新三角學講義精解

新三角學講義精解

朱 鳳 豪 著

習 題 一

1,2,3. 從略．

4. $v=10$周/秒$=20\pi$ 本位弧/秒，$t=2/v=2/20\pi=7/220$秒．

5. 高（弧長）$=R\times\theta=1760\times\dfrac{\pi}{180}=30.73$ 碼．

　(5,6,8 題均假定目的物為圓弧，觀測點為圓心．)

6. $R=6\Big/\dfrac{\pi}{180}=\dfrac{1080}{\pi}=344$ 呎．

7. 弧長$=R\theta=3950\times\dfrac{\pi}{180\times60}=1.149$ 哩．

8. 月之直徑（弧長）$=238793\times1868\times\dfrac{\pi}{180\times60\times60}$
$$=2162 \text{ 哩．}$$

9. ∵ 10秒間轉過之弧長為 $20\left(\dfrac{10}{3600}\right)=\dfrac{1}{18}$ 哩．

$$\theta=\left(\dfrac{1}{18}\Big/\dfrac{1}{2}\right)\text{rad}=\dfrac{1}{9}\left(\dfrac{180°}{\pi}\right)=6.37°$$

10. 今分針移過 30 格時，時針在第 $15+\dfrac{30}{12}=17.5$ 格．

∴ 兩針相差為 12.5 格．

又因每格為 $6°$，故兩針相差 $12.5\times6=75°=\dfrac{5}{12}\pi$ rad．

11. 設五段弧長依次為
$$x-2y,\; x-y,\; x,\; x+y,\; x+2y$$

則由題意知
$$\begin{cases} x-2y+x-y+x+x+y+x+2y=2\pi & (1) \\ 6(x-2y)=x+2y & (2) \end{cases}$$

由(1)得　　　$5x=2\pi$　　　$\therefore$　　$x=\dfrac{2}{5}\pi$

代入(2)　　　　　$y=\dfrac{5}{14}x=\dfrac{1}{7}\pi$

故最小弧所對中心角爲

$$x-2y=\left(\dfrac{2}{5}-\dfrac{2}{7}\right)\pi=\dfrac{4}{35}\pi\ \text{rad}.$$

12. 從幾何學知 n 邊正多邊形之一內角爲

$$\dfrac{(n-2)\cdot180^\circ}{n}=\dfrac{(n-2)\pi}{n}\text{rad}$$

故五邊形之內角爲 $\dfrac{3}{5}\pi\ \text{rad},\cdots\cdots$

13. $\because$ x 本位弧 $=\dfrac{180x}{\pi}$度,　　　x 百分度 $=\dfrac{180x}{200}$度

$$\therefore\quad x+\dfrac{180x}{\pi}+\dfrac{180x}{200}=180^\circ$$

$$\therefore\quad x=180\bigg/\left(1+\dfrac{180}{\pi}+\dfrac{9}{10}\right)$$

故最小角爲　　$\dfrac{9}{10}x=\dfrac{9}{10}\cdot\dfrac{180}{1+\dfrac{180}{\pi}+\dfrac{9}{10}}=2^\circ44'12''$

14. 灣曲之長度爲 $\left(2100\times\dfrac{37^\circ}{2}+2800\times21^\circ\right)\dfrac{\pi}{180^\circ}$

$$=9765\times\dfrac{\pi}{18}=1705\ \text{尺}.$$

習 題 二 (12頁)

1. 今 $a=\dfrac{2}{3}c$,　 又 $a^2+b^2=c^2$,　$\therefore$　$b=\sqrt{c^2-a^2}=\dfrac{\sqrt{5}}{3}c$

從此可求矣.

2.　今　$a=\sqrt{c^2-b^2}=\sqrt{p^2+q^2-q}=p$　下從略.

3.　$c=\sqrt{a^2+b^2}=\sqrt{\dfrac{4x^2y^2}{(x-y)^2}+(x+y)^2}=\sqrt{\dfrac{4x^2y^2+(x^2-y^2)^2}{(x-y)^2}}$

$\qquad =\dfrac{x^2+y^2}{x-y}$　下從略.

4.　$a-b=\dfrac{1}{4}c\cdots\cdots\cdots(1)$　　　　$a^2+b^2=c^2\cdots\cdots\cdots(2)$

$(1)^2-(2),\qquad\qquad -2ab=-\dfrac{15}{16}c^2\cdots\cdots\cdots(3)$

$\sqrt{(2)-(3)},\qquad\qquad a+b=\dfrac{\sqrt{31}}{4}c\cdots\cdots\cdots(4)$

從(1)(4)得　$a=\dfrac{1}{8}(\sqrt{31}+1)c,\quad b=\dfrac{1}{8}(\sqrt{31}-1)c$

故　　　　　$\sin A=\dfrac{a}{c}=\dfrac{1}{8}(\sqrt{31}+1)$

$\qquad\qquad\cos A=\dfrac{b}{c}=\dfrac{1}{8}(\sqrt{31}-1)$

$\qquad\qquad\tan A=\dfrac{\sqrt{31}+1}{\sqrt{31}-1}=\dfrac{16+\sqrt{31}}{15}$

$\qquad\qquad\cot A=\dfrac{\sqrt{31}-1}{\sqrt{31}+1}=\dfrac{16-\sqrt{31}}{15}$

$\qquad\sec A=\dfrac{4}{15}(\sqrt{31}+1)\qquad\qquad\csc A=\dfrac{4}{15}(\sqrt{31}-1)$

5.　$\because\quad b=\sqrt{c^2-a^2}=\sqrt{16-8+2\sqrt{12}}=\sqrt{8+2\sqrt{12}}$
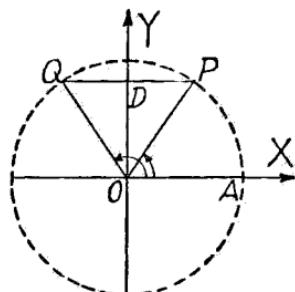
$\qquad =\sqrt{(\sqrt{6}+\sqrt{2})^2}=\sqrt{6}+\sqrt{2}$

$$\therefore \quad \cos A = \frac{b}{c} = \frac{1}{4}(\sqrt{6} + \sqrt{2})$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \frac{(\sqrt{6} - \sqrt{2})^2}{6-2}$$

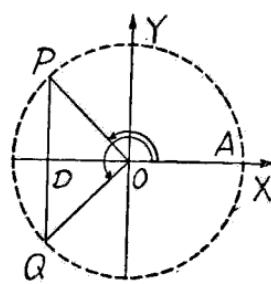$$= \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3}$$
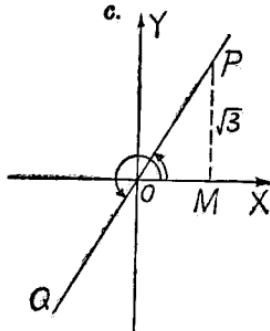
習　題　三 (17—18 頁)

1.
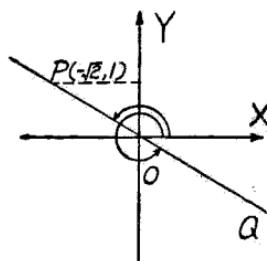
a.

$OA = 1, OD = \dfrac{4}{5}$

b.

$OA = 1, OD = \dfrac{1}{2}\sqrt{2}$

c.
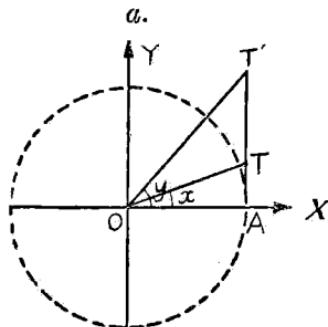
兩角 $\angle XOP, \angle XOQ$

d.

$\angle XOP$ 及 $\angle XOQ$

2.

a.

$AT' = 3AT$, y 爲 $\angle AOT$

b.

$DE = \dfrac{1}{m}DC$, y 爲　$\angle AOF$

c.

$OD = 2OC$

（設 $45° < x < 90°$）

y 爲 $\angle AOQ$

d.
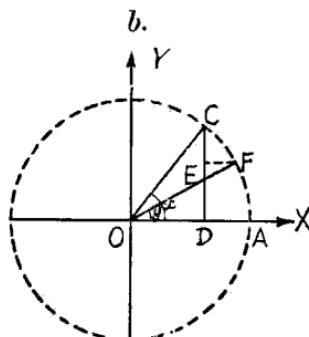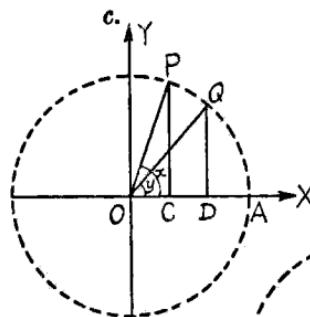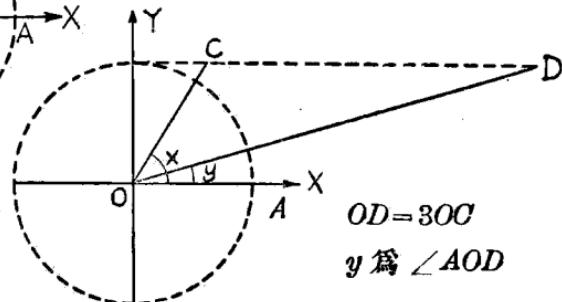
$OD = 3OC$

y 爲 $\angle AOD$

3. 今　$\sin\theta = x + \dfrac{1}{x}$,　　　即　　$x^2 - x\sin\theta + 1 = 0$

若 x 爲實數,　　　　　則　　　　$\triangle \geqq 0$,

即　　$\sin^2\theta - 4 \geqq 0$,　　即　　$|\sin\theta| \geqq 2$

但此不合理, 故 x 必不能爲實數.

或從證 $x + \dfrac{1}{x} = \dfrac{x^2 + 1}{x} > 1$, 因而得 $\sin\theta > 1$ 爲不可能.

4.　∵　$x^2+y^2 \geqq 2xy$　　　∴　$(x+y)^2 \geqq 4xy$

∴　$\dfrac{4xy}{(x+y)^2} \leqq 1$,　　　即　　$\sec^2\theta \leqq 1$

但 $\sec\theta$ 必大於1, 即 $\sec^2\theta$ 不能小於1, 至少等於 1.

此時　$4xy=(x+y)^2$,　即　$(x-y)^2=0$,　∴　$x=y$.

習　題　四 (22—23 頁)

1. (a) $\cos x = \dfrac{1}{\sec x} = -\dfrac{3}{5}$, $\sin x = \pm\sqrt{1-\left(-\dfrac{3}{5}\right)^2} = \pm\dfrac{4}{5}$,

$\tan x = \left(\pm\dfrac{4}{5}\right)\Big/\left(-\dfrac{3}{5}\right) = \mp\dfrac{4}{3}$, $\cot x = \dfrac{1}{\tan x} = \mp\dfrac{3}{4}$,

$\csc x = \dfrac{1}{\sin x} = \pm\dfrac{5}{4}$.　　　（注意本題之符號）

(c)　$\sin x = \dfrac{1}{\csc x} = \dfrac{1}{-1} = -1$,

$\cos x = \pm\sqrt{1-\sin^2 x} = 0$,　　$\tan x = \dfrac{-1}{0} = \infty$,

$\cot x = \dfrac{1}{\infty} = 0$,　　　　　$\sec x = \dfrac{1}{0} = \infty$.

(b)　(d)　從略.

(e)　在第三象限中, 除正切, 餘切外均爲負值解從略.

2. ∵　$\text{vers}\,x = 1-\cos x = \dfrac{\sqrt{2}-1}{\sqrt{2}} = 1-\dfrac{1}{\sqrt{2}}$

∴　$\cos x = \dfrac{1}{\sqrt{2}}$

故 x 在第一或第四象限內. 若 x 在第一象限內, 則可求其餘各函數之值, 代入右端後即得所求之答案; 若 x 在第四象限內, 則所得之答案爲 -2.

3. 左邊 $= (\sin\theta + \cos^2\theta/\sin\theta) + (\cos\theta + \sin^2\theta/\cos\theta) - \sec\theta$

$$= \frac{1}{\sin\theta} + \frac{1}{\cos\theta} - \sec\theta = \csc\theta = \sqrt{1+\cot^2\theta}$$

$$= \sqrt{1 + \frac{a^2-b^2}{b^2}} = \sqrt{\frac{a^2}{b^2}} = \frac{a}{b}.$$

4. $\qquad \sec^2\phi = 1 + \left(\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}\right)^2 = 1 + \frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{b^{\frac{2}{3}}},$

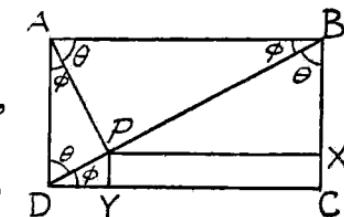$$\therefore \quad \sec\phi = \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{b^{\frac{1}{3}}}.$$

同理 $\qquad\qquad \csc\phi = \frac{(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}}{a^{\frac{1}{3}}}.$

故 $\qquad a\csc\phi + b\sec\phi = a^{\frac{2}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}} + b^{\frac{2}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}$

$$= (a^{\frac{2}{3}} + b^{\frac{2}{3}})(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{1}{2}}$$

$$= (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.$$

5. 如圖在 $\triangle PBX$, $\triangle PAB$, $\triangle ADB$ 中, $\qquad \sin\theta = \dfrac{PX}{PB}$,

$\sin\theta = \dfrac{PB}{BA}$, $\qquad \sin\theta = \dfrac{BA}{BD}$,

故 $\quad \sin^3\theta = \dfrac{PX}{BD}$, $\qquad\qquad \therefore \quad \sin\theta = \left(\dfrac{PX}{BD}\right)^{\frac{1}{3}}.$

又在 $\triangle PDY$, $\triangle PAD$, $\triangle ADB$ 中,

$\sin\phi = \dfrac{PY}{PD}$, $\qquad \sin\phi = \dfrac{PD}{AD}$, $\qquad \sin\phi = \dfrac{AD}{BD}.$

故　　　$\sin^3\phi = \dfrac{PY}{BD}$,　　　　$\therefore \sin\phi = \left(\dfrac{PY}{BD}\right)^{\frac{1}{3}}$

但在 $\triangle ADB$ 中, $\sin\theta = \dfrac{AB}{BD}$,　　$\sin\phi = \dfrac{AD}{BD}$

$\sin^2\theta + \sin^2\phi = \dfrac{\overline{AB}^2}{\overline{BD}^2} + \dfrac{\overline{AD}^2}{\overline{BD}^2} = \dfrac{\overline{AB}^2 + \overline{AD}^2}{\overline{BD}^2} = \dfrac{\overline{BD}^2}{\overline{BD}^2} = 1$

故　　　　　　$\left(\dfrac{PX}{BD}\right)^{\frac{2}{3}} + \left(\dfrac{PY}{BD}\right)^{\frac{2}{3}} = 1$

即　　$\overline{PX}^{\frac{2}{3}} + \overline{PY}^{\frac{2}{3}} = \overline{BD}^{\frac{2}{3}} = \overline{AC}^{\frac{2}{3}}$.　　（$\because$　$AC = BC$）

習　題　五 (34—36 頁)

1. 左邊 $= 1 + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

2. 右邊 $= \dfrac{1}{\sin^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = 1 + \cot^2 x$

3. 左邊 $= (\cos^2 3A + \sin^2 3A)(\cos^4 3A - \cos^2 3A \sin^2 3A$

$\qquad + \sin^4 3A)$

$\qquad = (\cos^2 3A + \sin^2 3A)^2 - 3\cos^2 3A \sin^2 3A$

$\qquad = 1 - 3\sin^2 3A \cos^2 3A =$ 右邊

4.　$\because$　$\sin^2(x+y) + \cos^2(x+y) = \sin^2(x-y)$

$\qquad\qquad\qquad + \cos^2(x-y) (=1)$

移項得　$\cos^2(x+y) - \sin^2(x-y)$

$\qquad\qquad = \cos^2(x-y) - \sin^2(x+y)$

5. 左邊 $=\dfrac{1-\tan A}{1+\tan A}\cdot\dfrac{\cot A}{\cot A}=\dfrac{\cot A-1}{\cot A+1}$

6 左邊 $=\dfrac{2\csc^2 A}{\csc^2 A-1}=\dfrac{2\csc^2 A}{\cot^2 A}=\dfrac{2}{\cos^2 A}=2\sec^2 A$

7. 左邊 $=\dfrac{(\sec A+\tan A)(\sec A-\tan A)}{\sec A-\tan A}$

$=\dfrac{\sec^2 A-\tan^2 A}{\sec A-\tan A}=\dfrac{1}{\sec A-\tan A}$

或從　$\sec^2 A-\tan^2 A=(\sec A-\tan A)(\sec A+\tan A)=1$

兩邊以　$\sec A-\tan A$ 除之卽得.

8. 左邊 $=\sin^2 x+2+\csc^2 x+\cos^2 x+2+\sec^2 x$

$=5+(1+\cot^2 x)+(1+\tan^2 x)=$右邊

9. 左邊 $=[\sec\beta(\sec x+\tan x)+\tan\beta(\sec x+\tan x)]$

$\times[\sec\beta(\sec x-\tan x)+\tan\beta(\tan x-\sec x)]$

$=(\sec\beta+\tan\beta)(\sec x+\tan x)(\sec\beta-\tan\beta)$

$\times(\sec x-\tan x)$

$=(\sec^2 x-\tan^2 x)(\sec^2\beta-\tan^2\beta)=1\cdot 1=1$

10. 左邊 $=\dfrac{(\csc x+\cot x)(\sec x-\tan x)}{\sec^2 x-\tan^2 x}$

$=\dfrac{(\csc^2 x-\cot^2 x)(\sec x-\tan x)}{\csc x-\cot x}=\dfrac{\sec x-\tan x}{\csc x-\cot x}$

或從　$\csc^2 x-\cot^2 x=\sec^2 x-\tan^2 x\ (=1)$ 做.

（用例一，證五法）

11. 左邊 $=\dfrac{(\cos x-\sin x)^2}{(\cos x+\sin x)(\cos x-\sin x)}$（以 $\sin^2 x+\cos^2 x$ 代 1）

$$=\frac{\cos x-\sin x}{\cos x+\sin x}=\frac{1-\tan x}{1+\tan x} \quad \text{(以 } \cos x \text{ 除分子,分母)}$$

12. 左邊 $=\dfrac{(\cos x-\sin x)^2}{(\cos x+\sin x)(\cos x-\sin x)}=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

$$=\frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}=\frac{\cos^2 x-\sin^2 x}{1+2\sin x\cos x}$$

13. 左邊 $=(2+\sin A)(\sec A-2\tan A)$

$$=2\sec A+\tan A-4\tan A-\frac{2\sin^2 A}{\cos A}$$

$$=2(1-\sin^2 A)\sec A-3\tan A$$

$$=2\cos A-3\tan A$$

14. 今　$\sec\theta=\dfrac{\sqrt{2}}{\cos x+\sin x}$　$\therefore$　$\cos\theta=\dfrac{\sqrt{2}(\sin x+\cos x)}{2}$

$\therefore$　$\sin\theta=\tan\theta\cos\theta=\dfrac{\sqrt{2}(\sin x-\cos x)}{2}$

$\therefore$　$\sqrt{2}\sin\theta=\sin x-\cos x$

15. 從巳知式移項得　$(\sqrt{2}+1)\sin\theta=\cos\theta$

兩邊以 $\sqrt{2}-1$ 乘之　$\sin\theta=(\sqrt{2}-1)\cos\theta$

移項得　　　　$\sin\theta+\cos\theta=\sqrt{2}\cos\theta$

16. 今　$2\tan x=m+n$,　　$2\sin x=m-n$

前式除後式得　$\cos x=(m-n)/(m+n)$

17.　$\therefore$　$\dfrac{\cos^3\theta}{\cos\alpha}+\dfrac{\sin^3\theta}{\sin\alpha}=\sin^2\alpha+\cos^2\alpha$

移項　$\dfrac{\cos^3\theta-\cos^3\alpha}{\cos\alpha}=\dfrac{\sin^3\alpha-\sin^3\theta}{\sin\alpha}$　　　　(1)

又因　$\dfrac{\cos^3\theta}{\cos\alpha}+\dfrac{\sin^3\theta}{\sin\alpha}=\sin^2\theta+\cos^2\theta$

$$\therefore \quad \frac{\cos^2\theta(\cos\theta-\cos\alpha)}{\cos\alpha}=\frac{\sin^2\theta(\sin\alpha-\sin\theta)}{\sin\alpha} \quad (2)$$

$$\frac{(1)}{(2)} \quad \frac{\cos^3\theta-\cos^3\alpha}{\cos^2\theta(\cos\theta-\cos\alpha)}=\frac{\sin^3\alpha-\sin^3\theta}{\sin^2\theta(\sin\alpha-\sin\theta)}$$

即

$$\frac{\cos^2\theta+\cos\theta\cos\alpha+\cos^2\alpha}{\cos^2\theta}$$

$$=\frac{\sin^2\alpha+\sin\alpha\sin\theta+\sin^2\theta}{\sin^2\theta}$$

即

$$\frac{\cos^2\alpha}{\cos^2\theta}-\frac{\sin^2\alpha}{\sin^2\theta}+\frac{\cos\alpha}{\cos\theta}-\frac{\sin\alpha}{\sin\theta}=0$$

即

$$\left(\frac{\cos\alpha}{\cos\theta}-\frac{\sin\alpha}{\sin\theta}\right)\left(\frac{\cos\alpha}{\cos\theta}+\frac{\sin\alpha}{\sin\theta}+1\right)=0$$

18. 左邊 $=\begin{vmatrix} 1 & \cos^4\theta & 0 \\ 1 & (1+\sin^2\theta)^2 & 0 \\ 1 & \cos^4\theta & (1+\cos^2\theta)^2-\sin^4\theta \end{vmatrix}$

$$=[(1+\cos^2\theta)^2-\sin^4\theta][(1+\sin^2\theta)^2-\cos^4\theta]$$

$$=(1+\cos^2\theta+\sin^2\theta)(1+\cos^2\theta-\sin^2\theta)$$

$$\times(1+\sin^2\theta+\cos^2\theta)(1+\sin^2\theta-\cos^2\theta)$$

$$=2\cdot2\cos^2\theta\cdot2\cdot2\sin^2\theta=16\sin^2\theta\cos^2\theta$$

習 題 六 (37—38 頁)
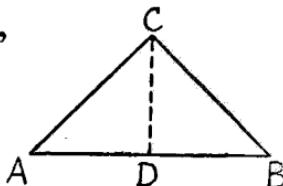
1. 2. 3. 4. 從略。

習 題 七 (42—43 頁)

1. a, b, c 從略。 (a 之答數 $c=3.1194$, $b=2.2223$)

2. 今 $\overline{AC}=30\sqrt{2}$ 尺, $AB=60$ 尺,

$AD=30$ 尺.

$\cos A = \dfrac{30}{30\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
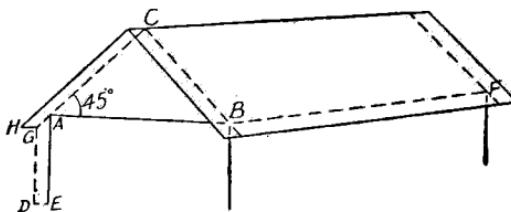
∴ $A=45°$ （屋頂之傾斜）

∴ $CD=30\tan 45°=30$ 尺 （屋脊距屋簷之高）

3. ∵ $AB=40$ 尺, $BF=80$ 尺, $\angle A=45°$

$DE=1$ 尺, $\angle H=\angle A=45°$

$AC=20\sqrt{2}$ 尺, $GA=\sqrt{2}$ 尺.

∴ 椽子 CG 長 $20\sqrt{2}+\sqrt{2}=21\sqrt{2}=29.694$ 尺.

全屋面之面積爲 $21\sqrt{2}\times82\times2=3444\sqrt{2}$

$=4870.5$ 平方尺.

4. 合力 $OR=\sqrt{170^2+120^2}=208$ 磅

∵ $\tan\theta=\dfrac{120}{170}=\dfrac{12}{17}=.7059$

∴ $\theta=35°13'$ （東 $35°13'$ 南）

5. ∵ $\angle AOC=32°3'$, ∴ $\angle AOB=57°57'$

$\therefore \quad \overline{AB} = \overline{AO}\, \text{s n}\, 57°57'$

$\qquad = 4000 \times .8476$

$\qquad = 3390.4$ 英里

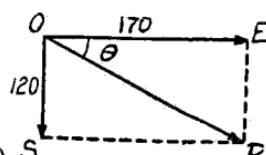$\therefore$ 緯度圈爲以 BA 爲半徑之

一圓周長爲

$2\pi \times 3390.4 = 21310$ 英里

6. 桿高 $= 80 \tan 50° = 80 \times 1.1918$

$\qquad = 95.34$ 尺.

7. 設 $\quad \overline{AB} = c, \quad \overline{BE} = b, \quad \overline{BD} = \dfrac{c}{2}$

則 $\quad \angle BOE = \angle AOE = \dfrac{180°}{n}$

今 $\quad \angle AOE$ 以 $\overset{\frown}{AE}$ 度之

卽 $\quad \angle EBA$ 以 $\dfrac{1}{2}\overset{\frown}{AE}$ 度之

故 $\quad \angle EBA = \dfrac{1}{2} \cdot \dfrac{180°}{n} = \dfrac{90°}{n}$

$\therefore \quad \dfrac{c/2}{b} = \cos \dfrac{90°}{n} \qquad\qquad \therefore \quad b = c \big/ 2\cos\dfrac{1}{n}90°$
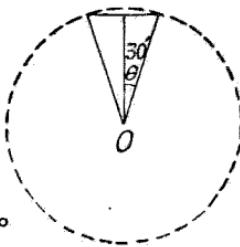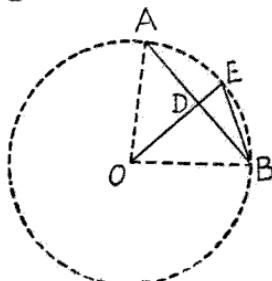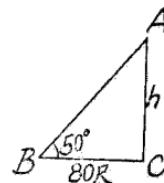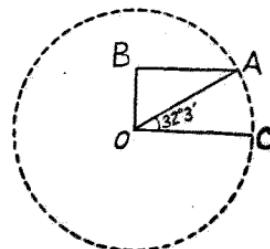
8. 設 p 爲周界, 則

$\dfrac{p}{720} \div 1 = \sin 30'$

($\because$ 中心角爲一度)

$\therefore \quad p = 720 \times 0.00873 = 6.283$

9. $\because \quad \overline{AB} = \overline{BO}\tan\dfrac{180°}{n} = r\tan\dfrac{180°}{n}$
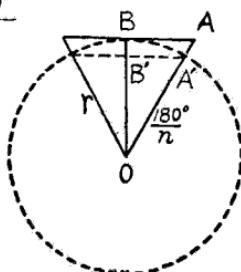
$$\therefore \quad S = nr \cdot r \tan \frac{180°}{n} = nr^2 \tan \frac{180°}{n}$$

$$\because \quad A'B' = r \sin \frac{180°}{n}$$

$$B'O' = r \cos \frac{180°}{n}$$
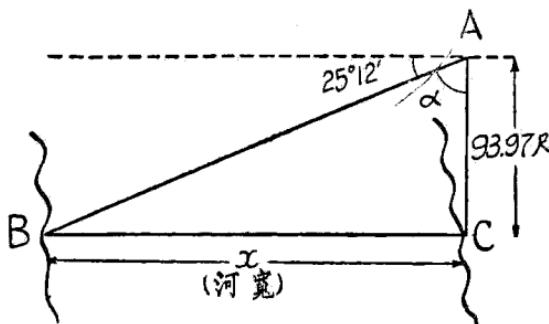
$$\therefore \quad S' = nr^2 \sin \frac{180°}{n} \cos \frac{180°}{n}$$

習　題　八 (53—57 頁)

1. $x = BC = $ 河寬 $= AC \tan \alpha = 93.97 \tan(90° - 25°12')$

$\qquad = 93.97 \tan 64°48' = 93.97 \times 2.1251 = 200$ 尺

2. 今 $\angle A$ 爲直角

$\qquad \angle L = 45° + 15° = 60°$

又　$LA = 4$ 浬
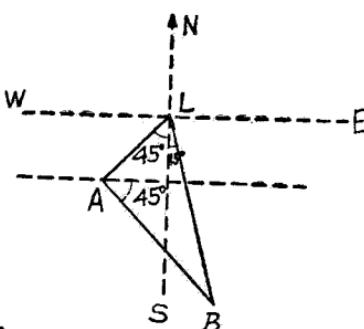
故二舟距離爲

$AB = LA \tan 60° = 4\sqrt{3}$

$\qquad = 6.928$

故 A, B 之距離爲 6.928 浬.

3. 今 AC (N.N.W.) 與 AB (W.S.W.) 成直角

$\angle N'BD = 37°30'$

$\angle N'BC = 7°30'$

故知,

$\angle CBD = 30°$

$\angle DBA = 90° - 37\frac{1}{2}° - 22\frac{1}{2}°$

$\qquad = 30°$

$\therefore \quad \angle BDA = 90° - 30° = 60°$

$\therefore \quad \angle DCB = \angle CBD,$

$\therefore \quad CD = DB = 1.5$ 哩,

$\therefore \quad AB = DB \cdot \cos DBA$

$\qquad = \dfrac{3}{2} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{4}$

$\therefore$ 火車之速率為 $\dfrac{3\sqrt{3}}{8}$ 哩/分 = 39 哩/時

4. 今 $\angle AOB = 180° - 28° - 62° = 90°$
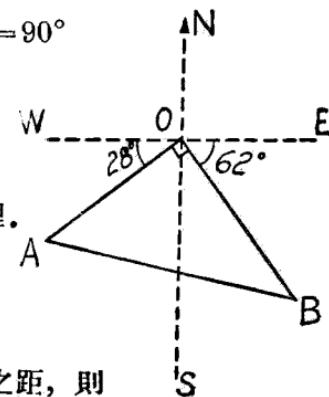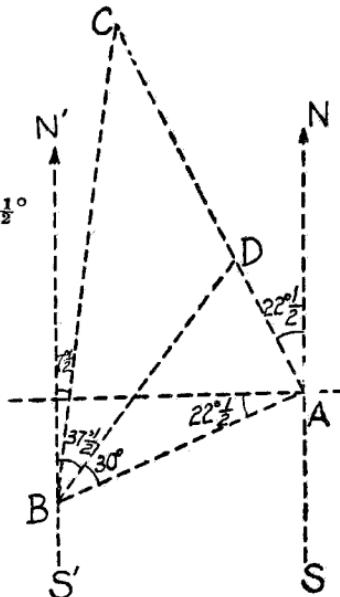
又 $\qquad OA = 2 \times 10 = 20$ 浬,

$\qquad OB = 2 \times 10\frac{1}{2} = 21$ 浬.

$\therefore \qquad AB = \sqrt{20^2 + 21^2} = 29$ 浬.

5. $\angle ACB = 54°24' - 27°12'$

$\qquad = 27°12'$
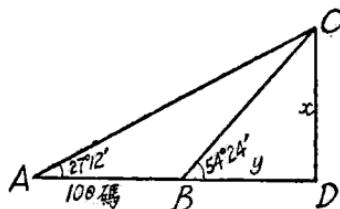
設 x 為山高; y 為近點至山之距, 則

$$x = \frac{300 \sin 27°12' \sin 54°24'}{\sin 27°12'} \qquad \text{〔從公式甲二（48 頁）〕}$$

$$= 300 \sin 54°24'$$

$$y = x \cot 54°24' = 300 \cos 54°24' = 174.6 \text{ 呎}$$

6. 從公式甲三 （48 頁）
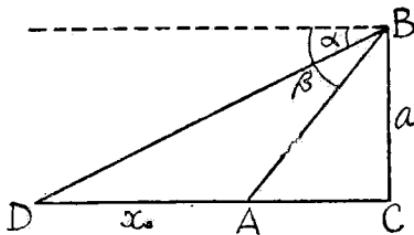
$$河闊 = \frac{30 \tan 45°}{\tan 60° - \tan 45°} = \frac{30}{1.732 - 1} = \frac{30}{.732} = 41 \text{ 尺}$$

7. 今 A, D 二點間之距離爲 x 尺

$$DC = a \tan(90° - a) = a \cot a$$

$$AC = a \tan(90° - \beta) = a \cot \beta$$

$$\therefore \quad x = DC - AC = a(\cot a - \cot \beta)$$

〔照例二之解二法可求得答數之另一形式〕

8. 距離 $= 150(\cot 30° - \cot 45°) = 150(\sqrt{3} - 1) = 109.8$ 尺

9. 設 $AB = a,$ 　　$BC = x,$ 　　$PC = y,$

$$\frac{x+a}{y}=\tan a, \qquad \frac{x}{y}=\tan \beta$$

則　　$\dfrac{x+a}{x}=\dfrac{\tan a}{\tan \beta}$
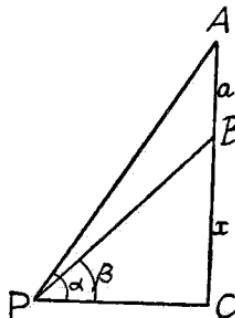
$$\therefore \quad x=\frac{a \tan \beta}{\tan a-\tan \beta}$$

10. 塔高$=\dfrac{25}{\sqrt{3}-1}=\dfrac{25}{2}(\sqrt{3}+1)$

$$=\frac{25}{2}\times 2.732=34.15 \text{ 尺.}$$
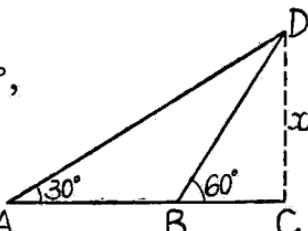
11. 今 $AB=1$ 哩，$\angle DAC=30°$，

$\angle DBC=60°$．

設此屋距路遠 x 哩。則

$$(1+x \cot 60°)=x \cot 30°$$

$$x=\frac{1}{\cot 30°-\cot 60°}=\frac{1}{\sqrt{3}-\frac{1}{3}\sqrt{3}}=\frac{\sqrt{3}}{2}$$
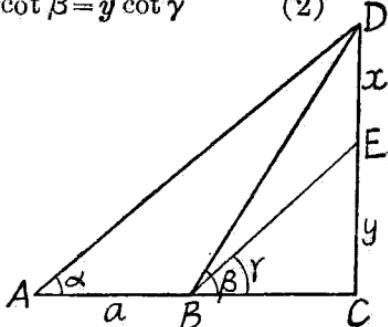
$$=0.866 \text{ 哩}\quad 1524 \text{ 碼.}$$

12. 設旗桿高 x 尺，屋高 y 尺，則

$$(x+y)\cot a-y \cot \gamma=a \qquad (1)$$

$$(x+y)\cot \beta=y \cot \gamma \qquad (2)$$

解 (1), (2) 得

$$x = \frac{a(\cot \gamma - \cot \beta)}{\cot \gamma(\cot a - \cot \beta)} = \frac{a \tan a(\tan \beta - \tan \gamma)}{\tan \beta - \tan a} \text{ 尺.}$$

13. 今　$a = 30°$,　　$\beta = 60°$,　　$\gamma = 50°$,　　$a = 100$ 尺.
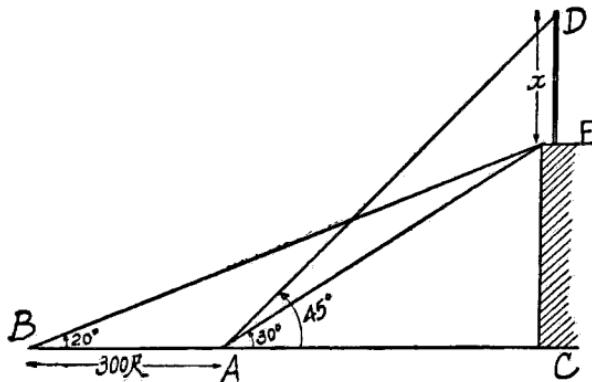
故旗桿長爲　$\dfrac{100(\tan 60° - \tan 50°)\tan 30°}{\tan 60° - \tan 30°}$

$$= \frac{100(\sqrt{3} - 1.1918)0.5774}{1.7321 - 0.5774} = 27 \text{ 尺.}$$

14. 在　$\triangle BCE$ 中從公式甲二知

$$EC = \frac{300 \sin 20° \sin 30°}{\sin 10°}$$

則　　$AC = EC \cot 30° = \dfrac{300 \sin 20° \cos 30°}{\sin 10°}$

又　　　　　　　　　$DC = AC \tan 45° = AC$

∴　$x = DC - EC = \dfrac{300 \sin 20°(\cos 30° - \sin 30°)}{\sin 10°}$

$$=\frac{300\times.3420\times(.866-.5)}{.1736}=216\ \text{尺}.$$

(此題如在 VI 章之後則應化爲對數式做)

15. 令此氣球高度爲 h 尺
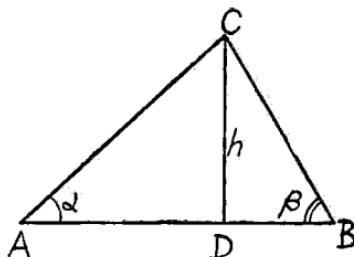
則 $\qquad AD=h\cot\alpha,\qquad\qquad BD=h\cot\beta$

$\therefore\quad h(\cot\alpha+\cot\beta)$

$\qquad=AD+BD=a$

$\therefore\quad h=\dfrac{a}{\cot\alpha+\cot\beta}$

$\qquad=\dfrac{a\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$
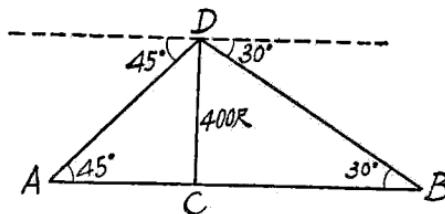
16. 今 $\alpha=44°21',\qquad\beta=62°30',\qquad AB=1800\ \text{尺}.$

代入得 $\qquad\qquad\qquad h=1166\ \text{尺}.$

(如在 VI 章之後則應化 $h=\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ 用對數做)

17. 今 $DC=$ 山高爲 400 尺, A,B 爲兩人立處.

$\qquad AC=400\cot45°,\qquad\qquad BC=400\cot30°$

$\therefore\quad AB=AC+BC=400(1+\sqrt{3})=1093$

卽兩人之直線距爲 1093 尺.
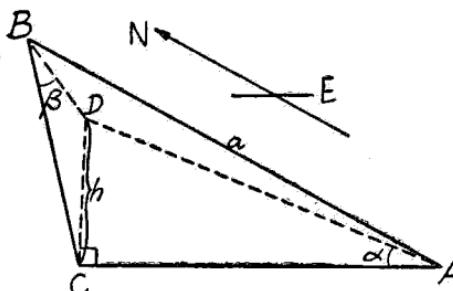
18. 設山高爲 h 尺, 則 CB 向北, CA 向東.

則 $AC = h \cot \alpha, \qquad BC = h \cot \beta$

∴ $h^2(\cot^2 \alpha + \cot^2 \beta) = \overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2 = a^2$

∴ $h = \dfrac{a}{\sqrt{\cot^2 \alpha + \cot^2 \beta}} = \dfrac{a \tan \alpha \tan \beta}{\sqrt{\tan^2 \alpha + \tan^2 \beta}}$
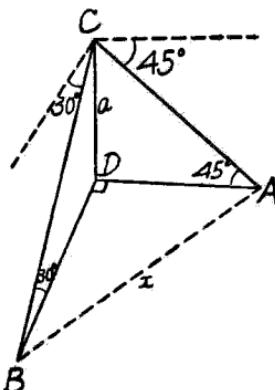
19.   山高 $= \dfrac{a}{\sqrt{\cot^2 \alpha + \cot^2 \beta}} = \dfrac{1000}{\sqrt{3}+1} = 500$ 尺．

20. 設 A, B 爲船之兩位置，其距離爲 x 尺

$AD = a \cot 45° = a, \qquad BD = a \cot 30° = a\sqrt{3}$

∴ $x^2 = \overline{AD}^2 + \overline{BD}^2 = a^2 + 3a^2 = 4a^2 \quad (AD \perp BD)$

∴　$x=2a$　即船之速度為每時 $2a$ 尺。

21. 設 D, E 為氣球之兩位置

高　　　$BD = CE$
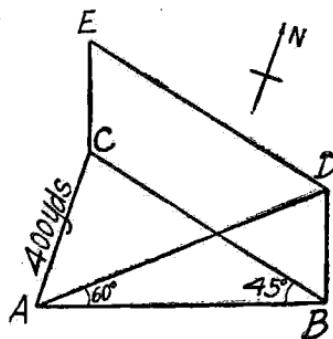
今　　　$AC = 400$ 碼向北,

　AB 向東,BC 向西北.

故　　　$\angle A = 90°$

　　　　$\angle CBA = 45°$

　　　　$\angle ACB = 45°$

∴　$AB = AC = 400$ 碼.

今　　$BD = AB\tan 60° = 400\sqrt{3} = 693$ 碼.

22. 今 $\alpha = 30°$,$\beta = 60°$,$a = 50$ 尺,代入公式兩三,則

屋高 $= \dfrac{50\tan 60°}{\tan 60° - \tan 30°} = \dfrac{50\sqrt{3}}{\sqrt{3} - \dfrac{1}{\sqrt{3}}} = 3 \times 25 = 75$ 尺.

23. 設 ED 為屋,BA 為塔 (高為 h 尺).

則 $h = BC + CA = DC(\tan\beta + \tan\alpha)$

　　　$= a(\tan\alpha + \tan\beta)$
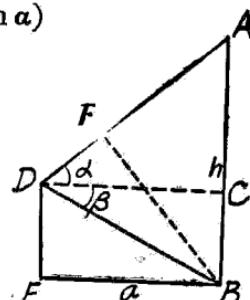
如作　$BF \perp AD$

則　$BD = DC/\cos\beta = a/\cos\beta$

　　$BF = BD\sin(\alpha + \beta)$

　　$AB = BF/\cos\alpha$

∴　$h = AB = \dfrac{a\sin(\alpha + \beta)}{\cos\alpha\cos\beta}$.

24. 如上圖中　$a = 60°$,　$\beta = 30°$,　$a = 30$ 尺.

則屋高　$AB = 30(\tan 60° + \tan 30°)$

$$= 40\sqrt{3} = 69.28 \text{ 尺}.$$

25. 設街闊為 DB,　A, C 為兩窗口.

$\therefore \quad DO = \sqrt{40^2 - 21^2}$
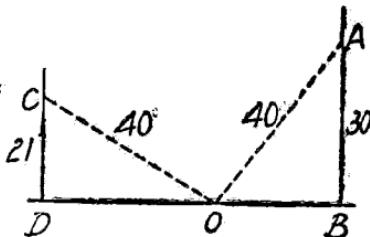
$$= \sqrt{1159} = 34.04$$

$\quad OB = \sqrt{40^2 - 30^2}$

$$= \sqrt{700} = 26.45$$

$\therefore$ 街闊 $= DO + OB$

$$= 26.45 + 34.04 = 60 \text{ 尺}.$$

26. 設 CD 為桿高, DA 為土堆高.

則　$AC = AB \tan 60° = \sqrt{3}\, AB$
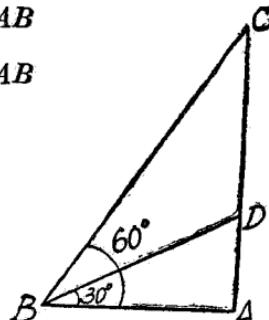
$\quad AD = AB \tan 30° = \dfrac{1}{\sqrt{3}} AB$

$$= \dfrac{\sqrt{3}}{3} AB$$

$\therefore \quad AC = 3AD$

但　　$AD + CD = 3AD$

故　　$CD = 2AD$.
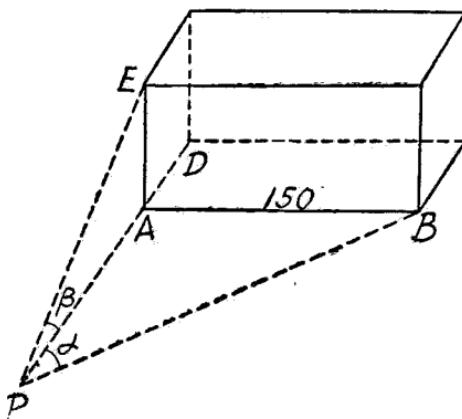
27. 如圖測點 P 在 DA 之延長線上，$AB = 150$ 尺, AE 為屋高.

今　$\cos a = \sqrt{\dfrac{1}{5}}$　　　$\therefore \quad \sin a = \dfrac{2}{\sqrt{5}}$
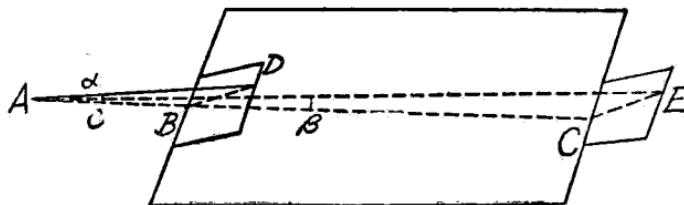
$\quad \tan a = 2 = \dfrac{AB}{AP} = \dfrac{150}{AP}$　$\therefore \quad AP = 75$ 尺.

再 $\sin \beta = \dfrac{3}{\sqrt{34}}$, $\qquad \cos \beta = \sqrt{1 - \dfrac{9}{34}} = \dfrac{5}{\sqrt{34}}$
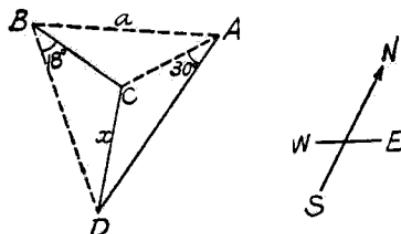
$\therefore \tan \beta = \dfrac{3}{5} = \dfrac{AE}{AP}$, $\quad \therefore AE = 75 \times \dfrac{3}{5} = 45$ 尺。

28. $\qquad DB = c \tan \alpha$, $\qquad DB = CE = (c + \overline{BC}) \tan \beta$

$\therefore \quad c \tan \alpha = (c + \overline{BC}) \tan \beta$

$\therefore \quad BC = 球場長度 = \dfrac{c \tan \alpha - c \tan \beta}{\tan \beta}$

$\qquad\qquad\qquad = c(\tan \alpha \cot \beta - 1)$ 尺。

29. 今 AD 南向, AB 西向. 故 $\angle BAD = 90°$

又設塔高 CD 爲 x 尺, 則 $\overline{BD}^2 - \overline{AD}^2 = \overline{AB}^2 = a^2$

但 $BD = x \cot 18^\circ = x\sqrt{\dfrac{5}{5-2\sqrt{5}}} = x\sqrt{5+2\sqrt{5}}$

$AD = x \cot 30^\circ = x\sqrt{3}$

代入上式得 $x^2(5+2\sqrt{5}-3) = a^2$,

$\therefore \quad x = \dfrac{a}{\sqrt{2\sqrt{5}+2}}$.

30. 今 $\angle CAD = 45^\circ$, $CD \perp AD$

$\therefore \quad AD = CD \cot 45^\circ = x$

在 rt $\triangle ABD$ 中,

$\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 = a^2 + x^2$

故 $\dfrac{CD}{BD} = \dfrac{x}{\sqrt{a^2+x^2}} = \tan 15^\circ$

$= 2 - \sqrt{3}$
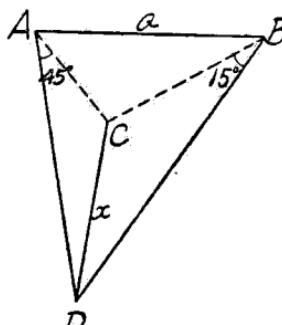
$\therefore \quad x^2 = (7-4\sqrt{3})(a^2+x^2)$

$\therefore \quad x^2 = \dfrac{a^2(7-4\sqrt{3})}{4\sqrt{3}-6} = \dfrac{a^2(7-4\sqrt{3})(4+2\sqrt{3})}{\sqrt{3}(4-2\sqrt{3})(4+2\sqrt{3})}$

$= \dfrac{a^2(4-2\sqrt{3})}{4\sqrt{3}} = \dfrac{a^2(\sqrt{3}-1)^2}{4\sqrt{3}}$

$\therefore \quad x = \dfrac{a}{2}\left(\dfrac{\sqrt{3}-1}{\sqrt[4]{3}}\right) = \dfrac{a}{2}(3^{\frac{1}{4}} - 3^{-\frac{1}{4}})$.

31. 今 $\angle ABD=60°$, $\quad\angle CBD=30°$,

$\quad\angle ABC=30°$, $\quad\angle BCA=135°$,

$\quad\angle CAB=15°$, $\quad\quad BC=5280$ 呎.

作 $\quad CF\perp AB$

則 $\quad BF=5280\cos 30°$

$\quad CF=5280\sin 30°=2640$,

$\quad AF=CF\cot 15°=2640\cot 15°$.
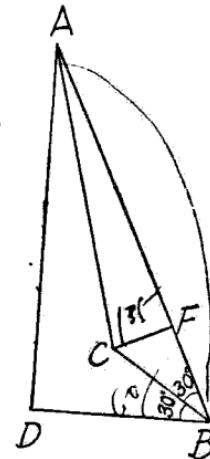
故山高 $AD=AB\sin 60°$

$\quad\quad =(BF+AF)\sin 60°$

$\quad\quad =2640(2\cos 30°+\cot 15°)\sin 60°$

$\quad\quad =2640\times 5.4641\times .866=12492.6$ 呎.

(參考 290 頁例十一)

32. E 爲 CD 之中點, O 爲底之中

心,故所求之角爲 AEO.
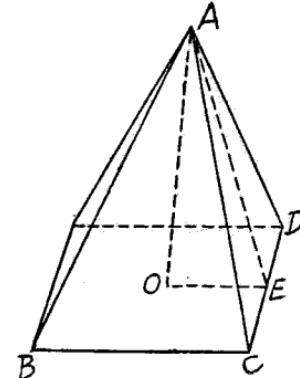
今 $\quad\quad AE=\sqrt{\overline{AC}^2-\overline{CE}^2}$

$\quad\quad\quad =\sqrt{150^2-100^2}$

$\quad\quad\quad =\sqrt{12500}$

$\cos AEO=\dfrac{OE}{AE}=\dfrac{100}{50\sqrt{5}}$

$\quad\quad\quad =\dfrac{2}{5}\sqrt{5}=.8944$
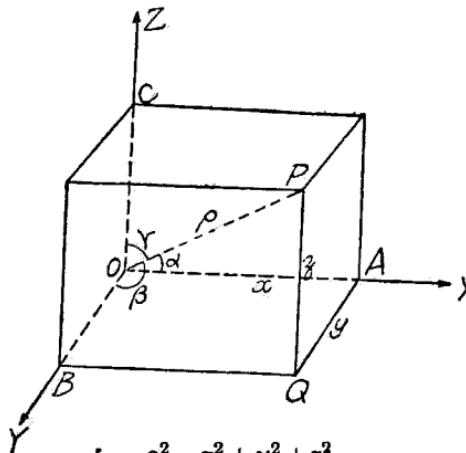
$\therefore\quad\angle AEO=26°34'$ (銳角).

33. 如圖 $OA=x$, $AQ=y$, $QP=z$, 又 $OP=\rho$ 爲長方體

$ABCP$ 之對角線, 今 $\angle OAP =$ 直角

$$\overline{OP}^2 = \overline{OA}^2 + \overline{AP}^2 = x^2 + \overline{AP}^2$$

但 $$\overline{AP}^2 = \overline{AQ}^2 + \overline{QP}^2 = y^2 + z^2 \quad (\angle PQA = 直角)$$

$$\therefore \quad \rho^2 = x^2 + y^2 + z^2 \qquad (1)$$

$$x = \rho \cos \alpha, \quad y = \rho \cos \beta, \quad z = \rho \cos \gamma$$

代入(1), 則 $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

設 $\alpha = \beta = \gamma$, 則 $3 \cos^2 \alpha = 1$

$$\therefore \quad \cos \alpha = \frac{1}{3}\sqrt{3} = .5773$$

$$\therefore \quad \alpha = 54°44'20''.$$
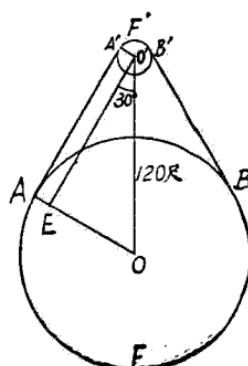
34. 如圖皮帶之長為

$$AA' + BB' + \overset{\frown}{AFB} + \overset{\frown}{A'F'B'}$$

今 $OA = 70,$ $O'A' = 10,$

作 $O'E /\!/ AA'$, 則 $AE = 10,$

$OE = 60,$ $OO' = 120$

故　　$\angle EO'O = 30°,$　　則　　$\angle EOO' = 60°$

$(\because\ \angle OEO' = 90°)$

故　　$\angle AOB = 120°,$　　　$\angle A'O'B' = 120°$

$(\because\ OA /\!/ O'A')$

則　　$\overset{\frown}{AFB} = 240° = \dfrac{2}{3}(大圓周) = \dfrac{280}{3}\pi$

$\overset{\frown}{A'F'B'} = 120° = \dfrac{1}{3}(小圓周) = \dfrac{20\pi}{3}$

又　　$AA' = BB' = OO' \cos 30° = 60\sqrt{3}$

故皮帶長爲　　$120\sqrt{3} + \dfrac{20\pi}{3} + \dfrac{280\pi}{3}$

$= 120\sqrt{3} + 100\pi = 522\ 呎.$

35. 如例八可得皮帶長爲

$\dfrac{4\pi}{3}(1+2) + 2(3\sqrt{3}) = 4\pi + 6\sqrt{3} = 22.96\ 尺.$

習　題　九 (66—68 頁)

1,2,3a, b……g 從略.

3h.　$\mathrm{Covers}(-300°) = 1 - \sin(-300°) = 1 + \sin 300°$

$= 1 + \sin(360° - 60°) = 1 - \sin 60°$

$= 1 - \dfrac{\sqrt{3}}{2} = \dfrac{1}{2}(2 - \sqrt{3})$

4. 化簡下式

a.　原式 $= a\cos(90° - x) - b\sin x - a\sin x + b\sin x$

$= a\sin x - a\sin x = 0$

b. 原式 $=(-\cos x)(-\sin x)+(-\tan y)\cos y-\tan x$

$\quad\quad\ =\sin x\cos x-\sin y-\tan x$

c. 原式 $=\dfrac{\cos x\sin x}{-\cos x}+\dfrac{-\sin x\sin x}{-\sin x}$

$\quad\quad\ =-\sin x+\sin x=0$

d. 原式 $=\dfrac{0}{-1}+\sin(720°+180°)-\cot(360°+90°)$

$\quad\quad\quad +\cos(5\cdot360°)$

$\quad\quad\ =\sin180°-\cot90°+\cos0°=0-0+1=1$

5. 略.

6. $\quad\quad\because\ \dfrac{11\pi}{4}=2\pi+\dfrac{3\pi}{4}$ 代入式中得

$\quad\quad\sin^2\dfrac{3\pi}{4}-\cos^2\dfrac{3\pi}{4}+2\tan\dfrac{3\pi}{4}-\sec^2\dfrac{3\pi}{4}$

$\quad\quad\quad\ =\dfrac{1}{2}-\dfrac{1}{2}-2-2=-4$

7. $\quad\because\ \cos134°42'=\cos(450°-315°18')=\sin315°18'$

$\quad\quad\quad\quad\quad =-.70$

$\therefore\ \cot315°18'=-\sqrt{\csc^2315°18'-1}$ （因在第四象限）

$\quad\quad\quad\quad\ =-\sqrt{1.021}=-1.01$

8. a. $\because\ A+B+C=180°\quad\quad\therefore\ A=180-\overline{B+C}$

$\quad\therefore\ \cos A=\cos(180°-\overline{B+C})=-\cos(B+C)$.

b. $\because\ 2A+B+C=180°+A$

$\quad\therefore\ \sin(2A+B+C)=\sin(180°+A)=-\sin A$.

c. $\therefore\ A+B+C=180°\quad\therefore\ \dfrac{3A}{2}=270°-\dfrac{3}{2}(B+C)$

$$\therefore \quad \sin\frac{3A}{2} = \sin\left[270° - \frac{3}{2}(B+C)\right]$$

$$= -\cos\frac{3}{2}(B+C).$$

d. $\qquad \because \quad \dfrac{A+B}{4} = \dfrac{\pi-C}{4}$

$$\therefore \quad \tan\frac{A+B}{4} = \tan\frac{\pi-C}{4} = \tan\left(\frac{\pi}{2} - \frac{\pi+C}{4}\right)$$

$$= \cot\frac{\pi+C}{4} = -\tan\left(\frac{\pi}{2} + \frac{\pi+C}{4}\right)$$

$$= -\tan\frac{3\pi+C}{4}.$$

e. $\quad \because \quad \cos\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right) = \sin\left(\dfrac{\pi}{2} - \overline{\dfrac{\pi}{4} - \dfrac{A}{2}}\right)$

$$= \sin\left(\frac{\pi}{4} + \frac{A}{2}\right)$$

又因 $\qquad \dfrac{A}{2} = \dfrac{\pi}{2} - \dfrac{1}{2}(B+C)$

$$\frac{\pi}{4} - \frac{A}{2} = -\frac{\pi}{4} + \frac{B+C}{2} = -\left(\frac{\pi}{4} - \frac{B+C}{2}\right)$$

$$\therefore \quad \cos\left(\frac{\pi}{4} - \frac{A}{2}\right) = \cos\left[-\left(\frac{\pi}{4} - \frac{B+C}{2}\right)\right]$$

$$= \cos\left(\frac{\pi}{4} - \frac{B+C}{2}\right)$$

故 $\quad \cos\left(\dfrac{\pi}{4} - \dfrac{A}{2}\right) = \sin\left(\dfrac{\pi}{4} + \dfrac{A}{2}\right) = \cos\left(\dfrac{\pi}{4} - \dfrac{B+C}{2}\right).$

習　題　十 (74—77 頁)

1. 在圖中設　$OA=1$,　　$\angle AOB=x$,　　$\angle COB=y$.

