NEW ANALYTIC GEOMETRY

SMITH AND GALE

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ANALYTIC GEOMETRY

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PREFACE

A glance at the Table of Contents of the present volume will reveal the fact that the subject matter differs in many respects from that included in current textbooks on analytic geometry. The authors have recognized the great importance, in the applications, of the exponential and trigonometric functions, of "setting up" and studying functions by their graphs, of parametric equations and the locus problem, and of "fitting" curves to points determined by empirically given data. To meet this need chapters have been included covering all these topics. The discussion in Chapter VI of transcendental curves and equations is intended to be thorough, and tables are provided, whenever useful, to lighten the labor of computation. A student loses interest in a function if he is unable to calculate rapidly its numerical values. The problems of Chapter VIII provide a large variety of functions arising from applied problems, and careful "graphing" and measurement of maximum and minimum values are emphasized. The text of Chapter XII on Parametric Equations and Loci is unusually complete, and care is taken to familiarize the student with those curves which occur in applied mathematics. The study of locus problems by means of parametric equations is amply illustrated. Chapter XX presents the topic of empirical equations and contains a wide variety of problems.

The authors have not neglected to provide an adequate and thorough drill in the use of coördinates and in the employment of analytic methods. It is acknowledged that this is the primary aim of analytic geometry. The proofs will be found simple and direct. The chapters devoted to the study of the conic sections (Chapters X and XI) are brief but contain all essential characteristics of these important curves. The examples are numerous, and many are given without answers in case any useful purpose is served by so doing. The book, like the authors' "Elements of Analytic Geometry," is essentially a drill book; but at the same time all difficulties are not smoothed out, though the student is aided in making his own way. He is taught to formulate rules descriptive of methods, and to summarize the main results. The appearance in the text of various Rules is designed expressly to encourage the student in the habit of formulating precise statements and of making clear to himself each new acquisition.

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THE AUTHORS

NEW HAVEN, CONNECTICUT

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NEW ANALYTIC GEOMETRY

CHAPTER I

FORMULAS AND TABLES FOR REFERENCE

1. Occasion will arise in later chapters to make use of the following formulas and theorems proved in geometry, algebra, and trigonometry.

1. Circumference of circle $= 2\pi r$.*

2. Area of circle $= \pi r^2$.

3. Volume of prism $= Ba$.

4. Volume of pyramid $= \frac{1}{3} Ba$.

5. Volume of right circular cylinder $= \pi r^2 a$.

6. Lateral surface of right circular cylinder $= 2\pi r a$.

7. Total surface of right circular cylinder $= 2\pi r(r + a)$.

8. Volume of right circular cone $= \frac{1}{3}\pi r^2 a$.

9. Lateral surface of right circular cone $= \pi r s$.

10. Total surface of right circular cone $= \pi r(r + s)$.

11. Volume of sphere $= \frac{4}{3}\pi r^3$.

12. Surface of sphere $= 4\pi r^2$.

13. In a geometrical series,

$$l = ar^{n-1};\quad s = \frac{rl - a}{r - 1} = \frac{a(r^n - 1)}{r - 1},$$

a = first term, r = common ratio, l = nth term, s = sum of n terms.

14. $\log ab = \log a + \log b$.

15. $\log \frac{a}{b} = \log a - \log b$.

16. $\log a^n = n \log a$.

17. $\log \sqrt[n]{a} = \frac{1}{n} \log a$.

18. $\log 1 = 0$.

19. $\log_a a = 1$.

20. $\log \frac{1}{a} = -\log a$.

* In formulas 1–12, r denotes radius, a altitude, B area of base, and s slant height.

Functions of an angle in a right triangle. In any right triangle one of whose acute angles is A, the functions of A are defined as follows:

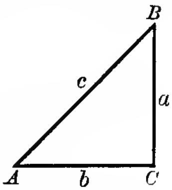
21. $\sin A = \frac{\text{opposite side}}{\text{hypotenuse}}$, $\csc A = \frac{\text{hypotenuse}}{\text{opposite side}}$,

$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}}$, $\sec A = \frac{\text{hypotenuse}}{\text{adjacent side}}$,

$\tan A = \frac{\text{opposite side}}{\text{adjacent side}}$, $\cot A = \frac{\text{adjacent side}}{\text{opposite side}}$.

From the above the theorem is easily derived:

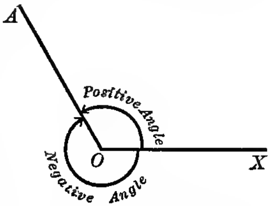
22. In a right triangle a side is equal to the product of the hypotenuse and the sine of the angle opposite to that side, or to the product of the hypotenuse and the cosine of the angle adjacent to that side.



Angles in general. In trigonometry an angle XOA is considered as generated by the line OA rotating from an initial position OX. The angle is positive when OA rotates from OX *counter-clockwise*, and negative when the direction of rotation of OA is *clockwise*.

The fixed line OX is called the *initial line*, the line OA the *terminal line*.

Measurement of angles. There are two important methods of measuring angular magnitude; that is, there are two unit angles.



Degree measure. The unit angle is $\frac{1}{360}$ of a complete revolution, and is called a *degree*.

Circular measure. The unit angle is an angle whose subtending arc is equal to the radius of that arc, and is called a *radian*.

The fundamental relation between the unit angles is given by the equation

23. $$180 \textit{ degrees} = \pi \textit{ radians} \quad (\pi = 3.14159 \cdots).$$

Or also, by solving this,

24. $$1 \text{ degree} = \frac{\pi}{180} = .0174 \cdots \text{ radians}.$$

25. $$1 \text{ radian} = \frac{180}{\pi} = 57.29 \cdots \text{ degrees}.$$

These equations enable us to change from one measurement to another. In the higher mathematics circular measure is always used, and will be adopted in this book.

The generating line is conceived of as rotating around O through as many revolutions as we choose. Hence the important result:

Any real number is the circular measure of some angle, and conversely, any angle is measured by a real number.

26. $\cot x = \dfrac{1}{\tan x}$; $\sec x = \dfrac{1}{\cos x}$; $\csc x = \dfrac{1}{\sin x}$.

27. $\tan x = \dfrac{\sin x}{\cos x}$; $\cot x = \dfrac{\cos x}{\sin x}$.

28. $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $1 + \cot^2 x = \csc^2 x$.

29. $\sin(-x) = -\sin x$; $\csc(-x) = -\csc x$;

$\cos(-x) = \cos x$; $\sec(-x) = \sec x$;

$\tan(-x) = -\tan x$; $\cot(-x) = -\cot x$.

30. $\sin(\pi - x) = \sin x$; $\sin(\pi + x) = -\sin x$;

$\cos(\pi - x) = -\cos x$; $\cos(\pi + x) = -\cos x$;

$\tan(\pi - x) = -\tan x$; $\tan(\pi + x) = \tan x$.

31. $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$; $\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$;

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$; $\cos\left(\dfrac{\pi}{2} + x\right) = -\sin x$;

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$; $\tan\left(\dfrac{\pi}{2} + x\right) = -\cot x$.

32. $\sin(2\pi - x) = \sin(-x) = -\sin x$, etc.

33. $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

34. $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

35. $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

36. $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

37. $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$.

38. $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$.

39. $\sin 2x = 2\sin x\cos x$; $\cos 2x = \cos^2 x - \sin^2 x$; $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$.

40. $\sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}}$; $\cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}}$; $\tan\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}$.

41. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$; $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

42. $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

43. $\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

44. *Theorem. Law of cosines.* In any triangle the square of a side equals the sum of the squares of the two other sides diminished by twice the product of those sides by the cosine of their included angle; that is,

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

45. *Theorem. Area of a triangle.* The area of any triangle equals one half the product of two sides by the sine of their included angle; that is,

$$area = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B.$$

2. Three-place table of common logarithms of numbers.

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	415	431	447	462
3	477	491	505	518	532	544	556	568	580	591
4	602	613	623	634	643	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	832	839
7	845	851	857	863	869	875	881	886	892	898
8	903	908	914	919	924	929	934	939	944	949
9	954	959	964	968	973	978	982	987	991	996
10	000	004	009	013	017	021	025	029	033	037
11	041	045	049	053	057	061	064	068	072	076
12	079	083	086	090	093	097	100	104	107	111
13	114	117	121	124	127	130	134	137	140	143
14	146	149	152	155	158	161	164	167	170	173
15	176	179	182	185	188	190	193	196	199	201
16	204	207	210	212	215	218	220	223	225	228
17	230	233	236	238	241	243	246	248	250	253
18	255	258	260	262	265	267	270	272	274	276
19	279	281	283	286	288	290	292	294	297	299

3. Squares and cubes; square roots and cube roots.

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

4. Natural values of trigonometric functions.

Angle in Radians	Angle in Degrees	sin	cos	tan	cot		
.000	0°	.000	1.000	.000	∞	90°	1.571
.017	1°	.017	.999	.017	57.29	89°	1.553
.035	2°	.035	.999	.035	28.64	88°	1.536
.052	3°	.052	.999	.052	19.08	87°	1.518
.070	4°	.070	.998	.070	14.30	86°	1.501
.087	5°	.087	.996	.088	11.43	85°	1.484
.174	10°	.174	.985	.176	5.67	80°	1.396
.262	15°	.259	.966	.268	3.73	75°	1.309
.349	20°	.342	.940	.364	2.75	70°	1.222
.436	25°	.423	.906	.466	2.14	65°	1.134
.524	30°	.500	.866	.577	1.73	60°	1.047
.611	35°	.574	.819	.700	1.43	55°	.960
.698	40°	.643	.766	.839	1.19	50°	.873
.785	45°	.707	.707	1.000	1.00	45°	.785
		cos	sin	cot	tan	Angle in Degrees	Angle in Radians

5. Logarithms of trigonometric functions.

Angle in Radians	Angle in Degrees	log sin	log cos	log tan	log cot		
.000	0°	....	0.000	....	....	90°	1.571
.017	1°	8.242	9.999	8.242	1.758	89°	1.553
.035	2°	8.543	9.999	8.543	1.457	88°	1.536
.052	3°	8.719	9.999	8.719	1.281	87°	1.518
.070	4°	8.844	9.999	8.845	1.155	86°	1.501
.087	5°	8.940	9.998	8.942	1.058	85°	1.484
.174	10°	9.240	9.993	9.246	0.754	80°	1.396
.262	15°	9.413	9.985	9.428	0.572	75°	1.309
.349	20°	9.534	9.973	9.561	0.439	70°	1.222
.436	25°	9.626	9.957	9.669	0.331	65°	1.134
.524	30°	9.699	9.938	9.761	0.239	60°	1.047
.611	35°	9.759	9.913	9.845	0.165	55°	0.960
.698	40°	9.808	9.884	9.924	0.086	50°	0.873
.785	45°	9.850	9.850	0.000	0.000	45°	0.785
		log cos	log sin	log cot	log tan	Angle in Degrees	Angle in Radians

6. Natural values. Special angles.

Angle in Radians	Angle in Degrees	sin	cos	tan	cot	sec	csc
0	0°	0	1	0	∞	1	∞
$\frac{1}{2}\pi$	90°	1	0	∞	0	∞	1
π	180°	0	-1	0	∞	-1	∞
$\frac{3}{2}\pi$	270°	-1	0	∞	0	∞	-1
2π	360°	0	1	0	∞	1	∞

Angle in Radians	Angle in Degrees	sin	cos	tan	cot	sec	csc
0	0°	0	1	0	∞	1	∞
$\frac{1}{6}\pi$	30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
$\frac{1}{4}\pi$	45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{1}{3}\pi$	60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
$\frac{1}{2}\pi$	90°	1	0	∞	0	∞	1

7. Rules for signs of the trigonometric functions.

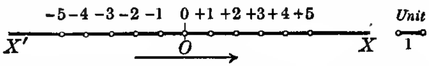
Quadrant	sin	cos	tan	cot	sec	csc
First	+	+	+	+	+	+
Second . . .	+	−	−	−	−	+
Third . . .	−	−	+	+	−	−
Fourth . . .	−	+	−	−	+	−

CHAPTER II

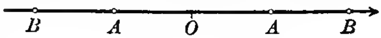
CARTESIAN COÖRDINATES

8. Directed line. Let $X'X$ be an indefinite straight line, and let a point O, which we shall call the **origin,** be chosen upon it. Let a unit of length be adopted, and assume that lengths measured from O to the right are *positive,* and to the left *negative.*

Then any *real* number, if taken as the measure of the length of a line OP, will determine a point P on the line. Conversely, to each point P on the line will correspond a real number, namely the measure of the length OP, with a positive or negative sign according as P is to the right or left of the origin.



The direction established upon $X'X$ by passing from the origin to the points corresponding to the positive numbers is called the **positive direction** on the line. A *directed line* is a straight line upon which an origin, a unit of length, and a positive direction have been assumed.



An arrowhead is usually placed upon a directed line to indicate the positive direction.

If A and B are any two points of a directed line such that

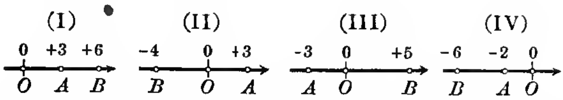
$$OA = a, \quad OB = b,$$

then the length of the segment AB is always given by $b - a$; that is, the length of AB is the difference of the numbers corresponding to B and A. This statement is evidently equivalent to the following definition:

For all positions of two points A and B on a directed line, the length AB is given by

(1) $$AB = OB - OA,$$

where O is the origin.



The above definition is illustrated in each of the four figures, as follows:

I. $AB = OB - OA = 6 - 3 = +3$; $BA = OA - OB = 3 - 6 = -3$;
II. $AB = OB - OA = -4 - 3 = -7$; $BA = OA - OB = 3 - (-4) = +7$;
III. $AB = OB - OA = +5 - (-3) = +8$; $BA = OA - OB = -3 - 5 = -8$;
IV. $AB = OB - OA = -6 - (-2) = -4$; $BA = OA - OB = -2 - (-6) = +4$.

The following properties of lengths on a directed line are obvious:

(2) $AB = - BA$.

(3) AB is positive if the direction from A to B agrees with the positive direction on the line, and negative if in the contrary direction.

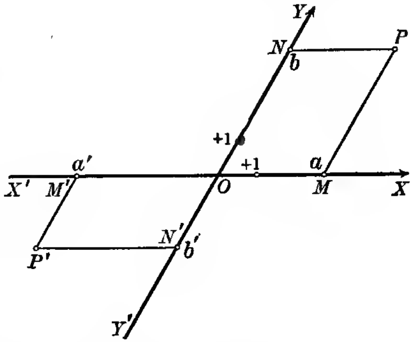
The phrase "distance between two points" should not be used if these points lie upon a directed line. Instead, we speak of the length AB, remembering that the lengths AB and BA are *not equal*, but that $AB = - BA$.

9. Cartesian* coördinates. Let $X'X$ and $Y'Y$ be two directed lines intersecting at O, and let P be any point in their plane. Draw lines through P parallel to $X'X$ and $Y'Y$ respectively. Then, if

$$OM = a, \quad ON = b,$$

* So called after René Descartes, 1596–1650, who first introduced the idea of coördinates into the study of geometry.

the numbers a, b are called the *Cartesian coördinates* of P, a the **abscissa** and b the **ordinate.** The directed lines $X'X$ and $Y'Y$ are called the **axes** of coördinates, $X'X$ the **axis of abscissas,** $Y'Y$ the **axis of ordinates,** and their intersection O the *origin.*



The coördinates a, b of P are written (a, b), and the symbol $P(a, b)$ is to be read, "The point P, whose coördinates are a and b."

Any point P in the plane determines two numbers, the coördinates of P. Conversely, given two real numbers a' and b', then a point P' in the plane may always be constructed whose coördinates are (a', b'). For lay off $OM' = a'$, $ON' = b'$, and draw lines parallel to the axes through M' and N'. These lines intersect at P' (a', b'). Hence

Every point determines a pair of real numbers, and, conversely, a pair of real numbers determines a point.

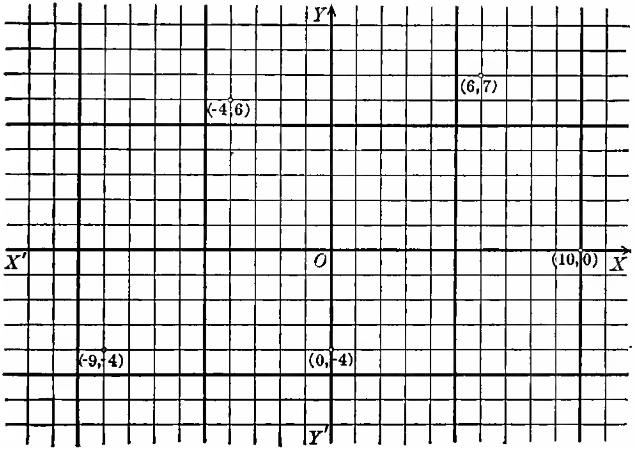
The imaginary numbers of algebra have no place in this representation, and for this reason elementary analytic geometry is concerned only with the real numbers of algebra.

10. Rectangular coördinates. A rectangular system of coördinates is determined when the axes $X'X$ and $Y'Y$ are perpendicular to each other. This is the usual case, and will be assumed unless otherwise stated.

The work of plotting points in a rectangular system is much simplified by the use of *coördinate* or *plotting paper*, constructed by ruling off the plane into equal squares, the sides beiug parallel to the axes.

In the figure several points are plotted, the unit of length being assumed equal to one division on each axis. The method is simply this:

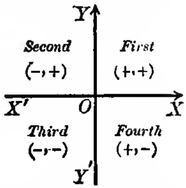
Count off from O along $X'X$ a number of divisions equal to the given abscissa, and then from the point so determined a



number of divisions up or down equal to the given ordinate, observing the

Rule *for signs:*

Abscissas are positive or negative according as they are laid off to the right or left of the origin. Ordinates are positive or negative according as they are laid off above or below the axis of x.



Rectangular axes divide the plane into four portions called **quadrants**; these are numbered as in the figure, in which the proper signs of the coördinates are also indicated.

As distinguished from rectangular coördinates, the term **oblique coördinates** is employed when the axes are not perpendicular, as in the figure of Art. 9. The rule of signs given above applies to this case also. Note, however, in plotting, that the ordinate MP is drawn *parallel* to OY.

In the following problems assume rectangular coördinates unless the contrary is stated.

PROBLEMS

1. Plot accurately the points (3, 2), (3, − 2), (− 4, 3), (6, 0), (− 5, 0), (0, 4).

2. What are the coördinates of the origin? *Ans.* (0, 0).

3. In what quadrants do the following points lie if a and b are positive numbers: $(-a, b)$? $(-a, -b)$? $(b, -a)$? (a, b)?

4. To what quadrants is a point limited if its abscissa is positive? negative? if its ordinate is positive? negative?

5. Draw the triangle whose vertices are (2, − 1), (− 2, 5), (− 8, − 4).

6. Plot the points whose oblique coördinates are as follows, when the angle between the axes is 60°: (2, − 3), (3, − 2), (4, 5), (− 6, − 7), (− 8, 0), (9, − 5), (− 6, 2).

7. Draw the quadrilateral whose vertices are (0, − 2), (4, 2), (0, 6), (− 4, 2), when the angle between the axes is 60°.

8. If a point moves parallel to the axis of x, which of its coördinates remains constant? If parallel to the axis of y?

9. Can a point move when its abscissa is zero? Where? Can it move when its ordinate is zero? Where? Can it move if both abscissa and ordinate are zero? Where will it be?

10. Where may a point be found if its abscissa is 2? if its ordinate is − 3?

11. Where do all those points lie whose abscissas and ordinates are equal?

12. Two sides of a rectangle of lengths a and b coincide with the axes of x and y respectively. What are the coördinates of the vertices of the rectangle if it lies in the first quadrant? in the second quadrant? in the third quadrant? in the fourth quadrant?

13. Construct the quadrilateral whose vertices are (− 3, 6), (− 3, 0), (3, 0), (3, 6). What kind of a quadrilateral is it? What kind of a quadrilateral is it when the axes are oblique?

14. Show that (x, y) and $(x, -y)$ are symmetrical with respect to $X'X$; (x, y) and $(-x, y)$ with respect to $Y'Y$; and (x, y) and $(-x, -y)$ with respect to the origin.

15. A line joining two points is bisected at the origin. If the coördinates of one end are $(a, -b)$, what will be the coördinates of the other end?

16. Consider the bisectors of the angles between the coördinate axes. What is the relation between the abscissa and ordinate of any point of the bisector in the first and third quadrants? second and fourth quadrants?

17. A square whose side is $2a$ has its center at the origin. What will be the coördinates of its vertices if the sides are parallel to the axes? if the diagonals coincide with the axes?

Ans. (a, a), $(a, -a)$, $(-a, -a)$, $(-a, a)$;
$(a\sqrt{2}, 0)$, $(-a\sqrt{2}, 0)$, $(0, a\sqrt{2})$, $(0, -a\sqrt{2})$.

18. An equilateral triangle whose side is a has its base on the axis of x and the opposite vertex above $X'X$. What are the vertices of the triangle if the center of the base is at the origin? if the lower left-hand vertex is at the origin?

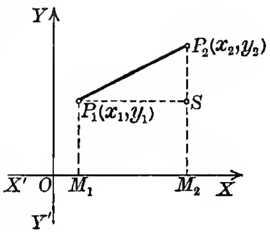
Ans. $\left(\frac{a}{2}, 0\right)$, $\left(-\frac{a}{2}, 0\right)$, $\left(0, \frac{a\sqrt{3}}{2}\right)$; $(0, 0)$, $(a, 0)$, $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$.

11. Lengths. Consider any two given points

$$P_1(x_1, y_1), \quad P_2(x_2, y_2).$$

Then in the figure $OM_1 = x_1$, $OM_2 = x_2$, $M_1P_1 = y_1$, $M_2P_2 = y_2$. We may now easily prove the important

Theorem. *The length l of the line joining two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ is given by the formula*



$$\text{(I)} \quad l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Proof. Draw lines through P_1 and P_2 parallel to the axes to form the right triangle P_1SP_2.

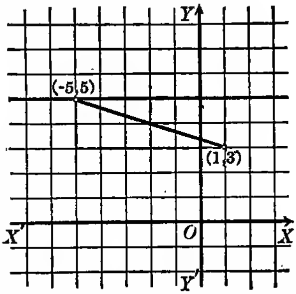
Then
$$P_1S = OM_2 - OM_1 = x_2 - x_1,$$
$$SP_2 = M_2P_2 - M_1P_1 = y_2 - y_1,$$
$$P_1P_2 = \sqrt{\overline{SP_2}^2 + \overline{P_1S}^2};$$

and hence
$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \qquad \text{Q. E. D.}$$

The same method is used in deriving (I) for *any* positions of P_1 and P_2; namely, we construct a right triangle by drawing lines parallel to the axes through P_1 and P_2. The horizontal side of this triangle is equal to the difference of the abscissas of P_1 and P_2, while the vertical side is equal to the difference of the ordinates. The required length is then the square root of the sum of the *squares* of these sides, which gives (I). A number of different figures should be drawn to make the method clear.

EXAMPLE

Find the length of the line joining the points (1, 3) and (– 5, 5).



Solution. Call (1, 3) P_1, and (–5, 5) P_2.

Then $x_1 = 1, \quad y_1 = 3,$

and $x_2 = -5, \quad y_2 = 5;$

and substituting in (I), we have

$$l = \sqrt{(1+5)^2 + (3-5)^2} = \sqrt{40} = 2\sqrt{10}.$$

It should be noticed that we are simply finding the hypotenuse of a right triangle whose sides are 6 and 2.

Remark. The fact that formula (I) is true for *all* positions of the points P_1 and P_2 is of fundamental importance. The application of this formula to any given problem is therefore simply a matter of direct substitution. In deriving such general formulas it is most convenient to draw the figure so that the points lie in the first quadrant, or, in general, so that *all the quantities assumed as known shall be positive.*

PROBLEMS

1. Find the lengths of the lines joining the following points:

(a) $(-4, -4)$ and $(1, 3)$. *Ans.* $\sqrt{74}$.

(b) $(-\sqrt{2}, \sqrt{3})$ and $(\sqrt{3}, \sqrt{2})$. *Ans.* $\sqrt{10}$.

(c) $(0, 0)$ and $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$. *Ans.* a.

(d) $(a+b, c+a)$ and $(c+a, b+c)$. *Ans.* $\sqrt{(b-c)^2 + (a-b)^2}$.

2. Find the lengths of the sides of the following triangles:

(a) (0, 6), (1, 2), (3, − 5).
(b) (1, 0), (− 1, − 5), (− 1, − 8).
(c) (a, b), (b, c), (c, d).
(d) $(a, -b)$, $(b, -c)$, $(c, -d)$.
(e) $(0, y)$, $(-x, -y)$, $(-x, 0)$.

3. Find the lengths of the sides of the triangle whose vertices are (4, 3), (2, − 2), (− 3, 5).

4. Show that the points (1, 4), (4, 1), (5, 5) are the vertices of an isosceles triangle.

5. Show that the points (2, 2), (− 2, − 2), $(2\sqrt{3}, -2\sqrt{3})$ are the vertices of an equilateral triangle.

6. Show that (3, 0), (6, 4), (− 1, 3) are the vertices of a right triangle. What is its area?

7. Prove that (− 4, − 2), (2, 0), (8, 6), (2, 4) are the vertices of a parallelogram. Also find the lengths of the diagonals.

8. Show that (11, 2), (6, − 10), (− 6, − 5), (− 1, 7) are the vertices of a square. Find its area.

9. Show that the points (1, 3), $(2, \sqrt{6})$, $(2, -\sqrt{6})$ are equidistant from the origin; that is, show that they lie on a circle with its center at the origin and its radius equal to the $\sqrt{10}$.

10. Show that the diagonals of any rectangle are equal.

11. Find the perimeter of the triangle whose vertices are (a, b), $(-a, b)$, $(-a, -b)$.

12. Find the perimeter of the polygon formed by joining the following points two by two in order: (6, 4), (4, − 3), (0, − 1), (− 5, − 4), (− 2, 1).

13. One end of a line whose length is 13 is the point (− 4, 8); the ordinate of the other end is 3. What is its abscissa? *Ans.* 8 or − 16.

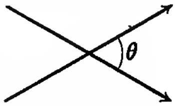
14. What equation must the coördinates of the point (x, y) satisfy if its distance from the point (7, − 2) is equal to 11?

15. What equation expresses algebraically the fact that the point (x, y) is equidistant from the points (2, 3) and (4, 5)?

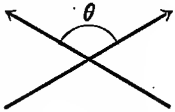
16. Find the length of the line joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ when the coördinates are oblique.

Hint. Use the law of cosines, 44, p. 4.

12. Inclination and slope. The angle between two intersecting directed lines is defined to be the angle made by their positive directions. In the figures the angle between the directed lines is the angle marked θ.

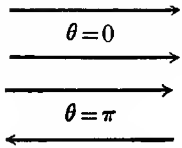


If the directed lines are parallel, then the angle between them is zero or 180°, according as the positive directions agree or do not agree.



Evidently the angle between two directed lines may have any value from 0 to 180° inclusive. Reversing the direction of either directed line changes θ to the supplement $180° - \theta$. If both directions are reversed, the angle is unchanged.

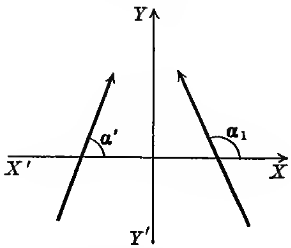
When it is desired to assign a positive direction to a line intersecting $X'X$, we shall always assume the *upward direction as positive.*



The *inclination* of a line is the angle between the axis of x and the line when the latter is given the upward direction.

This amounts to saying that the inclination is the angle *above* the x-axis and to the *right* of the given line, as in the figure.

The *slope* of a line is the tangent of its inclination.



The inclination of a line will be denoted by the Greek letter α, α_1, α_2, α', etc. ("alpha," etc.); its slope by m, m_1, m_2, m', etc., so that $m = \tan \alpha$, $m_1 = \tan \alpha_1$, etc.

The inclination may be any angle from 0 to 180° inclusive.

The slope may be any real number, since the tangent of an angle in the first two quadrants may be any number positive or negative. The slope of a line parallel to $X'X$ is of course zero, since the inclination is 0 or 180°. For a line parallel to $Y'Y$ the slope is infinite.

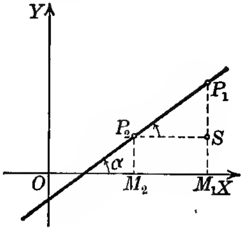
Theorem. *The slope m of the line passing through two points* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ *is given by*

(II) $$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Proof. In the figure

$$OM_1 = x_1, \quad OM_2 = x_2, \quad M_1P_1 = y_1, \quad M_2P_2 = y_2.$$

Draw P_2S parallel to OX. Then in the right triangle P_2SP_1, since angle $P_1P_2S = \alpha$, we have



(1) $$m = \tan \alpha = \frac{SP_1}{P_2S}.$$

But $SP_1 = M_1P_1 - M_1S$

$= M_1P_1 - M_2P_2 = y_1 - y_2;$

and $P_2S = M_2M_1$

$= OM_1 - OM_2 = x_1 - x_2.$

Substituting these values in (1) gives (II). Q.E.D.

The student should derive (II) when α is obtuse.*

We next derive the conditions for parallel lines and for perpendicular lines in terms of their slopes.

Theorem. *If two lines are parallel, their slopes are equal; if perpendicular, the slope of one is the negative reciprocal of the slope of the other, and conversely.*

Proof. Let α_1 and α_2 be the inclinations and m_1 and m_2 the slopes of the lines.

If the lines are parallel, $\alpha_1 = \alpha_2$. $\therefore m_1 = m_2$.

* To construct a line passing through a given point P_1 whose slope is a positive fraction $\frac{a}{b}$, we mark a point S b units to the right of P_1 and a point P_2 a units above S, and draw P_1P_2. If the slope is a negative fraction, $-\frac{a}{b}$, then plot S a units to the left of P_1.

If the lines are perpendicular, as in the figure,

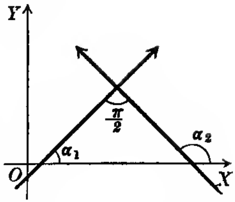
$$\alpha_2 = \frac{\pi}{2} + \alpha_1.$$

$$\therefore m_2 = \tan\alpha_2 = \tan\left(\frac{\pi}{2} + \alpha_1\right)$$

$$= -\cot\alpha_1 \quad \text{(by 31, p. 3)}$$

$$= -\frac{1}{\tan\alpha_1}. \quad \text{(By 26, p. 3)}$$

$$\therefore m_2 = -\frac{1}{m_1}.$$



Q. E. D.

The converse is proved by retracing the steps with the assumption, in the second part, that α_2 is greater than α_1.

PROBLEMS

1. Find the slope of the line joining (1, 3) and (2, 7). *Ans.* 4.

2. Find the slope of the line joining (2, 7) and (– 4, – 4). *Ans.* $\frac{11}{6}$.

3. Find the slope of the line joining $(\sqrt{3}, \sqrt{2})$ and $(-\sqrt{2}, \sqrt{3})$.
Ans. $2\sqrt{6} - 5$.

4. Find the slope of the line joining $(a + b, c + a)$, $(c + a, b + c)$.
Ans. $\frac{b - a}{c - b}$.

5. Find the slopes of the sides of the triangle whose vertices are (1, 1), (– 1, – 1), $(\sqrt{3}, -\sqrt{3})$.
Ans. 1, $\frac{1 + \sqrt{3}}{1 - \sqrt{3}}$, $\frac{1 - \sqrt{3}}{1 + \sqrt{3}}$.

6. Prove by means of slopes that (– 4, – 2), (2, 0), (8, 6), (2, 4) are the vertices of a parallelogram.

7. Prove by means of slopes that (3, 0), (6, 4), (– 1, 3) are the vertices of a right triangle.

8. Prove by means of slopes that (0, – 2), (4, 2), (0, 6), (– 4, 2) are the vertices of a rectangle, and hence, by (I), of a square.

9. Prove by means of their slopes that the diagonals of the square in Problem 8 are perpendicular.

10. Prove by means of slopes that (10, 0), (5, 5), (5, – 5), (– 5, 5) are the vertices of a trapezoid.

11. Show that the line joining (a, b) and $(c, -d)$ is parallel to the line joining $(-a, -b)$ and $(-c, d)$.

12. Show that the line joining the origin to (a, b) is perpendicular to the line joining the origin to $(-b, a)$.

13. What is the inclination of a line parallel to $Y'Y$? perpendicular to $Y'Y$?

14. What is the slope of a line parallel to $Y'Y$? perpendicular to $Y'Y$?

15. What is the inclination of the line joining (2, 2) and (– 2, – 2)?

Ans. $\frac{\pi}{4}$.

16. What is the inclination of the line joining (– 2, 0) and (– 5, 3)?

Ans. $\frac{3\pi}{4}$.

17. What is the inclination of the line joining (3, 0) and $(4, \sqrt{3})$?

Ans. $\frac{\pi}{3}$.

18. What is the inclination of the line joining (3, 0) and $(2, \sqrt{3})$?

Ans. $\frac{2\pi}{3}$.

19. What is the inclination of the line joining (0, – 4) and $(-\sqrt{3}, -5)$?

Ans. $\frac{\pi}{6}$.

20. What is the inclination of the line joining (0, 0) and $(-\sqrt{3}, 1)$?

Ans. $\frac{5\pi}{6}$.

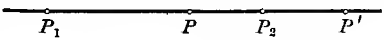
21. Prove by means of slopes that (2, 3), (1, – 3), (3, 9) lie on the same straight line.

22. Prove that the points $(a, b + c)$, $(b, c + a)$, and $(c, a + b)$ lie on the same straight line.

23. Prove that (1, 5) is on the line joining the points (0, 2) and (2, 8) and is equidistant from them.

24. Prove that the line joining (3, – 2) and (5, 1) is perpendicular to the line joining (10, 0) and (13, – 2).

13. Point of division. Let P_1 and P_2 be two fixed points on a directed line. Any third point on the line, as P or P', is said "to divide the line into two segments," and is called a **point of division.** The division is called **internal** or **external** according as the point falls within or without P_1P_2. The position of the point of division depends upon the *ratio*



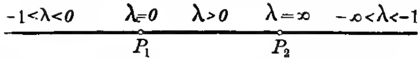
of its distances from P_1 and P_2. Since, however, the line is directed, some convention must be made as to the manner of reading these distances. We therefore adopt the rule:

If P is a point of division on a directed line passing through P_1 and P_2, then P is said to divide P_1P_2 into the segments P_1P and PP_2. The **ratio of division** is the value of the ratio * $\frac{P_1P}{PP_2}$.

We shall denote this ratio by λ (Greek letter "lambda"), that is,

$$\lambda = \frac{P_1P}{PP_2}.$$

If the division is internal, P_1P and PP_2 agree in direction and therefore in sign, and λ is therefore positive. In external division λ is negative. The *sign* of λ therefore indicates whether the point of division P is within or without the segment P_1P_2; and the *numerical value* determines whether P lies nearer P_1 or P_2. The distribution of λ is indicated in the figure.



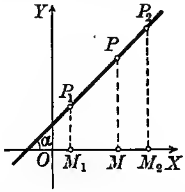
That is, λ may have any positive value between P_1 and P_2, any negative value between 0 and -1 to the left of P_1, and any negative value between -1 and $-\infty$ to the right of P_2. The value -1 for λ is excluded.

Introducing coördinates, we next prove the

Theorem. ***Point of division.*** *The coördinates (x, y) of the point of division P on the line joining $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, such that the ratio of the segments is*

$$\frac{P_1P}{PP_2} = \lambda,$$

are given by the formulas



$$\text{(III)} \qquad x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$

* To assist the memory in writing down this ratio, notice that the point of division P is written *last* in the numerator and *first* in the denominator.

Proof. Given $$\lambda = \frac{P_1P}{PP_2}.$$

Draw the ordinates M_1P_1, MP, and M_2P_2. Then, by geometry, these ordinates will intercept proportional segments on the transversals P_1P_2 and OX; that is,*

(1) $$\frac{M_1M}{MM_2} = \frac{P_1P}{PP_2}.$$

But $$M_1M = OM - OM_1 = x - x_1,$$
$$MM_2 = OM_2 - OM = x_2 - x,$$

and, by hypothesis, $$\frac{P_1P}{PP_2} = \lambda,$$

Substituting in (1), $\dfrac{x - x_1}{x_2 - x} = \lambda.$

Clearing of fractions and solving for x,

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}.$$

Similarly, by drawing the abscissas of P_1, P, and P_2 to the axis of y we may prove $$y = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$ Q.E.D.

Corollary. *Middle point.* *The coördinates (x, y) of the middle point of the line joining $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are found by taking the averages of the given abscissas and ordinates; that is,*

(IV) $$x = \tfrac{1}{2}(x_1 + x_2), \quad y = \tfrac{1}{2}(y_1 + y_2).$$

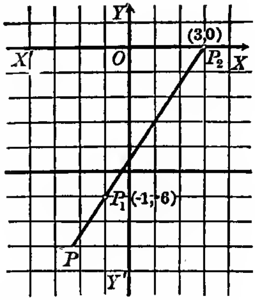
For if P is the middle point of P_1P_2, then $\lambda = \dfrac{P_1P}{PP_2} = 1$, and substituting $\lambda = 1$ in (III) gives (IV).

To apply (III), mark the point of division P, the extremities of the line to be divided P_1 and P_2, and make sure that the value of λ satisfies $\lambda = P_1P \div PP_2$.

* Care must be taken to read the segments on the transversals (since we are dealing with directed lines) so that they all have *positive* directions.

EXAMPLES

1. Find the point P dividing $P_1(-1, -6)$, $P_2(3, 0)$ in the ratio $\lambda = -\frac{1}{4}$.



Solution. By the statement,

$$\frac{P_1P}{PP_2} = -\frac{1}{4}.$$

Hence, applying (III), $x_1 = -1$, $y_1 = -6$,

$$x_2 = 3,\ y_2 = 0.$$

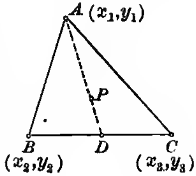
$$\therefore\ x = \frac{-1 - \frac{1}{4}\cdot 3}{1 - \frac{1}{4}} = \frac{-\frac{7}{4}}{\frac{3}{4}} = -2\tfrac{1}{3},$$

$$y = \frac{-6 - \frac{1}{4}\cdot 0}{1 - \frac{1}{4}} = \frac{-6}{\frac{3}{4}} = -8.$$

Hence P is $(-2\frac{1}{3}, -8)$. *Ans.*

The result is checked by plotting. The point P lies *outside* the segment P_1P_2 and the length of P_1P is four times that of PP_2.

2. **Center of gravity of a triangle.** Find the coördinates of the point of intersection of the medians of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3).



Solution. By plane geometry we have to find the point P on the median AD such that $AP = \frac{2}{3}AD$; that is, $AP : PD :: 2 : 1$, or $\lambda = 2$.

By the corollary, D is $[\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3)]$.

To find P, apply (III), remembering that A corresponds to (x_1, y_1) and D to (x_2, y_2).

This gives $x = \dfrac{x_1 + 2\cdot\frac{1}{2}(x_2 + x_3)}{1+2}$, $\quad y = \dfrac{y_1 + 2\cdot\frac{1}{2}(y_2 + y_3)}{1+2}$.

$$\therefore\ x = \tfrac{1}{3}(x_1 + x_2 + x_3), \qquad y = \tfrac{1}{3}(y_1 + y_2 + y_3).\ \textit{Ans.}$$

Hence the abscissa of the intersection of the medians of a triangle is the average of the abscissas of the vertices, and similarly for the ordinate.

The *symmetry* of these answers is evidence that the particular median chosen is immaterial, and the *formulas therefore prove the fact of the intersection of the medians.*

The result just found admits of extension to any polygon, and, with formulas (IV), illustrates the fact that the coördinates of centers of gravity are found by taking average values.

PROBLEMS

1. Find the coördinates of the middle point of the line joining $(4, -6)$ and $(-2, -4)$. *Ans.* $(1, -5)$.

2. Find the coördinates of the middle point of the line joining $(a+b, c+d)$ and $(a-b, d-c)$. *Ans.* (a, d).

3. Find the middle points of the sides of the triangle whose vertices are $(2, 3)$, $(4, -5)$, and $(-3, -6)$. Also find the lengths of the medians.

4. Find the coördinates of the point which divides the line joining $(-1, 4)$ and $(-5, -8)$ in the ratio $1:3$. *Ans.* $(-2, 1)$.

5. Find the coördinates of the point which divides the line joining $(-3, -5)$ and $(6, 9)$ in the ratio $2:5$. *Ans.* $(-\frac{8}{7}, -1)$.

6. Find the coördinates of the point which divides the line joining $(2, 6)$ and $(-4, 8)$ into segments whose ratio is $-\frac{4}{3}$. *Ans.* $(-22, 14)$.

7. Find the coördinates of the point which divides the line joining $(-3, -4)$ and $(5, 2)$ into segments whose ratio is $-\frac{2}{3}$. *Ans.* $(-19, -16)$.

8. Find the coördinates of the points which trisect the line joining the points $(-2, -1)$ and $(3, 2)$. *Ans.* $(-\frac{1}{3}, 0)$, $(\frac{4}{3}, 1)$.

9. Prove that the middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

10. Show that the diagonals of the parallelogram whose vertices are $(1, 2)$, $(-5, -3)$, $(7, -6)$, $(1, -11)$ bisect each other.

11. Prove that the diagonals of any parallelogram bisect each other.

12. Show that the lines joining the middle points of the opposite sides of the quadrilateral whose vertices are $(6, 8)$, $(-4, 0)$, $(-2, -6)$, $(4, -4)$ bisect each other.

13. In the quadrilateral of Problem 12 show by means of slopes that the lines joining the middle points of the adjacent sides form a parallelogram.

14. Show that in the trapezoid whose vertices are $(-8, 0)$, $(-4, -4)$, $(-4, 4)$, and $(4, -4)$ the length of the line joining the middle points of the nonparallel sides is equal to one half the sum of the lengths of the parallel sides. Also prove that it is parallel to the parallel sides.

15. In what ratio does the point $(-2, 3)$ divide the line joining the points $(-3, 5)$ and $(4, -9)$? *Ans.* $\frac{1}{6}$.

16. In what ratio does the point $(16, 3)$ divide the line joining the points $(-5, 0)$ and $(2, 1)$? *Ans.* $-\frac{3}{2}$.

17. In any triangle show that a line joining the middle points of any two sides is parallel to the third side and equal to one half of it.

18. If (2, 1), (3, 3), (6, 2) are the middle points of the sides of a triangle, what are the coördinates of the vertices of the triangle?

Ans. (– 1, 2), (5, 0), (7, 4).

19. Three vertices of a parallelogram are (1, 2), (– 5, – 3), (7, – 6). What are the coördinates of the fourth vertex?

Ans. (1, – 11), (– 11, 5), or (13, – 1).

20. The middle point of a line is (6, 4), and one end of the line is (5, 7). What are the coördinates of the other end? *Ans.* (7, 1).

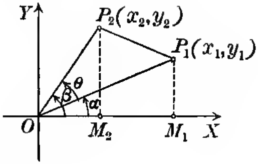
21. The vertices of a triangle are (2, 3), (4, – 5), (– 3, – 6). Find the coördinates of the point where the medians intersect (center of gravity).

14. Areas. In this section the problem of determining the area of any polygon, the coördinates of whose vertices are given, will be solved. We begin with the

Theorem. *The area of a triangle whose vertices are the origin,* $P_1(x_1, y_1)$, *and* $P_2(x_2, y_2)$ *is given by the formula*

(V) $$\textbf{Area of } \triangle\, OP_1P_2 = \tfrac{1}{2}(x_1y_2 - x_2y_1).$$

Proof. In the figure let



$$\alpha = \angle XOP_1,$$

$$\beta \text{ (Greek "beta")} = \angle XOP_2,$$

$$\theta \text{ (Greek "theta")} = \angle P_1OP_2.$$

(1) $$\therefore\ \theta = \beta - \alpha.$$

By 45, p. 4,

(2) $$\text{Area } \triangle\, OP_1P_2 = \tfrac{1}{2}\, OP_1 \cdot OP_2 \sin\theta$$

$$= \tfrac{1}{2}\, OP_1 \cdot OP_2 \sin(\beta - \alpha) \qquad \text{(by (1))}$$

(3) $$= \tfrac{1}{2}\, OP_1 \cdot OP_2 (\sin\beta\cos\alpha - \cos\beta\sin\alpha).$$

(By 34, p. 3)

But in the figure

$$\sin\beta = \frac{M_2P_2}{OP_2} = \frac{y_2}{OP_2}, \quad \cos\beta = \frac{OM_2}{OP_2} = \frac{x_2}{OP_2},$$

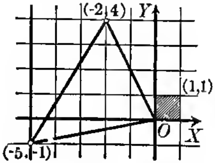
$$\sin\alpha = \frac{M_1P_1}{OP_1} = \frac{y_1}{OP_1}, \quad \cos\alpha = \frac{OM_1}{OP_1} = \frac{x_1}{OP_1}.$$

Substituting in (3) and reducing, we obtain

$$\text{Area } \triangle\, OP_1P_2 = \tfrac{1}{2}(x_1y_2 - x_2y_1). \qquad \text{Q.E.D.}$$

EXAMPLE

Find the area of the triangle whose vertices are the origin, $(-2, 4)$, and $(-5, -1)$.



Solution. Denote $(-2, 4)$ by P_1, $(-5, -1)$ by P_2. Then

$$x_1 = -2, \quad y_1 = 4, \quad x_2 = -5, \quad y_2 = -1.$$

Substituting in (V),

$$\text{Area} = \tfrac{1}{2}[-2 \cdot -1 - (-5) \cdot 4] = 11.$$

Then area $= 11$ *unit squares.*

If, however, the formula (V) is applied by denoting $(-2, 4)$ by P_2, and $(-5, -1)$ by P_1, the result will be -11.

The two figures for this example are drawn below.

The cases of *positive* and *negative* area are distinguished by the

Theorem. *Passing around the perimeter in the order of the vertices O, P_1, P_2,*

if the area is on the left, as in Fig. 1, then (V) *gives a positive result;*

if the area is on the right, as in Fig. 2, then (V) *gives a negative result.*

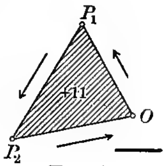


FIG. 1

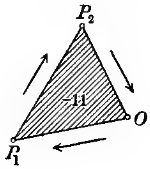


FIG. 2

Proof. In the formula

$$\text{(4)} \qquad \text{Area} \triangle OP_1P_2 = \tfrac{1}{2} OP_1 \cdot OP_2 \sin\theta$$

the angle θ is measured *from OP_1 to OP_2 within the triangle.* Hence θ is positive when the area lies to the left in passing around the perimeter O, P_1, P_2, as in Fig. 1, since θ is then measured counter-clockwise (p. 2). But in Fig. 2, θ is measured clockwise. Hence θ is *negative* and $\sin\theta$ in (4) is also negative. Q.E.D.

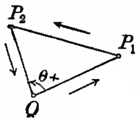


FIG. 1

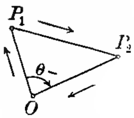


FIG. 2

We apply (V) to any triangle by regarding its area as made up of triangles with the origin as a common vertex.

Theorem. *The area of a triangle whose vertices are* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ *is given by*

(VI) $$\textbf{Area}\ \triangle P_1P_2P_3 = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

This formula gives a positive or negative result according as the area lies to the left or right in passing around the perimeter in the order $P_1P_2P_3$.

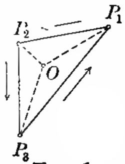


FIG. 1

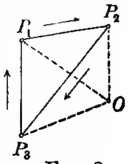


FIG. 2

Proof. Two cases must be distinguished according as the origin is within or without the triangle.

Fig. 1, *origin within the triangle.* By inspection,

(5) $$\text{Area}\ \triangle P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1,$$

since these areas all have the *same sign.*

Fig. 2, *origin without the triangle.* By inspection,

(6) $$\text{Area}\ \triangle P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1,$$

since OP_1P_2, OP_3P_1 have the *same* sign, but OP_2P_3 the opposite sign, the *algebraic sum* giving the desired area.

By (V), $$\triangle OP_1P_2 = \tfrac{1}{2}(x_1y_2 - x_2y_1),$$
$$\triangle OP_2P_3 = \tfrac{1}{2}(x_2y_3 - x_3y_2),$$
and $$\triangle OP_3P_1 = \tfrac{1}{2}(x_3y_1 - x_1y_3).$$

Substituting in (5) and (6), we have (VI).

Also in (5) the area is positive, in (6) negative. Q.E.D.

An easy way to apply (VI) is given by the following

Rule *for finding the area of a triangle.*

First step. *Write down the vertices in two columns, abscissas in one, ordinates in the other, repeating the coördinates of the first vertex.*

$$\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{matrix}$$

Second step. *Multiply each abscissa by the ordinate of the next row, and add results. This gives* $x_1y_2 + x_2y_3 + x_3y_1$.

Third step. *Multiply each ordinate by the abscissa of the next row, and add results. This gives* $y_1x_2 + y_2x_3 + y_3x_1$.

Fourth step. *Subtract the result of the third step from that of the second step, and divide by 2. This gives the required area,* namely formula (VI).

Formula (VI) may be readily memorized by remarking that the right-hand member is a determinant of simple form, namely

$$\text{Area } \triangle P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

In fact, when this determinant is expanded by the usual rule, the result, when divided by 2, is precisely (VI).

It is easy to show that the above rule applies to any polygon if the following caution be observed in the first step:

Write down the coördinates of the vertices in an order agreeing with that established by passing continuously around the perimeter, and repeat the coördinates of the first vertex.

EXAMPLE

Find the area of the quadrilateral whose vertices are (1, 6), (– 3, –4), (2, – 2), (– 1, 3).

1	6
– 1	3
– 3	– 4
2	– 2
1	6

Solution. Plotting, we have the figure from which we choose the *order* of the vertices as indicated by the arrows. Following the rule:

First step. Write down the vertices in order.

Second step. Multiply each abscissa by the ordinate of the next row, and add. This gives

$$1 \times 3 + (-1 \times -4) + (-3 \times -2) + 2 \times 6 = 25.$$

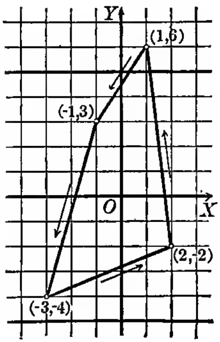
Third step. Multiply each ordinate by the abscissa of the next row, and add. This gives

$$6 \times -1 + 3 \times -3 + (-4 \times 2) + (-2 \times 1) = -25.$$

Fourth step. Subtract the result of the third step from the result of the second step, and divide by 2.

$$\therefore \text{Area} = \frac{25 + 25}{2} = 25 \text{ unit squares.} \quad \textit{Ans.}$$

The result has the positive sign, since the area is on the *left*.



PROBLEMS

1. Find the area of the triangle whose vertices are (2, 3), (1, 5), (– 1, –2). *Ans.* $\frac{11}{2}$.

2. Find the area of the triangle whose vertices are (2, 3), (4, – 5), (– 3, – 6). *Ans.* 29.

3. Find the area of the triangle whose vertices are (8, 3), (– 2, 3), (4, – 5). *Ans.* 40.

4. Find the area of the triangle whose vertices are $(a, 0)$, $(-a, 0)$, $(0, b)$. *Ans.* ab.

5. Find the area of the triangle whose vertices are (0, 0), (x_1, y_1), (x_2, y_2). *Ans.* $\frac{x_1y_2 - x_2y_1}{2}$.

6. Find the area of the triangle whose vertices are $(a, 1)$, $(0, b)$, $(c, 1)$. *Ans.* $\frac{(a-c)(b-1)}{2}$.

7. Find the area of the triangle whose vertices are (a, b), (b, a), $(c, -c)$. *Ans.* $\frac{1}{2}(a^2 - b^2)$.

8. Find the area of the triangle whose vertices are (3, 0), $(0, 3\sqrt{3})$, $(6, 3\sqrt{3})$. *Ans.* $9\sqrt{3}$.

9. Prove that the area of the triangle whose vertices are the points (2, 3), (5, 4), (– 4, 1) is zero, and hence that these points all lie on the same straight line.

10. Prove that the area of the triangle whose vertices are the points $(a, b + c)$, $(b, c + a)$, $(c, a + b)$ is zero, and hence that these points all lie on the same straight line.

11. Prove that the area of the triangle whose vertices are the points $(a, c + a)$, $(-c, 0)$, $(-a, c - a)$ is zero, and hence that these points all lie on the same straight line.

12. Find the area of the quadrilateral whose vertices are (– 2, 3), (– 3, – 4), (5, – 1), (2, 2). *Ans.* 31.

13. Find the area of the pentagon whose vertices are (1, 2), (3, – 1), (6, – 2), (2, 5), (4, 4). *Ans.* 18.

14. Find the area of the parallelogram whose vertices are (10, 5), (–2, 5), (– 5, – 3), (7, – 3). *Ans.* 96.

15. Find the area of the quadrilateral whose vertices are (0, 0), (5, 0), (9, 11), (0, 3). *Ans.* 41.

16. Find the area of the quadrilateral whose vertices are (7, 0), (11, 9), (0, 5), (0, 0). *Ans.* 59.

17. Show that the area of the triangle whose vertices are (4, 6), (2, − 4), (− 4, 2) is four times the area of the triangle formed by joining the middle points of the sides.

18. Show that the lines drawn from the vertices (3, − 8), (− 4, 6), (7, 0) to the point of intersection of the medians of the triangle divide it into three triangles of equal area.

19. Given the quadrilateral whose vertices are (0, 0), (6, 8), (10, − 2), (4, − 4); show that the area of the quadrilateral formed by joining the middle points of its adjacent sides is equal to one half the area of the given quadrilateral.

CHAPTER III

CURVE AND EQUATION

15. Locus of a point satisfying a given condition. The curve* (or group of curves) passing through all points which satisfy a given condition, and through no other points, is called the **locus** of the point satisfying that condition.

For example, in plane geometry, the following results are proved:

The perpendicular bisector of the line joining two fixed points is the locus of all points equidistant from these points.

The bisectors of the adjacent angles formed by two lines are the locus of all points equidistant from these lines.

To solve any locus problem involves two things:

1. To draw the locus by constructing a sufficient number of points satisfying the given condition and therefore lying on the locus.

2. To discuss the nature of the locus; that is, to determine properties of the curve.

Analytic geometry is peculiarly adapted to the solution of both parts of a locus problem.

16. Equation of the locus of a point satisfying a given condition. Let us take up the locus problem, making use of coördinates. We imagine the point $P(x, y)$ *moving* in such a manner that the given condition is fulfilled. Then the given condition will lead to an equation involving the variables x and y. The following example illustrates this.

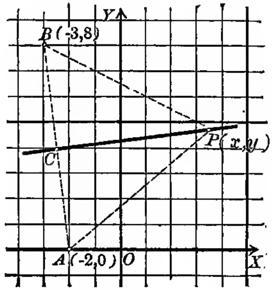
* The word "curve" will hereafter signify *any continuous line*, straight or curved.

EXAMPLE

The point $P(x, y)$ moves so that it is always equidistant from $A(-2, 0)$ and $B(-3, 8)$. Find the equation of the locus.

Solution. Let $P(x, y)$ be any point on the locus. Then by the given condition

(1) $$PA = PB.$$



But, by formula (I), p. 13,

$$PA = \sqrt{(x+2)^2 + (y-0)^2},$$

and $$PB = \sqrt{(x+3)^2 + (y-8)^2}.$$

Substituting in (1),

(2) $$\sqrt{(x+2)^2 + (y-0)^2} = \sqrt{(x+3)^2 + (y-8)^2}.$$

Squaring and reducing,

(3) $$2x - 16y + 69 = 0.$$

In the equation (3), x and y are *variables* representing the coördinates of any point on the locus; that is, of any point on the perpendicular bisector of the line AB. This equation is called the equation of the locus; that is, it is the equation of the perpendicular bisector CP. It has two important and characteristic properties:

1. The coördinates of any point on the locus may be substituted for x and y in the equation (3), and the result will be true.

For let $P_1(x_1, y_1)$ be any point on the locus. Then $P_1A = P_1B$, by definition. Hence, by formula (I), p. 13,

(4) $$\sqrt{(x_1+2)^2 + y_1^2} = \sqrt{(x_1+3)^2 + (y_1-8)^2},$$

or, squaring and reducing,

(5) $$2x_1 - 16y_1 + 69 = 0.$$

But this equation is obtained by substituting x_1 and y_1 for x and y respectively in (3). Therefore x_1 and y_1 satisfy (3).

2. Conversely, every point whose coördinates satisfy (3) will lie upon the locus.

For if $P_1(x_1, y_1)$ is a point whose coördinates satisfy (3), then (5) is true, and hence also (4) holds. Q.E.D.

In particular, the coördinates of the middle point C of A and B, namely, $x = -2\frac{1}{2}$, $y = 4$ (IV, p. 21), satisfy (3), since

$$2(-2\tfrac{1}{2}) - 16 \times 4 + 69 = 0.$$

This discussion leads to the definition:

The **equation of the locus** of a point satisfying a given condition is an equation in the variables x and y representing coördinates such that (1) the coördinates of every point on the locus will satisfy the equation; and (2) conversely, every point whose coördinates satisfy the equation will lie upon the locus.

This definition shows that the equation of the locus must be tested *in two ways* after derivation, as illustrated in the example of this section. The student should supply this test in the examples and problems of Art. 17.

From the above definition follows at once the

Corollary. *A point lies upon a curve when and only when its coördinates satisfy the equation of the curve.*

17. First fundamental problem. *To find the equation of a curve which is defined as the locus of a point satisfying a given condition.*

The following rule will suffice for the solution of this problem in many cases:

Rule. ***First step.*** *Assume that $P(x, y)$ is any point satisfying the given condition, and is therefore on the curve.*

Second step. *Write down the given condition.*

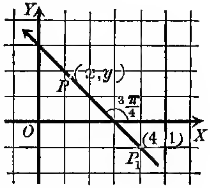
Third step. *Express the given condition in coördinates and simplify the result. The final equation, containing x, y, and the given constants of the problem, will be the required equation.*

EXAMPLES

1. Find the equation of the straight line passing through $P_1(4, -1)$ and having an inclination of $\frac{3\pi}{4}$.

Solution. *First step.* Assume $P(x, y)$ any point on the line.

Second step. The given condition, since the inclination α is $\frac{3\pi}{4}$, may be written



(1) $$\text{slope of } P_1P = \tan\alpha = -1.$$

Third step. From (II), p. 17,

(2) $$\text{slope of } P_1P = \tan\alpha = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y+1}{x-4}.$$

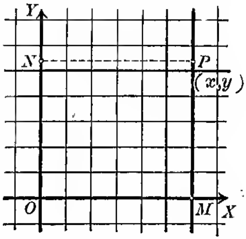
[By substituting (x, y) for (x_1, y_1), and $(4, -1)$ for (x_2, y_2).]

Therefore, from (1), $$\frac{y+1}{x-4} = -1, \text{ or}$$

(3) $$x + y - 3 = 0. \quad \textit{Ans.}$$

2. Find the equation of a straight line parallel to the axis of y and at a distance of 6 units to the right.

Solution. *First step.* Assume that $P(x, y)$ is any point on the line, and draw NP perpendicular to OY.



Second step. The given condition may be written

(4) $$NP = 6.$$

Third step. Since $NP = OM = x$, (4) becomes

(5) $$x = 6. \quad \textit{Ans.}$$

3. Find the equation of the locus of a point whose distance from $(-1, 2)$ is always equal to 4.

Solution. *First step.* Assume that $P(x, y)$ is any point on the locus.

Second step. Denoting $(-1, 2)$ by C, the given condition is

(6) $$PC = 4.$$

Third step. By formula (I), p. 13,

$$PC = \sqrt{(x+1)^2 + (y-2)^2}.$$

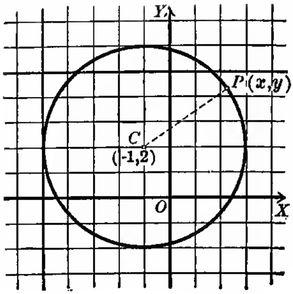
Substituting in (6),

$$\sqrt{(x+1)^2 + (y-2)^2} = 4.$$

Squaring and reducing,

(7) $$x^2 + y^2 + 2x - 4y - 11 = 0.$$

This is the required equation, namely, the equation of the circle whose center is $(-1, 2)$ and radius equal to 4.



PROBLEMS

1. Find the equation of a line parallel to OY and

(a) at a distance of 4 units to the right.
(b) at a distance of 7 units to the left.
(c) at a distance of 2 units to the right of (3, 2).
(d) at a distance of 5 units to the left of $(2, -2)$.

2. Find the equation of a line parallel to OX and

(a) at a distance of 3 units above OX.
(b) at a distance of 6 units below OX.
(c) at a distance of 7 units above $(-2, -3)$.
(d) at a distance of 5 units below $(4, -2)$.

3. What is the equation of XX'? of YY'?

4. Find the equation of a line parallel to the line $x = 4$ and 3 units to the right of it; 8 units to the left of it.

5. Find the equation of a line parallel to the line $y = -2$ and 4 units below it; 5 units above it.

6. What is the equation of the locus of a point which moves always at a distance of 2 units from the axis of x? from the axis of y? from the line $x = -5$? from the line $y = 4$?

7. What is the equation of the locus of a point which moves so as to be equidistant from the lines $x = 5$ and $x = 9$? equidistant from $y = 3$ and $y = -7$?

8. What are the equations of the sides of the rectangle whose vertices are (5, 2), (5, 5), (−2, 2), (−2, 5)?

In Problems 9 and 10, P_1 is a given point on the required line, m is the slope of the line, and α its inclination.

9. What is the equation of a line if

(a) P_1 is (0, 3) and $m = -3$? *Ans.* $3x + y - 3 = 0$.
(b) P_1 is (− 4, − 2) and $m = \frac{1}{3}$? *Ans.* $x - 3y - 2 = 0$.
(c) P_1 is (− 2, 3) and $m = \frac{\sqrt{2}}{2}$? *Ans.* $\sqrt{2}x - 2y + 6 + 2\sqrt{2} = 0$.
(d) P_1 is (0, 5) and $m = \frac{\sqrt{3}}{2}$? *Ans.* $\sqrt{3}x - 2y + 10 = 0$.
(e) P_1 is (0, 0) and $m = -\frac{2}{3}$? *Ans.* $2x + 3y = 0$.
(f) P_1 is (a, b) and $m = 0$? *Ans.* $y = b$.
(g) P_1 is (− a, b) and $m = \infty$? *Ans.* $x = -a$.

10. What is the equation of a line if

(a) P_1 is (2, 3) and $\alpha = 45°$? *Ans.* $x - y + 1 = 0$.
(b) P_1 is (− 1, 2) and $\alpha = 45°$? *Ans.* $x - y + 3 = 0$.
(c) P_1 is (− a, − b) and $\alpha = 45°$? *Ans.* $x - y = b - a$.
(d) P_1 is (5, 2) and $\alpha = 60°$? *Ans.* $\sqrt{3}x - y + 2 - 5\sqrt{3} = 0$.
(e) P_1 is (0, − 7) and $\alpha = 60°$? *Ans.* $\sqrt{3}x - y - 7 = 0$.
(f) P_1 is (− 4, 5) and $\alpha = 0°$? *Ans.* $y = 5$.
(g) P_1 is (2, − 3) and $\alpha = 90°$? *Ans.* $x = 2$.
(h) P_1 is (3, $-3\sqrt{3}$) and $\alpha = 120°$? *Ans.* $\sqrt{3}x + y = 0$.
(i) P_1 is (0, 3) and $\alpha = 150°$? *Ans.* $\sqrt{3}x + 3y - 9 = 0$.
(j) P_1 is (a, b) and $\alpha = 135°$? *Ans.* $x + y = a + b$.

11. Find the equation of the straight line which passes through the points

(a) (2, 3) and (− 4, − 5). *Ans.* $4x - 3y + 1 = 0$.

Hint. Find the slope by (II), p. 17, and then proceed as in Problem 9.

(b) (2, − 5) and (− 1, 9). *Ans.* $14x + 3y - 13 = 0$.
(c) (− 1, 6) and (6, − 2). *Ans.* $8x + 7y - 34 = 0$.
(d) (0, − 3) and (4, 0). *Ans.* $3x - 4y - 12 = 0$.
(e) (8, − 4) and (− 1, 2). *Ans.* $2x + 3y - 4 = 0$.

12. Find the equation of the circle with

(a) center at (3, 2) and radius = 4. *Ans.* $x^2 + y^2 - 6x - 4y - 3 = 0$.
(b) center at (12, − 5) and $r = 13$. *Ans.* $x^2 + y^2 - 24x + 10y = 0$.
(c) center at (0, 0) and radius = r. *Ans.* $x^2 + y^2 = r^2$.
(d) center at (0, 0) and $r = 5$. *Ans.* $x^2 + y^2 = 25$.
(e) center at ($3a$, $4a$) and $r = 5a$. *Ans.* $x^2 + y^2 - 2a(3x + 4y) = 0$.
(f) center at ($b + c$, $b - c$) and $r = c$.
Ans. $x^2 + y^2 - 2(b + c)x - 2(b - c)y + 2b^2 + c^2 = 0$.

13. Find the equation of a circle whose center is (5, – 4) and whose circumference passes through the point (– 2, 3).

14. Find the equation of a circle having the line joining (3, – 5) and (– 2, 2) as a diameter.

15. Find the equation of a circle touching each axis at a distance 6 units from the origin.

16. Find the equation of a circle whose center is the middle point of the line joining (– 6, 8) to the origin and whose circumference passes through the point (2, 3).

17. A point moves so that its distances from the two fixed points (2, – 3) and (– 1, 4) are equal. Find the equation of the locus.

Ans. $3x - 7y + 2 = 0$.

18. Find the equation of the perpendicular bisector of the line joining

(a) (2, 1), (– 3, – 3). *Ans.* $10x + 8y + 13 = 0$.

(b) (3, 1), (2, 4). *Ans.* $x - 3y + 5 = 0$.

(c) (– 1, – 1), (3, 7). *Ans.* $x + 2y - 7 = 0$.

(d) (0, 4), (3, 0). *Ans.* $6x - 8y + 7 = 0$.

(e) (x_1, y_1), (x_2, y_2).

Ans. $2(x_1 - x_2)x + 2(y_1 - y_2)y + x_2^2 - x_1^2 + y_2^2 - y_1^2 = 0$.

19. Show that in Problem 18 the coördinates of the middle point of the line joining the given points satisfy the equation of the perpendicular bisector.

20. Find the equations of the perpendicular bisectors of the sides of the triangle (4, 8), (10, 0), (6, 2). Show that they meet in the point (11, 7).

18. Locus of an equation. The preceding sections have illustrated the fact that a locus problem in analytic geometry leads at once to an equation in the variables x and y. This equation having been found or being given, the complete solution of the locus problem requires two things, as already noted in Art. 15 of this chapter, namely:

1. To draw the locus by plotting a sufficient number of points whose coördinates satisfy the given equation, and through which the locus therefore passes.

2. To discuss the nature of the locus; that is, to determine properties of the curve.

These two problems are respectively called:

1. Plotting the locus of an equation (second fundamental problem).

2. Discussing an equation (third fundamental problem).

For the present, then, we concentrate our attention upon some given equation in the variables x and y (one or both) and start out with the definition:

The **locus of an equation** in two variables representing coördinates is the curve or group of curves passing through all points whose coördinates satisfy that equation,* and through such points only.

From this definition the truth of the following theorem is at once apparent:

Theorem I. *If the form of the given equation be changed in any way (for example, by transposition, by multiplication by a constant, etc.), the locus is entirely unaffected.*

We now take up in order the solution of the second and third fundamental problems.

19. Second fundamental problem.

Rule *to plot the locus of a given equation.*

First step. *Solve the given equation for one of the variables in terms of the other.*†

* An equation in the variables x and y is not necessarily satisfied by the coördinates of any points. For coördinates are *real* numbers, and the form of the equation may be such that it is satisfied by no *real* values of x and y. For example, the equation

$$x^2+y^2+1=0$$

is of this sort, since, when x and y are real numbers, x^2 and y^2 are necessarily positive (or zero), and consequently x^2+y^2+1 is always a positive number greater than or equal to 1, and therefore *not* equal to zero. Such an equation therefore has *no locus*. The expression "the locus of the equation is imaginary" is also used.

An equation may be satisfied by the coördinates of a *finite* number of points only. For example, $x^2+y^2=0$ is satisfied by $x=0$, $y=0$, but by no other real values. In this case the group of points, one or more, whose coördinates satisfy the equation, is called the locus of the equation.

† The form of the given equation will often be such that solving for one variable is simpler than solving for the other. *Always choose the simpler solution.*

Second step. *By this formula compute the values of the variable for which the equation has been solved by assuming real values for the other variable.*

Third step. *Plot the points corresponding to the values so determined.*

Fourth step. *If the points are numerous enough to suggest the general shape of the locus, draw a smooth curve through the points.*

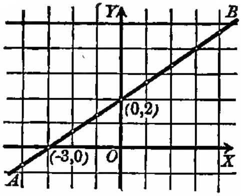
Since there is no limit to the number of points which may be computed in this way, it is evident that the locus may be drawn as *accurately as may be desired* by simply plotting a sufficiently large number of points.

Several examples will now be worked out. The arrangement of the work should be carefully noted.

EXAMPLES

1. Draw the locus of the equation

$$2x - 3y + 6 = 0.$$



Solution. *First step.* Solving for y,

$$y = \tfrac{2}{3}x + 2.$$

Second step. Assume values for x and compute y, arranging results in the form of the accompanying table:

x	y	x	y
0	2	0	2
1	$2\frac{2}{3}$	-1	$1\frac{1}{3}$
2	$3\frac{1}{3}$	-2	$\frac{2}{3}$
3	4	-3	0
4	$4\frac{2}{3}$	-4	$-\frac{2}{3}$
etc.	etc.	etc.	etc.

Thus, if

$$x = 1,\ y = \tfrac{2}{3}\cdot 1 + 2 = 2\tfrac{2}{3},$$
$$x = 2,\ y = \tfrac{2}{3}\cdot 2 + 2 = 3\tfrac{1}{3},$$

etc.

Third step. Plot the points found.

Fourth step. Draw a smooth curve through these points.

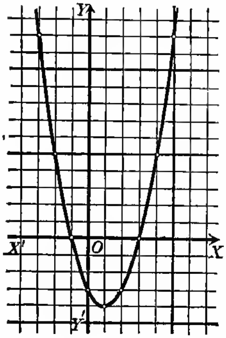
2 Plot the locus of the equation

$$y = x^2 - 2x - 3.$$

Solution. *First step.* The equation as given is solved for y.

Second step. Computing y by assuming values of x, we find the table of values below:

x	y	x	y
0	-3	0	-3
1	-4	-1	0
2	-3	-2	5
3	0	-3	12
4	5	-4	21
5	12	etc.	etc.
6	21		
etc.	etc.		



Third step. Plot the points.

Fourth step. Draw a smooth curve through these points. This gives the curve of the figure.

3. Plot the locus of the equation

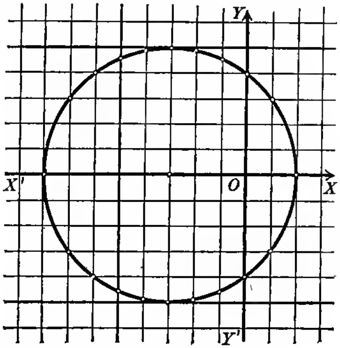
$$x^2 + y^2 + 6x - 16 = 0.$$

Solution. *First step.* Solving for y,

$$y = \pm\sqrt{16 - 6x - x^2}.$$

Second step. Compute y by assuming values of x. For this purpose the table of Art. 3 will be found convenient.

x	y	x	y
0	± 4	0	± 4
1	± 3	-1	± 4.6
2	0	-2	± 4.9
3	imag.	-3	± 5
4	"	-4	± 4.9
5	"	-5	± 4.6
6	"	-6	± 4
7	"	-7	± 3
		-8	0
		-9	imag.



For example, if $x = -1$, $y = \pm\sqrt{16 + 6 - 1} = \sqrt{21} = \pm 4.6$,

if $x = 3$, $y = \pm\sqrt{16 - 18 - 9} = \pm\sqrt{-11}$,

an imaginary number.

Third step. Plot the corresponding points.

Fourth step. Draw a smooth curve through these points.

The student will doubtless remark that the locus of Example 1, p. 38, *appears* to be a straight line, and also that the locus of Example 3, p. 39, *appears* to be a circle. This is, in fact, the case. But the *proof* must be reserved for later sections.

PROBLEMS

1. Plot the locus of each of the following equations:

(a) $x + 2y = 0$.
(b) $x + 2y = 3$.
(c) $3x - y + 5 = 0$.
(d) $y = 4x^2$.
(e) $x^2 + 4y = 0$.
(f) $y = x^2 - 3$.
(g) $x^2 + 4y - 5 = 0$.
(h) $y = x^2 + x + 1$.
(i) $x = y^2 + 2y - 3$.
(j) $4x = y^3$.
(k) $4x = y^3 - 1$.
(l) $y = x^3 - 1$.
(m) $y = x^3 - x$.
(n) $y = x^3 - x^2 - 5$.
(o) $x^2 + y^2 = 4$.
(p) $x^2 + y^2 = 9$.
(q) $x^2 + y^2 = 25$.
(r) $x^2 + y^2 + 9x = 0$.
(s) $x^2 + y^2 + 4y = 0$.
(t) $x^2 + y^2 - 6x - 16 = 0$.
(u) $x^2 + y^2 - 6y - 16 = 0$.
(v) $4y = x^4 - 8$.
(w) $4x = y^4 + 8$.
(x) $4y^2 = x^3 - 1$.

The following problem illustrates the

Theorem. *If an equation can be put in the form of a product of variable factors equal to zero, the locus is found by setting each factor equal to zero and plotting each equation separately.*

2. Draw the locus of $4x^2 - 9y^2 = 0$.

Solution. Factoring,

(1) $$(2x - 3y)(2x + 3y) = 0.$$

Then, by the theorem, *the locus consists of the straight lines* (p. 59)

(2) $$2x - 3y = 0, \text{ and}$$

(3) $$2x + 3y = 0.$$

Proof. 1. *The coördinates of any point* (x_1, y_1) *which satisfy* (1) *will satisfy either* (2) *or* (3).

For if (x_1, y_1) satisfies (1),

(4) $$(2x_1 - 3y_1)(2x_1 + 3y_1) = 0.$$

This product can vanish only when one of the factors is zero. Hence either
$$2x_1 - 3y_1 = 0,$$
and therefore (x_1, y_1) satisfies (2);
or
$$2x_1 + 3y_1 = 0,$$
and therefore (x_1, y_1) satisfies (3).

2. *A point* (x_1, y_1) *on either of the lines defined by* (2) *and* (3) *will also lie on the locus of* (1).

For if (x_1, y_1) is on the line $2x - 3y = 0$, then (Corollary, p. 32)

$$2x_1 - 3y_1 = 0. \tag{5}$$

Hence the product $(2x_1 - 3y_1)(2x_1 + 3y_1)$ also vanishes, since by (5) the first factor is zero, and therefore (x_1, y_1) satisfies (1).

Therefore every point on the locus of (1) is also on the locus of (2) and (3), and conversely. This proves the theorem for this example. Q. E. D.

3. Show that the locus of each of the following equations is a pair of straight lines, and plot the lines:

(a) $xy = 0$.
(b) $x^2 = 9y^2$.
(c) $x^2 - y^2 = 0$.
(d) $y^2 - 6y = 7$.
(e) $xy - 2x = 0$.
(f) $9x^2 - y^2 = 0$.
(g) $x^2 - 3xy = 0$.
(h) $y^2 + 4xy = 0$.
(i) $x^2 - 4x - 5 = 0$.
(j) $xy - 2x^2 - 3x = 0$.
(k) $y^2 - 5xy + 6y = 0$.
(l) $x^2 - y^2 + x + y = 0$.
(m) $x^2 - 3xy - 4y^2 = 0$.
(n) $x^2 - xy + 5x - 5y = 0$.
(o) $x^2 - 4y^2 + 5x + 10y = 0$.
(p) $x^2 + 2xy + y^2 + x + y = 0$.
(q) $x^2 + 3xy + 2y^2 + x + y = 0$.
(r) $x^2 - 2xy + y^2 + 6x - 6y = 0$.
(s) $3x^2 + xy - 2y^2 + 6x - 4y = 0$.
(t) $3x^2 - 2xy - y^2 + 5x - 5y = 0$.
(u) $x^2 - 4xy - 5y^2 + 2x - 10y = 0$.
(v) $x^2 + 4xy + 4y^2 + 5x + 10y + 6 = 0$.

4. Show that the locus of $Ax^2 + Bx + C = 0$ is a pair of parallel lines, a single line, or that there is no locus according as $\Delta = B^2 - 4AC$ is positive, zero, or negative.

5. Show that the locus of $Ax^2 + Bxy + Cy^2 = 0$ is a pair of intersecting lines, a single line, or a point according as $\Delta = B^2 - 4AC$ is positive, zero, or negative.

6. Show that the following equations have *no locus* (footnote, p. 37):

(a) $x^2 + y^2 + 1 = 0$.
(b) $2x^2 + 3y^2 = -8$.
(c) $x^2 + 4 = 0$.
(d) $x^4 + y^2 + 8 = 0$.
(e) $(x + 1)^2 + y^2 + 4 = 0$.
(f) $x^2 + y^2 + 2x + 2y + 3 = 0$.
(g) $4x^2 + y^2 + 8x + 5 = 0$.
(h) $y^4 + 2x^2 + 4 = 0$.
(i) $9x^2 + 4y^2 + 18x + 8y + 15 = 0$.

Hint. Write each equation in the form of a sum of squares, and reason as in the footnote on page 37.

20. Third fundamental problem. Discussion of an equation. The method explained of solving the second fundamental problem gives no knowledge of the required curve except that it passes through all the points whose coördinates are determined as satisfying the given equation. Joining these points gives a curve more or less like the exact locus. Serious errors may be made in this way, however, since *the nature of the curve between any two successive points plotted is not determined.* This objection is somewhat obviated by determining *before plotting* certain properties of the locus by a discussion of the given equation now to be explained.

The nature and properties of a locus depend upon the form of its equation, and hence the steps of any discussion must depend upon the particular problem. In every case, however, certain questions should be answered. These questions will now be presented.

1. *Is the curve symmetrical with respect to either axis of coördinates or with respect to the origin?*

To answer this question we may proceed as in the following example:

EXAMPLE

Discuss the symmetry of the locus of

(1) $$x^2 + 4y^2 = 16.$$

Solution. The equation contains no odd powers of x or y; hence it may be written in any one of the forms

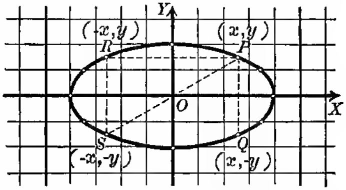
(2) $(x)^2 + 4(-y)^2 = 16$, replacing (x, y) by $(x, -y)$;

(3) $(-x)^2 + 4(y)^2 = 16$, replacing (x, y) by $(-x, y)$;

(4) $(-x)^2 + 4(-y)^2 = 16$, replacing (x, y) by $(-x, -y)$.

The transformation of (1) into (2) corresponds in the figure to replacing each point $P(x, y)$ on the curve by the point

$Q(x, -y)$. But the points P and Q are symmetrical with respect to XX', and (1) and (2) have the same locus (Theorem I, p. 37). Hence the locus of (1) is unchanged if each point is changed to a second point symmetrical to the first with respect to XX'. Therefore *the locus is symmetrical with respect to the axis of x.*



Similarly, from (3), *the locus is symmetrical with respect to the axis of y*, and from (4) *the locus is symmetrical with respect to the origin*, for the points $P(x, y)$ and $S(-x, -y)$ are symmetrical with respect to the origin, since $OP = OS$.

In plotting the equation we take advantage of our knowledge of the symmetry of the curve by limiting the calculation to points in the first quadrant, as in the table. We plot these points, mark off the points symmetrical to them with respect to the axes and the origin, and then draw the curve.

x	y
4	0
3.4	1
2.7	$1\frac{1}{2}$
0	2

The locus is called an **ellipse**.

The facts brought out in the example are stated in

Theorem II. *Symmetry.* *If the locus of an equation is unaffected by replacing y by $-y$ throughout its equation, the locus is symmetrical with respect to the axis of x.*

If the locus is unaffected by changing x to $-x$ throughout its equation, the locus is symmetrical with respect to the axis of y.

If the locus is unaffected by changing both x and y to $-x$ and $-y$ throughout its equation, the locus is symmetrical with respect to the origin.

These theorems may be made to assume a somewhat different form if the equation is *algebraic* in x and y. The locus of an algebraic equation in the variables x and y is called an **algebraic curve.** Then from Theorem II follows

Theorem III. ***Symmetry of an algebraic curve.*** *If no odd powers of y occur in an equation, the locus is symmetrical with respect to XX'; if no odd powers of x occur, the locus is symmetrical with respect to YY'. If every term is of even* degree, or every term of odd degree, the locus is symmetrical with respect to the origin.*

The second question arises from the following considerations:

Coördinates are *real* numbers. Hence all values of x which give imaginary values of y must be excluded in the calculation. Similarly, all values of y which lead to imaginary values of x must be excluded. The second question is, then:

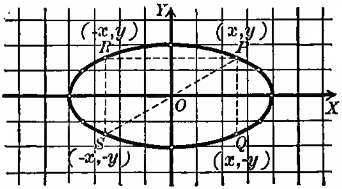
2. *What values, if any, of either coördinate will give imaginary values of the other coördinate?*

The following examples illustrate the method:

EXAMPLES

1. What values of x and y, if any, must be excluded in determining points on the locus of

(1) $$x^2 + 4y^2 = 16?$$



Solution. Solving for x in terms of y, and also for y in terms of x,

(2) $$x = \pm 2\sqrt{4 - y^2},$$

(3) $$y = \pm \tfrac{1}{2}\sqrt{16 - x^2}.$$

From the radical in (2) we see that all values of y *numerically* greater than 2 will make $4 - y^2$ negative, and hence make x imaginary. Hence all values of y greater than 2 or less than -2 must be excluded.

Similarly, from the radical in (3), it is clear that values of x greater than 4 or less than -4 must be excluded.

* A *constant term* is to be regarded as of zero (*even*) degree, as 16 in (1), p. 42.

Therefore in determining points on the locus, we need assume for y values only between 0 and 2, as on page 43, or values of x between 0 and 4 inclusive.

A further conclusion is this: The curve lies *entirely within* the rectangle bounded by the four lines

$$x = 4, \quad x = -4, \quad y = 2, \quad y = -2,$$

and is therefore a *closed curve*.

2. What values, if any, of the coördinates are to be excluded in determining the locus of

(4) $$y^2 - 4x + 15 = 0?$$

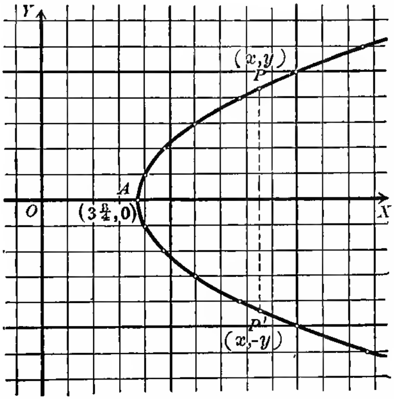
Solution. Solving for x in terms of y, and also for y in terms of x,

(5) $$x = \tfrac{1}{4}(15 + y^2),$$

(6) $$y = \pm\sqrt{4x - 15}.$$

From (5) any value of y will give a real value of x. Hence no values of y are excluded.

x	y
$3\frac{3}{4}$	0
4	± 1
$4\frac{3}{4}$	± 2
6	± 3
$7\frac{3}{4}$	± 4
10	± 5
$12\frac{3}{4}$	± 6
etc.	etc.



From the radical in (6) all values of x for which $4x - 15$ is negative must be excluded; that is, all values of x less than $3\frac{3}{4}$.

The locus therefore lies entirely to the right of the line $x = 3\frac{3}{4}$. Moreover, since no values of y are excluded, the locus extends to infinity, y increasing as x increases.

The locus is, by Theorem III, symmetrical with respect to the x-axis, and is called a **parabola**.

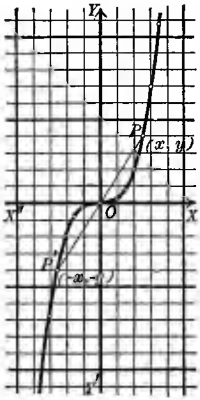
3. Determine what values of x and y, if any, must be excluded in determining the locus of

(7) $$4y = x^3.$$

Solution. Solving for x in terms of y, and also for y in terms of x

(8) $$x = \sqrt[3]{4y},$$

(9) $$y = \tfrac{1}{4}x^3.$$



From these equations it appears that no values of either coördinate need be excluded.

The locus is, by Theorem III, symmetrical with respect to the origin. The coördinates increase together; the curve extends to infinity and is called a **cubical parabola**.

The method illustrated in the examples is summed up in the

Rule *to determine all* ***values*** *of x and y which must be* ***excluded.***

Solve the equation for x in terms of y, and from this result determine all values of y for which the computed value of x would be imaginary. These values of y must be excluded.

Solve the equation for y in terms of x, and from this result determine all values of x for which the computed value of y would be imaginary. These values of x must be excluded.

