Stellar)	Inst.	2 ,,	Last photo. in totality
19	,, ,,	5 secs.	2 ,,	After totality
20	,, ,,	Inst.	5 ,,	,, ,,
21	,,	,,	10 ,,	,, ,,
22	,,	8 secs.	20 ,,	,, ,,
23	,,	Inst.	2 ,,	,, ,,
24	,,	,,	..	,, ,,

IV. Description of the African Photographs.

Photographs near Beginning of Totality.

From what has been stated with respect to the conditions of eclipse, it is clear that at the commencement of totality an arc of chromosphere and its associated prominences would be visible in the north-east quadrant. In other parts of the sun's edge, the chromosphere was hidden by the moon, but several prominences were large enough to show their outlying parts, with the result that the photographs exhibit nearly complete rings in the radiations common to chromosphere and prominences. Negative No. 7, taken with an instantaneous exposure about 10 seconds after the commencement of totality, is reproduced in Plate 11; as the photographic plate was not isochromatic, the spectrum only extends to F at the less refrangible end. The principal lines, or rather portions of rings, are obviously due to the H and K radiations of calcium and hydrogen radiations. The bright arc on the right is H_{β}, and the two very prominent rings near the violet end are H and K.

The photograph shows very distinctly the variation of spectrum in passing from one prominence or chromospheric region to another. One small prominence is specially remarkable for its complex spectrum, and there is nearly every gradation between this and prominences which show practically nothing but H and K. The forms of the prominences themselves are very clearly depicted, and it will be seen that the scale is sufficiently large to reduce confusion of images to a minimum. The general appearances on the photograph correspond very closely with those seen at the beginning of totality by RESPIGHI in 1871, and by RANYARD in 1878.

Negative No. 8, taken 2 seconds later, is hardly distinguishable from No. 7. In both plates the corona is represented chiefly by what appears to be continuous spectrum. As was to be expected from the projection consequent on the direction of dispersion, this continuous spectrum is brightest at the eastern and western limbs, and it is brighter on the eastern than the western side, this region of greater intensity corresponding to a lower part of the corona. Besides the continuous spectrum, however, there are a few feeble arcs, not seen in the reproduction, representing true coronal radiations; these are quite distinct from the chromospheric arcs.

Photographs about Mid-eclipse.

The arc of chromosphere, seen in Photographs Nos. 7 and 8, was covered by the moon when the next plate was exposed, but the associated prominences were not completely extinguished by the advancing moon until No. 17 was exposed. The upper part of the chromosphere in the south-western quadrant did not appear until No. 20 was exposed. Hence, Photographs 9–19 inclusive show no chromospheric spectrum, but they give a record of the spectra of the upper parts of numerous prominences, as well as of the spectrum of the corona.

A reproduction of Negative 17, taken on an isochromatic plate, is given in Plate 12. It will be seen that the spectra of the prominences are similar to those in No. 7, but they are simpler for the reason that the lower reaches were covered by the moon. At the extreme right in the photograph there is a feeble image of the bright group of prominences produced by the C radiation of hydrogen; as the plates are scarcely sensitive at all to the red, the image must in reality have been very intense; D_3 images of the prominences are also seen in the photographs.

The spectrum of the corona in Negatives 9–19 is to a large extent continuous, but, in addition, it is represented in some of the photographs by a nearly complete ring corresponding to the 1474 K line, and smaller portions of fainter rings.

The continuous spectrum is brightest near the photosphere, and fades out very gradually at heights depending upon the exposure and development of the plates and the wave-length of the light. The maximum intensity on the ordinary plates is about λ 450, while the isochromatic plates have another maximum about λ 560, and it is at these wave-lengths that the continuous spectrum extends furthest from the photosphere. The greatest extension is in Plates Nos. 13, 15, and 19, where it amounts to two-thirds of the sun's apparent diameter, corresponding to a height above the photosphere of nearly 600,000 miles.

From the points of maximum photographic action the continuous spectrum diminishes in height and intensity in both directions.

The 1474 ring appears on Photographs 9, 16, 17, and 18, the last three being isochromatic; its appearance on Photograph No. 9 indicates its great intensity in the lower corona. It is important to bear in mind that in successive photographs the conditions are different. Thus Photograph 9 was exposed between 18 and 23 seconds after the commencement of totality, so that at the beginning of the exposure the lowest part of the corona, in the north-east quadrant, was only about 3,000 miles from the photosphere, the relative movement of the moon when projected on the sun being about 166 miles per second. At the beginning of the exposure of Photograph 16, on the other hand, the lowest part of the corona photographed in the north-east quadrant was nearly 24,000 miles above the photosphere, and that in the south-west quadrant about 18,000 miles; in this photograph the 1474 ring is consequently brightest in the south-west quadrant. In the case of Photograph No. 17 the conditions at the beginning of the exposure were almost identical with those at the end of No. 16, but in this case the exposure was 40 seconds; at the middle of the exposure the moon's limb would be about 27,000 miles above the photosphere in the north-east quadrant, and in the south-west nearly 15,000 miles.

At the middle of the exposure of Photograph No. 18, the corresponding figures were about 31,000 and 11,000 miles.

The greatest height of the 1474 K ring occurs in No. 17, taken with a long exposure; in its brightest part it extends about 45″ above the moon's limb, corresponding to a distance of about 20,000 miles. Its intensity varies greatly, being scarcely visible in the region of the sun's poles, and brightest in the equatorial regions; the brightest parts of the ring correspond with the brightest regions in the photographs of the corona taken with the coronagraph. The ring is quite distinct from the prominence rings and is not intensified in the region of prominences, indeed there seems to be no trace of 1474 in the spectrum of any of the prominences.

In Negative No. 17, Plate 12, the 1474 ring is at the right of the photograph, just within the very broad ill-defined ring due, as shown elsewhere, to photographic causes.

Like 1474 K, the fainter coronal rings are as readily distinguished from chromosphere and prominence rings, as the latter are from the corona in an ordinary photograph, as their maximum luminosity never coincides with them; while the monochromatic images of chromosphere and prominences are sharply outlined, the true coronal rings are not thus defined. The intensities of the coronal radiations appear to diminish very rapidly in passing outwards from the sun's limb. The distribution of these rings in the African Negative No. 17 is shown by the dotted arcs in Plate 12, it being quite impossible to reproduce the rings themselves in photographic enlargements in consequence of their dimness.

Photographs taken near the end of Totality.

Two photographs, Nos. 20 and 21, were taken during the visibility of the chromosphere in the south-west quadrant just before the end of totality. No. 21, taken with an instantaneous exposure about 10 seconds before the end of totality, is reproduced in Plate 11, and it will be seen that the appearances are very similar to those photographed near the beginning of totality. The spectrum of a long arc of chromosphere and of numerous prominences are very strongly marked, and the full dimensions of the large group of prominences near the sun's south pole are at the same time revealed. This group was large enough to be visible throughout the whole of totality, but others are only visible in the photographs near the beginning and end. The spectra of the prominences are again of various degrees of complexity.

The corona is represented chiefly by continuous spectrum, but faint arcs representing coronal radiations can also be distinguished in the negatives; one, a little less refrangible than H, is sufficiently bright and defined to be seen in the reproduction of Photograph No. 21. The continuous spectrum of the corona in these photographs is brightest on the western (lower) edge, for the reason that at that part of the sun's limb the moon's edge was nearer to the photosphere than at the opposite limb, so that a brighter part of the corona was exposed.

The coronal ring corresponding to 1474 K is not seen in either No. 20 or 21, as the plates were not isochromatic.

Unfortunately, the attempt to make an exposure at the moment of third contact was not successful, for the reason already stated.

Photographs taken out of Totality.

Six photographs (Nos. 1–6) were taken before totality, and nine (Nos. 22–30) after totality.

The first five show nothing but a curved Fraunhofer spectrum, the visible crescent acting as a curved slit. No. 6 shows a similar spectrum with the addition of five short bright arcs at the cusp nearest to the north point in continuation of the dark arcs due to H_β, H_γ, H_δ, and H and K; it also shows faint impressions in H and K radiations of the prominences in the region of the sun's south pole. Photographs 27 to 30 inclusive also give nothing but a Fraunhofer spectrum, the curves being diametrically opposed to those appearing in the photographs taken before totality. Photographs Nos. 25 and 26 show in addition bright projecting arcs at H and K, H_γ, H_δ, and H_ζ.

The most valuable photographs taken out of totality are Nos. 22, 23, and 24, all taken within about 12 seconds from the end of totality. During this interval the photospheric light was sufficiently reduced to exhibit the radiation spectra of the surrounding vapours.

The visible crescent of photosphere when these three photographs were taken was in the south-west quadrant, and as the dispersion was in the north and south direction, the cusp nearest to the south point was the one most favourably situated for showing bright arcs. The light of the photosphere is drawn out into a continuous band by the action of the prism, and this band is of varying intensity in consequence of the irregularities of the moon's limb, the so-called BAILY's beads being drawn out into longitudinal streaks, as was seen in a photograph taken out of totality in 1882.

In Photograph No. 22 this photospheric spectrum is crossed by bright arcs which also project some distance beyond in both directions, and no Fraunhofer lines are visible; the bright arcs corresponding to H_β, H_γ, H_δ, H, and K, H_ζ, and other ultra-violet lines of hydrogen are especially intense. The group of prominences near the south pole is also depicted in these radiations, as well as in H_α and D_3. A very large number of short bright arcs is also seen at the cusp, but these cannot be traced across the spectrum of the photosphere.

An attempt to indicate the appearances of different typical arcs is made in fig. 7, as the detail is too minute to be satisfactorily shown in a photographic reproduction. The brighter ones show great irregularities throughout their lengths, and the fainter

Fig. 7.

Appearances of bright arcs in spectrum of cusp.

ones appear only in the parts corresponding to the brightest regions. None of the arcs are brightest close to the edge of the continuous spectrum, and the faintest of them have the appearance of being detached as in the case of Fe 4072 in the diagram. In the Photograph No. 22 the spectrum of the corona is also seen; as a faint continuous one at the eastern and western edges on the illuminated sky background, and an arc corresponding to 1474 K is visible in the south-western part. A portion of this photograph is given in Plate 13. It cannot be satisfactorily reproduced in its entirety owing to the great density of the negative. The general appearances of Photographs 23 and 24 (see Plate 13) are very similar to that of No. 22, but there are several important differences in detail. The photospheric spectrum, which is of increased width in each case, shows the longitudinal dark streaks due to BAILY's beads, and they both show bright chromospheric arcs crossing the continuous spectrum, as well as bright arcs at the cusp. In No. 23, however, distinct traces of the strongest FRAUNHOFER lines are seen, especially in the ultra-violet, and they are still more plainly marked in No. 24. The bright arcs at H and K, H_γ, &c., are not quite coincident with the corresponding dark ones, but lie on the outer edges, as might be expected. The number of bright arcs at the cusp in No. 23 is very much smaller than in No. 22, and in No. 24 it is smaller still. The continuous spectrum of the corona is faintly impressed at the eastern edge, but 1474 K and other coronal radiations are not seen.

In passing outwards from totality, the bright lines at the cusp become reduced in number and intensity, while the Fraunhofer spectrum becomes more and more distinct (see Photograph No. 25, Plate 13)* up to a certain limit; after which the dark arcs become broad and ill defined.

V. DESCRIPTION OF THE BRAZILIAN PHOTOGRAPHS.

Photographs near Beginning of Totality.

Photograph No. 2, which is reproduced in Plate 12, is as near as possible the commencement of totality. It shows rings of chromosphere and prominences in positions corresponding to H and K, H_β, H_γ, H_δ, H_ζ. Near the top of the spectrum is a strip much brighter than the other parts of the spectrum; it corresponds with the central part of the chromospheric arc visible when the plate was exposed. Projecting out of this strip are many bright arcs, less refrangible than F (H_β) and more refrangible than K; between these two wave-lengths, however, in consequence of the extreme brightness of the chromosphere the photograph is considerably over-exposed, and the bright arcs are masked. As the moon advanced and covered the chromosphere, other photographs were taken which show that it was obscured at the fifth photograph; hence, knowing the interval of time between these two photographs (from table = 25 seconds) and knowing the relative motion of the moon ($\cdot 253''$ per second), this gives 2,850 miles as the depth of the chromosphere, a very close agreement with that calculated from the African photographs. Photographs 3, 4, and 5 are very similar to the corresponding African photographs, but they do not show so much detail.

The spectrum of the corona is represented in Photographs 2, 3, 4, and 5, by an apparently continuous spectrum, which by reason of the projection of the broad ring is brightest along the top edge of the spectrum. The maximum intensity is about λ 448, and at this point the continuous spectrum can be traced to a distance of $6'\cdot 5$ above the moon's limb. At the bottom edge of the spectrum the intensity is

* It will be observed that this photograph is solarised, owing to over-exposure.

less, because the lower part of the corona was eclipsed when these photographs were taken.

Photographs about Mid-Eclipse.

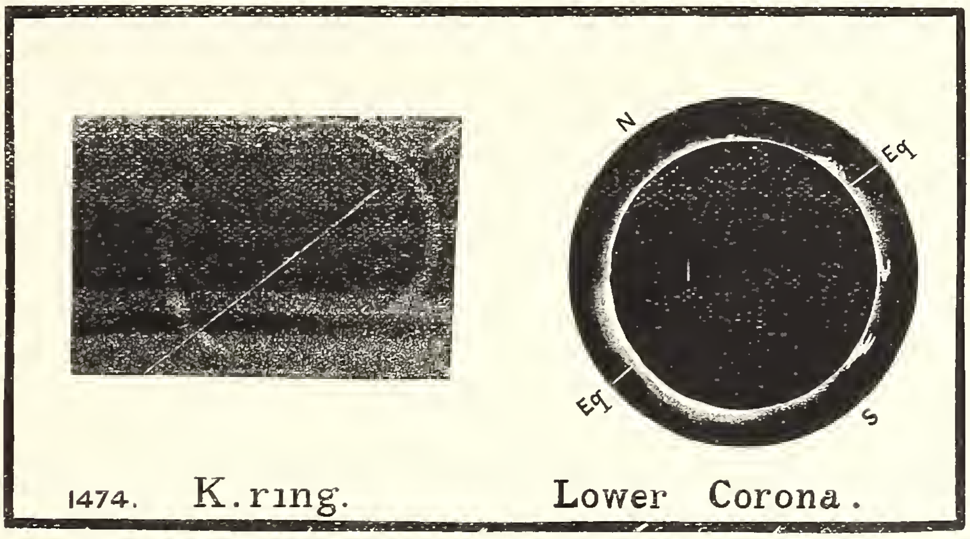
As in the case of the African series there are several of the photographs, Nos. 6 to 16, which show no chromosphere spectrum, although the spectrum of some of the larger prominences are present. Only the upper parts of such prominences are visible, and their spectra are very simple, consisting of hydrogen, helium, and the H and K radiations of calcium.

The continuous spectrum of the corona in these photographs is very similar in appearance to that described in the earlier photographs, except that in those of long exposure it extends to the considerable height of 12′ above the moon's edge.

By reason of the short focal length of the object glass used in Brazil the images of the coronal rings would be six and a quarter times brighter than those given with the African instrument, and for the same reason, together with the less dispersion, the continuous spectrum is five and a quarter times brighter.

Consequently, as will be seen on reference to the Brazilian negatives taken in mid-eclipse (Plate 12), they are remarkable for the great intensity and definition of the 1474 K-ring, which from measurements extend to a height of 2′ above the moon's edge. An enlargement of the ring as it appears in the Negative No. 12, is given in fig. 8, where it is placed alongside a reduced copy of a photograph taken

Fig. 8.



Comparison of the 1474 K-ring with the lower corona.

by SCHAEBERLE in Chili. The latter has been selected because the exposure was relatively short enough to make the lower corona thus obtained comparable with the spectrum ring at 1474 K. It will be seen that the prismatic camera has picked out the brightest parts of the corona, and where it is strongest the spectrum ring and the continuous spectrum at these points is most intense, whilst a prominence occurring at any part of the sun's limb does not appear to alter the intensity of the coronal ring at the corresponding part. Besides this principal ring we should expect to see some of the fainter ones shown on the African negatives, if their presence was not masked by the intense continuous spectrum, but there are indications of others in less actinic parts of the spectrum, which are not shown on the African plates.

Photographs about End of Totality.

The chromosphere begins to put in appearance again, but only feebly, in Photograph 16; between this and the next plate a change of slides was made, and a considerable time elapsed. In consequence of this rather large interval between the 16th and 17th exposures it is slightly difficult to estimate more than approximately the length of time the chromosphere was visible, but probably corresponds very nearly to that at the beginning of the eclipse. Photograph No. 18 was the last exposed during totality; the sun reappearing between exposing this plate and No. 19, the spectrum of the chromosphere is shown by long arcs corresponding to H_β, H_γ, H_δ, H_ζ, H_η, H_θ, some of the principal Ca and Sr lines and some He lines. The spectrum of the corona is represented by the continuous spectrum, intensified in the two strong bands at the top and bottom of the spectrum for reasons explained previously.

Photographs Out of Totality.

The first photo. out of totality was No. 1: it was given an instantaneous exposure, and a narrow band of continuous spectrum extends from one end of the plate to the other; besides this there seems to be no indications of any arcs from the prominences except those corresponding to H_β, H_γ, H_δ, H_ϵ, and K.

In this respect it is very similar to Photograph No. 22, taken at the African station.

The first photo. after totality, No. 19, was exposed for 5 seconds, probably commencing about two seconds after the sun broke out. During this time the sun was sufficiently uncovered to present a thin crescent, and the photograph shows a Fraunhofer spectrum with arcs in place of the lines observed in the ordinary way.

In consequence of the long exposure the Fraunhofer spectrum is solarised, and outside this the plate is much fogged with stray light, which was then becoming strong, but here and there, where a prominence was unusually high to show beyond the edge of the moon, and bright enough to be seen above the stray light, are a few dots from the top of a prominence, in positions corresponding to the hydrogen lines. The remaining five plates are very similar to the one described above, except that as the crescent of the sun became broader, the Fraunhofer spectrum is more ill defined.

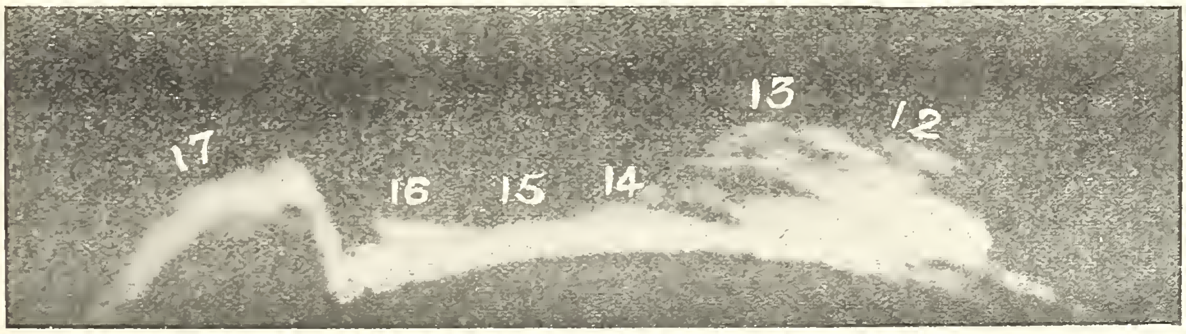
VI. Forms of the Prominences.

Monochromatic Images.

The results obtained clearly indicate that the best possible way of photographing the prominences during a total eclipse, so as to show their forms, is to secure monochromatic images of them by means of the prismatic camera. In the pictures taken with the coronagraph in the ordinary way the prominences are apt to be submerged in the intense light of the lower corona. In some of the African photographs taken with the prismatic camera the definition is very fine, and the structure can be minutely investigated by enlarging the photographs. The group of prominences near the sun's south pole shows a wealth of detail in most of the African negatives, as will be clear from the enlargement of the K image, which is reproduced in fig. 9.

By isolating the portions of the rings of prominences and chromosphere which correspond to any particular radiation, the distribution of the corresponding vapour can be represented at any phase of the eclipse; a composite photograph made up of two such isolated portions of rings, one as seen at the beginning and the other at the end of totality, indicates the distribution of the vapour throughout the chromosphere and prominences visible during the whole of totality. Fig. 10 is a copy of the K-ring, as built up from the African photographs Nos. 7 and 21, the light due to other radiations having been painted out.

Fig. 9.

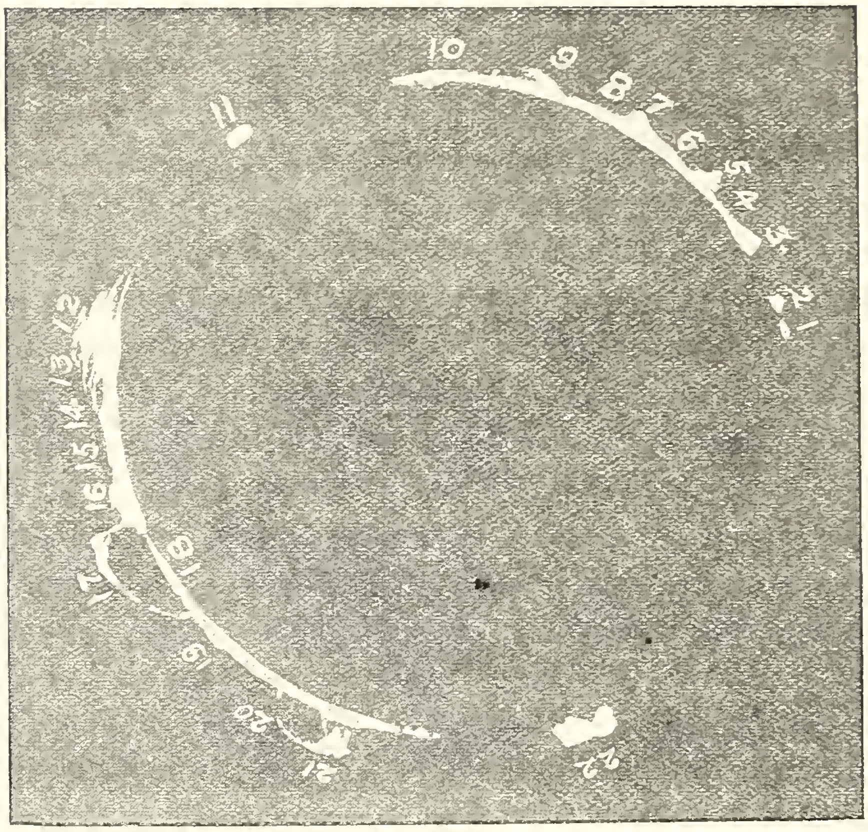


Group of prominences near sun's south pole, April 16, 1893.

The ring is not complete for the reason that some parts of the chromosphere—namely, those in the neighbourhood of the points on the sun's limb which are cut by a line perpendicular to the direction of the moon's motion—were covered by the moon during the whole of totality.

The various parts of the chromosphere and prominences have been numbered, as shown in the above diagram, for purposes of reference in the discussion of the photographs.

Fig. 10.



Chromosphere and prominences depicted in the K radiation of calcium.

Forms in Different Radiations.

Rings corresponding to various radiations can be isolated in the way indicated, and compared with each other. This has been done for H_β and, so far as it goes, it resembles the K ring, but some of the prominences depicted by K are not shown in H_β light. In the case of the metallic prominences, the images formed by the hydrogen radiations very closely resemble those formed by H and K, but they do not reach to so great a height. In the case of large prominences, such as that near the south pole, the hydrogen images are feeble and show only very small parts of those depicted in H and K light, although H and K are quite as intense as in the metallic prominences.

The images produced by the radiations of helium are similar to those given by hydrogen, but they are smaller and less intense.

Effects of Movement.

The forms of monochromatic images of the prominences may be produced in part by the movement in the line of sight of the vapours which give rise to them. Regions in which the vapours are approaching the earth will be displaced to the more refrangible side of their true positions with respect to the sun's limb, and in the case of receding vapours there would be displacements towards the less refrangible end. Such distortions can be determined, if they exist, by comparing the monochromatic images with those photographed at the same time with the coronagraph. For this purpose a photograph of the eclipsed sun was enlarged to exactly the same size as the K ring shown in fig. 10, and the comparison could be made very exactly by fitting a negative of one to a positive of the other. No differences of form, however, could be detected, so that the velocities in the line of sight must have been comparatively small. Movements across the line of sight, that is normal to the photosphere, would not affect the forms of the monochromatic images of the prominences.

PART II.—DISCUSSION OF THE OBSERVATIONS.

VII. The Interpretation of the Photographs.

Having now given a description of the phenomena actually photographed, I proceed to consider how we are to justly interpret them. For this purpose it is necessary first to deal generally with the results as contrasted with those given by slit-spectroscopes, in which an object glass is employed to form an image of the eclipsed sun upon the slit, and then to consider the phenomena which might be expected under the most probable conditions of solar structure.

Comparison of the results to be expected from Slit and Slitless Spectroscopes.

The considerations which led me, in 1871, to employ a spectroscope without collimator may be briefly stated. If in an ordinary spectroscope, the straight slit be replaced by a circular one, bright rings replace the bright lines which are ordinarily seen in radiation spectra, and since in the solar surroundings we have chiefly to deal with radiation phenomena, the chromosphere and corona themselves can be used during an eclipse as ring slits, and on account of their distance, a collimating lens can be dispensed with.

In the report on the eclipse of 1875, by Dr. Schuster and myself, the principles of the method, as applying to photographs taken during totality, were stated as follows:*

"Supposing that the corona and chromosphere only send out the same homogeneous light, one image only will appear on the sensitive plate, the only effect of the prism being to displace the image. As far as the protuberances are concerned we know they give a spectrum of bright lines, and we expect, therefore, to find on the plate each protuberance represented as many times as it contains lines in the photographic region. The different protuberances would be arranged in a circle round the sun, and

* 'Phil. Trans.,' 1878, Part 1, p. 139.

these circles would overlap or not, according to the dispersive power of the prism and the difference in refrangibility of the lines. . . . If the corona gives a series of bright lines we shall find a series of outlines on the photographs similar to that corresponding to the protuberances. . . . If we find that part of the corona gives a continuous spectrum, that part alone will be drawn out into a band."

To this it may be added, that successive photographs will differ on account of the difference of phase. One part of the chromosphere will be visible at the beginning of totality, and another part at the end. The smaller prominences visible at the beginning of totality are subsequently eclipsed by the moon, and their spectra are consequently absent from later photographs, while a new prominence region makes its appearance. In the same way, the part of the corona the spectrum of which is photographed will vary at different phases, but only in the lower parts.

Corona.

For the discussion of the advantages of the different methods of work in the case of the corona, it is necessary to consider the possible sources of spectra which are to be found in the neighbourhood of the corona. From previous experience, the chief sources may be stated as follows :—

(*a*) Intrinsic light of the corona, giving the so-called continuous spectrum.

(*b*) Intrinsic light of the corona, giving bright lines.

(*c*) Light of the sun reflected by the solid or liquid particles in the corona.

(*d*) Light scattered by the particles in our own atmosphere, giving frequently the lines of the chromosphere and prominences.

It is evident that a slit spectroscope must integrate all these spectra, so that in discussing any particular line it is very difficult to know to which origin it should be ascribed. For example, if we suppose the corona to give a spectrum of hydrogen, the lines will be superposed upon lines of hydrogen due to the light of the prominences scattered by our atmosphere, and it would not be safe to draw any conclusion as to the presence or absence of hydrogen in the corona.

The advantages of the slit spectroscope in regard to the corona may be stated as follows :—

(1.) If the spectrum of the corona consists of a large number of lines of nearly equal intensity, the slit spectroscope will show them more clearly than the prismatic camera, for the reason that, with the latter instrument, the overlapping of the rings would have a greater tendency to give the spectrum the appearance of being continuous. With a wide slit this advantage of the slit spectroscope would be diminished.

(2.) Feeble corona lines have a greater chance to show themselves with the slit spectroscope, since it only takes account of a very small area giving continuous spectrum, while, in the prismatic camera, the continuous spectrum from adjacent parts is superposed, with the result that there is a greater tendency to masking of bright lines. The use of greater dispersion in the case of the prismatic camera would tend to remove its deficiency in this respect.

(3.) The slit spectroscope will give well-defined spectra, due to the solar light reflected by the corona, as observed by JANSSEN in 1871, while, in the prismatic camera, the corresponding dark rings will be too diffuse to show themselves, for the reason that they come from so large an area.

Chromosphere and Prominences.

The advantages of the slitless spectroscope, in the case of the chromosphere and prominences, may be summarized as follows :—

(1.) Production of actual pictures. Unlike the slit spectroscope, it does not give the spectrum of one section only of the corona and prominences, but combines the functions of a telescope with those of a spectroscope, and gives actual views of the whole of the solar surroundings in each radiation strong enough to produce an image. Any chemical differences which there may be between different regions will be shown by the limitation of some of the spectral features to certain segments of the rings.

(2.) Prominences will be localised by the prismatic camera, so that their separation from the normal corona spectrum will be greatly facilitated. The superposition of such a spectrum upon those due to other sources, when a slit spectroscope is employed, prevents their recognition as local phenomena.

(3.) Elimination of the light from the prominences and corona scattered by the dust particles in our air. This light, though producing false lines in the spectrum of the corona, and even across the dark body of the moon itself when a slit spectroscope is employed, cannot by any possibility produce more than a general illumination of the field when viewed through a prismatic camera. Images of the corona can only be depicted by true coronal radiations, or by radiations reflected by the corona itself from the lower parts of the sun's atmosphere, if such reflected radiations be possible.

(4.) There is a great saving of light due to the absence of condensing lens, collimating lens, and slit, so that photographs may be obtained with shorter exposures, and, therefore, at a greater number of stages of the eclipse.

Comparison of Methods in the Case of Atmospheric Layers.

In a Paper communicated to the Society in 1882,* I pointed out the importance of considering the conditions under which we observe the phenomena of the sun's atmosphere. It was shown that whether the sun's atmosphere be composed of concentric layers of different composition, or of vapours, all of which rest on the photosphere and thin out at different heights, the phenomena observed with a slit spectroscope

* 'Roy. Soc. Proc.,' vol. 34, p. 292.

will, *in the main*, be the same in both cases, for the reason that we have to deal with the projection of a sphere and not with a section. The only criterion is that if the vapours rest on the photosphere, the lines will thicken towards the base, whereas, in the case of a separate higher layer they would not widen or brighten towards the base, but really be thickest at the top, if we do not take account of effects of temperature. Taking temperature into consideration, as the lines will be less bright as the distance from the sun is increased, and, therefore, the temperature is reduced, the lines produced by the higher layers will be of equal brightness throughout, and dimmer than the others. Accordingly the slit spectroscope can only give information as to the distribution of vapours in the sun's absorbing atmosphere by means of these very delicate observations of brightness, or widening of the bright lines observed.

Somewhat similar difficulties are met with in the case of the prismatic camera, when we attempt to distinguish between the two kinds of layers. First, consider the effects during totality. In the case of a vapour extending down to the photosphere, we should obtain spectrum rings decreasing in intensity as we pass outwards from the moon's limb (this is exemplified by the ring obtained in the case of 1474 K, to which reference will be made later, and the same appearance is seen in the case of the other coronal rings). The apparent internal diameters of such rings upon the photographs would be equal in every case, but their heights would depend upon the intensities of the radiations producing them. There should be no difficulty in detecting such rings unless they are of very nearly equal brightness, and very numerous, or unless they do not extend far enough to be visible beyond the moon.

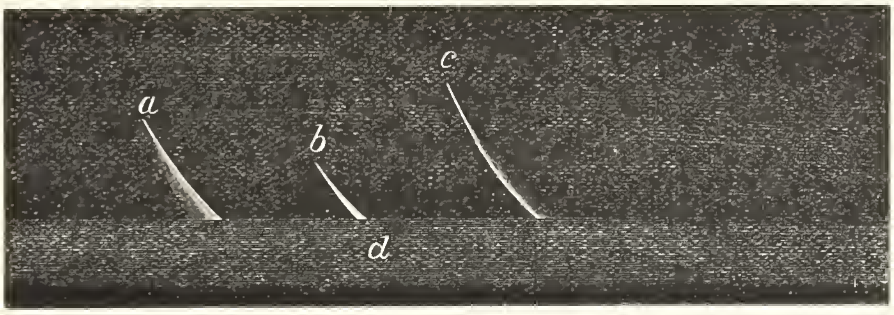
There is a very definite way in which the photographs taken with the prismatic camera may indicate the presence of layers of vapours concentric with the photosphere, but not reaching down to it. At a certain height above the photosphere, the chromosphere spectrum in a photograph of the chromosphere visible at any one instant beyond the edge of the moon, will show arcs with certain relative intensities. As the moon advances and gradually uncovers the base of the chromosphere, the same arcs will remain visible, but those produced by a layer which does not extend lower down will be reduced in intensity as compared with arcs produced by vapours which do reach lower down ; the latter will continue to get brighter, while the others remain of the same absolute intensity. As the lowest part of the chromosphere is shown in the photographs taken immediately after totality, or exactly at the end, it is only necessary to compare the relative intensities of the arcs in different photographs in order to investigate the general question as to the existence of layers. A large number of photographs taken in rapid succession with instantaneous exposures during the visibility of the chromosphere spectrum, either near the beginning or end of totality, would further enable us to determine the order of such layers in proceeding outwards from the photosphere.

Another way in which the prismatic camera may possibly help to determine the presence of layers is as follows. A layer concentric with, but separated from, the photosphere, will be shown in each of its radiations which is bright enough to be photographed, as a ring brightest at the outer edge, and dimming very rapidly towards the photosphere in consequence of its greater thickness in the line of sight near the tangent. If only the outer part were bright enough to be photographed, the ring would have a diameter greater than that of the moon. Layers considerably removed from the photosphere would be cool and dim, and their feeble images would tend to be lost in the general continuous spectrum. The diameters of the rings being different, true wave-lengths could not be assigned, and the superposition of a great number of them would give the appearance of a nearly continuous spectrum, in the part lying within the band equal in breadth to the moon's diameter. Outside this band, where the rings would be best visible, above and below the moon, on account of their different diameters, the direction of dispersion is such as to cause the greatest overlapping of images, and the consequent confusion would make the spectrum look nearly continuous, in the case of a considerable number of images. Hence there will be some difficulty in detecting the effects due to a very large number of concentric separate layers when the prismatic camera is employed during totality.

The conditions with regard to layers in the photographs taken out of totality with the prismatic camera, are somewhat as follows :—

If the vapours extend from the photosphere outwards, and are brightest at the base, the arcs due to them which appear at the cusps, provided they are bright

Fig. 11.



Possible appearances of bright arcs at cusp in photographs taken out of totality.

(*a*) Arc due to vapour extending from photosphere outwards, with gradually diminishing brightness.

(*b*) Arc due to a thin layer close to photosphere and equally bright throughout.

(*c*) Arc due to a shell of vapour, concentric with photosphere, but some distance from it.

(*d*) Continuous spectrum of photosphere.

enough to show themselves in the general illumination of the field will have a somewhat triangular appearance, with the maximum brightness nearest to the cusp as shown at *a*, in fig. 11.

The intensity of such an arc will gradually diminish in all directions from the cusp and its extension will depend upon intrinsic brightness. All the arcs will have the same internal radius.

A thin layer of vapour of equal brightness throughout, and resting on the photosphere, will give short arcs having the appearance shown at *b* in fig. 11 ; as we are observing a spherical shell, the appearance would not be different from that at *a*, but the edges of the arc might be expected to be more sharply marked.

In the case of a concentric shell some distance removed from the photosphere, its representative in the spectrum of the cusp will be a relatively long arc as at *c* in fig. 11, brightest near its outer edge, and suddenly dimming for the reason above stated.

Different layers of this kind will give arcs struck with different external radii. The arcs due to all but the brightest layers, however, will be lost in the general illumination of the field, and since the brightest vapours will be near to the photosphere, the differences of radii will not be very great, and as we are dealing with short arcs, these differences of radii will not help us to distinguish the outer shells in this way.

This suggests another consideration which may perhaps help us eventually, when higher dispersions are employed to distinguish concentric shells, if they exist. As the arcs due to them are brightest on the outside, they will appear to occupy places in the spectrum which do not correspond to their true wave-lengths. At the same time they will be longer, so that it is in the case of long arcs that we might expect to find departures from the true wave-lengths; if we found, for instance, three lines near iron lines with a constant difference from the true wave-lengths of the iron lines, we should be justified in regarding such lines as iron lines displaced in the manner indicated.

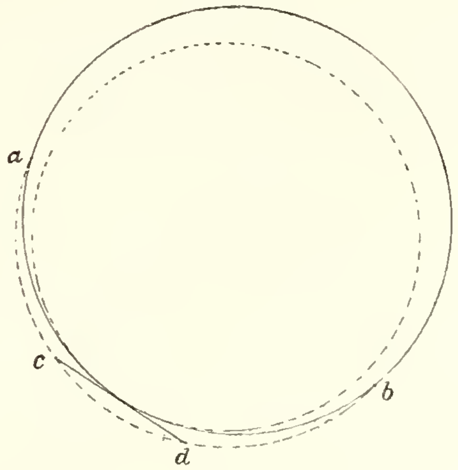
Comparison of the Methods in the case of Vapours close to the Photosphere.

The prismatic camera offers special advantages for the investigation of the vapours which lie nearest to the photosphere.

(1.) The spectrum of even a very shallow layer will be represented by arcs of considerable length in a photograph taken with the prismatic camera at the beginning or end of totality, the length of the arc corresponding to the whole of the layer uncovered by the moon at the time. With a slit spectroscope the lines will be relatively much shorter, even assuming that the slit can be placed exactly at a tangent to the moon's limb. This is shown in fig. 12, in which the continuous circle represents the moon, and the dotted one the disc of the sun. A layer of vapour resting on the photosphere is represented by the outer portion of a dotted circle. In the prismatic camera this layer would be represented by an arc of length *ab*, while, with the slit spectroscope, under the most favourable conditions, it will only be represented by a line of length *cd*.

No special adjustment of the instrument is necessary to enable such photographs to be obtained, whereas, with the slit spectroscope, the most accurate adjustment would be necessary.

Fig. 12.



Illustrating comparative lengths of arcs and lines photographed with prismatic camera and slit-spectroscope respectively, at beginning or end of totality.

(2.) The radiations of such vapours may be studied by means of photographs taken shortly before and after totality, for which no readjustment of the instrument is required. The bright arcs at the cusps are of greater length than the lines in similar photographs, which might be taken with slit spectroscopes.

Interpretation of Photographs taken during Totality.

The photographs taken during totality show the spectrum of the chromosphere and prominences as they appear at different heights above the photosphere.

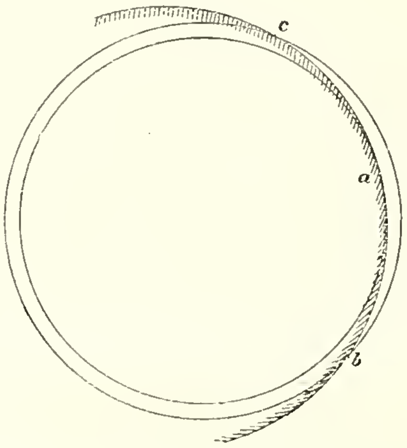
Thus, the metallic prominence shown in the upper right-hand quadrant in Photograph 7, Plate 11, appears also in Photographs 8, 9, 10, 11, the lower parts being gradually eclipsed, so that the last only shows the spectrum of the tip of the prominence.

In the case of the spectrum of the chromosphere, different parts of the arcs photographed correspond to the spectrum at different distances above the photosphere. Thus, at a position angle corresponding to the point of contact at the beginning or end of totality, the edge of the moon will reveal the chromosphere to a greater depth than at adjacent parts, as shown in fig. 13. If the inner circle represents the boundary of the photosphere, and the circle concentric with it represents the chromosphere, the edge of the moon, at the moment of contact, will be in some such position as *c a b*. At the point *a* we should at that moment get the spectrum of the base of the chromosphere, while at *b* and *c* we should only get the spectrum of the higher reaches.

In case the chromosphere consisted of concentric shells of vapour, the spectrum seen at the point *a* would be the integration of the spectra of all the shells of vapour, but at *c* and *b* only the outer shells would be effective in producing a spectrum.

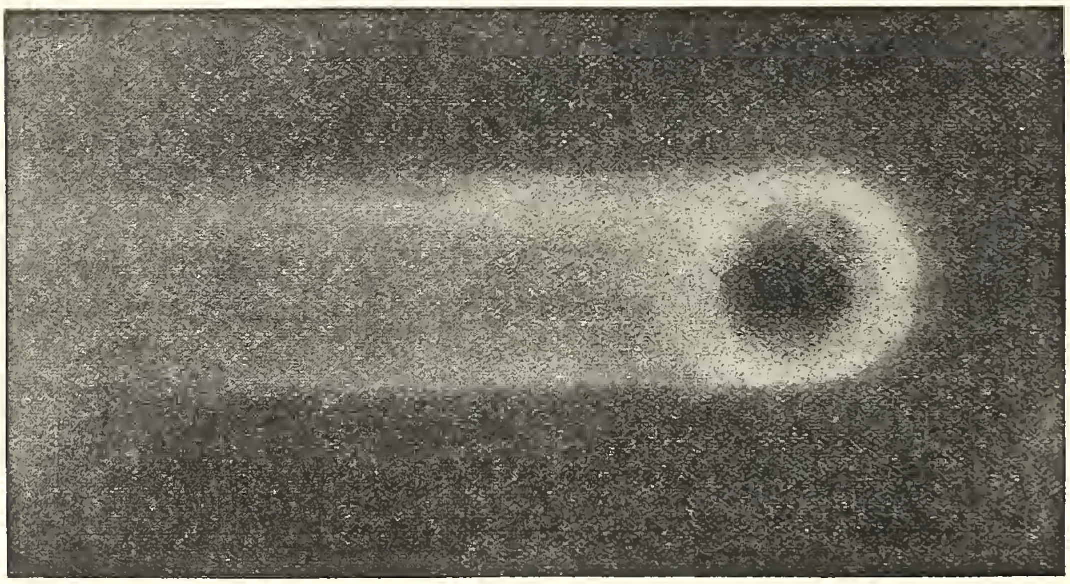
A point which it is important to bear in mind, when attempting to interpret the photographs taken during totality, is the production of rings by a purely continuous spectrum, when a slitless spectroscope is employed. Attention was drawn to this in the preliminary report,* and it was stated that the broad ill-defined ring, a little more refrangible than D_3 in some of the African photographs, as well as a less conspicuous one in the blue, and possibly even another in the violet, has its origin in the *continuous*

Fig. 13.



Illustrating different depths of chromosphere photographed with one exposure.

Fig. 14.



Appearance of continuous spectrum photographed on an isochromatic plate with a ring slit.

spectrum of the corona acting on a plate, which has one maximum of sensitiveness for the yellow rays and one or more maxima in the blue and violet. Experiments with ring slits, illuminated by a source of light giving a continuous spectrum, have confirmed this explanation. The photograph reproduced in fig. 14 was taken on an isochromatic plate with the 6-inch prismatic camera, temporarily provided with a $3\frac{1}{2}$-inch collimator in which the slit was replaced by a positive picture of the corona. A bull's-eye lamp

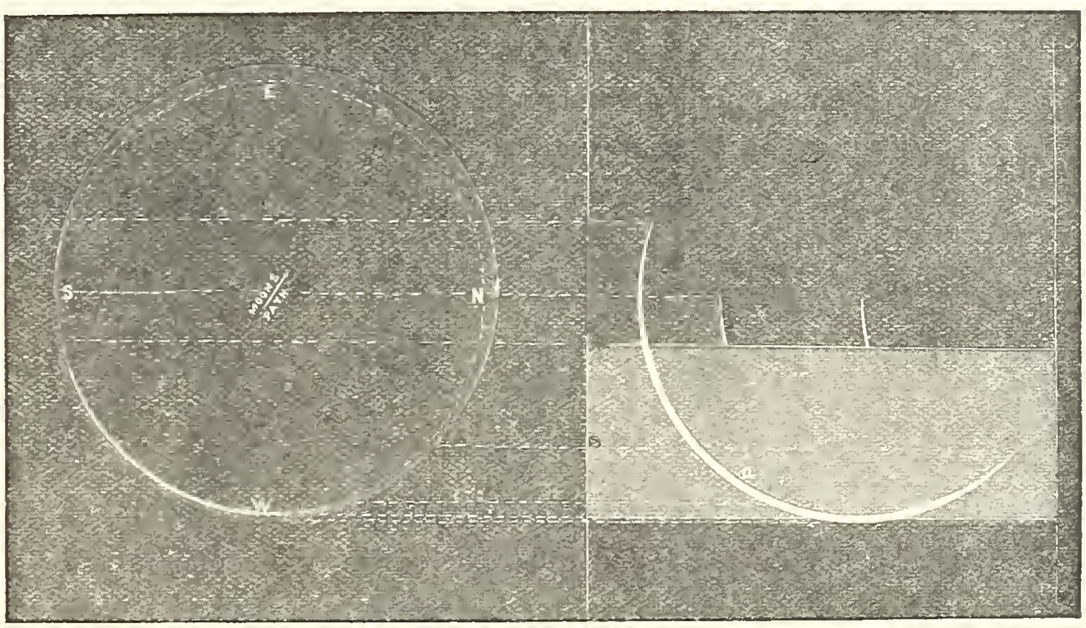
* 'Phil. Trans.,' 1894, vol. 185 A, p. 714.

was employed as a source of light, and it will be seen that although there was no suspicion of anything but continuous radiations, an appearance of rings was produced exactly resembling those to which reference has been made as taken during the eclipse.

Interpretation of the Photographs taken out of Totality.

The photographs taken out of totality greatly increase the chances of obtaining a record of the spectra of the vapours near the photosphere. It is evident that shallow strata can only be visible for a very short time after the beginning and before the end of totality, but they will be visible at the cusps out of totality, so long as the general illumination of the sky by the uneclipsed part of the photosphere is insufficient to mask them.

Fig. 15.



General explanation of photographs taken out of totality.

Fig. 15 will facilitate a general explanation of the photographs taken out of totality. The part of the diagram to the left represents the relative positions of the sun and moon at the African station about a minute after totality, there being a thin crescent of photosphere then visible in the south-west quadrant. Two imaginary layers of vapour are drawn round the sun. The direction of dispersion being north and south, the spectrum is drawn out in the direction indicated by the continuous spectrum in the part of the diagram on the right. It is evident that the radiations of the outer layer of vapour will be represented by long arcs in the spectrum of the cusp, while those proceeding from the lower vapour will be represented by shorter ones.

It may be remarked also that the lengths of the arcs at the cusps for any particular layer of vapour will depend upon the interval from totality at which the photograph is taken; even a shallow layer will be represented by long spectrum arcs at the moments of 2nd and 3rd Contacts, but the lengths of these will be gradually diminished in consequence of the less oblique section of them which is made by the moon as totality is departed from.

If the vapours are evenly distributed in the layers the arcs at the cusp will be geometrically regular, but if they are disturbed, as we have reason to believe they may be, the forms of the arcs may not be regular, and they may be of unequal brightness in different parts.

Although irregularities of the vapours may be distributed all along the sun's limb, the conditions under which they are presented to us will vary with the distance from the cusp, just as in the case of uniform layers. A knot near the cusp would be fully revealed to us, and the prismatic camera would give us the spectrum of all its parts. A knot a little removed from the cusp, however, will have its base covered by the moon's edge, and only the spectrum of its higher parts will be seen. Taking the spectrum generally, it will have four origins :—(1) the spectrum of the vapours of the cusps, giving bright arcs ; (2) the spectrum of the visible crescent of photosphere, consisting of a bright continuous spectrum crossed by dark arcs ; (3) the spectrum of the vapours surrounding the uneclipsed part of the sun, giving bright arcs ; (4) the spectrum of the corona, consisting of a relatively feeble continuous spectrum and a number of bright rings.

VIII. The Spectrum of the Corona.

List of Coronal Radiations.

Except in the case of 1474 K, the coronal rings are very feeble, and their wave-lengths can, perhaps, not be depended upon to within one-tenth metre. They were read off from the African negatives by direct comparison with the spectrum of Arcturus. Possibly, owing to the great intensity of the continuous spectrum, the Brazilian negatives show only 1474 K in the spectrum of the corona.

The accompanying table indicates the wave-length of the coronal radiations, and the number of the African photographs in which they have been detected. It is almost impossible to form any trustworthy estimate of the relative intensities of the rings, but it may be noted that the one at wave-length 3987 comes next to 1474 K in order of brightness

There are indications of other extremely faint rings, the positions of which cannot be determined with the necessary degree of accuracy to enable a useful statement to be made touching their wave-lengths.

Wave-length.	Numbers of the African photographs.								
	VII.	VIII.	IX.	XII.	XVI.	XVII.	XVIII.	XXI.	XXII.
3987	..	×	..	×	×	×	×	×	
4086	..	×	..	×	×	×			
4217	..	..	..	×	×				
4231	×	×	..	×	×	×	×		
4240	..	..	..	×					
4280	..	..	..	×	×				
4486	..	..	..	×	..	×	×		
5316·9	..	..	×	..	×	×	×	..	×

Comparison with Results Obtained by Slit Spectroscope.

Coronal rings, other than those due to 1474 K, or hydrogen, have not previously been recorded by the prismatic camera, though some of the lines corresponding to them appear to have been photographed with slit spectroscopes.

The coronal rings photographed in 1893 are compared in the following table with results obtained by the slit spectroscopes in the years 1882, 1883, 1886,* and 1893,† those lines only which are possibly common being included.

Prismatic Camera, 1893. λ R.	Slit Spectroscopes, 1893. λ A°.	Slit Spectroscopes, 1886. λ A°.	Slit Spectroscopes, 1883. λ A°.	Slit Spectroscopes, 1882. λ A°.
3987	3986·4	3985·0 (1)‡	3986 ?	
4086	..	4084·2 (4) 4089·3 (4)	} 4085	4085
4217	..	4216·5 (3)		
4231	..	4232·8 (5)		
4240	..	4241 (4)	..	4241
4280	4279·7	4280·6 (4)	4279	
4486	..	4485·6 (3)		

We see then that all the coronal radiations above referred to probably correspond with lines photographed by Dr. SCHUSTER in 1886. The intensities, however, are not the same.

The number of coronal rings recorded with the prismatic camera is very much smaller than the number of lines attributed to the corona photographed with the slit spectroscopes in this and previous eclipses. This is, no doubt, partly due to the rings

* 'Phil. Trans.,' 1889, A, p. 335.

† 'Roy. Soc. Proc.,' vol. 56, p. 20.

‡ Intensity on a scale where 6 = brightest line.

being submerged in continuous spectrum, which is relatively more intense in the case of the prismatic camera. Further, as already pointed out, it is not yet established that many of the lines recorded in the corona by the slit spectroscopes are not due to glare.

By a comparison of the results obtained with slit spectroscopes and prismatic cameras, it would seem to be possible to determine which of the lines recorded by the former instruments really belong to the coronal spectrum. The most intense light will give the strongest glare, and therefore the brightest lines of the chromosphere and prominences will become superposed on those due to the corona. As the results obtained with the prismatic cameras are so very definite with regard to the spectrum of the prominences, it seems only necessary to subtract the common lines of the spectra recorded by the two instruments from the total number recorded by the slit spectroscopes in order to determine those which certainly belong to the corona.

An attempt has been made to investigate the coronal spectrum in this way by reference to the slit spectra of 1886 and 1893, but no satisfactory results can be obtained in this way until slit spectra taken with greater dispersion become available.

In the case of the rings, the evidence that they truly belong to the corona is absolutely conclusive.

Comparison with Fraunhofer Spectrum.

In the absence of more exact knowledge of the wave-lengths of the radiations producing the rings, it is not yet possible to determine if they are represented by dark lines in the Fraunhofer spectrum, but it can already be stated that, if present at all, they are among the feeble lines.

Comparison with Prominence Spectrum.

A point of some importance is the apparent absence of the 1474 K ring from the spectra of the chromosphere and prominences. A similar absence of 1474 K from the prominence spectrum was noted by RESPIGHI in the eclipse of 1871. I am not aware of any observation in which the *form* of a prominence has been observed in 1474 light. All these facts seem to indicate that when the 1474 is observed at the sun's limb without an eclipse, the spectrum of the corona itself is under examination, under the same conditions as those recorded in the eclipse photographs.

Of the other coronal rings photographed in 1893, those at wave lengths 4217 and 4280 are approximately coincident with feeble prominence radiations, but since the other coronal rings are not represented in the prominences, the coincidences may be regarded as accidental.

Although H and K are by far the most intense of the radiations of the prominences, on no occasion have they been photographed as rings in the spectrum of the *corona* with the prismatic cameras. They have, however, been occasionally recorded as corona lines with slit spectroscopes, but it does not seem improbable that in most cases they were produced by prominence light scattered by our atmosphere, as before explained—light of which the prismatic camera takes no account.

Perhaps the most decided evidence in favour of the existence of H and K in the corona spectrum is that depending upon the photographs taken with slit spectroscopes in 1886; Dr. SCHUSTER states that "the lines end sharply with the corona, and we must conclude, therefore, that in spite of the unfavourable atmospheric conditions, there was but little light scattered by our own atmosphere in the neighbourhood of the sun."

But in spite of this observation, Dr. SCHUSTER has concluded that H and K "do not form part of the normal spectrum of the corona";* and I may add that the prismatic camera strengthens this conclusion.

The "Continuous" Spectrum.

The photographs taken with the prismatic cameras in 1893 show a pretty strong "continuous" spectrum, but it has already been explained that this appearance may have been produced by a very complicated spectrum, such as that which I observed in the corona in 1882. Concerning my observations, I wrote :—†

"Instead of the gradual smooth toning seen, say, in the spectrum of the limelight, there were maxima and minima, producing an appearance of ribbed structure, the lines of hydrogen and 1474 being, of course, over all. This observation, however, requires confirmation, for the look I had at the corona spectrum was instantaneous only."

IX. THE VARIABILITY OF THE SPECTRUM OF THE CORONA.

General Comparison of the 1893 *Results with Earlier Observations.*

A change in the spectrum of the corona was placed beyond all doubt in my own mind by my observations in 1871 and 1878. With reference to this I wrote as follows in 1878 :—‡

"The utter disappearance of the large bright red corona of former years in favour of a smaller and white one in this year of minimum struck everybody. Indeed, it is remarkable that after all our past study of eclipses this last one should have exhibited phenomena the least anticipated. It isolates the matter that gives us the continuous spectrum from the other known gaseous constituents. The present eclipse has accomplished, if nothing else, the excellent result of intensifying our knowledge concerning the running down of the solar energy. On the former

* 'Phil. Trans.,' vol. 180, A, p. 341.

† 'Roy. Soc. Proc.,' vol. 34, p. 299.

‡ 'Nature,' vol. 18, p. 460.

occasion in 1871, the 1474 ring was very bright, but in 1878 I did not see it at all.'' As the sun-spot period is one of about eleven years, it was to be expected that the conditions of 1871 would be repeated in 1882 and 1893, and during both these eclipses the 1474 ring was photographed with the prismatic cameras. The photographic plates employed in 1875 and 1886 were not sensitive to the green, and, since no eye observations were made, we have no evidence as to the visibility of the 1474 ring in those years.