　　今　　$\sin(x-y)=CF=PE-GC$

　　但　　$\dfrac{PE}{OP}=\sin x$

　　∴　　$PE=\sin x \cdot OP=\sin x \cos y$

　　　　　$\dfrac{GC}{CP}=\cos x$

　　∴　　$GC=\cos x \cdot CP=\cos x \sin y$

　　故　　　$\sin(x-y)=\sin x \cos y-\cos x \sin y$

　　又　　　$\cos(x-y)=OF=OE+PG$

　　但　　$\dfrac{OE}{OP}=\cos x$,　　**∴**　　$OE=\cos x \cdot OP=\cos x \cos y$

　　　　　$\dfrac{PG}{CP}=\sin x$,　　**∴**　　$PG=\sin x \cdot CP=\sin x \sin y$

　　故　　　$\cos(x-y)=\cos x \cos y+\sin x \sin y$

2.　$\sin x=\sin(x-y)\cos y+\cos(x-y)\sin y$　　　　(1)

　　$\cos x=\cos(x-y)\cos y-\sin(x-y)\sin y$　　　　(2)
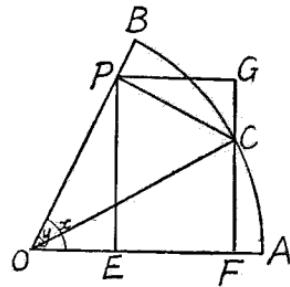
　$\cos y(1)-\sin y(2)$ 得

　　$\sin x \cos y-\cos x \sin y=\sin(x-y)(\sin^2 y+\cos^2 y)$

　　　∴　　$\sin(x-y)=\sin x \cos y-\cos x \sin y$

　$\sin y(1)+\cos y(2)$ 得

　　$\sin x \sin y+\cos x \cos y=\cos(x-y)(\sin^2 y+\cos^2 y)$

$$\therefore \quad \cos(x-y) = \sin x \sin y + \cos x \cos y$$

3. $\quad \sin a = \dfrac{1}{\sqrt{5}} \qquad \therefore \quad \cos a = \dfrac{2}{\sqrt{5}}$

$\qquad$ (∵　a　為銳角，$\cos a$　取正號)

$\qquad \cos \beta = \dfrac{1}{\sqrt{10}} \qquad \therefore \quad \sin \beta = \dfrac{3}{\sqrt{10}}$

$\qquad$ (∵　β　為銳角，$\sin \beta$　取正號)

$$\sin(\beta - a) = \sin \beta \cos a - \cos \beta \sin a$$

$$= \frac{3}{\sqrt{10}} \cdot \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \beta - a = \frac{\pi}{4}$$

4. 令 $\quad \sin A = \dfrac{1}{2}, \qquad\qquad \cos A = \dfrac{\sqrt{3}}{2},$

$\qquad\quad \sin \beta = \dfrac{1}{4}, \qquad\qquad \cos \beta = \dfrac{1}{4}\sqrt{15}.$

又因 $\quad \cos C = \sqrt{\dfrac{2}{3}} \qquad \therefore \quad \sin C = \sqrt{1 - \dfrac{2}{3}} = \dfrac{1}{\sqrt{3}}$

∵ $\quad \cos D = \dfrac{\sqrt{2} + \sqrt{3}}{2\sqrt{3}}$

∴ $\quad \sin D = \sqrt{1 - \dfrac{5 + 2\sqrt{6}}{12}} = \dfrac{\sqrt{7 - 2\sqrt{6}}}{2\sqrt{3}} = \dfrac{\sqrt{6} - 1}{2\sqrt{3}}$

故 $\quad \dfrac{\sin(A+B)}{\sin(C+D)} = \dfrac{\sin A \cos B + \cos A \sin B}{\sin C \cos D + \cos C \sin D}$

$$= \frac{\sqrt{15} + \sqrt{3}}{8} \cdot \frac{6}{\sqrt{2} + \sqrt{3} + \sqrt{12} - \sqrt{2}}$$

$$= \frac{\sqrt{3}(\sqrt{5} + 1)}{4} \cdot \frac{3}{3\sqrt{3}} = \frac{\sqrt{5} + 1}{4}.$$

5. $\tan(A-B) = \dfrac{\dfrac{\sqrt{3}}{4-\sqrt{3}} - \dfrac{\sqrt{3}}{4+\sqrt{3}}}{1 + \dfrac{\sqrt{3}}{4-\sqrt{3}} \cdot \dfrac{\sqrt{3}}{4+\sqrt{3}}} = \dfrac{\dfrac{6}{13}}{1 + \dfrac{3}{13}} = .375$

6. $\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \sin 30° \cos 45°$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}(\sqrt{6} + \sqrt{2})$$

$\tan 615° = \tan(720° - 105°) = -\tan 105°$

$$= -\tan(45° + 60°)$$

$$= -\frac{\tan 45° + \tan 60°}{1 - \tan 45° \tan 60°} = -\frac{1 + \sqrt{3}}{1 - \sqrt{3}} = 2 + \sqrt{3}$$

7. $\sin(2\pi - x) = \sin 2\pi \cos x - \cos 2\pi \sin x = -\sin x$

$\cos(270° - x) = \cos 270° \cos x + \sin 270° \sin x = -\sin x$

$$\cot(\pi - x) = \frac{\cot \pi \cot x + 1}{\cot x - \cot \pi} = \frac{\cot x + \dfrac{1}{\cot \pi}}{\dfrac{\cot x}{\cot \pi} - 1} = \frac{\cot x + \dfrac{1}{\infty}}{\dfrac{\cot x}{\infty} - 1}$$

$$= \frac{\cot x}{-1} = -\cot x$$

8. a. 左邊 $= \dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y} = \dfrac{\sin x \cos y + \sin y \cos x}{\cos x \cos y}$

$$= \frac{\sin(x+y)}{\cos x \cos y}$$

b. 左邊 $= (\sin x \cos y + \sin y \cos x)$

$$\times (\sin x \cos y - \sin y \cos x)$$

$$= \sin^2 x \cos^2 y - \sin^2 y \cos^2 x$$

$$= \sin^2 x (1 - \sin^2 y) - \sin^2 y \cos^2 x$$

$$= \sin^2 x - \sin^2 y (\cos^2 x + \sin^2 x)$$

$$= \sin^2 x - \sin^2 y$$

c.　左邊 $= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$

$$= \cos^2 x (1 - \sin^2 y) - \sin^2 x \sin^2 y$$

$$= \cos^2 x - \sin^2 y (\cos^2 x + \sin^2 x)$$

$$= \cos^2 x - \sin^2 y$$

d.　$\dfrac{a}{\cot \alpha - \cot \beta} = \dfrac{a}{\dfrac{1}{\tan \alpha} - \dfrac{1}{\tan \beta}} = \dfrac{a}{\dfrac{\tan \beta - \tan \alpha}{\tan \alpha \tan \beta}}$

$$= \dfrac{a \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$$

又　$\dfrac{a}{\cot \alpha - \cot \beta} = \dfrac{a}{\dfrac{\cos \alpha}{\sin \alpha} - \dfrac{\cos \beta}{\sin \beta}}$

$$= \dfrac{a \sin \alpha \sin \beta}{\sin \beta \cos \alpha - \sin \alpha \cos \beta} = \dfrac{a \sin \alpha \sin \beta}{\sin(\beta - \alpha)}$$

e.　左邊 $= \sin 45° \cos x + \cos 45° \sin x + \cos 45° \cos x$

$$- \sin 45° \sin x$$

$$= \dfrac{\sqrt{2}}{2} (2 \cos x) = \sqrt{2} \cos x$$

f.　$\sin x - \cos x = -(\cos x - \sin x)$

$$= -\sqrt{2} \left(\dfrac{\sqrt{2}}{2} \cos x - \dfrac{\sqrt{2}}{2} \sin x \right)$$

$$= -\sqrt{2} \left(\cos x \cos 45° - \sin x \sin 45° \right)$$

$$= -\sqrt{2} \cos \left(x + \dfrac{\pi}{4} \right)$$

g.　左邊 $= \dfrac{\tan\dfrac{\pi}{4} \pm \tan x}{1 \mp \tan\dfrac{\pi}{4}\tan x} = \dfrac{1 \pm \tan x}{1 \mp \tan x}$　$\left(\because \ \tan\dfrac{\pi}{4} = 1 \right)$

h.　左邊 $= \dfrac{\tan\dfrac{9\pi}{4} + \tan\theta}{1 - \tan\dfrac{9\pi}{4}\tan\theta} \cdot \dfrac{\tan\dfrac{3\pi}{4} + \tan\theta}{1 - \tan\dfrac{3\pi}{4}\tan\theta}$

$\because \ \ \tan\dfrac{9\pi}{4} = \tan\left(2\pi + \dfrac{\pi}{4} \right) = \tan\dfrac{\pi}{4} = 1$

$\tan\dfrac{3\pi}{4} = \tan\left(\pi - \dfrac{\pi}{4} \right) = -\tan\dfrac{\pi}{4} = -1$

$\therefore$　上式即爲 $\dfrac{1 + \tan\theta}{1 - \tan\theta} \cdot \dfrac{-1 + \tan\theta}{1 + \tan\theta} = -1$

或從 $\tan\left[\left(\dfrac{9\pi}{4} + \theta \right) - \left(\dfrac{3\pi}{4} + \theta \right) \right] = \tan\dfrac{5\pi}{4} = \infty$,

其展開式之分母應爲 0, 亦可證明.

i.　左邊 $= \cos A \cos B + \sin A \sin B - \sin A \cos B$

　　　　　$- \cos A \sin B$

　　　$= \cos A(\cos B - \sin B) - \sin A(\cos B - \sin B)$

　　　$= (\cos A - \sin A)(\cos B - \sin B)$

j.　左邊 $= \dfrac{\sin\beta\cos\gamma - \sin\gamma\cos\beta}{\cos\beta\cos\gamma}$

　　　$+ \dfrac{\sin\gamma\cos\alpha - \sin\alpha\cos\gamma}{\cos\alpha\cos\gamma}$

　　　$+ \dfrac{\sin\alpha\cos\beta - \sin\beta\cos\alpha}{\cos\alpha\cos\beta} = \dfrac{\sin\beta}{\cos\beta} - \dfrac{\sin\gamma}{\cos\gamma}$

　　　$+ \dfrac{\sin\gamma}{\cos\gamma} - \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta} = 0$

k. ∵　$\tan(5A-3A)=\dfrac{\tan 5A-\tan 3A}{1+\tan 5A\,\tan 3A}$

∴　左邊 $=\tan(5A-3A)=\tan 2A$

l. ∵　$\sin[(x+y)-y]=\sin(x+y)\cos y$
$$-\cos(x+y)\sin y$$

∴　左邊 $=\sin[(x+y)-y]=\sin x$

m. 今　$\cos a=\cos\left[\left(\dfrac{a}{2}+nx\right)+\left(\dfrac{a}{2}-nx\right)\right]$

$$=\cos\left(\dfrac{a}{2}+nx\right)\cos\left(\dfrac{a}{2}-nx\right)$$

$$-\sin\left(\dfrac{a}{2}+nx\right)\sin\left(\dfrac{a}{2}-nx\right)$$

n. 今　$\cos(\overline{a+\beta}+\gamma)=\cos(a+\beta)\cos\gamma$
$$-\sin(a+\beta)\sin\gamma$$
$$=(\cos a\cos\beta-\sin a\sin\beta)\cos\gamma$$
$$-(\sin a\cos\beta+\sin\beta\cos a)\sin\gamma$$
$$=\cos a\cos\beta\cos\gamma-\Sigma\sin a\sin\beta\cos\gamma$$

o. 今　$\dfrac{\sin(a-\beta+\gamma)}{\cos a\cos\beta\cos\gamma}$

$$=\dfrac{\sin(a-\beta)\cos\gamma+\cos(a-\beta)\sin\gamma}{\cos a\cos\beta\cos\gamma}$$

$$=\dfrac{(\sin a\cos\beta-\cos a\sin\beta)\cos\gamma}{\cos a\cos\beta\cos\gamma}$$

$$+\dfrac{(\cos a\cos\beta+\sin a\sin\beta)\sin\gamma}{\cos a\cos\beta\cos\gamma}$$

$$=\tan a-\tan\beta+\tan\gamma+\tan a\tan\beta\tan\gamma$$

9. a. $\cot(\alpha+\beta+\gamma)=\cot(\overline{\alpha+\beta}+\gamma)$

$$=\frac{\cot(\alpha+\beta)\cot\gamma-1}{\cot(\alpha+\beta)+\cot\gamma}=\frac{\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}\cdot\cot\gamma-1}{\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}+\cot\gamma}$$

$$=\frac{\cot\alpha\cot\beta\cot\gamma-\cot\alpha-\cot\beta-\cot\gamma}{\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha-1}$$

b. 從 (a)，以 $\alpha+\beta+\gamma=90°$ 代入

則得 $\cot 90°=0=\dfrac{\Pi\cot\alpha-\Sigma\cot\alpha}{\Sigma\cot\alpha\cot\beta-1}$

即 $\Pi\cot\alpha-\Sigma\cot\alpha=0$, 故 $\Sigma\cot\alpha=\Pi\cot\alpha$

10. 今 $\tan\dfrac{A+B+C}{2}=\tan\dfrac{\pi}{2}=\infty$

但 $\tan\left(\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}\right)=\dfrac{\Pi\tan\dfrac{A}{2}-\Sigma\tan\dfrac{A}{2}}{\Sigma\tan\dfrac{A}{2}\tan\dfrac{B}{2}-1}$ （公式 16）

故其分母必爲 0, 即 $\Sigma\tan\dfrac{A}{2}\tan\dfrac{B}{2}=1$
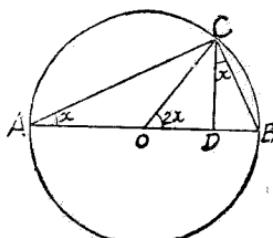
習 題 十 一 (84—87 頁)

1. 今半徑爲 1, $\therefore$ $AB=2$,

$AC=2\cos x$, $BC=2\sin x$

故 $\sin 2x=CD=AC\sin x$

$=2\cos x\sin x$

$\cos 2x=OD=OB-BD$

$=1-BC\sin x$

$$= 1 - 2\sin^2 x = \cos^2 x + \sin^2 x - 2\sin^2 x$$
$$= \cos^2 x - \sin^2 x$$

2. 今　$\cos 2010° = -\sqrt{1 - \dfrac{1}{4}} = -\sqrt{\dfrac{3}{4}} = -\dfrac{1}{2}\sqrt{3}$

（在第三象限取負號）

$\therefore$　$\tan 1005° = \tan\dfrac{1}{2}(2010°) = \dfrac{1 - \cos 2010°}{\sin 2010°}$

$$= \dfrac{1 + \frac{1}{2}\sqrt{3}}{-\frac{1}{2}} = -2 - \sqrt{3}$$

3. 今　$\cos x = \dfrac{2}{\sqrt{5}}$,

$\therefore$　$\tan\dfrac{x}{2} = \dfrac{1 - \cos x}{\sin x} = \dfrac{1 - \dfrac{2}{\sqrt{5}}}{\dfrac{1}{\sqrt{5}}} = \sqrt{5} - 2$

又　$\cos 2x = 1 - 2\sin^2 x = 1 - \dfrac{2}{5} = \dfrac{3}{5}$

$\because$　$\cos 3x = 4\cos^3 x - 3\cos x = \dfrac{32}{5\sqrt{5}} - \dfrac{6}{\sqrt{5}} = \dfrac{2\sqrt{5}}{25}$

故　$\sin\dfrac{3x}{2} = \sqrt{\dfrac{1 - \cos 3x}{2}} = \sqrt{\dfrac{25 - 2\sqrt{5}}{50}}$

$$= \dfrac{1}{10}\sqrt{50 - 4\sqrt{5}}$$

〔此題亦可從　$\sin\left(x + \dfrac{x}{2}\right)$着手〕

4. 今　$\tan^2(\alpha - \beta) = \dfrac{1 - \cos 2(\alpha - \beta}{1 + \cos 2(\alpha - \beta)}$

$$= \dfrac{1 - \cos 2\alpha\cos 2\beta - \sin 2\alpha\sin 2\beta}{1 + \cos 2\alpha\cos 2\beta + \sin 2\alpha\sin 2\beta}$$

$$= \frac{\sec 2\alpha \sec 2\beta - (1 + \tan 2\alpha \tan 2\beta)}{\sec 2\alpha \sec 2\beta + (1 + \tan 2\alpha \tan 2\beta)}$$

但　　$\sec 2\alpha = \sqrt{1 + \tan^2 2\alpha} = \sqrt{1 + \frac{4(ab + cd)^2}{(a^2 - b^2 + c^2 - d^2)}}$

$$= \frac{\sqrt{[(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]}}{(a^2 - d^2) - (b^2 - c^2)} \; *$$

$$\sec 2\beta = \frac{\sqrt{[(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]}}{(a^2 - d^2) + (b^2 - c^2)}$$

∴　　$\sec 2\alpha \sec 2\beta = \frac{[(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]}{(a^2 - d^2)^2 - (b^2 - c^2)^2}$

$$1 + \tan 2\alpha \tan 2\beta = 1 + \frac{4(ab + cd)(ac + bd)}{(a^2 - d^2)^2 - (b^2 - c^2)^2}$$

$$= \frac{[(a+d)^2 - (b-c)^2][(a-d)^2 + (b+c)^2]}{(a^2 - d^2)^2 - (b^2 - c^2)^2}$$

∴　　$\tan^2(\alpha - \beta) = \frac{2(b-c)^2[(a-d)^2 + (b+c)^2]}{2(a+d)^2[(a-d)^2 + (b+c)^2]}$

$$= \frac{(b-c)^2}{(a+d)^2}$$

∴　　$\tan(\alpha - \beta) = \frac{b-c}{a+d}$　　　　　（此題相當之繁）

∵　$(a^2 - b^2 + c^2 - d^2)^2 + 4(ab + cd)^2$

$= [a^2 + b^2 + c^2 + d^2 - 2(b^2 + d^2)]^2 + 4(ab + cd)^2$

$= (a^2 + b^2 + c^2 + d^2)^2 + 4(b^2 + d^2)(b^2 + d^2 - a^2 - b^2 - c^2 - d^2)$
　$+ 4(ab + cd)^2$

$= (a^2 + b^2 + c^2 + d^2)^2 + 4(ab + cd)^2 - 4(b^2 + d^2)(a^2 + c^2)$

$= (a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2$

$= [a^2 + b^2 + c^2 + d^2 - 2ad + 2bc][a^2 + b^2 + c^2 + d^2 + 2ad - 2bc]$

$= [(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]$

5. $\because \tan x = \dfrac{5}{12},$ $\therefore \cos x = \dfrac{12}{13},$ $\sin x = \dfrac{5}{13}.$

又 $\cos^2 y = \dfrac{1}{2}(1 + \cos 2y) = \dfrac{1}{2} \cdot \dfrac{1152}{625} = \dfrac{576}{625}$

$\therefore \cos y = \dfrac{24}{25},$ $\sin y = \dfrac{7}{25}$

$\therefore \cos(x + y) = \dfrac{12 \cdot 24}{13 \cdot 25} - \dfrac{5 \cdot 7}{13 \cdot 25} = \dfrac{253}{325}$

則 $\sin\dfrac{x+y}{2} = \sqrt{\dfrac{1 - \cos(x+y)}{2}}$

$$= \sqrt{\dfrac{1 - (253/325)}{2}} = \dfrac{6}{5\sqrt{13}}$$

故 $\csc\dfrac{x+y}{2} = \dfrac{5}{6}\sqrt{13}$

6. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin^2\theta}{\cos\theta\,\sin\theta} = \dfrac{2(1 - \cos^2\theta)}{2\sin\theta\cos\theta}$

$= \dfrac{1 - \cos 2\theta}{\sin 2\theta} = \dfrac{b}{a}$

$\therefore a(1 - \cos 2\theta) = b\sin 2\theta$

即 $a\cos 2\theta + b\sin 2\theta = a$

（或從求 $\sin x,\ \cos x$ 着手做亦可）

7.   a.   左邊 $= \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$

$= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$

$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$

$= 4\cos^3 x - 3\cos x$

b.　左邊 $= \sin 2(2x) = 2 \sin 2x \cos 2x$

$\quad = 4 \sin x \cos x(1 - 2 \sin^2 x)$

$\quad = 4 \sin x \cos x - 8 \sin^3 x \cos x$

c.　左邊 $= \tan(2x + x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

$\quad = \dfrac{\dfrac{2 \tan x}{1 - \tan^2 x} + \tan x}{1 - \dfrac{2 \tan x}{1 - \tan^2 x} \cdot \tan x} = \dfrac{\dfrac{3 \tan x - \tan^3 x}{1 - \tan^2 x}}{\dfrac{1 - 3 \tan^2 x}{1 - \tan^2 x}}$

$\quad = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

d.　$\cos 2x = \dfrac{\cos^2 x - \sin^2 x}{1} = \dfrac{(\cos^2 x - \sin^2 x) \div \cos^2 x}{(\cos^2 x + \sin^2 x) \div \cos^2 x}$

$\quad = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

e.　$\cos 4x = 1 - 2 \sin^2 2x = 1 - 8 \sin^2 x \cos^2 x$

$\quad = 1 - 8 \cos^2 x + 8 \cos^4 x \quad (\because \sin^2 x = 1 - \cos^2 x)$

同理可證　　　$\cos 4x = 1 - 8 \sin^2 x + 8 \sin^4 x.$

f.　今　$\cot \dfrac{x}{2} = \dfrac{\cos \dfrac{x}{2}}{\sin \dfrac{x}{2}} = \dfrac{\pm \sqrt{\dfrac{1 + \cos x}{2}}}{\pm \sqrt{\dfrac{1 - \cos x}{2}}} = \pm \sqrt{\dfrac{1 + \cos x}{1 - \cos x}}$

$\quad = \pm \dfrac{\sqrt{1 - \cos^2 x}}{1 - \cos x} = \dfrac{\sin x}{1 - \cos x}$

$\quad = \pm \dfrac{1 + \cos x}{\sqrt{1 - \cos^2 x}} = \dfrac{1 + \cos x}{\sin x}$

g.　左邊 $= \dfrac{2 \sin \frac{1}{4} x \cos \frac{1}{4} x}{1} = \dfrac{2 \sin \frac{1}{4} x \cos \frac{1}{4} x}{\cos^2 \frac{1}{4} x + \sin^2 \frac{1}{4} x}$

$$= \dfrac{\dfrac{2 \sin \frac{1}{4} x}{\cos \frac{1}{4} x}}{\dfrac{\cos^2 \frac{1}{4} x + \sin^2 \frac{1}{4} x}{\cos^2 \frac{1}{4} x}} = \dfrac{2 \tan \frac{x}{4}}{1 + \tan^2 \frac{1}{4} x}$$

h.　$\sin\frac{1}{2}x \pm \cos\frac{1}{2}x = \sqrt{\left(\sin \frac{1}{2} x \pm \cos \frac{1}{2} x \right)^2}$

$$= \sqrt{\sin^2 \frac{1}{2} x + \cos^2 \frac{1}{2} x \pm 2 \sin \frac{1}{2} x \cos \frac{1}{2} x}$$

$$= \sqrt{1 \pm \sin x}$$

i.　左邊 $= \sin \frac{1}{2} x + \cos \frac{1}{2} x$

$$= 2 \sin \frac{1}{4} x \cos \frac{1}{4} x + 1 - 2 \sin^2 \frac{1}{4} x$$

$$= 1 + 2 \sin \frac{1}{4} x \left(\cos \frac{1}{4} x - \sin \frac{1}{4} x \right)$$

$$= 1 + 2 \sin \frac{1}{4} x \sqrt{\left(\cos \frac{1}{4} x - \sin \frac{1}{4} x \right)^2}$$

$$= 1 + 2 \sin\frac{1}{4} x \sqrt{\cos^2 \frac{1}{4} x + \sin^2 \frac{1}{4} x - 2 \sin \frac{1}{4} x \cos \frac{1}{4} x}$$

$$= 1 + 2 \sin \frac{1}{4} x \sqrt{1 - \sin \frac{1}{2} x}$$

$$\therefore \quad 左邊 = 右邊$$

j.　左邊 $= 3 \sin x - (3 \sin x - 4 \sin^3 x)$

$$= 4 \sin^3 x = 2 \sin x (2 \sin^2 x)$$

$$= 2 \sin x (1 - \cos 2x)$$

k.　左邊 $= \dfrac{1}{\sin 2x} + \dfrac{\cos 4x}{\sin 4x} = \dfrac{2 \cos 2x + 2 \cos^2 2x - 1}{2 \sin 2x \cos 2x}$

$$= \dfrac{2 \cos 2x (1 + \cos 2x)}{2 \sin 2x \cos 2x} - \dfrac{1}{\sin 4x}$$

$$= \dfrac{1 + \cos 2x}{\sin 2x} - \csc 4x = \dfrac{2 \cos^2 x}{2 \sin x \cos x} - \csc 4x$$

$$= \cot x - \csc 4x$$

1. 右邊 $= 2 \cdot \dfrac{2+(1+\cos 4x)}{2\sin^2 2x} = 2 \cdot \dfrac{2+2\cos^2 2x}{8\sin^2 x \cos^2 x}$

$= \dfrac{1+\cos^2 2x}{2\sin^2 x \cos^2 x}$

$= \dfrac{(\cos^2 x + \sin^2 x)^2 + (\cos^2 x - \sin^2 x)^2}{2\sin^2 x \cos^2 x}$

$= \dfrac{2(\cos^4 x + \sin^4 x)}{2\sin^2 x \cos^2 x} = \dfrac{\cos^4 x + \sin^4 x}{\sin^2 x \cos^2 x}$

$= \tan^2 x + \cot^2 x$

m. 左邊 $= \dfrac{\cos^2 \dfrac{\theta}{2} - \sin^2 \dfrac{\theta}{2}}{\cos^2 \dfrac{\theta}{2} + \sin^2 \dfrac{\theta}{2} - 2\sin \dfrac{\theta}{2}\cos \dfrac{\theta}{2}}$

$= \dfrac{\cos^2 \dfrac{\theta}{2} - \sin^2 \dfrac{\theta}{2}}{\left(\cos \dfrac{\theta}{2} - \sin \dfrac{\theta}{2}\right)^2} = \dfrac{\cos \dfrac{\theta}{2} + \sin \dfrac{\theta}{2}}{\cos \dfrac{\theta}{2} - \sin \dfrac{\theta}{2}} = \dfrac{1+\tan \dfrac{\theta}{2}}{1-\tan \dfrac{\theta}{2}}$

n. $\cos x + \sin x = 2\sin \dfrac{x}{2}\cos \dfrac{x}{2} + 2\cos^2 \dfrac{x}{2} - 1$

$= 2\cos \dfrac{x}{2}\left(\sin \dfrac{x}{2} + \cos \dfrac{x}{2}\right) - 1$

$\left(同理化 \quad \sin \dfrac{x}{2} + \cos \dfrac{x}{2}\right)$

$= 2\cos \dfrac{x}{2}\left[2\cos \dfrac{x}{4}\left(\sin \dfrac{x}{4} + \cos \dfrac{x}{4}\right) - 1\right] - 1$

8. a. 今左邊 $= 8\sin \dfrac{\theta}{8}\cos \dfrac{\theta}{8}\cos \dfrac{\theta}{4}\cos \dfrac{\theta}{2}$

$= 4\sin \dfrac{\theta}{4}\cos \dfrac{\theta}{4}\cos \dfrac{\theta}{2} = 2\sin \dfrac{\theta}{2}\cos \dfrac{\theta}{2} = \sin \theta$

b. $\because$ $2\cos^2 x = 1 + \cos 2x$ $\therefore$ $4\cos^2 x = 2 + 2\cos 2x$

    則 $2\cos x = \sqrt{2 + 2\cos 2x}$

    同理推之 $2\cos 2x = \sqrt{2 + 2\cos 4x}$

    代入則 $2\cos x = \sqrt{2 + \sqrt{2 + 2\cos 4x}}$

c. $x = 11\frac{1}{4}°$ 則 $4x = 45°$ 代入上式卽得

d. $\because$ $2\cos\theta - 1 = \dfrac{4\cos^2\theta - 1}{2\cos\theta + 1} = \dfrac{2\cos 2\theta + 1}{2\cos\theta + 1}$

    同理 $2\cos 2\theta - 1 = \dfrac{2\cos 4\theta + 1}{2\cos 2\theta + 1},$

            $2\cos 4\theta - 1 = \dfrac{2\cos 8\theta + 1}{2\cos 4\theta + 1}$

    故 $(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 4\theta - 1)$

       $= \dfrac{2\cos 2\theta + 1}{2\cos\theta + 1} \cdot \dfrac{2\cos 4\theta + 1}{2\cos 2\theta + 1} \cdot \dfrac{2\cos 8\theta + 1}{2\cos 4\theta + 1}$

       $= \dfrac{2\cos 8\theta + 1}{2\cos\theta + 1}$

e. $1 + \sec\theta = \dfrac{1 + \cos\theta}{\cos\theta} = \dfrac{2\cos^2\dfrac{\theta}{2}}{\cos\theta} \cdot \dfrac{\sin\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}$

       $= \dfrac{\sin\theta \cdot \cos\dfrac{\theta}{2}}{\cos\theta \cdot \sin\dfrac{\theta}{2}} = \tan\theta\cot\dfrac{\theta}{2}$

    同理 $1 + \sec\dfrac{\theta}{2} = \tan\dfrac{\theta}{2}\cot\dfrac{\theta}{4},$

           $1 + \sec\dfrac{\theta}{4} = \tan\dfrac{\theta}{4}\cot\dfrac{\theta}{8}$

相乘卽得. 左邊 $= \tan \theta \cot \dfrac{\theta}{8} =$ 右邊.

f.　$\because \tan x - \cot x = \dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x} = -\dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x}$

$$= -2 \cot 2x$$

$$2 \tan 2x - 2 \cot 2x = -4 \cot 4x$$

$$4 \tan 4x - 4 \cot 4x = -8 \cot 8x$$

相加得

$$\tan x + 2 \tan 2x + 4 \tan 4x = \cot x - 8 \cot 8x$$

g.　設 $x = 11°15'$,　則 $2x = 22°30'$,　$4x = 45°$.

$$\therefore \quad \cot 8x = \cot 90° = 0$$

故從上題卽可證得　左邊 = 右邊

9.　a.　原式 $= [\cos x \cos a - \cos(a + x)]^2 + \cos^2 x (1 - \cos^2 a)$

$$= [\cos x \cos a - (\cos x \cos a - \sin x \sin a)]^2$$

$$+ \cos^2 x \sin^2 a = \sin^2 a (\cos^2 x + \sin^2 x) = \sin^2 a$$

b.　原式 $= \dfrac{1 + \cos 2x}{2} + \dfrac{1 + \cos(240° + 2x)}{2}$

$$+ \dfrac{1 + \cos(240° - 2x)}{2}$$

$$= \dfrac{3 + \cos 2x + 2 \cos 240° \cos 2x}{2}$$

（將餘弦複角函數展開合併）

$$= \dfrac{3 + \cos 2x - 2 \cos 60° \cos 2x}{2}$$

$$= \dfrac{3 + \cos 2x - \cos 2x}{2} = \dfrac{3}{2}$$

10. a. $\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

$$= \pm\sqrt{\dfrac{1-[(\cos u - e)/(1-e\cos u)]}{1+[(\cos u - e)/(1-e\cos u)]}}$$

$$= \pm\sqrt{\dfrac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)}} = \pm\sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{u}{2}$$

$$\left(\because\quad \tan\dfrac{u}{2} = \pm\sqrt{\dfrac{1-\cos u}{1+\cos u}}\right)$$

b. 在上題中以 $\cos a$ 換 $\cos u$，又以 $\cos\beta$ 換 e，卽可推得本題之結果．

習 題 十 二 (91—94 頁)

1. a 至 k 各題均可用公式 $(28-35)$ 一次或二次直接推得，解法從略．

2. a. $\sin 3x = (\sin 3x - \sin x) + \sin x$

$$= 2\cos 2x\sin x + \sin x = 2\sin x(\cos 2x + \tfrac{1}{2})$$

$$= 2\sin x(\cos 2x - \cos 120°)$$

$$= 2\sin x[2\sin(x+60°)\sin(60°-x)]$$

$$= 4\sin x\sin(60°+x)\sin(60°-x)$$

b. 右邊 $= 2\cos x(\cos 120° + \cos 2x)$

$$= -\cos x + 2\cos x\cos 2x$$

$$= -\cos x + (\cos 3x + \cos x) = \cos 3x$$

c. 左邊 $= -2[\cos(\theta-2a+ma) - \cos(\theta-ma)]$
$$\times\cos(\theta-ma)$$

$$= -2[\cos(\theta - 2a + ma)\cos(\theta - ma)$$

$$-\cos^2(\theta - ma)\,]$$

$$= -\cos(2\theta - 2a) - \cos(2ma - 2a)$$

$$+2\cos^2(\theta - ma)$$

$$= 1 + \cos(2\theta - 2ma) - \cos(2\theta - 2a)$$

$$-\cos(2ma - 2a)$$

d. 左邊$=\dfrac{\cos A + \cos 11A}{\cos A - \cos 11A}$ $\quad\left(\begin{array}{l}\text{分子,分母各用公式}\\ 30,31 \text{ 化出合併}\end{array}\right)$

$$=\dfrac{\cos 6A \cos 5A}{\sin 6A \sin 5A} = \cot 6A \cot 5A$$

e. 左邊$=\dfrac{2\cos(2\theta - A)\cos A + 2\cos(2\theta - B - C)\cos(B - C)}{2\sin(2\theta - A)\cos A + 2\cos(2\theta - B - C)\sin(B - C)}$

$$=\dfrac{\cos(B + C) + \cos(B - C)}{\sin(B + C) + \sin(B - C)}$$

$$=\dfrac{2\cos B \cos C}{2\sin B \cos C} = \cot B$$

f. 左邊$=\dfrac{2\sin nA \cos(A - n) \pm \sin nA}{2\cos nA \cos(A - n) \pm \sin nA}$

$$=\dfrac{\sin nA[2\cos(A - n) \pm 1]}{\cos nA[2\cos(A - n) \pm 1]} = \tan nA$$

g. 左邊$= \cos x + 2\cos 120°\cos x$

$$= \cos x + 2(-\tfrac{1}{2})\cos x = 0$$

h. 左邊$= 2\sin 30°\cos 20° + 2\sin 30°\cos 10°$

$$= 2\sin 30°(\cos 20° + \cos 10°)$$

$$= \cos 20° + \cos 10°$$

$$=\cos(90^\circ-70^\circ)+\cos(90^\circ-80^\circ)$$

$$=\sin 70^\circ+\sin 80^\circ$$

i. 左邊$=\dfrac{2\sin 2A\sin A+2\sin 4A\sin A+2\sin 6A\sin A}{2\sin A}$

$$=\dfrac{\cos A-\cos 3A+\cos 3A-\cos 5A+\cos 5A-\cos 7A}{2\sin A}$$

$$=\dfrac{\cos A-\cos 7A}{2\sin A}$$

j. 左邊$=\dfrac{\cos 2na+\cos 2a}{2}+\sin^2 a$

$$=\dfrac{(2\cos^2 na-1)+(1-2\sin^2 a)}{2}+\sin^2 a$$

$$=\cos^2 na$$

k. 左邊$=(\sin 5x+\sin x)-\sin x$

$$=2\sin 3x\cos 2x-\sin x$$

l. $\because\quad \cos 5x-\cos x=-2\sin 3x\sin 2x$

$\therefore\quad \cos 5x=\cos x-2\sin 3x\sin 2x$

m. 左邊$=\cos(x-y)-\cos\left[\dfrac{\pi}{2}+(x+y)\right]$

$$=2\sin\left(\dfrac{\pi}{4}+y\right)\sin\left(\dfrac{\pi}{4}+x\right)$$

n. 左邊$=\cos\left[\dfrac{\pi}{2}-(x+y)\right]-\cos(x-y)$

$$=2\sin\left(\dfrac{\pi}{2}-y\right)\sin\left(x-\dfrac{\pi}{4}\right)$$

$$=-2\sin\left(x-\dfrac{\pi}{4}\right)\sin\left(y-\dfrac{\pi}{4}\right)$$

o. 左邊 $= \cos\left(\frac{\pi}{2} - a\right) - \cos\beta$

$= 2\sin\left[\frac{\pi}{4} - \frac{1}{2}(a-\beta)\right]\sin\left[\frac{1}{2}(a+\beta) - \frac{\pi}{4}\right]$

又　$= -\left[\cos\left(\frac{\pi}{2} + a\right) + \cos\beta\right]$

$= -2\cos\left[\frac{\pi}{4} + \frac{1}{2}(a+\beta)\right]\cos\left[\frac{\pi}{4} + \frac{1}{2}(a-\beta)\right]$

p. 左邊 $= \frac{1}{2}(\cos 2\beta - \cos 2a)$

$= \frac{1}{2}[1 - 2\sin^2\beta - (1 - 2\sin^2 a)] = \sin^2 a - \sin^2\beta$

q. 左邊 $= \frac{1}{2}[\cos(2n+2)A + \cos 2nA]$

$= \frac{1}{2}[2\cos^2(n+1)A - 1 + 1 - 2\sin^2 nA]$

$= \cos^2(n+1)A - \sin^2 nA$

r. 左邊 $= \frac{1}{2}[\cos(a+\beta) + \cos(a-\beta)]$

$= \frac{1}{2}[2\cos^2\frac{1}{2}(a+\beta) + 2\cos^2\frac{1}{2}(a-\beta) - 2]$

$=$ 右邊

s. 設　$2a - \beta = A$,　　$2\beta - a = B$,

則　$A + B = a + \beta$,　$A - B = 3a - 3\beta$,　其餘如 2(p).

t. 左邊 $= 2\cos(x+y)\cos z + 2\cos z\cos(x-y)$

$= 2\cos z[\cos(x+y) + \cos(x-y)] = 4\Pi\cos x$

u. $\because$　$\cos a + \cos\beta + \cos\gamma + \cos(a+\beta+\gamma)$

$= 4\Pi\cos\frac{1}{2}(a+\beta)$　（見上題）

故　$\Sigma\cos a = 4\Pi\cos\frac{1}{2}(a+\beta) - \cos(a+\beta+\gamma)$

3. a. 原式 $= \dfrac{\sin a\cos\beta - \cos a\sin\beta}{\sin a\sin\beta} + \cdots\cdots$

$$=\cot\beta-\cot\alpha+\cot\gamma-\cot\beta+\cot\alpha-\cot\gamma=0$$

b. 左邊$=\frac{1}{2}[\sin(\alpha+\beta-\gamma-\delta)-\sin(\alpha+\gamma-\beta-\delta)$

$$+\cdots\cdots]=\frac{1}{2}\cdot 0=0$$

c. 原式$=\dfrac{\sin\beta\sin(\alpha+\beta)}{\cos\beta[2\sin^2\frac{1}{2}(\alpha+\beta)]}+\dfrac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}$

$$=\dfrac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}+\dfrac{\sin[\frac{1}{2}(\alpha+\beta)-\beta]}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}$$

$$=\dfrac{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}{\cos\beta\sin\frac{1}{2}(\alpha+\beta)}=1$$

習 題 十 三 (127—138 頁)

1. 左邊$=2\sin^2\alpha\cdot 2\sin\alpha\cos 3\alpha+2\cos^2\alpha\cdot 2\cos\alpha\sin 3\alpha$

$$=2\sin^2\alpha(\sin 4\alpha-\sin 2\alpha)+2\cos^2\alpha(\sin 4\alpha+\sin 2\alpha)$$

$$=2\sin 4\alpha(\sin^2\alpha+\cos^2\alpha)+2\sin 2\alpha(\cos^2\alpha-\sin^2\alpha)$$

$$=2\sin 4\alpha+2\sin 2\alpha\cos 2\alpha=3\sin 4\alpha$$

2. 左邊$=\dfrac{1}{\tan 3\theta+\tan\theta}-\dfrac{\tan 3\theta\tan\theta}{\tan 3\theta+\tan\theta}=\dfrac{1-\tan 3\theta\tan\theta}{\tan 3\theta+\tan\theta}$

$$=\dfrac{1}{\tan(3\theta+\theta)}=\dfrac{1}{\tan 4\theta}=\cot 4\theta$$

3. 左邊$=\dfrac{\sin\theta(\cos\theta-\sin\phi)}{\cos^2\theta-\sin^2\phi}+\dfrac{\sin\phi(\cos\phi+\sin\theta)}{\cos^2\phi-\sin^2\theta}$

$$\cos^2\theta-\sin^2\phi=1-\sin^2\theta-(1-\cos^2\phi)=\cos^2\phi-\sin^2\theta$$

因上式分母同,故可改變分子之排列。

$$\therefore\quad 左邊=\dfrac{\sin\theta(\cos\theta+\sin\phi)}{\cos^2\theta-\sin^2\phi}+\dfrac{\sin\phi(\cos\phi-\sin\theta)}{\cos^2\phi-\sin^2\theta}$$

$$= \frac{\sin\theta}{\cos\theta - \sin\phi} + \frac{\sin\phi}{\cos\phi + \sin\theta}$$

4. 左邊 $= \left(\dfrac{1 - \tan^2\dfrac{\theta}{2}}{\tan\dfrac{\theta}{2}}\right)^2 = 4\left(\dfrac{1 - \tan^2\dfrac{\theta}{2}}{2\tan\dfrac{\theta}{2}}\right)^2$

$$= 4\left(\frac{1}{\tan\theta}\right)^2 = \frac{4}{\tan^2\theta}$$

右邊 $= 4 \left/ \left(1 - 2\tan\theta \times \dfrac{1 - \tan^2\theta}{2\tan\theta}\right)\right. = \dfrac{4}{\tan^2\theta}$

5. 左邊 $= \tan(2x+x)\tan(2x-x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \cdot \frac{\tan 2x - \tan x}{1 + \tan 2x \tan x}$$

$$= \frac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x}$$

6. $\cos 2x = \dfrac{\cos x \cos 2x}{\cos(2x - x)} = \dfrac{\cos x \cos 2x}{\cos x \cos 2x + \sin x \sin 2x}$

$$= \frac{1}{1 + \tan x \tan 2x} \qquad \text{(分子,分母同以 } \cos x \cos 2x \text{ 除之)}$$

7. 左邊 $= \dfrac{\sin 3x \sin x}{\sin x} = \dfrac{\cos 2x - \cos 4x}{2\sin x}$

$$= \frac{1 - 2\sin^2 x - (1 - 2\sin^2 2x)}{2\sin x} = \frac{\sin^2 2x - \sin^2 x}{\sin x}$$

8. 左邊 $= \left(\dfrac{1}{\cos x} + \dfrac{\cos x + 1}{\sin x \cos x}\right) \cdot \dfrac{\cos x}{\cos^2\dfrac{x}{2}} \cdot \dfrac{\cos\dfrac{x}{2}}{\cos^2\dfrac{x}{4}}$

$$= \frac{\sin x + (\cos x + 1)}{\sin x \cos\dfrac{x}{2}} \cdot \frac{1}{\cos^2\dfrac{x}{4}}$$

$$= \frac{2\cos\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos^2\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{4}}$$

$$= \frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} \cdot \sec^2\frac{x}{4} = \left(\sec\frac{x}{2}+\csc\frac{x}{2}\right)\sec^2\frac{x}{4}$$

9. 左邊 $= \left(\frac{\sin^2 3A}{\cos^2 3A \sin^2 A}-\frac{1}{\cos^2 3A}\right) \cdot \frac{\cos 4A}{\sin 4A}$

$$= \frac{(\sin 3A-\sin A)(\sin 3A+\sin A)}{\cos^2 3A \sin^2 A} \cdot \frac{\cos 4A}{2\sin 2A \cos 2A}$$

$$= \frac{2\sin A \cos A \cos 4A}{\cos^2 3A \sin^2 A} = \frac{\cos 2A \cos 4A}{\cos^2 3A \sin^2 A} \cdot \frac{\sin 2A}{\cos 2A}$$

$$= \frac{\cos 6A + \cos 2A}{2\cos^2 3A \sin^2 A}\tan 2A = \frac{\cos^2 3A-\sin^2 A}{\cos^2 3A \sin^2 A}\tan 2A$$

$$= (\csc^2 A-\sec^2 3A)\tan 2A$$

$$= \left(\csc^2\frac{x}{6}-\sec^2\frac{x}{2}\right)\tan\frac{x}{3}$$

10. 左邊 $= \sqrt{(1-\cos \alpha)(1-\cos \beta)} = 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

$$= \cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}(\alpha+\beta)$$

$$= [1-\cos\frac{1}{2}(\alpha+\beta)]-[1-\cos\frac{1}{2}(\alpha-\beta)]$$

$$= \text{vers}\frac{1}{2}(\alpha+\beta)-\text{vers}\frac{1}{2}(\alpha-\beta)$$

11. $\because$　$\cos 5\alpha + \cos \alpha = 2\cos 3\alpha \cos 2\alpha$

$$= 2(4\cos^3\alpha-3\cos \alpha)(2\cos^2\alpha-1)$$

∴　$\cos 5\alpha = 16 \cos^5\alpha - 20 \cos^3\alpha + 5 \cos \alpha$

12.　∵　$4\cos^3 x = \cos 3x + 3 \cos x,\quad 2\cos^2 x = \cos 2x + 1$

∴　$16 \cos^5 x = 2 \cos 3x \cos 2x + 6 \cos 2x \cos x$

$$+ 2 \cos 3x + 6 \cos x$$

$$= \cos 5x + 5 \cos 3x + 10 \cos x$$

13.　左邊 $= 2 \cos 60° \cos 20° - \cos 20° = \cos 20° - \cos 20° = 0$

14.　左邊 $= \dfrac{\sin\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)}{\cos\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)} + \dfrac{\cos\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)}{\sin\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)}$

$$= \dfrac{\sin^2\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right) + \cos^2\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)}{\sin\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)}$$

$$= \dfrac{2}{\sin\left(\dfrac{\pi}{2} + x\right)} = \dfrac{2}{\cos x}$$

15.　右邊 $= \dfrac{2(\sin 2x + \cos 2x)}{2(\cos 2x \sin x + \sin 2x \sin x)} = \csc x$

$$= \dfrac{\sin 2x + \cos 2x}{\sin x(\sin 2x + \cos 2x)} = \dfrac{1}{\sin x} = \csc x$$

16.　左邊 $= \dfrac{1}{\cos x}(\sin 8x + \sin 6x + 8 \sin 2x \cos^2 x)$

$$= \dfrac{\sin 6x + 4\sin 2x \cos 2x(2\cos^2 2x - 1) + 8 \sin 2x \cos^2 x}{\cos x}$$

$$= \dfrac{1}{\cos x}[\sin 6x + 8 \sin 2x \cos^3 2x - 4 \sin 2x(2\cos^2 x$$

$$-1 - 2 \cos^2 x)]$$

$$= \frac{\sin 6x + 4\sin 2x(1 + 2\cos^3 2x)}{\cos x}$$

17. 左邊 $= \csc^2 x \sec^2 x (\sin nx \cos x - \cos nx \sin x)$

$$= \frac{4}{(2\sin x \cos x)^2} \cdot \sin(nx - x) = \frac{4}{\sin^2 2x} \cdot \sin(n-1)x$$

$$= 4\sin(n-1)x \cdot \csc^2 2x$$

18. 左邊 $= \dfrac{\sin \frac{1}{2}(\alpha+\beta)\sin \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)\cos \frac{1}{2}(\alpha-\beta)} = \dfrac{\cos\beta - \cos\alpha}{\cos\alpha + \cos\beta}$

$$= \frac{2\sin\alpha\sin\beta(\cos\beta - \cos\alpha)}{2\sin\alpha\sin\beta(\cos\alpha + \cos\beta)}$$

$$= \frac{\sin 2\beta \sin\alpha - \sin 2\alpha \sin\beta}{\sin 2\beta \sin\alpha + \sin 2\alpha \sin\beta}$$

$$= \frac{\csc 2\alpha \csc\beta - \csc 2\beta \csc\alpha}{\csc 2\alpha \csc\beta + \csc 2\beta \csc\alpha}$$

（分子分母同以 $\sin\alpha \sin\beta \sin 2\alpha \sin 2\beta$ 除之）

19. 左邊 $= \cdots\cdots\cdots\cdots$（乘出）

$$= x^2[\cos 2\alpha \cos 2\beta - \cos^2(\alpha+\beta)]$$

$$+ y^2[\sin 2\alpha \sin 2\beta - \sin^2(\alpha+\beta)] + \sin^2(\alpha-\beta)$$

$$= -x^2\sin^2(\alpha-\beta) - y^2\sin^2(\alpha-\beta) + \sin^2(\alpha-\beta)$$

（參考習題十，8. b, c.）

$$= (1 - x^2 - y^2)\sin^2(\alpha-\beta)$$

或從 x^2 之係數 $\frac{1}{2}\cos 2(\alpha+\beta) + \frac{1}{2}\cos 2(\alpha-\beta)$

$$-\cos^2(\alpha+\beta) = \cdots\cdots = -\sin^2(\alpha-\beta)$$ 做

20. 設 $2^{2n-2}\alpha = A$ 則 $2^{2n-1}\alpha = 2A$，　$2^{2n}\alpha = 4A$

故本題可變爲　$\dfrac{\sec 4A-1}{\sec 2A-1}=\dfrac{\tan 4A}{\tan A}$　便易做矣

註：例八（99頁）亦以用上法令 $2^{2n-1}=B$ 代入做爲簡便.

21. 今　$(\cos x+i\sin x)^7=\cos^7 x+7i\cos^6 x\sin x$
$$-21\cos^5 x\sin^2 x\cdots\cdots$$

但　$(\cos x+i\sin x)^7=\cos 7x+i\sin 7x$

比較含 i 之項，則得

$$\sin 7x=7\cos^6 x\sin x-35\cos^4 x\sin^3 x$$
$$+21\cos^2 x\sin^5 x-\sin^7 x$$
$$=7(1-\sin^2 x)^3\sin x-35(1-\sin^2 x)^2\sin^3 x$$
$$+21(1-\sin^2 x)\sin^5 x-\sin^7 x$$
$$=7\sin x-56\sin^3 x+112\sin^5 x-64\sin^7 x$$

22. 左邊 $=2\sin 3x\cos 3x$
$$=2(3\sin x-4\sin^3 x)(4\cos^3 x-3\cos x)$$
$$=2\sin x[3-4(1-\cos^2 x)](4\cos^3 x-3\cos x)$$
$$=2\sin x(16\cos^5 x-16\cos^3 x+3\cos x)$$

23. 設　$y=\cos x+i\sin x,$　　則　$y^{-1}=\cos x-i\sin x$

∴ $y-y^{-1}=2i\sin x,$　　又　$y^n+y^{-n}=2\cos nx$

今　$(y-y^{-1})^6=-64\sin^6 x=(y^6+y^{-6})-6(y^4+y^{-4})$
$$+15(y^2+y^{-2})-20$$

∴ $-32\sin^6 x=\cos 6x-6\cos 4x+15\cos 2x-10.$

24. 如上題用棣美弗定理做，從 $(4\cos^3 x)^2\cos^2 x$ 做亦可.

25. 左邊 $=\sin(4x+x)+\cos(4x+x)$

$$= \cos 4x(\sin x + \cos x) + \sin 4x(\cos x - \sin x)$$

$$= (\sin x + \cos x)[\cos 4x + 2 \sin 2x(\cos x - \sin x)^2]$$

$$= (\sin x + \cos x)[\cos 4x + 2 \sin 2x(1 - \sin 2x)]$$

$$= (\sin x + \cos x)[\cos 4x + 2 \sin 2x - 2 \sin^2 2x]$$

$$= (\sin x + \cos x)[\cos 4x + 2 \sin 2x - (1 - \cos 4x)]$$

$$= 左邊$$

26. 左邊 $= \{\sin \beta[\sin(\alpha+2\beta)\sin(\alpha+3\beta) + \sin \alpha \sin(\alpha+3\beta)$

$$+ \sin \alpha \sin(\alpha+\beta)]\} \div [\sin(\alpha+\beta)\sin(\alpha+2\beta)$$

$$\cdot \sin(\alpha+3\beta)]$$

被除數 $= \frac{1}{2} \sin \beta[\cos \beta - \cos(2\alpha+5\beta) + \cos 3\beta$

$$- \cos(2\alpha+3\beta) + \cos \beta - \cos(2\alpha+\beta)]$$

$$= \frac{1}{2} \sin \beta \sin(\alpha+\beta)[2 \sin(\alpha+2\beta)$$

$$+ 2 \sin(\alpha+4\beta) + 2 \sin \alpha]$$

$$= \frac{1}{2} \sin(\alpha+\beta)[\cos(\alpha-\beta) - \cos(\alpha+5\beta)]$$

$$= \sin(\alpha+\beta)\sin(\alpha+2\beta)\sin 3\beta$$

故原式左邊 $= \dfrac{\sin 3\beta}{\sin(\alpha+3\beta)} = \sin 3\beta \csc(\alpha+3\beta)$

27. 右邊 $= \dfrac{\sec^2 \frac{1}{2}\theta}{a \sec^2 \frac{1}{2}\theta + b(1 - \tan^2 \frac{1}{2}\theta)}$

$$= \dfrac{\sec^2 \frac{1}{2}\theta}{a \sec^2 \frac{1}{2}\theta + b(\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta)\sec^2 \frac{1}{2}\theta} = \dfrac{1}{a + b \cos \theta}$$

$$\therefore \quad 左邊 = 右邊$$

28. 左邊 $= \frac{1}{2} \cos 20° \cos 40° \cos 80°$

$$= \frac{1}{4} (\cos 60° + \cos 20°) \cos 80°$$

$$= \frac{1}{4} (\cos 60° \cos 80° + \cos 20° \cos 80°)$$

$$= \tfrac{1}{8}\left(\cos 80^\circ + \cos 100^\circ + \cos 60^\circ\right)$$

$$= \tfrac{1}{8}\left(\cos 80^\circ - \cos 80^\circ + \cos 60^\circ\right) = 1/16$$

29. ∵　$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = 3/16$ 　　　（例十三）

　　又　$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = 1/16$ 　　（28 題）

　　∴　$\tan 20^\circ \tan 40^\circ \tan 60^\circ \tan 80^\circ = 3$

30. 左邊 $= (\tan 9^\circ + \tan 81^\circ) - (\tan 27^\circ + \tan 63^\circ)$

$$= \frac{\sin 90^\circ}{\cos 9^\circ \cos 81^\circ} - \frac{\sin 90^\circ}{\cos 27^\circ \cos 63^\circ}$$

$$= \frac{2}{\cos 72^\circ} - \frac{2}{\cos 36^\circ} = 2\left(\frac{\cos 36^\circ - \cos 72^\circ}{\cos 72^\circ \cos 36^\circ}\right)$$

$$= 2\left(\frac{2 \sin 18^\circ \sin 54^\circ}{\sin 18^\circ \sin 54^\circ}\right) = 4$$

31. 左邊 $= \dfrac{\cos 10^\circ - \sqrt{3}\,\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$

$$= \frac{4(\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ)}{\sin 20^\circ}$$

$$= \frac{4 \sin 20^\circ}{\sin 20^\circ} = 4$$

32. 左邊 $= \dfrac{1}{2}\lbrack \cos 120^\circ + \cos 40^\circ + \cos 240^\circ + \cos 80^\circ$

　　　　　$+ \cos 200^\circ + \cos 120^\circ \rbrack$

$$= \cos 120^\circ + \frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2}\cos 200^\circ + \cos 60^\circ \cos 20^\circ$$

$$= -\frac{1}{2} - \frac{1}{4} - \frac{1}{2}\cos 20^\circ + \frac{1}{2}\cos 20^\circ = -\frac{3}{4}$$

33. ∵　$\sqrt{3} = \tan 60^\circ = \dfrac{\tan 20^\circ + \tan 40^\circ}{1 - \tan 20^\circ \tan 40^\circ}$ 　去分母移項卽得

34. $\cos 9° + \sin 9° = \sqrt{1 + \sin 18°} = \sqrt{1 + \frac{1}{4}(\sqrt{5} - 1)}$

$$= \frac{1}{2}\sqrt{3 + \sqrt{5}}$$

$\cos 9° - \sin 9° = \sqrt{1 - \sin 18°} = \frac{1}{2}\sqrt{5 - \sqrt{5}}$

$$\therefore \quad \cos 9° = \frac{1}{4}(\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}})$$

35. 左邊 $= \tan\left(\frac{\pi}{4} - \frac{\pi}{10}\right) = \dfrac{\tan\frac{\pi}{4} - \tan\frac{\pi}{10}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{10}} = \dfrac{1 - \tan\frac{\pi}{10}}{1 + \tan\frac{\pi}{10}}$

$$= \dfrac{\left(1 - \tan\frac{\pi}{10}\right)^2}{1 - \tan^2\frac{\pi}{10}} = \dfrac{(1 - \frac{1}{5}\sqrt{25 - 10\sqrt{5}})^2}{1 - \frac{1}{25}(25 - 10\sqrt{5})}$$

$$= \dfrac{2 - \frac{2\sqrt{5}}{5}\sqrt{5 - 2\sqrt{5}} - \frac{2}{5}\sqrt{5}}{\frac{2}{5}\sqrt{5}}$$

$$= \sqrt{5} - 1 - \sqrt{5 - 2\sqrt{5}} = 右邊$$

36. 左邊 $= \frac{1}{2}\{3 + \cos 2A + \cos(120° + 2A) + \cos(120° - 2A)\}$

$$= \frac{3}{2} + \frac{1}{2}(\cos 2A + 2\cos 120°\cos 2A) = \frac{3}{2}$$

37. 左邊 $= \cos^8\frac{\pi}{8} + \cos^8\left(\pi - \frac{3\pi}{8}\right) + \cos^8\frac{3\pi}{8} + \cos^8\left(\pi - \frac{\pi}{8}\right)$

$$= 2\left(\cos^8\frac{\pi}{8} + \cos^8\frac{3\pi}{8}\right)$$

$$= \frac{1}{8}\left[\left(1 + \cos\frac{\pi}{4}\right)^4 + \left(1 + \cos\frac{3\pi}{4}\right)^4\right]$$

$$= \frac{1}{8}\left[\left(\frac{2+\sqrt{2}}{2}\right)^4 + \left(\frac{2-\sqrt{2}}{2}\right)^4\right] = \frac{1}{8}\left(\frac{136}{16}\right) = \frac{17}{16}$$

38. 設　$\frac{2}{7}\pi = \alpha$，又設原式爲　x，兩邊乘　$2\sin\alpha$

則　$2\sin\alpha \cdot x = \sin 2\alpha + (\sin 3\alpha - \sin\alpha) + (\sin 4\alpha - \sin 2\alpha)$

$\qquad = \sin 3\alpha + \sin 4\alpha - \sin\alpha$

但　$\sin 4\alpha = \sin\dfrac{8}{7}\pi = -\sin\dfrac{1}{7}\pi = -\sin\dfrac{6}{7}\pi = -\sin 3\alpha$

$\therefore$　$2\sin\alpha \cdot x = -\sin\alpha$　　　　　$\therefore$　$x = -\dfrac{1}{2}$

或　　$x = 2\cos 2\alpha\cos\alpha + (2\cos^2\alpha - 1)$

$\qquad = 2\cos\alpha[\cos 2\alpha + \cos\alpha] - 1$

$\qquad = 2\cos\alpha(\cos 2\alpha + \cos 6\alpha) - 1$

$\qquad = 4\cos\alpha\cos 2\alpha\cos 4\alpha - 1$

$\qquad = \dfrac{\sin 8\alpha}{2\sin\alpha} - 1 = \dfrac{\sin\alpha}{2\sin\alpha} - 1 = \dfrac{1}{2} - 1 = -\dfrac{1}{2}$

註： 例十六亦以用上兩法較便，尤以第一法爲簡．

39. 左邊 $= \dfrac{1}{2}\left(2\cos\dfrac{2\pi}{7} - 2\cos\dfrac{\pi}{7} - 2\cos\dfrac{3\pi}{7}\right)$

$\qquad = -\left(\cos\dfrac{3\pi}{7} - \cos\dfrac{2\pi}{7} + \cos\dfrac{\pi}{7}\right)$　　以下如 38 題

40. 左邊 $= 2\cos\dfrac{\pi}{3}\cos\dfrac{\pi}{36} - \cos\dfrac{\pi}{36} = \cos\dfrac{\pi}{36} - \cos\dfrac{\pi}{36} = 0$

41. 左邊 $= \dfrac{1}{2}\left(\cos\dfrac{2\pi}{3} + \cos\dfrac{\pi}{18}\right)\left(-\cos\dfrac{\pi}{36}\right)$

$\qquad = \dfrac{1}{4}\cos\dfrac{\pi}{36} - \dfrac{1}{4}\left(\cos\dfrac{\pi}{12} + \cos\dfrac{\pi}{36}\right) = -\dfrac{1}{4}\cos\dfrac{\pi}{12}$

$$= -\frac{1}{4} \cdot \frac{\sqrt{6}+\sqrt{2}}{4} = -\frac{1+\sqrt{3}}{8\sqrt{2}} \quad (102 頁表 13)$$

42. 從 $\quad 8\cos^4\theta = 2(\cos 2\theta + 1)^2$

求到 $\quad \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

再以 $\quad \alpha, \ \frac{\pi}{3}+\alpha, \ \frac{\pi}{3}-\alpha$ 次第代 θ, 再合併之, 則原式

左邊 $= \frac{1}{8}[\Sigma \cos 4\alpha + 4\Sigma \cos 2\alpha + 9]$

$$= \frac{1}{8}[9 + \cos 4\alpha + 2\cos\frac{4\pi}{3}\cos 4\alpha + 4(\cos 2\alpha$$

$$+ 2\cos\frac{2\pi}{3}\cos 2\alpha)]$$

$$= \frac{1}{8}[9 + \cos 4\alpha - \cos 4\alpha + 4(\cos 2\alpha - \cos 2\alpha)] = \frac{9}{8}$$

43. 設 $\quad \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{9\pi}{13} = x,$

$$\cos\frac{5\pi}{13} + \cos\frac{7\pi}{13} + \cos\frac{11\pi}{13} = y.$$

則 $x+y = \cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} + \cos\frac{7\pi}{13} + \cos\frac{9\pi}{13} + \cos\frac{11\pi}{13}$

$$= 2\cos\frac{\pi}{13}\left(\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{10\pi}{13}\right)$$

$$= \frac{\sin\frac{2\pi}{13}\left(\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{10\pi}{13}\right)}{\sin\frac{\pi}{13}}$$

$$= \frac{\frac{1}{2}\left(\sin\frac{4\pi}{13} + \sin\frac{8\pi}{13} - \sin\frac{4\pi}{13} + \sin\frac{12\pi}{13} - \sin\frac{8\pi}{13}\right)}{\sin\frac{\pi}{13}}$$

$$= \frac{\frac{1}{2}\sin\frac{12\pi}{13}}{\sin\frac{\pi}{13}} = \frac{\frac{1}{2}\sin\frac{\pi}{13}}{\sin\frac{\pi}{13}} = \frac{1}{2}$$

$$xy = \left(\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{9\pi}{13}\right)\left(\cos\frac{5\pi}{13} + \cos\frac{7\pi}{13} + \cos\frac{11\pi}{13}\right)$$

$$= -\left(\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{9\pi}{13}\right)\left(\cos\frac{8\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{2\pi}{13}\right)$$

乘出再用公式(30)化積爲和式，又化其間之

$$\cos\frac{15\pi}{13} = \cos\frac{11\pi}{13}, \qquad \cos\frac{17\pi}{13} = \cos\frac{9\pi}{13}$$

合併之得

$$xy = -\frac{3}{2}\left(\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{9\pi}{13} + \cos\frac{5\pi}{13} + \cos\frac{7\pi}{13} + \cos\frac{11\pi}{13}\right)$$

$$= -\frac{3}{2}(x+y) = -\frac{3}{2}\left(\frac{1}{2}\right) = -\frac{3}{4}$$

因此解 x, y 得

$$x - y = \frac{1}{2}\sqrt{13} \quad (取+號)$$

$$\therefore \quad x = \frac{1}{4}(1 + \sqrt{13}),$$

$$y = \frac{1}{4}(1 - \sqrt{13}).$$

44. 左邊 $= 2\cos\frac{1}{2}(x-z)\cos\frac{1}{2}(x+z-2y) + 2\cos^2\frac{1}{2}(z-x) - 1$

$$= -1 + 2\cos\frac{1}{2}(x-z)\left[\cos\frac{1}{2}(x+z-2y)\right.$$

$$\left. + \cos\frac{1}{2}(x-z)\right]$$

$$= -1 + 4 \cos\frac{1}{2}(x-z) \cos\frac{1}{2}(x-y) \cos\frac{1}{2}(z-y)$$

45. 右邊 $= 2 \cos(\alpha+\beta)\cos \gamma + 2 \cos \gamma \cos(\alpha-\beta)$

$$= 2 \cos \gamma [\cos(\alpha+\beta) + \cos(\alpha-\beta)] = 4\Pi \cos \alpha$$

46. 左邊 $= 2 \cos(\alpha+\beta)\cos(\alpha-\beta) + 2 \cos(\alpha+\beta+2\gamma)$

$$\times \cos(\alpha+\beta)$$

$$= 2 \cos(\alpha+\beta)[\cos(\alpha-\beta) + \cos(\alpha+\beta+2\gamma)]$$

$$= 4\Pi \cos(\alpha+\beta)$$

47. 左邊 $= \dfrac{1}{2\sin(\beta-\gamma)}[\sin(2\beta-\alpha) + \sin(2\beta-2\gamma-\alpha)$

$$- \sin(2\gamma-\alpha) + \sin(2\beta-2\gamma+\alpha)]$$

$$= \frac{1}{\sin(\beta-\gamma)}[\sin(2\beta-\alpha-\gamma)\cos \gamma$$

$$+ \sin(\alpha+\beta-2\gamma)\cos \beta]$$

$$= \frac{1}{\sin(\beta-\gamma)}[\sin(\alpha+\beta-2\gamma)\cos \beta$$

$$- \sin(\alpha+\gamma-2\beta)\cos \gamma]$$

48. 左邊 $= \dfrac{\Sigma \sin \alpha \sin(\beta-\gamma)}{\Pi \sin(\alpha-\beta)}$

$$= \frac{\Sigma[\cos(\alpha+\gamma-\beta) - \cos(\alpha+\beta-\gamma)]}{2\Pi \sin(\alpha-\beta)} = 0$$

49. 原式 $= -\left[\dfrac{\tan \alpha \tan(\beta-\gamma) + \tan \beta \tan(\gamma-\alpha)}{\Pi \tan(\alpha-\beta)}\right.$

$$\left. + \frac{\tan \gamma \tan(\alpha-\beta)}{\Pi \tan(\alpha-\beta)}\right]$$

設分子爲 N, 又令 $a = \tan \alpha$, $b = \tan \beta$, $c = \tan \gamma$, 則

$$N = \frac{a(b-c)}{1+bc} + \frac{b(c-a)}{1+ca} + \frac{c(a-b)}{1+ab}$$

$$= \frac{a(b-c)(1+ca)(1+ab) + b(c-a)(1+bc)(1+ab)}{\Pi(1+bc)}$$

$$+ \frac{c(a-b)(1+bc)(1+ca)}{\Pi(1+bc)}$$

$$= \frac{[\Sigma a(b-c) + \Sigma a^2(b^2-c^2) + \Sigma a^3 bc(b-c)]}{\Pi(1+bc)}$$

$$= \frac{[o+o+abc \Sigma a^2(b-c)]}{\Pi(1+bc)}$$

但從代數知　$\Sigma a^2(b-c) = -\Pi(b-c)$

故 $N = \dfrac{-abc \, \Pi(b-c)}{\Pi(1+bc)} = -abc \cdot \dfrac{b-c}{1+bc} \cdot \dfrac{c-a}{1+ca} \cdot \dfrac{a-b}{1+ab}$

$$= -\tan \alpha \tan \beta \tan \gamma \tan(\beta-\gamma)\tan(\gamma-\alpha)$$

$$\times \tan(\alpha-\beta)$$

故原式 $= -\dfrac{-\Pi \tan \alpha \cdot \Pi \tan(\alpha-\beta)}{\Pi \tan(\alpha-\beta)} = \Pi \tan \alpha$

50. 左邊 $= \Sigma \dfrac{\sin(\alpha-\beta)[\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)]}{-2\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$

$$= -\Sigma \dfrac{\sin 2(\alpha-\beta) - \sin 2(\theta-\beta) + \sin 2(\theta-\alpha)}{4\Pi \sin(\alpha-\beta)}$$

$$= -\dfrac{1}{4\Pi \sin(\alpha-\beta)}[\sin 2(\alpha-\beta) + \sin 2(\beta-\gamma)$$

$$+ \sin 2(\gamma-\alpha)]$$

$$= -\dfrac{1}{4\Pi \sin(\alpha-\beta)}[-4\sin(\alpha-\beta)\sin(\beta-\gamma)$$

$$\times \sin(\gamma-\alpha)] = 1 \quad （見 108 頁例十七）$$

51. 左邊 $= \Sigma \dfrac{\tan^2\alpha - \tan^2\beta}{\tan\alpha\tan\beta} = \Sigma \dfrac{\sin^2\alpha\,\cos^2\beta - \cos^2\alpha\,\sin^2\beta}{\sin\alpha\,\sin\beta\,\cos\alpha\,\cos\beta}$

$= \Sigma \dfrac{4(\cos^2\beta - \cos^2\alpha)}{\sin 2\alpha\,\sin 2\beta} = \dfrac{2\Sigma\sin 2\gamma(\cos 2\beta - \cos 2\alpha)}{\Pi\sin 2\alpha}$

$= \dfrac{\Sigma[\sin 2(\gamma+\beta) + \sin 2(\gamma-\beta) + \sin 2(\alpha-\gamma) - \sin 2(\alpha+\gamma)]}{\Pi\sin 2\alpha}$

$= \dfrac{2\Sigma\sin 2(\beta-\alpha)}{\Pi\sin 2\alpha}$

52. 左邊 $= \Sigma \dfrac{\sin(\theta-\beta-\alpha)\sin(\theta-\gamma-\alpha)}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}$

$= -\Sigma \dfrac{\cos(\beta-\gamma) - \cos(2\theta-2\alpha-\beta-\gamma)}{2\sin(\alpha-\beta)\sin(\gamma-\alpha)}$

$= -\dfrac{\Sigma[\cos(\beta-\gamma) - \cos(2\theta-2\alpha-\beta-\gamma)]\sin(\beta-\gamma)}{2\Pi\sin(\beta-\gamma)}$

分子 $= -\tfrac{1}{2}[\sin 2(\beta-\gamma) - \sin 2(\theta-\alpha-\gamma)$

$\qquad + \sin 2(\theta-\alpha-\beta) + \cdots\cdots]$

$= -\tfrac{1}{2}\Sigma\sin 2(\beta-\gamma) = 2\Pi\sin(\beta-\gamma)$ （見例十七）

故原式爲 1

53. 原式 $= \sin(\beta-\gamma)[\sin^2\alpha\,\sin(\gamma-\delta)\sin(\delta-\beta)$

$\qquad - \sin^2\delta\,\sin(\alpha-\beta)\sin(\gamma-\alpha)]$

$\qquad + \sin(\delta-\alpha)[\cdots\cdots\cdots\cdots]$

$= \tfrac{1}{4}\sin(\beta-\gamma)\{[\cos(\alpha-\gamma+\delta) - \cos(\alpha+\gamma-\delta)]$

$\qquad \times [\cos(\alpha+\beta-\delta) - \cos(\alpha-\beta+\delta)] - \cdots\} + \cdots\cdots$

$= \tfrac{1}{4}\sin(\beta-\gamma)[\cos(\alpha-\gamma+\delta)\cos(\alpha+\beta-\delta)$