In determining excluded values of x and y we obtain also an answer to the question:

3. *Is the curve a closed curve, or does it extend to infinity?*

The points of intersection of the curve with the coördinate axes should be found.

The **intercepts** of a curve on the axis of x are the abscissas of the points of intersection of the curve and XX'.

The intercepts of a curve on the axis of y are the ordinates of the points of intersection of the curve and YY'.

Rule *to find the intercepts.*

Substitute $y = 0$ and solve for real values of x. This gives the intercepts on the axis of x.

Substitute $x = 0$ and solve for real values of y. This gives the intercepts on the axis of y.

The proof of the rule follows at once from the definitions.

The rule just given explains how to answer the question:

4. *What are the intercepts of the locus?*

In particular, the locus may pass through the origin, in which case one intercept on each axis will be zero. In this case the coördinates (0, 0) must satisfy the equation. When the equation is algebraic we have

Theorem IV. *The locus of an algebraic equation passes through the origin when there is no constant term in the equation.*

The proof is immediate.

21. Directions for discussing an equation. Given an equation, the following questions should be answered in order before plotting the locus.

1. *Is the origin on the locus?*
2. *What are the intercepts?*
3. *Is the locus symmetrical with respect to the axes or the origin?*
4. *What values of x and y must be excluded?*
5. *Is the curve closed, or does it pass off indefinitely far?*

Answering these questions constitutes what is called a **general discussion** of the given equation. The successive results should be immediately transferred to the figure. Thus when the intercepts have been determined, *mark them off on the axes.* Indicate which axes are axes of symmetry. The excluded values of x and y will determine lines parallel to the axes which the locus *will not cross.* Draw these lines.

EXAMPLE

Give a general discussion of the equation

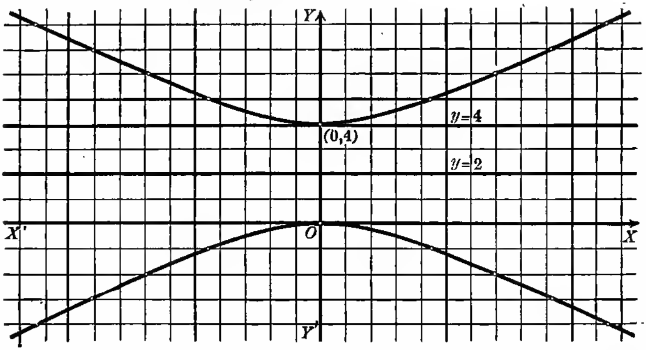
(1) $$x^2 - 4y^2 + 16y = 0.$$

Draw the locus.

Solution. 1. Since the equation contains no constant term, the origin is on the curve.

2. Putting $y = 0$, we find $x = 0$, the intercept on the axis of x. Putting $x = 0$, we find $y = 0$ and 4, the intercepts on the axis of y.

Lay off the intercepts on the axes.



3. The equation contains no odd powers of x; hence the locus is symmetrical with respect to YY'.

4. Solving for x,

(2) $$x = \pm 2\sqrt{y^2 - 4y}.$$

All values of y must be excluded which make the expression beneath the radical sign negative. Now the roots of $y^2 - 4y = 0$ are $y = 0$ and $y = 4$. For any value of y *between* these roots, $y^2 - 4y$ is negative. For example, $y = 2$ gives $4 - 8 = -4$. Hence all values of y between 0 and 4 must be excluded.

Draw the lines $y = 0$ and $y = 4$. The locus does not come between these lines.

Solving for y,

(3) $$y = 2 \pm \tfrac{1}{2}\sqrt{x^2 + 16}.$$

Hence no value of x is excluded, since $x^2 + 16$ is positive for all values of x.

5. From (3), y increases as x increases, and the curve extends out indefinitely far from both axes.

Plotting the locus, using (2), the curve is found to be as in the figure. The curve is a **hyperbola**.

Sign of a quadratic. In the preceding example it became necessary to determine for what values of y the quadratic expression $y^2 - 4y$ in (2) was positive.

The fact made use of is this:

If the sign of a quadratic expression is negative (or positive) for any *one* value of the unknown taken between the roots, it is also negative (or positive) for *every* value of the unknown between the roots.

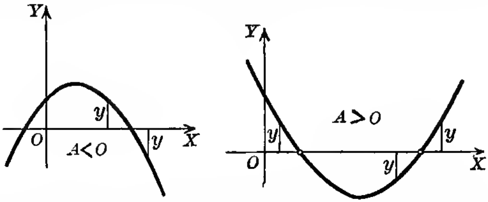
This is easily seen graphically. For take any quadratic

(4) $$Ax^2 + Bx + C.$$

Plot the equation

(5) $$y = Ax^2 + Bx + C.$$

The locus of (5) will be a parabola turned upward if A is positive, downward if A is negative (see Example 2, p. 38). The intercepts on the x-axis will be the roots of (4). The values of y from (5) will clearly all have one sign for all values of x between the intercepts, and the *opposite* sign for all other values of x. We see, then, that the values of the quadratic (4) will have one sign for all values of x taken between the roots, and the opposite sign for all other values.



To apply this, consider the locus of

(6) $$y = \sqrt{6 - 5x - x^2}.$$

What values of x must be excluded? To answer this, find the roots of $6 - 5x - x^2 = 0$. They are $x = -6$ and $x = 1$. Take any value of x between these roots, for example, $x = 0$. When $x = 0$, the quadratic $6 - 5x - x^2$ equals 6, a positive number. Hence $6 - 5x - x^2$ equals a positive number for *all* values of x between the roots -6 and 1. Then the quadratic is negative for all other values; hence we must exclude values of $x < -6$ and also $x > 1$.

PROBLEMS

1. Give a general discussion of each of the following equations and draw the locus. Make sure that the discussion and the figure agree.

(a) $x^2 - 4y = 0$.
(b) $y^2 - 4x + 3 = 0$.
(c) $x^2 + 4y^2 - 16 = 0$.
(d) $9x^2 + y^2 - 18 = 0$.
(e) $x^2 - 4y^2 - 16 = 0$.
(f) $x^2 - 4y^2 + 16 = 0$.
(g) $x^2 - y^2 + 4 = 0$.
(h) $x^2 - y + x = 0$.
(i) $xy - 4 = 0$.
(j) $9y + x^3 = 0$.
(k) $4x - y^3 = 0$.
(l) $6x - y^4 = 0$.
(m) $5x - y + y^3 = 0$.
(n) $9y^2 - x^3 = 0$.
(o) $9y^2 + x^3 = 0$.
(p) $2xy + 3x - 4 = 0$.
(q) $x^2 + 4xy + 3y^2 + 8 = 0$.
(r) $x^2 + xy + y^2 - 4 = 0$.
(s) $x^2 + 2xy - 3y^2 = 4$.
(t) $2xy - y^2 + 4x = 0$.
(u) $3x^2 - y + x = 0$.
(v) $4y^2 - 2x - y = 0$.
(w) $x^2 - y^2 + 6x = 0$.
(x) $x^2 + 4y^2 + 8y = 0$.
(y) $9x^2 + y^2 + 18x - 6y = 0$.
(z) $9x^2 - y^2 + 18x + 6y = 0$.

2. Determine the general nature of the locus in each of the following equations. In plotting, assume *particular* values for the arbitrary constants, but not *special* values; that is, values which give the equation an added peculiarity.*

(a) $y^2 = 2mx$.
(b) $x^2 - 2my = m^2$.
(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
(d) $2xy = a^2$.
(e) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
(f) $x^2 - y^2 = a^2$.
(g) $x^2 + y^2 = r^2$.
(h) $x^2 + y^2 = 2rx$.
(i) $x^2 + y^2 = 2ry$.
(j) $ay^2 = x^3$.
(k) $a^2y = x^3$.

The loci of the equations (a) to (i) in Problem 2 are all of the class known as *conics*, or *conic sections*, — curves following straight lines and circles in the matter of their simplicity. These curves are obtained when cross sections are taken of a right circular cone. Various definitions and properties will be given later. A definition often used is the following:

A conic section is the locus of a point whose distances from a fixed point and a fixed line are in a constant ratio.

* For example, in (a) and (b) $m = 0$ is a special value. In fact, in all these examples zero is a special value for any constant.

3. Show that every conic is represented by an equation of the second degree in x and y.

Hint. Take YY' to coincide with the fixed line, and draw XX' through the fixed point. Denote the fixed point by $(p, 0)$ and the constant ratio by e.

Ans. $(1 - e^2)x^2 + y^2 - 2px + p^2 = 0.$

4. Discuss and plot the locus of the equation of Problem 3:

(a) when $e = 1$. The conic is now called a *parabola* (see p. 45).

(b) when $e < 1$. The conic is now called an *ellipse* (see p. 43).

(c) when $e > 1$. The conic is now called a *hyperbola* (see p. 48).

5. A point moves so that the sum of its distances from the two fixed points (3, 0) and (– 3, 0) is constant and equal to 10. What is the locus?

Ans. Ellipse $16x^2 + 25y^2 = 400.$

6. A point moves so that the difference of its distances from the two fixed points (5, 0) and (– 5, 0) is constant and equal to 8. What is the locus?

Ans. Hyperbola $9x^2 - 16y^2 = 144.$

7. Find the equations of the following loci, and discuss and plot them.

(a) The distance of a point from the fixed point (0, 2) is equal to its distance from the x-axis increased by 2.

(b) The distance of a point from the fixed point (0, – 2) is equal to its distance from the y-axis increased by 2.

(c) The distance of a point from the origin is equal to its distance from the y-axis increased by 2.

(d) The distance of a point from the fixed point (2, – 4) is equal to its distance from the x-axis increased by 5. *Ans.* $2y = x^2 - 4x - 5.$

(e) The distance of a point from the point (3, 0) is equal to half its distance from the point (6, 0).

(f) The distance of a point from the point (8, – 4) is twice its distance from the point (2, – 1).

(g) One third of the distance of a point from the point (0, 3) is equal to its distance from the x-axis increased by unity. *Ans.* $x^2 - 8y^2 - 24y = 0.$

(h) The distances of a point to the fixed point (– 1, 0) and to the line $4x - 5 = 0$ are in the ratio $\frac{4}{5}$. *Ans.* $9x^2 + 25y^2 + 90x = 0.$

8. Prove the statement: If an equation is unaltered when x and y are interchanged, the locus is symmetrical with respect to the line $y = x$.

Make use of this result in drawing the loci of:

(a) $xy = 4$. (b) $x^2 + xy + y^2 = 9$. (c) $x^3 + y^3 = 1$. (d) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$.

22. Asymptotes. The following examples elucidate difficulties arising frequently in drawing the locus of an equation.

EXAMPLES

1. Plot the locus of the equation

(1) $$xy - 2y - 4 = 0.$$

Solution. Solving for y,

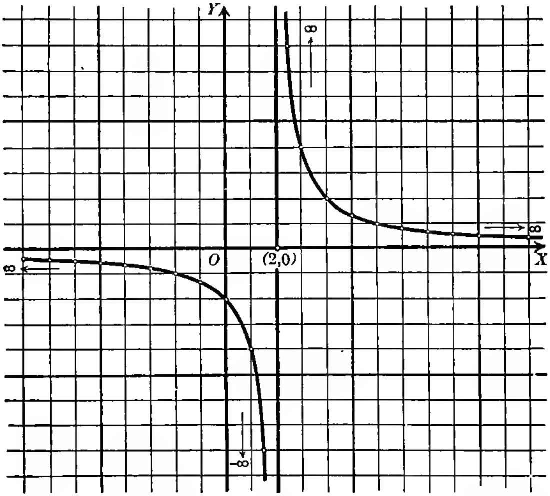
(2) $$y = \frac{4}{x-2}.$$

We observe at once, if $x = 2$, $y = \frac{4}{0} = \infty$. This is interpreted thus: The curve *approaches* the line $x = 2$ as it passes off to infinity. In fact, if we solve (1) for x and write the result in the form

$$x = 2 + \frac{4}{y},$$

it is evident that x approaches 2 as y increases indefinitely. Hence the locus extends both upward and downward indefinitely far, approaching in each case the line $x = 2$. The vertical line $x = 2$ is called a *vertical asymptote.*

x	y	x	y
0	-2	0	-2
1	-4	-1	$-\frac{4}{3}$
$1\frac{1}{2}$	-8	-2	-1
$1\frac{3}{4}$	-16	-4	$-\frac{2}{3}$
2	∞	-5	$-\frac{4}{7}$
$2\frac{1}{4}$	16	$\vdots$	$\vdots$
$2\frac{1}{2}$	8	-10	$-\frac{1}{3}$
3	4	etc.	etc.
4	2		
5	$\frac{4}{3}$		
6	1		
$\vdots$	$\vdots$		
12	0.4		
etc.	etc.		



In plotting, it is necessary to assume values of x differing slightly from 2, both less and greater, as in the table.

From (2) it appears that y diminishes and approaches zero as x increases indefinitely. The curve therefore extends indefinitely far to the right and left, approaching constantly the axis of x. The axis of x is therefore a *horizontal asymptote.**

This curve is called a *hyperbola.*

In the problem just discussed it was necessary to learn *what value x approached when y became very large,* and also what value y approached when x became very large. These questions, when important, are usually readily answered, as in the following examples:

2. Plot the locus of $$y = \frac{2x+3}{3x-4}. \quad \text{(Fig. 1)}$$

When x is very great, we may neglect the 3 in the numerator $(2x + 3)$ and the -4 in the denominator $(3x - 4)$. That is, when x is very large, $y = \frac{2x}{3x} = \frac{2}{3}$. Hence $y = \frac{2}{3}$ is a horizontal asymptote.

The equation shows directly that $3x - 4 = 0$ or $x = \frac{4}{3}$ is a vertical asymptote. Or we may solve the equation for x, which gives

$$x = \frac{4y+3}{3y-2}.$$

Hence, when y is very large, $x \doteq \frac{4y}{3y} = \frac{4}{3}.$

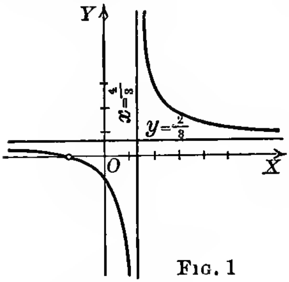


Fig. 1

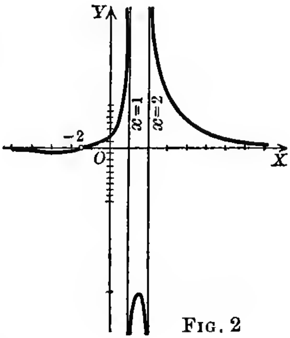


Fig. 2

3. The locus of $$y = \frac{2x+3}{x^2-3x+2}$$

is shown in Fig. 2. There are two vertical asymptotes, $x = 1$ and $x = 2$, since the denominator $x^2 - 3x + 2 = (x - 1)(x - 2)$. A branch of the

* For *oblique* asymptotes, that is, asymptotes not parallel to either axis, see Art. 66.

curve lies between these lines. Furthermore, when x becomes large, we may write the equation $y = \frac{2x}{x^2} = \frac{2}{x} = 0$. Hence the x-axis is a horizontal asymptote. A few points of the locus are given in the table. Note that *different* scales are used for ordinates and abscissas.

x	y
0	$\frac{3}{2}$
$-\frac{3}{2}$	0
$\frac{3}{2}$	-24
5	$\frac{13}{12}$
-5	$-\frac{1}{6}$

The determination of the vertical and horizontal asymptotes of a curve should be added to the discussion of the equation as outlined in Art. 21.

PROBLEMS

Plot each of the following, and determine the horizontal and vertical asymptotes:

1. (a) $xy + y - 8 = 0$.
   (b) $xy + x + 3 = 0$.
   (c) $2xy + 2x + 3y = 0$.
   (d) $x^2 + xy + 8 = 0$.
   (e) $2xy + 4x - 6y + 3 = 0$.
   (f) $y^2 + 2xy - 4 = 0$.
   (g) $xy + x + 2y - 3 = 0$.
   (h) $xy + y - x^2 + 2x = 0$.

2. (a) $x^2y - 5 = 0$.
   (b) $x^2y - y + 2x = 0$.
   (c) $xy^2 - 4x + 6 = 0$.
   (d) $x^3y - y + 8 = 0$.

(e) $y = \frac{5}{x^2 - 3x}$.

(f) $y = \frac{4x^2}{x^2 - 4}$.

(g) $y = \frac{x - 3}{x + 1}$.

(h) $y = \frac{x^2 - 4}{x^2 - 1}$.

(i) $y = \frac{(x-2)(x+3)}{(x+2)(x-3)}$.

(j) $y = \frac{x^2 - 4}{x^2 + x}$.

(k) $x = \frac{y^2}{y - 1}$.

(l) $x = \frac{y - 2}{y - 3}$.

(m) $y = \frac{x^2 - 1}{4 - x^2}$.

(n) $y = \frac{x^2 - 3x + 2}{x^2 + 3x + 2}$.

(o) $4x = \frac{y^2}{y^2 - 9}$.

(p) $12x = \frac{8y}{3 - y^2}$.

(q) $y = \left(\frac{x - 1}{x}\right)^2$.

(r) $y^2 = \frac{x^2}{x - 1}$.

(s) $y^2 = \frac{x^2}{x^2 - 3x + 2}$.

PROBLEMS FOR INDIVIDUAL STUDY

Discuss fully and draw carefully the following loci:

1. $y^2 - 4xy + x^3 = 0$.
2. $y^2 - 2xy - 2x^2 + x^3 = 0$.
3. $y^2 - x^2 + x^4 = 0$.
4. $(y - x)^2 - (a^2 - x^2) = 0$.
5. $(y - x)^2 - x^2(a^2 - x^2) = 0$.
6. $(y - x^2)^2 - (a^2 - x^2) = 0$.
7. $(y - x^2)^2 - x^2(a^2 - x^2) = 0$.
8. $y^2(a - x) - x^3 = 0$ (the cissoid).
9. $y^2(a - x) - x^2(a + x) = 0$ (the strophoid).
10. $x^4 + 2ax^2y - ay^3 = 0$.

11. $x^4 - axy^2 + y^4 = 0.$

12. $a^4y^2 - a^2x^4 + x^6 = 0.$

13. $ay^2 - bx^4 - x^5 = 0.$

14. $a^3y^2 - 2\,abx^2y - x^5 = 0.$

15. $y^2 - (a^2 - x^2)(b^2 - x^2)^2 = 0.$

16. $x^3y^2 - a^3x^2 + ay^4 = 0.$

17. $x(y - x)^2 - b^2y = 0.$

18. $(x^2 + y^2)^2 - a^2(x^2 - y^2) = 0$ (the **lemniscate**).

19. $(x^2 - a^2)^2 = ay^2(3\,a + 2\,y).$

20. $(x^2 + y^2 - 1)\,y - ax = 0.$

21. $y^2 - x^2 - x(x - 4)^2 = 0.$

22. $(x^2 + y^2 - 2\,ay)^2 = a^2(x^2 + y^2)$ (the **limaçon**).

23. $(x^4 + x^2y^2 + y^4) = x(ax^2 - 4\,ay^2).$

24. $(x^2 + y^2 + 4\,ay - a^2)(x^2 - a^2) + 4\,a^2y^2 = 0.$

25. $(y^2 - x^2)(x - 1)(x - \frac{3}{2}) = 2(y^2 + x^2 - 2\,x)^2.$

26. $(x^2 + y^2 + 4\,ay - a^2)(x^2 - a^2) + 4\,a^2y^2 = 0$ (the **cocked hat**).

23. Points of intersection. If two curves whose equations are given intersect, the coördinates of each point of intersection must satisfy both equations when substituted in them for the variables. In algebra it is shown that *all* values satisfying two equations in two unknowns may be found by regarding these equations as simultaneous in the unknowns and solving. Hence the

Rule *to find the points of intersection of two curves whose equations are given.*

Consider the equations as simultaneous in the coördinates and solve as in algebra.

Arrange the real solutions in corresponding pairs. These will be the coördinates of all the points of intersection.

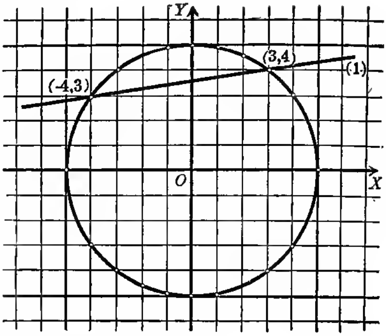
Notice that only *real* solutions correspond to common points of the two curves, since coördinates are always real numbers.

EXAMPLES

1. Find the points of intersection of

(1) $x - 7y + 25 = 0,$

(2) $x^2 + y^2 = 25.$



Solution. Solving (1) for x,

(3) $x = 7y - 25.$

Substituting in (2),

$(7y - 25)^2 + y^2 = 25.$

Reducing,

$y^2 - 7y + 12 = 0.$

$\therefore y = 3$ and 4.

Substituting in (3) [*not* in (2)],

$x = -4$ and $+3$.

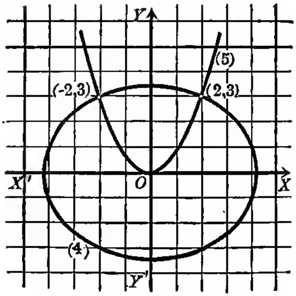
Arranging, the points of intersection are $(-4, 3)$ and $(3, 4)$. *Ans.*

In the figure the straight line (1) is the locus of equation (1), and the circle the locus of (2).

2. Find the points of intersection of the loci of

(4) $2x^2 + 3y^2 = 35,$

(5) $3x^2 - 4y = 0.$



Solution. Solving (5) for x^2,

(6) $x^2 = \frac{4}{3}y.$

Substituting in (4) and reducing,

$9y^2 + 8y - 105 = 0.$

$\therefore y = 3$ and $-\frac{35}{9}$.

Substituting in (6) and solving,

$x = \pm 2$ and $\pm \frac{1}{6}\sqrt{-210}.$

Arranging the *real* values, we find the points of intersection are $(+2, 3)$, $(-2, 3)$. *Ans.*

In the figure the ellipse (4) is the locus of (4), and the parabola (5) the locus of (5).

PROBLEMS

Find the points of intersection of the following loci:

1. $\left.\begin{matrix} 7x - 11y + 1 = 0 \\ x + y - 2 = 0 \end{matrix}\right\}$. *Ans.* $(\frac{7}{6}, \frac{5}{6})$.

2. $\left.\begin{matrix} x + y = 7 \\ x - y = 5 \end{matrix}\right\}$. *Ans.* (6, 1).

3. $\left.\begin{matrix} y = 3x + 2 \\ x^2 + y^2 = 4 \end{matrix}\right\}$. *Ans.* $(0, 2), (-\frac{6}{5}, -\frac{8}{5})$.

4. $\left.\begin{matrix} y^2 = 16x \\ y - x = 0 \end{matrix}\right\}$. *Ans.* (0, 0), (16, 16).

5. $\left.\begin{matrix} x^2 + y^2 = a^2 \\ 3x + y + a = 0 \end{matrix}\right\}$. *Ans.* $(0, -a), \left(-\frac{3a}{5}, \frac{4a}{5}\right)$.

6. $\left.\begin{matrix} x^2 - y^2 = 16 \\ x^2 = 8y \end{matrix}\right\}$. *Ans.* $(\pm 4\sqrt{2}, 4)$.

7. $\left.\begin{matrix} x^2 + y^2 = 41 \\ xy = 20 \end{matrix}\right\}$. *Ans.* $(\pm 5, \pm 4), (\pm 4, \pm 5)$.

8. $\left.\begin{matrix} y^2 = 2px \\ x^2 = 2py \end{matrix}\right\}$. *Ans.* $(0, 0), (2p, 2p)$.

9. $\left.\begin{matrix} 4x^2 + y^2 = 5 \\ y^2 = 8x \end{matrix}\right\}$. *Ans.* $(\frac{1}{2}, 2), (\frac{1}{2}, -2)$.

10. $\left.\begin{matrix} x^2 + y^2 = 100 \\ y^2 = \dfrac{9x}{2} \end{matrix}\right\}$. *Ans.* (8, 6), (8, − 6).

11. $\left.\begin{matrix} x^2 + y^2 = 5a^2 \\ x^2 = 4ay \end{matrix}\right\}$. *Ans.* $(2a, a), (-2a, a)$.

Find the area of the triangles and polygons whose sides are the loci of the following equations:

12. $3x + y + 4 = 0$, $3x - 5y + 34 = 0$, $3x - 2y + 1 = 0$. *Ans.* 36.

13. $x + 2y = 5$, $2x + y = 7$, $y = x + 1$. *Ans.* $\frac{3}{2}$.

14. $x + y = a$, $x - 2y = 4a$, $y - x + 7a = 0$. *Ans.* $12a^2$.

15. $x = 0$, $y = 0$, $x = 4$, $y = -6$. *Ans.* 24.

16. $x - y = 0$, $x + y = 0$, $x - y = a$, $x + y = b$. *Ans.* $\frac{ab}{2}$.

17. $y = 3x - 9$, $y = 3x + 5$, $2y = x - 6$, $2y = x + 14$. *Ans.* 56.

18. Find the distance between the points of intersection of the curves $3x - 2y + 6 = 0$, $x^2 + y^2 = 9$. *Ans.* $\frac{18}{13}\sqrt{13}$.

19. Does the locus of $y^2 = 4x$ intersect the locus of $2x + 3y + 2 = 0$? *Ans.* Yes.

20. For what value of a will the three lines $3x + y - 2 = 0$, $ax + 2y - 3 = 0$, $2x - y - 3 = 0$ meet in a point? *Ans.* $a = 5$.

21. Find the length of the common chord of $x^2 + y^2 = 13$ and $y^2 = 3x + 3$. *Ans.* 6.

22. If the equations of the sides of a triangle are $x + 7y + 11 = 0$, $3x + y - 7 = 0$, $x - 3y + 1 = 0$, find the length of each of the medians. *Ans.* $2\sqrt{5}$, $\frac{5}{2}\sqrt{2}$, $\frac{1}{2}\sqrt{170}$.

CHAPTER IV

THE STRAIGHT LINE

24. The degree of the equation of any straight line. It will now be shown that any straight line is represented by an equation of the first degree in the variable coördinates x and y.

Theorem. *The equation of the straight line passing through a point B $(0, b)$ on the axis of y and having its slope equal to m is*

(I) $$y = mx + b.$$

Proof. Assume that P (x, y) is any point on the line.

The given condition may be written

$$\text{slope of } PB = m.$$

Since by (II), p. 17,

$$\text{slope of } PB = \frac{y - b}{x - 0},$$

[Substituting (x, y) for (x_1, y_1) and $(0, b)$ for (x_2, y_2).]

then $$\frac{y - b}{x} = m, \text{ or } y = mx + b.$$ Q.E.D.

In equation (I), m and b may have any values, positive, negative, or zero.

Equation (I) will represent any straight line which intersects the y-axis. But the equation of any line *parallel* to the y-axis has the form $x =$ a constant, since the abscissas of all points on such a line are equal. The two forms, $y = mx + b$ and $x =$ constant, will therefore represent all lines. Each of these equations being of the first degree in x and y, we have the

Theorem. *The equation of any straight line is of the first degree in the coördinates x and y.*

25. Locus of any equation of the first degree. The question now arises: Given an equation of the first degree in the coördinates x and y, is the locus a straight line?

Consider, for example, the equation

(1) $$3x - 2y + 8 = 0.$$

Let us solve this equation for y. This gives

(2) $$y = \frac{3}{2}x + 4.$$

Comparing (2) with the formula (I),

$$y = mx + b,$$

we see that (2) is obtained from (I) if we set $m = \frac{3}{2}$, $b = 4$. Now in (I) m and b may have any values. The locus of (I) is, for all values of m and b, a straight line. Hence (2), or (1), is the equation of a straight line through (0, 4) with the slope equal to $\frac{3}{2}$. This discussion prepares the way for the general theorem.

The equation

(3) $$Ax + By + C = 0,$$

where A, B, and C are arbitrary constants, is called the **general equation of the first degree** in x and y because every equation of the first degree may be reduced to that form.

Equation (3) represents all straight lines.

For the equation $y = mx + b$ may be written $mx - y + b = 0$, which is of the form (3) if $A = m$, $B = -1$, $C = b$; and the equation $x =$ constant may be written $x -$ constant $= 0$, which is of the form (3) if $A = 1$, $B = 0$, $C = -$ constant.

Theorem. *The locus of the general equation of the first degree*

$$Ax + By + C = 0$$

is a straight line.

Proof. Solving (3) for y, we obtain

(4) $$y = -\frac{A}{B}x - \frac{C}{B}.$$

Comparison with (I) shows that the locus of (4) is the straight line for which

$$m = -\frac{A}{B}, \quad b = -\frac{C}{B}.$$

If, however, $B = 0$, the reasoning fails.

But if $B = 0$, (3) becomes

$$Ax + C = 0,$$

or

$$x = -\frac{C}{A}.$$

The locus of this equation is a straight line parallel to the y-axis. Hence in all cases the locus of (3) is a straight line.

Q. E. D.

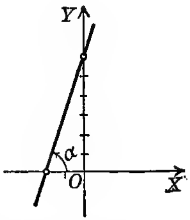
Corollary. *The slope of the line*

$$Ax + By + C = 0$$

is $m = -\frac{A}{B}$*; that is, the coefficient of x with its sign changed divided by the coefficient of y.*

26. Plotting straight lines. If the line does not pass through the origin (constant term not zero, p. 47), find the intercepts (p. 47), mark them off on the axes, and draw the line. If the line passes through the origin, find a second point whose coördinates satisfy the equation, and draw a line through this point and the origin.

EXAMPLE



Plot the locus of $3x - y + 6 = 0$. Find the slope.

Solution. Letting $y = 0$ and solving for x,

$x = -2 =$ intercept on x-axis.

Letting $x = 0$ and solving for y,

$y = 6 =$ intercept on y-axis.

The required line passes through the points $(-2, 0)$ and $(0, 6)$.

To find the slope: Comparison with the general equation (3) shows that $A = 3$, $B = -1$, $C = 6$. Hence $m = -\frac{A}{B} = 3$.

Otherwise thus: Reduce the given equation to the form $y = mx + b$ by solving it for y. This gives $y = 3x + 6$. Hence $m = 3$, $b = 6$, as before.

PROBLEMS

1. Find the intercepts and the slope of the following lines, and plot the lines:

(a) $2x + 3y = 6$. *Ans.* $3, 2;\ m = -\frac{2}{3}$.

(b) $x - 2y + 5 = 0$. *Ans.* $-5, 2\frac{1}{2};\ m = \frac{1}{2}$.

(c) $3x - y + 3 = 0$. *Ans.* $-1, 3;\ m = 3$.

(d) $5x + 2y - 6 = 0$. *Ans.* $\frac{6}{5}, 3;\ m = -\frac{5}{2}$.

2. Plot the following lines and find the slope:

(a) $2x - 3y = 0$.

(b) $y - 4x = 0$.

(c) $3x + 2y = 0$.

(d) $x - 3y = 0$.

3. Find the equations, and reduce them to the general form, of the lines for which

(a) $m = 2, b = -3$. *Ans.* $2x - y - 3 = 0$.

(b) $m = -\frac{1}{2}, b = \frac{3}{2}$. *Ans.* $x + 2y - 3 = 0$.

(c) $m = \frac{2}{5}, b = -\frac{5}{2}$. *Ans.* $4x - 10y - 25 = 0$.

(d) $\alpha = \frac{\pi}{4}, b = -2$. *Ans.* $x - y - 2 = 0$.

(e) $\alpha = \frac{3\pi}{4}, b = 3$. *Ans.* $x + y - 3 = 0$.

Hint. Substitute in $y = mx + b$ and transpose.

4. Select pairs of parallel and perpendicular lines from the following:

(a) $\begin{cases} L_1 : y = 2x - 3. \\ L_2 : y = -3x + 2. \\ L_3 : y = 2x + 7. \\ L_4 : y = \frac{1}{3}x + 4. \end{cases}$ *Ans.* $L_1 \parallel L_3;\ L_2 \perp L_4$.

(b) $\begin{cases} L_1 : x + 3y = 0. \\ L_2 : 8x + y + 1 = 0. \\ L_3 : 9x - 3y + 2 = 0. \end{cases}$ *Ans.* $L_1 \perp L_3$.

(c) $\begin{cases} L_1 : 2x - 5y = 8. \\ L_2 : 5y + 2x = 8. \\ L_3 : 35x - 14y = 8. \end{cases}$ *Ans.* $L_2 \perp L_3$.

5. Show that the quadrilateral whose sides are $2x - 3y + 4 = 0$, $3x - y - 2 = 0$, $4x - 6y - 9 = 0$, and $6x - 2y + 4 = 0$ is a parallelogram.

6. Find the equation of the line whose slope is -2, which passes through the point of intersection of $y = 3x + 4$ and $y = -x + 4$.

Ans. $2x + y - 4 = 0$.

7. Write an equation which will represent all lines parallel to the line

(a) $y = 2x + 7$. (c) $y - 3x - 4 = 0$.

(b) $y = -x + 9$. (d) $2y - 4x + 3 = 0$.

8. Find the equation of the line parallel to $2x - 3y = 0$ whose intercept on the Y-axis is -2. *Ans.* $2x - 3y - 6 = 0$.

9. Show that the following loci are straight lines and plot them:

(a) The locus of a point whose distances from the axes XX' and YY' are in a constant ratio equal to $\frac{2}{3}$. *Ans.* $2x - 3y = 0$.

(b) The locus of a point the sum of whose distances from the axes of coördinates is always equal to 10. *Ans.* $x + y - 10 = 0$.

(c) A point moves so as to be always equidistant from the axes of coördinates. *Ans.* $x - y = 0$.

(d) A point moves so that the difference of the squares of its distances from (3, 0) and (0, − 2) is always equal to 8.

Ans. The parallel straight lines $6x + 4y + 3 = 0$, $6x + 4y - 13 = 0$.

(e) A point moves so as to be always equidistant from the straight lines $x - 4 = 0$ and $y + 5 = 0$.

Ans. The perpendicular straight lines $x - y - 9 = 0$, $x + y + 1 = 0$.

10. A point moves so that the sum of its distances from two perpendicular lines is constant. Show that the locus is a straight line.

Hint. Choosing the axes of coördinates to coincide with the given lines, the equation is $x + y =$ constant.

11. A point moves so that the difference of the squares of its distances from two fixed points is constant. Show that the locus is a pair of straight lines.

Hint. Draw XX' through the fixed points, and YY' through their middle point. Then the fixed points may be written $(a, 0)$, $(-a, 0)$, and if the "constant difference" be denoted by k, we find for the locus $4ax = k$ and $4ax = -k$.

12. A point moves so that the difference of the squares of its distances from two perpendicular lines is zero. Show that the locus is a pair of perpendicular lines.

13. A point moves so that its distance from a fixed line is in a constant ratio to its distance from a fixed point on the line. For what values of the ratio is the locus real? What is the locus?

27. Point-slope form. If it is required that a straight line shall pass through a given point in a given direction, the line is determined.

The following problem is therefore definite:

To find the equation of the straight line passing through a given point $P_1(x_1, y_1)$ and having a given slope m.

Solution. Let $P(x, y)$ be any other point on the line. By the hypothesis,

$$\text{slope } PP_1 = m.$$

(1) $$\therefore \frac{y - y_1}{x - x_1} = m. \qquad \text{(II, p. 17)}$$

Clearing of fractions gives the formula

(II) $$y - y_1 = m(x - x_1).$$

28. Two-point form. A straight line is determined by two of its points. Let us then solve the problem:

To find the equation of the line passing through two given points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$.

Solution. The slope of the given line is

$$\text{slope } P_1P_2 = \frac{y_1 - y_2}{x_1 - x_2}.$$

Let $P(x, y)$ be any other point on the line P_1P_2. Then

$$\text{slope } PP_1 = \frac{y - y_1}{x - x_1}.$$

Since P, P_1, and P_2 are on one line, slope PP_1 = slope P_1P_2. Hence we have the formula

(III) $$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

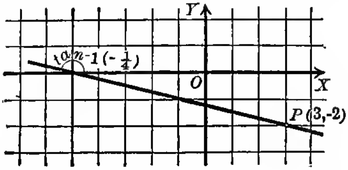
Equation (III) may be written in the determinant form

(2) $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

For the determinant, when expanded, is of the first degree in x and y. Hence (2) is the equation of a line. But (2) is satisfied when $x = x_1$, $y = y_1$, and also when $x = x_2$, $y = y_2$, for then two rows become identical and the determinant vanishes. Otherwise thus: Comparison of (2) with the formula at the close of Art. 14 shows that the area of the triangle PP_1P_2 is zero. Hence these three points lie on a line.

EXAMPLES

1. Find the equation of the line passing through P_1 $(3, -2)$ whose slope is $-\frac{1}{4}$.



Solution. Use the point-slope equation (II), substituting $x_1 = 3$, $y_1 = -2$, $m = -\frac{1}{4}$. This gives

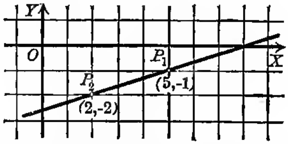
$$y + 2 = -\tfrac{1}{4}(x - 3).$$

Clearing and reducing,

$$x + 4y + 5 = 0.$$

2. Find the equation of the line through the two points P_1 $(5, -1)$ and P_2 $(2, -2)$.

Solution. Use the two-point equation (III), substituting $x_1 = 5$, $y_1 = -1$, $x_2 = 2$, $y_2 = -2$. This gives



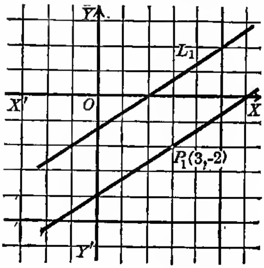
$$\frac{y + 1}{x - 5} = \frac{-1 + 2}{5 - 2} = \frac{1}{3}.$$

Clearing and reducing,

$$x - 3y - 8 = 0.$$

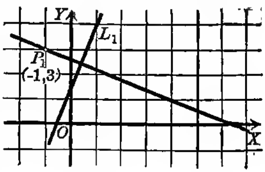
The answer should be *checked*. To do this, we must prove that the coördinates of the given points satisfy the answer. Thus for P_1, substituting $x = 5$, $y = -1$, the answer holds. Similarly for P_2. The student should supply checks for Examples 1, 3, and 4.

3. Find the equation of the line through the point P_1 $(3, -2)$ parallel to the line $L_1: 2x - 3y - 4 = 0$.



Solution. The slope of the given line L_1 equals $\frac{2}{3}$. Hence the slope of the required line also equals $\frac{2}{3}$ (Theorem, p. 17), and it passes through $P_1(3, -2)$. Using the point-slope equation (II), we have $y + 2 = \frac{2}{3}(x - 3)$, or $2x - 3y - 12 = 0$.

4. Find the equation of the line through the point P_1 $(-1, 3)$ perpendicular to the line $L_1: 5x - 2y + 3 = 0$.



Solution. The slope of the given line L_1 equals $\frac{5}{2}$. Hence the slope of the required line equals $-\frac{2}{5}$ (Theorem, p. 17). Since we know a point $P_1(-1, 3)$ on the line, we use the point-slope equation (II), and obtain $y - 3 = -\frac{2}{5}(x + 1)$, or $2x + 5y - 13 = 0$.

PROBLEMS

1. Find the equation of the line satisfying the following conditions, and plot the line. Check the answers:

(a) Passing through (0, 0) and (8, 2). *Ans.* $x - 4y = 0$.
(b) Passing through (– 1, 1) and (– 3, 1). *Ans.* $y - 1 = 0$.
(c) Passing through (– 3, 1) and slope = 2. *Ans.* $2x - y + 7 = 0$.
(d) Having the intercepts* $a = 3$ and $b = -2$. *Ans.* $2x - 3y - 6 = 0$.
(e) Slope = – 3, intercept on x-axis = 4. *Ans.* $3x + y - 12 = 0$.
(f) Intercepts $a = -3$ and $b = -4$. *Ans.* $4x + 3y + 12 = 0$.
(g) Passing through (2, 3) and (– 2, – 3). *Ans.* $3x - 2y = 0$.
(h) Passing through (3, 4) and (– 4, – 3). *Ans.* $x - y + 1 = 0$.
(i) Passing through (2, 3) and slope = – 2. *Ans.* $2x + y - 7 = 0$.

2. Find the equation of the line passing through the origin parallel to the line $2x - 3y = 4$. *Ans.* $2x - 3y = 0$.

3. Find the equation of the line passing through the origin perpendicular to the line $5x + y - 2 = 0$. *Ans.* $x - 5y = 0$.

4. Find the equation of the line passing through the point (3, 2) parallel to the line $4x - y - 3 = 0$. *Ans.* $4x - y - 10 = 0$.

5. Find the equation of the line passing through the point (3, 0) perpendicular to the line $2x + y - 5 = 0$. *Ans.* $x - 2y - 3 = 0$.

6. Find the equation of the line whose intercept on the y-axis is 5, which passes through the point (6, 3). *Ans.* $x + 3y - 15 = 0$.

7. Find the equation of the line whose intercept on the x-axis is 3, which is parallel to the line $x - 4y + 2 = 0$. *Ans.* $x - 4y - 3 = 0$.

8. Find the equation of the line passing through the origin and through the intersection of the lines $x - 2y + 3 = 0$ and $x + 2y - 9 = 0$.

Ans. $x - y = 0$.

9. Find the equations of the sides of the triangle whose vertices are (– 3, 2), (3, – 2), and (0, – 1).

Ans. $2x + 3y = 0$, $x + 3y + 3 = 0$, and $x + y + 1 = 0$.

10. Find the equations of the medians of the triangle in Problem 9, and show that they meet in a point.

Ans. $x = 0$, $7x + 9y + 3 = 0$, and $5x + 9y + 3 = 0$.

Hint. To show that three lines meet in a point, find the point of intersection of two of them and prove that it lies on the third.

* Intercept on x-axis = a, intercept on y-axis = b. The given points are (3, 0) and (0, – 2).

11. Determine whether or not the following sets of points lie on a straight line:

(a) (0, 0), (1, 1), (7, 7). *Ans.* Yes.
(b) (2, 3), (– 4, – 6), (8, 12). *Ans.* Yes.
(c) (3, 4), (1, 2), (5, 1). *Ans.* No.
(d) (3, – 1), (– 6, 2), ($-\frac{3}{2}$, 1). *Ans.* No.
(e) (5, 6), ($\frac{5}{6}$, 1), (– 1, $-\frac{6}{5}$). *Ans.* Yes.
(f) (7, 6), (2, 1), (6, – 2). *Ans.* No.
(g) (3, – 2), (6, – 4), (– 5, 4).
(h) (1, 0), (0, 1), (7, – 8).
(i) (– 3, – 1), (6, 2), (8, 3).

12. Find the equations of the lines joining the middle points of the sides of the triangle in Problem 9, and show that they are parallel to the sides.
Ans. $4x + 6y + 3 = 0$, $x + 3y = 0$, and $x + y = 0$.

13. Find the equation of the line passing through the origin and through the intersection of the lines $x + 2y = 1$ and $2x - 4y - 3 = 0$.
Ans. $x + 10y = 0$.

14. Show that the diagonals of a square are perpendicular.

Hint. Take two sides for the axes and let the length of a side be a.

15. Show that the line joining the middle points of two sides of a triangle is parallel to the third.

Hint. Choose the axes so that the vertices are (0, 0), (a, 0), and (b, c).

16. Two sides of a parallelogram are given by $2x + 3y - 7 = 0$ and $x - 3y + 4 = 0$. Find the other two sides if one vertex is the point (3, 2).
Ans. $2x + 3y - 12 = 0$ and $x - 3y + 3 = 0$.

17. Find the equations of the lines drawn through the vertices of the triangle whose vertices are (– 3, 2), (3, – 2), and (0, – 1), which are parallel to the opposite sides. Find the vertices of the new triangle.
Ans. $2x + 3y + 3 = 0$, $x + 3y - 3 = 0$, $x + y - 1 = 0$.

18. Find the equations of the lines drawn through the vertices of the triangle in Problem 17, which are perpendicular to the opposite sides, and show that they meet in a point.
Ans. $3x - 2y - 2 = 0$, $3x - y + 11 = 0$, $x - y - 5 = 0$.

19. Find the equations of the perpendicular bisectors of the sides of the triangle in Problem 17, and show that they meet in a point.
Ans. $3x - 2y = 0$, $3x - y - 6 = 0$, $x - y + 2 = 0$.

20. The equations of two sides of a parallelogram are $3x - 4y + 6 = 0$ and $x + 5y - 10 = 0$. Find the equations of the other two sides if one vertex is the point (4, 9). *Ans.* $3x - 4y + 24 = 0$ and $x + 5y - 49 = 0$.

21. The vertices of a triangle are (2, 1), (– 2, 3), and (4, – 1). Find the equations of (a) the sides of the triangle, (b) the perpendicular bisectors of the sides, and (c) the lines drawn through the vertices perpendicular to the opposite sides. Check the results by showing that the lines in (b) and (c) meet in a point.

29. Intercept form. A line is determined if its intercepts on the axes are given. If these intercepts are a on XX' and b on YY', then the line passes through $(a, 0)$ and $(0, b)$, and the two-point form (III) gives (writing $x_1 = a$, $y_1 = 0$, $x_2 = 0$, $y_2 = b$)

$$\frac{y-0}{x-a} = \frac{b-0}{0-a} = -\frac{b}{a}.$$

Clearing of fractions, transposing, and dividing by ab, we obtain

$$\text{(IV)} \qquad \frac{x}{a} + \frac{y}{b} = 1.$$

30. Condition that three lines shall intersect in a common point. It is shown in algebra that three linear equations in two unknowns x and y, for example,

$$(1) \quad Ax + By + C = 0, \quad A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0,$$

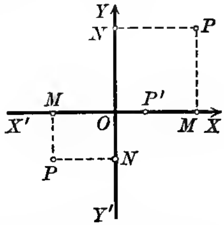
will have a common solution when and only when the determinant formed on the coefficients vanishes; that is, when

$$(2) \qquad \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0.$$

Hence the three lines (1) will intersect in a common point when and only when (2) holds, *provided always that the lines are not parallel*, however. But this latter fact may always be determined by inspection of the equations.

31. Theorems on projection. In preparation for deriving additional theorems of this and later chapters, some simple facts in regard to projection will now be discussed.

The **orthogonal projection** of a point upon a line is the foot of the perpendicular let fall from the point upon the line.



Thus in the figure

M is the orthogonal projection of P on $X'X$;

N is the orthogonal projection of P on $Y'Y$;

P' is the orthogonal projection of P' on $X'X$.

If A and B are two points of a directed line, and M and N their projections upon a second directed line CD, then MN is called the **projection of AB upon CD.**

FIRST THEOREM OF PROJECTION. *If A and B are points upon a directed line making an angle α with a second directed line CD, then the*

projection of the length AB upon $CD = AB \cos \alpha$.

Proof. In the figures

$$\text{projection of } AB \text{ upon } CD = MN.$$

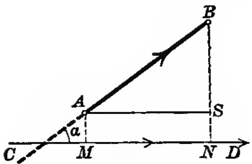


FIG. 1

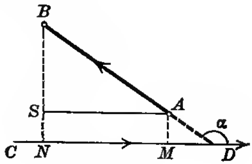


FIG. 2

Now in Fig. 1, from the right triangle BAS,

$$AS = AB \cos BAS.$$

But $AS = MN$, and $\angle BAS = \alpha$.

$$\therefore MN = AB \cos \alpha.$$

In Fig 2 (p. 68), α is obtuse and MN is a *negative* number. *Numerically*, AS and MN are equal, but they differ in sign, AS not being directed. As before, $AS = AB \cos BAS$. But

$$\angle BAS = 180^\circ - \alpha. \quad \therefore \cos BAS = - \cos \alpha \text{ (30, p. 3).}$$

Hence $AS = - AB \cos \alpha$.

$$\therefore MN = AB \cos \alpha. \qquad \text{Q.E.D.}$$

Consider next a broken line made up of directed parts, as in the figures. The line joining the first and last points of a broken line is called the **closing line.**

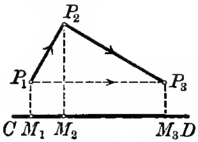


FIG. 1

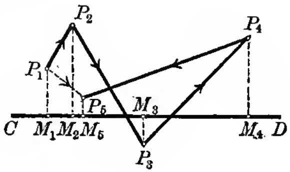


FIG. 2

Thus in Fig. 1 the closing line is P_1P_3; in Fig. 2 the closing line is P_1P_5.

With reference to such broken lines, the following theorem, of frequent application, holds.

SECOND THEOREM OF PROJECTION. *If each segment of a broken line be given the direction determined in passing continuously from one extremity to the other, then the algebraic sum of the projections of the segments upon any directed line equals the projection of the closing line.*

Proof. The proof results immediately. For, in Fig. 1,

$$M_1M_2 = \text{projection of } P_1P_2;$$
$$M_2M_3 = \text{projection of } P_2P_3;$$
$$M_1M_3 = \text{projection of closing line } P_1P_3.$$

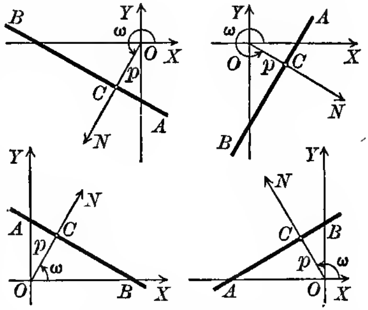
But obviously $\quad M_1M_2 + M_2M_3 = M_1M_3,$
and the theorem follows.

Similarly in Fig. 2. Q.E.D.

Corollary. *If the sides of a closed polygon be given the direction established by passing continuously around the perimeter, the sum of the projections of the sides upon any directed line is zero.*

For the closing line is now zero.

32. The normal equation of the straight line. In the preceding sections the lines considered were determined by two points or by a point and a direction. Both of these methods of determining a line are frequently used in elementary geometry, but we have now to consider a line determined by two conditions which belong essentially to analytic geometry. Let AB be any line, and let ON be drawn from the origin perpendicular to AB at C. *Let the positive direction on ON be from O toward N*, that is, *from* the origin *toward* the line, and denote the positive directed length OC by p, and the positive angle XON, measured, as in trigonometry, from OX as initial line to ON as terminal line, by ω (Greek letter "omega"). Then it is evident from the figures that *the position of any line is determined by a pair of values of p and ω, both p and ω being positive and $\omega < 360°$.*



On the other hand, every line which does not pass through the origin determines a single positive value of p and a single positive value of ω which is less than 360°.

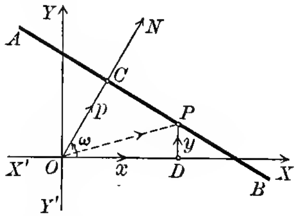
The problem now is this: Given for the line AB of the figure the perpendicular distance OC $(=p)$ from the origin and the angle XOC $(=\omega)$; to find the equation of AB.

Solution. Let $P(x, y)$ be any point on the given line AB.

Then since AB is perpendicular to ON, the projection of OP on ON is equal to p. Consider the broken line ODP. The closing line is OP. By the second theorem of projection (p. 69), the projection of OP on ON is equal to the sum of the projections of OD and DP on ON. Then

(1) projection of OD on ON + projection of DP on $ON = p$.

By the first theorem of projection (p. 68),



(2) projection of OD on

$ON = OD \cos \omega = x \cos \omega$, and

(3) projection of DP on

$ON = DP \cos\left(\frac{\pi}{2} - \omega\right) = y \sin \omega.$

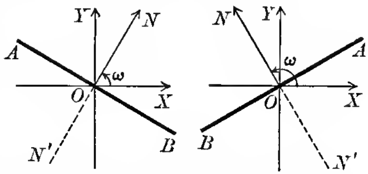
[For the angle between the directed lines DP and ON equals that between OY and $ON = \frac{\pi}{2} - \omega$.]

Substituting from (2) and (3) in (1),

(V) $$x \cos \omega + y \sin \omega - p = 0.$$ Q. E. D.

This equation is known as the *normal equation.*

When $p = 0$, however, AB passes through the origin, and the rule given above for the positive direction on ON becomes meaningless. From the figures we see that we can choose for ω either of the angles XON or XON'. *When $p = 0$ we shall always suppose that $\omega < 180°$ and that the positive direction on ON is the upward direction.*



33. Reduction to the normal form. In Art. 25 it appeared that the *slope* of any line could be found after its equation was reduced to the form $y = mx + b$. If now the equation of any line can be reduced to the normal form (V), we shall be able to find the perpendicular distance p from the origin to the line, as well as the angle ω which this perpendicular makes with OX.

To reduce a given equation

(1) $$Ax + By + C = 0$$

to the normal form, it is necessary to determine ω and p so that the locus of (1) is identical with the locus of

(2) $$x \cos \omega + y \sin \omega - p = 0.$$

This is the case when corresponding coefficients are proportional.* Hence we must have

$$\frac{\cos \omega}{A} = \frac{\sin \omega}{B} = \frac{-p}{C}.$$

Denote the common value of these ratios by r; then

(3) $$\cos \omega = rA,$$

(4) $$\sin \omega = rB, \text{ and}$$

(5) $$-p = rC.$$

To find r, square (3) and (4) and add; this gives

$$\sin^2 \omega + \cos^2 \omega = r^2(A^2 + B^2).$$

But $$\sin^2 \omega + \cos^2 \omega = 1; \qquad (28, \text{p. } 3)$$

and hence $$r^2(A^2 + B^2) = 1, \text{ or}$$

(6) $$r = \frac{1}{\pm\sqrt{A^2 + B^2}}.$$

Equation (5) shows which sign of the radical to use; for since p is positive, r and C must have *opposite* signs.

Substituting the value of r in (3), (4), and (5),

$$\cos \omega = \frac{A}{\pm\sqrt{A^2 + B^2}}, \quad \sin \omega = \frac{B}{\pm\sqrt{A^2 + B^2}}, \quad p = -\frac{C}{\pm\sqrt{A^2 + B^2}}.$$

Hence (2) becomes

(7) $$\frac{A}{\pm\sqrt{A^2 + B^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2}} = 0,$$

* The proof of this obvious fact is left to the student.

which is the normal form of (1). The result of the discussion may be stated in the following

Rule *to reduce* $Ax + By + C = 0$ *to the normal form.*

Find the numerical value of $\sqrt{A^2+B^2}$ *and give it the sign opposite to that of C. Divide the given equation by this number. The result is the required equation.*

For example, to reduce the equation

(8) $$3x - y + 10 = 0$$

to the normal form, divide the equation by $-\sqrt{10}$, since $A = 3$, $B = -1$, $\sqrt{A^2 + B^2} = \sqrt{10}$, and this radical must be given the negative sign, since $C\ (= 10)$ is positive. The normal form of (8) is accordingly

$$-\frac{3}{\sqrt{10}}x + \frac{1}{\sqrt{10}}y - \sqrt{10} = 0.$$

Here $\cos\omega = -\frac{3}{\sqrt{10}}$, $\sin\omega = \frac{1}{\sqrt{10}}$, $p = \sqrt{10} = 3.1 +$.

If $C = 0$, then $p = 0$, and hence $\omega < 180°$ (p. 71); then $\sin\omega$ is positive, and from (4) r and B must have the *same* signs.

The advantages of the normal form of the equation of the straight line over the other forms are twofold. In the first place, *every* line may have its equation put in the normal form; whether it is parallel to one of the axes or passes through the origin is immaterial. In the second place, as will be seen in the following section, it enables us to find immediately the perpendicular distance from a line to a point.

PROBLEMS

1. In what quadrant will ON (see figure on page 70) lie if $\sin\omega$ and $\cos\omega$ are both positive? both negative? if $\sin\omega$ is positive and $\cos\omega$ negative? if $\sin\omega$ is negative and $\cos\omega$ positive?

2. Find the equations and plot the lines for which

(a) $\omega = 0, p = 5.$ *Ans.* $x = 5.$

(b) $\omega = \frac{3\pi}{2}, p = 3.$ *Ans.* $y + 3 = 0.$

(c) $\omega = \frac{\pi}{4}, p = 3.$ *Ans.* $\sqrt{2}x + \sqrt{2}y - 6 = 0.$

(d) $\omega = \frac{2\pi}{3}$, $p = 2$. *Ans.* $x - \sqrt{3}y + 4 = 0$.

(e) $\omega = \frac{7\pi}{4}$, $p = 4$. *Ans.* $\sqrt{2}x - \sqrt{2}y - 8 = 0$.

3. Reduce the following equations to the normal form and find p and ω:

(a) $3x + 4y - 2 = 0$. *Ans.* $p = \frac{2}{5}$, $\omega = \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5}$.
(b) $3x - 4y - 2 = 0$. *Ans.* $p = \frac{2}{5}$, $\omega = \cos^{-1}\frac{3}{5} = \sin^{-1}(-\frac{4}{5})$.
(c) $12x - 5y = 0$. *Ans.* $p = 0$, $\omega = \cos^{-1}(-\frac{12}{13}) = \sin^{-1}\frac{5}{13}$.
(d) $2x + 5y + 7 = 0$.

$$Ans.\ p = \frac{7}{+\sqrt{29}},\ \omega = \cos^{-1}\left(\frac{2}{-\sqrt{29}}\right) = \sin^{-1}\left(\frac{5}{-\sqrt{29}}\right).$$

(e) $4x - 3y + 1 = 0$. *Ans.* $p = \frac{1}{5}$, $\omega = \cos^{-1}(-\frac{4}{5}) = \sin^{-1}\frac{3}{5}$.
(f) $4x - 5y + 6 = 0$.

$$Ans.\ p = \frac{6}{+\sqrt{41}},\ \omega = \cos^{-1}\left(\frac{4}{-\sqrt{41}}\right) = \sin^{-1}\left(\frac{5}{+\sqrt{41}}\right).$$

(g) $x - 4 = 0$. (h) $y - 3 = 0$. (i) $x + 2 = 0$. (j) $y + 4 = 0$.

4. Find the perpendicular distance from the origin to each of the following lines:

(a) $12x + 5y - 26 = 0$. *Ans.* 2.
(b) $x + y + 1 = 0$. *Ans.* $\frac{1}{2}\sqrt{2}$.
(c) $3x - 2y - 1 = 0$. *Ans.* $\frac{1}{13}\sqrt{13}$.
(d) $x + 4 = 0$.
(e) $y - 5 = 0$.

5. Derive (V) when (a) $\frac{\pi}{2} < \omega < \pi$; (b) $\pi < \omega < \frac{3\pi}{2}$; (c) $\frac{3\pi}{2} < \omega < 2\pi$; (d) $p = 0$, and $0 < \omega < \frac{\pi}{2}$.

6. For what values of p and ω will the locus of (V) be parallel to the x-axis? the y-axis? pass through the origin?

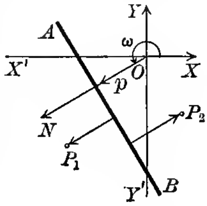
7. Find the equations of the lines whose slopes equal -2, which are at a distance of 5 from the origin.

Ans. $2\sqrt{5}x + \sqrt{5}y - 25 = 0$ and $2\sqrt{5}x + \sqrt{5}y + 25 = 0$.

8. Find the lines whose distance from the origin is 10, which pass through the point (5, 10). *Ans.* $y = 10$ and $4x + 3y = 50$.

9. Write an equation representing all lines whose perpendicular distance from the origin is 5.

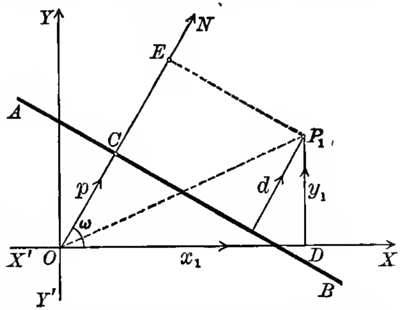
34. The perpendicular distance from a line to a point. The positive direction on the line ON drawn through the origin perpendicular to AB *is from O to AB* (Art. 32). The positive direction on ON will now be assumed to determine the *positive direction on all lines perpendicular to AB.* Hence *the perpendicular distance* ***from*** *the line AB* ***to*** *the point P_1 is positive if P_1 and the origin are on opposite sides of AB, and negative if P_1 and the origin are on the same side of AB.* Thus in the figure the distance from AB to P_1 is positive, and from AB to P_2 is negative.



Let us now solve this problem: Given the equation of any line AB and a point P_1; to find the perpendicular distance from AB to P_1.

Solution. Assume that the equation of AB is in the normal form

(1) $x \cos \omega + y \sin \omega - p = 0.$



Let the coördinates of P_1 be (x_1, y_1) and denote the perpendicular distance from AB to P_1 by d. In the figure project the broken line ODP_1 upon the normal ON. Then since OP_1 is the closing line, by the second theorem of projection (p. 69), projection of OP_1 on $ON =$ projection of OD on $ON +$ projection of DP_1 on ON.

From the figure,

$$\text{projection of } OP_1 \text{ on } ON = OE = p + d.$$

By the first theorem of projection (p. 68),

$$\text{projection of } OD \text{ on } ON = OD \cos \omega = x_1 \cos \omega,$$

$$\text{projection of } DP_1 \text{ on } ON = DP_1 \cos \left(\frac{\pi}{2} - \omega\right)$$
$$= y_1 \sin \omega.$$

Hence $$p + d = x_1 \cos \omega + y_1 \sin \omega,$$

and therefore $$d = x_1 \cos \omega + y_1 \sin \omega - p. \qquad \text{Q. E. D.}$$

In words: The perpendicular distance d is the result obtained by substituting the coördinates of P_1 for x and y in the left-hand member of the normal equation (1).

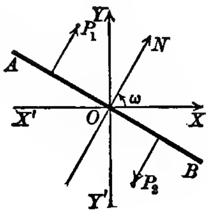
Hence the

Rule *to find the perpendicular distance d from a given line to a given point.*

Reduce the equation of the given line to the normal form (Art. 33), place d equal to the left-hand member of this equation, and then substitute the coördinates of the given point for x and y. The result is the required distance.

The sign of the result will show if the origin and the given point are on the same side (d is negative) or opposite sides (d is positive) of the line.

The perpendicular distance d from the line $Ax + By + C = 0$ to the point (x_1, y_1) will be, by this rule, equal to



(2) $$d = \frac{Ax_1 + By_1 + C}{\pm \sqrt{A^2 + B^2}},$$

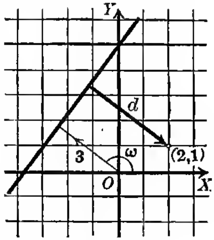
the sign of the radical being opposite to the sign of C.

When the given line AB passes through the origin, the positive direction on the normal ON is the *upward* direction. Hence the rule just stated will give a *positive* result for d when the perpendicular drawn **from** the line **to** the point has the upward direction, and a negative result in the contrary case. Thus in the figure the distance to P_1 is positive and to P_2 is negative.

Formula (2) may be used to find the perpendicular distance, but it is recommended that the rule be applied instead.

EXAMPLES

1. Find the perpendicular distance from the line $4x - 3y + 15 = 0$ to the point (2, 1).



Solution. The equation is reduced to the normal form by dividing by $-\sqrt{16+9} = -5$. Placing d equal to the left-hand member thus obtained,

$$d = \frac{4x - 3y + 15}{-5}.$$

Substituting $x = 2$, $y = 1$, then $d = \dfrac{8 - 3 + 15}{-5} = -4$.

Hence the *length* of the perpendicular distance is 4 and the point is on the same side of the line as the origin.

2. Find the equations of the bisectors of the angles formed by the lines

$$L_1 : x + 3y - 6 = 0,$$
$$L_2 : 3x + y + 2 = 0.$$

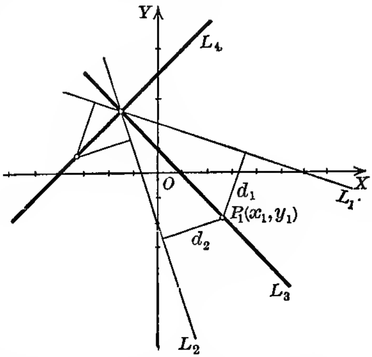
Solution. Let P_1 (x_1, y_1) be any point on the bisector L_3. Then, by geometry, P_1 is equally distant from the given lines. Thus, if

$$d_1 = \text{distance from } L_1 \text{ to } P_1,$$

and

$$d_2 = \text{distance from } L_2 \text{ to } P_1,$$

then d_1 and d_2 are *numerically* equal. Since, however, P_1 is on the same side of both lines as the origin, d_1 and d_2 are both negative. Hence for every point on the bisector L_3,



(1) $$d_1 = d_2.$$

By the rule for finding d,

$$d_1 = \frac{x_1 + 3y_1 - 6}{\sqrt{10}},$$

$$d_2 = \frac{3x_1 + y_1 + 2}{-\sqrt{10}}.$$

Substituting in (1) and reducing,

(2) $$x_1 + y_1 - 1 = 0.$$

Dropping the subscripts in order to follow the usual custom of having (x, y) denote any point on the line, we have for the equation of

(3) $$L_3 : x + y - 1 = 0. \quad \textit{Ans.}$$

For any point on the bisector L_4 the distances d_1 and d_2 will be equal numerically but will differ in sign. Hence, along L_4,

(4) $$d_1 = -d_2.$$

Proceeding as before, the equation of L_4 is found to be

(5) $$L_4 : x - y + 4 = 0. \quad \textit{Ans.}$$

We note that (3) and (5) represent perpendicular lines.

Regarded as a formal process, equations (3) and (5) of the bisectors are found by *reducing the equations of L_1 and L_2 to the normal form and then adding and subtracting these equations.*

PROBLEMS

1. Find the perpendicular distance from the line

(a) $x \cos 45° + y \sin 45° - \sqrt{2} = 0$ to $(5, -7)$. *Ans.* $-2\sqrt{2}$.

(b) $\frac{3}{5}x - \frac{4}{5}y - 1 = 0$ to $(2, 1)$. *Ans.* $-\frac{3}{5}$.

(c) $3x + 4y + 15 = 0$ to $(-2, 3)$. *Ans.* $-\frac{21}{5}$.

(d) $2x - 7y + 8 = 0$ to $(3, -5)$. *Ans.* $-\frac{49}{+\sqrt{53}}$.

(e) $x - 3y = 0$ to $(0, 4)$. *Ans.* $\frac{12}{+\sqrt{10}}$.

2. Do the origin and the point $(3, -2)$ lie on the same side of the line $x - y + 1 = 0$? *Ans.* Yes.

3. Does the line $2x + 3y + 2 = 0$ pass between the origin and the point $(-2, 3)$? *Ans.* No.

4. Find the lengths of the altitudes of the triangle formed by the lines $2x + 3y = 0$, $x + 3y + 3 = 0$, and $x + y + 1 = 0$.

Ans. $\frac{3}{\sqrt{13}}$, $\frac{6}{\sqrt{10}}$, and $\sqrt{2}$.

5. Find the length of the altitudes of the triangles whose vertices are

(a) $(7, 8)$, $(-8, 4)$, $(-2, -10)$.

(b) $(8, 0)$, $(0, -8)$, $(-3, -3)$.

(c) $(5, -4)$, $(-4, -5)$, $(0, 8)$.

6. Find the equations of the bisectors of the angles formed by

$$3x - 4y + 1 = 0 \text{ and } 4x + 3y - 1 = 0.$$

Ans. $7x - y = 0$ and $x + 7y - 2 = 0$.

7. Find the locus of all points which are twice as far from the line $12x + 5y - 1 = 0$ as from the y-axis. *Ans.* $14x - 5y + 1 = 0$.

8. Find the locus of points which are k times as far from $4x-3y+1=0$ as from $5x-12y=0$. *Ans.* $(52-25k)x-(39-60k)y+13=0$.

9. Find the bisectors of the angles formed by the lines in Problem 8. *Ans.* $77x-99y+13=0$ and $27x+21y+13=0$.

10. Find the distance between the parallel lines

(a) $\begin{cases} y=2x+5, \\ y=2x-3. \end{cases}$ *Ans.* $\dfrac{8}{\sqrt{5}}$.

(b) $\begin{cases} y=-3x+1, \\ y=-3x+4. \end{cases}$ *Ans.* $\dfrac{3}{\sqrt{10}}$.

(c) $\begin{cases} 2x-3y+4=0, \\ 4x-6y+9=0. \end{cases}$ *Ans.* $\dfrac{1}{2\sqrt{13}}$.

(d) $\begin{cases} y=mx+3, \\ y=mx-3. \end{cases}$ *Ans.* $\dfrac{6}{\sqrt{1+m^2}}$.

11. Find the equations of the bisectors of the angles of the following triangles, and prove that these bisectors meet in a common point:

(a) $x+2y-5=0$, $2x-y-5=0$, $2x+y+5=0$.
(b) $3x+y-1=0$, $x-3y-3=0$, $x+3y+11=0$.
(c) $3x+4y-22=0$, $4x-3y+29=0$, $y-5=0$.
(d) $x+2=0$, $y-3=0$, $x+y=0$.
(e) $x=0$, $y=0$, $x+y+3=0$.

12. Find the bisectors of the angles formed by the lines $4x-3y-1=0$ and $3x-4y+2=0$, and show that they are perpendicular. *Ans.* $7x-7y+1=0$ and $x+y-3=0$.

13. Find the equations of the bisectors of the angles formed by the lines $5x-12y+10=0$ and $12x-5y+15=0$.

14. Find the locus of a point the ratio of whose distances from the lines $4x-3y+4=0$ and $5x+12y-8=0$ is 13 to 5. *Ans.* $9x+9y-4=0$.

15. Find the bisectors of the interior angles of the triangle formed by the lines $4x-3y=12$, $5x-12y-4=0$, and $12x-5y-13=0$. Show that they meet in a point.
Ans. $7x-9y-16=0$, $7x+7y-9=0$, $112x-64y-221=0$.

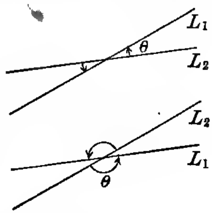
16. Find the bisectors of the interior angles of the triangle formed by the lines $5x-12y=0$, $5x+12y+60=0$, and $12x-5y-60=0$. Show that they meet in a point.
Ans. $2y+5=0$, $17x+7y=0$, $17x-17y-60=0$.

17. The sides of a triangle are $3x + 4y - 12 = 0$, $3x - 4y = 0$, and $4x + 3y + 24 = 0$. Show that the bisector of the interior angle at the vertex formed by the first two lines and the bisectors of the exterior angles at the other vertices meet in a point.

18. Find the equations of the lines parallel to $3x + 4y - 10 = 0$, and at a distance from it equal numerically to 3 units.

Ans. $3x + 4y = 25$ or -10.

35. The angle which a line makes with a second line. The angle between two directed lines has been defined (Art. 12) as the angle between their positive directions. When a line is given by means of its equation, no positive direction along the line is fixed. In order to distinguish between the two pairs of equal angles which two intersecting lines make with each other, we define the **angle which a line makes with a second line** to be the positive angle (p. 2) from the *second* line to the *first* line.



Thus the angle which L_1 makes with L_2 is the angle θ. We speak always of the "angle which one line makes with a second line," and the use of the phrase "the angle *between* two lines" should be avoided if those lines are not directed lines.

Theorem. *If m_1 and m_2 are the slopes of two lines, then the angle θ which the first line makes with the second is given by*

(VI) $$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Proof. Let α_1 and α_2 be the inclinations of L_1 and L_2 respectively. Then, since the exterior angle of a triangle equals the sum of the two opposite interior angles, we have

(Fig. 1) $$\alpha_1 = \theta + \alpha_2, \quad \text{or} \quad \theta = \alpha_1 - \alpha_2,$$

(Fig. 2) $$\alpha_2 = \pi - \theta + \alpha_1, \quad \text{or} \quad \theta = \pi + (\alpha_1 - \alpha_2).$$

And since (30, p. 3), $\tan(\pi + x) = \tan x$,

we have, in either case,

$$\tan\theta = \tan(\alpha_1 - \alpha_2)$$

$$= \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}. \qquad \text{(By 38, p. 3)}$$

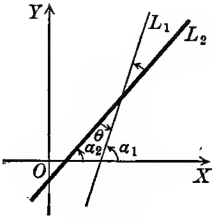


FIG. 1

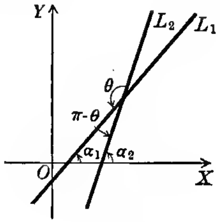


FIG. 2

But $\tan\alpha_1$ is the slope of L_1, and $\tan\alpha_2$ is the slope of L_2; hence, writing $\tan\alpha_1 = m_1$, $\tan\alpha_2 = m_2$, we have (VI).

In applying (VI) we remember that m_2 = slope of the line *from* which θ is measured in the *positive* direction. (The Greek letter θ used here is named "theta.")

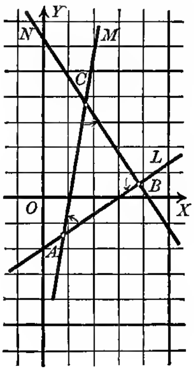
EXAMPLES

1. Find the angles of the triangle formed by the lines whose equations are

$$L: 2x - 3y - 6 = 0,$$

$$M: 6x - y - 6 = 0,$$

$$N: 6x + 4y - 25 = 0.$$



Solution. To see which angles formed by the given lines are the angles of the triangle, we plot the lines, obtaining the triangle ABC.

Let us find the angle A. In the figure, A is measured *from* the line L. Hence in (VI), m_2 = slope of $L = \frac{2}{3}$, m_1 = slope of $M = 6$.

$$\therefore \tan A = \frac{6 - \frac{2}{3}}{1 + 4} = \frac{16}{15}, \text{ and } A = \tan^{-1}\tfrac{16}{15}.$$

Next find the angle at B. In the figure, B is measured *from* N. Hence m_2 = slope of $N = -\frac{3}{2}$, m_1 = slope of $L = \frac{2}{3}$. Hence $m_2 = -\frac{1}{m_1}$, and $B = \frac{\pi}{2}$.

Finally, the angle at C is measured *from* the line M. Hence in (VI) m_2 = slope of $M = 6$, m_1 = slope of $N = -\frac{3}{2}$.

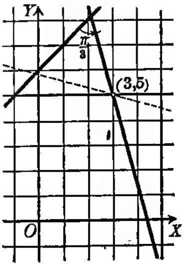
$$\therefore \tan C = \frac{-\frac{3}{2} - 6}{1 - 9} = \frac{15}{16}, \text{ and } C = \tan^{-1}\tfrac{15}{16}.$$

We may verify these results. For if $B = \frac{\pi}{2}$, then $A = \frac{\pi}{2} - C$; and hence (31, p. 3; and 26, p. 3) $\tan A = \cot C = \frac{1}{\tan C}$, which is true for the values found.

2. Find the equation of the line through (3, 5) which makes an angle of $\frac{\pi}{3}$ with the line $x - y + 6 = 0$.

Solution. Let m_1 be the slope of the required line. Then its equation is by (II), Art. 27,

(1) $$y - 5 = m_1(x - 3).$$



The slope of the given line is $m_2 = 1$, and since the angle which (1) makes with the given line is $\frac{\pi}{3}$, we have by (VI), since $\theta = \frac{\pi}{3} = 60°$,

$$\tan\frac{\pi}{3} = \frac{m_1 - 1}{1 + m_1},$$

or

$$\sqrt{3} = \frac{m_1 - 1}{1 + m_1}.$$

Whence

$$m_1 = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = -(2 + \sqrt{3}).$$

Substituting in (1), we obtain

$$y - 5 = -(2 + \sqrt{3})(x - 3),$$

or

$$(2 + \sqrt{3})x + y - (11 + 3\sqrt{3}) = 0.$$

In plane geometry there would be two solutions of this problem, — the line just obtained and the dotted line of the figure. Why must the latter be excluded here?

In working out the following problems the student should first draw the figure and mark by an arc the angle desired, remembering that this angle is measured from the second line to the first in the counterclockwise direction.

PROBLEMS

1. Find the angle which the line $3x - y + 2 = 0$ makes with $2x + y - 2 = 0$; also the angle which the second line makes with the first, and show that these angles are supplementary. *Ans.* $\frac{3\pi}{4}, \frac{\pi}{4}$.

2. Find the angle which the line

(a) $2x - 5y + 1 = 0$ makes with the line $x - 2y + 3 = 0$.
(b) $x + y + 1 = 0$ makes with the line $x - y + 1 = 0$.
(c) $3x - 4y + 2 = 0$ makes with the line $x + 3y - 7 = 0$.
(d) $6x - 3y + 3 = 0$ makes with the line $x = 6$.
(e) $x - 7y + 1 = 0$ makes with the line $x + 2y - 4 = 0$.

In each case plot the lines and mark the angle found by a small arc.

Ans. (a) $\tan^{-1}(-\frac{1}{12})$; (b) $\frac{\pi}{2}$; (c) $\tan^{-1}(\frac{13}{9})$; (d) $\tan^{-1}(-\frac{1}{2})$; (e) $\tan^{-1}(\frac{9}{13})$.

3. Find the angles of the triangle whose sides are $x + 3y - 4 = 0$, $3x - 2y + 1 = 0$, and $x - y + 3 = 0$.

Ans. $\tan^{-1}(-\frac{11}{3})$, $\tan^{-1}(\frac{1}{5})$, $\tan^{-1}(2)$.

Hint. Plot the triangle to see which angles formed by the given lines are the angles of the triangle.

4. Find the exterior angles of the triangle formed by the lines $5x - y + 3 = 0$, $y = 2$, $x - 4y + 3 = 0$.

Ans. $\tan^{-1}(5)$, $\tan^{-1}(-\frac{1}{4})$, $\tan^{-1}(-\frac{19}{9})$.

5. Find one exterior angle and the two opposite interior angles of the triangle formed by the lines $2x - 3y - 6 = 0$, $3x + 4y - 12 = 0$, $x - 3y + 6 = 0$. Verify the results by formula 37, p. 3.

6. Find the angles of the triangle formed by $3x + 2y - 4 = 0$, $x - 3y + 6 = 0$, and $4x - 3y - 10 = 0$.

7. Find the equation of the line passing through the given point and making the given angle with the given line.

(a) $(2, 1)$, $\frac{\pi}{4}$, $2x - 3y + 2 = 0$. *Ans.* $5x - y - 9 = 0$.

(b) $(1, -3)$, $\frac{3\pi}{4}$, $x + 2y + 4 = 0$. *Ans.* $3x + y = 0$.

(c) (x_1, y_1), ϕ, $y = mx + b$. *Ans.* $y - y_1 = \frac{m + \tan\phi}{1 - m\tan\phi}(x - x_1)$.

36. Systems of straight lines. An equation of the first degree in x and y which contains a single arbitrary constant will represent an infinite number of lines, for the locus of the equation will be a straight line for any value of the constant, and the locus will be different for different values of the constant.

The lines represented by an equation of the first degree which contains an arbitrary constant are said to form a *system*. The constant is called the **parameter** of the system.

Thus the equation $y = 2x + b$, where b is an arbitrary constant, represents the system of lines having the slope 2; and the equation $y - 5 = m(x - 3)$, where m is an arbitrary constant, represents the system of lines passing through (3, 5).

The methods already explained suffice for solving problems involving straight lines, but shorter methods result in some cases by using systems of lines, as will now be explained.

Given the line

(1) $$3x + 2y - 4 = 0.$$

Now every line of the system

(2) $$3x + 2y = k$$

is *parallel* to (1), for the slopes of (1) and (2) are equal.

Again, every line of the system

(3) $$2x - 3y = k$$

is *perpendicular* to (1); for the slope of (3) $= \frac{2}{3}$, the negative reciprocal of the slope of (1).

Note that the coefficients of x and y in (1) and (2) are the same, while the coefficients in (1) and (3) are interchanged and also the sign of one of them is changed.

Next, consider the line

$$y - 2 = 3(x + 2).$$

It passes through the point $(-2, 2)$. Now every line of the system

(4) $$y - 2 = k(x + 2)$$

passes through this point, since the equation is satisfied by its coördinates for all values of k.

Again, all the lines in the system

(5) $$x \cos k + y \sin k - 5 = 0$$

are at a distance of five units from the origin.

The value of the parameter k will depend upon the condition imposed upon the line (2), (3), (4), or (5).

Thus, if (2) must pass through $(1, -3)$, these coördinates must satisfy (2), and hence

$$3 - 6 = k. \quad \therefore k = -3.$$

That is, the equation of the line passing through $(1, -3)$ and parallel to $3x + 2y - 4 = 0$ is $3x + 2y + 3 = 0$.

Again, if (4) must form with the coördinate axes a triangle of unit area, we set one half the product of its intercepts equal to 1. Hence

$$\frac{1}{2}\left(\frac{-2k-2}{k}\right)(2k+2) = 1,$$

or

$$k^2 + \frac{5k}{2} + 1 = 0.$$

$$\therefore k = -2, -\tfrac{1}{2}.$$

Substituting these values in (4), we obtain

$$2x + y + 2 = 0, \quad x + 2y - 2 = 0,$$

both lines satisfying the above conditions.

Again, if (5) must pass through the point $(10, 0)$, then

$$10 \cos k = 5, \cos k = \tfrac{1}{2}, \sin k = \pm\sqrt{1 - \cos^2 k} = \pm\frac{\sqrt{3}}{2};$$

and substitution gives the two lines

$$x \pm \sqrt{3}\, y - 10 = 0.$$

In general, we may say this: *In finding the equation of a straight line defined by two conditions, we may begin by writing down the equation of the system of lines which satisfy one of these conditions, and then determine the value of the parameter so as to meet the second condition.*

PROBLEMS

1. Write the equations of the systems of lines defined by the conditions:

(a) Passing through $(-2, 3)$.
(b) Having the slope $-\frac{2}{5}$.
(c) Distance from the origin is 3.
(d) Having the intercept on the y-axis $= -3$.
(e) Passing through $(6, -1)$.
(f) Having the intercept on the x-axis $= 6$.
(g) Having the slope $\frac{1}{2}$.
(h) Having the intercept on the y-axis $= 5$.
(i) Distance from the origin $= 4$.
(j) Having one intercept double the other.
(k) Sum of the intercepts $= 4$.
(l) Length intercepted by the coördinate axes $= 3$.
(m) Forming a triangle of area 6 with the coördinate axes.

2. Determine k so that

(a) the line $2x - 3y + k = 0$ passes through $(-2, 1)$. *Ans.* $k = 7$.
(b) the line $2kx - 5y + 3 = 0$ has the slope 3. *Ans.* $k = \frac{15}{2}$.
(c) the line $x + y - k = 0$ passes through $(3, 4)$. *Ans.* $k = 7$.
(d) the line $3x - 4y + k = 0$ has intercept on the x-axis $= 2$.
Ans. $k = -6$.
(e) the line $x - 3ky + 4 = 0$ has intercept on the y-axis $= -3$.
Ans. $k = -\frac{4}{9}$.
(f) the line $4x - 3y + 6k = 0$ is distant three units from the origin.
Ans. $k = \pm\frac{5}{2}$.
(g) the line $2x + 7y - k = 0$ forms a triangle of area 3 with the coördinate axes. *Ans.* $k = \pm 2\sqrt{21}$.

3. Find the equation of the straight line which passes through the point

(a) $(0, 0)$ and is parallel to $x - 3y + 4 = 0$. *Ans.* $x - 3y = 0$.
(b) $(3, -2)$ and is parallel to $x + y + 2 = 0$. *Ans.* $x + y - 1 = 0$.
(c) $(-5, 6)$ and is parallel to $2x + 4y - 3 = 0$. *Ans.* $x + 2y - 7 = 0$.
(d) $(-1, 2)$ and is perpendicular to $3x - 4y + 1 = 0$.
Ans. $4x + 3y - 2 = 0$.
(e) $(-7, 2)$ and is perpendicular to $x - 3y + 4 = 0$.
Ans. $3x + y + 19 = 0$.

4. The equations of two sides of a parallelogram are $3x - 4y + 6 = 0$ and $x + 5y - 10 = 0$. Find the equations of the other two sides if one vertex is the point $(4, 9)$. *Ans.* $3x - 4y + 24 = 0$ and $x + 5y - 49 = 0$.

5. Find the equation of the straight line at a distance of three units from the origin, and which in addition satisfies the condition given:

(a) Parallel to the line $2x + y - 2 = 0$. *Ans.* $2x + y \pm 3\sqrt{5} = 0$.

(b) Perpendicular to the line $x + y - 1 = 0$. *Ans.* $x - y \pm 3\sqrt{2} = 0$.

(c) Passing through the point (0, 8). *Ans.* $\pm\sqrt{55}x + 3y - 24 = 0$.

(d) Passing through the point (1, 5).

(e) Forming a triangle with the coördinate axes whose area is 9.

(f) Making an angle of $\frac{3\pi}{4}$ with the line $x + 2y + 4 = 0$.

Ans. $3x + y \pm 3\sqrt{10} = 0$.

6. Find the equation of the straight line parallel to the line $3x + 4y - 7 = 0$, and satisfying the additional condition given:

(a) Passing through the point (2, $-$ 6).

(b) Forming a triangle of area 2 with the coördinate axes.

(c) Forming a triangle of perimeter 5 with the coördinate axes.

(d) The middle point of the intercepted part has unit abscissa.

(e) At a distance of three units from the origin.

(f) One unit nearer to the origin.

7. Find the equation of the line passing through the point (3, $-$ 2) and satisfying the additional condition:

(a) Parallel to the line $x - 7y - 8 = 0$.

(b) Perpendicular to the line $3x - 5y = 7$.

(c) Passing through the point (4, 1).

(d) Having the intercept ($-$ 7, 0).

(e) The sum of its intercepts is 6.

Ans. $x - 3y - 9 = 0$, $y + 2x - 4 = 0$.

(f) The given point bisects the part intercepted by the coördinate axes.

(g) Making an angle of 45° with the line $2x - 3y + 2 = 0$.