Although there can be no doubt as to a more or less regular change of intensity in the case of 1474 K, the evidence with regard to other radiations is less conclusive.

Confining our attention in the first instance to the results obtained with slitless spectroscopes, we have the following data :—

Date.	Observers.	Method.	Spectrum.
1871	RESPIGHI	Visual	C, 1474 K, F.
"	LOCKYER	"	" " F.G.
1875	SCHUSTER	Photographic	G.
1878	LOCKYER	Visual and Photographic	Continuous
"	RANYARD	Visual	"
"	DRAPER	Photographic	"
1882	SCHUSTER	"	D_3, 1474 K
"	LOCKYER	Visual	C, 1474 K
1883	LAWRENCE and WOODS	"	1474 K (middle of totality)
"	" "	Photographic	Continuous
1886	DARWIN	"	" (plate not sensitive to 1474 K)
1893	FOWLER	"	1474 K and 7 others
"	SHACKLETON	"	1474 K
"	"	Visual	"

Variation of Hydrogen.

Hydrogen rings have only been recorded in the spectrum of the corona on three occasions—1871, 1875, and 1882; it is probable that the rings corresponding to hydrogen seen in 1883 were produced by the chromosphere and prominences, since they were not seen in the middle of totality. If the appearance of the hydrogen rings in the corona have the same connection with the sun-spot period as the 1474 K line, their presence in 1893 might have been expected. As they were not photographed in this eclipse, the appearance of hydrogen rings might be regarded as subject to no law in relation to the sun-spot period, but, before coming to a decision on this point, it is worth while to consider very carefully the differences in the instruments employed in the various eclipses. To get at these facts Mr. FOWLER has made a number of experiments to determine to what extent false rings may be produced in the visible and photographic spectrum when a slitless spectroscope is employed, similar to those produced photographically in the manner already explained when certain plates are employed. A positive photograph of the corona of 1893, some 6′ high, illuminated by a large gas flame or electric lamp, forms an artificial eclipse, which has been viewed with instruments of different dispersions, at a distance such that the diameter subtended an angle of about 32′. With a very small dispersion, similar to that employed by RESPIGHI in 1871 in relation to the apparent size of the ring, one sees circular spectra, approximating very closely to those described and figured by RESPIGHI;* the effect is best seen when the green ring is intensified by introducing a salt of thallium between the poles of the electric arc, when the latter is employed as a source of light. A green ring, sharply defined on its inner edge, then becomes visible, a red ring is very distinct but ill-defined, and a blue ring is still less distinct. With increased dispersion, similar to that I employed in 1871, all traces of the blue and red rings disappear, and the green one is only visible so long as there is a special green radiation.

These experiments suggest that the very broad blue and red rings recorded by RESPIGHI in 1871 are possibly not conclusive evidence that the H_α and H_β radiations of hydrogen were very strong in the spectrum of the corona, as rings strikingly resembling them can be produced by a continuous spectrum alone when viewed with similar instrumental conditions. At the same time no such doubt is thrown upon my own observations of narrow rings which were made with the dispersion of five prisms, as false rings of red and blue light are not seen under these conditions. The red and blue hydrogen rings, as I saw them in 1871, were about 2′ high, and bright compared with 1474 K; my observations were made earlier in the eclipse than those of RESPIGHI, to which reference is made above, but the fact that I did not see D_3 seems conclusive evidence that I was not observing rings due to the chromosphere and prominences. Hydrogen was, therefore, certainly present in the corona of 1871 to a height of about 2′, but the observations suggesting its spectroscopic visibility at a greater height are not conclusive.

With regard to the photograph of 1875, in which H_γ was definitely ascribed by Dr. SCHUSTER and myself to the corona, the absence of corona rings corresponding to H and K, which are the strongest lines of the chromosphere and prominences, appears to support the idea that the H_γ ring was not due to chromosphere or prominences. A photograph of the artificial corona, illuminated by an incandescent lamp, under instrumental conditions closely approximating to those employed in 1875, shows only a *hazy ring* near H_γ, corresponding to the region of maximum sensitiveness of the plate; in this way a sharply defined ring similar to that on the photograph cannot be obtained.

It is probable, therefore, that in 1875, as well as in 1871, hydrogen was present in the inner corona.

My observations in 1882† also gave indications of hydrogen.

* 'Atti della Reale Accad. dei Lincei,' March 3, 1872, p. 17.

† 'Nature,' vol. 26, p. 101.

No hydrogen rings were seen during the eclipse of 1878, and as the slitless spectroscopes employed by DRAPER, RANYARD, and myself were of considerable dispersive power, no appearances of rings due to continuous spectrum could have been produced. The presence of hydrogen rings about the time of sun-spot maxima in the eclipses of 1871 and 1882, and their absence near the time of minimum in 1878, suggests a relation to the sun-spot period, similar to that which has been established in the case of 1474 K, but as the rings are not shown in the photographs taken at the maximum in 1893, a final verdict cannot yet be given as to whether hydrogen varies with the sun-spot period.

Reference to Helium.

As shown in the table, the D_3 ring was stated to have been photographed in Egypt in 1882, but not on any other occasion.* From the description of the ring given in the report of this eclipse,† it seems very probable that it is similar to the broad yellow ring photographed on the isochromatic plates in 1893, which, as already shown, is due to the action of continuous spectrum acting on plates with several maxima of intensity in different parts of the spectrum. It is described as being more uniform all round the sun than the 1474 ring, and this is the case also in the 1893 photographs, on which the ring in question is a little more refrangible than D_3; but in the smaller scale photograph of 1882 this difference of position may not have been evident, or the 1882 plate may have had one of its maxima of photographic action still nearer to D_3. In the absence of a copy of the photograph it is difficult to determine how far this explanation holds good, but, if it should be established, this, in combination with all the other eclipse records, would prove that D_3 does not form part of the coronal spectrum.

Reference to Unknown Radiations.

As to the variation in intensities of other radiations of the corona, no evidence is furnished by the prismatic camera results alone. The table of coronal rings already given (p. 593) suggests a similarity of the 1886 spectrum with that of 1893, but the conditions under which the two series of observations were made were so different, that it would be unwise to draw any conclusions as to the variation of the feebler radiations.

At present, then, there is no continuous and distinct instrumental evidence of a periodic change of specific parts of the spectrum, other than the varying brightness of 1474 K.

* As stated in the preliminary report on the eclipse of 1893 ('Phil. Trans.,' vol. 185, 1894, A, p. 716), a yellow ring was seen by Mr. SHACKLETON in Brazil; the subsequent discussion has shown that this must be ascribed without hesitation to chromosphere and prominences, and not to the corona.

† 'Phil. Trans.,' vol. 175, 1884, Part I., p. 264.

Variability of the Continuous Spectrum.

No data have been recorded in regard to the varying intensity of the so-called continuous spectrum of the corona, but, from my spectroscopic observations of 1871, 1878, and 1882, there is no question whatever in my mind that there is a considerable variation in brightness in this continuous spectrum at the maximum and minimum sun-spot periods. The absence of definite statements seems to suggest the desirability of having this question studied by means of the visibility of stars, and, if possible, numerical data should be obtained at each eclipse.

During the eclipse of 1886 the sun itself was clouded over, as seen from the station occupied by myself in the West Indies; but in other parts of the sky a great number of stars was visible—a much greater number than is visible at full moon. At the African station in 1893, the corona was so bright that only the planets Jupiter and Venus were seen by Mr. FOWLER.

There is, however, direct evidence of change of the total light of the corona plus the prominences and chromosphere from one eclipse to the other; and it is probable that the brightness of the so-called continuous spectrum varies in the same way.

Photometric measurements made in the eclipse of 1870 showed that the total light of the corona was represented by 0·42 of a standard candle at a distance of a foot.* Similar observations made in 1878 led Professor HARKNESS to conclude that the total light of the corona was 0·072 of the same unit.†

In 1886 the greatest value registered for the total light was 0·02 of the same unit, but in connection with this low estimate it is pointed out that the conditions of observation were not so favourable as in 1878.‡ Photometric observations were made in 1893, but the results have not yet been published.

Thus, the greatest brightness of the total light was recorded at a sun-spot maximum, while the light was very much less near the times of minima.

So far as they go, these facts agree with the view that the true corona is brightest near a sun-spot maximum.

The more solar physics is studied, the more an enormous change from maximum to minimum in all the phenomena is revealed. Besides this variability of the corona there is, in addition to the well-known eleven-yearly variation in the number of spots, faculæ, and prominences, a variation in the spectra of sun spots so marked, as I have shown in other communications, that there are few, if any, widened lines common to the maximum and minimum.

* 'U. S. Coast Survey Reports,' 1870, p. 172.

† 'Washington Observations,' 1876, App. III.

‡ 'Phil. Trans.,' vol. 180, A, p. 381.

X. Wave-lengths and Intensities of the Prominence and Chromosphere Lines.

Determination of Wave-lengths.

The wave-lengths throughout are expressed on Rowland's scale. In the region less refrangible than K, they have been determined from the African photographs, by comparison with the spectrum of Arcturus and other stars photographed with the same instrument, the wave-lengths of the lines in which were determined by reference to Rowland's photographic map. The spectrum of Arcturus is almost identical with that of the sun, so that the comparison lines were sufficiently numerous for the purpose. Stars like Bellatrix were employed as an additional check in the case of bright lines not coincident with prominent Fraunhofer lines.

Micrometric measurements of the lines were also made and reduced to wave-lengths in the usual way, by means of a curve; these furnished a check on the general accuracy.

In the case of the Brazilian negatives the wave-lengths were determined by means of micrometric measures and a curve, and checked by direct comparisons with a solar spectrum, photographed with the same spectroscope while it was temporarily provided with a slit and collimator.

For the reduction of the ultra-violet, in both series of photographs the wave-lengths of the hydrogen lines have been assumed as far as H_ϕ from those given by Hale,* with the exception of H_υ, which falls sufficiently near the calculated wave-length to be accepted as a hydrogen line.

With these as datum lines, wave-length curves were constructed, and the wave-lengths of the other lines found by interpolation.

The wave-lengths of the radiations more refrangible than H_ζ were determined from extrapolation curves, so that the degree of accuracy is necessarily less than in the case of the remaining lines.

The first column of Table I. summarizes the wave-lengths of the lines which have been measured in all the negatives in both series, whether occurring in the spectrum of the chromosphere or prominences. Although the total number of lines photographed in Brazil is smaller than that photographed in Africa, a few of them only appear in the Brazilian photographs; these are at wave-lengths 3674·2, 3679·5, 3748·5, 4934·2, 5018·6, and 5047·8.

* 'Astronomy and Astrophysics,' 1892, pp. 50, 602, 618.

Determination of Intensities.

The scale of intensities which has been adopted is such that 10 represents the brightest lines and 1 the faintest. This will facilitate comparisons with YOUNG's well-known list of chromospheric lines, in which 100 represents the maximum frequency and brightness. The intensities have been estimated by taking the strongest line in each negative as 10, irrespective of length of exposure.

Table I. gives the intensities of the various lines as they appear in different photographs of prominences Nos. 3 and 19, and in the spectrum of the cusp a few seconds after totality. The intensities of the lines in different photographs of the spectrum of another prominence, No. 13, are shown in Table II., and those at the outside of the chromosphere are contrasted with the intensities of the same lines at the base in Table III.

XI. LOCI OF ABSORPTION IN THE SOLAR ATMOSPHERE.

The Spectra of Prominences and Chromosphere at Different Heights.

If we consider a prominence on that part of the sun's limb where the second contact takes place, the first photograph taken during totality will show the spectrum of the whole prominence, and succeeding photographs will give the spectrum of the same prominence with the lower parts gradually cut off by the moon's edge. In the case of a prominence at the opposite limb, similar sections will be represented in successive photographs, and the last photograph taken during totality will show the spectrum of the greatest part of the prominence.

Prominences Nos. 3 and 19 (see fig. 10) have been investigated in this way, and particulars of their spectra at various heights are recorded in Table I. The first of them is shown in the African Photographs Nos. 7 to 13 inclusive, and the latter in Photographs 19 to 21 inclusive. The height above the photosphere, reckoned in seconds of arc, and in miles, at which each spectrum is given, is indicated beneath the numbers of the photograph in which the prominences appear. The relative intensities of the lines at different heights are shown by the figures ranged in horizontal lines with the wave-lengths. Some of the lines remain of the same relative intensity throughout all parts of the same prominence; others again dim rapidly in passing towards the upper parts. The two prominences in question are also seen to behave differently in respect to some of the lines; thus the line at λ 3856·5 disappears before a height of 2,000 miles is reached in prominence No. 3, but remains visible at a height of over 4,000 miles in prominence No. 19. Lines also occur in one prominence which do not appear in the other, *e.g.*, λ 4313·2. Other differences are also revealed by the table, but it may be remarked that too much stress should not be laid on the presence or absence of the very faintest lines in some of the photographs, as variations may be partially attributed to differences in the quality of the photographs.

The spectrum of a large prominence, No. 13 (fig. 9) at various heights from the photosphere, is shown in Table II. This prominence appears in all the African photographs taken during totality, and in some of those taken out of totality. In order that the changes of intensity of the various lines may be separated from the effects due to varying exposures, the individual observations are arranged in groups according to the exposure of the photographs. The total number of lines is much smaller than in the case of the two metallic prominences, but somewhat similar variations of intensity are noticeable.

The spectrum of the chromosphere at different heights can also be partially investigated in the eclipse photographs. A considerable arc of chromosphere was photographed in the African Negative, No. 21 (Plate 11). The photograph was taken about ten seconds before the end of totality, so that the lower reaches of the solar atmosphere within 1,660 miles of the photosphere were hidden. The bright arcs accordingly represent the spectrum of the chromosphere above that height. None of the photographs give us any information as to the spectrum lower down until we come to the part very near to the base which, as already explained, is shown at the cusp in Photograph 22 (Plate 13). The complete spectrum of the base of the chromosphere is given in Table I., and in Table III. the lines common to the outer part of the chromosphere and to the base are brought together for comparison.

Most of the lines become relatively brighter as the base of the chromosphere is approached, but some become dimmer. Further reference to these changes will be made later.

The changes in the spectrum of a prominence in passing from the top towards the base are illustrated in Plate 14. Spectra *a*, *b*, and *c* represent the spectrum of Prominence No. 3 as it appears in the African Photographs Nos. 11, 9, and 7 respectively, the first giving the spectrum of the upper part only, while the last shows the spectrum nearer the base. Accepting the time of commencement of totality in Africa as 2 hrs. 23 mins. 48 secs. by the deck watch, it is easily calculated that Spectrum 1 represents a part of the prominence 22″–26″ (9,950 to 11,600 miles) above the photosphere ; Spectrum 2, 6″·7–8″·5 (3,000 to 3,800 miles) ; and Spectrum 3, 3″·7 (1,660 miles) above the photosphere. Strip *d* is the spectrum of the base of the chromosphere as represented by the cusp in the African Photograph No. 22.

These enlarged spectra have been obtained by covering copies of the original negatives with tinfoil, leaving only narrow strips showing the prominence spectra, and giving them the necessary width by moving the photograph in a direction at right angles to the length of the spectrum, as described in a former paper.*

The want of exact coincidence of lines common to different horizons is due to the difficulty of obtaining enlargements on exactly the same scale. The difference in thickness of the same line in different photographs of a prominence is due to the varying sizes of the corresponding images of the prominence formed by the prismatic camera at different stages of the eclipse.

* 'Phil. Trans.,' 1893, vol. 184, A, p. 684.

The "Reversing Layer."

As a result of solar spectroscopic observations, combined with laboratory work, Dr. FRANKLAND and myself came to the conclusion, in 1869, that at least in one particular KIRCHHOFF'S theory of the solar constitution required modification. In that year we wrote as follows* :—" May not these facts indicate that the absorption to which the reversal of the spectrum and the Fraunhofer lines are due takes place in the photosphere itself, or extremely near to it, instead of in an extensive outer absorbing atmosphere?"

In an early observation of a prominence on April 17, 1870, I found hundreds of the Fraunhofer lines bright at the base, and remarked that "a more convincing proof of the theory of the solar constitution put forward by Dr. FRANKLAND and myself could scarcely have been furnished."†

During the eclipse of 1870, at the moment of disappearance of the sun, a similar reversal of lines was noticed; we had, to quote Professor YOUNG, "a sudden reversal into brightness and colour of the countless dark lines of the spectrum at the commencement of totality." On these observations was based the view that there was a region some 2″ high above the photosphere, which reversed for us *all* the lines visible in the solar spectrum; and on this ground the name "reversing layer" was given to it.

Continued observations, however, led me, in 1873, to abandon the view that the absorption phenomena of the solar spectrum are produced by any such thin stratum, and convinced me that the absorption took place at various levels above the photosphere. I need not give the evidence here; it is set forth in my 'Chemistry of the Sun.'‡ On the latter hypothesis, the different vapours exist normally at different distances above the photosphere according to their powers of resisting the dissociating effects of heat.§

My observations during the eclipse of 1882, in the seven minutes preceding totality, to my mind set the matter at rest. "We begin with one short and brilliant line constantly seen in prominences, never seen in spots. Next, another line appears, also short and brilliant, constantly seen in prominences; and now, for the first time, a *longer* and thinner line appears, occasionally noted as widened in spots; while, last of all, we get very long, very delicate relatively, two lines constantly seen widened in spots, and another line not seen in the spark and never yet recorded as widened in spots." ||

Similar observations in the same part of the spectrum were made by Professor

* 'Roy. Soc. Proc.,' vol. 17, p. 88.

† 'Roy. Soc. Proc.,' vol. 18, p. 358.

‡ Chapter xxii., pp. 303–309.

§ 'Roy. Soc. Proc.,' vol. 34, p. 292.

|| 'Roy. Soc. Proc.,' vol. 34, p. 297.

TURNER in 1886.* His observations were made under less favourable conditions than those in Egypt, and in the absence of statements as to the relative lengths of the lines observed, it is impossible to utilise them.

This is one of the most important points in solar physics, but there is not yet a consensus of opinion upon it. Professor YOUNG and others apparently still hold to the view first announced by Dr. FRANKLAND and myself in the infancy of the observations, that the Fraunhofer absorption takes place in a thin stratum, lying close to the photosphere.

It is, therefore, of the highest importance to determine what light is thrown on the subject by the photographs of 1893.

A study of the distribution of the lines recorded in Tables I., II., and III., throughout the different parts of the chromosphere and prominences as revealed at different phases, proves that the spectrum gradually becomes more complex as the photosphere is approached.

A partial photographic record of the reversal of the lines at the beginning of totality was secured in Brazil (Photograph No. 2), and of the reversal near the end in Africa (Photograph No. 22). Owing to over exposure, the Brazilian photograph (Plate 12) is very dense, and the lines are only distinctly seen in the regions F to *b*, and that more refrangible than K. The African photograph, taken just after totality (No. 22, Plate 13), shows the ends of the long bright arcs visible at the end of totality, and we seem to be justified in taking these arcs as representing the reversals seen in former eclipses. The number of these bright arcs, as well as the Brazilian photograph No. 2, indicates a spectrum of considerable complexity near the base of the sun's atmosphere.

The complexity of the spectrum of the sun's atmosphere in the neighbourhood of the photosphere is indicated in another way. In the African photographs, taken after totality, except at the southern cusp, the spectrum, as already pointed out, is due to the following four sources :—(1) The spectrum of the vapours of the cusps, giving bright arcs ; (2) the spectrum of the visible crescent of photosphere, consisting of a bright continuous spectrum crossed by dark arcs ; (3) the spectrum of the vapours surrounding the uneclipsed part of the sun, giving bright arcs ; (4) the spectrum of the corona, consisting of a relatively feeble continuous spectrum and a number of bright rings.

The results vary with the relative intensities of the different sources, the chief variable being the breadth of the photosphere, depending upon the interval from totality.

When the crescent is very thin, as in the African Photograph, No. 22, the bright chromospheric arcs are relatively strong and are seen crossing the continuous spectrum of the photosphere, while the dark arcs due to the photosphere are absent. RESPIGHI observed a similar absence of the Fraunhofer spectrum just after totality in

* 'Phil. Trans.,' 1889, vol. 180, A, p. 391.

1871, but he does not seem to have attached any special importance to it. It may be that the feeble Fraunhofer spectrum is under these conditions lost in the bright continuous spectrum of the lower corona; or, since the bright arcs due to hydrogen and calcium are strong enough to be seen superposed on the continuous spectrum of the chromosphere, it is conceivable that the Fraunhofer lines are masked by a complicated radiation spectrum from the base of the chromosphere such as would be furnished by the supposed reversing layer.

To produce the latter result, however, it is not necessary that every dark arc should have its counterpart among the bright ones due to the chromosphere, for the reason that during the exposure of this photograph the chromosphere was broader than the photosphere. The breadth of the chromosphere was about 8″, while that of the photosphere in its widest part varied from 1″ to 3″; thus some of the chromospheric arcs would be broad enough to mask several of the close dark ones in their vicinities, and the absence of the Fraunhofer lines in this photograph therefore does not necessarily favour the existence of a "reversing layer."

A consideration of the photographs disposes of any suggestion which may be made as to the possible production of an apparently continuous spectrum by the crowding together of a great number of short bright arcs, in the spectrum of the cusp, proceeding from the very lowest part of the chromosphere. The actual appearance of the African Photograph No. 22, Plate 13, did at first suggest that such was the case, the continuous spectrum of the photosphere having an excess of brightness immediately at the cusp where it joins the bright arcs. In the succeeding photographs, however, there is no such brighter band of continuous spectrum at the cusp, and the band which appears in Photograph 22 is seen to retain the same position as referred to the images of the prominences.

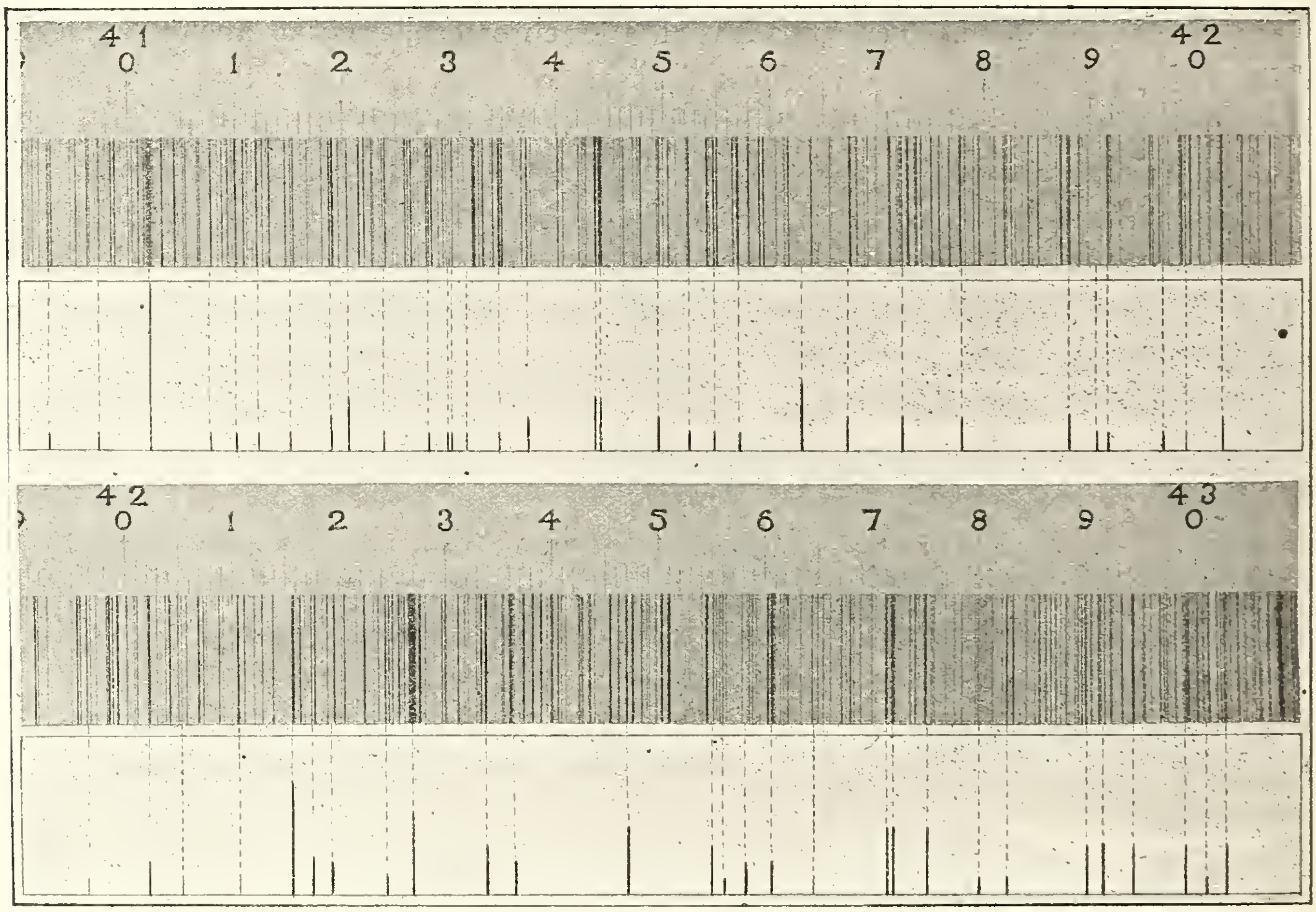
It seems probable, therefore, that the bright continuous band at the cusp, in No. 22, is due to an indentation of the moon's limb at the point where it was produced and that its coincidence with the cusp in the photograph is accidental.

In fig. 16, the bright arcs recorded in the African Photograph No. 22 are drawn in juxtaposition with a photographic map of a portion of the solar spectrum with lengths proportional to their intensities.

It becomes evident at once that the radiation spectrum is most distinctly *not* identical with the Fraunhofer spectrum; the most important point is that some of the strongest bright lines do not appear among the dark ones in the solar spectrum, while some of the strongest dark lines are not seen bright in the spectrum of the stratum of vapours in immediate contact with the photosphere. The region covered by the diagram lies between wave-lengths 4100 and 4300, but similar results follow when other regions are included in the inquiry.

These positive conclusions are not weakened by the consideration that the resolving power of the prismatic cameras employed is not sufficiently great to show all the lines of the Fraunhofer spectrum, which is used as a term of comparison; in fact, working under exactly the same conditions as during the eclipse, the instrument employed in Africa only shows 104 lines in the spectra of stars resembling the sun, in the region *h* to H, in place of 940 given in ROWLAND'S tables of lines in the solar spectrum. We, therefore, get a better term of comparison if we employ the spectrum of some star, such as Arcturus, which closely resembles the sun. Such a comparison is shown in Plate 14 ; out of 104 lines which the instrument is capable of depicting in the region *h* to H, only 40 are shown in the spectrum of the base of the sun's atmosphere. This comparison amply confirms the conclusion that the lines reversed

Fig. 16.



Comparison of the spectrum of the base of the chromosphere with Fraunhofer lines.

at the beginning or end of totality, though fairly numerous, do not correspond in intensity, though some of them correspond in position with the dark lines of the solar spectrum, and consequently that the so-called "reversing layer" close to the photosphere is incompetent to produce, by its absorption, the Fraunhofer lines. Further, while the chromosphere fails to show most of the lines which are present in the Fraunhofer spectrum, it shows many bright lines which are not represented among the dark ones. This again indicates that the chromosphere is not the origin of the Fraunhofer spectrum.

This striking result is absolutely in harmony with all the spectroscopic observations of the sun with which I am acquainted. The Italian observations of the quiet solar atmosphere and the Kensington observations of sun-spots may be especially mentioned. Not only is there no correspondence in intensity, but the variation in the sun-spot spectrum from maximum to minimum is enormous, while the Fraunhofer lines remain constant.

The Brightness of the Arcs at Different Levels.

The existence of a thin stratum competent to produce the Fraunhofer spectrum being thus disproved by the eclipse photographs, we have next to see if there are any indications as to the localization of the absorbing vapours which are not represented in the base of the chromosphere.

The most direct evidence which the eclipse photographs give as to the separation of the solar atmospheric vapours into layers is that afforded by the increased relative brightness of some of the lines in passing to higher levels.

As we have to deal with the projection of a sphere and not with a section of the sun's atmosphere, the spectrum arcs would brighten in passing outwards from the photosphere in consequence of the increased thickness of vapour presented to us, even if the radiation per unit volume remained constant. The spectroscopic differences recorded in Tables I., II., and III., however, show numerous inversions even in the behaviour of the same line in different prominences, so that the increased brightness observed cannot always be due to this cause alone.

Some of the lines are brightest at the base of the chromosphere, while others are brighter at greater elevations. As already explained (p. 586), lines which are brightest above the photosphere must be produced by vapours existing in layers concentric with, but detached from the photosphere. Those lines which become dimmer in passing outwards must owe their origin to vapours resting on the photosphere.

In contrasting the spectrum of the prominences with the spectrum of the cusp, it should be borne in mind that the cusp in the African Photograph No. 22 does not represent the base of the chromosphere immediately beneath either of the metallic prominences considered in Table I. Still the cusp is not far from prominence No. 19, and it is fair to consider the base of the chromosphere homogeneous. If so, the prominences cannot be fed from the base of the chromosphere, since they contain different vapours.

The preliminary discussion of individual substances has further abundantly shown that although some of the lines belonging to any particular metal may appear as dark lines in the solar spectrum on account of absorption by the chromosphere, other lines of the same substance are only represented among the dark lines because of absorption taking place elsewhere. This again is an indication of the stratification of the sun's absorbing atmosphere, which, if it exists, must furnish a very strong argument in favour of the dissociation of metallic vapours at solar temperatures. In fact, the eclipse phenomena have been found to be as bizarre, in relation to the non-dissociation hypothesis, as those which I have already discussed in relation to observation of sun spots and of the chromosphere and prominences without an eclipse.

The view I expressed in 1879,* and to which I adhere, is therefore strengthened by the eclipse work. I then wrote: "The discrepancy which I pointed out, six years ago, between the solar and terrestrial spectra of calcium is not an exceptional, but truly a typical, case. Variations of the same kind stare us in the face when the minute anatomy of the spectrum of almost every one of the so-called elements is studied. If, therefore, the arguments for the existence of our terrestrial elements in extra-terrestrial bodies, including the sun, is to depend upon the perfect matching of the wave-lengths and intensities of the metallic and Fraunhofer lines, then we are driven to the conclusion that the elements with which we are acquainted here do not exist in the sun."

XII. General Conclusions.

(1.) With the prismatic camera photographs may be obtained with short exposures, so that the phenomena can be recorded at short intervals during the eclipse.

(2.) The most intense images of the prominences are produced by the H and K radiations of calcium. Those depicted by the rays of hydrogen and helium are less intense and do not reach to so great a height.

(3.) The forms of the prominences photographed in monochromatic light (H and K) during the eclipse of 1893, do not differ sensibly from those photographed at the same time with the coronagraph.

(4.) The undoubted spectrum of the corona, in 1893, consisted of seven rings besides that due to 1474 K.

The evidence that these belong to the corona is absolutely conclusive. It is probable that they are only represented by feeble lines in the Fraunhofer spectrum, if present at all.

(5.) All the coronal rings recorded were most intense in the brightest coronal regions near the sun's equator as depicted by the coronagraph.

(6.) The strongest coronal line, 1474 K, is not represented in the spectrum of the chromosphere and prominences, while H and K do not appear in the spectrum of the corona, although they are the most intense radiations in the prominences.

(7.) A comparison of the results with those obtained in previous eclipses confirms the idea that 1474 K is brighter at the maximum than at the minimum sun-spot period.

(8.) Hydrogen rings were not photographed in the coronal spectrum of 1893.

(9.) D_3 was absent from the coronal spectrum of 1893, and reasons are given which suggest that its recorded appearance in 1882 was simply a photographic effect due to the unequal sensitiveness of the isochromatic plate employed.

* 'Roy. Soc. Proc.,' 1879, vol. 28, p. 13.

(10.) There is distinct evidence of periodic changes of the continuous spectrum of the corona.

(11.) Many lines hitherto unrecorded in the chromosphere and prominences were photographed by the prismatic cameras.

(12.) The preliminary investigation of the chemical origins of the chromosphere and prominence lines enables us to state generally that the chief lines are due to calcium, hydrogen, helium, strontium, iron, magnesium, manganese, barium, chromium, and aluminium. None of the lines appears to be due to nickel, cobalt, cadmium, tin, zinc, silicon, or carbon.

(13.) The spectra of the chromosphere and prominences become more complex as the photosphere is approached.

(14.) In passing from the chromosphere to the prominences some lines become relatively brighter but others dimmer. The same line sometimes behaves differently in this respect in different prominences.

(15.) The prominences must be fed from the outer parts of the solar atmosphere, since their spectra show lines which are absent from the spectrum of the chromosphere.

(16.) The absence of the Fraunhofer lines from the integrated spectra of the solar surroundings and uneclipsed photosphere shortly after totality need not necessarily imply the existence of a reversing layer.

(17.) The spectrum of the base of the sun's atmosphere, as recorded by the prismatic camera, contains only a small number of lines as compared with the Fraunhofer spectrum. Some of the strongest bright lines in the spectrum of the chromosphere are not represented by dark lines in the Fraunhofer spectrum, and some of the most intense Fraunhofer lines were not seen bright in the spectrum of the chromosphere. The so-called "reversing layer" is, therefore, incompetent to produce the Fraunhofer spectrum by its absorption.

(18.) Some of the Fraunhofer lines are produced by absorption taking place in the chromosphere, while others are produced by absorption at higher levels.

(19.) The eclipse work strengthens the view that chemical substances are dissociated at solar temperatures.

I have finally to express my thanks to Messrs. Fowler and Shackleton, not only for the admirable manner in which they performed the duties entrusted to them in securing the photographic records of the eclipse spectra, as will be gathered by a perusal of their reports, but for assistance in the preparation of the present memoir. Mr. Fowler has generally assisted in the discussion of the photographs, and Mr. Shackleton is mainly responsible for the determination of wave-lengths. Messrs. Baxandall and Butler have also rendered assistance in various ways, and the photographic enlargements have been made by Corporal Haslam, R.E

Table I.—List of Chromosphere and Prominence Lines.

Wave-length.	Intensities in Prominence No. 3.				Intensities in Prominence No. 19.			ntensities at cusp.
	Photo. 7.	Photo. 8.	Photo. 9.	Photo. 10.	Photo. 19.	Photo. 20.	Photo. 21.	Photo. 22.
	Exp. Inst. 3″·7 (1660 miles).	Exp. Inst. 4″·4 (2000 miles).	Exp. 5 secs. 6″·7–8″·5 (3000–3800 miles).	Exp. 25 secs. 12″·2–21″·5 (5480–9600 miles).	Exp. 25 secs. 18″·5–9″·3 (8251–4126 miles).	Exp. 20 secs. 8″·14–4″·44 (3630–1980 miles).	Exp. Inst. 3″·7 (1650 miles).	Exp. 5 secs. Base.
3602	..	..	..	..	Trace	..	..	2
3607	..	..	..	..	Trace	..	..	1
3609	..	..	..	..	1	..	..	1
3613·8	..	..	..	..	1	1	..	2
3619·0	..	..	..	..	1	1	..	1
3625·0	..	..	..	..	..	..	..	1
3630·7	..	..	1	..	2	1	..	4
3635·0	..	..	..	..	1	..	..	..
3641·8	..	..	1	..	1	1	..	2
3647	..	..	1	..	1	1	..	1
3650	..	..	..	..	1	1	..	1
3652	..	..	..	..	..	1	..	1
3659	..	..	..	..	..	..	..	1
3662·2	..	..	..	..	..	..	..	2
3669	..	..	..	..	..	..	..	1
3674·2*	..	..	..	..	..	..	..	..
3676·8	..	..	..	..	..	1	..	..
3679·5*	..	..	..	..	..	..	..	..
3681·1	..	..	..	..	2	2	..	..
3683·5	..	..	..	..	1	1	1	..
3685·2	4	4	6	6	8	6	5	6
3686·7	..	..	..	..	..	2	..	..
3691·5	1	1	1	1	2	3	1	1
3697·4	1	1	2	1	3	3	1	2
3704·0	1	2	4	3	3	5	3	3
3711·8	2	3	4	4	4	6	5	3
3715·5	..	..	..	..	..	1	..	..
3718·5	..	..	..	..	2	2	..	..
3721·9	4	4	5	4	5	5	7	4
3728·5	..	..	..	..	1	1	1	..
3734·2	5	5	6	5	6	6	7	5
3737·1	..	..	..	..	3	3	..	1
3741·7	..	..	..	..	2	2	..	1
3745·8	..	..	..	..	2	3	..	..
3748·5*	..	..	..	..	..	..	..	..
3750·2	6	6	6	5	7	7	8	6
3759·6	7	7	7	7	8	8	8	7
3761·6	7	7	7	7	8	8	8	7
3767·0	..	..	..	..	1	..	..	..
3770·8	6	6	6	6	7	8	8	6
3775·2	..	..	..	..	1	1	..	..
3788·4	..	..	..	..	1	1	..	2
3790·6	..	..	..	..	1	..	..	..

* Appears only in Brazilian photographs.

TABLE I.—List of Chromosphere and Prominence Lines—(continued).

Wave-length.	Intensities in Prominence No. 3.				Intensities in Prominence No. 19.			Intensities at cusp.
	Photo. 7.	Photo. 8.	Photo. 9.	Photo. 10.	Photo. 19.	Photo. 20.	Photo. 21.	Photo. 22.
	Exp. Inst. 3″·7 (1660 miles).	Exp. Inst. 4″·4 (2000 miles).	Exp. 25 secs. 6″·7–8″·5 (3000–3800 miles).	Exp. 25 secs. 12″·2–21″·5 (5480–9600 miles).	Exp. 25 secs. 18″·5–9″·3 (8251–4126 miles).	Exp. 20 secs. 8″·14–4″·44 (3630–1980 miles).	Exp. Inst. 3″·7 (1650 miles).	Exp. 5 secs. Base.
3793·6	..	..	..	..	1	..	..	..
3798·1	7	7	7	6	7	8	8	7
3811·0	..	..	..	..	1	..	..	..
3813·3	..	..	..	..	1	..	..	..
3815·6	..	..	..	..	1	..	1	2
3816·7	..	..	..	..	1	1	..	..
3820·4	2	2	3	3	3	3	3	2
3824·8* } 3826·1* }	..	..	1	..	2	2	1	2
3829·5	..	..	1	..	3	2	2	..
3832·6	..	..	2	2	4	3	3	..
3835·54	8	8	8	7	10	8	6	7
3838·3	4	3	3	3	5	3	4	1
3850·7	..	..	..	..	..	..	1	1
3856·5	1	..	..	..	2	2	1	1
3860·0	3	3	2	3	3	3	5	4
3871·7	..	..	..	..	..	..	1	1
3872·9	..	..	..	..	..	..	..	2
3873·5	..	..	..	..	..	1	1	1
3878·7	2	2	1	..	2	2	3	3
3882·5	..	..	..	..	..	..	..	2
3886·5	..	..	..	..	1	..	..	..
3889·14	10	10	9	9	10	10	8	9
3895·6	..	..	..	..	2	2	1	1
3900·7	4	4	2	3	3	2	5	4
3907·7	..	..	..	..	..	..	1	3
3913·5	4	4	2	3	3	3	5	4
3921·4	..	..	..	..	..	..	1	2
3924·2	..	..	..	..	..	..	1	..
3929·4	..	..	..	..	..	..	1	..
3933·86	10	10	10	10	10	10	10	10
3941·5	..	1	..	..	..	..	..	1
3944·2	3	3	2	2	1	2	3	2
3945·2	..	..	..	..	..	..	..	2
3947·8	..	..	..	..	..	..	..	1
3953·1	..	1	1	..	..	..	..	2
3956·8	..	..	..	..	..	..	..	2
3961·7	3	3	2	2	1	2	3	3
3964·7	..	..	..	..	..	..	..	2
3968·62 } 3970·25 }	10	10	10	10	10	10	10	10
3974	..	..	..	..	..	..	..	1
3978·0	..	..	..	..	..	..	..	3

* Appears only in Brazilian photographs.

Table I.—List of Chromosphere and Prominence Lines—(continued).

Wave-length.	Intensities in Prominence No. 3.				Intensities in Prominence No. 19.			Intensities at cusp.
	Photo. 7.	Photo. 8.	Photo. 9.	Photo. 10.	Photo. 19.	Photo. 20.	Photo. 21.	Photo. 22.
	Exp. Inst. 3″·7 (1660 miles).	Exp. Inst. 4″·4 (2000 miles).	Exp. 5 secs. 6″·7–8″·5 (3000–3800 miles).	Exp. 25 secs. 12″·2–21″·5 (5480–9600 miles).	Exp. 25 secs. 18″·5–9″·3 (8251–4126 miles).	Exp. 20 secs. 8″·14–4″·44 (3630–1980 miles).	Exp. Inst. 3″·7 (1650 miles).	Exp. 5 secs. Base.
3982	..	..	..	..	..	..	1	..
3984·2	..	..	..	..	..	..	..	3
3987·0	..	..	..	..	..	..	..	3
3990·1	..	..	..	..	..	..	1	3
3992	..	..	..	..	..	..	..	1
3996 }	..	..	..	..	..	..	..	1
3998 }	..	..	..	..	..	..	..	1
4001	..	..	..	..	..	..	..	1
4005·5	..	..	..	..	..	..	1	3
4007·0	..	..	..	..	..	..	..	1
4012	..	..	..	..	..	..	1	1
4017	..	..	..	..	..	..	..	1
4026·5	8	8	6	5	3	4	8	3
4030·91	1	1	1	1	..	..	3	3
4033·22	1	1	1	1	..	..	3	3
4034·64	1	1	1	1	..	..	3	3
4035·88	1	1	1	1	..	..	3	3
4040·8	..	1	..	..	.	..	..	1
4041·6	..	..	..	..	..	..	3	1
4044·1	..	..	..	..	..	..	..	3
4045·98	5	5	2	2	1	2	5	3
4049·0	..	..	..	..	..	..	..	3
4052·8	1	1	..	..	..	..	..	..
4055·1	1	1	1	..	..	1	3	3
4057·8	1	1	..	..	..	..	..	1
4059·1	..	..	..	..	..	..	..	1
4063·75	3	3	2	2	..	2	5	3
4066·8	..	1	..	..	..	..	..	1
4067·2	..	..	..	..	..	..	..	3
4070·7	..	..	..	..	..	..	..	3
4071·90	3	3	2	1	..	2	5	3
4077·88	8	8	7	6	4	6	8	7
4084·8	1	1	..	1	..	..	..	1
4087·0	1	1	..	1	..	..	..	1
4092·5	1	1	..	1	..	..	..	..
4097	..	..	..	1	..	..	..	1
4101·8	10	10	9	8	6	9	10	10
4107·5	..	..	..	..	..	..	..	1
4110	..	..	..	..	..	..	..	1
4112	..	1	..	..	..	..	..	..
4115	..	..	..	..	..	..	..	1
4118·7	..	..	..	..	..	..	..	1
4121·0	3	3	2	2	1	1	3	1
4123·7	..	..	..	..	..	..	..	1

TABLE I.—List of Chromosphere and Prominence Lines—(continued).

Wave-length.	Intensities in Prominence No. 3.				Intensities in Prominence No. 19.			Intensities at cusp.
	Photo. 7.	Photo. 8.	Photo. 9.	Photo. 10.	Photo. 19.	Photo. 20.	Photo. 21.	Photo. 22.
	Exp. Inst. 3″·7 (1660 miles).	Exp. Inst. 4″·4 (2000 miles).	Exp. 5 secs. 6″·7–8″·5 (3000–3800 miles).	Exp. 25 secs. 12″·2–21″·5 (5480–9600 miles).	Exp. 25 secs. 18″·5–9″·3 (8251–4126 miles).	Exp. 20 secs. 8″·14–4″·44 (3630–1980 miles).	Exp. Inst. 3″·7 (1650 miles).	Exp. 5 secs. Base.
4127·9	..	..	..	..	..	..	..	1
4129·8	..	..	..	..	..	..	1	1
4130·1	..	..	..	..	..	..	1	..
4131·4	..	.	..	..	..	..	1	1
4134·6	..	..	..	..	..	..	..	1
4137·2	..	..	..	..	..	..	1	1
4143·7 } 4144·0	3	3	2	..	1	1	3	5
4149·3	..	..	..	..	..	..	1	1
4152·2	..	..	..	..	..	..	..	1
4154·6	..	..	..	..	..	..	..	1
4157·0	..	..	..	..	..	..	1	1
4162·8	3	1	..	..	..	..	1	..
4167·2	3	3	..	..	..	..	..	3
4172·2	1	3	..	..	..	..	1	5
4177·8	1	..	..	..	..	..	3	5
4187·9	..	..	..	..	..	..	3	1
4190·5	..	..	..	..	..	..	1	..
4191·5	1	..	..	..	..	..	..	1
4198·8	..	1	..	..	..	..	1	1
4202·2	1	1	..	..	..	..	1	..
4205·2	..	..	..	..	..	..	1	1
4210·5	..	1	..	..	..	..	1	1
4215·69	8	7	6	5	2	4	8	7
4217·5	1	..	..	..	..	..	..	..
4219·4	1	..	..	..	..	..	..	1
4224·3	..	..	..	..	..	..	..	3
4226·89	7	5	4	3	2	3	7	5
4233·8	3	3	1	..	..	..	3	5
4236·4	1	..	..	..	..	..	1	1
4247·0	5	5	4	3	1	3	5	5
4254·8	3	3	2	..	..	1	3	1
4256	..	..	..	..	..	..	..	1
4258	..	..	..	..	..	..	..	1
4260·5	1	1	..	..	..	..	3	1
4264·5	1	..	..	..	..	1	..	..
4271·2 } 4271·9	1	5	3	..	..	1	5	3
4275·0	3	5	..	..	..	..	1	5
4280·0	..	1	..	..	..	..	..	1
4282·8	..	..	..	..	..	..	1	1
4290·0	3	3	..	..	..	1	5	5
4291·5	..	..	..	..	..	..	..	1
4294·2	3	3	..	1	..	1	3	..

TABLE I.—List of Chromosphere and Prominence Lines—(continued).

Wave-length.	Intensities in Prominence No. 3.				Intensities in Prominence No. 19.			Intensities at cusp.
	Photo. 7.	Photo. 8.	Photo. 9.	Photo. 10.	Photo. 19.	Photo. 20.	Photo. 21.	Photo. 22.
	Exp. Inst. 3″·7 (166 miles).	Exp. Inst. 4″·4 (2000 miles).	Exp. 5 secs. 6″·7–8″·5 (3000–3800 miles).	Exp. 25 secs. 12″·2–21″·5 (5480–9600 miles).	Exp. 25 secs. 18″·5–9″·3 (8[illegible]51–4126 miles).	Exp. 20 secs. 8″·14–4″·44 (3630–1980 miles).	Exp. Inst. 3″·7 (1650 miles).	Exp. 5 secs. Base.
4299·1	5 }	4 {	3	1	..	1	5	3
4301·0	3 }		..	.	..	..	..	1
4302·9	..	3	..	..	..	..	..	..
4308·0	4	5	2	1	..	1	3	3
4313·2	4	3	..	..	..	..	..	..
4315·2	3	5	..	1	..	1	3	3
4318·8	..	..	..	..	..	..	..	3
4321·0	..	1	..	..	..	..	3	3
4323·5	..	3	..	..	..	..	3	..
4325·8	3	5	..	1	..	1	..	3
4328	..	1	..	.	..	..	..	..
4331·0	..	...	..	..	..	..	..	3
4334	..	3	..	..	..	..	..	1
4340·66	10	10	10	9	8	10	10	10
4352·0	1	3	..	..	..	..	3	3
4352·9	..	1	1	..	..	..	..	..
4359·8	1	3	..	..	..	..	1	1
4374·2	1	5	1	..	..	1	3	5
4376·1	3	3	..	..	..	..	3	..
4383·8	5	5	2	1	1	1	5	5
4388·5	5	3	2	1	1	1	1	1
4394·2 } 4395·2 }	7	5	3	3	1	2	5	5
4398·7	..	1	..	..	..	..	..	1
4400·3	1	..	..	..	..	..	1	1
4404·92	3	2	1	1	..	..	3	2
4408·5	..	1	..	..	..	..	..	1
4415·3	3	3	1	..	..	1	3	3
4418·0	..	1	..	..	..	..	..	1
4422·8	..	1	..	..	..	..	..	..
4425·6	1	1	..	..	..	..	..	3
4435·2	..	1	1	1	..	1	..	3
4444·0	5	5	3	1	..	2	3	3
4447·5	..	..	..	..	..	..	..	1
4455·0	..	..	..	..	..	..	..	1
4461·8	..	1	..	..	..	1	..	..
4471·8	10	10	8	6	4	6	8	5
4480	..	..	..	..	..	..	1	..
4482·2	..	..	..	..	..	..	..	1
4490·5	..	..	..	..	..	..	1	..
4494·6	..	..	..	..	..	1	..	..
4499·1 } 4501·3 }	5	5	3	1	..	2	3	5
4507	..	..	..	..	..	..	..	1

TABLE I.—List of Chromosphere and Prominence Lines—(continued).

Wave-length.	Intensities in Prominence No. 3.				Intensities in Prominence No. 19.			Intensities at cusp.
	Photo. 7.	Photo. 8.	Photo. 9.	Photo. 10.	Photo. 19.	Photo. 20.	Photo. 21.	Photo. 22.
	Exp. Inst. 3″·7 (1660 miles).	Exp. Inst. 4″·4 (2000 miles).	Exp. 5 secs. 6″·7–8″·5 (3000–3800 miles).	Exp. 25 secs. 12″·2–21″·5 (5480–9600 miles).	Exp. 25 secs. 18″·5–9″·3 (8251–4126 miles).	Exp. 20 secs. 8″·14–4″·44 (3630–1980 miles).	Exp. Inst. 3″·7 (1650 miles).	Exp. 5 secs. Base.
4514·5	..	1	1	..	..	..	..	1
4518·6	..	1	..	..	..	..	..	..
4522·9	2	2	1	..	..	..	..	..
4525·6	..	..	..	..	..	..	..	1
4529	..	..	..	..	..	..	..	1
4534·2	5	5	3	2	..	1	3	2
4549·7	5	5	3	2	..	1	5	3
4554·2	..	..	..	..	..	} 1	5	{ 1
4556·2	3	3	..	..	..	}		{ 1
4565·8	4	4	1	1	..	1	3	3
4569	..	..	..	..	..	..	..	1
4572·2	5	5	2	2	..	1	3	1
4576·5	..	1	..	..	..	..	..	1
4581·7	..	1	..	..	..	..	..	..
4584·0	3	1	..	..	..	..	1	.
4588·2	..	1	..	..	..	..	..	..
4683·8	3	3	1	1	..	..	1	..
4713·2	5	5	4	3	..	1	5	..
4861·5	10	10	10	10	8	9	10	10
4922·1	3	3	4	2	..	..	4	3
4934·2*	..	..	..	..	..	..	..	..
5015·7	..	1	4	2	..	..	3	2
5018·6*	..	..	..	..	..	..	..	..
5047·8*	..	..	..	..	..	..	..	..
5167·57 } 5172·87	..	..	3	1	..	..	2	4
5183·79	..	..	4	2	..	..	1	2
5270	..	..	..	..	..	..	..	1
5276	..	..	..	..	..	..	..	1
5363	..	..	..	..	..	..	..	1
5371	..	..	..	..	..	..	..	1
5875·98	..	..	..	..	..	..	..	4
6563·05	..	..	..	..	..	..	..	5

* Appears only in Brazilian photographs.

TABLE II.—Wave-lengths and Intensities of the Lines in the Spectrum of Prominence No. 13.

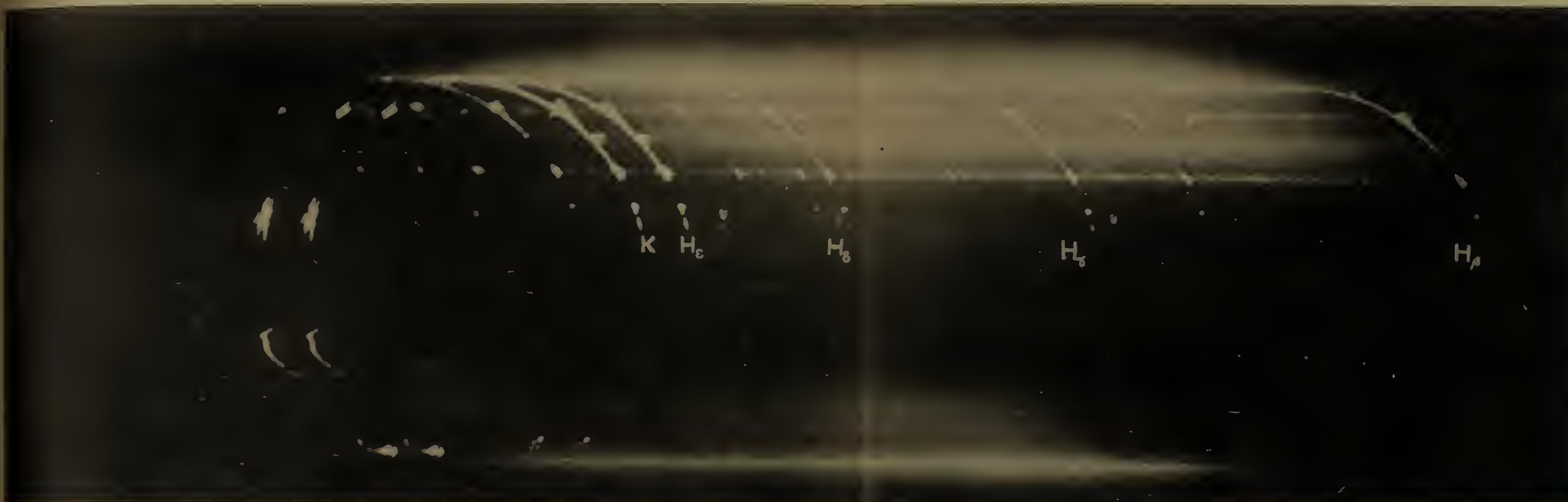
Wave-length.	Intensities in Prominence No. 13.															
	Exposure, instantaneous.					Exposure, 5 secs.				Exposure, 10 secs.			Exposure, 25 secs.			Exposure, 40 secs.
	Photo. 7. 70″.	Photo. 8. 69″·6.	Photo. 12. 57″·8.	Photo. 16. 44″·9.	Photo. 21. 25″·9.	Photo. 9. 68″·5 to 67″·5.	Photo. 13. 55″·5 to 54″·6.	Photo. 18. 36″·4 to 35″·4.	Photo. 22. 23″·4 to 22″·5.	Photo. 11. 60″·5 to 58″·6.	Photo. 15. 48″·5 to 46″·6.	Photo. 20. 28″·2 to 26″·3.	Photo. 10. 65″·6 to 60″·9.	Photo. 14. 53″·6 to 48″·9.	Photo. 19. 33″·5 to 28″·8.	Photo. 17. 44″·5 to 36″·9.
	30,600 miles.	30,400 miles.	25,300 miles.	19,600 miles.	11,300 miles.	29,900 to 29,500 miles.	24,200 to 23,800 miles.	15,900 to 15,500 miles.	10,200 to 9,800 miles.	26,500 to 25,600 miles.	21,200 to 20,400 miles.	12,300 to 11,500 miles.	28,700 to 26,600 miles.	23,400 to 21,400 miles.	14,600 to 12,600 miles.	19,400 to 16,100 miles.
3681·1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	1	..
3685·2	..	..	..	..	..	5	3	3	3	1	3	5	3	3	7	5
3691·5	..	..	..	..	..	..	..	1	..	..	1	1	..	1	1	1
3697·4	..	..	..	..	..	1	..	1	1	..	1	1	..	1	1	1
3704·0	..	..	..	..	..	1	1	1	1	..	3	3	1	3	3	1
3711·8	..	..	..	..	..	3	3	3	3	1	3	3	1	3	3	1
3721·9	..	..	..	..	..	3	3	3	3	1	3	3	3	3	5	3
3734·2	..	..	..	..	..	3	3	3	3	3	3	3	3	3	5	3
3750·2	..	..	..	..	..	3	3	3	3	3	3	3	3	3	5	3
3759·6	..	..	..	..	1	5	5	5	5	5	5	5	5	5	7	5
3761·6	..	..	..	..	1	5	5	5	5	5	5	5	5	5	7	5
3770·8	..	..	..	..	..	3	3	3	5	3	3	5	3	3	5	3
3798·1	..	..	..	..	1	5	3	4	5	3	3	5	3	3	7	3
3820·4	..	..	..	..	..	3	..	1	1	1	3	1	1	3	1	1
3835·54	..	..	..	..	3	5	5	5	5	5	5	5	5	5	7	5
3859·8	..	..	..	..	..	1	..	1	..	1	1	1	1	1	1	1
3889·14	3	3	5	3	5	7	8	8	7	7	7	8	7	7	8	8
3933·86	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
3968·62	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10
3970·25	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..
4026·5	..	..	1	1	5	3	3	3	5	3	3	5	3	3	3	3
4045·98	..	..	..	..	1	..	..	1	..	..	..	1	..	..	..	..
4063·75	..	..	..	..	1	..	..	..	..	..	..	1	..	..	..	..
4077·88	..	..	1	1	5	3	3	3	5	3	3	5	3	3	3	3
4101·8	..	..	5	5	7	5	5	7	7	5	5	7	5	5	7	7
4121·0	..	..	..	..	1	..	..	1	..	..	..	1	..	..	..	1
4144·0	..	..	..	..	1	..	..	1	..	..	..	1	..	..	..	1

TABLE II.—Wave-lengths and Intensities of the Lines in the Spectrum of Prominence No. 13—(continued).