$\qquad - \cos(\alpha+\gamma-\delta)\cos(\alpha+\beta-\delta) + \cdots\cdots] + \cdots\cdots$

$$= \tfrac{1}{8}\sin(\beta-\gamma)[\cos(2\alpha+\beta-\gamma)+2\cos(\beta+\gamma-2\delta)$$

$$-\cos(2\alpha+\beta+\gamma-2\delta)+\cos(2\alpha-\beta+\gamma)+\cdots]$$

$$+\cdots\cdots$$

$$=\tfrac{1}{16}[\sin 2(\alpha+\beta-\gamma)-\sin 2(\alpha+\beta-\delta)$$

$$+\sin 2(\alpha+\gamma-\delta)-\sin(\alpha-\beta+\gamma)+\cdots]$$

$$+\tfrac{1}{8}[\sin 2(\beta-\delta)-\sin 2(\gamma-\delta)-\sin 2(\alpha-\gamma)$$

$$+\sin 2(\alpha-\beta)]+\cdots\cdots=0$$

54. 因　$\Sigma A^3-3\Pi A=A^3+B^3+C^3-3ABC$

$$=\Sigma A\cdot(\Sigma A^2-\Sigma AB)$$

故如　$\Sigma A=0,$　　　　　則　　$\Sigma A^3=3\Pi A$

今　$\Sigma\cos(\beta+\gamma)\sin(\beta-\gamma)=\tfrac{1}{2}\Sigma(\sin 2\beta-\sin 2\gamma)=0$

故　$\Sigma\cos^3(\beta+\gamma)\sin^3(\beta-\gamma)=3\Pi\cos(\beta+\gamma)\sin(\beta-\gamma)$

55.

$$原式=\begin{vmatrix} 1 & 0 & 0 \\ \sin\alpha & \cos^2\alpha & \sin\gamma-\sin\alpha\sin\beta \\ \sin\beta & \sin\gamma-\sin\alpha\sin\beta & \cos^2\beta \end{vmatrix}$$

$$=\cos^2\alpha\cos^2\beta-(\sin\gamma-\sin\alpha\sin\beta)^2$$

$$=(\cos\alpha\cos\beta-\sin\gamma+\sin\alpha\sin\beta)$$

$$\times(\cos\alpha\cos\beta+\sin\gamma-\sin\alpha\sin\beta)$$

$$=[\cos(\alpha-\beta)-\sin\gamma][\cos(\alpha+\beta)+\sin\gamma]$$

$$\left[\cos(\alpha-\beta)+\cos\left(\tfrac{\pi}{2}+\gamma\right)\right]$$

$$\times\left[\cos(\alpha+\beta)+\cos\left(\tfrac{\pi}{2}-\gamma\right)\right]$$

$$= 4 \cos\left[\frac{1}{2}(\alpha + \beta + \gamma) - \frac{1}{4}\pi \right]$$

$$\times \Pi\left[\cos\frac{1}{2}(\alpha + \beta - \gamma) + \frac{1}{4}\pi \right]$$

56. 第一列乘 $\cos A$ 减第三列，又第二列乘 $\sin A$ 减第三列得

$$左边 = \frac{2}{\sin A \cos A} \begin{vmatrix} 0 & 0 & \sin A \cos A \\ \sin B(\cos A - \cos B) & \cos B(\sin A - \sin B) & \sin B \cos B \\ \sin C(\cos A - \cos C) & \cos C(\sin A - \sin C) & \sin C \cos C \end{vmatrix}$$

$$= 8 \begin{vmatrix} \sin B \sin\frac{1}{2}(A+B)\sin\frac{1}{2}(B-A) & \cos B \cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B) \\ \sin C \sin\frac{1}{2}(A+C)\sin\frac{1}{2}(C-A) & \cos C \cos\frac{1}{2}(A+C)\sin\frac{1}{2}(A-C) \end{vmatrix}$$

$$= 2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(C-A)\left\{ \left[\sin(B+C) + \sin(B-C) \right] \right.$$

$$\times \left[\sin\frac{2A+B+C}{2} + \sin\frac{B-C}{2} \right] - \left[\sin(B+C) - \sin(B-C) \right]$$

$$\left. \times \left[\sin\frac{2A+B+C}{2} - \sin\frac{B-C}{2} \right] \right\}$$

$$= 2\sin\frac{A-B}{2}\sin\frac{C-A}{2}\left[2\sin(B-C)\sin\frac{2A+B+C}{2} \right.$$

$$\left. + 2\sin(B+C)\sin\frac{B-C}{2} \right]$$

$$= 4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}\left[2\sin\frac{2A+B+C}{2}\cos\frac{B-C}{2} \right.$$

$$\left. + \sin(B+C) \right]$$

$$= 4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}\left[\sin(A+B) + \sin(B+C) \right.$$

$$\left. + \sin(C+A) \right]$$

57. 照左邊行列式之第一行展開之得

左邊 $=\cos \frac{1}{2}(B-C)[\cos \frac{1}{2}(C+A)\sin \frac{1}{2}(A+B)$

$\qquad -\sin \frac{1}{2}(C+A)\cos \frac{1}{2}(A+B)]-\cdots\cdots$

$\qquad =\cos\frac{1}{2}(B-C)\sin\frac{1}{2}(B-C)-\cos\frac{1}{2}(C-A)\sin\frac{1}{2}(A-C)$

$\qquad +\cos \frac{1}{2}(A-B)\sin \frac{1}{2}(A-B)$

$\qquad =\frac{1}{2}[\sin(B-C)-\sin(A-C)+\sin(A-B)]$

右邊 $=\frac{1}{2}\left\{\begin{vmatrix} \sin B & \cos B \\ \sin C & \cos C \end{vmatrix}-\begin{vmatrix} \sin A & \cos A \\ \sin C & \cos C \end{vmatrix}+\begin{vmatrix} \sin A & \cos A \\ \sin B & \cos B \end{vmatrix}\right\}$

$\qquad =\frac{1}{2}[\sin(B-C)-\sin(A-C)+\sin(A-B)]$

故兩邊相等。

58. 照分子之第一列展開之得

左邊分子 $=\cos(\beta-\gamma)\sin(\gamma-\beta)-\cos(\gamma-\alpha)\sin(\gamma-\alpha)$

$\qquad +\cos(\alpha-\beta)\sin(\beta-\alpha)$

$\qquad =-\frac{1}{2}[\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)$

$\qquad +\sin 2(\alpha-\beta)]$

$\qquad =2\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta)$ （見例十七）

$\qquad =16\Pi \sin \frac{1}{2}(\alpha-\beta)\cdot\Pi\cos\frac{1}{2}(\alpha-\beta)$

左邊分母 $=-[\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)]$

$\qquad$ （見上題之右邊化法）

$\qquad =4\Pi \sin \frac{1}{2}(\alpha-\beta)$ （見例十七）

故原式 $=4\Pi \cos \frac{1}{2}(\alpha-\beta)$

59. 今 $\sin 2^n\theta=\sin 2(2^{n-1}\theta)=2\sin 2^{n-1}\theta \cos 2^{n-1}\theta$

$$\sin 2^{n-1}\theta = 2 \sin 2^{n-2}\theta \cos 2^{n-2}\theta$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \cdots$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

相乘得 $\sin 2^{n}\theta = 2^{n} \cos 2^{n-1}\theta \cos 2^{n-2}\theta \cdots\cdots \cos \theta \sin \theta$

60. 從例二十七得 $\sin \theta = 2^{n} \cos \dfrac{\theta}{2} \cos \dfrac{\theta}{2^{2}} \cdots\cdots \cos \dfrac{\theta}{2^{n}} \sin \dfrac{\theta}{2^{n}}$

即 $\dfrac{\sin \theta}{\theta} = \cos \dfrac{\theta}{2} \cos \dfrac{\theta}{2^{2}} \cdots\cdots \cos \dfrac{\theta}{2^{n}}\left(\sin \dfrac{\theta}{2^{n}} \Big/ \dfrac{\theta}{2^{n}} \right)$

當 $n = \infty$ $\displaystyle\lim_{n \to \infty} \left(\sin \dfrac{\theta}{2^{n}} \Big/ \dfrac{\theta}{2^{n}} \right) = 1$

故 $\cos \dfrac{\theta}{2} \cos \dfrac{\theta}{2^{2}} \cdots\cdots \cos \dfrac{\theta}{2^{n}} = \dfrac{\sin \theta}{\theta}$

61. $\because$ $2 \cos^{2} \dfrac{45°}{2^{n}} = 2 \cos^{2} \dfrac{1}{2}\left(\dfrac{45°}{2^{n-1}} \right) = 1 + \cos \dfrac{45°}{2^{n-1}}$

$\therefore$ $2 \cos \dfrac{45°}{2^{n}} = \sqrt{2 + 2 \cos \dfrac{45°}{2^{n-1}}}$

同理 $2 \cos \dfrac{45°}{2^{n-1}} = \sqrt{2 + 2 \cos \dfrac{45°}{2^{n-2}}}$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

故 $2 \cos \dfrac{45°}{2^{n}} = \sqrt{2 + 2 \cos \dfrac{45°}{2^{n-1}}}$

$$= \sqrt{2 + \sqrt{2 + 2 \cos \dfrac{45°}{2^{n-2}}}}$$

$$= \sqrt{2 + \sqrt{2 + \cdots\cdots \sqrt{2 + 2 \cos 90°}}}$$

$$= \sqrt{2 + \sqrt{2 + \cdots\cdots + \sqrt{2}}}$$

62. $\because$ $2 \sin^{2} A = 1 - \cos 2A$ $\therefore$ $2 \sin A = \sqrt{2 - 2 \cos 2A}$

今　$2\cos\left(60°+\dfrac{30°}{2^n}\right)=2\sin\left(30°-\dfrac{30°}{2^n}\right)$

$\qquad =\sqrt{2-2\cos\left(60°-\dfrac{30°}{2^{n-1}}\right)}$

$\qquad =\sqrt{2-2\sin\left(30°+\dfrac{30°}{2^{n-1}}\right)}$

$\qquad =\sqrt{2-\sqrt{2-2\cos\left(60°+\dfrac{30°}{2^{n-2}}\right)}}$

設 n 爲偶數則至第 n 次得

$$2\sin\left(30°+\dfrac{30°}{2^{n-(n-1)}}\right)=\sqrt{2-2\cos\left(60°+\dfrac{30°}{2^{n-n}}\right)}$$

$$=\sqrt{2-2\cos 90°}=\sqrt{2}$$

故　$2\cos\left(60°+\dfrac{30°}{2^n}\right)=\sqrt{2-\sqrt{2-\cdots-\sqrt{2}}}$

63. 如例二十四，以 $d=x$ 代入卽行.

64. 第 n 項爲 $\sin\pi$，今加入計算，如例二十四，設 $x=d=\dfrac{\pi}{n}$，

卽得總和爲

$$\sin\dfrac{1}{2}n\left(\dfrac{\pi}{n}\right)\sin\left[\dfrac{\pi}{n}+\dfrac{1}{2}(n-1)\dfrac{\pi}{n}\right]\Big/\sin\dfrac{1}{2}\left(\dfrac{\pi}{n}\right)$$

$$=\sin\dfrac{1}{2}\pi\sin\left(\dfrac{\pi}{2}+\dfrac{\pi}{2n}\right)\Big/\sin\dfrac{\pi}{2n}$$

$$=\cos\dfrac{\pi}{2n}\Big/\sin\dfrac{\pi}{2n}=\cot\dfrac{\pi}{2n}$$

65. 設和爲 S

今　$2\cos x\sin\tfrac{1}{2}d=\sin(x+\tfrac{1}{2}d)-\sin(x-\tfrac{1}{2}d)$

$$2\cos(x+d)\sin \tfrac{1}{2}d=\sin(x+\tfrac{3}{2}d)-\sin(x+\tfrac{1}{2}d)$$

$$\dotfill$$

$$2\cos(x+\overline{n-1}d)\sin \tfrac{1}{2}d=\sin(x+\overline{n-\tfrac{1}{2}}d)$$

$$-(\sin x+\overline{n-\tfrac{3}{2}}d)$$

相加得 $2S\sin \tfrac{1}{2}d=\sin(x+\overline{n-\tfrac{1}{2}}d)-\sin(x-\tfrac{1}{2}d)$

∴ $S=\sin \tfrac{1}{2}nd\cos[x+\tfrac{1}{2}(n-1)d]/\sin \tfrac{1}{2}nd.$

66. ∵ $-\cos(A+B)=\cos(\pi+A+B),$

$$\cos(A+2B)=\cos(2\pi+A+2B),\cdots\cdots$$

故 $S=\cos A+\cos[A+(\pi+B)]$

$$+\cos[A+2(\pi+B)]+\cdots\cdots$$

在上題中,以 A 代 x, $\pi+B$ 代 d, 則得本題之證矣.

67. $\csc A=\dfrac{1}{\sin A}=\dfrac{\sin \tfrac{1}{2}A}{\sin A\sin \tfrac{1}{2}A}=\dfrac{\sin(A-\tfrac{1}{2}A)}{\sin A\sin \tfrac{1}{2}A}$

$$=\frac{\sin A\cos \tfrac{1}{2}A-\cos A\sin \tfrac{1}{2}A}{\sin A\sin \tfrac{1}{2}A}=\cot \tfrac{1}{2}A-\cot A$$

$$\csc 2A=\cot A-\cot 2A, \quad \csc 2^2 A=\cot 2A-\cot 2^2 A,$$

$$\dotfill$$

相加得 $S=\cot \tfrac{1}{2}A-\cot 2^{n-1}A$

68. ∵ $\tan \tfrac{1}{2}A\sec A=\dfrac{\sin \tfrac{1}{2}A}{\cos \tfrac{1}{2}A\cos A}=\dfrac{\sin(A-\tfrac{1}{2}A)}{\cos \tfrac{1}{2}A\cos A}$

$$=\frac{\sin A\cos \tfrac{1}{2}A-\sin \tfrac{1}{2}A\cos A}{\cos \tfrac{1}{2}A\cos A}$$

$$=\tan A-\tan \tfrac{1}{2}A$$

同理 $\tan\dfrac{1}{2^2}A\sec\dfrac{1}{2}A=\tan\dfrac{1}{2}A-\tan\dfrac{1}{2^2}A$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tan\frac{1}{2^n}A \; \sec\frac{1}{2^{n-1}}A = \tan\frac{A}{2^{n-1}} - \tan\frac{1}{2^n}A$$

相加得 $\qquad S = \tan A - \tan\frac{1}{2^n}A$

69. 因 $\quad 2\cos\theta - 1 = \dfrac{4\cos^2\theta - 1}{2\cos\theta + 1} = \dfrac{2\cos 2\theta + 1}{2\cos\theta + 1}$

同理得 $\qquad 2\cos\dfrac{\theta}{2} - 1 = \dfrac{2\cos\theta + 1}{2\cos\frac{1}{2}\theta + 1}$

$$2\cos\dfrac{\theta}{2^2} - 1 = \dfrac{2\cos\frac{1}{2}\theta + 1}{2\cos\frac{1}{2^2}\theta + 1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\cos\dfrac{\theta}{2^{n-1}} - 1 = \dfrac{2\cos\frac{1}{2^{n-2}}\theta + 1}{2\cos\frac{1}{2^{n-1}}\theta + 1}$$

相乘得 $(2\cos\theta - 1)(2\cos\frac{1}{2}\theta - 1)\cdots\cdots(2\cos\frac{1}{2^{n-1}}\theta - 1)$

$$= \dfrac{2\cos 2\theta + 1}{2\cos\frac{1}{2^{n-1}}\theta + 1}$$

70. 因 $\quad 1 + \sec 2\theta = \dfrac{\cos 2\theta + 1}{\cos 2\theta} = \dfrac{2\cos^2\theta}{\cos 2\theta} = \dfrac{\sin 2\theta \cos\theta}{\cos 2\theta \sin\theta}$

即 $\quad 1 + \sec 2\theta = \tan 2\theta \cot\theta, \quad 1 + \sec 4\theta = \tan 4\theta \cot 2\theta,$

$\qquad\cdots\cdots\cdots\cdots\cdots\cdots \quad 1 + \sec 2^n\theta = \tan 2^n\theta \cot 2^{n-1}\theta$

即 $\quad (1 + \sec 2\theta)(1 + \sec 4\theta)\cdots\cdots(1 + \sec 2^n\theta)$

$$= \tan 2^n\theta \cot\theta$$

71. $\quad\because\quad 8\cos^4 A = \cos 4A + 4\cos 2A + 3\quad$（從 20 節化法）

故　$8\times$左邊$=\left[\cos 4A + \cos 4\left(A+\dfrac{2\pi}{n}\right)\right.$

$\left. + \cos 4\left(A+\dfrac{4\pi}{n}\right) + \cdots\cdots\right]$

$+ 4\left[\cos 2A + \cos 2\left(A+\dfrac{2\pi}{n}\right)\right.$

$\left. + \cos 2\left(A+\dfrac{4\pi}{n}\right) + \cdots\cdots\right] + 3n$

$= \dfrac{\sin 4\pi \cos 4\left(A+\dfrac{n-1}{n}\pi\right)}{\sin\dfrac{4\pi}{n}}$

$\quad + \dfrac{4\sin 2\pi \cos 2\left(A+\dfrac{n-1}{n}\pi\right)}{\sin\dfrac{2\pi}{n}} + 3n\quad$（見65題）

$\therefore\quad$左邊$=\dfrac{3n}{8}\quad(\because\ \sin 4\pi=0,\quad \sin 2\pi=0)$

72. $\quad\because\quad \cos\dfrac{A}{2} + \cos\dfrac{B}{2} = \dfrac{\cos A - \cos B}{2\left(\cos\dfrac{A}{2} - \cos\dfrac{B}{2}\right)}$

同理　$\cos\dfrac{1}{2^2}A + \cos\dfrac{1}{2^2}B = \dfrac{\cos\dfrac{1}{2}A - \cos\dfrac{1}{2}B}{2\left(\cos\dfrac{1}{2^2}A - \cos\dfrac{1}{2^2}B\right)}$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cos\dfrac{1}{2^n}A + \cos\dfrac{1}{2^n}B = \dfrac{\cos\dfrac{1}{2^{n-1}}A - \cos\dfrac{1}{2^{n-1}}B}{2\left(\cos\dfrac{1}{2^n}A - \cos\dfrac{1}{2^n}B\right)}$

相乘得　左邊 $= \dfrac{\cos A - \cos B}{2^n\left(\cos\dfrac{1}{2^n}A - \cos\dfrac{1}{2^n}B\right)}$

73. 令　　$x_r = \cos\dfrac{y}{2^r}$,　　則　　$\cos\dfrac{y}{2^{r+1}} = \sqrt{\dfrac{1+x_r}{2}} = x_{r+1}$

即　　$\cos y = x_0,$　　$\cos\dfrac{y}{2} = x_1,$　　$\cos\dfrac{y}{2^2} = x_2 \cdots\cdots$

相乘得　$\cos\dfrac{y}{2}\cos\dfrac{y}{2^2}\cdots\cdots\cos\dfrac{y}{2^r} = x_1 \cdot x_2 \cdot x_3 \cdots\cdots x_r$

即　　$2^r \cos\dfrac{y}{2}\cos\dfrac{y}{2^2}\cdots\cdots\cos\dfrac{y}{2^r}\sin\dfrac{y}{2^r}$

$\qquad = 2^r x_1 \cdot x_2 \cdot x_3 \cdots\cdots x_r \sin\dfrac{y}{2^r}$

$\qquad = y x_1 \cdot x_2 \cdot x_3 \cdots\cdots x_r \sin\dfrac{y}{2^r} \Big/ \dfrac{y}{2^r}$

但　$\sin y = 2^r \cos\dfrac{y}{2}\cos\dfrac{y}{2^2}\cdots\cdots\cos\dfrac{y}{2^r}\sin\dfrac{y}{2^r}$　（見例二十七）

又　　　　　$\lim_{r \to \infty}\left(\sin\dfrac{y}{2^r} \Big/ \dfrac{y}{2^r}\right) = 1$

故　　　　　$\sin y = y x_1 x_2 x_3 \cdots\cdots$至$\infty$

$\because$　　　　$\cos y = x_0$

$\therefore$　$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x_0{}^2},$　$y = \cos^{-1} x_0$

即　　　　$\sqrt{1 - x_0{}^2} = \cos^{-1} x_0 [x_1 x_2 \cdots\cdots \infty]$

$\therefore$　　　$\cos^{-1} x_0 = \dfrac{\sqrt{1 - x_0{}^2}}{x_1 x_2 \cdots\cdots$至$\infty}$

74. 在61題設 $n = \infty$,　則 $\cos\dfrac{45°}{2^n} = \cos 0° = 1$,　即得此證矣

或令此式爲 x

則　　$x=\sqrt{2+x},$　　　　　即　　$x^2-x-2=0$

即　$(x-2)(x+1)=0,$　　$\therefore$　$x=2$　（-1根不合）

75. 從　　$\sin(x+y-z)-\sin(z+x-y)$

　　　　　$=\sin(z+x-y)-\sin(y+z-x)$

$\therefore$　　$2\cos x\sin(y-z)=2\cos z\sin(x-y)$

即　　$\cos x\sin y\cos z-\cos x\cos y\sin z$

　　　　　$=\sin x\cos y\cos z-\sin y\cos x\cos z$

兩邊各除以　　　$\cos x\cos y\cos z$

得　　　$\tan y-\tan z=\tan x-\tan y$

即　　　$\tan z-\tan y=\tan y-\tan x$　故如題云

76. 從　　　$\cot\gamma-\cot\beta=\cot\beta-\cot\alpha$　　　　（1）

亦即　　$\cot\gamma=2\cot\beta-\cot\alpha$　　　　　（2）

今　$\cot(\beta-\gamma)+\cot(\beta-\alpha)=\cot(\beta-\gamma)-\cot(\alpha-\beta)$

　　　　$=\dfrac{1+\cot\beta\cot\gamma}{\cot\gamma-\cot\beta}-\dfrac{1+\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}$

　　　　$=\dfrac{\cot\beta\,(\cot\gamma-\cot\alpha)}{\cot\beta-\cot\alpha}$　　〔從(1)式〕

　　　　$=\dfrac{2\cot\beta(\cot\beta-\cot\alpha)}{\cot\beta-\cot\alpha}$　　〔從(2)式〕

　　　　$=2\cot\beta$

故　　$\cot(\beta-\gamma)-\cot\beta=\cot\beta-\cot(\beta-\alpha)$

77. $\because$　　$z-y=y-x$　　　　$\therefore$　　　$y=\tfrac{1}{2}(x+z)$

即　　　$y-z=\tfrac{1}{2}(x+z)-z=\tfrac{1}{2}(x-z)$

故　　　　　　$\dfrac{\tan y}{\tan(y-z)} = \dfrac{\tan \frac{1}{2}(x+z)}{\tan \frac{1}{2}(x-z)}$

亦等於 $\dfrac{\sin\frac{1}{2}(x+z)\cos\frac{1}{2}(x-z)}{\cos\frac{1}{2}(x+z)\sin\frac{1}{2}(x-z)} = \dfrac{\sin x + \sin z}{\sin x - \sin z}$　故如題云

78. 因　$\tan y = \dfrac{\tan z}{2} = \dfrac{\tan\frac{z}{2}}{1-\tan^2\frac{z}{2}} = \dfrac{\tan\frac{z}{2}\left(1+\tan^2\frac{z}{2}\right)}{1-\tan^4\frac{z}{2}}$

$$= \dfrac{\tan\frac{z}{2}+\tan\frac{x}{2}}{1-\tan\frac{z}{2}\tan\frac{x}{2}} = \tan\frac{z+x}{2}, \quad 故 \quad y = \dfrac{z+x}{2}$$

79. 今　$\dfrac{1}{\cos\phi} - \dfrac{1}{\cos(\phi-\alpha)} = \dfrac{1}{\cos(\phi+\alpha)} - \dfrac{1}{\cos\phi}$

即　　$\cos\phi = \dfrac{2\cos(\phi-\alpha)\cos(\phi+\alpha)}{\cos(\phi-\alpha)+\cos(\phi+\alpha)} = \dfrac{\cos 2\phi + \cos 2\alpha}{2\cos\phi\cos\alpha}$

故　$2\cos\alpha\cos^2\phi = \cos 2\phi + \cos 2\alpha$

$$= (2\cos^2\phi - 1) + (2\cos^2\alpha - 1)$$

即　　　　$\cos^2\phi(1-\cos\alpha) = 1 - \cos^2\alpha$

$$\therefore \quad \cos^2\phi = 1 + \cos\alpha = 2\cos^2\frac{\alpha}{2}$$

$$\therefore \quad \cos\phi = \sqrt{2}\cos\frac{\alpha}{2}$$

80. 以 a, b 作爲未知數，從十字法(見附錄二)得

$$\frac{a}{\begin{vmatrix} \sin\beta & c\sin\gamma \\ \cos\beta & c\cos\gamma \end{vmatrix}} = \frac{b}{\begin{vmatrix} c\sin\gamma & \sin\alpha \\ c\cos\gamma & \cos\alpha \end{vmatrix}} = \frac{1}{\begin{vmatrix} \sin\alpha & \sin\beta \\ \cos\alpha & \cos\beta \end{vmatrix}}$$

即　　$\dfrac{a}{\sin(\beta-\gamma)} = \dfrac{b}{\sin(\gamma-\alpha)} = \dfrac{c}{\sin(\alpha-\beta)}$　　（化出且以
　　　　　　　　　　　　　　　　　　　　　　　　　　　　　c 乘之）

81. 平方第一式兩邊且以公式26代入得

$$\frac{1-\cos x}{1+\cos x}=\frac{(1+C)(1-\cos y)}{(1-C)(1+\cos y)}=\frac{1+C-\cos y-C\cos y}{1-C+\cos y-C\cos y}$$

即　　$\dfrac{2}{2\cos x}=\dfrac{2(1-C\cos y)}{2(\cos y-C)}$　　（附錄二比例中合分之理）

即　　$\cos x=\dfrac{\cos y-C}{1-C\cos y}$

82. 從第一式即　　$\dfrac{\tan A-\tan B}{\tan A(1+\tan A\tan B)}+\sin^2 C\csc^2 A=1$,

即　$\dfrac{\sin^2 C(1+\tan^2 A)}{\tan^2 A}=1-\dfrac{\tan A-\tan B}{\tan A(1+\tan A\tan B)}$

$$=\frac{\tan B(1+\tan^2 A)}{\tan A(1+\tan A\tan B)},$$

約去 $(1+\tan^2 A)/\tan A$ 且去分母移項得

$$\sin^2 C=\tan A\tan B(1-\sin^2 C)=\tan A\tan B\cos^2 C$$

$\therefore\quad \tan^2 C=\tan A\tan B$

83. 即　$\dfrac{1-\cos 2\theta}{1+\cos 2\theta}\cdot\dfrac{1-\cos 2\phi}{1+\cos 2\phi}=\dfrac{a-b}{a+b}$

$\therefore\quad (1-\cos 2\theta-\cos 2\phi+\cos 2\theta\cos 2\phi)(a+b)$

$$=(1+\cos 2\theta+\cos 2\phi+\cos 2\theta\cos 2\phi)(a-b)$$

即　$2a(\cos 2\theta+\cos 2\phi)=2b(1+\cos 2\theta\cos 2\phi)$

即　$a^2-ab(\cos 2\theta+\cos 2\phi)+b^2\cos 2\theta\cos 2\phi=-b^2+a^2$

$\therefore\quad (a-b\cos 2\theta)(a-b\cos 2\phi)=a^2-b^2$

84. $\because\quad \cos A=1-x,\quad \cos B=1-mx,\quad \cos C=m$

$\therefore\quad \sin A=\sqrt{2x-x^2},\quad \sin C=\sqrt{1-m^2}$

從　$B = C - A$,　則　$\cos B = \cos C \cos A + \sin C \sin A$

即　　$1 - mx = m(1-x) + \sqrt{1-m^2} \cdot \sqrt{2x-x^2}$

即　　$1 - m = \sqrt{(1-m^2)(2x-x^2)}$

即　　$1 - m = (1+m)(2x-x^2)$

$\therefore$　$x^2 - 2x + \dfrac{1-m}{1+m} = 0$,　　故　$x = 1 \pm \sqrt{\dfrac{2m}{1+m}}$

85.　$\because$　$\sin\theta = \dfrac{a}{b}\sin\phi$,　　　　$\cos\theta = \dfrac{c}{d}\cos\phi$

則　　　　$\dfrac{a^2}{b^2}\sin^2\phi + \dfrac{c^2}{d^2}\cos^2\phi = 1$

$\therefore$　　　　$\sin^2\phi = \dfrac{1 - \dfrac{c^2}{d^2}}{\dfrac{a^2}{b^2} - \dfrac{c^2}{d^2}} = \dfrac{b^2(d^2-c^2)}{a^2d^2 - b^2c^2}$

又　　　$\dfrac{a^2}{b^2} - \dfrac{a^2}{b^2}\cos^2\phi + \dfrac{c^2}{d^2}\cos^2\phi = 1$

$\therefore$　　　　$\cos^2\phi = \dfrac{\dfrac{a^2}{b^2} - 1}{\dfrac{a^2}{b^2} - \dfrac{c^2}{d^2}} = \dfrac{d^2(a^2 - b^2)}{a^2d^2 - b^2c^2}$

今　$\cos(\theta \mp \phi) = \cos\theta\cos\phi \pm \sin\phi\sin\theta$

$\qquad\qquad = \dfrac{c}{d}\cos^2\phi \pm \dfrac{a}{b}\sin^2\phi$

$\therefore$　$\cos(\theta \mp \phi) = \dfrac{c}{d} \cdot \dfrac{d^2(a^2-b^2)}{a^2d^2 - b^2c^2} \pm \dfrac{a}{b} \cdot \dfrac{b^2(d^2-c^2)}{a^2d^2 - b^2c^2}$

$\qquad\qquad = \dfrac{cd(a^2-b^2) \pm ab(d^2-c^2)}{a^2d^2 - b^2c^2}$

$$=\frac{(ad\mp bc)(ac\pm bd)}{(ad-bc)(ad+bc)}=\frac{ac\pm bd}{ad\pm bc}$$

（約去時同取上號或下號）

86. 今　$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\,\tan B}$

$$=\frac{\dfrac{\sin A}{\cos A}-\dfrac{n\sin A\cos A}{1-n\sin^2 A}}{1+\dfrac{\sin A}{\cos A}\cdot\dfrac{n\sin A\cos A}{1-n\sin^2 A}}$$

$$=\frac{\sin A(1-n\sin^2 A-n\cos^2 A)}{\cos A(1-n\sin^2 A+n\sin^2 A)}$$

$$=(1-n)\tan A$$

87. 今　$\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}=\dfrac{x(x+x^{-1}+1)-1}{(x+x^{-1}+1)^{\frac{1}{2}}(x+1)}$

$$=\frac{x(x+1)}{(x+1)(x+x^{-1}+1)^{\frac{1}{2}}}=x(x+x^{-1}+1)^{-\frac{1}{2}}$$

又　$\cot\gamma=1/\tan\gamma=(x^{-3}+x^{-2}+x^{-1})^{-\frac{1}{2}}$

$$=[x^{-2}(x+x^{-1}+1)]^{-\frac{1}{2}}$$

$$=x(x+x^{-1}+1)^{-\frac{1}{2}}$$

故　$\cot(\alpha+\beta)=\cot\gamma$　　　即　　　$\alpha+\beta=\gamma$

88. 化已知兩式爲

$$x=\frac{\tan\theta}{\tan\theta\cos\phi+\sin\phi},\qquad y=\frac{\tan\phi}{\tan\phi\cos\theta+\sin\theta}$$

故　$\dfrac{x}{y}=\dfrac{\tan\theta(\tan\phi\cos\theta+\sin\theta)}{\tan\phi(\tan\theta\cos\phi+\sin\phi)}$

$$=\frac{\sin\theta(\tan\phi+\tan\theta)}{\sin\phi(\tan\phi+\tan\theta)}=\frac{\sin\theta}{\sin\phi}$$

89. $\because\quad\dfrac{u_{n+2}-u_{n+4}}{u_n}=\dfrac{\sin^{n+2}\theta+\cos^{n+2}\theta-\sin^{n+4}\theta-\cos^{n+4}\theta}{\sin^n\theta+\cos^n\theta}$

$$=\frac{\sin^{n+2}\theta\cos^2\theta+\cos^{n+2}\theta\sin^2\theta}{\sin^n\theta+\cos^n\theta}$$

$$=\frac{\sin^2\theta\cos^2\theta(\sin^n\theta+\cos^n\theta)}{\sin^n\theta+\cos^n\theta}$$

$$=\sin^2\theta\cos^2\theta=定數$$

　　今以 $n=1$ **時爲左邊,** $n=3$ **時爲右邊故如題云.**

90.　左邊 $=(\cos\theta+i\sin\theta)(\cos2\theta+i\sin2\theta)\cdots\cdots$

　　　　$\cdots\cdots(\cos2n\theta+i\sin2n\theta)$

　　$=\cos(\theta+2\theta+\cdots\cdots+2n\theta)$

　　　　$+i\sin(\theta+2\theta+\cdots\cdots+2n\theta)$　　（從棣美弗定理）

　　但　$\theta+2\theta+\cdots\cdots+2n\theta=\dfrac{2n}{2}(1+2n)\theta=n(2n+1)\theta$

　　　　　　（$A.P.\ 2n$ 項之和）

　　故原式左邊 $=\cos n(2n+1)\theta+i\sin n(2n+1)\theta$

　　　　　$=U_{n(2n+1)}$

91.　$\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

$$=\frac{\dfrac{x\sin\alpha}{y-x\cos\alpha}+\dfrac{y\sin\alpha}{x-y\cos\alpha}}{1-\dfrac{x\sin\alpha}{y-x\cos\alpha}\cdot\dfrac{y\sin\alpha}{x-y\cos\alpha}}$$

$$=\frac{\sin\alpha[(x^2+y^2)-2xy\cos\alpha]}{xy-(x^2+y^2)\cos\alpha+xy(\cos^2\alpha-\sin^2\alpha)}$$

$$= \frac{[(x^2+y^2)-2xy\cos\alpha]\sin\alpha}{-[(x^2+y^2)-2xy\cos\alpha]\cos\alpha}$$

$$= -\tan\alpha, \quad 故如題云.$$

92. 今 $\sin^2\theta\cos(\theta-\alpha)\cos(\theta-\beta)$

$$= \cos^2\theta\sin(\theta-\alpha)\sin(\theta-\beta)$$

$\therefore$ $\sin^2\theta[\cos(2\theta-\alpha-\beta)+\cos(\alpha-\beta)]$

$$= \cos^2\theta[\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)]$$

故 $\cos(2\theta-\alpha-\beta)=\cos(\alpha-\beta)\cos2\theta$

即 $\cos2\theta\cos(\alpha+\beta)+\sin2\theta\sin(\alpha+\beta)$

$$= \cos2\theta\cos(\alpha-\beta)$$

$\therefore$ $\tan2\theta = \dfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\dfrac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$

93. $\tan z = \dfrac{2\tan\dfrac{z}{2}}{1-\tan^2\dfrac{z}{2}} = \dfrac{2\tan\dfrac{x}{2}\tan\dfrac{y}{2}}{1-\tan^2\dfrac{x}{2}\tan^2\dfrac{y}{2}}$

$$= \frac{\dfrac{2\sin\dfrac{x}{2}\sin\dfrac{y}{2}}{\cos\dfrac{x}{2}\cos\dfrac{y}{2}}}{\dfrac{\cos^2\dfrac{x}{2}\cos^2\dfrac{y}{2}-\sin^2\dfrac{x}{2}\sin^2\dfrac{y}{2}}{\cos^2\dfrac{x}{2}\cos^2\dfrac{y}{2}}}$$

$$= \frac{4\sin\dfrac{x}{2}\sin\dfrac{y}{2}\cos\dfrac{x}{2}\cos\dfrac{y}{2}}{2\cos\left(\dfrac{x}{2}+\dfrac{y}{2}\right)\cos\left(\dfrac{x}{2}-\dfrac{y}{2}\right)} = \frac{\sin x\sin y}{\cos x+\cos y}$$

94　今　$\sin^2\theta = \dfrac{\cos\beta - \cos\alpha}{\cos\beta}$, 　　$\sin^2\phi = \dfrac{\cos\beta - \cos\gamma}{\cos\beta}$

則　$\tan^2\theta = \dfrac{\cos\beta - \cos\alpha}{\cos\alpha}$, 　　$\tan^2\phi = \dfrac{\cos\beta - \cos\gamma}{\cos\gamma}$

∴　$\dfrac{\tan^2\theta}{\tan^2\phi} = \dfrac{\tan^2\alpha}{\tan^2\gamma} = \dfrac{\cos\gamma(\cos\beta - \cos\alpha)}{\cos\alpha(\cos\beta - \cos\gamma)}$

∴　$\sin^2\alpha \cos\gamma(\cos\beta - \cos\gamma)$

$= \sin^2\gamma \cos\alpha(\cos\beta - \cos\alpha)$

$\cos\beta = \dfrac{\sin^2\alpha \cos^2\gamma - \sin^2\gamma \cos^2\alpha}{\sin^2\alpha \cos\gamma - \sin^2\gamma \cos\alpha}$

$= \dfrac{\cos^2\gamma - \cos^2\alpha}{(\cos\gamma - \cos\alpha)(1 + \cos\alpha \cos\gamma)}$

$= \dfrac{\cos\gamma + \cos\alpha}{1 + \cos\alpha \cos\gamma}$

則　$\dfrac{1 - \cos\beta}{1 + \cos\beta} = \dfrac{(1 - \cos\alpha)(1 - \cos\gamma)}{(1 + \cos\alpha)(1 + \cos\gamma)}$

即　$\tan\dfrac{\beta}{2} = \tan\dfrac{\alpha}{2}\tan\dfrac{\gamma}{2}$

95.　∵　$\dfrac{\tan(\theta+\alpha)}{x} = \dfrac{\tan(\theta+\beta)}{y} = \dfrac{\tan(\theta+\alpha) + \tan(\theta+\beta)}{x+y}$

$= \dfrac{\tan(\theta+\alpha) - \tan(\theta+\beta)}{x-y}$

∴　$\dfrac{x+y}{x-y} = \dfrac{\tan(\theta+\alpha) + \tan(\theta+\beta)}{\tan(\theta+\alpha) - \tan(\theta+\beta)} = \dfrac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$

∴　$\dfrac{x+y}{x-y}\sin^2(\alpha-\beta) = \sin(\alpha-\beta)\sin(2\theta+\alpha+\beta)$

$= \dfrac{1}{2}[\cos 2(\theta+\beta) - \cos 2(\theta+\alpha)]$

同理　$\dfrac{y+z}{y-z}\sin^2(\beta-\gamma)=\dfrac{1}{2}[\cos 2(\theta+\gamma)-\cos 2(\theta+\beta)]$

$\qquad\dfrac{z+x}{z-x}\sin^2(\gamma-\alpha)=\dfrac{1}{2}[\cos 2(\theta+\alpha)-\cos 2(\theta+\gamma)]$

$\qquad\therefore\ \Sigma\dfrac{x+y}{x-y}\sin^2(\alpha-\beta)=0$

96. 今　$\dfrac{\sin \alpha}{\cos \alpha}=\dfrac{n}{m}$　　故　$\dfrac{m-n}{m+n}=\dfrac{\cos \alpha-\sin \alpha}{\cos \alpha+\sin \alpha}$

故　$\sqrt{\dfrac{m-n}{m+n}}+\sqrt{\dfrac{m+n}{m-n}}=\sqrt{\dfrac{\cos \alpha-\sin \alpha}{\cos \alpha+\sin \alpha}}+\sqrt{\dfrac{\cos \alpha+\sin \alpha}{\cos \alpha-\sin \alpha}}$

$\qquad=\dfrac{\cos \alpha-\sin \alpha}{\sqrt{\cos^2\alpha-\sin^2\alpha}}+\dfrac{\cos \alpha+\sin \alpha}{\sqrt{\cos^2\alpha-\sin^2\alpha}}=\dfrac{2\cos \alpha}{\sqrt{\cos 2\alpha}}$

97. $\because\ \cot^2x=(\sqrt 2\cot \theta \sin \phi-\cos \phi)^2$

$\qquad=\dfrac{(\sqrt 2\cos \theta \sin \phi-\cos \phi \sin \theta)^2}{\sin^2\theta}$

$\therefore\ \csc^2x=\dfrac{2\cos^2\theta \sin^2\phi-\frac{1}{2}\sqrt 2\sin 2\phi \sin 2\theta}{\sin^2\theta}$

$\qquad+\dfrac{\cos^2\phi \sin^2\theta+\sin^2\theta}{\sin^2\theta}$

$\therefore\ \cot^2y=(\sqrt 2\cot \phi \sin \theta-\cos \theta)^2$

$\qquad=\dfrac{(\sqrt 2\cos \phi \sin \theta-\cos \theta \sin \phi)^2}{\sin^2\phi}$

$\csc^2y=\dfrac{2\cos^2\phi \sin^2\theta-\frac{1}{2}\sqrt 2\sin 2\phi \sin 2\theta}{\sin \phi}$

$\qquad+\dfrac{\cos^2\theta \sin^2\phi+\sin^2\phi}{\sin^2\phi}$

$$= \frac{(1-\sin^2\phi)(1-\cos^2\theta) - \frac{1}{2}\sqrt{2}\sin 2\phi \sin 2\theta}{\sin \phi}$$

$$+ \frac{\cos^2\phi \sin^2\theta + \cos^2\theta \sin^2\phi + \sin^2\phi}{\sin^2\phi}$$

$$= \frac{2\cos^2\theta \sin^2\phi - \frac{1}{2}\sqrt{2}\sin 2\phi \sin 2\theta + \cos^2\phi \sin^2\theta + \sin^2\theta}{\sin^2\phi}$$

$$\therefore \quad \frac{\csc^2 y}{\csc^2 x} = \frac{\sin^2 x}{\sin^2 y} = \frac{\sin^2\theta}{\sin^2\phi}$$

98. 從　　$\tan(\overline{\alpha+\beta}-\beta) = m\tan(\overline{\alpha+\beta}-\alpha)$

即　　$\tan(\gamma-\beta) = m\tan(\gamma-\alpha)$

即　$\sin(\gamma-\beta)\cos(\gamma-\alpha) = m\sin(\gamma-\alpha)\cos(\gamma-\beta)$

即　$\sin(2\gamma-\alpha-\beta) + \sin(\alpha-\beta)$

$\qquad = m[\sin(2\gamma-\alpha-\beta) - \sin(\alpha-\beta)]$

即　$\sin\gamma + \sin(\alpha-\beta) = m[\sin\gamma - \sin(\alpha-\beta)]$

$$(\because \quad \alpha+\beta = \gamma)$$

$$\therefore \quad \sin\gamma = \frac{m+1}{m-1}\sin(\alpha-\beta)$$

99. 今　　$(1+\sin x)^2(1+\sin y)^2(1+\sin z)^2$

$\qquad = (1-\sin^2 x)(1-\sin^2 y)(1-\sin^2 z)$

故　　　$\Pi(1+\sin x) = \Pi(1-\sin x)$

乘出得　　　$\Sigma \sin x = -\Pi \sin x$

令　$\sin x = u,\qquad \sin y = v,\qquad \sin z = w,$

$\qquad \cos x = k,\qquad \cos y = m,\qquad \cos z = n.$

則　　　　　$u+v+w = -uvw$

如 126頁 例四十一可化到

$$1/n^2+1/k^2+1/m^2-2/kmn=1$$

即　　　　$\Sigma \sec^2 x = 1 + 2\Pi \sec x$

100. 設　$\tan A \neq \tan B$　　則　$(\tan A - \tan B)^2 > 0$

即　　　　$\tan^2 A + \tan^2 B > 2\tan A \tan B$

同樣得　$\tan^2 B + \tan^2 C > 2\tan B \tan C$

和　　　　$\tan^2 C + \tan^2 A > 2\tan C \tan A$

相加除以 2 得　$\Sigma \tan^2 A > \Sigma \tan A \tan B$

習 題 十 四 (150—155 頁)

1. $\Sigma \cos A = 1 + 4\Pi \sin \dfrac{1}{2} A$　　　（見 138 頁例一）

又以　$\sin\dfrac{1}{2} A = \cos\left(\dfrac{\pi}{2} - \dfrac{A}{2}\right) = \cos\dfrac{1}{2}(\pi - A)$

$$= \cos\dfrac{1}{2}(B+C), \quad 故如題云.$$

2. $\because$　$\dfrac{1}{2}(A+B) = \dfrac{1}{2}(\pi - C)$

$\therefore \sin\dfrac{1}{2}(A+B) = \cos\dfrac{1}{2}C, \ \cos\dfrac{1}{2}(A+B) = \sin\dfrac{1}{2}C$

左邊 $= 2\sin\dfrac{1}{2}(A+B)\cos\dfrac{1}{2}(A-B) - 2\sin\dfrac{1}{2}C\cos\dfrac{1}{2}C$

$$= 2\cos\dfrac{1}{2}C\left[\cos\dfrac{1}{2}(A-B) - \cos\dfrac{1}{2}(A+B)\right]$$

$$= 4\sin\dfrac{1}{2}A\sin\dfrac{1}{2}B\cos\dfrac{1}{2}C$$

3. $\because$　$A+B=\pi-C$

　$\therefore$　$\sin(A+B)=\sin C,$　$\cos C=-\cos(A+B)$

　今左邊 $=2\sin(A+B)\cos(A-B)+2\sin C\cos C$

　　　　$=2\sin C[\cos(A-B)-\cos(A+B)]=4\Pi\sin A$

4. 右邊 $=2\cos\dfrac{\pi-C}{4}\left[\cos\dfrac{2\pi+A+B}{4}+\cos\dfrac{A-B}{4}\right]$

　　　$=\cos\dfrac{3\pi+A+B-C}{4}+\cos\dfrac{\pi+A+B+C}{4}$

　　　$\quad+\cos\dfrac{\pi+A-B-C}{4}+\cos\dfrac{\pi+B-C-A}{4}$

　　　$=\cos\dfrac{4\pi-2C}{4}+\cos\dfrac{2\pi}{4}+\cos\dfrac{2A}{4}+\cos\dfrac{2B}{4}$

　　　$=\cos\left(\pi-\dfrac{C}{2}\right)+\cos\dfrac{A}{2}+\cos\dfrac{B}{2}$

　　　$=\cos\dfrac{A}{2}+\cos\dfrac{B}{2}-\cos\dfrac{C}{2}$

5. 左邊 $=\dfrac{1}{2}\Sigma(\cos A+\cos B)$

　　　$=\Sigma[\cos\dfrac{1}{2}(A+B)\cos\dfrac{1}{2}(A-B)]$

　　　$=\Sigma[\cos\dfrac{1}{2}(\pi-C)\cos\dfrac{1}{2}(A-B)]$

　　　$=\Sigma[\sin\dfrac{1}{2}C\cos\dfrac{1}{2}(A-B)]=$右邊

6. 今左邊 $=2\cos(A+B)\cos(A-B)+2\sin C\cos C$

　　　　$=2\cos C[\sin C-\cos(A-B)]$

　　　　$(\because\quad\cos\overline{A+B}=-\cos C)$

$$= 2 \cos C [\sin(A+B) - \cos(A-B)]$$

$$(\because \quad \sin C = \sin \overline{A+B})$$

$$= 2 \cos C \Big[\sin\Big(\frac{\pi}{4} + A + B - \frac{\pi}{4}\Big)$$

$$- \sin\Big(\frac{\pi}{4} + A - B + \frac{\pi}{4}\Big) \Big]$$

$$= 4 \cos C \cos\Big(\frac{\pi}{4} + A\Big) \sin\Big(B - \frac{\pi}{4}\Big)$$

7. $\because \quad \cos(A+B+C) = \cos 180° = -1$

$\therefore \quad \cos(A+B+C) = \cos A \cos B \cos C$

$$- \Sigma \sin A \sin B \cos C = -1 \qquad \text{(公式 15)}$$

或　　$\Sigma \sin A \sin B \cos C = \Pi \cos A + 1$

即　　$\dfrac{\Sigma \sin A \sin B \cos C}{\sin A \sin B \sin C} = \dfrac{\Pi \cos A + 1}{\sin A \sin B \sin C}$

故　　　　$\Sigma \cot A = \Pi \cot A + \Pi \csc A$

此係特別做法，普通可從化 $\Sigma\dfrac{\cos A}{\sin A}$ 着手。

8. 從　　　$\sin\frac{1}{2}(A+B+C) = \sin\frac{1}{2}\pi = 1$

故　　$\Sigma \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} - \Pi \sin\frac{A}{2} = 1$

$$\text{(公式 14, 見 73 頁)}$$

移 $\Pi \sin\dfrac{A}{2}$ 至右邊，再兩邊以 $\Pi \cos\frac{1}{2}A$ 除之

即得　　$\Sigma \tan\frac{1}{2}A = \Pi \tan\frac{1}{2}A + \Pi \sec\frac{1}{2}A$

9. $\Sigma \cos^2 \alpha = \Sigma \dfrac{1}{2}(1 + \cos 2\alpha)$

$\qquad = \dfrac{3}{2} + \dfrac{1}{2} \Sigma \cos 2\alpha \quad$ （參考 143 頁例七）

$\qquad = \dfrac{3}{2} + \dfrac{1}{2}(-1 - 4\Pi \cos \alpha) \quad$ （見 142 頁例六）

$\qquad = 1 - 2\Pi \cos \boldsymbol{\alpha}$

10. 今 $\sin A = \sin(B + C)$

$\qquad$ 故右邊 $= \sin A[\sin(B + C) + \sin(B - C)]$

$\qquad = \sin^2 A + \sin(B + C)\sin(B - C)$

$\qquad = \sin^2 A - \tfrac{1}{2}(\cos 2B - \cos 2C)$

$\qquad = \sin^2 A - \tfrac{1}{2}(1 - 2\sin^2 B - 1 + 2\sin^2 C)$

$\qquad = \sin^2 A + \sin^2 B - \sin^2 C \quad$ （此題亦可如 9 題法做）

11. $\Sigma \cos^2 \dfrac{1}{2} A = \dfrac{1}{2} \Sigma(1 + \cos A) = \dfrac{3}{2} + \dfrac{1}{2} \Sigma \cos A$

$\qquad = \dfrac{3}{2} + \dfrac{1}{2}\left(1 + 4\Pi \sin \dfrac{1}{2} A\right) \quad$ （見 138 頁例一）

$\qquad = 2 + 2\Pi \sin \dfrac{1}{2} A$

12. $\because \quad 3A + 3B + 3C = 3\pi \qquad \therefore \quad \dfrac{3}{2}(B + C) = \dfrac{3}{2}\pi - \dfrac{3}{2}A$

$\qquad$ 故 $\qquad \sin \dfrac{3}{2}(B + C) = -\cos \dfrac{3}{2}A,$

$\qquad\qquad\qquad \cos \dfrac{3}{2} A = -\sin \dfrac{3}{2}(B + C)$

$\qquad$ 今左邊 $= 2\sin \dfrac{3}{2} A \cos \dfrac{3}{2} A + 2\sin \dfrac{3}{2}(B + C)\cos \dfrac{3}{2}(B - C)$

$$= -2\cos\frac{3}{2}(B+C)\cos\frac{3}{2}A - 2\cos\frac{3}{2}A\cos\frac{3}{2}(B-C)$$

$$= -2\cos\frac{3}{2}A\left[\cos\frac{3}{2}(B+C) + \cos\frac{3}{2}(B-C)\right]$$

$$= -4\Pi\cos\frac{3}{2}A$$

13. ∵ $\cos(\beta+\gamma) = \cos(2\pi-\alpha) = \cos\alpha$

又 $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$

今 $\Sigma\sin^2\alpha = \frac{3}{2} - \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\gamma)$

$$= \frac{3}{2} - \frac{1}{2}[2\cos^2\alpha - 1 + 2\cos(\beta+\gamma)\cos(\beta-\gamma)]$$

$$= 2 - [\cos\alpha\cos(\beta+\gamma) + \cos\alpha\cos(\beta-\gamma)]$$

$$= 2 - \cos\alpha[\cos(\beta+\gamma) + \cos(\beta-\gamma)]$$

$$= 2 - 2\Pi\cos\alpha$$

14. ∵ $\cot(\alpha+\beta) = \cot(\frac{1}{2}\pi-\gamma) = \tan\gamma = 1/\cot\gamma$

即 $\dfrac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha} = \dfrac{1}{\cot\gamma}$

去分母移項即得 $\Sigma\cot\alpha = \Pi\cot\alpha$

15.   ∵ $\dfrac{1}{\cot\dfrac{C}{2}} = \tan\dfrac{C}{2} = \tan\left(\dfrac{\pi}{2} - \dfrac{A+B}{2}\right)$

$$= \cot\frac{A+B}{2} = \frac{\cot\dfrac{A}{2}\cot\dfrac{B}{2} - 1}{\cot\dfrac{B}{2} + \cot\dfrac{A}{2}}$$

$$\Sigma\cot\frac{A}{2} = \Pi\cot\frac{A}{2}$$

但　$\Sigma \tan\dfrac{A}{2} = \Pi \tan\dfrac{A}{2} + \Pi \sec\dfrac{A}{2}$　（由題 8）

故　$\left(\Sigma \tan\dfrac{A}{2}\right)\left(\Sigma \cot\dfrac{A}{2}\right) = 1 + \Pi \csc\dfrac{A}{2}$

16. 今　$\dfrac{\Sigma \tan\alpha}{(\Sigma \sin\alpha)^2} = \dfrac{\Pi \tan\alpha}{\left(4\Pi \cos\dfrac{\alpha}{2}\right)^2}$　（見 140 頁例 三及附錄一公式 50 A）

$= \dfrac{\Pi(\sin\alpha / \cos\alpha)}{16\left(\Pi \cos\dfrac{\alpha}{2}\right)^2}$

$= \dfrac{8\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\cos\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}\cos\dfrac{\gamma}{2}}{16\cos^2\dfrac{\alpha}{2}\cos^2\dfrac{\beta}{2}\cos^2\dfrac{\gamma}{2}(\Pi \cos\alpha)}$

$= \dfrac{1}{2}\left(\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}\right) \cdot \dfrac{1}{\Pi \cos\alpha}$

$= \dfrac{\Pi \tan\dfrac{\alpha}{2}}{2\Pi \cos\alpha}$

17. 左邊 $= \Sigma \dfrac{\cos A}{\sin B \sin C} = \dfrac{\Sigma \sin A \cos A}{\Pi \sin A} = \dfrac{\Sigma \sin 2A}{2\Pi \sin A}$

$= \dfrac{4\Pi \sin A}{2\Pi \sin A} = 2$　（見題 3）　　**故左邊＝右邊**

18. 左邊去括號，且以

$\sin A \cos B + \cos A \sin B = \sin(A+B),\cdots\cdots\cdots$

則左邊 $= \sin(A+B) + \sin(B+C) + \sin(C+A)$

$= \sin(\pi-C) + \sin(\pi-B) + \sin(\pi-A)$

$= \sin C + \sin B + \sin A = \Sigma \sin A$

如以左邊$=\frac{1}{2}[\Sigma(\sin\overline{A+B}+\sin\overline{A-B})$

$$+\Sigma(\sin\overline{A+C}+\sin\overline{A-C})]$$ 亦可做.

19. 左邊$=2\sin 2\alpha\cos 2\alpha+2\sin 2(\beta+\gamma)\cos 2(\beta-\gamma)$

$$=2\sin 2\alpha\cos(2\pi-2\beta-2\gamma)$$

$$+2\sin(2\pi-2\alpha)\cos 2(\beta-\gamma)$$

$$=2\sin 2\alpha[\cos 2(\beta+\gamma)-\cos 2(\beta-\gamma)]$$

$$=-4\Pi\sin 2\alpha$$

20. $\because \quad 2\alpha+2\beta+2\gamma=2\pi \qquad \therefore \quad \cos 2\overline{\alpha+\beta)}=\cos 2\gamma$

$\Sigma\sin^2 2\alpha=1-\frac{1}{2}(\cos 4\alpha+\cos 4\beta)+(1-\cos^2 2\gamma)$

$$=2-\cos 2(\alpha+\beta)\cos 2(\alpha-\beta)-\cos^2 2\gamma$$

$$=2-\cos 2\gamma[\cos 2(\alpha-\beta)+\cos 2\overline{\alpha+\beta)}]$$

$$=2-2\Pi\cos 2\alpha$$

21. 設 $\quad x+y+z=\frac{1}{2}\pi$

則 $\quad \cos z=\sin(x+y)$

又 $\quad \dfrac{\pi}{2}-x=y+z,$

$$\dfrac{\pi}{2}-y=z+x$$

故得

$\Sigma\cos x=2\cos\dfrac{1}{2}(x+y)\cos\dfrac{1}{2}(x-y)$

$$+2\sin\dfrac{1}{2}(x+y)\cos\dfrac{1}{2}(x+y)$$

$$=2\cos\dfrac{1}{2}(x+y)\left[\cos\dfrac{1}{2}(x-y)+\sin\dfrac{1}{2}(x+y)\right]$$

$$= 2\cos\frac{1}{2}(x+y)\left[\cos\frac{1}{2}(x-y)+\cos\left(\frac{\pi}{2}-\frac{x+y}{2}\right)\right]$$

$$= 2\cos\frac{x+y}{2}\left[2\cos\frac{1}{2}\left(\frac{\pi}{2}-x\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-y\right)\right]$$

$$= 4\Pi\cos\frac{x+y}{2}$$

今因

$$\left(\frac{3A}{2}+B-2C\right)+\left(\frac{3B}{2}+C-2A\right)+\left(\frac{3C}{2}+A-2B\right)=\frac{\pi}{2}$$

故以 $\dfrac{3A}{2}+B-2C$ 代 x, 又第二, 第三括號中之數各代 y

及 z, 即得本題之證也.

22. 從　$\cos^4\alpha=\dfrac{1}{8}(\cos4\alpha+4\cos2\alpha+3)$　　　見 20 節(100 頁)

故　$\Sigma\cos^4\alpha=\dfrac{1}{8}\Sigma\cos4\alpha+\dfrac{1}{2}\Sigma\cos2\alpha+\dfrac{9}{8}$

$$=\frac{1}{8}(4\Pi\cos2\alpha-1)+\frac{1}{2}(-4\Pi\cos\alpha-1)+\frac{9}{8}$$

（令題 25 中之 m 爲 2 及 1 而得）

$$=\frac{1}{2}(1-4\Pi\cos\alpha+\Pi\cos2\alpha)$$

23. 左邊 $=\frac{1}{2}\Sigma\left[\sin(3\alpha+\beta-\gamma)-\sin(3\alpha-\beta+\gamma)\right]$

$$=\frac{1}{2}\Sigma\left[\sin2(\gamma-\alpha)+\sin2(\alpha-\beta)\right]$$

（以 $\alpha+\beta+\gamma=\pi$ 之關係代入）

$$=\sin2(\gamma-\alpha)+\sin2(\beta-\gamma)+\sin2(\alpha-\beta)$$

$$=-4\Pi\sin(\alpha-\beta)$$　（見 108 頁例十七）

24. $\because$ $\dfrac{\tan\gamma}{\tan\beta}+\dfrac{\tan\alpha}{\tan\beta}=\dfrac{\sin(\gamma+\alpha)}{\tan\beta\cos\gamma\cos\alpha}=\dfrac{\sin(\pi-\beta)}{\tan\beta\cos\gamma\cos\alpha}$

$$=\dfrac{\sin\beta}{\tan\beta\cos\gamma\cos\alpha}=\dfrac{\cos\beta}{\cos\gamma\cos\alpha}$$

$\therefore$ 原式左邊 $=\dfrac{\Sigma\cos^2\alpha}{\Pi\cos\alpha}=\dfrac{1-2\Pi\cos\alpha}{\Pi\cos\alpha}$　（見題 9）

$$=\Pi\sec\alpha-2$$

25. $\because$ $\cos m\alpha=\cos(m\pi-m\overline{\beta+\gamma})=(-1)^m\cos m(\beta+\gamma)$

$\therefore$ $\cos m(\beta+\gamma)=(-1)^m\cos m\alpha$

故 $\Sigma\cos 2m\alpha=2\cos^2 m\alpha-1+2\cos m(\beta+\gamma)\cos m(\beta-\gamma)$

$$=2\cos m\alpha[(-1)^m\cos m(\beta+\gamma)]$$

$$+2[(-1)^m\cos m\alpha\cos m(\beta-\gamma)]-1$$

$$=(-1)^m\cdot 2\cos m\alpha[\cos m(\beta+\gamma)$$

$$+\cos m(\beta-\gamma)]-1$$

$$=(-1)^m\cdot 4\Pi\cos m\alpha-1$$

26. $\because$ $\sin\dfrac{n(\alpha+\beta)}{2}=\sin\dfrac{n(\pi-\gamma)}{2}=\sin\left(\dfrac{4m\pm 1}{2}\pi-\dfrac{n\gamma}{2}\right)$

$$=\sin\left[2m\pi\pm\left(\dfrac{\pi}{2}\mp\dfrac{n\gamma}{2}\right)\right]$$

$$=\pm\sin\left(\dfrac{\pi}{2}\mp\dfrac{n\gamma}{2}\right)=\pm\cos\dfrac{n\gamma}{2}$$

亦卽 $\cos\dfrac{n\gamma}{2}=\pm\sin\dfrac{n(\alpha+\beta)}{2}$

故 $\Sigma\sin n\alpha=2\sin\dfrac{n(\alpha+\beta)}{2}\cos\dfrac{n(\alpha-\beta)}{2}$

$$+2\sin\dfrac{n\gamma}{2}\cos\dfrac{n\gamma}{2}$$

$$= \pm 2 \cos\frac{n\gamma}{2}\Big[\cos\frac{n(\alpha - \beta)}{2} + \cos\frac{n(\alpha + \beta)}{2}\Big]$$

$$= \pm 4\Pi \cos\frac{n\alpha}{2} = 4 \sin\frac{n\pi}{2} \cdot \Pi \cos\frac{n\alpha}{2}$$

因　$\sin\dfrac{n}{2}\pi = \sin\Big(\dfrac{4m \pm 1}{2}\Big)\pi = \sin\Big(2m\pi \pm \dfrac{\pi}{2}\Big)$

$$= \sin\Big(\pm \frac{\pi}{2}\Big) = \pm 1 \text{ 也．}$$

27.　因　$\sin B + \sin C - \sin A = 4 \sin\dfrac{B}{2}\sin\dfrac{C}{2}\cos\dfrac{A}{2}$　（見題 2）

又　　　　　　$\Sigma \sin A = 4\Pi \cos\dfrac{A}{2}$

（公式 50A 或在 26 題中設 $n = 1$）

故　$(\Sigma \sin A)(\sin B + \sin C - \sin A)$

$$= 4 \sin B \sin C \cos^2\frac{A}{2}, \text{ 故如題云．}$$

28.　因　$\Sigma \sin^3\alpha = \dfrac{3}{4}\Sigma \sin\alpha - \dfrac{1}{4}\Sigma \sin 3\alpha$　（見 143 頁例八）

又因 $\alpha + \beta + \gamma = 2\pi$, 故於 145 頁例十一中, 以 α 代 $2A$, ……

並令 $m = 1$ 及 3, 則得

$\Sigma \sin\alpha = 4\Pi \sin\dfrac{\alpha}{2}$, $\Sigma \sin 3\alpha = 4\Pi \sin\dfrac{3\alpha}{2}$, 　代入之卽得．

29.　∵　$\Sigma \sin A(1 + 2\cos B) = \Sigma[\sin A + \sin(A+B)$

$$+ \sin(A - B)]$$

$$= \sin A - \sin C + \sin(A - B) + \cdots\cdots$$

$$[\because \sin\overline{A + B} = \sin(2\pi - C) = -\sin C, \cdots\cdots]$$

$$= \sin(A - B) + \sin(B - C) + \sin(C - A)$$

$$= -4\Pi \sin \tfrac{1}{2}(A-B) \qquad (\text{見 108 頁例十七})$$

30. 卽第 8 題也，不過以 α 易 $\tfrac{1}{2}A$ 耳．

31. 卽　$2\alpha+2\beta+2\gamma=\pi$

今　$\Sigma \sin^2\alpha = \dfrac{3}{2} - \dfrac{1}{2}\Sigma \cos 2\alpha$

$$= \dfrac{3}{2} - \dfrac{1}{2}(1+4\Pi \sin \alpha) \quad (\text{參閱 138 頁例一})$$

$$= 1 - 2\Pi \sin \alpha$$

32. 設　$6B+4C-8A=\alpha, \cdots\cdots$

故　$\alpha+\beta+\gamma=2(A+B+C)=\dfrac{\pi}{2}$

且　$\tfrac{1}{2}(\beta+\gamma)=5A-2B-C$，從 139 頁例二推之

$$\Sigma \cos \alpha = 4\Pi \cos \tfrac{1}{2}\left(\dfrac{\pi}{2}-\alpha\right) = 4\Pi \cos \tfrac{1}{2}(\beta+\gamma)$$

$$= 4\Pi \cos(5A-2B-C)$$

33. 今　$\Sigma\alpha=0$，　又　$\Sigma 2\alpha=0$，　卽每三角之和爲 0．

從 108 頁例十七得

$$\Sigma \sin 2\alpha = -4\Pi \sin \alpha, \quad \Sigma \sin \alpha = -4\Pi \sin \tfrac{1}{2}\alpha$$

又由 130 頁題 44 得　　$\Sigma \cos \alpha = -1+4\Pi \cos \tfrac{1}{2}\alpha$

故右邊 $= 2(-4\Pi \sin \tfrac{1}{2}\alpha)(4\Pi \cos \tfrac{1}{2}\alpha) = -4\Pi \sin \alpha =$ 左邊

34. $\because \ \sin(\alpha+\beta)=\sin[(2n+1)\pi-\gamma]=\sin(\pi-\gamma)=\sin \gamma$

故左邊 $= \sin^2\gamma + \tfrac{1}{2}(\cos 2\alpha - \cos 2\beta)$

$$= \sin \gamma \sin(\alpha+\beta) - \sin(\alpha+\beta)\sin(\alpha-\beta)$$

$$= \sin\gamma[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

$$= 2 \sin \gamma \sin \beta \cos \alpha$$

35. $\because\quad \dfrac{\alpha+\beta}{2}=\dfrac{2n+1}{2}\pi-\dfrac{\gamma}{2}=n\pi+\dfrac{\pi}{2}-\dfrac{\gamma}{2}$

$\therefore\quad \cos\dfrac{\alpha+\beta}{2}=\pm\cos\left(\dfrac{\pi}{2}-\dfrac{\gamma}{2}\right)=\pm\sin\dfrac{\gamma}{2}$

亦即　　$\sin\dfrac{\gamma}{2}=\pm\cos\dfrac{\alpha+\beta}{2}$

今原式$=\left(\cos^2\dfrac{\alpha}{2}+\cos^2\dfrac{\beta}{2}-\cos^2\dfrac{\gamma}{2}\right)^2-4\cos^2\dfrac{\alpha}{2}\cos^2\dfrac{\beta}{2}$

$\qquad\qquad +4\cos^2\dfrac{\alpha}{2}\cos^2\dfrac{\beta}{2}\cos^2\dfrac{\gamma}{2}$

$\qquad =\left(\cos^2\dfrac{\alpha}{2}+\cos^2\dfrac{\beta}{2}-\cos^2\dfrac{\gamma}{2}\right)^2$

$\qquad\qquad -4\sin^2\dfrac{\gamma}{2}\cos^2\dfrac{\alpha}{2}\cos^2\dfrac{\beta}{2}$

但括號內式$=1+\dfrac{1}{2}(\cos\alpha+\cos\beta)-\cos^2\dfrac{\gamma}{2}$

$\qquad\qquad =\dfrac{1}{2}(\cos\alpha+\cos\beta)+\sin^2\dfrac{\gamma}{2}$

$\qquad\qquad =\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}+\sin^2\dfrac{\gamma}{2}$

$\qquad\qquad =\pm\sin\dfrac{\gamma}{2}\left[\cos\dfrac{\alpha-\beta}{2}+\cos\dfrac{\alpha+\beta}{2}\right]$

$\qquad\qquad =\pm 2\sin\dfrac{\gamma}{2}\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}$

故原式$=\left(\pm 2\sin\dfrac{\gamma}{2}\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\right)^2$

$\qquad\qquad -4\sin^2\dfrac{\gamma}{2}\cos^2\dfrac{\alpha}{2}\cos^2\dfrac{\beta}{2}=0$

36. 將左邊乘出,再用公式 14,15(附錄一)得

$$2\Pi\sin\alpha+2\Pi\cos\alpha+\sin(\alpha+\beta+\gamma)-\cos(\alpha+\beta+\gamma)$$

因 $\quad\alpha+\beta+\gamma=\begin{cases}2n\pi\\(2n-\frac{1}{2})\pi\end{cases}$

則 $\quad\sin(\alpha+\beta+\gamma)-\cos(\alpha+\beta+\gamma)=\begin{Bmatrix}0-1\\-1+0\end{Bmatrix}=-1$

故如題云.