8. Find the equation of the line parallel to the line $3x + 4y - 15 = 0$, such that the point (2, $-$4) shall lie midway between the two lines.

Ans. $3x + 4y + 35 = 0$.

9. Find the equation of the straight line which forms a triangle of area 2 with the coördinate axes, and whose intercepts differ by 3.

37. System of lines passing through the intersection of two given lines. Given the two lines

$$(1) \qquad L_1 : x + 2y - 5 = 0,$$

$$(2) \qquad L_2 : 3x - y - 2 = 0.$$

Consider the system of lines whose equation is

(3) $$x + 2y - 5 + k(3x - y - 2) = 0,$$

where k is an arbitrary number.

It is easy to see that the line (3) will pass through the intersection of the given lines L_1 and L_2. In fact, by solving (1) and (2) for x and y, we find $x = 1$, $y = 2$, and these values satisfy (3).

Note that the equation (3) is formed from the left-hand members of (1) and (2) by multiplying one of them by the parameter k and adding. The method of forming (3) shows at once that the line it represents must pass through the intersection of the given lines.

Problems requiring the equation of a line which passes through the intersection of two given lines are often much shortened by forming the equation of the system (3) and determining k to meet the given condition. The advantage of this method is that *we do not need to know the coördinates of the point of intersection of L_1 and L_2.*

EXAMPLES

1. Find the equation of the line passing through P_1 (2, 1) and the intersection of $L_1: 3x - 5y - 10 = 0$ and $L_2: x + y + 1 = 0$.

Solution. The system of lines passing through the intersection of the given lines is represented by

$$3x - 5y - 10 + k(x + y + 1) = 0.$$

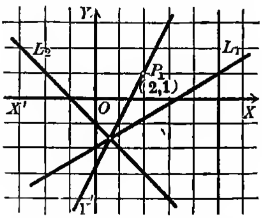
If P_1 lies on this line, then

$$6 - 5 - 10 + k(2 + 1 + 1) = 0;$$

whence $$k = \tfrac{9}{4}.$$

Substituting this value of k and simplifying, we have the required equation

$$21x - 11y - 31 = 0.$$

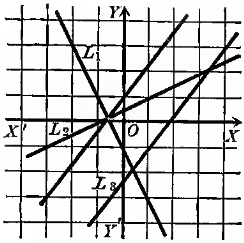


2. Find the equation of the line passing through the intersection of $L_1: 2x + y + 1 = 0$ and $L_2: x - 2y + 1 = 0$ and parallel to the line whose equation is $L_3: 4x - 3y - 7 = 0$.

Solution. The equation of every line through the intersection of the first two given lines has the form

$$2x + y + 1 + k(x - 2y + 1) = 0,$$

or $$(2 + k)x + (1 - 2k)y + (1 + k) = 0.$$



The slope of this line is $-\dfrac{2+k}{1-2k}$. This must equal the slope of L_3; that is, $\frac{4}{3}$.

$$\therefore -\frac{2+k}{1-2k} = \tfrac{4}{3}, \text{ or } k = 2.$$

Substituting and simplifying, we obtain

$$4x - 3y + 3 = 0. \quad \textit{Ans.}$$

Solve the following problems without finding the point of intersection of the two lines given.

PROBLEMS

1. Find the equation of the line passing through the intersection of $2x - 3y + 2 = 0$ and $3x - 4y - 2 = 0$, and which

(a) passes through the origin;
(b) is parallel to $5x - 2y + 3 = 0$;
(c) is perpendicular to $3x - 2y + 4 = 0$.

Ans. (a) $5x - 7y = 0$; (b) $5x - 2y - 50 = 0$; (c) $2x + 3y - 58 = 0$.

2. Find the equations of the lines which pass through the vertices of the triangle formed by the lines $2x - 3y + 1 = 0$, $x - y = 0$, and $3x + 4y - 2 = 0$, which are

(a) parallel to the opposite sides;
(b) perpendicular to the opposite sides.

Ans. (a) $3x + 4y - 7 = 0$, $14x - 21y + 2 = 0$, $17x - 17y + 5 = 0$;
(b) $4x - 3y - 1 = 0$, $21x + 14y - 10 = 0$, $17x + 17y - 9 = 0$.

3. Find the equation of the line passing through the intersection of $x + y - 2 = 0$ and $x - y + 6 = 0$ and through the intersection of $2x - y + 3 = 0$ and $x - 3y + 2 = 0$. *Ans.* $19x + 3y + 26 = 0$.

Hint. The systems of lines passing through the points of intersection of the two pairs of lines are

$$x + y - 2 + k(x - y + 6) = 0,$$

and $$2x - y + 3 + k'(x - 3y + 2) = 0.$$

These lines will coincide if the coefficients are proportional; that is, if

$$\frac{1+k}{2+k'}=\frac{1-k}{-1-3k'}=\frac{-2+6k}{3+2k'}.$$

Letting r be the common value of these ratios, we obtain

$$1+k=2r+rk',$$
$$1-k=-r-3rk',$$

and
$$-2+6k=3r+2rk'.$$

From these equations we can eliminate the terms in rk' and r, and thus find the value of k which gives that line of the first system which also belongs to the second system.

4. Find the equation of the line passing through the intersection of $2x+y-8=0$ and $3x+2y=0$ and

(a) parallel to the y-axis. *Ans.* $x-16=0$.
(b) parallel to the x-axis. *Ans.* $y+24=0$.

5. The equations of the sides of a parallelogram are $x+3y+2=0$, $x+3y-8=0$, $3x-2y=0$, $3x-2y-16=0$. Find the equations of the diagonals.

6. Find the equations of the lines through the point of intersection of the lines $x+3y-10=0$, $3x-y=0$, which are at unit distance from the origin. *Ans.* $x-1=0$, $4x-3y+5=0$.

7. Find the equations of the lines through the point of intersection of the two lines $7x+7y-24=0$, $x-y=0$, which form with the coördinate axes a triangle of perimeter 12.

Ans. $4x+3y-12=0$; $3x+4y-12=0$.

REVIEW. TRIANGLE PROBLEMS

1. In the following problems the coördinates of the vertices of a triangle are given. Find (1) the equations of the sides, (2) the equations of the perpendicular bisectors of the sides, (3) the equations of the medians, (4) the equations of the lines drawn from the vertices perpendicular to the opposite sides, (5) the equations of the lines drawn through the vertices parallel to the opposite sides, (6) the lengths of the three medians, (7) the lengths of the three altitudes, (8) the area, (9) the three angles, (10) the equation of the circumscribed circle.

(a) (8, 2), (6, 6), (– 1, 5).
(b) (– 4, 5), (– 3, 8), (4, 1).
(c) (4, 13), (16, 5), (– 1, – 12).
(d) (2, 4), (8, 4), (6, 0).
(e) (4, 0), (2, 4), (– 5, 3).
(f) (0, – 4), (6, – 2), (4, – 5).
(g) (– 3, – 3), (– 2, 0), (5, – 7).
(h) (0, 2), (8, 0), (5, 5).
(i) (3, – 1), (3, – 5), (0, – 2).
(j) (– 1, 15), (11, 7), (– 6, – 10).

2. In the following problems the coördinates of the vertices of a triangle are given. Find (1) the equations of the sides, (2) the equations of the perpendicular bisectors of the sides, (3) the equations of the bisectors of the interior angles, (4) the equation of the circumscribed circle, (5) the equation of the inscribed circle.

(a) $(8, 1), (2, 4), (-2, -4)$.
(b) $(6, 30), (36, -10), (-24, -10)$.
(c) $(3, 3), (-3, 6), (-7, -2)$.
(d) $(0, 32), (30, -8), (-30, -8)$.

3. In the following problems the equations of the sides of a triangle are given. Find (1) the angles, (2) the equations of the bisectors of the interior angles, (3) the equations of the bisectors of the exterior angles, (4) the equation of the inscribed circle.

(a) $4x - 3y - 4 = 0,\ 3x + 4y - 8 = 0,\ 5x - 12y - 60 = 0.$
(b) $5x + 12y - 24 = 0,\ 12x + 5y + 7 = 0,\ 5x - 12y - 48 = 0.$
(c) $5x - 12y - 42 = 0,\ 12x + 5y - 2 = 0,\ 5x + 12y - 66 = 0.$
(d) $12x + 5y + 50 = 0,\ 5x - 12y - 81 = 0,\ 5x + 12y - 33 = 0.$
(e) $4x - 3y + 25 = 0,\ 5x - 12y + 1 = 0,\ 3x + 4y - 5 = 0.$
(f) $5x + 12y - 123 = 0,\ 12x + 5y + 21 = 0,\ 5x - 12y - 27 = 0.$
(g) $5x - 12y - 3 = 0,\ 12x + 5y + 24 = 0,\ 5x + 12y - 75 = 0.$
(h) $12x + 5y + 50 = 0,\ 5x + 12y - 16 = 0,\ 5x - 12y - 16 = 0.$

CHAPTER V

THE CIRCLE

38. Equation of the circle. Every circle is determined when its center and radius are known.

Theorem. *The equation of the circle whose center is a given point* (α, β) *and whose radius equals* r *is*

(I) $$(x-\alpha)^2+(y-\beta)^2=r^2.$$

Proof. Assume that $P(x, y)$ is any point on the locus.

If the center (α, β) be denoted by C, the given condition is

$$PC = r.$$

By (I), p. 13, $$PC = \sqrt{(x-\alpha)^2+(y-\beta)^2}.$$

$$\therefore \sqrt{(x-\alpha)^2+(y-\beta)^2} = r.$$

Squaring, we have (I). Q.E.D.

Corollary. *The equation of the circle whose center is the origin* (0, 0) *and whose radius is* r *is*

$$x^2+y^2=r^2.$$

If (I) is expanded and transposed, we obtain

(1) $$x^2+y^2-2\,\alpha x-2\,\beta y+\alpha^2+\beta^2-r^2=0.$$

The form of this equation is clearly

$$x^2+y^2+\textit{terms of lower degree}=0.$$

In words: Any circle is defined by an equation of the *second degree* in the variables x and y, in which *the terms of the second degree consist of the sum of the squares of* x *and* y.

Equation (1) is of the form

(2) $$x^2+y^2+Dx+Ey+F=0,$$

where

(3) $$D=-2\,\alpha,\ E=-2\,\beta, \text{ and } F=\alpha^2+\beta^2-r^2.$$

Can we infer, conversely, that the locus of every equation of the form (2) is a circle? To decide this question transform (2) into the form of (I) as follows: Rewrite (2) by collecting the terms in x and the terms in y thus:

$$(4) \qquad x^2 + Dx + y^2 + Ey = -F.$$

Complete the square of the terms in x by adding $(\frac{1}{2}D)^2$ to both sides of (4), and do the same for the terms in y by adding $(\frac{1}{2}E)^2$ to both members.

Then (4) may be written

$$(5) \qquad (x + \tfrac{1}{2}D)^2 + (y + \tfrac{1}{2}E)^2 = \tfrac{1}{4}(D^2 + E^2 - 4F).$$

In (5) we distinguish three cases:

If $D^2 + E^2 - 4F$ is positive, (5) is in the form (I), and hence the locus of (2) is a circle whose center is $(-\frac{1}{2}D, -\frac{1}{2}E)$ and whose radius is $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

If $D^2 + E^2 - 4F = 0$, the only real values satisfying (5) are $x = -\frac{1}{2}D$, $y = -\frac{1}{2}E$ (footnote, p. 37). The locus, therefore, is the single point $(-\frac{1}{2}D, -\frac{1}{2}E)$. In this case the locus of (2) is often called a **point circle,** or a **circle whose radius is zero.**

If $D^2 + E^2 - 4F$ is negative, no real values satisfy (5), and hence (2) has no locus.

The expression $D^2 + E^2 - 4F$ is called the **discriminant** of (2), and is denoted by Θ (Greek letter "Theta"). The result is given by the

Theorem. *The locus of the equation*

$$\text{(II)} \qquad \boldsymbol{x^2 + y^2 + Dx + Ey + F = 0,}$$

whose discriminant is $\Theta = D^2 + E^2 - 4F$, *is determined as follows:*

When Θ *is positive, the locus is the circle whose center is* $(-\frac{1}{2}D, -\frac{1}{2}E)$ *and whose radius is* $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F} = \frac{1}{2}\sqrt{\Theta}$.

When Θ *is zero, the locus is the point circle* $(-\frac{1}{2}D, -\frac{1}{2}E)$.

When Θ *is negative, there is no locus.*

Corollary. *When* $E = 0$, *the center of* (II) *is on the x-axis, and when* $D = 0$, *the center is on the y-axis.*

EXAMPLE

Find the locus of the equation $x^2 + y^2 - 4x + 8y - 5 = 0$.

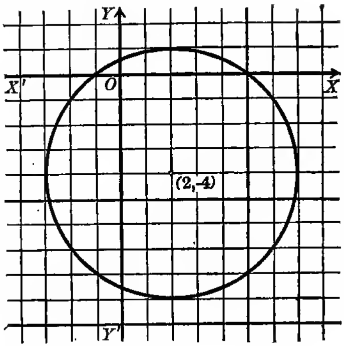
First solution. The given equation is of the form (II), where

$$D = -4, \quad E = 8, \quad F = -5,$$

and hence

$$\Theta = 16 + 64 + 20 = 100 > 0.$$

The locus is therefore a circle whose center is the point $(2, -4)$ and whose radius is $\frac{1}{2}\sqrt{100} = 5$.



Second solution. The problem may be solved without applying the theorem if we follow the method by which the theorem was established.

Collecting terms,

$$(x^2 - 4x) + (y^2 + 8y) = 5.$$

Completing the squares,

$$(x^2 - 4x + 4) + (y^2 + 8y + 16) = 25.$$

Or, also, $$(x - 2)^2 + (y + 4)^2 = 25.$$

Comparing with (I), $\alpha = 2, \beta = -4, r = 5.$

The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is called the **general equation of the second degree** in x and y because it contains all possible terms in x and y of the second and lower degrees. This equation can be reduced to the form (II) when and only when $A = C$ and $B = 0$. Hence the locus of an equation of the second degree is a circle only when the coefficients of x^2 and y^2 are equal and the xy-term is lacking.

39. Circles determined by three conditions. The equation of any circle may be written in either one of the forms

$$(x - \alpha)^2 + (y - \beta)^2 = r^2,$$

or

$$x^2 + y^2 + Dx + Ey + F = 0.$$

Each equation has three arbitrary constants. To determine these constants three equations are necessary. Such an equation means that the circle satisfies some geometrical condition. Hence a circle may be determined to satisfy three conditions.

Rule *to determine the equation of a circle satisfying three conditions.*

First step. *Let the required equation be*

$$(1)\qquad (x-\alpha)^2+(y-\beta)^2=r^2,$$

or

$$(2)\qquad x^2+y^2+Dx+Ey+F=0,$$

as may be more convenient.

Second step. *Find three equations between the constants α, β, and r [or D, E, and F] which express that the circle* (1) [*or* (2)] *satisfies the three given conditions.*

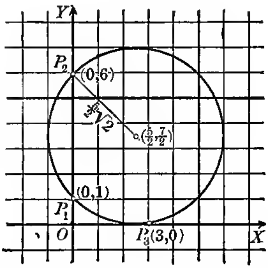
Third step. *Solve the equations found in the second step for α, β, and r [or D, E, and F].*

Fourth step. *Substitute the results of the third step in* (1) [*or* (2)]. *The result is the required equation.*

In some problems, however, a more direct method results by constructing the center of the required circle from the given conditions and then finding the equations and points of intersection of the lines of the figure.

EXAMPLES

1. Find the equation of the circle passing through the three points $P_1(0, 1)$, $P_2(0, 6)$, and $P_3(3, 0)$.



First solution. *First step.* Let the required equation be

$$(3)\qquad x^2+y^2+Dx+Ey+F=0.$$

Second step. Since P_1, P_2, and P_3 lie on (3), their coördinates must satisfy (3). Hence we have

$$(4)\qquad 1+E+F=0,$$

$$(5)\qquad 36+6E+F=0,$$

and

$$(6)\qquad 9+3D+F=0.$$

Third step. Solving (4), (5), and (6), we obtain

$$E=-7,\quad F=6,\quad D=-5.$$

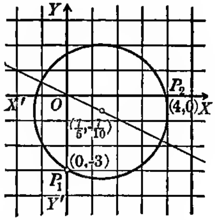
Fourth step. Substituting in (3), the required equation is

$$x^2 + y^2 - 5x - 7y + 6 = 0.$$

The center is $(\frac{5}{2}, \frac{7}{2})$, and the radius is $\frac{5}{2}\sqrt{2} = 3.5$.

Second solution. A second method which follows the geometrical construction for the circumscribed circle is the following. Find the equations of the perpendicular bisectors of P_1P_2 and P_1P_3. The point of intersection is the center. Then find the radius by the length formula.

2. Find the equation of the circle passing through the points $P_1(0, -3)$ and $P_2(4, 0)$ which has its center on the line $x + 2y = 0$.



First solution. *First step.* Let the required equation be

(7) $$x^2 + y^2 + Dx + Ey + F = 0.$$

Second step. Since P_1 and P_2 lie on the locus of (7), we have

(8) $$9 - 3E + F = 0,$$

and

(9) $$16 + 4D + F = 0.$$

The center of (7) is $\left(-\frac{D}{2}, -\frac{E}{2}\right)$, and since it lies on the given line,

$$-\frac{D}{2} + 2\left(-\frac{E}{2}\right) = 0,$$

or

(10) $$D + 2E = 0.$$

Third step. Solving (8), (9), and (10),

$$D = -\frac{14}{5}, \quad E = \frac{7}{5}, \quad F = -\frac{24}{5}.$$

Fourth step. Substituting in (7), we obtain the required equation,

$$x^2 + y^2 - \frac{14}{5}x + \frac{7}{5}y - \frac{24}{5} = 0,$$

or

$$5x^2 + 5y^2 - 14x + 7y - 24 = 0.$$

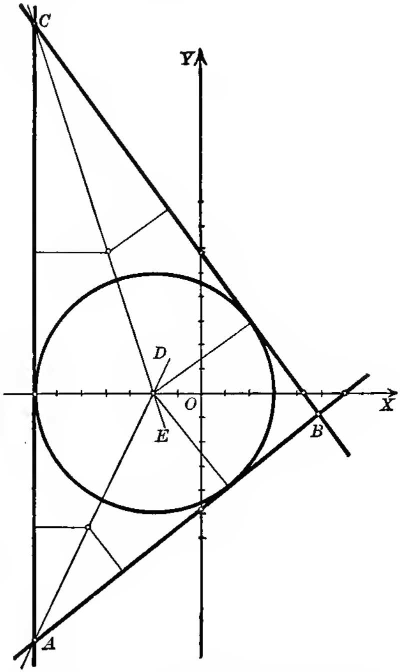
The center is the point $(\frac{7}{5}, -\frac{7}{10})$, and the radius is $\frac{1}{2}\sqrt{29}$.

Second solution. A second solution is suggested by geometry, as follows: Find the equation of the perpendicular bisector of P_1P_2. The point of intersection of this line and the given line is the center of the required circle. The radius is then found by the length formula.

3. Find the equation of the circle inscribed in the triangle whose sides are

$$AB: 3x - 4y - 19 = 0,$$
(11) $$BC: 4x + 3y - 17 = 0,$$
$$CA: x + 7 = 0.$$

Solution. The center is the point of intersection of the bisectors of the angles of the triangle. We therefore find the equations of the bisectors of the angles A and C.



Reducing equations (11) to the normal form,

(12) $$AB: \frac{3x - 4y - 19}{5} = 0;$$

$$BC: \frac{4x + 3y - 17}{5} = 0;$$

$$CA: \frac{x + 7}{-1} = 0.$$

Then, by Example 2, Art. 34, the bisectors are

(13) $$AD: \frac{3x-4y-19}{5} = \frac{x+7}{-1},$$

or $$2x - y + 4 = 0,$$

$$CE: \frac{4x + 3y - 17}{5} = \frac{x + 7}{-1},$$

or $$3x + y + 6 = 0.$$

The point of intersection of AD and CE is $(-2, 0)$. This is therefore the center of the inscribed circle. The radius is the perpendicular distance from any of the lines (11) to $(-2, 0)$. Taking the side AB, then, from (12),

$$r = \frac{3(-2) - 4(0) - 19}{5} = -5.$$

Hence, by (I), the equation of the required circle is

$$(x + 2)^2 + (y - 0)^2 = 25, \quad \text{or } x^2 + y^2 + 4x - 21 = 0. \quad \textit{Ans.}$$

PROBLEMS

1. Find the equation of the circle whose center is

(a) (0, 1) and whose radius is 3. *Ans.* $x^2 + y^2 - 2y - 8 = 0$.
(b) (– 2, 0) and whose radius is 2. *Ans.* $x^2 + y^2 + 4x = 0$.
(c) (– 3, 4) and whose radius is 5. *Ans.* $x^2 + y^2 + 6x - 8y = 0$.
(d) $(\alpha, 0)$ and whose radius is α. *Ans.* $x^2 + y^2 - 2\alpha x = 0$.
(e) $(0, \beta)$ and whose radius is β. *Ans.* $x^2 + y^2 - 2\beta x = 0$.
(f) $(0, -\beta)$ and whose radius is β. *Ans.* $x^2 + y^2 + 2\beta x = 0$.

2. Draw the locus of the following equations:

(a) $x^2 + y^2 - 6x - 16 = 0$.
(b) $3x^2 + 3y^2 - 10x - 24y = 0$.
(c) $x^2 + y^2 = 8x$.
(d) $x^2 + y^2 - 8x - 6y + 25 = 0$.
(e) $x^2 + y^2 - 2x + 2y + 5 = 0$.
(f) $x^2 + y^2 - 6x + 4y - 5 = 0$.
(g) $(x + 1)^2 + (y - 2)^2 = 0$.
(h) $7x^2 + 7y^2 - 4x - y = 3$.
(i) $x^2 + y^2 + 2ax + 2by + a^2 + b^2 = 0$.
(j) $x^2 + y^2 + 16x + 100 = 0$.

3. Show that the following loci are circles, and find the radius and the coördinates of the center in each case:

(a) A point moves so that the sum of the squares of its distances from (3, 0) and (– 3, 0) always equals 68. *Ans.* $x^2 + y^2 = 25$.

(b) A point moves so that its distances from (8, 0) and (2, 0) are always in a constant ratio equal to 2. *Ans.* $x^2 + y^2 = 16$.

(c) A point moves so that the ratio of its distances from (2, 1) and (– 4, 2) is always equal to $\frac{1}{2}$. *Ans.* $3x^2 + 3y^2 - 24x - 4y = 0$.

(d) The distance of a moving point from the fixed point (– 1, 2) is twice its distance from the origin. *Ans.* $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{3}$, $r = \dfrac{2\sqrt{5}}{3}$.

(e) The distance of a moving point from the fixed point $(2, -\frac{1}{2})$ is half its distance from the fixed point (0, 3).

(f) The square of the distance of a moving point from the origin is proportional to the sum of its distances from the coördinate axes.

(g) The square of the distance of a moving point from the fixed point (– 4, 3) is proportional to its distance from the line $3x - 4y - 5 = 0$.

(h) The sum of the squares of the distances of a point from the two lines $x - 2y = 0$, $2x + y - 10 = 0$, is unity.

4. Find the equation of a circle passing through any three of the following points:

(0, 2)	(3, 3)	(6, 2)	(7, 1)	(8, – 2)	(7, – 5)
(6, – 6)	(3, – 7)	(0, – 6)	(– 1, – 5)	(– 2, – 2)	(– 1, 1)

Ans. $x^2 + y^2 - 6x + 4y - 12 = 0$.

5. Find the equation of the circle which

(a) has the center (2, 3) and passes through (3, – 2).

Ans. $x^2 + y^2 - 4x - 6y - 13 = 0.$

(b) has the line joining (3, 2) and (– 7, 4) as a diameter.

Ans. $x^2 + y^2 + 4x - 6y - 13 = 0.$

(c) passes through the points (0, 0), (8, 0), (0, – 6).

Ans. $x^2 + y^2 - 8x + 6y = 0.$

(d) passes through (0, 1), (5, 1), (2, – 3).

Ans. $2x^2 + 2y^2 - 10x + y - 3 = 0.$

(e) circumscribes the triangle (4, 5), (3, – 2), (1, – 4).

(f) has the center (– 1, – 5) and is tangent to the x-axis.

Ans. $x^2 + y^2 + 2x + 10y + 1 = 0.$

(g) has the center (3, – 5) and is tangent to the line $x - 7y + 2 = 0.$

Ans. $x^2 + y^2 - 6x + 10y + 2 = 0.$

(h) passes through the points (3, 5) and (– 3, 7) and has its center on the x-axis. *Ans.* $x^2 + y^2 + 4x - 46 = 0.$

(i) passes through the points (4, 2) and (– 6, – 2) and has its center on the y-axis. *Ans.* $x^2 + y^2 + 5y - 30 = 0.$

(j) passes through the points (5, – 3) and (0, 6) and has its center on the line $2x - 3y - 6 = 0.$ *Ans.* $3x^2 + 3y^2 - 114x - 64y + 276 = 0.$

(k) passes through the points (0, 2), (– 1, 1) and has its center in the line $3y + 2x = 0.$ *Ans.* $x^2 + y^2 - 6x + 4y - 12 = 0.$

(l) circumscribes the triangle $x - 6 = 0$, $x + 2y = 0$, $x - 2y = 8.$

Ans. $2x^2 + 2y^2 - 21x + 8y + 60 = 0.$

(m) is inscribed in the triangle (0, 6), (8, 6), (0, 0).

Ans. $x^2 + y^2 - 4x - 8y + 16 = 0.$

(n) passes through (1, 0) and (5, 0) and is tangent to the y-axis.

Ans. $x^2 + y^2 - 6x \pm 2\sqrt{5}y + 5 = 0.$

(o) passes through the points (– 3, – 1), (1, 1) and is tangent to the line $4x + 3y + 25 = 0.$

6. Find the equations of the inscribed circles of the following triangles:

(a) $x + 2y - 5 = 0,$	$2x - y - 5 = 0,$	$2x + y + 5 = 0.$
(b) $3x + y - 1 = 0,$	$x - 3y - 3 = 0,$	$x + 3y + 11 = 0.$
(c) $3x + 4y - 22 = 0,$	$4x - 3y + 29 = 0,$	$y - 5 = 0.$
(d) $x + 2 = 0,$	$y - 3 = 0,$	$x + y = 0.$
(e) $x = 0,$	$y = 0,$	$x + y + 3 = 0.$

7. What is the equation of a circle whose radius is 10, if it is tangent to the line $4x + 3y - 70 = 0$ at the point whose abscissa is 10?

In the proofs of the following theorems the choice of the axes of coördinates is left to the student, since no mention is made of either coördinates or equations in the problem. In such cases always choose the axes in the most convenient manner possible.

8. A point moves so that the sum of the squares of its distances from two fixed points is constant. Prove that the locus is a circle.

9. A point moves so that the sum of the squares of its distances from two fixed perpendicular lines is constant. Prove that the locus is a circle.

10. A point moves so that the ratio of its distances from two fixed points is constant. Determine the nature of the locus.

Ans. A circle if the constant ratio is not equal to unity, and a straight line if it is.

11. A point moves so that the square of its distance from a fixed point is proportional to its distance from a fixed line. Show that the locus is a circle.

CHAPTER VI

TRANSCENDENTAL CURVES AND EQUATIONS

In the preceding chapters the emphasis has been laid chiefly on algebraic equations; that is, equations involving only powers of the coördinates. We now turn our attention to equations such as

$$y = \log x, \quad y = 2^x, \quad x = \sin y,$$

which are called *transcendental equations*, and their loci, *transcendental curves.*

40. Natural logarithms. The *common* logarithm of a given number N is the exponent x of the base 10 in the equation

(1) $$10^x = N; \text{ that is, } x = \log_{10} N.$$

A second system of logarithms, known as the *natural system*, is of fundamental importance in mathematics. The base of this system is denoted by e, and is called the *natural base.* Numerically to three decimal places, the natural base is always

(2) $$e = 2.718.$$

The *natural* logarithm of a given number N is the exponent y in the equation

(3) $$e^y = N; \text{ that is, } y = \log_e N.$$

To find the equation connecting the *common* and *natural* logarithms of a given number, we may take the logarithms of both members of (3) to the base 10, which gives

(4) $$\log_{10} e^y = \log_{10} N, \text{ or } y \log_{10} e = \log_{10} N. \qquad (16, \text{p. } 1)$$

(5) $$\therefore \log_{10} N = \log_{10} e \cdot \log_e N \text{ (using the value of } y \text{ in (3))}$$

The equation shows that the common logarithm of any number equals the product of the natural logarithm by the constant $\log_{10}e$. This constant is called the **modulus** ($= M$) of the common system. That is (Table, Art. 2),

(6) $$M = \log_{10} e = 0.434\text{; also } \frac{1}{M} = 2.302.$$

We may summarize in the equations,

(A) **Common log = natural log times 0.434,**
Natural log = common log times 2.302.

These equations show us how to find the natural logarithm from the common logarithm, or vice versa.

Exponential and logarithmic curves. The locus of the equation

(7) $$y = e^x$$

is called an *exponential curve*. From the preceding we may write (7) also in the form

(8) $$x = \log_e y = 2.302 \log_{10} y.$$

The locus of (7) is therefore the curve whose abscissas are the natural logarithms of the ordinates. Let us now discuss and plot (7). (Figure, p. 103.)

Discussion. Since negative numbers and zero have no logarithms, y is necessarily positive. Moreover, x increases as y increases. The coördinates of a few points on the locus are set down in the table. The discussion and figure illustrate the fact that

$$\log_e 0 = -\infty.$$

x	y	x	y
0	1	0	1
1	$e = 2.7$	-1	$\frac{1}{e} = .37$
2	$e^2 = 7.4$	-2	$\frac{1}{e^2} = .14$
etc.	etc.	etc.	etc.

For clearly, as y approaches zero, x becomes *negatively* larger and larger, without limit. Hence the x-axis is a horizontal asymptote.

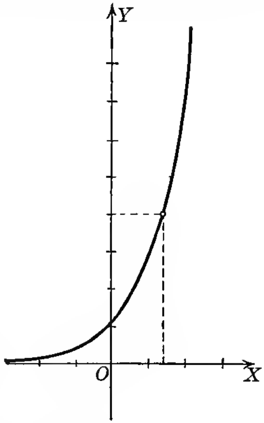
If the curve is carefully drawn, natural logarithms may be measured off. Thus, by measurement in the figure, if

$$y = 4,\ x = 1.38 = \log_e 4.$$

More generally, the locus of

(9) $$y = e^{kx},$$

where k is a given constant, is an *exponential curve.* The discussion of the difference of this locus from that in the figure is left to the reader.



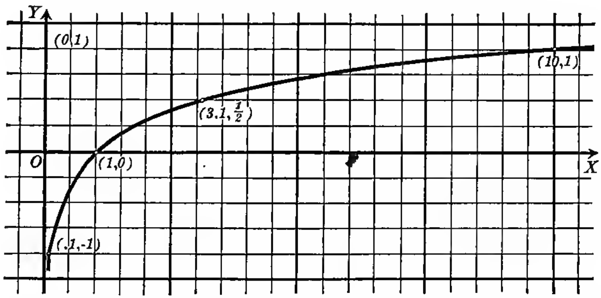
The locus of the equation

(10) $$y = \log_{10} x,$$

which is called a *logarithmic curve,* differs essentially from the locus of (7) only in its relation to the axes. In fact, both curves are exponential or logarithmic curves, depending upon the point of view.

The locus of (10) is given in the figure below. Clearly, since $\log_{10} 0 = -\infty$, the y-axis is a vertical asymptote. The scales chosen are

unit length on XX' equals 2 divisions,
unit length on YY' equals 4 divisions.



Compound interest curve. The problem of compound interest introduces exponential curves. For, if r = rate per cent of interest, and n = number of years, then the amount ($= A$) of one dollar in n years, if the interest is compounded annually, is given by the formula

$$A = (1 + r)^n.$$

For example, if the rate is 5 per cent, the formula is

(11) $$A = (1.05)^n.$$

If we plot years as abscissas and the amount as ordinates, the corresponding curve will be an exponential curve. For, by Art. 2, $\log_{10} 1.05 = .021$.

Hence, from (A), $$\log_e 1.05 = 2.302 \text{ times } .021$$
$$= .048 \text{ (to three decimal places).}$$

Hence, by (3), $e^{.048} = 1.05$, and the equation (11) becomes

(12) $$A = e^{.048n},$$

which is in the form of (9); that is, $k = .048$.

For convenience in plotting exponential curves accurately the following table is inserted.

Table of values of the exponential function e^x.

x	.0 e^x	.0 e^{-x}	.1 e^x	.1 e^{-x}	.2 e^x	.2 e^{-x}	.3 e^x	.3 e^{-x}	.4 e^x	.4 e^{-x}
0	1.00	1.00	1.11	0.90	1.22	0.82	1.35	0.74	1.49	0.67
1	2.72	0.37	3.00	0.33	3.32	0.30	3.67	0.27	4.06	0.25
2	7.39	0.14	8.17	0.12	9.03	0.11	9.97	0.10	11.0	0.09
3	20.1	0.05	22.2	0.05	24.5	0.04	27.1	0.04	30.0	0.03
4	54.6	0.02	60.3	0.02	66.7	0.01	73.7	0.01	81.5	0.01
5	148	0.01	164	0.01	181	0.01	200	0.00	221	0.00

x	.5 e^x	.5 e^{-x}	.6 e^x	.6 e^{-x}	.7 e^x	.7 e^{-x}	.8 e^x	.8 e^{-x}	.9 e^x	.9 e^{-x}
0	1.65	0.61	1.82	0.55	2.01	0.50	2.23	0.45	2.46	0.41
1	4.48	0.22	4.95	0.20	5.47	0.18	6.05	0.17	6.69	0.15
2	12.2	0.08	13.5	0.07	14.9	0.07	16.4	0.06	18.2	0.06
3	33.1	0.03	36.6	0.03	40.4	0.02	44.7	0.02	49.4	0.02
4	90.0	0.01	99.5	0.01	110	0.01	122	0.01	134	0.01
5	245	0.00	270	0.00	299	0.00	330	0.00	365	0.00

For example, to find the value of $e^{2.3}$, we look in the column with the caption x for the value 2 and then pass to the right under the caption .3. The value sought is found in the column under e^x to be 9.97. The next value to the right of this under e^{-x} is $e^{-2.3} = 0.10$.

PROBLEMS

Draw * the loci of each of the following:

1. $y = e^{-x}$.

2. $y = e^{-\frac{1}{2}x}$.

3. $y = e^{2x}$.

4. $y = e^{-2x}$.

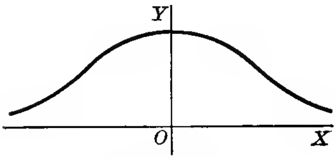
5. $y = 2e^{-x}$.

6. $y = 2e^{-\frac{1}{4}x}$.

7. $y = xe^{-x}$.

8. $s = t^2e^{-t}$.

9. $v = 2e^{-\frac{1}{2}u}$.



10. $y = e^{-x^2}$.

PROBABILITY CURVE

11. $y = 2\log_{10} x$.

12. $y = \log_e (1 + x)$.

13. $y = 2\log_{10} \frac{1}{2}x$.

14. $y = \log_{10}\sqrt{x}$.

15. $y = \log_e (1 + e^x)$.

16. $s = \log_{10} (1 + 2t)$.

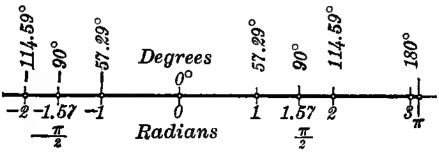
17. $v = \log_e (1 + t^2)$.

18. $x = \log_{10} (1 - y)$.

41. Sine curves. As already explained (p. 2), the two common methods of angular measurement, namely *circular measure* and *degree measure*, employ as units of measurement the *radian* and the *degree* respectively. The relation between these units is

$$(1)\qquad \begin{aligned} 1 \text{ radian} &= \frac{180}{\pi} \quad \text{or} \quad 57.29 \text{ degrees}, \\ 1 \text{ degree} &= 0.0174 \text{ radians} \quad \text{or} \quad \frac{\pi}{180}, \end{aligned}$$

in which $\pi = 3.14$ (or $\frac{22}{7}$ approximately), as usual.



Equations (1) may be written

$$(2)\qquad \pi \text{ radians} = 180 \text{ degrees}.$$

Thus $\frac{\pi}{2}$ radians $= 90°$, $\frac{\pi}{4}$ radians $= 45°$, etc. The two scales laid off on the same line give the figure.

* If the *shape* only of the curves 1–10 is desired, we may replace e by the approximate value 3.

In advanced mathematics it is assumed that circular measure is to be used. Thus the numerical values of

$$\sin 2x, \quad x \tan \frac{\pi x}{4}, \quad \frac{\cos \frac{\pi x}{6}}{2x}$$

for $x = 1$, are as follows:

$$\sin 2x = \sin 2 \text{ radians} = \sin 114°.59 = 0.909,$$

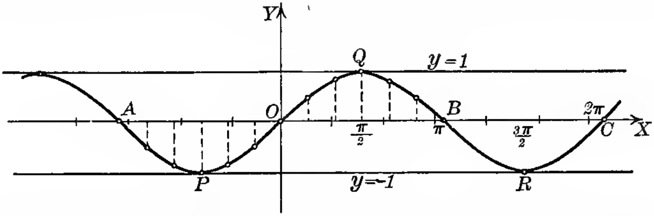
$$x \tan \frac{\pi x}{4} = 1 \cdot \tan\left(\frac{\pi}{4} \text{ radians}\right) = \tan 45° = 1,$$

$$\frac{\cos \frac{\pi x}{6}}{2x} = \frac{\cos\left(\frac{\pi}{6} \text{ radians}\right)}{2} = \frac{\cos 30°}{2} = 0.433.$$

Let us now draw the locus of the equation

(3) $$y = \sin x,$$

in which, as just remarked, x is the circular measure of an angle.



Solution. In making the calculation for plotting, it is convenient to choose angles at intervals of, say, 30°, and then find x, the circular measure of this angle, in radians, and y from the Table of Art. 4.

Angle in degrees	x radians	y	Angle in degrees	x radians	y
0	0	0	0	0	0
30	.52	.50	− 30	− .52	− .50
60	1.04	.86	− 60	− 1.04	− .86
90	1.56	1.00	− 90	− 1.56	− 1.00
120	2.08	.86	− 120	− 2.08	− .86
150	2.60	.50	− 150	− 2.60	− .50
180	3.14	0	− 180	− 3.14	0

Thus, for 30°, $y = \sin 30° = .50$. For 150°, $y = \sin 150° = \sin(180° - 30°) = \sin 30° = .50$ (30, p. 3).

To plot, choose a convenient unit of length on XX' to represent 1 radian, and use the same unit of length for ordinates. The divisions laid off on the x-axis in the figure are 1 radian, 2 radians, etc. Plotting the points (x, y) of the table, the curve $APOQB$ is the result.

The course of the curve beyond B is easily determined from the relation

$$\sin(2\pi + x) = \sin x.$$

Hence
$$y = \sin x = \sin(2\pi + x);$$

that is, the curve is *unchanged if* $x + 2\pi$ *be substituted for* x. This means, however, that every point is moved a distance 2π to the right. Hence the arc APO may be moved parallel to XX' until A falls on B, that is, into the position BRC, and *it will also be a part of the curve in its new position.* This property is expressed by the statement: The curve $y = \sin x$ is a periodic curve with a **period** equal to 2π. Also, the arc OQB may be displaced parallel to XX' until O falls upon C. In this way it is seen that the entire locus consists of an indefinite number of congruent arcs, alternately above and below XX'.

General discussion. 1. The curve passes through the origin, since (0, 0) satisfies the equation.

2. In (3), if $x = 0$, $y = \sin 0 = 0 =$ intercept on the axis of y.

Solving (3) for x,

(4)
$$x = \arc\sin y.$$

In (4), if $y = 0$, $x = \arc\sin 0 = n\pi$, n being any integer.

Hence the curve cuts the axis of x an indefinite number of times both on the right and left of O, these points being at a distance of π from one another.

3. Since $\sin(-x) = -\sin x$, changing signs in (3),

$$-y = -\sin x,$$

or
$$-y = \sin(-x).$$

Hence the locus is unchanged if (x, y) is replaced by $(-x, -y)$, and the curve is *symmetrical with respect to the origin* (Theorem II, p. 43).

4. In (3), x may have any value, since any number is the circular measure of an angle.

In (4), y may have values from -1 to $+1$ inclusive, since the sine of an angle has values only from -1 to $+1$ inclusive.

5. The curve extends out indefinitely along XX' in both directions, but is contained entirely between the lines $y = +1$, $y = -1$.

The locus is called the **wave curve**, from its shape, or the **sine curve**, from its equation (3). The maximum value of y is called the **amplitude.**

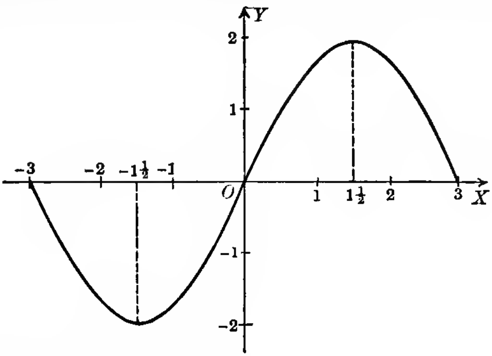
Again, let us construct the locus of

$$y = 2 \sin \frac{\pi x}{3}. \tag{5}$$

Solution. We now choose for x the values 0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, etc., radians, and arrange the work of calculation as in the table.

x radians	$\frac{1}{3}\pi x$ radians	$\frac{1}{3}\pi x$ degrees	$\sin \frac{\pi x}{3}$	y
0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{6}\pi$	30	.50	1.00
1	$\frac{1}{3}\pi$	60	.86	1.72
$1\frac{1}{2}$	$\frac{1}{2}\pi$	90	1.00	2.00
2	$\frac{2}{3}\pi$	120	.86	1.72
$2\frac{1}{2}$	$\frac{5}{6}\pi$	150	.50	1.00
3	π	180	0	0

The figure represents a sine curve of period 6 and amplitude 2. For the curve crosses the x-axis at intervals of 3, and the maximum value of y equals 2.



Equation (5) is of the form

$$y = a \sin kx.$$

The amplitude in this case equals a. To find the period, set $kx = 2\pi$. Solving for x, $x = \frac{2\pi}{k} =$ period.

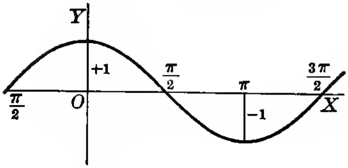
As it is important to sketch sine curves quickly, the following directions are useful:

1. *Find the amplitude and the period.*
2. *Choose the same scales on both axes.*
3. *Lay off points on XX' at intervals of a quarter-period. The highest and lowest points are at the odd quarter-periods. The intersections with XX' are at the even quarter-periods.*

PROBLEMS

Plot the loci of the equations:

1. *$y = \cos x$ (see figure).



2. $y = \sin 2x$.

3. $y = \cos 2x$.

4. $y = \sin \frac{1}{2} x$.

5. $y = \cos \frac{1}{2} x$.

6. $y = \cos \dfrac{\pi x}{3}$.

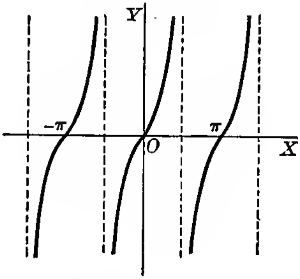
7. $y = \sin \dfrac{\pi x}{4}$.

8. $y = 3 \cos \dfrac{\pi x}{4}$.

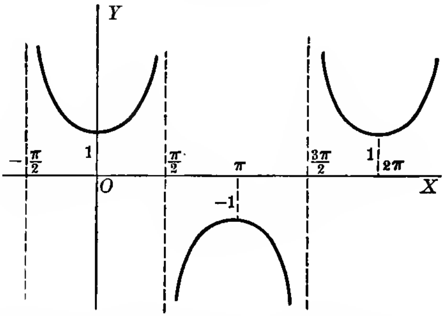
9. $y = 3 \sin \dfrac{\pi x}{5}$.

10. $y = 2 \sin \dfrac{\pi x}{2}$.

11. $y = \tan x$.



12. $y = \sec x$.



* The cosine curve differs from the sine curve only in the position of the y-axis. The highest and lowest points occur at half-periods and the intersections with OX at odd quarter-periods.

13. $y = \tan \frac{\pi x}{4}$.

14. $y = 2 \tan x$.

15. $y = 2 \tan \frac{\pi x}{3}$.

16. $y = 3 \tan \frac{\pi x}{4}$.

17. $y = \cot x$.

18. $y = \cot \frac{\pi x}{4}$.

19. $y = 4 \cot \frac{\pi x}{6}$.

20. $y = \csc x$.

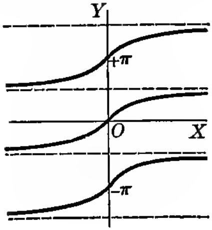
21. $y = \sec \frac{1}{2} x$.

22. $y = \csc \frac{1}{2} x$.

23. $y = \sec \frac{\pi x}{4}$.

24. $y = \csc \frac{\pi x}{4}$.

25. $x = \sin y$. Also written $y = \text{arc} \sin x$ or $\sin^{-1} x$, and read "the angle whose sine is x."



26. $x = 2 \cos y$, or $y = \text{arc} \cos \frac{1}{2} x$.

27. $x = \tan y$, or $y = \text{arc} \tan x$ (see figure).

28. $x = 2 \sin \frac{2}{3} \pi y$.

29. $x = \frac{1}{2} \cos \frac{1}{3} \pi y$.

30. $y = \text{arc} \tan \frac{1}{2} x$.

31. $y = 2 \text{ arc} \cos \frac{1}{3} x$.

The locus of the equation

$$(6) \qquad y = 2 \sin \left(\frac{\pi x}{3} + \frac{\pi}{6} \right)$$

is also a sine curve. For, by taking the coefficient of x, namely $\frac{\pi}{3}$, outside the parenthesis, (6) becomes

$$(7) \qquad y = 2 \sin \frac{\pi}{3} \left(x + \frac{1}{2} \right).$$

Now set $x + \frac{1}{2} = x'$. Substituting in (7), the latter becomes

$$(8) \qquad y = 2 \sin \frac{\pi x'}{3}.$$

But this is equation (5) except that x' takes the place of x. Hence to draw the locus of (6), proceed thus: Mark the point $x = -\frac{1}{2}$ (or $x' = 0$) on the x-axis. Using this point as the new origin, plot the locus of equation (8).

The figure obtained is obviously precisely that on page 108, if the y-axis is moved to the right a distance equal to $\frac{1}{2}$.

Observe that the period of (6) is determined, as before, by the coefficient of x. The added term $\left(\frac{\pi}{6} \right)$ simply affects the intercepts on the x-axis.

PROBLEMS

Plot the curves:

1. $y = \sin(x + \frac{1}{2})$.
2. $y = 2\cos(2x - 1)$.
3. $y = 2\sin\left(\frac{2\pi x}{3} + \frac{\pi}{4}\right)$.
4. $y = 2\sin\left(2\pi x + \frac{\pi}{6}\right)$.
5. $y = 3\cos\left(\frac{\pi x}{2} - \frac{2\pi}{3}\right)$.
6. $y = \tan\left(\frac{\pi x}{4} + \frac{\pi}{2}\right)$.
7. $y = \frac{1}{2}\sin(\frac{1}{2}x + \frac{1}{3})$.
8. $y = \cos\left(x + \frac{\pi}{3}\right)$.
9. $s = a\sin(kx + \pi)$.
10. $x = 2\cos\left(\frac{2\pi y}{5} - \frac{\pi}{2}\right)$.
11. $x = \sin\left(\frac{2\pi y}{3} - \pi\right)$.
12. $s = a\cos\left(\frac{2\pi t}{P} + \beta\right)$.

42. Addition of ordinates. When the equation of a curve has the form

y = the algebraic sum of two expressions,

as, for example, $y = \sin x + \cos x$, $y = \frac{1}{2}x + \sin^2 x$, $s = e^t + e^{-t}$, etc., the principle known as addition of ordinates may with advantage be employed. For example, to construct the locus of

$$y = 2\sin\frac{\pi x}{4} + \frac{1}{2}x, \tag{1}$$

we employ the auxiliary curves

$$y_1 = 2\sin\frac{\pi x}{4}, \tag{2}$$

$$y_2 = \tfrac{1}{2}x. \tag{3}$$

Plot these curves one below the other, keeping the y-axes in a straight line. The *same scales* must be used in both figures. The locus of (2) is the sine curve of Fig. 1, p. 112. The locus of (3) is the straight line in Fig. 2.

The ordinates of Fig. 1 are now added to the corresponding ones in Fig. 2, attention being given to the algebraic signs. The derived curve $A_1B_1OB_2A_2$ has the equation

$$y = y_1 + y_2 = 2\sin\frac{\pi x}{4} + \frac{1}{2}x \tag{4}$$

as required. The locus winds back and forth across the line $y = \frac{1}{2}x$, crossing the line at $x = 0, \pm 4, \pm 8, \pm 12$, etc.; that is, directly under the points where the sine curve in Fig. 1 crosses the x-axis.

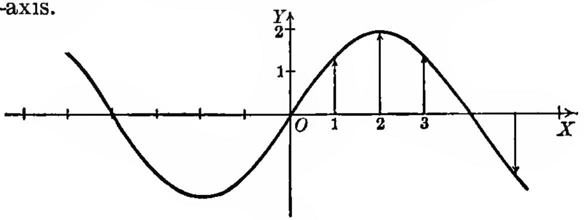


FIG. 1

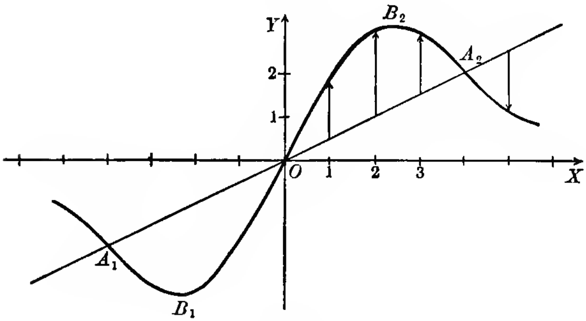


FIG. 2

PROBLEMS

Draw the following curves and calculate y accurately for the given value of x:

1. $y = \cos x + \frac{1}{3}x.$ $x = 1.$
2. $y = \sin 2x + \frac{x^2}{10}.$ $x = \frac{1}{2}.$
3. $y = \sin x + \cos x.$ $x = -\frac{1}{2}.$
4. $y = \frac{1}{4}x - 3\sin\frac{\pi x}{3}.$ $x = 2.$
5. $y = \frac{x^2}{16} - 4\cos\frac{\pi x}{4}.$ $x = -2.$
6. $y = \frac{e^x + e^{-x}}{2}.$ $x = \frac{3}{2}.$
7. $y = e^x - \sin 2x.$ $x = -\frac{1}{2}.$
8. $y = \frac{e^t - e^{-t}}{2}.$ $x = 3.$
9. $y = e^{\frac{x}{4}} - \cos 4x.$ $x = \frac{1}{2}\pi.$
10. $y = \sin x + \sin 2x.$ $x = 0.8.$

11. $y = \sin\frac{\pi x}{4} + \cos\frac{\pi x}{3}$. $x = -\frac{1}{2}$.

12. $y = \sin ax + \cos ax$. $x = \frac{9}{10}a$.

13. $y = 2\sin x + 5\cos x$. $x = 0.5$.

14. $y = 2\sin 2x + 3\cos\frac{1}{2}x$. $x = 2$.

15. $y = \sin ax + \sin bx$.

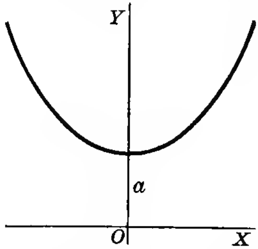
16. $y = \sqrt{9 - x^2} + \sin 2\pi x$. $x = 2\frac{1}{2}$.

17. $y = e^{-x} + 4x^2$. $x = -2.4$.

18. $y = \log_{10}x + \sin\frac{2\pi x}{3}$. $x = 2$.

19. $y = 2\sqrt{x} + \frac{1}{2}\cos\left(\frac{\pi x}{2} + \frac{\pi}{3}\right)$.

20. $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$. $x = 2\frac{1}{5}a$.



The locus in Problem 20 is called the **catenary** (see figure). The shape of the curve is that assumed by a heavy flexible cord freely suspended from its extremities.

The student may have observed from the preceding examples the truth of the following

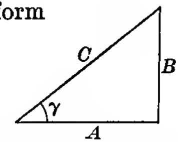
Theorem. *The curve obtained by adding corresponding ordinates of sine curves with the same period is also a sine curve with equal period.*

For example, consider the equation

$$(5) \qquad y = a\sin\left(\frac{2\pi t}{P} + \alpha\right) + b\sin\left(\frac{2\pi t}{P} + \beta\right),$$

in which α, β, and P are constants. The period of both sine curves equals P. Expand the right-hand member by the rule (33, p. 3) for $\sin(x + y)$ and collect the terms in $\sin\frac{2\pi t}{P}$ and $\cos\frac{2\pi t}{P}$. Then equation (5) assumes the form

$$(6) \qquad y = A\sin\frac{2\pi t}{P} + B\cos\frac{2\pi t}{P},$$



where A and B are constants, independent of t. Let us now introduce the angle γ of the right triangle whose legs are A and B. Let the hypotenuse $\sqrt{A^2 + B^2} = C$. Then $B = C\sin\gamma$,

$A = C\cos\gamma$. Substituting these values in (6) gives

$$(7)\quad y = C\left(\sin\frac{2\pi t}{P}\cos\gamma + \cos\frac{2\pi t}{P}\sin\gamma\right) = C\sin\left(\frac{2\pi t}{P} + \gamma\right).$$

This is a sine curve with period P and amplitude $C = \sqrt{A^2 + B^2}$.

Q.E.D.

The curve resulting from the addition of ordinates of sine curves with *unequal* periods is, however, *not* a sine curve.

43. Boundary curves. In plotting the locus of an equation of the form

$$(1)\qquad y = \textit{product of two factors}$$

one of which is a sine or cosine, as, for example,

$$y = e^x \sin x, \quad \text{or} \quad s = t^2 \cos\frac{\pi t}{4},$$

much aid is obtained by the following considerations:

For example, consider the locus of

$$(2)\qquad y = e^{-\frac{1}{4}x}\sin\frac{\pi x}{2}.$$

We now make the following observations:

1. Since the numerical value of the sine never exceeds unity, the values of y in (2) will not exceed in numerical value the value of the first factor $e^{-\frac{1}{4}x}$. Moreover, the extreme values of $\sin\frac{1}{2}\pi x$ are $+1$ and -1 respectively. *Hence y has the extreme values $e^{-\frac{1}{4}x}$ and $-e^{-\frac{1}{4}x}$.*

Consequently, if the curves

$$(3)\qquad y = e^{-\frac{1}{4}x} \quad \text{and} \quad y = -e^{-\frac{1}{4}x}$$

are drawn, the locus of (2) will lie entirely *between* these curves. They are accordingly called *boundary curves.*

Draw these curves. The second is obviously symmetrical to the first with respect to the x-axis. To plot, find three points on the first curve, as in the table. (Use the Table, p. 104.)

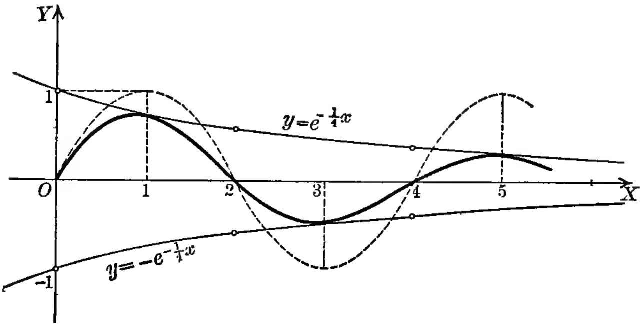
x	y
0	1
2	$e^{-\frac{1}{2}} = .61$
4	$e^{-1} = .37$

2. When $\sin \frac{1}{2}\pi x = 0$, then in (2) $y = 0$, since the first factor is always *finite*. Hence *the locus of* (2) *meets the x-axis in the same points as the auxiliary sine curve*

(4) $$y = \sin \tfrac{1}{2}\pi x.$$

3. The required curve *touches** the boundary curves when the second factor, $\sin \frac{1}{2}\pi x$, is $+1$ or -1; that is, when the ordinates of the auxiliary curve (4) have a maximum or minimum value.

Hence draw the sine curve (4). The period is 4 and the amplitude is 1. This curve is the dotted line of the figure.



The discussion shows these facts:

The locus of (2) *crosses XX' at $x = 0, \pm 2, \pm 4, \pm 6$, etc., and touches the boundary curves* (3) *at $x = \pm 1, \pm 3, \pm 5$, etc.*

We may then readily sketch the curve, as in the figure; that is, the winding curve between the boundary curves (3).

4. For a check remember that the ordinate of (2) is the *product* of the ordinates of the boundary curve $y = e^{-\frac{1}{4}x}$ and the sine curve (4). In the figure, for example, the required curve lies above XX' between $x = 0$ and $x = 2$, for the ordinates of $y = e^{-\frac{1}{4}x}$ and of the sine curve are now all positive. But between $x = 2$ and $x = 4$ the required curve lies below XX', for the ordinates of $y = e^{-\frac{1}{4}x}$ and the sine curve now have unlike signs.

*The discussion shows merely that the curve (2) *reaches* the boundary curves. *Tangency* is shown by calculus.

PROBLEMS

Draw the following loci and calculate y accurately for the given values of x:

1. $y = \frac{x}{4}\sin x.$ $x = 2;\ \frac{1}{2}\pi.$

2. $y = \frac{x^2}{16}\cos 2x.$ $x = 1;\ \pi.$

3. $y = \frac{x}{3}\sin\frac{\pi x}{3}.$ $x = 3;\ \frac{1}{2}.$

4. $y = \frac{x^2}{10}\cos\frac{\pi x}{5}.$ $x = 3;\ 2\frac{1}{2}.$

5. $y = e^{-x}\sin x.$ $x = \frac{1}{4}\pi;\ \frac{5}{4}\pi.$

6. $y = e^{-x}\cos 2x.$ $x = \frac{1}{2}\pi;\ 2.$

7. $y = e^{-\frac{1}{2}x}\sin\frac{\pi x}{4}.$ $x = -2;\ 3.$

8. $y = e^{-\frac{1}{4}x}\cos\frac{\pi x}{3}.$ $x = 3;\ -\frac{1}{2}.$

9. $y = 4e^{-\frac{1}{10}x}\cos\left(\frac{\pi x}{3} + \frac{\pi}{4}\right).$

10. $y = ae^{-a^2x}\cos\left(\frac{2\pi x}{P} + \alpha\right).$

11. $y = \frac{\sin x}{x}\left(=\frac{1}{x}\sin x\right).$ $x = 0.1.$

12. $y = \frac{\sin 2x}{2x}.$ $x = 0.1;\ 1.$

13. $y = \frac{\cos x}{x}.$ $x = 1;\ \pi.$

14. $y = \frac{\sin x}{x^2}.$ $x = 0.2;\ \frac{1}{2}\pi.$

15. $y = \frac{\sin\frac{\pi x}{4}}{x}.$ $x = 0.1;\ 2.$

16. $y = \sin\frac{1}{2}x\cos 2x.$ $x = \frac{1}{2}\pi.$

17. $y = \left(x + \frac{1}{2}\right)\sin\frac{1}{2}x.$

18. $y = \frac{x^2}{4}\cos\frac{1}{2}x - \frac{1}{4}\cos\frac{1}{2}x.$

19. $y = e^{-t}\sin\pi t + e^{-\frac{1}{2}t}\sin\frac{\pi t}{2}.$

20. Draw the two loci obtained (1) by adding and (2) by multiplying the ordinates in the following pairs of curves:

(a) $\begin{cases} y = x - \pi, \\ y = \sin x. \end{cases}$

(b) $\begin{cases} y = e^{\frac{x}{8}}, \\ y = \cos\pi x. \end{cases}$

(c) $\begin{cases} y = e^{-\frac{x}{8}}, \\ y = \sin\pi x. \end{cases}$

(d) $\begin{cases} y = 3 + \frac{x^2}{16}, \\ y = \sin\frac{\pi x}{2}. \end{cases}$

(e) $\begin{cases} y = 2 + \frac{x^2}{16}, \\ y = \cos\frac{\pi x}{3}. \end{cases}$

(f) $\begin{cases} y = \frac{16 - x^2}{8}, \\ y = \cos\frac{\pi x}{2}. \end{cases}$

44. Transcendental equations. Graphical solution. The solution of certain equations of frequent occurrence may be simplified by using graphical methods.

Consider the equation

(1) $\cot x = x$, or $\cot x - x = 0$.

To find values of x (in radians) for which this equation holds.

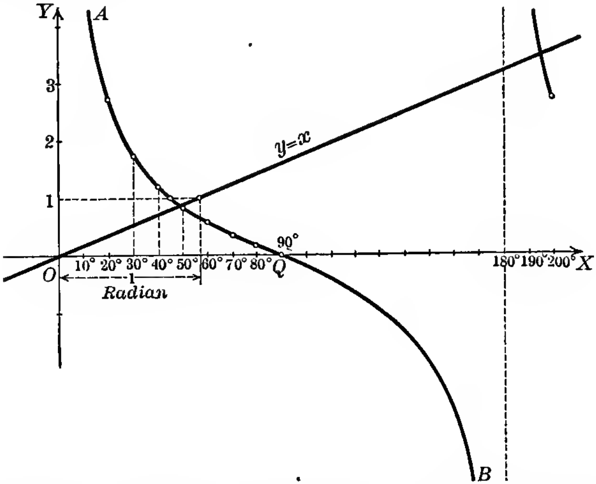
To aid in determining the roots, let us draw the curves

(2) $y = \cot x$ and $y = x$.

Now *the abscissa of each point of intersection is a root of equation* (1), for, obviously, at each point of intersection of the curves (2) we must have $\cot x = x$; that is, equation (1) is satisfied.

$y = \cot x$		
Degrees	**x radians**	**y**
0	0	∞
10	.174	5.67
20	.342	2.75
30	.524	1.73
40	.698	1.19
45	.785	1.000
50	.873	.839
60	1.047	.577
70	1.222	.364
80	1.396	.176
90	1.571	.0

In plotting it is well to lay off carefully both scales (degrees and radians) on OX.



Number of solutions. The curve $y = \cot x$ consists of an infinite number of branches congruent to AQB of the figure.

The line $y = x$ will obviously cross each branch. Hence the equation (1) has an infinite number of solutions.

Smallest solution. From the figure this solution lies between 45° and 50°, or, in radians, between $x = .785$ and $x = .873$. Hence the first significant figure of the smallest root is 0.8. Interpolation is necessary to determine subsequent figures.

For this purpose arrange the work thus, using the preceding table.

	x *(radians)*	$\cot x$	$\cot x - x$
	.873	.839	$-.034$
	.785	1.000	$+.215$
difference	$+.088$		$-.249$

We wish to know what change in x above .785 will produce a decrease in $\cot x - x$ equal to .215; that is, make $\cot x - x$ equal to zero. Call this change z. Then, by proportion,

$$\frac{z}{.088} = \frac{-.215}{-.249}. \qquad \therefore z = .076$$

Hence $x = .785 + .076 = .86$ (to two decimal places).

PROBLEMS

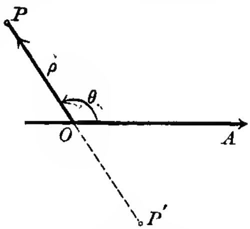
Determine graphically the number of solutions in each of the following, and find the smallest root (different from zero).

1. $\cos x = x$. *Ans.* One solution; $x = 0.74$.
2. $\sin 2x = x$. *Ans.* Three solutions; $x = 0.95$
3. $\tan x = x$. *Ans.* Infinite number.
4. $\sin x = \frac{1}{2}x$. *Ans.* Three.
5. $\sin x = x^2$. *Ans.* Two.
6. $\cos x = x^2$. *Ans.* Two.
7. $\tan x = x^2$.
8. $\cot x = x^2$.
9. $\cos x = \frac{x}{3}$.
10. $\tan x = 1 - x$.
11. $\cos x = 1 - x$.
12. $3 \sin x = \cos x - \frac{1}{2}$.
13. $3 \sin x = 2 \cos 4x$.
14. $2 \sin \frac{x}{2} = \cos 2x$.
15. $\sin 3x = \cos 2x$.
16. $e^{-x} = x$.
17. $e^{x} = \sin x$.
18. $e^{-x} = \cos x$.
19. $e^{x} = \tan x$.
20. $\sin x = \log_{10} x$.
21. $\cos x = \log_{10} x$.
22. $\tan x = \log_{10} x$.
23. $e^{-x} = \log_e x$.
24. $e^{-x^2} = x^2$.

CHAPTER VII

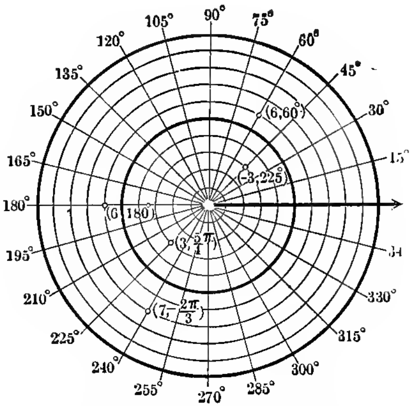
POLAR COÖRDINATES

45. Polar coördinates. In this chapter we shall consider a second method of determining points of the plane by pairs of real numbers. We suppose given a fixed point O, called the **pole**, and a fixed line OA, passing through O, called the **polar axis**. Then any point P determines a length $OP = \rho$ (Greek letter "rho") and an angle $AOP = \theta$. The numbers ρ and θ are called the **polar coördinates** of P. ρ is called the **radius vector** and θ the **vectorial angle.** The vectorial angle θ is *positive* or *negative* as in trigonometry. The radius vector is *positive* if P lies on the terminal line of θ, and *negative* if P lies on that line produced through the pole O.



Thus in the figure the radius vector of P is positive, and that of P' is negative.

It is evident that *every pair of real numbers* (ρ, θ) *determines a single point,* which may be plotted by the

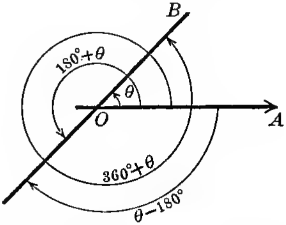


Rule *for plotting a point whose polar coördinates* (ρ, θ) *are given.*

First step. *Construct the terminal line of the vectorial angle θ, as in trigonometry.*

Second step. *If the radius vector is positive, lay off a length $OP = \rho$ on the terminal line of θ; if negative, produce the terminal line through the pole and lay off OP equal to the numerical value of ρ. Then P is the required point.*

In the figure on page 119 are plotted the points whose polar coördinates are $(6, 60°)$, $\left(3, \frac{5\pi}{4}\right)$, $(-3, 225°)$, $(6, 180°)$, and $\left(7, -\frac{2\pi}{3}\right)$.



Every point determines an infinite number of pairs of numbers (ρ, θ).

Thus, if $OB = \rho$, the coördinates of B may be written in any one of the forms (ρ, θ), $(-\rho, 180° + \theta)$, $(\rho, 360° + \theta)$, $(-\rho, \theta - 180°)$, etc.

Unless the contrary is stated, *we shall always suppose that θ is positive, or zero, and less than* 360°; that is, $0 \leqq \theta < 360°$.

PROBLEMS

1. Plot the points $(4, 45°)$, $(6, 120°)$, $\left(-2, \frac{2\pi}{3}\right)$, $\left(4, \frac{\pi}{3}\right)$, $(-4, -240°)$, $(5, \pi)$.

2. Plot the points $\left(6, \pm\frac{\pi}{4}\right)$, $\left(-2, \pm\frac{\pi}{2}\right)$, $(3, \pi)$, $(-4, \pi)$, $(6, 0)$, $(-6, 0)$.

3. Show that the points (ρ, θ) and $(\rho, -\theta)$ are symmetrical with respect to the polar axis.

4. Show that the points (ρ, θ), $(-\rho, \theta)$ are symmetrical with respect to the pole.

5. Show that the points $(-\rho, 180° - \theta)$ and (ρ, θ) are symmetrical with respect to the polar axis.

46. Locus of an equation. If we are given an equation in the variables ρ and θ, then the **locus of the equation** is a curve such that

1. Every point whose coördinates (ρ, θ) satisfy the equation lies on the curve.

2. The coördinates of every point on the curve satisfy the equation.

The curve may be plotted by solving the equation for ρ and finding the values of ρ for particular values of θ until the coördinates of enough points are obtained to determine the form of the curve.

The plotting is facilitated by the use of polar coördinate paper, which enables us to plot values of θ by lines drawn through the pole and values of ρ by circles having the pole as center. The tables on page 6 are to be used in constructing tables of values of ρ and θ.

EXAMPLES

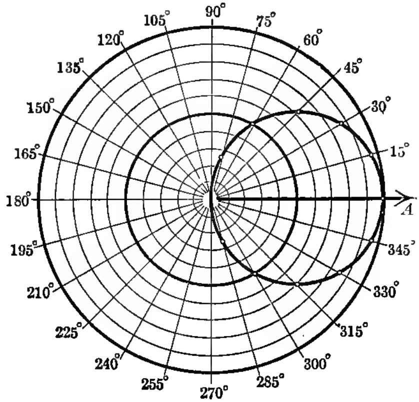
1. Plot the locus of the equation

(1) $$\rho = 10 \cos \theta.$$

Solution. The calculation is made by assuming values for θ, as in the table, and calculating ρ, making use of the natural values of the cosine given in Art. 4. For example, if

$$\theta = 105°, \ \rho = 10 \cos 105° = 10 \cos (180° - 75°) = -10 \cos 75° = -2.6.$$

$\rho = 10 \cos \theta$			
θ	ρ	θ	ρ
0	10	105°	− 2.6
15°	9.7	120°	− 5
30°	8.7	135°	− 7
45°	7	150°	− 8.7
60°	5	165°	− 9.7
75°	2.6	180°	−10
90°	0		



The complete locus is found in this example without going beyond 180° for θ. The curve is a circle (Art. 50).

Since $\cos(-\theta) = \cos\theta$ (29, p. 3), equation (1) may be written $\rho = 10 \cos(-\theta)$; that is, for every point (ρ, θ) on the locus there is also a second point $(\rho, -\theta)$ on the locus. Since these points are symmetrical with respect to the polar axis, we have the result: *The locus of* (1) *is symmetrical with respect to the polar axis.*

2. Draw the locus of

$$\rho^2 = a^2 \cos 2\theta. \tag{2}$$

Solution. Before plotting, we make the following observations:

1. Since the maximum value of $\cos 2\theta$ is 1, the maximum value of ρ is a, and the curve must be *closed.*

2. When $\cos 2\theta$ is negative, ρ will be imaginary. Now $\cos 2\theta$ is negative when 2θ is an angle in the second or third quadrant. That is, when

$$90^\circ < 2\theta < 270^\circ, \text{ that is, } 45^\circ < \theta < 135^\circ,$$

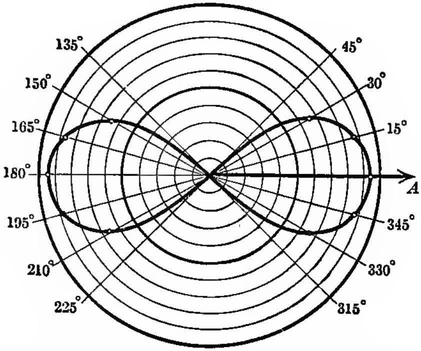
ρ is imaginary. There is no part of the curve between the 45° and 135° lines.

3. We may change θ to $-\theta$ in (2) without affecting the equation, and hence the locus is symmetrical with respect to the polar axis.

The complete curve is obtained if θ is given values from 0° to 45°, as in the table.

$\rho^2 = a^2 \cos 2\theta$

θ	2θ	$\cos 2\theta$	ρ
0	0	1	$\pm a$
15°	30°	.866	$\pm .93a$
30°	60°	.500	$\pm .7a$
45°	90°	0	0



The complete curve results by plotting these points and the points symmetrical to them with respect to the polar axis. The curve is called a **lemniscate**. In the figure a is taken equal to 9.5.

3. Discuss and plot the locus of the equation

$$\rho = a \sec^2 \tfrac{1}{3}\theta. \tag{3}$$

For convenience we change the form of the equation. Using (26), p. 3,

$$\rho = \frac{a}{\cos^2 \frac{1}{2}\theta}.$$

Then by (41), p. 4, $\cos^2 \frac{1}{2}\theta = \frac{1}{2} + \frac{1}{2}\cos\theta$. Hence the result:

$$\rho = \frac{2a}{1 + \cos\theta}.$$

$\rho = 2 \div (1 + \cos\theta)$							
θ	$\cos\theta$	$1+\cos\theta$	ρ	θ	$\cos\theta$	$1+\cos\theta$	ρ
0	1	2	1	105°	−.259	.741	2.7
15°	.966	1.966	1.02	120°	−.500	.500	4
30°	.866	1.866	1.07	135°	−.707	.293	6.7
45°	.707	1.707	1.2	150°	−.866	.134	14
60°	.500	1.500	1.3	165°	−.966	.034	50
75°	.259	1.259	1.6	180°	−1	0	∞
90°	0	1	2				

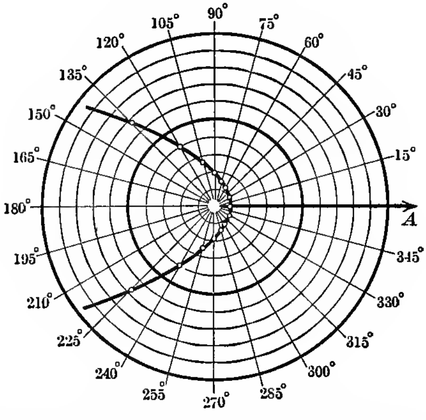
Solution. Before plotting, we note

1. The curve is symmetrical with respect to the polar axis, since θ may be replaced by $-\theta$.

2. ρ becomes infinite when $1 + \cos\theta = 0$, or $\cos\theta = -1$, and hence $\theta = 180°$. The curve recedes to infinity in the direction $\theta = 180°$.

3. ρ is never imaginary.

On account of 1 the table of values is computed only to $\theta = 180°$, and the rest of the curve is obtained from the symmetry with respect to the polar axis. Take $a = 1$. The locus is a parabola.



Before plotting polar equations, the student should establish such simple facts as result from a discussion, as illustrated above.

PROBLEMS

Plot the loci of the following equations:

1. $\rho = 10.$

2. $\theta = 45°.$

3. $\rho = 16 \cos\theta.$

4. $\rho \cos\theta = 6.$

5. $\rho \sin\theta = 4.$

6. $\rho = \dfrac{4}{1 - \cos\theta}.$

7. $\rho = \dfrac{8}{2 - \cos\theta}.$

8. $\rho = \dfrac{8}{1 - 2\cos\theta}.$

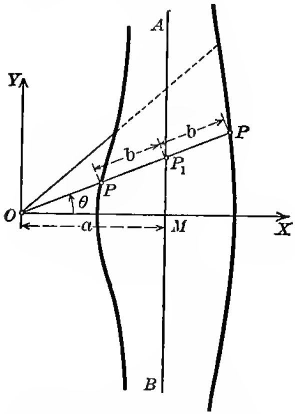
9. $\rho = a \sin\theta.$

10. $\rho = \dfrac{10}{1 + \tan\theta}.$

11. $\rho^2 \sin 2\theta = 16.$

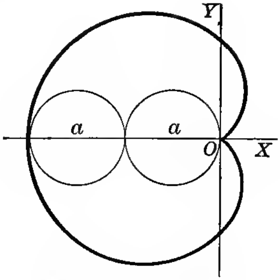
12. $\rho^2 \cos 2\theta = a^2.$

13. $\rho \cos\theta = a \sin^2\theta.$



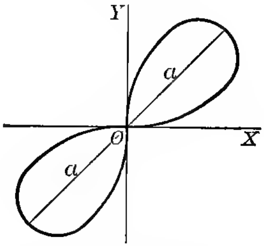
14. $\rho = a \sec\theta \pm b. \quad b < a.$

CONCHOID OF NICOMEDES



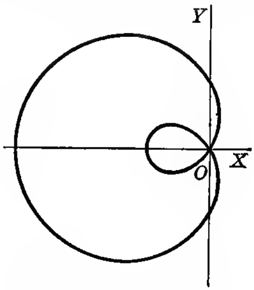
15. $\rho = a(1 - \cos\theta).$

CARDIOID



16. $\rho^2 = a^2 \sin 2\theta.$

TWO-LEAVED ROSE LEMNISCATE



17. $\rho = b - a\cos\theta. \quad b < a.$

LIMAÇON

18. Plot the conchoid (Problem 14) for $b = a$; $b > a$.

19. Plot the limaçon (Problem 17) for $b > a$.

47. The student should acquire skill in plotting polar equations rapidly when a rough diagram will serve.

For example, to draw the locus of

$$\rho = a \sin 3\theta, \tag{1}$$

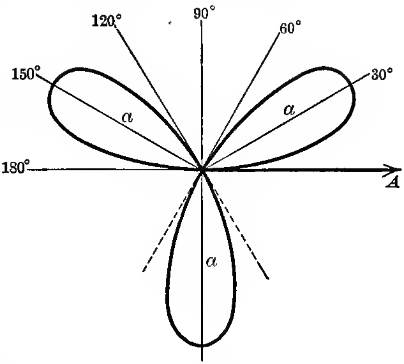
we proceed as follows:

Let θ increase from 0°. Follow the variation of ρ from (1) *as 3θ describes the successive quadrants.*

When 3θ varies from	0° to 90°	90° to 180°	180° to 270°	270° to 360°	360° to 450°	450° to 540°
then θ varies from	0° to 30°	30° to 60°	60° to 90	90° to 120°	120° to 150°	150° to 180°
and ρ varies from	0 to a	a to 0	0 to $-a$	$-a$ to 0	0 to a	a to 0

For example, when 3θ varies from 270° to 360°, that is, is an angle in the fourth quadrant, then ρ is negative and increases from $-a$ to 0.

Now draw the radial lines corresponding to the intervals of θ; that is, 0°, 30°, 60°, 90°, 120°, 150°, 180°.



Noting the variation of ρ, we sketch the curve as follows:

The curve starts from the pole in the direction 0°, crosses the 30° line perpendicularly at $\rho = a$, returns to and passes through the pole on the 60° line, crosses the 90° line produced at $\rho = -a$, returns to and passes through the pole on the 120° line (produced), crosses the 150° line at $\rho = a$, and returns to the pole on the 180° line.

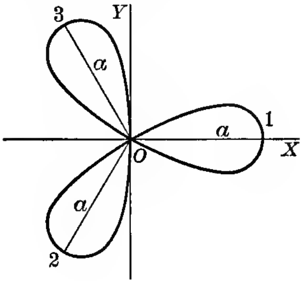
This gives the complete locus. The pencil point has moved continuously without *abrupt change in direction, and has returned to the original position and direction.*

The curve is called the **three-leaved rose.**

PROBLEMS

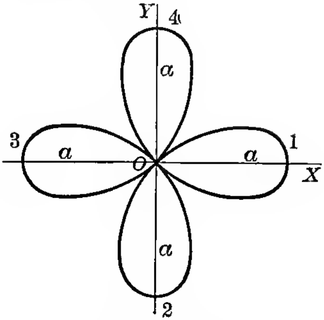
Draw rapidly the locus of each of the following equations:

1. $\rho = a \cos 3\theta$.



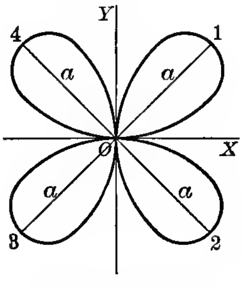
THREE-LEAVED ROSE

3. $\rho = a \cos 2\theta$.



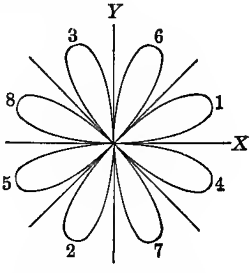
FOUR-LEAVED ROSE

2. $\rho = a \sin 2\theta$.



FOUR-LEAVED ROSE

4. $\rho = a \sin 4\theta$.



EIGHT-LEAVED ROSE

5. $\rho = a \cos 4\theta$.

6. $\rho = a \sin 5\theta$.

7. $\rho = a \cos 5\theta$.

8. $\rho = a(1 + \sin\theta)$.

9. $\rho = a(1 + \cos\theta)$.

10. $\rho = a \cos(\theta + 45°)$.

11. $\rho = a \sin\left(\theta + \frac{\pi}{6}\right)$.

12. $\rho = a \sin \frac{1}{2}\theta$.

13. $\rho = a \cos \frac{\theta}{3}$.

14. $\rho = a \sin 6\theta$.

15. $\rho = a \sin^2 \frac{1}{2}\theta$.

16. $\rho = a \cos^2 \frac{1}{2}\theta$.

17. $\rho = a \sin^3 \frac{1}{3}\theta$.

18. $\rho = a \cos^3 \frac{1}{3}\theta$.

48. Points of intersection. By a method analogous to that used in rectangular coördinates we find the coördinates of the points of intersection of two polar curves by solving their equations simultaneously. This is best done by eliminating ρ, which will give rise in general to a transcendental equation in θ which can be solved either by inspection or by the graphical method employed in Art. 44.

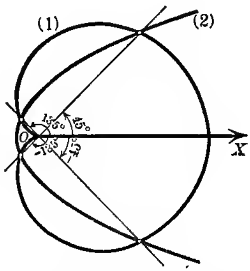
The following example will illustrate the method.

EXAMPLE

Find the points of intersection of

(1) $$\rho = 1 + \cos\theta,$$

(2) $$\rho = \frac{1}{2(1-\cos\theta)}.$$



Solution. Eliminating ρ,

$$1 + \cos\theta = \frac{1}{2(1-\cos\theta)},$$

or $$1 - \cos^2\theta = \tfrac{1}{2},$$

$$\cos\theta = \pm\frac{\sqrt{2}}{2}.$$

$$\therefore\ \theta = \pm 45°,\ \pm 135°.$$

Substituting these values in either equation, we obtain the following four points,

$$\left(1+\frac{\sqrt{2}}{2},\ \pm 45°\right),\ \left(1-\frac{\sqrt{2}}{2},\ \pm 135°\right).$$

The result checks in the figure. The locus of (1) is a cardioid; of (2), a parabola.

PROBLEMS

Find the points of intersection of the following pairs of curves and check by drawing the figure:

1. $\begin{cases} 4\rho\cos\theta = 3, \\ 2\rho = 3. \end{cases}$

2. $\begin{cases} 4\rho\cos\theta = 3, \\ \rho = 3\cos\theta. \end{cases}$

3. $\begin{cases} 2\rho = 3, \\ \rho = 3\sin\theta. \end{cases}$

4. $\begin{cases} \rho = \sqrt{3}, \\ \rho = 2\sin\theta. \end{cases}$

5. $\begin{cases} \rho = \cos\theta, \\ 4\rho = 3\sec\theta. \end{cases}$

6. $\begin{cases} \rho = 1+\cos\theta, \\ 2\rho = 3. \end{cases}$

7. $\begin{cases} 2\rho = \sec^2\dfrac{\theta}{2}, \\ \rho = 2. \end{cases}$

8. $\begin{cases} 3\rho = 4\cos\theta, \\ 2\rho\cos^2\dfrac{\theta}{2} = 1. \end{cases}$

9. $\begin{cases} \rho = \sin\theta, \\ \rho = \cos 2\theta. \end{cases}$ *Ans.* $(\frac{1}{2}, 30°)$, $(\frac{1}{2}, 150°)$.

10. $\begin{cases} \rho = 1 + \cos\theta, \\ \rho(1 + \cos\theta) = 1. \end{cases}$ *Ans.* $(1, \pm 90°)$.

11. $\begin{cases} \rho = 2(1 - \sin\theta), \\ \rho(1 + \sin\theta) = 1. \end{cases}$ *Ans.* $(2 \mp \sqrt{2}, \pm 45°)$, $(2 \mp \sqrt{2}, \pm 135°)$.

12. $\begin{cases} \rho = 4(1 + \cos\theta), \\ \rho(1 - \cos\theta) = 3. \end{cases}$ *Ans.* $(6, \pm 60°)$, $(2, \pm 120°)$.

13. $\begin{cases} \rho = 5 - 2\sin\theta, \\ \rho = \dfrac{6}{1 + \sin\theta}. \end{cases}$

14. $\begin{cases} \rho = 3 - 2\cos\theta, \\ \rho = \dfrac{8}{3 + 2\cos\theta}. \end{cases}$

15. $\begin{cases} \rho^2 = 9\cos 2\theta, \\ \rho = \sqrt{6}\cos\theta. \end{cases}$

16. $\begin{cases} \rho^2 = \sin 2\theta, \\ \rho = \sqrt{2}\sin\theta. \end{cases}$

17. $\begin{cases} \rho = \cos 3\theta, \\ 2\rho = \cos\theta. \end{cases}$

18. $\begin{cases} \rho = \theta, \\ \rho = \cos\theta. \end{cases}$

49. Transformation from rectangular to polar coördinates. Let OX and OY be the axes of a rectangular system of coördinates, and let O be the pole and OX the polar axis of a system of

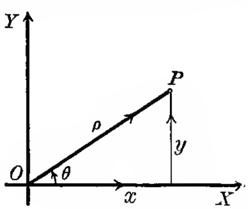


Fig. 1

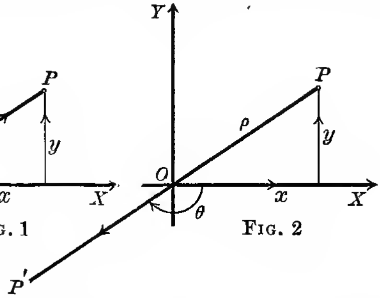


Fig. 2

polar coördinates. Let (x, y) and (ρ, θ) be respectively the rectangular and polar coördinates of any point P. It is necessary to distinguish two cases according as ρ is positive or negative.