Wave-length.	Intensities in Prominence No. 13.															
	Exposure, instantaneous.					Exposure, 5 secs.				Exposure, 10 secs.			Exposure, 25 secs.			Exposure, 40 secs.
	Photo. 7. 70″. 30,600 miles.	Photo. 8. 69″·6. 30,400 miles.	Photo. 12. 57″·8. 25,300 miles.	Photo. 16. 44″·9. 19,600 miles.	Photo. 21. 25″·9. 11,300 miles.	Photo. 9. 68″·5 to 67″·5. 29,900 to 29,500 miles.	Photo. 13. 55″·5 to 54″·6. 24,200 to 23,800 miles.	Photo. 18. 36″·4 to 35″·4. 15,900 to 15,500 miles.	Photo. 22. 23″·4 to 22″·5. 10,200 to 9,800 miles.	Photo. 11. 60″·5 to 58″·6. 26,500 to 25,600 miles.	Photo. 15. 48″·5 to 46″·6. 21,200 to 20,400 miles.	Photo. 20. 28″·2 to 26″·3. 12,300 to 11,500 miles.	Photo. 10. 65″·6 to 60″·9. 28,700 to 26,600 miles.	Photo. 14. 53″·6 to 48″·9. 23,400 to 21,400 miles.	Photo. 19. 33″·5 to 28″·8. 14,600 to 12,600 miles.	Photo. 17. 44″·5 to 36″·9. 19,400 to 16,100 miles.
4215·69	..	..	1	..	3	1	..	3	3	1	1	3	1	1	1	2
4226·89	..	..	..	..	1	..	..	2	..	..	..	1	..	..	..	..
4233·8	..	..	..	..	1	..	..	1	..	..	..	1	..	..	1	..
4247·0	..	..	..	..	1	..	..	2	..	..	..	1	..	..	..	..
4340·66	..	..	7	5	7	5	5	7	7	5	5	7	5	5	7	7
4376·1	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4383·8	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4394·2	..	..	..	..	..	..	..	..	..	..	..	1	..	..	..	..
4395·2	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4415·3	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4445·0	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4471·8	..	..	5	3	5	3	3	5	5	3	3	5	3	3	5	5
4501·3	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4534·2	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4549·7	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4556·2	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4683·8	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..
4713·2	..	..	..	..	3	..	..	..	..	..	..	1	..	..	..	1
4861·8	..	..	7	5	7	5	5	7	7	5	5	7	5	5	7	7
5875·98	..	..	..	..	..	..	..	5	..	..	..	..	..	..	..	5
6563·05	..	..	..	..	..	..	..	1	..	..	..	..	..	..	3	1

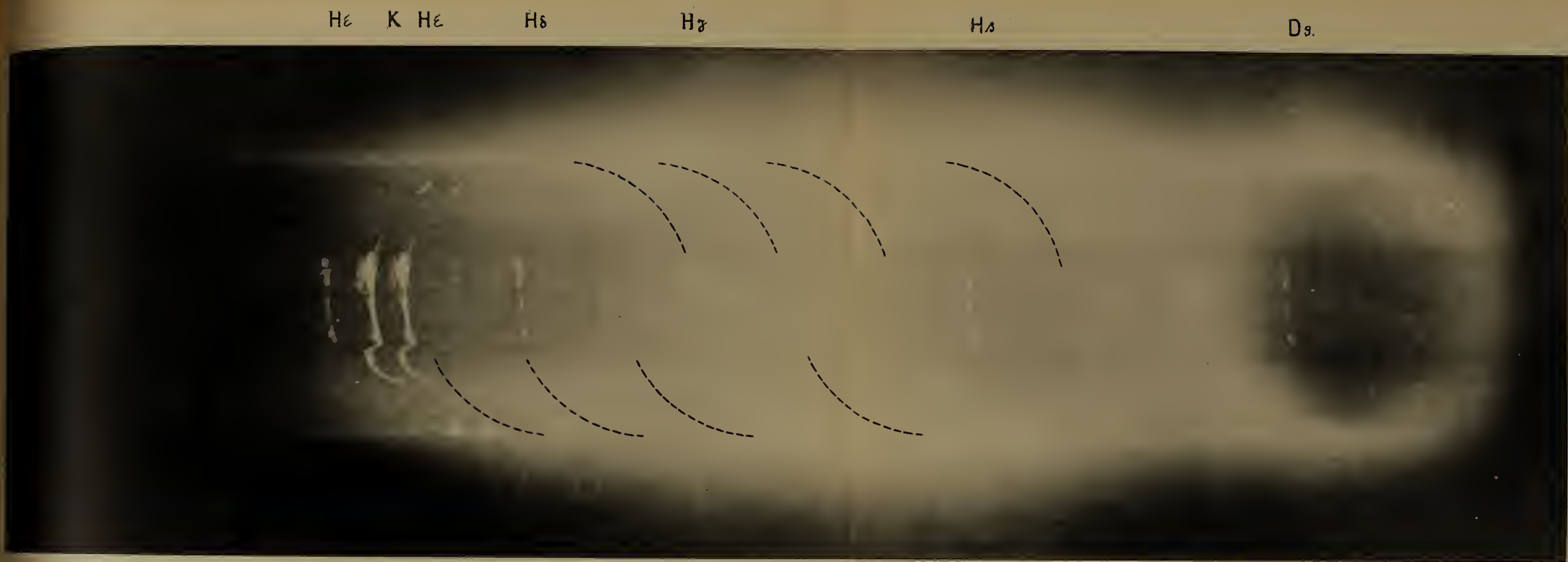
TABLE III.—Spectrum of Chromosphere at top and bottom.

Wave-length.	Intensity in chromospheric-arc No. 20. African photo. 21. Above 3″·7.	Intensity at base of chromosphere. (Cusp.)
3734·2	1	5
3750·2	1	6
3759·6	2	7
3761·6	2	7
3770·8	2	6
3798·1	4	7
3835·5	5	7
3838·3	2	1
3889·1	6	9
3900·7	1	4
3913·5	1	4
3933·9	10	10
3968·6 } 3970·2 }	10	10
4026·5	4	3
4030·9	1	3
4046·0	1	3
4063·8	1	3
4071·9	1	3
4077·9	4	7
4101·8	8	10
4121·0	1	1
4144·0	1	5
4215·7	3	7
4226·9	2	5
4233·8	1	5
4247·0	2	5
4340·7	8	10
4383·8	1	5
4394·2	2	5
4444·0	2	3
4471·8	5	5
4501·3	2	5
4534·2	1	2
4549·7	1	3
4683·8	1	. .
4713·2	3	. .
4861·8	8	10

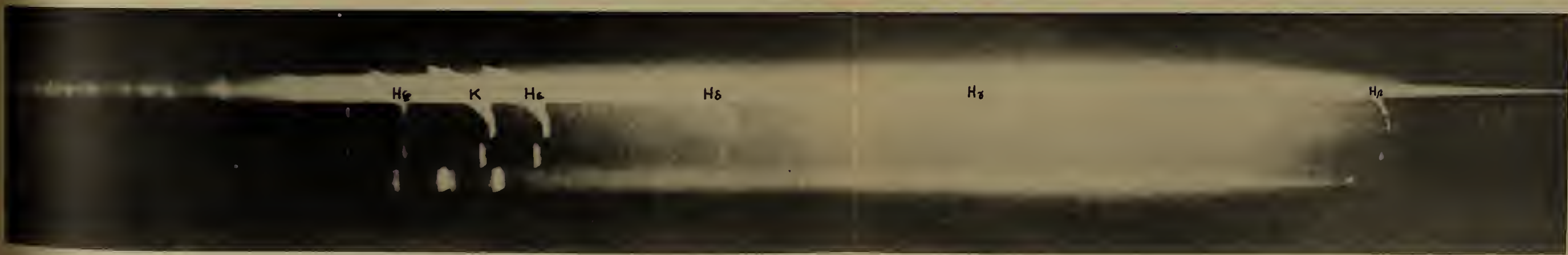


N° 7. ABOUT 10 SECS. AFTER COMMENCEMENT OF TOTALITY

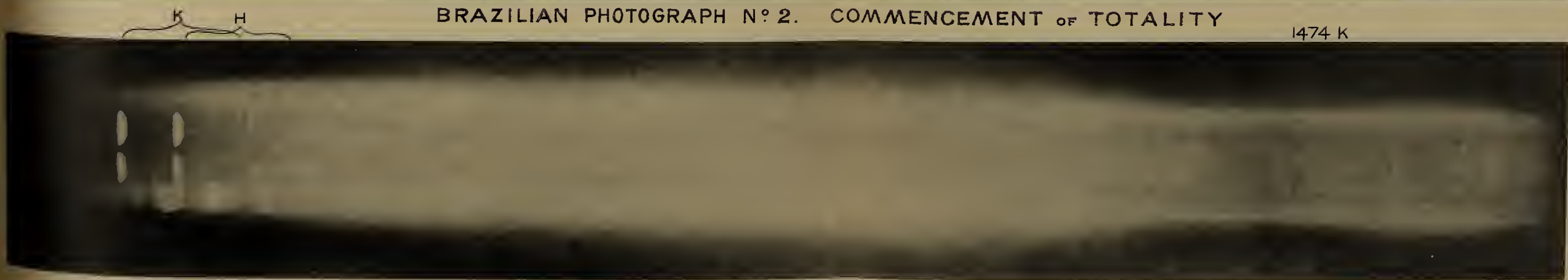
N° 21 ABOUT 10 SECS. BEFORE THE END OF TOTALITY



AFRICAN PHOTOGRAPH N° 17. SHOWING POSITIONS OF CORONAL RINGS.

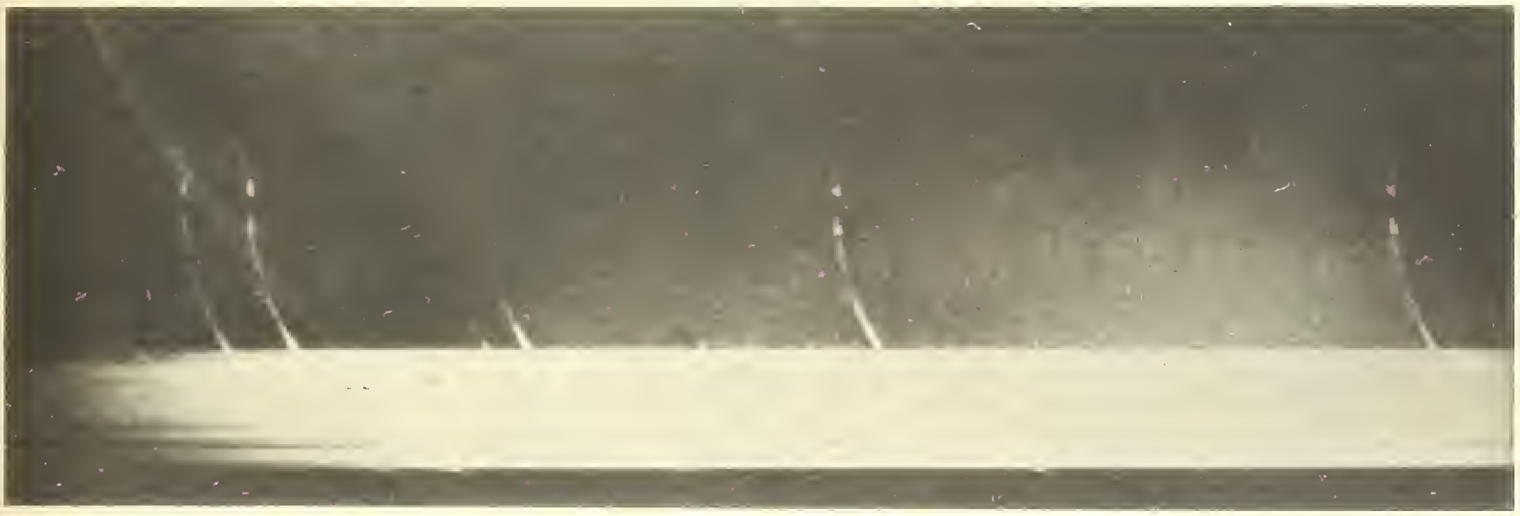


BRAZILIAN PHOTOGRAPH N° 2. COMMENCEMENT OF TOTALITY



BRAZILIAN PHOTOGRAPH N° 12. MIDDLE OF TOTALITY

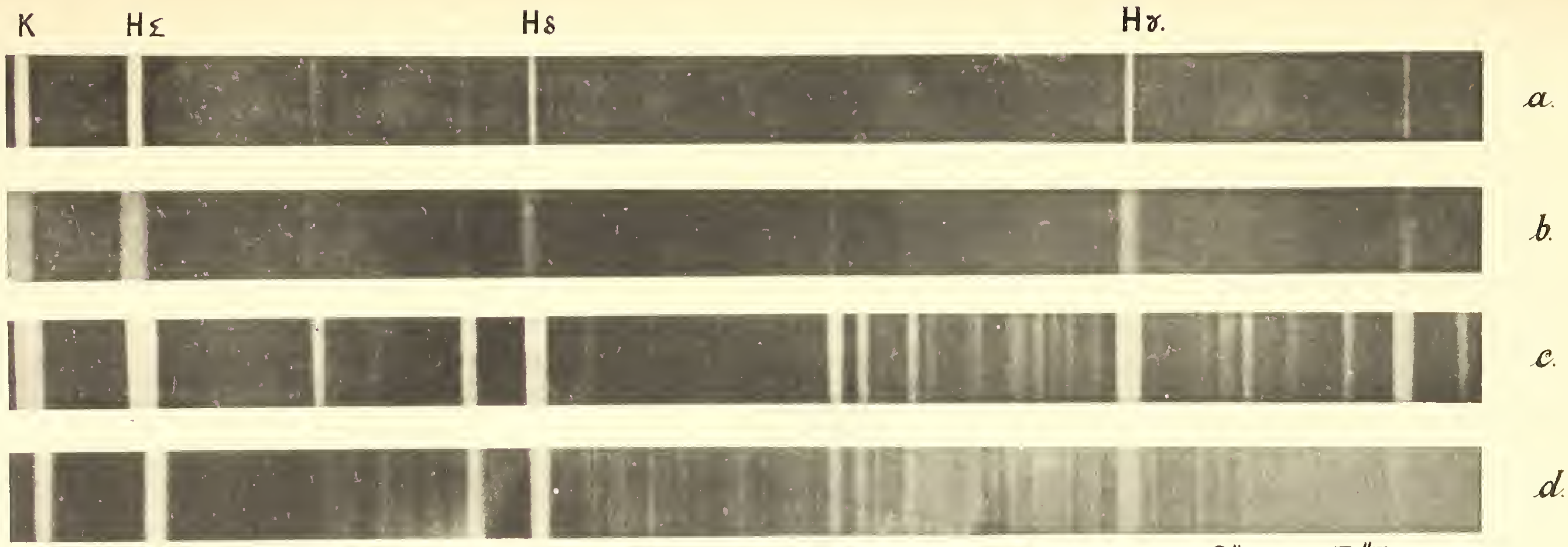
AFRICAN PHOTO No. 22. 3-8 SECONDS AFTER TOTALITY



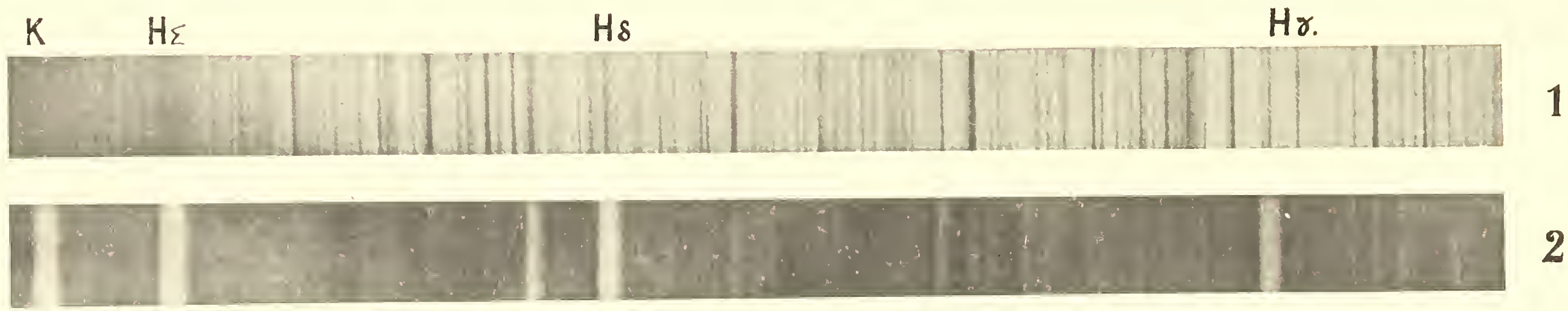
No. 23. 10 SECONDS AFTER TOTALITY

No. 24. 11 SECONDS AFTER TOTALITY

No. 25. 41-49 SECONDS AFTER TOTALITY



Spectrum of Prominence No. 3. (*a*) 22″—26″ above photosphere; (*b*) 7″—9″, (*c*) 3″.7; compared with spectrum of base of chromosphere (*d*).



The spectrum of the base of the chromosphere (2) compared with the spectrum of Arcturus (1).

INDEX SLIP.

XVI. *Memoir on the Theory of the Partition of Numbers.*—Part I.

By Major P. A. MacMahon, *R.A., F.R.S.*

Received December 31, 1895,—Read January 16, 1896.

§ 1.

Art. 1. I have under consideration multipartite numbers as defined in a former paper.*

I recall that the multipartite number

$$\overline{\alpha\beta\gamma\ldots},$$

may be regarded as specifying $\alpha + \beta + \gamma + \ldots$ things, α of one sort, β of a second, γ of a third, and so forth. If the things be of m different sorts the number is said to be multipartite of order m or briefly an m-partite number. It is convenient to call $\alpha, \beta, \gamma, \ldots$ the first, second, third . . . figures of the multipartite number. If such a number be divided into parts each part is regarded as being m-partite; if the order in which the parts are written from left to right is essential we obtain a composition of the multipartite number; whereas if the parts themselves are alone specified, and not the order of arrangement, we have a partition of the multipartite number. This, and much more, is explained in the paper quoted, which is concerned only with the compositions of multipartite numbers.

Art. 2. The far more difficult subject of partitions is taken up in the present paper. The compositions admitted of easy treatment by a graphical process. An m-partite reticulation or lattice is taken to be the graph of an m-partite number, and on this graph every composition can be satisfactorily depicted.

A suitable graphical representation of the partitions appears to be difficult of attainment. As the Memoir proceeds, the extent to which the difficulties have been overcome will appear. There are several bonds of connection between partitions and compositions; in general, these do not exist between m-partite partitions and m-partite compositions, but arise from a general survey of the partitions and compositions of multipartite numbers of all orders.

These bonds are of considerable service in the gradual evolution of a theory of partitions. Two bonds have already been made known (*loc. cit.*). They both have

* "Memoir on the Theory of the Composition of Numbers," 'Phil. Trans.,' R.S. of London, vol. 184 (1893), A, pp. 835–901.

reference to the perfect partitions of unipartite numbers.* Firstly, there is a one-to-one correspondence between the compositions of the multipartite

$$\overline{\alpha\beta\gamma\ \ldots}\ ,$$

and the perfect partitions of the unipartite

$$a^{\alpha}b^{\beta}c^{\gamma}\ldots - 1\ ;$$

$a, b, c, \ldots$ being any different primes.

Secondly, there is a one-to-one correspondence between the compositions of the unipartite number m and the perfect partitions, comprising m parts, of the whole assemblage of unipartite numbers.

These bonds are interesting but, for present purposes, trivial. We require some correspondence concerning partitions which are not subject to the restriction of being *perfect*.

Art. 3. I first proceed to explain an important link connecting unipartite with multipartite partitions arising from the notion of the separation of the partition of a number (whether unipartite or multipartite) into separates ; a notion which leads to a theory of the separations of a partition of a number* which was partially set forth in the series of Memoirs referred to in the foot-note. The theory of separations arises in a perfectly natural manner in the evolution of the theory of symmetrical algebra, and is, I venture to think, of considerable algebraical importance. Up to the present time the theory has been worked out as it was required for algebraical purposes ; the various definitions and theorems are scattered about several Memoirs in a manner which is inconvenient for reference, and it therefore will be proper, while explaining the connection with compositions, to bring the salient features of the theory together under the eye of the reader. It suffices for the most part to deal with unipartite numbers.

The Theory of Separations.

Definitions.

Art. 4. A number is partitioned into parts by writing down a set of positive numbers (it is convenient, but not necessary to assume positive parts, and occasionally to regard zero as a possible part) which, when added together, reproduce the original number.

The constituent numbers, termed parts, are written in descending numerical order from left to right and are usually enclosed in a bracket ( ). This succession of

* See also "The Perfect Partitions of Numbers and the Compositions of Multipartite Numbers." The Author, 'Messenger of Mathematics,' New Series, No. 235, November, 1890.

numbers is termed a partition of the original number; and this number, *quâ* partitions, is termed by SYLVESTER the partible number.

A partition of a number is separated into separates by writing down a set of partitions, each in its own brackets, such that when all the parts of the partitions are assembled in a single bracket and arranged in order, the partition which is separated is reproduced. The constituent partitions, which are the separates, are written down from left to right in descending numerical order as regards the weights of the partitions.

N.B. The partition (pqr) is said to have a weight $p+q+r$.

The partition separated may be termed the separable partition.

Taking as separable partition

$$(p_1p_2p_3p_4p_5),$$

two separations are

$$(p_1p_2)\,(p_3p_4)\,(p_5),$$

$$(p_1p_2p_3)\,(p_4p_5),$$

and there are many others.

If the successive weights of the separates be

$$w_1,\ w_2,\ w_3,\ \ldots;$$

the separation is said to have a specification

$$(w_1,\ w_2,\ w_3,\ \ldots);$$

the specification being denoted by a partition of the weight w of the separable partition.

The *degree* of a separation is the sum of the highest parts of the several separates.

If the separation be

$$(p_1\ldots)^{j_1}\,(p_s\ldots)^{j_2}\,(p_t\ldots)^{j_3}\ldots$$

the *multiplicity* of the separation is defined by the succession of indices

$$j_1,\ j_2,\ j_3,\ \ldots$$

The characteristics of a separation are

(i.) The weight.
(ii.) The separable partition.
(iii.) The specification.
(iv.) The degree.
(v.) The number of separates.
(vi.) The multiplicity.

Art. 5. The separations of a given partition may be grouped in a manner which is independent of their specifications.

Consider the separable partition

$$(p_1^3 p_2^2),$$

which is itself to be regarded as one amongst its own separations.

Viewed thus, it has *quâ* partition a multiplicity (32).

Write down any one of its separations, say

$$(p_1^2)\ (p_1 p_2)\ (p_2).$$

This separation may be regarded as being compounded of the two separations

$$(p_1^2)\ (p_1) \text{ of } (p_1^3)$$

and

$$(p_2)^2 \text{ of } (p_2^2).$$

Three other separations enjoy the same property, viz.,

$$\begin{array}{l} (p_1^2)\ (p_1)\ (p_2)^2, \\ (p_1^2 p_2)\ (p_1)\ (p_2), \\ (p_1^2 p_2)\ (p_1 p_2)\,; \end{array}$$

for, on suppressing p_2 in each, we are left with

$$(p_1^2)\,(p_1)\,;$$

and on suppressing p_1 there remains

$$(p_2)^2.$$

These four separations

$$\left.\begin{array}{l} (p_1^2)\ (p_1 p_2)\ (p_2) \\ (p_1^2)\ (p_1)\ (p_2)^2 \\ (p_1^2 p_2)\ (p_1)\ (p_2) \\ (p_1^2 p_2)\ (p_1 p_2) \end{array}\right\} \text{Set } \{(21), (1^2)\},$$

form a set which is defined by two partitions, one appertaining to each of the two numbers which define the multiplicity of the separable partition.

Thus the first number 3 of the multiplicity occurs in each separation of the group in the partition (21), and the second number 2 in the partition (1^2).

There are as many sets of separations as there are combinations of a partition of 3 with a partition of 2.

In the present instance there are six sets, viz.,

$$S\{(3),\ (2)\},$$
$$S\{(3),\ (1^2)\},$$
$$S\{(21),\ (2)\},$$
$$S\{(21),\ (1^2)\},$$
$$S\{(1^3),\ (2)\},$$
$$S\{(1^3),\ (1^2)\}.$$

In general, if

$$(p_1^{\pi_1}\, p_2^{\pi_2}\, p_3^{\pi_3} \ldots)$$

be the separable partition, the multiplicity is

$$(\pi_1\ \pi_2\ \pi_3 \ldots)\ ;$$

and if the unipartite π_s possess ρ_s partitions, the separations can be arranged in

$$\rho_1\ \rho_2\ \rho_3 \ldots$$

sets.

I observe that the notion of sets of separations enters in a fundamental manner into the theory of symmetric functions.

Art. 6. One of the first problems encountered in the arithmetical theory is the enumeration of the separations of a given partition. I shall prove that the number of separations of the partition

$$(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$$

is identical with the number of partitions of the multipartite number

$$(\overline{\pi_1 \pi_2 \pi_3 \ldots}),$$

formed from the multiplicity of the separable partition.*

If we separate $(p_1^{\pi_1} p_2^{\pi_2} p_3^{\pi_3} \ldots)$ so that one separate of the separation is

$$(q_1^{\chi_1} q_2^{\chi_2} q_3^{\chi_3} \ldots),$$

it is clear that we can partition the multipartite $(\overline{\pi_1 \pi_2 \pi_3 \ldots})$ in such wise that one part of the partition is

$$(\overline{\chi_1 \chi_2 \chi_3 \ldots}).$$

* Note that the rule ———— distinguishes a multipartite number from a partition of a unipartite number.

It is also manifest that to each separate of the separation corresponds a part of the partition, and that we obtain a partition

$$(\overline{\chi_1\chi_2\chi_3\ldots}\quad\ldots\quad\ldots)$$

of the multipartite

$$(\overline{\pi_1\pi_2\pi_3\ldots}),$$

in correspondence with each separation

$$(q_1^{\chi_1}q_2^{\chi_2}q_3^{\chi_3}\ldots)\ (\ldots)\ (\ldots)\ldots$$

of the partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots).$$

There is, therefore, identity of enumeration. Also, the enumeration of the separations into k separates is identical with that of the partitions into k parts.

Ex. gr., the subjoined correspondence :—

Separations.	Partitions.
(p^2q^2)	$(\overline{22})$,
$(p^2)\ (q^2)$,	$(\overline{20}\ \overline{02})$,
$(p^2q)\ (q)$,	$(\overline{21}\ \overline{01})$,
$(p)\ (pq^2)$,	$(\overline{10}\ \overline{12})$,
$(pq)^2$,	$(\overline{11}^2)$,
$(p^2)\ (q)^2$,	$(\overline{20}\ \overline{01}^2)$,
$(p)^2\ (q^2)$,	$(\overline{10}^2\ \overline{02})$,
$(pq)\ (p)\ (q)$,	$(\overline{11}\ \overline{10}\ \overline{01})$,
$(p)^2\ (q)^2$,	$(\overline{10}^2\ \overline{01}^2)$.

Art. 7. It is important to take note of the fact that the subject of the separations of partitions of unipartite numbers necessitates the consideration of the partitions of multipartite numbers.

The partitions of a multipartite number are divisible into sets in the same manner as the separations of a unipartite number. In the case of the number

$$(\pi_1\pi_2\pi_3\ldots)$$

there are $\rho_1\rho_2\rho_3\ldots$ sets, where ρ_s is the number of partitions of the unipartite π_s.

The same succession of numbers may be employed to denote either a multipartite number or the partition of a unipartite number, so that we naturally find great similarity between the theories of unipartite partitions and of multipartite numbers. We see above that the separations of partitions of unipartites are in co-relation with the partitions of multipartites.

Art. 8. In a natural manner the separations of a partition of a multipartite present themselves for consideration. In correspondence we find what may be termed a double separation of a partition of a unipartite.

Ex. gr. Consider the partition $(\overline{20}\ \overline{01}^2)$ of the multipartite $(\overline{22})$, which is co-related to the separation $(p^2)\ (q)^2$ of the partition (p^2q^2) of the unipartite $2p + 2q$.

Of this partition we find a separation

$$(\overline{20}\ \overline{01})\ (\overline{01})$$

in correspondence with a *double* separation

$$\{(p^2)\ (q),\ (q)\}$$

of the partition (p^2q^2).

Hence the enumeration of the separations of a multipartite partition is identical with that of the double separations of a unipartite partition.

In general n-tuple separations of unipartite numbers correspond one-to-one with $n - 1$-tuple separations of multipartite numbers.

For the present I leave the subject of separations, merely remarking that the theory was made the basis of all the memoirs on symmetric functions to which reference has been given, and that algebraically considered they are of extraordinary interest.

§ 2. The Graphical Representation of Partitions.

Art. 9. In an important contribution to the theory of unipartite partitions Sylvester* adopted a graphical method which threw great light on the subject, and was fruitful in algebraical results.

The method consisted in arranging rows of nodes, each row corresponding to a part of the partition and containing as many nodes as the number expressing the magnitude of the part.

Ex. gr., the partition (32^21) has the graph

• • •
• •
• •
•

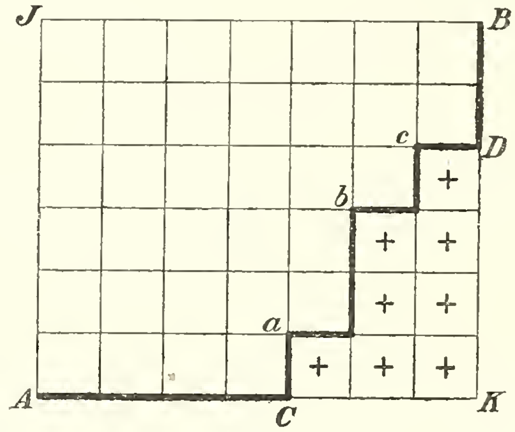
* "A Constructive Theory of Partitions," by J. J. Sylvester, with insertions by Dr. F. Franklin, 'American Journal of Mathematics,' vol. 5.

This method cannot be simply extended to the case of multipartite partitions, though some progress can be made in this direction as will be shown.

Art. 10. SYLVESTER's graphs naturally present themselves in the graphical representation of the compositions of bipartite numbers.

As the graph of a bipartite $(\overline{pq})$ we take $p + 1$ lines parallel and at equal distances apart and cut them by $q + 1$ other lines at equal distances apart and at right angles to the former; (N.B. The right angle is not essential,) thus forming a reticulation or lattice.

The figure represents the reticulation of the bipartite $(\overline{76})$.



I recall that A, B are the initial and final points of the graph, and that the remaining intersections are termed the "points" of the graph.

The lines of the graph have either the direction AK (called the α direction) or the direction AJ (called the β direction).

Each line is made up of segments, and we speak of α-segments and of β-segments, indicating that the lines, on which lie the segments, are in the α and β directions.

If the bipartite $(\overline{pq})$ have a composition

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3}\ \ldots),$$

the composition is delineated upon the graph, as follows:—

Starting from the point A we pass over p_1 α-segments and then over q_1 β-segments, and place a node at the point arrived at. Starting again from this node, we pass over p_2 α-segments and q_2 β-segments, and place a second node at the point then reached; we proceed similarly with the other parts of the composition until finally the point B is reached. At this point it is not necessary to place a node. In this manner the composition containing θ parts is represented by $\theta - 1$ nodes placed at $\theta - 1$ different "points" of the graph.

The segments, passed over in tracing the composition, form a line of route through the reticulation. In general many compositions have the same line of route. Along every line of route there are $p + q - 1$ "points," which *may* be nodes. A certain number of these points *must* be nodes. These occur at all points where there is a change from the β to the α direction. They are termed "essential nodes."

In the line of route traced in the figure the points a, b, c are essential nodes.

Along a line of route there is a composition which is depicted by the essential nodes alone. This is termed the "principal composition along the line of route."

In the figure the principal composition is

$$(\overline{41}\ \overline{12}\ \overline{11}\ \overline{12}).$$

I have shown (*loc. cit.*) that the number of lines of route which possess s essential nodes is

$$\binom{p}{s}\binom{q}{s};$$

and that the total number of lines of route is

$$\sum_{s=0}^{s=q}\binom{p}{s}\binom{q}{s} = \binom{p+q}{p} \qquad (p \geq q).$$

We remark that the line of route divides the reticulation into two portions, an upper portion AJBDC, and a lower portion CKD.

Placing a Sylvester-node in each square of the lower portion, we recognise at once SYLVESTER'S regularised graph of the partition

$$(32^21)$$

of the unipartite number 8.

Similarly, from the upper portion, we obtain SYLVESTER'S regularised graph of the partition

$$(7^265^24),$$

of the unipartite number 34 (thirty-four).

Whatever be the line of route, we simultaneously exhibit two of SYLVESTER'S regularised graphs, one of a partition of the unipartite N, and one of a partition of the unipartite $pq - \text{N}$.

The two partitions may be termed complementary in respect of the unipartite number pq.

Art. 11. This interesting bond between the partitions of unipartites and the compositions of bipartites, I propose to submit to a detailed examination. The partitions, with which we are concerned, are limited in magnitude of part to p, and in number of parts to q.

Moreover (as in the bipartite), we may suppose the numbers p, q, to be interchanged. This would simply amount to rotating the reticulation through a right angle.

The line of route partitions the reticulation into two parts, each of which may be regarded as a partition of a unipartite. In fact, a line of route graphically represents a pair of partitions.

These partitions can be equally depicted by the essential nodes only that occur along the line of route. It will be more convenient, for some purposes, to take the line of route, and not merely the essential nodes, to be the graphical representation. Attention for the present will be limited to the partition lying to the North-West of the line of route (*i.e.*, towards the point J). A line of route involves bends $\lrcorner$, termed "left-bends," and bends $\ulcorner$, termed "right-bends." Essential nodes occur at the angular points of the latter. The North-West partition has as many *different* parts as there are left-bends on the line of route. The number of lines of route which have s left-bends is equal to the number which have s right-bends. This may be seen by rotating the reticulation through two right angles.

This number is $\binom{p}{s}\binom{q}{s}$ (*loc. cit.*, Art. 22).

Hence :—

"The number of partitions of all numbers into s different parts limited in magnitude to p and in number to q is

$$\binom{p}{s}\binom{q}{s}."$$

Art. 12. The number of different partitions is equal to the number of lines of route, and this is

$$\sum_s \binom{p}{s}\binom{q}{s} = \binom{p+q}{p} \qquad (p \nless q).$$

Hence :—

"The number of partitions, of all numbers, into parts limited in magnitude to p, and in number to q is

$$\binom{p+q}{p}."$$

Art. 13. A line of route, with s left-bends, has either $s-1$, s or $s+1$ right-bends. If it commences by tracing an α-segment and ends by tracing a β-segment, the number is $s-1$. If it commences by tracing an α-segment and ends by tracing an α-segment, the number is s. If a β-segment begins and a β-segment ends the line of route, the number is s, and if a β-segment begins and an α-segment ends, the number is $s+1$.

If an α-segment begins the line, the North-West partition has exactly q parts. If an α-segment ends, the highest part is less than p. If a β-segment begins, the number of parts is less than q, and if a β-segment ends, the highest part is equal to p.

Inspection of the Memoir cited shows that the enumeration of the lines of route possessing s left-bends and $s-1$, s, $s+1$ right-bends respectively are given by

$$\binom{p-1}{s-1} \quad \binom{q-1}{s-1},$$

$$\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s},$$

$$\binom{p-1}{s}\binom{q-1}{s};$$

hence :—

"The number of partitions of all numbers which have exactly q parts, a highest part equal to p, and s different parts is

$$\binom{p-1}{s-1}\binom{q-1}{s-1}."$$

"The number of partitions of all numbers which have exactly q parts, a highest part less than p, and s different parts, *or* which have less than q parts, a highest part equal to p, and s different parts is

$$\binom{p-1}{s}\binom{q-1}{s-1}+\binom{p-1}{s-1}\binom{q-1}{s}."$$

"The number of partitions of all numbers which have less than q parts, a highest part less than p, and s different parts is

$$\binom{p-1}{s}\binom{q-1}{s}."$$

Ex. gr. If $p=q=3$, $s=2$ the partitions enumerated by these three theorems are

$$(31^2),\quad (3^21),\quad (32^2),\quad (3^22)$$
$$(31),\quad (21^2),\quad (32),\quad (2^21)$$
$$(21)$$

respectively.

Art. 14. It is clear that all identical relations between binomial coefficients yield results in this theory of partitions. *Ex. gr.*, such relations as

$$\binom{p+q}{p}-\binom{p+q-1}{p-1}=\binom{p+q-1}{p}$$

$$\binom{p-1}{s}\binom{q-1}{s}-\binom{p-2}{s}\binom{q-1}{s}=\binom{p-2}{s-1}\binom{q-1}{s}$$

admit of immediate interpretation.

Art. 15. A line of route has in general right and left bends. The whole number of bends may be even or uneven. To determine the number of lines, with a given number of bends, we must separate the two cases. If the bends be $2k$ in number, k must be right and k left and the enumeration gives

$$\binom{p-1}{k}\binom{q-1}{k-1}+\binom{p-1}{k-1}\binom{q-1}{k}.$$

If the bends be $2k+1$ in number we may have k right-bends and $k+1$ left-bends or $k+1$ right and k left. The enumeration gives

$$2\binom{p-1}{k}\binom{q-1}{k}.$$

Hence the number of pairs of complementary partitions which have each k different parts is

$$\binom{p-1}{k}\binom{q-1}{k-1}+\binom{p-1}{k-1}\binom{q-1}{k};$$

and the number of pairs in which the partitions have k and $k+1$ different parts respectively, is

$$2\binom{p-1}{k}\binom{q-1}{k}.$$

Adding the number of lines of route with $2k-1$ bends to the number with $2k$ bends we obtain the number

$$\frac{p+q}{k}\binom{p-1}{k-1}\binom{q-1}{k-1}.$$

Hence the identity,

$$(p+q)\sum_k \frac{1}{k}\binom{p-1}{k-1}\binom{q-1}{k-1}=\binom{p+q}{p};$$

or

$$\binom{p-1}{0}\binom{q-1}{0}+\tfrac{1}{2}\binom{p-1}{1}\binom{q-1}{1}+\tfrac{1}{3}\binom{p-1}{2}\binom{q-1}{2}+\ldots=\frac{(p+q-1)!}{p!\,q!};$$

which may be established independently.

Art. 16. Consider the lines of route which pass through a particular point P of the graph, say the point distant from A, a α-segments and b β-segments. In the reticulation AP we may draw any line of route, and any line also in the reticulation PB. In AB we can thus obtain

$$\binom{a+b}{a}\binom{p+q-a-b}{p-a}$$

lines of route passing through P.

In the associated North-West partitions, the highest part $\not< a$ and the number of parts $\not< q-b$.

Put $q-b=c$, and we obtain the theorem :—

"Of the partitions whose parts are limited in magnitude to p and in number to q, there are,

$$\binom{q+a-c}{a}\binom{p-a+c}{c},$$

such that the highest part $\not< a$ and number of parts $\not< c$."

In particular, if

$$a+b=p,$$

or

$$a-c=p-q,$$

the number is

$$\binom{p}{p-a}\binom{q}{p-a};$$

which enumerates the lines of route with $p-a$ left-bends. There is thus a one-to-one correspondence between the lines of route passing through the point for which $(a, b) = (a, p-a)$, and the lines of route with $p-a$ left-bends, and hence also between the partitions under consideration and those which involve parts of $p-a$ different kinds.

Art. 17. A line of route is a graphical representation of a principal composition of the bipartite. Such a composition being

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3} \ldots \overline{p_sq_s}),$$

the numbers

$$q_1, p_2, q_2, p_3, q_3 \ldots p_s$$

are all superior to zero. A positive number q_1 of the bipart $\overline{p_1q_1}$ is adjacent to a positive number p_2 of the bipart $\overline{p_2q_2}$, and we may assert of the composition that all its contacts are positive-positive (see Art. *loc. cit.*).

Each line of route represents a composition with positive-positive contacts, and there is a one-to-one correspondence. Hence:

"There is a one-to-one correspondence between the compositions, with positive-positive contacts, of the bipartite $\overline{pq}$ and the partitions of all unipartite numbers into parts limited in magnitude by p and in number by q."

Further, there are as many such compositions with $s+1$ parts as there are such partitions which involve s *different* parts.

Art. 18. Every line of route through the reticulation of $\overline{pq}$ may be represented by a permutation of the letters in $\alpha^p\beta^q$. We have merely to write down the α and β segments as they occur along the line of route to obtain such a permutation.

The composition

$$(\overline{p_1q_1}\,\overline{p_2q_2}\,\overline{p_3q_3} \ldots \overline{p_{s+1}\,q_{s+1}}),$$

gives the permutation

$$\alpha^{p_1}\beta^{q_1}\alpha^{p_2}\beta^{q_2}\alpha^{p_3}\beta^{q_3}\ldots\alpha^{p_{s+1}}\beta^{q_{s+1}}.$$

From what has gone before it will be seen that every permutation of $\alpha^p\beta^q$ corresponds to a partition of a unipartite number into parts limited in magnitude to p and in number to q. Every theorem in permutations of two *different* letters will thus yield a theorem in partitions of unipartite numbers.

The North-West partition associated with the above-written permutation is easily seen to be (writing the parts in ascending order as regards magnitude, viz.: in SYLVESTER's regularised orders reversed)

$$\left(p_1^{q_1}\,\overline{p_1+p_2}^{q_2}\,\overline{p_1+p_2+p_3}^{q_3}\ldots\overline{p_1+p_2+\ldots+p_{s+1}}^{q_{s+1}}\right).$$

This is a partition of the unipartite

$$q_1p_1+q_2(p_1+p_2)+q_3(p_1+p_2+p_3)+\ldots+q_{s+1}(p_1+p_2+\ldots+p_{s+1})$$

into $q_1+q_2+q_3+\ldots+q_{s+1}$ parts, the highest part being $p_1+p_2+p_3+\ldots+p_{s+1}$.

If p_1 be zero there will be less than q parts. If q_{s+1} be zero the highest part will be less than p. On the other hand if p_1 be not zero there are exactly q parts, and if q_{s+1} be not zero the highest part is p.

Art. 19. Observe that we have a fourfold correspondence, viz., between

(1.) The lines of route in the reticulation of the bipartite $\overline{pq}$.

(2.) The compositions, with positive-positive contacts, of the bipartite number $\overline{pq}$.

(3.) The permutations of the letters forming the product $\alpha^p\beta^q$.

(4.) The partitions of all unipartite numbers into parts limited in magnitude to p and in number to q.

And also, in particular, between—

(1.) The lines of route with s right-bends or with s left-bends.

(2.) The compositions into $s+1$ parts.

(3.) The permutations with s, $\alpha\beta$ or with s, $\beta\alpha$ contacts.

(4.) The partitions involving s *different* parts.

Art. 20. The generating function for the number of lines of route through the reticulation which possess s left-bends is

$$1+\binom{p}{1}\binom{q}{1}\mu+\binom{p}{2}\binom{q}{2}\mu^2+\ldots+\binom{p}{s}\binom{q}{s}\mu^s+\ldots+\binom{p}{q}\binom{q}{p}\mu^q$$

which is the coefficient of $\alpha^p\beta^q$ in the product

$$(\alpha+\mu\beta)^p\ (\alpha+\beta)^q;$$

(see *loc. cit.*, Art. 24),

and this is the coefficient of $\alpha^p\beta^q$ in the development of the fraction

$$\frac{1}{1-\alpha-\beta+(1-\mu)\,\alpha\beta};$$

(see "Memoir on a Certain Class of Generating Functions in the Theory of Numbers," 'Phil. Trans.,' Roy. Soc. of London, vol. 185 (1894), A, pp. 111–160), and this fraction may be written

$$\sum_0 \frac{\alpha^s\beta^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}}\mu^s.$$

Hence

$$\frac{\alpha^s\beta^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}}$$

is the generating function for the lines of route in all bipartite reticulations which possess s left-bends *or* s right-bends, and also for the other entities in correspondence therewith. In particular it enumerates all unipartite partitions into s *different* parts limited, in any desired manner, in regard to number and magnitude.

Art. 21. Various theorems in algebra are derivable from the foregoing theorems.

The generating function for the partitions of all unipartite numbers into parts limited in magnitude to p and in number to q is

$$\frac{1}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2\,.\,1-ax^3\,\ldots\,1-ax^p};$$

the enumeration being given by the coefficients of a^qx^{pq} in the ascending expansion.

The G.F. is redundant as we are only concerned with that portion, of the expanded form, which proceeds by powers of ax^p.

The foregoing theory enables us to isolate this portion, inasmuch as we know it to have the expression

$$1+\binom{p+1}{1}ax^p+\binom{p+2}{2}a^2x^{2p}+\binom{p+3}{3}a^3x^{3p}+\ldots+\binom{p+q}{p}a^qx^{qp}+\ldots$$

which may be written

$$(1-ax^p)^{-p-1}.$$

As a verification of the simplest cases we find the identities

$$\frac{1}{1-a\,.\,1-x\,.\,1-ax}=\frac{1}{(1-ax)^2}\left(1+\frac{a}{1-a}+\frac{x}{1-x}\right)$$

$$\frac{1}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2}=\frac{1}{(1-ax^2)^3}\left\{1+\frac{a}{1-a}+\frac{x}{1-x}+\frac{ax}{1-a}+\frac{ax\,(1-ax^2)}{1-a\,.\,1-ax}\right\},$$

a simple inspection of which demonstrates the validity of the theorem in these cases

To obtain a general formula write

$$\frac{(1-ax^p)^{p+1}}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2\ldots 1-ax^p\,.}$$

$$=\frac{(1-ax^{p-1})^p}{1-a\,.\,1-x\,.\,1-ax\,.\,1-ax^2\ldots 1-ax^{p-1}}+\mathrm{U}_p(x);$$

then

$$\mathrm{U}_p(x)=\frac{(1-ax^p)^{p+1}-(1-ax^p)(1-ax^{p-1})^p}{1-a\,.\,1-x\,.\,1-ax\,.\,.\,1-ax^2\ldots 1-ax^p}$$

$$=\frac{ax^{p-1}\{(1-ax^p)^{p-1}+(1-ax^p)^{p-2}(1-ax^{p-1})+\ldots+(1-ax^{p-1})^{p-1}\}}{1-a\,.\,1-ax\,.\,1-ax^2\ldots 1-ax^{p-1}}$$

and we have in succession

$$\mathrm{U}_0(x)=\frac{1}{1-x},$$

$$\mathrm{U}_1(x)=\frac{a}{1-a},$$

$$\mathrm{U}_2(x)=\frac{ax}{1-a}+\frac{ax(1-ax^2)}{1-a\,.1-ax},$$

$$\mathrm{U}_3(x)=\frac{ax^2(1-ax^2)}{1-a\,.1-ax}+\frac{ax^2(1-ax^3)}{1-a\,.1-ax}+\frac{ax^2(1-ax^3)^2}{1-a\,.1-ax\,.1-ax^2},$$

$$\mathrm{U}_4(x)=\frac{ax^3(1-ax^3)^2}{1-a\,.1-ax\,.1-ax^2}+\frac{ax^3(1-ax^3)(1-ax^4)}{1-a\,.1-ax\,.1-ax^2}$$

$$+\frac{ax^3(1-ax^4)^2}{1-a\,.1-ax\,.1-ax^2}+\frac{ax^3(1-ax^4)^3}{1-a\,.1-ax\,.1-ax^2\,.1-ax^3};$$

and, in general,

$$\mathrm{U}_p(x)=\frac{ax^{p-1}(1-ax^{p-1})^{p-2}}{1-a\,.1-ax\ldots 1-ax^{p-2}}+\frac{ax^{p-1}(1-ax^{p-1})^{p-3}(1-ax^p)}{1-a\,.1-ax\ldots 1-ax^{p-2}}$$

$$+\frac{ax^{p-1}(1-ax^{p-1})^{p-4}(1-ax^p)^2}{1-a\,.1-ax\,.\,.\,1-ax^{p-2}}+\ldots$$

$$+\frac{ax^{p-1}(1-ax^p)^{p-2}}{1-a\,.1-ax\ldots 1-ax^{p-2}}+\frac{ax^{p-1}(1-ax^p)^{p-1}}{1-a\,.1-ax\ldots 1-ax^{p-2}\,.1-ax^{p-1}}.$$

Hence,

$$\frac{1}{1-x\,.1-a\,.1-ax\,.1-ax^2\ldots 1-ax^p}=\frac{1}{(1-ax^p)^{p+1}}\sum_0^p \mathrm{U}_p(x);$$

a valuable expansion.

Simple inspection of this formula shows that $(1-ax^p)^{-p-1}$ represents that portion of the G.F. which is a function of ax^p only.

Art. 22. Again, the partitions of all unipartite numbers into s *different* parts, limited in magnitude to p and in number to q, are enumerated by the coefficient of $a^q b^s x^{pq}$ in the development of the product

$$\frac{1}{1-x}\cdot\frac{1}{1-a}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)\dots\left(1+\frac{abx^p}{1-ax^p}\right);$$

or by the coefficient of $a^q x^{pq}$ in

$$\frac{a^s}{1-x\,.\,1-a}\,\Sigma\,\frac{x^{k_1+k_2+k_3+\dots+k_s}}{(1-ax^{k_1})(1-ax^{k_2})(1-ax^{k_3})\dots(1-ax^{k_s})},$$

where $k_1, k_2, k_3, \dots k_s$ are any s different numbers drawn from the natural series

$$1, 2, 3, \;\dots p\,;$$

and the summation is in respect of all such selections. This is the coefficient of

$$a^{q-s}\,x^{pq-\binom{s+1}{2}}$$

in

$$\frac{1}{1-x\,.\,1-a}\,\Sigma\,\frac{x^{k_1+k_2+k_3+\dots+k_s-\binom{s+1}{2}}}{(1-ax^{k_1})(1-ax^{k_2})(1-ax^{k_3})\dots(1-ax^{k_s})}.$$

Taking the former of these two expressions; inasmuch as we know from the reticulation theory that the coefficient of $a^q x^{pq}$ is $\binom{p}{s}\binom{q}{s}$, we find that the effective portion of the generating function is

$$\binom{p}{s}\binom{s}{s}(ax^p)^s+\binom{p}{s}\binom{s+1}{s}(ax^p)^{s+1}+\dots\textit{ad inf.},$$

which is

$$\binom{p}{s}\frac{(ax^p)^s}{(1-ax^p)^{s+1}};$$

viz., we have succeeded in isolating that portion of the generating function which proceeds by powers of ax^p only.

The latter expression of the generating function also, is seen to have an effective portion

$$\binom{p}{s}\frac{x^{ps-\binom{s+1}{2}}}{(1-ax^p)^{s+1}}.$$

The isolations thus effected would, I believe, be difficult to accomplish algebraically.

Art. 23. Again, regarding p and q as constant and s as variable, we know that the coefficient of $(ax^p)^q$ in the product

$$\frac{1}{1-x.1-a}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)..\left(1+\frac{abx^p}{1-ax^p}\right),$$

is

$$\binom{p}{0}\binom{q}{0}+\binom{p}{1}\binom{q}{1}b+\binom{p}{2}\binom{q}{2}b^2+\ldots+\binom{p}{q}\binom{q}{q}b^q;$$

calling this expression B_q, the effective portion of the generating function is written

$$1+B_1ax^p+B_2(ax^p)^2+B_3(ax^p)^3+\ldots \text{ ad inf.}$$

Art. 24. I recall now the generating function which enumerates the partitions of all unipartite numbers into parts limited in magnitude by p and in number by q, viz. :—

$$\frac{1}{(1-x)(1-a)(1-ax)(1-ax^2)\ldots(1-ax^p)}$$

The coefficient of a^q in this development is well known to be

$$\frac{(1-x^{q+1})(1-x^{q+2})\ldots(1-x^{q+p})}{(1-x)^2(1-x^2)(1-x^3)\ldots(1-x^p)}.$$

Hence the coefficient of a^qx^{pq} in the former is the same as the coefficient of x^{pq} in the latter. This we know to have the value $\binom{p+q}{q}$.

Art. 25. Numerous theorems of *isolation*, in the senses in which the word is employed in this paper, may be obtained from the reticulation theorems. I proceed to give some of those which present features of interest.