37. 今 $\quad\delta=2\pi-(\alpha+\beta+\gamma)\quad\therefore\quad\sin\delta=-\sin(\alpha+\beta+\gamma)$

$\therefore\quad\sin\alpha+\sin\beta+\sin\gamma+\sin\delta$

$$=\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$$

$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta+2\gamma}{2}$$

$$=2\sin\frac{\alpha+\beta}{2}\left[\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta+2\gamma}{2}\right]$$

$$=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$$

38. 今 $\quad\dfrac{\alpha+\beta+\gamma+\delta}{2}=\pi\qquad$即$\qquad\dfrac{\alpha+\beta}{2}=\pi-\dfrac{\gamma+\delta}{2}$

$\therefore\quad\cos\dfrac{\alpha+\beta}{2}=\cos\left(\pi-\dfrac{\gamma+\delta}{2}\right)=-\cos\dfrac{\gamma+\delta}{2}$

即 $\quad\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}-\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}$

$$=-\cos\frac{\gamma}{2}\cos\frac{\delta}{2}+\sin\frac{\gamma}{2}\sin\frac{\delta}{2}\qquad\text{移項即得}$$

39.

左邊$=\begin{vmatrix}1&\cos\gamma&\cos\beta\\0&\sin^2\gamma&\cos\alpha-\cos\beta\cos\gamma\\0&\cos\alpha-\cos\beta\cos\gamma&\sin^2\beta\end{vmatrix}$

$$= \sin^2\gamma \sin^2\beta - (\cos\alpha - \cos\beta\cos\gamma)^2$$

$$= (\sin\gamma\sin\beta - \cos\beta\cos\gamma + \cos\alpha)$$

$$\times (\sin\gamma\sin\beta + \cos\beta\cos\gamma - \cos\alpha)$$

$$= [\cos\alpha - \cos(\beta+\gamma)][\cos(\beta-\gamma) - \cos\alpha]$$

$$= 4\sin\frac{\beta+\gamma+\alpha}{2}\sin\frac{\beta+\gamma-\alpha}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\alpha+\gamma-\beta}{2}$$

$$= 4\sin S \cdot \Pi \sin(S-\alpha)$$

$$(\because \quad \beta+\gamma-\alpha = 2(S-\alpha),\cdots\cdots\cdots\cdots)$$

40. 左邊 $= \dfrac{\sin(2S-\alpha-\beta)}{\cos(S-\alpha)\cos(S-\beta)} + \dfrac{\sin(S-\gamma-S)}{\cos(S-\gamma)\cos S}$

$$= \frac{2\sin\gamma}{\cos\gamma + \cos(\alpha-\beta)} - \frac{2\sin\gamma}{\cos(\alpha+\beta) + \cos\gamma}$$

$$= \frac{2\sin\gamma[\cos(\alpha+\beta) - \cos(\alpha-\beta)]}{\cos^2\gamma + \cos\gamma[\cos(\alpha-\beta) + \cos(\alpha+\beta)] + \cos(\alpha-\beta)\cos(\alpha+\beta)}$$

$$= \frac{4\Pi\sin\alpha}{1 - \Sigma\cos^2\alpha - 2\Pi\cos\alpha}$$

41. 化已知式得

$$2\cos(\gamma+\delta)\cos(\alpha-\beta) = 2\cos(\alpha+\beta)\cos(\gamma-\delta)$$

即 $\qquad \dfrac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \dfrac{\cos(\gamma-\delta)}{\cos(\gamma+\delta)}$

故 $\quad \dfrac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{\cos(\alpha-\beta) + \cos(\alpha+\beta)} = \dfrac{\cos(\gamma-\delta) - \cos(\gamma+\delta)}{\cos(\gamma-\delta) + \cos(\gamma+\delta)}$

(比例合分之理)

即 $\qquad \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} = \dfrac{\sin\gamma\sin\delta}{\cos\gamma\cos\delta}$

即 $\qquad \tan\alpha\tan\beta = \tan\gamma\tan\delta$

42. 今 $\sin\alpha+\sin\gamma=2\sin\beta=2\sin(\alpha+\gamma)$

即 $\cos\dfrac{\alpha-\gamma}{2}=2\cos\dfrac{\alpha+\gamma}{2}$

即 $\cos\dfrac{\alpha}{2}\cos\dfrac{\gamma}{2}+\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}$

$$=2\left(\cos\dfrac{\alpha}{2}\cos\dfrac{\gamma}{2}-\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}\right)$$

即 $3\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}=\cos\dfrac{\alpha}{2}\cos\dfrac{\gamma}{2}$

$\therefore$ $\tan\dfrac{\alpha}{2}\tan\dfrac{\gamma}{2}=\dfrac{1}{3}$

43. 今 $\tan\alpha+\tan\gamma=2\tan\beta=-2\tan(\alpha+\gamma)$

$$=\dfrac{-2(\tan\alpha+\tan\gamma)}{1-\tan\alpha\tan\gamma}$$

$\therefore$ $1-\tan\alpha\tan\gamma=-2$ 即 $\tan\alpha\tan\gamma=3$

$\therefore$ $\tan\alpha=3\cot\gamma$ 或 $\tan^2\alpha=9\cot^2\gamma$

故 $\dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}=\dfrac{9(1+\cos 2\gamma)}{1-\cos 2\gamma}$

即 $\dfrac{1}{\cos 2\alpha}=\dfrac{5+4\cos 2\gamma}{-(4+5\cos 2\gamma)}$ (合分之理)

但 $\cos 2\alpha=-\cos(\pi-2\alpha)=-\cos(\beta+\gamma-\alpha)$

$\therefore$ $\cos(\beta+\gamma-\alpha)=\dfrac{4+5\cos 2\gamma}{5+4\cos 2\gamma}$

44.

(1)左邊 $=\begin{vmatrix} \sin^2 A-\sin^2 B & \cot A-\cot B & 0 \\ \sin^2 B-\sin^2 C & \cot B-\cot C & 0 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$

$$=\begin{vmatrix} \sin(A+B)\sin(A-B) & \dfrac{\sin(B-A)}{\sin A \sin B} & 0 \\[2mm] \sin(B+C)\sin(B-C) & \dfrac{\sin(C-B)}{\sin C \sin B} & 0 \\[2mm] \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$=\frac{\sin(A-B)\sin(B-C)}{\sin A \sin B \sin C}\begin{vmatrix} \sin C & -\sin C \\ \sin A & -\sin A \end{vmatrix} \quad 0$$

(2)展開行列式得 $\qquad 2\sin^2\dfrac{A}{2}\sin^2\dfrac{B}{2}\sin^2\dfrac{C}{2}$

即 $\qquad\qquad\qquad 2\left(\Pi \sin\dfrac{1}{2}A\right)^2$

但從 138 頁例一得 $\quad \Sigma \cos A = 1 + 4\Pi \sin\dfrac{1}{2}A$

$$(因 \quad A+B+C=\pi)$$

故 $\quad \left(\Pi \sin\dfrac{1}{2}A\right)^2 = \dfrac{1}{16}(\Sigma \cos A - 1)^2 \quad$ 代入上式即得.

45. 令 $\quad x = \tan\alpha, \qquad y = \tan\beta, \qquad z = \tan\gamma$

如 147 頁例十五推到 $\quad \alpha + \beta + \gamma = n\pi$

則 $\qquad\qquad 3\alpha + 3\beta + 3\gamma = 3n\pi = m\pi$

再如 145 頁例十二推到 $\quad \Sigma \tan 3\alpha = \Pi \tan 3\alpha$

但 $\quad \tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = \dfrac{3x - x^3}{1 - x^2}, \cdots\cdots$ (公式 23)

故 $\qquad \Sigma\dfrac{3x - x^3}{1 - x^2} = \Pi\dfrac{3x - x^3}{1 - x^2}$

46. 設 $\quad x = \tan\alpha, \qquad y = \tan\beta, \qquad z = \tan\gamma$

則 $\qquad\qquad \Sigma \tan\alpha \tan\beta = 1$

從公式 16 得

$$\tan(\alpha + \beta + \gamma) = \frac{\Sigma \tan \alpha - \Pi \tan \alpha}{1 - \Sigma \tan \alpha \tan \beta} = \infty$$

∴ $\alpha + \beta + \gamma = n\pi + \frac{1}{2}\pi,$ $2(\alpha + \beta + \gamma) = (2n+1)\pi$

從 145 頁例十二 $\Sigma \tan 2\alpha = \Pi \tan 2\alpha$

但 $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2x}{1-x^2},$ ....

∴ $\Sigma \frac{2x}{1-x^2} = \Pi \frac{2x}{1-x^2},$ 去分母即得本題之證。

47. 今 $\dfrac{x}{\sin A} = \dfrac{y}{\sin B} = \dfrac{z}{\sin C} = \dfrac{x-y}{\sin A - \sin B}$

$$= \frac{y-z}{\sin B - \sin C} = \frac{z-x}{\sin C - \sin A} (= k)$$

則 $\Sigma(x-y)\cot\dfrac{C}{2} = k\Sigma 2\cos\dfrac{1}{2}(A+B)\sin\dfrac{1}{2}(A-B)$

$$\times \cot\frac{1}{2}(\pi - A - B)$$

$$= k\Sigma 2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$= k\Sigma(\cos B - \cos A) = 0$$

48. 今 $\cos^2 A = \dfrac{1+\cos 2A}{2} = \dfrac{ac+bd}{(d+a)(b+c)}$

則 $\sin^2 A = \dfrac{ab+cd}{(d+a)(b+c)}$

∴ $\tan^2 A = \dfrac{ab+cd}{ac+bd}$

同理 $\tan^2 B = \dfrac{bc+ad}{ab+cd},$ $\tan^2 C = \dfrac{ac+bd}{bc+ad}$

∴ $\Pi \tan^2 A = 1$ 則 $\Pi \tan A = \pm 1$

但 $A + B + C = \pi$

故從 140 頁例三 $\Sigma \tan A = \Pi \tan A = \pm 1$

49. 今從 148 頁例十七知 $\cot \gamma = \Sigma \cot A$

即 $\cot^2 \gamma = (\Sigma \cot A)^2 = \Sigma \cot^2 A + 2\Sigma \cot A \cot B$

即 $\csc^2 \gamma - 1 = \Sigma(\csc^2 A - 1) + 2\Sigma \cot A \cot B$

$$= \Sigma \csc^2 A - 3 + 2\Sigma \cot A \cot B$$

但因 $A + B + C = \pi$ ∴ $\Sigma \cot A \cot B = 1$

代入上式即得 $\csc^2 \gamma = \Sigma \csc^2 A$

　　註: 本題尚有其他證法,可不用例十七之結果着手,見

　　　　275 頁例十五之證四.

50. 設 $A \neq B \neq C$

則 $\left(\tan\dfrac{A}{2} - \tan\dfrac{B}{2}\right)^2 + \left(\tan\dfrac{B}{2} - \tan\dfrac{C}{2}\right)^2$

$$+ \left(\tan\dfrac{C}{2} - \tan\dfrac{A}{2}\right)^2 > 0$$

即 $2\Sigma \tan^2\dfrac{A}{2} > 2\Sigma \tan\dfrac{A}{2} \tan\dfrac{B}{2}$

即 $\Sigma \tan^2\dfrac{A}{2} > \Sigma \tan\dfrac{A}{2}\tan\dfrac{B}{2}$

今 $\dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2} = \dfrac{\pi}{2}$

從 144 頁例九得

$$\Sigma \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1   \text{ 故如題云.}$$

51. $\because\;\; A+B+C=\pi\;\;\therefore\;\; \sin C=\sin(\pi-C)=\sin(A+B)$

今　$x\sin A+y\sin B+z\sin C=0$　　　　　　(1)

卽　$x\sin A+y\sin B+z(\sin A\cos B+\cos A\sin B)=0$

卽　$\sin A(x+z\cos B)=-\sin B(y+z\cos A)$　　(2)

同理,因　$\sin B=\sin(C+A)$　及　$\sin A=\sin(B+C)$

代入(1)可得

　　$\sin C(z+y\cos A)=-\sin A(x+y\cos C)$　　(3)

且　$\sin B(y+x\cos C)=-\sin C(z+x\cos B)$　　(4)

$(2)\cdot(3)\cdot(4)$　　　$\Pi(x+z\cos B)=-\Pi(y+z\cos A)$

習 題 十 五 (165—167 頁)

1. a.　　　　　$x=n\cdot180°+(-1)^n210°$

　　$\therefore$　比 $360°$ 小之正角爲 $210°$, $330°$.

　b.　　　　$4x=2n\cdot180°\pm180°$

　　$\therefore$　$x=(2n\pm1)\cdot45°=(2m+1)\cdot45°$

　　$\therefore$　比 $360°$ 小之正角爲 $45°$, $135°$, $225°$, $315°$.

　c.　$\tan x=\pm\dfrac{\sqrt{3}}{3}$　　$\therefore$　$x=n\cdot180°\pm30°$

　　$\therefore$　比 $360°$ 小之正角爲 $30°$, $150°$, $210°$, $330°$.

　d.　$\frac{3}{2}x=n\cdot180°+135°$　　$\therefore$　$x=n\cdot120°+90°$

　　$\therefore$　比 $360°$ 小之正角爲 $90°$, $210°$, $330°$.

　e.　(1)　$2\sin x-3=0$,　　則　$\sin x=\frac{3}{2}$　(不可能)

　　(2)　$6\cos x+5=0$,　　則　$\cos x=-\frac{5}{6}$

$$\therefore \quad x = n \cdot 360° \pm 146°26'30''$$

$\therefore$ 比 $360°$ 小之正角爲 $146°26'30''$, $213°33'30''$.

(3) $\sec^2 x - 2 = 0$, 則 $\sec x = \pm\sqrt{2}$

$\therefore \quad x_1 = n \cdot 360° \pm 45°$, $\quad x_2 = n \cdot 360° \pm 135°$

$\therefore$ 比 $360°$ 小之正角爲 $45°$, $135°$, $225°$, $315°$

(4) $\sin^{-1} x - \dfrac{1}{2}\pi = 0$, 則 $x = \sin\dfrac{\pi}{2} = 1$.

2. a. $x = k\pi + (-1)^k \dfrac{\pi}{2} = \dfrac{1}{2}[2k + (-1)^k]\pi$

$\qquad = \dfrac{1}{2}(4n+1)\pi$

$\qquad (\because$ 設 $k=1$, $x = \dfrac{\pi}{2}$; $k=2$, $x = \dfrac{5\pi}{2}$; 均爲

$\qquad \dfrac{\pi}{2}$ 之奇數倍而差爲 4.)

b. $\because$ y 之主值爲 $\dfrac{\pi}{3}$ 及 $\pi + \dfrac{\pi}{3}$

$\therefore y = 2k\pi \pm \dfrac{\pi}{3}$ 或 $y = (2k\pm1)\pi \pm \dfrac{\pi}{3}$

即 k 無論爲奇數或偶數, $y = n\pi \pm \dfrac{\pi}{3}$.

c. $\because$ $3x = n\pi + \dfrac{\pi}{2} = \dfrac{1}{2}(2n+1)\pi$

$\therefore \qquad x = \dfrac{1}{6}(2n+1)\pi$

d. $\because$ $\operatorname{covers} A = 1 - \sin A$

$\therefore \qquad \sin A = 1 \mp \dfrac{1}{2} = \dfrac{1}{2}$ 或 $\dfrac{3}{2}$

但　$|\sin A|<1$　　　$\therefore$　$\sin A = \dfrac{1}{2}$

即　　　　　$A = n\pi + (-1)^n \cdot \dfrac{\pi}{6}$

e.　$\because$　$\theta + \tan^{-1}\dfrac{b}{a} = n\pi + (-1)^n\alpha$　$(\alpha = \sin^{-1}c)$

$\therefore$　　　　$\theta = n\pi + (-1)^n\alpha - \tan^{-1}\dfrac{b}{a}$

f.　當 m 爲偶數時，θ 之主值爲 $\dfrac{\pi}{3}$

故　　　　　$\theta = 2n\pi \pm \dfrac{\pi}{3}$　　　　　(1)

當 m 爲奇數時，θ 之主值爲 $\pi + \dfrac{\pi}{3}$

故　　　　$\theta = (2n \pm 1)\pi \pm \dfrac{\pi}{3}$　　　(2)

故　$\theta = (2n+m)\pi \pm \dfrac{\pi}{3}$　（m 不論爲偶數爲奇數）

因當 m 爲偶數則爲 (1) 之形式，m 爲奇數則爲 (2) 之形式也。

g.　$\because$　$\tan\theta = (-1)^m$　　$\therefore$　θ 之主角爲 $(-1)^m\dfrac{\pi}{4}$

故　　　　　$\theta = n\pi + (-1)^m\dfrac{\pi}{4}$

3. a.　$\because$　$\theta = n\pi \pm (-1)^n\dfrac{\pi}{4} = n\pi \pm \dfrac{\pi}{4}$

$\therefore$　比 $360°$ 小之正角爲 $45°,\ 135°,\ 225°,\ 315°$.

b.　$\dfrac{\theta}{3} = 2n\pi \pm \dfrac{\pi}{6}$　　$\therefore$　$\theta = 6n\pi \pm \dfrac{\pi}{2}$

$\therefore$　比 $360°$ 小之正角爲 $90°$.

c. 即 $\quad \sin x = \cos x = \sin\left(\dfrac{\pi}{2} - x\right)$

$\therefore \quad \dfrac{\pi}{2} - x = n\pi + (-1)^n x$

即 $\quad [1 + (-1)^n] x = \dfrac{\pi}{2}(1 - 2n)$

$\therefore \quad x = \dfrac{(1 - 2n)\pi}{2[1 + (-1)^n]}$

$\therefore$ 比 $360°$ 小之正角爲 $45°$, $225°$.

d. 今 $\tan 5x = \tan 3x \qquad$ 即 $\quad 5x = n\pi + 3x$

故 $\quad x = \dfrac{n\pi}{2}$, $\therefore$ 比 $360°$ 小之正角爲 $0, \dfrac{\pi}{2}, \pi, \dfrac{3}{2}\pi$.

e. 今 $\quad \sin 2x = -\cos 3x$

但 $\quad \sin 2x = -\cos\left(\dfrac{3}{2}\pi - 2x\right)$

$\therefore \quad \cos\left(\dfrac{3}{2}\pi - 2x\right) = \cos 3x$

$\therefore \quad \dfrac{3}{2}\pi - 2x = 2n\pi \pm 3x$

故得 (1) $\quad 5x = \dfrac{\pi}{2}(3 - 4n) = \dfrac{\pi}{2}(4k - 1)$

$\therefore \quad x = \dfrac{\pi}{10}(4k - 1)$

(2) $\quad x = 2n\pi - \dfrac{3}{2}\pi = \dfrac{\pi}{2}(4n - 3) = \dfrac{\pi}{2}(4k + 1)$

f. 今 $\sin mx = \sin nx \qquad$ 即 $\quad mx = k\pi + (-1)^k nx$

即 $\quad [m - n(-1)^k] x = k\pi$

$\therefore \quad x = \dfrac{k\pi}{m - n(-1)^k}$

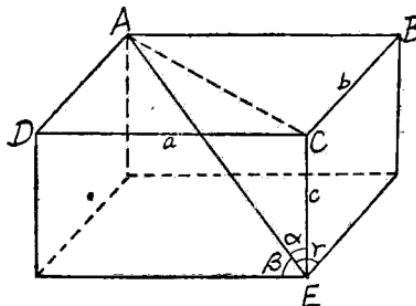g. $\qquad \tan mx = -\cot nx = \tan\left(nx + \dfrac{\pi}{2}\right)$

$\therefore \quad mx = k\pi + nx + \dfrac{\pi}{2}$ 　　即 $\quad (m-n)x = \dfrac{\pi}{2}(2k+1)$

$\therefore \quad x = \dfrac{(2k+1)\pi}{2(m-n)}$

4. 設 $\qquad DC = a, \qquad CB = b, \qquad CE = c$

則 $\quad AC = \sqrt{a^2+b^2} \qquad \therefore \quad AE = \sqrt{a^2+b^2+c^2} = \rho$ （設）

$\therefore \quad \cos\alpha = \dfrac{c}{\rho}, \qquad \cos\beta = \dfrac{a}{\rho}, \qquad \cos\gamma = \dfrac{b}{\rho}$

$\therefore$ 三角爲 $\cos^{-1}\dfrac{a}{\rho}, \qquad \cos^{-1}\dfrac{b}{\rho}, \qquad \cos^{-1}\dfrac{c}{\rho}$

若三度相等，每度爲 a，則其相夾所成之角爲 $\cos^{-1}\dfrac{\sqrt{3}}{3}$，

即 $54°44'20''$．

5. 即 $\qquad 4\sin^2\theta(1-\sin^2\theta) - \sin^2\theta = \tfrac{1}{4}$

即 $\qquad (4\sin^2\theta - 1)^2 - 4\sin^2\theta = 0$

即 $\quad (4\sin^2\theta + 2\sin\theta - 1)(4\sin^2\theta - 2\sin\theta - 1) = 0$

$\therefore \quad \sin\theta = \dfrac{-1\pm\sqrt{5}}{4}, \qquad \dfrac{1\pm\sqrt{5}}{4}$

$$\therefore \quad \theta \text{ 之主角爲 } \pm\frac{\pi}{10}, \ \pm\frac{3}{10}\pi$$

故　　　　$\theta = n\pi \pm \frac{1}{10}\pi, \qquad \theta = n\pi \pm \frac{3}{10}\pi$

今　　　$\left(n \pm \frac{1}{5} \pm \frac{1}{10}\right)\pi = \left(n+\frac{3}{10}\right)\pi, \ \left(n-\frac{3}{10}\right)\pi,$

$$\left(n+\frac{1}{10}\right)\pi, \ \left(n-\frac{1}{10}\right)\pi.$$

故本題 θ 之解均包含在 $\left(n \pm \frac{1}{5} \pm \frac{1}{10}\right)\pi$ 之內.

6. a.　$\because \quad \cot\left(\frac{\pi}{2\sqrt{2}}\cos\theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2\sqrt{2}}\cos\theta\right)$

$\therefore \quad \frac{\pi}{2\sqrt{2}}\sin\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2\sqrt{2}}\cos\theta$

（參考 146 頁例九）

$\therefore \quad \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) = 2n+1$

即　　　$\cos\left(\theta - \frac{\pi}{4}\right) = 2n+1$　　（73 頁例五）

但　$\left|\cos\left(\theta-\frac{\pi}{4}\right)\right| \leqq 1$, 故在 $n=0$ 時此式始能成立.

即　$\cos\left(\theta - \frac{\pi}{4}\right) = 1$, $\therefore \quad \theta - \frac{\pi}{4} = 2n\pi$, 故如題云.

b.　$\because \quad \cot(m\tan\theta) = \tan\left(\frac{\pi}{2} - m\tan\theta\right)$

$\therefore \quad m\cot\theta = p\pi + \frac{\pi}{2} - m\tan\theta$　　（p 爲任意整數）

即 $\quad m(\cot\theta + \tan\theta) = \dfrac{\pi}{2}(2p+1)$

$\therefore \quad \dfrac{2m}{\sin 2\theta} = \dfrac{\pi}{2}(2p+1)$

即 $\quad \sin 2\theta = \dfrac{4m}{(2p+1)\pi}$

故 $\quad \theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{1}{2}\sin^{-1}\dfrac{4m}{(2p+1)\pi}$

c. $\because \quad \cos(m\sin\theta) = \sin\left(\dfrac{\pi}{2} - m\sin\theta\right)$

$\therefore \quad m[\cos\theta + (-1)^n \sin\theta] = \dfrac{[2n + (-1)^n]\pi}{2}$

$\therefore \quad \cos\left[\theta - (-1)^n \dfrac{\pi}{4}\right] = \dfrac{[2n + (-1)^n]\pi}{2m\sqrt{2}}$

故 $\quad \theta - (-1)^n \dfrac{\pi}{4} = \cos^{-1}\dfrac{[2n + (-1)^n]\pi}{2m\sqrt{2}}$

即 $\quad \theta = (-1)^n \dfrac{\pi}{4} + \cos^{-1}\dfrac{[2n + (-1)^n]\pi}{2m\sqrt{2}}$

d. $\because \quad \sin(\pi\cos\theta) = \cos\left(\dfrac{\pi}{2} - \pi\cos\theta\right)$

$= \cos\left(\pi\cos\theta - \dfrac{\pi}{2}\right)$

$\therefore \quad \pi\cos\theta - \dfrac{\pi}{2} = 2n\pi \pm \pi\sin\theta$

$\therefore \quad \cos\theta \mp \sin\theta = 2n + \tfrac{1}{2}$

但 $\quad |\cos\theta \mp \sin\theta| \leqq 2$

故祇在 $n=0$ 時上式始成立

即 $\quad \cos\theta \mp \sin\theta = \dfrac{1}{2}$ 即 $\quad 1 \mp \sin 2\theta = \dfrac{1}{4}$

即　　$\pm \sin 2\theta = \dfrac{3}{4}$　　　故　$2\theta = \pm \sin^{-1}\dfrac{3}{4}$

e.　$\because$　$\sin(\pi \cot \theta) = \cos\left(\pi \cot \theta - \dfrac{\pi}{2}\right)$

$\therefore$　$\pi \cot \theta - \dfrac{\pi}{2} = 2n\pi \pm \pi \tan \theta$

$\therefore$　$\cot \theta \mp \tan \theta = 2n + \frac{1}{2}$

(1) 取正號時　$\dfrac{2}{\sin 2\theta} = 2n + \dfrac{1}{2}$

即　　　$\csc 2\theta = \frac{1}{4}(4n+1)$

(2) 取負號時　$2\cot 2\theta = 2n + \frac{1}{2}$

即　　　$\cot 2\theta = \frac{1}{4}(4n+1)$

習 題 十 六 (178—183 頁)

1. a.　從公式 39 (167 頁), 知原式為 $\sin \frac{1}{2}\pi = 1$

或以第一,第二角各為 α, β, 因此推到

$\sin \alpha = \dfrac{1}{2}$,　$\cos \alpha = \dfrac{\sqrt{3}}{2}$,　$\cos \beta = \dfrac{1}{2}$,　$\sin \beta = \dfrac{\sqrt{3}}{2}$

故　$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$= \dfrac{1}{2} \cdot \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{3}}{2} = 1$$

b.　設　　　$\cot^{-1}\dfrac{2a-b}{b\sqrt{3}} = \alpha$

則　$\cot \alpha = \dfrac{2a-b}{b\sqrt{3}}$,　　$\tan \alpha = \dfrac{b\sqrt{3}}{2a-b}$

設 $\qquad \cot^{-1}\dfrac{2b-a}{a\sqrt{3}}=\beta$

則　$\cot\beta=\dfrac{2b-a}{a\sqrt{3}}, \qquad \tan\beta=\dfrac{a\sqrt{3}}{2b-a}$

今原式$=\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\dfrac{\dfrac{b\sqrt{3}}{2a-b}+\dfrac{a\sqrt{3}}{2b-a}}{1-\dfrac{b\sqrt{3}}{2a-b}\cdot\dfrac{a\sqrt{3}}{2b-a}}$$

$$=\dfrac{2\sqrt{3}\,(a^2-ab+b^2)}{2(ab-a^2-b^2)}=-\sqrt{3}$$

c. 設　$\dfrac{1}{2}\sin^{-1}\dfrac{2x}{1+x^2}=\alpha; \quad \dfrac{1}{2}\cos^{-1}\dfrac{1-y^2}{1+y^2}=\beta$

則　$\sin 2\alpha=\dfrac{2x}{1+x^2}; \qquad \cos 2\beta=\dfrac{1-y^2}{1+y^2}$

$\therefore \quad \cos 2\alpha=\dfrac{1-x^2}{1+x^2}; \qquad \sin 2\beta=\dfrac{2y}{1+y^2}$

由公式 26 得　$\tan\alpha=x; \qquad \tan\beta=y$

故原式$=\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{x+y}{1-xy}$

d. 設第一角爲 α　　　則　$\cos\alpha=\dfrac{63}{65}, \qquad \sin\alpha=\dfrac{16}{65}$

又設　$2\tan^{-1}\dfrac{1}{5}=\beta$　　　　則　$\tan\dfrac{\beta}{2}=\dfrac{1}{5}$

卽 $\qquad \dfrac{1-\cos\beta}{1+\cos\beta}=\dfrac{1}{25}$

故　　　　$\cos\beta=\dfrac{12}{13},$　　　$\sin\beta=\dfrac{5}{13}$

今原式$=\sin(\alpha+\beta)=\dfrac{16}{65}\cdot\dfrac{12}{13}+\dfrac{63}{65}\cdot\dfrac{5}{13}=\dfrac{3}{5}$

e.　設　$\tan^{-1}\cot x=\alpha$　　　則　$\tan\alpha=\cot x$

　　今　$\csc 2\tan^{-1}\cot x=\csc 2\alpha=\dfrac{1}{\sin 2\alpha}$

　　　　　$=\dfrac{1+\tan^{2}\alpha}{2\tan\alpha}$　　（80 頁例四）

　　　　　$=\dfrac{1+\cot^{2}x}{2\cot x}=\dfrac{\tan^{2}x+1}{2\tan x}=\csc 2x$

f.　設　　$\tan^{-1}x=\alpha$　　　　則　$\tan\alpha=x$

　　$\therefore$　　　$\cot 2\alpha=\dfrac{1}{\tan 2\alpha}=\dfrac{1-x^{2}}{2x}$

　　又設　　　$\cos^{-1}\cot 2\alpha=\beta$

　　則　$\cos\beta=\cot 2\alpha=\dfrac{1-x^{2}}{2x};$　　$\sin\beta=\dfrac{\sqrt{6x^{2}-x^{4}-1}}{2x}$

　　今　$\sin 2\cos^{-1}\cot 2\tan^{-1}x=\sin 2\beta=2\sin\beta\cos\beta$

　　　　　$=2\cdot\dfrac{\sqrt{6x^{2}-x^{4}-1}}{2x}\cdot\dfrac{1-x^{2}}{2x}$

　　　　　$=\dfrac{(1-x^{2})\sqrt{6x^{2}-x^{4}-1}}{2x^{2}}$

2.　a.　設　　　　$\tan x=y$

　　則　　　$x=\tan^{-1}y=\tan^{-1}(\tan x)$　　（以 y 之值代入）

　　又設　　$\tan^{-1}x=\beta$

　　則　　　$x=\tan\beta=\tan(\tan^{-1}x)$　　（以 β 之值代入）

b.　設　　$\mathrm{vers}\,x = y$　　則　　$x = \mathrm{vers}^{-1}y = \mathrm{vers}^{-1}(\mathrm{vers}\,x)$

　　又設　$\mathrm{vers}^{-1}x = \beta$　　則　　$x = \mathrm{vers}\,\beta = \mathrm{vers}(\mathrm{vers}^{-1}x)$

c.　設　　$\sin^{-1}x = \alpha$　　　　則　　$\sin\alpha = x$

　　故　　　　　　　$\cos\alpha = \sqrt{1-x^2}$

　　又設　$\cos^{-1}x = \beta$　　　　則　　$\cos\beta = x$

　　故　　　　　　　$\sin\beta = \sqrt{1-x^2}$

　　$\therefore$　$\cos\alpha = \sin\beta$,　　即　$\cos\sin^{-1}x = \sin\cos^{-1}x$

d.　設　　$\sin^{-1}x = \alpha$　　　　則　　　　$x = \sin\alpha$

　　　　　　$\therefore$　$-x = -\sin(\sin^{-1}x)$

　　又設　　$y = -\sin x = \sin(-x)$

　　故　　　$-x = \sin^{-1}y = \sin^{-1}(-\sin x)$

　　故　　　　　　左邊＝右邊

e.　設　　　$\tan^{-1}x = \alpha$,　$\cot^{-1}x = \beta$

　　則　$\tan\alpha = x = \cot\beta$,　　即　$\tan\alpha = \tan\left(\dfrac{\pi}{2}-\beta\right)$

　　　　$\therefore$　$\alpha = \dfrac{\pi}{2}-\beta$,　　即　$\alpha+\beta = \dfrac{\pi}{2}$

　　故如題云.（參考 170 頁例五.）

f.　設　$\sec^{-1}\dfrac{y}{x} = \alpha$　　則　$\sec\alpha = \dfrac{y}{x}$　　$\therefore$　$\cos\alpha = \dfrac{x}{y}$

　　又設　$\sin^{-1}\dfrac{x}{y} = \beta$　則　$\sin\beta = \dfrac{x}{y}$ 以後做法同例五.

g.　設　　$\sin^{-1}x = \alpha$　　　　則　　　$\sin\alpha = x$

　　今　　$\cos 2\alpha = 1-2\sin^2\alpha = 1-2x^2$

　　　$\therefore$　$2\alpha = \cos^{-1}(1-2x^2)$　　故如題云.

h.　設　　$\cos^{-1}x = \alpha$　　　　則　　$\cos\alpha = x$

　　今　　$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha = 4x^3 - 3x$

　　　　　　$\therefore\ 3\alpha = \cos^{-1}(4x^3 - 3x)$

　　卽　　　$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

i.　設　　$\tan^{-1}x = \alpha$　　　　則　　$\tan\alpha = x$

　　今　　$\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = \dfrac{3x - x^3}{1 - 3x^2}$

　　$\therefore\ 3\alpha = \tan^{-1}[(3x - x^3):(1 - 3x^2)]$　故如題云.

j.　設第一及第二角各爲 α, β, 則 $\tan\alpha = \frac{1}{2},\ \tan\beta = \frac{1}{3}$

　　$\therefore\ \tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - (\frac{1}{2})(\frac{1}{3})}$

　　　　　　　　$= \dfrac{5/6}{5/6} = 1$

　　$\therefore\ \alpha + \beta = \dfrac{\pi}{4}$　　　故　$\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4}$

k.　設　$\tan^{-1}\dfrac{3}{5} = \alpha$　則 $\tan\alpha = \dfrac{3}{5}$　　$\therefore\ \cot\alpha = \dfrac{5}{3}$

　　又設　$\cot^{-1}\dfrac{7}{3} = \beta$　則　$\cot\beta = \dfrac{7}{3}$

　　代入 $\cot(\alpha - \beta)$ 公式得

　　　　$\cot(\alpha - \beta) = \dfrac{\frac{5}{3}\cdot\frac{7}{3} + 1}{\frac{7}{3} - \frac{5}{3}} = \dfrac{44}{6} = \dfrac{22}{3}$

　　　　$\therefore\ \alpha - \beta = \cot^{-1}\dfrac{22}{3}$　故如題云.

l.　設第一, 第二角各爲 α, β

　　則　$\tan\alpha = 2 + \sqrt{3},$　　$\tan\beta = 2 - \sqrt{3}$

$$\therefore \quad \tan(\alpha-\beta)=\cdots=\dfrac{(2+\sqrt{3})-(2-\sqrt{3})}{1+(2+\sqrt{3})(2-\sqrt{3})}$$

$$=\dfrac{2\sqrt{3}}{2}=\sqrt{3} \qquad .$$

故 $\quad \sec(\alpha-\beta)=\sqrt{1+3}=2 \qquad \therefore \quad \alpha-\beta=\sec^{-1}2$

故如題云. (此題若從求 $\sec(\alpha-\beta)$ 入手,則必先化

正切爲正餘弦,便無此簡)

m. 設三角依次爲 α,β,γ

則 $\quad \tan\alpha=\dfrac{1}{2}, \quad \tan\beta=\dfrac{1}{5}, \quad \tan\gamma=\dfrac{1}{8}$

則 $\qquad \tan(\alpha+\beta)=\dfrac{7}{9}$

又 $\qquad \tan(\overline{\alpha+\beta}+\gamma)=1 \quad$ (法同 2.j 題)

$\therefore \quad \alpha+\beta+\gamma=\dfrac{\pi}{4} \quad$ 故如題云. (參考 172 頁例八)

n. 設 $\qquad \tan^{-1}\dfrac{1}{408}=\alpha$

則 $\quad \tan\alpha=\dfrac{1}{408}, \quad \tan2\alpha=\dfrac{816}{166463}$,

又 $\quad \tan^{-1}\dfrac{1}{1393}=\beta, \quad$ 則 $\quad \tan\beta=\dfrac{1}{1393}$

今 $\quad \tan(2\alpha-\beta)=\dfrac{\dfrac{816}{166463}-\dfrac{1}{1393}}{1-\dfrac{816}{166463}\cdot\dfrac{1}{1393}}$

$$=\dfrac{970225}{231883775}=\dfrac{1}{239}$$

$$\therefore \quad 2\alpha - \beta = \tan^{-1}\frac{1}{239}$$

故如題云。（此題數字甚大容易算錯）

o. 設兩角爲 α, β

得　　$\tan(\alpha+\beta)=\dfrac{1}{2},$　　　　$\tan 2(\alpha+\beta)=\dfrac{4}{3}$

$\therefore \quad \sec 2(\alpha+\beta)=\dfrac{5}{3}$　　即　$\cos 2(\alpha+\beta)=\dfrac{3}{5}$

$\therefore \quad \alpha+\beta=\dfrac{1}{2}\cos^{-1}\dfrac{3}{5}$　　　如題所云。

p. 設四角依次爲 $\alpha, \beta, \gamma, \delta$

則　　$\tan(\alpha+\beta)=\dfrac{7}{11},$　　　　$\tan(\gamma+\delta)=\dfrac{2}{9}$

再求 $\tan(\overline{\alpha+\beta}+\overline{\gamma+\delta})$ 即得。（此題類似 172 頁例八）

q. 設第一,第二角各爲 α, β

則　$\cot\alpha=(a^3+a^2+a)^{\frac{1}{2}}=\sqrt{a(a^2+a+1)}$

$\cot\beta=(a+a^{-1}+1)^{\frac{1}{2}}=\sqrt{\dfrac{a^2+a+1}{a}}$

故　$\cot(\alpha+\beta)=\dfrac{(a^2+a+1)-1}{\left(\sqrt{a}+\dfrac{1}{\sqrt{a}}\right)\sqrt{a^2+a+1}}$

$$=\dfrac{a(a+1)\sqrt{a}}{(a+1)\sqrt{a^2+a+1}}$$

$\therefore \quad \tan(\alpha+\beta)=\dfrac{\sqrt{a^2+a+1}}{a\sqrt{a}}=\sqrt{\dfrac{a^2+a+1}{a^3}}$

$$=(a^{-1}+a^{-2}+a^{-3})^{\frac{1}{2}}$$

即　　$\alpha+\beta=\tan^{-1}(a^{-3}+a^{-2}+a^{-1})^{\frac{1}{2}}$ 如題所云。

r. 設兩角爲 A, B

則 $\tan A = (\sqrt{2}+1)\tan\alpha, \quad \tan B = (\sqrt{2}-1)\tan\alpha$

$\therefore \quad \tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \dfrac{2\tan\alpha}{1+\tan^2\alpha}$

$\qquad\qquad\qquad = \dfrac{2\sin\alpha/\cos\alpha}{\sec^2\alpha} = 2\sin\alpha\cos\alpha$

$\qquad\qquad\qquad = \sin 2\alpha$

$\therefore \quad A - B = \tan^{-1}(\sin 2\alpha) \quad$ 故如題云。

s. 設左邊之角爲 α

則 $\quad \sin\alpha = \sqrt{\dfrac{x-y}{x-z}}, \qquad \cos\alpha = \sqrt{\dfrac{y-z}{x-z}}$

故 $\quad \tan\alpha = \sqrt{\dfrac{x-y}{y-z}}$

$\therefore \quad \alpha = \tan^{-1}\sqrt{(x-y):(y-z)} \quad \therefore$ 左邊 $=$ 右邊

t. 設兩角依次爲 α, β, 則 $\sin\alpha = \dfrac{1}{\sqrt{82}}$,

$\cos\alpha = \dfrac{9}{\sqrt{82}}, \quad \sin\beta = \dfrac{4}{\sqrt{41}}, \quad \cos\beta = \dfrac{5}{\sqrt{41}}$

$\therefore \quad \tan\alpha = \dfrac{1}{9}, \qquad \tan\beta = \dfrac{4}{5}$

今 $\tan(\alpha+\beta) = \cdots\cdots = 1 \quad \therefore \quad \alpha+\beta = \dfrac{\pi}{4}$

故如題云。

此題之右邊爲 $\dfrac{\pi}{4}$，且 $\tan\alpha$ 等亦比 $\sin\alpha$ 等爲簡，故用 $\tan(\alpha+\beta)$ 做。否則仍以從 $\sin(\alpha+\beta)$ 或 $\cos(\alpha+\beta)$ 入手較爲適宜也。

u. 設第一,第二兩角爲 α, β

則　$\sec \alpha = \dfrac{5}{3}$,　$\cos \alpha = \dfrac{3}{5}$,　$\sin \alpha = \dfrac{4}{5}$;

$\sec \beta = \dfrac{13}{12}$,　$\cos \beta = \dfrac{12}{13}$,　$\sin \beta = \dfrac{5}{13}$

∴ $\sin(\alpha+\beta) = \dfrac{4}{5} \cdot \dfrac{12}{13} + \dfrac{3}{5} \cdot \dfrac{5}{13} = \dfrac{63}{65} = .9692$

故　　　　　$\alpha+\beta = 75°45'$

v. 設第一,第二兩角各爲 α, β

則　　　$\mathrm{vers}\,\alpha = 1 - \cos \alpha = a$

卽　　　$\cos \alpha = 1-a$,　$\sin \alpha = \sqrt{2a-a^2}$

同理　　$\cos \beta = 1-b$,　$\sin \beta = \sqrt{2b-b^2}$

今　$\mathrm{vers}(\alpha+\beta) = 1 - \cos(\alpha+\beta)$

$= 1 - [(1-a)(1-b) - \sqrt{(2a-a^2)(2b-b^2)}]$

$= a+b-ab+\sqrt{(2a-a^2)(2b-b^2)}$

∴ $\alpha+\beta = \mathrm{vers}^{-1}\left\{a+b-ab+\sqrt{(2a-a^2)(2b-b^2)}\right\}$

w. 令第一,第二角各爲 α, β, 再求 $\cos(\alpha-\beta)$ 卽得,法同上兩題.

x. 設　$\tan^{-1}x = a$　則　$\tan \alpha = x$,　$\cos \alpha = \dfrac{1}{\sqrt{x^2+1}}$

設　$\cot^{-1}\cos \alpha = \beta$　　　則　$\cot \beta = \dfrac{1}{\sqrt{x^2+1}}$

卽　$\csc \beta = \dfrac{\sqrt{x^2+2}}{\sqrt{x^2+1}}$　∴ $\sin \beta = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$

卽　$\sin \cot^{-1} \cos \tan^{-1} x = [(x^2+1)/(x^2+2)]^{\frac{1}{2}}$

y. 設第一,第二角各爲 α, β, 求 $\sin(\alpha-\beta)$ 卽得(法與 u,v,w 等題同).

z. 設第一,第二,第三角各爲 α,β,γ

則 $\quad \tan\alpha=\dfrac{1}{3}, \quad \tan\gamma=\dfrac{1}{26}, \quad \tan\beta=\dfrac{1}{7}$

則 $\quad \tan 3\beta=\dfrac{\frac{3}{7}-(\frac{1}{7})^3}{1-3(\frac{1}{7})^2}=\dfrac{147-1}{343-21}=\dfrac{73}{161}$

(見附錄公式 23)

$$\tan(\alpha+\gamma)=\cdots=\dfrac{29}{77}, \tan\left(3\beta-\dfrac{\pi}{4}\right)=\cdots=-\dfrac{44}{117}$$

$$\therefore \quad \tan\left(\alpha+\gamma+3\beta-\dfrac{\pi}{4}\right)=\dfrac{1}{2057}$$

卽 $\quad \alpha+\gamma+3\beta-\dfrac{\pi}{4}=\tan^{-1}\dfrac{1}{2057}$

故 $\quad \tan^{-1}\dfrac{1}{3}+3\tan^{-1}\dfrac{1}{7}+\tan^{-1}\dfrac{1}{26}=\dfrac{\pi}{4}-\tan^{-1}\dfrac{1}{2057}$

3. a. 設第一角爲 α, 則 $\tan\alpha=\sqrt{\dfrac{a(a+b+c)}{bc}}$

$$=\dfrac{1}{bc}\sqrt{abc(a+b+c)}$$

第二角爲 β,則 $\tan\beta=\sqrt{\dfrac{b(a+b+c)}{ca}}$

$$=\dfrac{1}{ca}\sqrt{abc(a+b+c)}$$

今 $\quad \tan(\alpha+\beta)=\dfrac{\dfrac{a+b}{abc}\sqrt{abc(a+b+c)}}{1-\dfrac{1}{abc^2}[abc(a+b+c)]}$

$$=\frac{\dfrac{a+b}{abc}\sqrt{abc(a+b+c)}}{-\dfrac{a+b}{c}}=-\frac{1}{ab}\sqrt{abc(a+b+c)}$$

$$=-\sqrt{\frac{c(a+b+c)}{ab}}$$

卽　　$\tan(\pi-\overline{\alpha+\beta})=\sqrt{\dfrac{c(a+b+c)}{ab}}$

$\because$　$\pi-\overline{\alpha+\beta}=\tan^{-1}\sqrt{\dfrac{c(a+b+c)}{ab}}$　　故如題云.

b.　設第一,第二,第三角各爲 α,β,γ

則　　$\tan\alpha=\dfrac{1}{5}$,　　$\tan 2\alpha=\dfrac{2/5}{1-1/25}=\dfrac{5}{12}$,

　　　$\tan 4\alpha=\dfrac{10/12}{1-25/144}=\dfrac{120}{119}$,　　$\tan(4\alpha-\beta)=\dfrac{49}{50}$

又　　　$\tan\left(\dfrac{\pi}{4}-\gamma\right)=\dfrac{1-1/99}{1+1/99}=\dfrac{49}{50}$

故　　　　　　$4\alpha-\beta=\dfrac{\pi}{4}-\gamma$　　故如題云.

c.　設角依次爲 $\alpha,\beta,\gamma,\delta$, 求得

$\tan(\alpha+\beta)=\cdots\cdots=x^2/2$,　$\tan(\gamma+\delta)=\cdots\cdots=2/x^2$

故　$\tan(\alpha+\beta+\gamma+\delta)=\dfrac{x^2/2+2/x^2}{1-x^2/2\cdot 2/x^2}=\infty$

$\therefore$　$\alpha+\beta+\gamma+\delta=n\pi+\dfrac{\pi}{2}$

d.　設第一,第二角各爲 α,β, 則 $\sin\alpha=4/5$,

$\cos\alpha=3/5$,　$\cos\beta=12/13$,　$\sin\beta=5/13$

故　$\sin(\alpha+\beta)=\dfrac{4}{5}\cdot\dfrac{12}{13}+\dfrac{3}{5}\cdot\dfrac{5}{13}=\dfrac{63}{65}$

卽　$\cos\left(\dfrac{\pi}{2}-\overline{\alpha+\beta}\right)=\dfrac{63}{65}$

$\therefore\ \dfrac{\pi}{2}-\alpha-\beta=\cos^{-1}\dfrac{63}{65}$　　故如題云.

e. 設角依次爲 α,β,γ　　　　則　$\cos\alpha=\dfrac{4}{5}$,

$\sin\beta=\dfrac{\sqrt{10}}{10},\quad \sin\alpha=\dfrac{3}{5},\quad \cos\beta=\dfrac{3\sqrt{10}}{10}$

故　$\tan\alpha=\dfrac{3}{4},\quad \tan\beta=\dfrac{1}{3}$　（以下做法同 $2.m$）

f. 設第一,二,三角各爲 α,β,γ,　又　$\tan^{-1}x=\delta$　✓

則　$\tan\beta=x-1,\qquad \tan\gamma=x+1,$

$$\tan\alpha=\dfrac{2x}{2+x^2+x^4}$$

故　$\tan(\beta+\gamma)=\dfrac{2x}{1-(x^2-1)}=\dfrac{2x}{2-x^2}$

$$\tan(\alpha+\beta+\gamma)=\dfrac{\dfrac{2x}{2+x^2+x^4}+\dfrac{2x}{2-x^2}}{1-\dfrac{2x}{2+x^2+x^4}\cdot\dfrac{2x}{2-x^2}}$$

$$=\dfrac{2x(4+x^4)}{4-4x^2+x^4-x^6}=\dfrac{2x(4+x^4)}{(1-x^2)(4+x^4)}$$

$$=\dfrac{2x}{1-x^2}\quad 又\quad \tan 2\delta=\dfrac{2x}{1-x^2}$$

$\therefore\ \alpha+\beta+\gamma=2\delta$　　故如題云.

g. 設第一,二,三角各爲 $A,\ B,\ C$

則　$\tan A = \dfrac{1}{2}\tan 2\alpha = \dfrac{\tan \alpha}{1-\tan^2\alpha},\ \cdots\cdots$

今　$\tan(B+C) = \dfrac{\cot \alpha + \cot^3\alpha}{1-\cot^4\alpha} = \dfrac{\cot \alpha}{1-\cot^2 \alpha}$

$\qquad = \dfrac{\tan \alpha}{\tan^2\alpha-1} = -\tan A = \tan(-A)$

∴　$B+C = -A$　即　$A+B+C=0$　故如題云.

h. 設　$\dfrac{1}{2}\cos^{-1}\dfrac{a}{b} = \alpha$　則　$\cos 2\alpha = \dfrac{a}{b}$

又因　$\dfrac{\pi}{2}-\left(\dfrac{\pi}{4}-\alpha\right) = \dfrac{\pi}{4}+\alpha$

故　$\tan\left(\dfrac{\pi}{4}+\alpha\right) + \tan\left(\dfrac{\pi}{4}-\alpha\right)$

$\qquad = \tan\left(\dfrac{\pi}{4}+\alpha\right) + \cot\left(\dfrac{\pi}{4}+\alpha\right)$

$\qquad = \dfrac{1}{\sin\left(\dfrac{\pi}{4}+\alpha\right)\cos\left(\dfrac{\pi}{4}+\alpha\right)} = \dfrac{2}{\sin\left(\dfrac{\pi}{2}+2\alpha\right)}$

$\qquad = \dfrac{2}{\cos 2\alpha} = \dfrac{2b}{a}$

i. 設第一,二角爲 $\alpha,\ \beta$

則　$\sin \alpha = \dfrac{2ab}{a^2+b^2},\qquad \cos \alpha = \dfrac{a^2-b^2}{a^2+b^2},\ \cdots\cdots$

∴　$\sin(\alpha+\beta) = \dfrac{2ab(c^2-d^2)}{(a^2+b^2)(c^2+d^2)} + \dfrac{2cd(a^2-b^2)}{(a^2+b^2)(c^2+d^2)}$

$\qquad = \dfrac{2(ac-bd)(bc+ad)}{(ac-bd)^2+(bc+ad)^2}$

$$\therefore \quad \alpha + \beta = \sin^{-1}\frac{2xy}{x^2+y^2} \qquad 故如題云.$$

j.　設第一,第二角爲 α, β

則　　$\tan \alpha = t,$　　　$\tan \beta = 2t/(1-t^2)$

今　$\tan(\alpha+\beta) = \dfrac{t+\dfrac{2t}{1-t^2}}{1-\dfrac{2t^2}{1-t^2}} = \dfrac{3t-t^3}{1-3t^2}$

$$= \frac{t(t+\sqrt{3})(t-\sqrt{3})}{(\sqrt{3}\,t+1)(\sqrt{3}\,t-1)} \quad (=x)$$

今　$t>0,$ 則　$t+\sqrt{3}>0,$　$\sqrt{3}\,t+1>0$

(1)設 $t>\sqrt{3}$ 或 $t<1/\sqrt{3},$ 則 $t-\sqrt{3}$ 與 $\sqrt{3}t-1$ 同號, 故 x 爲正.

$$\therefore \quad \tan(\alpha+\beta) = \frac{3t-t^3}{1-3t^2} \quad \therefore \quad \alpha+\beta = \tan^{-1}\frac{3t-t^3}{1-3t^2}$$

(2)設 $\sqrt{3}>t>1/\sqrt{3},$　則 $t-\sqrt{3}$ 與 $\sqrt{3}t-1$ 異號, 故 x 爲負. 但 $\alpha+\beta$ 通常假定爲銳角.

$$\therefore \quad \tan(\alpha+\beta) = -\frac{3t-t^3}{1-3t^2}$$

卽　　$\tan(\pi-\overline{\alpha+\beta}) = \dfrac{3t-t^3}{1-3t^2}$

卽　　$\pi-\alpha-\beta = \tan^{-1}\dfrac{3t-t^3}{1-3t^2}$

故　　$\alpha+\beta = \pi - \tan^{-1}\dfrac{3t-t^3}{1-3t^2}$

k.　設　$\tan^{-1}(\tan^3 a) = \alpha$　　$\therefore \quad \tan \alpha = \tan^3 a$

則　　$\tan(a+\alpha)=\dfrac{\tan a+\tan^3 a}{1-\tan^4 a}=\dfrac{\tan a}{1-\tan^2 a}$

又　　$2\tan(a+\alpha)=\dfrac{2\tan a}{1-\tan^2 a}=\tan 2a$

$\therefore$　$\tan^{-1}\{2\tan(a+\alpha)\}=2a$　　　故如題云.