When ρ is *positive* (Fig. 1) we have, by definition,

$$\cos\theta = \frac{x}{\rho}, \quad \sin\theta = \frac{y}{\rho},$$

whatever quadrant P is in.

Hence

(1) $$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$

When ρ is *negative* (Fig. 2) we consider the point P' symmetrical to P with respect to O, whose rectangular and polar coördinates are respectively $(-x, -y)$ and $(-\rho, \theta)$. The radius vector of P', $-\rho$, is positive, since ρ is negative, and we can therefore use equations (1). Hence for P'

$$-x = -\rho \cos\theta, \quad -y = -\rho \sin\theta;$$

and hence for P

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta,$$

as before.

Hence we have the

Theorem. *If the pole coincides with the origin and the polar axis with the positive x-axis, then*

$$\text{(I)} \qquad \begin{cases} x = \rho \cos\theta, \\ y = \rho \sin\theta, \end{cases}$$

where (x, y) are the rectangular coördinates and (ρ, θ) the polar coördinates of any point.

Equations (I) are called the **equations of transformation** from rectangular to polar coördinates. They express the rectangular coördinates of any point in terms of the polar coördinates of that point and enable us to find the equation of a curve in polar coördinates when its equation in rectangular coördinates is known, and vice versa.

From the figures we also have

$$\text{(2)} \qquad \begin{cases} \rho^2 = x^2 + y^2, & \theta = \tan^{-1}\dfrac{y}{x}, \\ \sin\theta = \dfrac{y}{\pm\sqrt{x^2+y^2}}, & \cos\theta = \dfrac{x}{\pm\sqrt{x^2+y^2}}. \end{cases}$$

These equations express the polar coördinates of any point in terms of the rectangular coördinates. They are not as convenient for use as (I), although the first one is at times very convenient.

EXAMPLES

1. Find the equation of the circle $x^2 + y^2 = 25$ in polar coördinates.

Solution. From the first equation of (2), we have at once $\rho^2 = 25$; hence $\rho = \pm 5$, which is the required equation. It expresses the fact that the point (ρ, θ) is five units from the origin.

2. Find the equation of the lemniscate (Ex. 2, p. 122) $\rho^2 = a^2 \cos 2\theta$ in rectangular coördinates.

Solution. By 39, p. 4, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\rho^2 = a^2(\cos^2\theta - \sin^2\theta).$$

Substituting from (2),

$$x^2 + y^2 = a^2\left(\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2}\right).$$

$$\therefore (x^2 + y^2)^2 = a^2(x^2 - y^2). \quad Ans.$$

50. Applications. Straight line and circle.

Theorem. *The general equation of the straight line in polar coördinates is*

$$\text{(II)} \qquad \rho(A \cos\theta + B \sin\theta) + C = 0,$$

where A, B, and C are arbitrary constants.

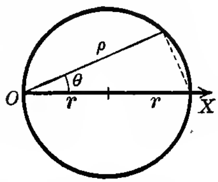
Proof. The general equation of the line in rectangular coördinates is

$$Ax + By + C = 0.$$

By substitution from (I) we obtain (II). Q.E.D.

Special cases of (II) are $\rho \cos\theta = a$, $\rho \sin\theta = b$, which result respectively when $B = 0$, or $A = 0$; that is, when the line is parallel to OY or OX.

In like manner we obtain from (II), p. 93, the



Theorem. *The general equation of the circle in polar coördinates is*

$$\text{(III)} \qquad \rho^2 + \rho(D \cos\theta + E \sin\theta) + F = 0,$$

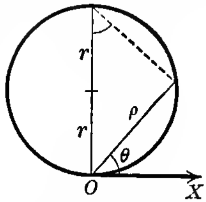
where D, E, and F are arbitrary constants.

We may easily show further that if the pole is on the circumference and the polar axis is a diameter, the equation of the circle is

$$\rho = 2r\cos\theta,$$

where r is the radius of the circle.

For if the center lies on the polar axis, or x-axis, $E = 0$, and if the circle passes through the pole, or origin, $F = 0$. The abscissa of the center equals the radius, and hence $-\frac{D}{2} = r$, or $D = -2r$. Substituting these values of D, E, and F in (III) gives $\rho - 2r\cos\theta = 0$.



This result is easily seen also directly from the figure on page 130.

Similarly, if the circle touches the polar axis at the pole, the equation is $\rho = 2r\sin\theta$.

Theorem. *The length l of the line joining two points $P_1(\rho_1, \theta_1)$ and $P_2(\rho_2, \theta_2)$ is given by*

$$\text{(IV)} \qquad l^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_1 - \theta_2).$$

Proof. Let the rectangular coördinates of P_1 and P_2 be respectively (x_1, y_1) and (x_2, y_2). Then by (I), p. 129,

$$x_1 = \rho_1\cos\theta_1, \quad x_2 = \rho_2\cos\theta_2,$$
$$y_1 = \rho_1\sin\theta_1, \quad y_2 = \rho_2\sin\theta_2.$$

But $\qquad l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$

and hence $\quad l^2 = (\rho_1\cos\theta_1 - \rho_2\cos\theta_2)^2 + (\rho_1\sin\theta_1 - \rho_2\sin\theta_2)^2.$

Removing parentheses and using 28 and 36, p. 3, we obtain (IV). Q. E. D.

Formula (IV) may also be derived directly from a figure by using the law of cosines (44, p. 4).

PROBLEMS

1. Find the polar coördinates of the points (3, 4), (−4, 3), (5, −12), (4, 5).

2. Find the rectangular coördinates of the points $\left(5, \frac{\pi}{2}\right)$ $\left(-2, \frac{3\pi}{4}\right)$, $(3, \pi)$.

3. Transform the following equations into polar coördinates and plot their loci:

(a) $x - 3y = 0$. *Ans.* $\theta = \tan^{-1}\frac{1}{3}$.

(b) $y^2 + 5x = 0$. *Ans.* $\rho = -5\cot\theta\operatorname{cosec}\theta$.

(c) $x^2 + y^2 = 16$. *Ans.* $\rho = \pm 4$.

(d) $x^2 + y^2 - ax = 0$. *Ans.* $\rho = a \cos \theta$.

(e) $2xy = 7$. *Ans.* $\rho^2 \sin 2\theta = 7$.

(f) $x^2 - y^2 = a^2$. *Ans.* $\rho^2 \cos 2\theta = a^2$.

(g) $x \cos \omega + y \sin \omega - p = 0$. *Ans.* $\rho \cos(\theta - \omega) - p = 0$.

4. Transform equations 1 to 18, p. 124, into rectangular coördinates.

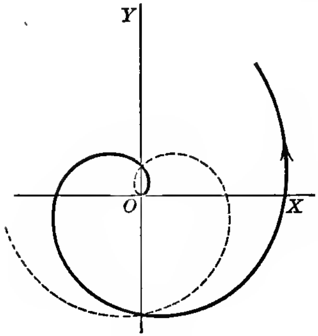
LOCUS PROBLEMS

The locus should be drawn in each case (see the figures below).

1. Find the locus of a point such that

(a) its radius vector is proportional to its vectorial angle.

Ans. The **spiral of Archimedes,** $\rho = a\theta$.

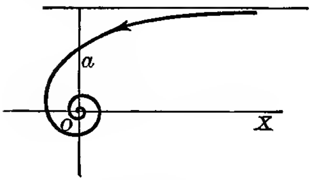


$\rho = a\theta$.

SPIRAL OF ARCHIMEDES

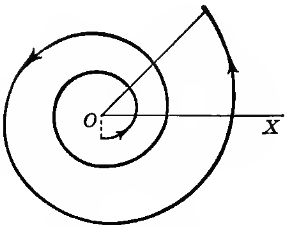
$\rho^2\theta = a^2$.

LITUUS



$\rho\theta = a$.

HYPERBOLIC OR RECIPROCAL SPIRAL



$\log \rho = a\theta$.

LOGARITHMIC OR EQUIANGULAR SPIRAL

(b) its radius vector is inversely proportional to its vectorial angle.

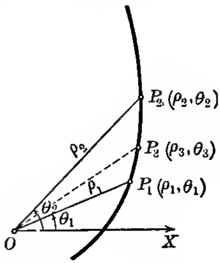
Ans. The **hyperbolic** or **reciprocal spiral**, $\rho\theta = a$.

(c) the square of its radius vector is inversely proportional to its vectorial angle. *Ans.* The **lituus**, $\rho^2\theta = a^2$.

(d) the logarithm of its radius vector is proportional to its vectorial angle. *Ans.* The **logarithmic spiral**, $\log \rho = a\theta$.

Theorem on the logarithmic spiral. When two points, P_1 and P_2, have been plotted on a logarithmic spiral, points between them on the locus may be constructed geometrically by the following theorem:

If the angle P_2OP_1 is bisected, and if on this bisector OP_3 is laid off equal to a mean proportional between OP_1 and OP_2, then P_3 is on the locus.



Proof. By hypothesis, since P_1 and P_3 are on the curve $\log \rho = a\theta$,

(1) $$\log \rho_1 = a\theta_1 \quad \text{and} \quad \log \rho_2 = a\theta_2.$$

Adding and dividing by 2,

$$\tfrac{1}{2} \log \rho_1 + \tfrac{1}{2} \log \rho_2 = a\left(\frac{\theta_1 + \theta_2}{2}\right), \text{ or}$$

(2) $$\log \sqrt{\rho_1\rho_2} = a\left(\frac{\theta_1 + \theta_2}{2}\right). \quad (14 \text{ and } 17, \text{p. } 1).$$

If P_3 is (ρ_3, θ_3), then, by construction,

$$\theta_2 - \theta_3 = \theta_3 - \theta_1, \text{ or } \theta_3 = \frac{\theta_1 + \theta_2}{2}, \text{ and } \rho_3 = \sqrt{\rho_1\rho_2}.$$

Hence, by (2), $\log \rho_3 = a\theta_3$, and P_3 is also on the locus. **Q.E.D.**

PROBLEMS FOR INDIVIDUAL STUDY

Plot carefully the following loci:

1. $\rho = a \sin\theta + b \sec\theta$.
2. $\left(\rho - \frac{a}{2}\right)^2 = a^2 \cos 2\theta$.
3. $\rho = a(\cos 2\theta + \sin 2\theta)$.
4. $\rho = a \cos 2\theta + \frac{a}{2} \sec\theta$.
5. $\rho = a \sin 2\theta + \frac{a}{2} \sec\theta$.
6. $\rho = a \cos 2\theta + b \cos\theta$.
7. $\rho = a \sin 2\theta + b \cos\theta$.
8. $\rho = a \cos 2\theta + b(\sin\theta + 1)$.
9. $\rho = a \cos 3\theta - b \cos\theta$.
10. $\rho = \cos 3\theta + \cos\theta + 1$.
11. $\rho = \cos 3\theta + \cos 2\theta$.
12. $\rho = \cos 3\theta - \sin 2\theta$.
13. $\rho = a \sin^3 \frac{\theta}{3}$.
14. $\rho = a \cos^2 \frac{\theta}{2}$.
15. $\rho^2 \cos\theta = a^2 \sin 3\theta$.
16. $\rho^2 = \frac{2 \cos\theta}{\cos 2\theta} + 1$.
17. $\rho^2 = \frac{2 \cos 2\theta}{\cos\theta + 2} + 1$.

CHAPTER VIII

FUNCTIONS AND GRAPHS

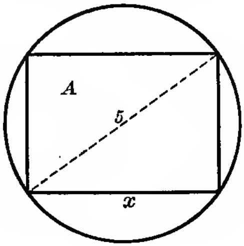
51. Functions. In many practical problems two variables are involved in such a manner that the value of one depends upon the value of the other. For example, given a large number of letters, the postage and the weight are variables, and the amount of the postage depends upon the weight. Again, the premium of a life-insurance policy depends upon the age of the applicant. Many other examples will occur to the student.

This relation between two variables is made precise by the definition:

A variable is said to be a function of a second variable when its value depends upon the value of the latter and is determined when a definite value is assumed for the second variable.

Thus the *postage* is *determined* when a *definite* weight is assumed; the premium is *determined* when a *definite* age is assumed.

Consider another example:



Draw a circle of diameter 5 in. An indefinite number of rectangles may be inscribed within this circle. But the student will notice that the *entire rectangle* is *determined* as soon as a *side* is drawn. Hence the *area* of the rectangle is a *function* of its side.

Let us now find the equation expressing the relation between a side and the area of the rectangle.

Draw any one of the rectangles and denote the length of its base by x in. Then by drawing a diagonal (which is, of course, a diameter of the circle), the altitude is found to be equal to $(25 - x^2)^{\frac{1}{2}}$. Hence if A denotes the area in *square inches*, we have

$$(1) \qquad A = x\,(25 - x^2)^{\frac{1}{2}}.$$

This equation gives the *functional* relation between the function A and the variable x. From it we are enabled to calculate the value of the function A corresponding to any value of the variable x. For example:

if $\quad x = 1$ in., $\quad A = (24)^{\frac{1}{2}} = 4.9$ sq. in.;
if $\quad x = 3$ in., $\quad A = 12$ sq. in.;
if $\quad x = 4$ in., $\quad A = 12$ sq. in.; etc.

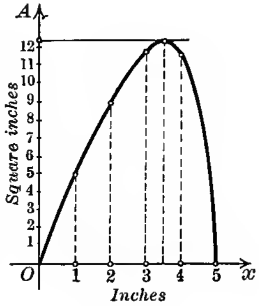
To obtain a representation of the equation (1) for *all* values of x, we draw a *graph* of the equation. This we do by drawing rectangular axes and plotting

the values of the variable (x) *as abscissas,*
the values of the function (A) *as ordinates.*

Any functional relation may be *graphed* in this way. We must, however, first *discuss* the equation (1).

The values of x and A are *positive* from the nature of the problem.

The values of x range from zero to 5, inclusive.



The student should now choose a suitable scale on *each* axis and draw the graph. In this case, unit length on the axis of abscissas represents 1 in., and unit length on the axis of ordinates represents 1 sq. in. These two unit lengths need not be the same.

What do we learn from the graph?

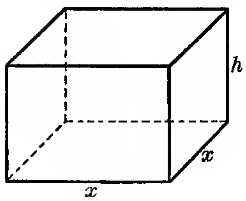
1. If carefully drawn, we may *measure from the graph* the *area* of the inscribed rectangle corresponding to any side we choose to assume.

2. There is *one horizontal* tangent. The ordinate at its point of contact is greater than any other ordinate. Hence this discovery: *One of the inscribed rectangles is greater in area than any of the others*; that is, there is a **maximum** *rectangle.* In other words, the function defined by equation (1) has a **maximum value.**

Careful measurement will give for the base of the maximum rectangle, $x = 3.5$, and for the area, $A = 12.5$. These results, as may be shown by the methods of the differential calculus, are, in fact, correct to one place of decimals. The maximum rectangle is a square; that is, of all rectangles inscribed in a given circle, the square has the greatest area.

The fact that a maximum rectangle *exists* can be seen in advance by reasoning thus: Let the base x increase from zero to 5 in. The area A will then begin with the value zero and return to zero. Since A is always positive, the graph must have a "highest point." Hence there is a maximum value of A, and therefore a maximum rectangle.

Take one more example: A wooden box, open at the top, is to be built to contain 108 cu. ft. The base must be square. This is the only condition. It is evident that under this condition any number of such boxes may be built, and that the number of square feet of lumber used will vary accordingly. If, however, we *choose* any length for a side of the square base, only one box with this dimension can be built, and the material used is determined. *Hence the material used is a function of a side of the square base.*



Let us now find the functional relation between the number of square feet of lumber necessary and the length of one side of the square base measured in feet.

Consider any one box.

Let M = amount of lumber in square feet,

x = length of side of the square base in feet,

and h = height of the box in feet.

Then area of base = x^2 sq. ft.,

and area of sides = $4hx$ sq. ft.

Hence $M = x^2 + 4hx.$

But a relation exists between h and x, for the value of M must depend upon the value of x alone. In fact, the volume equals 108 cu. ft.

Hence $hx^2 = 108$, and $h = \frac{108}{x^2}.$

Therefore

$$M = x^2 + \frac{432}{x}. \qquad (2)$$

This equation enables us to calculate the number of square feet of lumber in any box with a given square base which has a capacity of 108 cu. ft. The calculation is given in the table:

x	0	1	2	3	4	5	6	7	8	...	20	etc.	feet
M	∞	433	220	153	124	111	108	111	118	...	421	etc.	sq. ft.

Thus, if $x = 1$ ft., $M = 433$ sq. ft.;

if $x = 4$ ft., $M = 124$ sq. ft.;

if $x = 8$ ft., $M = 118$ sq. ft.; etc.

The student should now graph equation (2), choosing units thus:

unit length on the axis of abscissas represents 1 ft.;

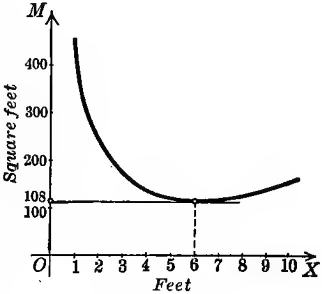
unit length on the axis of ordinates represents 1 sq. ft.

We must, however, choose a very small unit ordinate, since the values of M are large.

A preliminary discussion of (2) shows that x may have any value (positive).

What do we learn from the graph?

1. If carefully drawn, we may *measure from the graph* the number of square feet of lumber in any box which contains 108 cu. ft. and has a square base.



2. There is *one horizontal* tangent. The ordinate at its point of contact is less than any other ordinate. Hence this discovery: *One of the boxes takes less lumber than any other;* that is, M has a **minimum** value. This point on the graph can be determined exactly by calculus, but careful measurement will in this case give the correct values, namely, $x = 6$, $M = 108$. That is, the construction will take the least lumber (108 sq. ft.) if the base is 6 ft. square.

The fact that a least value of M must exist is seen thus. Let the base increase from a very small square to a very large one. In the former case the height must be very great, and hence the amount of lumber will be large. In the latter case, while the height is small, the base will take a great deal of lumber. Hence M varies from a large value to another large value, and the graph must have a "lowest point."

In the following problems the student will work out the functional relation, draw the graph, and state any conclusions to be drawn from the figure. Care should be exercised in the selection of suitable scales on the axes, especially in the scale adopted for plotting values of the function (compare p. 137). The graph should be neither very flat nor very steep. To avoid the latter we may select a large unit of length for the variable. The plot should be *accurate* and the maximum and minimum values of the function should be measured and calculated, additional values of the variable being used, if necessary.

PROBLEMS

1. Rectangles are inscribed in a circle of radius 2 in. Plot the perimeter P of the rectangles as a function of the breadth x.

Ans. $P = 2x + 2(16 - x^2)^{\frac{1}{2}}$.

2. Right triangles are constructed on a line of length 5 in. as hypotenuse. Plot (a) the area A and (b) the perimeter P as a function of the length x of one leg.

Ans. (a) $A = \frac{1}{2}x(25 - x^2)^{\frac{1}{2}}$; (b) $P = x + 5 + (25 - x^2)^{\frac{1}{2}}$.

3. Right cylinders* are inscribed in a sphere of radius r. Plot as functions of the altitude x of the cylinder, (a) the volume V of the cylinder, (b) the curved surface S.

Ans. (a) $V = \frac{\pi}{4}(4r^2x - x^3)$; (b) $S = \pi x(4r^2 - x^2)^{\frac{1}{2}}$.

4. Right cones* are inscribed in a sphere of radius r. Plot as functions of the altitude x of the cone, (a) the volume V of the cone, (b) the curved surface S.

Ans. (a) $V = \frac{\pi}{3}(2rx^2 - x^3)$; (b) $S = \pi(4r^2x^2 - 2rx^3)^{\frac{1}{2}}$.

5. Right cylinders are inscribed in a given right cone. If the height of the cone is h and the radius of the base r, plot (a) the volume V of the cylinder, (b) the curved surface S, (c) the entire surface T, as functions of the altitude x of the cylinder.

Ans. (a) $V = \frac{\pi r^2 x}{h^2}(h - x)^2$; (b) $S = \frac{2\pi r x}{h}(h - x)$;

(c) $T = \frac{2\pi r}{h^2}(h - x)[rh + (h - r)x]$.

6. Right cones are circumscribed about a sphere of radius r. Plot as a function of the altitude x of the cylinder, the volume V of the cone.

Ans. $V = \frac{1}{3}\pi\frac{r^2x^2}{x - 2r}$.

7. Right cones are constructed with a given slant height L. Plot as functions of the altitude x of the cone, (a) the volume V of the cone, (b) the curved surface S, (c) the entire surface T.

Ans. (a) $V = \frac{1}{3}\pi(L^2x - x^3)$; (b) $S = \pi L(L^2 - x^2)^{\frac{1}{2}}$.

8. A conical tent is to be constructed of given volume V. Plot the amount A of canvas required as a function of the radius x of the base.

Ans. $A = \frac{(\pi^2x^6 + 9V^2)^{\frac{1}{2}}}{x}$.

* Use formulas 5–9, p. 1.

9. A cylindrical tin can is to be constructed of given volume V. Plot the amount A of tin required as a function of the radius x of the can.

Ans. $A = 2\pi x^2 + \frac{2V}{x}$.

10. An open box is to be made from a sheet of pasteboard 12 in. square by cutting equal squares from the four corners and bending up the sides. Plot the volume V as a function of the side x of the square cut out. *Ans.* $V = x(12 - 2x)^2$.

11. The strength of a rectangular beam is proportional to the product of the cross section by the square of the depth. Plot the strength S as a function of the depth x for beams which are cut from a log 12 in. in diameter. *Ans.* $S = kx^3(144 - x^2)^{\frac{1}{2}}$.

12. A rectangular stockade is to be built to contain an area of 1000 sq. yd. A stone wall already constructed is available for one of the sides. Plot the length L of the wall to be built as a function of the length x of the side of the rectangle parallel to the wall. *Ans.* $L = \frac{2000}{x} + x$.

13. A tower is 100 ft. high. Plot the angle y subtended by the tower at a point on the ground as a function of the distance x from the foot of the tower.

Ans. $y = \tan^{-1}\frac{100}{x}$.

14. A tower 55 ft. high is surmounted by a statue 10 ft. high. If an observer's eyes are 5 ft. above the ground, plot the angle y subtended by the statue as a function of the observer's distance x from the tower.

Ans. $y = \tan^{-1}\frac{60}{x} - \tan^{-1}\frac{50}{x}$.

15. A line is drawn through a fixed point (a, b). Plot as a function of the intercept on XX' $(= x)$ of the line, the area A of the triangle formed with the coördinate axes.

Ans. $A = \frac{bx^2}{2(x - a)}$.

16. A ship is 41 mi. due north of a second ship. The first sails south at the rate of 8 mi. an hour, the second east at the rate of 10 mi. an hour. Plot their distance d apart as a function of the time t which has elapsed since they were in the position given. *Ans.* $d = (164t^2 - 656t + 1681)^{\frac{1}{2}}$.

17. Plot the distance e from the point $(4, 0)$ to the points (x, y) on the parabola $y^2 = 4x$. *Ans.* $e = (x^2 - 4x + 16)^{\frac{1}{2}}$.

18. A gutter is to be constructed whose cross section is a broken line made up of three pieces, each 4 in. long, the middle piece being horizontal, and the two sides being equally inclined. (a) Plot the area A of a cross section of the gutter as a function of the width x of the gutter across the top. (b) Plot the area A as a function of the angle of inclination of the sides to the horizontal.

Ans. (a) $A = \frac{1}{4}(x+4)(48+8x-x^2)^{\frac{1}{2}}$; (b) $A = 8(\sin 2\theta + 2\sin\theta)$.

19. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter P, plot the area A as a function of the width x.

Ans. $A = \frac{1}{2}xP - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$.

20. A person in a boat 9 mi. from the nearest point of the beach wishes to reach a place 15 mi. from that point along the shore. He can row at the rate of 4 mi. an hour and walk at the rate of 5 mi. an hour. The time it takes him to reach his destination depends on the place at which he lands. Plot the time as a function of the distance x of his landing place from the nearest point on the beach.

Ans. Time $= \frac{\sqrt{81+x^2}}{4} + \frac{15-x}{5}$.

21. The illumination of a plane surface by a luminous point varies directly as the cosine of the angle of incidence, and inversely as the square of the distance from the surface. Plot the illumination I at a point on the floor 10 ft. from the wall as a function of the height x of a gas burner on the wall.

Ans. $I = \frac{kx}{(100+x^2)^{\frac{3}{2}}}$.

22. A Gothic window has the shape of an equilateral triangle mounted on a rectangle. The base of the triangle is a chord of the window. The total length of the frame of the window is constant. Express, plot, and discuss the area of the window as a function of the width.

23. A printed page is to contain 24 sq. in. of printed matter. The top and bottom margins are each $1\frac{1}{2}$ in., the side margins 1 in. each. Express, plot, and discuss the area of the page as a function of the width.

24. A manufacturer has 96 sq. ft. of lumber with which to make a box with a square base and a top. Express, plot, and discuss the contents of the box as a function of the side of the base.

25. (a) Isosceles triangles of the same perimeter, 12 in., are cut out of rubber. Express, plot, and discuss the area as a function of the base. (b) Isosceles triangles of the same area, 10 sq. in., are cut out of rubber. Express, plot, and discuss the perimeter as a function of the base.

26. Small cylindrical boxes are made each with a cover whose breadth and height are equal. The cover slips on tight. Each box is to hold π cu. in. Express, plot, and discuss the amount of material used as a function of the length of the box.

27. A circular filter paper has a diameter of 11 in. It is folded into a conical shape. Express the volume of the cone as a function of the angle of the sector folded over. Plot and discuss this function.

28. Two sources of heat are at the points A and B. Remembering that the intensity of heat at a point varies inversely as the square of the distance from the source, express the intensity of heat at any point between A and B as a function of its distance from A. Plot and discuss this function.

29. A submarine telegraph cable consists of a central circular part, called the core, surrounded by a ring. If x denotes the ratio of the radius of the core to the thickness of the ring, it is known that the speed of signaling varies as $x^2 \log \frac{1}{x}$. Plot and discuss this function.

30. A wall 10 ft. high surrounds a square house which is 15 ft. from the wall. Express the length of a ladder placed without the wall, resting upon it and just reaching the house, as a function either of the distance of the foot of the ladder from the wall, or of the inclination of the ladder to the horizontal. Plot and discuss this function.

31. The volume of a right prism having an equilateral triangular base is 2. Express its total surface as a function of the edge of the base. Plot and discuss.

32. A letter Y stands a ft. high and measures b ft. across the top. Express the total length of the leg and two arms as a function of the length of the leg. Plot and discuss.

33. The sum of the perimeters of a square and a circle is constant. Express their combined areas as a function of the radius of the circle. Plot and discuss.

34. A water tank is to be constructed with a square base and open top, and is to hold 64 cu. yd. The cost of the sides is \$1 a square yard, and of the bottom \$2 a square yard. Plot and discuss the cost.

35. A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yd. wide along each straightaway is to be bought for grand stands, training quarters, etc. If the land costs \$200 an acre, plot and discuss the cost of the land required.

36. A cylindrical steam boiler is to be constructed having a capacity of 1000 cu. ft. The material for the side costs \$2 a square foot, and for the ends \$3 a square foot. Plot and discuss the cost.

37. In the corner of a field bounded by two perpendicular roads a spring is situated 6 rd. from one road and 8 rd. from the other. How should a straight road be run by this spring and across the corner so as to cut off as little of the field as possible?

Ans. 12 and 16 rd. from the corner.

38. When the resistance of air is taken into account, the inclination of a pendulum to the vertical is given by the formula

$$\theta = \alpha e^{-kt}\cos(nt + \epsilon).$$

Plot θ as a function of the time t.

52. Notation of functions. The symbol $f(x)$ is used to denote a function of x, and is read "*f of x*." In order to distinguish between different functions, the prefixed letter is changed, as $F(x)$, $\phi(x)$ (read "*phi of x*"), $f'(x)$, etc.

During any investigation the same functional symbol always indicates the same law of dependence of the function upon the variable. In the simpler cases this law takes the form of a series of analytical operations upon that variable. Hence, in such a case, the same functional symbol will indicate the same operations or series of operations, even though applied to different quantities. Thus, if

$$f(x) = x^2 - 9x + 14,$$

then

$$f(y) = y^2 - 9y + 14.$$

Also

$$f(a) = a^2 - 9a + 14,$$

$$f(b+1) = (b+1)^2 - 9(b+1) + 14 = b^2 - 7b + 6,$$

$$f(0) = 0^2 - 9\cdot 0 + 14 = 14,$$

$$f(-1) = (-1)^2 - 9(-1) + 14 = 24,$$

$$f(7) = 7^2 - 9\cdot 7 + 14 = 0, \text{ etc.}$$

PROBLEMS

1. Given $\phi(x) = \log_{10}x$. Find $\phi(2)$, $\phi(1)$, $\phi(5)$, $\phi(a-1)$, $\phi(b^2)$, $\phi(x+1)$, $\phi(\sqrt{x})$.

2. Given $\phi(x) = e^{2x}$. Find $\phi(0)$, $\phi(1)$, $\phi(-1)$, $\phi(2y)$, $\phi(-x)$.

3. Given $f(x) = \sin 2x$. Find $f\left(\frac{\pi}{2}\right)$, $f\left(\frac{\pi}{4}\right)$, $f(-\pi)$, $f(-x)$, $f(\pi - x)$, $f(\frac{1}{2}\pi - A)$, $f(\frac{3}{2}\pi + B)$.

4. Given $\theta(x) = \cos x$. Prove

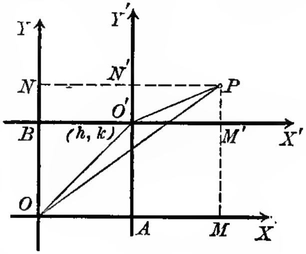
$$\theta(x) + \theta(y) = 2\,\theta\left(\frac{x+y}{2}\right)\theta\left(\frac{x-y}{2}\right).$$

CHAPTER IX

TRANSFORMATION OF COÖRDINATES

53. When we are at liberty to choose the axes as we please we generally choose them so that our results shall have the simplest possible form. When the axes are given, it is important to be able to find the equation of a given curve referred to some other axes. The operation of changing from one pair of axes to a second pair is known as a **transformation of coördinates.** We regard the axes as moved from their given position to a new position and we seek formulas which express the old coördinates in terms of the new coördinates.

54. Translation of the axes. If the axes be moved from a first position OX and OY to a second position $O'X'$ and $O'Y'$ such that $O'X'$ and $O'Y'$ are respectively parallel to OX and OY, then the axes are said to be **translated** from the first to the second position.



Let the new origin be $O'(h, k)$ and let the coördinates of any point P before and after the translation be respectively (x, y) and (x', y'). Then, in the figure,

$$OA = h, \quad OM = x, \quad O'M' = x',$$
$$OB = k, \quad MP = y, \quad M'P = y'.$$

Projecting OP and $OO'P$ on OX, we obtain (Art. 31)

$$OM = OA + O'M';$$
$$\therefore x = x' + h.$$

Similarly, $$y = y' + k.$$

Hence the

Theorem. *If the axes be translated to a new origin (h, k), and if (x, y) and (x', y') are respectively the coördinates of any point P before and after the translation, then*

$$\text{(I)} \qquad \begin{cases} x = x' + h, \\ y = y' + k. \end{cases}$$

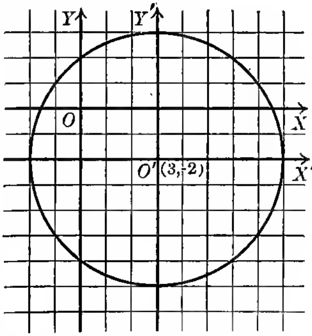
Equations (I) are called the **equations for translating the axes.** To find the equation of a curve referred to the new axes when its equation referred to the old axes is given, substitute in the given equation the values of x and y given by (I) and reduce.

EXAMPLE

Transform the equation

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

when the axes are translated to the new origin $(3, -2)$.



Solution. Here $h = 3$ and $k = -2$, so equations (I) become

$$x = x' + 3, \quad y = y' - 2.$$

Substituting in the given equation, we obtain

$$(x' + 3)^2 + (y' - 2)^2 - 6(x' + 3) + 4(y' - 2) - 12 = 0,$$

or, reducing, $x'^2 + y'^2 = 25.$

This result could easily be foreseen. For the locus of the given equation is a circle whose center is $(3, -2)$ and whose radius is 5. When the origin is translated to the center the equation of the circle must necessarily have the form obtained.

PROBLEMS

1. Find the new coördinates of the points $(3, -5)$ and $(-4, 2)$ when the axes are translated to the new origin $(3, 6)$.

2. Transform the following equations when the axes are translated to the new origin indicated and plot both pairs of axes and the curve :

(a) $3x - 4y = 6$, $(2, 0)$. *Ans.* $3x' - 4y' = 0.$

(b) $x^2 + y^2 - 4x - 2y = 0$, $(2, 1)$. *Ans.* $x'^2 + y'^2 = 5.$

(c) $y^2 - 6x + 9 = 0$, $(\frac{3}{2}, 0)$. *Ans.* $y'^2 = 6x'$.

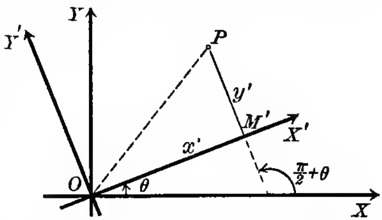
(d) $x^2 + y^2 - 1 = 0$, $(-3, -2)$. *Ans.* $x'^2 + y'^2 - 6x' - 4y' + 12 = 0$.

(e) $y^2 - 2kx + k^2 = 0$, $\left(\frac{k}{2}, 0\right)$. *Ans.* $y'^2 = 2kx'$.

(f) $x^2 - 4y^2 + 8x + 24y - 20 = 0$, $(-4, 3)$. *Ans.* $x'^2 - 4y'^2 = 0$.

3. Derive equations (I) if O' is in (a) the second quadrant; (b) the third quadrant; (c) the fourth quadrant.

55. Rotation of the axes. Let the axes OX and OY be rotated about O through an angle θ to the positions OX' and OY'. The equations giving the coördinates of any point referred to OX and OY in terms of its coördinates referred to OX' and OY' are called the **equations for rotating the axes.**



Theorem. *The equations for rotating the axes through an angle* θ *are*

$$\text{(II)} \qquad \begin{cases} x = x' \cos\theta - y' \sin\theta, \\ y = x' \sin\theta + y' \cos\theta. \end{cases}$$

Proof. Let P be any point whose old and new coördinates are respectively (x, y) and (x', y'). Draw OP, and draw PM' perpendicular to OX'. Project OP and $OM'P$ on OX.

The projection of OP on $OX = x$. (Art. 31)

The projection of OM' on $OX = x' \cos\theta$. (Art. 31)

The projection of $M'P$ on $OX = y' \cos\left(\frac{\pi}{2} + \theta\right)$ (Art. 31)

$= - y' \sin\theta$. (By 31, p. 3)

But by Art. 31,

projection of OP = projection of OM' + projection of $M'P$.

$$\therefore x = x' \cos\theta - y' \sin\theta.$$

In like manner, projecting OP and $OM'P$ on OY, we obtain

$$y = x' \cos\left(\frac{\pi}{2} - \theta\right) + y' \cos\theta$$
$$= x' \sin\theta + y' \cos\theta. \qquad \text{Q.E.D.}$$

If the equation of a curve in x and y is given, we substitute from (II) in order to find the equation of the same curve referred to OX' and OY'.

EXAMPLE

Transform the equation $x^2 - y^2 = 16$ when the axes are rotated through 45°.

Solution. Since

$$\sin 45° = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$$

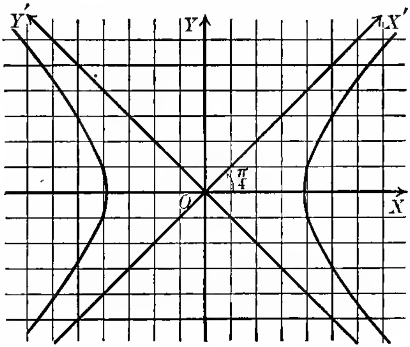
and $$\cos 45° = \frac{1}{\sqrt{2}},$$

equations (II) become

$$x = \frac{x' - y'}{\sqrt{2}}, \quad y = \frac{x' + y'}{\sqrt{2}}.$$

Substituting in the given equation, we obtain

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' + y'}{\sqrt{2}}\right)^2 = 16,$$



or, simplifying, $$x'y' + 8 = 0.$$

PROBLEMS

1. Find the coördinates of the points (3, 1), (– 2, 6), and (4, – 1) when the axes are rotated through $\frac{\pi}{2}$.

2. Transform the following equations when the axes are rotated through the indicated angle. Plot both pairs of axes and the curve.

(a) $x - y = 0$; $\frac{\pi}{4}$. *Ans.* $y' = 0$.

(b) $x^2 + 2xy + y^2 = 8$; $\frac{\pi}{4}$. *Ans.* $x'^2 = 4$.

(c) $y^2 = 4x$; $-\frac{\pi}{2}$. *Ans.* $x'^2 = 4y'$.

(d) $x^2 + 4xy + y^2 = 16$; $\frac{\pi}{4}$. *Ans.* $3x'^2 - y'^2 = 16$.

(e) $x^2 + y^2 = r^2$; θ. *Ans.* $x'^2 + y'^2 = r^2$.

(f) $x^2 + 2xy + y^2 + 4x - 4y = 0$; $-\frac{\pi}{4}$. *Ans.* $\sqrt{2}y'^2 + 4x' = 0$.

(g) $3x^2 - 4xy - 1 = 0$; arc tan 2. *Ans.* $x'^2 - 4y'^2 + 1 = 0$

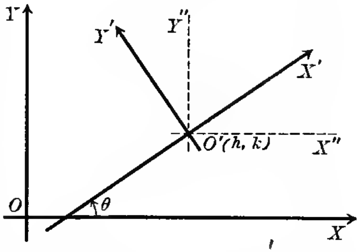
(h) $x^2 + 3xy - 3y^2 = 2$; arc tan $\frac{1}{3}$. *Ans.* $3x'^2 - 7y'^2 = 4$.

(i) $x^2 + 3xy + 5y^2 = 11$; arc tan 3. *Ans.* $11x'^2 + y'^2 = 22$.

(j) $3x^2 - 3xy - y^2 = 5$; arc tan 3.

(k) $x^2 + 4xy + 4y^2 + 12x - 6y = 0$; arc tan 2.

56. General transformation of coördinates. If the axes are moved in any manner, they may be brought from the old position to the new position by translating them to the new origin and then rotating them through the proper angle.



Theorem. *If the axes be translated to a new origin (h, k) and then rotated through an angle θ, the equations of the transformation of coördinates are*

(III) $$\begin{cases} x = x'\cos\theta - y'\sin\theta + h, \\ y = x'\sin\theta + y'\cos\theta + k. \end{cases}$$

Proof. To translate the axes to $O'X''$ and $O'Y''$ we have, by (I),

$$x = x'' + h,$$
$$y = y'' + k,$$

where (x'', y'') are the coördinates of any point P referred to $O'X''$ and $O'Y''$.

To rotate the axes we set, by (II),

$$x'' = x'\cos\theta - y'\sin\theta,$$
$$y'' = x'\sin\theta + y'\cos\theta.$$

Substituting these values of x'' and y'', we obtain (III). Q.E.D.

57. Classification of loci. The loci of algebraic equations are classified according to the *degree* of the equations. This classification is justified by the following theorem, which shows that the degree of the equation of a locus is the same, no matter how the axes are chosen.

Theorem. *The degree of the equation of a locus is unchanged by a transformation of coördinates.*

Proof. Since equations (III) are of the first degree in x' and y', the degree of an equation cannot be *raised* when the values of x and y given by (III) are substituted. Neither can the degree be *lowered;* for then the degree must be raised if we transform back to the old axes, and we have seen that it cannot be raised by changing the axes.*

As the degree can neither be raised nor lowered by a transformation of coördinates, it must remain unchanged. Q.E.D.

58. Simplification of equations by transformation of coördinates. The principal use made of transformation of coördinates is to simplify a given equation by choosing suitable new axes. The method of doing this is illustrated in the following examples.

EXAMPLES

1. Simplify the equation $y^2 - 8x + 6y + 17 = 0$ by translating the axes.

Solution. Set $x = x' + h$ and $y = y' + k$.

This gives $(y' + k)^2 - 8(x' + h) + 6(y' + k) + 17 = 0$, or

(1) $$y'^2 - 8x' + \left.\begin{array}{l}2k\\+6\end{array}\right| y' + \left.\begin{array}{l}k^2\\-8h\\+6k\\+17\end{array}\right| \dagger = 0.$$

If, now, we choose for h and k such numbers that the coefficient of y' shall be zero, that is,

(2) $$2k + 6 = 0,$$

and also the constant term shall be zero, that is,

(3) $$k^2 - 8h + 6k + 17 = 0,$$

the transformed equation is simply

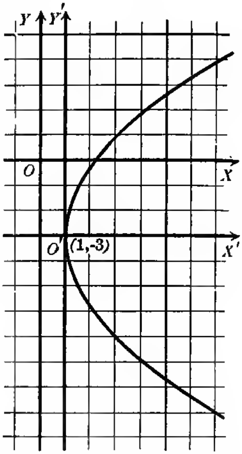
(4) $$y'^2 - 8x' = 0.$$

* This also follows from the fact that when equations (III) are solved for x' and y', the results are of the first degree in x and y.

† These vertical bars play the part of parentheses. Thus $2k + 6$ is the coefficient of y' and $k^2 - 8h + 6k + 17$ is the constant term. Their use enables us to collect like powers of x' and y' *at the same time* that we remove the parentheses in the preceding equation.

From (2) and (3) we obtain $h = 1$, $k = -3$, and these are the coördinates of the new origin.

The locus may be readily plotted by drawing the new axes and then plotting (4) on these axes.



A second method often used is the following:

Rewrite the given equation, collecting the terms in y,

(5) $$(y^2 + 6y) = 8x - 17.$$

Complete the square in the left-hand member,

(6) $$(y^2 + 6y + 9) = 8x - 17 + 9 = 8x - 8.$$

Writing this equation in the form

(7) $$(y + 3)^2 = 8(x - 1),$$

it is obvious *by inspection* that if we substitute in this equation

(8) $$x = x' + 1, \quad y = y' - 3,$$

the transformed equation is $y'^2 = 8x'$. But equations (8) translate the axes to the new origin $(1, -3)$, as before.

2. Simplify $x^2 + 4y^2 - 2x - 16y + 1 = 0$ by translating the axes.

Solution. Set $x = x' + h$ and $y = y' + k$. This gives

(9) $$x'^2 + 4y'^2 + \begin{array}{r|}2h \\ -2\end{array}\, x' + \begin{array}{r|}8k \\ -16\end{array}\, y' + \begin{array}{r|}h^2 \\ +4k^2 \\ -2h \\ -16k \\ +1\end{array} = 0.$$

Let us choose the new origin so that in (9) the coefficients of x' and y' shall be zero; that is, so that

(10) $$2h - 2 = 0 \quad \text{and} \quad 8k - 16 = 0.$$

From (10), $h = 1$, $k = 2$, and these values substituted in (9) give the transformed equation

(11) $$x'^2 + 4y'^2 = 16.$$

The locus of the given equation is now readily drawn by constructing parallel axes through (1, 2) and plotting equation (11) on these axes.

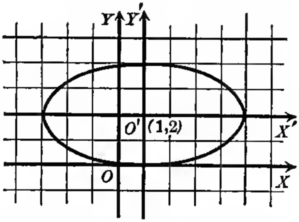
A second method is the following:

Collect corresponding terms in the given equation thus:

(12) $$(x^2 - 2x) + 4(y^2 - 4y) = -1.$$

Complete the squares within the parentheses, adding the corresponding numbers to the right-hand member,

(13) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4)$
$$= -1 + 1 + 16 = 16.$$



Writing (13) in the form
$$(x-1)^2 + 4(y-2)^2 = 16,$$
it is obvious *by inspection* that by substituting

(14) $x = x' + 1, \quad y = y' + 2,$

the simple new equation $x'^2 + 4y'^2 = 16$ results. But equations (14) translate the axes to the new origin (1, 2), the same as in the first method.

3. Remove the xy-term from $x^2 + 4xy + y^2 = 4$ by rotating the axes.

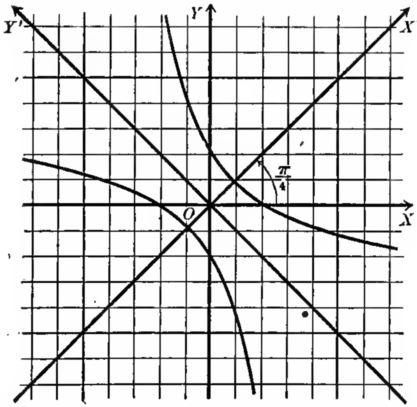
Solution. Set $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$, whence

$$\left.\begin{array}{l}\cos^2\theta\\ \quad + 4\sin\theta\cos\theta\\ \quad + \sin^2\theta\end{array}\right| x'^2 \left.\begin{array}{l}- 2\sin\theta\cos\theta\\ \quad + 4(\cos^2\theta - \sin^2\theta)\\ \quad + 2\sin\theta\cos\theta\end{array}\right| x'y' \left.\begin{array}{l}+ \sin^2\theta\\ \quad - 4\sin\theta\cos\theta\\ \quad + \cos^2\theta\end{array}\right| y'^2 = 4,$$

or, since $\quad 2\sin\theta\cos\theta = \sin 2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$,

(15) $\quad (1 + 2\sin 2\theta)x'^2 + 4\cos 2\theta \cdot x'y' + (1 - 2\sin 2\theta)y'^2 = 4.$

The new equation is to contain no $x'y'$-term. Hence, setting the coefficient of $x'y'$ equal to zero,
$$\cos 2\theta = 0.$$
$$\therefore 2\theta = \frac{\pi}{2} \quad \text{and} \quad \theta = \frac{\pi}{4}.$$



Substituting in (15), since $\sin\frac{\pi}{2} = 1$, the transformed equation is
$$3x'^2 - y'^2 = 4.$$

The locus of this equation is the hyperbola plotted on the new axes in the figure.

These examples show that it is often wise not to plot the locus of an equation as it stands, but rather to endeavor first to simplify by transformation to new axes.

PROBLEMS

1. Simplify the following equations by translating the axes. Plot both pairs of axes and the curve.

(a) $x^2 + 6x + 4y + 8 = 0$. *Ans.* $x'^2 + 4y' = 0$.
(b) $x^2 - 4y + 8 = 0$. *Ans.* $x'^2 = 4y'$.
(c) $x^2 + y^2 + 4x - 6y - 3 = 0$. *Ans.* $x'^2 + y'^2 = 16$.
(d) $y^2 - 6x - 10y + 19 = 0$. *Ans.* $y'^2 = 6x'$.
(e) $x^2 - y^2 + 8x - 14y - 35 = 0$. *Ans.* $x'^2 - y'^2 = 2$.
(f) $x^2 + 4y^2 - 16x + 24y + 84 = 0$. *Ans.* $x'^2 + 4y'^2 = 16$.
(g) $y^3 + 8x - 40 = 0$. *Ans.* $8x' + y'^3 = 0$.

2. Remove the xy-term from the following equations by rotating the axes. Plot both pairs of axes and the curve.

(a) $x^2 - 2xy + y^2 = 12$. *Ans.* $y'^2 = 6$.
(b) $x^2 - 2xy + y^2 + 8x + 8y = 0$. *Ans.* $\sqrt{2}y'^2 + 8x' = 0$.
(c) $xy = 18$. *Ans.* $x'^2 - y'^2 = 36$.
(d) $25x^2 + 14xy + 25y^2 = 288$. *Ans.* $16x'^2 + 9y'^2 = 144$.
(e) $3x^2 - 10xy + 3y^2 = 0$. *Ans.* $x'^2 - 4y'^2 = 0$.

3. Translate the axes so that each of the following equations is transformed into a new equation without any terms of the first degree in the new coördinates. Draw the locus.

(a) $x^2 - 4xy + 6y = 0$. *Ans.* $h = \frac{3}{2}$, $k = \frac{3}{4}$.
(b) $y^2 - 2xy + 3x = 0$.
(c) $x^2 + xy + y^2 + 6x = 0$.
(d) $x^2 - xy + 2y^2 + 6y = 0$.
(e) $3x^2 - xy - y^2 + 4x = 0$.
(f) $2xy + 6x - 8y = 0$.
(g) $3xy + 4y - 2 = 0$.

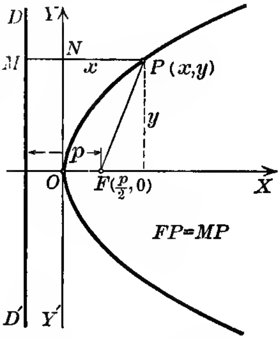
CHAPTER X

PARABOLA, ELLIPSE, AND HYPERBOLA

59. The parabola. Consider the following locus problem.

A point moves so that its distances from a fixed line and a fixed point are equal. Determine the nature of the locus.

Solution. Let DD' be the fixed line and F the fixed point. Draw the x-axis through F perpendicular to DD'. Take the origin midway between F and DD'.



Let

(1) *distance from F to $DD' = p$.*

Then, if $P(x, y)$ is any point on the locus,

(2) $$FP = MP.$$

But $FP = \sqrt{(x - \frac{1}{2}p)^2 + y^2}$, $MP = MN + NP = \frac{1}{2}p + x$.

Substituting in (2),

$$\sqrt{(x - \tfrac{1}{2}p)^2 + y^2} = \tfrac{1}{2}p + x.$$

Squaring and reducing,

(3) $$y^2 = 2px.$$

The locus is called a *parabola*. The fixed line DD' is called the **directrix,** the fixed point F, the **focus.** From (3), it is clear that the x-axis is an axis of symmetry. For this reason, the x-axis is called the **axis** of the parabola. Furthermore, the origin is on the curve. This point, midway between focus and directrix, is called the **vertex.**

Theorem. *If the origin is the vertex and the x-axis the axis of a parabola, then its equation is*

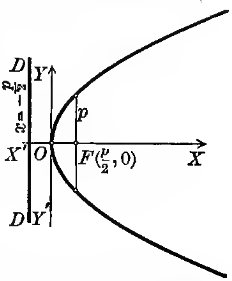
$$\text{(I)} \qquad y^2 = 2\,px.$$

The focus is the point $\left(\frac{p}{2}, 0\right)$, *and the equation of the directrix is* $x = -\frac{p}{2}$.

A discussion of (I) gives us the following properties of the parabola in addition to those already obtained.

1. Values of x having the sign opposite to that of p are to be excluded. Hence the curve lies to the *right* of YY' when p is *positive* and to the *left* when p is *negative.*

2. No values of y are to be excluded; hence the curve extends indefinitely up and down.



The chord drawn through the focus parallel to the directrix is called the **latus rectum.** To find its length, put $x = \frac{1}{2}p$ in (I). Then $y = \pm p$, and the length of the latus rectum $= 2\,p$; that is, *equals the coefficient of x in* (I).

It will be noted that equation (I) contains two terms only; namely, *the square of one coördinate and the first power of the other.* Obviously, the locus of

$$x^2 = 2\,py$$

is also a parabola, and thus we have the

Theorem. *If the origin is the vertex and the y-axis the axis of a parabola, then its equation is*

$$\text{(II)} \qquad x^2 = 2\,py.$$

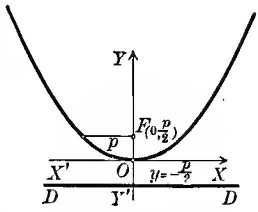
The focus is the point $\left(0, \frac{p}{2}\right)$, *and the equation of the directrix is* $y = -\frac{p}{2}$.

Equations (I) and (II) are called the *typical forms* of the equation of the parabola.

Equations of the forms

$$Ax^2 + Ey = 0 \quad \text{and} \quad Cy^2 - Dx = 0,$$

where A, E, C, and D are different from zero, may, by transposition and division, be written in one of the forms (I) or (II).



To plot a parabola quickly from its typical equation, its position (above or below XX', to the right or left of YY') is best determined by *discussion* of the equation. The value of $2p$ is found by comparison with (I) or (II), and the focus and directrix are then plotted.

EXAMPLES

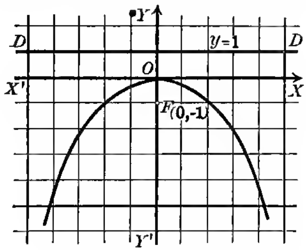
1. Plot the locus of $x^2 + 4y = 0$ and plot the focus and directrix.

Solution. The given equation may be written

$$x^2 = -4y.$$

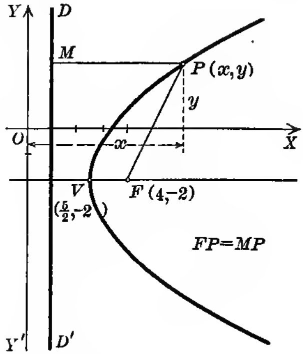
The y-axis is an axis of symmetry; positive values of y must be excluded. Hence the parabola lies below the x-axis. The table gives a few points on the curve.

x	y
0	0
± 2	-1
± 4	-4



Comparing with (II), $p = -2$. The focus is therefore the point $(0, -1)$ and the directrix the line $y = 1$. The length of the latus rectum is 4. Every point on the locus is equidistant from $(0, -1)$ and the line $y = 1$.

2. Find the equation of the parabola whose focus is $(4, -2)$ and directrix the line $x = 1$.



Solution. In the figure, by definition,

(1) $$FP = PM.$$

But $$FP = \sqrt{(x-4)^2 + (y+2)^2},$$

and $$PM = x - 1.$$

Substituting in (1) and reducing,

(2) $$y^2 - 6x + 4y + 19 = 0. \quad \textit{Ans.}$$

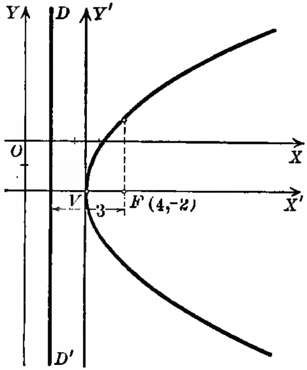
If the axes are translated to the vertex $(\frac{5}{2}, -2)$ as a new origin, that is, if we substitute in (2) $x = x' + \frac{5}{2}$ and $y = y' - 2$, the equation reduces to the typical form $y'^2 - 6x' = 0$.

A second and useful method is the following:

Draw the axis VX' of the parabola and the tangent VY' at the vertex. Referred to these lines as temporary axes, the equation must have the typical form

(3) $$y^2 = 6x,$$

since $p = 3$.



Now translate the temporary axes so that they will coincide with the given axes. The coördinates of O referred to the temporary axes are $(-\frac{5}{2}, 2)$. Substituting in (3) $x = x' - \frac{5}{2}$, $y = y' + 2$, and reducing, we obtain the equation (2).

PROBLEMS

1. Plot the locus of the following equations. Draw the focus and directrix in each case and find the length of the latus rectum.

(a) $y^2 = 4x$.

(b) $y^2 + 4x = 0$.

(c) $x^2 - 8y = 0$.

(d) $y^2 - 6x = 0$.

(e) $x^2 + 10y = 0$.

(f) $y^2 + x = 0$.

2. Find the equations of the following parabolas:

(a) directrix $x = 0$, vertex $(3, 4)$. *Ans.* $(y-4)^2 = 12(x-3)$.

(b) focus $(0, -3)$, vertex $(2, -3)$. *Ans.* $y^2 + 8x + 6y - 7 = 0$.

(c) axis $x = 0$, vertex $(0, -4)$, passes through $(6, 0)$. *Ans.* $x^2 = 9y + 36$.
(d) axis $y = 0$, vertex $(6, 0)$, passes through $(0, 4)$.
Ans. $3y^2 = 48 - 8x$.
(e) directrix $x + 2y - 1 = 0$, focus $(0, 0)$.
Ans. $(2x - y)^2 + 2x + 4y - 1 = 0$.

3. Transform each of the following equations to one of the typical forms (I) or (II) by translation of the axes. Draw the figure in each case.

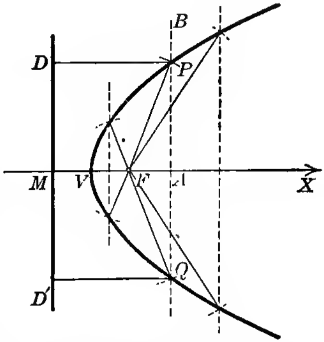
(a) $y^2 + 4x + 4y - 2 = 0$. *Ans.* $y'^2 + 4x' = 0$.
(b) $x^2 + 6x + y - 2 = 0$. *Ans.* $x'^2 + y' = 0$.
(c) $x^2 + 3x + 4y - 1 = 0$.
(d) $y^2 + 3x + 8y = 0$.
(e) $2x^2 + 5y + 4 = 0$.
(f) $y^2 + 6x - 9 = 0$.
(g) $7x^2 + 8y + 10 = 0$.
(h) $x^2 + 4y + 4 = 0$.
(i) $2y^2 + 3x - 8 = 0$.
(j) $5x^2 + 10y + 12 = 0$.
(k) $3x^2 - 6y + 8 = 0$.
(l) $2x^2 - 6x + y = 0$.

4. Show that abscissas of points on the parabola (I) are proportional to the squares of the ordinates.

5. Find the equation in polar coördinates of a parabola if the focus is the pole, and if the axis of the parabola is the polar axis.

Ans. $\rho = \dfrac{p}{1 - \cos\theta}$.

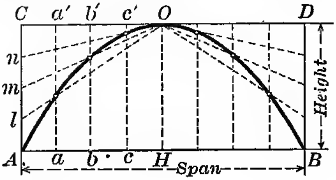
60. Construction of the parabola. A parabola whose focus and directrix are given is readily constructed by rule and compasses as follows:



Draw the axis MX. Construct the vertex V, the middle point of MF. Through any point A to the right of V draw a line AB parallel to the directrix. From F as a center with a radius equal to MA strike arcs to intersect AB at P and Q. Then P and Q are points on the parabola. For $FP = MA$, by construction, and hence P is equidistant from focus and directrix.

By changing the position of A we may construct as many points on the curve as desired.

Parabolic arch. When the span AB and height OH of a parabolic arch are given, points on the arch may be constructed as follows:



Draw the rectangle $ABCD$.

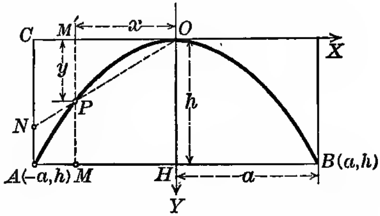
Divide AH and AC into the same number of equal parts.

Starting from A, let the successive points of division be

on AH, a, b, c,

on AC, l, m, n.

Now draw the perpendicular aa' to AB, and draw Ol. Mark the intersection. Do likewise for the points b and m, c and n. The intersections are points on the parabola required.



Proof. Take axes OX and OY, as in the figure. Let

(1) $OM' = x,\quad M'P = y,$

$AB = 2a,\quad OH = h.$

By construction, NC and MH are equal parts of AC and AH respectively.

(2) $$\therefore\ \frac{NC}{AC} = \frac{MH}{AH}, \quad \text{or} \quad \frac{NC}{h} = \frac{x}{a}.$$

From the similar triangles $OM'P$ and OCN,

(3) $$\frac{y}{x} = \frac{NC}{OC} = \frac{NC}{a}.$$

Substituting the value of NC from (2) into (3), and reducing,

(4) $$x^2 = \frac{a^2}{h} y.$$

This is the typical form (II), and the locus passes through O, $A(-a, h)$ and $B(a, h)$, as required.

Solving (4) for y, we get

(5) $$y = \frac{h}{a^2}x^2,$$

x	$\frac{1}{4}a$	$\frac{1}{2}a$	$\frac{3}{4}a$	a
y	$\frac{1}{16}h$	$\frac{1}{4}h$	$\frac{9}{16}h$	h

showing that y varies as the square of x, and giving a simple formula for computing y, as in the table.

61. Equations (I) and (II) are extraordinarily simple types of equations of the second degree. The question,

To derive a test for determining if the locus of a given equation of the second degree is a parabola,

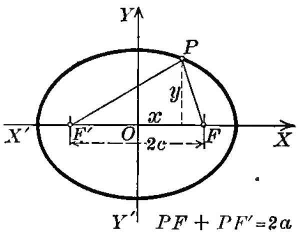
will be answered in Art. 70; but at this point, if the previous results are borne in mind, we may state the

Theorem. *The locus of an equation of the second degree is a parabola if the only term of the second degree * is the square of one coördinate, and if also the first power of the other coördinate is present in the equation.*

For illustration, see Problem 3, p. 157.

62. The ellipse. Let us solve the following locus problem:

Given two fixed points F and F'. A point P moves so that the **sum** of its distances from F and F' remains constant. Determine the nature of the locus.



$PF + PF' = 2a$

Solution. Draw the x-axis through F and F', and take for origin the middle point of $F'F$. By definition,

(1) $$PF + PF' = \text{a constant.}$$

* The student should not forget that the product xy is a term of the second degree.

Let us denote this constant by $2a$. Then (1) becomes

(2) $$PF + PF' = 2a.$$

Let $FF' = 2c$. Then

$$PF = \sqrt{(x-c)^2 + y^2}, \quad PF' = \sqrt{(x+c)^2 + y^2},$$

since the coördinates of F are $(c, 0)$, and of F', $(-c, 0)$.

Hence (2) becomes

(3) $$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

Transposing one of the radicals, squaring and reducing, the result is

(4) $$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

For added simplicity, set *

(5) $$a^2 - c^2 = b^2.$$

Then (4) becomes the simple equation

(6) $$b^2x^2 + a^2y^2 = a^2b^2.$$

Discussion. The intercepts are,

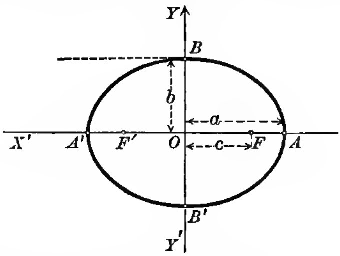
on XX', $\pm a$; on YY', $\pm b$.

The axes XX' and YY' are axes of symmetry and O is a center of symmetry.

Solving (6) for x and for y,

$$x = \pm \frac{a}{b}\sqrt{b^2 - y^2},$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$



Hence the values of x cannot exceed a numerically, nor can the values of y exceed b numerically. The curve is therefore closed.

The locus is called an *ellipse*. The point O, which bisects every chord passing through it, is called the **center**. The given fixed points F and F' are called the **foci**. The longest chord

* This is permissible. For $PF + PF' > FF'$, or $2a > 2c$; that is, $a > c$, and $a^2 - c^2$ is a *positive* number.

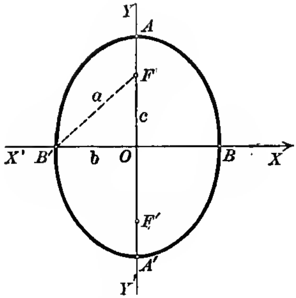
AA' through O is called the **major axis**; the shortest chord BB', the **minor axis**. Obviously,

(7) $$\textit{major axis} = 2a, \quad \textit{minor axis} = 2b.$$

Dividing (6) through by a^2b^2, and summarizing, gives the

Theorem. *The equation of an ellipse whose center is the origin and whose foci are on the x-axis is*

(III) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$



where $2a$ *is the major axis and* $2b$ *the minor axis. If* $c^2 = a^2 - b^2$*, then the foci are* $(\pm c, 0)$.

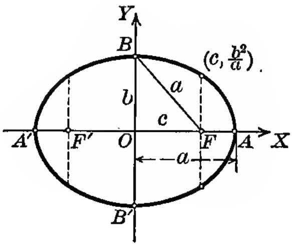
If the foci are on the y-axis, and if we keep the above notation, the equation of the ellipse is obviously

(8) $$a^2x^2 + b^2y^2 = a^2b^2, \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Equations (6), (8), and (III) are *typical* equations of the ellipse, and are of the form

(9) $$Ax^2 + By^2 = C,$$

where A, B, *and* C *agree in sign.*



In the figure $\overline{BF}^2 = b^2 + c^2$. Substituting the value of c^2 from (5), then $\overline{BF}^2 = a^2$. Hence the property: *The distance from either focus to the end of the minor axis equals the semimajor axis.*

The chord drawn through either focus perpendicular to the major axis is called the **latus rectum**. Its length is determined by setting $x = c$ in (III), and solving for y. This gives $y = \frac{b}{a}\sqrt{a^2 - c^2} = \frac{b^2}{a}$. Hence

(10) $$\textit{length of latus rectum} = \frac{2b^2}{a}.$$

Eccentricity. When the foci are very near together the ellipse differs but little from a circle. The value of the ratio $OF : OA$ may, in fact, be said to determine the divergence of the ellipse from a circle. The value of this ratio is called the *eccentricity* of the ellipse, and is denoted by e. Hence

$$(11) \qquad e = \frac{OF}{OA} = \frac{c}{a}.$$

The value of e varies from 0 to 1. If the major axis AA' remains of fixed length, then the "flatness" of the ellipse increases as e increases from 0 to 1, the limiting forms being a circle of diameter AA' and the line segment AA'.

From (11) and (5),

$$(12) \qquad b^2 = a^2 - c^2 = a^2(1 - e^2).$$

To draw an ellipse quickly when its equation is in the typical form, proceed thus:

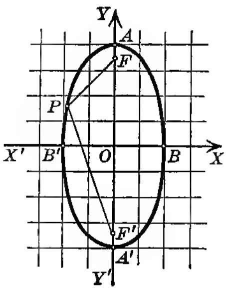
1. Find the intercepts, mark them off on the coördinate axes, and set the larger one equal to a, the smaller equal to b. Letter the major axis AA' and the minor axis BB'.

2. Find c from $c^2 = a^2 - b^2$. Mark the foci F and F' on the major axis.

3. Calculate directly one or more sets of values of the coördinates, and sketch in the curve.

EXAMPLE

Draw the ellipse $4x^2 + y^2 = 16$.



Solution. The intercepts are, on XX', ± 2; on YY', ± 4. Hence the major axis falls on YY', and $a = 4$, $b = 2$, $c = \sqrt{12} = 2\sqrt{3} = 3.4$. The foci are on the y-axis. The length of the latus rectum equals $\frac{2b^2}{a} = 2$. The eccentricity $e = \frac{c}{a} = \frac{1}{2}\sqrt{3}$. The points found in the table are the ends of the latus rectum. If P is any point on the ellipse, then $PF + PF' = 2a = 8$.

x	y
± 1	± 3.4

PROBLEMS

1. Plot each of the following equations. Letter the axes and mark the foci. Find the eccentricity, the length of the latus rectum, and draw the latus rectum.

(a) $x^2 + 9y^2 = 9$.
(b) $9x^2 + 16y^2 = 144$.
(c) $2x^2 + y^2 = 4$.
(d) $4x^2 + 9y^2 = 36$.
(e) $9x^2 + 4y^2 = 36$.
(f) $2x^2 + y^2 = 25$.
(g) $4x^2 + 8y^2 = 32$.
(h) $7x^2 + 3y^2 = 21$.

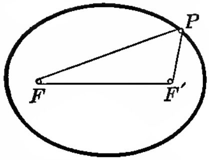
2. Transform each of the following equations by translation of the axes so that the transformed equation shall lack terms of the first degree in the new coördinates. Draw the figure.

(a) $x^2 + 4y^2 + 6x - 8y = 0$. *Ans.* $x'^2 + 4y'^2 = 13$.
(b) $9x^2 + 4y^2 + 36x - 4y + 1 = 0$.
(c) $x^2 + 5y^2 + 10y = 20$.
(d) $5x^2 + y^2 + 10x + 4y = 6$.
(e) $3x^2 + y^2 + 6x - 4y = 2$.
(f) $4x^2 + 5y^2 + 4x + 20y = 20$.

3. Find the equation of each of the following ellipses:

(a) major axis = 8, foci (5, 2) and (– 1, 2).
Ans. $7(x - 2)^2 + 16(y - 2)^2 = 112$.
(b) major axis = 10, foci (0, 0) and (0, 6).
Ans. $25x^2 + 16(y - 3)^2 = 400$.
(c) minor axis = 8, foci (– 1, 0) and (4, 0).
(d) minor axis = 4, foci (0, – 2) and (0, 4).

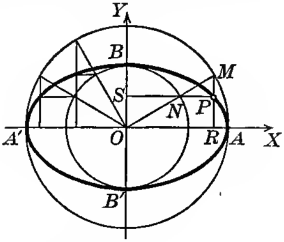
63. Construction of the ellipse. The definition (2) of the preceding section affords a simple method of drawing an ellipse.



Place two tacks in the drawing board at the foci F and F' and wind a string about them as indicated. If now a pencil be placed in the loop FPF' and be moved so as to keep the string taut, then $PF + PF'$ is constant and P describes an ellipse. If the major axis is to be $2a$, then the length of the loop FPF' must be $2a + 2c$.

A useful construction of an ellipse by rule and compasses is the following:

Draw circles on the axes AA' and BB' as diameters. From the center O draw any radius intersecting these circles in M and N respectively. From M draw a line MR parallel to the minor axis, and from N a line NS parallel to the major axis. These lines will intersect in a point P on the ellipse.



Proof. Take the coördinate axes as in the figure below. Let

$$OA = x, \quad AP = y = OD, \quad \angle MOX = \phi.$$

Clearly,

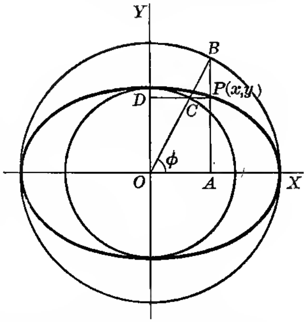
$$OB = \text{semimajor axis} = a,$$
$$OC = \text{semiminor axis} = b.$$

Then in the right triangle OAB,

$$(1) \qquad \cos\phi = \frac{OA}{OB} = \frac{x}{a}.$$

Similarly, in the right triangle ODC, $\angle OCD = \angle COA = \phi$, and

$$(2) \qquad \sin\phi = \frac{OD}{OC} = \frac{y}{b}.$$



But $\cos^2\phi + \sin^2\phi = 1$. Hence, from (1) and (2), $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and $P(x, y)$ lies on the ellipse whose semiaxes are a and b. Q.E.D.

The angle ϕ is called the **eccentric angle** of P.

The construction circles used in this problem are called, respectively, the major and minor **auxiliary circles**.

64. Equations (6) and (8) of Art. 62 are simple equations of the second degree. We may ask the question,

What is the test that the locus of a given equation of the second degree shall be an ellipse?

Reserving for a later section the answer to this question, we have, however, some light on it now. For we have observed in Problem 2, p. 163, that the locus was in each case an ellipse. These equations agree in the respect that *there is no xy-term, and the squares of x and y have unequal positive coefficients.* Consider such an equation, for example,

(1) $$x^2 + 4y^2 + 4x - 8y + N = 0,$$

where N is some number. If we translate the axes to the new origin $(-2, 1)$, the transformed equation is

(2) $$x'^2 + 4y'^2 = 8 - N.$$

If N is less than 8, the locus is an ellipse.

If $N = 8$, the locus is the single point (0, 0), often called a **point-ellipse.**

If N is greater than 8, there is no locus.

This discussion is general, and may be summarized in the

Theorem. *If an equation of the second degree contains no xy-term, and if x^2 and y^2 occur with coefficients having like signs, the locus is necessarily an ellipse or point-ellipse.*

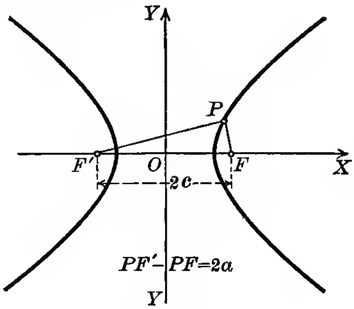
The case when x^2 and y^2 have equal coefficients has been discussed in Art. 38. The circle and point-circle may, of course, be regarded as special cases of the ellipse and point-ellipse.

65. **The hyperbola.** Let us next turn our attention to a third locus problem.

Given two fixed points F and F'. A point P moves so that the **difference** of its distance from F and F' remains constant. Determine the nature of the locus.

Solution. Draw the x-axis through the fixed points, and take for origin the middle point of $F'F$. By definition

(1) $$PF' - PF = \text{a constant.}$$



Let us denote this constant by $2a$. Then (1) becomes

(2) $$PF' - PF = 2a.$$

Let $FF' = 2c.$

Then $PF = \sqrt{(x-c)^2 + y^2}$,

and $PF' = \sqrt{(x+c)^2 + y^2}$,

since the coördinates of F are $(c, 0)$, and of F', $(-c, 0)$.

Substituting in (2),

(3) $$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a.$$

Transposing either radical, squaring and reducing, the result is

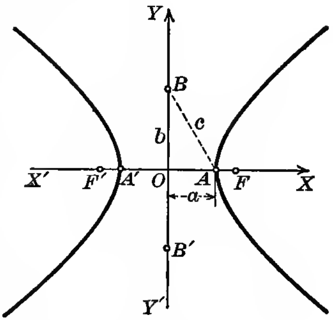
(4) $$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

For added simplicity,* set

(5) $$a^2 - c^2 = -b^2, \quad \text{or} \quad c^2 - a^2 = b^2.$$

Then (4) becomes the simple equation

(6) $$b^2x^2 - a^2y^2 = a^2b^2.$$



Discussion. The intercepts are, on XX', $\pm a$; on YY', $\pm b\sqrt{-1}$; that is, the locus does not cross the y-axis. The coefficient of the $\sqrt{-1}$ in the imaginary intercept on the y-axis is, however, b. The axes XX' and YY' are axes of symmetry and O is a center of symmetry.

* This is permissible. For in the figure, $PF' - PF < F'F$, or $2a < 2c$; that is, $a < c$, and $a^2 - c^2$ is a *negative* number.

Solving (6) for x and for y,

$$x = \pm \frac{a}{b}\sqrt{b^2 + y^2}, \quad y = \pm \frac{b}{a}\sqrt{x^2 - a^2},$$

whence we conclude that all values of x between $-a$ and a must be excluded, but no values of y.

When x increases, y also increases, and the curve extends out to infinity, consisting of two distinct **branches**.*

The locus is called a *hyperbola*, the point O, which bisects every chord drawn through it, is called the **center**. The given fixed points F and F' are the **foci**. The chord AA' is named the **transverse axis**. Marking off on YY' from O the lengths $\pm b$, the line BB' (Fig. p. 166) is called the **conjugate axis**. Thus the

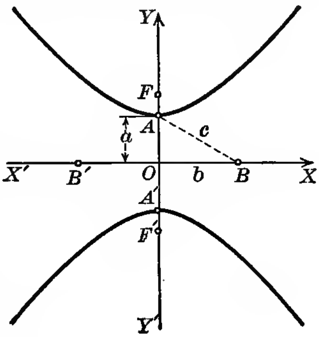
(7) *transverse axis* $= 2a$, *conjugate axis* $= 2b$.

Dividing (6) through by a^2b^2, and summarizing, gives the

Theorem. *The equation of a hyperbola whose center is the origin and whose foci are on the x-axis is*

(IV) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $2a$ *is the transverse axis and* $2b$ *the conjugate axis. If* $c^2 = a^2 + b^2$, *then the foci are* $(\pm c, 0)$.



If the foci are on the y-axis, and if we preserve the notation, the equation of the hyperbola is obviously

(8) $$a^2x^2 - b^2y^2 = -a^2b^2, \quad \text{or} \quad \frac{x^2}{b^2} - \frac{y^2}{a^2} = -1.$$

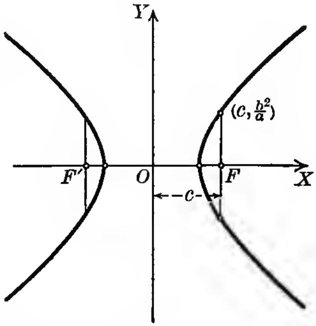
Equations (6) and (8) are typical equations of the hyperbola. They are of the form

(9) $$Ax^2 + By^2 = C,$$

where *A and B differ in sign.*

* On the left-hand branch, (2) is replaced by $PF - PF' = 2a$.

In the preceding figures $\overline{AB}^2 = a^2 + b^2$. Substituting the value of b^2 from (5), $\overline{AB}^2 = c^2$. Hence the property: *The distance between the extremities of the axes equals half the distance between the foci.*



The chord drawn through a focus and perpendicular to the transverse axis is called the **latus rectum.** We may determine its length by setting $x = c$ in (IV) and solving for y. Thus, by (5) we obtain $y = \pm \frac{b}{a}\sqrt{c^2 - a^2} = \pm \frac{b^2}{a}$. Hence

(10) **length of latus rectum** $= \frac{2\,b^2}{a}$.

Eccentricity. The value of the ratio $OF : OA$ in the hyperbola is called the *eccentricity* of the curve, as in the case of the ellipse. Denoting the eccentricity by e, then

(11) $$e = \frac{OF}{OA} = \frac{c}{a}.$$

For a hyperbola, $e > 1$. The relation of the value of e to the shape of the curve will be made clear later. From (5) and (11),

(12) $$b^2 = c^2 - a^2 = a^2(e^2 - 1).$$

To draw a hyperbola quickly when its equation is in the typical form (9), proceed thus:

1. Find the intercepts and mark them off on the proper axis. Set a equal to the real intercept and b equal to the coefficient of $\sqrt{-1}$ in the imaginary intercept. Lay off the conjugate axis; letter it BB' and the transverse axis AA'.

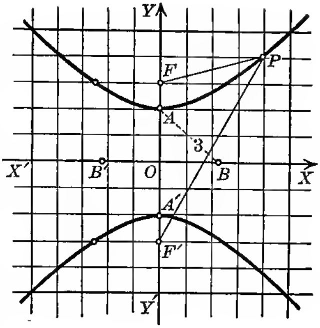
2. Find c from $c^2 = a^2 + b^2$. Mark the foci F and F' on the transverse axis.

3. Calculate directly one or more sets of values of the coördinates, and sketch the curve.

EXAMPLE

Draw the hyperbola

$$4x^2 - 5y^2 + 20 = 0.$$



Solution. The intercepts are, on XX', $\pm\sqrt{-5} = \pm\sqrt{5}\sqrt{-1}$; on YY', ± 2. Hence $b = \sqrt{5}$, $a = 2$, $c = \sqrt{a^2 + b^2} = 3$, and the transverse axis and the foci are on YY'. The eccentricity is $\frac{3}{2}$. The length of the latus rectum is $\frac{2b^2}{a} = 5$.

If P is any point on the hyperbola, then $PF' - PF = 4$.

x	y
0	± 2
$\pm\frac{5}{2}$	± 3

PROBLEMS

1. Plot each of the following equations, letter the axes, and mark the foci. Find the eccentricity, the length of the latus rectum, and draw the latus rectum.

(a) $5x^2 - 4y^2 = 20$.
(b) $x^2 - 8y^2 + 8 = 0$.
(c) $9x^2 - y^2 = 9$.
(d) $3x^2 - y^2 = 12$.
(e) $x^2 - 3y^2 + 3 = 0$.
(f) $7x^2 - 9y^2 = 63$.
(g) $2x^2 - 7y^2 = 18$.
(h) $7x^2 - 2y^2 = -8$.

2. Transform each of the following equations by translation of the axes so that the transformed equation shall lack terms of the first degree in the new coördinates. Draw the figure.

(a) $4x^2 - y^2 + 8x - 2y - 1 = 0$.
(b) $9x^2 - y^2 + 18x - 4y + 14 = 0$.
(c) $3x^2 - y^2 + 12x + 2y + 14 = 0$.
(d) $4x^2 - y^2 - 6x - 4y = 0$.
(e) $x^2 - 5y^2 + 6x - 10y = 0$.
(f) $x^2 - 2y^2 + 10y = 0$.

3. Find the equations of the following hyperbolas:

(a) transverse axis = 6, foci $(-2, 0)$ and $(6, 0)$.
Ans. $7(x-2)^2 - 9y^2 = 63$

(b) conjugate axis = 6, foci $(0, 2)$ and $(0, -8)$.
(c) conjugate axis = 4, foci $(1, 2)$ and $(-4, 2)$.
(d) transverse axis = 2, foci $(0, 0)$ and $(-4, 0)$.

66. Conjugate hyperbolas and asymptotes. Two hyperbolas are called **conjugate hyperbolas** if the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other.

If the equation of a hyperbola is given in typical form, then *the equation of the conjugate hyperbola is found by changing the signs of the coefficients of x^2 and y^2 in the given equation.*

Thus the loci of the equations

(1) $$16x^2 - y^2 = 16 \quad \text{and} \quad -16x^2 + y^2 = 16$$

are conjugate hyperbolas. They may be written

$$\frac{x^2}{1} - \frac{y^2}{16} = 1 \quad \text{and} \quad -\frac{x^2}{1} + \frac{y^2}{16} = 1.$$

The foci of the first are on the x-axis, those of the second on the y-axis. The transverse axis of the first and the conjugate axis of the second are equal to 2, while the conjugate axis of the first and the transverse axis of the second are equal to 8.

The foci of two conjugate hyperbolas are equally distant from the origin. For c^2 equals the sum of the squares of the semitransverse and semiconjugate axes, and that sum is the same for two conjugate hyperbolas.

Thus in the first of the hyperbolas above $c^2 = 1 + 16$, while in the second $c^2 = 16 + 1$.

If in one of the typical forms of the equation of a hyperbola we replace the constant term by zero, then the locus of the new equation is a pair of lines (Theorem, p. 40) which are called the **asymptotes** of the hyperbola.

Thus the asymptotes of the hyperbola

(2) $$b^2x^2 - a^2y^2 = a^2b^2$$

are the lines

(3) $$b^2x^2 - a^2y^2 = 0,$$

or

(4) $$bx + ay = 0 \quad \text{and} \quad bx - ay = 0.$$

These may be written

(5) $$y = -\frac{b}{a}x \text{ and } y = \frac{b}{a}x.$$

They pass through the origin and their slopes are respectively $-\frac{b}{a}$ and $\frac{b}{a}$.

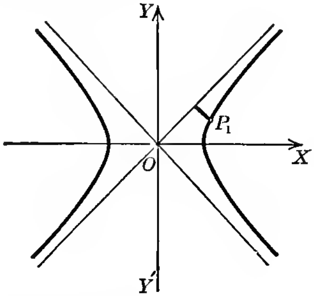
The property of these lines which they have in common with the vertical or horizontal asymptotes of Art. 22 is expressed in the

Theorem. *The branches of the hyperbola approach indefinitely near its asymptotes as they recede to infinity.*

Proof. Let $P_1(x_1, y_1)$ be a point on either branch of (2) near the asymptote $$bx - ay = 0.$$

The perpendicular distance from this line to P_1 is

(6) $$d = \frac{bx_1 - ay_1}{-\sqrt{b^2 + a^2}}.$$



We may find a value for the numerator as follows:

Since P_1 lies on (2),

$$b^2x_1^2 - a^2y_1^2 = a^2b^2.$$

Factoring and dividing,

$$bx_1 - ay_1 = \frac{a^2b^2}{bx_1 + ay_1}.$$

Substituting in (6), $$d = \frac{a^2b^2}{-\sqrt{b^2 + a^2}\,(bx_1 + ay_1)}.$$

As P_1 recedes to infinity in the first quadrant, x_1 and y_1 become infinite and d approaches zero.

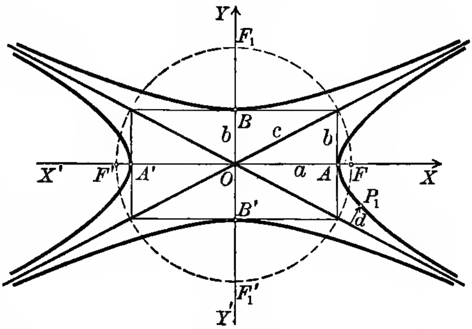
Hence the curve approaches closer and closer to its asymptotes. Q.E.D.

Two conjugate hyperbolas have the same asymptotes.

Thus the asymptotes of the conjugate hyperbolas (1) are respectively the loci of $$16x^2 - y^2 = 0 \quad \text{and} \quad -16x^2 + y^2 = 0,$$
which are the same.

A hyperbola may be drawn with fair accuracy by the following

Construction. Lay off $OA = OA' = a$ on the axis on which the foci lie, and $OB = OB' = b$ on the other axis. Draw lines through A, A', B, B', parallel to the axes, forming a rectangle. Draw the diagonals of the rectangle. Then the length of each diagonal is obviously $2c$ (since $a^2 + b^2 = c^2$). Moreover, the diagonals produced are the asymptotes. For the equations of the diagonals are readily seen to be $bx - ay = 0$ and $bx + ay = 0$, and these are the same as (4). Construct the circle which circumscribes the rectangle. Draw the branches of the hyperbola tangent to the sides of the rectangle at A and A' and approaching nearer and nearer to the diagonals. The conjugate hyperbola may be drawn tangent to the sides of the rectangle at B and B' and approaching the diagonals. The foci of both are the points in which the circle cuts the axes.