A previous result was to the effect that the number of partitions of all numbers which have exactly q parts, a highest part equal to p and s *different* parts is enumerated by

$$\binom{p-1}{s-1}\binom{q-1}{s-1}.$$

Hence, without specification of s, the number is

$$\sum_s\binom{p-1}{s}\binom{q-1}{s}=\binom{p+q-2}{p-1}.$$

This enumeration is also given by the coefficient of $a^{q-1}x^{p(q-1)}$ in the function

$$\frac{1}{(1-x)(1-ax)\ldots(1-ax^p)}$$

Hence we can isolate that portion of the generating function which contains only powers of ax^p. It is

$$1 + \binom{p}{1} ax^p + \binom{p+1}{2} (ax^p)^2 + \binom{p+2}{3} (ax^p)^3 + \ldots\ldots;$$

or

$$\frac{1}{(1-ax^p)^p}.$$

This fact leads as before to an expansion theorem.

Putting

$$\frac{(1-ax^p)^p}{1-x\,.\,1-ax\ldots.1-ax^p} = \frac{(1-ax^{p-1})^{p-1}}{1-x\,.\,1-ax\ldots.1-ax^{p-1}} + \mathrm{V}_p(x),$$

then of course

$$\frac{1}{1-x\,.\,1-ax\ldots.1-ax^p} = \frac{1}{(1-ax^p)^p}\sum_{p=1}^{p=p} \mathrm{V}_p(x),$$

and we find in succession

$$\mathrm{V}_1(x) = \frac{1}{1-x},$$

$$\mathrm{V}_2(x) = \frac{ax}{1-ax},$$

$$\mathrm{V}_3(x) = \frac{ax^2}{1-ax} + \frac{ax^2(1-ax^3)}{1-ax\,.\,1-ax^2}$$

$$\mathrm{V}_4(x) = \frac{ax^3(1-ax^3)}{1-ax\,.\,1-ax^2} + \frac{ax^3(1-ax^4)}{1-ax\,.\,1-ax^2} + \frac{ax^3(1-ax^4)^2}{1-ax\,.\,1-ax^2\,.\,1-ax^3},$$

and in general

$$\mathrm{V}_p(x) = \frac{ax^{p-1}(1-ax^{p-1})^{p-3}}{1-ax\,.\,1-ax^2\ldots..1-ax^{p-2}} + \frac{ax^{p-1}(1-ax^{p-1})^{p-4}(1-ax^p)}{1-ax\,.\,1-ax^2\ldots.1-ax^{p-2}}$$

$$+\ldots\ldots+\frac{ax^{p-1}(1-ax^p)^{p-3}}{1-ax\,.\,1-ax^2\ldots.1-ax^{p-2}} + \frac{ax^{p-1}(1-ax^p)^{p-2}}{1-ax\,.\,1-ax^2\ldots.1-ax^{p-2}\,.\,1-ax^{p-1}}.$$

The simplest cases, omitting the trivial one corresponding to $p=1$, are

$$\frac{1}{1-x\,.\,1-ax\,.\,1-ax^2} = \frac{1}{(1-ax^2)^2}\left\{\frac{1}{1-x} + \frac{ax}{1-ax}\right\};$$

$$\frac{1}{1-x\,.\,1-ax\,.\,1-ax^2\,.\,1-ax^3} = \frac{1}{(1-ax^3)^3}\left\{\frac{1}{1-x} + \frac{ax}{1-ax} + \frac{ax^2}{1-ax} + \frac{ax^2(1-ax^3)}{1-ax.1-ax^2}\right\};$$

$$\frac{1}{1-x.1-ax.1-ax^2.1-ax^3.1-ax^4} = \frac{1}{(1-ax^4)^4}\left\{\frac{1}{1-x}+\frac{ax}{1-ax}+\frac{ax^2}{1-ax}+\frac{ax^2(1-ax^3)}{1-ax.1-ax^2}\right.$$
$$\left.+\frac{ax^3(1-ax^3)}{1-ax.1-ax^2}+\frac{ax^3(1-ax^4)}{1-ax.1-ax^2}+\frac{ax^3(1-ax^4)^2}{1-ax.1-ax^2.1-ax^3}\right\}.$$

The fractions, in the brackets { }, may be united in batches, but I prefer to leave them as written, as the law of development is shown the better. Also the fraction $\frac{1}{1-ax^p}$ may be cancelled if desired. I have not done so, in order to keep in touch with the arithmetic.

Inspection of these expansions establishes the isolation therein independently.

Art. 26. In the function

$$\frac{1}{1-x.1-ax.1-ax^2....1-ax^p},$$

the coefficient of a^{q-1} is, as is well known,

$$\frac{1-x^q.1-x^{q+1}....1-x^{q+p-2}}{1-x.1-x^2....1-x^{p-1}}x^{q-1}.$$

Hence the coefficient of $a^{q-1}x^{p(q-1)}$ in the former is equal to the coefficient of $x^{p(q-1)}$ in the latter; *i.e.*, to the coefficient of $x^{(p-1)(q-1)}$ in

$$\frac{1-x^q.1-x^{q+1}....1-x^{q+p-2}}{(1-x)^2 1-x^2....1-x^{p-1}},$$

which we know to be $\binom{p+q-2}{p-1}$, verifying our result.

For a given value of s the partitions are enumerated by the coefficient of $a^q b^s x^{pq}$ in the product

$$\frac{1}{1-x}\left(1+\frac{abx}{1-ax}\right)\left(1+\frac{abx^2}{1-ax^2}\right)...\left(\frac{abx^p}{1-ax^p}\right);$$

or, the same thing, by the coefficient of $(ax^p)^{q-1}$ in the function

$$\frac{a^{s-1}}{1-x.1-ax^p}\sum\frac{x^{k_1+k_2+...+k_{s-1}}}{(1-ax^{k_1})(1-ax^{k_2})...(1-ax^{k_{s-1}})},$$

wherein $k_1, k_2, ... k_{s-1}$ denote any selection of $s-1$ different integers drawn from the natural series 1, 2, 3, ... $p-1$.

This coefficient, from previous work, has the value

$$\binom{p-1}{s-1}\binom{q-1}{s-1}$$

hence the portion of this function which consists only of powers of ax^p is

$$\binom{p-1}{s-1}\left\{(ax^p)^{s-1} + \binom{s}{1}(ax^p)^s + \binom{s+1}{2}(ax^p)^{s+1} + \ldots\right\}$$

or

$$\binom{p-1}{s-1}\frac{(ax^p)^{s+1}}{(1-ax^p)^s}.$$

Hence also from the function

$$\frac{a^{s-1}}{1-x}\Sigma\frac{x^{k_1+k_2+\ldots+k_{s-1}}}{(1-ax^{k_1})(1-ax^{k_2})\ldots(1-ax^{k_{s-1}})}$$

we can isolate the portion

$$\binom{p-1}{s-1}\left(\frac{ax^p}{1-ax^p}\right)^{s-1}.$$

Ex. gr. for $p = 3$, $s = 2$ we can verify that

$$2\frac{ax^3}{1-ax^3}$$

can be isolated from

$$\frac{ax}{1-x\,.\,1-ax} + \frac{ax^2}{1-x\,.\,1-ax^2}.$$

Art. 27. Before generalising the foregoing it will be proper to give another correspondence between the compositions and partitions of unipartite numbers which leads readily to theorems concerning the generating functions of partitions when the parts are *unrepeated.*

Writing down any composition of the unipartite p, viz. :

$$(p_1p_2p_3\ldots\ldots ; p_s),$$

we can at once construct a regularised partition, viz. :—

$$(p_1, p_1+p_2, p_1+p_2+p_3, \ldots\ldots, p)$$

of the number $sp_1 + (s-1)p_2 + (s-2)p_3 + \ldots\ldots + p_s$.

The correspondence is between the compositions of p into s parts and the partitions of all unipartite numbers into s *unequal* parts limited in magnitude to p and possessing a part p.

The numbers whose partitions appear are the natural series extending from

$$p + \binom{s}{2}, \text{ to } sp - \binom{s}{2}.$$

For the enumeration we must take the coefficient of $a^s x^{sp-\binom{s}{2}}$ in the development of the generating function

$$\frac{ax^p}{1-x}(1+ax)(1+ax^2)(1+ax^3)\dots(1+ax^{p-1});$$

or the coefficient of $(ax^{p-\frac{1}{2}s})^{s-1}$ in

$$\frac{1}{1-x}(1+ax)(1+ax^2)(1+ax^3)\dots(1+ax^{p-1}).$$

But the number of compositions of p into s parts is

$$\binom{p-1}{s-1},$$

and thence we see that a term in the development of

$$\frac{1}{1-x}(1+ax)(1+ax^2)(1+ax^3)\dots(1+ax^{p-1}),$$

is

$$\binom{p-1}{s-1} a^{s-1} x^{(p-\frac{1}{2}s)(s-1)};$$

and, giving s successive values, we can isolate, from this product, a portion

$$1+\binom{p-1}{1}ax^{p-1}+\binom{p-1}{2}a^2x^{2p-3}+\binom{p-1}{3}a^3x^{3p-6}+\dots+a^{p-1}x^{\binom{p}{2}};$$

or symbolically $(1+ax^p)^{p-1}$, where after expansion $x^{\mu p}$ is to be replaced by $x^{\mu p-\frac{1}{2}\mu(\mu+1)}$.

Art. 28. By leaving s unspecified we can readily reach a theorem concerning the product,

$$\frac{1}{1-x}(1+x)(1+x^2)(1+x^3)\dots(1+x^{p-1}).$$

It is easy to show that the coefficient of $x^{\binom{p}{2}}$, in the development, is 2^{p-1}. We may say that the number of partitions of $\binom{p}{2}$ and all lower numbers into unrepeated parts not exceeding $p-1$ in magnitude is 2^{p-1}. For $p=5$ these partitions are:

4321				
432	43	31	4	
431	42	21	3	, 16 in number.
421	41		2	
321	32		1	
			0	

Art. 29. It is obvious that, by the same process, we can obtain a correspondence

between the compositions and partitions of multipartite numbers. In the bipartite case we pass from any composition

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ \overline{p_3q_3} \cdot \ \cdot \overline{p_sq_s})$$

to the regularised partition

$$(\overline{p_1q_1}\ \overline{p_1+p_2,\ q_1+q_2},\ \overline{p_1+p_2+p_3,\ q_1+q_2+q_3} \ldots \overline{pq})$$

of a certain bipartite number.

The correspondence is between the compositions of $\overline{pq}$ into s parts and the partitions of all bipartite numbers into s *unrepeated* biparts, the parts of the biparts being limited in magnitude to p and q respectively, and the highest bipart being $\overline{pq}$.

Or, we may strike out the highest bipart $\overline{pq}$, and then the partition is into $s-1$ unrepeated biparts, the parts of the biparts being limited as before. The partitions are subject to the further restriction that they are regularised in the sense that the unipartite partitions of p and q, that appear in the bipartition, are separately regularised.

Art. 30. Instead of insisting upon this two-fold regularisation, we may, starting from the composition

$$(\overline{p_1q_1},\ \overline{p_2q_2},\ \overline{p_3q_3}, \ldots \overline{p_sq_s}),$$

proceed to the singly regularised partition

$$(\overline{p_1q_1}\ \overline{p_1+p_2,\ q_2}\ \overline{p_1+p_2+p_3,\ q_3} \ldots \overline{pq_s}).$$

There are, in fact, various ways of forming connecting links between compositions and partitions of multipartite numbers whatever the order of multiplicity. These methods may be pursued at pleasure so as to obtain results of more or less interest.

§ 3.

Art. 31. The correspondence set forth between unipartite partitions and bipartite compositions naturally suggests the possibility of a similar correspondence between bipartite partitions and tripartite compositions, and generally between m-partite partitions and $m+1$-partite compositions.

For the graph of the tripartite number $\overline{pqr}$, we take $r+1$ similar graphs of the bipartite $\overline{pq}$, and place them similarly with corresponding lines parallel, and like points lying on straight lines; the graph is completed by drawing these straight lines, which are in a new direction, say the γ direction.

There are three directions through each point of the graph (see *loc. cit.*, Art. 31). There are $\binom{p+q+r}{p,\,q}$* lines of route along which the tripartite compositions are

* This notation explains $\dfrac{(p+q+r)!}{p!\ q!\ r!}$ and so in similar cases.

depicted, one line of route for each permutation of the symbols in the product $\alpha^p\beta^q\gamma^r$. A study of these permutations shows the connection with a certain class of bipartite partitions.

Consider a permutation

$$\alpha^{p_1}\beta^{q_1}\gamma^{r_1}\ \alpha^{p_2}\beta^{q_2}\gamma^{r_2}\ \alpha^{p_3}\beta^{q_3}\gamma^{r_3} \ldots \alpha^{p_s}\beta^{q_s}\gamma^{r_s},$$

which is *not* the most general permutation, but such that, in regard to any section

$$\alpha^{p_k}\beta^{q_k}\gamma^{r_k}$$

of the permutation

(1.) r_k must be superior to zero except when $k = s$.
(2.) p_k, q_k may be either, but not both, zero, except when $k = 1$.

The permutation has $\gamma\alpha$ and $\gamma\beta$ contacts, but no $\beta\alpha$ contact.

In the reticulation corresponding thereto, we have lines of route with $\gamma\alpha$ and $\gamma\beta$ bends but *not* with $\beta\alpha$ bends. All the lines of route with $\beta\alpha$ bends are excluded from consideration. From the permutation we can form a bipartite partition.

$$(\overline{p_1q_1}^{r_1}\ \overline{p_1+p_2, q_1+q_2}^{r_2}\ \overline{p_1+p_2+p_3, q_1+q_2+q_3}^{r_3} \ldots \overline{p_1+p_2+\ldots+p_s, q_1+q_2+\ldots+q_s}^{r_s}),$$

which is regularised in the sense that the partitions of the unipartites p, q, that appear are each separately regularised.

The two parts of the bipartite number thus partitioned are

$$r_1p_1 + r_2(p_1 + p_2) + r_3(p_1 + p_2 + p_3) + \ldots + r_s(p_1 + p_2 + \ldots + p_s),$$
$$r_1q_1 + r_2(q_1 + q_2) + r_3(q_1 + q_2 + q_3) + \ldots + r_s(q_1 + q_2 + \ldots + q_s).$$

The associated principal composition is

$$(\overline{p_1q_1r_1}\ \overline{p_2q_2r_2}\ \overline{p_3q_3r_3} \ldots \overline{p_sq_sr_s}).$$

As before, consider the contacts r_1p_2, r_2p_3, &c. . . . Looking at the whole of the principal compositions, observe that a $\gamma\alpha$ contact in the permutation yields a contact r_kp_{k+1} in the composition in which r_k and p_{k+1} are both superior to zero, say a positive-positive contact. A $\gamma\beta$ contact yields a positive-zero contact and a $\beta\alpha$ contact a zero-positive contact. Hence the present correspondence is only concerned with compositions which possess positive-positive and positive-zero contacts, and not with those which involve contacts of other natures. The bipartite partitions are those of all bipartite numbers into biparts whose parts are limited to p and q respectively in magnitude and whose biparts are limited to r in number.

Art. 32. We have then a one-to-one correspondence. Each bipartite partition of the nature considered is represented graphically by a line of route in a tripartite reticulation. If we please we may regard a pair of bipartite partitions as represented by a line of route, for from the permutation we are also led to the complementary partition,

$$(\overline{p - p_1, q - q_1}^{\,r_1} \quad \overline{p - p_1 - p_2, q - q_1 - q_2}^{\,r_2} \ldots),$$

in which p_1, q_1 *may* be both zero.

Art. 33. It has been shown that the number of lines of route which possess

$$\begin{array}{l} s_{21}\ \beta\alpha \text{ bends,} \\ s_{32}\ \gamma\beta \quad ,, \quad , \\ s_{31}\ \gamma\alpha \quad ,, \quad , \end{array}$$

is

$$\binom{s_{21} + s_{31}}{s_{21}} \binom{p}{s_{21} + s_{31}} \binom{q}{s_{32}} \binom{q + s_{31}}{s_{21} + s_{31}} \binom{r}{s_{32} + s_{31}};*$$

and that this number is the coefficient of

$$\lambda_{21}^{s_{21}} \lambda_{32}^{s_{32}} \lambda_{31}^{s_{31}} \alpha^p \beta^q \gamma^r$$

in the development of

$$(\alpha + \lambda_{21} \beta + \lambda_{31} \gamma)^p (\alpha + \beta + \lambda_{32})^q (\alpha + \beta + \gamma)^r.$$

Here $s_{21} = 0$, and the number in question becomes

$$\binom{p}{s_{31}} \binom{q}{s_{32}} \binom{q + s_{31}}{s_{31}} \binom{r}{s_{32} + s_{31}},$$

whilst the generating function becomes

$$(\alpha + \lambda_{31} \gamma)^p (\alpha + \beta + \lambda_{32} \gamma)^q (\alpha + \beta + \gamma)^r.$$

In this the coefficient of

$$\lambda_{32}^{s_{32}} \lambda_{31}^{s_{31}} \alpha^p \beta^q \gamma^r$$

is equal to the coefficient of the same term in the expansion of the fraction

$$\frac{1}{1 - \alpha - \beta - \gamma + \alpha\beta + (1 - \lambda_{32}) \beta\gamma + (1 - \lambda_{31}) \alpha\gamma - (1 - \lambda_{32}) \alpha\beta\gamma},$$

which is

$$\frac{1}{(1 - \alpha)(1 - \beta)(1 - \gamma) - \lambda_{31} \alpha\gamma - \lambda_{32} \beta\gamma (1 - \alpha)},$$

and the verification is readily carried out.

* 'Phil. Trans.,' vol. 184 (*loc. cit.*), Arts. 34, *et seq.*

Art. 34. The number of lines of route which possess exactly s_{31} $\gamma\alpha$ bends and no $\beta\alpha$ bends is

$$\binom{p}{s_{31}}\binom{q+s_{31}}{s_{31}}\sum_{s_{32}}\binom{q}{s_{32}}\binom{r}{s_{32}+s_{31}}$$

or

$$\binom{p}{s_{31}}\binom{q+s_{31}}{s_{31}}\binom{q+r}{q+s_{31}} \text{ or } \binom{p}{s_{31}}\binom{r}{s_{31}}\binom{q+r}{r}.$$

Also, by putting $\lambda_{31} = \lambda_{32} = \lambda$ in the generating function, we find that the number of lines of route which have s $\gamma\beta$ and $\gamma\alpha$ bends but no $\beta\alpha$ bend is given by the coefficient of $\alpha^p\beta^q\gamma^r$ in

$$\frac{(\alpha\gamma + \beta\gamma - \alpha\beta\gamma)^s}{(1-\alpha)^{s+1}(1-\beta)^{s+1}(1-\gamma)^{s+1}}.$$

Art. 35. It will be convenient to give the complete correspondence in the case of some simple tripartite number, say, $\overline{222}$.

S_{31}	S_{32}	Permutation.	Composition.	Partition.	Number partitioned.
0	0	$\alpha^2\beta^2\gamma^2$	$(\overline{222})$	$(\overline{22}^2)$	$\overline{44}$
1	0	$\alpha\beta^2\gamma\alpha\gamma$	$(\overline{121}\ \overline{101})$	$(\overline{12}\ \overline{22})$	$\overline{34}$
1	0	$\alpha\beta^2\gamma^2\alpha$	$(\overline{122}\ \overline{100})$	$(\overline{12}^2)$	$\overline{24}$
1	0	$\beta^2\gamma\alpha^2\gamma$	$(\overline{021}\ \overline{201})$	$(\overline{02}\ \overline{22})$	$\overline{24}$
1	0	$\alpha\beta\gamma\alpha\beta\gamma$	$(\overline{111}\ \overline{111})$	$(\overline{11}\ \overline{22})$	$\overline{33}$
1	0	$\alpha\gamma\alpha\beta^2\gamma$	$(\overline{101}\ \overline{121})$	$(\overline{10}\ \overline{22})$	$\overline{32}$
1	0	$\beta\gamma\alpha^2\beta\gamma$	$(\overline{011}\ \overline{211})$	$(\overline{01}\ \overline{22})$	$\overline{23}$
1	0	$\beta^2\gamma^2\alpha^2$	$(\overline{022}\ \overline{200})$	$(\overline{02}^2)$	$\overline{04}$
1	0	$\alpha\beta\gamma^2\alpha\beta$	$(\overline{112}\ \overline{110})$	$(\overline{11}^2)$	$\overline{22}$
1	0	$\gamma\alpha^2\beta^2\gamma$	$(\overline{001}\ \overline{221})$	$(\overline{00}\ \overline{22})$	$\overline{22}$
1	0	$\alpha\gamma^2\alpha\beta^2$	$(\overline{102}\ \overline{120})$	$(\overline{10}^2)$	$\overline{20}$
1	0	$\beta\gamma^2\alpha^2\beta$	$(\overline{012}\ \overline{210})$	$(\overline{01}^2)$	$\overline{02}$
1	0	$\gamma^2\alpha^2\beta^2$	$(\overline{002}\ \overline{220})$	$(\overline{00}^2)$	$\overline{00}$
0	1	$\alpha^2\beta\gamma\beta\gamma$	$(\overline{211}\ \overline{011})$	$(\overline{21}\ \overline{22})$	$\overline{43}$
0	1	$\alpha^2\beta\gamma^2\beta$	$(\overline{212}\ \overline{010})$	$(\overline{21}^2)$	$\overline{42}$
0	1	$\alpha^2\gamma\beta^2\gamma$	$(\overline{201}\ \overline{021})$	$(\overline{20}\ \overline{22})$	$\overline{42}$
0	1	$\alpha^2\gamma^2\beta^2$	$(\overline{202}\ \overline{020})$	$(\overline{20}^2)$	$\overline{40}$
1	1	$\alpha\beta\gamma\alpha\gamma\beta$	$(\overline{111}\ \overline{101}\ \overline{010})$	$(\overline{11}\ \overline{21})$	$\overline{32}$
1	1	$\alpha\beta\gamma\beta\gamma\alpha$	$(\overline{111}\ \overline{011}\ \overline{100})$	$(\overline{11}\ \overline{12})$	$\overline{23}$

S_{31}	S_{32}	Permutation.	Composition.	Partition.	Number partitioned.
1	1	$\alpha\gamma\alpha\beta\gamma\beta$	$(\overline{101}\ \overline{111}\ \overline{010})$	$(\overline{10}\ \overline{21})$	$\overline{31}$
1	1	$\beta\gamma\alpha^2\gamma\beta$	$(\overline{011}\ \overline{201}\ \overline{010})$	$(\overline{01}\ \overline{21})$	$\overline{22}$
1	1	$\alpha\gamma\beta^2\gamma\alpha$	$(\overline{101}\ \overline{021}\ \overline{100})$	$(\overline{10}\ \overline{12})$	$\overline{22}$
1	1	$\alpha\gamma\alpha\gamma\beta^2$	$(\overline{101}\ \overline{101}\ \overline{020})$	$(\overline{10}\ \overline{20})$	$\overline{30}$
1	1	$\beta\gamma\beta\gamma\alpha^2$	$(\overline{011}\ \overline{011}\ \overline{200})$	$(\overline{01}\ \overline{02})$	$\overline{03}$
1	1	$\gamma\alpha^2\beta\gamma\beta$	$(\overline{001}\ \overline{211}\ \overline{010})$	$(\overline{00}\ \overline{21})$	$\overline{21}$
1	1	$\alpha\gamma\beta\gamma\alpha\beta$	$(\overline{101}\ \overline{011}\ \overline{110})$	$(\overline{10}\ \overline{11})$	$\overline{21}$
1	1	$\gamma\alpha^2\gamma\beta^2$	$(\overline{001}\ \overline{201}\ \overline{020})$	$(\overline{00}\ \overline{20})$	$\overline{20}$
1	1	$\gamma\beta^2\gamma\alpha^2$	$(\overline{001}\ \overline{021}\ \overline{200})$	$(\overline{00}\ \overline{02})$	$\overline{02}$
1	1	$\gamma\beta\gamma\alpha^2\beta$	$(\overline{001}\ \overline{011}\ \overline{210})$	$(\overline{00}\ \overline{01})$	$\overline{01}$
2	0	$\beta^2\gamma\alpha\gamma\alpha$	$(\overline{021}\ \overline{101}\ \overline{100})$	$(\overline{02}\ \overline{12})$	$\overline{14}$
2	0	$\beta\gamma\alpha\beta\gamma\alpha$	$(\overline{011}\ \overline{111}\ \overline{100})$	$(\overline{01}\ \overline{12})$	$\overline{13}$
2	0	$\gamma\alpha\beta^2\gamma\alpha$	$(\overline{001}\ \overline{121}\ \overline{100})$	$(\overline{00}\ \overline{12})$	$\overline{12}$
2	0	$\beta\gamma\alpha\gamma\alpha\beta$	$(\overline{011}\ \overline{101}\ \overline{110})$	$(\overline{01}\ \overline{11})$	$\overline{12}$
2	0	$\gamma\alpha\beta\gamma\alpha\beta$	$(\overline{001}\ \overline{111}\ \overline{110})$	$(\overline{00}\ \overline{11})$	$\overline{11}$
2	0	$\gamma\alpha\gamma\alpha\beta^2$	$(\overline{001}\ \overline{101}\ \overline{120})$	$(\overline{00}\ \overline{10})$	$\overline{10}$
0	2	$\alpha^2\gamma\beta\gamma\beta$	$(\overline{201}\ \overline{011}\ \overline{010})$	$(\overline{20}\ \overline{21})$	$\overline{41}$

There are 36 partitions.

The first two columns show the nature of the permutation in regard to $\gamma\alpha$ and $\gamma\beta$ contacts and the nature of the composition in regard to positive-positive and positive-zero contacts. The partitions are into two parts, zero not excluded, and have regard to bipartite numbers extending from $\overline{44}$ to $\overline{00}$. They are doubly regularised by ascending magnitude, and the figures of the parts do not exceed 2, 2 the first two figures of the tripartite.

If we write down the partitions of 4 into two parts, zeros not excluded, limited not to exceed 2 in magnitude, viz. :—

$$22,\ 12,\ 02,\ 11,\ 01,\ 00,$$

the ascending order of part magnitude being adhered to, we can obtain one of the 36 partitions by combining any one of these partitions with itself or any other of the 6.

Thus the fourth of the above partitions is obtained by combining the unipartite partitions

$$02, \quad 22,$$

and from any two unipartite partitions

$$ab, \quad cd,$$

we proceed to the bipartite partition

$$(\overline{ac}\quad \overline{bd}).$$

The number is thus shown to be $6 \times 6 = 36$.

Art. 36. In general, when the tripartite is $\overline{pqr}$, the partitions are into r parts, zeros not excluded, the first and second figures of the biparts being limited to p and q respectively.

The bipartite numbers partitioned extend from

$$\overline{p \times r, q \times r} \quad \text{to} \quad \overline{00}.$$

The partitions are doubly regularised and may be enumerated by observing that we have to combine every partition of $p \times r$ and lower unipartite numbers into r parts, zeros not excluded, and no part exceeding p in magnitude, with every partition of $q \times r$ and lower numbers into r parts, zeros not excluded, and no part exceeding q in magnitude.

Hence (see *ante*, Art. 12) the number of partitions is

$$\binom{p+r}{r}\binom{q+r}{r}.$$

This expression also enumerates (1) the compositions which have only positive-positive and positive-zero contacts; (2) the lines of route in the tripartite reticulation which are without $\beta\alpha$ bends; (3) the permutations of $\alpha^p\beta^q\gamma^r$ which are without $\beta\alpha$ contacts.

Art. 37. The truth of the theorem may be seen also as follows :—Suppose a solid reticulation and take the directions α, β, γ as axes of x, y, and z meeting at the origin of the lines of route. The face of the solid in the plane xz is a *bipartite* reticulation in which $\binom{p+r}{r}$ lines of route may be drawn; similarly $\binom{q+r}{r}$ lines of route may be drawn in the bipartite reticulation which lies in the plane yz. One of the former lines of route is an orthogonal projection of a tripartite line of route on the plane xz; one of the latter is an orthogonal projection on the plane yz; any one of the former may be associated with any one of the latter, and such a pair uniquely determines a tripartite line of route which does not possess $\beta\alpha$ bends. This may be clearly seen by considering the permutation

$$\alpha^{p_1}\beta^{q_1}\gamma^{r_1}\ \alpha^{p_2}\beta^{q_2}\gamma^{r_2}\ldots;$$

suppression alternately of the letters β and α yields two permutations, viz. :—

$$\alpha^{p_1}\gamma^{r_1}\alpha^{p_2}\gamma^{r_2}\ldots$$
$$\beta^{q_1}\gamma^{r_1}\beta^{q_2}\gamma^{r_2}\ldots$$

which express the bipartite lines of route which are the projections on the planes xz, yz respectively. Since the tripartite permutation involves no $\beta\alpha$ contacts, we see that these two permutations uniquely determine the permutation

$$\alpha^{p_1}\beta^{q_1}\gamma^{r_1}\alpha^{p_2}\beta^{q_2}\gamma^{r_2}\ldots$$

Hence the number of lines of route in question is

$$\binom{p+r}{r}\binom{q+r}{r}.$$

Art. 38. Hence also the interesting summation formula

$$\sum_{s_{31}}\sum_{s_{32}}\binom{p}{s_{31}}\binom{q}{s_{32}}\binom{q+s_{31}}{s_{31}}\binom{r}{s_{31}+s_{32}}=\binom{p+r}{r}\binom{q+r}{r}.$$

Observe that the expression further enumerates the lines of route with r, $\beta\alpha$ bends in the reticulation of the bipartite $\overline{p+r,\ q+r}$.

A generating function which enumerates these partitions is

$$\frac{1}{1-x\,.\,1-a\,.\,1-ax\ldots1-ax^p\,1-y\,.\,1-b\,.\,1-by\ldots1-by^q},$$

in which the coefficient of $(abx^py^q)^r$ must be sought.

The compositions that appear are the principal ones along lines of route which have no $\beta\alpha$ bends. We may strike out the last part of the composition whenever its last figure is zero, and then the compositions are not of the single tripartite $\overline{222}$, but of the 9 tripartites extending from $\overline{222}$ to $\overline{002}$, the last figure being 2, and the first two figures not exceeding 2, 2 respectively. The compositions are into 2, or fewer parts. Generally the compositions appear of the $(p+1)(q+1)$ tripartites extending from $\overline{pqr}$ to $\overline{00r}$, the last figure being r, and the first two figures not exceeding p, q, respectively. The compositions are into r, or fewer parts, no part having the last figure zero.

The partitions present themselves in complementary pairs. To every partition $(\overline{ab}\ \overline{cd}\ldots)$ corresponds another $(\overline{p-a,\ q-b},\ \overline{p-c,\ q-d}\ldots)$ the numbers partitioned being respectively $\overline{a+c+\ldots,\ b+d+\ldots}$ and $\overline{rp-a-c-\ldots,\ rq-b-d-\ldots}$ *Ex. gr.*, the complementary partitions $(\overline{02}\ \overline{22})$, $(\overline{20}\ \overline{00})$ of the bipartites $\overline{24}$, $\overline{20}$ Certain partitions are self-complementary. The number partitioned is then $\overline{\tfrac{1}{2}rp,\ \tfrac{1}{2}rq}$.

Art 39. We may enumerate the partitions which, excluding zero, involve k different parts. Let s_{23}, s_{13}, represent the number of $\beta\gamma$ and $\alpha\gamma$ contracts in a permutation. If $s_{23} + s_{13} = k$, the corresponding partition possesses k different parts other than zero. The lines of route are such as have no $\beta\alpha$ bends, s_{13} $\alpha\gamma$ bends and s_{23} $\beta\gamma$ bends. Reversing the permutation we have a similar number of lines of route which have no $\alpha\beta$ bends, s_{13} $\gamma\alpha$ bends, and s_{23} $\gamma\alpha$ bends. Now interchange α and β and replace the reticulation of the tripartite $\overline{pqr}$ by that of $\overline{qpr}$. In this new reticulation we have the same number of lines of route which have no $\beta\alpha$ bends, s_{13} $\gamma\beta$ bends, and s_{23} $\gamma\alpha$ bends. This number has been shown to be

$$\binom{q}{s_{23}}\binom{p+s_{23}}{s_{23}}\binom{p}{s_{13}}\binom{r}{s_{13}+s_{23}}.$$

Art. 40. Hence the bipartite partitions possessing k different parts other than zero are enumerated by

$$\binom{r}{k}\sum_{s_{23}}\binom{q}{s_{23}}\binom{p+s_{23}}{s_{23}}\binom{p}{k-s_{23}}.$$

Theorem.—Having under consideration the doubly-regularised partitions of all bipartite numbers into r parts, zero parts included, such that the figures of the parts are limited in magnitude to p and q respectively, the number of partitions which possess exactly k different parts, other than zero, is

$$\binom{r}{k}\sum_{s_{23}}\binom{q}{s_{23}}\binom{p+s_{23}}{s_{23}}\binom{p}{k-s_{23}},$$

s_{23} assuming all compatible values.

This result may be verified in the case of the tripartite $\overline{222}$ from the table given above. As an additional verification, consider the tripartite $\overline{123}$. For $k = 2$, we have

Permutations.	Compositions.	Partitions.
$\gamma\alpha\beta\gamma\beta\gamma$	$(\overline{001}\ \overline{111}\ \overline{011})$	$(\overline{00}\ \overline{11}\ \overline{12})$
$\gamma\alpha\gamma\beta^2\gamma$	$(\overline{001}\ \overline{101}\ \overline{021})$	$(\overline{00}\ \overline{10}\ \overline{12})$
$\gamma\alpha\gamma\beta\gamma\beta$	$(\overline{001}\ \overline{101}\ \overline{011})$	$(\overline{00}\ \overline{10}\ \overline{11})$
$\alpha\beta\gamma^2\beta\gamma$	$(\overline{112}\ \overline{011})$	$(\overline{11}^2\ \overline{12})$
$\alpha\beta\gamma\beta\gamma^2$	$(\overline{111}\ \overline{012})$	$(\overline{11}\ \overline{12}^2)$
$\alpha\gamma^2\beta^2\gamma$	$(\overline{102}\ \overline{021})$	$(\overline{10}^2\ \overline{12})$
$\alpha\gamma^2\beta\gamma\beta$	$(\overline{102}\ \overline{011})$	$(\overline{10}^2\ \overline{11})$
$\alpha\gamma\beta\gamma^2\beta$	$(\overline{101}\ \overline{012})$	$(\overline{10}\ \overline{11}^2)$
$\alpha\gamma\beta^2\gamma^2$	$(\overline{101}\ \overline{022})$	$(\overline{10}\ \overline{12}^2)$

Permutations.	Compositions.	Partitions.
$\gamma\beta\gamma\beta\gamma\alpha$	$(\overline{001}\ \overline{011}\ \overline{011})$	$(\overline{00}\ \overline{01}\ \overline{02})$
$\gamma\beta\gamma\alpha\beta\gamma$	$(\overline{001}\ \overline{011}\ \overline{111})$	$(\overline{00}\ \overline{01}\ \overline{12})$
$\beta^2\gamma^2\alpha\gamma$	$(\overline{022}\ \overline{101})$	$(\overline{02}^2\ \overline{12})$
$\beta^2\gamma\alpha\gamma^2$	$(\overline{021}\ \overline{102})$	$(\overline{02}\ \overline{12}^2)$
$\beta\gamma^2\alpha\beta\gamma$	$(\overline{012}\ \overline{111})$	$(\overline{01}^2\ \overline{12})$
$\beta\gamma^2\alpha\gamma\beta$	$(\overline{012}\ \overline{101})$	$(\overline{01}^2\ \overline{11})$
$\beta\gamma^2\beta\gamma\alpha$	$(\overline{012}\ \overline{011})$	$(\overline{01}^2\ \overline{02})$
$\beta\gamma\alpha\beta\gamma^2$	$(\overline{011}\ \overline{112})$	$(\overline{01}\ \overline{12}^2)$
$\beta\gamma\alpha\gamma^2\beta$	$(\overline{011}\ \overline{102})$	$(\overline{01}\ \overline{11}^2)$
$\beta\gamma\beta\gamma^2\alpha$	$(\overline{011}\ \overline{012})$	$(\overline{01}\ \overline{02}^2)$
$\gamma\beta^2\gamma\alpha\gamma$	$(\overline{001}\ \overline{021}\ \overline{101})$	$(\overline{00}\ \overline{02}\ \overline{12})$
$\gamma\beta\gamma\alpha\gamma\beta$	$(\overline{001}\ \overline{011}\ \overline{101})$	$(\overline{00}\ \overline{01}\ \overline{11})$

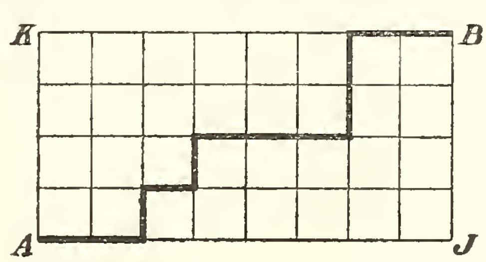
21 partitions; while for the enumeration, giving s_{23} the values 0, 1, 2 in succession with $k = 2$, $p = 1$, $q = 2$, $r = 3$.

$$\binom{3}{2}\left\{\binom{2}{0}\binom{1+0}{0}\binom{1}{2}+\binom{2}{1}\binom{1+1}{1}\binom{1}{1}+\binom{2}{2}\binom{1+2}{2}\binom{1}{0}\right\}$$
$$= 3\,(1 \times 1 \times 0 + 2 \times 2 \times 1 + 1 \times 3 \times 1) = 3 \times 7 = 21.$$

The foregoing particular theory of the correspondence that exists between tripartite compositions and bipartite partitions is, for present purposes, sufficiently indicative of the general correspondence between $(m+1)$-partite compositions and a certain regularised class of m-partite partitions.

§ 4. Constructive Theory.

Art. 41. Given a line of route in a bipartite reticulation it may be necessary to enumerate the lines of route which lie altogether on either side of it.



Thus in respect of the line of route delineated in the reticulation AB, lines of route exist which, throughout their entire course, are either coincident with it, or lie on the side of it towards J. Such lines of route may be termed inferior or subjacent to the given line of route. Similarly those lines of route which, everywhere, are either coincident with the given line, or on the side remote from J, may be termed superior or superjacent lines of route in respect of the given line. All lines are thus accounted for with the exception of those which cross the given line passing from the side towards J to the side remote from J, or *vice versâ*; these may be termed transverse lines in respect of the given line.

Art. 42. I am concerned, at present, with those lines which are subjacent to a given line, though it will be remarked that the superjacent and transverse lines also suggest questions of interest. A given line of route defines a bipartite principal composition

$$(\overline{p_1q_1}\ \overline{p_2q_2}\ldots),$$

and a unipartite south-easterly partition

$$(\overline{p-p_1}^{q_1}\ \overline{p-p_1-p_2}^{q_2}\ldots).$$

The bipartite compositions and the unipartite partitions, defined by the subjacent lines of route, are termed subjacent to the given composition and the given partition respectively.

We may draw a number of lines of route, each of which is subjacent to the given line and not transverse to any other of the number. We thus obtain what may be termed a subjacent succession of lines giving rise to a subjacent succession of unipartite partitions.

These regularised partitions may be

$$(a_1a_2a_3\ldots),\quad (b_1b_2b_3\ldots),\quad (c_1c_2c_3\ldots)\ldots.$$

and they are such that the partitions

$$(a_1b_1c_1\ldots),\ (a_2b_2c_2\ldots),\ (a_3b_3c_3\ldots).$$

are also regularised.

It is clear also that the subjacent succession of lines represents the multipartite partition

$$(\overline{a_1a_2a_3\ldots},\ \overline{b_1b_2b_3\ldots},\ \overline{c_1c_2c_3\ldots},\ \overline{\ldots\ldots})$$

of the multipartite numbers

$$(\overline{a_1+b_1+c_1+\ldots},\ \overline{a_2+b_2+c_2+\ldots},\ \overline{a_3+b_3+c_3+\ldots},\ \overline{\ldots}).$$

This partition may be termed "graphically regularised" by reason of its origination in a subjacent succession of lines in the bipartite graph. This species of regularisation is the natural extension to three dimensions of SYLVESTER's graphical method in two dimensions.

Art. 43. SYLVESTER represents the partition $(a_1 a_2 a_3 \ldots)$ of a unipartite number A by the graph

the lines containing $a_1, a_2, a_3 \ldots$ nodes successively.

The same graph also represents a multipartite number $(\overline{a_1 a_2 a_3 \ldots})$ whose content is A, viz.,

$$a_1 + a_2 + a_3 + \ldots = \text{A}.$$

SYLVESTER's theory is, in fact, not only a theory of the partitions of a number A, but also a theory of the multipartite numbers whose content is A. For purpose of generalization I prefer to regard it from the latter point of view.

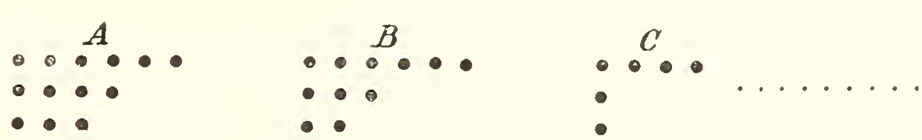
If we consider the graphically regularised partition

$$(\overline{a_1 a_2 a_3 \ldots}, \overline{b_1 b_2 b_3 \ldots}, \overline{c_1 c_2 c_3 \ldots}, \overline{\ldots\ldots})$$

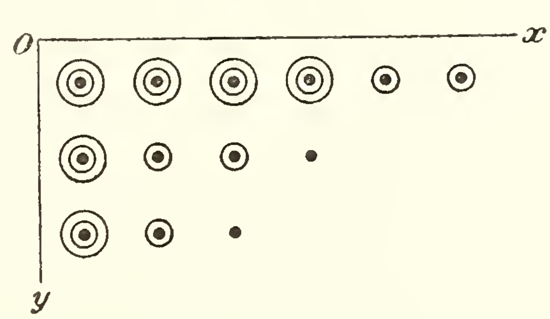
of the multipartite number

$$(\overline{a_1 + b_1 + c_1 + \ldots}, \overline{a_2 + b_2 + c_2 + \ldots}, \overline{a_3 + b_3 + c_3 + \ldots}, \overline{\ldots\ldots})$$

and write down the Sylvester-graphs of the multipartite numbers which are the parts of the partition



it is clear that we may pile B upon A, and then C upon B, &c., and thus form a three-dimensional graph of the partition



which is regularised in three-dimensions just as the Sylvester-graphs are regularised in two.

This representation is only possible when the subjacent succession of lines is insisted upon.

Art. 44. Every Sylvester-graph in two dimensions is representative of two unipartite partitions; it may, in fact, be read by lines or by columns, and when the two readings are identical the graph is said to be self conjugate.

In this enlarged theory every graph denotes 3 ! graphically-regularised multipartite partitions; of the same total content, but not, as a rule, appertaining to the same multipartite number.

Take coordinate axes as shown, the axis of z being perpendicular to the plane of the paper. We read as follows:—

Planes parallel to the plane xy and in direction Ox

$$(\overline{643}\ \overline{632}\ \overline{411}).$$

Planes parallel to plane xy and in direction Oy

$$(\overline{333211}\ \overline{332111}\ \overline{311100}).$$

Planes parallel to plane yz and in direction Oy

$$(\overline{333}\ \overline{331}\ \overline{321}\ \overline{211}\ \overline{110}\ \overline{110}).$$

Planes parallel to plane yz and in direction Oz

$$(\overline{333}\ \overline{322}\ \overline{321}\ \overline{310}\ \overline{200}\ \overline{200}).$$

Planes parallel to plane zx and in direction Oz

$$(\overline{333322}\ \overline{322100}\ \overline{321000}).$$

Planes parallel to plane zx and in direction Ox

$$(\overline{664}\ \overline{431}\ \overline{321}),$$

the multipartite numbers, of which these are partitions, being

$(\overline{16, 8, 6})$,
$(\overline{976422})$, content
$(\overline{13, 11, 6})$, 30
$(\overline{16, 8, 6})$,
$(\overline{976422})$,
$(\overline{13, 11, 6})$.

The graph is therefore representative of three multipartite numbers and of two partitions of each.

Art. 45. A multipartite number has two characteristics. It may be r-partite, *i.e.*, it may consist of r figures, and its highest figure may be p. A multipartite partition has three characteristics. Each part may be r-partite ; the highest figure may be p ; the number of parts may be q. If the graph be formed of a multipartite partition with characteristics

$$r, p, q,$$

the five other readings yield partitions with characteristics :—

$$\begin{array}{c} p,\ r,\ q \\ q,\ r,\ p \\ r,\ q,\ p \\ p,\ q,\ r \\ q,\ p,\ r. \end{array}$$

The six partitions correspond to the six permutations of the three symbols p, q, r.

The two partitions which are r-partite appertain to the same multipartite number ; similarly for the pairs which are p-partite and q-partite respectively. Hence the three multipartite numbers involved correspond to the three pairs of permutations so formed that in any pair the commencing symbol of each permutation is the same.

Art. 46. The consideration of graphs formed with a *given number* of nodes now leads to the theorem : "The enumeration of the graphically regularised r-partite partitions, into q parts and having p for the highest figure, gives the same number for each of the six ways in which the numbers p, q, r may be permuted."

Also the theorem :—

"The enumeration of the graphically regularised partitions which are at most r-partite, into q or fewer parts, the highest figure not exceeding p, gives the same number for each of the six ways in which the numbers p, q, r may be permuted."

The first theorem is concerned with fixed values of p, q, and r ; the second with restricted values of these numbers. It is also clear that we may fix one or two of the numbers and leave the remaining two or one restricted.

Observe that this six-fold conjugation obtains even though equalities exist between the numbers p, q, r ; they must be regarded always as different numbers. Sometimes, as we shall see, the correspondence is less than six-fold, but this does not depend solely upon the assignment of the numbers p, q, r.

If we regard the multipartite number appertaining to a partition and not merely the total content, we find that the partitions occur in pairs.

Quâ a given multipartite number, a partition which has q parts and a highest figure p is in association with one which has p parts and a highest figure q.

Thus of the multipartite number $(\overline{13.11.6})$ we have the partitions

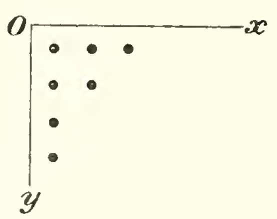
$$(\overline{333}\ \overline{331}\ \overline{321}\ \overline{211}\ \overline{110}\ \overline{110})$$

$$(\overline{664}\ \overline{431}\ \overline{321})$$

derived from the above written graph.

Art. 47. It is interesting to view the two-dimensional Sylvester-graphs from the three-dimensional standpoint.

Consider the graph



which, following SYLVESTER, denotes the unipartite partition (3211) of the unipartite number 7.

In this paper, the graph, read Sylvester-wise in the plane xy and in direction Ox, denotes the multipartite number $(\overline{3211})$ of content 7. SYLVESTER'S conjugate reading, plane xy and direction Oy, gives the partition (421) but here denotes the multipartite number $(\overline{421})$. There are four other readings in this theory. The six readings are

Plane xy	Direction	Ox	$(\overline{3211})$	$(p, q, r) = (3, 1, 4)$
,, xy	,,	Oy	$(\overline{421})$	$(p, q, r) = (4, 1, 3)$
,, yz	,,	Oy	(421)	$(p, q, r) = (4, 3, 1)$
,, yz	,,	Oz	$(\overline{1111}\ \overline{1100}\ \overline{1000})$	$(p, q, r) = (1, 3, 4)$
,, zx	,,	Oz	(3211)	$(p, q, r) = (3, 4, 1)$
,, zx	,,	Ox	$(\overline{111}\ \overline{110}\ \overline{100}\ \overline{100})$	$(p, q, r) = (1, 4, 3)$.

The three multipartite numbers

$$(7),\ \overline{421},\ (\overline{3211})$$

appear each in two partitions.

In general we establish, in regard to Sylvester-graphs, the six-fold correspondence between

(1) r-partite partitions, containing 1 part and a highest figure p.

(2) p-partite partitions, containing 1 part and a highest figure r.

(3) unipartite partitions, containing p parts and a highest figure r.
(4) r-partite partitions, containing p parts and a highest figure 1.
(5) unipartite partitions, containing r parts and a highest figure p.
(6) p-partite partitions, containing r parts and a highest figure 1.

In this enunciation we may substitute for r or p, or for both, the phrases "not exceeding r," "not exceeding p."

Art. 48. For a given number of nodes, in the simplest cases, it will be suitable to view the graphs of the graphically regularised partitions.

Omitting the trivial case of a single node, we have

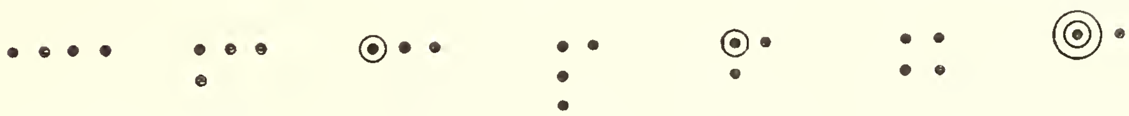
Number 2.

(2) $(\overline{11})$ (11)

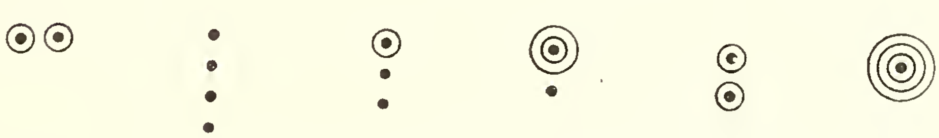
Number 3.

(3) $(\overline{21})$ (21) $(\overline{111})$ $(\overline{11}\;\overline{10})$ (111)

Number 4.



(4) $(\overline{31})$ (31) $(\overline{211})$ $(\overline{21}\;\overline{10})$ $2\overline{2}$ (211)

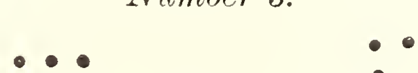


(22) $(\overline{1111})$ $(\overline{111}\;\overline{100})$ $(\overline{11}\;\overline{10}\;\overline{10})$ $(\overline{11}\;\overline{11})$ (1111)

The table is continued in an obvious manner. The *essentially* distinct graphs are for the

Number 2.

Number 3.



Number 4.

 *

and the whole of the partitions are obtainable by reading them in the various ways above explained.

Art. 49. It will be convenient to adopt in future another notation for the graphs; the number m will denote a vertical column of m nodes piled upon one another. The 13 graphs appertaining to the number 4 are written

1111	111 1	211	11 1 1	21 1	11 11	31
(4)	$(\overline{31})$	(31)	$(\overline{211})$	$(\overline{21}\ \overline{10})$	$(\overline{22})$	(211)

22	1 1 1 1	2 1 1	3 1	2 2	4
(22)	$(\overline{1111})$	$(\overline{111}\ \overline{100})$	$(\overline{11}\ \overline{10}\ \overline{10})$	$(\overline{11}\ \overline{11})$	(1111).

The essentially distinct graphs with the partitions appertaining to them are

1111	111 1	21 1	11 11
(4) $(\overline{1111})$ (1111)	$(\overline{31})$ (31) $(\overline{211})$ (211) $(\overline{111}\ \overline{100})$ $(\overline{11}\ \overline{10}\ \overline{10})$	$(\overline{21}\ \overline{10})$	$(\overline{22})$ (22) $(\overline{11}\ \overline{11})$

Art. 50. Such graphs are either symmetrical, quasi-symmetrical, or unsymmetrical. The symmetrical graphs have three dimensional symmetry, and yield only one partition each.

The quasi-symmetrical have two-dimensional symmetry and yield three partitions each. The unsymmetrical yield each six partitions.

If F (x) be the enumerating generating function to a given content, we may write

$$F(x) = f_1(x) + 3f_2(x) + 6f_3(x),$$

* The interesting question arises as to the enumeration of the essentially distinct graphs of given content.

$f_1(x), f_2(x), f_3(x)$ being the generating functions for the essentially distinct graphs which are symmetrical, quasi-symmetrical, and unsymmetrical respectively.

Also we may write

$$F(x) = F_1(x) + F_2(x) + F_3(x),$$

where $F_1(x)$, $F_2(x)$, $F_3(x)$ are the generating functions of the partitions of the three natures.

The present theory is really the solidification of SYLVESTER's theory given in the 'American Journal of Mathematics' (*loc. cit.*). Already we have seen that the Sylvester-graphs are susceptible of a far wider interpretation than was at first anticipated. If we view these graphs from a two-dimensional standpoint, every graph is either symmetrical or unsymmetrical, the symmetrical class comprising all graphs which are self-conjugate. If, however, our standpoint be three-dimensional, there are no longer any symmetrical graphs. The two classes are the quasi-symmetrical and the unsymmetrical. A single exception to the above occurs where the graph is of unity. Moreover, the classes now do not comprise the same members. Certain graphs which were unsymmetrical from the first standpoint appear as quasi-symmetrical from the second.

Omitting the trivial symmetrical graph of unity every two-dimensional graph can be read either in three or six ways. The quasi-symmetrical class giving three readings, comprises the self-conjugate graphs and also those which consist of either a single line or a single column of nodes. The remaining graphs give six readings.

Ex. gr. The graph

11111

yields the three partitions (5), $(\overline{11111})$, (11111) being quasi-symmetrical from the three-dimensional standpoint although it is unsymmetrical in SYLVESTER's theory.

Also the self-conjugate graph

111
1
1

yields the three partitions $(\overline{311})$, (311), $(\overline{111}\ \overline{100}\ \overline{100})$.

Such a graph as

111
111

being unsymmetrical in both theories yields six partitions

$$(\overline{33}),\ (33),\ (\overline{222}),\ (222),\ (\overline{111}\ \overline{111}),\ (\overline{11}\ \overline{11}\ \overline{11}).$$

Art. 51. The enumeration of the three-dimensional graphs that can be formed with a given number of nodes, corresponding to the regularised partitions of all multipartite numbers of given content, is a weighty problem. I have verified to a high order that the generating function of the complete system is

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}(1-x^4)^{-4}\ldots \textit{ad inf.},$$

and, so far as my investigations have proceeded, everything tends to confirm the truth of this conjecture.

I observe that, to negative signs *près*, the exponents are

$$1,\quad 2,\quad 3,\quad 4,\quad 5,\ldots$$

viz., the figurate numbers of order 2.

The generating function which enumerates the two-dimensional graphs, is

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}\ldots$$

where (notice) the exponents are

$$1,\quad 1,\quad 1,\quad 1,\quad 1,\ldots$$

the figurate numbers of order 1.

Proceeding further back, we find that one-dimensional graphs are enumerated by

$$(1-x)^{-1}(1-x^2)^{0}(1-x^3)^{0}(1-x^4)^{0}\ldots$$

the numbers

$$1,\quad 0,\quad 0,\quad 0,\quad 0,\ldots$$

being the figurate numbers of order zero. Going forward again it is easy to verify up to a certain point that four-dimensional graphs (which it is quite easy to graphically realise in two dimensions) are enumerated by

$$(1-x)^{-1}(1-x^2)^{-3}(1-x^3)^{-6}\ldots,$$

where the exponents involve the figurate numbers of order 3.

The law of enumeration appears, conjecturally, to involve the successive series of figurate numbers.

Art. 52. Before proceeding to establish certain results, it may be proper, as illustrating the method pursued in this difficult investigation, to give other results which, at first mere conjectures, are gradually having the mark of truth stamped upon them.