1.　設　$\cot^{-1}x=\alpha$　$\therefore$　$\cot\alpha=x$,　$\tan\alpha=\dfrac{1}{x}$,

　　　　$\tan 3\alpha=\dfrac{3/x-1/x^3}{1-3/x^2}=\dfrac{3x^2-1}{x^3-3x}$

設　　$\cos^{-1}\tan 3\alpha=\beta$　　　$\therefore$　$\cos\beta=\dfrac{3x^2-1}{x^3-3x}$

則　　$\sin\beta=\dfrac{\sqrt{x^6-15x^4+15x^2-1}}{x^3-3x}$

$\therefore$　$\sin 2\beta=\dfrac{2(3x^2-1)\sqrt{x^6-15x^4+15x^2-1}}{x^2(x^2-3)^2}$

故如題云.

m.　設　$\tan^{-1}x=\alpha$　　則　$\tan\alpha=x=\cot\left(\dfrac{\pi}{2}-\alpha\right)$

故　$\cot^{-1}x=\dfrac{\pi}{2}-\alpha$,　　$\csc\alpha=\dfrac{\sqrt{x^2+1}}{x}$,

　　　　$\tan\left(\dfrac{\pi}{2}-\alpha\right)=\cot\alpha=\dfrac{1}{x}$

$\therefore$　$\csc\tan^{-1}x-\tan\cot^{-1}x=(\sqrt{x^2+1}-1)/x$

再設　　$\tan^{-1}\dfrac{\sqrt{x^2+1}-1}{x}=\beta$

　　$\therefore$　$\tan\beta=\dfrac{\sqrt{x^2+1}-1}{x}$

即 $\tan 2\beta = \dfrac{2\left(\dfrac{\sqrt{x^2+1}-1}{x}\right)}{1-\dfrac{x^2+2-2\sqrt{x^2+1}}{x^2}}$

$$= \dfrac{2x(\sqrt{x^2+1}-1)}{2(\sqrt{x^2+1}-1)} = x$$

即 $2\beta = \tan^{-1}x$

故 $\tan^{-1}x = 2\tan^{-1}[\csc\tan^{-1}x - \tan\cot^{-1}x]$

n. 設 $\tan^{-1}\dfrac{a}{b}=2\alpha$, $\tan^{-1}\dfrac{b}{a}=2\beta$

則 $\tan 2\alpha = \dfrac{a}{b}$, $\tan 2\beta = \dfrac{b}{a}$, $\cos 2\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$

$\therefore$ $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2} = \dfrac{\sqrt{a^2+b^2}+b}{2\sqrt{a^2+b^2}}$

$\therefore$ $\sec^2\alpha = \dfrac{2\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}+b} = \dfrac{2\sqrt{a^2+b^2}}{a^2}(\sqrt{a^2+b^2}-b)$

同理 $\csc^2\beta = \dfrac{2\sqrt{a^2+b^2}}{b^2}(\sqrt{a^2+b^2}+a)$

今原式左邊 $= \dfrac{a^3}{2}\sec^2\alpha + \dfrac{b^3}{2}\csc^2\beta = [a(\sqrt{a^2+b^2}-b)$

$$+ b(\sqrt{a^2+b^2}+a)]\sqrt{a^2+b^2}$$

$$= (a+b)(a^2+b^2)$$

o. 設 $\dfrac{\sin^{-1}(3\sin x)+x}{4} = A$

則 $\dfrac{\sin^{-1}(3\sin x)-3x}{4} = A-x$

今　　　　$\sin^{-1}(3\sin x)+x=4A$

∴　　$3\sin x=\sin(4A-x)$

即　$2\sin x=\sin(4A-x)-\sin x$

　　　　$=2\cos 2A\sin(2A-x)$

即　$\sin x(\sin^2A+\cos^2A)$

　　　　$=(\cos^2A-\sin^2A)\sin(2A-x)$　♀

∴　$\sin^2A[\sin x+\sin(2A-x)]$

　　　　$=\cos^2A[\sin(2A-x)-\sin x]$

即　$\sin^2A[2\sin A\cos(A-x)]$

　　　　$=\cos^2A[2\cos A\sin(A-x)]$

故　$\tan^3x=\tan(A-x)$　　　故如題云。

p.　設　$\dfrac{2\pi}{3}+\cos^{-1}\dfrac{a}{b}=\alpha,\quad \dfrac{2\pi}{3}-\cos^{-1}\dfrac{a}{b}=\beta$

則原式左邊 $=\sin\alpha\sin\beta-\cos\alpha\cos\beta$

　　　　$=-\cos(\alpha+\beta)=-\cos\dfrac{4\pi}{3}$

　　　　$=-\left[-\cos\dfrac{\pi}{3}\right]=\cos\dfrac{\pi}{3}=\dfrac{1}{2}$　故如題云。

q.　設　$\tan^{-1}x=\alpha$　　　則　　$\tan\alpha=x$

∴　$\tan^2\alpha=\dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}=x^2$　　故　$\cos 2\alpha=\dfrac{1-x^2}{1+x^2}$

今　$\cos 6\alpha=4\cos^3 2\alpha-3\cos 2\alpha$　　代入卽得

r.　設左邊之角爲 A,　則　$\cos A=\dfrac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}$

$$\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}} = \sqrt{\frac{1+\tan\alpha\tan\beta-\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta+\tan\alpha+\tan\beta}}$$

$$= \sqrt{\frac{(1-\tan\alpha)(1-\tan\beta)}{(1+\tan\alpha)(1+\tan\beta)}}$$

$$= \sqrt{\tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-\beta\right)} \qquad (76\ \text{頁}\ g)$$

$$\therefore \quad A = 2\tan^{-1}\sqrt{\tan\left(\frac{\pi}{4}-\alpha\right)\tan\left(\frac{\pi}{4}-\beta\right)} \quad \text{故如題云.}$$

s.　設右邊之角為 x, 則 $\cos x = \dfrac{\sin 2\alpha + \cos\beta}{1+\sin 2\alpha\cos\beta}$

$$\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x} = \frac{(1-\sin 2\alpha)(1-\cos\beta)}{(1+\sin 2\alpha)(1+\cos\beta)}$$

$$= \frac{(\cos\alpha-\sin\alpha)^2}{(\cos\alpha+\sin\alpha)^2}\tan^2\frac{\beta}{2}$$

$$= \left(\frac{1-\tan x}{1+\tan x}\right)^2 \tan^2\frac{\beta}{2}$$

$$\therefore \quad \tan\frac{x}{2} = \tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}$$

$$\therefore \quad x = 2\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}\right] \quad \text{故如題云.}$$

t.　本題做法與上題一樣,亦是從右邊推起.

u.　設　$\tan^{-1}(\cot x) = \alpha$ 　　則　$\tan\alpha = \cot x$

　　設　$\tan^{-1}(\tan x) = \beta$ 　　則　$\tan\beta = \tan x$

　　今　$\tan(\alpha-\beta) = \dfrac{\cot x - \tan x}{1+1} = \dfrac{1-\tan^2 x}{2\tan x} = \cot 2x$

　　又　$\tan(2x + \overline{\alpha-\beta}) = \dfrac{\tan 2x + \cot 2x}{1-1} = \infty$

$$\therefore \quad 2x + \alpha - \beta = n\pi + \frac{\pi}{2} = \frac{\pi}{2}(2n+1)$$

$$\alpha + 2x = \beta + \frac{1}{2}(\ n+1)\pi \qquad 故如題云.$$

v. 今 $\tan^{-1}\dfrac{c_1 x - y}{c_1 y + x} = \tan^{-1}\dfrac{\dfrac{x}{y} - \dfrac{1}{c_1}}{1 + \dfrac{1}{c_1} \cdot \dfrac{x}{y}}$

$$= \tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1}$$

又 $\tan^{-1}\dfrac{c_2 - c_1}{c_2 c_1 + 1} = \tan^{-1}\dfrac{\dfrac{1}{c_1} - \dfrac{1}{c_2}}{1 + \dfrac{1}{c_2} \cdot \dfrac{1}{c_1}}$

$$= \tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2}$$

同理 $\tan^{-1}\dfrac{c_3 - c_2}{c_3 c_2 + 1} = \tan^{-1}\dfrac{1}{c_2} - \tan^{-1}\dfrac{1}{c_3}$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

又 $\tan^{-1}\dfrac{c_n - c_{n-1}}{c_n c_{n-1} + 1} = \tan^{-1}\dfrac{1}{c_{n-1}} - \tan^{-1}\dfrac{1}{c_n}$

$\therefore$ 原式左邊 $= \left(\tan^{-1}\dfrac{x}{y} - \tan^{-1}\dfrac{1}{c_1}\right)$

$$+ \left(\tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2}\right)$$

$$+ \left(\tan^{-1}\frac{1}{c_2} - \tan^{-1}\frac{1}{c_3}\right) + \cdots\cdots$$

$$+ \left(\tan^{-1}\frac{1}{c_{n-1}} - \tan^{-1}\frac{1}{c_n}\right) + \tan^{-1}\frac{1}{c_n}$$

$$= \tan^{-1}\frac{x}{y}$$

w. $\because$ $\tan^{-1}\dfrac{1}{2n^2} = \tan^{-1}\dfrac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}$

$$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

如 $n = 1, 2 \cdots\cdots\cdots n$,   則

$\tan^{-1}\frac{1}{2} = \tan^{-1}3 - \tan^{-1}1$, $\tan^{-1}\frac{1}{8} = \tan^{-1}5 - \tan^{-1}3$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tan^{-1}\frac{1}{2n^2} = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

相加,得 $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{8} + \cdots\cdots + \tan^{-1}\dfrac{1}{2n^2}$

$$= \tan^{-1}(2n+1) - \tan^{-1}1$$

$$= \tan^{-1}(2n+1) - \frac{\pi}{4}$$

x. $\because$ $\tan^{-1}\dfrac{x}{1+(n-1)nx^2} = \tan^{-1}\dfrac{nx-(n-1)x}{1+n(n-1)x^2}$

$$= \tan^{-1}nx - \tan^{-1}(n-1)x$$

如 $n = 1, 2, \cdots\cdots\cdots n$

則 $\tan^{-1}x = \tan^{-1}x$

$$\tan^{-1}\frac{x}{1+1\cdot 2x^2} = \tan^{-1}2x - \tan^{-1}x, \cdots\cdots\cdots\cdots$$

$$\tan^{-1}\frac{x}{1+(n-1)nx^2} = \tan^{-1}nx - \tan^{-1}(n-1)x$$

相加,得原式左邊 $= \tan^{-1}nx$

4. 設 $\cot^{-1}\sqrt{\cos\alpha} = A$ 則 $\cot A = \sqrt{\cos\alpha}$

$\therefore$ $\sin A = \dfrac{1}{\sqrt{1+\cos\alpha}}$, $\cos A = \dfrac{\sqrt{\cos\alpha}}{\sqrt{1+\cos\alpha}}$

又設 $\tan^{-1}\sqrt{\cos\alpha}=B$ 則 $\tan\beta=\sqrt{\cos\alpha}$

$\therefore \sin B=\dfrac{\sqrt{\cos\alpha}}{\sqrt{1+\cos\alpha}};$ $\cos B=\dfrac{1}{\sqrt{1+\cos\alpha}}$

$\therefore \sin u=\sin A\cos B-\cos A\sin B$

$$=\frac{1}{1+\cos\alpha}-\frac{\cos\alpha}{1+\cos\alpha}=\left(\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\right)^2=\tan^2\frac{\alpha}{2}$$

$$\therefore \sin u=\tan^2\frac{\alpha}{2}$$

5. 設三角爲 α,β,γ, 則 $\tan\alpha=a$, $\tan\beta=b$, $\tan\gamma=c$

因 $\alpha+\beta+\gamma=\pi$

則 $\Sigma\tan\alpha=\Pi\tan\alpha$ （見 140 頁例三）

$\therefore$ $a+b+c=abc$

6. 設三角依次爲 α,β,γ

則 $\sin\alpha=\dfrac{x}{a}$, $\sin\beta=\dfrac{y}{b}$, $\sin\gamma=\dfrac{c^2}{ab}$,

$$\cos\gamma=\frac{\sqrt{a^2b^2-c^4}}{ab}$$

今 $\alpha+\beta=\gamma$ $\therefore$ $\cos\alpha\cos\beta=\cos\gamma+\sin\alpha\sin\beta$

卽 $(1-\sin^2\alpha)(1-\sin^2\beta)=\cos^2\gamma+\sin^2\alpha\sin^2\beta$

$$+2\cos\gamma\sin\alpha\sin\beta$$

卽 $-\sin^2\alpha-\sin^2\beta+\sin^2\gamma=2\sin\alpha\sin\beta\cos\gamma$

$\therefore$ $-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{c^4}{a^2b^2}=\dfrac{2xy}{ab}\cdot\dfrac{\sqrt{a^2b^2-c^4}}{ab}$

故 $b^2x^2+a^2y^2+2xy(a^2b^2-c^4)^{\frac{1}{2}}=c^4$

7. 設兩角爲 A,B, 則 $\cos A=\dfrac{x}{a}$, $\cos B=\dfrac{y}{b}$

令 $A+B=\alpha$ ∴ $\cos A \cos B - \sin A \sin B = \cos \alpha$

即 $\cos A \cos B - \cos \alpha = \sin A \sin B$

即 $\cos^2 A \cos^2 B + \cos^2 \alpha - 2 \cos A \cos B \cos \alpha$

$= (1 - \cos^2 A)(1 - \cos^2 B)$

∴ $\cos^2 A + \cos^2 B - 2 \cos A \cos B \cos \alpha$

$= 1 - \cos^2 \alpha = \sin^2 \alpha$

即 $\dfrac{x^2}{a^2} - \dfrac{2xy}{ab} \cos \alpha + \dfrac{y^2}{b^2} = \sin^2 \alpha$

8. 設 $\cos^{-1} \dfrac{x}{a} = \alpha$, $\sin^{-1} \dfrac{y}{a} = \beta$

則 $\cos \alpha = \dfrac{x}{a}$, $\sin \beta = \dfrac{y}{a}$

令 $\alpha = 2\beta$ 故 $\cos \alpha = \cos 2\beta = 1 - 2 \sin^2 \beta$

即 $x/a = 1 - 2y^2/a^2$ 故 $a^2 = ax + 2y^2$

9. 令 $\dfrac{\tan(\theta - \alpha)}{\tan \theta} = \dfrac{\tan \theta}{\tan(\theta - \beta)}$

即 $\dfrac{\tan(\theta - a) + \tan \theta}{\tan \theta - \tan(\theta - a)} = \dfrac{\tan \theta + \tan(\theta - \beta)}{\tan(\theta - \beta) - \tan \theta}$

即 $\dfrac{\sin(2\theta - \alpha)}{\sin \alpha} = \dfrac{\sin(2\theta - \beta)}{\sin(-\beta)} = -\dfrac{\sin(2\theta - \beta)}{\sin \beta}$

故 $-\sin \beta(\sin 2\theta \cos \alpha - \cos 2\theta \sin \alpha)$

$= \sin \alpha(\sin 2\theta \cos \beta - \cos 2\theta \sin \beta)$

∴ $\sin 2\theta(\sin \alpha \cos \beta + \sin \beta \cos \alpha)$

$= 2 \cos 2\theta \sin \alpha \sin \beta$

∴ $\tan 2\theta = \dfrac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$

即 $\qquad \theta = \dfrac{1}{2}\tan^{-1}\dfrac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$

10. 設第一，二，三角依次爲 α, β, γ

則　vers $\alpha = 1-\cos\alpha = \dfrac{x}{a}$　　　$\therefore$　$\cos\alpha = 1-\dfrac{x}{a}$

同理　$\cos\beta = 1-\dfrac{bx}{a}$,　　　$\cos\gamma = b$

今　　$\alpha-\beta = \gamma$　　　即　$\gamma-\alpha = -\beta$

故　　$\cos\gamma\cos\alpha + \sin\gamma\sin\alpha = \cos\beta$

即　$\left(1-\dfrac{x}{a}\right)b + \sqrt{\dfrac{2x}{a}-\dfrac{x^2}{a^2}}\sqrt{1-b^2} = 1-\dfrac{bx}{a}$

即　　　$\sqrt{\dfrac{2x}{a}-\dfrac{x^2}{a^2}} = \dfrac{\sqrt{1-b}}{\sqrt{1+b}}$

即　　　$\dfrac{x^2}{a^2} - 2\left(\dfrac{x}{a}\right) + \dfrac{1-b}{1+b} = 0$

依 $\dfrac{x}{a}$ 之二次方程式解得　$\dfrac{x}{a} = 1\pm\sqrt{\dfrac{2b}{1+b}}$

11. 設第一，二角爲 A, B

則　　$\sin A = \sin\theta + \sin\phi$,　　$\sin B = \sin\theta - \sin\phi$

故　　$\sin^2 A + \sin^2 B = 2(\sin^2\theta + \sin^2\phi) = 2(\tfrac{1}{2}) = 1$

　　　$\therefore$　$\sin^2 A = 1-\sin^2 B = \cos^2 B$

即　　　　$\sin A = \cos B$　　（取正號）

即　　，　　$\sin A = \sin\left(\dfrac{\pi}{2}-B\right)$

即　　　　$A + B = \dfrac{\pi}{2}$　　（取主值）　故如題云．

12. 今．$\tan\theta = x\sqrt{3}/(2c-x)$,　　$\tan\phi = (2x-c)/c\sqrt{3}$

故　$\tan(\theta-\phi)=\dfrac{x\sqrt{3}\,/(2c-x)-(2x-c)/c\sqrt{3}}{1+x\sqrt{3}\,(2x-c)/c\sqrt{3}\,(2c-x)}$

$$=\dfrac{2(c^2-cx+x^2)}{2\sqrt{3}\,(c^2-cx+x^2)}=\dfrac{1}{\sqrt{3}}$$

即　　$\theta-\phi=\dfrac{\pi}{6}$　　（取主值）

習 題 十 七 (210—219 頁)

1. 化 $\cos 3x$ 爲 $\cos x$ 函數，則得　$12\cos^3 x-3\cos x=0$

即　　$3\cos x(2\cos x-1)(2\cos x+1)=0$

$$\therefore\quad \cos x=0,\ \tfrac{1}{2},\ -\tfrac{1}{2}$$

$\therefore\quad x=n\cdot360°\pm90°,\ n\cdot360°\pm60°,\ n\cdot360°\pm120°$

故比 $360°$ 小之正角爲 $60°,90°,120°,240°,270°,300°$.

2. 因　　　　$\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$　　（見 80 頁例 4）

故　　　　$\tan^2 x=\dfrac{2\tan x}{1+\tan^2 x}$

即　　$\tan x(\tan x-1)(\tan^2 x+\tan x+2)=0$

　　1.　$\tan x=0$　　　　　$\therefore\quad x=n\cdot180°$

　　2.　$\tan x=1$　　　　　$\therefore\quad x=n\cdot180°+45°$

　　3.　$\tan^2 x+\tan x+2=0$.　　$\triangle=1-8<0$, 故爲虛數.

故比 $360°$ 小之正角爲 $0°,45°,180°,225°$.

3. 即　$\cos 4x+\cos 2x+1=0$　　即　$2\cos^2 2x+\cos 2x=0$

即　$\cos 2x(2\cos 2x+1)=0$　　$\therefore\quad \cos 2x=0,\ -\tfrac{1}{2}$

即 $\begin{cases} 2x = 2n\cdot180°\pm90° \\ 2x = 2n\cdot180°\pm120° \end{cases}$　　　　即 $\begin{cases} x = n\cdot180°\pm45° \\ x = n\cdot180°\pm60° \end{cases}$

小於 $360°$ 正角有 $45°,60°,120°,135°,225°,240°,300°,315$

4. 即　$\dfrac{1}{\cos x} - \dfrac{1}{\sin x} = \dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x} = \dfrac{\cos^2 x - \sin^2 x}{\sin x \cos x}$

設　　　　　$\sin x \cos x \neq 0$

則　　　$(\cos x - \sin x)(\sin x + \cos x + 1) = 0$

1.　$\cos x - \sin x = 0$　　　　$\therefore$　$\tan x = 1$

　　　　$\therefore$　$x = n\cdot180° + 45°$

2.　$\sin x + \cos x + 1 = 0$　　$\therefore$　$\sin x + \cos x = -1$

　　平方化簡得 $\sin 2x = 0$, 即 $\sin x \cos x = 0$　不合.

故比 $360°$ 小之正角有 $45°, 225°$.

5. 今　　　$2\cos^2 x - 1 = a(1 - \cos x)$

即　　　$2\cos^2 x + a\cos x - (1 + a) = 0$

$\therefore$　　　$\cos x = \dfrac{1}{4}[-a \pm \sqrt{a^2 + 8(1 + a)}]$

即　　　$x = \cos^{-1}\dfrac{1}{4}(-a \pm \sqrt{a^2 + 8a + 8})$

6. $\because$　　$\tan\left(\dfrac{1}{4}\pi + x\right) = \cdots\cdots\cdots = \dfrac{1 + \tan x}{1 - \tan x}$

　　　$\tan\left(\dfrac{1}{4}\pi - x\right) = \dfrac{1 - \tan x}{1 + \tan x}$ ✓

代入去分母得

$(1 + \tan x)^2 + (1 - \tan x)^2 = 4(1 - \tan x)(1 + \tan x)$

即　$\tan^2 x = 1/3$　　　即　$\tan x = \pm\sqrt{3}/3$

$$\because \quad x = n \cdot 180° \pm 30°$$

故比 $360°$ 小之正角爲 $30°$, $150°$, $210°$, $330°$. (或可從公式 27 着手)

7. 即 $\dfrac{\cos x - \sin x}{\cos x + \sin x} = \cos^2 x - \sin^2 x$

即 $(\cos x - \sin x)[(\cos x + \sin x)^2 - 1] = 0$

  1. $\cos x - \sin x = 0$ $\therefore \quad \tan x = 1$

$$\therefore \quad x = n \cdot 180° + 45°$$

  2. $(\cos x + \sin x)^2 = 1$ 即 $\sin 2x = 0$

即 $\sin x \cos x = 0$ $\therefore \quad x = n \cdot 180°$, $n \cdot 360° \pm 90°$

故小於 $360°$ 之正角爲 $0°$, $45°$, $90°$, $180°$, $225°$, $270°$.

8. 即 $(\cos x - \sin x)(\cos^2 x - \sin^2 x) = a(\cos x + \sin x)$

即 $(\cos x + \sin x)[(\cos x - \sin x)^2 - a] = 0$

  1. $\tan x = -1$ $\therefore x = n \cdot 180° - 45°$ 爲 $135°$ 及 $315°$

  2. $\sin 2x = 1 - a$ $\therefore x = \frac{1}{2}[n\pi + (-1)^n \sin^{-1}(1-a)]$

9. 今左邊 $= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)$

$$= 1 - 3\sin^2 x \cos^2 x = 1 - \frac{3}{4}\sin^2 2x$$

故 $1 - \dfrac{3}{4}\sin^2 2x = \dfrac{7}{12}\sin^2 2x$

即 $\sin^2 2x = 3/4$ $\therefore \quad \sin 2x = \pm\sqrt{3}/2$

$\therefore \quad 2x = n \cdot 180° \pm 60°$ $\therefore \quad x = n \cdot 90° \pm 30°$

所求之角爲 $30°$, $60°$, $120°$, $150°$, $210°$, $240°$, $300°$, $330°$

10. 今 $\tan(a - x) = -1/\tan x = -\cot x = \tan(\frac{1}{2}\pi + x)$

$$a-x=k\pi+(\tfrac{1}{2}\pi+x)$$

即　$2x=a-\tfrac{1}{4}(2k+1)\pi$　　故　$x=\tfrac{1}{4}[(2n+1)\pi+2a]$

〔因　$(2k+1)$, $2n+1$　均表示奇數也.〕

11. 今左邊 $=\dfrac{\sin x\sin 3x}{\cos x\cos 3x}=-\dfrac{\cos 4x-\cos 2x}{\cos 4x+\cos 2x}=-\dfrac{2}{5}$

即　　　$3\cos 4x-7\cos 2x=0$

即　　　$6\cos^2 2x-7\cos 2x-3=0$

即　　　$(2\cos 2x-3)(3\cos 2x+1)=0$

∴　$\cos 2x=-\tfrac{1}{3}$, $\tfrac{3}{2}$ （但 $\tfrac{3}{2}$ 不合）

當　$\cos 2x=-\tfrac{1}{3}$, 即 $\cos(180°-2x)=\tfrac{1}{3}=0.3333$ 時,

　　主角爲 $70°32'$.

故 $2x$ 之主角爲 $180°-70°32'=109°28'$

故　　　　$2x=n\cdot 360°\pm 109°28'$

∴　$x=n\cdot 180°\pm 54°44'$

所求之角爲 $54°44'$, $125°16'$, $234°44'$, $305°16'$.

12. 平方兩邊得　$2-2\sqrt{\cos^2 x}=4\cos^2 x$

即　　　　$2\cos^2 x+\cos x-1=0$

即　　　　$(2\cos x-1)(\cos x+1)=0$

∴　$\cos x=\tfrac{1}{2}$, -1

∴　$x=n\cdot 360°\pm 60°$, $n\cdot 360°\pm 180°$

比 $360°$ 小之正角適合原式者只有 $60°$.

13. 即　$2\sin(\tfrac{1}{2}\pi+x)\cos\tfrac{1}{6}\pi=\tfrac{3}{2}$　　即　$\cos x=\tfrac{1}{2}\sqrt{3}$

∴　$x=n\cdot 360°\pm 30°$　　故所求角爲 $30°$, $330°$.

14. 即　$\sin 2x + \sin 24° = \sin 90° + \sin 24°$　∴　$\sin 2x = 1$

∴　$x = n \cdot 90° + (-1)^n \cdot 45°$,　故所求角爲　$45°, 225°$.

15. 左邊 $= \dfrac{\cos(x+120°) + \cos(x-120°)}{\cos(x+120°)\cos(x-120°)} = \dfrac{4\cos x \cos 120°}{\cos 2x + \cos 240°}$

$$= \dfrac{-2\cos x}{2\cos^2 x - 1 - \frac{1}{2}} = \dfrac{-4\cos x}{4\cos^2 x - 3}$$

$$\therefore \quad \dfrac{-4\cos x}{4\cos^2 x - 3} = 2\cos x$$

即　$2\cos x(4\cos^2 x - 1) = 0$　　∴　$\cos x = 0,\ \pm \frac{1}{2}$

∴　$x = n \cdot 360° \pm 90°,\ n \cdot 360° \pm 60°,\ n \cdot 360° \pm 120°$

比 $360°$ 小之正角有　$60°, 90°, 120°, 240°, 270°, 300°$.

16. 即　$\dfrac{\sin^2 x \cos^2 2x}{\cos^2 x \sin^2 2x} = 1$　　　　即　$\tan^2 2x = \tan^2 x$

即　　　　$\tan 2x = \pm \tan x = \tan(\pm x)$

∴　$2x = n\pi \pm x$　　　　　∴　$x = n\pi,\ \frac{1}{3}n\pi$

故所求角爲　$60°, 120°, 240°, 300°$（$0°$ 及 $180°$ 不合）.

17. 即　　　　$2\cos 3x \cos 2x + \cos 2x = 0$

即　　　　　$\cos 2x (2\cos 3x + 1) = 0$

　1.　$\cos 2x = 0$　　　　　∴　$2x = n \cdot 360° \pm 90°$

　　故　　　$x = n \cdot 180° \pm 45°$

　2.　$\cos 3x = -\frac{1}{2}$　　　∴　$3x = n \cdot 360° \pm 120°$

　　故　　　$x = n \cdot 120° \pm 40°$

比 $360°$ 小之正角爲　$40°, 45°, 80°, 135°, 160°, 200°,$
$225°, 280°, 315°, 320°$.

18. 即 $2(\cos 4x \cos 3x + \cos 4x \cos x) = 0$

即 $(\cos 3x + \cos x)\cos 4x = 0$

即 $\cos x \cos 2x \cos 4x = 0$

∴ $x = n \cdot 360° \pm 90°,\ n \cdot 180° \pm 45°,\ n \cdot 90° \pm 22°30'$

故比 $360°$ 小之正角爲 $22°30', 45°, 67°30', 90°, 112°30',$

$135°, 157°30', 202°30', 225°, 247°30', 270°, 292°30',$

$315°, 337°30'$ 十四個角.

19. 即 $2 \sin 3\theta \cos \alpha - 2 \sin \theta \cos \alpha = \cos \alpha$

因 $\cos \alpha$ 爲定數,設不爲 0, 則 $2(\sin 3\theta - \sin \theta) = 1$

即 $8 \sin^3 \theta - 4 \sin \theta + 1 = 0$

即 $(2 \sin \theta - 1)(4 \sin^2 \theta + 2 \sin \theta - 1) = 0$

  1. $\sin \theta = \tfrac{1}{2}$ ∴ $\theta = n \cdot 180° + (-1)^n \cdot 30°$

  2. $\sin \theta = \dfrac{-1 \pm \sqrt{5}}{4}$ ∴ $\theta = n \cdot 180° + (-1)^n \cdot 18°$

      或 $\theta = n \cdot 180° - (-1)^n \cdot 54°$

故小於 $360°$ 之正角爲 $18°, 30°, 150°, 162°, 234°, 306°$.

20. 即 $\cos 6\theta + \cos 2\alpha + \cos 10\theta + \cos 2\alpha = 2 \cos 2\alpha$

即 $2 \cos 8\theta \cos 2\theta = 0$

∴ $\theta = \dfrac{1}{8}\left(2n\pi \pm \dfrac{\pi}{2}\right)$ 或 $\dfrac{1}{2}\left(2n\pi \pm \dfrac{\pi}{2}\right)$

上式亦可寫爲 $\dfrac{1}{16}(2k+1)\pi,\ \ \dfrac{1}{4}(2k+1)\pi$

故比 2π 小之正角爲 $\dfrac{\pi}{16},\ \dfrac{3\pi}{16}$ 等 20 角.

21. 即　　　　　　$4\cos^2\theta - 3 = 2\sin\theta$

即　　　　　　$4\sin^2\theta + 2\sin\theta - 1 = 0$

$$\therefore\quad \sin\theta = \frac{-1\pm\sqrt{5}}{4}$$

$\therefore\quad \theta = n\cdot 180° + (-1)^n\cdot 18°$ 或 $n\cdot 180° - (-1)^n\cdot 54°$

故比 $360°$ 小之正角爲 $18°, 162°, 234°, 306°$。

22. 即　　　　　$-2\sin(n-1)\theta\sin\theta = \sin\theta$

即　　　　　$\sin\theta[2\sin(n-1)\theta + 1] = 0$

   1.　$\sin\theta = 0$　　　　　　$\therefore\quad \theta = m\pi$

   2.　$\sin(n-1)\theta = -\frac{1}{2}$

$$\therefore\quad \theta = \frac{1}{n-1}\Big[m\pi + (-1)^m\cdot\frac{7\pi}{6}\Big]$$

23. 即　$[\sin n\theta + \sin(n-1)\theta][\sin n\theta - \sin(n-1)\theta] = \sin^2\theta$

即　$2\sin\tfrac{1}{2}(2n-1)\theta\cos\tfrac{1}{2}\theta\cdot 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}(2n-1)\theta$
$$= \sin^2\theta$$

即　　　　$\sin\theta[\sin(2n-1)\theta - \sin\theta] = 0$

即　　　　$2\sin\theta\sin(n-1)\theta\cos n\theta = 0$

故　　$\theta = m\pi,\quad \frac{1}{n-1}m\pi\quad$ 或 $\frac{1}{n}\Big[2m\pi\pm\frac{\pi}{2}\Big]$

24. 即　$\sin\theta + 4\sin\theta\cos\theta + 3(3\sin\theta - 4\sin^3\theta) = 0$

即　　　　$\sin\theta(10 + 4\cos\theta - 12\sin^2\theta) = 0$

即　　　　$\sin\theta(6\cos^2\theta + 2\cos\theta - 1) = 0$

   1.　$\sin\theta = 0$　　　　　$\therefore\quad \theta = n\pi$

   2.　$\cos\theta = \dfrac{-1\pm\sqrt{7}}{6} = 0.2743,\quad -0.6076$

$$\therefore \quad \theta = \begin{cases} n \cdot 360° \pm 74°5' \\ n \cdot 360° \pm 127°25' \end{cases}$$

故六角爲 $0°,\ 74°5',\ 127°25',\ 180°,\ 232°35',\ 288°55'$.

25. 因右邊 $= \sin\theta\left(1 + \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1-\cos\theta}{\sin\theta}\right) = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

故原式爲 $\tan\theta + 2\cot2\theta = \tan\theta$ 　　即 　$\cot2\theta = 0$

$$\therefore \quad 2\theta = k\pi \pm \frac{\pi}{2} = \frac{1}{2}(2k\pm1)\pi = \frac{1}{2}(2n+1)\pi$$

$$\therefore \quad \theta = \tfrac{1}{4}(2n+1)\pi$$

26. 即 　　$\tan\theta = 1$ 　　　　　$\therefore \quad \theta = n \cdot 180° + 45°$

比 $540°$ 小之正角爲 $45°,\ 225°,\ 405°$.

27. 即 　　　$\sin(\alpha - x)\cos(\alpha - x) = \sin x\cos x$

即 　　　　　$\sin(2\alpha - 2x) = \sin2x$

$$\therefore \quad 2x = k\pi + (-1)^k(2\alpha - 2x)$$

1.　$k = 2n+1$ 　則 　$0 = k\pi - 2\alpha$ 　　不合理

2.　$k = 2n$ 　　　則 $4x = 2n\pi + 2\alpha$ 　$\therefore \quad x = \tfrac{1}{2}(n\pi + \alpha)$

28. 即 　$\cos^2 x - \sin^2 x = \sin x\cos x(\sin x + \cos x)$

即 　$(\cos x + \sin x)(\cos x - \sin x - \sin x\cos x) = 0$

1.　$\cos x + \sin x = 0$ 　　　　即 　　$\tan x = -1$

$$\therefore \quad x = n \cdot 180° + 45°$$

2.　$\cos x - \sin x = \sin x\cos x$

即 $1 - \sin2x = \tfrac{1}{4}\sin^2 2x$ 　即 $\sin^2 2x + 4\sin2x - 4 = 0$

$\therefore \quad \sin2\alpha = -2 + 2\sqrt{2}$ 　（負號不合因小於 -1 也）

即 $\sin2x = 0.8284$ $\therefore \quad 2x = n \cdot 180° + (-1)^n \cdot 55°56'$

$$\therefore \quad x = n \cdot 90° + (-1)^n \cdot 27°58'$$

故小於 360° 之正角爲 27°58', 135°, 242°2', 315°.（有

不適合之二角已棄去）

29. 卽　$\dfrac{1+\tan\theta}{1-\tan\theta} = 8\tan\theta$　　卽　$8\tan^2\theta - 7\tan\theta + 1 = 0$

$$\therefore \quad \tan\theta = \frac{1}{16}(7 \pm \sqrt{17}) = 0.1798, 0.6951$$

$$\therefore \quad \theta = n \cdot 180° + 10°12' \text{ 或 } n \cdot 180° + 34°48'$$

故比 360° 小之正角有 10°12', 34°48', 190°12', 214°48'.

30. 卽　$\dfrac{\tan\theta + \tan^2\theta}{1-\tan\theta} = 2$　　卽　$\tan^2\theta + 3\tan\theta - 2 = 0$

$$\therefore \quad \tan\theta = \tfrac{1}{2}(-3 \pm \sqrt{17}) = 0.5616, -3.5616$$

$$\therefore \quad \theta = n \cdot 180° + 29°19', \quad \theta = n \cdot 180° + 105°41'$$

故比 360° 小之正角有 29°19', 105°41', 209°19', 285°41'.

31 因　$\tan 3\theta = \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \dfrac{\tan 3\theta}{1 - \tan\theta\tan 2\theta}$

卽　$\tan\theta\tan 2\theta\tan 3\theta = 0$　$\therefore$　$\theta = n\pi, \tfrac{1}{2}n\pi$ 或 $\tfrac{1}{3}n\pi$

卽爲 $n \cdot 90°$ 及 $n \cdot 60°$（$n \cdot 180°$ 包含於 $n \cdot 90°$ 中）

故角爲 0°, 60°, 90°, 120°, 180°, 240°, 270°, 300°.

32. 卽　$2\sin^2\theta = \cos^2\dfrac{3\theta}{2}$　　卽　$4(1-\cos^2\theta) = 1 + \cos 3\theta$

卽　　$4\cos^3\theta + 4\cos^2\theta - 3\cos\theta - 3 = 0$

卽　　$(\cos\theta + 1)(4\cos^2\theta - 3) = 0$

$$\therefore \quad \cos\theta = -1, \ \pm\tfrac{1}{2}\sqrt{3}$$

$$\therefore \quad \theta = 2k\pi \pm \pi, \ 2k\pi \pm \tfrac{1}{6}\pi \text{ 或 } 2k\pi \pm \tfrac{5}{6}\pi$$

故比 $360°$ 小之正角為 $30°$, $150°$, $180°$, $210°$, $330°$.

33. 即　　　$(2\sin\theta-1)(2\sin\theta-\sqrt{3})=0$

$$\therefore\ \ \sin\theta=1/2,\ \ \sqrt{3}/2$$

$$\therefore\ \ \theta=n\cdot180°+(-1)^n\cdot30°\ 或\ n\cdot180°+(-1)^n\cdot60°$$

故比 $360°$ 小之正角為 $30°$, $60°$, $120°$, $150°$.

34. $(4-\sqrt{3})(\sin\theta+\cos\theta)=4(\sin^3\theta+\cos^3\theta)$

$$=4(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)$$

$$=(\sin\theta+\cos\theta)(4-2\sin2\theta)$$

即　　　$(\sin\theta+\cos\theta)(2\sin2\theta-\sqrt{3})=0$

　　1.　$\tan\theta=-1$　　　　$\therefore\ \theta=n\cdot180°-45°$

　　2.　$\sin2\theta=\sqrt{3}/2$　　　$\therefore\ \theta=n\cdot90°+(-1)^n\cdot30°$

故小於 $360°$ 之正角為 $30°$, $60°$, $135°$, $210°$, $240°$, $315°$.

35. $\sin(\theta+\alpha)=\cos(\theta-\alpha)=\sin\left(\dfrac{\pi}{2}-\theta+\alpha\right)$

$$\therefore\ \theta+\alpha=k\pi+(-1)^k\left(\tfrac{1}{2}\pi-\theta+\alpha\right)\ (參考 195 頁例十六)$$

k 為奇數時不合, 故 k 必為偶數, 設為 $2n$, 則

$$\theta+\alpha=2n\pi+\tfrac{1}{2}\pi-\theta+\alpha\ \ \ \therefore\ \theta=\tfrac{1}{4}(4n+1)\pi$$

36. 今　$\cos\left(\dfrac{3x}{2}-\dfrac{\pi}{4}\right)=-\sin\left(\dfrac{2x}{3}-\dfrac{\pi}{4}\right)=\cos\left(\dfrac{2x}{3}+\dfrac{\pi}{4}\right)$

$$\therefore\ \ \dfrac{3x}{2}-\dfrac{\pi}{4}=2n\pi\pm\left(\dfrac{2x}{3}+\dfrac{\pi}{4}\right)$$

　　1.　$\dfrac{5x}{6}=2n\pi+\dfrac{\pi}{2}$　　　　$\therefore\ x=\dfrac{3}{5}(4n+1)\pi$

　　2.　$\dfrac{13x}{6}=2n\pi$　　　　　$\therefore\ x=\dfrac{12}{13}n\pi$

37. 卽　$\tan m\theta = \tan n\theta$　　　　$\therefore\quad m\theta = k\pi + n\theta$

　　卽　$(m-n)\theta = k\pi$　　　　　$\therefore\quad \theta = \dfrac{k\pi}{m-n}$

38. 今　$\pi\theta = n\pi + \dfrac{\pi}{\theta}$　　　　　$\therefore\quad \theta^2 - n\theta - 1 = 0$

　　　　$\therefore\quad \theta = \dfrac{1}{2}(n \pm \sqrt{n^2+4})$

39. 卽　$\cos\left(x+\dfrac{\pi}{4}\right) = \pm\sin(\pi-2x)$

　　今於正負兩號分別討論之

　　　1.　$\cos\left(x+\dfrac{\pi}{4}\right) = \sin(\pi-2x) = \cos\left(2x-\dfrac{\pi}{2}\right)$

　　　　　$\therefore\quad x+\dfrac{\pi}{4} = 2k\pi \pm \left(2x-\dfrac{\pi}{2}\right)$

　　　故　　　　$x = \dfrac{3}{4}\pi - 2k\pi,\ \dfrac{2k\pi}{3}+\dfrac{\pi}{12}$

　　　2.　$\cos\left(x+\dfrac{\pi}{4}\right) = -\sin(\pi-2x) = \sin(2x-\pi)$

　　　　　　　　$= \cos\left(\dfrac{3\pi}{2}-2x\right)$

　　　　　$\therefore\quad x+\dfrac{\pi}{4} = 2k\pi \pm \left(\dfrac{3\pi}{2}-2x\right)$

　　　故　　　　$x = \dfrac{2k\pi}{3}+\dfrac{5}{12}\pi,\ \ \dfrac{7\pi}{4}-2k\pi$

　　故比 2π 小之角爲　$\dfrac{\pi}{12},\ \ \dfrac{5\pi}{12},\ \ \dfrac{3\pi}{4},\ \ \dfrac{13\pi}{12},\ \ \dfrac{17\pi}{12},\ \ \dfrac{7\pi}{4}.$

40. 今　$\sin 3\theta + \sin 2\theta = 2\sin 2\theta \cos\theta = \sin 3\theta + \sin\theta$

　　卽　$\sin 2\theta - \sin\theta = 0$　　　　卽　$2\sin\dfrac{\theta}{2}\cos\dfrac{3\theta}{2} = 0$

$$\therefore \quad \theta = 2k\pi \ \ \text{或} \ \ \frac{2}{3}\left(2k\pi \pm \frac{\pi}{2}\right) = \frac{1}{3}(4k \pm 1)\pi = \frac{1}{3}(2n+1)\pi$$

比 2π 小之正角爲 $0,\ \frac{1}{3}\pi,\ \pi,\ \frac{5}{3}\pi.$（如從 $\sin 2\theta = \sin \theta$

由例十六亦可做）

41. $\because \quad 2\csc^2\theta = \frac{2}{\sin^2\theta} = 1 \Big/ 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}$

故去原式分母得　$\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{\sqrt{3}}{2}$

即　$\cos\theta = \frac{\sqrt{3}}{2}$　　$\therefore \quad \theta = n \cdot 360° \pm 30°$

故所求角爲 $30°, 330°$.

42. 即　　　　$\frac{1}{\sqrt{2}}(\cos 3\theta + \sin 3\theta) = 1$

即　　　　$\cos 3\theta \cos\frac{\pi}{4} + \sin 3\theta \sin\frac{\pi}{4} = 1$

即　$\cos\left(3\theta - \frac{\pi}{4}\right) = 1$　　　　即　$3\theta - \frac{\pi}{4} = 2n\pi$

$$\therefore \quad \theta = \frac{1}{3}\left(2n + \frac{1}{4}\right)\pi$$

比 2π 小之正角爲 $\frac{1}{12}\pi,\ \frac{3}{4}\pi,\ \frac{17}{12}\pi.$（或從原式兩邊平

方亦可做）

43. 即　　$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 3\tan\theta$

即　$8\tan^3\theta = 0$　　　　即　$\tan\theta = 0$

$\therefore \quad \theta = n\pi$　　　　　　故所求角爲 $0, \pi$.

44. $\dfrac{8\cos\theta}{\sin\theta}=\dfrac{1}{\sin^2\frac{1}{2}\theta}+\dfrac{1}{\cos^2\frac{1}{2}\theta}=\dfrac{1}{\sin^2\frac{1}{2}\theta\,\cos^2\frac{1}{2}\theta}=\dfrac{4}{\sin^2\theta}$

即　$2\sin\theta\cos\theta=1$　　　　即　$\sin2\theta=1$

$\therefore\ \theta=\dfrac{1}{2}\Big[n\pi+(-1)^n\cdot\dfrac{\pi}{2}\Big]=\dfrac{\pi}{4}[2n+(-1)^n]$

$\qquad=\dfrac{\pi}{4}[4k+1]=k\pi+\dfrac{\pi}{4}$

〔$\because$　在 $2n+(-1)^n$ 中,若 $n=2k$, 則爲　$4k+1$,

若 $n=2k+1$, 則亦爲 $4k+1$ 也〕

故比 2π 小之正角爲 $\dfrac{1}{4}\pi$, $\dfrac{5}{4}\pi$.

45. 今　　$16\cos^5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$

(見 128 頁題 11)

即　　$\cos\theta(2\cos\theta-1)(2\cos\theta+1)=0$

$\therefore\ \theta=n\cdot360°\pm90°,\ \ n\cdot360°\pm60°,\ \ n\cdot360°\pm120°$

比 $360°$ 小之正角爲 $60°$, $90°$, $120°$, $240°$, $270°$, $300°$.

46. 即　　$2\sin4\theta\cos\theta+2\sin(\theta+45°)\cos\theta=0$

(參考 188 頁例二解三)

即　　$2\cos\theta[\sin4\theta+\sin(\theta+45°)]=0$

即　$4\cos\theta\sin\frac{1}{2}(5\theta+45°)\cos\frac{1}{2}(3\theta-45°)=0$

1.　$\theta=n\cdot360°\pm90°$

2.　$5\theta+45°=2n\cdot180°$　　　　$\therefore\ \theta=n\cdot72°-9°$

3.　$3\theta-45°=2n\cdot360°\pm180°$

$\therefore\ \ \theta=n\cdot240°\pm60°+15°$

比 360° 小之正角爲 63°, 75°, 90°, 135°, 195°, 207°,
270°, 279°, 315°, 351°.

47. 即　$2\sin\left(\theta-\frac{1}{3}\pi\right)(\cos 3\theta+3\cos\theta)+2\cos\left(\theta-\frac{1}{3}\pi\right)$

$$\times(3\sin\theta-\sin 3\theta)-6\sin\left(2\theta-\frac{1}{3}\pi\right)=\sqrt{3}$$

〔用 $\cos^3\theta=\frac{1}{4}(\cos 3\theta+3\cos\theta)$, ……〕

即　　$-2\sin\left(2\theta+\frac{1}{3}\pi\right)+6\sin\left(2\theta-\frac{1}{3}\pi\right)$

$$-6\sin\left(2\theta-\frac{1}{3}\pi\right)=\sqrt{3}$$

即　　　$\sin\left(2\theta+\frac{1}{3}\pi\right)=-\frac{1}{2}\sqrt{3}$

$$\therefore\ 2\theta+\frac{1}{3}\pi=n\pi+(-1)^n\frac{4\pi}{3}$$

$$\therefore\ \theta=\frac{1}{6}(3n-1)\pi+\frac{2}{3}(-1)^n\pi$$

48. $\because\ \cos(\alpha-\beta)=\cos(\alpha-\beta)+\cos\alpha-\cos\alpha$

$$=2\cos\left(\alpha-\frac{\beta}{2}\right)\cos\frac{\beta}{2}-\cos\alpha$$

代入原式得

$$x^2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)+x\left[2\cos\left(\alpha-\frac{\beta}{2}\right)\cos\frac{\beta}{2}-\cos\alpha\right]$$

$$-2\cos\frac{\beta}{2}=0$$

即　$x\cos\left(\alpha-\frac{\beta}{2}\right)\left(x\cos\alpha+2\cos\frac{\beta}{2}\right)$

$$-\left(x\cos\alpha+2\cos\frac{\beta}{2}\right)=0$$

即　$\left[x\cos\left(\alpha-\dfrac{\beta}{2}\right)-1\right]\left[x\cos\alpha+2\cos\dfrac{\beta}{2}\right]=0$

$\therefore$　$x=\sec\left(\alpha-\dfrac{\beta}{2}\right)$　或　$2\sec\alpha\cos\dfrac{\beta}{2}$

〔或即解二次方程式再化簡〕

49.　即　$2(\cos^2\theta-\cos^2\alpha)=(\cos3\theta+3\cos\theta)(\cos\theta-\cos\alpha)$

$$+(\sin3\theta-3\sin\theta)(\sin\theta-\sin\alpha)$$

即　$\cos2\theta-\cos2\alpha=(\cos3\theta\cos\theta+\sin3\theta\sin\theta)$

$$+3(\cos^2\theta-\sin^2\theta)$$

$$-(\cos3\theta\cos\alpha+\sin3\theta\sin\alpha)$$

$$-3(\cos\theta\cos\alpha-\sin\theta\sin\alpha)$$

$$=\cos2\theta+3\cos2\theta-\cos(3\theta-\alpha)$$

$$-3\cos(\theta+\alpha)$$

即　$\cos(3\theta-\alpha)-\cos2\alpha-3[\cos2\theta-\cos(\theta+\alpha)]=0$

即　$2\sin\tfrac{1}{2}(3\theta+\alpha)[\sin\tfrac{1}{2}(3\alpha-3\theta)-3\sin\tfrac{1}{2}(\alpha-\theta)]=0$

　　1.　$\sin\tfrac{1}{2}(3\theta+\alpha)=0$　　　　$\therefore$　$3\theta+\alpha=2n\pi$

　　　　　　$\therefore$　$\theta=\tfrac{1}{3}(2n\pi-\alpha)$

　　2.　$3\sin\tfrac{1}{2}(\alpha-\theta)-4\sin^3\tfrac{1}{2}(\alpha-\theta)=3\sin\tfrac{1}{2}(\alpha-\theta)$

　　即　$\sin\tfrac{1}{2}(\alpha-\theta)=0$　　　　$\therefore$　$\theta-\alpha=2n\pi$

　　　　　　$\therefore$　$\theta=2n\pi+\alpha$

50.　從提示　$4\sin^3\tfrac{1}{6}x-3\sin\tfrac{1}{6}x=4\sin^2\tfrac{1}{6}x$

令　$\tfrac{1}{6}x=A$，　則　$4\sin^3A-4\sin^2A-3\sin A=0$

即　$\sin A(2\sin A-3)(2\sin A+1)=0$

$$\sin A = \frac{3}{2} 不合 \qquad \therefore \quad A = k\pi \ 或 \ k\pi - (-1)^k \cdot \frac{\pi}{6}$$

$$\therefore \quad x = 6k\pi \ 或 \ 6k\pi - (-1)^k \pi$$

51. 卽　　$\tan\theta + \dfrac{\sqrt{3} + \tan\theta}{1 - \sqrt{3}\tan\theta} + \dfrac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = 3$

　　卽　　　　$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$

　　卽　　　$(\tan\theta + 1)(\tan^2\theta - 4\tan\theta + 1) = 0$

$$\therefore \quad \tan\theta = -1, \ 2 \pm \sqrt{3}$$

$$\therefore \quad \theta = n\pi + \frac{3\pi}{4}, \ n\pi + \tan^{-1}(2 \pm \sqrt{3})$$

52. $\because \quad \tan(\overline{x+\alpha} + \overline{x+\beta} + \overline{x+\gamma})$

$$= \frac{\Sigma\tan(x+\alpha) - \Pi\tan(x+\alpha)}{1 - \Sigma\tan(x+\alpha)\tan(x+\beta)}$$

　今分母爲 0, 故 $\tan(x+\alpha+x+\beta+x+\gamma) = \infty$

$$\therefore \quad 3x + \alpha + \beta + \gamma = n\pi + \frac{\pi}{2}$$

$$\therefore \quad x = \frac{1}{3}\left(n\pi + \frac{\pi}{2} - \alpha - \beta - \gamma\right)$$

53. 卽　　　$\dfrac{\sin 5\theta}{\cos\theta\cos 4\theta} + \dfrac{\sin 5\theta}{\cos 2\theta\cos 3\theta} = 0$

　　卽　$\sin 5\theta(\cos\theta\cos 4\theta + \cos 2\theta\cos 3\theta) = 0$

　　卽　$\sin 5\theta\cos\theta[(2\cos^2 2\theta - 1) + \cos 2\theta(4\cos^2\theta - 3)] = 0$

　　卽　$\sin 5\theta\cos\theta(4\cos^2 2\theta - \cos 2\theta - 1) = 0$

$$\therefore \quad \theta = \frac{n\pi}{5}, \ 2k\pi \pm \frac{\pi}{2}\left(卽 \ \frac{1}{2}\overline{2n+1}\pi\right)$$

$$或 \ \frac{1}{2}\cos^{-1}\frac{1 \pm \sqrt{17}}{8}$$

54. 令　　$\theta/5 = \alpha$　　　　則　　　$\tan 5\alpha = 5 \tan \alpha$

即　　　　　$\sin 5\alpha \cos \alpha = 5 \sin \alpha \cos 5\alpha$

即　　　　　$3 \sin 4\alpha = 2 \sin 6\alpha$

即　　　　$3 \sin 2\alpha \cos 2\alpha = 3 \sin 2\alpha - 4 \sin^3 2\alpha$

即　　　　$\sin 2\alpha (\cos 2\alpha - 1)(4 \cos 2\alpha + 1) = 0$

$\therefore$　　　$\alpha = \dfrac{1}{2} n\pi,\ n\pi \left(\text{此包含在} \dfrac{1}{2} n\pi \ \text{內}\right)$

　　　　或 $\dfrac{1}{2}\left[2n\pi + \cos^{-1}\left(-\dfrac{1}{4}\right)\right]$

即　　　$\theta = \dfrac{5}{2} n\pi,\ \ 5\left[n\pi + \dfrac{1}{2}\cos^{-1}\left(-\dfrac{1}{4}\right)\right]$

55. 即　　$\dfrac{\sin 2^x \alpha}{\sin 2^x \alpha \sin 2^{x+1}\alpha} = \dfrac{1}{\sin 2^3 \alpha}$　　（參考 197 頁例十九）

即　　　　　$\sin 2^{x+1}\alpha = \sin 2^3 \alpha$

故　　　$2^{x+1}\alpha = n\pi + (-1)^n \cdot (2^3 \alpha)$

當 $n = 0$ 時，　　　　$2^{x+1}\alpha = 2^3 \alpha$

　　　　$\therefore \ \ x + 1 = 3,\ \ $ 即　　$x = 2$

56. 即　$7346 \cdot 7^{\sec x} + 7 \cdot 7^{\sec x} - 7010 \cdot 7^{2 \sec x} - 343 \cdot 7^{2 \sec x}$

　　　　　　$+ 147 \cdot 7^{3 \sec x} = 147$

即　$49 \cdot 7^{3 \sec x} - 2451 \cdot 7^{2 \sec x} + 2451 \cdot 7^{\sec x} - 49 = 0$

即　$(7^{\sec x} - 1)(7^{\sec x} - 49)(49 \cdot 7^{\sec x} - 1) = 0$

　　　　$\therefore \ 7^{\sec x} = 7^0,\ 7^2,\ 7^{-2}$

即　$\sec x = 0, 2, -2$　　　即　$\cos x = \infty,\ \dfrac{1}{2},\ -\dfrac{1}{2}$

今　$\cos x = \infty$ 不合，因 $|\cos x| \leqq 1$ 也

$$\therefore \quad x = 2n\pi \pm \frac{2}{3}\pi \quad \text{或} \quad 2n\pi \pm \frac{1}{3}\pi$$

57. 即 $\qquad 5 \cdot 5^{\tan x} + 5/5^{\tan x} = 26$

即 $\qquad 5 \cdot 5^{2 \tan x} - 26 \cdot 5^{\tan x} + 5 = 0$

即 $\qquad (5^{\tan x} - 5)(5 \cdot 5^{\tan x} - 1) = 0$

即 $\qquad 5^{\tan x} = 5^1,\ 5^{-1}$

$\therefore \ \tan x = \pm 1 \qquad\qquad \therefore \quad x = n \cdot 180° \pm 45°$

58. $\because \quad 4 - 4\cos 2\theta = 8\sin^2\theta \quad 又 \quad 6 - 4\cos^2\theta = 2 + 4\sin^2\theta$

即 $\qquad 2^{8\sin^2\theta} - 3 \cdot 2^{2+4\sin^2\theta} + 32 = 0$

即 $\qquad (2^{4\sin^2\theta})^2 - 12 \cdot 2^{4\sin^2\theta} + 32 = 0 \qquad$ （參考 197 頁 例二十）

即 $\qquad (2^{4\sin^2\theta} - 4)(2^{4\sin^2\theta} - 8) = 0$
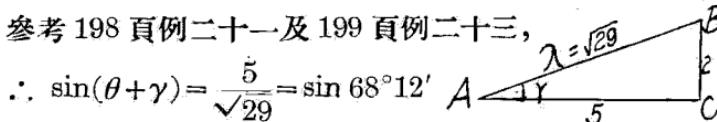
$\therefore \quad 2^{4\sin^2\theta} = 2^2,\ 2^3 \qquad\qquad \therefore \quad \sin^2\theta = \frac{1}{2},\ \frac{3}{4}$

$$\therefore \quad \sin\theta = \pm \frac{\sqrt{2}}{2},\ \frac{\sqrt{3}}{2}$$

$$\therefore \quad \theta = (n \pm 1/4)\pi \quad \text{或} \quad (n \pm 1/3)\pi$$

59. 如圖 $\lambda = \sqrt{29}, \quad \tan\gamma = \frac{2}{5} = .4 \quad \therefore \quad \gamma = 21°48'$

參考 198 頁例二十一及 199 頁例二十三，

$\therefore \ \sin(\theta + \gamma) = \dfrac{5}{\sqrt{29}} = \sin 68°12'$

$\therefore \quad \theta = -21°48' + n \cdot 180° + (-1)^n (68°12')$

故角爲 $46°24', 90°$.

60. 即 $\qquad 1 + \frac{1}{2}(1 - \cos 2\theta) = \frac{3}{2}\sin 2\theta$
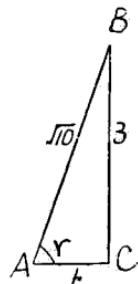
即　　　　　　　$3 \sin 2\theta + \cos 2\theta = 3$

以下做法如上題可求得

$$\cos(2\theta - \alpha) = \frac{3}{\sqrt{10}} = \cos 18°26'$$

$$\therefore \quad 2\theta - \alpha = n \cdot 360° \pm 18°26'$$

$$\therefore \quad \theta = 35°47' + n \cdot 180° \pm 9°13'$$

故比 $360°$ 小之正角為 $26°34', 45°, 206°34', 225°$.

61.　$\because$　　　　　　$2 + \sqrt{3} = \tan 75°$

即　　　$\sin\theta \cos 75° + \cos\theta \sin 75° = \cos 75°$

即　　　　　　$\sin(\theta + 75°) = \sin 15°$

$\therefore$　　　　$\theta + 75° = n \cdot 180° + (-1)^n \cdot 15°$

故比 $360°$ 小之正角為 $90°, 300°$. （此題或可照上兩題法做）

62.　今　$\tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta \tan 2\theta)$

$\therefore \quad \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta} = \sqrt{3}$　　　即　　$\tan 3\theta = \sqrt{3}$

$\therefore \quad 3\theta = n\pi + \dfrac{\pi}{3}$　　　　　　即　$\theta = \dfrac{1}{3}\left(n + \dfrac{1}{3}\right)\pi$

故比 2π 小之正角有 $\dfrac{\pi}{9}, \cdots\cdots \dfrac{16\pi}{9}$ 六角.

63.　即　　　　　$m \sin 2(\alpha - \theta) = n \sin 2\theta$

即　$m(\sin 2\alpha \cos 2\theta - \cos 2\alpha \sin 2\theta) = n \sin 2\theta$

即　$\sin 2\theta(m \sin 2\alpha \cot 2\theta - m \cos 2\alpha - n) = 0$

　1.　$\sin 2\theta = 0$　　　　即　$2 \sin\theta \cos\theta = 0$

　　但　$\cos\theta$ 不可為 0　$\therefore$　$\sin\theta = 0$　$\therefore$　$\theta = n\pi$

2. $\cot 2\theta = \dfrac{m \cos 2\alpha + n}{m \sin 2\alpha}$

$\therefore \quad \theta = \dfrac{1}{2}\tan^{-1}\dfrac{m \sin 2\alpha}{m \cos 2\alpha + n}$

64. 即 $\dfrac{\tan(\theta - \alpha)}{\tan \theta} = \dfrac{\tan \theta}{\tan(\theta - \beta)}$

即 $\dfrac{\tan(\theta - \alpha) + \tan \theta}{\tan \theta + \tan(\theta - \beta)} = \dfrac{\tan(\theta - \alpha) - \tan \theta}{\tan \theta - \tan(\theta - \beta)}$

即 $\dfrac{\sin(2\theta - \alpha)}{\sin(-\alpha)} = \dfrac{\sin(2\theta - \beta)}{\sin \beta}$

即 $\sin \beta(\sin 2\theta \cos \alpha - \cos 2\theta \sin \alpha)$

$\qquad = -\sin \alpha(\sin 2\theta \cos \beta - \cos 2\theta \sin \beta)$

即 $\sin(\alpha + \beta)\sin 2\theta = 2 \sin \alpha \sin \beta \cos 2\theta$

$\therefore \qquad \tan 2\theta = \dfrac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$

$\therefore \quad \theta = \dfrac{1}{2}n\pi + \dfrac{1}{2}\tan^{-1}\dfrac{2 \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$

65. 今 $4 \cos \theta \sin(\theta - \alpha)\sin^3\alpha = \cos^2\alpha(\sin^2\alpha - \cos^2\theta)$

$\qquad = \cos^2\alpha(\sin^2\alpha \sin^2\theta - \cos^2\theta\cos^2\alpha)$

即 $4 \cos \theta \sin^3\alpha(\sin \theta \cos \alpha - \cos \theta \sin \alpha)$

$\qquad = \cos^2\alpha(\sin^2\alpha \sin^2\theta - \cos^2\theta \cos^2\alpha)$

全式除以 $\cos^4\alpha \cos^2\theta$, 得

$\qquad 4 \tan^3\alpha(\tan \theta - \tan \alpha) = \tan^2\alpha \tan^2\theta - 1$

即 $\tan^2\theta \tan^2\alpha - 4 \tan^3\alpha \tan \theta + 4 \tan^4\alpha - 1 = 0$

$\therefore \qquad \tan \theta = 2 \tan \alpha \pm \cot \alpha$

$$\therefore \quad \theta = n\pi + \tan^{-1}(2\tan\alpha \pm \cot\alpha)$$

66. 即　$35(3\sin\theta - 4\sin^3\theta) - 40\sin 2\theta\sin\theta + 39\sin\theta = 0$

即　$4\sin\theta(36 - 10\sin 2\theta - 35\sin^2\theta) = 0$

　　1.　$\sin\theta = 0 \qquad\qquad \therefore \quad \theta = n\pi$

　　2.　$36(\sin^2\theta + \cos^2\theta) - 20\sin\theta\cos\theta - 35\sin^2\theta = 0$

　　　即　$\tan^2\theta - 20\tan\theta + 36 = 0$

　　　即　$(\tan\theta - 2)(\tan\theta - 18) = 0$

　　　$\therefore \qquad \tan\theta = 2, \quad \tan\theta = 18$

　　　故　$\theta = n\pi + \tan^{-1}2, \quad n\pi + \tan^{-1}18$

67. 即　$\left[1 + \dfrac{\cos(\theta+\alpha) + \cos(\theta-\alpha)}{2}\right] \cdot \dfrac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)}$

$$+ \frac{\sin 2\alpha}{2} + \frac{\sin(\theta+\alpha) - \sin(\theta-\alpha)}{2} = 0$$

即　$\sin(\theta+\alpha) + \frac{1}{4}\sin(2\theta+2\alpha) + \frac{1}{2}\sin(\theta+\alpha)\cos(\theta-\alpha)$

$$+ \tfrac{1}{2}\cos(\theta+\alpha)\sin 2\alpha + \tfrac{1}{4}\sin(2\theta+2\alpha)$$

$$- \tfrac{1}{2}\cos(\theta+\alpha)\sin(\theta-\alpha) = 0$$

即　$2\sin(\theta+\alpha) + 2\sin(\theta+\alpha)\cos(\theta+\alpha)$

$$+ \sin 2\alpha + \cos(\theta+\alpha)\sin 2\alpha = 0$$

即　$2\sin(\theta+\alpha)[1 + \cos(\theta+\alpha)]$

$$+ \sin 2\alpha[1 + \cos(\theta+\alpha)] = 0$$

即　$[2\sin(\theta+\alpha) + \sin 2\alpha][1 + \cos(\theta+\alpha)] = 0$

$\therefore \quad \theta + \alpha = n\pi + (-1)^n \sin^{-1}(-\tfrac{1}{2}\sin 2\alpha)$ 或 $2n\pi \pm \pi$

68. 左邊 $= [\tan(\alpha+\theta) - \tan(\alpha-\theta)][\tan(\alpha+\theta) + \tan(\alpha-\theta)]$

$$= \frac{\sin 2\alpha \sin 2\theta}{\cos^2(\alpha+\theta)\cos^2(\alpha-\theta)} = \frac{4\sin 2\alpha \sin 2\theta}{(\cos 2\alpha + \cos 2\theta)^2}$$

故　$\sin \alpha \sin \theta [16 \cos^2\alpha \cos^2\theta - (\cos 2\alpha + \cos 2\theta)^2] = 0$

1. $\sin\theta = 0$ 　∴ $\theta = n\pi$ 　　（設　$\sin \alpha \neq 0$）

2. $4\cos \alpha \cos \theta = \pm 2(\cos^2\alpha + \cos^2\theta - 1)$

∴　$(\cos \theta \mp \cos \alpha)^2 = 1$,　即　$\cos \theta \mp \cos \alpha = 1$

∴　$\theta = \cos^{-1}(1 \pm \cos \alpha)$

69. 即　$\sec^2\theta - 1 = 2\tan \alpha \tan \beta \sec \theta + \tan^2\alpha + \tan^2\beta$

即　$(\sec \theta - \tan \alpha \tan \beta)^2 = (1+\tan^2\alpha)(1+\tan^2\beta)$

即　$\sec \theta - \tan \alpha \tan \beta = \pm \sec \alpha \sec \beta$

∴　$\theta = \sec^{-1}(\tan \alpha \tan \beta \pm \sec \alpha \sec \beta)$

70. 設　$\theta = 6\alpha$ 　　則　　$\cos 6\alpha = -1$

今　　　$\cos 3\alpha = \pm \sqrt{\dfrac{1+\cos 6\alpha}{2}} = 0$

即　$4\cos^3\alpha - 3\cos \alpha = 0$ 　　即　$\cos \alpha(4\cos^2\alpha - 3) = 0$

∴　$\cos \alpha = 0,\ \pm\dfrac{\sqrt{3}}{2}$ 　　故　$\cos\dfrac{\theta}{6} = 0,\ \pm\dfrac{\sqrt{3}}{2}$

本題亦可從 $\theta = (2n+1)\pi$ 着手, 但若化 $\cos \theta$ 為

$\cos(\theta/6)$ 函數做則繁矣.

71. 令五角依次為 $x-2y,\ x-y,\ x,\ x+y,\ x+2y$（公差為 y）

則　$\cos x = \cos(x-2y) + \cos(x-y) + \cos(x+y)$
　　　　$+ \cos(x+2y)$

∴　$\cos x = 2\cos x \cos 2y + 2\cos x \cos y$

即　$\cos x[1 - 2\cos y - 2(2\cos^2y - 1)] = 0$ 　　$\cos x$ 不能為 0

$\therefore\quad 4\cos^2 y + 2\cos y - 3 = 0\quad\therefore\quad \cos y = \dfrac{\pm\sqrt{13}-1}{4}$

但　$\dfrac{-\sqrt{13}-1}{4}<-1,\quad$ 故公差爲 $y=\cos^{-1}\dfrac{\sqrt{13}-1}{4}.$

72. 從 $(1),(2),\ (2),(3)$

$$x = \frac{\cos(\beta-\gamma)-\cos(\alpha-\beta)}{\cos(\alpha+\beta)-\cos(\beta+\gamma)} = \frac{\cos(\gamma-\alpha)-\cos(\beta-\gamma)}{\cos(\beta+\gamma)-\cos(\gamma+\alpha)}$$

即　$\dfrac{\sin\left(\beta-\dfrac{\alpha+\gamma}{2}\right)}{\sin\left(\beta+\dfrac{\alpha+\gamma}{2}\right)} = \dfrac{\sin\left(\gamma-\dfrac{\alpha+\beta}{2}\right)}{\sin\left(\gamma+\dfrac{\alpha+\beta}{2}\right)}$

即　$\dfrac{\sin\beta\cos\dfrac{\alpha+\gamma}{2}-\cos\beta\sin\dfrac{\alpha+\gamma}{2}}{\sin\beta\cos\dfrac{\alpha+\gamma}{2}+\cos\beta\sin\dfrac{\alpha+\gamma}{2}}$

$$= \dfrac{\sin\gamma\cos\dfrac{\alpha+\beta}{2}-\cos\gamma\sin\dfrac{\alpha+\beta}{2}}{\sin\gamma\cos\dfrac{\alpha+\beta}{2}+\cos\gamma\sin\dfrac{\alpha+\beta}{2}}$$

即　$\dfrac{\tan\beta-\tan\frac{1}{2}(\alpha+\gamma)}{\tan\beta+\tan\frac{1}{2}(\alpha+\gamma)} = \dfrac{\tan\gamma-\tan\frac{1}{2}(\alpha+\beta)}{\tan\gamma+\tan\frac{1}{2}(\alpha+\beta)}$

從比例合分之理得

$$\frac{\tan\beta}{\tan\frac{1}{2}(\gamma+\alpha)} = \frac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}$$

再從 $(1),(2);\ (1),(3)$ 即得本題之證.