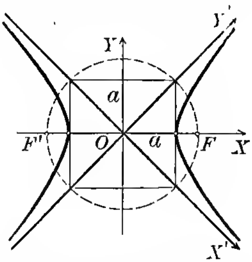
From this construction the influence of the value of the eccentricity upon the shape of the hyperbola can be easily discussed. In the figure, let AA' be fixed. Now from (12), Art. 65,

$$b^2 = a^2(e^2 - 1).$$

When e diminishes towards unity, b decreases, the altitude BB' of the rectangle diminishes, the asymptotes turn towards the x-axis, and the hyperbola flattens.

When e increases, the asymptotes turn from the x-axis, and the hyperbola broadens.

67. Equilateral or rectangular hyperbola. When the axes of a hyperbola are equal ($a = b$), the hyperbola is said to be *equilateral*. If we set $a = b$ in equation (IV), we obtain



(1) $$x^2 - y^2 = a^2,$$

which is accordingly the equation of an equilateral hyperbola whose transverse axis lies on XX'.

Its asymptotes are the lines

$$x - y = 0 \text{ and } x + y = 0.$$

These lines are perpendicular, and hence they may be used as coördinate axes. The designation "rectangular" hyperbola arises from this fact.

Theorem. *The equation of an equilateral hyperbola referred to its asymptotes is*

(V) $$2xy = a^2.$$

Proof. The axes must be rotated through $-45°$ to coincide with the asymptotes. Hence we substitute (Art. 55)

$$x = \frac{x' + y'}{\sqrt{2}}, \quad y = \frac{-x' + y'}{\sqrt{2}}$$

in (1). This gives

$$\frac{(x' + y')^2}{2} - \frac{(-x' + y')^2}{2} = a^2.$$

Reducing and dropping primes we have (V). Q.E.D.

It is important to observe that (V) has the simple form

(2) $$xy = \text{a constant}.$$

68. Construction of the hyperbola. A mechanical construction, depending upon the definition (1) of Art. 65, is the following:

Fasten thumb tacks at the foci. Pass *over* F' and *around* F a string whose ends are held together (Fig. 1, p. 174).

If a pencil be tied to the string at P, and both strings be pulled in or let out together, then $PF' - PF$ will be constant and P will describe a hyperbola. If the transverse axis is to be $2a$, the strings must be adjusted at the start so that the difference between PF' and PF equals $2a$.

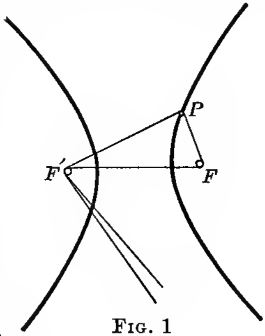


FIG. 1

A construction often used for an equilateral hyperbola when the asymptotes and one point A are given, is as follows (Fig. 2):

Let OX and OY be the asymptotes and A the given point. Draw any line through A to meet OX at M and OY at N. Lay off $MP = AN$. Then P is a point on the required hyperbola.

Proof. Choose the asymptotes as axes. Let the coördinates of A be (a, b) and of P, (x, y). Then

$OS = x$, $SP = y$, $OB = b$, $BA = a$.

By construction, $AN = MP$.

$\therefore$ triangle PSM = triangle NBA,

and $BN = SP = y$, $SM = AB = a$.

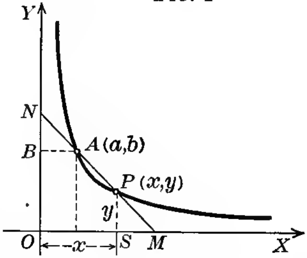


FIG. 2

Since the triangles OMN and ABN are similar,

$$\therefore \frac{BN}{AB} = \frac{ON}{OM} = \frac{OB + BN}{OS + SM}.$$

Substituting,

$$\frac{y}{a} = \frac{b + y}{a + x}, \quad \text{or} \quad xy = ab.$$

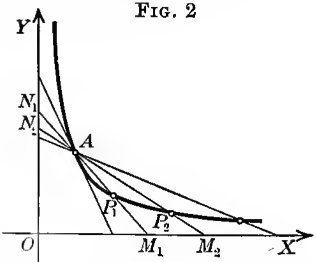


FIG. 3

Comparing with (V), we see that $P(x, y)$ lies upon an equilateral hyperbola which has OX and OY for its asymptotes and which passes through (a, b). Q.E.D.

By drawing different lines through A, and laying off $M_1P_1 = AN_1$, $M_2P_2 = AN_2$, etc., we determine as many points P_1, P_2, etc., as we wish on the hyperbola (Fig. 3).

PROBLEMS

1. Find the equations of the asymptotes and of the hyperbolas conjugate to the following hyperbolas, and plot:

(a) $4x^2 - y^2 = 36$.
(b) $9x^2 - 25y^2 = 100$.
(c) $16x^2 - y^2 + 64 = 0$.
(d) $8x^2 - 16y^2 + 25 = 0$.

2. The distance from an asymptote of a hyperbola to either focus is numerically equal to b.

3. The distance from the center to a line drawn through a focus of a hyperbola perpendicular to an asymptote is numerically equal to a.

4. The product of the distances from the asymptotes to any point on the hyperbola is constant.

5. The focal radius of a point $P_1(x_1, y_1)$ on the parabola $y^2 = 2px$ is $\frac{p}{2} + x_1$.

6. The ordinates of points on an ellipse and the major auxiliary circle which have the same abscissas are in the ratio of $b:a$.

7. The area of an ellipse is πab.

Hint. Divide the major axis into equal parts. With these as bases inscribe rectangles in the ellipse and major auxiliary circle (p. 164). Apply Problem 6 and increase the number of rectangles indefinitely.

69. The examples of Problem 2, p. 169, illustrated the fact that any equation of the second degree lacking an xy-term, but containing x^2 and y^2 with coefficients of unlike signs, can by translation of the axes be transformed into the form (9)

$$Ax^2 + By^2 = C,$$

in which A and B differ in sign.

From the preceding it is clear that the locus of this equation is a hyperbola if C is not zero, and a pair of intersecting lines if C is zero. Hence the

Theorem. *If an equation of the second degree contains no xy-term, and if x^2 and y^2 occur with coefficients differing in sign, the locus is either a hyperbola or a pair of intersecting lines.*

70. Locus of any equation of the second degree. The locus problems of this chapter have led to the equations of the second degree,

(1) $$y^2 = 2px \quad \text{and} \quad x^2 = 2py,$$

(2) $$b^2x^2 + a^2y^2 = a^2b^2 \quad \text{and} \quad b^2x^2 - a^2y^2 = a^2b^2.$$

These are simple types, of course. The question is, however, this:

Given an equation of the second degree, can the equation be transformed by translating and rotating the axes so that the transformed equation will reduce to one of these simple types?

To answer this question, take the **general equation of the second degree,** namely,

(3) $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

This equation contains every term that can appear in an equation of the second degree.

We begin by rotating the axes through an angle θ. To do this, set in (3),

$$x = x'\cos\theta - y'\sin\theta,$$

and $$y = x'\sin\theta + y'\cos\theta.$$

This gives, after squaring, multiplying, and collecting, the transformed equation

(4) $$\left.\begin{array}{l} A\cos^2\theta \\ +B\sin\theta\cos\theta \\ +C\sin^2\theta \end{array}\right| x'^2 \left.\begin{array}{l} -2A\sin\theta\cos\theta \\ +B(\cos^2\theta - \sin^2\theta) \\ +2C\sin\theta\cos\theta \end{array}\right| x'y' \left.\begin{array}{l} +A\sin^2\theta \\ -B\sin\theta\cos\theta \\ +C\cos^2\theta \end{array}\right| y'^2$$
$$\left.\begin{array}{l} +D\cos\theta \\ +E\sin\theta \end{array}\right| x' \left.\begin{array}{l} -D\sin\theta \\ +E\cos\theta \end{array}\right| y' + F = 0.$$

The angle θ is, as yet, any angle at all. But let us now, if possible, choose this angle so that the equation (4) shall not contain the $x'y'$-term. To do this, we must set the coefficient of $x'y'$ equal to zero; that is,

(5) $$-2A\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) + 2C\sin\theta\cos\theta = 0.$$

But $2\sin\theta\cos\theta = \sin 2\theta, \quad \cos^2\theta - \sin^2\theta = \cos 2\theta.$

Hence (5) becomes

$$(6)\qquad (C-A)\sin 2\theta + B\cos 2\theta = 0.$$

Dividing through by $\cos 2\theta$, and transposing,

$$(7)\qquad \tan 2\theta = \frac{B}{A-C}.$$

Since any number may be the tangent of an angle, it is always possible to find a value for θ from this equation. If, then, the axes are rotated through the angle θ determined by (7), equation (3) reduces to

$$(8)\qquad A'x'^2 + C'y'^2 + D'x' + E'y' + F = 0,$$

where from (4),

$$(9)\qquad A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta,$$

$$(10)\qquad C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta.$$

The discussion gives the

Theorem. *The term in xy may always be removed from an equation of the second degree,*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

by rotating the axes through an angle θ such that

$$(\text{VI})\qquad \tan 2\theta = \frac{B}{A-C}.$$

Now equation (8) is of a form which we have met frequently in this chapter, and we have learned to simplify it by translation of the axes. We saw in Art. 61 that if only one square ($A' = 0$, or $C' = 0$) and the first power of the other coördinate were present, the equation could be transformed into one of the typical forms (1) of the parabola.

Suppose, however, that the first power of the other coördinate does not appear. For example, suppose in (8) that $A' = 0$ and $D' = 0$. Then the equation is

$$C'y'^2 + E'y' + F = 0.$$

This is an ordinary quadratic in y. If the roots are real, the locus will be two lines parallel to the x'-axis. These lines will coincide if the roots are equal. There will be no locus if the roots are imaginary.

If neither A' nor C' is zero, we may, by translation to the new origin $\left(-\frac{D'}{2A'}, -\frac{E'}{2C'}\right)$, transform the equation into

$$(11) \qquad A'x''^2 + C'y''^2 + F' = 0.$$

The locus of this equation has been discussed in Arts. 64 and 69.

The result we have established is expressed in the

Theorem. *The locus of an equation of the second degree is either a parabola, an ellipse, a hyperbola, two straight lines (which may coincide), or a point.*

The following conclusion also may be drawn: *The presence of the xy-term indicates that the axes of the curve are not parallel to the axes of coördinates.*

We seek now a test to apply to an equation containing an xy-term in order to decide in advance the nature of the locus. To do this we eliminate the angle θ from equations (9) and (10), making use of (6). The result is the simple equation,

$$\text{(VII)} \qquad -4A'C' = B^2 - 4AC.$$

The steps in the elimination process are as follows:

Adding and subtracting (9) and (10),

$$(12) \qquad A' + C' = A + C \quad (\text{since } \sin^2\theta + \cos^2\theta = 1).$$

$$(13) \qquad A' - C' = (A - C)\cos 2\theta + B\sin 2\theta.$$

Squaring (13),

$$(14) \quad (A' - C')^2 = (A - C)^2\cos^2 2\theta + 2B(A - C)\sin 2\theta\cos 2\theta + B^2\sin^2 2\theta.$$

Squaring (6),

$$(15) \quad 0 = (A - C)^2\sin^2 2\theta + 2B(C - A)\sin 2\theta\cos 2\theta + B^2\cos^2 2\theta.$$

Adding (14) and (15),

$$(16) \qquad (A' - C')^2 = (A - C)^2 + B^2.$$

Squaring (12),

(17) $$(A' + C')^2 = (A + C)^2.$$

Subtracting (16) and (17), we obtain (VII).

If the locus of (8) is a parabola, $A' = 0$ or $C' = 0$. Hence from (VII), $B^2 - 4AC = 0$.

If the locus of (8) is an ellipse, A' and C' agree in sign. Hence $A'C'$ is positive, and from (VII), $B^2 - 4AC$ is negative.

If the locus of (8) is a hyperbola, A' and C' differ in sign. Hence $A'C'$ is negative, and from (VII), $B^2 - 4AC$ is a positive number.

Collecting all the results in tabular form, we have the

Theorem. *Given any equation of the second degree,*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The possible loci may be classified thus:

Test	General case	Exceptional cases *
$B^2 - 4AC$ † zero	parabola	two parallel lines one line
$B^2 - 4AC$ negative	ellipse	point-ellipse
$B^2 - 4AC$ positive	hyperbola	two intersecting lines

A point-ellipse is often called a "degenerate ellipse," two intersecting lines a "degenerate hyperbola," and two parallel lines a "degenerate parabola."

Note that $B^2 - 4AC$ is the *discriminant* of the terms of the second degree in the equation.

* For tests to distinguish the exceptional cases, see Smith and Gale's "Elements of Analytic Geometry," p. 277.

† This case is recognizable by inspection, for the terms of the second degree, $Ax^2 + Bxy + Cy^2$, now will form a *perfect square*.

The exceptional cases are recognizable by the condition that the equation is then *factorable into two factors of the first degree in x and y.* A number of problems of this kind were given on page 41. When the equation is not readily factored by trial, it may appear by the first method of the following section (Art. 71) that factors do nevertheless exist. Moreover, under the two first cases in the table (parabola and ellipse) there may be no locus. This fact will also readily appear by the first method of Art. 71.

71. Plotting the locus of an equation of the second degree. In this section we discuss methods of plotting second-degree equations which contain xy-terms.

First Method. *By direct plotting.* Test by the theorem at the end of the preceding section, and then plot the equation directly.

EXAMPLES

1. Plot the locus of

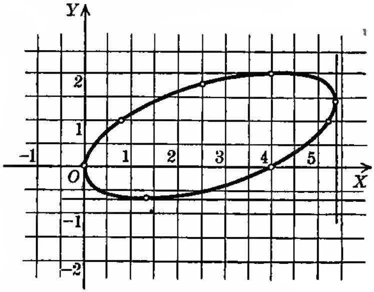
(1) $$x^2 - 2xy + 4y^2 - 4x = 0.$$

Solution. Here $A = 1, \quad B = -2, \quad C = 4.$

$$\therefore B^2 - 4AC = 4 - 16 = -12 = \text{a negative number.}$$

Hence the locus is an ellipse.

x	y
$\frac{4}{3}$	$-\frac{2}{3}$
$0, 4$	0
$3 \pm \sqrt{5}$	1
4	2
$5\frac{1}{3}$	$\frac{4}{3}$



Solve the equation for x as follows:

(2) $$x^2 - (2y + 4)x + \left(\frac{2y + 4}{2}\right)^2 = -4y^2 + \left(\frac{2y + 4}{2}\right)^2.$$

[Collecting terms in x and completing the square.]

(3) $$\therefore x = y + 2 \pm \sqrt{(2 - y)(2 + 3y)}.$$

Solving also for y,

(4) $$y = \frac{1}{4}x \pm \frac{1}{4}\sqrt{x(16 - 3x)}.$$

From the radicals in (3) and (4) we see that (see p. 49)

y may have values from $-\frac{2}{3}$ to 2 inclusive ;

x may have values from 0 to $\frac{16}{3}$ inclusive.

Hence the ellipse lies within the rectangle

$$y = -\tfrac{2}{3}, \quad y = 2, \quad x = 0, \quad x = \tfrac{16}{3}.$$

Points on the locus may be found from (3) as in the table.

2. Determine the locus of

$$5x^2 + 4xy - y^2 + 24x - 6y - 5 = 0.$$

Solution. $A = 5, \quad B = 4, \quad C = -1. \quad \therefore B^2 - 4AC = 16 + 20 = 36.$ Hence, from the table of Art. 70, we may expect a hyperbola or a pair of intersecting lines.

Solve the equation for y as follows :

$$\begin{aligned} y^2 - (4x - 6)y + (2x - 3)^2 &= 5x^2 + 24x - 5 + (2x - 3)^2 \\ &= 9x^2 + 12x + 4 = (3x + 2)^2. \end{aligned}$$

[Collecting terms in y and completing the square.]

$$\therefore y - (2x - 3) = \pm(3x + 2).$$

Hence the locus is the intersecting lines

$$y = 5x - 1 \quad \text{and} \quad y = -x - 5.$$

PROBLEMS

1. Test and plot the following equations :

(a) $x^2 - 2xy + y^2 - 5x = 0.$ (c) $4xy + 4y^2 + 4y + 4 = 0.$

(b) $4xy + 4y^2 - 2x + 3 = 0.$ (d) $2x^2 + 4xy + 4y^2 + 2x - 3 = 0.$

(e) $x^2 + 2xy + 2y^2 + 2x + 2y - 1 = 0.$

(f) $3x^2 - 12xy + 9y^2 + 8x - 12y + 5 = 0.$

(g) $5x^2 - 12xy + 9y^2 + 8x - 12y + 3 = 0.$

(h) $x^2 + xy + y^2 + 3y = 0.$

(i) $x^2 + 2xy + 4y^2 + 6y = 0.$

(j) $4x^2 + 4xy + y^2 + 6x - 9 = 0.$

(k) $3x^2 - 2xy + y^2 - 4x - 6 = 0.$

(l) $x^2 - 2xy + 5y^2 - 8y = 0.$

(m) $x^2 - 4xy + 4y^2 + 4x + 2y = 0.$

(n) $3x^2 + 4xy + y^2 - 2x - 1 = 0.$

(o) $3x^2 + 8xy + 4y^2 + 2x + 4y = 0.$

SECOND METHOD. *By transformation.* If the xy-term is lacking, we have seen that the equation may be simplified by translating the axes. The transformed equation is then readily plotted on the new axes.

When the xy-term is present, rotate the axes through the angle θ given by (VI),

$$(5) \qquad \tan 2\theta = \frac{B}{A - C}.$$

The term in xy will then disappear and further simplification is accomplished by translation.

To rotate, we substitute

$$(6) \qquad x = x' \cos\theta - y \sin\theta, \quad y = x' \sin\theta + y' \cos\theta.$$

We find $\sin\theta$ and $\cos\theta$ as follows. First compute $\cos 2\theta$ from

$$(7) \qquad \cos 2\theta = \pm\frac{1}{\sqrt{1 + \tan^2 2\theta}}. \quad \text{(26 and 28, p. 3)}$$

From (5), 2θ must lie in the first or second quadrant, so the *sign* in (7) must be the same as in (5). θ will then be acute; and from 40, p. 4, we have

$$(8) \qquad \sin\theta = +\sqrt{\frac{1 - \cos 2\theta}{2}}, \quad \cos\theta = +\sqrt{\frac{1 + \cos 2\theta}{2}}.$$

EXAMPLES

1. Construct and discuss the locus of

$$(9) \qquad x^2 + 4xy + 4y^2 + 12x - 6y = 0.$$

Solution. Here $A = 1, \quad B = 4, \quad C = 4.$

$$\therefore B^2 - 4AC = 0, \text{ and the locus is a parabola.}$$

Write the equation (9) in the form

$$(10) \qquad (x + 2y)^2 + 12x - 6y = 0.$$

We rotate the axes through an angle θ, such that

$$\tan 2\theta = \frac{4}{1-4} = -\frac{4}{3}.$$

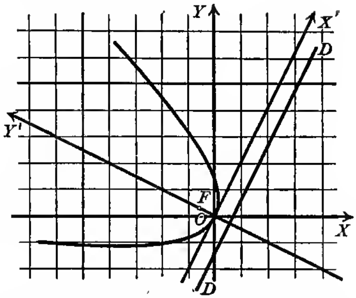
Then by (7), $\cos 2\theta = -\frac{3}{5}$,

and by (8),

(11) $$\sin\theta = \frac{2}{\sqrt{5}} \text{ and } \cos\theta = \frac{1}{\sqrt{5}}.$$

The equations for rotating the axes are therefore

$$x = \frac{x' - 2y'}{\sqrt{5}}, \quad y = \frac{2x' + y'}{\sqrt{5}}.$$



Substituting in the equation (10), we obtain

$$x'^2 - \frac{6}{\sqrt{5}}y' = 0.$$

Hence the locus is a parabola for which $p = \frac{3}{\sqrt{5}}$, and whose focus is on the y'-axis.

The figure shows both sets of axes, the parabola, its focus and directrix. The axis OX' has the slope $\tan\theta = \frac{\sin\theta}{\cos\theta} = 2$, from (11). Hence to draw OX', simply draw a line through the origin whose slope equals 2.

In the new coördinates the focus is the point $\left(0, \frac{3}{2\sqrt{5}}\right)$ and the directrix is the line $y' = -\frac{3}{2\sqrt{5}}$.

2. Construct the locus of

$$5x^2 + 6xy + 5y^2 + 22x - 6y + 21 = 0.$$

Solution. Here $A = 5, \quad B = 6, \quad C = 5.$

$$\therefore B^2 - 4AC = 36 - 100 = -64 = \text{a negative number.}$$

Hence the locus is an ellipse.

We rotate the axes through the angle θ, given by

$$\tan 2\theta = \frac{6}{5-5} = \infty.$$

$$\therefore 2\theta = 90°, \quad \theta^* = 45°.$$

Hence the equations of the transformation are

$$x = \frac{x' - y'}{\sqrt{2}}, \quad y = \frac{x' + y'}{\sqrt{2}}.$$

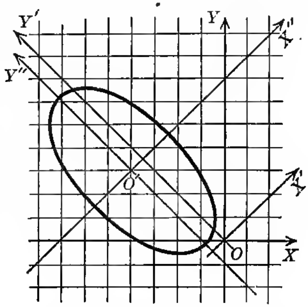
* If $A = C$, the angle θ always equals 45°.

Substituting in the given equation and reducing,

$$4x'^2 + y'^2 + 4\sqrt{2}x' - 7\sqrt{2}y' + \tfrac{21}{2} = 0.$$

Translating to the new origin $\left(-\frac{1}{2}\sqrt{2}, \frac{7}{2}\sqrt{2}\right)$, the final equation is

$$4x''^2 + y''^2 = 16.$$



Hence the locus is an ellipse whose major axis is 8, whose minor axis is 4, and whose foci are on the Y''-axis.

The figure shows the three sets of axes and the ellipse. The coördinates of the new origin $O'\left(-\frac{1}{2}\sqrt{2}, \frac{7}{2}\sqrt{2}\right)$ refer to the axes OX' and OY', and this must be remembered in plotting.

The equation

(12) $\quad Bxy + Dx + Ey + F = 0,$

in which x^2 *and* y^2 *are lacking*, offers an exception to the above process, for, by translation, the equation may be reduced to

(13) $$Bx'y' + F' = 0;$$

and the locus of (13) is, by (V), Art. 67, an equilateral hyperbola referred to its asymptotes as axes. Hence to plot (12), translate so that the terms of the first degree disappear and then plot the new equation.

To show that (12) may be transformed into (13) by translation, proceed thus:

Substitute $x = x' + h$, $y = y' + k$, in (12), multiply out and collect the terms. We obtain

(14) $$\left.\begin{array}{l|l|l} Bx'y' + Bk & x' + Bh & y' + Bhk \\ \qquad\quad + D & \quad + E & \quad + Dh \\ & & \quad + Ek \\ & & \quad + F \end{array}\right| = 0.$$

Choose the new origin (h, k) so that the

coefficient of x' vanishes; that is, $Bk + D = 0$,

coefficient of y' vanishes; that is, $Bh + E = 0$.

Solving these equations, $h = -\dfrac{E}{B}$, $k = -\dfrac{D}{B}$, and (14) reduces to the form (13).

PROBLEMS

1. Simplify the following equations and construct the loci. Check the figure by finding the intercepts on the original axes.

(a) $x^2 + xy + y^2 = 3.$ *Ans.* $3x'^2 + y'^2 = 6.$
(b) $x^2 + 3xy + y^2 + 4y = 0.$ *Ans.* $25x''^2 - 5y''^2 + 32 = 0.$
(c) $x^2 + 2xy + y^2 + 3x - 3y = 0.$ *Ans.* $2x'^2 - 3\sqrt{2}y' = 0.$
(d) $3x^2 - 4xy + 8x - 1 = 0.$ *Ans.* $x''^2 - 4y''^2 + 1 = 0.$
(e) $4x^2 + 4xy + y^2 + 8x - 16y = 0.$ *Ans.* $5x'^2 - 8\sqrt{5}y' = 0.$
(f) $3xy + 4x + 6y + 1 = 0.$ *Ans.* $3x'y' - 7 = 0.$
(g) $17x^2 - 12xy + 8y^2 - 68x + 24y - 12 = 0.$
Ans. $x''^2 + 4y''^2 - 16 = 0.$
(h) $y^2 + 6x - 6y + 21 = 0.$ *Ans.* $y'^2 + 6x' = 0.$
(i) $6xy + 4x - 12y + 3 = 0.$ *Ans.* $6x'y' + 11 = 0.$
(j) $12xy - 5y^2 + 48y - 36 = 0.$ *Ans.* $4x''^2 - 9y''^2 = 36.$
(k) $4x^2 - 12xy + 9y^2 + 2x - 3y - 12 = 0.$
Ans. $52y''^2 - 49 = 0.$
(l) $12x^2 + 8xy + 18y^2 + 48x + 16y + 43 = 0.$
Ans. $4x^2 + 2y^2 = 1.$
(m) $7x^2 + 50xy + 7y^2 = 50.$ *Ans.* $16x'^2 - 9y'^2 = 25.$
(n) $x^2 + 3xy - 3y^2 + 6x = 0.$ *Ans.* $21x''^2 - 49y''^2 = 72.$
(o) $16x^2 - 24xy + 9y^2 - 60x - 80y + 400 = 0.$
Ans. $y''^2 - 4x'' = 0.$

2. Show that the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

may be simplified by translation only, so that the new equation contains no terms of the first degree in x and y, if the coördinates of the new origin (h, k) satisfy the equations

$$2Ah + Bk + D = 0, \quad Bh + 2Ck + E = 0.$$

Hence show that the new origin (h, k) is the *center* of the locus, unless $B^2 - 4AC = 0$. In the latter case the transformation fails.

72. Conic sections. Historically, the parabola, ellipse, and hyperbola were discovered as plane sections of a right circular cone. Hence the generic term used for them, — *conic sections* or **conics.**

A definition often used, which will include all conic sections, is the following: *When a point P moves so that its distances* *from a given fixed point and a given fixed line are in a constant ratio, the locus is a conic.*

The given fixed line is called the **directrix,** the fixed point the **focus,** and the number representing the ratio of the distances of P from the focus and directrix is called the **eccentricity.**

In Problem 3, p. 51, we found the equation for any conic to be

$$(1) \qquad (1-e^2)x^2 + y^2 - 2px + p^2 = 0,$$

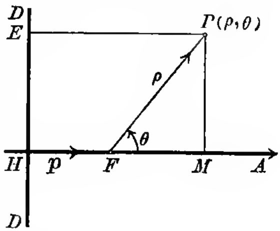
if e is the eccentricity, YY' is the directrix, and $(p, 0)$ is the focus. Now (1) has no xy-term. Hence we see at once by comparison with our previous results that a conic is

a parabola when $e = 1$,
an ellipse when $e < 1$,
a hyperbola when $e > 1$.

Clearly, when $e = 1$ the definition of the conic agrees with that already given for the parabola.

The ellipse and hyperbola, each having a center, are called **central conics.**

Focus and eccentricity, as used in this section, agree with these terms as already introduced. This fact is left to the student to prove in the following problems.



The equation of a conic in polar coordinates is readily found. We may show that if the pole is the focus and the polar axis the principal axis of a conic section, then the polar equation of the conic is

$$(2) \qquad \rho = \frac{ep}{1 - e\cos\theta},$$

where e is the eccentricity and p is the distance from the directrix to the focus.

For let P be any point on the conic. Then, by definition,

$$\frac{FP}{EP} = e.$$

From the figure, $FP = \rho,$

and $EP = HM = p + \rho \cos\theta.$

Substituting these values of FP and EP, we have

$$\frac{\rho}{p + \rho\cos\theta} = e;$$

or, solving for ρ, $\rho = \dfrac{ep}{1 - e\cos\theta}.$ Q.E.D.

PROBLEMS

1. Simplify (1), p. 186, by translation of the axes when $e \neq 1$.

Ans. $(1 - e^2)x^2 + y^2 = \dfrac{e^2p^2}{1 - e^2}.$

2. Show that in a central conic the focus coincides with the focus already adopted. Hence show that a central conic has two directrices, one associated by the above definition with each focus.

3. Prove that e in Problem 1 agrees with e as defined in Arts. 62 and 65.

4. Prove that the focal radii of a point (x, y) on the ellipse (III), p. 161, are $a + ex$ and $a - ex$.

5. Prove that the focal radii of a point on the hyperbola (IV), p. 167, are $ex - a$ and $ex + a$.

LOCUS PROBLEMS

It is expected that the locus in each problem will be constructed and discussed after its equation is found.

1. The base of a triangle is fixed in length and position. Find the locus of the opposite vertex if

(a) the sum of the other sides is constant. *Ans.* An ellipse.
(b) the difference of the other sides is constant. *Ans.* A hyperbola.
(c) one base angle is double the other. *Ans.* A hyperbola.
(d) the sum of the base angles is constant. *Ans.* A circle.
(e) the difference of the base angles is constant. *Ans.* A conic.
(f) the product of the tangents of the base angles is constant.
Ans. A conic.

(g) the product of the other sides is equal to the square of half the base. *Ans.* A lemniscate (Ex. 2, p. 122).

(h) the median to one of the other sides is constant. *Ans.* A circle.

2. Find the locus of a point the sum of the squares of whose distances from (a) the sides of a square, (b) the vertices of a square, is constant.

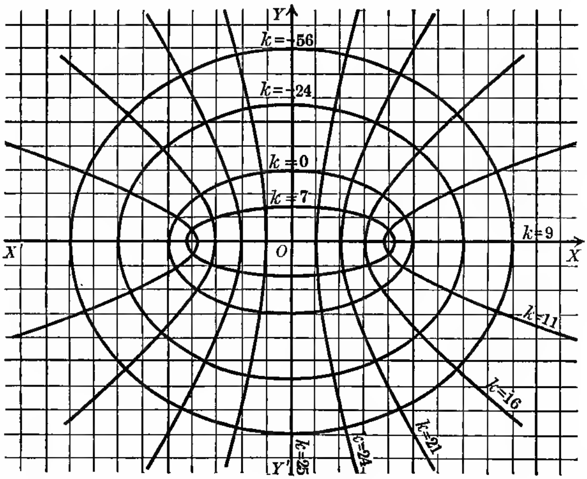
Ans. A circle in each case.

3. Find the locus of a point such that the ratio of its distance from a fixed point $P_1(x_1, y_1)$ to its distance from a given line $Ax + By + C = 0$ is equal to a constant k.

Ans.
$$(A^2 + B^2 - k^2A^2)x^2 - 2k^2ABxy + (A^2 + B^2 - k^2B^2)y^2 - 2(A^2x_1 + B^2x_1 + k^2AC)x - 2(A^2y_1 + B^2y_1 + k^2BC)y + (x_1^2 + y_1^2)(A^2 + B^2) - k^2C^2 = 0.$$

4. Find the locus of a point such that the ratio of the square of its distance from a fixed line to its distance from a fixed point equals a constant k.

Ans. $x^4 - k^2(x - p)^2 - k^2y^2 = 0$ if the y-axis is the fixed line and the x-axis passes through the fixed point, p being the distance from the line to the point.



Systems of conics. When an equation of the second degree contains one arbitrary constant, the locus is a system of conics.

EXAMPLE

Discuss the system represented by $\dfrac{x^2}{25-k}+\dfrac{y^2}{9-k}=1.$

Solution. When $k<9$ the locus is an ellipse whose foci are $(\pm c, 0)$, where $c^2=(25-k)-(9-k)=16$. When $9<k<25$ the locus is an hyperbola whose foci are $(\pm c, 0)$, where $c^2=(25-k)-(9-k)=16$. When $k>25$ there is no locus. Since the ellipses and hyperbolas have the same foci $(\pm 4, 0)$, they are called **confocal.**

In the figure the locus is plotted for $k=-56, -24, 0, 7, 9, 11, 16, 21, 24, 25$. As k increases and approaches 9, the ellipses flatten out and finally degenerate into the x-axis, and as k decreases and approaches 9, the hyperbolas flatten out and degenerate into the x-axis. As k increases and approaches 25, the two branches of the hyperbolas lie closer to the y-axis, and in the limit they coincide with the y-axis.

PROBLEMS

1. Plot the following systems of conics and show that the conics of each system belong to the same type. Draw enough conics so that the degenerate conics of the system appear as limiting cases.

(a) $\dfrac{x^2}{16}+\dfrac{y^2}{9}=k.$

(b) $y^2=2\,kx.$

(c) $\dfrac{x^2}{16}-\dfrac{y^2}{9}=k.$

(d) $x^2=2\,ky-6.$

2. Plot the following systems of conics and show that all of the conics of each system are confocal. Discuss degenerate cases and show that two conics of each system pass through every point in the plane.

(a) $\dfrac{x^2}{16-k}+\dfrac{y^2}{36-k}=1.$

(b) $y^2=2\,kx+k^2.$

(c) $\dfrac{x^2}{64-k}+\dfrac{y^2}{16-k}=1.$

(d) $x^2=2\,ky+k^2.$

3. Plot and discuss the systems:

(a) $16\,(x-k)^2+9\,y^2=144.$

(b) $xy=k.$

(c) $(y-k)^2=4\,x.$

(d) $4\,(x-k)^2-9\,(y-k)^2=36.$

4. Plot the following systems and discuss the locus as k approaches zero and infinity:

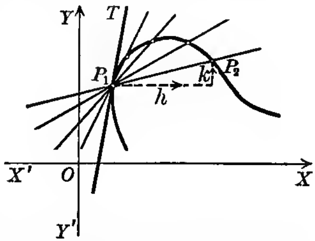
(a) $\dfrac{(x-k)^2}{k^2}+\dfrac{y^2}{36}=1.$

(b) $\dfrac{(x-k)^2}{k^2}-\dfrac{y^2}{36}=1.$

CHAPTER XI

TANGENTS

73. Equation of the tangent. A tangent to a curve at a point P_1 is obtained as follows. Take a second point P_2 on the curve near P_1. Draw the secant through P_1 and P_2.



Now let P_2 move along the curve toward P_1. The secant will turn around P_1. The limiting position of the secant when P_2 reaches P_1 is called the *tangent at* P_1.

We wish to calculate the *slope* of the tangent at a point on a curve. Let the coördinates of P_1 be (x_1, y_1) and of P_2 $(x_1 + h, y_1 + k)$. Then

$$\textit{slope of secant } P_1P_2 = \frac{k}{h}.$$

To find the slope of the tangent, we begin by finding a value for $\frac{k}{h}$, the slope of the secant, as in the following example.

EXAMPLE

Find the slope of the tangent to the curve $C : 8y = x^3$ at any point $P_1(x_1, y_1)$ on C (see figure on page 191).

Solution. Let $P_1(x_1, y_1)$ and $P_2(x_1+h, y_1+k)$ be two points on C.

Then since these coördinates must satisfy the equation of C,

(2) $$8y_1 = x_1^2,$$

and $$8(y_1+k) = (x_1 + h)^3;$$

or

(3) $$8y_1 + 8k = x_1^3 + 3x_1^2h + 3x_1h^2 + h^3.$$

Subtracting (2) from (3), we obtain

$$8k = 3x_1^2 h + 3x_1h^2 + h^3.$$

Factoring, $8k = h(3x_1^2 + 3x_1h + h^2)$;

and hence

$$\frac{k}{h} = \frac{3x_1^2 + 3x_1h + h^2}{8}$$

$$= \text{slope of secant } P_1P_2.$$

Now as P_2 approaches P_1, h and k approach zero, and when the secant becomes a tangent to the curve, h and k are both equal to zero.

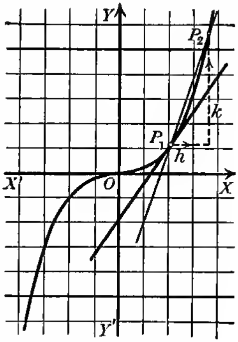
Hence the slope m of the tangent at P_1 will be obtained from the above value of the slope of the secant, namely,

$$\frac{3x_1^2 + 3x_1h + h^2}{8},$$

by setting $h = 0$ and also $k = 0$, if k appeared in the expression. Hence

$$m = \frac{3x_1^2}{8}. \quad \textit{Ans.}$$

The method employed in this example is general and may be formulated in the following



Rule *to determine the slope of the tangent to a curve C at a point P_1 on C.*

First step. *Let $P_1(x_1, y_1)$ and $P_2(x_1 + h, y_1 + k)$ be two points on C. Substitute their coördinates in the equation of C and subtract.*

Second step. *Find a value for $\frac{k}{h}$, the slope of the secant through P_1 and P_2.*

Third step. *Find the limiting value of the result of the second step when h and k approach zero. This value is the required slope.*

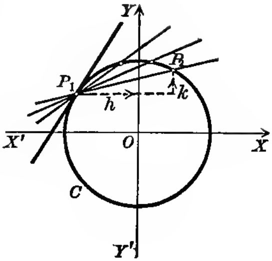
Having found the slope of the tangent at P_1, its equation is found at once by the point-slope formula. The point P_1 is called the **point of contact.**

EXAMPLE

Find the equation of the tangent to the circle

$$x^2 + y^2 = r^2$$

at the point of contact (x_1, y_1).



Solution. Let $P_1(x_1, y_1)$ and $P_2(x_1 + h, y_1 + k)$ be two points on the circle C.

Then these coördinates must satisfy the equation of the circle. Therefore

(1) $$x_1^2 + y_1^2 = r^2,$$

and $$(x_1 + h)^2 + (y_1 + k)^2 = r^2;$$

or

(2) $$x_1^2 + 2x_1h + h^2 + y_1^2 + 2y_1k + k^2 = r^2.$$

Subtracting (1) from (2), we have

$$2x_1h + h^2 + 2y_1k + k^2 = 0.$$

Transposing and factoring, this becomes

$$k(2y_1 + k) = -h(2x_1 + h).$$

Whence $$\frac{k}{h} = -\frac{2x_1 + h}{2y_1 + k}$$

$$= \text{slope of the secant through } P_1 \text{ and } P_2.$$

Letting P_2 approach P_1, h and k approach zero, so that m, the slope of the tangent at P_1, is

$$m = -\frac{x_1}{y_1}.$$

The equation of the tangent at P_1 is then

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1),$$

or $$x_1x + y_1y = x_1^2 + y_1^2.$$

This equation may be simplified. For by (1),

$$x_1^2 + y_1^2 = r^2,$$

so that the required equation is

$$x_1x + y_1y = r^2. \qquad \text{Q. E. D.}$$

Theorem. *The equation of the tangent to the circle*

$$C : x^2 + y^2 = r^2$$

at the point of contact $P_1(x_1, y_1)$ *is*

$$\text{(I)} \qquad \mathbf{x_1 x + y_1 y = r^2}.$$

The point to be observed in this proof is this:

Always simplify the equation of the tangent by making use of the equation obtained when x_1 *and* y_1 *are substituted for* x *and* y *in the equation of the given curve.*

In the equation (I) the point of contact is (x_1, y_1), while (x, y) is any point on the tangent.

In like manner we may prove the following

Theorem. *The equation of the tangent at the point of contact* $P_1(x_1, y_1)$ *to the*

ellipse $b^2x^2 + a^2y^2 = a^2b^2$ *is* $\mathbf{b^2x_1x + a^2y_1y = a^2b^2}$;

hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ *is* $\mathbf{b^2x_1x - a^2y_1y = a^2b^2}$;

parabola $y^2 = 2px$ *is* $\mathbf{y_1y = p(x + x_1)}$.

PROBLEMS

1. Find the equations of the tangent to each of the following curves at the point of contact (x_1, y_1):

(a) $x^2 = 2py$. *Ans.* $x_1x = p(y + y_1)$.

(b) $x^2 + y^2 = 2rx$. *Ans.* $x_1x + y_1y = r(x + x_1)$.

(c) $y^2 = 4x + 3$. *Ans.* $y_1y = 2x + 2x_1 + 3$.

(d) $xy = a^2$. *Ans.* $x_1y + y_1x = 2a^2$.

(e) $x^2 + xy = 4$. *Ans.* $2x_1x + x_1y + y_1x = 8$.

(f) $x^2 + y^2 + Dx + Ey + F = 0$.

Ans. $x_1x + y_1y + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0$.

(g) $y = x^3$. *Ans.* $3x_1^2x - y + 2y_1 = 0$.

(h) $y^2 = x^3$.

(i) $y = Ax^2 + Bx + C$.

(j) $Ax^2 + By^2 + Cx = 0$.

(k) $Ax^2 + By^3 = 0$.

(l) $Axy + Bx + Cy = 0$.

(m) $xy^2 + a = 0$.

(n) $x^2y + b = 0$.

(o) $xy^2 + a^2x - a^2b = 0$.

(p) $y^2(2a - x) = x^3$.

74. Taking next any equation of the second degree, we may prove the

Theorem. *The equation of the tangent to the locus of*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

at the point of contact $P_1(x_1, y_1)$ *is*

$$\boldsymbol{Ax_1x + B\frac{y_1x + x_1y}{2} + Cy_1y + D\frac{x + x_1}{2} + E\frac{y + y_1}{2} + F = 0.}$$

Proof. Let $P_1(x_1, y_1)$ and $P_2(x_1 + h, y_1 + k)$ be two points on the conic. Then

(1) $$Ax_1^2 + Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 + F = 0 \text{ and}$$

$$A(x_1 + h)^2 + B(x_1 + h)(y_1 + k) + C(y_1 + k)^2 + D(x_1 + h) + E(y_1 + k) + F = 0.$$

Clearing of parentheses,

(2) $$Ax_1^2 + 2Ax_1h + Ah^2 + Bx_1y_1 + Bx_1k + By_1h + Bhk + Cy_1^2 + 2Cy_1k + Ck^2 + Dx_1 + Dh + Ey_1 + Ek + F = 0.$$

Subtracting (1) from (2),

(3) $$2Ax_1h + Ah^2 + Bx_1k + By_1h + Bhk + 2Cy_1k + Ck^2 + Dh + Ek = 0.$$

Transposing all the terms containing h, and factoring, (3) becomes

$$k(Bx_1 + 2Cy_1 + Ck + E) = -h(2Ax_1 + Ah + By_1 + Bk + D);$$

whence

$$\frac{k}{h} = -\frac{2Ax_1 + By_1 + D + Ah + Bk}{Bx_1 + 2Cy_1 + E + Ck}.$$

This is the slope of the secant P_1P_2.

Letting P_2 approach P_1, h and k will approach zero and the slope of the tangent is

$$m = -\frac{2Ax_1 + By_1 + D}{Bx_1 + 2Cy_1 + E}.$$

The equation of the tangent line is then

$$y - y_1 = -\frac{2Ax_1 + By_1 + D}{Bx_1 + 2Cy_1 + E}(x - x_1).$$

To reduce this equation to the required form we first clear of fractions and transpose. This gives

$$(2Ax_1 + By_1 + D)x + (Bx_1 + 2Cy_1 + E)y - (2Ax_1^2 + 2Bx_1y_1 + 2Cy_1^2 + Dx_1 + Ey_1) = 0.$$

But from (1) the last parenthesis in this equation equals

$$-(Dx_1 + Ey_1 + 2F).$$

Substituting, the equation of the tangent line is

$$(2Ax_1 + By_1 + D)x + (Bx_1 + 2Cy_1 + E)y + (Dx_1 + Ey_1 + 2F) = 0.$$

Removing the parentheses, collecting the coefficients of A, B, C, D, E, and F, and dividing by (2), we obtain the equation of the theorem. Q. E. D.

The above result enables us to write down the equation of the tangent to the locus of any equation of the second degree. For by comparing the equation of the curve and the equation of the tangent we obtain the following

Rule *to write the equation of the tangent at the point of contact* $P_1(x_1, y_1)$ *to the locus of an equation of the second degree.*

Substitute x_1x *and* y_1y *for* x^2 *and* y^2, $\frac{y_1x + x_1y}{2}$ *for* xy, *and* $\frac{x + x_1}{2}$ *and* $\frac{y + y_1}{2}$ *for* x *and* y *in the given equation.*

For example, the equation of the tangent at the point of contact (x_1, y_1) to the conic $x^2 + 3xy - 4y + 5 = 0$ is

$$x_1x + \tfrac{3}{2}(x_1y + y_1x) - \tfrac{4}{2}(y + y_1) + 5 = 0;$$

or, also, $$(2x_1 + 3y_1)x + (3x_1 - 4)y - 4y_1 + 10 = 0.$$

75. Equation of the normal. The normal to a curve at a point P_1 is the line drawn through P_1 perpendicular to the tangent at P_1. When the equation of the tangent has been found, we may find at once the equation of the normal in the manner of Chapter IV. Thus, using the equations of the tangents given on page 193, we find the

Theorem. *The equation of the normal at* $P_1(x_1, y_1)$ *to the*

ellipse $b^2x^2 + a^2y^2 = a^2b^2$ *is* $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$;

hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ *is* $a^2y_1x + b^2x_1y = (a^2 + b^2)x_1y_1$;

parabola $y^2 = 2px$ *is* $y_1x + py = x_1y_1 + py_1$.

For example, for the ellipse:

The slope of the tangent

$$b^2x_1x + a^2y_1y = a^2b^2$$

is $m = -\frac{A}{B} = -\frac{b^2x_1}{a^2y_1}$. Hence the equation of the normal is

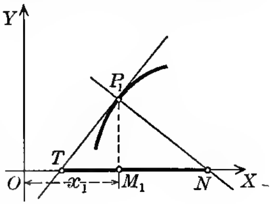
$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1),$$

and this reduces to the equation in the theorem.

In numerical examples the student should use the Rule given to write down the equation of the tangent, find the normal as a perpendicular line, and not use the special formulas.

76. Subtangent and subnormal. If the tangent and normal at P_1 intersect the x-axis in T and N respectively, then we define

(1) $P_1T =$ length of tangent at P_1,
$P_1N =$ length of normal at P_1.



The projections on XX' of P_1T and P_1N are called respectively the *subtangent* and *subnormal* at P_1. That is, in the figure,

(2) $M_1T =$ *subtangent* at P_1,
$M_1N =$ *subnormal* at P_1.

The subtangent and subnormal are readily found when the equations of the tangent and normal are known. For, from the figure,

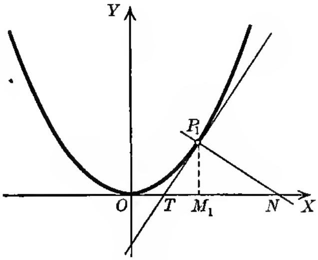
(3) $M_1T = OT - OM_1$,
$M_1N = ON - OM_1$,

and $OM_1 = x_1$,

while OT and ON are respectively the intercepts on XX' of the tangent and normal at P_1. Since the subtangent and subnormal are measured in opposite directions from the foot of the ordinate M_1P_1, they will have opposite signs.

EXAMPLE

Find the equations of tangent and normal, and the lengths of subtangent and subnormal at the point on the parabola $x^2 = 4y$ whose abscissa equals 3.



Solution. The point of contact (x_1, y_1) is

$$x_1 = 3, \quad y_1 = \tfrac{9}{4}.$$

The formula for the tangent at (x_1, y_1) is, by the Rule, p. 195,

$$x_1 x = 2(y + y_1).$$

Substituting the values of x_1 and y_1,

$$3x = 2(y + \tfrac{9}{4}) \text{ or } 6x - 4y - 9 = 0.$$

This is the required equation of the tangent.

The slope of this line is $\frac{3}{2}$. Hence the equation of normal at $(3, \frac{9}{4})$ is

$$y - \tfrac{9}{4} = -\tfrac{2}{3}(x - 3), \text{ or } 8x + 12y - 51 = 0.$$

The intercept on XX' of the tangent is $\frac{3}{2}$; of the normal $\frac{51}{8}$. Also $x_1 = 3$.

$$\therefore \text{ subtangent} = \tfrac{3}{2} - 3 = -\tfrac{3}{2},$$

and
$$\text{subnormal} = \tfrac{51}{8} - 3 = \tfrac{27}{8}.$$

The lengths of the tangents and normals may be found by geometry, for the lengths of the legs of the triangles P_1M_1T and P_1M_1N are now known.

PROBLEMS

1. Find the equations of the tangent and normal at the point indicated to each of the following. Find also the lengths of subtangent and subnormal. Draw a figure in each case.

(a) $2x^2 + 3y^2 = 35$, $x_1 = 2$, y_1 positive.*

Ans. Tangent, $4x + 9y = 35$; normal, $9x - 4y = 6$.
Subtangent $= \frac{27}{4}$; subnormal $= -\frac{4}{3}$.

(b) $x^2 - 4y^2 + 15 = 0$, $x_1 = 1$, y_1 negative.

(c) $y^2 = 4x + 3$, $y_1 = 2$.

(d) $xy = 4$, $x_1 = 2$.

(e) $x^2 + y^2 - 4x - 3 = 0$, $x_1 = 3$.

(f) $x^2 + 4y^2 + 5x = 0$, $y_1 = 1$.

(g) $4x^2 + 3y^2 = 1$; positive extremity of latus rectum.

* Substituting $x = 2$ in the given equation, we find $y = \pm 3$. Hence $y_1 = +3$.

(h) $x^2 + xy + 4 = 0$, $x_1 = 2$.
(i) $y^2 + 2xy - 3 = 0$, $y_1 = -1$.
(j) $x^2 - 3xy - 4y^2 + 9 = 0$, x_1 positive, $y_1 = 2$.
(k) $x^2 + xy + y^2 = 4$, $x_1 = 0$, y_1 negative.
(l) $x^2 + 4y^2 + 4x - 8y = 0$, $x_1 = 0$.
(m) $4y = x^3$, $x_1 = 2$.
(n) $4y^2 = x^3$, $x_1 = 2$.

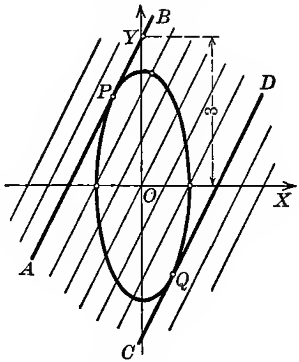
2. Show that the subtangent in the parabola $y^2 = 2px$ is bisected at the vertex, and that the subnormal is constant and equals p.

77. Tangent whose slope is given. Let it be required to find the equation of a tangent to the ellipse

(1) $$5x^2 + y^2 = 5$$

whose slope equals 2.

Solution. Draw the system of lines whose slope equals 2 (Art. 36). We observe that some of the lines intersect the ellipse in two points, and also that some of them do not intersect the ellipse at all. Furthermore, two of them are tangent. We wish to find the equations of these two tangents.



The equation of the system of lines whose slope equals 2 is

(2) $$y = 2x + k,$$

where k is an arbitrary parameter.

Let us now *start to solve for the points of intersection*. Substituting from (2) into (1),

(3) $$5x^2 + (2x + k)^2 = 5.$$

Squaring and collecting terms,

(4) $$9x^2 + 4kx + k^2 - 5 = 0.$$

If the line (2) is the tangent AB of the figure, by solving equation (4) we shall obtain the abscissa of the point of contact. But (4) is a quadratic and has two roots. Hence *these roots must be equal.*

We learn in algebra that the roots of the quadratic

$$(5) \qquad Ax^2 + Bx + C = 0$$

are equal when

$$(6) \qquad B^2 - 4\,AC = 0.$$

Comparing (4) with (5),

$$A = 9, \quad B = 4\,k, \quad C = k^2 - 5.$$

Substituting in (6),

$$(7) \qquad 16\,k^2 - 36\,(k^2 - 5) = 0, \quad \text{or} \quad k = \pm\, 3.$$

Hence the equations of the required tangents are

$$AB : y = 2\,x + 3 \quad \text{and} \quad CD : y = 2\,x - 3.$$

Check. Writing $k = 3$ in (4), it becomes

$$9\,x^2 + 12\,x + 4 = 0, \quad \text{or} \quad (3\,x + 2)^2 = 0.$$

The equation is now a *perfect square*, and this fact constitutes the check desired. Hence the equal roots have the common value $x = -\frac{2}{3}$. This is the abscissa of the point of contact P. The ordinate is found from $y = 2\,x + 3$ to be $y = \frac{5}{3}$. Hence P is $(-\frac{2}{3}, \frac{5}{3})$.

Similarly, putting $k = -3$ in (4), we find Q to be $(\frac{2}{3}, -\frac{5}{3})$.

The method followed in the preceding may be thus outlined.

To find the equation of the tangent to a conic when the slope of the tangent is given.

1. Write down the equation of the system of lines with the given slope ($y = mx + k$). This equation contains a parameter (k) whose value must be found.

2. Eliminate x or y from the equations of the line and conic and arrange the result in the form of a quadratic

$$(8) \qquad Ay^2 + By + C = 0, \quad \text{or} \quad Ax^2 + Bx + C = 0.$$

3. The roots of this quadratic must be equal. Hence set

(9) $$B^2 - 4\,AC = 0,$$

and solve this for the parameter k.

4. Substitute the values of the parameter k in the equation of the system of lines.

5. *Check*. When each value of the parameter satisfying (9) is substituted in (8), the quadratic becomes a perfect square.

PROBLEMS

1. Find the equations of the tangents to the following conics which satisfy the condition indicated, check, and find the points of contact. Verify by constructing the figure.

(a) $y^2 = 4\,x$, slope $= \frac{1}{2}$. *Ans.* $x - 2\,y + 4 = 0$.

(b) $x^2 + y^2 = 16$, slope $= -\frac{4}{3}$. *Ans.* $5\,x + 3\,y \pm 20 = 0$.

(c) $9\,x^2 + 16\,y^2 = 144$, slope $= -\frac{1}{4}$. *Ans.* $x + 4\,y \pm 4\sqrt{10} = 0$.

(d) $x^2 - 4\,y^2 = 36$, perpendicular to $6\,x - 4\,y + 9 = 0$. *Ans.* $2\,x + 3\,y \pm 3\sqrt{7} = 0$.

(e) $x^2 + 2\,y^2 - x + y = 0$, slope $= -1$. *Ans.* $x + y = 1$, $2x + 2y + 1 = 0$.

(f) $xy + y^2 - 4\,x + 8\,y = 0$, parallel to $2\,x - 4\,y = 7$. *Ans.* $x = 2\,y$, $x - 2\,y + 48 = 0$.

(g) $x^2 + 2\,xy + y^2 + 8\,x - 6\,y = 0$, slope $= \frac{4}{3}$. *Ans.* $4\,x - 3\,y = 0$.

(h) $x^2 + 2\,xy - 4\,x + 2\,y = 0$, slope $= 2$. *Ans.* $y = 2\,x$, $2\,x - y + 10 = 0$.

(i) $2\,x^2 + 3\,y^2 = 35$, slope $= \frac{4}{9}$.

(j) $x^2 + y^2 = 25$, slope $= -\frac{3}{4}$.

(k) $x^2 + 4\,y - 8 = 0$, slope $= 2$.

(l) $y^2 + 4\,x - 9 = 0$, slope $= -1$.

(m) $x^2 - y^2 = 16$, slope $= \frac{5}{3}$.

(n) $xy - 4 = 0$, slope $= -\frac{8}{2}$.

78. Formulas for tangents when the slope is given. For later reference we collect in this section formulas giving the equations of tangents to the conics in terms of the slope m of the tangent. The student should derive these formulas, following the method of the preceding section.

Theorem. *The equation of a tangent in terms of its slope m to the*

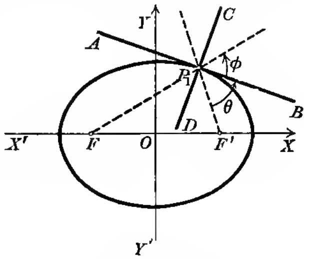
circle $x^2 + y^2 = r^2$ *is* $y = mx \pm r\sqrt{1 + m^2}$;

ellipse $b^2x^2 + a^2y^2 = a^2b^2$ *is* $y = mx \pm \sqrt{a^2m^2 + b^2}$;

hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ *is* $y = mx \pm \sqrt{a^2m^2 - b^2}$;

parabola $y^2 = 2\,px$ *is* $y = mx + \dfrac{p}{2\,m}$.

79. Properties of tangents and normals to conics. If we draw the tangent AB and the normal CD at any point P_1 on the ellipse, and if we draw also the **focal radii** P_1F and P_1F', we may prove the property:



The tangent and normal to an ellipse bisect respectively the external and internal angles formed by the focal radii of the point of contact.

Proof. In the figure we wish to prove $\theta = \phi$. To do this we find $\tan\phi$ and $\tan\theta$ by (VI), Art. 35.

The slopes of the lines joining $P_1(x_1, y_1)$ on the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

to the foci $F'(c, 0)$ and $F(-c, 0)$ are

$$\text{slope of } F'P_1 = \frac{y_1}{x_1 - c};$$

$$\text{slope of } FP_1 = \frac{y_1}{x_1 + c}.$$

The equation of the tangent AB is (Theorem, Art. 73)

$$b^2x_1x + a^2y_1y = a^2b^2.$$

$$\therefore \text{ slope of } AB = -\frac{b^2x_1}{a^2y_1}.$$

Now $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1m_2}$, where $m_1 =$ slope of AB, $m_2 =$ slope of P_1F'.

Substituting the above values of the slopes,

$$\tan\theta = \frac{-\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1 - c}}{1 - \dfrac{b^2x_1y_1}{a^2y_1(x_1 - c)}} = \frac{-b^2x_1^2 + b^2cx_1 - a^2y_1^2}{a^2x_1y_1 - a^2cy_1 - b^2x_1y_1}$$

$$= \frac{(a^2y_1^2 + b^2x_1^2) - b^2cx_1}{a^2cy_1 - (a^2 - b^2)x_1y_1}.$$

But since P_1 lies on the ellipse,

$$a^2y_1^2 + b^2x_1^2 = a^2b$$

and also

$$a^2 - b^2 = c^2.$$

Hence

$$\tan\theta = \frac{a^2b^2 - b^2cx_1}{a^2cy_1 - c^2x_1y_1} = \frac{b^2(a^2 - cx_1)}{cy_1(a^2 - cx_1)} = \frac{b^2}{cy_1}.$$

In like manner,

$$\tan\phi = \frac{-b^2x_1^2 - b^2cx_1 - a^2y_1^2}{-a^2x_1y_1 - a^2cy_1 + b^2x_1y_1} = \frac{(b^2x_1^2 + a^2y_1^2) + b^2cx_1}{a^2cy_1 + (a^2 - b^2)x_1y_1}$$

$$= \frac{a^2b^2 + b^2cx_1}{a^2cy_1 + c^2x_1y_1} = \frac{b^2}{cy_1}.$$

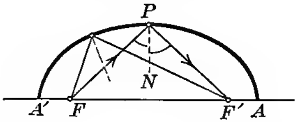
Hence $\tan\theta = \tan\phi$; and since θ and ϕ are both less than π, $\theta = \phi$. That is, AB bisects the external angle of FP_1 and $F'P_1$, and hence, also, CD bisects the internal angle. Q. E. D.

An obvious application of this theorem is to the problem:

To draw a tangent and normal at a given point P_1 on an ellipse.

This is accomplished by connecting P_1 with the foci and bisecting the internal and external angles formed by these lines.

The phenomenon observed in "whispering galleries" depends upon this property; namely, let the elliptic arc $A'PA$ be a vertical section of such a gallery. The waves of sound from a person's voice at the focus F will, after meeting the ceiling of the gallery, be reflected in the direction F'.

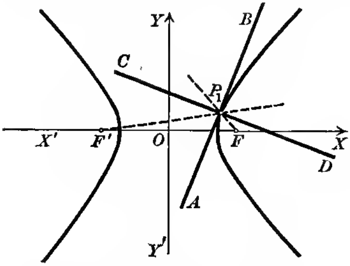


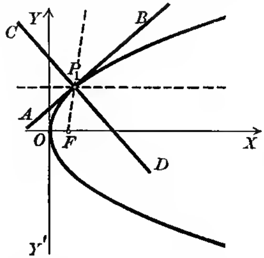
For if PN is the normal at P, angle NPF = angle NPF', and the law of reflection of sound waves is precisely that the angles of incidence ($= \angle NPF$) and reflection ($= \angle NPF'$) are equal. Hence sound waves emanating from F in all directions will converge at F'. A whisper at F, which would not carry over the distance FF', might consequently, through reflection, be audible at F'.

In like manner we prove the following properties:

The tangent and normal to a hyperbola bisect respectively the internal and external angles formed by the focal radii of the point of contact.

The tangent and normal to a parabola bisect respectively the internal and external angles formed by the focal radius of the point of contact and the line through that point parallel to the axis.





These theorems give rules for constructing the tangent and normal to these conics by means of ruler and compasses.

Construction. To construct the tangent and normal to a hyperbola at any point, join that point to the foci and bisect the angles formed by these lines. To construct the tangent and normal to a parabola at any point, draw lines through it to the focus and parallel to the axis, and bisect the angles formed by these lines.

The principle of parabolic reflectors depends upon the property of tangent and normal just enunciated; namely, the reflecting surface of such a reflector is obtained by revolving a parabolic arc about its axis. If, now, a light be placed at the focus, the rays of light which meet the surface of the reflector will all be reflected in the direction of the axis of the parabola; for a ray meeting the surface at P_1 in the figure will be reflected in a direction making with the normal PD an angle equal to the angle FP_1D. But this direction is, by the above property, parallel to the axis OX of the parabola.

PROBLEMS

1. Tangents to an ellipse and its major auxiliary circle (p. 164) at points with the same abscissa intersect on the x-axis.

2. The point of contact of a tangent to a hyperbola is midway between the points in which the tangent meets the asymptotes.

3. The foot of the perpendicular from the focus of a parabola to a tangent lies on the tangent at the vertex.

4. The foot of the perpendicular from a focus of an ellipse to a tangent lies on the major auxiliary circle (p. 164).

5. Tangents to a parabola from a point on the directrix are perpendicular to each other.

6. Tangents to a parabola at the extremities of a chord which passes through the focus are perpendicular to each other.

7. The ordinate of the point of intersection of the directrix of a parabola and the line through the focus perpendicular to a tangent is the same as that of the point of contact.

8. How may Problem 7 be used to draw a tangent to a parabola?

9. The line drawn perpendicular to a tangent to a central conic from a focus, and the line passing through the center and the point of contact intersect on the corresponding directrix (Art. 72).

10. The angle which one tangent to a parabola makes with a second is half the angle which the focal radius drawn to the point of contact of the first makes with that drawn to the point of contact of the second.

11. The product of the distances from a tangent to a central conic to the foci is constant.

12. Tangents to any conic at the ends of the latus rectum pass through the intersection of the directrix and principal axis.

13. Tangents to a parabola at the extremities of the latus rectum are perpendicular.

14. The equation of the parabola referred to the tangents in Problem 13 is

$$x^2 - 2xy + y^2 - 2\sqrt{2}p(x + y) + 2p^2 = 0.$$

Show that this equation has the form $x^{\frac{1}{2}} + y^{\frac{1}{2}} = \sqrt{p\sqrt{2}}$.

15. The area of the triangle formed by a tangent to a hyperbola and the asymptotes is constant.

16. An ellipse and a hyperbola which are confocal intersect at right angles.

CHAPTER XII

PARAMETRIC EQUATIONS AND LOCI

80. If x and y are rectangular coördinates, and if each is expressed as a function of a variable parameter, as, for example,

(1) $$x = \tfrac{1}{2}t^2, \quad y = \tfrac{1}{4}t^3,$$

in which t is a variable, then these equations are called the *parametric equations* of the curve, — the locus of (x, y).

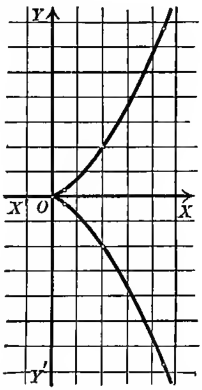
To plot the curve, give values to t and compute values of x and y, arranging the work in a table. When the computation is finished, plot the points (x, y) and draw a smooth curve through them.

EXAMPLES

1. Plot the curve whose parametric equations are

(2) $$x = \tfrac{1}{2}t^2, \quad y = \tfrac{1}{4}t^3.$$

t	x	y
0	0	0
1	.5	.25
2	2	2
3	4.5	6.75
etc.	etc.	etc.
− 1	.5	− .25
− 2	2	− 2
− 3	4.5	− 6.75
etc.	etc.	etc.



Solution. The table is easily made. For example, if $t = 2$, then $x = 2$, $y = 2$, etc.

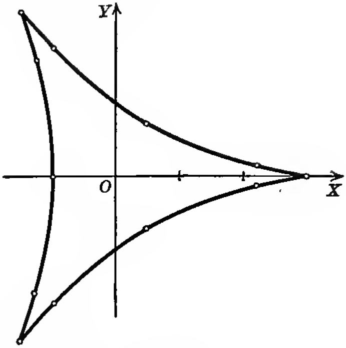
The curve is called a **semicubical parabola**.

2. Draw the locus of the equations

(3) $x = 2r\cos\theta + r\cos 2\theta,$

$y = 2r\sin\theta - r\sin 2\theta,$

where θ is a variable parameter.



Solution. Take $r = 5$. Arrange the computation as below:

The three-pointed curve thus obtained is called a **hypocycloid of three cusps.**

$x=10\cos\theta+5\cos 2\theta,\quad y=10\sin\theta-5\sin 2\theta$							
θ	$\cos\theta$	2θ	$\cos 2\theta$	x	$\sin\theta$	$\sin 2\theta$	y
0	1	0	1	15	0	0	0
30°	.86	60°	.50	11.1	.50	.86	0.7
60°	.50	120°	− .50	2.5	.86	.86	4.3
90°	0	180°	− 1	− 5	1	0	10
120°	− .50	240°	− .50	− 7.5	.86	− .86	12.9
150°	− .86	300°	.50	− 6.1	.50	− .86	9.3
180°	− 1	360°	1	− 5	0	0	0
210°	− .86	420°	.50	− 6.1	− .50	.86	− 9.3
240°	− .50	480°	− .50	− 7.5	− .86	.86	− 12.9
270°	0	540°	− 1	− 5	− 1	0	− 10
300°	.50	600°	− .50	2.5	− .86	− .86	− 4.3
330°	.86	660°	.50	11.1	− .50	− .86	− 0.7
360°	1	720°	1	15	0	0	0

To obtain the rectangular equation from the parametric equations, the parameter must be eliminated. The method used depends upon the example.

EXAMPLES

1. Find the rectangular equation of the curve whose parametric equations are

(4) $$x = 2t + 3, \quad y = \tfrac{1}{2}t^2 - 4.$$

Solution. The first equation may be solved readily for t. We find $t = \frac{1}{2}(x - 3)$, and substituting in the second equation gives $y = \frac{1}{8}(x - 3)^2 - 4$; or, expanding and simplifying, $x^2 - 6x - 8y - 23 = 0$, a parabola.

2. Find the rectangular equation of the curve whose parametric equations are

(5) $$x = 3 + 4\cos\theta, \quad y = 3\sin\theta.$$

Solution. Remembering that $\sin^2\theta + \cos^2\theta = 1$, we solve the first equation for $\cos\theta$, the second for $\sin\theta$. This gives

(6) $$\cos\theta = \tfrac{1}{4}(x - 3), \quad \sin\theta = \tfrac{1}{3}y.$$

Hence the rectangular equation is

(7) $$\frac{(x-3)^2}{16} + \frac{y^2}{9} = 1,$$

an ellipse.

PROBLEMS

1. Plot the following parametric equations, t and θ being variable parameters. Find the rectangular equation in each case:

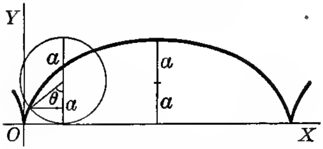
(a) $x = t - 1,\ y = 4 - t^2$.
(b) $x = 2t^2 - 2,\ y = t - 3$.
(c) $x = 3\cos\theta,\ y = \sin\theta$.
(d) $x = 3\tan\theta,\ y = \sec\theta$.
(e) $x = 2t,\ y = \dfrac{4}{t}$.
(f) $x = 2 + \sin\theta,\ y = 2\cos\theta$.
(g) $x = \frac{1}{2}t^3,\ y = \frac{1}{4}t$.
(h) $x = t^2 - 2t,\ y = 1 - t^2$.
(i) $x = \cos\theta,\ y = \cos 2\theta$.
(j) $x = \frac{1}{2}\sin\theta,\ y = \sin 2\theta$.
(k) $x = 1 - \cos\theta,\ y = \frac{1}{2}\sin\frac{1}{2}\theta$.
(l) $x = 3t^2,\ y = 3t - t^3$.
(m) $x = 2\sin\theta + 3\cos\theta,\ y = \sin\theta$.
(n) $x = 2\cos\theta + 1,\ y = \sin\theta + 4\cos\theta$.
(o) $x = t - t^2,\ y = t + t^2$.
(p) $x = 3 - 2t,\ y = 1 + \dfrac{2}{t}$.

2. Plot the following parametric equations:

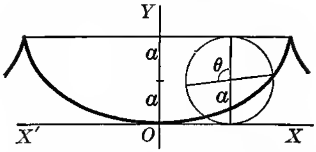
(a) $x = 2r\cos\theta - r\cos 2\theta, \quad y = 2r\sin\theta - r\sin 2\theta$.
(b) $x = 3r\cos\theta + r\cos 3\theta, \quad y = 3r\sin\theta - r\sin 3\theta$.
(c) $x = 3r\cos\theta - r\cos 3\theta, \quad y = 3r\sin\theta - r\sin 3\theta$.
(d) $x = r\cos\theta - r\cos 2\theta, \quad y = r\sin\theta - r\sin 2\theta$.
(e) $x = 2r\cos\theta + \frac{1}{2}r\cos 2\theta, \quad y = 2r\sin\theta - \frac{1}{2}r\sin 2\theta$.

(f) $\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta). \end{cases}$

(g) $\begin{cases} x = a(\theta + \sin\theta), \\ y = a(1 - \cos\theta). \end{cases}$



CYCLOID, CUSP AT ORIGIN



CYCLOID, VERTEX AT ORIGIN

(h) $x = a\theta - \frac{1}{2} a \sin\theta,\ y = a - \frac{1}{2} a \cos\theta.$

(i) $x = a\theta - 2a\sin\theta,\ y = a - 2a\cos\theta.$

(j) $x = r\cos\theta + r\theta\sin\theta,\ y = r\sin\theta - r\theta\cos\theta.$

(k) $x = 4r\cos\theta - r\cos 4\theta,\ y = 4r\sin\theta - r\sin 4\theta.$

(l) $x = a\log t,\ y = \frac{1}{2} a\left(t + \frac{1}{t}\right).$

(m) $x = t + \sin t,\ y = 1 + \cos t.$

(n) $x = 2\cos t + t,\ y = 3\cos t + \sin 2t.$

(o) $x = b\cos^2\theta,\ y = a\tan\theta.$

81. Various parametric equations for the same curve. When the rectangular equation of a curve is given, any number of parametric equations may be obtained for the curve.

For example, given the ellipse

(1) $$4x^2 + y^2 = 16.$$

Let $x = 2\cos\theta$, where θ is a variable parameter. Substituting in (1),

$$16\cos^2\theta + y^2 = 16, \quad \text{or} \quad y^2 = 16(1 - \cos^2\theta) = 16\sin^2\theta.$$

Hence the equations

(2) $$x = 2\cos\theta, \quad y = 4\sin\theta,$$

are parametric equations of the ellipse (1).

Again, substitute in (1),

$$y = tx + 4,$$

where t is a variable parameter.

This gives

(3) $$4x^2 + t^2x^2 + 8tx + 16 = 16, \quad \text{or } (4 + t^2)x^2 + 8tx = 0.$$

(4) $$\therefore\ x = -\frac{8t}{4 + t^2}.$$

Substituting this value in (3) and reducing,

(5) $$y = -\frac{4t^2}{4+t^2}.$$

Hence the equations (4) and (5) are also parametric equations of the ellipse.

The point is: *We obtain parametric equations by setting one of the coördinates equal to a function of a parameter, substituting in the given rectangular equation and solving for the other coördinate in terms of the parameter.*

To obtain *simple* parametric equations we must, of course, assume the *right function* for one coördinate. No general rule applicable to all cases can be given for this purpose, but the study of the problems below will aid the student.

Many rectangular equations difficult to plot are treated by deriving parametric equations and plotting the latter.

EXAMPLES

1. Draw the locus of the equation

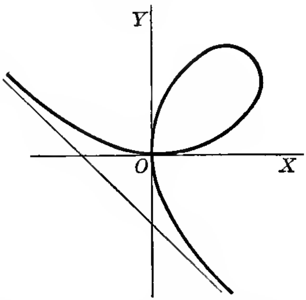
(6) $$x^3 + y^3 - 3axy = 0.$$

Solution. Set $y = tx$, where t is the parameter. Then, from (6),

(7) $$x^3 + t^3x^3 - 3atx^2 = 0.$$

Dividing out the x^2, solving for x, and remembering that $y = tx$, we obtain the desired parametric equations

(8) $$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$



The locus is the curve of the figure, called the **folium of Descartes.**

The line drawn in the figure is an oblique asymptote. Its equation is $x + y + a = 0$.

The parameter t in (7) is obviously the slope of the line $y = tx$; that is, of the line joining a point on the curve and the origin.

The *reason* for assuming the relation $y = tx$ in the preceding example is that x^2 divides out in (7), leaving an equation of the first degree to solve for x. Problems 1 (a), (d), (e), (f), and (j) below are worked on the same principle. In many cases trigonometric functions are employed with advantage, as in (b) and (c).

PROBLEMS

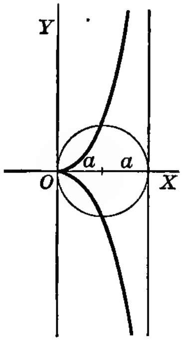
1. Find parametric equations for each of the following curves by making the substitution indicated in the given equation. The parameter is t or θ, as the case may be. Plot the locus.

(a) $y^2 = 4x^2 - x^3$, $y = tx$. *Ans.* $x = 4 - t^2$, $y = 4t - t^3$.

(b) $x^2y^2 = b^2x^2 + a^2y^2$, $x = a\sec\theta$. *Ans.* $y = b\csc\theta$.

(c) $x^2y^2 = a^2y^2 - b^2x^2$, $x = a\sin\theta$. *Ans.* $y = b\tan\theta$.

(d) $y^3 = 2ax^2 - x^3$, $y = tx$.



(e) $y^2(2a - x) = x^3$, $y = tx$.

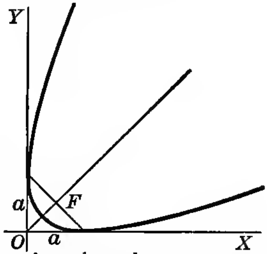
CISSOID OF DIOCLES

(f) $y^2 = x^2\dfrac{2+x}{2-x}$, $y = tx$.

Ans. $x = \dfrac{2t^2 - 2}{1 + t^2}$, $y = \dfrac{2t^3 - 2t}{1 + t^2}$.

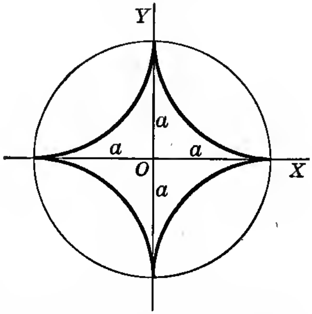
(g) $x^2 + xy + 2y^2 + 2x + 1 = 0$, $x = ty - 1$.

Ans. $x = -\dfrac{2 + t^2}{t^2 + t + 2}$, $y = \dfrac{1}{t^2 + t + 2}$.



(h) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, $x = a\cos^4\theta$.

PARABOLA



(i) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $x = a\sin^3\theta$.

HYPOCYCLOID OF FOUR CUSPS

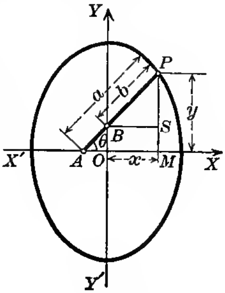
(j) $x^4 + 2ax^2y - ay^3 = 0$, $y = tx$.
(k) $(x^2 + y^2 + 4ay - a^2)(x^2 - a^2) + 4a^2y^2 = 0$, $x^2 = t^2y^2 + a^2$.
(l) $x^2 = y(y-2)^2$, $y - 2 = tx$.
(m) $(x^2 - \frac{1}{2}b^2)^2 + y^2(x^2 - b^2) = 0$, $x^2 = \frac{1}{2}b^2 + ty$.

82. Locus problems solved by parametric equations. Parametric equations are important because it is sometimes easy in locus problems to express the coördinates of a point on the locus in terms of a parameter, when it is otherwise difficult to obtain the equation of the locus. The following examples illustrate this statement:

EXAMPLES

1. ABP is a rigid line. The points A and B move along two perpendicular intersecting lines. What is the locus of the point P on AB?

In the figure, A moves on XX', B moves on YY'; required the locus of the point $P(x, y)$.



Solution. Take the coördinate axes as indicated, and consider the line in any one of its positions. Choose for parameter the angle $XAB = \theta$.

Let $AP = a, \quad PB = b.$

Now $OM = x, \quad MP = y.$

In the right triangle MPA,

(1) $$\sin\theta = \frac{MP}{PA} = \frac{y}{a}.$$

In the right triangle BSP, $\angle PBS = \theta$.

(2) $$\therefore \cos PBS = \cos\theta = \frac{BS}{BP} = \frac{x}{b}.$$

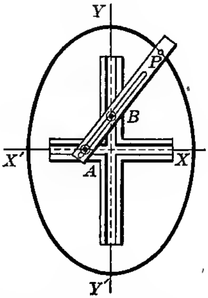
From (1) and (2),

(3) $$x = b\cos\theta, \quad y = a\sin\theta.$$

These are the parametric equations of the locus.

Squaring (1) and (2) and adding,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

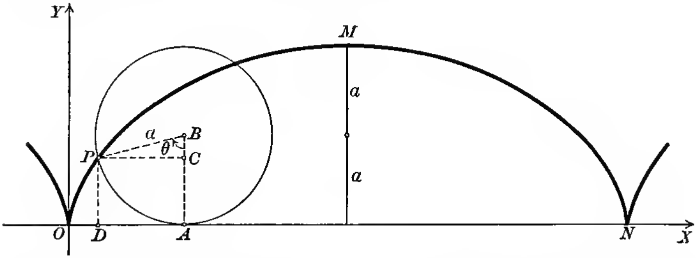


Hence the point P moves on an ellipse whose axes $2a$ and $2b$ lie along the given perpendicular lines.

A method commonly employed for drawing ellipses depends upon this result. The instrument consists of two grooved perpendicular bars $X'X$

and YY' and a crossbar ABP. At A and B are screw nuts fitting the grooves and adjustable along ABP. If the crossbar is moved, a pencil at P will describe an ellipse whose semiaxes are PA and PB.

2. The cycloid. Find the parametric equations of the locus of a point P on a circle which rolls along the axis of x.



Solution. Take for origin a point O at which the moving point P touched the axis of x. Let the circle drawn be any position of the rolling circle. Let a be the radius of the circle and take for the variable parameter θ the variable angle CBP. Then

$$PC = a \sin \theta, \quad CB = a \cos \theta.$$

By definition, $\quad OA = \text{arc}\, AP = a\theta.$

[For an arc of a circle equals its radius times the subtended angle, from the definition of a radian.]

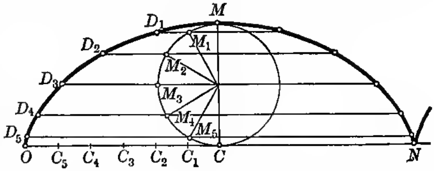
Hence from the figure, if (x, y) are the coördinates of P,

$$x = OD = OA - PC = a\theta - a \sin \theta, \; y = DP = AB - CB = a - a \cos \theta.$$

(4) $$\therefore \begin{cases} x = a(\theta - \sin \theta), \\ y = a(1 - \cos \theta). \end{cases}$$

These are the parametric equations of the *cycloid.*

The cycloid extends indefinitely to the right and left and consists of arcs equal to OMN.



Construction of the cycloid. The definition of the cycloid suggests the following simple construction:

Lay off $ON = 2\pi a =$ circumference of the generating circle. Draw the latter touching at C, the middle point of ON. Divide OC into any number of equal parts, and the semicircle CM into the same number of equal arcs. Letter as in the figure. Through M_1, M_2, etc., draw lines parallel to ON. Lay off

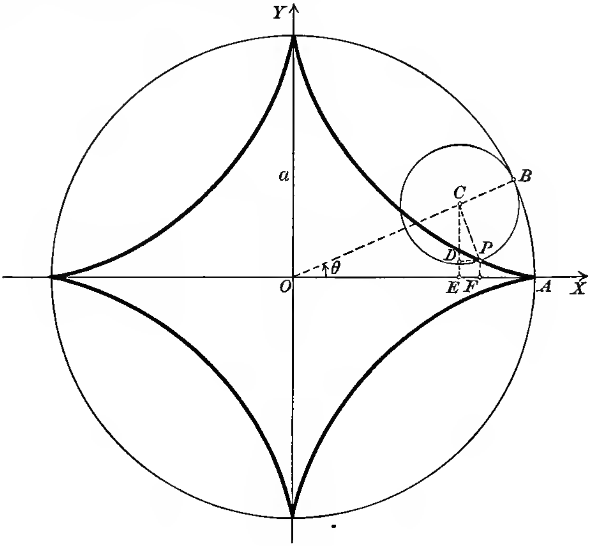
$$M_1D_1 = CC_1, \quad M_2D_2 = CC_2, \quad M_3D_3 = CC_3, \text{ etc.}$$

Then D_1, D_2, D_3, etc. are points on the cycloid.

For, let the generating circle roll to the left, the point M tracing the curve. When the circle touches ON at C_1, M will lie on a level with M_1, and at a distance to the left of M_1 equal to CC_1. Similarly for D_2, D_3, etc.

The arc MN of the cycloid may be constructed by using CM as an axis of symmetry.

3. The hypocycloid of four cusps. Find the parametric equations of the locus of a point P on a circle which rolls on the inside of a fixed circle of four times the radius.



Solution. Take the center of the fixed circle for the origin and let the x-axis pass through a point A where the tracing point P touched the large circle. Then $OA = 4\,CB$, by hypothesis. $\therefore\ CB = \frac{OA}{4} = \frac{a}{4}\cdot$ Draw the rolling circle in any of its positions. Take for the variable parameter θ the $\angle AOB$. Then $\angle BCP = 4\,\theta$.

[For, by hypothesis, arc $PB =$ arc AB; and, from the definition of a radian, arc $PB = \frac{a}{4} \angle BCP$, arc $AB = a\theta$. $\therefore \frac{a}{4} \angle BCP = a\theta$, or $\angle BCP = 4\theta$.]

But $$\angle OCE + \angle ECP + \angle PCB = \pi.$$

$$\therefore \frac{\pi}{2} - \theta + \angle ECP + 4\theta = \pi.$$

Whence $$\angle ECP = \frac{\pi}{2} - 3\theta.$$

Now $$OF = x, \ FP = y.$$

From the figure,

(5) $$OF = OE + DP,$$
$$FP = EC - CD.$$

Finding the lengths of the segments in the right-hand members,

$$OE = OC \cos\theta = \frac{3a}{4}\cos\theta, \ EC = OC \sin\theta = \frac{3a}{4}\sin\theta.$$

$$DC = CP \cos\left(\frac{\pi}{2} - 3\theta\right) = \frac{a}{4}\sin 3\theta, \qquad \text{(by 31, p. 3)}$$

$$DP = CP \sin\left(\frac{\pi}{2} - 3\theta\right) = \frac{a}{4}\cos 3\theta. \qquad \text{(by 31, p. 3)}$$

Substituting in (5),

(6) $$\begin{cases} x = \frac{3}{4} a \cos\theta + \frac{1}{4} a \cos 3\theta, \\ y = \frac{3}{4} a \sin\theta - \frac{1}{4} a \sin 3\theta. \end{cases}$$

These are parametric equations for the *hypocycloid of four cusps.*

Another form of (6) from which the rectangular equation may easily be derived is obtained by expressing $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ respectively. Thus,

$$\begin{aligned} \cos 3\theta &= \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \qquad \text{(by 35, p. 3)} \\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta \cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

$$\begin{aligned} \sin 3\theta &= \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \qquad \text{(by 33, p. 3)} \\ &= 2\sin\theta \cos^2\theta + (1 - 2\sin^2\theta)\sin\theta \\ &= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta. \end{aligned}$$

Substituting in (6) and reducing, the result is

(7) $$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

From these, $x^{\frac{2}{3}} = a^{\frac{2}{3}}\cos^2\theta$, $y^{\frac{2}{3}} = a^{\frac{2}{3}}\sin^2\theta$. Adding,

(8) $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

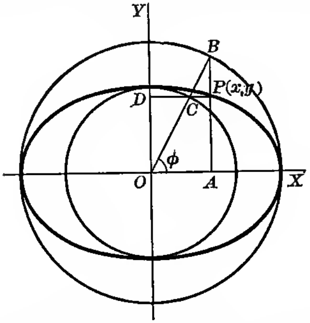
which is the *rectangular equation of the hypocycloid of four cusps*

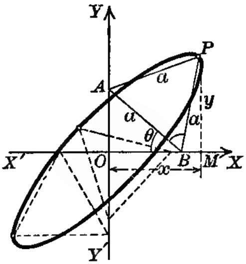
PROBLEMS

In the following problems express x and y in terms of the parameter and the lengths of the given lines of the figure. Sketch the locus.

1. Find the parametric equations of the ellipse, using as parameter the **eccentric angle** ϕ, that is, the angle between the major axis and the radius of the point B on the major auxiliary circle (p. 164) which has the same abscissa as the point P (x, y) on the ellipse. (See figure.)