Consider graphs in which only the numbers 1 and 2 appear. These are two-layer partitions. The enumeration to a high order is given by the generating function

$$(2\,;\infty\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-2}(1-x^4)^{-2}\ldots$$

where the notation $(l\,;m\,;n)$ is employed to represent the generating function of partitions whose graphs are limited in height, breadth, and length by l, m, n respectively.

Similarly we shall find :—

$$(3\,;\infty\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}[(1-x^3)(1-x^4)\ldots]^{-3},$$

$$(4\,;\infty\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}[(1-x^4)(1-x^5)\ldots]^{-4},$$

$$(l\,;\infty\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}\ldots(1-x^{l-1})^{-(l-1)}[(1-x^l)(1-x^{l+1})\ldots]^{-l},$$

$$(l\,;1\,;\infty) = (1-x)^{-1}(1-x^2)^{-1}\ldots(1-x^l)^{-1},$$

$$(l\,;2\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-2}\ldots(1-x^l)^{-2}(1-x^{l+1})^{-1},$$

$$(l\,;3\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}\ldots(1-x^l)^{-3}(1-x^{l+1})^{-2}(1-x^{l+2})^{-1},$$

$$(l\,;\mathrm{m}\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}\ldots(1-x^{m-1})^{-(m-1)}\times[(1-x^m)\ldots(1-x^l)]^{-m},$$
$$\times\,(1-x^{l+1})^{-(m-1)}(1-x^{l+2})^{-(m-2)}\ldots(1-x^{l+m-1})^{-1},$$

if m be not greater than l ;

with an equivalent form

$$(l\,;m\,;\infty) = (1-x)^{-1}(1-x^2)^{-2}\ldots(1-x^{l-1})^{-l-1}\times[(1-x^l)\ldots(1-x^m)]^{-l}$$
$$\times\,(1-x^{m+1})^{-(l-1)}(1-x^{m+2})^{-(l-2)}\ldots(1-x^{l+m-1})^{-1},$$

if m be greater than l ;

and finally

$$(l\,;m\,;n) = \frac{1-x^{n+1}}{1-x}\cdot\frac{(1-x^{n+2})^2}{(1-x^2)^2}\cdots\frac{(1-x^{n+l-1})^{l-1}}{(1-x^{l-1})^{l-1}}$$
$$\times\left[\frac{1-x^{n+l}}{1-x^l}\cdot\frac{1-x^{n+l+1}}{1-x^{l+1}}\cdots\frac{1-x^{n+m}}{1-x^m}\right]^l$$
$$\times\,\frac{(1-x^{n+m+1})^{l-1}}{(1-x^{m+1})^{l-1}}\cdot\frac{(1-x^{n+m+2})^{l-2}}{(1-x^{m+2})^{l-2}}\cdots\frac{1-x^{n+l+m-1}}{1-x^{l+m-1}},$$

a result which can be shown to be symmetrical in l, m and n, as ought, of course, to be the case.

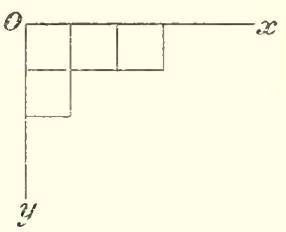
This expression for $(l\,;\,m\,;\,n)$ can be exhibited in a more suggestive form, viz. :—
Writing $1-x^s=(s)$

$$(l;\ m;\ n) = \frac{(n+1)(n+2)\ldots\ldots(l+m+n-1)}{(1)\quad(2)\quad\ldots\ldots\quad(l+m-1)}$$
$$\times\,\frac{(n+2)(n+3)\ldots\ldots(l+m+n-2)}{(2)\quad(3)\quad\ldots\ldots\quad(l+m-2)}\times\frac{(n+3)(n+4)\ldots\ldots(l+m+n-3)}{(3)\quad(4)\quad\ldots\ldots\quad(l+m-3)},$$

$\times$ . . . . to l factors or m factors, according as m or l is the greater.

Art. 53. In attempting to establish these results, it is easy to construct a generating function which contains implicitly the complete solution of the problems.

The problem itself may be enunciated in another manner which has points of great interest. A two-dimensional graph of SYLVESTER may be supposed formed by pushing



a number of cubes into a flat rectangular corner $y0x$ in such wise that the arrangement is immovable under the action of forces applied in the directions $x0$, $y0$.

It is clear that the number of such arrangements of n cubes is the number of two-dimensional graphs of n, or the number of partitions of the unipartite number n. Similarly, we may push a number of cubes into a three-dimensional rectangular corner, piling of cubes permissible, and such that the arrangement is immovable for forces applied in the three directions $x0$, $y0$, $z0$. The enumeration of these arrangements is the same as that in the problem under discussion.

Art. 54. First consider arrangements limited in the manner $(l\,;\ m\,;\ n) = (2\,;\ 1\,;\ \infty)$.

We have such a graph as

$$\begin{matrix} 2 \\ 2 \\ 2 \\ 1 \\ 1, \end{matrix}$$

obtained by writing a column of nodes, and over it another column of nodes, not exceeding the former in number.

We may take, as the generating function,

$$\frac{1}{(1 - ax)\,(1 - x/a)},$$

in which we are only concerned with that portion of the expansion which is integral as regards a. The function is, in fact, redundant since it involves terms which are superfluous, and we obtain the reduced or condensed generating function by putting a equal to unity in the portion we retain.

Since

$$\frac{1}{(1 - ax)\left(1 - \dfrac{x}{a}\right)} = \frac{1}{1 - x^2}\left\{\frac{1}{1 - ax} + \frac{\dfrac{x}{a}}{1 - \dfrac{x}{a}}\right\}$$

the reduced generating function is

$$\frac{1}{(1-x)(1-x^2)},$$

and this is obviously correct, because from the form of the graph we have merely to enumerate the ways of partitioning numbers with the parts 1 and 2.

Art. 55. Again if $(l\,;\ m\,;\ n) = (2\,;\ 2\,;\ \infty)$, we have graphs like

2 2
2 2
2 1
1 1
1
1

We are led to construct the function

$$\frac{1}{(1-ax)\left(1-\frac{x}{a}\right)(1-abx^2)\left(1-\frac{x^2}{ab}\right)},$$

in the expansion of which all terms involving negative powers of a and b have to be rejected. Isolating the integral portion and putting $a = b = 1$, we find the reduced generating function

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)}$$

a result which, unlike the previous one, is not obvious.

For the case $(l\,;\ m\,;\ n) = (2\,;\ 3\,;\ \infty)$ we introduce additional denominator factors

$$(1-abcx^3)\left(1-\frac{x^3}{abc}\right),$$

and with increasing labour of algebraical performance we arrive at the reduced generating function

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}.$$

Art. 56. In general for the case

$$(l\,;\ m\,;\ n) = (2\,;\ m\,;\ \infty)$$

the generating function is the reciprocal of the product of the $2m$ factors

$$(1 - a_1x)\left(1 - \frac{x}{a_1}\right)$$

$$(1 - a_1a_2x^2)\left(1 - \frac{x^2}{a_1a_2}\right)$$

$$(1 - a_1a_2a_3x^3)\left(1 - \frac{x^3}{a_1a_2a_3}\right)$$

$$\vdots \qquad \vdots$$

$$(1 - a_1a_2 \dots a_mx^m)\left(1 - \frac{x^m}{a_1a_2 \dots a_m}\right)$$

We have to expand this fraction in ascending powers of x, reject all terms containing negative powers of $a_1, a_2, \dots a_m$, and then obtain the reduced generating function by putting $a_1 = a_2 = \dots = a_m = 1$ in the portion retained. This has proved a difficult algebraical problem. I am indebted to Professor FORSYTH for a beautiful solution which he will publish elsewhere.* He establishes that the reduced generating function is

$$\frac{1}{(1 - x)\{(1 - x^2)(1 - x^3)\dots(1 - x^m)\}^2(1 - x^{m+1})},$$

a result which agrees with the prediction. His method is that of selective summation. He forms the general term of the expansion

$$\left.\begin{array}{l} a_1^{m_1-n_1+m_2-n_2+m_3-n_3+\dots} \\ a_2^{m_2-n_2+m_3-n_3+\dots} \\ a_3^{m_3-n_3+\dots} \\ \vdots \end{array}\right\} x^{m_1+n_1+2(m_2+n_2)+3(m_3+n_3)+\dots}$$

and performs the enumeration

$$\sum_{m_1}\sum_{n_1}\sum_{m_2}\sum_{n_2}\sum_{m_3}\sum_{n_3}\dots x^{m_1+n_1+2(m_2+n_2)+3(m_3+n)_3+\dots}$$

for all values of $m_1, n_1, m_2, n_2, m_3, n_3, \dots\dots$ which make each of the m expressions

$$\begin{array}{r} m_1 - n_1 + m_2 - n_2 + m_3 - n_3 + \dots \\ m_2 - n_2 + m_3 - n_3 + \dots \\ m_3 - n_2 + \dots \\ \dots\dots \end{array}$$

not less than zero.

The case $(l\,;\,m\,;\,n) = (2\,;\,m\,;\,\infty)$ is thus completely solved by a method which

* 'Proc. L.M.S.,' vol. 27.

appears to be of general application if the difficulties presented by the algebra can be surmounted.

Art. 57. At this point we may enquire into the meaning of the reduced generating function which has been so happily and ingeniously established. We may write it as the product of two fractions :—

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} + \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^{m+1})}$$

the indication being that every two-layer arrangement is derivable from a combination of two ordinary single-layer partitions whose parts are drawn from the two series of numbers,

$$1, 2, 3, \ldots m,$$
$$2, 3, \ldots m, m+1,$$

respectively. Otherwise we may say that a number N possesses as many two-layer partitions $(2; m; \infty)$ as there are modes of partitionment employing the parts

$$1_1, 2_1, 2_2, 3_1, 3_2, \ldots m_1, m_2, m+1_2.$$

Ex. gr. If $N = 4$ and $m = 3$, the graphs are 9 in number

111	11	11	1	211	21	2	22	2
1	11	1	1		1	1		2
		1	1			1		
			1					

and employing parts

$$\begin{array}{cccc} 1_1 & 2_1 & 3_1 & \\ & 2_2 & 3_2 & 4_2 \end{array}$$

we can form 9 partitions, viz. :—

$$(1_1^4),\ (2_1 1_1^2),\ (2_2 1_1^2),\ (2_1^2),\ (2_1 2_2),\ (2_2^2),\ (3_1 1_1),\ (3_2 1_1),\ (4_2).*$$

Art. 58. The problem is therefore reduced to establishing a one-to-one correspondence, between the graphs and the partitions of the kind indicated, of general application. I will in part establish this correspondence, which is not very simple in character, later on. At present it is convenient to take a further survey of the general problems in order to obtain ideas concerning the difficulties that confront us.

I form a tableau of algebraic factors.

* The solution thus shows that the two-layer graphs may be exhibited as a one-layer graph by nodes of two colours, say black and red; nodes of different colours not appearing in any single line.

$$(1-p_1x)\left(1-\frac{p_2}{p_1}x\right)\left(1-\frac{p_3}{p_2}x\right)\ldots\left(1-\frac{p_{l-1}}{p_{l-2}}x\right)\left(1-\frac{x}{p_{l-1}}\right),$$

$$(1-p_1q_1x^2)\left(1-\frac{p_2q_2}{p_1q_1}x^2\right)\left(1-\frac{p_3q_3}{p_2q_2}x^2\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}}{p_{l-2}q_{l-2}}x^2\right)\left(1-\frac{x^2}{p_{l-1}q_{l-1}}\right),$$

$$(1-p_1q_1r_1x^3)\left(1-\frac{p_2q_2r_2}{p_1q_1r_1}x^3\right)\left(1-\frac{p_3q_3r_3}{p_2q_2r_2}x^3\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}r_{l-1}}{p_{l-2}q_{l-2}r_{l-2}}x^3\right)\left(1-\frac{x^3}{p_{l-1}q_{l-1}r_{l-1}}\right),$$

$$\text{"} \qquad \text{"} \qquad \text{"} \qquad \ldots \qquad \text{"} \qquad \text{"}$$

$$\text{"} \qquad \text{"} \qquad \text{"} \qquad \ldots \qquad \text{"} \qquad \text{"}$$

$$(1-p_1q_1\ldots x^m)\left(1-\frac{p_2q_2\ldots}{p_1q_1\ldots}x^m\right)\left(1-\frac{p_3q_3\ldots}{p_2q_2\ldots}x^m\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}\ldots}{p_{l-2}q_{l-2}\ldots}x^m\right)\left(1-\frac{x^m}{p_{l-1}q_{l-1}\ldots}\right),$$

forming a rectangle of m rows and l columns, the letters p, q, r, ..., m in number, each occurring with $l-1$ different suffixes.

I say that forming a fraction with unit numerator, having the product of these factors for denominator, we obtain a generating function for the arrangements defined by $(l\,;\ m\,;\ \infty)$.

The number of layers is restricted to l (*i.e.*, l or less), and the breadth to m (*i.e.*, m or less), but the graphs are otherwise unrestricted. Reasoning of the same nature as that employed in the simple case of two layers, enables us readily to construct this function. The function is redundant, as we only require that portion of the expansion whose terms are altogether integral. In this portion we put the letters p, q, r, ... all equal to unity, and thus arrive at the reduced generating function.

I recall that the predicted result is the reciprocal of

$$\begin{aligned}&(1-x)(1-x^2)(1-x^3)\ldots(1-x^{m-2})(1-x^{m-1})(1-x^m)\\&\quad\times(1-x^2)(1-x^3)\ldots\ldots\ldots\ldots(1-x^{m-1})(1-x^m)(1-x^{m+1})\\&\quad\quad\times(1-x^3)\ldots\ldots\ldots\ldots\ldots\ldots(1-x^m)(1-x^{m+1})(1-x^{m+2})\\&\quad\quad\quad\times\ldots\\&\quad\quad\quad\quad\times(1-x^l)(1-x^{l+1})\ldots\ldots\ldots\ldots(1-x^{l+m-1}).\end{aligned}$$

Professor FORSYTH has not yet succeeded in obtaining this result from his powerful method of selective summation. I hear from him that he has verified it in numerous particular cases, but that, so far, he has not been able to surmount the algebraic difficulties presented by the general case.

As regards the final result, the tableau of factors possesses row and column symmetry.

Simple rotation of the graphs through a right angle in the plane xy establishes this intuitively.

We get the same result from m rows and l columns as from l rows and m columns. Taking only the first row, we find that the fraction

$$\frac{1}{(1-p_1x)\left(1-\frac{p_2}{p_1}x\right)\ldots\left(1-\frac{p_{l-1}}{p_{l-2}}x\right)\left(1-\frac{x}{p_{l-1}}\right)}$$

leads to the same reduced generating function as the fraction

$$\frac{1}{(1-p_1x)(1-p_1q_1x^2)\ldots(1-p_1q_1\ldots x^l)},$$

and this is obviously

$$\frac{1}{(1-x)(1-x^2)\ldots(1-x^l)}.$$

Art. 59. If the result predicted be the true result we should be able to establish it by means of a one-to-one correspondence between the graphs and partitions of a certain kind. This presents difficulties to which I will advert in a moment.

Finally I construct the generating function for the case $(l\,;m\,;n)$, the graphs being restricted in all three dimensions.

The numerator is unity and the denominator the product of the factors exhibited in the subjoined tableau :—

$$(1-gp_1x)\left(1-\frac{p_2}{p_1}x\right)\ldots\left(1-\frac{p_{l-1}}{p_{l-2}}x\right)\left(1-\frac{x}{p_{l-1}}\right)$$

$$(1-gp_1q_1x^2)\left(1-\frac{p_2q_2}{p_1q_1}x^2\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}}{p_{l-2}q_{l-2}}x^2\right)\left(1-\frac{x^2}{p_{l-1}q_{l-1}}\right)$$

$$\text{''} \qquad \text{''} \qquad \ldots \qquad \text{''} \qquad \text{''}$$

$$\text{''} \qquad \text{''} \qquad \ldots \qquad \text{''} \qquad \text{''}$$

$$(1-gp_1q_1\ldots x^m)\left(1-\frac{p_2q_2\ldots}{p_1q_1\ldots}x^m\right)\ldots\left(1-\frac{p_{l-1}q_{l-1}\ldots}{p_{l-2}q_{l-2}\ldots}x^m\right)\left(1-\frac{x^m}{p_{l-1}q_{l-1}\ldots}\right)$$

in which the occurrence of the symbol g in the first column will be noticed.

We have as usual to neglect all terms in the expansion which involve negative powers of symbols and in addition we must now neglect all terms which involve g raised to a greater power than n.

This construction prevents the lower layer of the graph from having a greater extent than n in the direction Oy, and thus the whole graph is similarly restricted.

The reduced generating functions can be shown in simple instances to agree with the predicted results.

Ex. gr. take $(l\,;\ m\,;\ n) = (2\,;\ 2\,;\ 1)$; the fraction is

$$\frac{1}{(1-gp_1x)\left(1-\frac{x}{p_1}\right)(1-gp_1q_1x^2)\left(1-\frac{x^2}{p_1q_1}\right)}.$$

We have to retain the integral portion of

$$(1+p_1x+p_1q_1x^2)\left(1+\frac{x}{p_1}+\frac{x^2}{p_1q_1}\right);$$

selecting this and putting $p_1 = q_1 = 1$, we obtain

$$1 + x + 2x^2 + x^3 + x^4,$$

which is

$$\frac{(1-x^2)(1-x^3)^2(1-x^4)}{(1-x)(1-x^2)^2(1-x^3)}.$$

As in simpler cases I have not been able to overcome the algebraic difficulties, it is perhaps needless to say that in this most general case I cannot establish the form of the reduced generating function.

Art. 60. I return to consider various particular points of the problem. When the number of layers of nodes is restricted to two, we have seen that the generating function which enumerates the graphs that can be formed with a given number of nodes is

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-2}(1-x^4)^{-2}\ldots\ldots$$

In correspondence we have the regularised bipartitions (including uni-partitions) of multipartite numbers of given content.

Also if the breadth of the graph do not exceed m *or* the multipartite numbers be not more than m-partite the generating function is

$$(1-x)^{-1}\{(1-x^2)(1-x^3)\ldots(1-x^m)\}^{-2}(1-x^{m+1})^{-1}.$$

I propose to give another proof of these results based upon a certain mode of dissection of the graph.

In the notation that has been used, a graph may be written

$$\begin{matrix} 2^{\lambda} & 1^{\mu} \\ 2^{\lambda'} & 1^{\mu'} \\ 2^{\lambda''} & 1^{\mu''} \\ \vdots & \end{matrix}$$

where

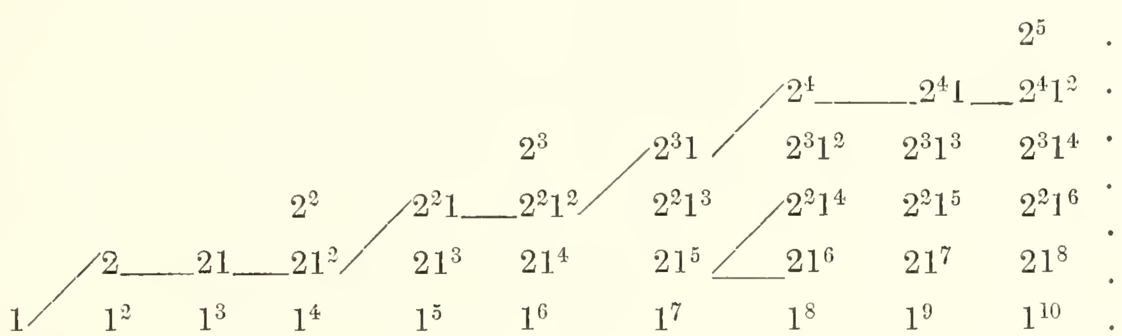
$$\lambda + \mu, \quad \lambda' + \mu', \quad \lambda'' + \mu'', \dots .$$

and also

$$\lambda, \quad \lambda' \quad \lambda'', \dots .$$

are in descending order of magnitude.

A line of the graph has a certain weight $2\lambda + \mu$. Any number of lines may be identical and consequently of the same weight, but no two *different* lines may have the same weight in the same graph. Let us form a graph, beginning at the lowest line, taken to be of weight unity, and proceeding upwards through every superior weight. We will find that such a graph may have a variety of forms. Construct the subjoined scheme of graph lines.



In each column every graph line has the same weight. In each line every graph line has the same number of twos. From any graph line, say $2^\lambda 1^\mu$ ($\mu > 0$), of weight $2\lambda + \mu$, we can pass to a graph line of weight $2\lambda + \mu + 1$ in two ways; viz., by taking $2^\lambda 1^{\mu+1}$ by horizontal progression or $2^{\lambda+1} 1^{\mu-1}$ by diagonal progression. From 2^λ we can only pass to $2^\lambda 1$ by horizontal progression. In accordance with these laws we can form a graph consisting of graph lines of all weights, from unity upwards, in a definite number of ways, depending upon the weight of the highest graph line. For example, we can select the graph whose successive lines are

$$1, 2, 21, 21^2, 2^2 1, 2^2 1^2, 2^3 1, 2^4, 2^4 1, 2^4 1^2 \dots .$$

The progression from graph line to graph line is either horizontal or diagonal, which we can denote by A and B respectively. Then the graph may be denoted by

B A A B A B B A A.

The specification of the selected graph may be taken to be a collection of line graphs, each of which is reached by diagonal progression, and which proceeds by horizontal progression.

Thus, in the particular case before us, the specification is

$$2, \quad 2^2 1, \quad 2^4.$$

To find the form of the graph between any two graph lines, say $2^2 1$, 2^4, we have merely to proceed forwards horizontally from $2^2 1$ and backwards diagonally from 2^4 till a junction is effected in the manner

$$2^2 1 \,\_\_\_\, 2^2 1^2 \diagup 2^3 1 \diagup 2^4$$

In the A, B notation the specification is given by the position of the BA contacts.

BAABABBAA.

Every graph that can be selected has its own specification. To enumerate the graphs we must enumerate the specifications.

We must discover the properties of the succession of graph lines which are able to constitute a specification.

If $2^\lambda 1^\mu$ be a graph line in a specification, λ must be greater than zero since the lower line of the scheme cannot be reached by diagonal succession. Also $2^\lambda 1^\mu$ may be followed by a graph line $2^{\lambda_1} 1^{\mu_1}$ so long as $\lambda_1 > \lambda$ and $\lambda_1 + \mu_1 > \lambda + \mu$; and may be preceded by a graph line $2^{\lambda_0} 1^{\mu_0}$ so long as $\lambda > \lambda_0$, $\lambda + \mu > \lambda_0 + \mu_0$.

These laws may be gathered by simple inspection of the scheme.

Hence we may form the graph lines of the specification into a graph of the specification, viz.:—

$$\begin{array}{c} 2^\lambda 1^\mu \\ 2^{\lambda_1} 1^{\mu_1} \\ 2^{\lambda_2 \mu_2} \\ \vdots \end{array}$$

which has the simple properties—

(i) $\lambda > \lambda_1 > \lambda_2 > \ldots$

(ii) $\lambda + \mu > \lambda_1 + \mu_1 > \lambda_2 + \mu_2 > \ldots$

(iii) The lowest graph line contains, at least, one 2.

Art. 61. Any selected graph has a certain specification graph. From the former we may suppose any graph lines, which do not also belong to the specification, to be absent, and thus obtain a number of graphs which all have the same specification, and may be considered to follow the same line of route through the scheme. Further, the graph lines, whether belonging to the specification or no, may occur any number of times, repeated without the graph of the specification being changed.

Finally, we may associate any partition of a unipartite number with any specification graph whatever, so as to form a two-layer graph. The partition of the unipartite must be interpreted upon that line of route through the scheme which is associated with the given specification graph.

Ex. gr. Suppose the specification graph to be

2 2 2 2
2 2 1
2

and the unipartite partition to be

(9 7 7 6 5 5 4 3 2 2).

Interpreted on the line of route concerned, which is that marked upon the scheme above, we obtain

  2 2 2 2 1
S 2 2 2 2
  2 2 2 1
  2 2 2 1
  2 2 1 1
  2 2 1
  2 2 1
S 2 2 1
  2 1 1
  2 1
  2
  2
S 2,

in which the specification graph lines, marked S, have been interpolated.

Hence, if F (x) be the generating function which enumerates specification graphs,

$$\frac{F(x)}{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots \textit{ad inf.}}$$

will be the generating function of all two-layer graphs—that is of forms specified by $(2\,;\ \infty\,;\ \infty)$.

We have next to determine the form of F (x).

Art. 62. A specification graph may contain no graph lines ; this will be the case when the line of route through the scheme is the lowest horizontal line. There is only *one* such graph ; generating function 1. If it contains one line, this line must be of the form $2^\lambda 1^\mu$ $(\lambda > 0)$, and the number of such graphs is given by the generating function

$$\frac{x^2}{(1-x)(1-x^2)}.$$

We may also take the following view of the matter. Let $k_1(x)$ be the generating function for the number of two-layer graphs in which the number of rows is limited to unity; that is, of the graphs specified by $(2;\ 1;\ \infty)$. The graph either does not or does contain a specification graph line. The former are enumerated by

$$\frac{1}{1-x};$$

the latter by $x^2 k_1(x)$.

Hence,

$$k_1(x) = \frac{1}{1-x} + x^2 k_1(x);$$

or

$$k_1(x) = \frac{1}{(1-x)(1-x^2)};$$

and the number of specification graphs containing one graph line is

$$x^2 k_1(x) \quad \text{or} \quad \frac{x^2}{(1-x)(1-x^2)}.$$

Next let $k_2(x)$ denote the number of two-layer graphs in which the number of rows is restricted to two. If it contain no specification graph line it must be of the form

$$\begin{matrix} 1^{\lambda} & \\ 1^{\mu} & \qquad \lambda \overline{\gtrless} \mu. \end{matrix}$$

For these the generating function is

$$\frac{1}{(1-x)(1-x^2)}.$$

If it contain *one* specification graph line it must be of the form

$$\begin{matrix} 2^{\lambda+1}\, 1^{\mu} \\ 1^{\nu} \end{matrix} \quad (\lambda + \mu + 1 \overline{\gtrless} \nu)$$

and these are enumerated by

$$\frac{x^2 k_1(x)}{1-x}.$$

If it contain two specification graph lines its form must be

$$\begin{matrix} 2^{\lambda+2}\, 1^{\mu} \\ 2^{\lambda'+1}\, 1^{\mu'} \end{matrix}$$

where

$$\lambda \gtreqless \lambda' \text{ and } \lambda + \mu + 1 > \lambda' + \mu'.$$

These are enumerated by

$$x^6 k_2(x).$$

Hence

$$k_2(x) = \frac{1}{(1-x)(1-x^2)} + \frac{x^2 k_1(x)}{1-x} + x^6 k_2(x).$$

Therefore

$$k_2(x) = \frac{1}{(1-x)(1-x^2)^2(1-x^3)}$$

and the number of specification graphs containing two graph lines is

$$x^6 k_2(x) \quad \text{or} \quad \frac{x^6}{(1-x)(1-x^2)^2(1-x^3)}.$$

Similarly we shall find that $k_3(x)$ is composed of four parts corresponding to the occurrence of 0, 1, 2, or 3 specification graph lines. The first three are readily seen to be enumerated by generating functions

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}, \quad \frac{x^2 k_1(x)}{(1-x)(1-x^2)}, \quad \frac{x^6 k_2(x)}{1-x};$$

When three specification graph lines occur, the form must be

$$\begin{matrix} 2^{\lambda+3}\ 1^{\mu} \\ 2^{\lambda'+2}\ 1^{\mu'} \\ 2^{\lambda''+1}\ 1^{\mu''} \end{matrix}$$

$$\lambda \gtreqless \lambda' \gtreqless \lambda'' \text{ and } \lambda + \mu + 2 > \lambda' + \mu' + 1 > \lambda'' + \mu'',$$

and the generating function $x^{12} k_3(x)$.

Hence

$$k_3(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)} + \frac{x^2 k_1(x)}{(1-x)(1-x^2)} + \frac{x^6 k_2(x)}{1-x} + x^{12} k_3(x),$$

and we can show that

$$k_3(x) = \frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)},$$

and the specification graphs containing three graph lines are given by

$$x^{12}k_3(x) \quad \text{or} \quad \frac{x^{12}}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}.$$

In general we obtain the relation

$$k_m(x) = \frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} + \frac{x^2k_1(x)}{(1-x)(1-x^2)\ldots(1-x^{m-1})}$$

$$+\frac{x^6k_2(x)}{(1-x)(1-x^2)\ldots(1-x^{m-2})}+\ldots+\frac{x^{s(s+1)}k_s(x)}{(1-x)(1-x^2)\ldots(1-x^{m-s})}+\ldots+x^{m(m+1)}k_m(x),$$

and also

$$k_\infty(x) = \frac{1+x^2k_1(x)+x^6k_2(x)+\ldots+x^{s(s+1)}k_s(x)+\ldots}{(1-x)(1-x^2)(1-x^3)\ldots},$$

where the numerator is the generating function for specification graphs of given content.

Art. 63. We can now establish that $k_s(x)$ is the expression

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2\ldots(1-x^s)^2(1-x^{s+1})};$$

for assume the law true for values of s equal and inferior to $m-1$; substitute in the foregoing identity and writing $1-x^s=(s)$,

$$(1)(2)\ldots(m)(1-x^{m^2+m})k_m(x)$$

$$=1+\frac{x^2(m)}{(1)(2)}+\frac{x^6(m)(m-1)}{(1)(2)^2(3)}+\ldots+\frac{x^{m^2-m}(m)(m-1)\ldots(2)}{(1)(2)^2\ldots(m-1)^2(m)}.$$

Recalling the well-known identity

$$\frac{1}{(1-ax)(1-ax^2)\ldots(1-ax^m)} = 1+\frac{(m)}{(1)}\cdot\frac{ax}{1-ax}+\frac{(m)(m-1)}{(1)(2)}\frac{a^2x^4}{1-ax\,.\,1-ax^2}$$

$$+\frac{(m)(m-1)(m-2)}{(1)(2)(3)}\frac{a^3x^9}{1-ax\,.\,1-ax^2\,.\,1-ax^3}+\ldots$$

and putting therein $a=x$, we find

$$\frac{1}{(2)(3)\ldots(m+1)} = 1+\frac{x^2(m)}{(1)(2)}+\frac{x^6(m)(m-1)}{(1)(2)^2(3)}+\ldots$$

$$+\frac{x^{m^2-m}(m)(m-1)\ldots(2)}{(1)(2)^2(3)^2\ldots(m-1^2)(m)}+\frac{x^{m^2+m}(m)(m-1)\ldots(1)}{(1)(2)^2(3)^2\ldots(m)^2(m+1)}.$$

Hence

$$(1)(2)\ldots(m)(1-x^{m^2+m})\,k_m(x) = \frac{1}{(2)(3)\ldots(m+1)} - \frac{x^{m^2+m}}{(2)(3)\ldots(m+1)}.$$

Therefore

$$k_m(x) = \frac{1}{(1)(2)^2(3)^2\ldots(m-1)^2(m)^2(m+1)}.$$

Hence, by induction, it has been established that $k_m(x)$ has this expression for all values of m.

Therefore the result

$$(2\,;\,m\,;\,\infty) = \frac{1}{(1-x)(1-x^2)^2(1-x^3)^2\ldots(1-x^m)^2(1-x^{m+1})}$$

agreeing with that obtained in a totally different manner by FORSYTH.

I hope to continue the theory, adumbrated in this paper, in a future communication to the Royal Society.

INDEX SLIP.

XVII. *Problems in Electric Convection.*

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Communicated by Professor J. J. Thomson, *F.R.S.*

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Contents.

Introduction.

1. The following paper is occupied by an attempt to investigate the distribution of the electric and magnetic forces which are called into play when certain electromagnetic systems are made to move with uniform velocity through the ether. Maxwell's theory will be employed throughout, and will be applied to the exact solution of several problems, and to the establishment of some results of a general

nature. Professor J. J. THOMSON was, I believe, the first to consider a problem of this sort, his paper* giving the solution for the motion of a single electrical point-charge at a speed small compared with that of light. Mr. OLIVER HEAVISIDE next considered the question, and in his paper† obtained an *exact* solution for the motion of a point-charge. He got the solution by means of the "vector potential" of the convection current formed by the moving charge. The mathematical analysis is of the symbolical kind, but it is shown that the result obtained by means of it satisfies all the necessary conditions. By integrations Mr. HEAVISIDE obtains solutions for the motion of some simple cases of electrical distribution. Mr. HEAVISIDE's expression for the vector potential is

$$\mathbf{A} = \frac{4\pi\rho\mathbf{u}}{p^2/v^2 - \nabla^2},$$

which he re-writes in the form

$$\mathbf{A} = \frac{-4\pi\rho\mathbf{u}/\nabla^2}{1 - \dfrac{p^2}{v^2\nabla^2}},$$

where $\mathbf{A}$ is the vector symbol for the vector potential, $\mathbf{u}$ the vector symbol for the velocity of the electricity, whose volume density is ρ, and v is the velocity of light, while $p^2 = u^2 d^2/dz^2$ and $\nabla^2 = d^2/dx^2 + d^2/dy^2 + d^2/dz^2$; the motion is supposed to take place parallel to the axis of z.

Mr. HEAVISIDE performs the operation $1/\nabla^2$ first, and obtains

$$\mathbf{A}_0 = \frac{4\pi\rho\mathbf{u}}{-\nabla^2} = \Sigma\,\frac{\rho\mathbf{u}}{r},$$

so that

$$\mathbf{A} = \frac{1}{1 - \dfrac{p^2}{v^2\nabla^2}}\,\mathbf{A}_0.$$

The operation here indicated is then performed for the special case in which $\mathbf{A}_0 = q\mathbf{u}/r$, corresponding to the motion of a single point-charge, and a correct value of $\mathbf{A}$ is obtained.

[August 20, 1896.—But except in this simple case, there appears to be some difficulty in the interpretation of the two operations $\dfrac{1}{\nabla^2}$ and $\left\{1 - \dfrac{p^2}{v^2\nabla^2}\right\}^{-1}$ when they are performed separately. For if the operations are performed separately and in Mr. HEAVISIDE's order for a uniformly charged sphere of radius a, the result is the same as for a point-charge at its centre, since A_0 varies simply as the reciprocal of r

* 'Phil. Mag.,' April, 1881.

† 'Phil. Mag.,' April, 1889, or 'Electrical Papers,' vol. 2, p. 504.

in both cases. This result is not correct, for it can be shown* that it is not a point-charge, but a uniformly charged line of length $2au/v$, which produces the same effect as the uniformly charged sphere.]

Professor J. J. THOMSON has also obtained the exact solution for a point-charge in two different ways. In his first treatment† he adopts MAXWELL'S equations involving the Vector Potential, and an electrostatic potential Ψ. In his last paper‡ he finds the solution by the aid of his novel method of considering the phenomena of the electromagnetic field as being brought about by the motion of tubes of electric force. This paper may be considered as an attempt to take a step beyond MAXWELL'S analytical theory, and to give a sort of material representation of the mechanism of the electromagnetic field.

The result of all these investigations is that while the electric force due to a moving point-charge is still *radial*, the intensity of the force, for a given distance from the charge, gradually increases as the radius vector turns from the direction of motion to a perpendicular direction. There is also a distribution of magnetic force, in which the lines of force are circles centred on the axis of motion, the planes of the circles being perpendicular thereto.

The fact that the electric force is radial led Mr. HEAVISIDE to form the conclusion that the expression for the electric force due to a point-charge is the same as that due to a charged sphere in motion carrying an equal charge, the distribution on the sphere being such that $\sigma = KE_n/4\pi$, where E_n is the electric force normal to the surface which would be due to the point-charge placed at the centre of the sphere. But the surface which gives rise to a field the same as that due to a point-charge is an ellipsoid of revolution, whose minor axis, which is also the axis of figure, lies along the direction of motion, and whose axes are in the ratios $1 : 1 : (1 - u^2/v^2)^{\frac{1}{2}}$, where u is the velocity of the point and v is the velocity of light through the dielectric.* The charge is distributed in the same way as if the ellipsoid were statically charged, *i.e.*, the surface density is proportioned to the perpendicular from the centre on the tangent plane. This surface I call the "Heaviside" ellipsoid.

Mr. HEAVISIDE appears to have thought that if there is no disturbance within a closed surface, then the surface condition is that the electric force just outside the surface should be *normal* to the surface. As this led to the supposed equivalence of the sphere and the point, and as I convinced myself that this equivalence does not exist, I asked Mr. HEAVISIDE about the matter. This led him to reconsider the conditions which obtain at a surface bounding a region of zero disturbance, and he showed§ that it is not the electric force which is perpendicular to the surface, but a certain vector **F**. This vector **F**, I have shown, is simply the mechanical force

* This can be readily shown by the use of the auxiliary coordinates ξ, η, ζ of § 16 below.

† 'Phil. Mag.,' July, 1889.

‡ 'Phil. Mag.,' March, 1891, and 'Recent Researches in El. and Mag.,' p. 16.

§ 'Electrical Papers," vol. 2, p. 514, and 'Electromagnetic Theory,' vol. 1, p. 273.

experienced by a unit of positive electricity *in motion at the same speed as the rest of the system.* This mechanical force experienced by the unit charge consists not only of the part due to the existence of electric force in the field, but also of a part due to the fact that the moving unit charge is acted on like a current element by the magnetic induction.

Mathematical Abbreviations.

2. Certain mathematical forms occur so frequently in the theory of electromagnetism that it is convenient to have some compact method of indicating them. The following are the abbreviations which will be employed in this essay.

(1.) The vector quantity whose components are A_1, A_2, A_3, will be written **A** in **clarendon** type, while its magnitude without regard to direction will be denoted by A.

(2.) The scalar quantity $AB \cos \theta \equiv A_1B_1 + A_2B_2 + A_3B_3$, where θ is the angle between **A** and **B**, is called the Scalar Product of **A** and **B**, and is denoted by **SAB**. Of course **SAB** = **SBA**. If **A** and **B** are parallel and in the same sense, **SAB** = AB simply. If they are perpendicular to each other, **SAB** = 0.

(3.) The vector **C**, whose components are

$$C_1 = A_2B_3 - A_3B_2 \qquad C_2 = A_3B_1 - A_1B_3 \qquad C_3 = A_1B_2 - A_2B_1,$$

is called the Vector Product of **A** and **B**, and is denoted by **C** = **VAB**. If θ be the angle between **A** and **B**, then $C = AB \sin \theta$. Moreover, **C** is perpendicular to both **A** and **B**, and its positive direction is such that right-handed rotation about **C** carries **A** to **B**. It is plain that **VAB** = − **VBA**, and that if **A** and **B** are parallel, then **VAB** = 0.

(4.) If **D** = **VAB**, then **VCVAB** stands for **VCD**. By working out the three components of **VCD**, it is easily found that

$$\mathbf{VCVAB} = \mathbf{ASBC} - \mathbf{BSCA}.$$

(5.) If **D** = **VAB**, then **SCVAB** stands for **SCD**.

(6.) The vector **A**, whose components are $\frac{d\Psi}{dx}, \frac{d\Psi}{dy}, \frac{d\Psi}{dz}$, where Ψ is any scalar quantity, is called the Slope of Ψ and is denoted by $\mathbf{A} = \nabla\Psi$. The vector **A** points in the direction in which Ψ increases most rapidly, and is normal to the surface Ψ = constant.

(7.) The value of the surface integral of the normal component (reckoned outwards) of a vector **A**, when applied to any infinitesimal closed surface, is

$$\frac{dA_1}{dx} + \frac{dA_2}{dy} + \frac{dA_3}{dz}$$

per unit volume of the enclosed space. This is called the Divergence of **A**, and will be denoted by div **A**. It is a scalar quantity.

(8.) The vector **B**, whose components are the values per unit of area of the line integrals of the vector **A** taken in right-handed directions round three infinitesimal areas in planes perpendicular to x, y, z, is called the Curl of **A**, and is denoted by $\mathbf{B} = \text{curl}\ \mathbf{A}$. Its components are

$$B_1 = \frac{dA_3}{dy} - \frac{dA_2}{dz}, \qquad B_2 = \frac{dA_1}{dz} - \frac{dA_3}{dx}, \qquad B_3 = \frac{dA_2}{dx} - \frac{dA_1}{dy}.$$

The positive direction of the integration and the positive direction of **B** are related in the same way as the rotation and translation of a right-handed screw working in a fixed nut.

(9). From the definition it is plain that $\text{div curl}\ \mathbf{A} = 0$.

(10). When any vector **A** changes with the time, the vector $\frac{d\mathbf{A}}{dt}$, or $\dot{\mathbf{A}}$, denotes that vector which would have to be compounded with the vector **A** at any instant in order to obtain the new value which **A** has after unit time, if the change is uniform. Its components are of course

$$\frac{dA_1}{dt}, \quad \frac{dA_2}{dt}, \quad \frac{dA_3}{dt}.$$

Mr. HEAVISIDE has given a very useful chapter on the elementary parts of Vector Algebra and Analysis in vol. 1 of his 'Electromagnetic Theory.' I have followed his notation with the exception of writing the scalar product **SAB** instead of **AB** simply. But I gather that he would not object to this change.* An account of vector analysis on the same lines has also been given by Dr. A. FÖPPL.†

Statement of Principles.

3. In my investigation I shall follow MAXWELL'S theory, but shall adopt the method, employed by HEAVISIDE, HERTZ, and others, of stating the fundamental equations in terms of the electric and magnetic forces. Quantities of the nature of potentials will be introduced during the course of the work, but this will be done simply to facilitate the calculations.

The four principal quantities to be dealt with are: (1) the electric force **E**, (2) the electric displacement **D**, (3) the magnetic force **H**, and (4) the magnetic induction **B**. Since the medium is supposed to be homogeneous and isotropic, we have the relations

$$\mathbf{D} = \frac{1}{4\pi} K\mathbf{E} \quad \ldots \ldots \ldots \quad (1),$$

$$\mathbf{B} = \mu\mathbf{H} \quad \ldots \ldots \ldots \quad (2),$$

* 'Electromagnetic Theory,' vol. 1, p. 304, also 'Electrical Papers,' vol. 2, p. 528.

† 'Einführung in die Maxwell'sche Theorie,' Leipzig, 1894.

where K is the specific inductive capacity, and μ the magnetic permeability of the medium. The components of the electric force **E** parallel to the axes of x, y, and z will be denoted by E_1, E_2, E_3 respectively. The same notation will also be applied to the other quantities.

If **E**, **D**, **H**, **B** are all measured in the same system of units, then the principles to be employed in the formation of the fundamental equations may be expressed as follows:—(1). The line-integral of the magnetic force taken once round any closed circuit fixed in space is equal to 4π times the total amount of electric current flowing through the circuit, the positive directions of the current and of the integration being related to each other in the same way as the translation and rotation of a right-handed screw working in a fixed nut.

If **H** and **C** are the magnetic force and the electric current respectively, the set of differential equations which expresses this result may be written

$$\text{curl}\,\mathbf{H} = 4\pi\mathbf{C} \qquad (3).$$

Now it was an essential part of the theory, as MAXWELL left it, that the variation of the electric displacement constitutes a true current whose amount and direction is expressed by $\frac{d\mathbf{D}}{dt}$ or $\frac{K}{4\pi}\frac{d\mathbf{E}}{dt}$. But Professor G. F. FITZGERALD* has shown that there ought also to be included the convection current $\rho\mathbf{u}$, where ρ is the volume density of electrification and $\mathbf{u}$ is its velocity. Since we are not concerned with conduction currents we may leave them out of account. We have, then,

$$\text{curl}\,\mathbf{H} = 4\pi\frac{d\mathbf{D}}{dt} + 4\pi\rho\mathbf{u} = K\frac{d\mathbf{E}}{dt} + 4\pi\rho\mathbf{u} \qquad (4).$$

(2.) The line-integral of the electric force taken once round any closed circuit fixed in space is equal to *minus* the total amount of magnetic current through the circuit, the positive directions of the magnetic current and of the integration being related as in (1).

There is no evidence for the existence of a magnetic conduction current, involving a waste of energy. The only constituent of the magnetic current is that which arises from the increase in the magnetic induction, viz., $\frac{d\mathbf{B}}{dt}$ or $\mu\frac{d\mathbf{H}}{dt}$ when μ is constant. We might include a fictitious magnetic convection current $\tau\mathbf{u}$ when τ is the volume density of magnetism, but for the present we omit it. The relation may thus be expressed by

$$\text{curl}\,\mathbf{E} = -\frac{d\mathbf{B}}{dt} = -\mu\frac{d\mathbf{H}}{dt} \qquad (5).$$

Equations (4) and (5) must be satisfied at all points of the field. They at once

* 'B. A. Report,' 1883, p. 404.

lead to two important results, for if the *divergence* of each of these equations be taken we have

$$\text{div curl } \mathbf{H} = 4\pi \frac{d}{dt} (\text{div } \mathbf{D}) + 4\pi \text{ div } (\rho \mathbf{u}) \quad \ldots \ldots \quad (6),$$

$$\text{div curl } \mathbf{E} = - \frac{d}{dt} (\text{div } \mathbf{B}) \quad \ldots \ldots \quad (7).$$

But div curl **H** and div curl **E** both vanish identically, so that

$$\frac{d}{dt} (\text{div } \mathbf{D}) = - \text{ div } (\rho \mathbf{u}) \quad \ldots \ldots \quad (8).$$

$$\frac{d}{dt} (\text{div } \mathbf{B}) = 0 \quad \ldots \ldots \quad (9).$$

But div $\mathbf{D} = \rho$, so that (8) becomes

$$d\rho/dt = - \text{ div } (\rho \mathbf{u}) \quad \ldots \ldots \quad (10).$$

Thus the density of electrification at any point can only be changed by the convection of electrification to or from the place. If a body has a charge q, no change in q can be produced by the motion of other charged bodies or of magnets in its neighbourhood. In the ordinary parts of the field ρ is zero initially, and therefore continues zero.

From equation (9) we find that div B = constant. But we already know that div $\mathbf{B} = 0$.

If K and μ are constant, then at all points of the field we have,

$$\text{div } \mathbf{E} = \frac{4\pi\rho}{\text{K}} \quad \ldots \ldots \quad (11),$$

$$\text{div } \mathbf{H} = 0 \quad \ldots \ldots \quad (12).$$

Application to Steady Motion.

4. I shall now apply the principles already stated to the case of the steady motion of any system through the field. The coordinates x, y, z will be supposed measured from a system of axes moving forwards with the system, without rotation. The motion of the axes will introduce no difficulty, for the values of the line and surface integrals are the same whether the axes are at rest or in motion.

For the sake of greater generality, I shall first suppose that the velocity **u** of the system has the components u_1, u_2, u_3. Then since the motion is steady, we have

$$\frac{d}{dt} = -\left(u_1\frac{d}{dx} + u_2\frac{d}{dy} + u_3\frac{d}{dz}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Taking the first of each of the two sets of equations represented by (5) and (4) and substituting for d/dt, we have

$$\frac{dE_3}{dy} - \frac{dE_2}{dz} = \mu\left(u_1\frac{dH_1}{dx} + u_2\frac{dH_1}{dy} + u_3\frac{dH_1}{dz}\right) \quad . \quad . \quad . \quad . \quad . \quad (14).$$

$$\frac{dH_3}{dy} - \frac{dH_2}{dz} = -K\left(u_1\frac{dE_1}{dx} + u_2\frac{dE_1}{dy} + u_3\frac{dE_1}{dz}\right) + 4\pi\rho u_1 \quad . \quad . \quad . \quad (15).$$

But div $\mathbf{E} = 4\pi\rho/K$ and div $\mathbf{H} = 0$. Using the latter, (14) becomes

$$\begin{aligned}\frac{dE_3}{dy} - \frac{dE_2}{dz} &= \mu\left(u_2\frac{dH_1}{dy} + u_3\frac{dH_1}{dz} - u_1\frac{dH_2}{dy} - u_1\frac{dH_3}{dz}\right)\\ &= \mu\left\{\frac{d}{dy}(H_1u_2 - H_2u_1) - \frac{d}{dz}(H_3u_1 - H_1u_3)\right\}\\ &= dP_3/dy - dP_2/dz,\end{aligned}$$

if $\mathbf{P} = \mu V\mathbf{Hu}$.

Hence

$$\frac{d}{dy}(E_3 - P_3) - \frac{d}{dz}(E_2 - P_2) = 0.$$

The remaining two equations symbolised by (5) may be treated in the same manner, and the resulting equations may be symbolised by

$$\text{curl}\,(\mathbf{E} - \mu V\mathbf{Hu}) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Similarly from (4) we find

$$\text{curl}\,(\mathbf{H} + KV\mathbf{Eu}) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

ρ disappearing from the equations.

These two equations take the place of (5) and (4) for the case of steady motion, and must be satisfied throughout the field.

It follows from (16) and (17) that we can write

$$\mathbf{E} - \mu V\mathbf{Hu} = -\nabla\Psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

$$\mathbf{H} + KV\mathbf{Eu} = -\nabla\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

or

$$E_1 - \mu(H_2u_3 - H_3u_2) = -\frac{d\Psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$H_1 + K(E_2u_3 - E_3u_2) = -\frac{d\Omega}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

together with four other equations of the same type. The quantities Ψ and Ω are, as we shall see, sufficient to determine the state of the field at every point, and must be found in order to get a solution of any problem.

If we solve this set of six equations for the components of **E** and **H**, and remember that $K\mu v^2 = 1$, where v is the velocity of an electromagnetic disturbance through the medium, we obtain

$$E_1\left(1 - \frac{u^2}{v^2}\right) = -\left(1 - \frac{u_1^2}{v^2}\right)\frac{d\Psi}{dx} + \frac{u_1u_2}{v^2}\frac{d\Psi}{dy} + \frac{u_1u_3}{v^2}\frac{d\Psi}{dz} + \mu\left(u_2\frac{d\Omega}{dz} - u_3\frac{d\Omega}{dy}\right) \quad . \quad (22),$$

$$H_1\left(1 - \frac{u^2}{v^2}\right) = -\left(1 - \frac{u_1^2}{v^2}\right)\frac{d\Omega}{dx} + \frac{u_1u_2}{v^2}\frac{d\Omega}{dy} + \frac{u_1u_3}{v^2}\frac{d\Omega}{dz} - K\left(u_2\frac{d\Psi}{dz} - u_3\frac{d\Psi}{dy}\right) \quad . \quad (23),$$

together with four similar equations.

By differentiating these equations and using $\operatorname{div}\mathbf{E} = 4\pi\rho/K$ and $\operatorname{div}\mathbf{H} = 0$, we find

$$\nabla^2\Psi - \frac{1}{v^2}\left(u_1\frac{d}{dx} + u_2\frac{d}{dy} + u_3\frac{d}{dz}\right)^2\Psi = -\frac{4\pi}{K}\left(1 - \frac{u^2}{v^2}\right)\rho \quad . \quad . \quad (24),$$

$$\nabla^2\Omega - \frac{1}{v^2}\left(u_1\frac{d}{dx} + u_2\frac{d}{dy} + u_3\frac{d}{dz}\right)^2\Omega = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

These equations become much simpler when the motion takes place parallel to the axis of x. We then have $u_1 = u$, $u_2 = u_3 = 0$, and thus obtain

$$E_1 = -\frac{d\Psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

$$E_2\left(1 - \frac{u^2}{v^2}\right) = -\frac{d\Psi}{dy} - \mu u\frac{d\Omega}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

$$E_3\left(1 - \frac{u^2}{v^2}\right) = -\frac{d\Psi}{dz} + \mu u\frac{d\Omega}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

$$H_1 = -\frac{d\Omega}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

$$H_2\left(1-\frac{u^2}{v^2}\right)=-\frac{d\Omega}{dy}+Ku\frac{d\Psi}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

$$H_3\left(1-\frac{u^2}{v^2}\right)=-\frac{d\Omega}{dz}-Ku\frac{d\Psi}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

while the equations satisfied by Ψ and Ω become

$$\left(1-\frac{u^2}{v^2}\right)\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}+\frac{d^2\Psi}{dz^2}=-\frac{4\pi}{K}\left(1-\frac{u^2}{v^2}\right)\rho \quad . \quad . \quad . \quad . \quad (32),$$

$$\left(1-\frac{u^2}{v^2}\right)\frac{d^2\Omega}{dx^2}+\frac{d^2\Omega}{dy^2}+\frac{d^2\Omega}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$

The solution of any problem depends upon finding functions which satisfy (32) and (33), and which fit in with the particular electromagnetic system which is supposed to be moving. In all ordinary parts of the field we shall have $\rho = 0$, and thus generally we have

$$\left(1-\frac{u^2}{v^2}\right)\frac{d^2\Psi}{dx^2}+\frac{d^2\Psi}{dy^2}+\frac{d^2\Psi}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

Our knowledge of functions which satisfy LAPLACE's equation helps us to find solutions, for if $f(x, y, z)$ satisfies $\nabla^2 f = 0$, it follows that $f\{x/\sqrt{1-u^2/v^2}, y, z\}$ satisfies (34). When in this manner values of Ψ and Ω have been found, the values of **E** and **H** are at once deduced from equations (26) to (31).

The quantity $1 - u^2/v^2$ occurs continually in the course of the work, and will always be denoted by α. The motion will always be supposed to take place parallel to the axis of x, unless it is otherwise stated.

Application of Vector Methods.

5. The solution of the six equations typified by (20) and (21) is tedious by ordinary algebraical processes. But the solution is readily obtained by simple vector analysis, and affords a good example of the great saving of labour effected by Mr. HEAVISIDE's methods. Thus, let $\mathbf{F} = -\nabla\Psi$ and $\mathbf{R} = -\nabla\Omega$, so that, by (18) and (19)

$$\mathbf{E} - \mu \mathrm{V}\mathbf{H}\mathbf{u} = \mathbf{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

$$\mathbf{H} + K\mathrm{V}\mathbf{E}\mathbf{u} = \mathbf{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Then we have to find **E** and **H** in terms of Ψ and Ω, or in terms of **F** and **R**. Substituting from (36) in (35) we have

$$\mathbf{E} + \mu \mathbf{V}\mathbf{u}(\mathbf{R} - \mathbf{K}\mathbf{V}\mathbf{E}\mathbf{u}) = \mathbf{F},$$

where we have used

$$\mathbf{V}\mathbf{H}\mathbf{u} = -\mathbf{V}\mathbf{u}\mathbf{H}.$$

Thus,

$$\mathbf{E} + \mu \mathbf{V}\mathbf{u}\mathbf{R} - \frac{1}{v^2}\mathbf{V}\mathbf{u}\mathbf{V}\mathbf{E}\mathbf{u} = \mathbf{F} \quad \ldots \ldots \quad (37).$$

But,

$$\mathbf{V}\mathbf{u}\mathbf{V}\mathbf{E}\mathbf{u} = \mathbf{E}\mathbf{S}\mathbf{u}\mathbf{u} - \mathbf{u}\mathbf{S}\mathbf{u}\mathbf{E} = u^2\mathbf{E} - \mathbf{u}\mathbf{S}\mathbf{u}\mathbf{E}.$$

Again, by (35),

$$\mathbf{S}\mathbf{u}\mathbf{E} - \mu \mathbf{S}\mathbf{u}\mathbf{V}\mathbf{H}\mathbf{u} = \mathbf{S}\mathbf{u}\mathbf{F}.$$

But **VHu** is at right angles to **u** (and also to **H**), and hence the "scalar product" of **u** and **VHu** vanishes. Thus **SuE** = **SuF**, and therefore (37) becomes

$$\mathbf{E} + \mu \mathbf{V}\mathbf{u}\mathbf{R} - \frac{u^2}{v^2}\mathbf{E} - \frac{\mathbf{u}}{v^2}\mathbf{S}\mathbf{u}\mathbf{F} = \mathbf{F},$$

so that

$$\mathbf{E}\left(1 - \frac{u^2}{v^2}\right) = \mathbf{F} - \frac{\mathbf{u}}{v^2}\mathbf{S}\mathbf{u}\mathbf{F} - \mu \mathbf{V}\mathbf{u}\mathbf{R} \quad \ldots \ldots \quad (38).$$

Similarly,

$$\mathbf{H}\left(1 - \frac{u^2}{v^2}\right) = \mathbf{R} - \frac{\mathbf{u}}{v^2}\mathbf{S}\mathbf{u}\mathbf{R} + \mathbf{K}\mathbf{V}\mathbf{u}\mathbf{F} \quad \ldots \ldots \quad (39).$$

The last pair of equations are easily seen to be equivalent to the set of six typified by (22) and (23).