73. 以 β,γ 之值代 x 則 $\begin{cases} bc\cos\alpha\cos\beta+ca\sin\alpha\sin\beta=ab \\ bc\cos\alpha\cos\gamma+ca\sin\alpha\sin\gamma=ab \end{cases}$

就 $\cos\alpha,\sin\alpha$ 解之（見附錄二代數 2）

$$\frac{\cos\alpha}{a^2bc(\sin\beta-\sin\gamma)}=\frac{\sin\alpha}{ab^2c(\cos\gamma-\cos\beta)}$$

$$=\frac{-1}{abc^2\sin(\gamma-\beta)}$$

即　$\dfrac{\cos\alpha}{a\cos\frac{1}{2}(\beta+\gamma)}=\dfrac{\sin\alpha}{b\sin\frac{1}{2}(\beta+\gamma)}=\dfrac{1}{c\cos\frac{1}{2}(\beta-\gamma)}$

即　$\dfrac{\cos^2\alpha+\sin^2\alpha}{a^2\cos^2\frac{1}{2}(\beta+\gamma)+b^2\sin^2\frac{1}{2}(\beta+\gamma)}=\dfrac{1}{c^2\cos^2\frac{1}{2}(\beta-\gamma)}$

即　$a^2[1+\cos(\beta+\gamma)]+b^2[1-\cos(\beta+\gamma)]$

$$=c^2[1+\cos(\beta-\gamma)]$$

展開複角函數整理之卽得本題之證。

74. 今　　　$\sin x\cos\alpha+\cos x\sin\alpha=m\sin\alpha$

$$\therefore\quad\tan\alpha=\frac{\sin x}{m-\cos x}$$

以 β,γ 代入則　$\dfrac{\sin\beta}{m-\cos\beta}=\dfrac{\sin\gamma}{m-\cos\gamma}$

即　　　$\sin(\beta-\gamma)=m(\sin\beta-\sin\gamma)$

故　　　$\cos\frac{1}{2}(\beta-\gamma)=m\cos\frac{1}{2}(\beta+\gamma)$

〔設　$\sin\frac{1}{2}(\beta-\gamma)\neq0$〕

75. 從根與係數之關係得 $\Sigma\tan\alpha=0$　　（見附錄二代數4）

$$\therefore\quad\tan\gamma=-(\tan\alpha+\tan\beta)=-h$$

今 $\tan\gamma$ 為 $\tan\theta$ 之一值，亦卽 $-h$ 代 $\tan\theta$ 必適合。

故　　　　$ah^3+(2a-x)h=y$

76. 今　　　$\dfrac{1+\tan\theta}{1-\tan\theta}=3\cdot\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

即 $3\tan^4\theta - 6\tan^2\theta + 8\tan\theta - 1 = 0$

故由根與係數之關係得 $\Sigma\tan\alpha = 0$

(因三次項之係數為 0)

77. 今 $\sin\theta\cos\lambda + \cos\theta\sin\lambda = 2\sin\theta\cos\theta + b$

令 $\tan\tfrac{1}{2}\theta = x$

則 $\sin\theta = \dfrac{2x}{1+x^2}, \qquad \cos\theta = \dfrac{1-x^2}{1+x^2}$

則 $(b+\sin\lambda)x^4 - 2(2+\cos\lambda)x^3 + 2bx^2 + 2(2-\cos\lambda)x$

       $+ (b-\sin\lambda) = 0$ (參考 202 頁例二十七)

則 $S_1 = \Sigma\tan\dfrac{\alpha}{2} = \dfrac{2(2+\cos\lambda)}{b+\sin\lambda}, \qquad S_2 = \dfrac{2b}{b+\sin\lambda}$

     $S_3 = \dfrac{2(\cos\lambda-2)}{b+\sin\lambda}, \qquad\qquad\qquad S_4 = \dfrac{b-\sin\lambda}{b+\sin\lambda}$

今 $\tan\dfrac{\alpha+\beta+\gamma+\delta}{2} = \dfrac{S_1-S_3}{1-S_2+S_4} = \cdots\cdots = \dfrac{8}{0} = \infty$

      $\therefore\ \Sigma\dfrac{\alpha}{2} = n\pi + \dfrac{\pi}{2} = (2n+1)\dfrac{\pi}{2}$

        $\therefore\ \Sigma\alpha = (2n+1)\pi$

78. $\because\ \cot(\theta+\alpha) - \csc(\theta+\alpha) = \dfrac{\cos(\theta+\alpha)-1}{\sin(\theta+\alpha)}$

                     $= -\tan\dfrac{\theta+\alpha}{2}$

故原式為 $\Sigma\tan\dfrac{\theta+\alpha}{2} = 0$ 即 $\Sigma\dfrac{\tan\dfrac{\theta}{2}+\tan\dfrac{\alpha}{2}}{\tan\dfrac{\theta}{2}\tan\dfrac{\alpha}{2}-1} = 0$

設 $\tan\dfrac{\theta}{2} = x$

故 $\quad \Sigma\left(x+\tan\dfrac{\alpha}{2}\right)\left(x\tan\dfrac{\beta}{2}-1\right)\left(x\tan\dfrac{\gamma}{2}-1\right)=0$

即 $\quad \Sigma\left[x^3\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}+\left(\Pi\tan\dfrac{\alpha}{2}-\tan\dfrac{\beta}{2}-\tan\dfrac{\gamma}{2}\right)x^2\right.$

$$+\left.\left(1-\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}-\tan\dfrac{\alpha}{2}\tan\dfrac{\gamma}{2}\right)x+\tan\dfrac{\alpha}{2}\right]=0$$

$$\therefore \ \Sigma\tan\dfrac{\theta_1}{2}=-\dfrac{2\Sigma\tan\dfrac{\alpha}{2}-3\Pi\tan\dfrac{\alpha}{2}}{\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}}$$

$$\Sigma\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}=\dfrac{3-2\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}}{\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}}$$

$$\Pi\tan\dfrac{\theta_1}{2}=-\dfrac{\Sigma\tan\dfrac{\alpha}{2}}{\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}}$$

$$\tan\dfrac{\theta_1+\theta_2+\theta_3}{2}=\dfrac{\Sigma\tan\dfrac{\theta_1}{2}-\Pi\tan\dfrac{\theta_1}{2}}{1-\Sigma\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}}$$

$$=\dfrac{2\Sigma\tan\dfrac{\alpha}{2}-3\Pi\tan\dfrac{\alpha}{2}+\Sigma\tan\dfrac{\alpha}{2}}{\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}-3+2\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}}$$

$$=\dfrac{3\left(\Sigma\tan\dfrac{\alpha}{2}-\Pi\tan\dfrac{\alpha}{2}\right)}{-3\left(1-\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\right)}$$

$$= -\tan\frac{\alpha+\beta+\gamma}{2} = \tan\left(-\frac{\alpha+\beta+\gamma}{2}\right)$$

$$\therefore\quad \frac{\theta_1+\theta_2+\theta_3}{2} = k\pi - \frac{\alpha+\beta+\gamma}{2}$$

$$\therefore\quad \theta_1+\theta_2+\theta_3+\alpha+\beta+\gamma = 2k\pi$$

79. 令　　　　　　$\tan\tfrac{1}{2}x = t$

則　　　　$\cos x = \dfrac{1-t^2}{1+t^2}, \quad \sin x = \dfrac{2t}{1+t^2}$

卽　　　　$A\left(\dfrac{2t}{1+t^2}\right)^3 + B\left(\dfrac{1-t^2}{1+t^2}\right)^3 + C = 0$

卽　　$F(t) = (C-B)t^6 + 3(B+C)t^4 + 8At^3 + 3(C-B)t^2$
$$+ (C+B) = 0$$

$$F'(t) = 6(C-B)t^5 + 12(B+C)t^3 + 24At^2 + 6(C-B)t$$

今　$F(t), F'(t)$ 無公因式，故本題有六個不同之根．

設六根爲 $t_1, t_2 \cdots\cdots t_6$，從根與係數關係得：

$$S_1 = 0, \quad S_2 = 3(B+C)/(C-B), \quad S_3 = -8A/(C-B),$$

$$S_4 = 3, \quad S_5 = 0, \quad S_6 = (C+B)/(C-B)$$

故　　　$\tan\dfrac{\Sigma t_1}{2} = \dfrac{S_1 - S_3 + S_5}{1 - S_2 + S_4 - S_6} = -\dfrac{A}{B}$

80. 今以 θ 代 x 可適合，且 θ 與 α, β, γ 相差均不爲 2π，故
知 θ 爲此方程式之一根．今以 $\tan(x/2) = A$，則

$$\sin x = 2A/(1+A^2), \quad \cos x = (1-A^2)/(1+A^2)$$

代入原式則　$\sin 2\theta\left[\dfrac{2aA}{1+A^2} + \dfrac{b(1-A^2)}{1+A^2}\right] = \dfrac{2A(1-A^2)}{(1+A^2)^2}$
$$\times (a\sin\theta + b\cos\theta)$$

即　　$\sin 2\theta(1+A^2)(b+2aA-bA^2)$

　　　　$=2A(1-A^2)(a\sin\theta+b\cos\theta)$

即　　$-b\sin 2\theta \cdot A^4 + \cdots\cdots + b\sin 2\theta = 0$

$\therefore$　$\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}\tan\dfrac{\theta}{2} = \dfrac{b\sin 2\theta}{-b\sin 2\theta} = -1$

故如題云.

81. 設　$\tan\dfrac{\theta}{2}=x$　　則　$\sin\theta=\dfrac{2x}{1+x^2}$,　$\cos\theta=\dfrac{1-x^2}{1+x^2}$

則　$a\left[\dfrac{(1-x^2)^2-4x^2}{(1+x)^2}\right]+b\dfrac{4x(1-x^2)}{(1+x^2)^2}+c\dfrac{1-x^2}{1+x^2}+d=0$

即　　$(a-c+d)x^4-4bx^3+(-6a+2d)x^2$

　　　　$+4bx+(a+c+d)=0$

$\therefore$　$\Sigma\tan\dfrac{\alpha}{2}=\dfrac{4b}{a-c+d}$

及　$\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}=-\dfrac{4b}{a-c+d}$

$\therefore$　$\Sigma\tan\dfrac{\alpha}{2}+\Sigma\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}=0$

即　$\left(\tan\dfrac{\alpha}{2}+\tan\dfrac{\delta}{2}\right)\left(1+\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}\right)$

　　　　$+\left(\tan\dfrac{\beta}{2}+\tan\dfrac{\gamma}{2}\right)\left(1+\tan\dfrac{\alpha}{2}\tan\dfrac{\delta}{2}\right)=0$

即　　$\left(\sin\dfrac{\alpha}{2}\cos\dfrac{\delta}{2}+\cos\dfrac{\alpha}{2}\sin\dfrac{\delta}{2}\right)$

　　　　$\times\left(\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}+\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}\right)$

$$+\left(\sin\frac{\beta}{2}\cos\frac{\gamma}{2}+\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\right)$$

$$\times\left(\cos\frac{\alpha}{2}\cos\frac{\delta}{2}+\sin\frac{\alpha}{2}\sin\frac{\delta}{2}\right)=0$$

即　$\sin\dfrac{\alpha+\delta}{2}\cos\dfrac{\beta-\gamma}{2}+\sin\dfrac{\beta+\gamma}{2}\cos\dfrac{\alpha-\delta}{2}=0$

即　$\sin\dfrac{\alpha+\delta+\beta-\gamma}{2}+\sin\dfrac{\alpha+\delta-\beta+\gamma}{2}+\sin\dfrac{\beta+\gamma+\alpha-\delta}{2}$

$$+\sin\dfrac{\beta+\gamma-\alpha+\delta}{2}=0$$

82. (1) 令　$\sin x(2\cos x-n)=m\cos x-k$

即　$(1-\cos^2x)(4\cos^2x-4n\cos x+n^2)$

$$=m^2\cos^2x-2km\cos x+k^2$$

即　$4\cos^4x-4n\cos^3x+\cdots\cdots=0$

$\therefore$　$\Sigma\cos\alpha=4n/4=n$

(2) 令　$(\sin 2x+k)^2=(m\cos x+n\sin x)^2$

$$=\tfrac{1}{2}m^2(1+\cos 2x)$$

$$+\tfrac{1}{2}n^2(1-\cos 2x)+mn\sin 2x$$

設　$\sin 2x=y$

則　$2(y+k)^2-2mny=m^2+n^2+(m^2-n^2)\cos 2x$

即　$2(y+k)^2-2mny-m^2-n^2=(m^2-n^2)\cos 2x$

即　$[2y^2+(4k-2mn)y+2k^2-m^2-n^2]^2$

$$=(m^2-n^2)^2(1-y^2)$$

即　$4y^4+8(2k-mn)y^3+\cdots\cdots\cdots=0$

$\therefore$　$\Sigma\sin\alpha=-2(2k-mn)=2mn-4k$

83. 今　　　　　$\sin\theta+\cos\theta=c\sin\theta\cos\theta$

即　　　　$1+2\sin\theta\cos\theta=c^2\sin^2\theta\cos^2\theta$

$$c^2\sin^2 2\theta-4\sin 2\theta-4=0$$

故　$\sin 2\theta=\dfrac{2\pm 2\sqrt{1+c^2}}{c^2}$　　　　$\because\ 1+c^2>0$

$\therefore$　$\sin 2\theta$ 可有兩解, 惟 $|\sin 2\theta|<1$, 故必須

$$-c^2<2+2\sqrt{1+c^2}<c^2\ (1)$$

或　$-c^2<2-2\sqrt{1+c^2}<c^2\ (2)$

(1) 就(1)言, 則　$2+2\sqrt{1+c^2}>-c^2$　　必可成立

（因左邊必為正）

故只須　　$c^2>2+2\sqrt{1+c^2}$

即　　　　$c^2-2>2\sqrt{1+c^2}$

即　　$(c^2-2)^2>4(1+c^2)$　　即　$c^2(c^2-8)>0$

$\therefore$　只須　$c^2>8$

即在 $c^2>8$ 時, 必可以成立, 而 $\sin 2\theta$ 可有一解.

(2) 就(2)言,　則　$c^2>2-2\sqrt{1+c^2}$　　必可成立

（因右邊必為負）

故只須　　$2-2\sqrt{1+c^2}>-c^2$

即　　　　$c^2+2>2\sqrt{1+c^2}$

即　　$(c^2+2)^2>4(1+c^2)$　　即　$c^4>0$

不論 $c^2\gtrless 8$ 均可成立.

綜上所述則 $c^2>8$ 時, $\sin 2\theta$ 有兩解, 即在 $0,2\pi$ 間

θ 有四根.

又如 $e^x < 8$，則 θ 在 0 與 2π 之間祇有兩根（根指主角而言）。

84. 卽　　　　　$32\cos^3 x - 24\cos x + 3\sqrt{6} = 0$

卽　　$(4\cos x - \sqrt{6})(8\cos^2 x + 2\sqrt{6}\cos x - 3) = 0$

(1)　　$\cos x = \dfrac{\sqrt{6}}{4} = \dfrac{\sqrt{6}}{2}\sin\dfrac{\pi}{6}$

(2)　　$8\cos^2 x + 2\sqrt{6}\cos x - 3 = 0$

$\therefore \cos x = \dfrac{-\sqrt{6} \pm \sqrt{30}}{8} = \dfrac{\sqrt{6}}{2}\left[\dfrac{-1 \pm \sqrt{5}}{4}\right]$

$= \dfrac{\sqrt{6}}{2}\sin\dfrac{\pi}{10}, \quad -\dfrac{\sqrt{6}}{2}\sin\dfrac{3\pi}{10} \qquad$ 故如題云．

85. 如例二十九，令 $x = \rho\cos\theta$ 比較下兩式之相當係數

$$4\cos^3\theta - 3\cos\theta - \cos 3\theta = 0$$

$$\rho^3\cos^3\theta - 3\rho\cos\theta + 1 = 0$$

得　$\dfrac{\rho^3}{4} = \dfrac{3\rho}{3} = \dfrac{1}{-\cos 3\theta} \quad \therefore \rho = 2, \quad \cos 3\theta = -\dfrac{1}{2}$

$$\therefore \theta = n \cdot 120° \pm 40°$$

故當 θ 爲 $80°, 160°, 320°$ 時，$\cos\theta$ 有不同之值

故　　$x = 2\cos 80°, \ 2\cos 160°$ 或 $2\cos 320°$

卽爲　$2\cos 80°, \ -2\cos 20°$ 或 $2\cos 40°$．

（此係三角解法）

86. 如上題得　$\rho = 2\sqrt{2}, \qquad \cos 3\theta = \dfrac{1}{2}\sqrt{2}$

$$\therefore \theta = n \cdot 120° \pm 15°$$

故當 θ 爲 $15°, 105°, 135°$ 時，$\cos\theta$ 有不同之值

故　$x = 2\sqrt{2}\cos 15°,\ \ 2\sqrt{2}\cos 105°,\ \ 2\sqrt{2}\cos 135°$

卽　$x = 1 \pm \sqrt{3},\ \ \ 1 - \sqrt{3},\ \ \ 及\ -2.$

〔因　$\cos 15° = \frac{1}{4}(\sqrt{6} + \sqrt{2})$〕

87. 如上兩題求到　　$\rho = 2a,\ \ \ \ \ \cos 3\theta = \cos 3A$

$$\therefore\ \ \theta = n \cdot 120° \pm A$$

$\theta = A,\ 120° \pm 1\ \ \ (\cos\theta\ 不同之值)$

故　$x = 2a\cos A,\ \ \ \ 2a\cos(120° \pm A)$

88. 今　　$(\tan x - \sqrt{3})(\tan x - 1) < 0$

卽　　　　　$\sqrt{3} > \tan x > 1$

可知 $\tan x$ 恆爲正，故 x 必在第一，三象限內．且正切之值隨角而增．

(1) 設在第一象限：

因　$\tan\dfrac{\pi}{3} = \sqrt{3},\ \ \ \tan\dfrac{\pi}{4} = 1\ \ \ 故\ \ \dfrac{\pi}{3} > x > \dfrac{\pi}{4}$

(2) 設在第三象限：

因　$\tan\dfrac{4\pi}{3} = \sqrt{3},\ \ \ \tan\dfrac{5\pi}{4} = 1\ \ \ 故\ \ \dfrac{4\pi}{3} > x > \dfrac{5\pi}{4}$

89. 卽　　$\dfrac{(\sin x + 1)(2\sin x - 1)}{(\sqrt{2}\sin x + 1)(\sqrt{2}\sin x - 1)} < 0$

因　$|\sin x| \leqq 1$　　　　故　$\sin x + 1 > 0$

今以 $(2\sin^2 x - 1)^2$ 乘兩邊，則

$$(2\sin x - 1)(\sqrt{2}\sin x - 1)(\sqrt{2}\sin x + 1) < 0$$

$\therefore\ \ \dfrac{1}{2} < \sin x < \dfrac{1}{\sqrt{2}}\ \ 或\ \ -1 < \sin x < -\dfrac{1}{\sqrt{2}}$

（因 $\sin x$ 必大於 -1）

本題假定 $x < 180°$，即 $\sin x$ 必爲正,故祇有

$$\frac{1}{2} < \sin x < \frac{1}{\sqrt{2}}$$

(1) 在第一象限：　$\pi/6 < x < \pi/4$

（第一象限內角增正弦亦增加）

(2) 在第二象限：　$3\pi/4 < x < 5\pi/6$

（第二象限內角增正弦反減小）

90. 今　$\dfrac{-4\sin^2 x + 2(\sqrt{3} - \sqrt{2})\sin x - (\sqrt{6} - 2)}{4\sin^2 x - 1} < 0$

今分子之二次項之係數爲負且其判別式

$\triangle = 4(\sqrt{3} - \sqrt{2})^2 - 16(\sqrt{6} - 2) = 52 - 24\sqrt{6} < 0$

故其分子必爲負　　故　$\dfrac{1}{4\sin^2 x - 1} > 0$

兩邊乘以　$(4\sin^2 x - 1)^2$

則　　　　$(2\sin x - 1)(2\sin x + 1) > 0$

$\therefore$　$\sin x > \frac{1}{2}$ 或 $\sin x < -\frac{1}{2}$

故　$150° > x > 30°$ 或 $330° > x > 210°$　（今 $x < 360°$）

91. 設 x 有不等之實根,則此式之判別式大於 0

即　　　$24^2 \sin^2 \alpha - 64(-3\sin\alpha + 2) > 0$

即　　　$(3\sin\alpha + 2)(3\sin\alpha - 1) > 0$

$\therefore$　$\sin\alpha > \frac{1}{3}$ 或 $\sin\alpha < -\frac{2}{3}$

但 α 限於 $180°$ 之內,即 $\sin\alpha$ 恆爲正,故只有 $\sin\alpha > \frac{1}{3}$

$\therefore$　$90° > \alpha > \sin^{-1}\frac{1}{3}$ 及 $\pi - \sin^{-1}\frac{1}{3} > \alpha > 90°$

即　α 在 $\pi - \sin^{-1}\frac{1}{3}$ 與 $\sin^{-1}\frac{1}{3}$ 之間。

92, 今條件有二：

(1) $\triangle = 4b^2 - 4c > 0$　　　即　　　$b^2 > c$

(2) $\because$ $|\sin x| \leqq 1$　　　$\therefore$ $\left| \dfrac{-2b \pm 2\sqrt{b^2-c}}{2} \right| \leqq 1$

即　　　　　　$| -b \pm \sqrt{b^2-c} | \leqq 1$

今 b 爲正, 故　$-b \pm \sqrt{b^2-c}$ 爲負

故即　　　　$| b \mp \sqrt{b^2-c} | \leqq 1$

惟　　　　　$b + \sqrt{b^2-c} \geqq b - \sqrt{b^2-c}$

故第二條件爲　$| b + \sqrt{b^2-c} | \leqq 1$

而　　　　　$b + \sqrt{b^2-c}$ 必爲正

故即　　　　$b + \sqrt{b^2-c} \leqq 1$

是故原式中設 $\sin x$ 有兩不等之實根時其條件爲：

$$b^2 \geqq c \ \text{及} \ b + \sqrt{b^2-c} \leqq 1$$

93, 今　　　$\tan \theta = \tan \beta \dfrac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha}$

即　$\tan \alpha \tan^2\theta + \tan \theta (\tan \beta - 1) + \tan \alpha \tan \beta = 0$

今 θ 爲實數, 即 $\tan \theta$ 爲實數, 故其判別式爲正

即　　　$(\tan \beta - 1)^2 - 4 \tan^2\alpha \tan \beta > 0$

$\tan^2\beta - 2(1 + 2\tan^2\alpha)\tan \beta + (\sec^2\alpha - \tan^2\alpha)^2 > 0$

故　　　$[\tan \beta - (\sec \alpha - \tan \alpha)^2]$

$\times [\tan \beta - (\sec \alpha + \tan \alpha)^2] > 0$

即 $\tan \beta$ 不能介于 $(\sec \alpha - \tan \alpha)^2$ 及 $(\sec \alpha + \tan \alpha)^2$

之間.

94. 令

$$x = \frac{\tan(\theta+\alpha)}{\tan(\theta-\alpha)}$$

則

$$\frac{x+1}{x-1} = \frac{\tan(\theta+\alpha)+\tan(\theta-\alpha)}{\tan(\theta+\alpha)-\tan(\theta-\alpha)}$$

即

$$\frac{x+1}{x-1} = \frac{\sin 2\theta}{\sin 2\alpha}$$

因 θ 爲實數, 則 $\left|\frac{(x+1)\sin 2\alpha}{x-1}\right| < 1$

即

$$\frac{(x+1)^2\sin^2 2\alpha}{(x-1)^2} < 1$$

即

$$(x+1)^2\sin^2 2\alpha < (x-1)^2$$

即 $(1-\sin^2 2\alpha)x^2 - 2(1+\sin^2 2\alpha)x + (1-\sin^2 2\alpha) > 0$

即

$$[(1-\sin 2\alpha)x - (1+\sin 2\alpha)]$$
$$\times [(1+\sin 2\alpha)x - (1-\sin 2\alpha)] > 0$$

即 x 之值不能介於 $\dfrac{1+\sin 2\alpha}{1-\sin 2\alpha}$ 及 $\dfrac{1-\sin 2\alpha}{1+\sin 2\alpha}$ 之間.

95. 設原式爲 x, 則 $(a-x)\tan^2\theta + b\tan\theta + (c-x) = 0$

今 θ 爲實值, 則 $\triangle = b^2 - 4(a-x)(c-x) > 0$

即

$$4x^2 - 4(a+c)x + (4ac-b^2) < 0$$

即

$$[2x - (a+c) + \sqrt{b^2+(a-c)^2}]$$
$$\times [2x - (a+c) - \sqrt{b^2+(a-c)^2}] < 0$$

故 x 之值在 $\frac{1}{2}[a+c-\sqrt{b^2+(a-c)^2}]$ 及 $\frac{1}{2}[a+c+\sqrt{b^2+(a-c)^2}]$ 之間.

96. a. 設

$$x = p\cot\theta + q\tan\theta$$

則

$$q\tan^2\theta - x\tan\theta + p = 0$$

$$\therefore \quad \tan \theta = \frac{x \pm \sqrt{x^2 - 4pq}}{2q}$$

∵ $\quad \triangle \gtreqqless 0 \qquad \therefore \quad x^2 - 4pq \gtreqqless 0$

故 $\quad |x| \gtreqqless 2\sqrt{pq} \qquad x$ 之極小值爲 $2\sqrt{pq}$

則 $\qquad \tan \theta = \frac{x}{2q} = \sqrt{\frac{p}{q}}$

b. 設 $\qquad x = 3 - 2\cos\theta + \cos^2\theta$

則 $\qquad \cos^2\theta - 2\cos\theta + 3 - x = 0$

∴ $\quad \cos\theta = \frac{2 \pm \sqrt{4 - 4(3-x)}}{2} = 1 \pm \sqrt{x-2}$

∵ $\quad \triangle \gtreqqless 0 \quad$ 則 $\quad x - 2 \gtreqqless 0 \quad \therefore \quad x \gtreqqless 2$

故 x 之極小值爲 2, 則 $\cos\theta = 1$, 故 $\theta = 2n\pi$.

97. a. 設 $\qquad x = \sin\theta + \cos\theta$

則 $\qquad (x - \sin\theta)^2 = \cos^2\theta = 1 - \sin^2\theta$

卽 $\qquad 2\sin^2\theta - 2x\sin\theta + x^2 - 1 = 0$

因 $\sin\theta$ 爲實數

故 $\quad 4x^2 - 8(x^2 - 1) \gtreqqless 0 \qquad$ 卽 $\quad x^2 \lesseqqgtr 2$

故 x 之極大值爲 $\sqrt{2}$

則 $\qquad 2\sin^2\theta - 2\sqrt{2}\sin\theta + 1 = 0$

∴ $\qquad \sin\theta = \frac{2\sqrt{2} \pm \sqrt{8 - 8}}{4} = \frac{1}{\sqrt{2}}$

∴ $\quad \theta = n\pi + (-1)^n \cdot \frac{\pi}{4}$

b. 設 $\qquad x = \sin\theta\cos(\alpha - \theta)$

則 $2x = \sin\alpha + \sin(2\theta - \alpha)$

設 x 為極大,則 $\sin(2\theta - \alpha)$ 亦為極大,而 $\sin(2\theta - \alpha)$

之極大值為 1

即 $2\theta - \alpha = n\pi + (-1)^n \cdot \dfrac{\pi}{2}$

$\therefore \quad \theta = n \cdot \dfrac{\pi}{2} + (-1)^n \cdot \dfrac{\pi}{4} + \dfrac{\alpha}{2}$

代入原式得極大值為 $x = \frac{1}{2}(\sin\alpha + 1)$

98. a. 設原式為 x,則 $4\cos^2\theta - 2\sqrt{3}\, x\cos\theta + 3 = 0$

$\because \quad \cos\theta$ 為實數,則 $12x^2 - 12 \cdot 4 \geqq 0$

故 $x \geqq 2$ 或 $x \leqq -2$

故極小值為 2,極大值為 -2.

b. 設原式為 x

則 $(x-1)\tan^2\theta + (x+1)\tan\theta + (x-1) = 0$

今 $\tan\theta$ 為實數

故 $(1+x)^2 - 4(x-1)^2 > 0$

即 $3x^2 - 10x + 3 < 0$ 即 $(3x-1)(x-3) < 0$

$\therefore \quad 3 > x > \frac{1}{3}$ 故極大值為 3,極小值為 $\frac{1}{3}$.

99. 照第一列展開

$1 + 2\Pi\cos\alpha - \Sigma\cos^2\alpha - \cos^2\theta(1 - \cos^2\gamma) = 0$

即 $\sin^2\gamma\sin^2\theta = \cos^2\alpha + \cos^2\beta - 2\Pi\cos\alpha$

則 $\sin\theta = \pm\dfrac{\sqrt{\cos^2\alpha + \cos^2\beta - 2\Pi\cos\alpha}}{\sin\gamma}$

故 $\qquad \theta = n\pi \pm \sin^{-1}\dfrac{\sqrt{\cos^2\alpha + \cos^2\beta - 2\Pi\cos\alpha}}{\sin\gamma}$

100. 設此爲圓錐曲線之特例時(卽退化曲線 degenerate curve), 則

$$\ominus = \frac{1}{2}\begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = 0$$

卽
$$\begin{vmatrix} 2 & 2\sin\alpha & 2\sin\alpha \\ 2\sin\alpha & 2 & 2\sin\alpha \\ 2\sin\alpha & 2\sin\alpha & 2 \end{vmatrix}$$

$$= 8\begin{vmatrix} 1 & \sin\alpha & \sin\alpha \\ \sin\alpha & 1 & \sin\alpha \\ \sin\alpha & \sin\alpha & 1 \end{vmatrix} = 0$$

卽 $\qquad 2\sin^3\alpha - 3\sin^2\alpha + 1 = 0$

卽 $\qquad (\sin\alpha - 1)^2(2\sin\alpha + 1) = 0$

$\qquad \therefore \sin\alpha = 1,\ 1,\ -\frac{1}{2}$

$\therefore \alpha = n\pi + (-1)^n \cdot \dfrac{\pi}{2}$ 或 $n\pi - (-1)^n \cdot \dfrac{\pi}{6}$

習 題 十 八 (226-228 頁)

1. 設第一,第二角爲 α, β 則 $\alpha + \beta = \pi/4$

$\qquad \therefore \tan(\alpha + \beta) = 1$

卽 $\qquad \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$

即 $\dfrac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{(x-1)(x+1)}{(x-2)(x+2)}}=1$

即 $\dfrac{(x-1)(x+2)+(x+1)(x-2)}{(x^2-4)-(x^2-1)}=1$

即 $2x^2=1$　　　　　　$\therefore\ x=\pm\dfrac{\sqrt{2}}{2}$

2. 設第一, 二, 三角依次爲 $\alpha,\ \beta,\ \gamma$

則　$\alpha+\beta=\gamma$　　　　　　即　$\cot(\alpha+\beta)=\cot\gamma$

即 $\dfrac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}=a-1$

即 $\dfrac{x(a^2-x+1)-1}{x+(a^2-x+1)}=a-1$

即 $(x-a)[x-(a^2-a+1)]=0$

$\therefore\ x=a,\quad a^2-a+1$

3. 設四個角依次爲 $\alpha,\ \beta,\ \gamma,\ \delta$

則　$\tan\alpha=x+1,$　　　　　$\tan\beta=1/(x-1)$

又　$\sin\gamma=\dfrac{4}{5}$　　　　　$\cos\gamma=\dfrac{3}{5}$

則　$\tan\gamma=\dfrac{4}{3}$　　　　　$\tan\delta=\dfrac{4}{3}$

今　$\alpha+\beta=\gamma+\delta$　$\therefore\ \tan(\alpha+\beta)=\tan(\gamma+\delta)$

即 $\dfrac{x+1+\dfrac{1}{x-1}}{1-\dfrac{x+1}{x-1}}=\dfrac{\dfrac{4}{3}+\dfrac{4}{3}}{1-\dfrac{4}{3}\cdot\dfrac{4}{3}}$

即 $\qquad -\dfrac{x^2}{2} = -\dfrac{24}{7}$ $\qquad \therefore \quad 7x^2 = 48$

$$\therefore \quad x = \pm\sqrt{\dfrac{48}{7}} = \pm\dfrac{4}{7}\sqrt{21}$$

4. 兩邊取正切,則 $\lambda + 1 = \dfrac{3(\lambda-1)-(\lambda-1)^3}{1-3(\lambda-1)^2}$

則 $\quad \lambda^3 - 2\lambda = 0$ $\qquad$ 即 $\quad \lambda(\lambda^2-2) = 0$

$$\therefore \quad \lambda = 0, \ \pm\sqrt{2}$$

5. 設四角依次爲 $\alpha,\beta,\gamma,\delta$ $\quad$ 則 $\quad \alpha+\beta = \delta-\gamma$

即 $\qquad\qquad \tan(\alpha+\beta) = \tan(\delta-\gamma)$

即 $\qquad \dfrac{x+1+x}{1-x(x+1)} = \dfrac{3x-(x-1)}{1+3x(x-1)}$

即 $\quad 2x(4x^2-1) = 0$ $\qquad \therefore \quad 2x(2x-1)(2x+1) = 0$

$$\cdot \quad x = 0, \ \pm\tfrac{1}{2}$$

6. 設三角依次爲 α, β, γ

則 $\quad \cos\alpha = x, \quad \cos\beta = \sqrt{1-x^2}, \quad \cos\gamma = x\sqrt{3}$

$\therefore \quad \sin\alpha = \sqrt{1-x^2}, \quad \sin\beta = x$ $\qquad$ 兩邊取餘弦

則 $\quad x\sqrt{1-x^2} + x\sqrt{1-x^2} = 2\sqrt{3}$ $\quad \therefore \quad 4x^2(1-x^2) = 3x^2$

即 $\quad x^2(4x^2-1) = 0$ $\qquad \therefore \quad x = 0, \ \pm\tfrac{1}{2}$

7. 設兩角爲 α,β $\quad$ 則 $\quad \sin\alpha = 5/x, \ \sin\beta = 12/x$

今 $\quad \alpha = \pi/2 - \beta$ $\qquad$ 則 $\qquad \sin\alpha = \cos\beta$

即 $\quad \dfrac{5}{x} = \sqrt{1-\dfrac{144}{x^2}}$ $\qquad$ 即 $\qquad 25 = x^2 - 144$

$\therefore \quad x^2 = 169$ $\quad \therefore \quad x = 13$ (負號不合因假定爲銳角也)

8. 設第一,第二角各爲 α, β

則　　$\sin \alpha = \dfrac{2x}{1+x^2}$,　　　　$\cos \beta = \dfrac{1-x^2}{1+x^2}$

　　　$\cos \alpha = \sqrt{1-\sin^2\alpha} = \dfrac{1-x^2}{1+x^2} = \cos \beta$

$\therefore$ $\alpha = \beta$　　$\therefore$ $2\alpha = \dfrac{\pi}{4}$　　　即　$\tan\dfrac{\alpha}{2} = \tan\dfrac{\pi}{16}$

但　　　　　$\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos \alpha}{1+\cos \alpha}} = x$

　　　　　$\therefore$ $x = \tan\dfrac{\pi}{16}$

9. 設三個角各爲 α, β, γ

則　$\csc \alpha = \lambda$,　　　$\sin \alpha = \dfrac{1}{\lambda}$,　　　$\sin \beta = \dfrac{1}{a}$,

　$\cos \beta = \dfrac{\sqrt{a^2-1}}{a}$,　$\sin \gamma = \dfrac{1}{b}$,　　$\cos \gamma = \dfrac{\sqrt{b^2-1}}{b}$

兩邊取正弦,則

　　$\dfrac{1}{\lambda} = \dfrac{\sqrt{a^2-1}}{ab} + \dfrac{\sqrt{b^2-1}}{ab} = \dfrac{\sqrt{a^2-1}+\sqrt{b^2-1}}{ab}$

$\therefore$ $\lambda = \dfrac{ab}{\sqrt{a^2-1}+\sqrt{b^2-1}} = \dfrac{ab}{a^2-b^2}(\sqrt{a^2-1}-\sqrt{b^2-1})$

10. 今　　　$3\sin^{-1}\lambda - 2\cos^{-1}\lambda = \dfrac{2}{3}\pi$　　　　(1)

由公式(39)　$\sin^{-1}\lambda + \cos^{-1}\lambda = \dfrac{\pi}{2}$　　　　　(2)

$3(2)-(1)$　　$5\cos^{-1}\lambda = \dfrac{5}{6}\pi$　　　$\therefore$ $\cos^{-1}\lambda = \dfrac{\pi}{6}$

則　　　　　　　　　$\lambda = \cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$

11. 今　　　　　　$\tan^{-1}x + m\cot^{-1}x = 135°$

由公式(39)　　$\tan^{-1}x + \cot^{-1}x = 90°$

相減得　　　　　$(m-1)\cot^{-1}x = 45°$

$\therefore$　$\cot^{-1}x = \dfrac{45°}{m-1}$　　　　　故　$x = \cot\dfrac{45°}{m-1}$

12. 設　$\tan^{-1}\dfrac{1}{4}$, $\tan^{-1}\dfrac{1}{5}$, $\tan^{-1}\dfrac{1}{6}$, $\tan^{-1}\dfrac{1}{x}$ 各爲 α, β, γ, δ

則　$\tan\alpha = \dfrac{1}{4}$,　$\tan\beta = \dfrac{1}{5}$,　$\tan\gamma = \dfrac{1}{6}$,　$\tan\delta = \dfrac{1}{x}$

又　　　$\tan 2\beta = \dfrac{5}{12}$,　　　$\tan(\alpha+\gamma) = \dfrac{10}{23}$

今　$\alpha + 2\beta + \gamma = \dfrac{\pi}{4} - \delta$　　　兩邊取正切, 則

$$\frac{\dfrac{10}{23}+\dfrac{5}{12}}{1-\dfrac{10}{23}\cdot\dfrac{5}{12}} = \frac{1-\dfrac{1}{x}}{1+\dfrac{1}{x}}　　　即　\frac{x-1}{x+1} = \frac{235}{226}$$

$\therefore$　$235(x+1) = 226(x-1)$　　　$\therefore$　$x = -461/9$

13. 設角依次爲 $\alpha, \beta, \gamma, \delta$　　　則　$\alpha + \beta = \dfrac{\pi}{2} - (\gamma + \delta)$

即　　　　　$\cot(\alpha+\beta) = \tan(\gamma+\delta)$

今　　　　　$\cot(\alpha+\beta) = \cdots\cdots = \dfrac{x^2 - ab}{(a+b)x}$,

　　　　　　$\tan(\gamma+\delta) = \cdots\cdots = \dfrac{(c+d)x}{x^2 - cd}$

代入去分母得　$(x^2-ab)(x^2-cd)=(a+b)(c+d)x^2$

即　$x^4-(\Sigma ab)x^2+abcd=0$　　　　此方程之根即爲答案.

14. 設　$\tan^{-1}x=\alpha$　　　　又右邊兩角各爲　β,γ

則　　$\tan\alpha=x,$　　　　$\cos\beta=\dfrac{1-a^2}{1+a^2}$

$\therefore$　　$\tan\dfrac{\beta}{2}=a$　　　同理　$\tan\dfrac{\gamma}{2}=b$　(見題 8)

今　　　　$2\alpha=\beta-\gamma$

即　　$\alpha=\dfrac{\beta}{2}-\dfrac{\gamma}{2}$　　　$\therefore$　$\tan\alpha=\tan\left(\dfrac{\beta}{2}-\dfrac{\gamma}{2}\right)$

$\therefore$　$x=\dfrac{\tan\frac{1}{2}\beta-\tan\frac{1}{2}\gamma}{1+\tan\frac{1}{2}\beta\tan\frac{1}{2}\gamma}=\dfrac{a-b}{1+ab}$

15. 設兩角爲　α,β　　　　　　則　　$\cos\alpha=\dfrac{x^2-1}{x^2+1}$

$\therefore$　$\tan\dfrac{\alpha}{2}=\dfrac{1}{x}$（見題 8）　又　$\tan\beta=\dfrac{2x}{x^2-1}$

但　$\tan\alpha=\dfrac{2x}{x^2-1}$　　　　$\therefore$　$\alpha=\beta$（設均爲銳角）

今　$\alpha+\beta=\dfrac{2\pi}{3}$　　　$\therefore$　$2\alpha=\dfrac{2\pi}{3}$　　即　$\dfrac{\alpha}{2}=\dfrac{\pi}{6}$

即　$\tan\dfrac{\alpha}{2}=\tan\dfrac{\pi}{6}$　　$\therefore$　$\dfrac{1}{x}=\dfrac{1}{\sqrt{3}}$　　$\therefore$　$x=\sqrt{3}$

若 α,β 並非均爲銳角,但皆小於 $360°$, 則由 $\tan\alpha$

$=\tan\beta$ 更可得　　$\alpha=\pi+\beta$

即　$\pi+\beta+\beta=\dfrac{2\pi}{3}$　　　$\therefore$　$\dfrac{\beta}{2}=-\dfrac{\pi}{12}$

$$\therefore \quad \tan\frac{\beta}{2} = \tan\left(-\frac{\pi}{12}\right) = -\tan\frac{\pi}{12}$$

$$\therefore \quad x = 2 - \sqrt{3} \quad (102 \text{ 頁表 } 13)$$

16. 設　$\arctan x = \alpha$,　　則　$\tan\alpha = x$,　$\tan 2\alpha = 2x/(1-x^2)$

又設　　　　　　$\frac{1}{2}\operatorname{arcsec} 5x = \beta$

則　　$\sec 2\beta = 5x$,　　$\tan 2\beta = \sqrt{25x^2-1}$

今　　$\alpha + \beta = \frac{\pi}{4}$　　　　　$\therefore \quad 2\beta = \frac{\pi}{2} - 2\alpha$

則　$\tan 2\beta = \cot 2\alpha$　　　　即　$\sqrt{25x^2-1} = (1-x^2)/2x$

即　$99x^4 - 2x^2 - 1 = 0$　　　即　$(9x^2-1)(11x^2+1) = 0$

$$\therefore \quad x = \frac{1}{3} \quad \left(-\frac{1}{3}\text{不合}\right)$$

17. 設　$\operatorname{vers}^{-1}(1+x) = \alpha$,　　　　$\operatorname{vers}^{-1}(1-x) = \beta$

則　$\cos\alpha = 1 - (1+x) = -x$,　　$\tan\alpha = \pm\dfrac{\sqrt{1-x^2}}{-x}$,

$\cos\beta = 1 - (1-x) = x$,　　　$\tan\beta = \pm\dfrac{\sqrt{1-x^2}}{x}$

(1) 取正號, 則　$\tan(\alpha-\beta) = 2\sqrt{1-x^2}$

$$\therefore \quad \frac{2x\sqrt{1-x^2}}{2x^2-1} = 2\sqrt{1-x^2}$$

$$\therefore \quad 2\sqrt{1-x^2}(2x^2-x-1) = 0$$

$$\therefore \quad 2\sqrt{1-x^2}(2x+1)(x-1) = 0$$

$$\therefore \quad x = \pm 1, \ -\frac{1}{2}, \ 1$$

(2) 取負號，同理可得 x 之四根爲 ± 1, $\frac{1}{2}$, -1

故　　$x = \pm 1$, $\pm \frac{1}{2}$ （代入均合）

18. 今　$\tan \cos^{-1} \theta = \sin \cos^{-1} \frac{1}{2}$　　令　$\cos^{-1} \theta = \alpha$

則　　$\cos \alpha = \theta$　　　　　　　故　$\tan \alpha = \dfrac{\sqrt{1-\theta^2}}{\theta}$

又令　$\cos^{-1} \frac{1}{2} = \beta$　　　　　則　$\cos \beta = \frac{1}{2}$

故　　　　$\sin \beta = \sqrt{1 - \dfrac{1}{4}} = \dfrac{\sqrt{3}}{2}$

今　　$\tan \alpha = \sin \beta$　　　$\therefore$　$\dfrac{\sqrt{1-\theta^2}}{\theta} = \dfrac{\sqrt{3}}{2}$

即　　$\dfrac{1-\theta^2}{\theta^2} = \dfrac{3}{4}$　　　　　$\therefore$　$7\theta^2 = 4$

$\therefore$　$\theta = \dfrac{2}{7}\sqrt{7}$ （負號　取）

19. 令第一角爲 A，第二角爲 $2B$

則　$\cot A = \frac{1}{2} \cot^2 x$　　　　　　$\therefore$　$\tan A = 2 \tan^2 x$

又　$\sin 2B = \dfrac{3 \sin 2x}{5 + 4 \cos 2x} = \dfrac{6 \sin x \cos x}{9 \cos^2 x + \sin^2 x} = \dfrac{6 \tan x}{9 + \tan^2 x}$

$\therefore$　$\cos 2B = \pm \sqrt{1 - \dfrac{36 \tan^2 x}{(9 + \tan^2 x)^2}} = \dfrac{9 - \tan^2 x}{9 + \tan^2 x}$ （取正號）

$\therefore$　$\tan B = \dfrac{1 - \cos 2B}{\sin 2B} = \dfrac{9 + \tan^2 x - 9 + \tan^2 x}{6 \tan x} = \dfrac{1}{3} \tan x$

今　　$x = A - B$

$\therefore$　$\tan x = \tan(A - B) = \dfrac{2 \tan^2 x - \frac{1}{3} \tan x}{1 + \frac{2}{3} \tan^3 x}$

$= \dfrac{6 \tan^2 x - \tan x}{3 + 2 \tan^3 x}$

即 $3 \tan x + 2 \tan^4 x = 6 \tan^2 x - \tan x$

即 $2 \tan x(\tan^3 x - 3 \tan x + 2) = 0$

即 $2 \tan x(\tan x - 1)^2(\tan x + 2) = 0$

$\therefore$ $\tan x = 0, \ 1, \ -2$

20. 令此式為 x，又第一，第二兩角各為 α β

則 $x = \alpha - \beta$ $\therefore \dfrac{x}{2} = \dfrac{\alpha}{2} - \dfrac{\beta}{2}$

今 $\cos \alpha = \dfrac{1 - a^2 \cos 2c - 2a \sin c}{1 + a^2 - 2a \sin c}$

$$= \dfrac{1 + a^2 - 2a \sin c - 2a^2 \cos^2 c}{1 + a^2 - 2a \sin c}$$

即 $1 - 2 \sin^2 \dfrac{\alpha}{2} = 1 - \dfrac{2a^2 \cos^2 c}{1 + a^2 - 2a \sin c}$

$\therefore$ $\sin \dfrac{\alpha}{2} = \dfrac{a \cos c}{\sqrt{1 + a^2 - 2a \sin c}}$

$\cos^2 \dfrac{\alpha}{2} = 1 - \dfrac{a^2 \cos^2 c}{1 + a^2 - 2a \sin c}$

$$= \dfrac{1 + a^2(1 - \cos^2 c) - 2a \sin c}{1 + a^2 - 2a \sin c}$$

$$= \dfrac{1 + a^2 \sin^2 c - 2a \sin c}{1 + a^2 - 2a \sin c}$$

$\therefore$ $\cos \dfrac{\alpha}{2} = \dfrac{1 - a \sin c}{\sqrt{1 + a^2 - 2a \sin c}}$

又 $\cos \beta = \dfrac{\cos 2c + 2a \sin c - a^2}{1 + a^2 - 2a \sin c}$

$$= \cdots\cdots = \dfrac{2 \cos^2 c}{1 + a^2 - 2a \sin c} - 1$$

$$\therefore \quad \cos\frac{\beta}{2} = \frac{\cos c}{\sqrt{1+a^2-2a\sin c}}$$

$$\sin\frac{\beta}{2} = \frac{a-\sin c}{\sqrt{1+a^2-2a\sin c}}$$

今 $\quad \cos\frac{x}{2} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

$$= \frac{\cos c(1-a\sin c) + a\cos c(a-\sin c)}{1+a^2-2a\sin c}$$

$$= \frac{\cos c(1+a^2-2a\sin c)}{1+a^2-2a\sin c} = \cos c$$

$$\therefore \quad \frac{x}{2} = c \qquad \therefore \quad x = 2c \qquad\qquad 故如題云.$$

21. 設 $\quad \tan^{-1}\alpha = A, \ \tan^{-1}\beta = B, \ \tan^{-1}\gamma = C, \ \tan^{-1}\delta = D$

則 $\quad \alpha = \tan A, \quad \beta = \tan B, \quad \gamma = \tan C, \quad \delta = \tan D$

今從根與係數關係得

$$S_1 = \Sigma\alpha = \Sigma\tan A = \sin 2\lambda$$

$$S_2 = \Sigma\alpha\beta = \Sigma\tan A\tan\beta = \cos 2\lambda$$

$$S_3 = \Sigma\alpha\beta\gamma = \Sigma\tan A\tan B\tan C = \cos\lambda$$

$$S_4 = \alpha\beta\gamma\delta = \tan A\tan B\tan C\tan D = -\sin\lambda$$

$$\therefore \quad \tan(A+B+C+D) = \frac{S_1-S_3}{1-S_2+S_4} = \frac{\sin 2\lambda - \cos\lambda}{1-\cos 2\lambda - \sin\lambda}$$

$$= \frac{\cos\lambda(2\sin\lambda-1)}{\sin\lambda(2\sin\lambda-1)}$$

$$= \cot\lambda = \tan\left(\frac{\pi}{2}-\lambda\right)$$

$$\therefore \quad \Sigma A = n\pi + \frac{\pi}{2}-\lambda \qquad . \qquad \Sigma\tan^{-1}\alpha = n\pi + \frac{1}{2}\pi - \lambda$$

習題十九 (236--238 頁)

1. 從(1)，　　　　　$x = 1 - \cos\phi$

代入(2)，　$1 - \sin^2\phi = 1 - 2\cos\phi + \cos^2\phi$

即　　$2\cos\phi = 1$　　　　　即　　$\cos\phi = \dfrac{1}{2}$

$$\therefore \quad \phi = \frac{\pi}{3}, \qquad x = \frac{1}{2}$$

2. 從十字法得

$$\left. \begin{aligned} \sin x &= \frac{b'c + bc'}{ab' + a'b} \\ \sin y &= \frac{a'c - ac'}{ab' + a'b} \end{aligned} \right\} \quad 即 \quad \left. \begin{aligned} x &= \sin^{-1}\frac{b'c + bc'}{ab' + a'b} \\ y &= \sin^{-1}\frac{a\,c - ac'}{ab' + a'b} \end{aligned} \right\}$$

3. $(1)/(2)$,　　$\cos x = \dfrac{\sqrt{2}}{\sqrt{3}}\cos y$　　　(3)

$(1)^2 + (3)^2$,　　$1 = 2\sin^2 y + \dfrac{2}{3}\cos^2 y$

即　　　　　　　　$\left. \begin{aligned} \cos^2 y &= \frac{3}{4} \\ 以\,y\,值代入(3),\quad \cos^2 x &= \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2} \end{aligned} \right\}$

即　　$\left\{ \begin{aligned} y &= \cos^{-1}\left(\pm \frac{\sqrt{3}}{2}\right) = n\pi \pm \frac{\pi}{6} \\ x &= \cos^{-1}\left(\pm \frac{\sqrt{2}}{2}\right) = n\pi \pm \frac{\pi}{4} \end{aligned} \right.$

故一組主值爲 $\dfrac{\pi}{4},\ \dfrac{\pi}{6}$．

4. 從(1)，　　$2 \sin \tfrac{1}{2}(x+y)\cos \tfrac{1}{2}(x-y)=\sin \alpha$　　　(3)

從(2)，　　$2 \cos \tfrac{1}{2}(x+y)\cos \tfrac{1}{2}(x-y)=1+\cos \alpha$　　(4)

$(3)/(4)$，　$\tan \tfrac{1}{2}(x+y)=\tan \tfrac{1}{2}\alpha$　　　$\therefore$　$x+y=\alpha$

代入(3)，　$\cos \tfrac{1}{2}(x-y)=\dfrac{\sin \alpha}{2 \sin \tfrac{1}{2}\alpha}=\dfrac{2 \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha}{2 \sin \tfrac{1}{2}\alpha}$

$$=\cos \tfrac{1}{2}\alpha$$

$\therefore$　$x-y=\pm\alpha$　　二組主值爲　$\alpha,\ o;\ o,\ \alpha$

5. $(1)^2+(2)^2$，$2+2(\sin \theta \cos \phi+\sin \phi \cos \theta)=a^2+b^2$

即　　　　　　　$\sin(\theta+\phi)=\dfrac{a^2+b^2-2}{2}$

$\therefore$　$\theta+\phi=\sin^{-1}\dfrac{a^2+b^2-2}{2}$

$(1)\cdot(2)$，$\tfrac{1}{2}(\sin 2\theta+\sin 2\phi)+\cos(\theta-\phi)=ab$

即　　　$\sin(\theta+\phi)\cos(\theta-\phi)+\cos(\theta-\phi)=ab$

$\cos(\theta-\phi)[\sin(\theta+\phi)+1]=ab$

即　$\cos(\theta-\phi)=\dfrac{2ab}{a^2+b^2}$　　　$\therefore$　$\theta-\phi=\cos^{-1}\dfrac{2ab}{a^2+b^2}$

$\therefore$　$\theta=\dfrac{1}{2}\left[\sin^{-1}\dfrac{a^2+b^2-2}{2}+\cos^{-1}\dfrac{2ab}{a^2+b^2}\right]$

$\phi=\dfrac{1}{2}\left[\sin^{-1}\dfrac{a^2+b^2-2}{2}-\cos^{-1}\dfrac{2ab}{a^2+b^2}\right]$

6. 從(1)，　　　　$1-m \cos \theta=n \cos \phi$　　　(3)

從(2)，　　　$-1+m \sin \theta=n \sin \phi$　　　(4)

$(3)^2+(4)^2$，$2-2m \cos \theta-2m \sin \theta+m^2=n^2$

即　　　　$2m \cos \theta+2m \sin \theta=m^2-n^2+2$

$\therefore \qquad \sin\dfrac{\pi}{4}\ \cos\theta + \cos\dfrac{\pi}{4}\ \sin\theta = \dfrac{m^2-n^2+2}{2m\sqrt{2}}$

即 $\qquad\qquad \sin\left(\dfrac{\pi}{4}+\theta\right) = \dfrac{m^2-n^2+2}{2m\sqrt{2}}$

設令 $\qquad\qquad \dfrac{m^2-n^2+2}{2m\sqrt{2}} = k$

則 $\qquad \dfrac{\pi}{4}+\theta = \sin^{-1}k \qquad$ 即 $\qquad \theta = \sin^{-1}k - \dfrac{\pi}{4}$

從 (1),(2) 兩式移 ϕ 之函數至右邊,兩邊平方再加,得

$\qquad\qquad 2n\cos\phi - 2n\sin\phi = n^2-m^2+2$

即 $\qquad \cos\dfrac{\pi}{4}\ \cos\phi - \sin\dfrac{\pi}{4}\ \sin\phi = \dfrac{n^2-m^2+2}{2n\sqrt{2}}$

即 $\qquad\qquad \cos\left(\dfrac{\pi}{4}+\phi\right) = \dfrac{n^2-m^2+2}{2n\sqrt{2}}$

令 $\qquad\qquad \dfrac{n^2-m^2+2}{2n\sqrt{2}} = k'$

則 $\qquad \dfrac{\pi}{4}+\phi = \cos^{-1}k' \qquad \therefore \quad \phi = \cos^{-1}k' - \dfrac{\pi}{4}$

7. 即 $\qquad 2\theta+3\phi = \cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{3} \qquad (1)$ $\left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\}$ 主角

$\qquad\qquad 3\theta+2\phi = \cos^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{6} \qquad (2)$

$2(1)-3(2),\ 5\theta = -\dfrac{1}{6}\pi \quad \therefore \ \theta = -\dfrac{1}{30}\pi$ $\left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\}$ 一組主值

代入 (1), $\qquad\qquad\qquad\qquad \phi = \dfrac{2}{15}\pi$

8. 從(1)得　　$a(\sec^2\theta-2)=b\cos\phi$　　　　(3)

從(2)得　　$b(\cos^2\theta-2)=a\sec\phi$　　　　(4)

(3)·(4),　　$(\sec^2\theta-2)(\cos^2\theta-2)=1$

即　　　　$\cos^4\theta-2\cos^2\theta+1=0$

即　$\cos^2\theta=1$　　∴　$\cos\theta=\pm1$　　∴　$\left.\begin{array}{l}\theta=n\pi\\[4pt]\end{array}\right\}$

代入(1)　　　　　　　　　　　$\phi=\cos^{-1}\left(-\dfrac{a}{b}\right)$

9. 今　　　　$\sin(x-60°)=\cos(x-30°)$

即　$\dfrac{1}{2}\sin x-\dfrac{\sqrt{3}}{2}\cos x=\dfrac{\sqrt{3}}{2}\cos x+\dfrac{1}{2}\sin x$

∴　　$\cos x=0$

∴　　$x=2m\pi\pm\dfrac{\pi}{2}=\dfrac{1}{2}(4m\pm1)\pi=\dfrac{1}{2}(2n+1)\pi$

代入(1)得　$\rho=2\Big/\sin\left(\dfrac{2n+1}{2}\pi-\dfrac{\pi}{3}\right)=2\Big/\cos\dfrac{\pi}{3}=4$

一組主值爲　$4,90°$　　　　註: 本題亦可從例四法解.

10. 從(2)化得　　$\cos^2x+\cos^2y=(b+2)/2$　　　(3)

$(1)^2-(3)$,　　$2\cos x\cos y=a^2-(b+2)/2$　　(4)

從(3),(4),　　$\cos x-\cos y=\pm\sqrt{b+2-a^2}=\pm k$

∴　$\begin{cases}\cos x=\frac{1}{2}(a\pm k)\\[4pt]\cos y=\frac{1}{2}(a\mp k)\end{cases}$　　故　$\begin{cases}x=\cos^{-1}\frac{1}{2}(a\pm k)\\[4pt]y=\cos^{-1}\frac{1}{2}(a\mp k)\end{cases}$

11. 由(1)　　　$a\sin\phi\cos\theta-b\sin\theta\cos\phi=0$　　(3)

由(2)　　　$\sin\phi\cos\theta+\sin\theta\cos\phi=a+b$　　(4)

從(3),(4), $\sin\phi\cos\theta=b$,　$\sin\theta\cos\phi=a$

故 $\qquad \sin\theta\cos\phi - \cos\theta\sin\phi = a - b$

$\therefore \quad \sin(\theta - \phi) = a - b \qquad \therefore \quad \theta - \phi = \sin^{-1}(a - b)$

從(2) $\qquad\qquad\qquad\qquad \theta + \phi = \sin^{-1}(a + b)$

$\therefore \quad \theta = \frac{1}{2}[\sin^{-1}(a + b) + \sin^{-1}(a - b)]$

$\qquad\quad \phi = \frac{1}{2}[\sin^{-1}(a + b) - \sin^{-1}(a - b)]$

12. 舊版題爲: $\quad n\sin x - m\cos x = 2m\sin y \qquad\qquad (1)$

$\qquad\qquad\qquad n\sin 2x - m\cos 2y = m \qquad\qquad (2)$

從(2)得 $\qquad n\sin 2x - 2m\cos^2 y = 0 \qquad\qquad (3)$

$2\cos x \cdot (1) - (2), \quad 2m(\cos^2 y - \cos^2 x) = 4m\cos x\cos y$

卽 $\qquad\qquad\qquad (\cos x + \cos y)^2 = 1$

$\therefore \quad \sin y = 1 - \cos x \qquad (\text{取正號}) \quad (4)$

以(4)代入(1) $\quad n\sin x + m\cos x - 2m = 0$

平方化簡得 $\quad (m^2 + n^2)\cos^2 x - 4m^2\cos x + 4m^2 - n^2 = 0$

$\therefore \quad \cos x = \dfrac{2m^2 \pm \sqrt{4m^4 - (m^2 + n^2)(4m^2 - n^2)}}{m^2 + n^2}$

$\qquad\qquad = (2m^2 \pm n\sqrt{n^2 - 3m^2})/(m^2 + n^2) = A \qquad (5)$

以(5)代入(4) $\quad \sin y = 1 - \dfrac{2m^2 \pm n\sqrt{n^2 - 3m^2}}{m^2 + n^2}$

$\qquad\qquad = \dfrac{n^2 - m^2 \mp n\sqrt{n^2 - 3m^2}}{m^2 + n^2} = B$

$\therefore \quad x = \cos^{-1} A, \qquad y = \sin^{-1} B$

卽得 $\quad x = \cos^{-1}\dfrac{2 \pm 2}{5} = \cos^{-1}\dfrac{4}{5} \left(\text{或} \quad \tan^{-1}\dfrac{3}{4}\right),$

$\qquad\qquad\qquad \cos^{-1} 0 \left(\text{或} \quad 2n\pi \pm \dfrac{\pi}{2}\right)$

$$y=\sin^{-1}\frac{3\mp 2}{5}=\sin^{-1}\frac{1}{5}, \quad \sin^{-1} 1 \left(n\pi+(-1)^n\cdot\frac{\pi}{2}\right)$$

新版之 12 題則可照下法做

從(2) $4 \sin x \cos x=2 \cos^2 y$ (3)

$2(3)+(1)^2$, $4 \sin^2 x+4 \sin x \cos x+\cos^2 x=4$

卽 $\cos x(4 \sin x-3 \cos x)=0$

∴ $\cos x=0,$ $\tan x=\frac{3}{4}$ ………

13. 從(2) $\cot(x+y)=\dfrac{1}{\tan(x+y)}=\dfrac{1-\tan x \tan y}{\tan x+\tan y}=b$

以(1)之値代入得 $\tan x \tan y=1-ab$ (3)

$\sqrt{(1)^2-4(3)}$, $\tan x-\tan y=\pm\sqrt{a^2+4ab-4}=\pm k$

∴ $\begin{cases}\tan x=\frac{1}{2}(a\pm k)\\ \tan y=\frac{1}{2}(a\mp k)\end{cases}$ 故 $\begin{cases}x=\tan^{-1}\frac{1}{2}(a\pm k)\\ y=\tan^{-1}\frac{1}{2}(a\mp k)\end{cases}$

14. $\cot x\cdot(1)-\cot y\cdot(2),$ $\cot x=\cot y$

代入(1) $\tan x+\cot x=2,$ 卽 $(\tan x-1)^2=0$

∴ $\tan x=1$ 及 $\tan y=1$

故 $x=n\pi+\dfrac{1}{4}\pi,$ $y=n\pi+\dfrac{1}{4}\pi$

一組主値爲 $\dfrac{\pi}{4},$ $\dfrac{\pi}{4}$

15. 由(1),(2)得 $\sin(x+y)=\dfrac{\sqrt{2}}{2}$

故 $\cos(x+y)=\dfrac{\sqrt{2}}{2}$ (3)

以(2)代入(3) $\sin x \sin y=0$

$$\therefore \quad \tan x \tan y = 0 \qquad (4)$$

$$\sqrt{(1)^2 - 4(4)}, \qquad \tan x - \tan y = \pm 1$$

$$\therefore \quad \left.\begin{array}{l} \tan x = 1 \\ \tan y = 0 \end{array}\right\} \quad \left.\begin{array}{l} 0 \\ 1 \end{array}\right\} \quad 故 \quad \left.\begin{array}{l} x = n\pi + \tfrac{1}{4}\pi \\ y = m\pi \end{array}\right\} \quad \left.\begin{array}{l} m\pi \\ n\pi + \tfrac{1}{4}\pi \end{array}\right\}$$

16. 從(1)　　$2 \sin \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y) = 1 \qquad (3)$

從(2)　　$\cos(x+y) + \cos(x-y) = -\tfrac{3}{2}$

卽　　$\cos^2 \tfrac{1}{2}(x-y) - \sin^2 \tfrac{1}{2}(x+y) = -\tfrac{3}{4} \qquad (4)$

以　$\sin \tfrac{1}{2}(x+y) = z, \ \cos \tfrac{1}{2}(x-y) = t$ 代入(3),(4)得

$$t^2 - z^2 = -\tfrac{3}{4}, \quad 2tz = 1$$

解之得　　$z = \pm 1, \quad t = \pm \tfrac{1}{2}$

$$\therefore \quad \left.\begin{array}{l} x - y = 2\cos^{-1}(\pm \tfrac{1}{2}) \qquad\qquad = 2(2m\pi \pm \tfrac{1}{3}\pi) \\ x + y = 2\sin^{-1}(\pm 1) = 2\cos^{-1}0 = 2(2n\pi \pm \tfrac{1}{2}\pi) \end{array}\right\} (第一組)$$

$$或 \quad \left.\begin{array}{l} 2(2m\pi \pm \tfrac{2}{3}\pi) \\ 2(2n\pi \pm \tfrac{3}{2}\pi) \end{array}\right\} (第二組)$$

$$\therefore \quad \left.\begin{array}{l} x = 2k\pi \pm \dfrac{5\pi}{6} \\[2mm] y = 2k'\pi \pm \dfrac{\pi}{6} \end{array}\right\} \quad \left.\begin{array}{l} 2k\pi \pm \dfrac{\pi}{6} \\[2mm] 2k'\pi \pm \dfrac{5\pi}{6} \end{array}\right\} \quad 主值 \quad \left.\begin{array}{l} \dfrac{5\pi}{6} \\[2mm] \dfrac{\pi}{6} \end{array}\right\} \cdot \left.\begin{array}{l} \dfrac{\pi}{6} \\[2mm] \dfrac{5\pi}{6} \end{array}\right\}$$

第二組爲 $2\left[(2m'+1)\pi \pm \dfrac{\pi}{3}\right], \ 2\left[(2n'+1)\pi \pm \dfrac{1}{2}\pi\right]$

所得 x, y 之解與第一組相同.

17. 從(1)得　　$3\cos^2 2A - \cos^2 2B - 2 = 0 \qquad (3)$

從(2)得　　　　$\sqrt{3}\cos 2A + \cos 2B - 2 = 0$　　　(4)

以(4)代入(3)，$(2-\cos 2B)^2 - \cos^2 2B - 2 = 0$

即　$\cos 2B = 1/2$　　代入(4)得　$\cos 2A = \sqrt{3}/2$

$\therefore$ $\left.\begin{array}{l} 2A = 2n\pi \pm \pi/6 \\ 2B = 2n\pi \pm \pi/3 \end{array}\right\}$　　故　$\left.\begin{array}{l} A = n\pi \pm \pi/12 \\ B = n\pi \pm \pi/6 \end{array}\right\}$

主值為　$\left.\begin{array}{l} 15° \\ 30° \end{array}\right\}$（上書答數誤）

18. 設　$x = \tan\theta$，$y = \tan\phi$，　　代入原式得

$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} = 2 \quad (1), \qquad x + y = 4 \quad (2)$$

從(1)　$3x^2y^2(x+y) - 18x^2y^2 - (x^3+y^3) - 9xy(x+y)$

$$+ 6(x^2+y^2) + 3(x+y) - 2 = 0$$

$$\therefore \quad xy = -7, \ 1 \quad (3)$$

從(2),(3)得　$\left.\begin{array}{l} x = 2 \pm \sqrt{11} \\ y = 2 \mp \sqrt{11} \end{array}\right\}$　　$\left.\begin{array}{l} 2 \pm \sqrt{3} \\ 2 \mp \sqrt{3} \end{array}\right\}$

即　$\left.\begin{array}{l} \theta = \tan^{-1}(2 \pm \sqrt{11}) \\ \phi = \tan^{-1}(2 \mp \sqrt{11}) \end{array}\right\}$ $\left.\begin{array}{l} \dfrac{\pi}{12} \\ \dfrac{5\pi}{12} \end{array}\right\}$ $\left.\begin{array}{l} \dfrac{5\pi}{12} \\ \dfrac{\pi}{12} \end{array}\right\}$

19. 設　$\tan^{-1}x = \alpha$，$\tan^{-1}y = \beta$，$\sin^{-1}x = \gamma$，$\cos^{-1}y = \delta$

則　　　$\cos\gamma = \sqrt{1-x^2}$，　　$\sin\delta = \sqrt{1-y^2}$

(1)式兩邊取正切得　$\dfrac{x+y}{1-xy} = 1$

即　　　　　　　$x + y = 1 - xy$　　　　(3)

(2)式兩邊取正弦得　$xy + \sqrt{(1-x^2)(1-y^2)} = 0$

即 $\qquad x^2+y^2=1 \qquad\qquad (4)$

$(4)+2(3),\ \ (x+y)^2+2(x+y)-3=0$

即 $\qquad (x+y-1)(x+y+3)=0 \qquad (5)$

從(3),(5)得 $\quad \begin{array}{l} x+y=1\\ xy=0 \end{array}\bigg\} \qquad \begin{array}{l} -3\\ 4 \end{array}\bigg\}$

$\therefore\quad \begin{array}{l} x=1\\ y=0 \end{array}\bigg\} \quad \begin{array}{l} 0\\ 1 \end{array}\bigg\} \quad \begin{array}{l} \frac{1}{2}(-3\pm i\sqrt{7})\\ \frac{1}{2}(-3\mp i\sqrt{7}) \end{array}\bigg\}$

只有一組值 1,0, 第二解代入不合,第三解爲虛數均去之.