Ans. $x = a \cos \phi$, $y = b \sin \phi$.



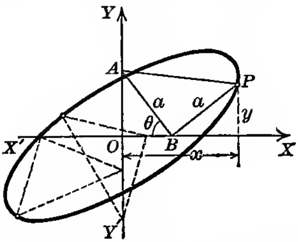


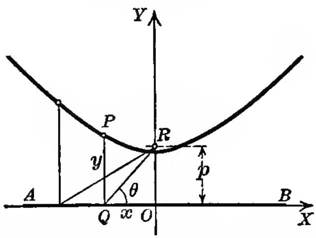
2. In the figure, ABP is a rigid equilateral triangle. A moves on YY', B moves on XX'. Find the locus of the vertex P.

Ans. $x = a \cos \theta + a \cos (120° - \theta)$, $y = a \sin (120° - \theta)$.
Ellipse, $x^2 - \sqrt{3}\, xy + y^2 = \frac{1}{4} a^2$

3. Two vertices A and B of a rigid right triangle ABP move on perpendicular lines. Find the locus of the vertex P.

Ans. $x = a \cos \theta + a \sin \theta$, $y = a \cos \theta$. *Ellipse*, $x^2 - 2\,xy + 2\,y^2 = a^2$.



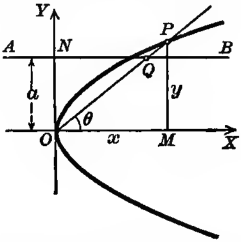


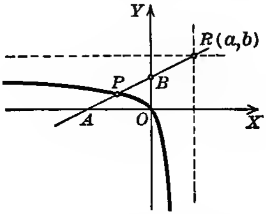
4. AB is a fixed line and R a fixed point. Draw RQ to any point Q in AB and erect the perpendicular QP, making $QP \div QR$ equal to a constant e. What is the locus of P?

Ans. $x = p \cot \theta$, $y = ep \csc \theta$. *Hyperbola*, $\frac{x^2}{p^2} - \frac{y^2}{p^2 e^2} = 1$.

5. AB is a fixed line and O a fixed point. Through O draw OX parallel to AB and ON perpendicular to AB. Draw a line from O through any point Q in AB. Mark on this line a point P such that $MP = NQ$, MP being $\perp$ to OX. What is the locus of P?

Ans. $x = a \cot^2 \theta$, $y = a \cot \theta$. *Parabola*, $y^2 = ax$.





6. Through the fixed point R (a, b) lines are drawn meeting the coördinate axes in A and B. What is the locus of the middle point of AB?

Ans. $x = a - \dfrac{b}{t}$, $y = b - at$, where t = slope of AB.

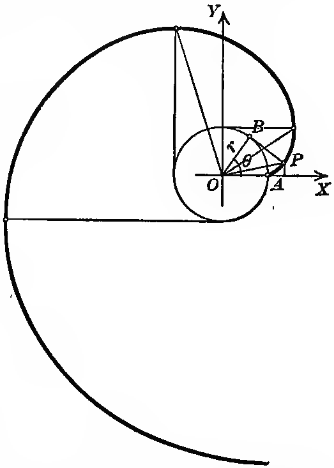
Equilateral hyperbola, $(x - a)(y - b) = ab$.

7. Find the locus of a point Q on the radius BP (Fig., Ex. 2, p. 212) if $BQ = b$.

Ans. $\begin{cases} x = a\theta - b \sin \theta, \\ y = a - b \cos \theta. \end{cases}$

The locus is called a **prolate** or **curtate cycloid** according as b is greater or less than a.

Describe a construction for the curve analogous to that given for the cycloid in Art. 82.



8. Given a string wrapped around a circle; find the locus of the end of the string as it is unwound.

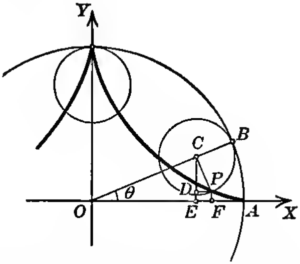
Hint. Take the center of the circle for origin and let the x-axis pass through the point A at which the end of the string rests. If the string is unwound to a point B, let $\angle AOB = \theta$. (See figure.)

Ans. The **involute of a circle** $\begin{cases} x = r \cos \theta + r\theta \sin \theta, \\ y = r \sin \theta - r\theta \cos \theta. \end{cases}$

9. A circle of radius r rolls on the inside of a circle whose radius is r'. Find the locus of a point on the rolling circle.

Ans. The **hypocycloid**

$$\begin{cases} x = (r' - r)\cos\theta + r\cos\dfrac{r'-r}{r}\theta, \\ y = (r' - r)\sin\theta - r\sin\dfrac{r'-r}{r}\theta. \end{cases}$$



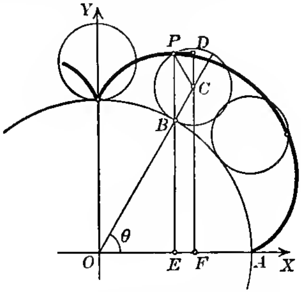
The curve is closed when r and r' are commensurable. The hypocycloid of four cusps, p. 213, is a special case.

Describe a construction for the curve analogous to that given for the cycloid in Art. 82.

10. A circle of radius r rolls on the outside of a circle whose radius is r'. Find the locus of a point on the rolling circle.

Ans. The **epicycloid**

$$\begin{cases} x = (r' + r)\cos\theta - r\cos\dfrac{r'+r}{r}\theta, \\ y = (r' + r)\sin\theta - r\sin\dfrac{r'+r}{r}\theta. \end{cases}$$



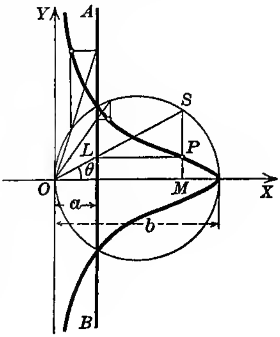
The curve is closed when r and r' are commensurable.

Describe a construction for the curve analogous to that given for the cycloid in Art. 82.

11. Given a fixed point O on a fixed circle and a fixed line AB. Draw the x-axis through O perpendicular to AB and the y-axis through O parallel to AB. Draw any line through O to meet AB in L and the fixed circle in S. Draw $LP \parallel$ to OX to meet SM drawn $\parallel$ to OY. Required the locus of P.

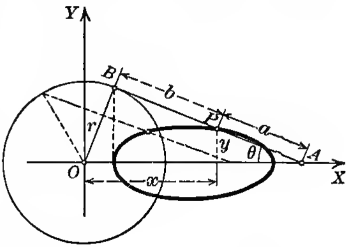
Ans. $x = b\cos^2\theta,\ y = a\tan\theta.$

Cubic, $xy^2 + a^2x - a^2b = 0.$



Give a full discussion of the equation. Show that the y-axis is an asymptote. What modifications, if any, are necessary in the equations when AB is a tangent? when AB does not intersect the circle?

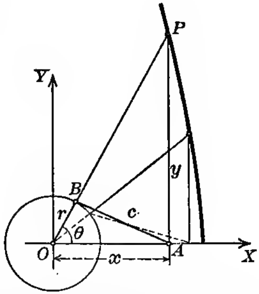
12. OB is the crank of an engine and AB the connecting rod. B moves on the crank circle whose center is O, and A moves on the fixed line OX. What is the locus of any point P on AB?



Ans. $x = b\cos\theta + \sqrt{r^2 - (a+b)^2\sin^2\theta},\ y = a\sin\theta.$

Ellipse, when $r = a + b$; otherwise an *egg-shaped curve*.

13. OB is an engine crank revolving about O, and AB is the connecting rod, the point A moving on OX. Draw $AP \perp$ to OX to meet OB produced at P.* What is the locus of P?



Ans. $x = r\cos\theta + \sqrt{c^2 - r^2\sin^2\theta},$

$y = r\sin\theta + \tan\theta\sqrt{c^2 - r^2\sin^2\theta}.$

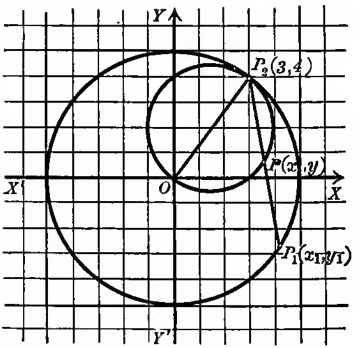
When $c = r$, the locus is the circle

$$x^2 + y^2 = 4r^2.$$

83. Loci derived by a construction from a given curve. Many important loci are defined as the locus of a point obtained by a given construction from a given curve. The method of treatment of such loci is illustrated in the following examples.

EXAMPLES

1. Find the locus of the middle points of the chords of the circle $x^2 + y^2 = 25$ which pass through $P_2(3, 4)$.



Solution. Let $P_1(x_1, y_1)$ be any point on the circle.

(1) $\therefore\ x_1^2 + y_1^2 = 25.$

Then a point $P(x, y)$ on the locus is obtained by bisecting P_1P_2. By (IV), Art. 13,

$$x = \tfrac{1}{2}(x_1 + 3), \quad y = \tfrac{1}{2}(y_1 + 4).$$

$$\therefore\ x_1 = 2x - 3, \qquad y_1 = 2y - 4.$$

* P is the "instantaneous center" of the motion of the connecting rod.

Substituting in (1),

$$(2x-3)^2+(2y-4)^2=25,$$

or

$$x^2+y^2-3x-4y=0. \quad Ans.$$

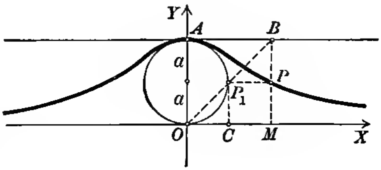
The locus is a circle constructed upon OP_2 as a diameter.

2. The witch. Find the equation of the locus of a point P constructed as follows: Let OA be a diameter of the circle $x^2+y^2-2ay=0$, and let any line OB be drawn through O to meet the circle at P_1 and the tangent at A at B. Draw $P_1P \perp$ to OA and $BP \parallel$ to OA. Required the locus of P.

Solution. Let (x, y) be the coördinates of P and (x_1, y_1) of P_1. Then the coördinates of $P_1(x_1, y_1)$ must satisfy the equation

$$x^2+y^2-2ay=0.$$

(2) $\therefore x_1^2+y_1^2-2ay_1=0.$



From the figure,

(3) $y_1=y.$

From the similar triangles OCP_1 and OMB we have

(4) $$\frac{OC}{OM}=\frac{CP_1}{MB}, \quad \text{or} \quad \frac{x_1}{x}=\frac{y_1}{2a}.$$

[For $OC=x_1$, $OM=x$, $CP_1=y_1$, $MB=2a$.]

Solving (3) and (4) for x_1 and y_1, we obtain

(5) $$x_1=\frac{xy}{2a}, \quad y_1=y.$$

Substituting from (5) in (2),

$$\frac{x^2y^2}{4a^2}+y^2-2ay=0,$$

or

(6) $$y(x^2+4a^2)=8a^3.$$

The locus of this equation is known as the *witch of Agnesi*.

The method followed in Examples 1 and 2 may evidently be described as follows:

Rule *for finding the equation of a locus derived by a construction from a given curve.*

First step. *The construction will give rise to a figure from which we may find expressions for the coördinates of any point* $P_1(x_1, y_1)$ *on the given curve in terms of a point* $P(x, y)$ *on the required curve.*

Second step. *Substitute the results of the first step for the coördinates* x_1 *and* y_1 *in the equation of the given curve and simplify. The result is the required equation.*

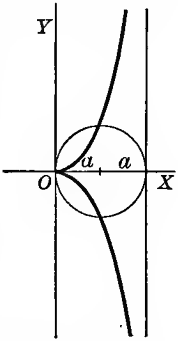
PROBLEMS

1. Find the locus of a point whose ordinate is half the ordinate of a point on the circle $x^2 + y^2 = 64$. *Ans.* The ellipse $x^2 + 4y^2 = 64$.

2. Find the locus of a point which cuts off a part of an ordinate of the circle $x^2 + y^2 = a^2$ whose ratio to the whole ordinate is $b : a$.
Ans. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

3. Find the locus of the middle points of the chords of (a) an ellipse, (b) a parabola, (c) a hyperbola which pass through a fixed point $P_2(x_2, y_2)$ on the curve.

Ans. A conic of the same type for which the values of a and b or of p are half the values of those constants for the given conic.



4. Lines are drawn from the point (0, 4) to the hyperbola $x^2 - 4y^2 = 16$. Find the locus of the points which divide these lines in the ratio 1 : 2.
Ans. $3x^2 - 12y^2 + 64y - 90\frac{2}{3} = 0$.

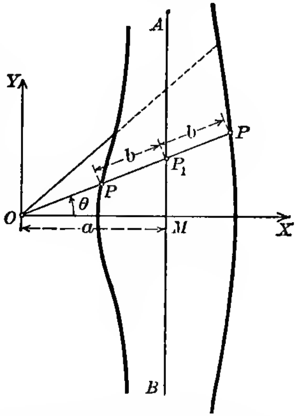
5. A chord OP_1 of the circle $x^2 + y^2 - 2ax = 0$ meets the line $x = 2a$ at a point A. Find the locus of a point P on the line OP_1 such that $OP = P_1A$.
Ans. The **cissoid of Diocles** $y^2(2a - x) = x^3$ (see figure).

6. DD' is the directrix and F the focus of a given conic (Art. 72). Q is any point on the conic. Through Q draw $QN \perp$ to the axis of the conic and construct P on NQ so that $NP = FQ$. What is the locus of P?
Ans. A straight line.

84. Loci using polar coördinates. When the required locus is described by the end-point of a line of variable length whose other extremity is fixed, polar coördinates may be employed to advantage.

EXAMPLE

The conchoid. Find the locus of a point P constructed as follows: Through a fixed point O, a line is drawn cutting a fixed line AB at P_1. On this line a point P is taken so that $P_1P = \pm b$, where b is a constant.



Solution. The required locus is the locus of the end-point P of the line OP, and O is fixed. Hence we use polar coördinates, taking O for the pole and the perpendicular OM to AB for the polar axis. Then

(1) $OP = \rho, \quad \angle MOP = \theta.$

By construction,

(2) $\rho = OP = OP_1 \pm b.$

But in the right triangle OMP_1,

(3) $OP_1 = OM \sec \angle MOP_1 = a \sec \theta.$

Substituting from (3) in (2),

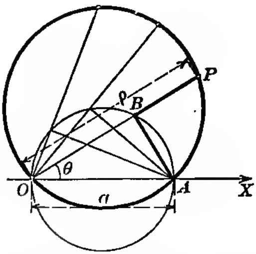
(4) $\rho = a \sec \theta \pm b.$

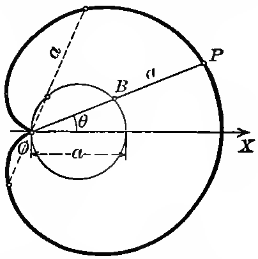
The locus of this equation is called the *conchoid of Nicomedes*. It has three distinct forms according as a is greater, equal to, or less than b.

PROBLEMS

1. OA is a diameter of a fixed circle, and OB is any chord drawn from the fixed point O. In the figure below, $BP = AB$. Find the locus of P.

Ans. The circle $\rho = a(\sin\theta + \cos\theta)$.



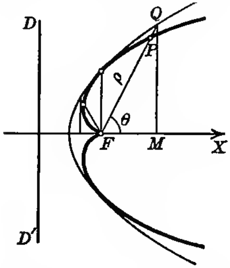


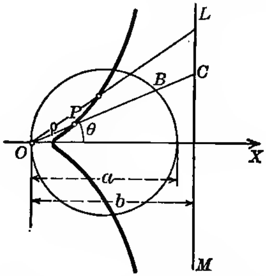
2. The chord OB of a fixed circle drawn from O is produced to P, making BP = diameter = a. What is the locus of P?

Ans. The cardioid $\rho = a(1 + \cos\theta)$.

3. In problem 2, if BP = any length $= b$, the locus of P is the **limaçon of Pascal,** $\rho = b + a\cos\theta$. The limaçon has three distinct forms according as $b \gtreqless a$. In the figure on p. 124, $b < a$. The rectangular equation is $(x^2 + y^2 + ax)^2 = b^2(x^2 + y^2)$.

4. F is the focus and DD' the directrix of a conic (figure below). Q is any point on the conic. On the focal radius FQ lay off $FP = QM$, where QM is $\parallel$ to DD'. Find the locus of P (see Art. 72). *Ans.* $\rho = \dfrac{ep\sin\theta}{1 - e\cos\theta}$.





5. Lines are drawn from the fixed point O on a fixed circle to meet a fixed line LM which is $\perp$ to the diameter through O. On any such line OC lay off $OP = BC$. What is the locus of P? *Ans.* $\rho = b\sec\theta - a\cos\theta$.

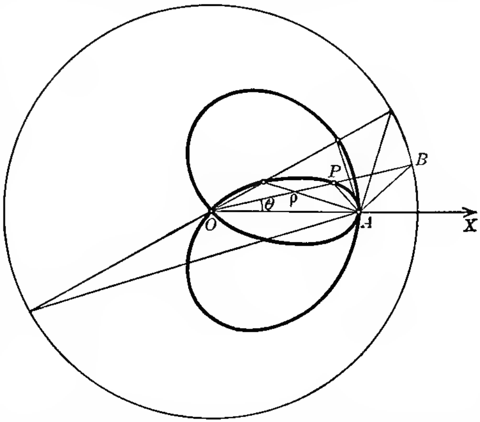
Draw the locus for $b > a$, $b < a$, and $b = a$. In the last case the curve is the cissoid (Problem 5, p. 220).

6. O is the center of a fixed circle and A a fixed interior point. Draw any radius OB, connect A and B, and draw $AP \perp$ to AB to meet OB at P. Required the locus of P.

Ans. $\rho = e\dfrac{e - a\cos\theta}{e\cos\theta - a}$,

if $OB = a$, $OA = e$.

Draw the locus.

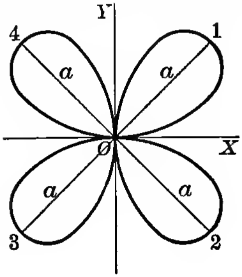


7. A line is drawn from a fixed point O meeting a fixed line in P_1. Find the locus of a point P on this line such that $OP_1 \cdot OP = a^2$. *Ans.* A circle.

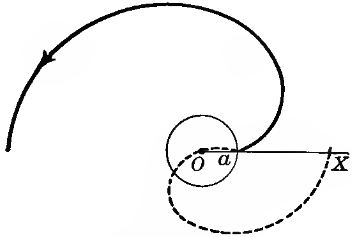
8. A line is drawn through a fixed point O, meeting a fixed circle in P_1 and P_2. Find the locus of a point P on this line such that

$$OP = 2\,OP_1 \cdot OP_2 \div (OP_1 + OP_2). \quad \textit{Ans.} \text{ A straight line.}$$

9. In Ex. 1, Art. 82, find the locus of the foot of the perpendicular from the origin upon AB. *Ans.* The **four-leaved rose** $\rho = a \sin 2\theta$ (see figure).



FOUR-LEAVED ROSE



PARABOLIC SPIRAL

10. Let the x-axis cut the circle $x^2 + y^2 = a^2$ at A. An arc AB is laid off on the circle equal to the abscissa x_0 of a point (x_0, y_0) on the parabola $y^2 = 4\,cx$, and the radius OB is produced to P making $BP = y_0$. Show that the locus of P is the **parabolic spiral** $(\rho - a)^2 = 4\,ac\,\theta$ (see figure).

85. Loci defined by the points of intersection of systems of lines. If two systems of lines involve the same parameter, the lines belonging to the same value of the parameter are called **corresponding lines.** Many loci are defined, or may be easily regarded as the locus of the points of intersection of such lines. The method of treatment will now be illustrated.

EXAMPLES

1. Find the locus of the foot of the perpendicular drawn from the vertex of a parabola to the tangent. (See the figure on p. 224).

Solution. Taking the typical equation $y^2 = 2\,px$, the equation of a tangent AB in terms of the slope t is (Art. 78)

$$(1) \qquad y = tx + \frac{p}{2\,t}.$$

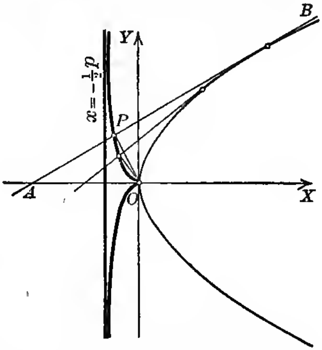
The equation of the perpendicular OP is

$$(2) \qquad y = -\frac{1}{t}x.$$

Equations (1) and (2) define the two systems of lines in the parameter t. The locus of the point of intersection P of corresponding lines is required.

Solving (1) and (2) for x and y,

(3) $x = -\dfrac{p}{2(1+t^2)}, \quad y = \dfrac{p}{2t + 2t^3}.$



These are the *parametric equations* of the required locus.

The rectangular equation is found thus:

From (2), $t = -\dfrac{x}{y}$. Substituting in the first equation of (3) and reducing,

$$y^2(x + \tfrac{1}{2}p) = -x^3.$$

Comparison with the answer to Problem 5, p. 220, shows that the locus is a cissoid.

The method of solving Example 1 may be summed up in the

Rule *to find the equation of the locus of the points of intersection of corresponding lines of two systems.*

First step. *Find the equations of the two systems of lines defining the locus in terms of the same parameter.*

Second step. *Solve these equations for x and y in terms of the parameter. This gives the parametric equations of the locus.*

If only the equation in rectangular coördinates is required, it may be obtained by eliminating the parameter from the equations found in the first step, for the result will be the same as that obtained by eliminating the parameter from the equations found in the second step.

2. Find the locus of the points of intersection of two perpendicular tangents to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$.

Solution. *First step.* The equation of a tangent in terms of its slope t is (Art. 78)

(4) $$y = tx + \sqrt{a^2t^2 + b^2}.$$

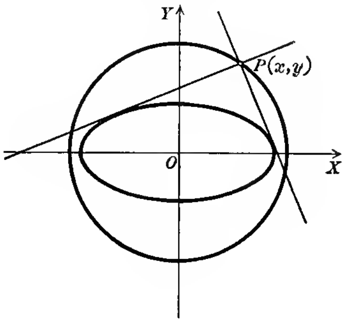
The slope of the tangent perpendicular to (4) is $-\dfrac{1}{t}$. By replacing t in (4) by $-\dfrac{1}{t}$, we find the equation of the perpendicular tangent to be

(5) $$y = -\frac{x}{t} + \sqrt{\frac{a^2}{t^2} + b^2}.$$

Second step. As the parametric equations are not required, this step may be omitted.

To eliminate t from (4) and (5) we write them in the forms

$$tx - y = -\sqrt{a^2t^2 + b^2},$$
$$x + ty = \sqrt{a^2 + b^2t^2}.$$



Squaring these equations, we obtain

$$t^2x^2 - 2\,txy + y^2 = a^2t^2 + b^2,$$
$$x^2 + 2\,txy + t^2y^2 = a^2 + b^2t^2.$$

Adding,

$$(1 + t^2)x^2 + (1 + t^2)y^2 = (1 + t^2)a^2 + (1 + t^2)b^2.$$

Dividing by $1 + t^2$, the required equation is

$$x^2 + y^2 = a^2 + b^2.$$

The locus is therefore a circle whose center is the center of the ellipse, and whose radius is $\sqrt{a^2 + b^2}$. It is called the **director circle.**

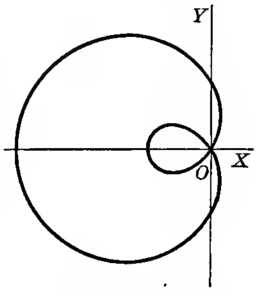
PROBLEMS

1. Find the locus of the intersections of perpendicular tangents to (a) the parabola, (b) the hyperbola (IV), p. 167.

Ans. (a) The *directrix*; (b) $x^2 + y^2 = a^2 - b^2$.

2. Find the locus of the point of intersection of a tangent to (a) an ellipse, (b) a parabola, (c) a hyperbola with the line drawn through a focus perpendicular to the tangent.

Ans. (a) $x^2 + y^2 = a^2$; (b) $x = 0$; (c) $x^2 + y^2 = a^2$.



3. Find the locus of the point of intersection of a tangent to an equilateral hyperbola and the line drawn through the center perpendicular to that tangent.

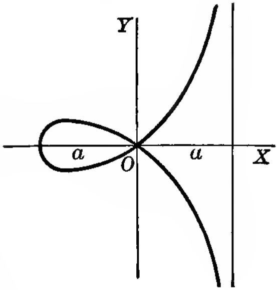
Ans. The **lemniscate** $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ (Ex. 2, Art. 46).

4. Find the locus of the point of intersection of a tangent to the circle $x^2 + y^2 + 2\,ax + a^2 - b^2 = 0$ and the line drawn through the origin perpendicular to it.

Ans. The **limaçon** $(x^2 + y^2 + ax)^2 = b^2(x^2 + y^2)$ (Problem 3, p. 222).

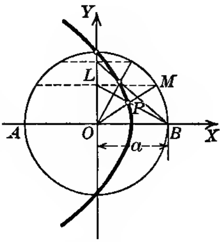
5. Find the locus of the foot of the perpendicular drawn from the origin to a tangent to the parabola $y^2 + 4ax + 4a^2 = 0$.

Ans. The **strophoid** $y^2 = x^2 \frac{a+x}{a-x}$ (see figure).



6. Find the locus of the intersection of the normals drawn at points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and major auxiliary circle $x^2 + y^2 = a^2$ which have the same abscissas. *Ans. Circle* $x^2 + y^2 = (a + b)^2$.

7. In the figure, LM is any half chord of the circle parallel to the diameter AB. Find the locus of P, the intersection of BL and OM. *Ans. Parabola* $y^2 = a^2 - 2ax$.



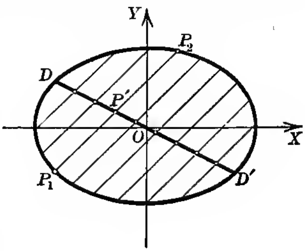
8. A tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ meets the axes of x and y in A and B respectively. From A draw a line ∥ to OY, and from B a line ∥ to OX. What is the locus of their point of intersection?
Ans. $x^2y^2 = a^2y^2 + b^2x^2$. (Problem 1, (*b*), p. 210.)

9. Work out Problem 8 when the ellipse is replaced by a hyperbola.

A somewhat different class of locus problems is illustrated in the following example.

EXAMPLE

What is the locus of the middle points of a system of parallel chords of an ellipse?



Solution. Let the equation of the system of parallel chords be

(6) $$y = mx + k,$$

where k is a parameter and m = slope of chords. Let the value of k for the chord P_1P_2 be k_1; that is,

(7) $$y = mx + k_1$$

is the equation of P_1P_2. Assume that the coördinates of P_1 are (x_1, y_1), and of P_2 (x_2, y_2)

If P' (x', y') is the middle point of P_1P_2, then

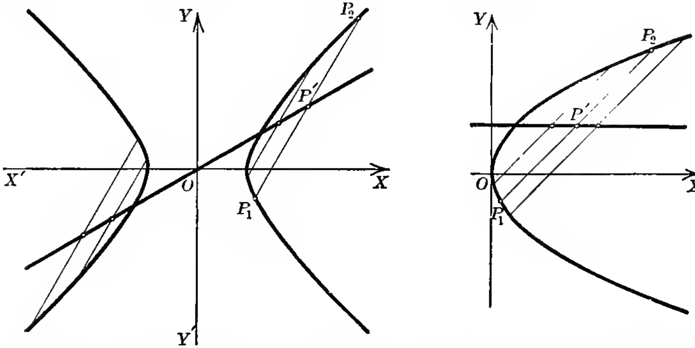
(8) $$x' = \tfrac{1}{2}(x_1 + x_2), \quad y' = \tfrac{1}{2}(y_1 + y_2).$$

Since (x_1, y_1) and (x_2, y_2) are the points of intersection of the chord (7) and the ellipse, we shall find their values by solving

(9) $$y = mx + k_1 \quad \text{and} \quad b^2x^2 + a^2y^2 = a^2b^2.$$

Eliminating y, we obtain the equation

(10) $$(a^2m^2 + b^2)x^2 + 2a^2k_1mx + a^2k_1^2 - a^2b^2 = 0.$$



The roots of this equation are x_1 and x_2, and, from (8), x' equals one half the sum of these roots. Hence *we need to know in* (10) *only the sum of the roots.* But, by algebra,*

(11) $$x_1 + x_2 = -\frac{2a^2k_1m}{a^2m^2 + b^2}.$$

Hence, from (8),

(12) $$x' = -\frac{a^2m}{a^2m^2 + b^2}k_1.$$

Since (x', y') satisfy (7),

(13) $$y' = mx' + k_1 = \frac{-a^2m^2k_1}{a^2m^2 + b^2} + k_1 = \frac{b^2}{a^2m^2 + b^2}k_1.$$

Eliminating k_1, from (12) and (13),

(14) $$b^2x' + a^2my' = 0.$$

Dropping the accents gives the equation of the locus,

(15) $$b^2x + a^2my = 0.$$

The locus is the straight line DD' in the figure.

* In the quadratic $Ax^2 + Bx + C = 0$, *sum of roots* $= -\frac{B}{A}$; *product of roots* $= \frac{C}{A}$

In a circle a diameter may be defined as the locus of the middle points of a series of parallel chords. The corresponding locus for a conic section is also called a **diameter** of the conic.

Hence we have the

Theorem. *The diameter of the ellipse*

$$b^2x^2 + a^2y^2 = a^2b^2$$

which bisects all chords with the slope m is

$$\mathbf{b^2x + a^2my = 0.}$$

In like manner (see the figures on p. 227) we may prove the

Theorem. *The diameter which bisects all chords with the slope m of the*

hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ *is* $\mathbf{b^2x - a^2my = 0}$;

parabola $y^2 = 2px$ *is* $\mathbf{my = p.}$

Every line through the center of an ellipse or hyperbola is a diameter, while in a parabola every line parallel to the axis is a diameter.

PROBLEMS

1. Find the equation of the diameter of each of the following conics which bisects the chords with the given slope m.

(a) $x^2 - 4y^2 = 16$, $m = 2$. *Ans.* $x - 8y = 0$.
(b) $y^2 = 4x$, $m = -\frac{1}{2}$. *Ans.* $y + 4 = 0$.
(c) $xy = 6$, $m = 3$. *Ans.* $y + 3x = 0$.
(d) $x^2 + xy - 8 = 0$, $m = -3$. *Ans.* $x - y = 0$.
(e) $x^2 - 4y^2 + 4x - 16 = 0$, $m = -1$. *Ans.* $x + 4y + 2 = 0$.
(f) $xy + 2y^2 - 4x - 2y + 6 = 0$, $m = \frac{2}{3}$. *Ans.* $2x + 11y - 16 = 0$.

2. Find the equation of that diameter of

(a) $4x^2 + 9y^2 = 36$ passing through (3, 2). *Ans.* $2x - 3y = 0$.
(b) $y^2 = 4x$ passing through (2, 1). *Ans.* $y = 1$.
(c) $xy = 8$ passing through (– 2, 3). *Ans.* $3x + 2y = 0$.
(d) $x^2 - 4y + 6 = 0$ passing through (3, – 4). *Ans.* $x = 3$.
(e) $xy - y^2 + 2x - 4 = 0$ passing through (5, 2). *Ans.* $4x - 9y - 2 = 0$.

3. Find the equation of the chord of the locus of

(a) $x^2 + y^2 = 25$ which is bisected at the point (2, 1).
Ans. $2x + y - 5 = 0$.

(b) $4x^2 - y^2 = 9$ which is bisected at the point (4, 2).
Ans. $8x - y - 30 = 0$.

(c) $xy = 4$ which is bisected at the point (5, 3). *Ans.* $3x + 5y - 30 = 0$.

(d) $x^2 - xy - 8 = 0$ which is bisected at the point (4, 0).
Ans. $2x - y - 8 = 0$.

4. Show that if two lines through the center of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

have slopes m and m' such that $mm' = -\frac{b^2}{a^2}$, then each line bisects all chords parallel to the other.

Draw two such lines. They are called **conjugate diameters.**

5. Through the point (x_0, y_0) on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ a diameter is drawn; prove that the coördinates of the extremities of its conjugate diameter are $x = \pm\frac{ay_0}{b}$, $y = \mp\frac{bx_0}{a}$.

6. If a' and b' are the lengths of two conjugate semidiameters of the ellipse, prove that $a'^2 + b'^2 = a^2 + b^2$ (use Example 5).

7. Prove that the tangent at any point of the ellipse is parallel to the diameter which is conjugate to the diameter through the given point; and hence that the tangents at the extremities of two conjugate diameters form a parallelogram.

8. Prove that the area of the parallelogram formed by the tangents at the extremities of two conjugate diameters of an ellipse is constant and is equal to $4ab$.

Hint. The area in question is eight times the area of the triangle whose vertices are (0, 0), (x_0, y_0), and $\left(\frac{ay_0}{b}, -\frac{bx_0}{a}\right)$ (see Example 5).

9. Two tangents with the slopes m_1 and m_2 are drawn from a point P to an ellipse $b^2x^2 + a^2y^2 = a^2b^2$. Find the locus of P

(a) when $m_1 + m_2 = 0$. *Ans.* $x = 0$ and $y = 0$.

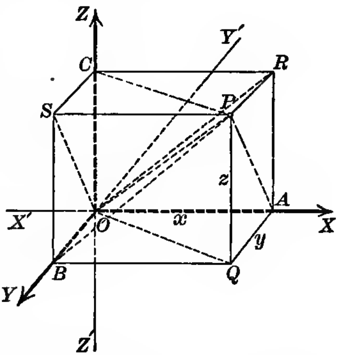
(b) when $m_1 + m_2 = 1$. *Ans.* $x^2 - 2xy - a^2 = 0$.

(c) when $m_1m_2 = 1$. *Ans.* $x^2 - y^2 = a^2 - b^2$.

CHAPTER XIII

CARTESIAN COÖRDINATES IN SPACE

86. Cartesian coördinates. The foundation of plane analytic geometry depends upon the possibility of determining a point in the plane by a pair of real numbers (x, y). The study of solid analytic geometry is based on the determination of a point in space by a set of *three* real numbers x, y, and z. This determination is accomplished as follows:



Let there be given three mutually perpendicular planes intersecting in the lines XX', YY', and ZZ', which will also be mutually perpendicular. These three planes are called the **coördinate planes** and may be distinguished as the XY-plane, the YZ-plane, and the ZX-plane. Their lines of intersection are called the **axes of coördinates,** and the positive directions on them are indicated by the arrowheads.* The point of intersection of the coördinate planes is called the **origin.**

Let P be any point in space and let three planes be drawn through P parallel to the coördinate planes and cutting the axes at A, B, and C. These three planes together with the

* XX' and ZZ' are supposed to be in the plane of the paper, the positive direction on XX' being to the *right*, that on ZZ' being *upward*. YY' is supposed to be perpendicular to the plane of the paper, the positive direction being *in front* of the paper, that is, from the plane of the paper toward the reader.

coördinate planes form a rectangular parallelepiped, of which P and the origin O are opposite vertices, as in the figure. The three edges $OA = x$, $OB = y$, and $OC = z$ are called the **rectangular coördinates** of P.

Any point P in space determines three numbers, the coördinates of P. Conversely, given any three real numbers x, y, and z, a point P in space may always be constructed whose coördinates are x, y, and z. For if we lay off $OA = x$, $OB = y$, and $OC = z$, and draw planes through A, B, and C parallel to the coördinate planes, they will intersect in a point P. Hence

Every point determines three real numbers, and conversely, three real numbers determine a point.

The coördinates of P are written (x, y, z), and the symbol $P(x, y, z)$ is to be read, "The point P whose coördinates are x, y, and z."

From the figure we have the relations

$$AP = OS = \sqrt{(OB)^2 + (OC)^2};$$

$$BP = OR = \sqrt{(OC)^2 + (OA)^2};$$

$$CP = OQ = \sqrt{(OA)^2 + (OB)^2};$$

$$OP = \sqrt{(OA)^2 + (OB)^2 + (OC)^2}.$$

Hence, let $P(x, y, z)$ be any point in space; then its distance

from the XY-plane is z,

from the YZ-plane is x,

from the ZX-plane is y,

from the X-axis is $\sqrt{y^2 + z^2}$,

from the Y-axis is $\sqrt{z^2 + x^2}$,

from the Z-axis is $\sqrt{x^2 + y^2}$,

from the origin is $\sqrt{x^2 + y^2 + z^2}$.

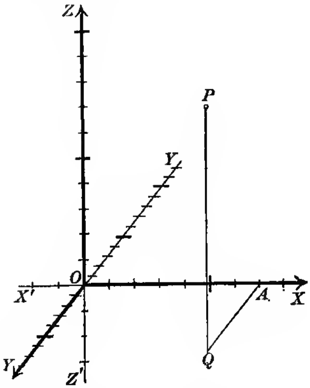
The coördinate planes divide all space into eight parts called **octants**, designated by $O\text{-}XYZ$, $O\text{-}X'YZ$, etc. The signs of the coördinates of a point in any octant may be determined by the

Rule for signs.

x is positive or negative according as P lies to the right or left of the YZ-plane.

y is positive or negative according as P lies in front or in back of the ZX-plane.

z is positive or negative according as P lies above or below the XY-plane.



Points in space may be conveniently plotted by marking the same scale on XX' and ZZ' and a somewhat smaller scale on YY' Then to plot any point, for example (7, 6, 10), we lay off $OA = 7$ on OX, draw AQ parallel to OY and equal to 6 units on OY, and QP parallel to OZ and equal to 10 units on OZ.

PROBLEMS

1. What are the coördinates of the origin?

2. Plot the following sets of points:

(a) (8, 0, 2), (– 3, 4, 7), (0, 0, 5).
(b) (4, – 3, 6), (– 4, 6, 0), (0, 8, 0).
(c) (10, 3, – 4), (– 4, 0, 0), (0, 8, 4).
(d) (3, – 4, – 8), (– 5, – 6, 4), (8, 6, 0).
(e) (– 4, – 8, – 6), (3, 0, 7), (6, – 4, 2).
(f) (– 6, 4, – 4), (0, – 4, 6), (9, 7, – 2).

3. Calculate the distances of each of the following points to each of the coördinate planes and axes and to the origin:

(a) $(2, -2, 1)$, (b) $(3, -4, -3)$, (c) $\left(\frac{1}{\sqrt{3}}, -1, \frac{1}{2}\right)$.

4. Show that the following points lie on a sphere whose center is the origin and whose radius is 3:

$(\sqrt{3}, -2, \sqrt{2})$, $(2\sqrt{2}, 0, -1)$, $(-2, 2, 1)$, $(-\sqrt{5}, \sqrt{3}, 1)$.

5. Show that the following points lie on a circular cylinder of radius 5 whose axis is the Y-axis:

$$(3, -8, 4), (2\sqrt{5}, 6, \sqrt{5}), (-4, 0, -3), (1, \tfrac{1}{3}, 2\sqrt{6}).$$

6. Where can a point move if $x = 0$? if $y = 0$? if $z = 0$?

7. Where can a point move if $x = 0$ and $y = 0$? if $y = 0$ and $z = 0$? if $z = 0$ and $x = 0$?

8. Show that the points (x, y, z) and $(-x, y, z)$ are symmetrical with respect to the YZ-plane; (x, y, z) and $(x, -y, z)$ with respect to the ZX-plane; (x, y, z) and $(x, y, -z)$ with respect to the XY-plane.

9. Show that the points (x, y, z) and $(-x, -y, z)$ are symmetrical with respect to ZZ'; (x, y, z) and $(x, -y, -z)$ with respect to XX'; (x, y, z) and $(-x, y, -z)$ with respect to YY'; (x, y, z) and $(-x, -y, -z)$ with respect to the origin.

10. What is the value of z if $P(x, y, z)$ is in the XY-plane? of x if P is in the YZ-plane? of y if P is in the ZX-plane?

11. What are the values of y and z if $P(x, y, z)$ is on the X-axis? of z and x if P is on the Y-axis? of x and y if P is on the Z-axis?

12. A rectangular parallelepiped lies in the octant O-XYZ with three faces in the coördinate planes. If its dimensions are a, b, and c, what are the coördinates of its vertices?

87. Orthogonal projections. To extend the first theorem of projection, Art. 31, we define the **angle between two directed lines** in space which do not intersect to be the angle between two intersecting directed lines drawn parallel to the given lines and having their positive directions agreeing with those of the given lines.

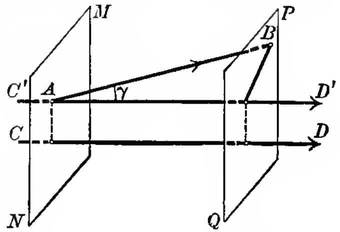
The definitions of the orthogonal projection of a point upon a line and of a directed length AB upon a directed line hold when the points and lines lie in space instead of in the plane. It is evident that the projection of a point upon a line may also be regarded as the point of intersection of the line and the plane passed through the point perpendicular to the line. As two parallel planes are equidistant, then *the projections of a directed length AB upon two parallel lines whose positive directions agree are equal.*

First Theorem of Projection. *If A and B are points upon a directed line making an angle of γ with a directed line CD, then the*

(I) **projection of the length AB upon $CD = AB \cos \gamma$.**

Proof. Draw $C'D'$ through A parallel to CD. Then, by definition, the angle between AB and $C'D'$ equals γ. Since $C'D'$ and AB intersect we may apply the first theorem of projection in the plane, and hence the

projection of the length AB upon $C'D' = AB \cos \gamma$.



Since the projection of AB on CD equals the projection of AB upon $C'D'$ we get (I). Q.E.D.

Second Theorem of Projection. *If each segment of a broken line in space be given the direction determined in passing continuously from one extremity to the other, then the algebraic sum of the projections of the segments upon any directed line equals the projection of the closing line.*

The proof given on page 69 holds whether the broken line lies in the plane or in space.

Corollary I. *The projections of the line joining the origin to any point P on the axes of coördinates are respectively the coördinates of P.*

For the projection of OP (Fig., p. 230) upon OX equals OA, since A is the projection of P on OX. Similarly for the projections on OY and OZ.

Corollary II. *Given any two points P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2),* then

$$x_2 - x_1 = \text{projection of } P_1P_2 \text{ upon } XX',$$
$$y_2 - y_1 = \text{projection of } P_1P_2 \text{ upon } YY',$$
$$z_2 - z_1 = \text{projection of } P_1P_2 \text{ upon } ZZ'.$$

For if we project P_1OP_2 and P_1P_2 upon XX', we have the projection of P_1O + projection of OP_2 = projection of P_1P_2.

But by Corollary I,

$$\text{projection of } P_1O = -x_1, \quad \text{projection of } OP_2 = x_2.$$

$$\therefore\ x_2 - x_1 = \text{projection of } P_1P_2 \text{ upon } XX'.$$

In like manner the other formulas are proved.

Corollary III. *If the sides of a polygon be given the direction established by passing continuously around the perimeter, the sum of the projections of the sides upon any directed line is zero.*

PROBLEMS

1. Find the projections upon each of the axes of the sides of the triangles whose vertices are the following points, and verify the results by Corollary III.

(a) (−3, 4, − 8), (5, − 6, 4), (8, 6, 0).
(b) (− 4, − 8, − 6), (3, 0, 7), (6, 4, − 2).
(c) (10, 3, − 4), (− 4, 0, 2), (0, 8, 4).
(d) (− 6, 4, − 4), (0, − 4, 6), (9, 7, − 2).

2. If the projections of P_1P_2 on the axes are respectively 3, − 2, and 7, and if the coördinates of P_1 are (− 4, 3, 2), find the coördinates of P_2.

Ans. (− 1, 1, 9).

3. A broken line joins continuously the points (6, 0, 0), (0, 4, 3), (− 4, 0, 0), and (0, 0, 8). Find the sum of the projections of the segments and the projection of the closing line on (a) the X-axis, (b) the Y-axis, (c) the Z-axis, and verify the results. Construct the figure.

4. A broken line joins continuously the points (6, 8, − 3), (0, 0, − 3), (0, 0, 6), (− 8, 0, 2), and (− 8, 4, 0). Find the sum of the projections of the segments and the projection of the closing line on (a) the X-axis, (b) the Y-axis, (c) the Z-axis, and verify the results. Construct the figure.

5. Find the projections on the axes of the line joining the origin to each of the points in Problem 1.

6. Find the angle between each axis and the line drawn from the origin to

(a) the point (8, 6, 0). *Ans.* $\cos^{-1}\frac{4}{5}, \cos^{-1}\frac{3}{5}, \frac{\pi}{2}$.

(b) the point (2, − 1, − 2). *Ans.* $\cos^{-1}\frac{2}{3}, \cos^{-1}\left(-\frac{1}{3}\right), \cos^{-1}\left(-\frac{2}{3}\right)$.

7. Find two expressions for the projections upon the axes of the line drawn from the origin to the point $P(x, y, z)$, if the length of the line is ρ and the angles between the line and the axes are α, β, and γ.

8. Find the projections of the coördinates of $P(x, y, z)$ upon the line drawn from the origin to P if the angles between that line and the axes are α, β, and γ. *Ans.* $x \cos\alpha$, $y \cos\beta$, $z \cos\gamma$.

88. Direction cosines of a line. The angles α, β, and γ between a directed line and the axes of coördinates are called the **direction angles** of the line.

If the line does not intersect the axes, then α, β, and γ are the angles between the axes and a line drawn through the origin parallel to the given line and agreeing with it in direction.

The cosines of the direction angles of a line are called the **direction cosines** of the line.

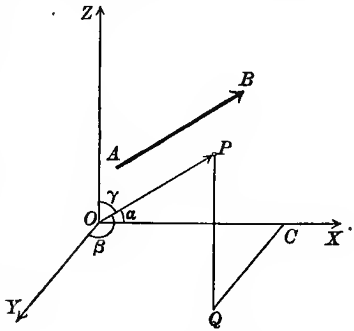
Reversing the direction of a line changes the signs of the direction cosines of the line.

For reversing the direction of a line changes α, β, and γ into $\pi - \alpha$, $\pi - \beta$, and $\pi - \gamma$ respectively, and (30, p. 3) $\cos(\pi - x) = -\cos x$.

Theorem. *If α, β, and γ are the direction angles of a line, then*

(II) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

That is, the sum of the squares of the direction cosines of a line is unity.



Proof. Let AB be a line whose direction angles are α, β, and γ. Through O draw OP parallel to AB and let $OP = \rho$. By definition $\angle XOP = \alpha$, $\angle YOP = \beta$, $\angle ZOP = \gamma$. Projecting OP on the axes,

(1) $$x = \rho \cos\alpha, \quad y = \rho\cos\beta, \quad z = \rho\cos\gamma.$$

Projecting OP and $OCQP$ on OP,

(2) $$\rho = x \cos \alpha + y \cos \beta + z \cos \gamma.$$

Substituting from (1) in (2) and dividing by ρ, we obtain (II). Q.E.D.

Corollary. *If* $\frac{\cos \alpha}{a} = \frac{\cos \beta}{b} = \frac{\cos \gamma}{c}$, *then*

(III) $$\cos \alpha = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, \quad \cos \beta = \frac{b}{\pm\sqrt{a^2+b^2+c^2}},$$

$$\cos \gamma = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}.$$

That is, if the direction cosines of a line are proportional to three numbers, they are respectively equal to these numbers each divided by the square root of the sum of their squares.

For if r denotes the common value of the given ratios, then

(3) $$\cos \alpha = ar, \quad \cos \beta = br, \quad \cos \gamma = cr.$$

Squaring, adding, and applying (II),

$$1 = r^2(a^2 + b^2 + c^2).$$

$$\therefore\ r = \frac{1}{\pm\sqrt{a^2+b^2+c^2}}.$$

Substituting in (3), we get the values of $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ to be derived.

The important conclusion just derived may be thus stated:

Any three numbers a, b, and c determine the direction of a line in space. This direction is the same as that of the line joining the origin and the point (a, b, c).

If a line cuts the XY-plane, it will be *directed upward* or *downward* according as $\cos \gamma$ is *positive* or *negative*.

If a line is parallel to the XY-plane, $\cos \gamma = 0$, and it will be *directed in front* or *in back of the ZX-plane* according as $\cos \beta$ is *positive* or *negative*.

If a line is parallel to the X-axis, $\cos \beta = \cos \gamma = 0$, and its positive direction will *agree or disagree with that of the X-axis* according as $\cos \alpha = 1$ or -1.

These considerations enable us to choose the sign of the radical in the Corollary so that the positive direction on the line shall be that given in advance.

89. Lengths.

Theorem. *The length l of the line joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by*

(IV) $$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Proof. Let the direction angles of the line P_1P_2 be α, β, and γ.

Projecting P_1P_2 on the axes, we get, by the first theorem of projection and Corollary II, p. 234,

(1) $$l \cos \alpha = x_2 - x_1, \quad l \cos \beta = y_2 - y_1, \quad l \cos \gamma = z_2 - z_1.$$

Squaring and adding,

$$l^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$
$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

Applying (II), and taking the square root, we have (IV).

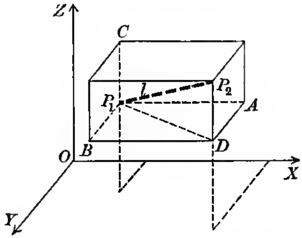
Q.E.D.

Corollary. *The direction cosines of the line drawn from P_1 to P_2 are proportional to the projections of P_1P_2 on the axes.*

For, from (1),

$$\frac{\cos \alpha}{x_2 - x_1} = \frac{\cos \beta}{y_2 - y_1} = \frac{\cos \gamma}{z_2 - z_1},$$

since each ratio equals $\frac{1}{l}$. Also the denominators are the projections of P_1P_2 on the axes.



If we construct a rectangular parallelepiped by passing planes through P_1 and P_2 parallel to the coördinate planes, its edges will be parallel to the axes and equal numerically to the projections of P_1P_2 upon the axes. P_1P_2 will be a diagonal of this parallelepiped, and hence l^2 will equal the sum of the squares of its three dimensions. We have thus a second method of deriving (IV).

PROBLEMS

1. Find the length and the direction cosines of the line drawn from

(a) $P_1\,(4, 3, -2)$ to $P_2\,(-2, 1, -5)$. *Ans.* $7, -\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7}$.
(b) $P_1\,(4, 7, -2)$ to $P_2\,(3, 5, -4)$. *Ans.* $3, -\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$.
(c) $P_1\,(3, -8, 6)$ to $P_2\,(6, -4, 6)$. *Ans.* $5, \frac{3}{5}, \frac{4}{5}, 0$.

2. Find the direction cosines of a line directed upward if they are proportional to (a) 3, 6, and 2; (b) 2, 1, and -4; (c) 1, -2, and 3.

Ans. (a) $\frac{3}{7}, \frac{6}{7}, \frac{2}{7}$; (b) $\frac{2}{-\sqrt{21}}, \frac{1}{-\sqrt{21}}, \frac{4}{+\sqrt{21}}$; (c) $\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

3. Find the lengths and direction cosines of the sides of the triangles whose vertices are the following points; then find the projections of the sides upon the axes by the first theorem of projection and verify by Corollary III, p. 235.

(a) $(0, 0, 3), (4, 0, 0), (8, 0, 0)$.
(b) $(3, 2, 0), (-2, 5, 7), (1, -3, -5)$.
(c) $(-4, 0, 6), (8, 2, -1), (2, 4, 6)$.
(d) $(3, -3, -3), (4, 2, 7), (-1, -2, -5)$.

4. In what octant (O-XYZ, O-$X'YZ$, etc.) will the positive part of a line through O lie if

(a) $\cos\alpha > 0, \cos\beta > 0, \cos\gamma > 0$?
(b) $\cos\alpha > 0, \cos\beta > 0, \cos\gamma < 0$?
(c) $\cos\alpha > 0, \cos\beta < 0, \cos\gamma < 0$?
(d) $\cos\alpha > 0, \cos\beta < 0, \cos\gamma > 0$?
(e) $\cos\alpha < 0, \cos\beta > 0, \cos\gamma > 0$?
(f) $\cos\alpha < 0, \cos\beta < 0, \cos\gamma > 0$?
(g) $\cos\alpha < 0, \cos\beta < 0, \cos\gamma < 0$?
(h) $\cos\alpha < 0, \cos\beta > 0, \cos\gamma < 0$?

5. What is the direction of a line if $\cos\alpha = 0$? $\cos\beta = 0$? $\cos\gamma = 0$? $\cos\alpha = \cos\beta = 0$? $\cos\beta = \cos\gamma = 0$? $\cos\gamma = \cos\alpha = 0$?

6. Find the projection of the line drawn from the origin to $P_1\,(5, -7, 6)$ upon a line whose direction cosines are $\frac{6}{7}$, $-\frac{3}{7}$, and $\frac{2}{7}$. *Ans.* 9.

Hint. The projection of OP_1 on any line equals the projection of a broken line whose segments equal the coördinates of P_1.

7. Find the projection of the line drawn from the origin to $P_1\,(x_1, y_1, z_1)$ upon a line whose direction angles are α, β, and γ.

Ans. $x_1 \cos\alpha + y_1 \cos\beta + z_1 \cos\gamma$.

8. Show that the points $(-3, 2, -7)$, $(2, 2, -3)$, and $(-3, 6, -2)$ are the vertices of an isosceles triangle.

9. Show that the points $(4, 3, -4)$, $(-2, 9, -4)$, and $(-2, 3, 2)$ are the vertices of an equilateral triangle.

10. Show that the points $(-4, 0, 2)$, $(-1, 3\sqrt{3}, 2)$, $(2, 0, 2)$, and $(-1, \sqrt{3}, 2 + 2\sqrt{6})$ are the vertices of a regular tetrahedron.

11. What does formula (IV) become if P_1 and P_2 lie in the XY-plane? in a plane parallel to the XY-plane?

12. Show that the direction cosines of the lines joining each of the points $(4, -8, 6)$ and $(-2, 4, -3)$ to the point $(12, -24, 18)$ are the same. How are the three points situated?

13. Show by means of direction cosines that the three points $(3, -2, 7)$, $(6, 4, -2)$, and $(5, 2, 1)$ lie on a straight line.

14. What are the direction cosines of a line parallel to the X-axis? to the Y-axis? to the Z-axis?

15. What is the value of one of the direction cosines of a line parallel to the XY-plane? the YZ-plane? the ZX-plane? What relation exists between the other two?

16. Show that the point $(-1, -2, -1)$ is on the line joining the points $(4, -7, 3)$ and $(-6, 3, -5)$ and is equally distant from them.

17. If two of the direction angles of a line are $\frac{\pi}{3}$ and $\frac{\pi}{4}$, what is the third?

Ans. $\frac{\pi}{3}$ or $\frac{2\pi}{3}$.

18. Find the direction angles of a line which is equally inclined to the three coördinate axes. *Ans.* $\alpha = \beta = \gamma = \cos^{-1} \frac{1}{3}\sqrt{3}$.

19. Find the length of a line whose projections on the axes are respectively

(a) $6, -3$, and 2. *Ans.* 7.

(b) $12, 4$, and -3. *Ans.* 13.

(c) $-2, -1$, and 2. *Ans.* 3.

90. Angle between two directed lines.

Theorem. *If α, β, γ and α', β', γ' are the direction angles of two directed lines, then the angle θ between them is given by*

$$\text{(V)} \qquad \cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

Proof. Draw OP and OP' (figure, p. 241) parallel to the given lines and let $OP = \rho$. Then, by definition,

$$\angle POP' = \theta.$$

Now, if the coördinates of P are (x, y, z), then, in the figure,

$$OA = x, \quad AB = y, \quad BP = z.$$

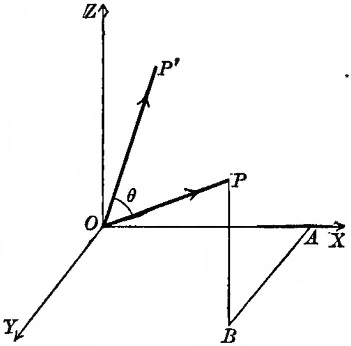
Project OP and $OABP$ on OP'. Then

(1) $$\rho \cos \theta = x \cos \alpha' + y \cos \beta' + z \cos \gamma'.$$

Projecting OP on the axes,

(2) $$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma.$$

Substituting in (1) from (2) and dividing by ρ, we obtain (V). Q.E.D.



Theorem. *If α, β, γ and α', β', γ' are the direction angles of two lines, then the lines are*

(a) *parallel and in the same direction* when and only when*

$$\alpha = \alpha', \quad \beta = \beta', \quad \gamma = \gamma';$$

(b) *perpendicular† when and only when*

$$\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma' = 0.$$

That is, two lines are parallel and in the same direction when and only when their direction angles are equal, and perpendicular when and only when the sum of the products of their direction cosines is zero.

Proof. The condition for parallelism follows from the fact that both lines will be parallel to and agree in direction with the same line through the origin when and only when their direction angles are equal.

The condition for perpendicularity follows from (V), for if $\theta = \frac{\pi}{2}$, then $\cos \theta = 0$, and conversely Q.E.D.

* They will be parallel and have opposite directions when and only when the direction angles are supplementary.

† Two lines in space are said to be perpendicular when the angle between them is $\frac{\pi}{2}$, but the lines do not necessarily intersect.

In the applications we usually have given not the direction cosines, but three numbers to which they are proportional. Hence the importance of the following

Corollary. *If the direction cosines of two lines are proportional to a, b, c and a′, b′, c′, then the conditions for parallelism and perpendicularity are respectively*

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}, \quad aa' + bb' + cc' = 0.$$

91. Point of division.

Theorem. *The coördinates (x, y, z) of the point of division P on the line joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ such that the ratio of the segments is*

$$\frac{P_1P}{PP_2} = \lambda$$

are given by the formulas

$$\text{(VI)} \qquad x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad z = \frac{z_1 + \lambda z_2}{1 + \lambda}.$$

This is proved as in Art. 13.

Corollary. *The coördinates (x, y, z) of the middle point P of the line joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are*

$$x = \tfrac{1}{2}(x_1 + x_2), \quad y = \tfrac{1}{2}(y_1 + y_2), \quad z = \tfrac{1}{2}(z_1 + z_2).$$

PROBLEMS

1. Find the angle between two lines whose direction cosines are respectively

(a) $\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}$ and $\frac{3}{7}, -\frac{2}{7}, \frac{6}{7}$. *Ans.* $\frac{\pi}{2}$.

(b) $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$ and $-\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$. *Ans.* $\cos^{-1}\frac{14}{39}$.

(c) $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ and $\frac{3}{7}, \frac{6}{7}, \frac{2}{7}$. *Ans.* $\cos^{-1}(-\frac{4}{21})$.

2. Show that the lines whose direction cosines are $\frac{3}{7}, \frac{6}{7}, \frac{2}{7}$; $-\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$; and $-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$ are mutually perpendicular.

3. Show that the lines joining the following pairs of points are either parallel or perpendicular.

(a) (3, 2, 7), (1, 4, 6) and (7, − 5, 9), (5, − 3, 8).

(b) (13, 4, 9), (1, 7, 13) and (7, 16, − 6), (3, 4, − 9).

(c) (− 6, 4, − 3), (1, 2, 7) and (8, − 5, 10), (15, − 7, 20).

4. Find the coördinates of the point dividing the line joining the following points in the ratio given.

(a) (3, 4, 2), (7, – 6, 4), $\lambda = \frac{1}{2}$. *Ans.* $(\frac{13}{3}, \frac{2}{3}, \frac{8}{3})$.
(b) (– 1, 4, – 6), (2, 3, – 7), $\lambda = -3$. *Ans.* $(\frac{7}{2}, \frac{5}{2}, -\frac{15}{2})$.
(c) (8, 4, 2), (3, 9, 6), $\lambda = -\frac{1}{3}$. *Ans.* $(\frac{21}{2}, \frac{3}{2}, 0)$.
(d) (7, 3, 9), (2, 1, 2), $\lambda = 4$. *Ans.* $(3, \frac{7}{5}, \frac{17}{5})$.

5. Show that the points (7, 3, 4), (1, 0, 6), and (4, 5, – 2) are the vertices of a right triangle.

6. Show that the points (– 6, 3, 2), (3, – 2, 4), (5, 7, 3), and (– 13, 17, – 1) are the vertices of a trapezoid.

7. Show that the points (3, 7, 2), (4, 3, 1), (1, 6, 3), and (2, 2, 2) are the vertices of a parallelogram.

8. Show that the points (6, 7, 3), (3, 11, 1), (0, 3, 4), and (– 3, 7, 2) are the vertices of a rectangle.

9. Show that the points (6, – 6, 0), (3, – 4, 4), (2, – 9, 2), and (– 1, – 7, 6) are the vertices of a rhombus.

10. Show that the points (7, 2, 4), (4, – 4, 2), (9, – 1, 10), and (6, – 7, 8) are the vertices of a square.

11. Show that each of the following sets of points lies on a straight line, and find the ratio of the segments in which the third divides the line joining the first to the second.

(a) (4, 13, 3), (3, 6, 4), and (2, – 1, 5). *Ans.* – 2.
(b) (4, – 5, – 12), (– 2, 4, 6), and (2, – 2, – 6). *Ans.* $\frac{1}{2}$.
(c) (– 3, 4, 2), (7, – 2, 6), and (2, 1, 4). *Ans.* 1.

12. Find the lengths of the medians of the triangle whose vertices are the points (3, 4, – 2), (7, 0, 8), and (– 5, 4, 6). *Ans.* $\sqrt{113}$, $\sqrt{89}$, $2\sqrt{29}$.

13. Show that the lines joining the middle points of the opposite sides of the quadrilaterals whose vertices are the following points bisect each other.

(a) (8, 4, 2), (0, 2, 5), (– 3, 2, 4), and (8, 0, – 6).
(b) (0, 0, 9), (2, 6, 8), (– 8, 0, 4), and (0, – 8, 6).
(c) $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$.

14. Show that the lines joining successively the middle points of the sides of any quadrilateral form a parallelogram.

15. Find the projection of the line drawn from $P_1(3, 2, -6)$ to $P_2(-3, 5, -4)$ upon a line directed upward whose direction cosines are proportional to 2, 1, and – 2. *Ans.* $4\frac{1}{3}$.

16. Find the projection of the line drawn from $P_1(6, 3, 2)$ to $P_2(4, 2, 0)$ upon the line drawn from $P_3(7, -6, 0)$ to $P_4(-5, -2, 3)$. *Ans.* $\frac{14}{13}$.

17. Find the coördinates of the point of intersection of the medians of the triangle whose vertices are $(3, 6, -2)$, $(7, -4, 3)$, and $(-1, 4, -7)$.
Ans. $(3, 2, -2)$.

18. Find the coördinates of the point of intersection of the medians of the triangle whose vertices are any three points P_1, P_2, and P_3.
Ans. $[\frac{1}{3}(x_1 + x_2 + x_3), \frac{1}{3}(y_1 + y_2 + y_3), \frac{1}{3}(z_1 + z_2 + z_3)]$.

19. The three lines joining the middle points of the opposite edges of a tetrahedron pass through the same point and are bisected at that point.

20. The four lines drawn from the vertices of any tetrahedron to the point of intersection of the medians of the opposite face meet in a point which is three fourths of the distance from each vertex to the opposite face (the center of gravity of the tetrahedron).

CHAPTER XIV

SURFACES, CURVES, AND EQUATIONS

92. Loci in space. In solid geometry it is necessary to consider two kinds of loci:

1. The locus of a point in space which satisfies *one* given condition is, in general, a *surface.*

Thus the locus of a point at a given distance from a fixed point is a sphere, and the locus of a point equidistant from two fixed points is the plane which is perpendicular to the line joining the given points at its middle point.

2. The locus of a point in space which satisfies *two* conditions* is, in general, a *curve.* For the locus of a point which satisfies either condition is a surface, and hence the points which satisfy both conditions lie on two surfaces, that is, on their curve of intersection.

Thus the locus of a point which is at a given distance r from a fixed point P_1 and is equally distant from two fixed points P_2 and P_3 is the circle in which the sphere whose center is P_1 and whose radius is r intersects the plane which is perpendicular to P_2P_3 at its middle point.

These two kinds of loci must be carefully distinguished.

93. Equation of a surface. First fundamental problem. If any point P which lies on a given surface be given the coördinates (x, y, z), then the condition which defines the surface as a locus will lead to an equation involving the variables x, y, and z.

* The number of conditions must be counted carefully. Thus if a point is to be equidistant from three fixed points P_1, P_2, and P_3, it satisfies *two* conditions, namely, of being equidistant from P_1 and P_2 and from P_2 and P_3.

The **equation of a surface** is an equation in the variables x, y, and z representing coördinates such that:

1. The coördinates of every point on the surface will satisfy the equation.

2. Every point whose coördinates satisfy the equation will lie upon the surface.

If the surface is defined as the locus of a point satisfying one condition, its equation may be found in many cases by a Rule analogous to that in Art. 17.

EXAMPLE

Find the equation of the locus of a point whose distance from $P_1(3, 0, -2)$ is 4.

Solution. Let $P(x, y, z)$ be any point on the locus. The given condition may be written

$$P_1P = 4.$$

By (IV),

$$P_1P = \sqrt{(x-3)^2 + y^2 + (z+2)^2}.$$

$$\therefore \sqrt{(x-3)^2 + y^2 + (z+2)^2} = 4.$$

Simplifying, we obtain as the required equation

$$x^2 + y^2 + z^2 - 6x + 4z - 3 = 0.$$

That this is indeed the equation of the locus should be verified as on page 31.

PROBLEMS

1. Find the equation of the locus of a point which is
   (a) 3 units above the XY-plane.
   (b) 4 units to the right of the YZ-plane.
   (c) 5 units below the XY-plane.
   (d) 10 units back of the ZX-plane.
   (e) 7 units to the left of the YZ-plane.
   (f) 2 units in front of the ZX-plane.

2. Find the equation of the plane which is parallel to
   (a) the XY-plane and 4 units above it.
   (b) the XY-plane and 5 units below it.
   (c) the ZX-plane and 3 units in front of it.
   (d) the YZ-plane and 7 units to the left of it.
   (e) the ZX-plane and 2 units back of it.
   (f) the YZ-plane and 4 units to the right of it.

3. What are the equations of the coördinate planes?

4. What is the form of the equation of a plane which is parallel to the XY-plane? the YZ-plane? the ZX-plane?

5. What are the equations of the faces of the rectangular parallelepiped which has one vertex at the origin, three edges lying along the coördinate axes, and one vertex at the point (3, 5, 7)?

6. Find the equation of the locus of a point whose distance from the point

(a) $(2, -2, 1)$ is 3.
(b) $(0, \frac{1}{2}, -2)$ is $\frac{1}{2}$.
(c) $(-1, 3, \frac{2}{3})$ is $\sqrt{3}$.
(d) $(-2, \frac{1}{3}, 0)$ is $\sqrt{5}$.
(e) (a, b, c) is d.
(f) (α, β, γ) is r.

7. Find the equation of the sphere whose center is the point

(a) $(3, 0, 4)$ and whose radius is 5.
Ans. $x^2 + y^2 + z^2 - 6x - 8z = 0$.

(b) $(-3, 2, 1)$ and whose radius is 4.
Ans. $x^2 + y^2 + z^2 + 6x - 4y - 2z - 2 = 0$.

(c) $(6, 4, 0)$ and whose radius is 7.

(d) (α, β, γ) and whose radius is r.
Ans. $x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z + \alpha^2 + \beta^2 + \gamma^2 - r^2 = 0$.

8. Find the equation of a sphere

(a) having the line joining $(3, 0, 7)$ and $(1, -2, -1)$ for a diameter.

(b) of radius 2, which is tangent to all three coördinate planes in the first octant.

(c) of radius 3, which is tangent to all three coördinate planes in the third octant.

(d) whose center is the point $(3, 1, -2)$ and which is tangent to the XY-plane.

(e) whose center is $(6, 2, 3)$ and which passes through the origin.

(f) passing through the four points $(-2, 0, 0), (0, -4, 0), (0, 0, 4), (8, 0, 0)$.

9. Find the equation of the locus of a point which is equally distant from the points

(a) $(3, 2, -1)$ and $(4, -3, 0)$. *Ans.* $2x - 10y + 2z - 11 = 0$.
(b) $(4, -3, 6)$ and $(2, -4, 2)$. *Ans.* $4x + 2y + 8z - 37 = 0$.
(c) $(1, 3, 2)$ and $(4, -1, 1)$. *Ans.* $3x - 4y - z - 2 = 0$.
(d) $(4, -6, -8)$ and $(-2, 7, 9)$. *Ans.* $6x - 13y - 17z + 9 = 0$.

10. Find the equation of a plane perpendicular at the middle point to the line joining

(a) $(1, -2, 1)$ and $(2, -1, 0)$.
(b) $(-3, \frac{1}{2}, 0)$ and $(0, 0, \frac{1}{3})$.
(c) $(-2, \frac{1}{3}, \frac{1}{5})$ and $(\frac{1}{2}, 0, 0)$.

11. Find the equations of the six planes drawn through the middle points of the edges of the tetrahedron whose vertices are the points (5, 4, 0), (2, − 5, − 4), (1, 7, − 5), and (− 4, 3, 4), which are perpendicular to the respective edges, and show that they all pass through the point (− 1, 1, − 2).

12. Find the equation of the locus of a point which is three times as far from the point (2, 6, 8) as from (4, − 2, 4), and determine the nature of the locus by comparison with the answer to Problem 7 (d).

13. Find the equation of the locus of a point the sum of the squares of whose distances from (1, 3, − 2) and (6, − 4, 2) is 50, and determine the nature of the locus by comparison with the answer to Problem 7 (d).

14. Find the equation of the locus of a point whose distance

(a) from the X-axis is 3.
(b) from the Y-axis is $\frac{1}{3}$.
(c) from the Z-axis is $\sqrt{5}$.

15. Find the equation of a circular cylinder

(a) whose axis is the Y-axis and whose radius is 2.
(b) whose axis is the Z-axis and whose radius is $\sqrt{3}$.
(c) whose axis is the X-axis and whose diameter is $\sqrt{7}$.

16. A point moves so that the sum of its distances to the two fixed points ($\sqrt{3}$, 0, 0) and (− $\sqrt{3}$, 0, 0) is always equal to 4. Find the equation of its locus. *Ans.* $x^2 + 4z^2 + 4y^2 - 4 = 0$.

17. Find the equation of the locus of a point

(a) whose distance from the point (1, 0, 0) equals its distance from the YZ-plane. *Ans.* $y^2 + z^2 - 2x + 1 = 0$.

(b) whose distance from the point (1, 0, 0) equals its distance from the Z-axis. *Ans.* $z^2 - 2x + 1 = 0$.

(c) whose distance from the X-axis is one half of its distance from the YZ-plane. *Ans.* $4y^2 + 4z^2 - x^2 = 0$.

(d) whose distance from the Z-axis is twice its distance from the Y-axis.

(e) whose distance from the origin equals the sum of its distances from the XZ-plane and the YZ-plane. *Ans.* $z^2 - 2xy = 0$.

(f) the sum of whose distances from the three coördinate planes is constant.

(g) whose distance from the origin equals the sum of its distances from the three coördinate planes. *Ans.* $xy + yz + zx = 0$.

(h) whose distance from the X-axis is half the difference of its distances from the XY-plane and the XZ-plane.

(i) whose distance from the point (0, 0, 1) equals its distance from the XY-plane increased by 1.

(j) whose distance from the Z-axis equals its distance from the point (1, 1, 0).

18. Find the equation of the locus of a point the sum of whose distances from the X-axis and the Y-axis is unity.

19. Find the equation of the locus of a point the sum of whose distances from the three coördinate axes is unity.

94. Planes parallel to the coördinate planes. We may easily prove the

Theorem. *The equation of a plane which is*

parallel to the XY-plane has the form $z = constant;$

parallel to the YZ-plane has the form $x = constant;$

parallel to the ZX-plane has the form $y = constant.$

95. Equations of a curve. First fundamental problem. If any point P which lies on a given curve be given the coördinates (x, y, z), then the two conditions which define the curve as a locus will lead to two equations involving the variables x, y, and z.

The **equations of a curve** are *two* equations in the variables x, y, and z representing coördinates such that:

1. The coördinates of every point on the curve will satisfy both equations.

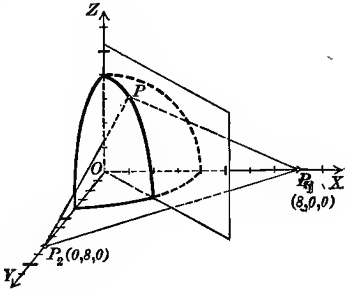
2. Every point whose coördinates satisfy both equations will lie on the curve.

If the curve is defined as the locus of a point satisfying two conditions, the equations of the surfaces defined by each condition separately may be found in many cases by a Rule analogous to that of Art. 17. These equations will be the equations of the curve.

It will appear later that the equations of the same curve may have an endless variety of forms.

EXAMPLES

1. Find the equations of the locus of a point whose distance from the origin is 4 and which is equally distant from the points $P_1(8, 0, 0)$ and $P_2(0, 8, 0)$.



Solution. Let $P(x, y, z)$ be any point on the locus.

The given conditions are

(1) $\quad PO = 4, \quad PP_1 = PP_2.$

By (IV),

$$PO = \sqrt{x^2 + y^2 + z^2},$$

$$PP_1 = \sqrt{(x-8)^2 + y^2 + z^2},$$

$$PP_2 = \sqrt{x^2 + (y-8)^2 + z^2}.$$

Substituting in (1), we get

$$\sqrt{x^2 + y^2 + z^2} = 4, \quad \sqrt{(x-8)^2 + y^2 + z^2} = \sqrt{x^2 + (y-8)^2 + z^2}.$$

Squaring and reducing, we have the required equations, namely,

$$x^2 + y^2 + z^2 = 16, \quad x - y = 0.$$

These equations should be verified as in Art. 16.

2. Find the equations of the circle lying in the XY-plane whose center is the origin and whose radius is 5.

Solution. In plane analytic geometry the equation of the circle is

(2) $$x^2 + y^2 = 25.$$

Regarded as a problem in solid analytic geometry we must have *two* equations which the coördinates of any point $P(x, y, z)$ which lies on the circle must satisfy. Since P lies in the XY-plane,

(3) $$z = 0.$$

Hence equations (2) and (3) together express that the point P lies in the XY-plane and on the given circle. The equations of the circle are therefore

$$x^2 + y^2 = 25, \quad z = 0.$$

The reasoning in Ex. 2 is general. Hence

If the equation of a curve in the XY-plane is known, then the equations of that curve regarded as a curve in space are the given equation and $z = 0$.

An analogous statement evidently applies to the equations of a curve lying in one of the other coördinate planes.

From Art. 94 we have at once the

Theorem. *The equations of a line which is parallel to*

the X-axis have the form $y = constant, \quad z = constant$;

the Y-axis have the form $z = constant, \quad x = constant$;

the Z-axis have the form $x = constant, \quad y = constant$.

PROBLEMS

1. Find the equations of the locus of a point which is

(a) 3 units above the XY-plane and 4 units to the right of the YZ-plane.

(b) 5 units to the left of the YZ-plane and 2 units in front of the ZX-plane.

(c) 4 units back of the ZX-plane and 7 units to the left of the YZ-plane.

(d) 9 units below the XY-plane and 4 units to the right of the YZ-plane.

2. Find the equations of the straight line which is

(a) 5 units above the XY-plane and 2 units in front of the ZX-plane

(b) 2 units to the left of the YZ-plane and 8 units below the XY-plane.

(c) 3 units to the right of the YZ-plane and 5 units from the Z-axis.

(d) 13 units from the X-axis and 5 units back of the ZX-plane.

(e) parallel to the Y-axis and passing through $(3, 7, -5)$.

(f) parallel to the Z-axis and passing through $(-4, 7, 6)$.

3. What are the equations of the axes of coördinates?

4. What are the equations of the edges of a rectangular parallelepiped whose dimensions are a, b, and c, if three of its faces coincide with the coördinate planes and one vertex lies in $O\text{-}XYZ$? in $O\text{-}XY'Z$? in $O\text{-}X'Y'Z$?

5. Find the equations of the locus of a point which is

(a) 5 units from the origin and 3 units above the XY-plane.

(b) 5 units from the origin and 3 units from the X-axis.

(c) 6 units from the Y-axis and 3 units behind the XZ-plane.

(d) 7 units from the Z-axis and 2 units below the XY-plane.

6. Find the equations of a circle defined as follows:

(a) center on the Z-axis, radius 4, and lying in the XY-plane.

(b) center on the X-axis, radius 7, and lying in a plane parallel to the YZ-plane and 3 units to the right of it.

(c) center on the Y-axis, radius 2, and lying in a plane 2 units behind the XZ-plane.

(d) center at the point (1, 0, 1), parallel to the XY-plane, and cutting the Z-axis.

7. The following equations are the equations of curves lying in one of the coördinate planes. What are the equations of the same curves regarded as curves in space?

(a) $y^2 = 4x$.
(b) $x^2 + z^2 = 16$.
(c) $8x^2 - y^2 = 64$.
(d) $4z^2 + 9y^2 = 36$.
(e) $x^2 + 4z + 6x = 0$.
(f) $y^2 - z^2 - 4y = 0$.
(g) $yz^2 + z^2 - 6y = 0$.
(h) $z^2 - 4x^2 + 8z = 0$.

8. Find the equations of the locus of a point which is

(a) 5 units above the XY-plane and 3 units from (3, 7, 1).
Ans. $z = 5$, $x^2 + y^2 + z^2 - 6x - 14y - 2z + 50 = 0$.

(b) 2 units from (3, 7, 6) and 4 units from (2, 5, 4).
Ans. $x^2 + y^2 + z^2 - 6x - 14y - 12z + 90 = 0$,
$x^2 + y^2 + z^2 - 4x - 10y - 8z + 29 = 0$.

(c) 5 units from the origin and equidistant from (3, 7, 2) and (− 3, − 7, − 2). *Ans.* $x^2 + y^2 + z^2 - 25 = 0$, $3x + 7y + 2z = 0$.

(d) equidistant from (3, 5, − 4) and (− 7, 1, 6), and also from (4, − 6, 3) and (− 2, 8, 5).
Ans. $5x + 2y - 5z + 9 = 0$, $3x - 7y - z + 8 = 0$.

(e) equidistant from (2, 3, 7), (3, − 4, 6), and (4, 3, − 2).
Ans. $2x - 14y - 2z + 1 = 0$, $x + y - 8z + 16 = 0$.

9. Find the equations of the locus of a point which is equally distant from the points (6, 4, 3) and (6, 4, 9), and also from (− 5, 8, 3) and (− 5, 0, 3), and determine the nature of the locus. *Ans.* $z = 6$, $y = 4$.

10. Find the equations of the locus of a point which is equally distant from the points (3, 7, − 4), (− 5, 7, − 4), and (− 5, 1, − 4), and determine the nature of the locus. *Ans.* $x = -1$, $y = 4$.

11. Determine the nature of each of the following loci after finding their equations. The moving point is equidistant from

(a) the three coördinate planes.
(b) the three coördinate axes.
(c) the three points (1, 0, 0), (0, 1, 0), and (0, 0, 1).
(d) the XY-plane, the Z-axis, and the point (0, 0, 1).
(e) the XY-plane, the X-axis, and the point (0, 0, 1).
(f) the points (1, 0, 0), (0, 1, 0), and the Z-axis.
(g) the X-axis, the Y-axis, and the point (1, 0, 0).
(h) the Z-axis, the XY-plane, and the YZ-plane.

96. Locus of one equation. Second fundamental problem. The **locus of one equation** in three variables (one or two may be lacking) representing coördinates in space is the *surface* passing through all points whose coördinates satisfy that equation and through such points only.

The coördinates of points on the surface may be obtained as follows:

Solve the equation for one of the variables, say z, assume pairs of values of x and y, and compute the corresponding values of z.

A rough model of the surface might then be constructed by taking a thin board for the XY-plane, sticking needles into it at the assumed points (x, y) whose lengths are the computed values of z, and stretching a sheet of rubber over their extremities.

97. Locus of two equations. Second fundamental problem. The **locus of two equations** in three variables representing coördinates in space is the *curve* passing through all points whose coördinates satisfy both equations and through such points only. That is, the locus is the curve of intersection of the surfaces defined by the two given equations.

The coördinates of points on the curve may be obtained as follows:

Solve the equations for two of the variables, say x and y, in terms of the third, z, assume values for z, and compute the corresponding values of x and y.

98. Discussion of the equations of a curve. Third fundamental problem. The discussion of curves in elementary analytic geometry is largely confined to curves which lie entirely in a plane which is usually parallel to one of the coördinate planes. Such a curve is defined as the intersection of a given surface with a plane parallel to one of the coördinate planes. The method of determining its nature is illustrated as follows:

EXAMPLE

Determine the nature of the curve in which the plane $z = 4$ intersects the surface whose equation is $y^2 + z^2 = 4x$.

Solution. The equations of the curve are, by definition,

(1) $$y^2 + z^2 = 4x, \quad z = 4.$$

Eliminate z by substituting from the second equation in the first. This gives

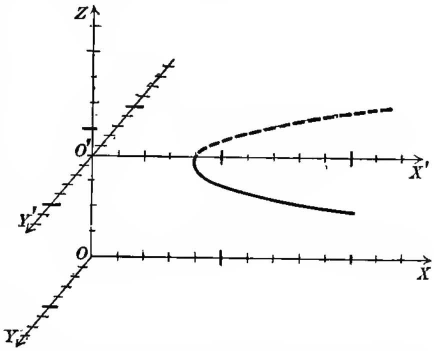
(2) $$y^2 - 4x + 16 = 0, \quad z = 4.$$

Equations (2) are also the equations of the curve.

For every set of values of (x, y, z) which satisfy both of equations (1) will evidently satisfy both of equations (2), and conversely.

If we take as axes in the plane $z = 4$ the lines $O'X'$ and $O'Y'$ in which the plane cuts the ZX-plane and the YZ-plane, then the equation of the curve when *referred to these axes* is the first of equations (2), namely,

(3) $y^2 - 4x + 16 = 0.$



The locus of (3) is a parabola. The vertex, in the plane $z = 4$, is the point $(4, 0)$; also $p = 2$.

In plotting the locus of (3) in the plane $X'O'Y'$ the values of x and y must be laid off *parallel* to $O'X'$ and $O'Y'$ respectively, as in plotting oblique coördinates (Art. 9).

From the preceding example we may state the

Rule *to determine the nature of the curve in which a plane parallel to one of the coördinate planes cuts a given surface.*

Eliminate the variable occurring in the equation of the plane from the equations of the plane and surface. The result is the equation of the curve referred to the lines in which the given plane cuts the other two coördinate planes as axes. Discuss this curve by the methods of plane analytic geometry.

PROBLEMS

1. Determine the nature of the following curves and construct their loci:

(a) $x^2 - 4y^2 = 8z,\ z = 8.$

(b) $x^2 + 9y^2 = 9z^2,\ z = 2.$

(c) $x^2 - 4y^2 = 4z,\ y = -2.$

(d) $x^2 + y^2 + z^2 = 25,\ x = 3.$

(e) $x^2 + 4y^2 + 9z^2 = 36,\ y = 1.$

(f) $x^2 - 4y^2 + z^2 = 25,\ x = -3.$

(g) $x^2 - y^2 - 4z^2 + 6x = 0,\ x = 2.$

(h) $y^2 + z^2 - 4x + 8 = 0,\ y = 4.$

2. Construct the curves in which each of the following surfaces intersects the coördinate planes:

(a) $x^2 + 4y^2 + 16z^2 = 64.$

(b) $x^2 + 4y^2 - 16z^2 = 64.$

(c) $x^2 - 4y^2 - 16z^2 = 64.$

(d) $x^2 + 9y^2 = 10z.$

(e) $x^2 - 9y^2 = 10z.$

(f) $x^2 + 4y^2 - 16z^2 = 0.$

3. Show that the curves of intersection of each of the surfaces in Problem 2 with a system of planes parallel to one of the coördinate planes are conics of the same species (see Art. 70).

4. Determine the nature of the intersection of the surface $x^2 + y^2 + 4z^2 = 64$ with the plane $z = k$. How does the curve change as k increases from 0 to 4? from -4 to 0? What idea of the appearance of the surface is thus obtained?

5. Determine the nature of the intersection of the surface $4x - 2y = 4$ with the plane $y = k$; with the plane $z = k'$. How does the intersection change as k or k' changes? What idea of the form of the surface is obtained?

6. In each of the following find the equations of the locus, determine its nature, and construct it:

(a) A point is 5 units from the origin and 3 units from the Z-axis.

(b) A point is 3 units from both the X-axis and the Z-axis.

(c) The distance of a point from the Z-axis is equal to twice its distance from the XY-plane and its distance from the origin is 2.

(d) A point is 5 units from the X-axis and 4 units from the XZ-plane.

(e) A point is equidistant from the YZ-plane and the XZ-plane and its distance from the X-axis is 7. *Ans.* An ellipse.

(f) A point is equidistant from the Z-axis, the YZ-plane, and the point (2, 0, 0). *Ans.* A parabola.

7. The ratio of the distances of a point to the Z-axis and the Y-axis respectively is $\frac{3}{5}$. Determine the nature of its locus if it is also

(a) one unit above the XY-plane.
(b) one unit in front of the XZ-plane.
(c) one unit to the left of the YZ-plane.
(d) in the XZ-plane.
(e) equidistant from the XZ-plane and the YZ-plane.
(f) in the plane $4x - 3z - 12 = 0$.