The forms which **E** and **H** assume when **u** is parallel to x, are given in equations (76) and (77) below.

Motion of a Point-Charge.

6. The problem of a moving point-charge has been solved by Mr. HEAVISIDE and Professor J. J. THOMSON, but as the solution will often be needed in other parts of the work, it will be useful to put it down.

If, in the ordinary case of electrostatics, there is a point-charge q at the origin, the electrostatic potential is $q\,\{x^2 + y^2 + z^2\}^{-\frac{1}{2}}$.

Guided by this, let us put

$$\Psi = \mathrm{A}\left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{1}{2}}. \qquad \Omega = 0.$$

Since these values satisfy equations (33), (34), they form the solution of *some* problem in the case of motion. We have now to find what that problem is.

From equations (26) to (31) we have at once

$$\frac{E_1}{x} = \frac{E_2}{y} = \frac{E_3}{z} = \frac{A}{\alpha}\left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

$$H_1 = 0, \qquad \frac{H_2}{-z} = \frac{H_3}{y} = Ku\,\frac{A}{\alpha}\left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{3}{2}} \quad . \quad . \quad . \quad . \quad (41).$$

From (40) it follows that the lines of electric force are radii drawn from the origin. From (41) it appears that the lines of magnetic force are circles having their centres on the axis of x, and their planes perpendicular thereto. Since the electric force is radial, there will be a definite amount of electric displacement outwards through any closed surface, however small, which encloses the origin. Hence the field given by our solution can be produced by the motion of a definite point-charge at the origin. If q is this charge, we can find the value of A corresponding to it from the consideration that the surface integral of the normal electric displacement, taken over any surface enclosing the origin, is equal to q. For the closed surface we may take an infinite cylinder of radius c coaxial with x. Hence

$$q = \frac{K}{4\pi}\int_{-\infty}^{+\infty}\frac{2\pi A c^2\,dx}{\alpha\,\{x^2/\alpha + c^2\}^{\frac{3}{2}}} = \frac{KA}{\sqrt{\alpha}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

Thus $A = \dfrac{q\sqrt{\alpha}}{\kappa}$, so that

$$\Psi = \frac{q\sqrt{\alpha}}{K}\left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43),$$

and the values of the electric and magnetic forces now become

$$\frac{E_1}{x} = \frac{E_2}{y} = \frac{E_3}{z} = \frac{q}{K\sqrt{\alpha}}\left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44).$$

$$H_1 = 0, \qquad \frac{H_2}{-z} = \frac{H_3}{y} = \frac{uq}{\sqrt{\alpha}}\left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad (45).$$

These values are the same as those obtained by HEAVISIDE and J. J. THOMSON.

If r denote the radius vector from the origin, and θ its inclination to the axis of x, then we have for the resultant forces

$$E = \frac{q\,(1 - u^2/v^2)}{Kr^2\,\{1 - \sin^2\theta\,u^2/v^2\}^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

$$H = \frac{qu \sin\theta\,(1 - u^2/v^2)}{r^2\,\{1 - \sin^2\theta\, u^2/v^2\}^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47).$$

From (46) we see that the electric force varies inversely as the square of the distance for any given direction, but that for any given distance it gradually increases as θ increases from 0 to $\frac{1}{2}\pi$. As the speed increases the electric force tends to become more and more concentrated about the plane through the origin at right angles to the axis of x. When $u = v$, there is no electric force except in that plane; we have, in fact, a plane electric wave moving forward at the speed of light.

The expression for H shows that at low speeds, where u^2/v^2 may be neglected in comparison with unity, the magnetic force is the same as that attributed by AMPÈRE'S formula to a current element of "moment" uq. By the moment of an element is meant the product of its length by the strength of the current in it. When the speed of light is attained, the magnetic force is confined to the plane yz, and the lines of force are circles in that plane with their common centre at the origin.

Mr. HEAVISIDE has *stated** the result when u is greater than v, but has not up to the present (March 14, 1896) divulged the manner in which he has obtained the solution in this case. I confine this paper to the case in which u is not greater than v.

As the charge moves along, the electric displacement at each point varies, giving rise to a current, and I shall now investigate the form of the current lines in the case under consideration. The currents evidently flow in planes drawn through the axis of x, so that it will be sufficient to find the form of the current lines in the plane xy. The x and y components of the current at any point are $\frac{K}{4\pi}\frac{dE_1}{dt}$ and $\frac{K}{4\pi}\frac{dE_2}{dt}$, or, since the motion is steady, $-\frac{Ku}{4\pi}\frac{dE_1}{dx}$ and $-\frac{Ku}{4\pi}\frac{dE_2}{dx}$. Hence, if dy/dx refer to one of the current lines, we have

$$\frac{dy}{dx} = \frac{dE_2}{dx} \Big/ \frac{dE_1}{dx}.$$

Performing the differentiations, we find that for points in the plane xy

$$\frac{dy}{dx} = \frac{3xy}{2x^2 - \alpha y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48),$$

the solution of which is

$$cy^2 = (x^2 + \alpha y^2)^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49),$$

or in polar coordinates

$$r = \frac{c \sin^2\theta}{\left(1 - \frac{u^2}{v^2}\sin^2\theta\right)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (50).$$

The form of the lines of flow is given by equation (49) or (50).

* 'Electrical Papers,' vol. 2, p. 516.

When the motion is very slow, the equation becomes

$$r = c \sin^2 \theta,$$

the same as the equation to the lines of flow of a doublet, consisting of a "source" of current and a "sink" of equal strength, placed infinitely near each other in an infinite conducting medium. The currents flow along these curves, being "closed" by the convection current formed by the moving charge. At the speed of light the currents are confined to the plane of yz, and then take the form of outward radial currents in the front face of that plane and inward radial currents in the back face of the plane.

Motion of a Line-Charge.

7. Mr. HEAVISIDE has obtained* the solution for the motion of a uniformly charged straight line, both in its own line and also transversely, by integration of the result for a point-charge.

The method I have employed for the point-charge can be easily applied to these problems, when the line is infinite in length.

(1.) Motion in its own line.

The line coincides with the axis of x, and is supposed to have a charge q per unit length. In the electrostatic problem the potential is $- q \log (y^2 + z^2)$.

Hence a solution in the case of motion is given by

$$\Psi = - \mathrm{A} \log (y^2 + z^2) \qquad \Omega = 0.$$

From this, by equations (26) to (28),

$$\mathrm{E}_1 = 0 \qquad \frac{\mathrm{E}_2}{y} = \frac{\mathrm{E}_3}{z} = \frac{2\mathrm{A}}{\alpha (y^2 + z^2)} \quad \ldots \ldots \quad (51).$$

The electric force is therefore everywhere perpendicular to the charged line, and the resultant is given by

$$\mathrm{E} = \frac{2\mathrm{A}}{\alpha\rho} = \frac{2\mathrm{A}}{\alpha\sqrt{y^2 + z^2}} \quad \ldots \ldots \quad (52).$$

To find A, integrate $\mathrm{KE}/4\pi$ over unit length of a cylinder of radius ρ coaxial with the charged line, and equate the result to q. Thus

$$q = 2\pi\rho \cdot \frac{\mathrm{K}}{4\pi} \cdot \frac{2\mathrm{A}}{\alpha\rho} = \frac{\mathrm{KA}}{\alpha}.$$

* 'Electrical Papers,' vol. 2, p. 516.

Hence

$$A = q\alpha/K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

The state of the field is therefore given by the equations

$$\left.\begin{aligned} E_1 &= 0 \\ E_2 &= \frac{2qy}{K(y^2+z^2)} \\ E_3 &= \frac{2qz}{K(y^2+z^2)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).$$

$$\left.\begin{aligned} H_1 &= 0 \\ H_2 &= -\frac{2quz}{y^2+z^2} \\ H_3 &= \frac{2quy}{y^2+z^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

The resultant electric force is perpendicular to the charged line. Its value is

$$E = 2q/K\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56),$$

the same as if the charge were at rest.

The resultant magnetic force is in circles round the wire. Its value is

$$H = 2qu/\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

the same as that due to a current qu. Thus the motion introduces a magnetic force without affecting the electric force at all.

(2.) Motion perpendicular to its own line.

Let the charged line coincide with the axis of z. The potential in the electrostatic problem is $-q\log(x^2+y^2)$. Hence a solution in the case of motion is given by

$$\Psi = -A\log\left(\frac{x^2}{\alpha}+y^2\right) \qquad \Omega = 0.$$

From this, by equations (26) to (28),

$$\frac{E_1}{x} = \frac{E_2}{y} = \frac{2A}{\alpha\left(\dfrac{x^2}{\alpha}+y^2\right)}, \qquad E_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

The lines of electric force are thus everywhere straight and at right angles to the charged line. To find A we must equate to q the surface integral of the normal electric displacement taken over unit length of a cylinder of any form, enclosing the charged line, the generating lines of the cylinder and the charged lines being parallel. The most convenient surface is that formed by the two infinite planes $x = a$ and $x = -a$ respectively.

Thus

$$q = 4\frac{K}{4\pi}\int_0^\infty \frac{2Aa\,dy}{\alpha\left(\frac{a^2}{\alpha} + y^2\right)} = \frac{KA}{\sqrt{\alpha}}$$

or

$$A = \frac{q\sqrt{\alpha}}{K}.$$

Hence

$$\frac{E_1}{x} = \frac{E_2}{y} = \frac{2q}{K\sqrt{\alpha}\left(\frac{x^2}{\alpha} + y^2\right)}, \qquad E_3 = 0 \quad . \quad . \quad . \quad . \quad (59).$$

Equations (29) to (31) give us

$$H_1 = H_2 = 0, \qquad H_3 = \frac{2quy}{\sqrt{\alpha}\left(\frac{x^2}{\alpha} + y^2\right)} \quad . \quad . \quad . \quad . \quad . \quad (60).$$

If ρ is written for $x^2 + y^2$, and θ is measured from the axis of x in the plane xy, the resultant electric and magnetic forces may be written

$$E = \frac{2q\sqrt{\alpha}}{K\rho\left(1 - \frac{u^2}{v^2}\sin^2\theta\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

$$H = \frac{2qu\sqrt{\alpha}\sin\theta}{\rho\left(1 - \frac{u^2}{v^2}\sin^2\theta\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62),$$

so that the forces vary inversely as the distance from the charged line. When $u = v$, the electric force, and also the magnetic force, is confined to the plane yz just as in the case of the point-charge.

Mechanical Force due to Electromagnetic Action.

8. The mechanical force experienced by any very small portion of the electromagnetic medium, when reckoned *per unit of volume*, has the following constituents :—

(1.) A force $\mathbf{E}\rho$, where $\mathbf{E}$ is the electric force and ρ the volume density of positive electrification.

(2.) A force $\mathbf{H}\tau$, where $\mathbf{H}$ is the magnetic force and τ the volume-density of positive imaginary magnetic matter.

These two forces follow from the ordinary laws of electrostatics and magnetism.

(3.) A force $\mathrm{V}\mathbf{CB}$ (Electromagnetic force), where $\mathbf{C}$ is the electric current density and $\mathbf{B}$ the magnetic induction. As far as I know no satisfactory proof of the formula has been given. MAXWELL obtains this formula in § 602, vol. 2, of his 'Electricity and Magnetism,' but he assumes (practically) the result he is going to obtain, for he assumes that the force "corresponding to the element ds," actually acts on ds. The formula gives absolutely correct results when applied to find the force experienced by a complete circuit, and has besides the merit of simplicity.

The expression can be deduced from MAXWELL's expression for the magnetic stresses in the field, but apart from the harmony which results when all the forces due to magnetic actions can be obtained from a single formula, no confirmation of its correctness is obtained, for the Maxwell stress was *constructed* so as to give the force $\mathrm{V}\mathbf{CB}$.

(4.) A force $-\mathrm{V}\mathbf{GD}$ (Magneto-electric force) where $\mathbf{G}$ is the rate of increase of the magnetic induction, or the "magnetic current," and $\mathbf{D}$ is the electric displacement. This force, as Mr. HEAVISIDE has remarked, can be deduced from the Maxwell electric stress provided that we assume that the stress is the same whether the electric force has a potential or not. The force has never been experimentally observed.

Mechanical Stress between Two Systems.

9. We shall now suppose the complete system to be made up of two separate systems of sources of disturbance, and will write down the force experienced by one of these systems due to the other. Since the sum of any number of solutions of the differential equations of the electromagnetic field is also a solution, it follows that if one of the systems of sources of disturbance gives rise by itself to a field characterized by $\mathbf{E}'$, $\mathbf{H}'$ and the other system gives rise by itself to the field $\mathbf{E}''$, $\mathbf{H}''$ and if $\mathbf{E}$, $\mathbf{H}$ denote the field when both systems are present, then

$$\mathbf{E} = \mathbf{E}' + \mathbf{E}'' \qquad \mathbf{H} = \mathbf{H}' + \mathbf{H}''.$$

The force experienced by any portion of the medium per unit of volume is therefore

$$(\mathbf{E}'+\mathbf{E}'')(\rho'+\rho'')+(\mathbf{H}'+\mathbf{H}'')(\tau'+\tau'')+\mathrm{V}(\mathbf{C}'+\mathbf{C}'')(\mathbf{B}'+\mathbf{B}'')-\mathrm{V}(\mathbf{G}'+\mathbf{G}'')(\mathbf{D}'+\mathbf{D}'').$$

If the force per unit volume which is due to the mutual action of the two systems be denoted by $\mathbf{P}$, then

$$\mathbf{P} = \mathbf{E}'\rho'' + \mathbf{E}''\rho' + \mathbf{H}'\tau'' + \mathbf{H}''\tau' + \mathrm{V}\mathbf{C}'\mathbf{B}'' + \mathrm{V}\mathbf{C}''\mathbf{B}' - \mathrm{V}\mathbf{G}'\mathbf{D}'' - \mathrm{V}\mathbf{G}''\mathbf{D}' \quad . \quad (63).$$

Mechanical Force Experienced by a Moving Point-Charge.

10. The first case I shall consider will be that of the motion of a point-charge, the mount of the charge being q. I shall deduce the mechanical force experienced by the charge.

Since the charge is supposed to be concentrated into an infinitely small volume, and since the values of the quantities E′, H′, . . . belonging to the system which is acting upon q, do not in general change at infinitely rapid rates from one point of space to the other, we may regard those values as constant throughout the space occupied by q. We suppose, of course, also that none of the charges, electric or magnetic, due to the influencing system are within the small volume occupied by q. Thus $\rho' = 0$ and $\tau' = 0$. Again, since by equation (45) the magnetic force **H**″, and, therefore also the magnetic induction **B**″, due to the charge q, is in circles round the axis of motion of q, it follows that the volume-integrals of B''_1, B''_2, B''_3, taken throughout any portion of space bounded by a surface of revolution having the axis of motion for its axis, are all zero. Thus, since in general, **C**′ is not infinite, the volume-integrals of the three components of $\mathbf{VC'B''}$ taken throughout the space bounded by an infinitely small surface of revolution enclosing q and having the axis of motion for its axis of figure, are all zero. If the surface of revolution is symmetrical fore and aft of the charge, then the volume-integrals of the components of $\mathbf{VG'D''}$ all vanish because **D**″ is radial. By supposition τ'' vanishes also.

Thus if **P** now stand for the force experienced by the small region (of the form just mentioned) surrounding q, we have simply

$$\mathbf{P} = \int \mathbf{E}'\rho''d\omega + \int \mathbf{VC''B'}d\omega - \int \mathbf{VG''D'}d\omega$$

where the integrations are to be understood *vectorially*, and $d\omega$ denotes an element of volume.

On account of the constancy of **E**′, **B**′, and **D**′ within the space considered, we have

$$\mathbf{P} = \mathbf{E}'\int \rho''d\omega + \mathrm{V}\left(\int \mathbf{C}''d\omega\right)\mathbf{B}' - \mathrm{V}\left(\int \mathbf{G}''d\omega\right)\mathbf{D}'. \quad . \quad . \quad . \quad . \quad (64).$$

The value of $\int \rho''d\omega$ is q.

Since **B**″ is in circles about the axis of motion **G**″ is also in similar circles. Hence $\int \mathbf{G}''d\omega$ vanishes when applied to a region bounded by a surface of revolution. Thus the last term vanishes.

In finding the value of $\int \mathbf{C}''d\omega$, the form of the bounding surface is important. If, for instance, we take a small sphere whose centre is at q, its polar axis coinciding with the axis of motion, then there is a positive x-component of the displacement current at points near the axis, but a negative x-component at points near the equatorial plane. The volume-integral is therefore less than it would be in a more suitably chosen space. If we take a very small circular cylinder, whose axis is in the axis of motion and whose length is very great compared with its radius, we shall clearly get rid of the "back" current. The volume-integral of the x-component of the current in such a cylinder can best be calculated by means of the theorem that the line-integral of the magnetic force round any circuit is 4π times the current flowing through any surface bounded by the circuit.

Let the charge q be in motion at speed u along the axis of x. Then by (45) the resultant magnetic force at the point x, ρ is

$$H'' = \frac{qu\rho}{\sqrt{\alpha}} \left\{ \frac{x^2}{\alpha} + \rho^2 \right\}^{-\frac{3}{2}}$$

where $\rho^2 = y^2 + z^2$.

The total x-current flowing across the section of the cylinder of radius ρ made by the plane $x = x$, is therefore

$$\frac{1}{4\pi} \cdot 2\pi\rho \cdot \frac{qu\rho}{\sqrt{\alpha}} \left\{ \frac{x^2}{\alpha} + \rho^2 \right\}^{-\frac{3}{2}}.$$

The volume-integral, when $2l$ is the length of the cylinder, is therefore

$$\frac{qu\rho^2}{2\sqrt{\alpha}} \int_{-l}^{+l} \frac{dx}{\left(\frac{x^2}{\alpha} + \rho^2 \right)^{\frac{3}{2}}} = qu \frac{l}{\sqrt{l^2 + \alpha\rho^2}} .$$

When ρ is infinitely small compared with l we have simply

$$\int C_1'' \, d\omega = qu,$$

as we should have expected.

The volume integrals of the other components of $\mathbf{C}''$ are clearly zero, so that

$$\int C_2'' \, d\omega = 0, \quad \int C_3'' \, d\omega = 0.$$

If, now, $\mathbf{F}$ denote the force per *unit* charge, we see from (64) that its value is*

$$\mathbf{F} = \mathbf{E} + \mathrm{V}u\mathbf{B} \quad \ldots \ldots \ldots \quad (65),$$

and its components are

* The accents, being no longer needed, have been omitted. The quantities $\mathbf{E}$ and $\mathbf{B}$ are the values which would obtain at any point if the unit charge, which has been supposed to be placed there, were removed.

$$\left.\begin{aligned} F_1 &= E_1 \\ F_2 &= E_2 - uB_3 \\ F_3 &= E_3 + uB_2 \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (66),$$

since **u** is parallel to the axis of x.

Writing the equations in terms of H instead of B we have

$$\mathbf{F} = \mathbf{E} + \mu \mathrm{V}\mathbf{u}\mathbf{H} \quad \ldots \ldots \ldots \quad (67),$$

$$F_1 = E_1, \quad F_2 = E_2 - \mu u H_3, \quad F_3 = E_3 + \mu u H_2 \quad \ldots \quad (68).$$

Inserting the values of E and H in terms of Ψ and Ω from equations (26) to (31), we find at once

$$F_1 = -\frac{d\Psi}{dx}, \quad F_2 = -\frac{d\Psi}{dy}; \quad F_3 = -\frac{d\Psi}{dz} \quad \ldots \ldots \quad (69),$$

or,

$$\mathbf{F} = -\nabla\Psi \quad \ldots \ldots \ldots \quad (70).$$

Thus it appears that though there is no proper potential from which the electric force can be derived, yet there is a potential for the mechanical force experienced by a moving charge. The electric force is really the mechanical force experienced by a unit charge *at rest*, while the force $-\nabla\Psi$ is the mechanical force experienced by a unit charge moving at the same speed as the system which gives rise to **E** and **H**.

Mechanical Force on a Moving Pole.

11. In exactly the same manner we should find that if the mechanical force experienced by a unit magnetic pole moving with the system be denoted by **R**, then,

$$\mathbf{R} = \mathbf{H} - 4\pi \mathrm{V}\mathbf{u}\mathbf{D} = \mathbf{H} - \mathrm{K}\mathrm{V}\mathbf{u}\mathbf{E} \quad \ldots \ldots \ldots \quad (71),$$

so that its components are

$$R_1 = H_1, \qquad R_2 = H_2 + KuE_3, \qquad R_3 = H_3 - KuE_2 \quad \ldots \quad (72).$$

Inserting the values of **E** and **H** in terms of Ψ and Ω from equations (26) to (31), we find

$$R_1 = -\frac{d\Omega}{dx}, \qquad R_2 = -\frac{d\Omega}{dy}, \qquad R_3 = -\frac{d\Omega}{dz} \quad \ldots \ldots \quad (73),$$

or

$$\mathbf{R} = -\nabla\Omega \quad \ldots \ldots \ldots \quad (74).$$

Thus, in this case also, although there is no true magnetic potential, still the mechanical force on a moving pole has a potential.

Mechanical Force on a Moving Electric Current.

12. If **c** denote the current, the force on it per unit length is simply

$$\mathbf{VcB} \quad \text{or} \quad \mu\mathbf{VcH} \qquad \qquad (75).$$

Values of **E** *and* **H** *in terms of* **F** *and* **R**.

13. The electric and magnetic forces **E** and **H** can now be at once expressed in terms of the mechanical forces **F** and **R** experienced by a moving unit electric charge and by a moving unit magnetic pole respectively. For, since $\mathbf{F} = -\nabla\Psi$ and $\mathbf{R} = -\nabla\Omega$, equations (26) to (31) become

$$\left.\begin{aligned} E_1 &= F_1 \\ E_2 &= \frac{1}{\alpha} F_2 + \frac{\mu u}{\alpha} R_3 \\ E_3 &= \frac{1}{\alpha} F_3 - \frac{\mu u}{\alpha} R_2 \end{aligned}\right\} \qquad \qquad (76),$$

$$\left.\begin{aligned} H_1 &= R_1 \\ H_2 &= \frac{1}{\alpha} R_2 - \frac{Ku}{\alpha} F_3 \\ H_3 &= \frac{1}{\alpha} R_3 + \frac{Ku}{\alpha} F_2 \end{aligned}\right\} \qquad \qquad (77).$$

Meaning of curl **F** = 0 *and curl* **R** = 0.

14. We now perceive the true meaning of the two equations (16) and (17), viz.,—

$$\text{curl}\,(\mathbf{E} - \mu\mathbf{VHu}) = 0, \qquad \text{curl}\,(\mathbf{H} + \mathrm{K}\mathbf{VEu}) = 0,$$

or, as we may now write them,

$$\text{curl}\,\mathbf{F} = 0, \qquad \text{curl}\,\mathbf{R} = 0.$$

They simply express the fact that the work done in taking a unit quantity of either electricity or magnetism round any closed path is zero, the path itself moving forward with the velocity **u** which is common to the whole system. This implies that **F** and

R are derivable from potential functions and that two functions, Ψ and Ω can be found such that

$$\mathbf{F} = -\nabla\Psi, \qquad \mathbf{R} = -\nabla\Omega.$$

Mr. HEAVISIDE* has shown by a method, of which mine (in § 4) is only a translation into Cartesian symbols, that the two vectors $\mathbf{E} - \mu\mathrm{V}\mathbf{Hu}$ and $\mathbf{H} + \mathrm{KV}\mathbf{Eu}$ are derivable from potential functions, and has deduced from them the values of **E** and **H** in terms of what I have denoted by Ψ and Ω. And he has shown that if we take an eolotropic medium in which the specific inductive capacities and magnetic permeabilities parallel to the three axes are

$$\mathrm{K}, \frac{\mathrm{K}}{1-\frac{u^2}{v^2}}, \frac{\mathrm{K}}{1-\frac{u^2}{v^2}} \text{ and } \mu, \frac{\mu}{1-\frac{u^2}{v^2}}, \frac{\mu}{1-\frac{u^2}{v^2}},$$

and suppose that the same functions Ψ and Ω now represent the electric and magnetic potentials respectively, then the electric displacement at any point in the electrostatic problem is in the same direction as, and $\mathrm{K}/4\pi$ times as great as, the electric force **E** at the same point in the problem of a moving charged body. Similarly the magnetic induction in the statical problem is μ times the magnetic force in the problem of a moving magnet. The analogy breaks down however when $\nabla\Psi$ and $\nabla\Omega$ exist together. It is obvious that the electric and magnetic forces in the statical problems are identical with **F** and **R** in the problem of motion, for in both cases they are the negative "slopes" of Ψ and Ω. But I believe that I have not been anticipated in giving the true explantion of the meaning of the vectors **F** and **R**.

Equilibrium Conditions.

15. I shall now consider the circumstances of a charged surface in motion, and shall begin by stating the nature of the surface upon which the charge is supposed to be deposited. The equations employed are those relating to the free ether, and would not necessarily apply to the interior of a mass of copper or other conducting substance. I do not know what happens at the surface, or at points in the interior, of a lump of copper when it is caused to move rapidly through the ether. The equations for a conductor at rest or in motion at a speed very small compared with that of light are well known, but very little is known for certain as to their form in rapidly moving masses of matter. The surface then is supposed to be formed of a thin film of some non-conducting substance whose electric and magnetic properties do not differ appreciably from those of free ether, and the charge is supposed to be deposited upon this surface.

* 'Electromagnetic Theory,' pp. 271, 276. 'Electrical Papers,' vol. 2, p. 499, and foot-note to p. 514.

In order to understand the conditions of equilibrium which apply to such a surface, it is necessary to make a careful distinction between the mechanical force experienced by any portion of the charged surface, and the tendency to convection experienced by the charge upon that portion.

Now, according to the statement of § 8, the mechanical force per unit volume, at any point where the magnetic density τ is zero, is given by

$$\mathbf{P} = \mathbf{E}\rho + \mathrm{V}\mathbf{CB} - \mathrm{V}\mathbf{GD},$$

where **C** includes both the convection and the displacement currents so that

$$\mathbf{C} = \rho\mathbf{u} + \frac{\mathrm{K}}{4\pi}\frac{d\mathbf{E}}{dt}.$$

If we take unit area of the charged surface and suppose it enclosed by a very short cylinder whose ends are parallel and infinitely close to the tangent plane to the surface, and integrate **P** throughout the cylinder, we shall obtain the force experienced by unit area of the charged surface. An equivalent method is to find the difference in the Maxwell stress in the medium on the two sides of the surface.

But when we consider the charge itself, we have to ask whether all the constituents of **P** are effective in tending to make the charge move relatively to the surface. When calculating, in § 10, the force experienced by a point-charge in motion, we were able to disregard the term $-\mathrm{V}\mathbf{GD}$ because the "magnetic current" **G** was in circles about the axis of motion, and thus the force on a unit charge was reduced to $\mathbf{E} + \mu\mathrm{V}\mathbf{uH}$. But generally at a moving charged surface there will be a discontinuity in the magnetic induction and in consequence a surface "magnetic current," and it would seem at first sight as if this ought to be taken account of. But although the electric displacement **D** acts upon the magnetic current **G**, giving rise to the mechanical force $-\mathrm{V}\mathbf{GD}$, at right angles to both **G** and **D**, still there will be no change produced in the amount or distribution of the "magnetic current." And if there is no change in the "magnetic current," there can be none in the magnetic induction whose variations constitute that magnetic current. And still less will there be any change in what causes the magnetic induction, viz., the displacement and convection currents. We need not consider here the magnetic force which may arise from magnets or electric currents flowing in conductors, and which would be represented in terms of the differential coefficients of Ω, for there will be no discontinuities in this part of the magnetic force, since we have supposed that at all points on the surface τ is zero and that there are no surface conduction currents. The action of the electric displacement upon this surface "magnetic current" will therefore avail nothing in producing convection of the charge from one part of the surface to another. The direct effect of the electric force upon the charge is taken account of in the first term of **P**, viz., the term $\mathbf{E}\rho$. Now we have already seen in § 3 that the only way in which alterations in the electrical distribution can be produced, when there is no conduction, is by convection, and hence since the term $-$ VGD can produce no changes in the electrical distribution, it must be omitted in estimating the tendency to convection. This is only what we might expect if we notice that the "magnetic current" is not a necessary accompaniment of a moving charged surface. For in the case of an infinite cylinder uniformly charged and in motion along its length there are no "magnetic currents" at all, since there is no change in the magnetic induction along any line parallel to the length of the cylinder.

An example of a somewhat similar kind occurs when an electric current flows through a conductor in a magnetic field. The magnetic field gives rise to a mechanical force which is experienced by the conductor, but there is no change produced in either the strength of the current or in its distribution, provided, at least, that the conductor is not of bismuth (when its resistance would be altered by the magnetic field) and that the "Hall effect" is disregarded.

The convection current $\rho\mathbf{u}$ is a true part of the electric current. The substance upon which the charge is deposited experiences, therefore, the force $\rho\mathrm{V}\mathbf{uB}$ per unit volume, or $\mu\mathrm{V}\mathbf{uH}$ per unit of charge. But the charge must move when the substance conveying it moves, and thus we may regard the charge as experiencing the force. Hence the term must be included in estimating the tendency to convection. In contrast to the "magnetic current," the convection current $\rho\mathbf{u}$ depends only upon ρ and $\mathbf{u}$, and is not dependent upon the manner in which ρ is distributed.

If we use $\mathbf{F}$ to denote the "tendency to convection," we have finally

$$\mathbf{F} = \mathbf{E} + \mu\mathrm{V}\mathbf{uH}.$$

But $\mathbf{E} - \mu\mathrm{V}\mathbf{Hu} = -\nabla\Psi$, so that $\mathbf{F} = -\nabla\Psi$.

Since Ψ is the potential whose "slope" is the "tendency to convection," it will be convenient to call Ψ the "electric convection potential." In the same way Ω may be called the "magnetic convection potential."

Equilibrium Surfaces.

16. Since the "tendency to convection" experienced by a unit moving charge is given by $\mathbf{F} = -\nabla\Psi$, it follows at once that $\mathbf{F}$ is everywhere perpendicular to the surface $\Psi = constant$. The surface $\Psi = constant$ may therefore be termed an equilibrium surface, for a small concentrated charge which is constrained to remain upon the surface will not tend to move about upon it. And the result of § 15 shows us that this statement remains true for each part of the charge, even when a charge is distributed over the whole of the surface.

Now consider what happens in the case of a charged surface in motion when the charge has acquired an equilibrium distribution, it being supposed that there are no charges within the surface itself. Since the charge is in equilibrium the "tendency to convection" **F** must be everywhere perpendicular to the surface. This can only be when Ψ is constant all over the surface. From this I shall now show that, when the surface is closed, Ψ is constant throughout the interior of the surface, and that consequently **F** is zero there. If $\Omega = 0$ this last result implies that both **E** and **H** vanish also, as may be seen from equations (76) and (77).

Let $\Psi = f(x, y, z)$ be the value of the "electric convection potential" at any point outside the charged surface S, and $\Psi = f'(x, y, z)$ its value at any point inside S. The surface S is by supposition an equilibrium surface, so that Ψ is constant at all points on it, and consequently $f(x, y, z) = f'(x, y, z) = c$, a constant, when x, y, z lies on S. If there are no charges in the interior of S then f' satisfies the equation

$$\alpha \frac{d^2 f'}{dx^2} + \frac{d^2 f'}{dy^2} + \frac{d^2 f'}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78).$$

Now, corresponding to the point x, y, z take in a new system of coordinates, the point ξ, η, ζ such that

$$\xi = x, \qquad \eta = y\sqrt{\alpha}, \qquad \zeta = z\sqrt{\alpha}.$$

Then, corresponding to the surface S, we shall have a new surface Σ whose equation in terms of ξ, η, ζ is

$$\phi(\xi, \eta, \zeta) \equiv f(\xi, \eta/\sqrt{\alpha}, \zeta/\sqrt{\alpha}) = c.$$

If we have also $\phi'(\xi, \eta, \zeta) \equiv f'(\xi, \eta/\sqrt{\alpha}, \zeta/\sqrt{\alpha})$, then the values of f and f' at any point x, y, z are the same as those of ϕ and ϕ' at the point ξ, η, ζ. Now, since at all internal points f' satisfies (78), it follows that at all points internal to Σ, ϕ' satisfies

$$\frac{d^2\phi'}{d\xi^2} + \frac{d^2\phi'}{d\eta^2} + \frac{d^2\phi'}{d\zeta^2} = \nabla^2\phi' = 0.$$

Moreover ϕ' is constant at all points on the surface Σ. Hence ϕ' is the value of the electrostatic potential due to a distribution of electricity at rest, such that the surface Σ is an equipotential surface, and such that there are no charges within it. But in this case we know that ϕ' is constant at all internal points. It follows, therefore, that f' is constant at all points internal to the surface S. Hence, when a charged surface is in motion, and the charge has acquired an equilibrium distribution, the "convection potential" is constant throughout the interior of the surface. If there are no sources of magnetic disturbance in the field, so that $\Omega = 0$, the constancy of Ψ implies that both the electric and the magnetic forces vanish at all points internal to the charged surface. Thus if the only source of disturbance is the charged surface itself, the electric and magnetic forces due to it are entirely on the outside of the surface. There is no disturbance within it. The same is true when there are other electrical disturbances outside the surface, and the surface is still an equilibrium surface for the whole system.

But if there are sources of magnetic disturbance in the neighbourhood of the surface, and Ψ is still constant over the surface, it will also be constant throughout the interior of the surface. Yet the electric and magnetic forces do not now vanish at internal points, for parts of these forces are derived from the "magnetic convection potential" Ω. Since Ψ is constant inside the surface, it follows from equations (26) to (31) that the field there is given by

$$E_1 = 0. \qquad H_1 = -\frac{d\Omega}{dx}.$$

$$E_2 = -\frac{\mu u}{\alpha}\frac{d\Omega}{dz}. \qquad H_2 = -\frac{1}{\alpha}\frac{d\Omega}{dy}.$$

$$E_3 = \frac{\mu u}{\alpha}\frac{d\Omega}{dy}. \qquad H_3 = -\frac{1}{\alpha}\frac{d\Omega}{dz}.$$

But though there is now both electric and magnetic force inside the surface there is no mechanical force on a small moving charge since **F** is zero because Ψ is constant. Outside the surface the field is the resultant of the fields due, the one to Ω and the other to Ψ.

We have already seen that when $\Omega = 0$ there is neither electric nor magnetic force inside an equilibrium surface. The lines of magnetic force just outside the surface must be tangential to it since there is no magnetic force inside the surface and no distribution of magnetism upon it. The lines of magnetic force are also in planes perpendicular to the axis of x since $H_1 = 0$ when $\Omega = 0$. Hence the lines of magnetic force on the surface itself are the lines in which the surface is cut by planes perpendicular to the axis of x.

It is easy to show by analysis that throughout the field, as long as $\Omega = 0$, the lines of magnetic force are given by the sections of the surfaces $\Psi = constant$ by the planes $x = constant$.

For by (29), (30), and (31), when $\Omega = 0$,

$$H_1 = 0 \quad H_2 = \frac{Ku}{\alpha}\frac{d\Psi}{dz} \quad H_3 = -\frac{Ku}{\alpha}\frac{d\Psi}{dy}.$$

Hence, if dy/dz refer to one of the lines of force,

$$\frac{dy}{dz} = \frac{H_2}{H_3} = -\frac{d\Psi}{dz}\Big/\frac{d\Psi}{dy}.$$

But at all points of the section of the surface $\Psi = c$ made by the plane $x = c'$, we have y and z connected by the relation $\Psi = c$, while, of course, x is constant.

Hence,

$$\frac{d\Psi}{dz} + \frac{d\Psi}{dy}\frac{dy}{dz} = 0,$$

or

$$\frac{dy}{dz} = -\frac{d\Psi}{dz}\bigg/\frac{d\Psi}{dy}.$$

Thus the lines of magnetic force are given by the lines defined by $\Psi = c$, $x = c'$, where c and c' are variable parameters.

Electrical Distribution on an Equilibrium Surface.

17. The surface density at any point of a charged surface on which the electricity is in equilibrium is found from the fact that the surface-integral of the normal electric displacement taken over any closed surface is equal to the quantity of electricity within that surface. Hence, if E_n denote the electric force normal to the surface, and σ the surface density, we have, just as in electrostatics, when there are no charges inside the surface,

$$\frac{1}{4\pi} KE_n = \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79),$$

since by § 16 the electric force vanishes inside the surface.

The above statement refers to the case in which $\Omega = 0$.

When Ω does not vanish, we have instead,

$$\frac{1}{4\pi} K \{E_n \text{ (outside)} - E_n \text{ (inside)}\} = \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (80),$$

where the electric forces are both reckoned in the same direction, *i.e.*, along the outward drawn normal.

Since no sources of magnetic disturbance reside on the surface, the part of **E** which is derived from Ω is unchanged in passing through the surface, and the difference between the normal electric forces inside and outside may be computed from Ψ alone. Since Ψ is constant throughout the interior of the surface, the part of **E** due to it vanishes inside the surface.

Hence, if l, m, n be the direction cosines of the outward normal to the surface, we have by (26) to (28),

$$\sigma = -\frac{K}{4\pi}\left\{l\frac{d\Psi}{dx} + \frac{m}{\alpha}\frac{d\Psi}{dy} + \frac{n}{\alpha}\frac{d\Psi}{dz}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (81),$$

whether $\nabla\Omega$ vanish or not.

Since $\mathbf{F} = -\nabla\Psi$ and is normal to the surface we have

$$\sigma = \frac{K}{4\pi}\left(l^2 + \frac{m^2}{\alpha} + \frac{n^2}{\alpha}\right)F = \frac{K}{4\pi\alpha}\left(1 - \frac{u^2}{v^2}l^2\right)F \quad . \quad . \quad . \quad . \quad (82).$$

But by (76) when $\nabla\Omega$ vanishes,

$$E_1 = F_1, \quad E_2 = \frac{1}{\alpha}F_2, \quad E_3 = \frac{1}{\alpha}F_3$$

so that

$$E = \frac{1}{\alpha}\sqrt{1 - l^2(1-\alpha^2)}\, F \quad \ldots \ldots \ldots \quad (83).$$

Hence

$$E = \frac{4\pi\sigma}{K}\frac{\sqrt{1 - l^2(1-\alpha^2)}}{1 - \dfrac{u^2}{v^2}l^2} \quad \ldots \ldots \ldots \quad (84).$$

[October 23, 1896.

The direction cosines of **E** are

$$\frac{\alpha l}{\sqrt{1 - l^2(1-\alpha^2)}}, \quad \frac{m}{\sqrt{1 - l^2(1-\alpha^2)}}, \quad \frac{n}{\sqrt{1 - l^2(1-\alpha^2)}}.$$

We have now obtained the value of **E** as far as it depends upon the moving electric charges. If $\nabla\Omega$ does not vanish we have simply to add on the electric force whose components are

$$E_1 = 0, \quad E_2 = -\frac{\mu u}{\alpha}\frac{d\Omega}{dz}, \quad E_3 = \frac{\mu u}{\alpha}\frac{d\Omega}{dy}.\Big]$$

The magnetic force near a moving charged surface is compounded of two parts, one due to Ψ, the other due to Ω, and these are quite independent. I shall now calculate the value of **H** when $\nabla\Omega$ is zero. If in any case $\nabla\Omega$ is not zero, we have simply to add on the part of **H** which is due to Ω.

Now when $\nabla\Omega$ vanishes (and therefore also **R**) equations (77) give

$$H_1 = 0, \quad H_2 = -\frac{Ku}{\alpha}F_3, \quad H_3 = \frac{Ku}{\alpha}F_2$$

so that

$$H = \frac{Ku}{\alpha}\sqrt{F_2^2 + F_3^2} = \frac{Ku}{\alpha}F\sqrt{1 - l^2} \quad \ldots \ldots \quad (85).$$

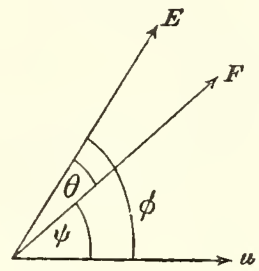
But making use of (83) and (84) we have

$$H = Ku\frac{\sqrt{1-l^2}}{\sqrt{1 - l^2(1-\alpha^2)}}E = 4\pi u\sigma\frac{\sqrt{1-l^2}}{1 - \dfrac{u^2}{v^2}l^2} \quad \ldots \ldots \quad (86).$$

Now when $\mathbf{R} = 0$ we have by (39) that **H** is perpendicular to both **u** and **F**. Again, by (36) **H** is perpendicular to **E** and **u**. Hence **E**, **F**, and **u** are co-planar, and since **F** has already been proved to be normal to the surface, **E** lies in the plane containing the direction of motion and the normal to the surface.

Let the angle between **E** and **F** be θ, that between **F** and **u** ψ, and that between **E** and **u** ϕ. Then the following relations will be found useful :—

Fig. 1.



By § 5

$$\mathbf{SuE} = \mathbf{SuF}$$

so that

$$\mathrm{E} \cos \phi = \mathrm{F} \cos \psi.$$

Again by (38) after multiplying vectorially by **u**

$$\alpha \mathbf{VEu} = \mathbf{VFu}$$

so that

$$\mathrm{E} \sin \phi = \frac{1}{\alpha} \mathrm{F} \sin \psi,$$

whence

$$\cos \phi = \frac{\alpha \cos \psi}{\sqrt{\alpha^2 \cos^2 \psi + \sin^2 \psi}}, \qquad \sin \phi = \frac{\sin \psi}{\sqrt{\alpha^2 \cos^2 \psi + \sin^2 \psi}}.$$

But by (36) since $\mathbf{R} = 0$

$$\mathbf{H} = \mathrm{K}\mathbf{VuE}$$

so that

$$\mathrm{H} = \mathrm{K}u\mathrm{E} \sin \phi = \frac{\mathrm{K}u}{\alpha} \mathrm{F} \sin \psi.$$

Lastly by (38) after multiplying vectorially by **E**

$$\mathbf{VEF} - \frac{1}{v^2} \mathbf{VEu} \,.\, \mathbf{SuF} = 0$$

so that

$$\sin \theta = \frac{u^2}{v^2} \sin \phi \cos \psi.$$

Since $\theta = \phi - \psi$ this is the same as

$$\alpha \sin \theta = \frac{u^2}{v^2} \cos \phi \sin \psi.$$

Also

$$\cos \theta = \cos (\phi - \psi) = \frac{\alpha \cos^2 \psi + \sin^2 \psi}{\sqrt{\alpha^2 \cos^2 \psi + \sin^2 \psi}}.$$

The expressions for **E** and **H** hold good for a point just outside any equilibrium surface. But they plainly hold good for *any* point between a pair of parallel plates bearing complementary charges.

Mechanical Force on a Charged Surface.

18. The mechanical force experienced by any portion of a charged surface may be found by considering the difference of the Maxwell stress on the two sides of the surface. If the surface is an equilibrium surface, and if $\nabla\Omega$ is zero, there is neither electric nor magnetic force inside the surface, and consequently the Maxwell stress on the inner side vanishes. Let **E** and **H** be the electric and magnetic forces at a point just outside the surface. Then the Maxwell stress gives a normal outward force

$$\frac{\mathrm{KE}^2}{8\pi}(\cos^2\theta - \sin^2\theta) - \frac{\mu\mathrm{H}^2}{8\pi}$$

per unit area of the surface. Note that **H** lies in the tangent plane.

There is also a tangential force in the plane containing **E**, **u**, and the normal, and it acts towards the **E** side of **F**; its amount per unit of area is

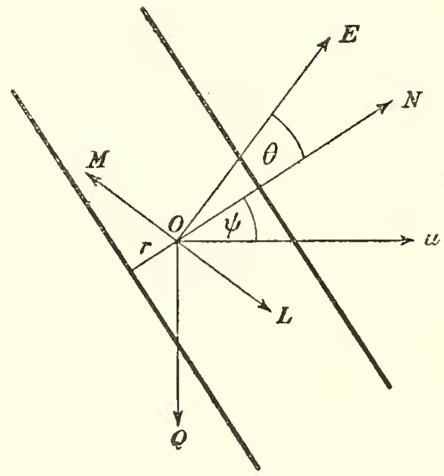
$$\frac{\mathrm{KE}^2}{4\pi}\cos\theta\sin\theta.$$

The force experienced by the medium per unit volume is, by § 8, or by § 15,

$$\mathbf{P} = \mathbf{E}\rho + \rho\mathrm{V}\mathbf{u}\mathbf{B} + \frac{\mathrm{K}}{4\pi}\mathrm{V}\frac{d\mathbf{E}}{dt}\mathbf{B} - \mathrm{V}\mathbf{G}\mathbf{D}.$$

The application of this formula to calculate the force experienced by the charged surface affords a good example of electromagnetic principles. We shall suppose that the electricity is uniformly distributed through a layer of small but finite thickness, a,

Fig. 2.



the volume-density being ρ, so that $\rho a = \sigma$. Now, if there is no disturbance on the side of the layer away from which it is moving, it follows that if $\mathbf{E}_0$ and $\mathbf{H}_0$ are the electric and magnetic forces at a point on the front of the layer, then the forces at any point O whose distance from the back of the layer is r, are

$$\mathbf{E} = \frac{r}{a}\mathbf{E}_0 \qquad \mathbf{H} = \frac{r}{a}\mathbf{H}_0.$$

At the fixed point through which O is passing, the forces are increasing at the rates

$$\frac{\mathbf{E}}{dt} = -\frac{u}{a}\cos\psi\mathbf{E}_0 \qquad \frac{d\mathbf{H}}{dt} = -\frac{u}{a}\cos\psi\mathbf{H}_0.$$

The direction of $d\mathbf{E}/dt$ is along OE, and that of $d\mathbf{H}/dt$ is along $\mathbf{H}$, *i.e.*, perpendicular to the plane of the paper and towards the reader. Thus $\mathbf{E}\rho$ acts along OE, and has the value $\rho E_0 r/a$; $\rho\mathrm{V}\mathbf{uB}$ acts along OQ in the plane of the paper, and has the value $\rho u\mu H_0 r/a$; $\frac{K}{4\pi}\mathrm{V}\frac{d\mathbf{E}}{dt}\mathbf{B}$ acts along OL at right angles to OE, and has the value $-K\mu u\cos\psi E_0 H_0 r/4\pi a^2$. The direction of $\mathfrak{G} \equiv \mu d\mathbf{H}/dt$ is outwards from the paper, so that $\mathrm{V}\mathfrak{G}\mathbf{D}$ acts along OM at right angles to OE. The value of $\mathrm{V}\mathfrak{G}\mathbf{D}$ is $-K\mu u\cos\psi E_0 H_0 r/4\pi a^2$.

Integrating these forces with regard to r, and remembering that $\rho a = \sigma$, we find for the normal pull

$$N = \tfrac{1}{2}E_0\sigma\cos\theta - \tfrac{1}{2}\sigma\mu u H_0\sin\psi - K\mu u E_0 H_0\sin\theta\cos\psi/4\pi.$$

Making use of $\sigma = KE_0\cos\theta/4\pi$ and employing the relations given in § 17 between E, H, θ, ϕ, and ψ, and noting that $\psi + \theta = \phi$ the expression easily reduces to

$$N = \frac{KE_0^2}{8\pi}(\cos^2\theta - \sin^2\theta) - \frac{\mu H_0^2}{8\pi}.$$

This is identical with the result obtained from the Maxwell stress.

In finding the tangential stress, we know that $\mathbf{E} + \mathrm{V}\mathbf{uB}$ is normal to the surface, so that the first two terms may be disregarded. For the tangential stress we thus obtain

$$T = K\mu u\cos\psi E_0 H_0\cos\theta/4\pi = Ku^2E_0^2\cos\psi\cos\theta\sin\phi/4\pi v^2$$
$$= \frac{KE_0^2}{4\pi}\cos\theta\sin\theta.$$

This acts on the $\mathbf{E}$ side of $\mathbf{F}$, and is therefore identical in direction and magnitude with the force derived from the Maxwell stress.

Normal Pull.—If we express H in terms of E and ψ, and θ in terms of ψ, and write β for u^2/v^2, we shall find that

$$N = \text{normal pull} = \frac{KE_0^2}{8\pi}\,\frac{2\beta^2\cos^4\psi - \cos^2\psi(\beta + \beta^2) + 1 - \beta}{1 - (2\beta - \beta^2)\cos^2\psi},$$

or in terms of σ

$$N = \frac{2\pi\sigma^2}{K}\,\frac{2\beta^2\cos^4\psi - \cos^2\psi(\beta + \beta^2) + 1 - \beta}{(1 - \beta\cos^2\psi)^2}.$$

When $\psi = 0$ so that the normal is along the direction of motion, the normal pull is $\mathrm{KE}_0^2/8\pi$, or $2\pi\sigma^2/\mathrm{K}$, and is independent of β.

When $\psi = \pi/2$, so that the tangent plane is parallel to the direction of motion, the normal pull is $\mathrm{KE}_0^2\,(1 - u^2/v^2)/8\pi$, vanishing when the speed of light is reached, *i.e.*, when $u = v$.

Now for all real values of β the denominator is positive. Thus, if β is large enough, the normal pull, N, may be *negative* over a certain range of values of ψ. For a given value of β, as ψ increases from 0 to $\pi/2$, N changes from positive to negative and from negative to positive again as ψ passes through the values given by

$$2\beta^2 \cos^4\psi - \cos^2\psi\,(\beta + \beta^2) + 1 - \beta = 0,$$

or

$$\cos^2\psi = \frac{1}{4\beta}\{1 + \beta \pm \sqrt{\beta^2 + 10\beta - 7}\}.$$

The value of ψ given by this is not real till

$$\beta^2 + 10\beta - 7 = 0,$$

i.e., till $\beta = -5 + \sqrt{32} = \cdot 6568542$ (β must be positive).

The value of ψ corresponding to this value of β is $37^\circ\ 25'\ 45''\cdot 4$. Thus, if an electrified sphere is in motion along its polar axis, the normal pull is positive all over it till $\beta = \cdot 6568542$, or $u/v = \cdot 810465$. At this speed the pull vanishes at the points whose co-latitude is $37^\circ\ 25'\ 45''\cdot 4$. As the speed increases, there are two lines of latitude along which the pull vanishes, and between which the pull is *negative*. If ψ_1 denote the value ψ where the pull changes from positive to negative, and ψ_2 the value where it changes from negative to positive, then the values of ψ_1 and ψ_2 are given in the following table :—

β.	u/v.	ψ_1.	ψ_2.
·6568542	·810465	37° 25′ 45″·4	37° 25′ 45″·4
·7	·8367	22 13	53 18
·75	·8660	15 41	60 42
·8	·8944	11 6	66 15
·85	·9220	7 42	71 2
·9	·9487	4 42	75 33
·95	·9747	2 8	80 25
1·00	1·00	0 0	90 0

When $u = v$, we have already seen that N vanishes when $\psi = \pi/2$, so that there is then no real change from a negative value to a positive one. Now when $\psi = 0$, N is always positive whatever the value of u/v; thus when $u = v$, N changes from positive to negative for an infinitely small increase in ψ.

When $u = v$ we have

$$N = -\frac{KE_0^2}{4\pi}\cos^2\psi,$$

or, in terms of σ, since by § 17 $\cos\theta = \sin\psi$ when $u = v$,

$$N = -\frac{4\pi\sigma^2}{K}\cot^2\psi.$$

Thus apparently when $\psi = 0$, $N = -\infty$. On the other hand, when ψ was put zero *before* u was made equal to v, we found $N = 2\pi\sigma^2/K$. The reason of the discrepancy appears to be as follows :—If the surface is one of two parallel planes of *absolutely infinite* extent, and the motion is along the normal, the only possible direction of $\mathbf{E}$ is also along the normal. But if the surfaces are not infinite, *e.g.*, a pair of circular parallel plates, at all ordinary points there is a definite direction, at right angles to the motion, along which the electric force must lie. And if the charge is supposed confined to an infinitely thin layer there will consequently be a finite amount of displacement through an infinitely small area, thus producing infinite electric force. When, as in the case of a moving ellipsoid, we are able to take a proper account of the distribution the discrepancy disappears.

Tangential Pull.—The tangential force per unit area is

$$T = \frac{KE^2}{4\pi}\cos\theta\sin\theta = \frac{KE^2}{4\pi}\frac{u^2}{v^2}\sin\psi\cos\psi\,\frac{\alpha\cos^2\psi + \sin^2\psi}{\alpha^2\cos^2\psi + \sin^2\psi},$$

or, in terms of σ,

$$T = \frac{4\pi\sigma^2}{K}\frac{u^2}{v^2}\frac{\sin\psi\cos\psi}{\alpha\cos^2\psi + \sin^2\psi}.$$

When $\psi = 0$, or when $\psi = \frac{\pi}{2}$, $T = 0$.

When $u = v$, $T = \frac{4\pi\sigma^2}{K}\cot\psi$.

There is thus a discrepancy when $\psi = 0$ and $u = v$. The explanation is the same as for the normal pull.

Stress between a Pair of Moving Charges.