20. 設 $\qquad x=\tan\alpha,\qquad y=\tan\beta$

由(1)及(2)得 $\quad \tan(\alpha+\beta)+\cot(\alpha-\beta)=a \qquad (3)$

$\qquad\qquad\qquad \tan(\alpha-\beta)+\cot(\alpha+\beta)=b \qquad (4)$

從(3)得 $\qquad \dfrac{\cos 2\beta}{\cos(\alpha+\beta)\sin(\alpha-\beta)}=a$

即 $\qquad \dfrac{2\cos 2\beta}{\sin 2\alpha-\sin 2\beta}=a$

即 $\qquad \dfrac{\sin 2\alpha}{\cos 2\beta}-\tan 2\beta=\dfrac{2}{a}$

同理化(4)爲 $\quad \dfrac{\sin 2\alpha}{\cos 2\beta}+\tan 2\beta=\dfrac{2}{b}$

故 $\quad \tan 2\beta=\dfrac{a-b}{ab} \qquad \therefore\quad \beta=\dfrac{1}{2}\tan^{-1}\dfrac{a-b}{ab}$

則 $\qquad \cos 2\beta=\dfrac{ab}{\sqrt{a^2b^2+(a-b)^2}}$

$\therefore\quad \sin 2\alpha=\dfrac{a+b}{\sqrt{a^2b^2+(a-b)^2}}$

$$\alpha = \frac{1}{2}\sin^{-1}\frac{a+b}{\sqrt{a^2b^2+(a-b)^2}}$$

$\tan\alpha$ 及 $\tan\beta$ 卽爲 x 及 y 之值。

或用下法求 α 及 β 亦可,從(3),(4)消去 $(\alpha-\beta)$ 得

$$\tan(\alpha+\beta)=A/2b \quad (A=ab\pm\sqrt{a^2b^2-4ab})$$

再從(3),(4)消去 $(\alpha+\beta)$, 得　$\tan(\alpha-\beta)=A/2a$

故　　$\alpha=\frac{1}{2}[\tan^{-1}(A/2b)+\tan^{-1}(A/2a)]$

$$\beta=\frac{1}{2}[\tan^{-1}(A/2b)-\tan^{-1}(A/2a)]$$

21. 設 a,b 爲實數,又 $|x|<1, |y|<1$

令　　　　$x=\sin^2\theta, \qquad y=\sin^2\phi$

則　　　　$\sin\theta\cos\phi+\sin\phi\cos\theta=a,$

$$\sin\theta\sin\phi+\cos\theta\cos\phi=b$$

卽　　$\begin{cases}\sin(\theta+\phi)=a & \therefore \quad \theta+\phi=\sin^{-1}a\\ \cos(\theta-\phi)=b & \theta-\phi=\cos^{-1}b\end{cases}$

$\therefore \quad \theta=\frac{1}{2}(\sin^{-1}a+\cos^{-1}b), \quad \phi=\frac{1}{2}(\sin^{-1}a-\cos^{-1}b)$

$\therefore \quad \begin{cases}x=\sin^2\theta=\sin^2\frac{1}{2}(\sin^{-1}a+\cos^{-1}b)\\ y=\sin^2\phi=\sin^2\frac{1}{2}(\sin^{-1}a-\cos^{-1}b)\end{cases}$

22. 設 $|z|<1$, 而令 $z=\sin\theta$, 則

$$x=a\cos\theta, \qquad y=b\cos\theta$$

代入(3),得 $\sin\theta=\sqrt{(1-a^2\cos^2\theta)(1-b^2\cos^2\theta)}$

平方之,以 $1-\cos^2\theta$ 代 $\sin^2\theta$, 則得

$$\cos^2\theta[a^2b^2\cos^2\theta-(a^2+b^2-1)]=0$$

$\therefore \quad \cos\theta=\sqrt{a^2+b^2-1}/ab, \; 0$

故求得 x, y, z 之值爲 $\sqrt{a^2+b^2-1}/b$, $\sqrt{a^2+b^2-1}/a$,

$\quad \sqrt{(a^2-1)(b^2-1)}/ab$; $\quad 0, 0, 1$.

23. 令 (2) 式之比值爲 t

則 $\quad \tan x = t, \quad \tan y = 2t, \quad \tan z = 3t$ $\qquad$ (3)

又因 $\quad x + y + z = \pi$ $\qquad$ 則 $\quad \Sigma \tan x = \Pi \tan x$

故 $\quad t + 2t + 3t = 6t^3$ $\qquad$ 即 $\quad t(t^2 - 1) = 0$

$\qquad\qquad \therefore \quad t = 0, \quad \pm 1$

故 $\qquad \left.\begin{array}{l} \tan x = 0 \\ \tan y = 0 \\ \tan z = 0 \end{array}\right\} \begin{array}{l} 1 \\ 2 \\ 3 \end{array}\right\}$ (t 不取負號)

$\qquad \therefore \quad \left.\begin{array}{l} x = \pi \\ y = 0 \\ z = 0 \end{array}\right\} \left.\begin{array}{l} 0 \\ \pi \\ 0 \end{array}\right\} \left.\begin{array}{l} 0 \\ 0 \\ \pi \end{array}\right\} \left.\begin{array}{l} \tan^{-1}1 \\ \tan^{-1}2 \\ \tan^{-1}3 \end{array}\right\}$

24. 從 (1) $\quad \dfrac{\sin(\theta - \beta)\cos\alpha}{\cos(\theta + \alpha)\sin\beta} = -\dfrac{\sin(\phi - \alpha)\cos\beta}{\cos(\phi - \beta)\sin\alpha}$

即 $\quad \dfrac{\sin\theta\cos\beta\cos\alpha - \cos\theta\sin\beta\cos\alpha}{\cos\theta\cos\alpha\sin\beta - \sin\theta\sin\alpha\sin\beta}$

$\qquad = \dfrac{\cos\phi\sin\alpha\cos\beta - \sin\phi\cos\alpha\cos\beta}{\cos\phi\cos\beta\sin\alpha + \sin\alpha\sin\beta\sin\phi}$

即 $\quad \dfrac{\tan\theta - \tan\beta}{\tan\beta - \tan\alpha\tan\beta\tan\theta} = \dfrac{\tan\alpha - \tan\phi}{\tan\alpha + \tan\alpha\tan\beta\tan\phi}$

即 $\quad (\tan\alpha\tan\beta + 1)\tan\alpha\tan\theta$

$\qquad - (\tan\alpha\tan\beta - 1)\tan\beta\tan\phi$

$\qquad - 2\tan\alpha\tan\beta = 0$ $\qquad$ (3)

從(2)　$\dfrac{\tan\theta\tan\alpha}{\tan\phi\tan\beta}=-\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$

卽　$\dfrac{\tan\theta\tan\alpha}{\tan\phi\tan\beta}=\dfrac{1+\tan\alpha\tan\beta}{\tan\alpha\tan\beta-1}$

卽　$(\tan\alpha\tan\beta-1)\tan\alpha\tan\theta$

$\qquad -(\tan\alpha\tan\beta+1)\tan\beta\tan\phi=0$　　(4)

$(3)+(4)$，化簡得

$\qquad \tan\alpha\tan\theta-\tan\beta\tan\phi-1=0$　　(5)

$(3)-(4)$，化簡得

$\tan\alpha\tan\theta+\tan\beta\tan\phi-\tan\alpha\tan\beta=0$　　(6)

$(5)+(6)$，　$2\tan\alpha\tan\theta=1+\tan\alpha\tan\beta$

$\qquad\therefore\quad \tan\theta=\tfrac12(\tan\beta+\cot\alpha)$

$(6)-(5)$，　$2\tan\beta\tan\phi=\tan\alpha\tan\beta-1$

$\qquad\therefore\quad \tan\phi=\tfrac12(\tan\alpha-\cot\beta)$

25.

$x=\begin{vmatrix}\cos(\beta-\gamma) & \cos\alpha & \sin\alpha\\ \cos(\gamma-\alpha) & \cos\beta & \sin\beta\\ \cos(\alpha-\beta) & \cos\gamma & \sin\gamma\end{vmatrix}\div\begin{vmatrix}1 & \cos\alpha & \sin\alpha\\ 1 & \cos\beta & \sin\beta\\ 1 & \cos\gamma & \sin\gamma\end{vmatrix}$

$\quad=\dfrac{\Sigma\cos(\beta-\gamma)\sin(\beta-\gamma)}{\Sigma\sin(\beta-\gamma)}=\dfrac{\Sigma\sin2(\beta-\gamma)}{2\Sigma\sin(\beta-\gamma)}$

但　$\Sigma\sin(\beta-\gamma)=-4\Pi\sin\tfrac12(\beta-\gamma)$

又　$\Sigma\sin2(\beta-\gamma)=-4\Pi\sin(\beta-\gamma)$

$\qquad\qquad =-32[\Pi\sin\tfrac12(\beta-\gamma)\Pi\cos\tfrac12(\beta-\gamma)]$

故　$\qquad\qquad x=4\Pi\cos\tfrac12(\beta-\gamma).$

習 題 二 十 (249--250 頁)

1. $B = 180° - (A + C) = 180° - 73°45' = 106°15'$

$$a = \frac{b \sin A}{\sin B} \qquad\qquad c = \frac{b \sin C}{\sin B}$$

$\log b = 3.21801$	$\log b = 3.21801$
$\log \sin A = 9.64953 - 10$	$\log \sin C = 9.86589 - 10$
$\text{colog } \sin B = 0.01771$	$\text{colog } \sin B = 0.01771$
$\log a = 2.88525$	$\log c = 3.10161$
$\therefore\ \ a = 767.8$	$\therefore\ \ c = 1263.6$

2. $C = 180° - (A + B) = 180° - (75° + 30°) = 75°$

$\because\quad A = C = 75° \qquad\qquad \therefore\quad a = c$

$$a = \frac{b \sin A}{\sin B} = \frac{\sqrt{8}\,(\sqrt{6} + \sqrt{2})/4}{1/2} = 2(\sqrt{3} + 1)$$

$$c = 2(\sqrt{3} + 1)$$

3. 今　$\tan\dfrac{A}{2} = \dfrac{r}{s-a}, \quad \tan\dfrac{B}{2} = \dfrac{r}{s-b}, \quad \tan\dfrac{C}{2} = \dfrac{r}{s-c}$

$$\left(r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \right)$$

$a = 3.41$	$s = 3.7975$
$b = 2.605$	$s - a = 0.3875$
	$s - b = 1.1925$
$c = 1.58$	$s - c = 2.2175$
$2s = 7.595$	$s = 3.7975$(驗)

$$\text{colog } s = 9.42050 - 10$$
$$\log(s-a) = 9.58827 - 10$$
$$\log(s-b) = 0.07644$$
$$\underline{\log(s-c) = 0.34587}$$
$$2 \log r = 19.43108 - 20$$
$$\log r = 9.71554 - 10$$

$$\log \tan \frac{A}{2} {}^{*} = 0.12727$$

$$\log \tan \frac{B}{2} = 9.63910 - 10$$

$$\log \tan \frac{C}{2} = 9.36967 - 10$$

$$\frac{A}{2} = 53°16'39'', \quad A = 106°33'18''$$

$$\frac{B}{2} = 23°32'19'', \quad B = 47°4'38''$$

$$\frac{C}{2} = 13°11'2'', \quad C = 26°22'4''$$

驗　　$A + B + C = 180°$

(*求 $\log \tan \dfrac{A}{2}$ 時即以 $\log r$ 減 $\log(s-a)$，不必另行寫

出，餘以此類推)。

4. 今　$a:b:c = \dfrac{1}{4}(\sqrt{6} - \sqrt{2}) : \dfrac{1}{2}\sqrt{2} : \dfrac{1}{2}\sqrt{3}$

$$= \sqrt{3} - 1 : 2 : \sqrt{6}$$

$\therefore \quad a = k(\sqrt{3} - 1), \qquad b = 2k, \qquad c = k\sqrt{6}$

$\therefore \quad \cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{4 - 2\sqrt{3} + 4 - 6}{4(\sqrt{3} - 1)} = -\dfrac{1}{2}$

$$C = 120°$$

$$\cos B = \frac{6+4-2\sqrt{3}-4}{2\sqrt{6}(\sqrt{3}-1)} = \frac{2(3-\sqrt{3})}{2\sqrt{2}(3-\sqrt{3})} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad B = 45°, \quad A = 15°$$

5.

$a = 681$	$B = 50°42'$
$c = 243$	$A + C = 129°18'$
$a - c = 438$	$\frac{1}{2}(A+C) = 64°39'$
$a + c = 924$	

$$\log(a-c) = 2.64147$$

$$\text{colog}(a+c) = 7.03433 - 10$$

$$\underline{\log \tan \frac{A+C}{2} = 10.32444} \quad (+$$

$$\log \tan \frac{A-C}{2} = 10.00024$$

$$\frac{1}{2}(A-C) = 45°0'57''$$

$$A = 109°39'57'', \quad C = 19°38'3''$$

$$b = \frac{c \sin B}{\sin C}$$

$$\log c = 2.38561$$

$$\underline{\log \sin B = 9.88865} \quad (+$$

$$12.27426$$

$$\underline{\log \sin C = 9.52636} \quad (-$$

$$\log b = 2.74790$$

$$\therefore \quad b = 559.63$$

6. 今　$c<b$, $C<90°$, 而　$b \sin C = \frac{5}{2} > c$, 故無解.

7. 今　$a<b$, $A<90'$, 而　$b>a>b \sin A$, 故有兩解.

$$\therefore \quad \sin B = \frac{b \sin A}{a} = \frac{1}{2}\sqrt{3} \quad \therefore \quad B = 60° \text{ 或 } 120°$$

$$C = 180° - (A+B) = 90° \text{ 或 } 30°$$

$$c = \frac{a \sin C}{\sin A} = 2 \text{ 或 } 1$$

解爲 $\begin{cases} B = 60°, & C = 90°, & c = 2; \\ B' = 120°, & C' = 30°, & c' = 1. \end{cases}$

8. 今 $c < b$, 則 $C < B$, 但 $C = 160° > 90°$, 故不可解.

9. 今 $\sin C = \frac{c \sin A}{a} = \frac{\sqrt{3}+1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}(\sqrt{6}+\sqrt{2})$

$$\therefore \quad C = 105°, \quad C' = 75° \quad\quad 故 \quad B = 30°, \quad B' = 60°$$

$$b = a \sin B / \sin A \quad\quad 故 \quad b = \sqrt{2}, \quad b' = \sqrt{6}$$

10. 今 $\cos A = \frac{1}{4}\sqrt{7} \quad \therefore \quad \sin A = \frac{3}{4}$ （負號不合因 $A < 180°$）

$$\sin B = \frac{b \sin A}{a} = \frac{1}{5}\left[7\left(\frac{3}{4}\right)\right] = \frac{21}{20} > 1 \quad 故此題無解.$$

或從 $a < b$, $\quad a = 5 < b \sin A\left(= \frac{21}{4}\right) \quad\quad$ 亦可知其無解.

11. $\qquad a = 767, \qquad b = 242, \qquad A = 36°53'2''$

$\sin B = \dfrac{b \sin A}{a}$	$c = \dfrac{a \sin C}{\sin A}$
$\log 242 = 2.38382$	$\log 767 = 2.88480$
$\log \sin A = 9.77830 - 10$	$\log \sin C = 9.86970 - 10$
$\operatorname{colog} 767 = 7.11520 - 10$	$\operatorname{colog} \sin A = 0.22170$
$\log \sin B = 19.27732 - 20$	$\log c = 12.97620 - 10$
$\therefore \quad B = 10°54'58''$	$\therefore \quad c = 946.68$
$C = 132°12'$	

12. $a = 117.48,$ $b = 726.3,$ $A = 80°0'50''$

$$\sin B = \frac{b \sin A}{a}$$

$$\log b = 2.86112$$

$$\log \sin A = 9.99337 - 10$$

$$\mathrm{colog}\, a = 7.93003 - 10 \quad (+$$

$$\log \sin B = 0.78452$$

∴ $\sin B > 1$ 故此題不能解.

13. $a = 177.01,$ $b = 216.45,$ $A = 35°36'20''$

$$\sin B = \frac{b \sin A}{a}$$

$$\log b = 2.33536$$

$$\log \sin A = 9.76507 - 10$$

$$\mathrm{colog}\, a = 7.75200 - 10 \quad (+$$

$$\log \sin B = 9.85243 - 10$$

今 $a < b,$ 而 $\log \sin B < 0,$ 故有兩解.

$B = 45°23'28''$	$B' = 134°36'32''$
$C = 99°0'12''$	$C' = 9°47'8''$
$\log a = 2.24800$	$\log a' = 2.24800$
$\log \sin C = 9.99462 - 10$	$\log \sin C' = 9.23035 - 10$
$\mathrm{colog} \sin A = 0.23493$	$\mathrm{colog} \sin A' = 0.23493$
$\log c = 2.47755$	$\log c' = 1.71328$
∴ $c = 300.29$	$c' = 51.675$

14. $a = 13.2,$ $b = 15.7,$ $A = 57°18'15'' < 90°,$ $b < a$

$$\sin B = \frac{b \sin A}{a}$$

$$\log b = 1.19590$$

$$\log \sin A = 9.92467 - 10$$

$$\text{colog } a = 8.87943 - 10$$

$$\log \sin B = 0$$

$$\therefore \quad B = 90°$$

$$C = 32°46'45''$$

$$c = b \sin C$$

$$\log b = 1.19590$$

$$\log \sin C = 9.73352 - 10$$

$$\log c = 0.92942$$

$$\therefore \quad c = 8.5$$

15. 設　$a = 2x + 3$,　$b = x^2 + 3x + 3$,　$c = x^2 + 2x$

今　$b - a = x^2 + x > 0$,　$b - c = x + 3 > 0$　　（$\because$　$x > 0$）

故 b 爲最大邊, 故知對 b 之角 B 爲最大。

今　$\cos B = \dfrac{(2x+3)^2 + (x^2+2x)^2 - (x^2+3x+3)^2}{2(2x+3)(x^2+2x)}$

$$= \frac{-2x^3 - 7x^2 - 6x}{2(2x^3 + 7x^2 + 6x)} = -\frac{1}{2}$$

故　$B = 120°$　　　此爲最大角。

16. 設　$a = 242$,　$b = 188$,　$c = 270$,　　故 C 角最大

$\therefore$　$s = \frac{1}{2}(a+b+c) = 350$,　　　$s - a = 108$,

$$s - b = 162, \qquad s - c = 80$$

今　$\tan \dfrac{1}{2} C = \sqrt{\dfrac{(s-b)(s-a)}{s(s-c)}} = \sqrt{\dfrac{162 \cdot 108}{350 \cdot 80}} = \sqrt{\dfrac{2 \cdot 3^7}{7 \cdot 10^3}}$

$\therefore$　$\log \tan \frac{1}{2} C = \frac{1}{2}(\log 2 + 7 \log 3 - \log 7 - 3)$

$$= \frac{1}{2}(.30103 + 3.33984 - .84510 - 3)$$

$$= -.10211 = 9.89789 - 10$$

$\therefore$　$\frac{1}{2} C = 38°19'32.3''$　　　　$\therefore$　$C = 76°39'5''$

17. 今　$c^2 = (\sqrt{3}+1)^2 + (\sqrt{3}-1)^2$

$\qquad -2(\sqrt{3}+1)(\sqrt{3}-1)\cos 60°$

$\qquad = 3+2\sqrt{3}+1+3-2\sqrt{3}+1-2=6$

$\qquad\qquad \therefore\quad c=\sqrt{6}$

18. 設比值爲 k, 則

$\qquad a = (m+n)k, \quad b = (m-n)k, \quad c = \sqrt{2(m^2+n^2)}k$

$\therefore\quad \cos C = \dfrac{(m+n)^2 k^2 + (m-n)^2 k^2 - 2(m^2+n^2)k^2}{2(m+n)(m-n)k^2} = 0$

$\qquad\qquad \therefore\quad C = 90°$

習 題 二 十 一 (261—263 頁)

1. 設比值爲 k, 則　$a=5k, \quad b=7k, \quad c=8k, \quad$ 今 A 最小

$\qquad \cos A = \dfrac{(49+64-25)k^2}{2\times 7\times 8 k^2} = \dfrac{11}{14} = .7857 \qquad \therefore\quad A = 38°12'$

2. 今邊順次爲 a, b, c, 則　$S = \dfrac{1}{4}(2\sqrt{3}+3\sqrt{2}+\sqrt{6})$,

$\qquad\qquad\qquad S-a = \dfrac{1}{4}(3\sqrt{2}+\sqrt{6}-2\sqrt{3})$,

$\qquad\qquad\qquad S-b = \dfrac{1}{4}(\sqrt{6}+2\sqrt{3}-\sqrt{2})$,

$\qquad\qquad\qquad S-c = \dfrac{1}{4}(2\sqrt{3}+\sqrt{2}-\sqrt{6})$.

$\triangle = \dfrac{1}{16}\times$

$\sqrt{(2\sqrt{3}+3\sqrt{2}+\sqrt{6})(3\sqrt{2}+\sqrt{6}-2\sqrt{3})(\sqrt{6}+2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2}-\sqrt{6})}$

$= \dfrac{1}{16}\sqrt{[(3\sqrt{2}+\sqrt{6})^2-12][12-(\sqrt{2}-\sqrt{6})^2]} = \dfrac{1}{4}\sqrt{(3+3\sqrt{3})(1+\sqrt{3})}$

$= \dfrac{1}{4}(1+\sqrt{3})\sqrt{3} = \dfrac{1}{4}(3+\sqrt{3}) = \dfrac{1}{4}(4.7320) = 1.183$

3. 今 $\quad s = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}, \quad s-a = \dfrac{z}{x}, \quad s-b = \dfrac{x}{y}, \quad s-c = \dfrac{y}{z}$

$\therefore \quad \triangle = \sqrt{\left(\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}\right)\left(\dfrac{x}{y}\right)\left(\dfrac{y}{z}\right)\left(\dfrac{z}{x}\right)}$

$\qquad = \sqrt{\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}}$

4. 今 $\quad \triangle = \dfrac{a^2 \sin B \sin C}{2 \sin(B+C)} = \dfrac{4(\sqrt{3}+1)^2 \sin 45° \sin 60°}{2 \sin(45°+60°)}$

$\qquad = \dfrac{4(2+\sqrt{3})\sqrt{6}}{\sqrt{6}+\sqrt{2}}$

$\qquad \left[\because \quad \sin 75° = \dfrac{1}{4}(\sqrt{6}+\sqrt{2})\right]$

$\qquad = (2+\sqrt{3})(6-2\sqrt{3}) = 6+2\sqrt{3}$

5. 從 $\quad S = \dfrac{1}{2} bc \sin A \qquad$ 即 $\quad \dfrac{\sqrt{3}}{4} bc = 10\sqrt{3}$

$\qquad\qquad\qquad \therefore \quad bc = 40 \qquad (1)$

又從 $\quad \cos 60° = \dfrac{b^2+c^2-a^2}{2bc} \qquad$ 及 $\quad a+b+c = 20$

即 $\qquad b^2+c^2 - [20-(b+c)]^2 = bc$

即 $\qquad 40(b+c) - 400 = 3bc = 120$

$\qquad\qquad \therefore \quad b+c = 13 \qquad (2)$

解(1),(2)得兩邊爲 $8,5 \qquad$ 則 $\qquad a = 7$

故三邊爲 8 尺,5 尺,7 尺.

6. 設三角爲 $A = B-y, \; B, \; C = B+y$

又三邊爲 $a = 16$ 尺, $b, \; c = 24$ 尺

今　$A+B+C=180°$　　$\therefore$　$3B=180°$ 或 $B=60°$

又　$b^2=a^2+c^2-2ac\cos B=256+576-2\cdot16\cdot24\cdot\frac{1}{2}$

$\qquad\therefore$　$b^2=448$　　　故　$b=8\sqrt{7}$ 尺

$\therefore$　$\dfrac{c-a}{c+a}=\dfrac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)}=\dfrac{\tan y}{\tan 60°}$

$\therefore$　$\tan y=\dfrac{8}{40}\sqrt{3}=\dfrac{2\sqrt{3}}{10}=\dfrac{\sqrt{3}}{5}$

$\qquad\log\tan y=\log 2+\frac{1}{2}\log 3-1=\cdot53959-1$

$\qquad\qquad=9\cdot53959-10$　　$\therefore$　$y=19°6'24''$

故三角爲 $40°53'36''$, $60°$, $79°6'24''$,　　第三邊則爲 $8\sqrt{7}$尺.

7. $\therefore$　　$\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

今　　$s=\dfrac{1}{2}(a+91+125)=\dfrac{a}{2}+108$

$\qquad s-a=108-\dfrac{a}{2},\quad s-b=\dfrac{a}{2}+17,\quad s-c=\dfrac{a}{2}-17$

$\therefore$　$\left(\dfrac{17}{6}\right)^2=\dfrac{\left(\dfrac{a}{2}\right)^2-289}{-\left(\dfrac{a}{2}\right)^2+108^2}$　　　解之得　$a=204$

8. 因　$\cot\dfrac{C}{2}=\tan\dfrac{1}{2}(A+B)=\dfrac{\dfrac{5}{6}+\dfrac{20}{37}}{1-\dfrac{5\cdot20}{6\cdot37}}=\dfrac{305}{122}=\dfrac{5}{2}$

$\qquad\qquad\therefore$　$\tan\dfrac{C}{2}=\dfrac{2}{5}$

$$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}=\frac{\left(\frac{20}{37}-\frac{2}{5}\right)\bigg/\left(1+\frac{20}{37}\cdot\frac{2}{5}\right)}{\cot\frac{1}{2}A}$$

$$=\frac{\frac{26}{225}}{\frac{6}{5}}=\frac{13}{135}$$

又　$$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}=\frac{\left(\frac{5}{6}-\frac{20}{37}\right)\bigg/\left(1+\frac{5}{6}\cdot\frac{20}{37}\right)}{\cot\frac{1}{2}C}$$

$$=\frac{\frac{65}{322}}{\frac{5}{2}}=\frac{13}{161}$$

從上兩式推得　　$c=\dfrac{61}{74}b,$　　$a=\dfrac{87}{74}b$

$$\therefore\quad a+c=\frac{87}{74}b+\frac{61}{74}b=2b$$

9. 今　　$\dfrac{A}{1}=\dfrac{B}{2}=\dfrac{C}{3}=\dfrac{A+B+C}{6}=\dfrac{180°}{6}=30°$

$$\therefore\quad A=30°,\quad B=60°,\quad C=90°$$

從正弦定律得　$a:b:c=\sin30°:\sin60°:\sin90°$

$$=\frac{1}{2}:\frac{\sqrt{3}}{2}:1=1:\sqrt{3}:2$$

10. 今　$\cos A=\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{(a+b+c)(b+c-a)-2bc}{2bc}$

但　　　　$(a+b+c)(b+c-a)=3bc$

$$\therefore\quad \cos A=\tfrac{1}{2}\quad\text{故}\quad A=60°$$

或從　$\cos A = 2\cos^2\dfrac{A}{2} - 1 = \dfrac{2s(s-a)}{bc} - 1$

$\qquad\qquad = \dfrac{3}{2} - 1 = \dfrac{1}{2}\qquad [\because\quad 4s(s-a) = 3bc]$

$\qquad\therefore\quad A = 60°$

11. 作　$DE \perp AB$　則　$DE = a\sin\theta$

今 □ 面積 $= \overline{DE} \cdot \overline{AB} = a\sin\theta \cdot b$

$\qquad\qquad = ab\sin\theta$

12. 如圖設　$BE \perp AC,\ DF \perp AC$

$\qquad AC = d_1,\qquad BD = d_2$

則　$BE = BG\sin\theta,$

$\qquad DF = DG\sin\theta$

今　$\triangle ABC = \frac{1}{2}\overline{AC} \cdot \overline{BE}$

$\qquad\qquad = \frac{1}{2}d_1\overline{BG}\sin\theta$

$\qquad \triangle ADC = \frac{1}{2}\overline{AC} \cdot \overline{DF} = \frac{1}{2}d_1\overline{DG}\sin\theta$

故　$ABCD = \frac{1}{2}d_1\sin\theta[\overline{BG} + \overline{DG}] = \frac{1}{2}d_1 d_2\sin\theta$

13. 設　$\triangle ABC$ 之三邊為 $a',\ b',\ c'$

又周界之半為 s，則 $s = 3a$,
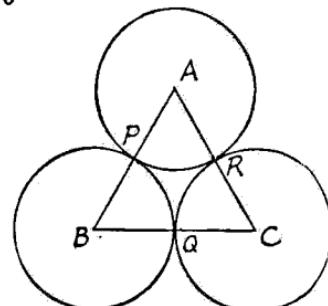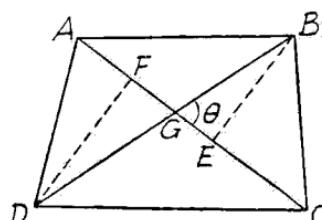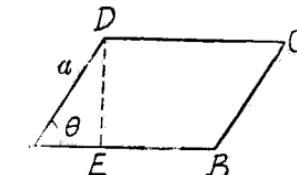
$\qquad s - a' = s - b' = s - c' = a$

故　$\triangle = a^2\sqrt{3}$

又　$A = B = C = 60°$

故扇形 $APR = BPQ = CQR$

$\qquad\qquad = \frac{1}{6}$ 圓 $= \frac{1}{6}\pi a^2$
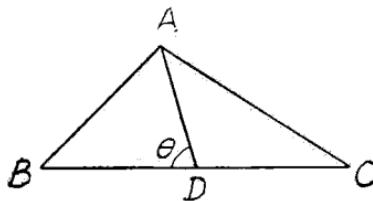
故空隙面積 $= \triangle - 3($ 扇形 $AFR)$

$$= a^2 \sqrt{3} - \tfrac{1}{2}\pi a^2 = (\sqrt{3} - \tfrac{1}{2}\pi) a^2$$

14. 如圖在 $\triangle ABD, ACD$ 中，設 $\angle ADB = \theta$

則　　$\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2\overline{AD} \cdot \overline{BD} \cos\theta$

$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2 + 2\overline{AD} \cdot \overline{DC} \cos\theta$$

因　　　　　　$BD = DC = \tfrac{1}{2} BC$

代入上兩式再相加之，又兩邊乘以 2

則　　　　$2\overline{AB}^2 + 2\overline{AC}^2 = \overline{BC}^2 + 4\overline{AD}^2$

15. 即　$(a^2 + b^2)^2 - 2(a^2 + b^2)c^2 + c^4 = a^2 b^2$

即　$(a^2 + b^2 - c^2)^2 = a^2 b^2$　　　$\therefore$　$a^2 + b^2 - c^2 = \pm ab$

從　　　　$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \pm \dfrac{1}{2}$

$\therefore$　$C = 60°$ 或 $120°$

16. $\because$　$\tan B = \dfrac{\cos(C-B)}{\sin(C+B) + \sin(C-B)}$

$$= \dfrac{\cos C \cos B + \sin C \sin B}{2 \sin C \cos B}$$

$$= \dfrac{1}{2}(\tan B + \cot C)$$

$$\therefore \quad \tan B = \cot C = \tan\left(\frac{\pi}{2} - C\right)$$

$$\therefore \quad B = \frac{\pi}{2} - C \qquad \therefore \quad B + C = \frac{\pi}{2} \qquad 卽 \quad A = \frac{\pi}{2}$$

故如題云．

17. 卽　　　　　$\sin(B+C) = 2\sin B \cos C$

卽　　　　　$\sin(B-C) = 0$　　　　　$\therefore \quad B = C$

18. 今　　$c(a+b)\sqrt{\dfrac{s(s-b)}{ac}} = b(c+a)\sqrt{\dfrac{s(s-c)}{ab}}$

卽　$\dfrac{c(a+b)}{b(c+a)} = \sqrt{\dfrac{c(s-c)}{b(s-b)}}$　　　卽　$\dfrac{c(a+b)^2}{b(c+a)^2} = \dfrac{a+b-c}{a-b+c}$

去分母整理之：

$$ca^3 + c^2a^2 - cb^3 = 3ab^2c - 3abc^2 - bc^3 + ba^3 + b^2a^2$$

卽　$a^3(c-b) + a^2(c-b)(c+b) + bc(c-b)(c+b)$

　　　　$+ 3abc(c-b) = 0$

卽　$(c-b)[a^3 + a^2(c+b) + bc(c+b) + 3abc] = 0$

　　　$\therefore \quad c - b = 0$　　　　　$\therefore \quad b = c$

19. 卽　$3b^2 - 6bc + 3c^2 = b^2 + c^2 - 2bc\cos 60° = b^2 + c^2 - bc$

卽　$2b^2 - 5bc + 2c^2 = 0$　　　卽　$(2b-c)(b-2c) = 0$

$\because \quad b-c$ 爲正　　　卽　$b > c$　　　故　$b = 2c$

故　　　　　$a = (2c-c)\sqrt{3} = c\sqrt{3}$

今　　　　　$a^2 + c^2 = 3c^2 + c^2 = 4c^2 = b^2$

故　　$B = 90°$　　　　　則　　　$C = 30°$

【此題亦可用正弦定律從角着手】

20　即　右邊 $= \dfrac{2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}{2 \cos \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}$

$$= \frac{\sin \frac{1}{2}(B+C)}{\cos \frac{1}{2}(B+C)} = \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A}$$

即　　$\dfrac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A} = \sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$

（$\cos \frac{1}{2}A = 0$　無意義）

$\therefore$　$\sin^2 \dfrac{A}{2} = \dfrac{1}{2}$　　　　$\therefore$　$\sin \dfrac{A}{2} = \dfrac{1}{2}\sqrt{2}$

$\therefore$　$\dfrac{A}{2} = 45°$　　　　$\therefore$　$A = 90°$　　故如題云。

21　即　$(1-\cos^2 A) + (1-\cos^2 B) + (1-\cos^2 C) = 2$

即　$\Sigma \cos^2 A = 1$　　　即　$\Sigma \cos 2A = -1$

但因　　　　　$A+B+C = \pi$

故從 142 頁例六知　$\Sigma \cos 2A = -1 - 4\Pi \cos A$

代入上式得　　　　$\cos A \cos B \cos C = 0$

中間任一個因式為 0，　如 $\cos A = 0$，　則　$A = 90°$

故如題云。

　　註： 此題化至 $\Sigma \cos^2 A = 1$ 時，如用餘弦定律，則得

　　　$\Pi(a^2 + b^2 - c^2) = 0$　　惟甚繁也。

22　即　$a(\tan A - \cot \frac{1}{2}C) = b(\cot \frac{1}{2}C - \tan B)$

　　　　$(\because\ \tan \frac{1}{2}(A+B) = \cot \frac{1}{2}C)$

即　$a\left(\dfrac{\sin A}{\cos A} - \dfrac{1+\cos C}{\sin C}\right) = b\left(\dfrac{1+\cos C}{\sin C} - \dfrac{\sin B}{\cos B}\right)$

即　$\dfrac{a}{\cos A}[-\cos(A+C)-\cos A]$　　　　【此題用正弦定律便甚繁】

　　　$=\dfrac{b}{\cos B}[\cos(B+C)+\cos B]$

即　$a\cos B(\cos B-\cos A)=b\cos A(\cos B-\cos A)$

若　$\cos B-\cos A=0$　　　　則　$\cos A=\cos B$

$\therefore\ A=B$　　　　　　　　又　$a\cos B=b\cos A$

即　$\dfrac{a^2+c^2-b^2}{2c}=\dfrac{b^2+c^2-a^2}{2c}$　　$\therefore\ a^2=b^2$　　即　$a=b$

故此題必爲一等腰三角形。

23. 因　$\dfrac{\sin A\cos B}{\cos A\sin B}=\dfrac{\sin A}{\sin B}$　　　　即　$\cos B=\cos A$

　　　　$\therefore\ \ A=B$　　　　　　故如題云。

24. 從合分之理得　　$\dfrac{\tan A}{\tan B}=\dfrac{2c-b}{b}$

即　$\dfrac{\sin A\cos B}{\cos A\sin B}=\dfrac{2c-b}{b}$　　以　$\sin A=\dfrac{a\sin B}{b}$ 代入

則得　　　　$a\cos B=2c\cos A-b\cos A$

但　　　　　$a\cos B+b\cos A=c$　　〔公式$(41A)$〕

故　　　　　$\cos A=\tfrac12$　　　　$\therefore\ \ A=60°$

25. 即　$\cos A\sin C+2\cos C\sin C$

　　　　$=\sin B\cos A+2\cos B\sin B$

即　$\cos A(\sin C-\sin B)+\sin 2C-\sin 2B=0$

即　$\cos A(\sin C-\sin B)-2\sin(C-B)\cos A=0$

即　$\cos A[\sin C-\sin B-2\sin(C-B)]=0$

即　$\cos A\left[2\sin\dfrac{C-B}{2}\cos\dfrac{C+B}{2}-4\sin\dfrac{C-B}{2}\cos\dfrac{C-B}{2}\right]=0$

即　$2\cos A\sin\dfrac{C-B}{2}\left[\cos\dfrac{C+B}{2}-2\cos\dfrac{C-B}{2}\right]=0$

$\therefore$　$\cos A=0$　　則　$A=90°$　　（此爲一直角三角形）

或　$\sin\dfrac{C-B}{2}=0$　　　則　$\dfrac{C-B}{2}=0$

$\therefore$　$C=B$　（此爲一等腰三角形）

26. 今　$\tan\dfrac{A}{2}\tan\dfrac{C}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}\cdot\dfrac{(s-a)(s-b)}{s(s-c)}}$

$$=\dfrac{c+a-b}{a+b+c}=\dfrac{b}{3b}=\dfrac{1}{3}$$

$$(\because\quad a+b=2b)$$

註：此題如用正弦定律做便甚繁。

27. 設　$A=8\theta$,　$B=4\theta$,　$C=2\theta$

則　$14\theta=180°$　　$\therefore$　$7\theta=90°$

今　$\dfrac{a}{\sin 8\theta}=\dfrac{b}{\sin 4\theta}=\dfrac{c}{\sin 2\theta}=\dfrac{a+b+c}{\sin 8\theta+\sin 4\theta+\sin 2\theta}$

但　$\sin 8\theta=2\sin 4\theta\cos 4\theta=\cdots\cdots\cdots$

$$=8\sin\theta\cos\theta\cos 2\theta\cos 4\theta$$

又　$\sin 8\theta+\sin 4\theta+\sin 2\theta=4\cos 4\theta\cos 2\theta\cos\theta$

（見附錄一公式 50A）

$\therefore$　$\dfrac{a}{a+b+c}=2\sin\theta=2\sin\dfrac{90°}{7}:1$

28. 設三邊爲 $k-r$, k, $k+r$, 卽爲 $k(1-x)$, k, $k(1+x)$ 之

形式,則　　　$a:b:c = 1-x:1:1+x$

今　　　$s = \dfrac{3k}{2},$　　$s-a = \left(\dfrac{1}{2}+x\right)k,$　$\cdots\cdots\cdots$

故　$\cos\dfrac{\alpha}{2} = \cos\dfrac{1}{2}(A-C) = \cos\dfrac{A}{2}\cos\dfrac{C}{2} + \sin\dfrac{A}{2}\sin\dfrac{C}{2}$

$$= \dfrac{s}{b}\sqrt{\dfrac{(s-a)(s-c)}{ac}} + \dfrac{s-b}{b}\sqrt{\dfrac{(s-a)(s-c)}{ac}}$$

$$= \dfrac{2s-b}{b}\sqrt{\dfrac{(s-a)(s-c)}{ac}} = 2\sqrt{\dfrac{\frac{1}{4}-x^2}{1-x^2}}$$

$$= \sqrt{\dfrac{1-4x^2}{1-x^2}}　　　　\therefore\ \cos\alpha = \dfrac{1-7x^2}{1-x^2}$$

卽　$(7-\cos\alpha)x^2 = 1-\cos\alpha$　　　$\therefore\ x = \sqrt{\dfrac{1-\cos\alpha}{7-\cos\alpha}}$

29. 設周界爲 p,　則　$p = 2\pi r$

$$\therefore\ r = \dfrac{p}{2\pi},\ S(圓面積) = \pi r^2 = \dfrac{p^2}{4\pi}$$

又設正 n 邊形之邊心距爲 d, 則因每邊長爲 $\dfrac{p}{n}$,所對之

圓心角爲 $\dfrac{2\pi}{n}$

$$\therefore\ \dfrac{\frac{p}{2n}}{d} = \tan\dfrac{\pi}{n}　　　\therefore\ d = \dfrac{\frac{p}{2n}}{\tan\dfrac{\pi}{n}}$$

$$\therefore\ S'(正\,n\,邊形面積) = \dfrac{\frac{p}{2n}}{\tan\dfrac{\pi}{n}}\cdot\dfrac{p}{2} = \dfrac{p^2}{4n\tan\dfrac{\pi}{n}}$$

$$\therefore \quad S:S'=\frac{p^2}{4\pi} : \frac{p^2}{4n\tan\dfrac{\pi}{n}}=\tan\frac{\pi}{n} : \frac{\pi}{n}$$

30 設 Ω 爲 $ABCD$ 之面積

則　$\Omega=\triangle ABD+\triangle BCD$

$\therefore \quad \Omega=\frac{1}{2}ab\sin A+\frac{1}{2}cd\sin A$

$\qquad =\frac{1}{2}(ab+cd)\sin A$

今 $ABCD$ 內接於一圓

則　$A+C=180°$

$\therefore \quad \cos C=-\cos A \qquad 故 \quad x^2=c^2+d^2-2cd\cos A$

又　$x^2=a^2+b^2-2ab\cos C=a^2+b^2+2ab\cos A$

消去 x 得　　$\cos A=\dfrac{c^2+d^2-a^2-b^2}{2(ab+cd)}$
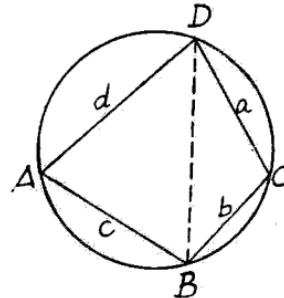
則　　$\sin^2 A=1-\left[\dfrac{c^2+d^2-a^2-b^2}{2(ab+cd)}\right]^2$

$\qquad =\dfrac{4(ab+cd)^2-(c^2+d^2-a^2-b^2)^2}{4(ab+cd)^2}$

分子爲 $(2ab+2cd+c^2+d^2-a^2-b^2)$

$\qquad \times(2ab+2cd-c^2-d^2+a^2+b^2)$

$\quad =[(c+d)^2-(a-b)^2][(a+b)^2-(c-d)^2]$

$\quad =(-a+b+c+d)(a-b+c+d)$

$\qquad \times(a+b-c+d)(a+b+c-d)$

設　$s=\frac{1}{2}(a+b+c+d)$　　　則　$s-a=\frac{1}{2}(-a+b+c+d)$

$$\therefore \quad \sin A=\frac{2\sqrt{\Pi(s-a)}}{ab+cd}$$

故　$\Omega = \dfrac{1}{2}(ab+cd) \cdot \dfrac{2\sqrt{\Pi(s-a)}}{ab+cd} = \sqrt{\Pi(s-a)}$

習 題 二 十 二 (279—282 頁)

1. 今　$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = \dfrac{ma}{m\sin A}$

$\qquad = \dfrac{ma+b+c}{m\sin A+\sin B+\sin C}$

$\qquad = \dfrac{ma-b+c}{m\sin A-\sin B+\sin C}$

$\therefore \quad \dfrac{ma+b+c}{ma-b+c} = \dfrac{m\sin A+\sin B+\sin C}{m\sin A-\sin B+\sin C}$

2. 今　$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

即　$\dfrac{ma^2}{m\sin^2 A} = \dfrac{nab}{n\sin A \sin B} = \dfrac{pb^2}{p\sin^2 B}$

$\qquad = \dfrac{ma^2+nab+pb^2}{m\sin^2 A+n\sin A\sin B+p\sin^2 B}$

$\qquad = \dfrac{ma^2-nab+pb^2}{m\sin^2 A-n\sin A\sin B+p\sin^2 B}$

或　$\dfrac{ma^2+nab+pb^2}{ma^2-nab+pb^2} = \dfrac{m\sin^2 A+n\sin A\sin B+p\sin^2 B}{m\sin^2 A-n\sin A\sin B+p\sin^2 B}$

$\therefore \quad \sqrt{\dfrac{ma^2+nab+pb^2}{ma^2-nab+pb^2}}$

$\qquad = \sqrt{\dfrac{m\sin^2 A+n\sin A\sin B+p\sin^2 B}{m\sin^2 A-n\sin A\sin B+p\sin^2 B}}$

3 $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = \dfrac{bc}{c\sin B} = \dfrac{bc}{b\sin C}$

$\qquad = \dfrac{bc}{\sqrt{bc\sin B\sin C}} = \dfrac{bc^2+b^2c}{c^2\sin B+b^2\sin C}$

故 $(b+c)\sqrt{bc\sin B\sin C} = b^2\sin C + c^2\sin B$

4. ∵ $b = a\cos C + c\cos A,\qquad c = a\cos B + b\cos A$

∴ $b - c = a(\cos C - \cos B) + (c - b)\cos A$

即 $(b-c)(1+\cos A) = a(\cos C - \cos B)$

即 $(b-c)\left(2\cos^2\dfrac{A}{2}\right) = a\left(2\sin\dfrac{B+C}{2}\sin\dfrac{B-C}{2}\right)$

∴ $(b-c)\cos\dfrac{A}{2} = a\sin\dfrac{B-C}{2}$

【從正弦定理亦可證,參考 265 頁例二】

5. 從 $b = a\cos C + c\cos A,\qquad c = b\cos A + a\cos B$

得 $\cos B = \dfrac{c - b\cos A}{a},\qquad \cos C = \dfrac{b - c\cos A}{a}$

故 $\cos B : \cos C = c - b\cos A : b - c\cos A$

【從正弦定律亦可做,正常方法是由右至左】

6. 做法同題 4 不過易 $b-c$ 為 $b+c$(即易減為加)耳。

7. ∵ $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ ∴ $c\sin B = b\sin C$

左邊 $= 2(b^2\sin C\cos C + c^2\sin B\cos B)$

$\qquad = 2(b\cos C\cdot c\sin B + c\cos B\cdot b\sin C)$

$\qquad = 2bc(\cos C\sin B + \sin C\cos B)$

$\qquad = 2bc\sin(B+C) = 2bc\sin A$

8. 從 $a = b\cos C + c\cos B,$ $\qquad b = c\cos A + a\cos C$

$\therefore$ $(a+b)(1-\cos C) = c(\cos A + \cos B)$

$\therefore$ $\cos A + \cos B = 2\left(\dfrac{a+b}{c}\right)\sin^2\dfrac{C}{2}$

$\qquad (\because\ 1 - \cos C = 2\sin^2\dfrac{C}{2}$ 也$)$

【由左邊 $= 2\sin^2\dfrac{B}{2} - 2\sin^2\dfrac{C}{2}$ 做便繁】

9. 因 $\quad a = \dfrac{c\sin A}{\sin C},$ $\qquad b = \dfrac{c\sin B}{\sin C}$

故左邊 $= \dfrac{c\sin A\cos A}{\sin C} + \dfrac{c\sin B\cos B}{\sin C}$

$\qquad = \dfrac{c}{2\sin C}(\sin 2A + \sin 2B)$

$\qquad = \dfrac{c}{\sin C}[\sin(A+B)\cos(A-B)] = c\cos(A-B)$

$\qquad (\because\ \sin(A+B) = \sin C$ 也$)$

10. 左邊 $= \dfrac{1-\cos A}{1-\cos B} = \dfrac{\sin^2\dfrac{A}{2}}{\sin^2\dfrac{B}{2}} = \dfrac{\dfrac{(s-b)(s-c)}{bc}}{\dfrac{(s-a)(s-c)}{ac}}$

$\qquad = \dfrac{a(s-b)}{b(s-a)} = \dfrac{a(a-b+c)}{b(b+c-a)}$

11. 從 $\qquad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$\therefore$ $a = 2R\sin A,$ $\qquad b = 2R\sin B,$ $\qquad c = 2R\sin C$

故 $\quad \Sigma(a-b)\sin C = 2R\Sigma\sin C(\sin A - \sin B)$

$\qquad\qquad = 2R\cdot 0 = 0$

12. 右邊 $=\frac{1}{2}\sin A(2R\sin B)(2R\sin C)=\frac{1}{2}bc\sin A=S$

【參考上題】

13 從公式 41A 得

$$a+b+c=2s=(a+b)\cos C+(b+c)\cos A+(c+a)\cos B$$

【此題如用 41B 做便甚繁】

14. 左邊 $=\Sigma\left(2bc\cdot\dfrac{b^2+c^2-a^2}{2bc}\right)=\Sigma(b^2+c^2-a^2)=\Sigma a^2$

15. 左邊 $=\dfrac{b^2(b^2+c^2-a^2)}{2abc}+\dfrac{c^2(a^2+c^2-b^2)}{2abc}+\dfrac{a^2(a^2+b^2-c^2)}{2abc}$

$\qquad =\dfrac{\Sigma a^4}{2\Pi a}$

16. $\because$ $\qquad a=b\cos C+c\cos B$ $\qquad$ (1)

$\qquad\qquad b=c\cos A+a\cos C$ $\qquad$ (2)

$a(1)-b(2),\quad a^2-b^2=c(a\cos B-b\cos A)$

【用 41B 做亦不繁】

17. $c^2=a^2+b^2-2ab\cos C$

$\qquad =a^2(\sin^2\tfrac{1}{2}C+\cos^2\tfrac{1}{2}C)+b^2(\sin^2\tfrac{1}{2}C+\cos^2\tfrac{1}{2}C)$

$\qquad\quad -2ab(\cos^2\tfrac{1}{2}C-\sin^2\tfrac{1}{2}C)$

$\qquad =(a^2+2ab+b^2)\sin^2\tfrac{1}{2}C+(a^2-2ab+b^2)\cos^2\tfrac{1}{2}C$

$\qquad =(a+b)^2\sin^2\tfrac{1}{2}C+(a-b)^2\cos^2\tfrac{1}{2}C$

18. 從 $\quad\tan\dfrac{A}{2}=\dfrac{r}{s-a},\quad\tan\dfrac{B}{2}=\dfrac{r}{s-b},\quad\tan\dfrac{C}{2}=\dfrac{r}{s-c}$

$\therefore\quad (s-a)\tan\dfrac{A}{2}=(s-b)\tan\dfrac{B}{2}=(s-c)\tan\dfrac{C}{2}\ (=r)$

19. 從
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\therefore \quad \frac{\Pi a}{\Pi \sin A} = 8R^3 \qquad \therefore \quad R = \frac{1}{2}\left(\frac{\Pi a}{\Pi \sin A}\right)^{\frac{1}{3}}$$

20. $\because \quad \tan\frac{A}{2} = \frac{r}{s-a}, \quad \tan\frac{B}{2} = \frac{r}{s-b}, \quad \tan\frac{C}{2} = \frac{r}{s-c}$

$$\therefore \quad \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{r^3}{(s-a)(s-b)(s-c)} = \frac{sr^3}{S^2}$$

$$(\because \quad S^2 = s(s-a)(s-b)(s-c) \text{也})$$

21. 左邊 $= \dfrac{s(s-b)(s-c)+s(s-a)(s-c)}{\triangle^2}$

$$+\frac{s(s-a)(s-b)-(s-a)(s-b)(s-c)}{\triangle^2}$$

$$= \frac{2s^3-as^2-bs^2-cs^2+abc}{\triangle^2}$$

$$= \frac{s^2(a+b+c-a-b-c)+abc}{\triangle^2}$$

$$= \frac{abc}{\triangle^2} = \frac{abc}{\triangle} \cdot \frac{1}{\triangle} = \frac{4R}{\triangle}$$

22. $\because \quad$ 左邊 $= \dfrac{2-2\cos(A-B)\cos(A+B)}{2-2\cos(A-C)\cos(A+C)}$

$$= \frac{2-(\cos 2A+\cos 2B)}{2-(\cos 2A+\cos 2C)} = \frac{2\sin^2 A+2\sin^2 B}{2\sin^2 A+2\sin^2 C}$$

$$= \frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C} = \frac{a^2+b^2}{a^2+c^2}$$

$$\left(\because \quad \frac{a^2}{\sin^2 A} = \frac{b^2}{\sin^2 B} = \frac{c^2}{\sin^2 C} = \frac{a^2+b^2}{\sin^2 A+\sin^2 B}\right.$$

$$= \frac{a^2+c^2}{\sin^2 A+\sin^2 C} \quad \text{也}\bigg)$$

23 令
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

則 $a = k \sin A, \quad b = k \sin B, \quad c = k \sin C$

左邊 $= k^2 \Sigma \sin A \sin (B-C)$

$= \frac{1}{2} k^2 \Sigma 2 \sin(B+C) \sin(B-C)$

$= \frac{1}{2} k^2 \Sigma (\cos 2C - \cos 2B) = \frac{1}{2} k^2 \cdot 0 = 0$

或化左邊 $= k \Sigma a \sin(B-C) = k \cdot 0 = 0$

(從 266 頁例三)

24. 今 $\dfrac{a^2 \sin(B-C)}{\sin B + \sin C} = \dfrac{k^2 \sin^2 A \sin(B-C)}{\sin B + \sin C}$

$$= \frac{k^2 \sin A \sin(B+C) \sin(B-C)}{\sin B + \sin C}$$

$$= \frac{k^2 \sin A (\cos 2C - \cos 2B)}{2(\sin B + \sin C)}$$

$$= \frac{k^2 \sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C}$$

$$= k^2 \sin A (\sin B - \sin C)$$

$\therefore \ \Sigma \dfrac{a^2 \sin(B-C)}{\sin B + \sin C} = k^2 \Sigma \sin A (\sin B - \sin C) = k^2 \cdot 0 = 0$

25.

左邊 $= \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B - \sin^2 A & \cot B - \cot A & 0 \\ \sin^2 C - \sin^2 A & \cot C - \cot A & 0 \end{vmatrix}$

$\begin{vmatrix} \sin^2 B - \sin^2 A & \cot B - \cot A \\ \sin^2 C - \sin^2 A & \cot C - \cot A \end{vmatrix}$

$$= -\begin{vmatrix} \sin(A+B)\sin(A-B) & \dfrac{\sin(A-B)}{\sin A \sin B} \\ \sin(A+C)\sin(A-C) & \dfrac{\sin(A-C)}{\sin A \sin C} \end{vmatrix}^{*}$$

$$= -\frac{\sin(A-B)\sin(A-C)}{\sin A}\begin{vmatrix} \sin C & \dfrac{1}{\sin B} \\ \sin B & \dfrac{1}{\sin C} \end{vmatrix} = 0$$

或從 $\Sigma \sin^2 A(\cot B - \cot C) = \cdots\cdots\cdots$

$$= \Sigma \sin^3 A \sin(C-B)/\sin A \sin B \sin C$$

$$= \frac{\Sigma(1-\cos 2A)(\cos 2B - \cos 2C)}{4\sin A \sin B \sin C} = \cdots\cdots = 0$$

26.
$$左邊 = \begin{vmatrix} a & a^2 & \dfrac{s(s-a)a}{abc} \\ b & b^2 & \dfrac{s(s-b)b}{abc} \\ c & c^2 & \dfrac{s(s-c)c}{abc} \end{vmatrix} = \frac{s\,abc}{abc}\begin{vmatrix} 1 & a & s-a \\ 1 & b & s-b \\ 1 & c & s-c \end{vmatrix}$$

$$= s\begin{vmatrix} 1 & a & s \\ 1 & b & s \\ 1 & c & s \end{vmatrix} = s^2\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0$$

27. 原式 $= \Sigma(a-b)\cdot\dfrac{s-c}{r}$ $\quad\left(\because \cot\dfrac{C}{2} = 1 \ \tan\dfrac{C}{2} = \dfrac{s-c}{r}\right)$

$$= \frac{1}{r}\Sigma(a-b)(s-c) = \frac{1}{r}\Sigma[s(a-b) - c(a-b)] = 0$$

---

* 因　$\sin^2 B - \sin^2 A = \tfrac{1}{2}[(1-2\sin^2 A) - (1-2\sin^2 B)]$

$$= \tfrac{1}{2}(\cos 2A - \cos 2B) = -\sin(A+B)\sin(A-B)$$

從原式 $=2R\Sigma(\sin A - \sin B)\cot\dfrac{C}{2}$ 做較繁，若以附錄一

公式 $43C$ 代入做更繁．

28. 左邊 $=\Sigma\dfrac{a\cos C+c\cos A-2a\cos C}{a\sin C}=\Sigma\dfrac{c\cos A-a\cos C}{a\sin C}$

$\qquad =\Sigma\left(\dfrac{\sin C\cos A}{\sin A\sin C}-\dfrac{\cos C}{\sin C}\right)=\Sigma(\cot A-\cot C)=0$

【或用公式 $41B$ 可推到 $\dfrac{2R}{abc}\Sigma(c^2-a^2)$】

29. 左邊 $=\Sigma\dfrac{a\cdot 2\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(B+C)}{\sin A}$

$\qquad =2R\Sigma[2\sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(B+C)]$

$\qquad =2R\Sigma(\cos C-\cos B)=2R\cdot 0=0$

30. $\Sigma a\cos A=\dfrac{1}{2abc}\Sigma a^2(b^2+c^2-a^2)=\dfrac{1}{2abc}(2\Sigma a^2 b^2-\Sigma a^4)$

$\qquad =\dfrac{(a+b+c)(a-b+c)(a+b-c)(-a+b+c)}{2abc}$

$\qquad =\dfrac{16s(s-a)(s-b)(s-c)}{2abc}=\dfrac{16S^2}{2abc}=\dfrac{2S}{R}$

從公式 $41A$ 做較繁且須用到 $\Sigma\cos A=4\Pi\sin\frac{1}{2}A+1$

31. $\Sigma a\sin A=\dfrac{1}{2R}\Sigma a\cdot 2R\sin A=\dfrac{1}{2R}\Sigma a\cdot a=\dfrac{\Sigma a^2}{2R}$

32. 今 左邊 $=\dfrac{2R\Sigma\sin A\sin A}{2R\Sigma\sin A\cos A}=\dfrac{\Sigma 2\sin^2 A}{\Sigma\sin 2A}$

但 $\Sigma 2\sin^2 A=4+4\cos A\cos B\cos C$ （143 頁例七）

又 $\Sigma\sin 2A=4\sin A\sin B\sin C$ （145 頁例十一）

$$\therefore \quad 左邊 = \frac{4 + 4\Pi \cos A}{4\Pi \sin A}$$

$$= \frac{1 + [\cos(A+B+C) + \Sigma \sin A \sin B \cos C]^*}{\Pi \sin A}$$

$$= \frac{\Sigma \sin A \sin B \cos C}{\Pi \sin A} = \Sigma \cot A$$

$$* \quad \because \quad \cos(A+B+C) = \Pi \cos A - \Sigma \sin A \sin B \cos C$$

$$(77 \ 頁題 \ n \ 公式 \ 15)$$

33. $\quad 左邊 = \Sigma \dfrac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)}{ab \cdot 2bc \cdot 2ac}$

$$= \Sigma \frac{c^4 - (a^2 - b^2)^2}{4a^2b^2c^2} = \frac{-\Sigma a^4 + 2\Sigma a^2 b^2}{4a^2 b^2 c^2}$$

$$= \frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4a^2 b^2 c^2}$$

$$= \frac{16s(s-a)(s-b)(s-c)}{4 \cdot 16R^2 S^2} = \frac{4S^2}{16R^2 S^2} = \frac{1}{4R^2}$$

$$\left(\because \quad S = \frac{abc}{4R} \right)$$

34. $\quad 左邊 = 2R\Sigma a^2 \sin(B+C)\cos(B-C)$

$$= R\Sigma a^2 (\sin 2B + \sin 2C)$$

$$= 2R\Sigma a^2 (\sin B \cos B + \sin C \cos C)$$

$$= \Sigma a^2 (b \cos B + c \cos C)$$

$$= \Sigma a^2 \left[\frac{b^2(c^2 + a^2 - b^2)}{2abc} + \frac{c^2(a^2 + b^2 - c^2)}{2abc} \right]$$

$$= \frac{\Sigma [a^2 (2b^2 c^2 - b^4 - c^4) + a^4 (b^2 + c^2)]}{2abc}$$

$$= \frac{\Sigma(2a^2 b^2 c^2)}{2abc} = \frac{6a^2 b^2 c^2}{2abc} = 3abc$$

35. 今 $(a+b+c)^2(\Sigma \cot A) = (\Sigma a)^2 \cdot \dfrac{\Sigma a \sin A}{\Sigma a \cos A}$ （見題 32）

$$= (\Sigma a)^2 \cdot \dfrac{(\Sigma a^2) \cdot R}{2R \cdot 2S} \qquad (\text{見題 30 及 31})$$

$$= (\Sigma a)^2 \cdot \dfrac{\Sigma a^2}{4S} = (\Sigma a^2) \cdot \dfrac{(\Sigma a)^2}{4S} = (\Sigma a^2) \cdot \dfrac{s(\Sigma a)^2}{4s^2 r}$$

$$= (\Sigma a^2) \cdot \dfrac{s}{r} = (\Sigma a^2) \cdot \dfrac{(s-a)+(s-b)+(s-c)}{r}$$

$$= (\Sigma a^2)\left(\Sigma \cot \dfrac{A}{2}\right) \qquad\qquad \text{故如題云.}$$

36. 今 $\Sigma a^2 \cot A = 2R\Sigma a \cos A = 2R\Sigma a \cdot \dfrac{b^2+c^2-a^2}{2bc}$

$$= \dfrac{2R}{2abc} \Sigma(a^2 b^2 + a^2 c^2 - a^4)$$

$$= \dfrac{2R}{2abc}(2\Sigma a^2 b^2 - \Sigma a^4)$$

$$= \dfrac{2R}{2abc}(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

$$= \dfrac{16R\triangle^2}{abc} = \dfrac{16R\triangle^2}{4R\triangle} = 4\triangle$$

或原式 $= \dfrac{a}{\sin A} \cdot \Sigma a \cos A = 2R(2\triangle/R) = 4\triangle$

（見題 30）

37. 右邊 $= \dfrac{abc}{S} \cdot \dfrac{S}{s} \cdot \sqrt{\dfrac{s^3(s-a)(s-b)(s-c)}{a^2 b^2 c^2}}$

$$= \sqrt{s(s-a)(s-b)(s-c)} = S \qquad \text{故如題云.}$$

或從 $\quad S = sr = r \cdot \dfrac{a+b+c}{2}$

$$= \frac{r}{2}(2R\sin A + 2R\sin B + 2R\sin C)$$

$$= Rr(\Sigma\sin A) = Rr\left(4\Pi\cos\frac{A}{2}\right) \quad \text{(公式 50 A)}$$

$$= 4Rr\Pi\cos\frac{A}{2}$$

38. $\Sigma a\cos A = 2R\Sigma\sin A\cos A = R\Sigma\sin 2A$
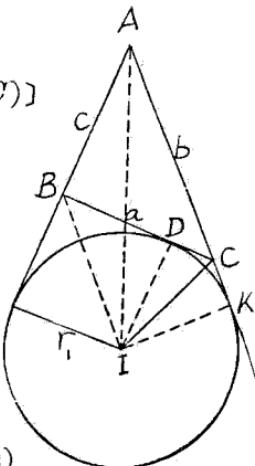
$$= R[2\sin A\cos A$$

$$+ 2\sin(B+C)\cos(B-C)]$$

$$= R(2\sin A)[\cos(B-C)$$

$$- \cos(B+C)]$$

$$= 4R\sin A\sin B\sin C$$

39. $\triangle = \triangle IAB + \triangle IAC - \triangle IBC$

$$= \frac{1}{2}(r_1 b + r_1 c - r_1 a) = r_1(s-a)$$

同理　$\triangle = r_2(s-b) = r_3(s-c)$

故　$r_1(s-a) = r_2(s-b) = r_3(s-c)$

40. 由前題　$r_1 = \frac{\triangle}{s-a}$,　$r_2 = \frac{\triangle}{s-b}$,　$r_3 = \frac{\triangle}{s-c}$,　$r = \frac{\triangle}{s}$

$$\therefore \quad r_1 + r_2 + r_3 - r = \frac{\triangle}{s-a} + \frac{\triangle}{s-b} + \frac{\triangle}{s-c} - \frac{\triangle}{s}$$

$$= \triangle \cdot \frac{\Sigma s(s-b)(s-c) - \Pi(s-a)}{s(s-a)(s-b)(s-c)}$$

$$= \frac{abc}{\triangle} = \frac{4R\triangle}{\triangle} = 4R \quad \text{(見題 21)}$$

41. $\because \quad \triangle = r_1(s-a)$ 　　　$\therefore \quad \frac{1}{r_1} = \frac{s-a}{\triangle}$

$$\Sigma \frac{1}{r_1} = \frac{(s-a)+(s-b)+(s-c)}{\triangle} = \frac{s}{\triangle} = \frac{s}{sr} = \frac{1}{r}$$

或由題 39 之圖 $a = r_1 \cot ICD + r_1 \cot IBD$

$$= r_1 \left(\cot \frac{\pi - C}{2} + \cot \frac{\pi - B}{2} \right)$$

$$= r_1 \left(\tan \frac{C}{2} + \tan \frac{B}{2} \right)$$

$$= r_1 \left(\frac{r}{s-b} + \frac{r}{s-c} \right) = \frac{ar_1 r}{(s-b)(s-c)}$$

$$\therefore \quad \frac{1}{r_1} = \frac{r}{(s-b)(s-c)}, \quad \cdots\cdots\cdots$$

$$\therefore \quad \Sigma \frac{1}{r_1} = \Sigma \frac{r}{(s-b)(s-c)} = \frac{r(3s-a-b-c)}{\Pi(s-a)}$$

$$= \frac{rs}{\Pi(s-a)} = r \cdot \frac{1}{r^2} = \frac{1}{r}$$

42. $\qquad \because \quad \dfrac{a^2}{\sin A} = a \cdot \dfrac{a}{\sin A} = a \cdot 2R$

$$\therefore \quad \Sigma \frac{a^2}{\sin A} = 2R \cdot \Sigma a = 4sR$$

又 $\quad \because \quad \sin \dfrac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

$$\therefore \quad \Pi \sin \frac{1}{2} A = \frac{(s-a)(s-b)(s-c)}{abc}$$

故 $\quad \left(\Sigma \dfrac{a^2}{\sin A} \right) \Pi \sin \dfrac{1}{2} A = \dfrac{4s(s-a)(s-b)(s-c)R}{abc}$

$$\sqrt{s(s-a)(s-b)(s-c)} = \triangle, \quad \frac{abc}{4R} = \triangle$$

$$\therefore \quad \left(\Sigma \frac{a^2}{\sin A} \right) \Pi \sin \frac{1}{2} A = \triangle^2 \cdot \frac{1}{\triangle} = \triangle$$

43. $\Sigma(b^2-c^2)\cot^2\dfrac{A}{2}=\Sigma(b^2-c^2)\dfrac{(s-a)^2}{r^2}$

$$=\dfrac{1}{r^2}\Sigma(b^2-c^2)(-2as)\quad *$$

$$=-\dfrac{2s}{r^2}\Sigma(b^2-c^2)a=-\dfrac{2s}{r^2}\Pi(a-b)$$

$$=-\dfrac{2s^3\Pi(a-b)}{s\,r^2}=-\dfrac{2s^3\Pi(a-b)}{\triangle^2}$$

$$\therefore\quad\Sigma(b^2-c^2)\cot^2\dfrac{A}{2}+\dfrac{2s^3\Pi(a-b)}{\triangle^2}=0$$

$*\quad\because\quad\Sigma(b^2-c^2)s^2=0,\quad\Sigma(b^2-c^2)a^2=0$

44. $\because\quad\cot\gamma=\cot A+\cot B+\cot C\qquad$（見 275 頁例 15）

$\therefore\quad\cot^2\gamma=(\Sigma\cot A)^2$
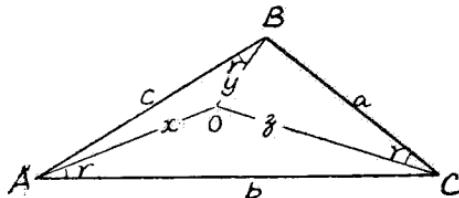
故　$\csc^2\gamma-1=\Sigma\cot^2A+2\Sigma\cot A\cot B$

$$=\Sigma\cot^2A+2=\Sigma\csc^2A-3+2$$

$$\therefore\quad\csc^2\gamma=\Sigma\csc^2A$$

或從　$\dfrac{\triangle OAB}{\triangle ABC}=\dfrac{cy\sin\gamma}{ca\sin B}=\dfrac{\sin^2\gamma}{\sin^2B}=\dfrac{\csc^2B}{\csc^2\gamma}$

同理　$\dfrac{\triangle OBC}{\triangle ABC}=\dfrac{\csc^2C}{\csc^2\gamma},\quad\dfrac{\triangle OCA}{\triangle ABC}=\dfrac{\csc^2A}{\csc^2\gamma}$

$$\therefore\quad \frac{\triangle OAB+\triangle OBC+\triangle OCA}{\triangle ABC}$$

$$=1=\frac{\csc^2 A+\csc^2 B+\csc^2 C}{\csc^2 \gamma}\qquad \text{故如題云．}$$

45. （1）　$\cot\dfrac{A}{2}-\cot\dfrac{B}{2}=\dfrac{s-a}{r}-\dfrac{s-b}{r}=\dfrac{b-a}{r}$

$\cot\dfrac{B}{2}-\cot\dfrac{C}{2}=\dfrac{s-b}{r}-\dfrac{s-c}{r}=\dfrac{c-b}{r}$

今　a,b,c 成 $A.P.$　　則　$b-a=c-b$

$\therefore\ \cot\dfrac{A}{2}-\cot\dfrac{B}{2}=\cot\dfrac{B}{2}-\cot\dfrac{C}{2}$　　故如題云．

（2）$\cos A\cot\dfrac{A}{2}-\cos B\cot\dfrac{B}{2}$

$$=\left(1-2\sin^2\dfrac{A}{2}\right)\cot\dfrac{A}{2}-\left(1-2\sin^2\dfrac{B}{2}\right)\cot\dfrac{B}{2}$$

$$=\left(\cot\dfrac{A}{2}-\cot\dfrac{B}{2}\right)-\left(2\sin\dfrac{A}{2}\cos\dfrac{A}{2}-2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right)$$

$$=\left(\cot\dfrac{A}{2}-\cot\dfrac{B}{2}\right)-(\sin A-\sin B)$$

同理　$\cos B\cot\dfrac{B}{2}-\cos C\cot\dfrac{C}{2}$

$$=\left(\cot\dfrac{B}{2}-\cot\dfrac{C}{2}\right)-(\sin B-\sin C)$$

今　a,b,c 成 $A.P.$　　即　$b-a=c-b$

$\therefore\ 2R(\sin A-\sin B)=2R(\sin B-\sin C)$

即　　　$\sin A-\sin B=\sin B-\sin C$

且由（1）知　$\cot\dfrac{A}{2}-\cot\dfrac{B}{2}=\cot\dfrac{B}{2}-\cot\dfrac{C}{2}$

故　　　　$\cos A \cot\dfrac{A}{2} - \cos B \ \mathrm{cot}\dfrac{B}{2}$

$$= \cos B \cot\dfrac{B}{2} - \cos C \cot\dfrac{C}{2} \qquad \text{故如題云.}$$

46. $\dfrac{1}{a \sec A} - \dfrac{1}{b \sec B} = \dfrac{\cos A}{a} - \dfrac{\cos B}{b}$

$$= \dfrac{b^2 + c^2 - a^2 - c^2 - a^2 + b^2}{2abc} = \dfrac{b^2 - a^2}{abc}$$

同理　　　　$\dfrac{1}{b \sec B} - \dfrac{1}{c \sec C} = \dfrac{c^2 - b^2}{abc}$

今 a^2, b^2, c^2 成 $A.P.$　　　　$\therefore \quad b^2 - a^2 = c^2 - b^2$

$\therefore \quad \dfrac{1}{a \sec A} - \dfrac{1}{b \sec B} = \dfrac{1}{b \sec B} - \dfrac{1}{c \sec C}$　　　　故如題云.