8. Find the equations of the locus of a point whose distance from the point (2, 0, 0) is always equal to three times its distance from the Z-axis, and whose distance from the YZ-plane is always unity. Name and draw the locus.

9. Find the equations of the locus of a point which is equidistant from the point (1, − 2, 0) and the Z-axis, and which is $3\frac{1}{2}$ units behind the XZ-plane. Name and draw the locus.

10. Find the equations of the locus of a point which is equidistant from the Y-axis and the XZ-plane and equidistant from the origin and the point (0, 0, − 4). Name and draw the locus.

99. Discussion of the equation of a surface. Third fundamental problem.

Theorem. *The locus of an algebraic equation passes through the origin if there is no constant term in the equation.*

The proof is analogous to that on page 47.

Theorem. *If the locus of an equation is unaffected by changing the sign of one variable throughout its equation, then the locus is symmetrical with respect to the coördinate plane from which that variable is measured.*

If the locus is unaffected by changing the signs of two variables throughout its equation, it is symmetrical with respect to the axis along which the third variable is measured.

If the locus is unaffected by changing the signs of all three variables throughout its equation, it is symmetrical with respect to the origin.

The proof is analogous to that on page 42.

Rule *to find the intercepts of a surface on the axes of coördinates.* *Set each pair of variables equal to zero and solve for real values of the third.*

The curves in which a surface intersects the coördinate planes are called its **traces** on the coördinate planes. From the Rule, p. 254, it is seen that

The equations of the traces of a surface are obtained by successively setting $x = 0$, $y = 0$, *and* $z = 0$ *in the equation of the surface.*

By these means we can determine some properties of the surface. The *general appearance of a surface* is determined by considering the curves in which it is cut by a system of planes parallel to each of the coördinate planes. This also enables us *to determine whether the surface is closed or recedes to infinity.*

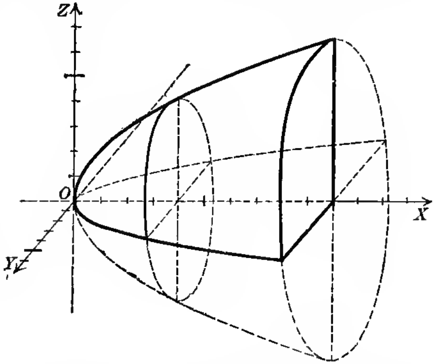
EXAMPLE

Discuss the locus of the equation $y^2 + z^2 = 4x$.

Solution. 1. The surface passes through the origin since there is no constant term in its equation.

2. The surface is symmetrical with respect to the XY-plane, the ZX-plane, and the X-axis.

For the locus of the given equation is unaffected by changing the sign of z, of y, or of both together.



3. It cuts the axes at the origin only.

4. Its traces are respectively the point-circle $y^2 + z^2 = 0$ and the parabolas $z^2 = 4x$ and $y^2 = 4x$.

5. It intersects the plane $x = k$ in the curve

$$y^2 + z^2 = 4k.$$

This curve is a circle whose center is the origin, that is, is on the X-axis, and whose radius is $2\sqrt{k}$ if $k > 0$, but there is no locus if $k < 0$. Hence the surface lies entirely to the right of the YZ-plane.

If k increases from zero to infinity, the radius of the circle increases from zero to infinity while the plane $x = k$ recedes from the YZ-plane.

The intersection with a plane $z = k$ or $y = k'$, parallel to the XY- or the ZX-plane, is seen to be a parabola whose equation is

$$y^2 = 4x - k^2 \quad \text{or} \quad z^2 = 4x - k'^2.$$

These parabolas have the same value of p, namely $p = 2$, and their vertices recede from the YZ- or the ZX-plane as k or k' increases numerically.

PROBLEMS

1. Discuss and draw the loci of the following equations:

(a) $x^2 + z^2 = 4x$.
(b) $x^2 + y^2 + 4z^2 = 16$.
(c) $x^2 + y^2 - 4z^2 = 16$.
(d) $6x + 4y + 3z = 12$.
(e) $3x + 2y + z = 12$.
(f) $x + 2z - 4 = 0$.
(g) $x^2 + y^2 - 2z = 0$.
(h) $x^2 + y^2 - 2x = 0$.
(i) $x^2 + y^2 - 4 = 0$.
(j) $y^2 + z^2 - x - 4 = 0$.
(k) $x^2 + y^2 - z^2 = 0$.
(l) $x^2 - y^2 - z^2 = 9$.
(m) $x^2 + y^2 - z^2 + 2xy = 0$.
(n) $x + y - 6z = 6$.
(o) $y^2 + z^2 = 25$.
(p) $x^2 + y^2 - z^2 - 1 = 0$.
(q) $x^2 + y^2 - z^2 + 1 = 0$.
(r) $4x^2 - y^2 - z^2 = 0$.
(s) $z^2 - x - y = 0$.
(t) $x^2 + y^2 - 2zx = 0$.

2. Show that the locus of $Ax + By + Cz + D = 0$ is a plane by considering its traces on the coördinate planes and the sections made by planes parallel to one of the coördinate planes.

3. In each of the following find the equation of the locus of the point and draw and discuss it:

(a) The sum of the distances of a point from the XZ-plane and the YZ-plane equals twice its distance from the XY-plane increased by 4.

(b) The square of its distance from the Z-axis is equal to four times its distance from the XY-plane.

(c) Its distance from the Z-axis is double its distance from the XY-plane.

(d) Its distance from the Y-axis is twice the square root of its distance from the YZ-plane.

(e) It is equally distant from the point (2, 0, 0) and the YZ-plane.
Ans. $y^2 + z^2 - 4x + 4 = 0$.

(f) It is equally distant from the point (0, 2, 0) and the X-axis.

(g) Its distance from the Z-axis is equal to its distance from the YZ-plane increased by 2.

(h) Its distance from the point $(0, 0, -2)$ is equal to double its distance from the XY-plane increased by unity.

(i) Its distance from the point $(\frac{1}{2}, 0, 0)$ is equal to half its distance from the YZ-plane diminished by one. *Ans.* $3x^2 + 4y^2 + 4z^2 - 3 = 0.$

(j) The product of the sum and the difference of its distances from the XZ-plane and the YZ-plane respectively is equal to twice its distance from the XY-plane.

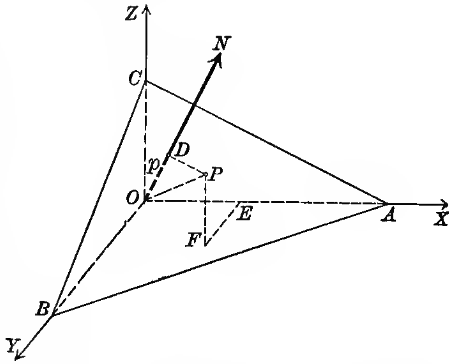
4. Find the equation of the locus of a point whose distance from the point $(0, 0, 3)$ is twice its distance from the XY-plane, and discuss the locus. *Ans.* $x^2 + y^2 - 3z^2 - 6z + 9 = 0.$

5. Find the equation of the locus of a point whose distance from the point $(0, 4, 0)$ is three fifths its distance from the ZX-plane, and discuss the locus. *Ans.* $25x^2 + 16y^2 + 25z^2 - 200y + 400 = 0.$

CHAPTER XV

THE PLANE AND THE GENERAL EQUATION OF THE FIRST DEGREE IN THREE VARIABLES

100. The normal form of the equation of the plane. Let ABC be any plane, and let ON be drawn from the origin perpendicular to ABC at D. *Let the positive direction on ON be from O toward N*, that is, *from* the origin *toward* the plane, and denote the directed length OD by p and the direction angles of ON by α, β, and γ. Then *the position of any plane is determined by given positive values of p, α, β, and γ.*



If $p = 0$, the positive direction on ON, as just defined, becomes meaningless. *If $p = 0$, we shall suppose that ON is directed upward*, and hence $\cos\gamma > 0$ since $\gamma < \frac{\pi}{2}$. If the plane passes through OZ, then ON lies in the XY-plane and $\cos\gamma = 0$; *in this case we shall suppose ON so directed that $\beta < \frac{\pi}{2}$* and hence $\cos\beta > 0$. Finally, if the plane coincides with the YZ-plane, the positive direction on ON shall be that on OX.

Let us now solve the problem:

Given the perpendicular distance p from the origin to a plane and the direction angles α, β, γ of this perpendicular, to find the equation of the plane.

Solution. Let $P(x, y, z)$ be any point on the given plane ABC. Draw the coördinates $OE = x$, $EF = y$, $FP = z$ of P. Project $OEFP$ and OP on the line ON. By the second theorem of projection,

$$\text{projection of } OE + \text{projection of } EF + \text{projection of } FP$$
$$= \text{projection of } OP.$$

Then by the first theorem of projection and by the definition of p,

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

Transposing, we obtain the

Theorem. *Normal form. The equation of a plane is*

(I) $$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0,$$

where p is the perpendicular distance from the origin to the plane, and α, β, and γ are the direction cosines of that perpendicular.

Corollary. *The equation of any plane is of the first degree in x, y, and z.*

101. The general equation of the first degree, $Ax + By + Cz + D = 0$. The question now arises: Given an equation of the first degree in the coördinates x, y, z; what is the locus? This question is answered by the

Theorem. *The locus of any equation of the first degree in x, y, and z,*

(II) $$Ax + By + Cz + D = 0,$$

is a plane.

Proof. We shall prove the theorem by showing that (II) may be reduced to the normal form (I) by multiplying by a proper constant. To determine this constant, multiply (II) by k, which gives

(1) $$kAx + kBy + kCz + kD = 0.$$

Equating corresponding coefficients of (1) and (I),

(2) $kA = \cos\alpha, \quad kB = \cos\beta, \quad kC = \cos\gamma, \quad kD = -p.$

Squaring the first three of equations (2) and adding,

$$k^2(A^2 + B^2 + C^2) = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

(3) $$\therefore k = \frac{1}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

From the last of equations (2) we see that the sign of the radical must be *opposite to that of D* in order that p shall be positive.

Substituting from (3) in (2), we get

(4) $$\begin{cases} \cos\alpha = \dfrac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, & \cos\beta = \dfrac{B}{\pm\sqrt{A^2 + B^2 + C^2}}, \\ \cos\gamma = \dfrac{C}{\pm\sqrt{A^2 + B^2 + C^2}}, & p = \dfrac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}. \end{cases}$$

We have thus determined values of α, β, γ, and p such that (I) and (II) have the same locus. Hence the locus of (II) is a plane. Q. E. D.

If $D = 0$, then $p = 0$; and from the third of equations (2) the sign of the radical must be the *same as that of C*, since when $p = 0$, $\cos\gamma > 0$. If $D = 0$ and $C = 0$, then $p = 0$ and $\cos\gamma = 0$; and from the second of equations (2) the sign of the radical must be the *same as that of B*, since when $p = 0$ and $\cos\gamma = 0$, $\cos\beta > 0$.

Equation (II) is called the **general equation of the first degree** in x, y, and z. The discussion gives the

Rule *to reduce the equation of a plane to the normal form.*

Divide the equation by $\pm\sqrt{A^2 + B^2 + C^2}$, *choosing the sign of the radical opposite to that of D.*

When $D = 0$, the sign of the radical must be the same as that of C, the same as that of B if $C = D = 0$, or the same as that of A if $B = C = D = 0$.

From (4) we have the important

Theorem. *The coefficients of x, y, and z in the equation of a plane are proportional to the direction cosines of any line perpendicular to the plane.*

From this theorem and Art. 90 we easily prove the following:

Corollary I. *Two planes whose equations are*

$$Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0$$

are parallel when and only when the coefficients of x, y, and z are proportional, that is,

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

Corollary II. *Two planes are perpendicular when and only when*

$$AA' + BB' + CC' = 0.$$

Corollary III. *A plane whose equation has the form*

$Ax + By + D = 0$ *is perpendicular to the XY-plane;*

$By + Cz + D = 0$ *is perpendicular to the YZ-plane;*

$Ax + Cz + D = 0$ *is perpendicular to the ZX-plane.*

That is, if one variable is lacking, the plane is perpendicular to the coördinate plane corresponding to the two variables which occur in the equation.

For these planes are respectively perpendicular to the planes $z = 0$, $x = 0$, and $y = 0$ by Corollary II.

Corollary IV. *A plane whose equation has the form*

$Ax + D = 0$ *is perpendicular to the axis of x;*

$By + D = 0$ *is perpendicular to the axis of y;*

$Cz + D = 0$ *is perpendicular to the axis of z.*

That is, if two variables are lacking, the plane is perpendicular to the axis corresponding to the variable which occurs in the equation.

For two of the direction cosines of a perpendicular to the plane are now zero, and hence this line is parallel to one of the axes and the plane is therefore perpendicular to that axis.

PROBLEMS

1. Find the intercepts on the axes and the traces on the coördinate planes of each of the following planes and construct the figures:

(a) $2x + 3y + 4z - 24 = 0.$
(b) $7x - 3y + z - 21 = 0.$
(c) $9x - 7y - 9z + 63 = 0.$
(d) $6x + 4y - z + 12 = 0.$
(e) $5x - 7y - 35 = 0.$
(f) $4x + 3z + 36 = 0.$
(g) $5y - 8z - 40 = 0.$
(h) $3x + 5z + 45 = 0.$

2. What are the intercepts and the equations of the traces on the coördinate planes of the plane $Ax + By + Cz + D = 0$?

3. Find the equations of the planes and construct them by drawing their traces, for which

(a) $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{3}, \gamma = \frac{\pi}{3}, p = 6.$ *Ans.* $\sqrt{2}x + y + z - 12 = 0.$

(b) $\alpha = \frac{2\pi}{3}, \beta = \frac{3\pi}{4}, \gamma = \frac{\pi}{3}, p = 8.$ *Ans.* $x + \sqrt{2}y - z + 16 = 0.$

(c) $\frac{\cos\alpha}{6} = \frac{\cos\beta}{-2} = \frac{\cos\gamma}{3}, p = 4.$ *Ans.* $6x - 2y + 3z - 28 = 0.$

(d) $\frac{\cos\alpha}{-2} = \frac{\cos\beta}{-1} = \frac{\cos\gamma}{-2}, p = 2.$ *Ans.* $2x + y + 2z + 6 = 0.$

4. Find the equation of the plane such that the foot of the perpendicular from the origin to the plane is the point

(a) $(-3, 2, 6)$. *Ans.* $3x - 2y - 6z + 49 = 0.$
(b) $(4, 3, -12)$. *Ans.* $4x + 3y - 12z - 169 = 0.$
(c) $(2, 2, -1)$. *Ans.* $2x + 2y - z - 9 = 0.$

5. Reduce the following equations to the normal form and find α, β, γ, and p:

(a) $6x - 3y + 2z - 7 = 0.$ *Ans.* $\cos^{-1}\frac{6}{7}, \cos^{-1}(-\frac{3}{7}), \cos^{-1}\frac{2}{7}, 1.$

(b) $x - \sqrt{2}y + z + 8 = 0.$ *Ans.* $\frac{2\pi}{3}, \frac{\pi}{4}, \frac{2\pi}{3}, 4.$

(c) $2x - 2y - z + 12 = 0.$ *Ans.* $\cos^{-1}(-\frac{2}{3}), \cos^{-1}\frac{2}{3}, \cos^{-1}\frac{1}{3}, 4.$

(d) $y - z + 10 = 0.$ *Ans.* $\frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{4}, 5\sqrt{2}.$

(e) $3x + 2y - 6z = 0.$ *Ans.* $\cos^{-1}(-\frac{3}{7}), \cos^{-1}(-\frac{2}{7}), \cos^{-1}\frac{6}{7}, 0.$

6. Find the distance from the origin to the plane $12x - 4y + 3z - 39 = 0.$

7. Find the area of the triangle which the three coördinate planes cut from each of the following planes:

(a) $2x + 2y + z - 12 = 0$. *Ans.* 54.

(b) $6x - 2y - 3z + 21 = 0$.

(c) $12x - 3y + 4z - 13 = 0$.

(d) $x + 5y + 7z - 3 = 0$. *Ans.* $\frac{27\sqrt{5}}{70}$.

(e) $x - 2y + 3z - 6 = 0$.

(f) $9x + 2y - z + 18 = 0$.

Hint. Find the volume of the tetrahedron formed by the four planes by finding the intercepts. Set this equal to the product of the required area by one third the distance of the given plane from the origin, and solve.

8. Find the distance between the parallel planes $6x + 2y - 3z - 63 = 0$ and $6x + 2y - 3z + 49 = 0$. *Ans.* 16.

9. Find the equation of a plane parallel to the plane $2x + 2y + z - 15 = 0$ and two units nearer to the origin.

10. Show that the following pairs of planes are either parallel or perpendicular:

(a) $\begin{cases} 2x + 5y - 6z + 8 = 0, \\ 6x + 15y - 18z - 5 = 0. \end{cases}$

(b) $\begin{cases} 3x - 5y - 4z + 7 = 0, \\ 6x + 2y + 2z - 7 = 0. \end{cases}$

(c) $\begin{cases} 6x - 3y + 2z - 7 = 0, \\ 3x + 2y - 6z + 28 = 0. \end{cases}$

(d) $\begin{cases} 14x - 7y - 21z - 50 = 0, \\ 2x - y - 3z + 12 = 0. \end{cases}$

11. What may be said of the position of the plane (I), Art. 100, if

(a) $\cos\alpha = 0$? (b) $\cos\beta = 0$? (c) $\cos\gamma = 0$? (d) $\cos\alpha = \cos\beta = 0$? (e) $\cos\beta = \cos\gamma = 0$? (f) $\cos\gamma = \cos\alpha = 0$?

12. For what values of α, β, γ, and p will the locus of (1), Art. 100, be parallel to the XY-plane? the YZ-plane? the ZX-plane? coincide with one of these planes?

13. For what values of α, β, γ, and p will the locus of (1), Art. 100, pass through the X-axis? the Y-axis? the Z-axis?

14. Find the coördinates of the point of intersection of the planes $x + 2y + z = 0$, $x - 2y - 8 = 0$, $x + y + z - 3 = 0$. *Ans.* $(2, -3, 4)$.

15. Show that the plane $x + 2y - 2z - 9 = 0$ passes through the point of intersection of the planes $x + y + z - 1 = 0$, $x - y - z - 1 = 0$, and $2x + 3y - 8 = 0$.

16. Show that the four planes $x + y + 2z - 2 = 0$, $x + y - 2z + 2 = 0$, $x - y + 8 = 0$, and $3x - y - 2z + 18 = 0$ pass through the same point.

17. Show that the planes $2x - y + z + 3 = 0$, $x - y + 4z = 0$, $3x + y - 2z + 8 = 0$, $4x - 2y + 2z - 5 = 0$, $9x + 3y - 6z - 7 = 0$, and $7x - 7y + 28z - 6 = 0$ bound a parallelepiped.

18. Show that the planes $6x - 3y + 2z = 4$, $3x + 2y - 6z = 10$, $2x + 6y + 3z = 9$, $3x + 2y - 6z = 0$, $12x + 36y + 18z - 11 = 0$, and $12x - 6y + 4z - 17 = 0$ bound a rectangular parallelepiped.

19. Show that the planes $x + 2y - z = 0$, $y + 7z - 2 = 0$, $x - 2y - z - 4 = 0$, $2x + y - 8 = 0$, and $3x + 3y - z - 8 = 0$ bound a quadrangular pyramid.

20. Derive the conditions for parallelism of two planes from the fact that two planes are parallel if all their traces are parallel lines.

102. Planes determined by three conditions. The equation

$$Ax + By + Cz + D = 0 \tag{1}$$

represents, as we know, all planes. The statement of a problem, to find the equation of a certain plane, may be such that we are able to write down three homogeneous equations in the coefficients A, B, C, D, which we can then solve for three coefficients in terms of the fourth. When these values are substituted in (1), the fourth coefficient will divide out, giving the required equation.

EXAMPLES

1. Find the equation of the plane which passes through the point $P_1(2, -7, \frac{3}{2})$ and is parallel to the plane $21x - 12y + 28z - 84 = 0$.

Solution. Let the equation of the required plane be

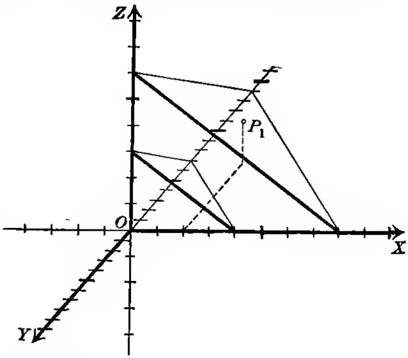
(2) $Ax + By + Cz + D = 0.$

Since P_1 lies on (2), we may substitute $x = 2$, $y = -7$, $z = \frac{3}{2}$, giving

(3) $2A - 7B + \frac{3}{2}C + D = 0.$

Since (2) is parallel to the given plane (Corollary I, p. 263),

(4) $\frac{A}{21} = \frac{B}{-12} = \frac{C}{28}.$



Equations (3) and (4) are three homogeneous equations in A, B, C, D.

Solving (3) and (4) for A, B, and D in terms of C,

$$A = \frac{3}{4}C, \quad B = -\frac{3}{7}C, \quad D = -6C.$$

Substituting in (2), $\frac{3}{4}Cx - \frac{3}{7}Cy + Cz - 6C = 0.$

Clearing of fractions and dividing by C,

$$21x - 12y + 28z - 168 = 0. \quad \textit{Ans.}$$

The answer should be checked by testing whether the coördinates of P_1 satisfy the answer.

2. To find the equation of a plane passing through three points, substitute for x, y, and z in (1) the coördinates of each of the three points. Then *three* equations involving A, B, C, and D will be obtained, which may be solved for three of these coefficients in terms of the fourth.

It is convenient to write down the equation of a plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) in the form of a determinant. This is

$$(5) \qquad \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

In fact, when (5) is expanded in terms of the elements of the first row, an equation of the first degree in x, y, and z results. Hence (5) is the equation of a plane. Further, (5) is satisfied when the coördinates of any one of the three given points are substituted for x, y, and z, since then two rows become identical. Hence the plane (5) passes through the given points.

The equation (5) may be used also to determine whether four given points lie in a plane.

If we write (5), when expanded, in the form

$$Ax + By + Cz + D = 0,$$

then the coefficients are the determinants of the third order,

$$A = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \quad B = -\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \quad C = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad D = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

PROBLEMS

Check the answer in each of the following:

1. Find the equation of the plane which passes through the points (2, 3, 0), (– 2, – 3, 4), and (0, 6, 0). *Ans.* $3x + 2y + 6z - 12 = 0.$

2. Find the equation of the plane which passes through the points (1, 1, – 1), (– 2, – 2, 2), and (1, – 1, 2). *Ans.* $x - 3y - 2z = 0.$

3. Find the equation of the plane which passes through the point (3, – 3, 2) and is parallel to the plane $3x - y + z - 6 = 0$.

Ans. $3x - y + z - 14 = 0.$

4. Find the equation of the plane which passes through the points (0, 3, 0) and (4, 0, 0) and is perpendicular to the plane $4x - 6y - z = 12$.
Ans. $3x + 4y - 12z - 12 = 0$.

5. Find the equation of the plane which passes through the point (0, 0, 4) and is perpendicular to each of the planes $2x - 3y = 5$ and $x - 4z = 3$.
Ans. $12x + 8y + 3z - 12 = 0$.

6. Find the equation of the plane whose intercepts on the axes are 3, 5, and 4.
Ans. $20x + 12y + 15z - 60 = 0$.

7. Find the equation of the plane which passes through the point (2, − 1, 6) and is parallel to the plane $x - 2y - 3z + 4 = 0$.
Ans. $x - 2y - 3z + 14 = 0$.

8. Find the equation of the plane which passes through the points (2, − 1, 6) and (1, − 2, 4) and is perpendicular to the plane $x - 2y - 2z + 9 = 0$.
Ans. $2x + 4y - 3z + 18 = 0$.

9. Find the equation of the plane whose intercepts are − 1, − 1, and 4.
Ans. $4x + 4y - z + 4 = 0$.

10. Find the equation of the plane which passes through the point (4, − 2, 0) and is perpendicular to the planes $x + y - z = 0$ and $2x - 4y + z = 5$.
Ans. $x + y + 2z - 2 = 0$.

11. Show that the four points (2, − 3, 4), (1, 0, 2), (2, − 1, 2), and (1, − 1, 3) lie in a plane.

12. Show that the four points (1, 0, − 1), (3, 4, − 3), (8, − 2, 6), and (2, 2, − 2) lie in a plane.

13. Find the equation of the plane which is perpendicular to the line joining (3, 4, − 1) and (5, 2, 7) at its middle point.
Ans. $x - y + 4z - 13 = 0$.

14. Find the equations of the faces of the tetrahedron whose vertices are the points (0, 3, 1), (2, − 7, 1), (0, 5, − 4), and (2, 0, 1).
Ans. $25x + 5y + 2z = 17$, $5x - 2z = 8$, $z = 1$, $15x + 10y + 4z = 34$.

15. The equations of three faces of a parallelepiped are $x - 4y = 3$, $2x - y + z = 3$, and $3x + y - 2z = 0$, and one vertex is the point (3, 7, − 2). What are the equations of the other three faces?
Ans. $x - 4y + 25 = 0$, $2x - y + z + 3 = 0$, $3x + y - 2z = 20$.

16. Find the equation of the plane whose intercepts are a, b, c.
Ans. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

17. What are the equations of the traces of the plane in Problem 16? How might these equations have been anticipated from plane analytic geometry?

18. Find the equation of the plane which passes through the point $P_1(x_1, y_1, z_1)$ and is parallel to the plane $A_1x + B_1y + C_1z + D_1 = 0$.

Ans. $A_1(x - x_1) + B_1(y - y_1) + C_1(z - z_1) = 0$.

19. Find the equation of the plane which passes through the origin and $P_1(x_1, y_1, z_1)$ and is perpendicular to the plane $A_1x + B_1y + C_1z + D_1 = 0$.

Ans. $(B_1z_1 - C_1y_1)x + (C_1x_1 - A_1z_1)y + (A_1y_1 - B_1x_1)z = 0$.

103. The equation of a plane in terms of its intercepts.

Theorem. *If a, b, and c are respectively the intercepts of a plane on the axes of X, Y, and Z, then the equation of the plane is*

$$\text{(III)} \qquad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Proof. Let the equation of the required plane be

$$(1) \qquad Ax + By + Cz + D = 0.$$

Then we know three points in the plane, namely

$$(a, 0, 0), \quad (0, b, 0), \quad (0, 0, c).$$

These coördinates must satisfy (1). Hence

$$Aa + D = 0, \quad Bb + D = 0, \quad Cc + D = 0.$$

Whence $\qquad A = -\frac{D}{a}, \quad B = -\frac{D}{b}, \quad C = -\frac{D}{c}.$

Substituting in (1), dividing by $-D$, and transposing, we obtain (III). Q.E.D.

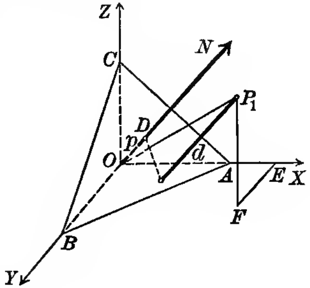
104. The perpendicular distance from a plane to a point. The positive direction on any line perpendicular to a plane is assumed to agree with that on the line drawn through the origin perpendicular to the plane (Art. 100). Hence the distance *from* a plane *to* the point P_1 is *positive* or *negative* according as P_1 and the origin are on opposite sides of the plane or not.

If the plane passes through the origin, the sign of the distance from the plane to P_1 must be determined by the conventions for the special cases in Art. 100.

We now solve the problem: Given the equation of a plane and a point, to find the perpendicular distance from the plane to the point.

Solution. Let the point be $P_1(x_1, y_1, z_1)$ and assume that the equation of the given plane is in the normal form

(1) $x \cos\alpha + y \cos\beta + z \cos\gamma - p = 0.$



Let d equal the required distance. Draw OP_1. Projecting OP_1 on ON, we evidently get $p + d$.

Projecting OE, EF, and FP_1 on ON, we get respectively $x_1 \cos\alpha$, $y_1 \cos\beta$, and $z_1 \cos\gamma$.

Then, by the second theorem of projection,

$$p + d = x_1 \cos\alpha + y_1 \cos\beta + z_1 \cos\gamma.$$

$$\therefore d = x_1 \cos\alpha + y_1 \cos\beta + z_1 \cos\gamma - p.$$

Hence the perpendicular distance d is the number obtained by substituting the coördinates of the given point for x, y, and z in the left-hand member of (1). Whence the

Rule *to find the perpendicular distance d from a given plane to a given point.*

Reduce the equation of the plane to the normal form. Place d equal to the left-hand member of this equation.

Substitute the coördinates of the given point for x, y, and z. The result is the required distance.

For example: To find the perpendicular distance from the plane $2x + y - 2z + 8 = 0$ to the point $(-1, 2, 3)$. Dividing the equation by -3, we have

$$d = \frac{2x + y - 2z + 8}{-3} = \frac{2(-1) + 2 - 2(3) + 8}{-3} = -\tfrac{2}{3}. \quad Ans.$$

Hence the given point is on the same side of the plane as the origin.

The rule gives for the perpendicular distance d from the plane

$$Ax + By + Cz + D = 0$$

to the point (x_1, y_1, z_1) the result

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}, \tag{2}$$

the sign of the radical being determined as above (Art. 101).

105. The angle between two planes. The plane angle of one pair of dihedral angles formed by two intersecting planes is evidently equal to the angle between the positive directions of the perpendiculars to the planes. That angle is called **the angle between the planes.**

Theorem. *The angle θ between the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by*

$$\text{(IV)} \qquad \cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2} \times \pm\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

the signs of the radicals being chosen as in Art. 101.

Proof. By definition the angle θ between the planes is the angle between their normals.

The direction cosines of the normals to the planes are

$$\cos\alpha_1 = \frac{A_1}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}}, \quad \cos\alpha_2 = \frac{A_2}{\pm\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

$$\cos\beta_1 = \frac{B_1}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}}, \quad \cos\beta_2 = \frac{B_2}{\pm\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

$$\cos\gamma_1 = \frac{C_1}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}}, \quad \cos\gamma_2 = \frac{C_2}{\pm\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

By (V), Art. 90, we have

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$$

Substituting the values of the direction cosines of the normals, we obtain (IV). Q.E.D.

PROBLEMS

1. Find the distance from the plane

(a) $6x - 3y + 2z - 10 = 0$ to the point (4, 2, 10). *Ans.* 4.
(b) $x + 2y - 2z - 12 = 0$ to the point (1, − 2, 3). *Ans.* − 7.
(c) $4x + 3y + 12z + 6 = 0$ to the point (9, − 1, 0). *Ans.* − 3.
(d) $2x - 5y + 3z - 4 = 0$ to the point (− 2, 1, 7). *Ans.* $\frac{4}{19}\sqrt{38}$.

2. Do the origin and the point (3, 5, − 2) lie on the same side of the plane $7x - y - 3z + 6 = 0$? *Ans.* Yes.

3. Does the point (1, 6, 0) lie on the same side of the plane $x + 2y - 3z = 6$ as the origin?

4. Find the length of the altitude which is drawn from the first vertex of the tetrahedron whose vertices are (0, 3, 1), (2, − 7, 1), (0, 5, − 4), and (2, 0, 1). *Ans.* $\frac{10}{29}\sqrt{29}$.

5. Find the volume of the tetrahedron formed by the point (1, 2, 1) and the points where the plane $3x + 4y + 2z - 12 = 0$ intersects the coördinate axes.

6. Find the volumes of the tetrahedrons having the following vertices:

(a) (3, 4, 0), (4, − 1, 0), (1, 2, 0), (6, − 1, 4). *Ans.* 8.
(b) (0, 0, 4), (3, 0, 0), (0, 2, 0), (7, 7, 3).
(c) (4, 0, 0), (0, 4, 0), (0, 0, 4), (7, 3, 2).
(d) (3, 0, 0), (0, − 2, 0), (0, 0, − 1), (3, − 1, − 1). *Ans.* 3.
(e) (1, 0, 0), (0, 1, 0), (0, 0, − 2), (4, − 1, 3).
(f) (3, 0, 0), (0, 5, 0), (0, 0, − 1), (3, − 4, 0).

7. Find the angles between the following pairs of planes:

(a) $2x + y - 2z - 9 = 0,\ x - 2y + 2z = 0$. *Ans.* $\cos^{-1}(-\frac{4}{9})$.
(b) $x + y - 4z = 0,\ 3y - 3z + 7 = 0$. *Ans.* $\cos^{-1}\frac{5}{6}$.
(c) $4x + 2y + 4z - 7 = 0,\ 3x - 4y = 0$. *Ans.* $\cos^{-1}(-\frac{2}{15})$.
(d) $2x - y + z = 7,\ x + y + 2z = 11$. *Ans.* $\frac{\pi}{3}$.
(e) $3x - 2y + 6z = 0,\ x + 2y - 2z + 5 = 0$.
(f) $x + 5y - 3z + 8 = 0,\ 2x - 3y + z - 5 = 0$.

8. Show that the angle given by (V) is that angle formed by the planes which does not contain the origin.

9. Find the vertex and the dihedral angles of that trihedral angle formed by the planes $x + y + z = 2$, $x - y - 2z = 4$, and $2x + y - z = 2$ in which the origin lies. *Ans.* (4, − 4, 2), $\cos^{-1}\frac{1}{3}\sqrt{2}$, $\frac{2\pi}{3}$, $\cos^{-1}\left(-\frac{1}{3}\sqrt{2}\right)$.

10. Find the equation of the plane which passes through the points $(0, -1, 0)$ and $(0, 0, -1)$ and which makes an angle of $\frac{2\pi}{3}$ with the plane $y + z = 7$.

Ans. $\pm\sqrt{6}x + y + z + 1 = 0.$

11. Find the locus of points which are equally distant from the planes $2x - y - 2z - 3 = 0$ and $6x - 3y + 2z + 4 = 0$.

Ans. $32x - 16y - 8z - 9 = 0.$

12. Find the locus of a point which is three times as far from the plane $3x - 6y - 2z = 0$ as from the plane $2x - y + 2z = 9$.

Ans. $17x - 13y + 12z - 63 = 0.$

13. Find the equation of the locus of a point whose distance from the plane $x + y + z - 1 = 0$ is equal to its distance from the origin.

14. Find the equation of the locus of a point whose distance from the plane $x + y = 1$ equals its distance from the Z-axis.

Ans. $(x - y)^2 + 2(x + y) - 1 = 0.$

15. Find the equation of the locus of a point, the sum of the squares of whose distances from the planes $x + y - z - 1 = 0$ and $x + y + z + 1 = 0$ is equal to unity.

Ans. $2(x + y)^2 + 2z(z + 2) - 1 = 0.$

106. Systems of planes. The equation of a plane which satisfies *two* conditions will, in general, contain an arbitrary constant, for it takes three conditions to determine a plane. Such an equation therefore represents a *system* of planes.

Systems of planes are used to find the equation of a plane satisfying three conditions in the same manner that systems of lines are used to find the equation of a line (Art. 36).

Three important systems of planes are the following:

The system of planes parallel to a given plane

$$Ax + By + Cz + D = 0$$

is represented by

(V) $$\mathbf{Ax + By + Cz + k = 0,}$$

where k is an arbitrary constant.

The plane (V) is obviously parallel to the given plane (Corollary I, Art. 101).

The system of planes passing through the line of intersection of two given planes

$$A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0$$

is represented by

$$\text{(VI)} \quad \boldsymbol{A_1x + B_1y + C_1z + D_1 + k(A_2x + B_2y + C_2z + D_2) = 0},$$

where k is an arbitrary constant.

Clearly, the coördinates of any point on the line of intersection will satisfy the equations of both of the given planes, and hence will satisfy (VI) also.

The equation of a system of planes which satisfy a single condition must contain two arbitrary constants. One of the most important systems of this sort is the following:

The system of planes passing through a given point $P_1(x_1, y_1, z_1)$ is represented by

$$\text{(VII)} \qquad \boldsymbol{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}.$$

Equation (VII) is the equation of a plane which passes through P_1, for the coördinates of P_1 obviously satisfy it. Again, if any plane whose equation is

$$Ax + By + Cz + D = 0$$

passes through P_1, then

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Subtracting, we get (VII). Hence (VII) represents all planes passing through P_1.

Equation (VII) contains two arbitrary constants, namely, the ratio of any two coefficients to the third.

In the following problems write down the equation of the appropriate system of planes and then determine the unknown parameters from the remaining data.

PROBLEMS

1. Determine the value of k such that the plane $x + ky - 2z - 9 = 0$ shall

(a) pass through the point $(5, -4, -6)$. *Ans.* 2.

(b) be parallel to the plane $6x - 2y - 12z = 7$. *Ans.* $-\frac{1}{3}$.

(c) be perpendicular to the plane $2x - 4y + z = 3$. *Ans.* 0.

(d) be 3 units from the origin. *Ans.* ± 2.

(e) make an angle of $\frac{\pi}{3}$ with the plane $2x - 2y + z = 0$.

Ans. $-\frac{3}{7}\sqrt{35}$.

2. Find the equation of the plane which passes through the point $(3, 2, -1)$ and is parallel to the plane $7x - y + z = 14$.

Ans. $7x - y + z - 18 = 0$.

3. Find the equation of the plane which passes through the intersection of the planes $2x + y - 4 = 0$ and $y + 2z = 0$, and which (a) passes through the point $(2, -1, 1)$; (b) is perpendicular to the plane $3x + 2y - 3z = 6$.

Ans. (a) $x + y + z - 2 = 0$; (b) $2x + 3y + 4z - 4 = 0$.

4. Find the equations of the planes which bisect the angles formed by the planes

(a) $2x - y + 2z = 0$ and $x + 2y - 2z = 6$.

Ans. $3x + y - 6 = 0$, $x - 3y + 4z + 6 = 0$.

(b) $6x - 2y - 3z = 0$ and $4x + 3y - 13z = 10$.

5. Find the equations of the planes passing through the line of intersection of the planes $2x + y - z = 4$ and $x - y + 2z = 0$ which are perpendicular to the coördinate planes.

Ans. $5x + y = 8$, $3x + z = 4$, $3y - 5z = 4$.

6. Find the equation of a plane parallel to the plane $6x - 3y + 2z + 21 = 0$ and tangent to a sphere of unit radius whose center is the origin.

7. Find the equation of a plane parallel to the plane $6x - 2y - 3z + 35 = 0$ and such that the point $(0, -2, -1)$ lies midway between the two planes.

8. Find the equation of a plane through the point $(2, -3, 0)$, and having the same trace on the XZ-plane as the plane $x - 3y + 7z - 2 = 0$.

9. Find the equation of a plane parallel to the plane $2x + y + 2z + 5 = 0$, and forming a tetrahedron of unit volume with the three coördinate planes.

10. Find the equation of a plane parallel to the plane $5x + 3y + z - 7 = 0$ if the sum of its intercepts is 23.

11. Find the equation of a plane parallel to the plane $2x + 6y + 3z - 8 = 0$, upon which the area intercepted by the coördinate planes in the first octant is $\frac{7}{8}$. *Ans.* $2x + 6y + 3z - 3 = 0.$

12. Find the equation of a plane parallel to the plane $2x + y + 2z - 5 = 0$ and such that the entire surface of the tetrahedron which it forms with the coördinate planes is unity. *Ans.* $2x + y + 2z \pm 1 = 0.$

13. Find the equation of a plane having the trace $x + 3y - 2 = 0$ and forming a tetrahedron of volume $\frac{4}{3}$ with the coördinate planes.

Ans. $3x + 9y + z - 6 = 0.$

14. Find the equation of a plane passing through the intersection of the two planes $6x + 2y + 3z - 6 = 0$ and $x + y + z - 1 = 0$ and forming a tetrahedron of unit volume with the coördinate planes.

Ans. $12x - 8y - 3z - 12 = 0.$

15. A point moves so that the volume of the tetrahedron which it forms with the three points (2, 0, 0), (0, 6, 0), and (0, 0, 4) is always equal to 2. Find the equation of its locus.

16. A point moves so that the sum of its distances from the three coördinate planes is unity. Determine the equation of the locus of a second point which bisects the line joining the first with the origin.

17. Find the equation of the plane passing through the intersection of the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ which passes through the origin.

Ans. $(A_1D_2 - A_2D_1)x + (B_1D_2 - B_2D_1)y + (C_1D_2 - C_2D_1)z = 0.$

18. Find the equations of the planes which bisect the angles formed by the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0.$

Ans. $$\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm \frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

19. Find the equations of the planes passing through the intersection of the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ which are perpendicular to the coördinate planes.

Ans. $(A_1B_2 - A_2B_1)y - (C_1A_2 - C_2A_1)z + A_1D_2 - A_2D_1 = 0,$
$(A_1B_2 - A_2B_1)x - (B_1C_2 - B_2C_1)z - (B_1D_2 - B_2D_1) = 0,$
$(C_1A_2 - C_2A_1)x - (B_1C_2 - B_2C_1)y + C_1D_2 - C_2D_1 = 0.$

CHAPTER XVI

THE STRAIGHT LINE IN SPACE

107. General equations of the straight line. A straight line may be regarded as the intersection of any two planes which pass through it. The equations of the planes regarded as simultaneous are the equations of the line of intersection, and hence the

Theorem. *The equations of a straight line are of the first degree in* x, y, *and* z.

Conversely, the locus of two equations of the first degree is a straight line unless the planes which are the loci of the separate equations are parallel. Hence we have the

Theorem. *The locus of two equations of the first degree,*

$$\text{(I)} \qquad \begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

is a straight line unless the coefficients of x, y, *and* z *are proportional.*

To plot a straight line we need to know only the coördinates of two points on the line. The easiest points to obtain are usually those lying in the coördinate planes, which we get by setting one of the variables equal to zero and solving for the other two, as in the following example.

The direction of a line is known when its direction cosines are known. The method of obtaining these will now be illustrated.

EXAMPLES

1. Find the direction cosines of the line whose equations are

(1) $$3x + 2y - z - 1 = 0, \quad 2x - y + 2z - 3 = 0.$$

Solution. Let us find the point where the line pierces the XY-plane. To do this, let $z = 0$ in both equations. Then solving the resulting equations $3x + 2y - 1 = 0$ and $2x - y - 3 = 0$ for x and y, we find the required point is $(1, -1, 0)$. Similarly, putting $y = 0$, the point on the line in the ZX-plane is $(\frac{5}{8}, 0, \frac{7}{8})$.

Hence $A\,(1, -1, 0)$ and $B\,(\frac{5}{8}, 0, \frac{7}{8})$ are two points on the line.

Let the required direction cosines of AB be $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. Then, by the corollary of Art. 89,

(2) $$\frac{\cos\alpha}{1-\frac{5}{8}} = \frac{\cos\beta}{-1-0} = \frac{\cos\gamma}{0-\frac{7}{8}};$$

or, reducing (multiplying the denominators by 8),

(3) $$\frac{\cos\alpha}{3} = \frac{\cos\beta}{-8} = \frac{\cos\gamma}{-7}.$$

The direction cosines may now be found as usual (Art. 88).

A second method is the following:

(4) Assume $$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c}.$$

The coefficients 3, 2, and -1 in the first plane of (1) are proportional to the direction cosines of a perpendicular to that plane. The required line lies in this plane. Hence (corollary, Art. 90)

(5) $$3a + 2b - c = 0.$$

For the same reason, using the second plane in (1),

(6) $$2a - b + 2c = 0.$$

Solving (5) and (6) for the ratios of a, b, and c, the result is

$$8a = -3b, \quad 7a = -3c.$$

(7) $$\therefore \frac{a}{3} = \frac{b}{-8} = \frac{c}{-7}.$$

Combining (7) and (4), we have the previous result (3).

2. Find the direction cosines of the line (I).

Solution. The direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ must satisfy

$$A_1\cos\alpha + B_1\cos\beta + C_1\cos\gamma = 0, \quad A_2\cos\alpha + B_2\cos\beta + C_2\cos\gamma = 0,$$

reasoning as in Ex. 1.

Solving these equations for the ratios, we have the

Theorem. *If α, β, and γ are the direction angles of the line (I), then*

$$\frac{\cos\alpha}{B_1C_2-B_2C_1}=\frac{\cos\beta}{C_1A_2-C_2A_1}=\frac{\cos\gamma}{A_1B_2-A_2B_1}.$$

The denominators are readily remembered as the three determinants of the second order

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix},\quad \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix},\quad \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix},$$

formed from the coefficients of x, y, and z in (1).

PROBLEMS

1. Find the points in which the following lines pierce the coördinate planes, and construct the lines:

(a) $2x+y-z=2$, $x-y+2z=4$.
(b) $4x+3y-6z=12$, $4x-3y=2$.
(c) $x+2y=8$, $2x-4y=7$.
(d) $y+z=4$, $x-y+2z=10$.

2. Find the direction cosines of the following lines:

(a) $2x-y+2z=0$, $x+2y-2z=4$. *Ans.* $\pm\frac{2}{65}\sqrt{65}$, $\mp\frac{6}{65}\sqrt{65}$, $\mp\frac{1}{13}\sqrt{65}$.
(b) $x+y+z=5$, $x-y+z=3$. *Ans.* $\pm\frac{1}{2}\sqrt{2}$, 0, $\mp\frac{1}{2}\sqrt{2}$.
(c) $3x+2y-z=4$, $x-2y-2z=5$. *Ans.* $\pm\frac{6}{25}\sqrt{5}$, $\mp\frac{1}{5}\sqrt{5}$, $\pm\frac{8}{25}\sqrt{5}$.
(d) $x+y-3z=6$, $2x-y+3z=3$. *Ans.* 0, $\pm\frac{3}{10}\sqrt{10}$, $\pm\frac{1}{10}\sqrt{10}$.
(e) $x+y=6$, $2x-3z=5$. *Ans.* $\pm\frac{3}{22}\sqrt{22}$, $\mp\frac{3}{22}\sqrt{22}$, $\pm\frac{1}{11}\sqrt{22}$.
(f) $y+3z=4$, $3y-5z=1$. *Ans.* ±1, 0, 0.
(g) $2x-3y+z=0$, $2x-3y-2z=6$. *Ans.* $\pm\frac{3}{13}\sqrt{13}$, $\pm\frac{2}{13}\sqrt{13}$, 0.
(h) $5x-14z-7=0$, $2x+7z=19$. *Ans.* 0, ±1, 0.

3. Show that the lines of the following pairs are parallel and construct the lines:

(a) $2y+z=0$, $3y-4z=7$; and $5y-2z=8$, $4y+11z=44$.
(b) $x+2y-z=7$, $y+z-2x=6$; and $3x+6y-3z=8$, $2x-y-z=0$.
(c) $3x+z=4$, $y+2z=9$; and $6x-y=7$, $3y+6z=1$.

4. Show that the lines of the following pairs meet in a point and are perpendicular:

(a) $x+2y=1$, $2y-z=1$; and $x-y=1$, $x-2z=3$.
(b) $4x+y-3z+24=0$, $z=5$; and $x+y+3=0$, $x+2=0$.
(c) $3x+y-z=1$, $2x-z=2$; and $2x-y+2z=4$, $x-y+2z=3$.

5. Find the angles between the following lines, assuming that they are directed upward, or in front of the ZX-plane:

(a) $x + y - z = 0,\ y + z = 0$; and $x - y = 1,\ x - 3y + z = 0$. *Ans.* $\frac{\pi}{3}$.

(b) $x + 2y + 2z = 1,\ x - 2z = 1$; and $4x + 3y - z + 1 = 0,\ 2x + 3y = 0$. *Ans.* $\cos^{-1}\frac{16}{21}$.

(c) $x - 2y + z = 2,\ 2y - z = 1$; and $x - 2y + z = 2,\ x - 2y + 2z = 4$. *Ans.* $\cos^{-1}\frac{1}{5}$.

6. Find the equations of the planes through the line

$$x + y - z = 0,\ 2x - y + 3z = 5,$$

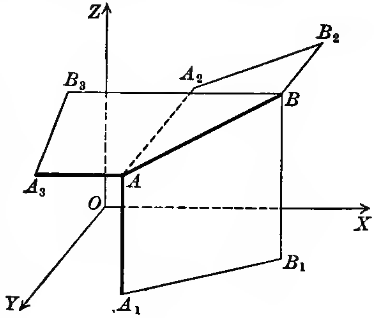
which are perpendicular to the coördinate planes.

Ans. $3x + 2z = 5,\ 3y - 5z + 5 = 0,\ 5x + 2y = 5$.

7. Show analytically that the intersections of the planes $x - 2y - z = 3$ and $2x - 4y - 2z = 5$ with the plane $x + y - 3z = 0$ are parallel lines.

8. Verify analytically that the intersections of any two parallel planes with a third plane are parallel lines.

108. The projecting planes of a line. The three planes passing through a given line and perpendicular to the coördinate planes are called the **projecting planes** of the line.



If the line is perpendicular to one of the coördinate planes, any plane containing the line is perpendicular to that plane. In this case we speak of but two projecting planes, namely, those drawn through the line perpendicular to the other coördinate planes.

If the line is parallel to one of the coördinate planes, two of the projecting planes coincide.

By (VI), Art. 106, the equation of any plane through the line

(1) $$3x + 2y - z - 1 = 0,\quad 2x - y + 2z - 3 = 0$$

has the form

$$3x + 2y - z - 1 + k(2x - y + 2z - 3) = 0.$$

Multiplying out and collecting terms,

(2) $(3+2k)x+(2-k)y+(-1+2k)z-1-3k=0.$

This plane will be perpendicular to the XY-plane when the coefficient of z equals zero, that is, if $k=\frac{1}{2}$. Writing this value of k in (2) and reducing,

(3) $$4x+\tfrac{3}{2}y-\tfrac{5}{2}=0, \quad \text{or } 8x+3y-5=0.$$

This is therefore the equation of the projecting plane of the line (1) on XY, that is, of the plane ABA_1B_1 of the figure.

Now equation (3) is simply the result obtained by *eliminating z from the equations* (1); namely, we multiply the first of equations (1) by 2 and add it to the second. Hence the result:

To find the equations of the projecting planes of a line, eliminate x, y, and z in turn from the given equations.

Thus, to finish the example begun, eliminating y from (1), we find $7x+3z-7=0$ for the projecting plane on XZ. Eliminating x, we get $7y-8z+7=0$ for the equation of the projecting plane on YZ.

Special forms of the projecting planes will indicate special positions of the line relative to the coördinate planes. These cases should be noted in the following problems.

PROBLEMS

1. Find the equations of the projecting planes of the following lines:

(a) $2x+y-z=0$, $x-y+2z=3$.
Ans. $5x+y=3$, $3x+z=3$, $3y-5z+6=0$.

(b) $x+y+z=6$, $x-y-2z=2$.
Ans. $3x+y=14$, $2x-z=8$, $2y+3z=4$.

(c) $2x+y-z=1$, $x-y+z=2$.
Ans. Line parallel to YZ. $x=1$, $y-z+1=0$.

(d) $x+y-4z=1$, $2x+2y+z=0$.
Ans. Line parallel to XY. $9x+9y=1$, $9z+2=0$.

(e) $2y+3z=6$, $2y-3z=18$.
Ans. Line parallel to OX. $y=6$, $z=-2$.

(f) $2x-y+z=0$, $4x+3y+2z=6$. *Ans.* $5y=6$, $10x+5z=6$.

(g) $x+z=1$, $x-z=3$. *Ans.* $x=2$, $z=-1$.

2. Reduce the equations of the following lines to the given answers and construct the lines:

(a) $x + y - 2z = 0,\ x - y + z = 4.$ *Ans.* $x = \frac{1}{2}z + 2,\ y = \frac{3}{2}z - 2.$

(b) $x + 2y - z = 2,\ 2x + 4y + 2z = 5.$ *Ans.* $z = \frac{1}{4},\ y = -\frac{1}{2}x + \frac{9}{8}.$

(c) $x - 2y + z = 4,\ x + 2y - z = 6.$ *Ans.* $x = 5,\ y = \frac{1}{2}z + \frac{1}{2}.$

(d) $x + 3z = 6,\ 2x + 5z = 8.$ *Ans.* $z = 4,\ x = -6.$

(e) $x + 2y - 2z = 2,\ 2x + y - 4z = 1.$ *Ans.* $x = 2z,\ y = 1.$

(f) $x - y + z = 3,\ 3x - 3y + 2z = 6.$ *Ans.* $z = 3,\ y = x.$

3. Find the equations of the line passing through the points $(-2, 2, 1)$ and $(-8, 5, -2)$. *Ans.* $x = 2z - 4,\ y = -z + 3.$

4. Find the equations of the projection of the line $x = z + 2,\ y = 2z - 4$ upon the plane $x + y - z = 0$. *Ans.* $x = \frac{1}{5}z + \frac{14}{5},\ y = \frac{4}{5}z - \frac{14}{5}.$

5. Find the equations of the projection of the line $z = 2,\ y = x - 2$ upon the plane $x - 2y - 3z = 4$. *Ans.* $x = -5z + 4,\ y = -4z.$

6. Show that the equations of a line may be written in one of the forms

$$\begin{cases} y = mx + a, \\ z = nx + b, \end{cases} \qquad \begin{cases} x = a, \\ z = my + b, \end{cases} \qquad \begin{cases} x = a, \\ y = b, \end{cases}$$

according as it pierces the YZ-plane, is parallel to the YZ-plane, or is parallel to the Z-axis.

7. Show that the condition that the line $x = mz + a,\ y = nz + b$ should intersect the line $x = m'z + a',\ y = n'z + b'$ is $\dfrac{a - a'}{m - m'} = \dfrac{b - b'}{n - n'}$.

109. Various forms of the equations of a straight line.

Theorem. ***Parametric form.*** *The coördinates of any point $P(x, y, z)$ on the line through a given point $P_1(x_1, y_1, z_1)$ whose direction angles are α, β, and γ are given by*

$$\text{(II)} \qquad x = x_1 + \rho\cos\alpha, \quad y = y_1 + \rho\cos\beta, \quad z = z_1 + \rho\cos\gamma,$$

where ρ denotes the variable directed length P_1P.

Proof. The projections of P_1P on the axes are respectively

$$x - x_1, \quad y - y_1, \quad z - z_1.$$

But, by the first theorem of projection, these are also equal to

$$\rho\cos\alpha, \quad \rho\cos\beta, \quad \rho\cos\gamma.$$

Hence

$$x - x_1 = \rho\cos\alpha, \quad y - y_1 = \rho\cos\beta, \quad z - z_1 = \rho\cos\gamma.$$

Solving for x, y, and z, we obtain (II). Q. E. D.

Theorem. ***Symmetric form.*** *The equations of the line passing through the point $P_1(x_1, y_1, z_1)$ whose direction angles are α, β, and γ have the form*

$$\text{(III)} \qquad \frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}.$$

To obtain (III), solve each of the equations of (II) for ρ and equate results.

Corollary. *If $\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c}$, then the symmetric equations of the line may be written in the form*

$$\text{(IV)} \qquad \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Theorem. ***Two-point form.*** *The equations of the straight line passing through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are*

$$\text{(V)} \qquad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Proof. The line (III) passes through P_1. If it also passes through P_2, then the coördinates x_2, y_2, and z_2 may be substituted for x, y, and z, and therefore

$$\frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma}.$$

Dividing (III) by this result, we obtain (V). Q. E. D.

Equations (III)–(V) each involve three equations, namely those obtained by neglecting in turn one of the three ratios. These equations are, in different form, the equations of the projecting planes, since one variable is lacking in each. Any two of the three equations are independent and may be used as the equations of the line, but all three are usually retained for the sake of their symmetry. In (IV) and (V), note that the denominators are numbers proportional to the direction cosines of the line.

PROBLEMS

1. Find the equations of the lines which pass through the following pairs of points, reduce them to the given answers, and construct the lines:

(a) (3, 2, − 1), (2, − 3, 4). *Ans.* $x = -\frac{1}{5}z + \frac{14}{5}$, $y = -z + 1$.

(b) (1, 6, 3), (3, 2, 3).* *Ans.* $z = 3$, $y = -2x + 8$.

(c) (1, − 4, 2), (3, 0, 3). *Ans.* $x = 2z - 3$, $y = 4z - 12$.

(d) (2, − 2, − 1), (3, 1, − 1). *Ans.* $z = -1$, $y = 3x - 8$.

(e) (2, 3, 5), (2, − 7, 5). *Ans.* $z = 5$, $x = 2$.

2. Show that the two-point form of the equations of a line becomes $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$, $z = z_1$, if $z_1 = z_2$. What do they become if $y_1 = y_2$? if $x_1 = x_2$?

3. What do the two-point equations of a line become if $x_1 = x_2$ and $y_1 = y_2$? if $y_1 = y_2$ and $z_1 = z_2$? if $z_1 = z_2$ and $x_1 = x_2$?

4. Do the following sets of points lie on straight lines?

(a) (3, 2, − 4), (5, 4, − 6), and (9, 8, − 10). *Ans.* Yes.

(b) (3, 0, 1), (0, − 3, 2), and (6, 3, 0). *Ans.* Yes.

(c) (2, 5, 7), (− 3, 8, 1), and (0, 0, 3). *Ans.* No.

5. Show that the conditions that the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ should lie on a straight line are $\frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1} = \frac{z_3 - z_1}{z_2 - z_1}$.

6. Find the equations of the line passing through the point (2, − 1, − 3) whose direction cosines are proportional to 3, 2, and 7, and reduce them to the given answer. *Ans.* $x = \frac{3}{7}z + \frac{23}{7}$, $y = \frac{2}{7}z - \frac{1}{7}$.

7. Find the equations of the line passing through the point (0, − 3, 2) which is parallel to the line joining the points (3, 4, 7) and (2, 7, 5).

Ans. $\frac{x}{1} = \frac{y + 3}{-3} = \frac{z - 2}{2}$.

8. Show that the lines $\frac{x - 2}{3} = \frac{y + 2}{-2} = \frac{z}{4}$ and $\frac{x + 1}{-3} = \frac{y - 5}{2} = \frac{z + 3}{-4}$ are parallel.

* From (V), $\frac{x-1}{3-1} = \frac{y-6}{2-6} = \frac{z-3}{3-3}$. The value of the last ratio is infinite unless $z - 3 = 0$. If $z - 3 = 0$, then the last ratio may have any value and may be equal to the first two. Hence the equations of the line become $\frac{x-1}{2} = \frac{y-6}{-4}$, $z = 3$. Geometrically it is evident that the two points lie in the plane $z = 3$, and hence the line joining them also lies in that plane.

9. Find the equations of the line through the point $(-2, 4, 0)$ which is parallel to the line $\frac{x}{4} = \frac{y+2}{3} = \frac{z-4}{-1}$, and reduce them to the answer.

Ans. $x = -4z - 2, y = -3z + 4.$

10. Show that the lines $\frac{x+2}{6} = \frac{y-3}{-3} = \frac{z-1}{2}$ and $\frac{x-3}{2} = \frac{y}{6} = \frac{z+3}{3}$ are perpendicular.

11. Find the angle between the lines $\frac{x-3}{2} = \frac{y+1}{1} = \frac{z-3}{-1}$ and $\frac{x+2}{1} = \frac{y-7}{2} = \frac{z}{1}$, if both are directed upward.

Ans. $\frac{2\pi}{3}$.

12. Find the parametric equations of the line passing through the point $(2, -3, 4)$ whose direction cosines are proportional to $1, -2$, and 2.

Ans. $x = 2 + \frac{1}{3}\rho, y = -3 - \frac{2}{3}\rho, z = 4 + \frac{2}{3}\rho.$

13. Construct the lines whose parametric equations are

(a) $x = 2 + \frac{2}{3}\rho, \quad y = 4 - \frac{1}{3}\rho, \quad z = 6 + \frac{2}{3}\rho.$

(b) $x = -3 - \frac{2}{7}\rho, \quad y = 6 - \frac{6}{7}\rho, \quad z = 4 + \frac{3}{7}\rho.$

14. Find the distance, measured along the line $x = 2 - \frac{3}{13}\rho, y = 4 + \frac{12}{13}\rho, z = -3 + \frac{4}{13}\rho$, from the point $(2, 4, -3)$ to the intersection of the line with the plane $4x - y - 2z = 6$.

Ans. $1\frac{5}{8}$.

15. Show that the symmetric equations of the straight line become $\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta}, z = z_1$, if $\cos\gamma = 0$. What do they become if $\cos\alpha = 0$? if $\cos\beta = 0$?

16. Show that the symmetric equations of the straight line become $z = z_1, x = x_1$, if $\cos\gamma = \cos\alpha = 0$. What do they become if $\cos\alpha = \cos\beta = 0$? if $\cos\beta = \cos\gamma = 0$?

17. Reduce the equations of the following lines to the symmetric form (IV).

(a) $x - 2y + z = 8, 2x - 3y = 13.$

Ans. $\frac{x - \frac{13}{2}}{3} = \frac{y}{2} = \frac{z - \frac{3}{2}}{1}.$

Solution. Find the equations of two projecting planes. The second plane is already the projecting plane on XY. Eliminating x, we get $y - 2z = -3$. Now in the two projecting planes thus found,

(1) $$2x - 3y = 13 \quad \text{and} \quad y - 2z = -3,$$

solving each for y and equating results,

(2) $$\frac{2x - 13}{3} = \frac{y}{1} = \frac{2z - 3}{1}.$$

Multiplying the numerators through by $\frac{1}{2}$, we have the answer.

Comparison with (IV) gives $x_1 = \frac{13}{2}$, $y_1 = 0$, $z_1 = \frac{3}{2}$, $a = 3$, $b = 2$, $c = 1$. Hence the line passes through $(\frac{13}{2}, 0, \frac{3}{2})$ and its direction cosines are proportional to 3, 2, 1.

A remark here is important. In (IV), x_1, y_1, and z_1 are the coördinates of any fixed point on the line. Hence for a given line the numerators in (IV) may be quite different. For example, putting $z = 0$ in (1), we find $x = 2$, $y = -3$. Hence the equations $\frac{x-2}{3} = \frac{y+3}{2} = \frac{z}{1}$ represent the given line also. Notice that in equations (IV) the coefficients of x, y, and z *must be unity*. This explains the step, after deriving (1), of removing the 2 from the $2x$ and the $2z$.

(b) $4x - 5y + 3z = 3$, $4x - 5y + z + 9 = 0$. *Ans.* $\frac{x}{5} = \frac{y-3}{4}$, $z = 6$.

(c) $2x + z + 5 = 0$, $x + 3z - 5 = 0$. *Ans.* $z = 3$, $x = -4$.

(d) $x + 2y + 6z = 5$, $3x - 2y - 10z = 7$. *Ans.* $\frac{x-3}{2} = \frac{y-1}{-7} = \frac{z}{2}$.

(e) $3x - y - 2z = 0$, $6x - 3y - 4z + 9 = 0$. *Ans.* $\frac{x-3}{2} = \frac{z}{3}$, $y = 9$.

(f) $3x - 4y = 7$, $x + 3y = 11$. *Ans.* $x = 5$, $y = 2$.

(g) $2x + y + 2z = 7$, $x + 3y + 6z = 11$. *Ans.* $\frac{y-3}{2} = \frac{z}{-1}$, $x = 2$

(h) $2x - 3y + z = 4$, $4x - 6y - z = 5$. *Ans.* $\frac{x}{3} = \frac{y+1}{2}$, $z = 1$.

(i) $3z + y = 1$, $4z - 3y = 10$. *Ans.* $y = -2$, $z = 1$.

(j) $x = mz + a$, $y = nz + b$. *Ans.* $\frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1}$.

18. Find the equations of the line passing through the point (2, 0, −2), which is perpendicular to each of the lines $\frac{x-3}{2} = \frac{y}{1} = \frac{z+1}{2}$ and $\frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2}$. *Ans.* $\frac{x-2}{4} = \frac{y}{2} = \frac{z+2}{-5}$.

19. Find the equations of the line passing through the point (3, −1, 2) which is perpendicular to each of the lines $x = 2z - 1$, $y = z + 3$, and $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$. *Ans.* $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-2}{4}$.

20. Find the equations of the line through $P_1(x_1, y_1, z_1)$ parallel to

(a) $\frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$. *Ans.* $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

(b) $x = mz + a$, $y = nz + b$. *Ans.* $\frac{x-x_1}{m} = \frac{y-y_1}{n} = \frac{z-z_1}{1}$

(c) $z = a$, $y = mx + b$. *Ans.* $\frac{x - x_1}{1} = \frac{y - y_1}{m}$, $z = z_1$.

(d) $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$.

Ans. $$\frac{x - x_1}{B_1C_2 - B_2C_1} = \frac{y - y_1}{C_1A_2 - A_2C_1} = \frac{z - z_1}{A_1B_2 - A_2B_1}.$$

21. Find the equations of the line passing through $P_1(x_1, y_1, z_1)$ which is perpendicular to each of the lines

$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \text{ and } \frac{x - x_3}{a_3} = \frac{y - y_3}{b_3} = \frac{z - z_3}{c_3}.$$

Ans. $$\frac{x - x_1}{b_2c_3 - b_3c_2} = \frac{y - y_1}{c_2a_3 - c_3a_2} = \frac{z - z_1}{a_2b_3 - a_3b_2}.$$

110. Relative positions of a line and plane. If the equations of the line have the form (IV), and if we substitute the values of two of the variables given by (IV) in the equation of the plane, then if the result is true for *all* values of the third variable, the line lies in the plane.

We next easily prove the

Theorem. *A line whose direction angles are* α, β, and γ *and the plane* $Ax + By + Cz + D = 0$ *are*

(a) *parallel when and only when*

$$A \cos \alpha + B \cos \beta + C \cos \gamma = 0;$$

(b) *perpendicular when and only when*

$$\frac{A}{\cos \alpha} = \frac{B}{\cos \beta} = \frac{C}{\cos \gamma}.$$

Proof. The direction cosines of a perpendicular L_2 to the plane are proportional to A, B, and C.

The line and plane are parallel when and only when the line is perpendicular to the line L_2; that is, when and only when

$$A \cos \alpha + B \cos \beta + C \cos \gamma = 0.$$

The line and plane are perpendicular when and only when the line is parallel to L_2; that is, when and only when

$$\frac{\cos \alpha}{A} = \frac{\cos \beta}{B} = \frac{\cos \gamma}{C}.$$

PROBLEMS

1. Show that the line $\frac{x+3}{2} = \frac{y-4}{-7} = \frac{z}{3}$ is parallel to the plane $4x + 2y + 2z = 9$.

2. Show that the line $\frac{x}{3} = \frac{y}{2} = \frac{z}{7}$ is perpendicular to the plane $3x + 2y + 7z = 8$.

3. Show that the line $x = z - 4$, $y = 2z - 3$ lies in the plane $2x - 3y + 4z - 1 = 0$.

4. Find the equations of the line passing through (1, − 6, 2) and perpendicular to the plane $2x - y + 6z = 0$. *Ans.* $\frac{x-1}{2} = \frac{y+6}{-1} = \frac{z-2}{6}$.

5. Show that the lines $x = 2z + 1$, $y = 3z + 2$, and $2x = z + 2$, $3y = 6 - z$ intersect, and find the equation of the plane determined by them. *Ans.* $20x - 9y - 13z - 2 = 0$.

6. Show that the line $\frac{x-2}{3} = \frac{y+2}{-1} = \frac{z-3}{4}$ lies in the plane $2x + 2y - z + 3 = 0$.

7. Find the equations of the line passing through the point (3, 2, − 6) which is perpendicular to the plane $4x - y + 3z = 5$.

Ans. $\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+6}{3}$.

8. Find the equations of the line passing through the point (4, − 6, 2) which is perpendicular to the plane $x + 2y - 3z = 8$.

Ans. $\frac{x-4}{1} = \frac{y+6}{2} = \frac{z-2}{-3}$.

9. Find the equations of the line passing through the point (− 2, 3, 2) which is parallel to each of the planes $3x - y + z = 0$ and $x - z = 0$.

Ans. $\frac{x+2}{1} = \frac{y-3}{4} = \frac{z-2}{1}$.

10. Find the equation of the plane passing through the point (1, 3, − 2) which is perpendicular to the line $\frac{x-3}{2} = \frac{y-4}{5} = \frac{z}{-1}$.

Ans. $2x + 5y - z = 19$.

11. Find the equation of the plane passing through the point (2, − 2, 0) which is perpendicular to the line $z = 3$, $y = 2x - 4$. *Ans.* $x + 2y + 2 = 0$.

12. Find the equation of the plane passing through the line $x + 2z = 4$, $y - z = 8$ which is parallel to the line $\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4}$.

Ans. $x + 10y - 8z - 84 = 0$.

13. Find the equation of the plane passing through the point $(3, 6, -12)$ which is parallel to each of the lines $\frac{x+3}{3} = \frac{y-2}{-1} = \frac{z+1}{3}$ and $\frac{x-4}{2} = \frac{z+2}{4}$, $y = 3$.

Ans. $2x + 3y - z = 36$.

14. Find the equations of the line passing through the point $(3, 1, -2)$ which is perpendicular to the plane $2x - y - 5z = 6$.

Ans. $x = -\frac{2}{5}z + \frac{11}{5}$, $y = \frac{1}{5}z + \frac{7}{5}$.

15. Show that the lines $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z}{-2}$ and $\frac{x-2}{-1} = \frac{y+1}{3} = \frac{z}{2}$ intersect, and find the equation of the plane determined by them.

Ans. $14x - 4y + 13z = 32$.

16. Find the equation of the plane determined by the line $\frac{x-2}{2} = \frac{y+3}{-2} = \frac{z-1}{1}$ and the point $(0, 3, -4)$.

Ans. $x + 2y + 2z + 2 = 0$.

17. Find the equation of the plane determined by the parallel lines $\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1}$ and $\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}$.

Ans. $8x + y - 26z + 6 = 0$.

18. Find the equations of a line lying in the plane $x + 3y - 2z + 4 = 0$ and perpendicular to the line $\frac{x-4}{3} = \frac{y+2}{2} = \frac{z-2}{3}$ at the point where it meets the plane.

19. Find the equations of a line tangent to the sphere $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, -2)$, and parallel to the plane $x + 3y - 5z - 1 = 0$.

20. Find the equations of a line tangent to the sphere $x^2 + y^2 + z^2 = 9$ at the point $(2, 2, -1)$, and perpendicular to the line $\frac{x-2}{3} = \frac{y+1}{1} = \frac{z}{5}$.

21. Find the equations of the line passing through $P_1(x_1, y_1, z_1)$ which is perpendicular to the plane $Ax + By + Cz + D = 0$.

Ans. $\frac{x-x_1}{A} = \frac{y-y_1}{B} = \frac{z-z_1}{C}$.

22. Find the equation of the plane passing through the point $P_1(x_1, y_1, z_1)$ which is perpendicular to the line $\frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$.

Ans. $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

23. Find the angle θ between the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and the plane $Ax+By+Cz+D=0$.

Ans. $\sin\theta=\frac{Aa+Bb+Cc}{\sqrt{A^2+B^2+C^2}\sqrt{a^2+b^2+c^2}}$.

Hint. The angle between a line and a plane is the acute angle between the line and its projection on the plane. This angle equals $\frac{\pi}{2}$ increased or decreased by the angle between the line and the normal to the plane.

24. Find the equation of the plane passing through $P_3(x_3, y_3, z_3)$ which is parallel to each of the lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$.

Ans. $(b_1c_2-b_2c_1)(x-x_3)+(c_1a_2-a_2c_1)(y-y_3)+(a_1b_2-a_2b_1)(z-z_3)=0$.

25. Find the condition that the plane $A_1x+B_1y+C_1z+D_1=0$ should be parallel to the line $A_2x+B_2y+C_2z+D_2=0$, $A_3x+B_3y+C_3z+D_3=0$.

Ans. $A_1(B_2C_3-B_3C_2)+B_1(C_2A_3-C_3A_2)+C_1(A_2B_3-A_3B_2)=0$.

26. Find the equation of the plane determined by the point $P_1(x_1, y_1, z_1)$ and the line $A_1x+B_1y+C_1z+D_1=0$, $A_2x+B_2y+C_2z+D_2=0$.

Ans. $(A_2x_1+B_2y_1+C_2z_1+D_2)(A_1x+B_1y+C_1z+D_1)$
$=(A_1x_1+B_1y_1+C_1z_1+D_1)(A_2x+B_2y+C_2z+D_2)$.

27. Find the equation of the plane determined by the intersecting lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_1}{a_2}=\frac{y-y_1}{b_2}=\frac{z-z_1}{c_2}$.

Ans. $(b_1c_2-b_2c_1)(x-x_1)+(c_1a_2-c_2a_1)(y-y_1)+(a_1b_2-a_2b_1)(z-z_1)=0$.

28. Find the equation of the plane determined by the parallel lines $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ and $\frac{x-x_2}{a}=\frac{y-y_2}{b}=\frac{z-z_2}{c}$.

Ans. $[(y_1-y_2)c-(z_1-z_2)b]x+[(z_1-z_2)a-(x_1-x_2)c]y$
$+[(x_1-x_2)b-(y_1-y_2)a]z+(y_1z_2-y_2z_1)a$
$+(z_1x_2-z_2x_1)b+(x_1y_2-x_2y_1)c=0$.

29. Find the conditions that the line $x=mz+a$, $y=nz+b$ should lie in the plane $Ax+By+Cz+D=0$.

Ans. $Aa+Bb+D=0$, $Am+Bn+C=0$.

30. Find the equation of the plane passing through the line $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ which is parallel to the line $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$.

Ans. $(b_1c_2-b_2c_1)(x-x_1)+(c_1a_2-c_2a_1)(y-y_1)+(a_1b_2-a_2b_1)(z-z_1)=0$.

CHAPTER XVII

SPECIAL SURFACES

111. In this chapter we shall consider spheres, cylinders, and cones* (surfaces considered in elementary geometry), and surfaces which may be generated by revolving a curve about one of the coördinate axes, or by moving a straight line.

112. The sphere. We begin with the

Theorem. *The equation of the sphere whose center is the point* (α, β, γ) *and whose radius is* r *is*

$$(\text{I}) \qquad (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2.$$

Proof. Let $P(x, y, z)$ be any point on the sphere, and denote the center of the sphere by C. Then, by definition, $PC = r$. Substituting the value of PC given by the length formula, and squaring, we obtain (I). Q.E.D.

When (I) is multiplied out, it is

$$x^2+y^2+z^2-2\,\alpha x-2\,\beta y-2\,\gamma z+\alpha^2+\beta^2+\gamma^2-r^2=0;$$

that is, it is in the form

$$x^2+y^2+z^2+Gx+Hy+Iz+K=0.$$

The question now is, *When is the locus of this equation a sphere?*

To answer this, collect the terms thus:

$$(x^2+Gx)+(y^2+Hy)+(z^2+Iz)=-K.$$

* In analytic geometry the terms "sphere," "cylinder," and "cone" are usually used to denote the spherical surface, cylindrical surface, and conical surface of elementary geometry, and not the solids bounded wholly or in part by such surfaces.

Completing the squares within the parentheses, we obtain

$$(x + \tfrac{1}{2} G)^2 + (y + \tfrac{1}{2} H)^2 + (z + \tfrac{1}{2} I)^2 = \tfrac{1}{4}(G^2 + H^2 + I^2 - 4K)$$

Comparing with (I), we have at once the

Theorem. *The locus of an equation of the form*

$$\text{(II)} \qquad x^2 + y^2 + z^2 + Gx + Hy + Iz + K = 0$$

is determined as follows:

(a) *When* $G^2 + H^2 + I^2 - 4K > 0$, *the locus is a sphere whose center is* $\left(-\frac{G}{2}, -\frac{H}{2}, -\frac{I}{2}\right)$ *and whose radius is*

$$r = \tfrac{1}{2}\sqrt{G^2 + H^2 + I^2 - 4K}.$$

(b) *When* $G^2 + H^2 + I^2 - 4K = 0$, *the locus is the point-sphere** $\left(-\frac{G}{2}, -\frac{H}{2}, -\frac{I}{2}\right)$.

(c) *When* $G^2 + H^2 + I^2 - 4K < 0$, *there is no locus.*

In numerical examples it is recommended that the theorem be not used, but that the squares be completed as in the proof, and the center and radius be found by comparison with (I).

EXAMPLE

What is the locus of the equation

$$x^2 + y^2 + z^2 - 2x + 3y + 1 = 0\,?$$

Solution. Collecting terms,

$$(x^2 - 2x) + (y^2 + 3y) + z^2 = -1.$$

Completing the squares,

$$(x^2 - 2x + 1) + (y^2 + 3y + \tfrac{9}{4}) + z^2 = -1 + 1 + \tfrac{9}{4},$$

or

$$(x - 1)^2 + (y + \tfrac{3}{2})^2 + z^2 = \tfrac{9}{4}.$$

This equation is in the form (I); $r = \frac{3}{2}$, $\alpha = 1$, $\beta = -\frac{3}{2}$, $\gamma = 0$. That is, the locus is a sphere of radius $\frac{3}{2}$ and center $(1, -\frac{3}{2}, 0)$.

* That is, a point or sphere of radius zero.

PROBLEMS

1. Find the equation of the sphere whose center is the point

(a) $(\alpha, 0, 0)$ and whose radius is α. *Ans.* $x^2 + y^2 + z^2 - 2\alpha x = 0$.

(b) $(0, \beta, 0)$ and whose radius is β. *Ans.* $x^2 + y^2 + z^2 - 2\beta y = 0$.

(c) $(0, 0, \gamma)$ and whose radius is γ. *Ans.* $x^2 + y^2 + z^2 - 2\gamma z = 0$.

2. Determine the nature of the loci of the following equations and find the center and radius if the locus is a sphere, or the coördinates of the point-sphere if the locus is a point-sphere.

(a) $x^2 + y^2 + z^2 - 6x + 4z = 0$.

(b) $x^2 + y^2 + z^2 + 2x - 4y - 5 = 0$.

(c) $x^2 + y^2 + z^2 + 4x - z + 7 = 0$.

(d) $x^2 + y^2 + z^2 - 12x + 6y + 4z = 0$.

3. Where will the center of (II) lie if

(a) $G = 0$?

(b) $H = 0$?

(c) $I = 0$?

(d) $G = H = 0$?

(e) $H = I = 0$?

(f) $I = G = 0$?

4. Prove that each of the following loci is a sphere, and find its radius and the coördinates of its center.

(a) The distance of a point from the origin is proportional to the square root of the sum of its distances from the three coördinate planes.

(b) The sum of the squares of the distances of a point from two fixed points $(2, 4, -8)$ and $(-4, 0, 2)$ is equal to 52.

Ans. $\alpha = -1, \beta = 2, \gamma = -3, r = \sqrt{14}$.

(c) The distance of a point from the origin is half its distance from the point $(3, -6, 9)$.

(d) The distance of a point from the point $(7, 1, -3)$ is twice its distance from the point $(-\frac{5}{4}, -2, \frac{3}{2})$.

Ans. $\alpha = -4, \beta = -3, \gamma = 1, r = \frac{\sqrt{141}}{2}$.

(e) The sum of the squares of the distances of a point from the three planes $-x + 2y + 2z - 1 = 0$, $2x - y + 2z - 1 = 0$, $2x + 2y - z - 1 = 0$ is unity.

5. Show that a sphere is determined by four conditions and formulate a rule by which to find its equation.

6. Find the equation of a sphere passing through the three points in any one of the following columns and through a fourth point selected from the other two.

$A(-1, -1, 1)$,	$D(0, 0, 1)$,	$G(0, -4, 5)$,
$B(-1, -3, 1)$,	$E(3, 0, 2)$,	$H(2, -4, 5)$,
$C(-1, -4, 4)$;	$F(2, 0, 1)$;	$I(3, -1, 5)$.

Ans. $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$.

7. Find the equation of a sphere which

(a) has the center (3, 0, − 2) and passes through (1, 6, − 5). *Ans.* $x^2 + y^2 + z^2 - 6x + 4z - 36 = 0.$

(b) passes through the points (0, 0, 0), (0, 2, 0), (4, 0, 0), and (0, 0, − 6). *Ans.* $x^2 + y^2 + z^2 - 4x - 2y + 6z = 0.$

(c) is concentric with the sphere $x^2 + y^2 + z^2 - 6x + 4z = 0$ and passes through the point (3, 1, 0).

(d) has the line joining (4, − 6, 5) and (2, 0, 2) as a diameter.

(e) has the center (2, 2, − 2) and is tangent to the plane $2x + y - 3z + 2 = 0.$

(f) has a unit radius and is tangent to each of the coördinate planes in the first octant.

(g) passes through the three points (1, 0, 2), (1, 3, 1), and (−3, 0, 0) and has the center in the XZ-plane. *Ans.* $x^2 + y^2 + z^2 - 2x + 6z - 15 = 0.$

(h) passes through the three points (1, − 3, 4), (1, − 5, 2), and (1, − 3, 0) and has its center in the plane $x + y + z = 0.$ *Ans.* $x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0.$

(i) has its center on the Y-axis and passes through the points (0, 2, 2) and (4, 0, 0). *Ans.* $x^2 + y^2 + z^2 + 4y - 16 = 0.$

(j) passes through the points (1, 5, − 3) and (− 3, 0, 0), and whose center lies on the line of intersection of the planes $3x + y + z = 0$, $x + 2y - 1 = 0.$ *Ans.* $x^2 + y^2 + z^2 - 2x + 6z - 15 = 0.$

(k) is tangent to the three coördinate planes and to the plane $6x + 2y + 3z - 4 = 0.$ *Ans.* $x^2 + y^2 + z^2 - 2x - 2y - 2z - 4 = 0.$

(l) has its center at (3, 1, 1) and is tangent to the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z + 2 = 0.$ *Ans.* $x^2 + y^2 + z^2 - 6x - 2y - 2z + 10 = 0;$ $x^2 + y^2 + z^2 - 6x - 2y - 2z - 14 = 0.$

(m) passes through the points (1, 1, 0), (0, 1, 1), and (1, 0, 1) and whose radius is 11. *Ans.* $x^2 + y^2 + z^2 - 14x - 14y - 14z + 26 = 0.$

(n) is tangent to the plane $x + y - z + 1 = 0$ at the point (3, − 2, 2) and has its center in the XY-plane.

(o) passes through the three points (2, 0, 1), (2, − 1, 0), and (1, − 1, 1) and is tangent to the plane $2x + 2y - z + 2 = 0.$ *Ans.* $x^2 + y^2 + z^2 - 4x + 2y - 2z + 5 = 0.$

(p) passes through the intersection of the two spheres $x^2 + y^2 + z^2 - 6x = 0$, $x^2 + y^2 + z^2 + 9y - 5z - 7 = 0$, and through the point (0, 1, 1).

8. Find the equations of the tangent plane and the normal line to the sphere $x^2 + y^2 + z^2 - 14 = 0$ at the point (3, − 2, 1).

9. Find the equations of the tangent plane and normal line to the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$ at the point (3, − 4, 2).

10. Find the equations of the planes tangent to the sphere $x^2+y^2+z^2-10x+5y-2z-24=0$ at the points where it intersects the coördinate axes.

11. Find the equation of a sphere inscribed in the tetrahedron formed by any four of the following planes:

$$14x+\ 5y-2z-168=0,\qquad 10x+11y+2z+\ 88=0,$$
$$14x-\ 5y+2z+\ 28=0,\qquad 2x-\ y-2z+\ 12=0,$$
$$10x-11y+2z+\ 33=0,\qquad 2x-\ y+2z+\ 8=0.$$

12. Find the equation of the smallest sphere tangent to the two spheres $x^2+y^2+z^2-2x-6y+1=0$, $x^2+y^2+z^2+6x+2y-4z+5=0$.

Ans. $x^2+y^2+z^2+2x-2y-2z+3=0.$

113. Cylinders. A surface which is generated by a straight line which moves parallel to itself and intersects a given fixed curve is called a *cylinder*. The fixed curve is called the **directrix**. We now consider equations whose loci are cylinders.

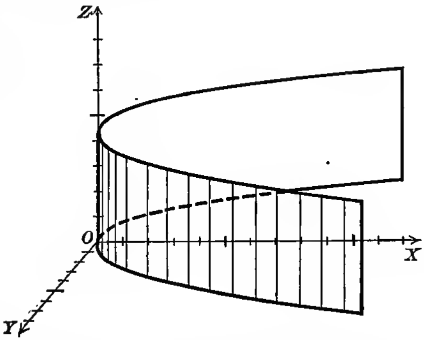
EXAMPLES

1. Determine the nature of the locus of $y^2=4x$.

Solution. The intersection of the surface with a plane $x=k$, parallel to the YZ-plane, is the pair of lines

(1) $$x=k,\quad y=\pm 2\sqrt{k},$$

which are parallel to the Z-axis. If $k>0$, the locus of equations (1) is a pair of lines; if $k=0$, it is a single line (the Z-axis); and if $k<0$, equations (1) have no locus.



Similarly, the intersection with a plane $y=k$, parallel to the ZX-plane, is a straight line whose equations are

$$x=\tfrac{1}{4}k^2,\quad y=k,$$

and which is therefore parallel to the Z-axis.

The intersection with a plane $z=k$ parallel to the XY-plane is the parabola

$$z=k,\quad y^2=4x.$$

For different values of k these parabolas are equal and placed one above another. The surface is therefore a cylinder whose elements are parallel to the Z-axis and intersect the parabola $y^2=4x,\ \ z=0$.