19. The theory of the mechanical force experienced by a moving charged particle can be readily applied to calculate the stress between two charges which are both moving parallel to x with velocity u. Let there be a charge q' at the origin and a charge q at the point x, y, z. Then by (43) the value of the "convection potential" due to q' is

$$\Psi = \frac{q'\sqrt{\alpha}}{K}\left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{1}{2}}.$$

Hence, if **P** denote the mechanical force on q so that $\mathbf{P} = q\mathbf{F}$, we have by (69)

$$\left.\begin{aligned} P_1 &= \frac{qq'}{K\sqrt{\alpha}}\, x \left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{3}{2}} \\ P_2 &= \frac{qq'\sqrt{\alpha}}{K}\, y \left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{3}{2}} \\ P_3 &= \frac{qq'\sqrt{\alpha}}{K}\, z \left\{\frac{x^2}{\alpha} + y^2 + z^2\right\}^{-\frac{3}{2}} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (87).$$

This set of forces is equivalent to a repulsion

$$\frac{qq'\left(1 - \dfrac{u^2}{v^2}\right)}{Kr^2\left(1 - \dfrac{u^2}{v^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

together with a force perpendicular to the axis of x, and towards it, of amount

$$\frac{\mu qq'u^2\left(1 - \dfrac{u^2}{v^2}\right)\sin\theta}{r^2\left(1 - \dfrac{u^2}{v^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

where r is the radius from q' to q, and θ the angle it makes with the direction of motion, and $\kappa\mu$ has been put for $1/v^2$. Taking the two charges as a complete system, the last force gives rise to a couple

$$\frac{\mu qq'u^2\sin\theta\cos\theta\left(1 - \dfrac{u^2}{v^2}\right)}{r\left(1 - \dfrac{u^2}{v^2}\sin^2\theta\right)^{\frac{3}{2}}}$$

tending to make r coincide with x.

The resultant force is perpendicular to the surface $\Psi =$ constant, which passes through the point x, y, z. It is, therefore, normal to the ellipsoid $x^2/\alpha + y^2 + z^2 = c^2$, where c is a proper parameter.

When the charges move at the speed of light, the disturbance due to q' is entirely confined to the plane of yz, and the stress vanishes unless the charge q lies in this plane.

In terms of r and θ the component of **P** perpendicular to x is, in any case

$$\frac{qq'\sin\theta\left(1 - \dfrac{u^2}{v^2}\right)^2}{Kr^2\left\{1 - \dfrac{u^2}{v^2}\sin^2\theta\right\}^{\frac{3}{2}}},$$

which vanishes when $u = v$, even when $\sin\theta = 1$. There is, therefore, no stress at all between a pair of charges moving parallel to each other at the speed of light.

Motion of a Charge in a Uniform Magnetic Field.

20. It has often been thought that some of the peculiar effects produced by a magnet upon vacuum tube discharges are to be explained by supposing that the discharge consists of charged particles flung off with considerable velocities from the negative electrode, and that each charged particle in motion is acted on mechanically by the magnetic field. It will, therefore, be of some interest to write down the forces experienced by a charged particle when moving through a uniform magnetic field.

The system which produces the field may be in motion, but it is supposed to be a purely magnetic system, *i.e.*, one in which $\mathbf{F} = 0$, so that a charged particle moving with the system experiences no force. The velocity of this system will be supposed to be u parallel to the axis of x. The moving charge q is supposed to have a velocity $\mathbf{w}$, whose components are w_1, w_2, w_3. Then, if $\mathbf{P}$ denote the mechanical force on q, we have, by (65),

$$\mathbf{P} = q\,(\mathbf{E} + \mu \mathrm{V}\mathbf{wH}) \quad \ldots \ldots \ldots \quad (88).$$

But the state of the field must be determined experimentally by estimating the force on a unit magnetic pole, which we shall suppose is moving with the magnetic system. This is exactly what we find when we make experiments to find the intensity of any magnetic field by means of a magnet, for this field and the magnet are both carried along by the rapid motion of the Earth. It is, in fact, $\mathbf{R}$ which we measure, and not $\mathbf{H}$. We must, therefore, determine $\mathbf{E}$ and $\mathbf{H}$ in terms of $\mathbf{R}$, the only quantity which we can observe. This has already been done in equations (76) and (77), where we have now to put $\mathbf{F} = 0$, so that,

$$\mathrm{E}_1 = 0, \qquad \mathrm{E}_2 = \frac{\mu u}{\alpha}\mathrm{R}_3, \qquad \mathrm{E}_3 = -\frac{\mu u}{\alpha}\mathrm{R}_2,$$

$$\mathrm{H}_1 = \mathrm{R}_1, \qquad \mathrm{H}_2 = \frac{1}{\alpha}\mathrm{R}_2, \qquad \mathrm{H}_3 = \frac{1}{\alpha}\mathrm{R}_3.$$

Expanding $\mathbf{P}$ into its three components and substituting the above values of $\mathbf{E}$ and $\mathbf{H}$, we find

$$\left.\begin{aligned} \mathrm{P}_1 &= \mu q\left\{\frac{w_2}{\alpha}\mathrm{R}_3 - \frac{w_3}{\alpha}\mathrm{R}_2\right\} \\ \mathrm{P}_2 &= \mu q\left\{w_3\mathrm{R}_1 + \frac{u - w_1}{\alpha}\mathrm{R}_3\right\} \\ \mathrm{P}_3 &= \mu q\left\{-\frac{u - w_1}{\alpha}\mathrm{R}_2 - w_2\mathrm{R}_1\right\} \end{aligned}\right\} \quad \ldots \ldots \quad (89).$$

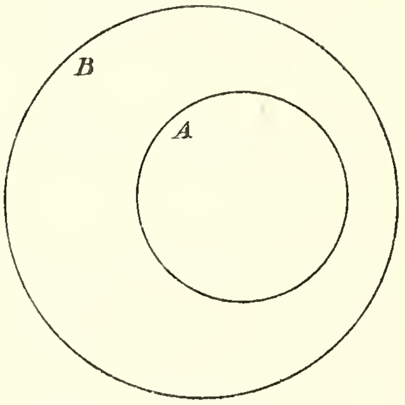
When the charge and the magnetic system are moving together so that $w_1 = u$, $w_2 = w_3 = 0$, then P vanishes. There is thus no force on a charged body when it is placed near a magnet, and both are carried through the ether by the motion of the earth.

Equivalent Distributions.

21. The following simple proposition, now to be proved, I have found of great service when investigating the properties of a moving charged ellipsoid.

Take any electrical system in motion and draw the series of equilibrium surfaces corresponding to successive values of Ψ. Let Ψ_1, Ψ_2 be the values of the "convection potential" corresponding to two of these surfaces, and suppose that the surface Ψ_1 lies within Ψ_2. Then if the same charge q be given to either of these two surfaces and be allowed to acquire its equilibrium distribution, then at all points not within the surface Ψ_2 the effects of the two charged surfaces are identical.

Fig. 3.



Let A be any electrified surface in motion having a charge q with an equilibrium distribution, and suppose for the moment that this distribution is rigidly fixed. Let Ψ_A be the "convection potential" due to A at any point. Let B be any one of the equilibrium surfaces surrounding A. Now suppose that such a distribution is imparted to B, that at all points outside B there is no disturbance due to the pair of charged surfaces A and B. The electric force due to A and B therefore vanishes, and hence so also does the surface integral of normal electric displacement when taken over any surface enclosing both A and B. Any electric force due to Ω contributes nothing to this integral, since, as is seen from equations (26) to (28), it satisfies $\text{div}\,\mathbf{E} = 0$ identically. The charge on B is therefore equal in amount and opposite in sign to that upon A, *i.e.*, B has a charge $-q$. Let Ψ_B denote the convection potential due to this distribution on B.

Now since there is no disturbance due to A and B outside B, it follows that Ψ has a constant value at all points outside B, and that, since Ψ vanishes at infinity, this constant value is zero. Now $\Psi = \Psi_A + \Psi_B$. But outside B, $\Psi = 0$. Hence outside B and at all points on B $\Psi_B = -\Psi_A$. Now B was taken to be an equilibrium surface for A, so that Ψ_A is constant all over it. Hence Ψ_B is also constant all over B, and therefore the distribution on B is the same as if B had been "freely" charged, except, of course, that the charge is now on the inner side of the surface B, whereas if B were freely charged it would be on the outer side of the surface. Since B has an equilibrium distribution, Ψ_B is constant throughout the interior of B, and hence the field between A and B and inside A is the same as if A alone were present. The restriction that the charge on A should be rigidly fixed may therefore be removed. There is no disburbance inside A since there both Ψ_A and Ψ_B are constant.

We now see at once that if the distribution on B be changed in sign and that on A be removed, then at all points outside B the field is exactly the same as that due to A. We have now only to substitute another equilibrium surface C for B in order to complete the proof of the proposition.

The electric force just outside B is, of course, the same whether it is produced either by the charge on B or by that on A. Thus, if E_n be the normal component reckoned outwards of that part of the electric force which is not derived from Ω, then

$$\sigma = \frac{K}{4\pi} E_n.$$

Energy of a system of Moving Charges.

22. If it be allowed that there is energy stored in the ether when it sustains electric and magnetic stresses, and that the amount of energy per unit volume does not depend upon the manner in which those stresses are produced, but only upon the values of the stresses themselves, then, as is well known, it follows that if U be the total energy due to electric stress, and T the total energy due to magnetic stress, the values of U and T are

$$U = \frac{1}{8\pi} \iiint KE^2 \, dx \, dy \, dz \quad \ldots \ldots \ldots \quad (90),$$

$$T = \frac{1}{8\pi} \iiint \mu H^2 \, dx \, dy \, dz \quad \ldots \ldots \ldots \quad (91),$$

the integration extending through all space, or, what is equivalent, throughout the whole of those regions where E and H do not vanish.

If W be the total energy of the system, then

$$W = U + T \quad \ldots \ldots \ldots \ldots \quad (92).$$

When $\Omega = 0$, and the electricity is distributed over surfaces which form the boundaries of regions of no disturbance, the expression for the energy admits of an important transformation. I have not succeeded in effecting any simplification in the case in which both Ω and Ψ exist.

If we take the values of E and H given by (26) to (31) in terms of Ψ when $\Omega = 0$, and remember that $K\mu v^2 = 1$, we find that

$$W = \frac{1}{8\pi}\iiint\left\{K\left(\frac{d\Psi}{dx}\right)^2 + \frac{K + K^2\mu u^2}{\alpha^2}\left(\frac{d\Psi}{dy}\right)^2 + \frac{K + K^2\mu u^2}{\alpha^2}\left(\frac{d\Psi}{dz}\right)^2\right\} dx\,dy\,dz$$

$$= \frac{K}{8\pi}\iiint\left\{\left(\frac{d\Psi}{dx}\right)^2 + \frac{1}{\alpha}\left(\frac{d\Psi}{dy}\right)^2 + \frac{1}{\alpha}\left(\frac{d\Psi}{dz}\right)^2\right\} dx\,dy\,dz$$

$$+ \frac{2K^2\mu u^2}{8\pi\alpha^2}\iiint\left\{\left(\frac{d\Psi}{dy}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2\right\} dx\,dy\,dz.$$

The second integral is by (30) and (31) simply

$$2\,\frac{1}{8\pi}\iiint \mu H^2\,dx\,dy\,dz = 2T.$$

The system will be supposed to consist of two surfaces bearing complementary charges so distributed that it is only in the space between the two surfaces that E and H do not vanish. If the "Equilibrium Conditions" of § 15 are correct, these distributions are also equilibrium distributions. If we integrate the first integral term by term "by parts" and remember that Ψ satisfies the differential equation

$$\alpha\frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2} = 0$$

at all points between the surfaces, we find

$$W = 2T - \frac{K}{8\pi}\int\Psi\left\{l\frac{d\Psi}{dx} + \frac{m}{\alpha}\frac{d\Psi}{dy} + \frac{n}{\alpha}\frac{d\Psi}{dz}\right\} dS,$$

where dS is an element of one of the surfaces, and l, m, n are the direction cosines of the outward normal to dS, the integration extending over both surfaces. Over the whole of each surface Ψ is constant, because the surface is the boundary of a region of zero disturbance. Also if σ be the surface density, we have by (81),

$$\sigma = \frac{-K}{4\pi}\left\{l\frac{d\Psi}{dx} + \frac{m}{\alpha}\frac{d\Psi}{dy} + \frac{n}{\alpha}\frac{d\Psi}{dz}\right\},$$

so that if q be the charge upon the surface Ψ_1 and $-q$ that on the surface Ψ_2, we have

$$W = 2T + \tfrac{1}{2}q\Psi_1 - \tfrac{1}{2}q\Psi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93).$$

The quantity $\frac{1}{2}q\Psi_1 - \frac{1}{2}q\Psi_2$ is evidently the mechanical work which must be spent in bringing the system together from a state of diffusion, in which, however, each part is still *moving with velocity u parallel to x*. It might, perhaps, have been thought that this amount of work would be equal to the sum of the electric and magnetic parts of the energy of the system, *i.e.*, to U + T. But instead, we have

$$\tfrac{1}{2}q\Psi_1 - \tfrac{1}{2}q\Psi_2 = \mathrm{U} - \mathrm{T}.$$

The discrepancy arises from the fact that there must be an expenditure of direct electric and magnetic energy while the system is being collected, in order to maintain in its proper strength the system of displacement and magnetic currents which accompany each moving elementary charge.

This transformation enables us to determine the energy of any system for which Ψ is known, with only one troublesome integration, viz., the space summation of $\mu\mathrm{H}^2/4\pi$.

If the surface corresponding to the suffix 2 is at an infinite distance from the surface corresponding to the suffix 1, it will generally happen that $\Psi_2 = 0$, and then we have for the energy

$$\mathrm{W} = 2\mathrm{T} + \tfrac{1}{2}q\Psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

where Ψ now refers to the finite surface.

In conclusion, I have much pleasure in expressing my best thanks to my friend, Mr. Oliver Heaviside, F.R.S. Besides giving me some personal instruction in Electromagnetic Theory on several occasions, he has constantly encouraged me during the progress of this investigation.

INDEX SLIP.

XVIII. *The Hysteresis of Iron and Steel in a Rotating Magnetic Field.*

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Communicated by Professor O. Lodge, *F.R.S.*

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In a paper on Dynamo Electric Machinery by Dr. John Hopkinson ('Phil. Trans.,' 1896), the suggestion is made that the value of the hysteresis of the iron core of a rotating dynamo armature need not be identical with the value obtained when the magnetising force is reversed by passing through a zero value. It was subsequently pointed out by Mr. Swinburne that as a necessary deduction from Professor Ewing's molecular theory of magnetism, the hysteresis of iron in a rotating field, or of iron rotating in a constant field, should show a distinct diminution in value below that in an alternating field, when the magnetic condition of the iron approaches saturation.

According to Ewing's theory hysteresis is due to the formation of stable magnetic combinations between adjacent molecules which tend to resist any movement of the molecular magnets caused by change of direction or magnitude in the magnetising force. On the breaking up of these combinations by such a change in the magnetising force, new arrangements are formed, and the potential energy of position is transformed into kinetic energy of partial rotational movement round the fixed axis of the molecule, which is damped out with or without oscillations above the axis of rotation. It has been further suggested that the damping process may be due to eddy currents induced by the movement of the magnets, but the precise nature of these eddy currents, or the extent to which other retarding influences akin to mechanical friction or viscosity may act has not been determined.

In the alternating field there is no continuity of position of the molecules; the magnetising force passes through a zero value at each alternation, and the molecular combinations may vary in successive alternations, although the average value of the induction and hysteresis remains sensibly constant. Increasing magnetising force and induction result in increased movement of the molecules with increased momentum and consequent dissipation of energy. But in a rotating field of constant value there is no diminution of the strength of the magnetising force. The molecules are always under the same restraint, and the movements impressed upon them will be more uniform and unidirectional. It is true that with small magnetising forces, when but few combinations are dissolved and the movement is of a quasi-elastic nature, the amount of energy dissipated will not be greatly different from that dissipated in an alternating field of equal strength. In fact, since the molecular changes will be forced to take place in one direction only, there will be on the average a greater resistance to movement, and in consequence an increased dissipation of energy. When, however, the field becomes stronger, each molecule will develop a tendency to rotate in synchronism with the field and be less affected by the magnetic influences of surrounding molecules. At this point the hysteresis will show a considerable diminution which will become more marked as the field increases in strength, until finally every molecule will rotate in unison with the field with complete absence of oscillatory movement. The value of the hysteresis under these last conditions will furnish an important clue to the nature of hysteresis. If the hysteresis sensibly vanishes at this point it will be strong evidence that the damping of the movements of the molecules is not due even partially to mechanical friction, but must be produced by some action which is called into play by the rapid oscillations of the molecular magnets, but not by the comparatively slow motion of their rotation with the field. Since the difference between these speeds must be enormous, the damping may be due either to some form of eddy currents or possibly to some form of fluid friction.

The first experimental work upon hysteresis in a rotating field was carried out by Professor FERRARIS ('Atti d. R. Acc. di Torino,' No. 23, 1888). Producing a rotating field by means of two coils at right angles supplied with alternating currents of approximately one quarter difference of phase, he showed that a laminated iron core would rotate by reason of its hysteresis. Beyond proving that at low speeds the hysteresis was independent of the speed, his results were not quantitative, owing to irregularities caused by vibration in the apparatus.

In order to investigate the matter the following apparatus was designed. A powerful electro-magnet is caused to rotate on fixed spindles arranged so that the axis of rotation passes between the pole pieces, which are bored out to a cylindrical form concentric with the axis. A cylindrical armature is held on pivot bearings in the fixed spindles concentrically between the poles. The direction of magnetisation rotates in a plane at right angles to the axis of the armature and concentric with its axis. The armature, though free to rotate in its bearings, is prevented from continuous rotation by a spring attached at one end to its spindle, and at the other end to the fixed spindles of the magnet. Movement of the armature is indicated by a beam of light reflected from a small mirror attached to the armature on to a circular scale concentric with the armature axis.

The apparatus is shown in fig. 1. The electro-magnet is of Swedish iron, 8 sq. centims. in cross-section inside the coils. The pole pieces are bored out to a diameter of 2·3 centims., and subtend an angle of 120°. They are axially of the same length as the armature. The excitation is produced by two coils each of 316 turns of No. 16 wire, one end of the exciting circuit being attached to an insulated ring, the other to the magnet. The current is led in by brush contacts. One of the bearings of the electro-magnet is in the yoke piece, the other is in a gun-metal bracket bolted to the ends of the pole pieces. This bracket carries the insulated collecting ring. A third bearing is arranged in a bracket placed between the pole pieces to prevent any bending of the extended end of the spindle. The spindles are of steel, and are held by set screws in upright gun-metal standards bolted to the bed plate. The rigidity of the electro-magnet was ensured by strong bolts passing through the yoke piece, by the bracket across the ends of the pole piece, and by additional brass brackets of L-section across the sides of the pole pieces. This was found to be quite satisfactory, not the slightest movement of the pole pieces being detected. The armatures were cylindrical in form, and were made of very thin plates strung on a manganese steel spindle 2·8 millims. diameter, and held in place by ebonite washers at each end. Some difficulty was experienced in obtaining satisfactory sheets of charcoal iron, but finally some exceptionally thin plates, ·081 millim. in thickness, were obtained from Messrs. KNIGHT and CROWTHER, who very kindly had them specially rolled for the purpose.

The metal was almost pure iron, containing ·14 per cent. of carbon and only traces of silicon. Its specific resistance was $1·2 \times 10^{-5}$ ohm. The plates were insulated from each other by a layer of tissue paper. The armature was 2·60 centims. in length and consisted of 250 plates. The diameter was 1·75 centims. A second armature of hard cold rolled high carbon steel was made with plates ·141 millim. in thickness, consisting of 147 plates similarly insulated with tissue paper. Its specific resistance was $1·5 \times 10^{-5}$ ohm.

The springs used were of non-magnetic material in the form of a flat helix. To avoid disturbances the spring was enclosed in an ebonite box through which the armature spindle passed. The armature was supported by the pointed and hardened ends of the steel spindle, which rested on hollow coned bearings of hardened steel, giving sufficient strength while causing a minimum of friction.

The magnet was driven by a small leather belt from an electric motor. Owing to the construction of the apparatus it was impossible to read the speed directly at high speeds, and the value was accordingly calculated from that of the driving motor. The magnetising current was read on a SIEMENS' electro-dynamometer, which had been calibrated from a Kelvin balance.

In order to ascertain whether the presence of windage or eddy currents in the spring attachment or spring or steel spindle produced any appreciable action, a preliminary test was performed with a dummy armature in which the iron was replaced by ebonite, everything else remaining the same. At the highest speeds and with the strongest fields, the effect was inappreciable, and errors from these causes were certainly absent.

When the magnet rotates, the armature revolves with it until the torque exerted by the spring becomes equal to the torque due to the hysteresis and eddy currents in the armature. The rate at which energy is dissipated in the armature is then easily measured, for

Power absorbed = torque × 2π speed of rotation, or
Energy absorbed per revolution = 2π × torque.

Since the restoring force of the spring was found to be proportional to the angle of rotation, the hysteresis and eddy current loss per revolution is proportional to the deflexions of the armature as shown by the reflected beam of light. The reflected beam of light was focussed on to a semicircular translucent scale concentric with, and perpendicular to, the axis of revolution of the electro-magnet. The readings of the movement of the spot of light are, therefore, proportional to the energy absorbed per revolution in the armature.

The springs were calibrated by measuring the deflexion produced by a small weight at the end of a light steel arm attached to the armature. They were calibrated frequently and were found to change only very slightly, even after considerable damage by being wound up, as once or twice happened. The hysteresis per revolution is thus measured directly in terms of the force of gravity acting through a known space.

If l is the length of the arm,
m is the mass of the weight and equivalent weight of the arm,
v is the volume of the iron,
d is the deflexion caused by the weight.

Then a deflexion of 1 division against the spring $= \frac{2\pi . l . m . 981}{d . v.}$ ergs per cub. centim. per revolution of energy loss in the armature.

Determination of Eddy Currents.

The only correction of any importance that is required is the loss due to eddy currents. In most tests on hysteresis at a high speed of alternation the eddy current loss has introduced an element of some uncertainty, and therefore great attention was paid to this point. There are two ways in which errors may be caused. Eddy currents will be set up in the armature plates by the movement of the lines of force through them, which will produce a torque on the armature in the same direction as that due to hysteresis. There may also be eddy currents set up in the framework of the apparatus which will modify the magnetic field through the armature, so that it will have a value slightly differing from that produced by the magnetising current alone when the magnet is standing still. This, however, may be shown to be very small. In the rotating parts there will be no eddy currents, since the direction of the lines of force will travel round with the electro-magnet. In the fixed spindles there will be currents set up due to the leakage lines of force which pass through them, which currents will flow along the sides of the spindles and across the ends. The latter portion will be parallel to the lines of force of the field, and hence will produce a magnetic field at right angles to this, causing a slight distortion, but not altering the total magnetic flux between the pole pieces and through the armature. Since the spindles were some distance from the pole pieces, and since no perceptible heating was produced, it may be assumed that no distortion was produced of sufficient magnitude to produce an appreciable change in the value of the hysteresis in different parts of the armature.

The eddy currents in the armature plates may be divided into two portions, (1) those due to the lines of force which pass through the plates parallel to their planes; (2) those which are produced by the fringe of lines of force at the ends of the pole pieces, passing into the armature at the ends in a direction perpendicular to the plane of the plates. The first set affect all the plates equally, and can be calculated with fair accuracy, but the second set only affect the end plates and are not easy to calculate. The distribution of these latter lines of force and currents are shown in fig. 1. It is clear that these lines of force will not penetrate far into the armature, owing to the numerous small air gaps between the plates, and hence the currents will be limited to only a few plates at each end.

Fig. 1.

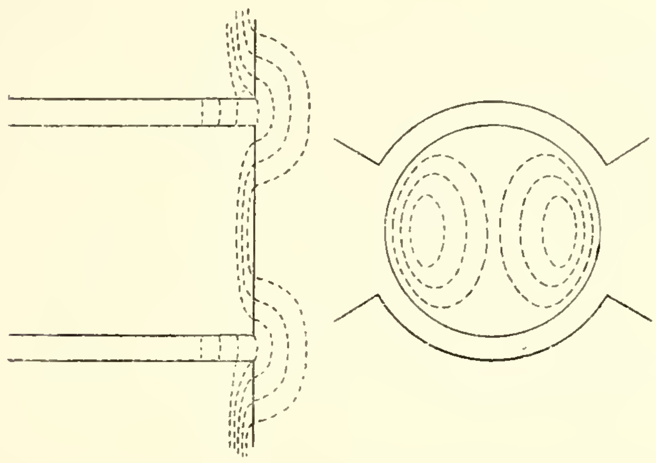
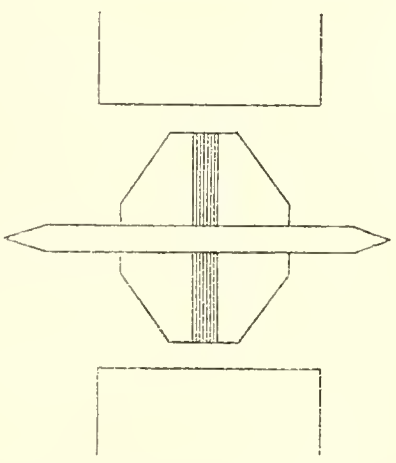


Fig. 2.



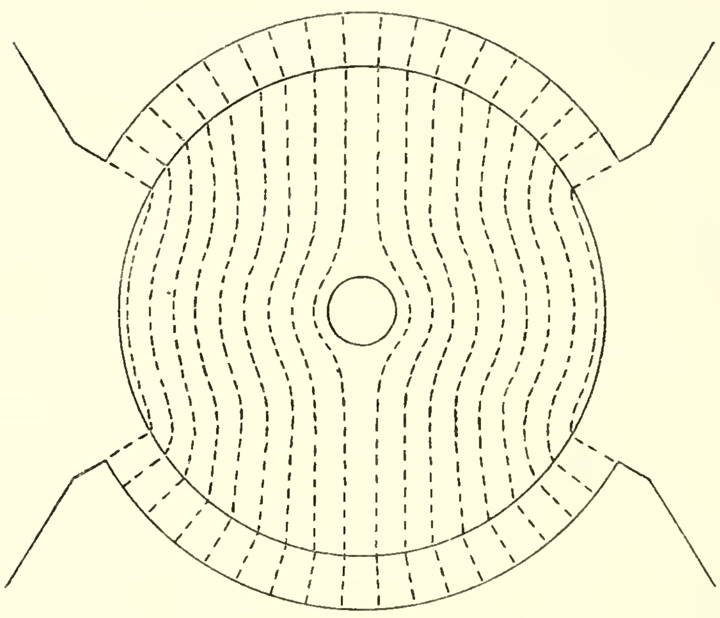
In order to determine the magnitude of this error, an armature was prepared, consisting of only 20 steel plates, insulated with paper as before, and 1·7 centim. diameter (fig. 2). It was placed in the middle of the polar area. If the effect of the eddy currents in the end plates were appreciable, it would be proportionally much increased in this armature, and, at a high increase of speed, there should be an appreciable increase in the deflexion. (It will subsequently be shown that the hysteresis in the steel is independent of the speed.) On experiment, it was found that there was no normal increase in the deflexion. It may therefore be concluded that these end plate eddy currents may be altogether neglected. The test was made with various exciting currents, from a low induction up to the maximum obtainable.

Calculation of Eddy Currents.

As the armature consists of a row of plates insulated from each other, in which the lines of induction lie in the plane of each plate, the problem is reduced to the estimation of the eddy currents induced in a thin plate rotating on a central axis perpendicular to the plane of the plates in a magnetic field, the direction of which is at all points parallel to the plane of the plate. The distribution of the lines of induction in the plate is a matter of some uncertainty, and will vary slightly with different degrees of magnetisation, owing to the variations in the permeability. Since it will be shown, however, that the effect of eddy currents is but small, an approximately accurate distribution will be assumed that will allow of mathematical treatment.

As it has been shown that even at the ends the lines of force do not pass from plate to plate, it will be assumed that their direction is entirely in the plane of the plate.

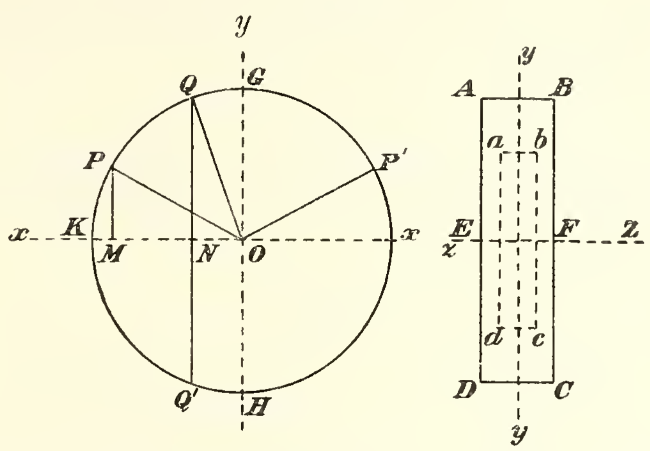
Fig. 3.



Since the magnetic reluctance of the air gap is considerably higher than that of the armature or pole pieces, the distribution of the lines of force in the air gap will be sensibly uniform and radial. It will be assumed that the whole of the lines of force pass in this way to the armature, and the fringe from the sides of the poles will be neglected. This assumption will somewhat increase the value of the eddy currents, as it gives a density in the outer layers slightly in excess of the actual value.

Inside the plate the paths of the lines will be curved, owing to the curve of the circumference and the space occupied by the spindle. Let a distribution be assumed such that the rate of cutting lines of force is proportional to the distance from the centre (neglecting the hole), and such that the portion of each revolution during which any point is cutting lines of force is the same, viz., $\frac{2}{3}$ of each revolution, and that the rate of cutting lines of force during that time is uniform. For the remaining $\frac{1}{3}$ revolution, the point is travelling along the lines of force. A reference to the fig. 3, illustrating this distribution, will show that such a distribution is very approximately the actual condition.

Fig. 4.



Let QP Q′P′ be a plate in the armature, fig. 4, and let ABCD be a section of this plate at some point QQ′ between G and P, this part being covered by the pole piece and PK being between the pole pieces.

Let the thickness of the plate AB = $2d$, and let the diameter GH = $2md$, and let the angle subtended by the pole piece = 120°. In the samples used, m has a value of 150 to 200.

In any section QQ′ the path of the current induced in the plate by its revolution in a magnetic field parallel to the plane of the plate, will be approximately rectangular and similar to ABCD.

Let ON = x, and let $abcd$ be the path of the current through a point in OF at a distance z from O.

Then the length of the path

$$= 4z + 4mz \frac{\mathrm{QN}}{\mathrm{QO}} = 4z\left(1 + m\frac{\sqrt{(m^2d^2 - x^2)}}{md}\right).$$

Let the thickness of the element be dx, and the breadth along $bc = dz$. Then breadth along $ab = mdz$. Let ρ = specific resistance of the metal in ohms.

Then the resistance of element

$$= \frac{4z\rho}{dz\,.\,dx}\left(\frac{1}{m} + m\frac{\sqrt{(m^2d^2 - x^2)}}{md}\right) = \frac{4z\rho}{m\,.\,d\,.\,dz\,.\,dx}\left(d + m\sqrt{m^2d^2 - x^2}\right).$$

Let the induction across diameter of plate = B. Then the induction in the air gap = $\mathrm{B}\frac{3}{\pi}$ assuming a distribution of magnetisation as above.

And according to the previous assumption the induction at right angles to the tangent at any point in the path as far as the line OP will be also = $\frac{3\mathrm{B}}{\pi}$.

Let n = number of revolutions per second of the armature.

Then EMF in the circuit $= \frac{3B}{\pi} . 2\pi nmz . 4z.$

$= 24\ Bnmz^2\ 10^{-8}$ volts.

Then the energy spent in eddy currents per second (from OP′ to OP) in the elementary circuit $abcd . dx . dz$

$$= \frac{E^2}{R} = \frac{24^2\ B^2 . n^2 . m^2 . z^4 . 10^{-16}\ m . d . dz . dx}{4\rho z\,(d + m\ \sqrt{m^2 d^2 - x^2})} = \frac{144\ B^2 . n^2 . m^3 . z^3 . d . dz . dx}{10^{16}\rho\,(d + m\ \sqrt{m^2 d^2 - x^2})}.$$

Therefore the loss in the whole section Q Q′

$$= \frac{144 B^2 n^2 m^3\, d}{10^{16}\rho} \cdot \frac{dx}{d + m\sqrt{m^2 d^2 - x^2}} \cdot \int_0^d z^3\, dz,$$

$$= \frac{36 B^2 n^2 m^3\, d^5}{10^{16}\rho} \cdot \frac{dx}{d + m\sqrt{m^2 d^2 - x^2}}.$$

Over the whole plate the loss is obtained by taking sections from G to P; *i.e.*, integrating from $x = 0$ to $x = \mathrm{OM} = \mathrm{OP} \sin 60° = md \sin \pi/3$.

The integral $\int_0^{md \sin \pi/3} \frac{dx}{d + m\sqrt{m^2 d^2 - x^2}}$ may be solved by writing $x = m\,d \sin \vartheta$, when it becomes $\int_0^{\pi/3} \frac{m \cos \vartheta}{1 + m^2 \cos \vartheta}\, d\,\vartheta$, but since m is a large number and d is small, it may be simplified to the form $\frac{1}{m} \int \frac{dx}{\sqrt{m^2 d^2 - x^2}}$.

This

$$= \frac{1}{m} \left[\sin^{-1} \frac{x}{m\,d}\right]_0^{md \sin \pi/3} = \frac{\pi}{3m}.$$

Therefore the eddy current losses in each plate

$$= 2 . \frac{36 B^2 n^2 m^3\, d^5}{10^{16}\rho} \cdot \frac{\pi}{3m} \text{ watts.}$$

The volume of each plate $= 2\pi m^2\, d^3$, therefore the eddy current loss per cubic centimetre

$$= \frac{12 B^2 n^2\, d^2}{10^{16}\rho} \text{ watts.}$$

For calculation it is more convenient to replace d the half thickness of the plate by t the thickness. Then the loss per cub. centim. $= \frac{3 B^2 n^2 t^2}{10^{16}\rho}$. The expression given by Professor J. J. Thomson for the loss due to eddy currents in iron plates in an alternating magnetic field $= 1{\cdot}67 \frac{B^2n^2t^2}{10^{16}\rho}$, the difference in the constant being due to the fact that the distribution of E.M.F. is different in the two cases, the average being larger for the rotating field, and also that in the latter the average length of path and resistance of the eddy current circuits are smaller.

It is more convenient to estimate the loss due to eddy currents in ergs per revolution. This $= \frac{3B^2nt^2}{10^9\rho}$ ergs per cub. centim. per revolution.

For the soft iron $t = {\cdot}0081$ and $\rho = 1{\cdot}2 \times 10^{-5}$; $\therefore$ eddy losses $= 1{\cdot}64 \times 10^{-8}\, B^2n$.

For the hard steel $t = {\cdot}0141$ and $\rho = 1{\cdot}5 \times 10^{-5}$; $\therefore$ eddy losses $= 4{\cdot}0 \times 10^{-8}\, B^2n$.

Determination of Induction in Armature.

The induction in the armature was measured by winding a few turns of wire round the iron, through holes in the ebonite washers. The electro-magnet was excited by a measured current. The coil was placed in series with a ballistic galvanometer, and the throw observed when the coil was suddenly moved half round. By this means all effects of the stray field of the magnet on the galvanometer and leading-in wires were avoided. These were, however, very small, since the galvanometer was 30 feet distant, and the connecting wires were carefully stranded and insulated.

The galvanometer was standardised by a Clark's standard cell and a Muirhead standard condenser. The resistance of the galvanometer circuit and coil was made large, to avoid damping due to currents induced in the coils by the swing of the needle. Other causes of damping being the same both for the induction throw and the condenser throw, it was not necessary to allow for this. In the soft iron armature, and to a less extent in the steel armature, the paper occupies some space, and with a strong field the lines of force through the paper are an appreciable quantity. The strength of the field was determined by winding an air coil round the ebonite washers at the end, and measuring the throw. This gives a fairly accurate value for the strength of the field, as it is just outside the iron. The area occupied by paper and air in the armature was calculated, and the proportional correction subtracted from the total number of lines of force as given by the coil round the iron. It did not amount to more than 5 per cent. in the strongest field. The results of the calibrations are given in the Diagrams 2 and 3, from which the values for the induction in the tables of readings are taken.

In the value of the area of the iron, the small hole in the centre was subtracted from the total cross-section of the armature, so that over a part of the armature the value of B will be really somewhat *less* than that given. But for the part under the pole-pieces the value will be nearly correct.

Test of Permeability and Hysteresis in a Reversing Field.

The method adopted was that used by EWING in his more recent experiments and described by him in his paper ('Phil. Trans.,' 1894), in which a continuous ring of iron is wound uniformly round with a magnetising coil, and a secondary coil is connected to a ballistic galvanometer. The current is always brought to the maximum value, and the throw of the galvanometer taken when the current is reduced (counting the sign algebraically), until the value of the reversed initial current is reached. Each change being practically instantaneous, it is probable that this method gives much the same value for the hysteresis as a rapidly alternating current does, but the point has not been experimentally proved. Dr. JOHN HOPKINSON'S tests* on hardened steel show close similarity in the permeability with an alternating current and with a step-by-step method, but his experiments are not conclusive for the hysteresis value. As the iron was well laminated, there would be, according to EWING, very little creeping effect in the magnetisation of the ring. For this purpose rings were made of the iron and steel, consisting of a narrow strip wound in several layers, the samples being taken from portions of the sheets of material used for making the armatures, and the physical conditions being kept as nearly as possible unchanged. Primary and secondary windings were wound on, the secondary being of fine wire placed immediately on the iron, so that no correction for air space was necessary. The dimensions of the rings were :—

Soft iron.—Area of metal . . . . . = ·0730 sq. centim.
Mean circumference of ring = 20·5 centims.

Steel.—Area of metal . . . . . = ·0366 sq. centim.
Mean circumference . . . = 22·8 centims.

The method being well known requires no further description.

The hysteresis is given by the expression $\frac{1}{4\pi}\int \mathrm{H}d\mathrm{B}$, where $\int \mathrm{H}d\mathrm{B}$ is the area of the curve. B and H being in absolute measure, the hysteresis is expressed in ergs per cubic centimetre per cycle.

By measuring the areas of the closed curves and dividing by 4π the following values were obtained :—

	B =	Hysteresis =
Soft Iron.	3,060	1,280 ergs.
	6,850	4,140 ,,
	10,300	7,450 ,,
	11,800	9,100 ,,
	14,100	12,600 ,,

* 'Electrician,' September, 1892, and May, 1893.

Hard Steel.—B =	2,720	Hysteresis =	3,580 ergs.
	6,600		13,850 ,,
	9,800		24,000 ,,
	11,900		33,300 ,,
	14,400		68,000 ,,

These values are shown in the curves on Diagrams 4 and 5.

Test of Variation of Hysteresis with Induction in Rotating Field.

The following tests were taken to determine the nature of the variation of hysteresis as the induction is increased. Both hard cold-rolled steel, and soft charcoal iron were used.

The machine was kept running continuously through the test, and the current was kept on continuously and gradually increased, except when very heavy currents were used. Some 10 or 15 secs. were allowed to pass before the reading was taken to give the deflection time to become steady. The effect of not complying with these conditions will be discussed later.

The shape of the pole pieces was varied twice to obtain a stronger field, and the armature was reduced in length. The first change was to cut away the poles, making the length of the polar area 1·8 centim. instead of 2·5 centims., and later coned ends were fitted to give a stronger field, the latter shape being shown in the Diagram (1).

The calibration curves for the three shapes are shown in Diagram 2.

The hard steel armature was used.

Readings were taken at 28 revolutions and at 56 revolutions per second, the agreement being fairly close.

With a shortened armature and stronger field, a set was taken at 34 revolutions per second.

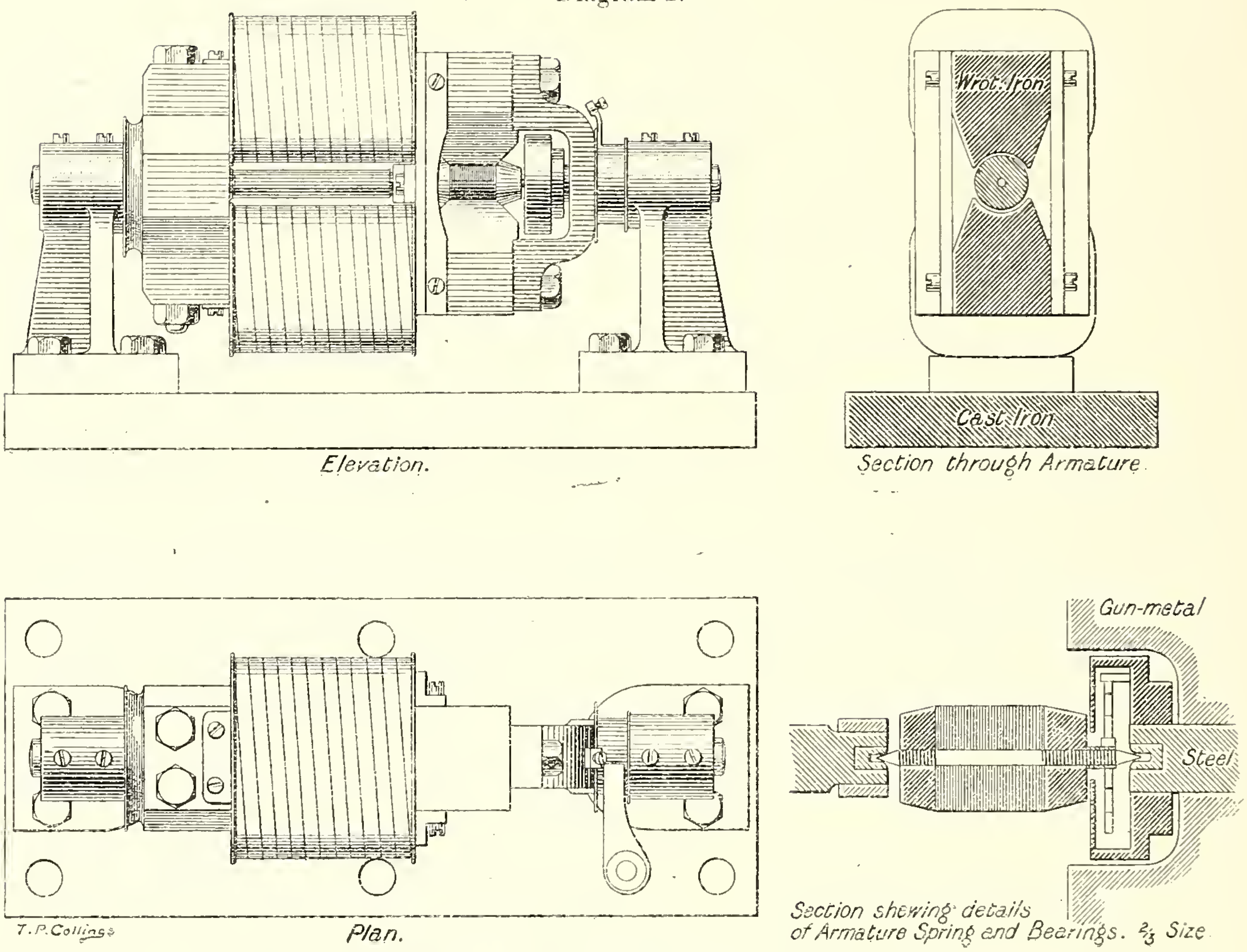
With the shortened armature and cone-shaped pole pieces a set was taken at 26 revolutions per second. This was the strongest field that could be obtained.

The results are given in the adjoining table, and the curves are plotted in Diagram 4, together with the hysteresis curve, by the method of reversals for a piece of the same sample of steel.

The whole curve is shown, from very low values of B up to nearly 20,000. It will be seen that all four sets of readings are very fairly concordant. A maximum value of 44,000 ergs is reached at an induction which varies between 14,000 and 15,000.

At the beginning the curve rises very slowly and then turns up sharply along an almost straight line. Between 9,000 and 10,000 there is a slight flexure in all of them and another straight piece until the maximum is reached. Here there is an abrupt bend and a more rapid descent on the other side, again almost straight, the

Diagram 1.



HYSTERESIS MEASURING APPARATUS.

Diagram 2.

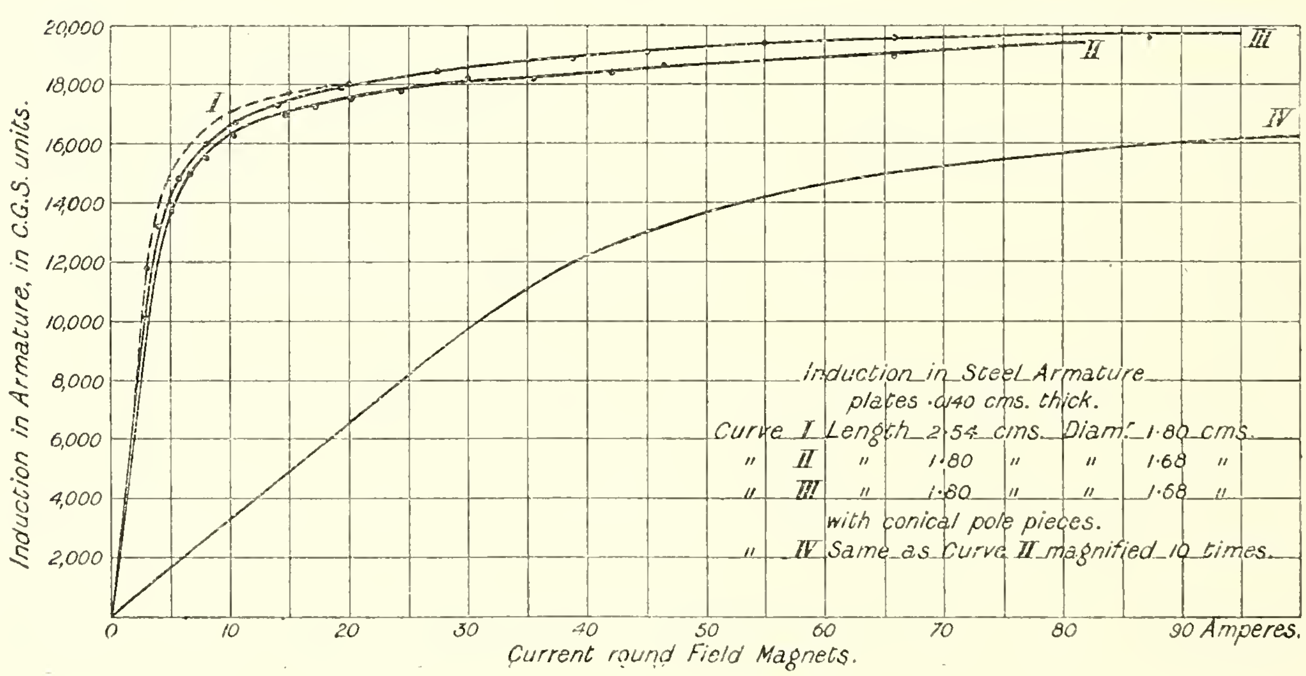


Diagram 3.

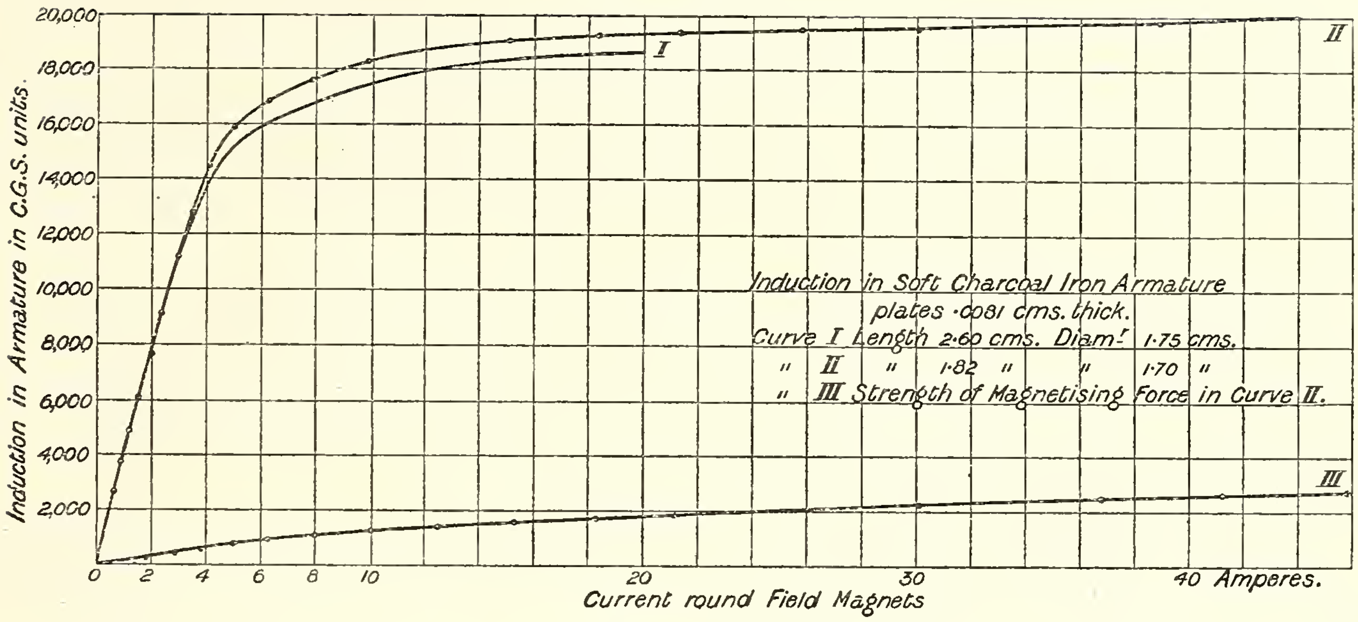
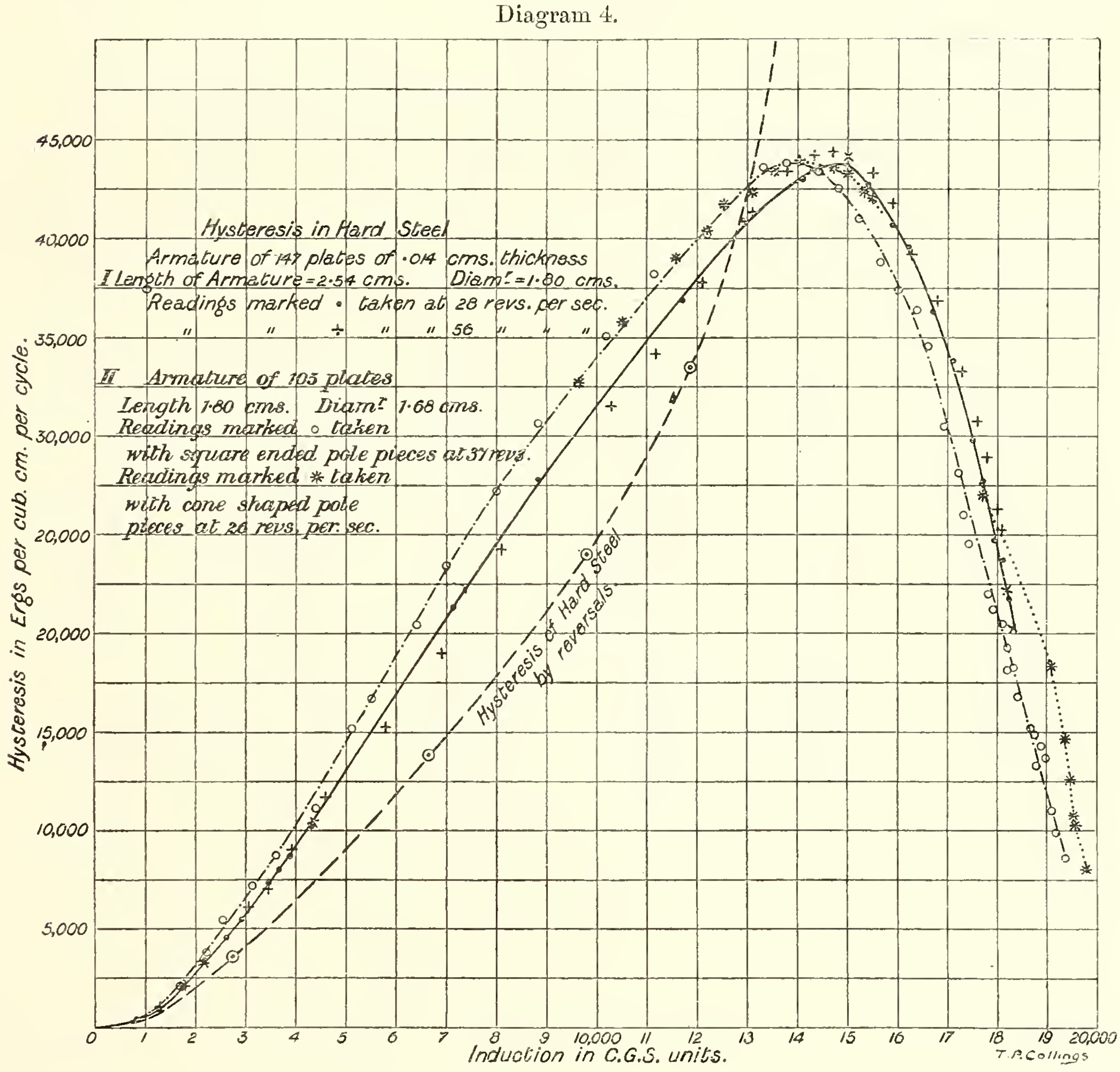


Diagram 4.



lowest value, 8,000 ergs, less than one-fifth of the maximum, being reached without any sign of another flexure.

This corresponds perfectly to the three stages in the B/H curve of hard steel, the rapid rise occurring during the rapid rise of the induction, and the maximum value being reached just at the bend before the approach to saturation. As saturation is more nearly reached, the hysteresis becomes rapidly smaller, the curve showing no sign of bending off asymptotically.

The values given in the tables for hysteresis are corrected for eddy currents, the magnitude of the correction being given at the foot. The maximum value of the correction at 56 revolutions per second, when B = 18,400 and the hysteresis is small, is only 4 per cent. of the hysteresis, and about the same for the smaller value of the hysteresis in the stronger field at the slower speed.

It will be noted that the ballistic curve lies considerably below the rotating field curve up to a point close to the maximum, after which it rises far above the other. The meaning of this will be discussed later.

TABLE I.—Hard Steel Armature of 105 Plates of thickness ·0141 centim. Diagram 4. Square pole pieces. Speed, 37 revs. per sec.

B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
1,700	2,100	14,800	42,500	18,900	14,200
2,200	3,600	15,200	41,000	19,000	13,800
2,520	5,500	15,600	38,800	19,100	10,900
3,100	7,200	16,000	37,400		
3,600	8,850	16,400	36,500		
4,400	12,300	16,600	34,600	18,200	18,100
5,100	15,200	16,900	30,500	18,400	16,700
5,500	16,800	17,200	28,100	18,700	15,100
6,400	20,500	17,300	26,000	18,800	13,200
7,000	23,600	17,400	24,500	19,000	10,400
8,000	27,300	17,500	24,300	19,200	9,860
8,800	30,800	17,800	22,000	19,400	8,580
10,200	35,100	17,900	21,200		
11,100	38,300	18,100	20,500		
12,200	42,000	18,200	19,300		
13,300	43,700	18,300	18,200	Eddy currents = $1{\cdot}5 \times 10^{-6} B^2$ ergs per cub. centim. per rev.	
13,800	43,800	18,500	16,700		
14,400	43,300	18,800	14,900		

Table II.—Hard Steel Armature of 147 Plates of thickness ·0141 centim. Variation of Hysteresis with induction. Diagram 4.

Speed, 28 revs. per sec.				Speed, 56 revs. per sec.			
B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
865	400	16,200	39,600	1,300	1,000	14,700	44,400
1,300	950	16,700	36,300	1,780	2,010	15,000	44,100
1,730	2,010	17,100	33,800	2,160	3,230	15,500	43,300
2,160	3,420	17,500	29,800	2,610	4,520	15,900	41,800
2,610	4,580	17,700	27,700	3,040	6,040	16,300	39,200
2,940	5,500	17,900	24,700	3,460	7,050	16,800	36,900
3,450	7,300	18,100	23,700	3,900	9,050	17,300	33,400
3,680	7,800	18,200	21,700	4,320	10,400	17,600	30,800
3,900	8,700			4,600	11,750	17,800	29,000
4,320	10,200			5,800	15,200	18,000	27,600
7,160	21,400			6,900	18,900	18,100	25,300
8,850	27,800			8,100	24,200	18,200	22,200
11,700	36,900			10,300	31,400	18,300	20,300
12,900	40,800	Eddy currents = $1·1 \times 10^{-6} B^2$ ergs per cub. centim. per rev.		11,200	34,100		
14,100	43,000			12,100	37,800		
14,300	43,500			13,100	41,300	Eddy currents = $2·2 \times 10^{-6} B^2$ ergs per cub. centim. per rev.	
15,000	43,400			13,800	43,500		
15,400	42,700			14,000	43,900		
15,900	40,700			14,300	44 200		

Table III.—Hard Steel Armature of 105 Plates of thickness ·0141 centim. Diagram 4. Coned pole pieces. Speed, 26 revs. per sec.

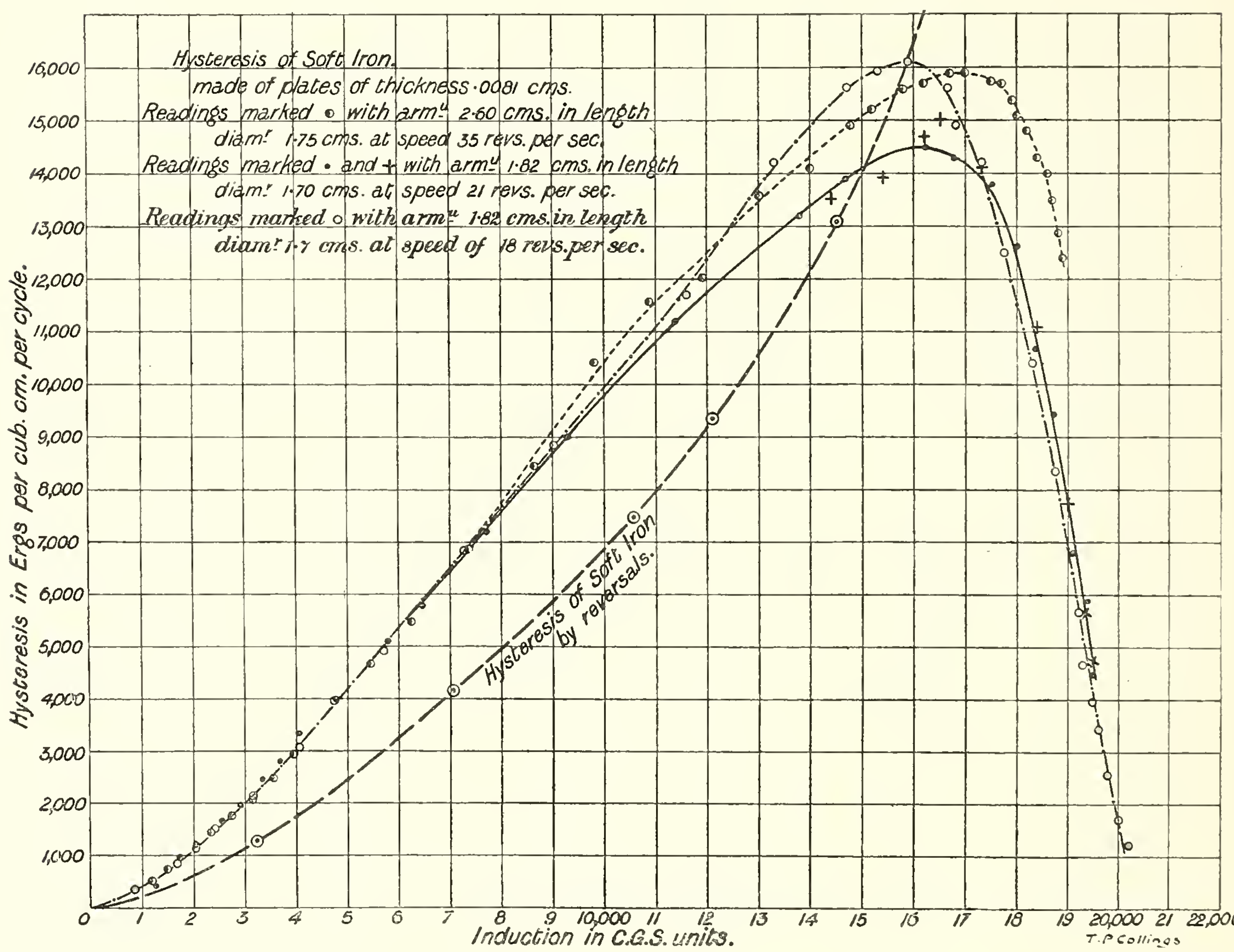
B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
9,700	32,700	15,100	43,300	19,650	10,700
10,600	36,300	15,400	42,400	19,700	10,200
11,600	39,100	15,600	42,000	19,800	7,900
12,200	40,700	17,800	29,500		
12,600	41,800	18,200	24,700	Eddy currents = $1·0 \times 10^{-6} B^2$ ergs per cub. centim. per rev.	
13,200	42,300	19,200	18,300		
13,600	43,600	19,500	14,800		
14,800	43,600	19,600	12,500		

Hysteresis of Soft Charcoal Iron.

The steel armature was replaced by the soft iron armature of 250 plates, ·0081 centim. thick. This was subsequently reduced to 156 plates, the calibration curves being given in Diagram 3. Curve 2 was carried further in later experiments, the maximum induction attained being 20,200.

Several sets of readings were taken at different speeds with both sizes of armature, some of the results being shown in Diagram 5, together with the ballistic curve. The general shape of the curve is similar to that of the steel, the three stages being

Diagram 5.



clearly marked—the slow beginning, then a long straight rise, with a less distinctly shown flexure in the middle, and a sharply defined maximum, followed by a sudden drop. The maximum occurs at a higher induction than in the steel, and the drop afterwards is more rapid.

It may be noted that, while the first and last parts of the curve are in good agreement, there is considerable divergence near the maximum, and repeated experiment failed to remove this. In fact, when kept running at the maximum value, the deflection was by no means steady, showing that it is a genuine phenomenon, and not due to errors in the apparatus. Occasionally, in the latter half of the curve, the same phenomenon occurred, the value altering slightly for a few readings, and then coming back on to the original curve, without any alteration or stoppage of the apparatus.

In these experiments the induction was pushed as high as possible, and a consistent continuous curve was obtained down to a very low value of the hysteresis. The minimum value obtained at an induction of 20,200 is only one-thirteenth of the maximum value, and the curve shows no signs of turning off again. It is therefore highly probable that the hysteresis vanishes altogether at a slightly higher induction, although the saturation point has hardly been reached. These last readings are very difficult to obtain, as the smallest irregularity or incidental error entirely vitiates them, and for some time the results were not good. However, the curve given was obtained several times, and the points marked are the mean of several readings, all in good agreement, so that it may be taken as correct.

The comparison with the ballistic curve exhibits the same features as in the case of hard steel, the relation between the two curves being singularly alike. It may therefore be concluded that the positions will be the same for other samples of iron and steel, since these two occupy extreme positions among the various types.

That the hysteresis in a rotating armature at low induction should be greater than in an alternating field is quite intelligible, since the movement is more gradual, and is free from sudden shocks. There is also not so much choice in the direction of movement, and hence some of the molecular combinations will offer more resistance to dissociation. The point is of considerable importance in the design of large dynamo armatures, which are usually worked at an induction of about 10,000 or 8,000 C.G.S. At this part of the curve the value of the hysteresis is some 50 per cent. higher than that given by a ballistic test, and allowance must be made accordingly for the larger amount of heating. On the other hand, in small ring armatures, which are worked at a high induction, the hysteresis will be considerably lower than the value given by the ballistic method.

Diagram 6.

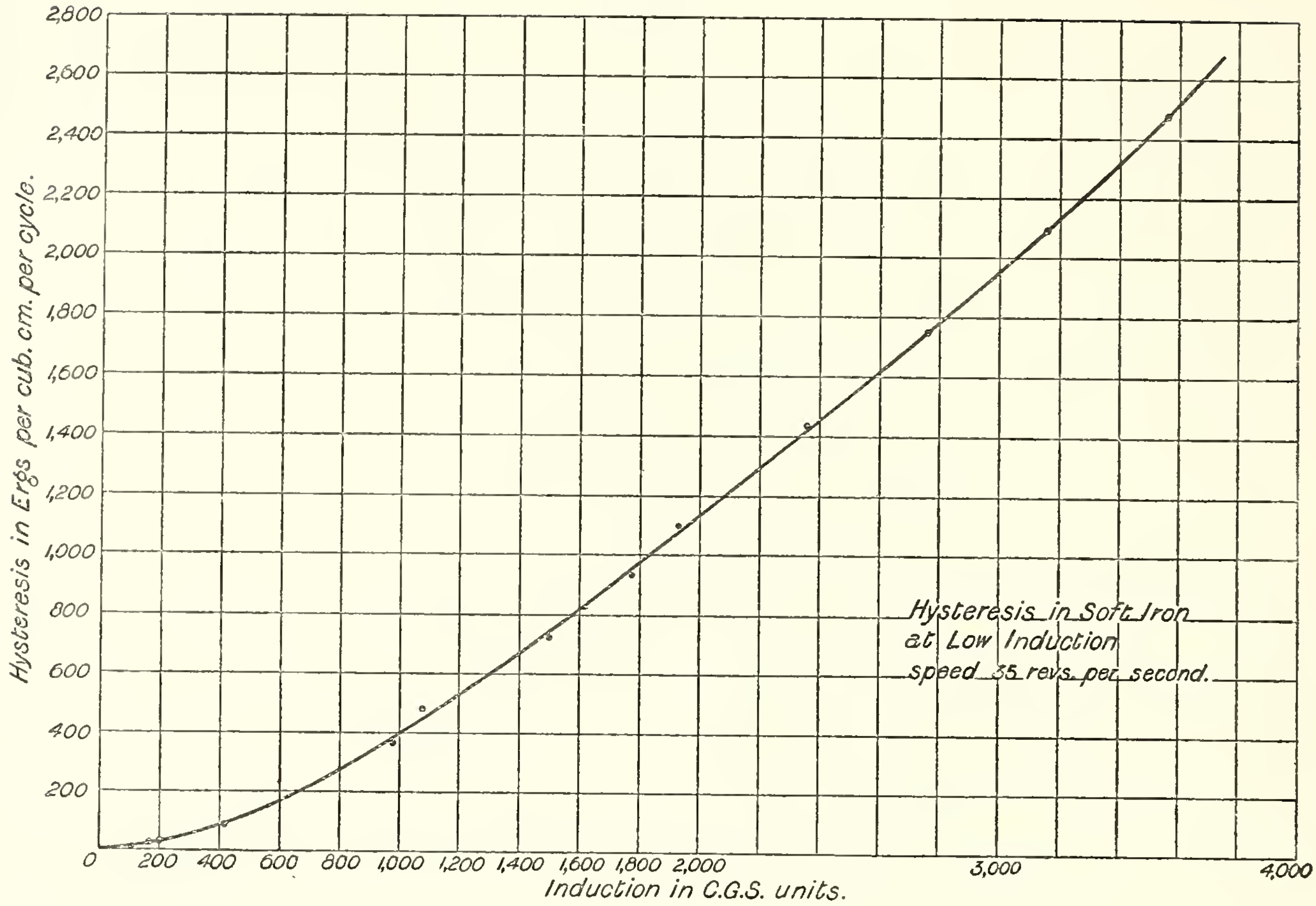


Diagram 7.

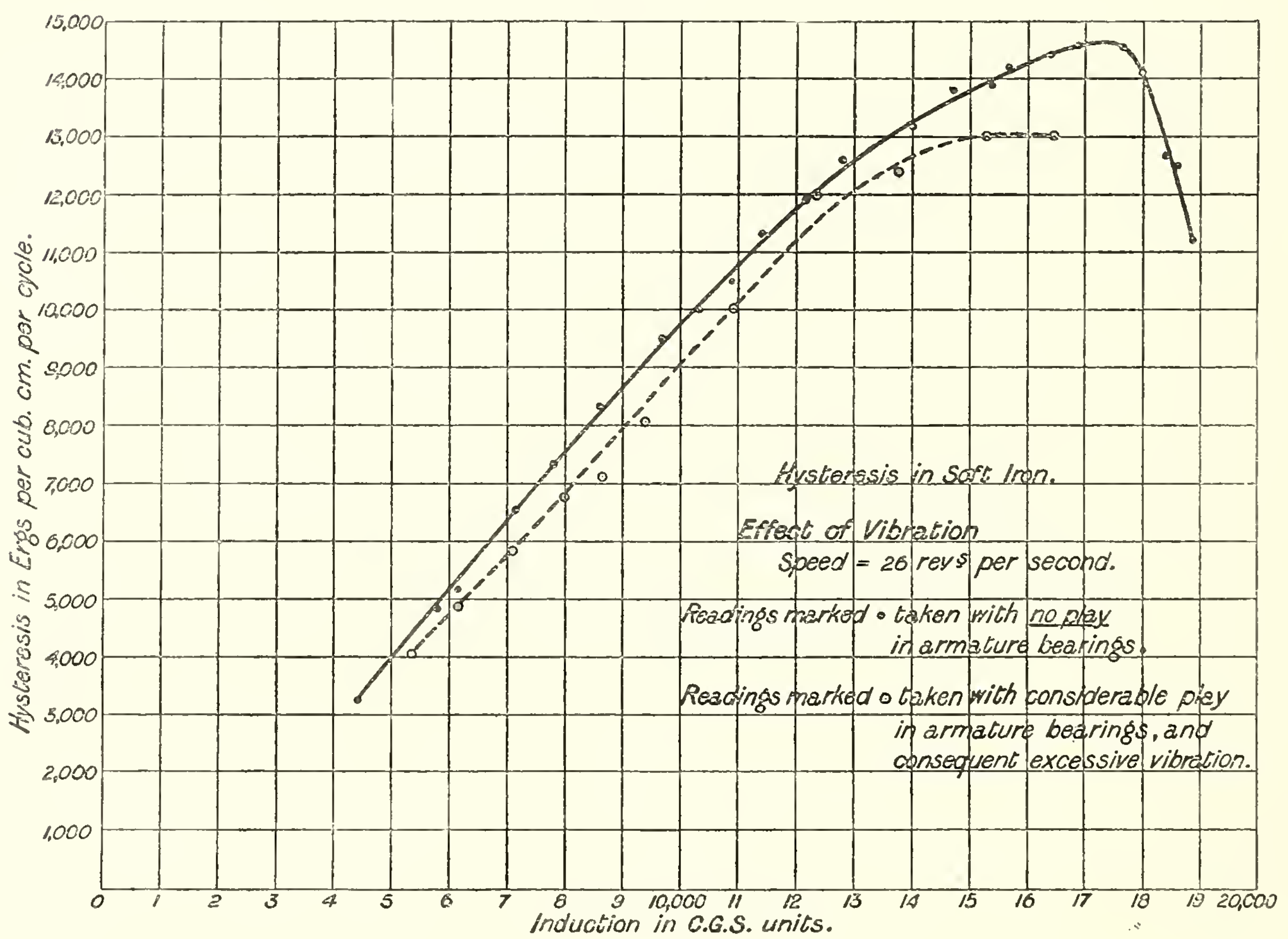


Diagram 8.

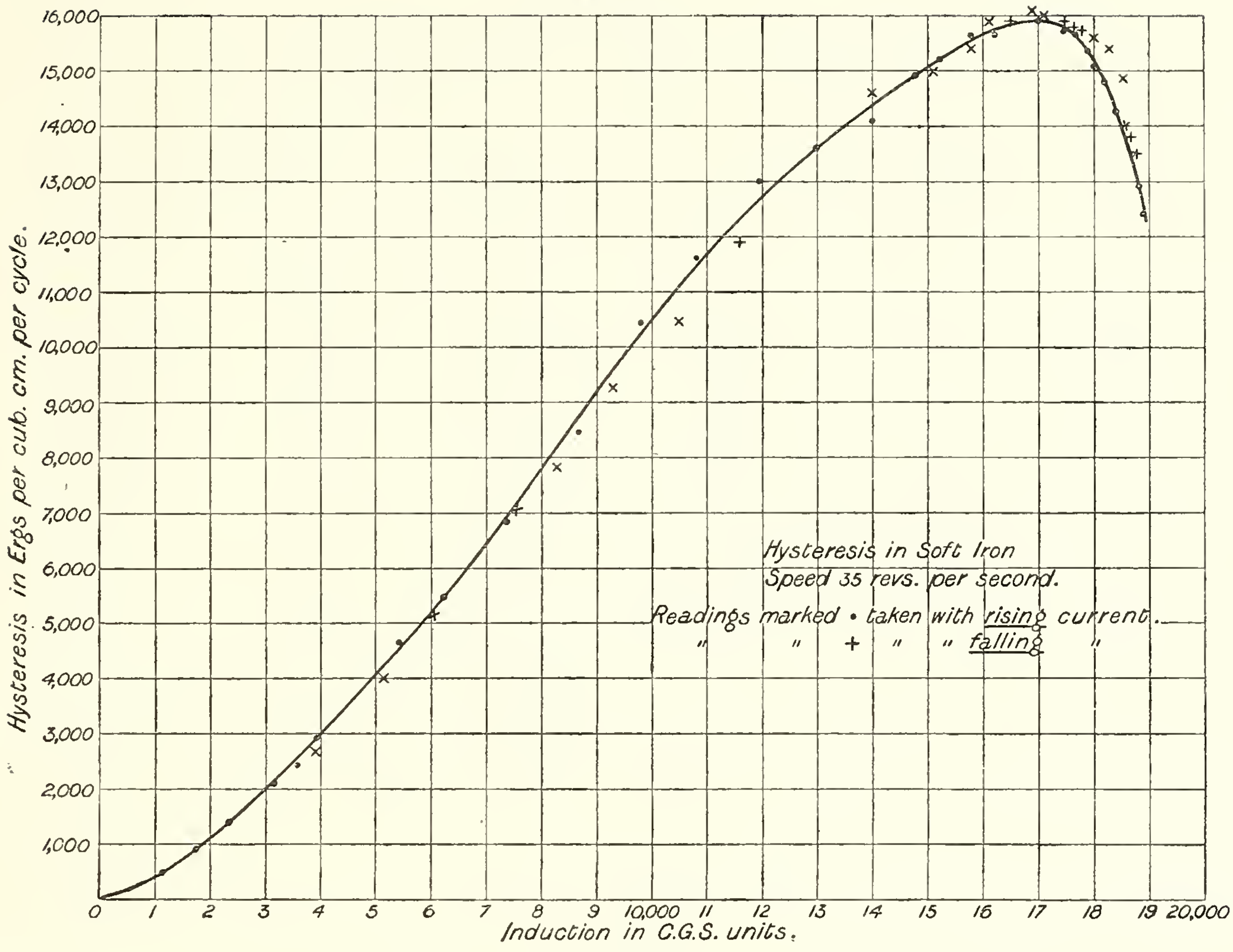


Diagram 9.

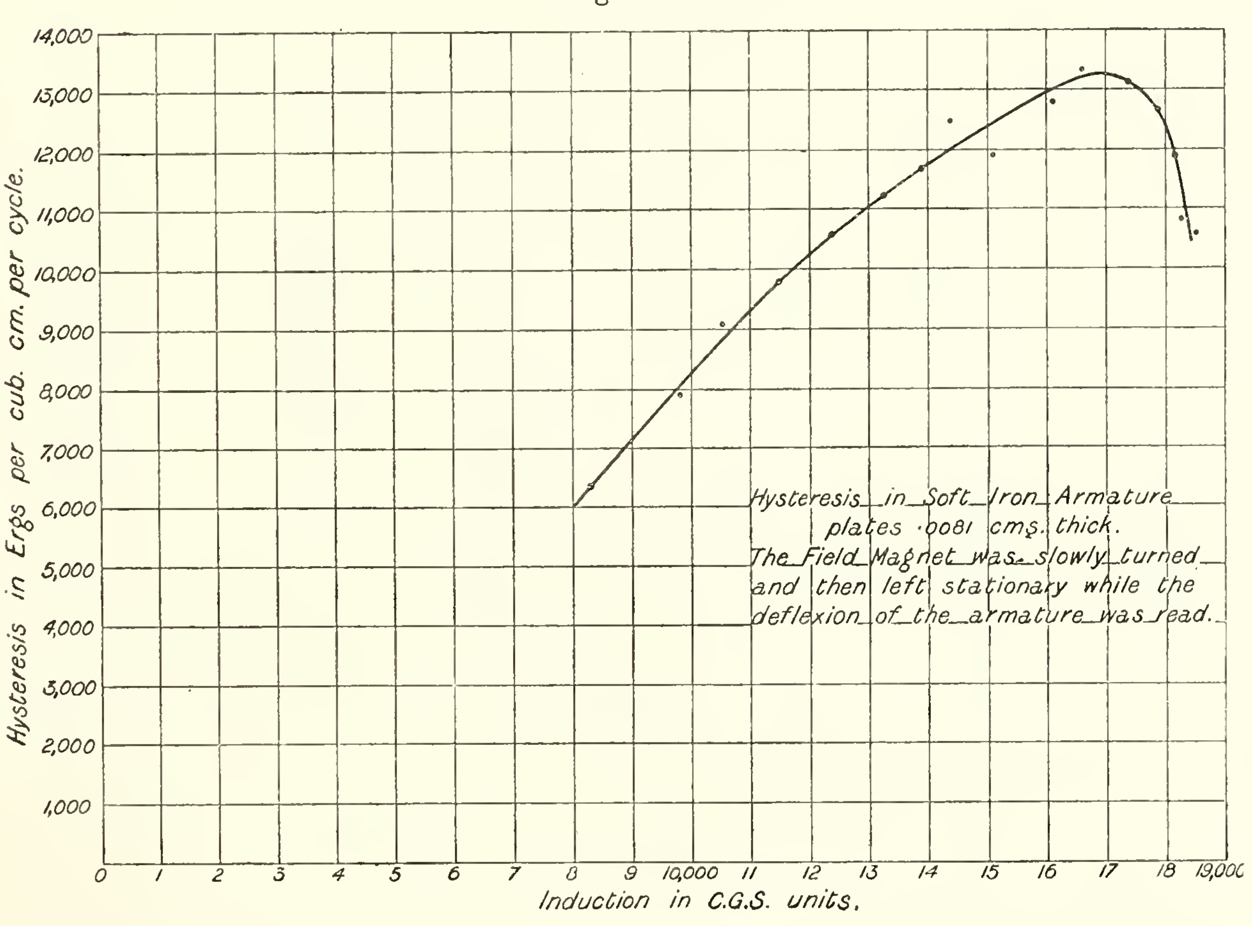


Table IV.—Soft Iron Armature of 250 Plates of thickness ·0081 centim. Diagram 5. Speed, 35 revs. per. sec.

B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
102	9·6	5,420	4,650	18,200	14,800
166	22	6,250	5,430	18,400	14,300
193	26	7,360	6,850	18,600	14,000
410	87	8,640	8,450	18,700	13,500
593	97	9,800	10,450	18,800	12,900
980	367	10,850	11,600	18,900	12,400
1,180	483	11,950	13,000		
1,500	725	13,000	13,600		
1,770	933	14,000	14,100		
1,930	1,090	14,800	14,900		
2,360	1,430	15,200	15,200		
2,760	1,750	15,800	15,600		
3,160	2,090	16,200	15,700	Eddies = $6·0 \times 10^{-7} B^2$.	
3,560	2,480	17,000	15,900		
3,940	2,960	17,500	15,750		
		17,700	15,700	Also plotted on Diagram 8.	
		17,900	15,400		
Also plotted on Diagram 6.		18,000	15,100		

Table V.—Hysteresis in Soft Iron Armature of 156 Plates of thickness ·0081 centim. Square pole pieces. Diagram 5. Speed, 21 revs. per sec.

B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
1,290	390	13,800	13,200	16,200	14,700
1,660	1,000	14,700	13,900	16,500	15,000
2,020	1,160	16,200	14,500	17,300	14,200
2,580	1,630	16,800	14,300	18,000	12,600
2,940	1,940	17,500	13,800	18,400	11,100
3,310	2,480	18,000	12,600	19,000	8,800
3,680	2,800	18,400	12,700	19,100	6,800
4,050	3,340	18,700	9,500	19,350	5,700
4,150	3,410	19,100	6,800	19,500	4,800
4,780	3,950	19,400	5,900		
5,800	5,100	19,500	4,500		
6,450	5,800				
7,700	7,200			Eddy currents = $3·5 \times 10^{-7} B^2$ ergs per cub. centim. per rev.	
9,300	9,000	14,400	13,500		
10,400	11,200	15,400	13,850		

TABLE VI.—Hysteresis of Soft Iron Armature with 156 Plates of thickness ·0081 centim. Speed, 18 revs. per sec. Diagram 5.

B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
830	350	11,600	11,700	19,200	5,700
1,200	545	13,300	14,200	19,300	4,600
1,660	816	14,700	15,600	19,500	3,900
2,020	1,110	15,300	15,900	19,600	3,400
2,400	1,500	15,900	16,100	19,800	2,600
3,130	2,180	16,600	15,600	20,000	1,700
4,050	3,060	16,800	14,900	20,200	1,200
4,700	3,940	17,300	14,200		
5,700	4,900	17,800	12,500	Eddy currents = $3{\cdot}0 \times 10^{-7}$ B².	
7,260	6,800	18,300	10,400		
9,020	8,850	18,800	8,400		

TABLE VII.—Soft Iron Armature of 250 Plates of thickness ·0081 centim. Variation of Hysteresis with Excessive Vibration. Diagram 7. Speed, 26 revs. per sec.

Armature almost tight in bearings.				Armature very loose in bearings. Excessive vibration.	
B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
4,420	3,280	15,400	13,900	5,340	4,050
5,700	4,840	15,700	14,200	6,160	4,820
6,160	5,200	16,400	14,400	7,100	5,800
7,170	6,550	16,900	14,600	8,000	6,750
7,820	7,330	17,700	14,600	8,650	7,100
8,650	8,300	18,000	14,100	9,400	8,050
9,700	9,500	18,400	12,700	10,900	10,000
10,300	10,000	18,600	12,500	12,300	12,000
10,900	10,500	18,900	11,200	13,800	12,400
11,400	11,300			15,300	13,000
12,200	11,900			16,500	13,000
12,800	12,600	Eddy currents = $4{\cdot}3 \times 10^{-7}$ B².			
14,000	13,200				
14,700	13,800				

Higher values of B were not possible as the vibrations threatened to break the pivots of the armature.

TABLE VIII.—Soft Iron Armature of 250 Plates of thickness ·0081 centim. Speed, 35 revs. per second. Diagram 8. Starting with maximum current and decreasing.

B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
18,800	13,500	17,100	16,000	9,300	9,250
18,700	13,800	16,900	16,100	8,300	7,800
18,600	14,000	16,500	15,900	7,550	7,050
18,500	14,900	16,100	15,900	6,070	5,120
18,300	15,400	15,800	15,400	5,150	4,000
18,000	15,600	15,100	15,000	3,960	2,680
17,800	15,750	14,000	14,600		
17,600	15,800	11,600	11,900		
17,500	15,900	10,500	10,450	Eddy currents = $6{\cdot}0 \times 10^{-7}$ B^2.	

TABLE IX.—Hysteresis and Induction in Soft Iron Armature of 250 Plates of thickness ·0081 centim. The Armature slowly turned and then left at rest and the reading taken. Diagram 9.

B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.	B.	Hyst. in ergs per cub. centim. per rev.
8,300	6,370	13,300	11,200	16,600	13,300
9,800	7,900	13,900	11,700	17,400	12,700
10,500	9,100	14,400	12,500	18,200	11,900
11,500	9,800	15,100	11,900	18,300	10,800
12,400	10,600	16,100	12,800	18,500	10,600

In Diagram 6 is given the first part of the curve plotted on a magnified scale, and determined by means of a very weak spring, so that the deflexions were large. The initial values are small, but the increase is rapid up to an induction of 1,400, after which the curve becomes more nearly straight.

The effect of violent vibration was tested with this armature by comparing the readings taken with the armature almost tight in its bearings, and the magnet running very smoothly, with the readings taken when the armature was very loose. This has the effect of causing excessive vibration, the armature rattling to and fro at every revolution. As might be expected, the hysteresis is diminished (Diagram 7), but not by much, and it may therefore be concluded that the small vibration present, under normal working, does not influence the values to any appreciable extent.

A similar experience of the comparatively small effect produced by violent mechanical vibration is mentioned by EVERSHED and VIGNOLLES ('Electrician,' 15th and 22nd May, 1891).

The effect of passing from a high induction to a low one, allowing a short time between each reading, was examined, the result being shown in Diagram 8. There was no definite change in the hysteresis, the curves of rising and falling currents being practically identical. This point will be referred to below, when more rapid changes are taken.

The effect of speed on hysteresis will be examined later, when experiments will be described in which the induction is kept constant and the speed varied.

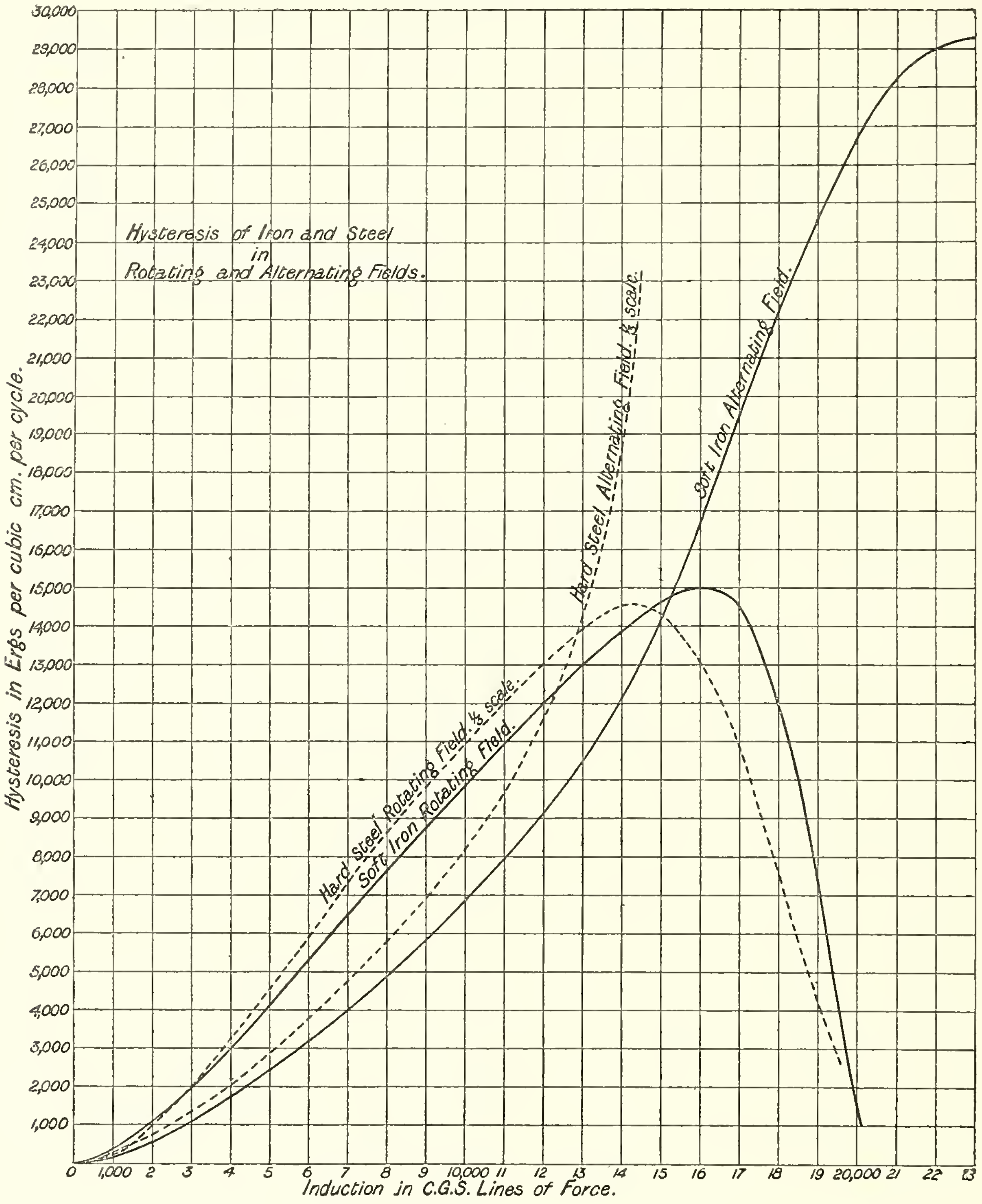
Finally, a test was made at an indefinitely slow speed by slowly turning the armature through a short distance and then stopping it. The deflexion was then read, while it was standing still. Since the smallest error of centring produced an additional pull either forwards or backwards, there are some irregularities in the readings, and a high induction could not be reached. The result is in complete accordance with the test at higher speeds, but it will be noted that the maximum value, though occurring at the same induction, viz., between 16,000 and 17,000, is not so high as before. This is due to a small backward movement of the armature immediately the magnet stopped, perhaps a kind of readjustment, which does not have time to take place when running. It is only a small effect, about 10 per cent. of the value of the hysteresis at that point, and is less marked at points beyond the maximum. The smallest actual rotation was sufficient to prevent its occurrence, the values then being closely in accordance with those at higher speeds. The curve is shown in Diagram 9.

[As it was found that the speed of rotation was without influence on the hysteresis, and that the eddy currents were almost negligible at moderate speeds, the greater number of readings were taken at speeds of 15 to 30 revolutions per second, since the deflexions were very much steadier than at low speeds. With the latter there was a tendency for the spot of light to oscillate, especially at high inductions, owing to small errors in centring, which it was almost impossible to eliminate altogether. The results obtained at low speeds, down to a speed of one revolution per second, were identical with those at high speeds.—2nd September, 1896.]

The above experiments prove very clearly the nature of hysteresis in a rotating field, and bear out most completely Mr. SWINBURNE's deduction from the molecular theory of magnetism. The agreement of the phenomena with the previously suggested deduction forms a strong verification of the truth of this theory while it also easily explains various other points which have been noted. The three stages of the curve correspond precisely with the three stages of the molecular movement, and the difference between the curves for iron and steel readily follow from the difference in their structure. In the soft iron, as has been shown by EWING, the molecules move in larger combinations than in steel, and hence it is not until a higher induction is reached that uniformity of movement is produced. When once past the maximum the decrease is very rapid, and both metals appear to give curves reaching approximately a zero value at an induction between 20,000 and 21,000.

It is an important part of Professor EWING's theory of magnetism that hysteresis is caused solely by intermolecular magnetic control, and not by any friction on the moving molecule, although there are great temptations to assume some such friction

Diagram 10.



in certain phenomena. But the almost complete disappearance of hysteresis in the soft iron proves that friction must be almost entirely absent, since an individual rotary motion would not avoid this.

The curve for hard steel is not so conclusive on this point, as the induction has not been pushed far enough, but so far there is no sign of any large amount of frictional control. This is noteworthy, as we are accustomed to consider the steel molecules as subject to some form of mechanical constriction.

In Diagram 10 are given the mean values of the curves obtained for iron and steel, the latter being plotted with a smaller vertical scale to admit of easier comparison. With them are also plotted the curves of hysteresis in an alternating field, that for soft iron being obtained by a calorimetric method with iron from the same sample. ('Electrician,' November 22, 1895.)

Part II.

Effect of Speed on Hysteresis.

The question of the relation between the speed of reversal of magnetization and the value of the hysteresis per reversal has been investigated by several experimentalists, by the employment of an alternating current of varying periodicity.

The experiments of Tanakadaté ('Phil. Mag.,' September, 1889) by the calorimetric method, indicate that there is no change in the hysteresis between the values 27 and 400 cycles per second. His method, however, is liable to error, and he does not measure the induction at the same time as the hysteresis, but finds it by a ballistic test. As his magnetic circuit has only a small reluctance and no air gap, this assumption of identity is doubtful.

The experiments of Mr. A. Siemens ('Proc. I.E.E.,' February, 1892) and Mr. C. P. Steinmetz ('E.T.Z.,' 1891 and 1892, and 'Electrician,' February, 1892) indicate a change of hysteresis in this respect, but their results are not consistent. By the magnetic curve tracer Professor Ewing has made some tests at low speeds, but the presence of eddy currents in the pole pieces obscure the results, and mechanical lag and vibration in the moving parts prevent the use of any but very moderate speeds. Dr. John Hopkinson ('Electrician,' *loc. cit.*) finds that in hard steel wires, while the permeability is not changed, there is a small increase in the hysteresis as the speed is increased. The experiment is, however, obscured in regard to this point by the changes in the form of the current curve consequent upon change of speed. Mr. T. Gray ('Roy. Soc. Proc.,' May, 1894) finds no variation in the hysteresis between the speeds of 3 and 8,000 cycles per minute, but his experiments are somewhat contradictory. Recently the author has shown ('Electrician,' *loc. cit.*) that the change of hysteresis, if it exists at all, must be very slight and that on theoretical grounds it is probably independent of speed.

In the foregoing experiments with a rotating field it has been indicated that there is at most very little change of hysteresis at different speeds. The point was more carefully worked out as follows :—

The machine was run at a gradually increasing speed, and the current was kept perfectly constant. As has been pointed out, this ensures a practically perfect constancy in the value of B in the armature.

At first with the steel armature, a distinct variation was found, the sign of the change depending on the induction; but the effect was traced to errors in the truth of the armature. When these were eliminated the result of a large number of tests is to show that up to a speed of 70 revolutions per second there is no regular or definite change in the hysteresis when tested in the rotating magnet machine.

All the experiments were performed by keeping the current constant and unbroken, and increasing the speed step by step, waiting at each reading until the reading had become steady.

The tests of the steel are shown in Diagram 11. There are irregularities which could not be eliminated, and which frequently repeated themselves at the same points, but on the whole there is a clear indication of no sensible change. The readings are corrected for eddy currents.

The values at the maximum, or near to it, tend to be irregular, as was seen also in the previous curves. This may well be expected, the metal being to some extent in a critical condition.

The values at the highest speeds were rendered slightly uncertain owing to vibration, which shook the spot of light, but there is scarcely a definite change in one direction more than the other.

The soft iron shows the same kind of curves, and the same kind of small variations, but it may be noted that the readings given in Tables XI. to XIV. are not so regular as in the case of hard steel, and repetition of the series does not reproduce the same irregularities.

At the value B = 17,200, four series were taken at different times; two of them show slightly rising values, another falling values, while a fourth gives an approximately constant value. An example of a rising curve is shown at B = 7,800, but on repetition this gave a practically horizontal line.

On repeating the tests with a smaller armature, the same results appear. At the maximum value of the hysteresis when B = 16,500, the readings were very irregular, as was anticipated, since the iron was in a critical condition. The same effect is noticed to a lesser degree near the maximum, when B = 15,000 and 17,200.

A large number of series have been given because the actual readings show irregularities which might prevent a generalization; but after constant repetition it is seen that the variations obey no regular law, and are probably due partially to errors in the machine, though to some extent there is strong ground for concluding that they are due to actual variations in the value of the hysteresis.

The method is more accurate than those which entail the accumulating of the waste of energy, since in them the amount measured is proportional to the speed, and the conditions are changed. The deflections here are quite unaffected by the speed, and depend only on the value of the hysteresis at that moment. Hence the small evanescent variations may be readily detected, whereas by the cumulative methods they would be merged in the general average.

There are a few points concerning the effect of time and sudden changes which are interesting. As has been mentioned, the curves were taken with steadily increasing current or speed, a short time being allowed before a reading was taken.

When, however, a sudden increase is made in the current, the value of the hysteresis is higher than the normal for that current and the value slowly decreases until it reaches the normal steady condition. The effect is not large, about 3 or 4 per cent., but is by no means regular. The time of its disappearance is usually a few seconds and the effect is greater at medium and high induction than at low induction. The reverse effect is shown in suddenly decreasing the current, the hysteresis having too low a value, and not rising up to its normal for upwards of a minute, the arrival at the normal being slower than with a rising current.

A similar effect is observed in changes of speed. A rapid rise in speed gives an increased value of the hysteresis, which dies away in a few seconds. But a rapid reduction in speed gives a diminished value of the hysteresis which lasts often more than a minute, during which time the deflexion is very irregular, jumping up to the normal and going back again. The change is also greater than any of the preceding ones, and is frequently as much as 10–15 per cent.

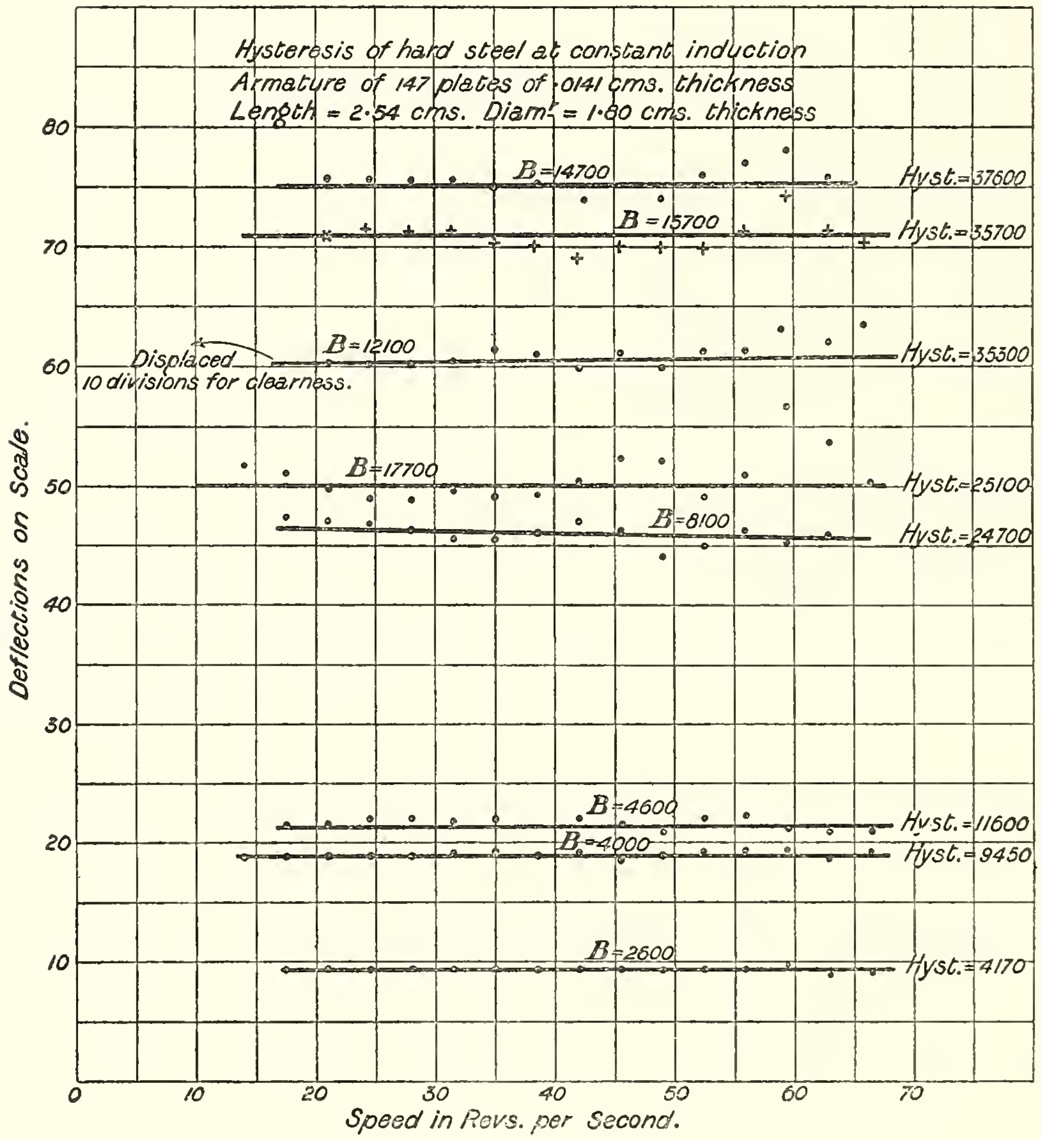
The effect of rapid makes and breaks in the magnetising current in quick succession is to increase the hysteresis. It recovers its normal value after a time but the readings are very irregular during the recovery. In all these effects, the machine is kept constantly running, and they occur more or less at all speeds. They are, however, not regular enough in amount to allow of any connection with the speed being established.

Effects somewhat similar to these have been noticed with alternating currents. TOMLINSON mentions ('Proc. Roy. Soc.,' Dec. 1889), that repetition of the cycle reduces the hysteresis, and he names it the "accommodation" of the molecules, supposing that the molecules become arranged in a manner in which it is more easy for them to reverse, and it is not surprising that with the more uniform motion of a rotating field, the same effect should be observed. The effect of suddenly diminishing the speed is more obscure; but in these experiments it seems to be due to a disarrangement and an absence of recombination for a short time, giving rise to great irregularity of hysteresis. The length of time over which the effect lasts is remarkable, as it often extended to over 1,000 cycles. It was completely stopped by a temporary stoppage of the current, while TOMLINSON found that even slight mechanical shocks were sufficient to stop the effect of "accommodation."

The effect of reducing the current is probably due to the same cause. The molecules are arranged in regular order, and it is some time before they disarrange themselves, while during the process the hysteresis is lower than the normal. An increase in the current is not immediately followed by complete arrangement of the molecules, and during the process the hysteresis is too high. On breaking and remaking the circuit after a sudden reduction in the current, the deflexion of the armature, previously lower than the normal, rises to a value higher than the normal and then drops to its normal value, showing that by breaking the circuit, the regular arrangement is completely disturbed.

Professor EWING in his book ("Magnetism in Iron," &c.) mentions that rapid makes and breaks increase the permeability, but increase still more the residual magnetism,

Diagram 11.



but he does not give the hysteresis effect, which may or may not have been increased. This is clearly a case of "accommodation." When, however, the current is reversed, the increased amount of combinations renders the iron less permeable with more "coercive force" necessary, and therefore with presumably more hysteresis. The result does not seem to agree very well with that of TOMLINSON'S.

In the rotating field it is possible that rapid makes and breaks tend to disarrange the symmetrical arrangement produced by continued rotation, and to increase the number of combinations. This would account for the increase in the hysteresis, and also for its irregular value before it again reached its normal value.

[It will be noted that while the average values of hysteresis obtained in the Tables X. to XIV. agree for the most part very closely with those in the previous tables; the values at the maximum are distinctly lower, both in iron and steel, indicating a greater "accommodation" under the long continued running in the critical condition.—2nd September, 1896.]

In the hard steel these effects are very much diminished, the deflexions being more uniform and consistent. The same difference is shown in the constancy of the hysteresis at the maximum point. In the steel, all the curves agree very closely in the maximum value attained, Diagram 4, and the speed curve at that point, Diagram 11, although somewhat irregular, is much less so than the curve from soft iron. This follows from the supposition that in steel the aggregations of molecules are smaller and less dependent on each other, and hence the average behaviour is more regular.

TABLE X.—Hard Steel Armature of 147 Plates, of thickness ·0141 centim. Variation of Hysteresis with Speed; Diagram 11; Deflexions on Scale: 1 division = 502 ergs per cub. centim. per rev.

Speed in revs. per sec.	B=2600.	4000.	4600.	8100.	12,100.	14,700.	15,700.	17,700.
·14	9·3	18·8	..	..	..	..	..	51·6
17·5	9·3	18·8	21·5	47·5	..	..	..	51·0
21	9·3	18·8	21·5	47·0	70·2	75·6	71·0	49·9
24·5	9·3	18·8	22·0	46·7	70·1	75·5	71·5	48·9
28	9·3	18·8	22·0	46·3	70·1	75·4	71·4	48·8
31·5	9·3	18·8	21·7	45·7	70·5	75·4	71·4	49·7
35	9·3	19·0	22·0	45·5	71·5	75·1	70·3	49·1
38·5	9·3	18·8	..	46·0	71·0	75·4	70·2	49·5
42	9·3	19·0	22·0	47·0	69·8	73·8	69·1	50·4
45·5	9·3	18·7	21·6	46·0	71·2	75·2	70·2	52·3
49	9·3	18·8	21·0	44·0	71·3	74·1	70·2	52·1
52·5	9·3	19·0	22·0	45·0	..	76·2	70·0	49·1
56	9·3	19·1	22·3	46·0	71·3	77·3	71·4	51·0
59·5	9·5	19·1	21·3	45·5	73·2	78·4	74·8	56·9
63	8·9	18·5	21·0	45·5	72·2	75·9	71·7	53·8
66·5	9·0	19·0	21·0	..	73·6	..	70·6	50·2
Value of eddies in scale division	·0005n	·0013n	·0016	·0050	·012n	·017n	·020n	·025n
Average value of hysteresis	4170	9450	11,600*	24,700*	35,300	37,600	35,100	25,100

* Value of spring 1 division = 537 ergs per cub. centim. per division.

TABLE XI.—Soft Iron Armature of 250 Plates, of thickness ·0081 centim. Variation of Hysteresis with Speed.

Speed in revs. per sec.	Deflexion.	Speed in revs. per sec.	Deflexion.
7·5	109	7	69
10	108	10	69
23	109	14	67·5
27	109·5	17	68
31·5	110·5	21	67·5
37	112	23	70·5
43	111·5	26·5	70
52·5	111	34	71
61	110	40	71
71	107·5	45·5	71
		51	71·5
		57	71
		62	69
		65	67·5
		70·5	69
B = 4800 Hysteresis = 3500 Eddy currents $= 1·2 \times 10^{-2}\, n$ divisions 1 scale division = 32 ergs per cub. centim. per rev.		B = 3400 Hysteresis = 2200 Eddy currents $= 6 \times 10^{-3}\, n$ divisions	

TABLE XII.—Soft Iron Armature of 250 Plates, of thickness ·0081 centim. Variation of Hysteresis with Speed.

Speed in revs. per sec.	Deflexion.	Speed.	Deflexion.	Speed.	Deflexion.	Speed. Repeated.*	Deflexion. Repeated.*
0	8·0	0	35·5	15	41·2	0	31·5
23·5	8·5	25	35·5	18·5	40·7	29	32·0
30	8·5	32	35·6	23	40·1	32	32·4
36	8·5	38	35·6	27	38·6	40	31·8
43	9·0	44·5	35·5	30	38·1	42	31·3
50	8·5	51	36·1	34	38·2	45	31·2
57	8·5	57·5	36·0	38	37·6	54	32·1
62	9·0	64	35·9	41	37·5	59	31·5
				49	38·0	62	31·4
				57·5	38·2	67	31·4
				62	38·8		
				66·5	38·2		
B = 4800 Hysteresis = 3800 Eddy currents $=1·1 \times 10^{-3}\, n$ divisions		B = 14,000 Hysteresis = 14,000 Eddy currents $=9·0 \times 10^{-3}\, n$ divisions		B = 17,100 Hysteresis = 14,700 Eddies $=1·3 \times 10^{-2}\, n$ divisions		B = 17,100	
1 division = 385 ergs per cub. centim. per rev.							

* This series was repeated later, and the spring was not calibrated. It had evidently changed in strength or method of applying restoring force.

TABLE XIII.—Soft Iron Armature of 250 Plates, of thickness ·0081 centim. Variation of Hysteresis with Speed; Deflexions.

Speed in revs. per sec.	B=2000.	2750	4400	7800	11,400	15,000	17,200	17,200
2	..	4·0	..	..	31·5			
7	..	4·0	7·8	18·2	31·4	37·6	38·8	39·1
14	2·8	4·1	7·9	18·3	31·4	38·2	38·7	39·3
17·5	2·6	4·2	7·9	..	..	..	38·8	38·8
21	2·6	4·2	8·1	18·7	32·1	38·6	39·1	38·1
24·5	2·4	4·3	8·1	19·0	31·8	39·1	39·5	38·1
28	2·6	4·4	8·2	19·1	31·2	38·7	40·1	40·1
31·5	2·6	4·4	..	19·3	31·4	37·9	40·1	39·0
35	2·7	4·4	8·2	19·2	31·5	38·1	39·7	40·6
38·5	2·7	..	..	19·6	31·8	38·8	40·2	39·9
42	2·7	4·3	8·2	19·4	30·8	37·9	39·5	38·6
45·5	2·7	4·3	8·5	19·7	30·8	38·4	39·9	40·2
49	..	4·4	8·4	20·3	30·5	37·6	39·1	41·2
52·5	..	4·2	..	20·5	30·7	38·4	39·1	39·8
56	..	..	..	20·5	31·5	38·9	40·2	40·2
59·5	..	..	..	21·2	31·9	40·4	41·2	41·6
63	..	..	..	20·6	32·2	39·9	41·7	41·2
66·5	..	..	..	..	32·6	..	43·6	
Hyst. ..	1050	1530	3060	7660	12,100	14,500	14,700	14,700
Eddy currents (div.)		·001n	·002n	·003n	·006n	·010n	·014n	·014n

TABLE XIV.—Soft Iron Armature of 156 Plates, of thickness ·0081 centim. Variation of Hysteresis with Speed; Diagram 17; Deflexions: 1 division = 775 ergs per cub. centim. per revolution.

Speed.	B = 5150	11,000	16,500	18,600
14	..	..	..	16·9
17·5	5·5			
21	5·5	13·9	16·1	16·9
24·5	5·5	13·9	16·6	
28	5·6	14·4	18·0	17·3
31·5	5·6	14·7	18·0	
35	5·6	14·9	16·0	16·8
38·5	5·6	14·8		
42	5·3	14·6	18·4	16·2
45·5	5·2	14·6	19·3	
49	5·3	14·8	18·9	16·7
52·5	5·7	14·7	19·9	
56	6·0	14·4	18·8	17·1
59·5	6·2	14·7	17·8	
63	5·9	15·2	16·9	17·6
66·5	5·8	..	17·3	17·0
Value of Eddy currents in scale divisions . . .	= ·0006n	= ·003n	= ·006n	= ·007n
Hysteresis mean value in ergs . .	4350	11,200	13,700	13,000

The foregoing experiments were carried out in the electro-technical laboratory in University College, Liverpool. My best thanks are due to Mr. E. H. MORGAN, my assistant, for his help in the construction of the apparatus and the performance of the experiments.

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