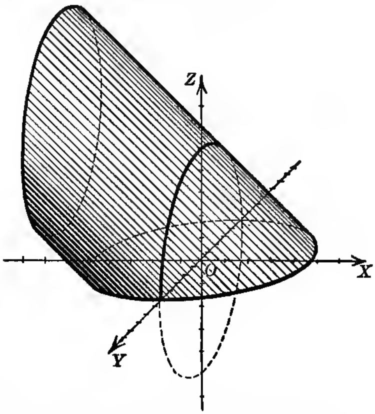
It is evident from Ex. 1 that the locus of any equation which contains but two of the variables x, y, and z will intersect planes parallel to two of the coördinate planes in one or more straight lines parallel to one of the axes, and planes parallel to the third coördinate plane in congruent curves. Such a surface is evidently a cylinder. Hence the

Theorem. *The locus of an equation in which one variable is lacking is a cylinder whose elements are parallel to the axis along which that variable is measured.*

The student should not infer from this statement that the equations of all cylinders have one variable lacking. In case the elements are inclined, all three variables will appear in the equation. This is illustrated by the following example:

2. Determine the nature of the locus of

$$x^2 + 2xz + z^2 = 1 - y^2.$$



The intersection of this locus by the plane $y = k$ is

$$y = k, \quad x + z = \pm\sqrt{1 - k^2},$$

a pair of parallel lines whose direction is independent of k. In fact, the direction cosines of these lines are proportional to $-1, 0, 1$; that is, they are parallel to the line joining the point $(-1, 0, 1)$ to the origin. We conclude then that the surface is a cylinder. To construct the surface, draw its traces and pass lines through them having the above direction. The trace in the YZ-plane is the circle

$$y^2 + z^2 = 1;$$

in the XY-plane, the circle

$$x^2 + y^2 = 1.$$

It is evident from Ex. 2 that in order to prove that a surface is cylindrical it is only necessary to find a system of planes which cut from it a system of parallel lines.

PROBLEMS

1. Determine the nature of the following loci, and discuss and construct them:

(a) $x^2 + y^2 = 36$.
(b) $x^2 + y^2 = 3x$.
(c) $x^2 - z^2 = 16$.
(d) $y^2 + 4z^2 = 0$.
(e) $x^2 + 2y - 4 = 0$.
(f) $z^2 + x^2 = r^2$.
(g) $x^2 + 6y = 0$.
(h) $yz - 4 = 0$.
(i) $y^2 + z - 4 = 0$.
(j) $y^2 - x^3 = 0$.

2. Find the equations of the cylinders whose directrices are the following curves and whose elements are parallel to one of the axes:

(a) $y^2 + z^2 - 4y = 0$, $x = 0$.
(b) $z^2 + 2x = 8$, $y = 0$.
(c) $b^2x^2 - a^2y^2 = a^2b^2$, $z = 0$.
(d) $y^2 + 2pz = 0$, $x = 0$.

3. Prove that the following loci are cylinders. Discuss and construct them.

(a) $x + y - z^2 = 0$.
(b) $xz + yz - 1 = 0$.
(c) $y^2 = 3x + z$.
(d) $x^2 - 4(z + y) + 8 = 0$.
(e) $x^2 + 2xy + y^2 = z$.
(f) $x^2 - 2xy + y^2 = 1 - z^2$.

4. A point moves so that its distance from a fixed point is always equal to its distance from a fixed line. Prove that the locus is a parabolic cylinder.

5. A point moves so that the difference of the squares of its distances from two intersecting perpendicular lines is constant. Prove that the locus is a hyperbolic cylinder.

6. A point moves so that the sum of its distances from two planes is equal to the square of its distance from a third plane. The three planes are mutually perpendicular. Prove that the locus is a parabolic cylinder.

7. A point moves so that the sum of its distances from two planes is equal to the square root of its distance from a third plane. Prove that the locus is a parabolic cylinder when the three planes are mutually perpendicular.

114. The projecting cylinders of a curve. The cylinders whose elements intersect a given curve and are parallel to one of the coördinate axes are called the **projecting cylinders** of the curve. The equations may be found by eliminating in turn each of the variables x, y, and z from the equations of the curve. For if we eliminate z, for example, the result, by the preceding section, is the equation of a cylinder which passes through the curve, since values of x, y, and z which satisfy each of two equations satisfy an equation obtained from them by eliminating one variable.

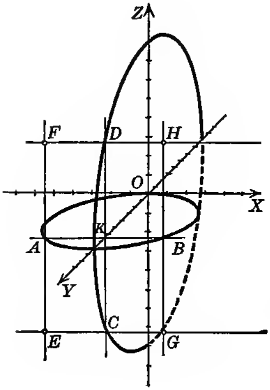
The equations of two of the projecting cylinders may be conveniently used as the equations of the curve.* Hence the problem of constructing the original curve reduces to that of constructing the curve of intersection of two cylinders whose elements are parallel to the coördinate axes. The method is illustrated in the following examples.

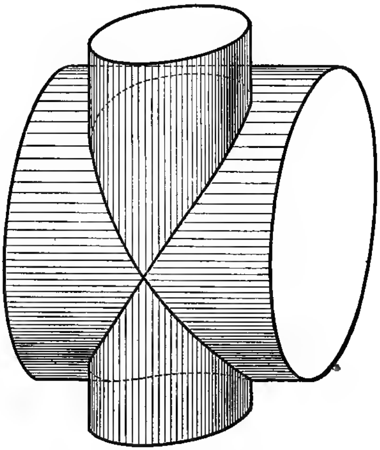
EXAMPLES

1. Construct the curve of intersection of the two cylinders

$$x^2 + y^2 - 2y = 0, \quad y^2 + z^2 - 4 = 0.$$

Solution. Draw the trace of each cylinder on the coördinate plane to which its elements are perpendicular. Then consider a plane perpendicu-





lar to the coördinate axis to which the elements of neither cylinder are parallel. In this case such a plane is $y = k$. Let this plane intersect the

* In general, the equations of a curve may be replaced by any two independent equations to which they are equivalent; that is, by two independent equations which are derived by combining the given equations.

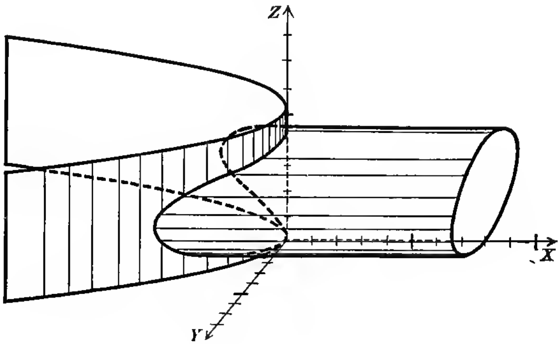
axis at the point K. It will intersect the traces at the points A, B, C, and D. Through each of these points will pass an element of the corresponding cylinder, all four elements lying in this plane. The points of intersection E, F, G, and H of these elements are points on the curve of intersection of the two cylinders. By taking several positions of the plane $y = k$, we obtain a sufficient number of points to construct the entire curve as shown in the second figure on page 298.

2. Construct the curve whose equations are

$$2y^2 + z^2 + 4x = 4z, \quad y^2 + 3z^2 - 8x = 12z.$$

Eliminating x, y, and z in turn, we obtain the equations of the projecting cylinders

$$y^2 + z^2 = 4z, \quad z^2 - 4x = 4z, \quad y^2 + 4x = 0.$$



The figure shows the first and third of these cylinders, intersecting in the original curve constructed by the method explained in the previous example.

It is usually wise to deduce the equations of all three of the projecting cylinders, for it may be that two of them are distinguished for simplicity and hence are most convenient to construct.

If the curve lies in a plane parallel to one of the coördinate planes, then two of its projecting cylinders coincide with the plane of the curve, or part of it.

For a straight line the projecting cylinders are the projecting planes.

PROBLEMS

1. Construct the curve in which the following, in each case a plane and a cylinder, intersect:

(a) $\begin{cases} x^2 + y^2 - 25 = 0, \\ y + z = 0. \end{cases}$

(b) $\begin{cases} y^2 + 4z^2 - 16 = 0, \\ x + z - 1 = 0. \end{cases}$

(c) $\begin{cases} x^2 + y^2 - 4x = 0, \\ x + 2z + 2y - 4 = 0. \end{cases}$

(d) $\begin{cases} x^2 - y^2 - 4 = 0, \\ y + z + x = 0. \end{cases}$

2. Construct the curve in which the following pairs of cylinders intersect:

(a) $\begin{cases} x^2 - 4y = 0, \\ y^2 + 4z = 0. \end{cases}$

(b) $\begin{cases} y^2 + 4z = 0, \\ x^2 + y^2 - 4 = 0. \end{cases}$

(c) $\begin{cases} x^2 - 9z + 36 = 0, \\ x^2 + y^2 - 36 = 0. \end{cases}$

(d) $\begin{cases} y^2 + 4z = 0, \\ x^2 + y^2 - 4y = 0. \end{cases}$

(e) $\begin{cases} x^2 + z^2 - 25 = 0, \\ y^2 - z = 0. \end{cases}$

(f) $\begin{cases} x^2 + y^2 = 25, \\ 5z + y^2 + 10y = 0. \end{cases}$

(g) $\begin{cases} y^2 + z^2 - 36 = 0, \\ x^2 + y^2 - 7y = 0. \end{cases}$

(h) $\begin{cases} y^2 + z^2 - 36 = 0, \\ x^2 + y^2 - 5y = 0. \end{cases}$

(i) $\begin{cases} y^2 + x^2 - 36 = 0, \\ z^2 + y^2 - 6y = 0. \end{cases}$

(j) $\begin{cases} zy = 12, \\ x^2 + y^2 - 7y + 6 = 0. \end{cases}$

3. Find the equations of the projecting cylinders of the following curves and construct the curve as the intersection of two of these cylinders:

(a) $x^2 + y^2 + z^2 = 25,\ x^2 + 4y^2 - z^2 = 0.$

(b) $x^2 + 4y^2 - z^2 = 16,\ 4x^2 + y^2 + z^2 = 16.$

(c) $x^2 + y^2 = 4z,\ x^2 - y^2 = 8z.$

(d) $x^2 + 2y^2 + 4z^2 = 32,\ x^2 + 4y^2 = 4z.$

(e) $y^2 + zx = 0,\ y^2 + 2x + y - z = 0.$

(f) $\begin{cases} x^2 - 10y - 5z - 25 = 0, \\ x^2 + 2y^2 + 5z + 10y - 25 = 0. \end{cases}$

(g) $\begin{cases} x^2 + 2y^2 + 4z - 4 = 0, \\ 2x^2 + 5y^2 + 12z - 8 = 0. \end{cases}$

(h) $\begin{cases} y^2 - x^2 + 2z^2 + 7y - 72 = 0, \\ x^2 - z^2 - 7y + 36 = 0. \end{cases}$

(i) $\begin{cases} 2x^2 + y^2 - 9z = 0, \\ y^2 + 9z - 72 = 0. \end{cases}$

(j) $\begin{cases} 2x^2 + 2y^2 + zy - 14y = 0, \\ x^2 + y^2 + 2zy - 7y - 18 = 0. \end{cases}$

4. A point is two units from the Z-axis and the sum of its distances from the *XY*-plane and the *YZ*-plane is equal to its distance from the *ZX*-plane increased by 2. Construct its locus.

5. A point is equidistant from the Z-axis and the *XY*-plane, and its distance from the origin is equal to its distance from the *YZ*-plane increased by 2. Construct the locus.

115. Parametric equations of curves in space. If the coördinates x, y, and z of a point P in space are functions of a variable parameter, then the locus of P is a curve (compare Art. 80).

For example, if

(1) $$x = \tfrac{1}{4}t^2, \quad y = 1 - 2t, \quad z = 3t^3 + 2,$$

where t is a variable parameter, then the locus of (x, y, z) is a curve in space. This curve may be drawn by assuming values for t, computing x, y, and z, plotting the points, and then joining these points in order by a continuous curve. Equations (1) are called the *parametric equations* of the curve.

The equations of the projecting cylinders of the curve, the locus of (1), result when the parameter t is eliminated from each pair of the equations. Thus, taking the first two,

(2) $$x = \tfrac{1}{4}t^2, \quad y = 1 - 2t,$$

we find from the second, $t = \tfrac{1}{2}(1 - y)$, and substituting in the first,

(3) $$4x = \tfrac{1}{4}(1 - y)^2, \quad \text{or } (y - 1)^2 - 16x = 0,$$

and the locus lies on this parabolic cylinder.

Similarly, eliminating t from the first and third equations of (1),

$$x = \tfrac{1}{4}t^2, \quad z = 3t^3 + 2,$$

we obtain the cubic cylinder

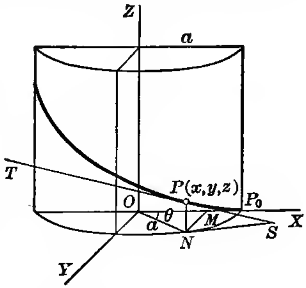
(4) $$(z - 2)^2 = 576x^3.$$

Hence the curve (1) is the curve of intersection of the cylinders (3) and (4).

In some cases it is convenient to find the equations of a curve in space by using a parameter.

EXAMPLE

Equations of the helix. A point moves on a right cylinder in such a manner that the distance it moves parallel to the axis varies directly as the angle it turns through around the axis. Find the equations of the locus.



Solution. Choose the axes of coördinates so that the equation of the cylinder is

(5) $$x^2 + y^2 = a^2,$$

as in the figure.

Let P_0 on OX be one position of the moving point, and P any other position. Then, by definition, the distance NP $(= z)$ varies as the angle XON $(= \theta)$; that is, $z = b\theta$, where b is a constant. Furthermore, from the figure,

$$x = OM = ON\cos\theta = a\cos\theta,$$
$$y = MN = ON\sin\theta = a\sin\theta.$$

Hence the equations of the helix are:

(6) $$x = a\cos\theta, \quad y = a\sin\theta, \quad z = b\theta,$$

where θ is a variable parameter. *Ans.*

Eliminating θ from the first two of equations (6), we obtain (5), as we should.

Given the equations of the projecting cylinders, to find parametric equations for the curve. It was shown in Art. 81 that an indefinite number of parametric equations could be obtained for the same plane curve. The same statement holds for space curves, as illustrated in the following example.

EXAMPLE

Find parametric equations for the curve of intersection of the surfaces (see Example 2, Art. 114),

$$2y^2 + z^2 + 4x = 4z, \quad y^2 + 3z^2 - 8x = 12z.$$

Solution. The projecting cylinders are

(7) $$y^2 + z^2 = 4z, \quad z^2 - 4x = 4z, \quad y^2 + 4x = 0.$$

If we assume $y = 2t$, then the last equation will give $x = -t^2$. From either of the other two cylinders we find

$$z = 2 \pm 2\sqrt{1 - t^2}.$$

Hence the given curve is the locus of

(8) $$x = -t^2, \quad y = 2t, \quad z = 2 \pm 2\sqrt{1 - t^2}.$$

Other parametric equations result when we set one of the coördinates in (7) equal to some other function of a parameter. The aim is, of course, to find *simple* parametric equations. The method adopted must depend upon the given problem.

PROBLEM

Find simple parametric equations for the curves of Problems 2 and 3, p. 300.

Ans. For Problem 2. (a) $x = 2t, \quad y = t^2, \quad z = -\frac{1}{4}t^4.$

(b) $x = 2\cos\theta, \quad y = 2\sin\theta, \quad z = -\sin^2\theta.$

(c) $x = 6\cos\theta, \quad y = 6\sin\theta, \quad z = \frac{1}{4}(1 + \cos^2\theta).$

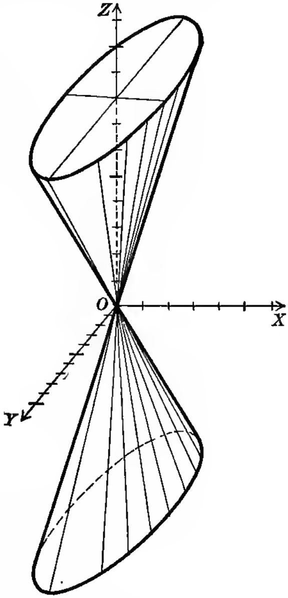
116. Cones. The surface generated by a straight line turning around one of its points and intersecting a fixed curve is called a *cone*.

EXAMPLE

Determine the nature of the locus of the equation $16x^2 + y^2 - z^2 = 0$.

Solution. Let $P_1(x_1, y_1, z_1)$ be a point on a curve C on the surface in which the locus intersects a plane, for example $z = k$. Then

(1) $$16x_1^2 + y_1^2 - z_1^2 = 0, \quad z_1 = k.$$



Now the origin O lies on the surface. We shall show that the line OP_1 lies entirely on the surface. The direction cosines of OP_1 are $\frac{x_1}{\rho_1}$, $\frac{y_1}{\rho_1}$, and $\frac{z_1}{\rho_1}$, where $\rho_1^2 = x_1^2 + y_1^2 + z_1^2 = OP_1^2$. Hence the coördinates of any point on OP_1 are, by (II), Art. 109,

(2) $$x = \frac{x_1}{\rho_1}\rho, \quad y = \frac{y_1}{\rho_1}\rho, \quad z = \frac{z_1}{\rho_1}\rho.$$

Substituting these values of x, y, and z in the left-hand member of the given equation, we obtain

(3) $$16\frac{x_1^2\rho^2}{\rho_1^2} + \frac{y_1^2\rho^2}{\rho_1^2} - \frac{z_1^2\rho^2}{\rho_1^2};$$

or also

(4) $$\frac{\rho^2}{\rho_1^2}(16x_1^2 + y_1^2 - z_1^2).$$

But from the first of equations (1) the expression in the parenthesis in (4) equals zero. Hence the product in (4) also vanishes for *any value of* ρ. This means that every *point* (x, y, z) on the line (2) lies on the surface, that is, the entire line lies on the surface. Hence the surface is a cone whose vertex is the origin.

The essential thing in the above solution is that (4) may be obtained from the first of equations (1) by multiplying by a power of $\frac{\rho}{\rho_1}$. This may be done whenever the equation of the surface is *homogeneous* * in the variables x, y, and z. Hence the

Theorem. *The locus of an equation which is homogeneous in the variables x, y, and z is a cone whose vertex is the origin.*

* An equation is homogeneous in x, y, and z when all the terms in the equation are of the same degree.

To construct the locus of the equation of a cone, find the intersection of the cone with a suitably chosen plane parallel to one of the coördinate planes, construct this plane curve, and then draw the elements from the points on this curve to the vertex of the cone.

Thus in the figure for the preceding example, the cone is cut by the plane $z = 8$, and the curve of intersection, namely the ellipse $16x^2 + y^2 - 64 = 0$, is drawn in this plane.

PROBLEMS

1. Determine the nature of the following loci, and discuss and construct them:

(a) $x^2 - y^2 + 36z^2 = 0$.
(b) $y^2 - 16x^2 + 4z^2 = 0$.
(c) $x^2 + y^2 - 2zx = 0$.
(d) $x + y + z = 0$.
(e) $x^2 + 9y^2 - 4z^2 = 0$.
(f) $x^2 + yz = 0$.
(g) $xy + yz + zx = 0$.
(h) $x^2 + yz + xz = 0$.

2. Discuss the following loci:

(a) $x^2 + y^2 = z^2 \tan^2 \gamma$. (b) $y^2 + z^2 = x^2 \tan^2 \alpha$. (c) $z^2 + x^2 = y^2 \tan^2 \beta$.

3. Find the equation of the cone whose vertex is the origin and whose elements cut the circle $x^2 + y^2 = 16, z = 2$. *Ans.* $x^2 + y^2 - 4z^2 = 0$.

4. A point is equidistant from a plane and a line perpendicular to the plane. Prove that the locus is a cone.

5. A point moves so that the ratio of its distances from two lines intersecting at right angles is constant. Prove that the locus is a cone. What is the nature of the locus when the ratio is unity?

6. The sum of the distances of a point from three mutually perpendicular planes is equal to its distance from their common point of intersection. Show that the locus is a cone.

117. Surfaces of revolution. The surface generated by revolving a curve about a line lying in its plane is called a **surface of revolution.**

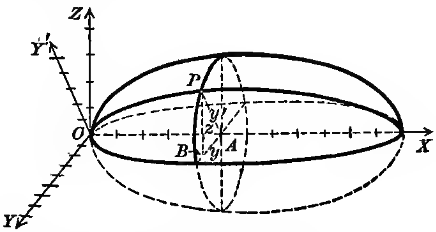
Familiar examples are afforded by the sphere, and the right cylinder and cone.

EXAMPLE

Find the equation of the surface of revolution generated by revolving the ellipse $x^2 + 4y^2 - 12x = 0$, $z = 0$, about the X-axis.

Solution. Let P (x, y, z) be any point on the surface. Pass a plane through P and OX which cuts the surface along one position of the ellipse, and in this plane draw OY' perpendicular to OX. Referred to OX and OY' as axes, the equation of the ellipse is evidently

(1) $x^2 + 4y'^2 - 12x = 0.$



But from the right triangle PAB we get

$$y'^2 = y^2 + z^2.$$

Substituting in (1),

(2) $x^2 + 4y^2 + 4z^2 - 12x = 0.$

This equation expresses the relation which any point on the surface must satisfy, and it is therefore the equation of the surface.

The method of the solution enables us to state the

Rule *to find the equation of the surface generated by revolving a curve in one of the coördinate planes about one of the axes in that plane.*

Substitute in the equation of the curve the square root of the sum of the squares of the two variables not measured along the axis of revolution for that one of these two variables which occurs in the equation of the curve.

The line about which the given curve is revolved is called the **axis** of the surface. Sections of the surface by planes perpendicular to its axis are obviously circles whose centers lie on the axis.

If the sections of a surface by all planes perpendicular to one of the coördinate axes are circles whose centers lie on that axis, then the surface is evidently a surface of revolution whose axis is this coördinate axis. This enables us to determine whether or not a given surface is a surface of revolution whose axis is one of the coördinate axes.

PROBLEMS

1. Find the equations of the surfaces of revolution generated by revolving each of the following curves about the axis indicated, and construct the figures:

(a) $y^2 = 4x - 16$, X-axis. *Ans.* $y^2 + z^2 = 4x - 16$.
(b) $x^2 + 4y^2 = 16$, Y-axis. *Ans.* $x^2 + 4y^2 + z^2 = 16$.
(c) $x^2 = 4z$, Z-axis. *Ans.* $x^2 + y^2 = 4z$.
(d) $x^2 - y^2 = 16$, Y-axis. *Ans.* $x^2 - y^2 + z^2 = 16$.
(e) $x^2 - y^2 = 16$, X-axis. *Ans.* $x^2 - y^2 - z^2 = 16$.
(f) $y^2 + z^2 = 25$, Z-axis. *Ans.* $x^2 + y^2 + z^2 = 25$.
(g) $y^2 = 2pz$, Z-axis. *Ans.* A **paraboloid of revolution**, $x^2 + y^2 = 2pz$.
(h) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, X-axis. *Ans.* An **ellipsoid of revolution**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$.
(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, Y-axis.
Ans. A **hyperboloid of revolution of one sheet**, $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$.
(j) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, X-axis.
Ans. A **hyperboloid of revolution of two sheets**, $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1$.

2. Find the equations of the surfaces of revolution generated by revolving each of the following curves about the axis indicated, and construct the figures:

(a) $x^2 = 4z$; X-axis.
(b) $y^2 = x^3$; X-axis.
(c) $x^2 = z + 4$; X-axis.
(d) $z^2 = x - 3$; Z-axis.
(e) $xz = 4$; X-axis.
(f) $xz = 4$; Z-axis.
(g) $y = x^3 - x$; X-axis.
(h) $z = \sin x$; X-axis.

3. Find the equation of and construct the surface formed by revolving the curve $z = e^x$ about (a) the X-axis; (b) the Z-axis.

4. Verify analytically that a sphere is generated by revolving a circle about a diameter.

5. Find the equation of the surface of revolution generated by revolving the circle $x^2 + y^2 - 2\alpha x + \alpha^2 - r^2 = 0$ about the Y-axis. Discuss the surface when $\alpha > r$, $\alpha = r$, and $\alpha < r$.

Ans. $(x^2 + y^2 + z^2 + \alpha^2 - r^2)^2 = 4\alpha^2(x^2 + z^2)$. When $\alpha > r$ the surface is called an **anchor ring** or **torus**.

6. Find the equations of the cylinders of revolution whose axes are the coördinate axes and whose radii equal r.

Ans. $y^2 + z^2 = r^2$; $z^2 + x^2 = r^2$; $x^2 + y^2 = r^2$.

7. Find the equations of the cones of revolution whose axes are the coördinate axes and whose elements make an angle of ϕ with the axis of revolution. *Ans.* $y^2+z^2=x^2\tan^2\phi$; $z^2+x^2=y^2\tan^2\phi$; $x^2+y^2=z^2\tan^2\phi$.

8. Show that the following loci are surfaces of revolution:

(a) $y^2+z^2=4x$.

(b) $x^2-4y^2+z^2=0$.

(c) $4x^2+4y^2-z^2=16$.

(d) $x^2-4y^2+z^2-3y=0$.

(e) $xz^2+xy^2=3$.

(f) $(x^2+z^2)y=4a^2(2a-y)$.

(g) $x^2+y^2+zx^2+zy^2-z+3=0$.

(h) $x^4-y^4+z^4+2x^2z^2=1$.

(i) $x^2+y^2+z^3-2y+1=0$.

9. A point moves so that its distance from a fixed plane is in a constant ratio to its distance from a fixed point. Show that the locus is a surface of revolution.

10. A point moves so that its distance from a fixed line is in a constant ratio to its distance from a fixed point on that line. Prove analytically that the locus is a cone of revolution. What values of the ratio are excluded?

118. Ruled surfaces. A surface generated by a moving straight line is called a **ruled surface.** If the equations of a straight line involve an arbitrary constant, then the equations represent a system of lines which form a ruled surface. If we eliminate the parameter from the equations of the line, the result will be the equation of the ruled surface.

For if (x_1, y_1, z_1) satisfy the given equations for some value of the parameter, they will satisfy the equation obtained by eliminating the parameter; that is, the coördinates of every point on every line of the system satisfy that equation.

Cylinders and cones are the simplest ruled surfaces.

EXAMPLES

1. Find the equation of the surface generated by the line whose equations are

$$x+y=kz, \quad x-y=\frac{1}{k}z.$$

Solution. We may eliminate k from these equations of the line by multiplying them. This gives

$$(1) \qquad x^2-y^2=z^2.$$

This is the equation of a cone (Art. 116) whose vertex is the origin. As the sections made by the planes $x=k$ are circles, it is a cone of revolution whose axis is the X-axis.

We may verify that the given line lies on the surface (1) for all values of k as follows:

Solving the equations of the line for x and y in terms of z, we get

$$x = \frac{1}{2}\left(k + \frac{1}{k}\right)z, \quad y = \frac{1}{2}\left(k - \frac{1}{k}\right)z.$$

Substituting in (1),

$$\frac{1}{4}\left(k + \frac{1}{k}\right)^2 z^2 - \frac{1}{4}\left(k - \frac{1}{k}\right)^2 z^2 = z^2,$$

an equation which is true for all values of k and z, as is seen by removing the parentheses. Hence every point on any line of the system lies on (1), since its coördinates satisfy (1).

2. Determine the nature of the surface $z^3 - 3zx + 8y = 0$.

Solution. The intersection of the surface with the plane $z = k$ is the straight line

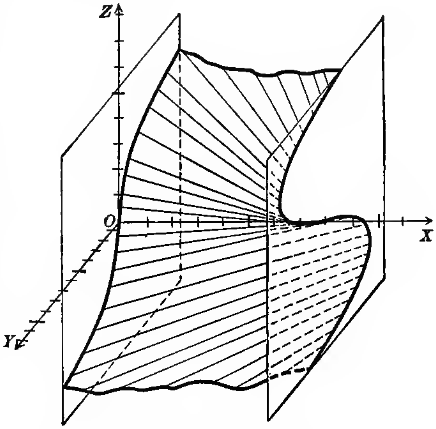
$$k^3 - 3kx + 8y = 0, \quad z = k.$$

Hence the surface is the ruled surface generated by this line as k varies. To construct the surface consider the intersections with the planes $x = 0$ and $x = 8$. Their equations are respectively

$$x = 0, \quad 8y + z^3 = 0;$$

and

$$x = 8, \quad 8y - 24z + z^3 = 0.$$



Joining the points on these curves which have the same value of z gives the lines generating the surface.

The method used in Ex. 2 is adapted to the determination and construction of ruled surfaces. An examination of the equation of such a surface will suggest a system of planes whose intersections with the surface are a system of lines, as illustrated in Problem 2 on the following page.

PROBLEMS

1. Show that the following loci are ruled surfaces whose generators are parallel to one of the coördinate planes. Construct and discuss the loci:

(a) $z - xy = 0.$
(b) $x^2y - z^2 = 0.$
(c) $z^2 - zx + y = 0.$
(d) $x^2y + xz = y.$
(e) $y - xz^2 = 0.$
(f) $y^2 = x^2z.$
(g) $y = xz(2-z)^2.$
(h) $y^2 = x^2(z^2+1).$
(i) $y^2 = x^2(z^2-1).$
(j) $y^2 = x^2(1-z^2).$

Remark. The surfaces may be easily constructed from string and cardboard.

2. Show that the following loci are ruled surfaces:

(a) $(x + y)z + (x + y)^2 - 1 = 0.$
(b) $x^2 - 2xz - y^2 + z^2 = 3.$
(c) $y^2 + 4z^2 + xy - 4yz - 2xz + 3 = 0.$
(d) $x^3 + 3yx^2 - xz^2 - 3yz^2 - x^2 + z^2 = 0.$
(e) $x^2 - y^2 = z.$
(f) $x^2 - y^2 = z^2 - 1.$

Hint. Find a system of planes which cut the surface in a system of straight lines.

3. Find the equations of the ruled surfaces whose generators are the following systems of lines, and discuss the surfaces:

(a) $x + y = k,\ k(x - y) = a^2.$ *Ans.* $x^2 - y^2 = a^2.$
(b) $4x - 2y = kz,\ k(4x + 2y) = z.$ *Ans.* $16x^2 - 4y^2 = z^2.$
(c) $x - 2y = 4kz,\ k(x - 2y) = 4.$ *Ans.* $x^2 - 4y^2 = 16z.$
(d) $x + ky + 4z = 4k,\ kx - y - 4kz = 4.$ *Ans.* $x^2 + y^2 - 16z^2 = 16.$
(e) $x - y - kz = 0,\ x - z - ky = 0.$
(f) $3x - z - k = 0,\ ky - z = 0.$

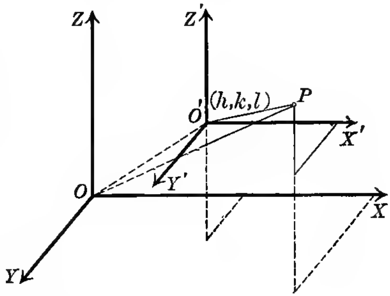
4. Given two planes, one with a variable intercept on the X-axis, the other with a variable intercept on the Y-axis. The remaining intercepts being unity, find the equation of the ruled surface generated by the line of intersection of these planes

(a) when their variable intercepts are in the ratio 1 : 2.
(b) when their distances from the origin are in the ratio 1 : 3.
Ans. $[y(z + y)]^2 - [3x(z + x)]^2 = (4xy)^2.$
(c) when the sum of their distances from the origin is unity.

CHAPTER XVIII

TRANSFORMATION OF COÖRDINATES. DIFFERENT SYSTEMS OF COÖRDINATES

119. Translation of the axes. Formulas applicable to space, entirely analogous to those established in Chapter IX for the plane, are derived as explained below.



Theorem. *The equations for translating the axes to a new origin O' (h, k, l) are*

(I) $$x = x' + h,$$
$$y = y' + k,$$
$$z = z' + l.$$

Proof. Let the coördinates of any point before and after the translation of the axes be (x, y, z) and (x', y', z') respectively. Projecting OP and $OO'P$ on each of the axes, we get equations (I). Q.E.D.

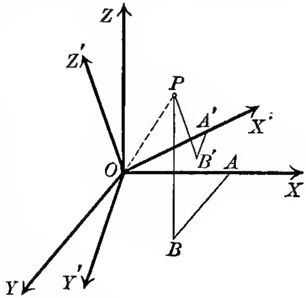
120. Rotation of the axes. Simple formulas for rotation arise if two of the axes are rotated about the third. For example, when the axes OX and OY are turned through an angle θ about the Z-axis, the z-coördinate of any point P does not change, and the new x- and y-coördinates are given by formulas (II), Art. 55. Hence the

Theorem. *The equations for rotating the axes about the Z-axis through an angle θ are*

(II) $$x = x' \cos\theta - y' \sin\theta, \quad y = x' \sin\theta + y' \cos\theta, \quad z = z'.$$

Similar formulas result when the axes are rotated about OY or OX.

If the axes are rotated about the origin into the new position $O\text{-}X'Y'Z'$, and if the coördinates of any point P before and after the rotation are respectively (x, y, z) and (x', y', z'), we have the



Theorem. *If* $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; *and* $\alpha_3, \beta_3, \gamma_3$, *are respectively the direction angles of the three mutually perpendicular lines* OX', OY', *and* OZ', *then the equations for rotating the axes to the position* $O\text{-}X'Y'Z'$ *are*

$$\text{(III)} \qquad \begin{cases} x = x' \cos \alpha_1 + y' \cos \alpha_2 + z' \cos \alpha_3, \\ y = x' \cos \beta_1 + y' \cos \beta_2 + z' \cos \beta_3, \\ z = x' \cos \gamma_1 + y' \cos \gamma_2 + z' \cos \gamma_3.* \end{cases}$$

Proof. Projecting OP and $OA'B'P$ on each of the axes OX, OY, and OZ, we obtain immediately equations (III). Q.E.D.

Theorem. *The degree of an equation is unchanged by a transformation of coördinates.*

This may be shown by reasoning as in Art. 57.

PROBLEMS

1. Transform the equation $x^2 + y^2 - 4x + 2y - 4z + 1 = 0$ by translating the origin to the point $(2, -1, -1)$. *Ans.* $x^2 + y^2 - 4z = 0$.

2. Derive the equations for rotating the axes through an angle θ about (a) the X-axis; (b) the Y-axis.

* The direction cosines of OX', OY', and OZ' obviously satisfy the six equations

$$\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1 = 1, \quad \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0,$$
$$\cos^2\alpha_2 + \cos^2\beta_2 + \cos^2\gamma_2 = 1, \quad \cos\alpha_2\cos\alpha_3 + \cos\beta_2\cos\beta_3 + \cos\gamma_2\cos\gamma_3 = 0,$$
$$\cos^2\alpha_3 + \cos^2\beta_3 + \cos^2\gamma_3 = 1, \quad \cos\alpha_3\cos\alpha_1 + \cos\beta_3\cos\beta_1 + \cos\gamma_3\cos\gamma_1 = 0.$$

Hence only *three* of the nine constants in (III) are independent.

3. Show that the following equations may be transformed into the given answers by translating the axes, or by rotating them about one of the coördinate axes (see Art. 71):

(a) $x^2 + y^2 - z^2 - 6x - 8y + 10z = 0.$ *Ans.* $x^2 + y^2 - z^2 = 0.$

(b) $3x^2 - 8xy + 3y^2 - 5z^2 + 5 = 0.$ *Ans.* $x^2 - 7y^2 + 5z^2 = 5.$

(c) $y^2 + 4z^2 - 16x - 6y + 16z + 9 = 0.$ *Ans.* $y^2 + 4z^2 = 16x.$

(d) $2x^2 - 5y^2 - 5z^2 - 6yz = 0.$ *Ans.* $x^2 - 4y^2 - z^2 = 0.$

(e) $9x^2 - 25y^2 + 16z^2 - 24zx - 80x - 60z = 0.$ *Ans.* $x^2 - y^2 = 4z.$

4. Show that $Ax + By + Cz + D = 0$ may be reduced to the form $x = 0$ by a transformation of coördinates.

Hint. Remove the constant term by translating the axes, then remove the z-term by rotating the axes about the Y-axis, and finally remove the y-term by rotating about the Z-axis.

5. Transform the equation $5x^2 + 8y^2 + 5z^2 - 4yz + 8zx + 4xy - 4x + 2y + 4z = 0$ by rotating the axes to a position in which their direction cosines are respectively $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$; $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$; $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$.

Ans. $3x^2 + 3y^2 = 2z.$

6. Show that the xy-term may always be removed from the equation $Ax^2 + By^2 + Cz^2 + Fxy + K = 0$ by a rotation about the Z-axis.

7. Show that the yz-term may always be removed from the equation $Ax^2 + By^2 + Cz^2 + Dyz + K = 0$ by rotating about the X-axis.

8. What are the direction cosines of OX, OY, and OZ (Fig., p. 311) referred to OX', OY', and OZ'? What six equations do they satisfy?

9. Show that the six equations obtained in Problem 8 are equivalent to the six equations in the footnote, p. 311.

10. If (x, y, z) and (x', y', z') are respectively the coördinates of a point before and after a rotation of the axes, show that

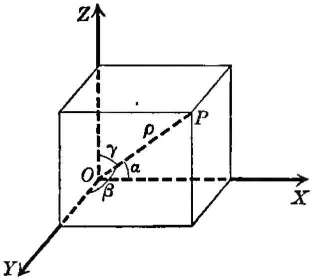
$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2.$$

11. The possibilities of simplifying an equation by rotation of the axes appear in the following example. Consider the equation of the second degree

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy + Gx + Hy + Iz + K = 0.$$

If the axes are rotated about OZ through the angle θ given by $\tan 2\theta = \frac{F}{A - B}$, the transformed equation will contain no xy-term (Art. 70). We may then rotate about OX and remove the yz-term, and finally about OY and remove the xz-term. Thus the terms in xy, yz, and zx can be made to disappear.

121. Polar coördinates. The line OP drawn from the origin to any point P is called the **radius vector** of P. Any point P determines *four* numbers, its radius vector ρ, and the direction angles of OP, namely α, β, and γ, which are called the **polar coördinates** of P.



These numbers are not all independent, since α, β, and γ satisfy (II), Art. 88. If two are known, the third may then be found, but all three are retained for the sake of symmetry.

Conversely, any set of values of ρ, α, β, and γ which satisfy (II), Art. 88, determine a point whose polar coördinates are ρ, α, β, and γ.

Projecting OP on each of the axes, we get the

Theorem. *The equations of transformation from rectangular to polar coördinates are*

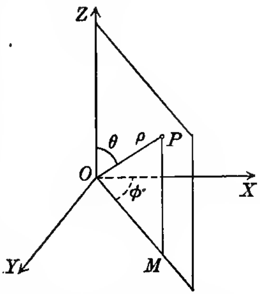
(IV) $$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma.$$

Obviously

(1) $$\rho^2 = x^2 + y^2 + z^2,$$

which expresses the radius vector in terms of x, y, and z.

122. Spherical coördinates. Any point P determines three numbers, namely, its radius vector ρ, the angle θ between the radius vector and the Z-axis, and the angle ϕ between the projection of its radius vector on the XY-plane and the X-axis. These numbers are called the **spherical coördinates** of P. θ is called the **colatitude** and ϕ the **longitude.**



Conversely, given values of ρ, θ, and ϕ determine a point P whose spherical coördinates are (ρ, θ, ϕ).

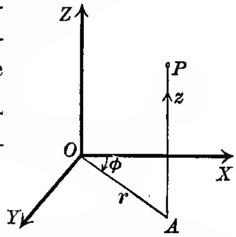
Projecting OP on OA, $OM = \rho \sin \theta$,
and projecting OP and OMP on each of the axes, we prove the

Theorem. *The equations of transformation from rectangular to spherical coördinates are*

$$\text{(V)} \qquad x = \rho \sin \theta \cos \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \theta.$$

The equations of transformation from spherical to rectangular coördinates may be obtained by solving (V) for ρ, θ, and ϕ.

123. Cylindrical coördinates. Any point $P\ (x, y, z)$ determines three numbers, its distance z from the XY-plane and the polar coördinates (r, ϕ) of its projection $(x, y, 0)$ on the XY-plane. These three numbers are called the **cylindrical coördinates** of P. Conversely, given values of r, ϕ, and z determine a point whose cylindrical coördinates are (r, ϕ, z). Then we have at once the



Theorem. *The equations of transformation from rectangular to cylindrical coördinates are*

$$\text{(VI)} \quad x = r \cos \phi, \quad y = r \sin \phi, \quad z = z.$$

The equations of transformation from cylindrical to rectangular coördinates may be obtained by solving (VI) for r, ϕ, and z.

PROBLEMS

1. What is meant by the "locus of an equation" in the polar coördinates ρ, α, β, and γ? in the spherical coördinates ρ, θ, and ϕ? in the cylindrical coördinates r, ϕ, and z?

2. How may the intercepts of a surface on the rectangular axes be found if its equation in polar coördinates is given? if its equation in spherical coördinates is given? if its equation in cylindrical coördinates is given?

3. Transform the following equations into polar coördinates:

(a) $x^2 + y^2 + z^2 = 25.$ *Ans.* $\rho = 5.$

(b) $x^2 + y^2 - z^2 = 0.$ *Ans.* $\gamma = \frac{\pi}{4}.$

(c) $2x^2 - y^2 - z^2 = 0.$ *Ans.* $\alpha = \cos^{-1} \frac{1}{3}\sqrt{3}.$

4. Transform the following equations into spherical coördinates:

(a) $x^2 + y^2 + z^2 = 16$. *Ans.* $\rho = 4$.

(b) $2x + 3y = 0$. *Ans.* $\phi = \tan^{-1}(-\frac{2}{3})$.

(c) $3x^2 + 3y^2 = 7z^2$. *Ans.* $\theta = \tan^{-1}\frac{1}{3}\sqrt{21}$.

5. Transform the following equations into cylindrical coördinates:

(a) $5x - y = 0$. *Ans.* $\phi = \tan^{-1} 5$.

(b) $x^2 + y^2 = 4$. *Ans.* $r = 2$.

6. Find the equation in polar coördinates of

(a) a sphere whose center is the pole.

(b) a cone of revolution whose axis is one of the coördinate axes.

Ans. (a) $\rho =$ constant; (b) $\alpha =$ constant, $\beta =$ constant, or $\gamma =$ constant.

7. Find the equation in spherical coördinates of

(a) a sphere whose center is the origin.

(b) a plane through the Z-axis.

(c) a cone of revolution whose axis is the Z-axis.

Ans. (a) $\rho =$ constant; (b) $\phi =$ constant; (c) $\theta =$ constant.

8. Find the equation in cylindrical coördinates of

(a) a plane parallel to the XY-plane.

(b) a plane through the Z-axis.

(c) a cylinder of revolution whose axis is the Z-axis.

Ans. (a) $z =$ constant; (b) $\phi =$ constant; (c) $r =$ constant.

9. In rectangular coördinates a point is determined as the intersection of three mutually perpendicular planes. Show that

(a) in polar coördinates a point is regarded as the intersection of a sphere and three cones of revolution which have an element in common.

(b) in spherical coördinates a point is regarded as the intersection of a sphere, a plane, and a cone of revolution which are mutually orthogonal.

(c) in cylindrical coördinates a point is regarded as the intersection of two planes and a cylinder of revolution which are mutually orthogonal.

10. Show that the square of the distance r between two points whose polar coördinates are $(\rho_1, \alpha_1, \beta_1, \gamma_1)$ and $(\rho_2, \alpha_2, \beta_2, \gamma^2)$ is

$$r^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2).$$

11. Find the general equation of a plane in polar coördinates.

Ans. $\rho(A\cos\alpha + B\cos\beta + C\cos\gamma) + D = 0$.

12. Find the general equation of a sphere in polar coördinates.

Ans. $\rho^2 + \rho(G\cos\alpha + H\cos\beta + I\cos\gamma) + K = 0$.

CHAPTER XIX

QUADRIC SURFACES AND EQUATIONS OF THE SECOND DEGREE IN THREE VARIABLES

124. Quadric surfaces. The locus of an equation of the second degree in x, y, and z, of which the most general form is

(1) $Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy + Gx + Hy + Iz + K = 0,$

is called a **quadric surface** or **conicoid**. We may learn something of the nature of such a surface by taking cross sections. We first obtain

Theorem I. *The intersection of a quadric with any plane is a conic or a degenerate conic.*

Proof. By a transformation of coördinates any plane may be made the XY-plane, $z = 0$. Referred to any axes the equation of a quadric has the form (1) (Theorem, p. 311). Hence the equation of the curve of intersection referred to axes in its own plane $z = 0$ is

$$Ax^2 + Fxy + By^2 + Gx + Hy + K = 0,$$

and the locus is therefore a conic or a degenerate conic, by Art. 70. Q.E.D.

As already pointed out in Art. 71, the parabola, ellipse, and hyperbola were originally studied as conic sections,—plane sections of a conical surface. From the preceding theorem and by intuition, the truth of the following statement is manifest.

Corollary. *The curve of intersection of a cone of revolution with a plane is an ellipse, hyperbola, or parabola, according as the plane cuts all of the elements, is parallel to two elements* *(cutting the other elements — some on one side of the vertex and some on the other), or is parallel to one element (cutting all the others on the same side of the vertex).*

For sections of a quadric by a set of parallel planes, the following result is important:

Theorem II. *The sections of a quadric with a system of parallel planes are conics of the same species.*

The truth of this statement is established in the following sections. The meaning of the theorem is this: A set of parallel sections will all be ellipses, or all hyperbolas, or all parabolas, the exceptional cases (Art. 70) under each species being included.

125. Simplification of the general equation of the second degree in three variables. If equation (1) be transformed by rotating the axes, it can be shown that the new axes may be so chosen that the terms in yz, zx, and xy will drop out (Problem 11, p. 312). Hence (1) reduces to the form

$$A'x^2 + B'y^2 + C'z^2 + G'x + H'y + I'z + K' = 0.$$

Transforming this equation by translating the axes, it is easy to show that the axes may be so chosen that the transformed equation will have one of the two forms

$$A''x^2 + B''y^2 + C''z^2 + K'' = 0, \tag{1}$$

$$A''x^2 + B''y^2 + I''z = 0. \tag{2}$$

Note the difference in (1) and (2). In (1) all the squares and no first powers are represented, in (2) only two squares and the first power of the other variable.

If all of the coefficients in (1) and (2) are different from zero, they may, with a change in notation, be respectively written in the forms

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1, \tag{3}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 2\,cz. \tag{4}$$

The purpose of the following sections is to discuss the loci of these equations, which are called **central** and **noncentral quadrics** respectively.

If one or more of the coefficients in (1) or (2) are zero, the locus is called a **degenerate quadric.**

Certain cases are readily disposed of by means of former results.

If $K''=0$, the locus of (1) is a *cone* (Theorem, Art. 116) unless the signs of A'', B'', and C'' are the same, in which case the locus is a *point*, namely the origin.

If *one* of the coefficients A'', B'', and C'' is zero, the locus is a *cylinder* whose elements are parallel to one of the axes and whose directrix is a conic of the elliptic or hyperbolic type. If also $K''=0$, the locus will be a *pair of intersecting planes.*

If *two* of the coefficients A'', B'', and C'' are zero, the locus is a *pair of parallel planes* (coincident if $K''=0$), or there is *no locus.*

If *one* of the coefficients in (2) is zero, the locus is a *cylinder* whose directrix is a parabola, or a *pair of intersecting planes.*

If *two* of the coefficients are zero, the locus is a *pair of coincident planes.* (A'' and B'' cannot be zero simultaneously, as the equation would cease to be of the second degree.)

PROBLEMS

1. Construct and discuss the loci of the following equations:

(a) $9x^2 - 36y^2 + 4z^2 = 0.$

(b) $16x^2 - 4y^2 - z^2 = 0.$

(c) $4x^2 + z^2 - 16 = 0.$

(d) $y^2 - 9z^2 + 36 = 0.$

(e) $4y^2 - 25 = 0.$

(f) $3y^2 + 7z^2 = 0.$

(g) $8y^2 + 25z = 0.$

(h) $z^2 + 16 = 0.$

2. Show by transformation of coördinates that the following quadrics are degenerate:

(a) $x^2 - y^2 + z^2 - 6z + 9 = 0.$

(b) $x^2 + 4y^2 - z^2 - 2x + 8y + 5 = 0.$

(c) $x^2 + y^2 + z^2 + 2x - 2y + 4z - 6 = 0.$

(d) $x^2 + y^2 - 2z^2 + 2y + 4z - 1 = 0.$

(e) $x^2 + yz = 0.$

126. The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. If all of the coefficients in (3), Art. 125, are positive, the locus is called an **ellipsoid.** A discussion of its equation gives us the following properties:

1. The ellipsoid is symmetrical with respect to each of the coördinate planes and axes and the origin. These planes of symmetry are called the **principal planes** of the ellipsoid.

2. Its intercepts on the axes are respectively

$$x=\pm a, \quad y=\pm b, \quad z=\pm c.$$

The lines $AA'=2a$, $BB'=2b$, $CC'=2c$, are called the **axes** of the ellipsoid (see figure below).

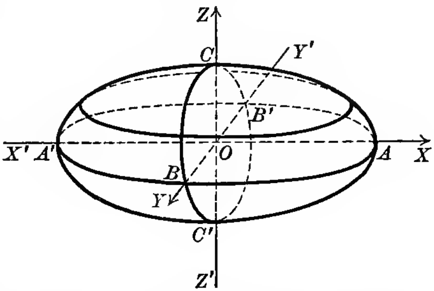
3. Its traces on the principal planes are the ellipses $ABA'B'$, $BCB'C'$, and $ACA'C'$, whose equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{x^2}{a^2}+\frac{z^2}{c^2}=1.$$

4. The equation of the curve in which a plane parallel to the XY-plane, $z=k$, intersects the ellipsoid is

(1) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}, \quad \text{or} \quad \frac{x^2}{\frac{a^2}{c^2}(c^2-k^2)}+\frac{y^2}{\frac{b^2}{c^2}(c^2-k^2)}=1.$$

The locus of this equation is an ellipse. If k increases from 0 to c, or decreases from 0 to $-c$, the plane recedes from the XY-plane, and the axes of the ellipse decrease from $2a$ and $2b$ respectively to 0, when the ellipse degenerates into a point. If $k>c$ or $k<-c$, there is no locus. Hence the ellipsoid lies entirely between the planes $z=\pm c$.



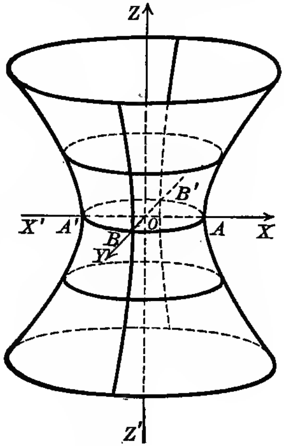
In like manner the sections parallel to the YZ-plane and the ZX-plane are ellipses whose axis decrease as the planes recede. Hence the ellipsoid lies entirely between the planes $x = \pm a$ and $y = \pm b$. The ellipsoid is therefore a closed surface.

If $a = b$, the section (1) is a circle for values of k such that $-c < k < c$, and hence the ellipsoid is now an ellipsoid of revolution whose axis is the Z-axis. If $b = c$ or $c = a$, it is an ellipsoid of revolution whose axis is the X- or Y-axis.

If $a = b = c$, the ellipsoid is a sphere, for its equation may be written in the form $x^2 + y^2 + z^2 = a^2$.

127. The hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. If two of the coefficients in (3), Art. 125, are positive and one is negative, the locus is called a **hyperboloid of one sheet.** Consider first the equation

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$



A discussion of this equation gives us the following properties:

1. The hyperboloid is symmetrical with respect to each of the coördinate planes and axes and the origin.

2. Its intercepts on the X-axis and the Y-axis are respectively

$$x = \pm a, \quad y = \pm b,$$

but it does not meet the Z-axis.

3. Its traces on the coördinate planes are the conics

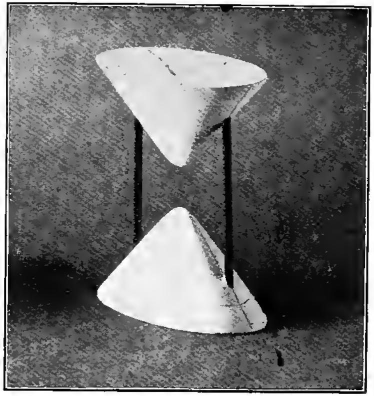
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

of which the first is the ellipse whose axes are $AA' = 2a$ and $BB' = 2b$, and the others are the hyperbolas whose transverse axes are BB' and AA' respectively.

PLATE I


Ellipsoid


Hyperboloid of one sheet


Hyperboloid of two sheets

CENTRAL QUADRICS

4. The equation of the curve in which a plane parallel to the XY-plane, $z = k$, intersects the hyperboloid is

$$(2) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \quad \text{or} \quad \frac{x^2}{\frac{a^2}{c^2}(c^2 + k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2 + k^2)} = 1.$$

The locus of this equation is an ellipse. If k increases from 0 to ∞, or decreases from 0 to $-\infty$, the plane recedes from the XY-plane, and the axes of the ellipse increase indefinitely from $2a$ and $2b$ respectively. Hence the surface recedes indefinitely from the XY-plane and from the Z-axis.

In like manner the sections formed by the planes $x = k'$ and $y = k''$ are seen to be hyperbolas. As k' and k'' increase numerically, the axes of the hyperbolas decrease, and when $k' = \pm a$ or $k'' = \pm b$, the hyperbolas degenerate into intersecting lines. As k' and k'' increase beyond this point, the directions of the transverse and conjugate axes are interchanged, and the lengths of these axes increase indefinitely.

The hyperboloid (1) is said to "lie along the Z-axis."

The equations

$$(3) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

are the equations of hyperboloids of one sheet which lie along the Y-axis and the X-axis respectively.

If $a = b$, the hyperboloid (1) is a surface of revolution whose axis is the Z-axis, because the section (2) becomes a circle. The hyperboloids (3) will be hyperboloids of revolution if $a = c$ and $b = c$ respectively.

128. The hyperboloid of two sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. If only one of the coefficients in (3), Art. 125, is positive, the locus is called a **hyperboloid of two sheets.** Consider first the equation

$$(1) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

1. The hyperboloid is symmetrical with respect to each of the coördinate planes and axes and the origin.

2. Its intercepts on the X-axis are $x = \pm a$, but it does not cut the Y-axis and the Z-axis.

3. Its traces on the XY-plane and the XZ-plane are respectively the hyperbolas

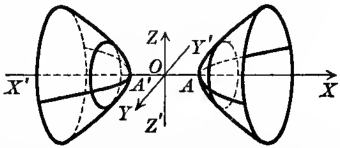
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

which have the same transverse axis $AA' = 2 \cdot a$, but it does not cut the YZ-plane.

4. The equation of the curve in which a plane parallel to the YZ-plane, $x = k$, intersects the hyperboloid (1) is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1, \quad \text{or} \quad \frac{y^2}{\frac{b^2}{a^2}(k^2 - a^2)} + \frac{z^2}{\frac{c^2}{a^2}(k^2 - a^2)} = 1.$$

This equation has no locus if $-a < k < a$. If $k = \pm a$, the locus is a point ellipse, and as k increases from a to ∞, or decreases from $-a$ to $-\infty$, the locus is an ellipse whose axes increase indefinitely. Hence the surface consists of two branches or sheets which recede indefinitely from the YZ-plane and from the X-axis.



In like manner the sections formed by *all* planes parallel to the XY-plane and the ZX-plane are hyperbolas whose axes increase indefinitely as their planes recede from the coördinate planes.

The hyperboloid (1) is said to "lie along the X-axis."

The equations

$$(2) \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

are the equations of hyperboloids of two sheets which lie along the Y-axis and the Z-axis respectively.

If $b = c$, $c = a$, or $a = b$, the hyperboloids (1) and (2) are hyperboloids of revolution.

It should be noticed that *the locus of* (3), *Art. 125, is an ellipsoid if all the terms on the left are positive, a hyperboloid of one sheet if but one term is negative, and a hyperboloid of two sheets if two terms are negative.* If all the terms on the left are negative, there is no locus. If the locus is a hyperboloid, it will lie along the axis corresponding to the term whose sign differs from that of the other two terms.

PROBLEMS

1. Discuss and construct the loci of the following equations:

(a) $4x^2 + 9y^2 + 16z^2 = 144$.

(b) $4x^2 + 9y^2 - 16z^2 = 144$.

(c) $4x^2 - 9y^2 - 16z^2 = 144$.

(d) $x^2 + 16y^2 + z^2 = 64$.

(e) $9x^2 - y^2 + 9z^2 = 36$.

(f) $z^2 - 4x^2 - 4y^2 = 16$.

(g) $16x^2 + y^2 + 16z^2 = 64$.

(h) $x^2 + y^2 - z^2 = 25$.

2. Reduce, by translation of the axes, each of the following to a standard form and determine the type of central quadric it represents:

(a) $x^2 + 2y^2 + 2z^2 - 2x + 4y - 8z + 10 = 0$.

(b) $x^2 - y^2 + 2z^2 - 6x + 2y + 4z + 9 = 0$.

(c) $y^2 - x^2 - 2z^2 + 6x - 2y - 4z + 6 = 0$.

(d) $x^2 - 2y^2 - 4z^2 - 2x - 8y - 8 = 0$.

(e) $4x^2 - y^2 - z^2 - 8x - 2y + 6 = 0$.

(f) $4x^2 - y^2 - z^2 - 8x - 2y + 4 = 0$.

(g) $3x^2 + 4y^2 - 8y - z^2 = 0$.

3. Find the equations of the planes whose intersections with the ellipsoid $9x^2 + 25y^2 + 169z^2 = 1$ are circles. *Ans.* $4x = \pm 12z + k$.

4. The square of the distance of a point from a line is equal to the square of its distance from a perpendicular plane (a) increased by a constant; (b) diminished by a constant. How do the two loci differ? What property have they in common?

5. A point moves so that its distances from a fixed point and a fixed line are in constant ratio μ. Determine and name the locus

(a) when $\mu < 1$.

(b) when $\mu > 1$.

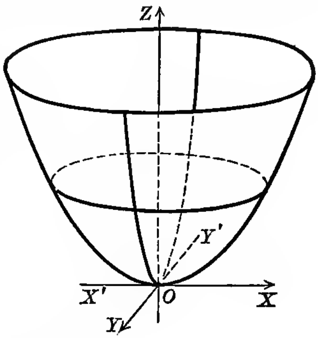
(c) when $\mu = 1$.

(d) when the point is on the line.

6. A point moves so that its distances from a fixed point and a fixed plane are in a constant ratio. Prove that the locus is an ellipsoid of revolution when the ratio is less than unity, and a hyperboloid of revolution when greater than unity.

7. A point moves so that the sum of the squares of its distances from two intersecting perpendicular lines in space is constant. Prove that the locus is an ellipsoid of revolution.

129. The elliptic paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2cz$. If the coefficient of y^2 in (4), Art. 125, is positive, the locus is called an **elliptic paraboloid.** A discussion of its equation gives us the following properties:



1. The elliptic paraboloid is symmetrical with respect to the YZ-plane and the ZX-plane and the Z-axis.

2. It passes through the origin, but does not intersect the axes elsewhere.

3. Its traces on the coördinate planes are respectively the conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0, \quad \frac{x^2}{a^2}=2\,cz, \quad \frac{y^2}{b^2}=2\,cz,$$

of which the first is a point-ellipse and the other two are parabolas.

4. The equation of the curve in which a plane parallel to the XY-plane, $z=k$, cuts the paraboloid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,ck, \quad \text{or} \quad \frac{x^2}{2\,a^2ck}+\frac{y^2}{2\,b^2ck}=1.$$

The curve is an ellipse if c and k have the same sign, but there is no locus if c and k have opposite signs. Hence, if c is positive, the surface lies entirely above the XY-plane. If k increases from 0 to ∞, the plane recedes from the XY-plane and the axes of the ellipse increase indefinitely. Hence the surface recedes indefinitely from the XY-plane and from the Z-axis.

In like manner the sections parallel to the YZ-plane and the ZX-plane are parabolas whose vertices recede from the XY-plane as their planes recede from the coördinate planes.

The paraboloid is said to "lie along the Z-axis."

The loci of the equations

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 2\,ax, \quad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 2\,by, \tag{1}$$

are elliptic paraboloids which lie along the X-axis and the Y-axis respectively.

If $a = b$, the first surface considered is a paraboloid of revolution whose axis is the Z-axis; and if $b = c$ and $a = c$, the paraboloids (1) are surfaces of revolution whose axes are respectively the X-axis and the Y-axis.

An elliptic paraboloid lies along the axis corresponding to the term of the first degree in its equation, and in the positive or negative direction of the axis according as that term is positive or negative.

130. The hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,cz$. If the coefficient of y^2 in (4), Art. 125, is negative, the locus is called a **hyperbolic paraboloid.**

1. The hyperbolic paraboloid is symmetrical with respect to the YZ-plane and the ZX-plane and the Z-axis.

2. It passes through the origin, but does not cut the axes elsewhere.

3. Its traces on the coördinate planes are respectively the conics

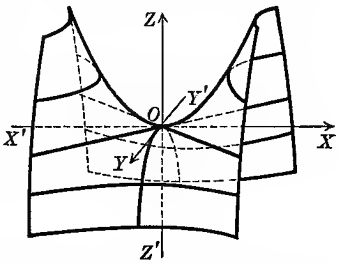
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \frac{x^2}{a^2} = 2\,cz, \quad -\frac{y^2}{b^2} = 2\,cz,$$

of which the first is a pair of intersecting lines and the other two are parabolas.

4. The equation of the curve in which a plane parallel to the XY-plane, $z = k$, cuts the paraboloid is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,ck, \text{ or } \frac{x^2}{2\,a^2ck} - \frac{y^2}{2\,b^2ck} = 1.$$

The locus is a hyperbola. If c is positive, the transverse axis of the hyperbola is parallel to the X- or Y-axis according as k is positive or negative. If k increases from 0 to ∞, or decreases from 0 to $-\infty$, the plane recedes from the XY-plane and the axes of the hyperbolas increase indefinitely. Hence the surface recedes indefinitely from the XY-plane and the Z-axis. The surface has approximately the shape of a saddle.



In like manner the sections parallel to the other coördinate planes are parabolas whose vertices recede from the XY-plane as their planes recede from the coördinate planes.

The surface is said to "lie along the Z-axis."

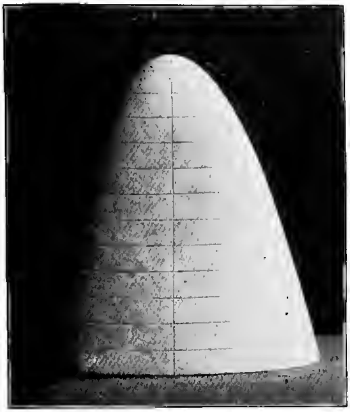
The loci of the equations

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 2\,by, \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 2\,ax,$$

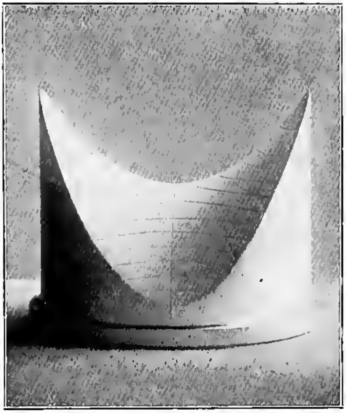
are hyperbolic paraboloids lying along the Y-axis and the X-axis respectively. A hyperbolic paraboloid also lies along the axis which corresponds to the first-degree term in its equation.

A plane of symmetry of a quadric is called a **principal plane.** Each paraboloid has two principal planes; each central quadric, three. Axes of symmetry are called **principal axes.** A paraboloid possesses one such axis; a central quadric, three. The existence of a center of symmetry for a *central* quadric explains the designation "central quadric."

PLATE II



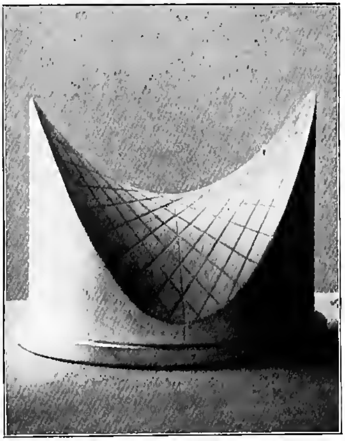
Elliptic Paraboloid



Hyperbolic Paraboloid

NONCENTRAL QUADRICS

Hyperboloid of one sheet



Hyperbolic Paraboloid

RULED QUADRICS

PROBLEMS

1. Discuss and construct the following loci:

(a) $y^2 + z^2 = 4x$.

(b) $y^2 - z^2 = 4x$.

(c) $x^2 - 4z^2 = 16y$.

(d) $x^2 + y^2 = 8z$.

(e) $9z^2 - 4x^2 = 288y$.

(f) $16x^2 + z^2 = 64y$.

(g) $y^2 - x^2 = 10z$.

(h) $y^2 + 16z^2 + x = 0$.

2. Reduce by transformation of coördinates each of the following to a standard form and determine the type of paraboloid it represents:

(a) $z = xy$.

(b) $z = x^2 + xy + y^2$.

(c) $x^2 + 2y^2 - 6x + 4y + 3z + 11 = 0$.

(d) $z^2 - 3y^2 - 4x + 2z - 6y + 1 = 0$.

3. A point is equidistant from a fixed plane and a fixed point. Show that the locus is an elliptic paraboloid of revolution.

4. A point is equidistant from two nonintersecting perpendicular lines. Show that the locus is a hyperbolic paraboloid.

5. Prove that the parabolas obtained by cutting (a) an elliptic paraboloid, and (b) a hyperbolic paraboloid by planes parallel to one of the principal planes, are all congruent.

6. Show analytically that any plane parallel to the axis along which (a) an elliptic paraboloid, and (b) a hyperbolic paraboloid lies, intersects the surface in a parabola.

131. Rectilinear generators. The equation of the hyperboloid of one sheet, Art. 127, may be written in the form

(1) $$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}.$$

As this equation is the result of eliminating k from the equations of the system of lines

$$\frac{x}{a} + \frac{z}{c} = k\left(1 + \frac{y}{b}\right), \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{k}\left(1 - \frac{y}{b}\right),$$

the hyperboloid is a ruled surface. Equation (1) is also the result of eliminating k from the equations of the system of lines

$$\frac{x}{a} + \frac{z}{c} = k\left(1 - \frac{y}{b}\right), \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{k}\left(1 + \frac{y}{b}\right),$$

and the hyperboloid may therefore be regarded in two ways as a ruled surface.

In like manner the hyperbolic paraboloid contains the two systems of lines

$$\frac{x}{a}+\frac{y}{b}=2\,ck,\quad \frac{x}{a}-\frac{y}{b}=\frac{z}{k},$$

and

$$\frac{x}{a}+\frac{y}{b}=kz,\quad \frac{x}{a}-\frac{y}{b}=\frac{2\,c}{k}.$$

These lines are called the **rectilinear generators** of these surfaces. Hence the

Theorem. *The hyperboloid of one sheet and the hyperbolic paraboloid have two systems of rectilinear generators, that is, they may be regarded in two ways as ruled surfaces.*

The two systems of generators are shown in Plate II.

REVIEW PROBLEMS

Name and draw the surfaces in each of the following groups, giving in detail all their characteristics:

1. (a) $xy = 0$.
(b) $xy = 1$.
(c) $xy = z$.
(d) $xy = z^2$.
(e) $xy = z^2 + 1$.
(f) $xy = z^2 + z$.

2. (a) $x^2 + y^2 = 0$.
(b) $x^2 + y^2 = 1$.
(c) $x^2 + y^2 = x$.
(d) $x^2 + y^2 = z$.
(e) $x^2 + y^2 = z^2$.
(f) $x^2 + y^2 = 2\,xy$.

3. (a) $x + y = 0$.
(b) $x + y = 1$.
(c) $x + y = z$.
(d) $x + y = z^2$.
(e) $x + y = xy$.

4. (a) $x^2 + y^2 = z^2 + 1$.
(b) $x^2 + y^2 = z^2 - 1$.
(c) $x^2 + y^2 = 1 - z^2$.

5. (a) $x^2 + y^2 = z^2 + 2\,z$.
(b) $x^2 + y^2 = z^2 - 2\,z$.
(c) $x^2 + y^2 = 2\,z - z^2$.

6. (a) $x^2 + 2\,y^2 + 3\,z^2 = 0$.
(b) $x^2 + 2\,y^2 + 3\,z^2 = 1$.
(c) $x^2 + 2\,y^2 + 3\,z^2 = 2\,x$.
(d) $x^2 + 2\,y^2 + 3\,z^2 = 2\,x - 1$.

7. (a) $x^2 + 2\,y^2 - 3\,z^2 = 0$.
(b) $x^2 + 2\,y^2 - 3\,z^2 = 1$.
(c) $x^2 + 2\,y^2 - 3\,z^2 = 2\,x$.
(d) $x^2 + 2\,y^2 - 3\,z^2 = 2\,x + 1$.
(e) $x^2 + 2\,y^2 - 3\,z^2 = 2\,x - 1$.
(f) $x^2 + 2\,y^2 - 3\,z^2 = 2\,x - 2$.

8. (a) $xy + yz + zx = 0$.
(b) $z^2 + yz + zx = 0$.
(c) $z + yz + zx = 0$.
(d) $z^2 + x^2 + zx = 0$.
(e) $z^2 + x^2 + xy = 0$.
(f) $z^2 + xy + x = 0$.

MISCELLANEOUS PROBLEMS

1. Construct the following surfaces and shade that part of the first intercepted by the second :

(a) $x^2 + 4y^2 + 9z^2 = 36$, $x^2 + y^2 + z^2 = 16$.
(b) $x^2 + y^2 + z^2 = 64$, $x^2 + y^2 - 8x = 0$.
(c) $4x^2 + y^2 - 4z = 0$, $x^2 + 4y^2 - z^2 = 0$.

2. Construct the solids bounded by the surfaces (a) $x^2 + y^2 = a^2$, $z = mx$, $z = 0$; (b) $x^2 + y^2 = az$, $x^2 + y^2 = 2ax$, $z = 0$.

3. Show that two rectilinear generators of (a) a hyperbolic paraboloid, and (b) a hyperboloid of one sheet, pass through each point of the surface.

4. If a plane passes through a rectilinear generator of a quadric, show that it will also pass through a second generator, and that these generators do not belong to the same system.

5. The equation of the hyperboloid of one sheet may be written in the form $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$. By treating this equation as in Art. 131, we obtain the equations of two systems of lines on the surface. Show that these systems of lines are identical with those already obtained.

6. Show that a quadric may, in general, be passed through any nine points.

7. If $a > b > c$, what is the nature of the locus of

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1$$

if $\lambda > a^2$? if $a^2 > \lambda > b^2$? if $b^2 > \lambda > c^2$? if $\lambda < c^2$?

8. Show that the traces of the system of quadrics in Problem 7 are confocal conics.

9. Show that every rectilinear generator of the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2cz$ is parallel to one of the planes $\frac{x}{a} \pm \frac{y}{b} = 0$.

10. Prove that the projections of the rectilinear generators of (a) the hyperboloid of one sheet, (b) the hyperbolic paraboloid, on the principal planes are tangent to the traces of the surface on those planes.

11. A plane passed through the center and a generator of a hyperboloid of one sheet intersects the surface in a second generator which is parallel to the first.

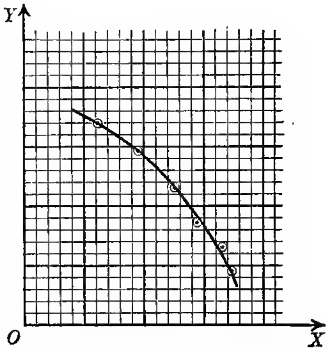
12. Show how to generate each of the central quadrics by moving an ellipse whose axes are variable.

13. Show how to generate each of the paraboloids by moving a parabola.

CHAPTER XX

EMPIRICAL EQUATIONS

132. A problem quite distinct from any thus far treated in this text arises when it is required to *find the equation of a curve which shall pass through a series of empirically given points.* That is, we suppose that certain values of the variable and of the function are known from an actual experiment, and the corresponding points are plotted on cross-section paper. A smooth curve is then drawn to "fit" these points, and an equation for this curve is required.



The general treatment of this important problem is beyond the scope of an elementary text, and the following sections are concerned with simple cases only.

133. Straight-line law. If the curve suggested by the plotted points is a straight line, assume the law

(1) $$y = mx + b,$$

and determine the values of m and b from the observed data. The straight line representing the required law will not necessarily pass *through* all the points plotted, for experimental work is subject to error. It is sufficient if the line fits the points within the limits of accuracy of the experiment. In general, the straight line may be drawn through two of the plotted points, and m and b may be calculated from their coördinates.

EXAMPLE

In an experiment with a pulley, the effort, E lb., required to raise a load of W lb. was found to be as follows:

W	10	20	30	40	50	60	70	80	90	100
E	$3\frac{1}{4}$	$4\frac{7}{8}$	$6\frac{1}{4}$	$7\frac{1}{2}$	9	$10\frac{1}{2}$	$12\frac{1}{4}$	$13\frac{3}{4}$	15	$16\frac{1}{2}$

Find a straight-line law to fit these data.

Solution. Plotting the points as in the figure, it is seen that the straight line drawn through $(30, 6\frac{1}{4})$ and $(100, 16\frac{1}{2})$ fits the observed data very well. To find its equation, substitute these values in the equation

(2) $\qquad E = mW + b.$

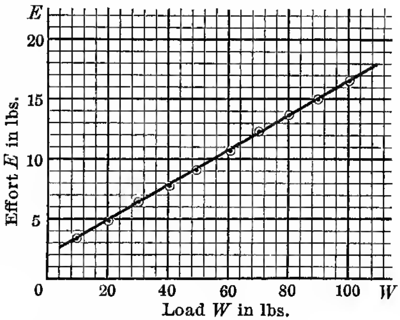
This gives $6\frac{1}{4} = 30\,m + b$, $16\frac{1}{2} = 100\,m + b$. Solving for m and b, we find $m = \frac{41}{280}$, $b = \frac{12}{7}$. Substituting in (2),

(3) $\qquad E = \frac{41}{280}W + \frac{12}{7},$

the required equation.

For the purpose of calculation we write (3) in the form

(4) $\quad E = 0.146\,W + 1.86,$



keeping three decimal places in the coefficient of W in order to secure three-figure accuracy.

We now test (4) by comparing the observed values and calculated values.

W	10	20	30	40	50	60	70	80	90	100
E, observed	3.25	4.87	6.25	7.50	9	10.5	12.2	13.7	15	16.5
E, calculated	3.32	4.98	6.24	7.68	9.16	10.6	12.1	13.5	15	16.5

The formula (4) may be used for calculating values of E for values of W intermediate between 10 and 100, and not given in the table. For example, the effort required to raise a load of 25 lb. is

$$E = 0.146 \times 25 + 1.86 = 5.51 \text{ lb.}$$

PROBLEMS

The following data treated in the same way will yield laws represented by the formula $y = mx + b$.

1. V is the volume in cubic centimeters of a certain quantity of gas at the temperature $t°$ C., the pressure being constant. Find the law connecting V and t.

t	27	33	40	55	68
V	109.9	112.0	114.7	120.1	125.

Ans. $V = 100 + 0.367\,t$.

2. V is the volume of a certain quantity of mercury at a temperature of $\theta°$ C. Find the law connecting V and θ.

$\theta°$ C.	18	36	60	72	90
V (cc.)	100.32	100.65	101.07	101.30	101.61

Ans. $V = 100 + 0.018\,\theta$.

3. S is the weight of sodium nitrate dissolved by 100 g. of water at the temperature $t°$ C. Find the law connecting S and t.

S	68.8	72.9	87.5	102
t	− 6	0	20	40

Ans. $S = 73.0 + 0.73\,t$.

4. The following are corresponding values of the speed and induced volts in an arc-light dynamo. Find the law connecting volts and revolutions per minute.

Rev. per minute, n	200	320	495	621	744
Volts induced, v	165	270	410	525	625

5. S is the weight of potassium bromide which will dissolve in 100 g. of water at the temperature $t°$ C. Find the law connecting S and t.

t	0	20	40	60	80
S	53.4	64.6	74.6	84.7	93.5

6. Find the equation of the straight lines that best fit the following data.

(a)

x	0.5	1	1.5	2	2.5	3
y	0.31	0.82	1.29	1.85	2.51	3.02

(b)

t	0	5	10	15	20	25	30
T	15	20	24.4	28.4	32	35.2	38.2

(c)

P	250	400	500	600	750	800	900
C	0.64	0.80	0.91	0.99	1.12	1.15	1.22

(d)

W	3	13	23	33	43
F	$\frac{3}{8}$	1	$1\frac{5}{8}$	$2\frac{1}{8}$	$2\frac{5}{8}$

134. Laws reduced to straight-line laws. By suitable treatment of the given data many laws can be transformed into a linear relation. Some cases of this kind of frequent occurrence will now be given.

1. The law $$y = a + bx^2.$$

When the points plotted suggest a vertical parabola with its vertex on the y-axis, the assumed equation will have the above form. If now we set $x^2 = t$, and plot the values of t and y, these values satisfy the relation $y = a + bt$, that is, a straight-line law.

EXAMPLE

An experiment to determine the coasting resistance R in pounds per ton of a motor wagon for the speed V miles per hour gave the following data:

V	0	$2\frac{1}{2}$	5	$7\frac{1}{2}$	10	$12\frac{1}{2}$	15
R	40	40	42	45	50	55	63

Plotting the points, the curve suggested (Fig. 1) appears to be a parabola with the equation

(1) $$R = a + bV^2.$$

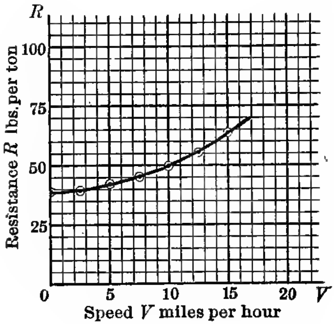


FIG. 1

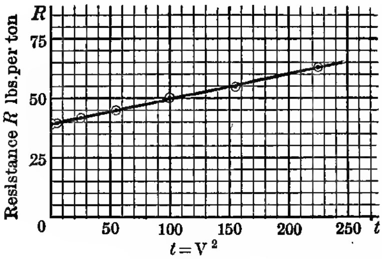


FIG. 2

To check this, calculate the values of V^2 (table, Art. 3), set $V^2 = t$, and retabulate the data thus:

$t(=V^2)$	0	$6\frac{1}{4}$	25	$56\frac{1}{4}$	100	$156\frac{1}{4}$	225
R	40	40	42	45	50	55	63

Plotting these points (Fig. 2), it appears that they are fitted by a straight line. By the preceding section we find the equation of this line to be $R = 39.3 + .107\,t$. Hence the required law is

(2) $$R = 39.3 + .107\,V^2.$$

PROBLEMS

The following data satisfy laws of the form $y = a + bx^2$. Determine the values of a and b.

1.

x	19	25	31	38	44
y	1900	3230	4900	7330	9780

Ans. $y = 5.09x^2 - 10.$

2.

S	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	2
P	2	$2\frac{3}{4}$	$3\frac{1}{2}$	5	$7\frac{1}{2}$	$10\frac{3}{4}$	$13\frac{3}{4}$	$22\frac{1}{2}$

3.

V	10	20	30	40	50
R	7	9.1	14.5	20	29

4.

d	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
S	663	1178	1841	2651	3608	4712

2. The law $\qquad y = ax^n.$

Taking logarithms, we have

(3) $$\log y = \log a + n \log x,$$

that is, the *logarithms of the given data satisfy a straight-line law.* Hence in this case tabulate the logarithms of the given data, determine the straight-line law to fit them,* compare with (3) to find a and n, and substitute in $y = ax^n$.

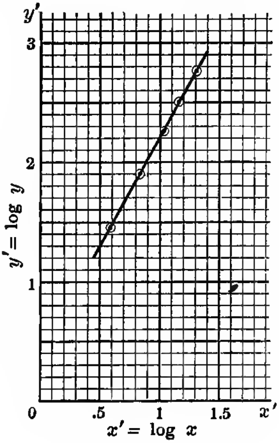
This law, as the study of the problems on page 337 will show, has wide application.

* Logarithmic squared-paper is of great convenience in this case.

EXAMPLE

The following data satisfy a law of the form $y = ax^n$. Find the values of a and n.

x	4	7	11	15	21
y	28.6	79.4	182	318	589



Solution. Tabulating the values of $\log x$ and $\log y$ (table, p. 4),

$\log x$	0.602	0.845	1.041	1.176	1.322
$\log y$	1.456	1.900	2.26	2.50	2.77

Plotting $x' = \log x$, $y' = \log y$, it appears that x' and y' satisfy a straight-line law. The equation is found to be

(4) $$y' = .364 + 1.82\,x'.$$

Hence, comparing with (3), $n = 1.82$, $\log a = .364$. Then $a = 2.3$, and the required law is

(5) $$y = 2.3\,x^{1.82}.\quad Ans.$$

PROBLEMS

1. Find a law of the form $y = ax^n$ for the following data:

x	2000	4000	6000	8000	10,000
y	2869	8700	16,660	26,370	37,660

2. Find a law of the form $p = av^n$ to fit the following data:

v	4	4.5	5	5.5	6	7
p	110	97.1	86.8	78.4	71.5	60.7

3. The time, t seconds, that it took for water to flow through a triangular notch, under a pressure of h feet, until the same quantity was in each case discharged, was found by experiment to be as follows:

h	.043	.057	.077	.094	.100
t	1260	540	275	170	135

Find the law.

4. The indicated horse power I required to drive a vessel with a displacement of D tons at a ten-knot speed is given by the following data. Find the law connecting I and D.

D	1720	2300	3200	4100
I	655	789	1000	1164

5. u is the volume in cubic feet of 1 lb. of saturated steam at a pressure of p lb. per square inch. Find the law of the form $pu^n = \text{const.}$ connecting p and u.

u	26.43	22.40	19.08	16.32	14.04
p	14.7	17.53	20.80	24.54	28.83

6. F is the force between two magnetic poles at a distance of d centimeters. Find the law connecting F and d.

d cm.	1.2	1.9	2.3	3.2	4.5
F dynes	4.44	1.77	1.21	0.625	0.316

7. D is the diameter in inches of wrought-iron shafting required to transmit H horse power when running 70 revolutions per minute.

Find a formula.

H	10	20	30	·40	50	60	70	80
D	2.11	2.67	3.04	3.36	3.61	3.82	4.02	4.22

8. Q is the quantity of water in cubic feet per second flowing through a right-angled isosceles notch when the surface of quiet water is H feet above the bottom of the notch. Find the law.

H	1	2	3	4
Q	2.63	15	41	84.4

9. A certain ship draws h feet of water and displaces V cubic feet. What is the law connecting h and V?

Draft h	18	13	11	9.5
Displacement V	107,200	65,800	51,200	41,100

135. Miscellaneous laws. In many experiments the analytic form of the law is known in advance. If, however, such fact is unknown, and if the preceding methods fail, the points determined by the data should in any case be plotted, and then the shape of the required curve may suggest an equation to be tried. The following problems furnish a variety in this regard.

PROBLEMS

1. The curve suggested in each of the following is a vertical parabola

$$y = a + bx + cx^2.$$

The values of a, b, and c may be found from three pairs of values of the data. Determine the law in each case.

(a)

x	1	2	3	4	5	6	7
y	25	41	55	67	77	85	91

(b) The resistance, R ohms, of a wire at $t°$ C. is given by the following table:

t	0	5	10	15	20	25
R	25	25.49	25.98	26.48	26.99	27.51

(c)

x	0	0.5	1	1.5	2	2.5	3
y	5.4	6.3	6.6	6.1	5.0	3.2	0.6

(d)

u	0	20	40	60	80	100
v	290	253	215	176	136	94

2. The points may be fitted by a branch of an equilateral hyperbola whose equation is

$$y = a + \frac{b}{x}.$$

In this case plot y and $\frac{1}{x} = t$, thus transforming into a straight-line law. Find the law after this manner for the following data:

(a)

x	4	5	6	7	8
y	4410	3530	2940	2520	2210

(b)

A	1.96	2.46	2.97	3.45	3.96	4.97	5.97
V	50.25	48.7	47.9	47.5	46.8	45.7	45

(c)

S	10	11	12	13	14
W	8370	4880	– 1970	– 490	– 2600

3. In some cases a branch of the rectangular hyperbola ((12), p. 184),

$$xy = bx + ay$$

will fit the points. Dividing through by xy, this becomes $1 = \frac{a}{x} + \frac{b}{y}$. Hence if $\frac{1}{x} = u$, $\frac{1}{y} = v$, are plotted, u and v will satisfy a straight-line law.

Show that this is the case in the following and find the law.

(a)

x	10	20	30	40	50	60	70	80
y	12.8	17.1	20	22.2	23.1	23.8	23.8	24.2

(b)

s	1	2	3	4	5	6	7	8
t	2.05	3.23	3.95	4.49	4.87	5.20	5.40	5.60

4. *Compound-interest law* (p. 103). If the law sought is of the form

$$y = ae^{kx},$$

taking logarithms,

$$\log y = \log a + kx \log e,$$

where $\log e = 0.434$, as usual. Hence $\log y$ and x satisfy a linear relation, and we accordingly plot x and $\log y$, determine the straight line, the values of a and k, and substitute.

Proceed in this manner in the following:

(a)

x	0	3.45	10.85	19.30	28.8	40.1	53.75
y	19.9	18.9	16.9	14.9	12.9	10.9	8.9

Hint. Plot x and $\log y$.

(b)

h	0	886	2753	4763	6942	10,593
p	30	29	27	25	23	20

Hint. Plot h and $\log p$.

(c)

t	0	10	27.4	42.1
s	61.5	62.1	66.3	70.3

x	0	2.1	5.6	9.3	11.5
y	20	18.92	17.34	15.8	14.96

136. The problem under discussion requires for thorough solution the method of least squares, and for an exposition of this theory the student is referred to treatises on that subject.

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