47. $\because \quad \cos(A+B) + \cos(B+C) = -\cos C - \cos A$

$\therefore \quad \cos A \cos B - \sin A \sin B + \cos B \cos C - \sin B \sin C$

$$= -\cos C - \cos A$$

$\therefore \quad (1 + \cos B)(\cos C + \cos A) = \sin B(\sin A + \sin C)$

$\therefore \quad 2 \cos^2\dfrac{B}{2} = \dfrac{\sin B(\sin A + \sin C)}{\cos A + \cos C} = \dfrac{2 \sin A \sin C}{\cos A + \cos C}$ *

$$\therefore \quad \cos\dfrac{B}{2} = \sqrt{\dfrac{\sin A \sin C}{\cos A + \cos C}}$$

$${}^{*}\!\left(\because \quad b = \dfrac{2ac}{a+c} \qquad \therefore \quad \sin B = \dfrac{2 \sin A \sin C}{\sin A + \sin C} \right)$$

【此題從右邊推起較簡但不及上法有趣也】

48. $\because \quad c = \dfrac{a-b}{\cos\theta} \qquad \therefore \quad \cos\theta = \dfrac{a-b}{c}, \ \sin\theta = \dfrac{\sqrt{c^2 - (a-b)^2}}{c}$

$$\therefore \quad \tan \theta = \frac{\sqrt{c - (a - b)^2}}{a - b}$$

$$\therefore \quad (a - b)\tan \theta = \sqrt{(c + a - b)(c - a + b)}$$

$$= \sqrt{4(s - b)(s - a)}$$

$$= 2\sqrt{\frac{(s - b)(s - a)}{ba}} \cdot \sqrt{ba}$$

$$= 2\sqrt{ba}\, \sin\frac{C}{2}$$

$$\therefore \quad \sin\frac{C}{2} = \frac{(a - b)\tan \theta}{2\sqrt{ab}}$$

49. $\Pi \sin\dfrac{A}{2} = \dfrac{(s - a)(s - b)(s - c)}{abc}$

$$= \frac{(-a + b + c)(a - b + c)(a + b - c)}{8abc}$$

從代數不等式　　$(x + y)(y + z)(z + x) > 8xyz$

設　$x = -a + b + c, \quad y = a - b + c, \quad z = a + b - c$

則　$x + y = 2c, \qquad y + z = 2a, \qquad z + x = 2b$

$\therefore \quad 8abc > 8(-a + b + c)(a - b + c)(a + b - c)$

故　$\dfrac{(-a + b + c)(a - b + c)(a + b - c)}{8abc} < \dfrac{1}{8}$

卽　　　　　　　　$\Pi \sin\dfrac{A}{2} < \dfrac{1}{8}$

50　$\Sigma \sin 2A = 2\sin A \cos A + 2\sin(B + C)\cos(B - C)$

$$= 2\sin A[\cos(B - C) - \cos(B + C)]$$

$$= 4\Pi \sin A \tag{1}$$

設　$\sin^{-1}\alpha = A, \quad \sin^{-1}\beta = B, \quad \sin^{-1}\gamma = C$

∴　$\sin A = \alpha$,　　　$\sin B = \beta$,　　　$\sin C = \gamma$,

$\cos A = \sqrt{1-\alpha^2}$, $\cos B = \sqrt{1-\beta^2}$, $\cos C = \sqrt{1-\gamma^2}$

今　$A+B+C = \pi$　∴　$\sin^{-1}\alpha + \sin^{-1}\beta + \sin^{-1}\gamma = \pi$

從(1),知　$\Sigma \sin A \cos A = 2\Pi \text{ in } A$

∴　$\Sigma \alpha \sqrt{1-\alpha^2} = 2\alpha\beta\gamma$

習 題 二 十 三 (307-317 頁)
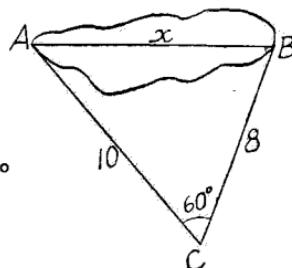
1. 設湖長 $= AB = x$ 里, C 爲測點

則　$\angle ACB = 60°$,

$CA = 10$ 里, $CB = 8$ 里

∴　$x^2 = 10^2 + 8^2 - 2 \cdot 10 \cdot 8 \cos 60°$

$= 164 - 80 = 84$

∴　$x = \sqrt{84} = 9.165$ 里　　故湖長 9.165 里.

2. 設 B, C 爲兩燈塔, x 浬爲 BC
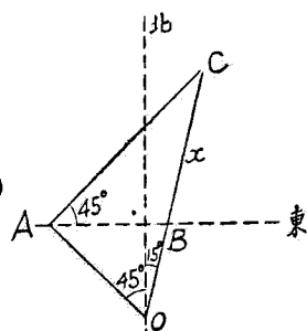
之距離, 則　$AO = 5$ 浬

今　　$x = OC - OB$

又　$OC = \dfrac{5}{\cos \widehat{AOC}} = \dfrac{5}{\cos 60°} = 10$

$OB = 5 \cdot \dfrac{\sin 45°}{\sin 75°} = 5 \times \dfrac{1}{\sqrt{2}}$
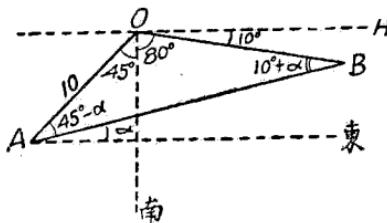
$\times \dfrac{2\sqrt{2}}{\sqrt{3}+1} = 5(\sqrt{3}-1)$

∴　$x = 10 - 5\sqrt{3} + 5 = 5(3 - \sqrt{3}) = 6.3395$ 浬

3　設 O 為港口，OB 為敵船行程，B 為敵船被捕之處，A 為艦所在處，設 x 浬為兵艦每時之速度，其方向為東 $\alpha°$ 北，今於 3/2 小時追及，則

$$AB = \frac{3x}{2} \text{浬}, \quad OB = \frac{3}{2} \cdot 9 = 27/2 \text{浬}, \quad AO = 10 \text{浬}$$

$\because \quad \angle AOB = 45° + 80° = 125°$

$\therefore \quad \overline{AB}^2 = \overline{AO}^2 + \overline{OB}^2 - 2\overline{AO} \cdot \overline{OB} \cos 125°$

即 $\quad \dfrac{9}{4}x^2 = 100 + \left(\dfrac{27}{2}\right)^2 + 270 \cos 55°$

$$x^2 = \frac{1}{9}(400 + 27^2 + 1080 \sin 35°)$$

$$= \frac{1}{9}(1129 + 619.488) = 194.3$$

$\therefore \quad x = \sqrt{194.3} = 13.9$

又 $\quad \dfrac{AO}{\sin(10° + \alpha)} = \dfrac{AB}{\sin 125°} = \dfrac{3}{2}(13.9)\dfrac{1}{.8192}$

$\therefore \quad \sin(10° + \alpha) = \dfrac{2}{3}\left(\dfrac{.8192}{13.9}\right)10 = 0.3918$

$\therefore \quad 10° + \alpha = 23°4' \qquad$ 即 $\quad \alpha = 13°4'$

故速度為 13.9 浬/時，其方向為東 $13°4'$ 北(或北 $76°56'$ 東)。

4. 設塔高 AB 爲 h 尺，又 C,D 爲兩測點

在 $\triangle CBA$ 中，$AC = h \cot 45° = h$

在 $\triangle BAD$ 中，$AD = h \cot 30°$

$$= \sqrt{3}\,h$$
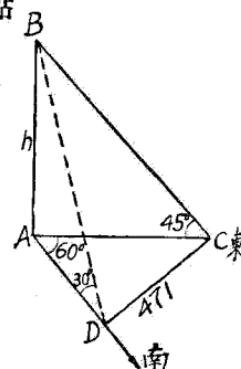
今 $\overline{CD}^2 = \overline{AD}^2 + \overline{AC}^2$

$$- 2\overline{AD} \cdot \overline{AC} \cos 60°$$

即 $471^2 = 3h^2 + h^2 - \sqrt{3}\,h^2$

$$= (4 - \sqrt{3})h^2$$

故所求塔高 $h = \dfrac{471}{\sqrt{4 - \sqrt{3}}} = \dfrac{471}{1.5} = 314$ 尺

5. 設測點之高 $DA = x$ 尺

山高 $EB = x + l = 6400$ 尺
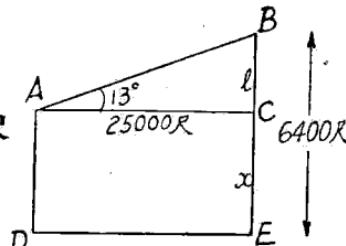
$AC = .5(50000) = 25000$ 尺

$l = 25000 \tan 13°$

$$= 25000(.2309)$$

$$= 5772$ 尺$$

$\therefore x = 6400 - 5772 = 628$ 尺

6. 設山高 AB 爲 h; x 爲 C,D 兩
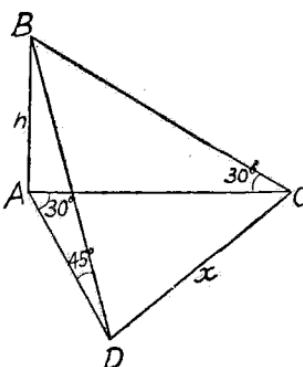
點之距離，則

$\angle BCA = 30°$, $\angle BDA = 45°$,

及 $\angle CAD = 30°$

$\therefore AD = h \tan 45° = h$,

$AC = h \sqrt{3}$

$$\therefore \quad x^2 = h^2 + 3h^2 - 2\sqrt{3}\,h^2\cos 30°$$

$$= 4h^2 - 2\sqrt{3}\,h^2 \cdot \frac{\sqrt{3}}{2} = h^2$$
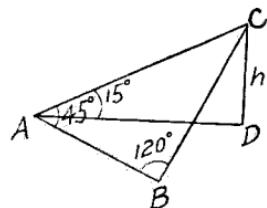
$$\therefore \quad x = h \qquad 即 \quad AB = CD$$

7. 今 $\angle ACB = 15°$, $AB = 1$ 里

設山高 $DC = h$ 里

則在 $\triangle ABC$ 中

$$AC = \frac{AB \sin 120°}{\sin 15°} = \frac{\frac{1}{2}\sqrt{3}}{\sin 15°}$$
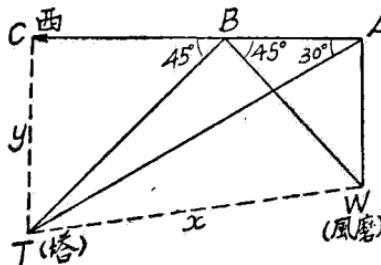
又 $h = AC \sin 15° = \frac{1}{2}\sqrt{3} = 0.866$ 里

8. 設塔與風磨之距離 $TW = x$ 里,塔與路之距離 $TC = y$ 里

今 $\angle ABT = 135°$, $\angle ATB = 15°$, $AB = 1$ 里

在 $\triangle TAB$ 中, $\dfrac{bT}{\sin 30°} = \dfrac{AB}{\sin 15°}$

$$\therefore \quad BT = \frac{1}{2\sin 15°} = \frac{1}{.5176} = 1.9319$$
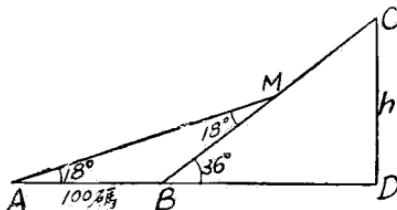
故塔與路之距 $y = BT \sin 45° = 1.9319 \times .7071 = 1.37$ 里

又 $\angle TBW = 180° - 45° - 45° = 90°$

故塔與風磨之距　$x = \sqrt{(BT)^2 + (WB)^2}$

$$= \sqrt{(1.932)^2 + (\sqrt{2})^2} = 2.39 \ \text{里}$$

9. 設山高 $DC = h$ 碼

今　$MC = MB$，　$\angle AMB = 36° - 18° = 18° = \angle MAB$

$\therefore$　$BM = AB = 100$ 碼，　　則　$BC = 200$ 碼

故　$h = BC \sin 36° = 200 \times .5878 = 117.56$

故所求山高爲 117.6 碼.

10. 今　$\angle ASC = 135°$，$AS = 1000$ 尺

設　$BC = h$ 尺

$\therefore$　$\dfrac{AC}{\sin 135°} = \dfrac{AS}{\sin 30°}$

$\therefore$　$AC = 1000\sqrt{2}$

$\therefore$　$h = AC \sin 47° = 1000\sqrt{2}$

　　　　　$\times .731 = 1034$ 尺
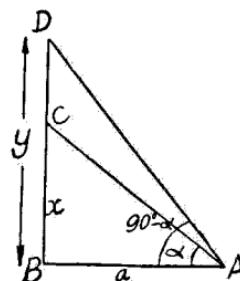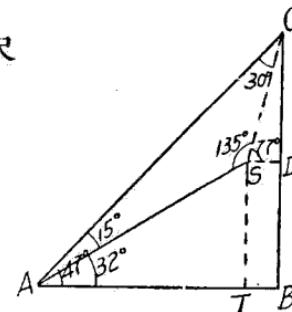
（參考 290 頁例十一）

11. 設塔高 $BC = x$ 尺，

　　桿頂高 $BD = y$ 尺

　　則　$x = a \tan a,$

　　　　$y = a \tan(90° - a) = \cot a$

$$\text{故桿長}=y-x=a(\cot\alpha-\tan\alpha)=\frac{a(\cot^2\alpha-1)}{\cot\alpha}$$

$$=2a\cot 2\alpha\ \text{尺}$$

12. 設 C 爲峭壁之頂,又

$$AB=x$$

則　$x=AD-BD$

$$=h(\cot\beta-\cot\alpha)$$

即　$x=h\left(\dfrac{\cos\beta}{\sin\beta}-\dfrac{\cos\alpha}{\sin\alpha}\right)$

$$=h\sin(\alpha-\beta)\csc\alpha\csc\beta$$

13. 今　$a=xE+Ey$

$$=h(\cot A+\cot B)$$

$$\therefore\ h=\frac{a}{\cot A+\cot B}$$

或　$h=a\Big/\left(\dfrac{\cos A}{\sin A}+\dfrac{\cos B}{\sin B}\right)$

$$=\frac{a\sin A\sin B}{\sin(A+B)}$$

14. 設塔高 DC 爲 h 尺
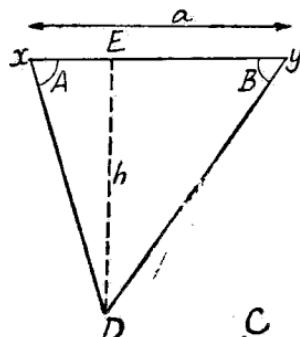
今　$AD=100$ 尺, $BD=100\sqrt{2}$ 尺

又設　$\angle DAC=\alpha$

則　　$\angle DBC=90°-\alpha$

今　$h=BD\tan(90°-\alpha)$

$$=100\sqrt{2}\cot\alpha$$

又　$h=100\tan\alpha$

兩式相乘得　　　$h^2 = 10000\sqrt{2}$

$\therefore\ h = 100\sqrt[4]{2} = 119.1$ 尺
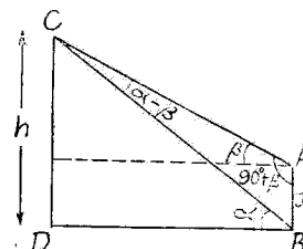
15. 如圖 BA 爲窗高，$DC = h$ 爲桿高

今　　$\angle CAB = 90° + \beta$

在 $\triangle ABC$ 中（參考 49 頁例四）

$$\frac{BC}{\sin(90° + \beta)} = \frac{a}{\sin(\alpha - \beta)}$$

$\therefore\ \ EC = \dfrac{a\cos\beta}{\sin(\alpha - \beta)}$
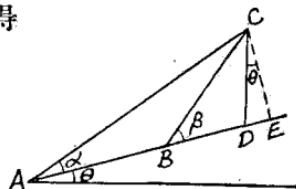
故　　　$h = \overline{BC}\sin\alpha = \dfrac{a\sin\alpha\cos\beta}{\sin(\alpha - \beta)}$

16. 設 $CD = h$，則由 288 頁例七得

$$\cos\theta = \frac{a\sin\alpha\sin\beta}{h\sin(\beta - \alpha)}$$

$$\therefore\ \ \theta = \cos^{-1}\frac{a\sin\alpha\sin\beta}{h\sin(\beta - \alpha)}$$

17. 設汽球高 DC 爲 h
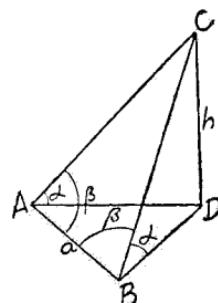
今　　$\angle CAD = \angle CBD\ \ \alpha$

$\therefore\ \ AC = BC(= h\csc\alpha)$

故　　$\triangle ACB$ 爲等腰三角形

$\therefore\ \ AC = \dfrac{a}{2}\sec\beta$

今　　$h = AC\sin\alpha = \dfrac{a}{2}\sin\alpha\sec\beta$

18. 今各邊於 P 所張角　$APB = BPC = CPA = 120°$

故兩鈍角 $\triangle APC$，BPC 全相等（SSA）
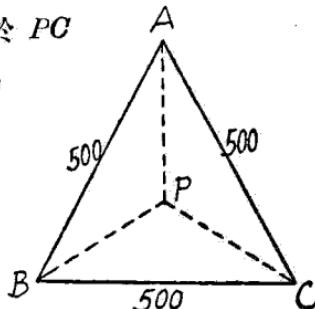
故 $PA = PB$，同理可證亦等於 PC

設此值爲 x 尺，則在 $\triangle PAC$ 中

$$\overline{AC}^2 = x^2 + x^2 - 2x^2 \cos 120°$$

卽　　　$3x^2 = 500^2$

$$\therefore \quad x = \frac{500}{3}\sqrt{3} = 288.67 \text{ 尺}$$
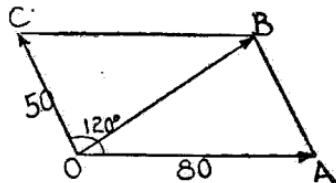
19 設 OA 代 80 磅之力，OB 代 50 磅

則 $\square OABC$ 中

$$\angle O = 120°, \quad \angle A = 60°$$

OB 表其合力之大小

則在 $\triangle OAB$ 中

$$\overline{OB}^2 = \overline{OA}^2 + \overline{OC}^2 - 2\overline{OA} \cdot \overline{OC} \cos 60°$$

$$= 6400 + 2500 - 2 \times 80 \times 50 \times \tfrac{1}{2} = 4900$$

$\therefore \quad OB = 70$（合力爲 70 磅）　　【參考 286 頁例四】

20. 如圖 $\angle CBD = 120° - 45° = 75°$，

$$\angle ACB = 15°$$
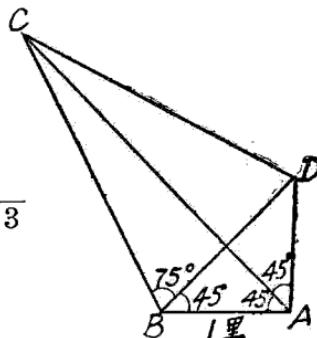
$$\therefore \quad BC = \frac{AB \sin 45°}{\sin 15°}$$

$$= 3 - 4 \sin^2 15°$$

$$= 1 + 2 \cos 30° = 1 + \sqrt{3}$$

又　$BD = \dfrac{\sin 90°}{\sin 45°} = \sqrt{2}$

今　$\overline{CD}^2 = \overline{BC}^2 + \overline{BD}^2 - 2\overline{BC} \cdot \overline{BD} \cos 75°$

$$=(1+\sqrt{3}\,)^2+(\sqrt{2}\,)^2-2(\sqrt{2}\,)\cdot\frac{\sin 45^\circ}{\sin 15^\circ}\cdot\sin 15^\circ$$

$$=(1+\sqrt{3}\,)^2+2-2=(1+\sqrt{3}\,)^2$$

$$\therefore\quad CD=1+\sqrt{3}=2.7\ \text{里}$$
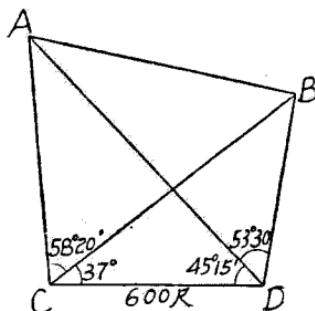
21. 今 $\angle CAD=39^\circ 25'$

在 $\triangle ACD,\ \triangle BCD$ 中

$$AC=\frac{600\sin 45^\circ 15'}{\sin 39^\circ 25'}=671.1$$

（此應用對數做）

$$BC=\frac{600\sin 98^\circ 45'}{\sin 44^\circ 15'}=849.8$$

在 $\triangle ABC$ 中今兩邊一夾角已知,可用正切定律

$$\tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B)=\frac{178.7}{1520.9}\tan 60^\circ 50'$$

$$A-B=23^\circ 46'\qquad\therefore\quad A=72^\circ 43',\qquad B=48^\circ 57'$$

再 $AB=\dfrac{AC\sin 58^\circ 20'}{\sin 48^\circ 57'}=\cdots\cdots\cdots=757.5\ \text{尺}$

22. 今 $AB=500,\ DA=EB=100,$

$DB=560,\ EA=550$
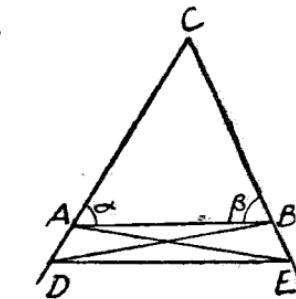
於 $\triangle ABD$ 中

$$560^2=100^2+500^2$$
$$-2\cdot 100\cdot 500\cos(\pi-a)$$

$$\therefore\quad 1000\cos a=536$$

$$\therefore\quad \cos a=0.536 \qquad\qquad 故\quad a=57^\circ 35'$$

又　　$550^2 = 500^2 + 100^2 - 2 \cdot 500 \cdot 100 \cos(\pi - \beta)$

∴　　$\cos \beta = .425$　　　　故　$\beta = 64°51'$,　　$C = 57°34'$

於 $\triangle ABC$ 中,　　　$\dfrac{AB}{\sin C} = \dfrac{AC}{\sin B}$

∴　　$AC = 500 \left(\dfrac{\sin 64°51'}{\sin 57°34'} \right) = 536.3$

23. 如圖 $EA = 2a$ 爲底之對角線之延長線

今　$YE = AE \tan 45° = AE = 2a$

卽　　$XD = ZF = 2a$

$AD = XD \cot 30° = 2\sqrt{3}\,a$

又　　$\angle DEA = 135°$

故在 $\triangle ADE$ 中

$\overline{AD}^2 = \overline{DE}^2 + \overline{EA}^2 - 2\overline{DE}$
　　　　$\times \overline{EA} \cos 135°$

卽　　$12a^2 = x^2 + 4a^2 + 2\sqrt{2}\,ax$

　　　（設　$DE = x$）

卽　　$x^2 + 2\sqrt{2}\,ax + 2a^2 = 10a^2$
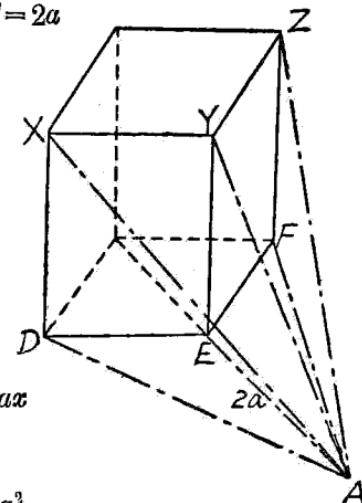
卽　　$(x + a\sqrt{2})^2 = 10a^2$　　　∴　$x + a\sqrt{2} = a\sqrt{10}$

∴　　$x = a(\sqrt{10} - \sqrt{2})$　　　　（負根不合）

24 設峯高 $MP = h$, 今從幾何知 $AM = BM$, 則中線 MC

垂直於 AB, 故

$$\overline{AM}^2 - \overline{MC}^2 = \overline{AC}^2 = a^2 \qquad (1)$$
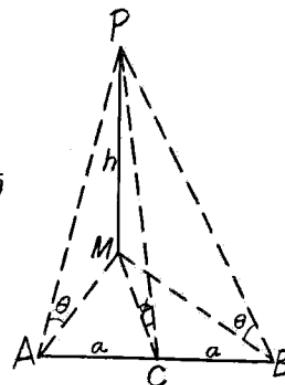
但　　　$AM = h \cot \theta,$　　　$MC = h \cot \phi$

故　$h^2(\cot^2\theta - \cot^2\phi) = a^2$

$\therefore\ h^2 = a^2 \Big/ \Big(\dfrac{\cos^2\theta}{\sin^2\theta} - \dfrac{\cos^2\phi}{\sin^2\phi}\Big)$

$\qquad = \dfrac{a^2\sin^2\theta\,\sin^2\phi}{\cos^2\theta\,\sin^2\phi - \cos^2\phi\,\sin^2\theta}$

$\qquad = \dfrac{a^2\sin^2\theta\,\sin^2\phi}{\sin(\theta+\phi)\sin(\phi-\theta)}$

故　$h = a\sin\theta\sin\phi$

$\qquad \times \sqrt{\csc(\phi+\theta)\csc(\phi-\theta)}$

25. 設　$AC = y$ 尺，烟囱高　$AB = CD = x$ 尺

則　$PC = x\cot 60° = x/\sqrt{3}$

$\therefore\ EC = x$

如圖　$PE = 80$ 尺

故　$x^2 = \dfrac{x^2}{3} + 80^2$

$\therefore\ x = 80\sqrt{\dfrac{3}{2}} = 40\sqrt{6}$ 尺

則　$PC = 80\sqrt{\dfrac{3}{2}}\Big(\dfrac{1}{\sqrt{3}}\Big) = 40\sqrt{2} = 56.57$ 尺
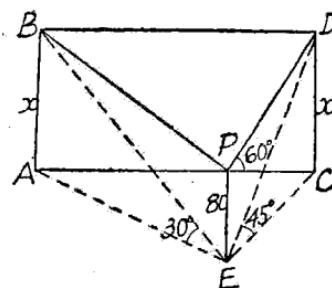
又　$AE = x\cot 30° = 120\sqrt{2}$
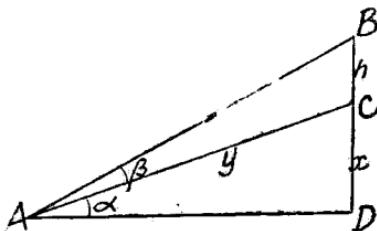
$\therefore\ AP = \sqrt{\overline{AE}^2 - \overline{PE}^2} = 40\sqrt{14} = 149.64$ 尺

故　$y = AC = AP + PC = 149.64 + 56.57 = 206.2$ 尺

26. 設塔高　CD 爲 x 尺，旗桿 CB 長 h 尺

又設　$AC = y$，今　$ACD = 90° - \alpha$，$B = 90° - \alpha - \beta$

$$\therefore \quad \frac{h}{\sin \beta} = \frac{y}{\sin[90° - (\alpha + \beta)]}$$

$$\therefore \quad y = h \csc \beta \cos(\alpha + \beta)$$

故 $x = y \sin \alpha = h \sin \alpha \csc \beta \cos(\alpha + \beta)$

27. 設 $EF = x$ 尺

今塔高爲 DC
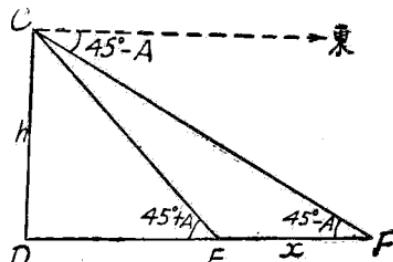
$\therefore \quad LF = h \cot(45° - A),$

$\quad DE = h \cot(45° + A)$

$\therefore \quad x = LF - DE$
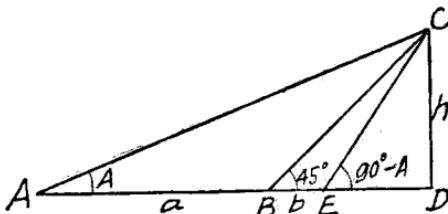
$$= h \left[\frac{\cos(45° - A)}{\sin(45° - A)} - \frac{\cos(45° + A)}{\sin(45° + A)} \right]$$

$$= \frac{2h \sin 2A}{\cos 2A - \cos 90°} = 2h \tan 2A \ \text{尺}$$

28 如圖 $ED = BD - BE = h - b$

又 $ED = h \cot(90° - A) = h \tan A$

$$\therefore \quad \tan A = (h-b)/h \qquad (1)$$
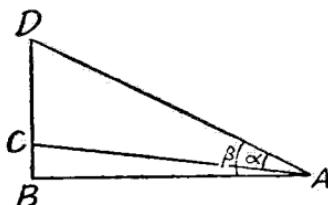
但　　　　　$\tan A = h/AD = h/(a+h) \qquad (2)$

故　$(h-b)/h = h/(a+h) \qquad \therefore \quad h = ab/(a-b)$ 尺

29. 今　$CD = \dfrac{3}{4}BD$

$$\therefore \quad BC = \dfrac{1}{4}BD$$

如圖　$\tan \alpha = 0.6$,

$$AB = BC \cot(\beta - \alpha) = BD \cot \beta$$

$$\therefore \quad \tan \beta = \frac{BD}{BC}\tan(\beta - \alpha), \quad 即 \quad \frac{1}{4}\tan \beta = \frac{\tan \beta - 0.6}{1 + 0.6\tan \beta}$$

即　　　　　$\tan^2\beta - 5\tan \beta + 4 = 0$

即　$(\tan \beta - 1)(\tan \beta - 4) = 0 \qquad \therefore \quad \tan \beta = 1, 4$

30. 設　$CA = x$ 尺,　$CB = y$ 尺,

塔高 $CD = h$ 尺

因　$\dfrac{h}{6} = \dfrac{x+24}{24}$

$\therefore \quad x = 4h - 24$

又　$\dfrac{h}{6} = \dfrac{y+30}{30}$
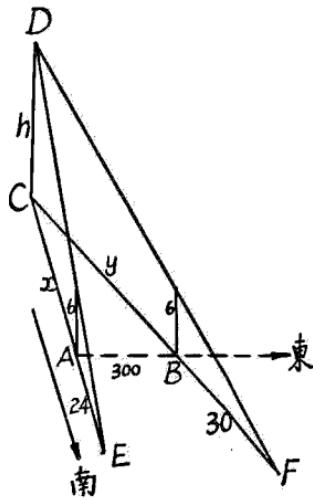
$\therefore \quad y = 5h - 30$

惟　$x^2 + 300^2 = y^2$

$(\because \quad \angle CAB = 90°)$

代入得 $16(h-6)^2 + 300^2$

$$= 25(h-6)^2$$

即　$h^2 - 12h - 9964 = 0$　　　即　$(h - 106)(h + 94) = 0$

∴　$h = 106$　　　（負值不合）

故燈塔之高度爲 106 尺.

31. 如圖塔高 OT 爲 h 尺，池半徑爲 r 尺，$BN = 45$ 尺，

$AE = 120$ 尺，則

$r = h \cot 60° = h / \sqrt{3}$

今　$\angle NOE = 90°$

∴　$\overline{ON}^2 + \overline{OE}^2 = \overline{NE}^2$

即　$\left(45 + \dfrac{h}{\sqrt{3}} \right)^2$

$+ \left(120 + \dfrac{h}{\sqrt{3}} \right)^2 = 375^2$

即　　$\dfrac{2h^2}{3} + \dfrac{330\sqrt{3}\,h}{3} = 124200$

∴　$h^2 + 165\sqrt{3}\,h - 186300 \quad 0$

即　　$(h - 180\sqrt{3})(h + 345\sqrt{3}) = 0$

∴　$h = 180\sqrt{3}$ 尺　　∴　$r = 180$ 尺
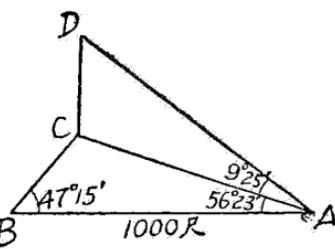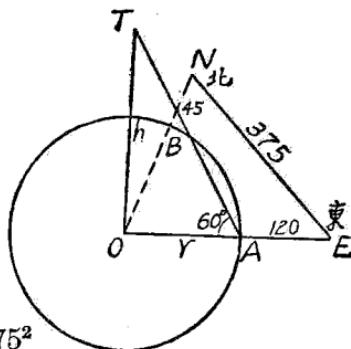
故圓池直徑爲 360 尺，塔高 $180\sqrt{3}$ 尺.

32. 設塔高 CD 爲 x 尺

今　$\angle ACB = 76°22'$

∴　$\dfrac{1000}{\sin 76°22'} = \dfrac{AC}{\sin 47°15'}$

∴　$AC = \dfrac{1000 \sin 47°15'}{\sin 76°22'}$

又　$h = AC \tan 9°25' = \dfrac{1000 \times .7343 \times .1658}{.9718} = 125.3$ 尺
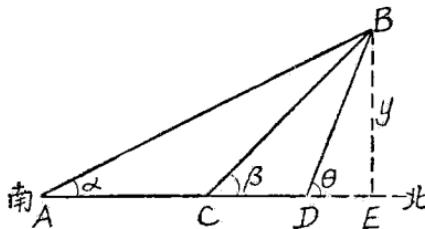
〔註〕　此題普通應用對數做

33. DB 為塔, 傾斜向北 θ 角, $CD = b$, $AD = a$

作 BE 垂直 AD 之延長線, 令 $EB = y$

則　　　　$AE = y \cot \alpha$,　　　$CE = y \cot \beta$

$\therefore$　　$AE - CE = AC = a - b = y(\cot \alpha - \cot \beta)$

又　$DE = y \cot \theta$　　$\therefore CE - DE = b = y(\cot \beta - \cot \theta)$

$\therefore$　$\dfrac{a - b}{b} = \dfrac{\cot \alpha - \cot \beta}{\cot \beta - \cot \theta}$

$\therefore$　$\cot \theta = \dfrac{b \cot \alpha - a \cot \beta}{b - a}$　　故如題云.
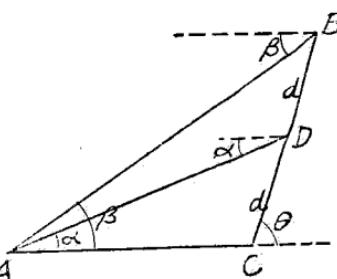
34. 設所求之傾斜角為 θ

今　$\dfrac{AC}{\sin(\theta - \alpha)} = \dfrac{d}{\sin \alpha}$,

$\dfrac{AC}{\sin(\theta - \beta)} = \dfrac{2d}{\sin \beta}$

$\therefore$　$\dfrac{\sin(\theta - \alpha)}{\sin \alpha} = \dfrac{2 \sin(\theta - \beta)}{\sin \beta}$

即　$\sin \theta \cot \alpha - \cos \theta = 2 \sin \theta \cot \beta - 2 \cos \theta$

$$\therefore \quad \cot \theta = 2 \cot \beta - \cot \alpha$$

$$\therefore \quad \theta = \cot^{-1}(2 \cot \beta - \cot \alpha)$$

35. 今 C 爲測點, $AC = h$ 尺

作 $CF \parallel AB$

設雲高 $BD = x$ 尺

則 $FD = x - h$

又以 E 爲 D 在水中之影

則 $BE = x$, $EF = x + h$

今 $AB = CF = FD \cot \alpha$, $CF = EF \cot \beta$

故 $FD \cot \alpha = EF \cot \beta$

卽 $(x - h) \cot \alpha = (x + h) \cot \beta$

$$\therefore \quad x(\cot \alpha - \cot \beta) = h(\cot \alpha + \cot \beta)$$

$$\therefore \quad x = \frac{h(\cos \alpha \sin \beta + \cos \beta \sin \alpha)}{\cos \alpha \sin \beta - \cos \beta \sin \alpha} = \frac{h \sin(\alpha + \beta)}{\sin(\beta - \alpha)}$$

故白雲離地之高爲 $\dfrac{h \sin(\alpha + \beta)}{\sin(\beta - \alpha)}$.
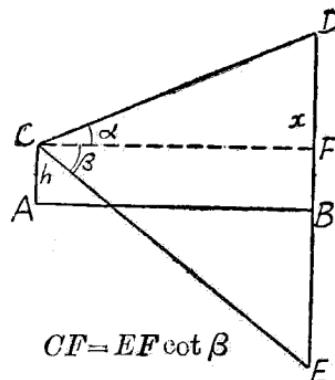
36. 設上升速爲每分鐘 v 哩, 則兩分鐘內上升 $GF = 2v$ 哩

又經水平距離 $EG = 2 \cdot 80 / 60 = \dfrac{8}{3}$ 哩

今因 $\angle PE'F' = 90°$, $\angle E'PF' = 45°$

則 $PE' = E'F' = \dfrac{8}{3}$ 哩, $PF' = \dfrac{8}{3}\sqrt{2}$ 哩

今 $\tan 8° = \dfrac{h}{PE'}$ $\therefore \quad h = \dfrac{8}{3} \tan 8°$

又　　$\tan 8° = \dfrac{2v+h}{PF''}$

∴　　$2v+h = \dfrac{8\sqrt{2}}{3}\tan 8°$
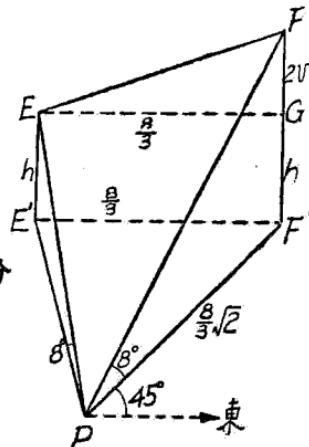
∴　　$v = \dfrac{4}{3}(\sqrt{2}-1)\tan 8°$

　　　$= \dfrac{.4142 \times 4}{3} \times .1405$ 哩/分

　　　$= .4142 \times 4 \times .1405$

　　　$\times 1760$ 呎/分

　　　$= 410$ 呎/分

37. 設房屋爲 H, 風磨爲 W, $ABDC$ 爲路, 且 $AB = 1$ 里,

　　$BC = 3$ 里, θ 爲所證之角

今　　$HB = AB\sin 30° = 1/2$ 里

又　　$BE = BC\cos 60° = 3/2$ 里

∴　　　$HE = 2$ 里

∵　　$\angle WBE = 45° = \angle BWE$

∴　　$WE = BE = 3/2$ 里

設　　$\angle WHE = \phi$
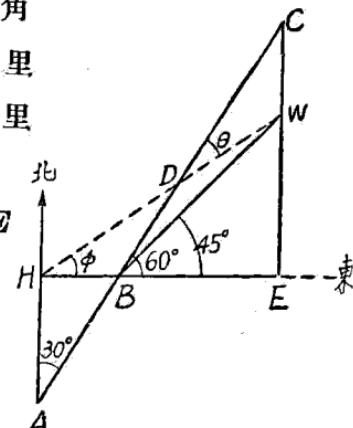
∴　　$\tan\phi = 3/4$

今　　$\theta = 60° - \phi$

∴　　$\tan\theta = \tan(60° - \phi)$

即　　$\tan\theta = \dfrac{\sqrt{3}-3/4}{1+3\sqrt{3}/4} = \dfrac{4\sqrt{3}-3}{4+3\sqrt{3}} = \dfrac{48-25\sqrt{3}}{11}$

　　∴　　$\theta = \tan^{-1}[(48-25\sqrt{3})/11]$

38 作　　$PQ \perp ABC$

且令　$PQ = x$

又令　$\angle BPQ = \theta$

及　　$\tan \theta = t$

今　　$AB = BC = 1$ 里,

$\angle CPQ = 30° + \theta$,

$\angle APQ = 45° - \theta$

故　$\begin{cases} AB = QB + AQ = x \tan \theta + x \tan(45° - \theta) \\ BC = QC - QB = x \tan(30° + \theta) - x \tan \theta \end{cases}$

即　　$1 = x \left(t + \dfrac{1-t}{1+t} \right) = \left(\dfrac{1+t^2}{1+t} \right) x$ **(1)**

$1 = x \left(\dfrac{1 + t\sqrt{3}}{\sqrt{3} - t} - t \right) = \left(\dfrac{1+t^2}{\sqrt{3} - t} \right) x$ **(2)**

從(1),(2)得　$1 + t = \sqrt{3} - t$ ∴ $t = \tfrac{1}{2}(\sqrt{3} - 1)$ **(3)**

代入(1),得　$x = \dfrac{1+t}{1+t^2} = \dfrac{1+\sqrt{3}}{4 - \sqrt{3}} = \dfrac{7 + 5\sqrt{3}}{13}$

39 ∵　$\alpha + \beta + \gamma = 180°$

∴　$\Sigma \tan \alpha = \Pi \tan \alpha$
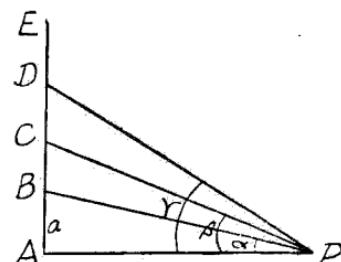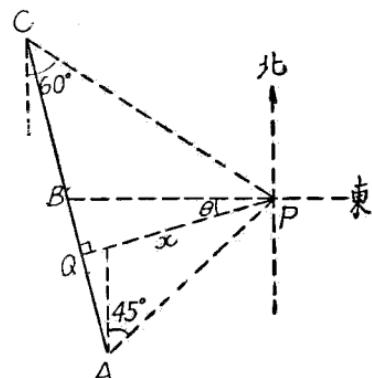
(公式 50 C)

今　$\tan \alpha = \dfrac{a}{x}$, $\tan \beta = \dfrac{b}{x}$,

$\tan \gamma = \dfrac{c}{x}$

∴　$\dfrac{a+b+c}{x} = \dfrac{abc}{x^3}$　　$x^2(a+b+c) = abc$

40. 設塔高 CD 爲 h，$\angle ADB = \theta$

今 　$\overline{AB}^2 = \overline{CB}^2 + \overline{CA}^2 - 2\overline{CB} \cdot \overline{CA} \cos \alpha$

又 　$\overline{AB}^2 = \overline{DB}^2 + \overline{DA}^2 - 2\overline{DB} \cdot \overline{DA} \cos \theta$

但 　$CB = h \csc \gamma$

　　$CA = h \csc \beta$

又 　$DB = h \cot \gamma$

　　$DA = h \cot \beta$

故 　$h^2(\csc^2\gamma + \csc^2\beta)$

　　　　$- 2h^2 \csc \gamma \csc \beta \cos \alpha$

　　$= h^2(\cot^2\beta + \cot^2\gamma) - 2h^2\cot \beta \cot \gamma \cos \theta$

即 　$1 - \csc \beta \csc \gamma \cos \alpha = -\cot \beta \cot \gamma \cos \theta$

即 　$\sin \beta \sin \gamma - \cos \alpha = -\cos \beta \cos \gamma \cos \theta$

故 　$\cos \theta = (\cos \alpha - \sin \beta \sin \gamma)/\cos \beta \cos \gamma$
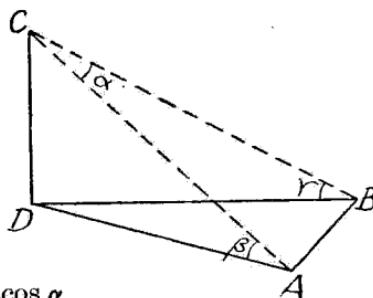
41. 設山高 DP 爲 h 尺，聯 AC

在 $\triangle ABC$ 中，

　　$AB = BC = 800$ 尺
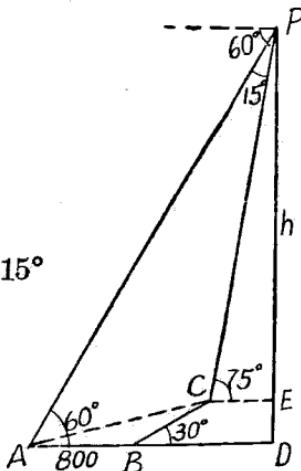
故 　$\angle CAB = \angle BCA = 15°$

∴ 　$AC = \dfrac{800 \sin 30°}{\sin 15°} = 400 \csc 15°$

在 $\triangle ACP$ 中，

　　$\angle CAP = 45°$，$\angle APC = 15°$，

　　$\angle ACP = 120°$

$\therefore$ $AP = \dfrac{AC \sin 120°}{\sin 15°} = 400 \sin 60° \csc^2 15°$

今 $h = AP \sin 60° = 400 \times 3 \times 3.7321 = 4478.5$ 尺

42. 設 $QP = x$ 碼，又坡之斜度爲 θ

在 $\triangle OAQ$ 中，$\dfrac{320}{\sin 25°} = \dfrac{500}{\sin(\theta - 25°)}$

$\therefore$ $\sin(\theta - 25°) = .6603$

$\therefore$ $\theta - 25° = 41°19'33''$

故 $\theta = 66°19'33''$

又在 $\triangle OAP$ 中，

$\dfrac{x + 320}{\sin 38°} = \dfrac{500}{\sin(\theta - 38°)}$

$\therefore$ $x + 320 = \dfrac{500 \times .6157}{.4748} = 649$ $\therefore$ $x = 329$ 碼

43. 設塔高 DC 爲 x 尺，又 $\angle CAD = \theta$

則 $\tan \theta = x/b$

今 $\dfrac{x + a}{b} = \tan(\theta + \alpha) = \dfrac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha}$

$= \dfrac{x + b \tan \alpha}{b - x \tan \alpha}$
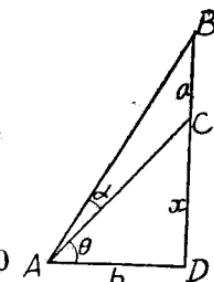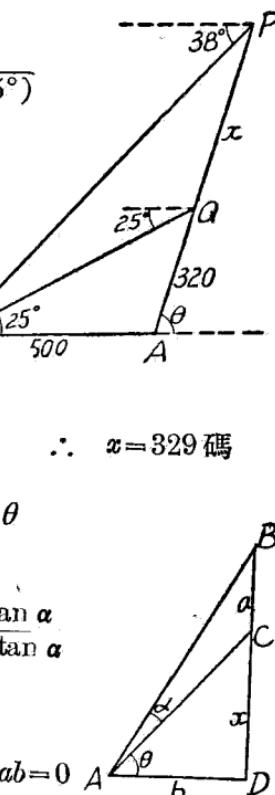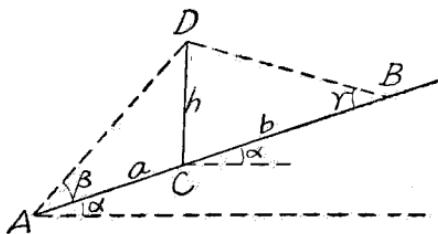
故 $x^2 \tan \alpha + ax \tan \alpha + b^2 \tan \alpha - ab = 0$

卽 $x^2 + ax + b^2 - ab \cot \alpha = 0$

$\therefore$ $x = \frac{1}{2}(-a + \sqrt{a^2 + 4ab \cot \alpha - 4b^2})$ （負值不合）

44. 設塔高 CD 爲 h，則在 $\triangle ABD$ 中

$$\dfrac{a + b}{\sin[\pi - (\beta + \gamma)]} = \dfrac{AD}{\sin \gamma}$$

$$\therefore \quad AD = \frac{(a+b)\sin\gamma}{\sin(\beta+\gamma)}$$

且在 $\triangle ADC$ 中， $\angle ACD = 90° + \alpha$

故 　　　$\dfrac{AD}{\sin(90°+\alpha)} = \dfrac{h}{\sin\beta}$

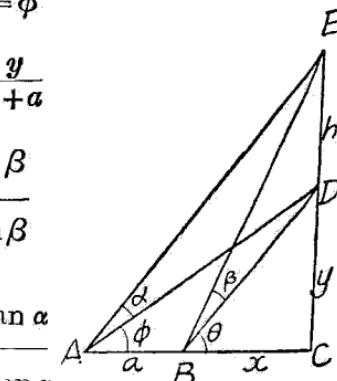$$\therefore \quad h = \frac{AD\sin\beta}{\cos\alpha} = \frac{(a+b)\sin\beta}{\cos\alpha}\frac{\sin\gamma}{\sin(\beta+\gamma)} \quad （塔高）$$

45. 設所求距離 BC 爲 x 尺, 柱高 CD 爲 y 尺

又設 　$\angle DBC = \theta$， $\angle DAC = \phi$

則 　$\tan\theta = \dfrac{y}{x}$， $\tan\phi = \dfrac{y}{x+a}$

故 $\begin{cases} \dfrac{y+h}{x} = \tan(\theta+\beta) = \dfrac{\dfrac{y}{x} + \tan\beta}{1 - \dfrac{y}{x}\tan\beta} \\[4mm] \dfrac{y+h}{x+a} = \tan(\phi°+\alpha) = \dfrac{\dfrac{y}{x+a} + \tan\alpha}{1 - \dfrac{y}{x+a}\tan\alpha} \end{cases}$

即 $\begin{cases} (y+h)(x - y\tan\beta) = x(y + x\tan\beta) \\ (y+h)(a+x - y\tan\alpha) = (x+a)[y + (a+x)\tan\alpha] \end{cases}$

即 $\begin{cases} x^2+y^2-hv\,c\ t\,\beta+hy=0 \\ x^2+y^2+(2a-h\cot\alpha)x+hy=ah\cot\alpha-a^2 \end{cases}$

相減得 $x(h\cot\alpha-2a-h\cot\beta)=a^2-ah\cot\alpha$
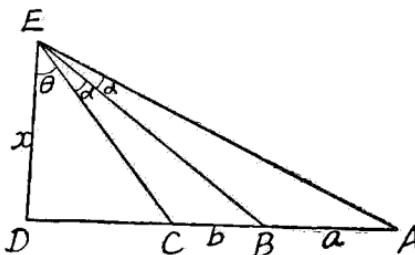
$\therefore$ $x=(a^2-ah\cot\alpha)/(h\cot\alpha-2a-h\cot\beta)$

46. 如圖設塔高 $DE=x$

則因 $EC=x/\cos\theta,$ $EA=x/\cos(\theta+2\alpha)$

但從幾何知 $EC:b=EA:a$

即 $b\cos\theta:x=a\cos(\theta+2\alpha):x$

$b\cos\theta=a(\cos\theta\cos2\alpha-\sin\theta\sin2\alpha)$

$\therefore$ $\tan\theta=(a\cos2\alpha-b)/a\sin2\alpha$

$\therefore$ $\sin\theta=\dfrac{a\cos2\alpha-b}{\sqrt{a^2+b^2-2ab\cos2\alpha}},$

$\cos\theta=\dfrac{a\sin2\alpha}{\sqrt{a^2+b^2-2ab\cos2\alpha}}$

今 $\dfrac{b}{\sin\alpha}=\dfrac{EC}{\sin(90°-\theta-\alpha)}=\dfrac{EC}{\cos(\theta+\alpha)}$

$EC=\dfrac{b}{\sin\alpha}(\cos\theta\cos\alpha-\sin\theta\sin\alpha),$

$$DE = \frac{b}{\sin \alpha}\left[\cos \alpha \cdot \frac{a^2\sin^2 2\alpha}{a^2+b^2-2ab\cos 2\alpha}\right.$$

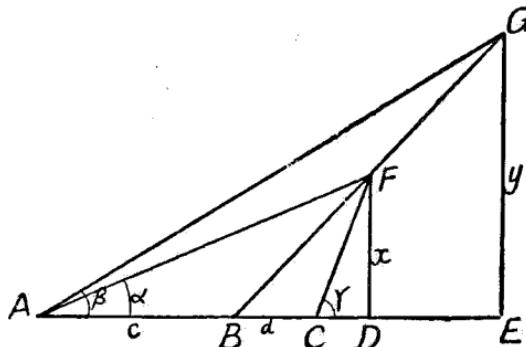$$\left. -\frac{a\sin \alpha \sin 2\alpha(a\cos 2\alpha-b)}{a^2+b^2-2ab\cos 2\alpha}\right]$$

即　$x = \dfrac{b}{\sin \alpha}\left[\dfrac{ab\sin \alpha \sin 2\alpha+a^2\sin 2\alpha(\sin \alpha)}{a^2+b^2-2ab\cos 2\alpha}\right]$

$$= \frac{ab(a+b)\sin 2\alpha}{a^2+b^2-2ab\cos 2\alpha} = \frac{2ab(a+b)\sin \alpha \cos \alpha}{a^2+b^2-2ab(2\cos^2\alpha-1)}$$

$$= \frac{2ab(a+b)\sin \alpha \cos \alpha}{(a+b)^2-4ab\cos^2\alpha} = \frac{2ab(a+b)\tan \alpha}{(a+b)^2\sec^2\alpha-4ab}$$

$$= \frac{2ab(a+b)\tan \alpha}{(a+b)^2(1+\tan^2\alpha)-4ab} = \frac{2ab(a+b)\tan \alpha}{(a-b)^2+(a+b)^2\tan^2\alpha}$$

47. 設低山 DF 高 x 里，高山 EG 高 y 里

則　　$\dfrac{CF}{\sin \alpha} = \dfrac{AC}{\sin AFC} = \dfrac{c+d}{\sin(\gamma-\alpha)}$

$\therefore\ \ CF = \dfrac{(c+d)\sin \alpha}{\sin(\gamma-\alpha)}$　　（參考 287 頁例六第一法）

$\therefore\ \ x = CF\sin \gamma = \dfrac{(c+d)\sin \alpha \sin \gamma}{\sin(\gamma-\alpha)}$　　（低山之高）

又　$y:x=BE:BD=(AE-AB):(AD-AB)$

　　　$=y\cot\beta-c:x\cot\alpha-c$

∴　　$y=\dfrac{cx}{c+x(\cot\beta-\cot\alpha)}$

　　　$=\dfrac{c(c+d)}{c(\cot\alpha-\cot\gamma)+(c+d)(\cot\beta-\cot\alpha)}$

或　　$y=\dfrac{c(c+d)\sin\alpha\sin\beta\sin\gamma}{c\sin\beta\sin(\gamma-\alpha)+(c+d)\sin\gamma\sin(\alpha-\beta)}$

　　　　　　　　　　　（高山之高）

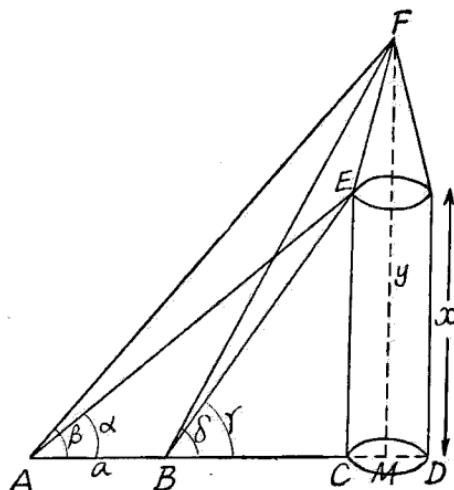**48.** 設塔高 $CE=x$ 尺, 实頂離地 MF 爲 y 尺

　則　　$a=AB=AM-BM=y\cot\beta-y\cot\delta$

　∴　　$y=a/(\cot\beta-\cot\delta)$

　　　　$=a\sin\beta\sin\delta\csc(\delta-\beta)$　　（尖頂高）

　又　　$a=AC-BC=x\cot\alpha-x\cot\gamma$

$$\therefore \quad x = a/(\cot\alpha - \cot\gamma)$$

$$= a\sin\alpha\sin\gamma\csc(\gamma-\alpha) \qquad （塔高）$$

塔之直徑 $= CD = 2CM = 2(BM-BC)$

$$= 2(y\cot\delta - x\cot\gamma)$$

$$= 2a[\sin\beta\cos\delta\csc(\delta-\beta)$$

$$-\sin\alpha\cos\gamma\csc(\gamma-\alpha)]$$

從 $2(AM-AC)$，直徑亦可爲

$$2a[\sin\delta\cos\beta\csc(\delta-\beta) - \sin\gamma\cos\alpha\csc(\gamma-\alpha)]$$

49. 設　$AD=h$,　今　$\angle EDC = \angle BDC$

$$\therefore \quad DE:DB = CE:BC$$

即　$\overline{DE}^2 : \overline{DB}^2 = \overline{CE}^2 : \overline{BC}^2$
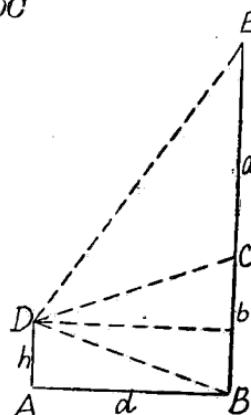
但　$\overline{DE}^2 = d^2 + (a+b-h)^2$,

$$\overline{DB}^2 = d^2 + h^2$$

$$\therefore \quad d^2+(a+b-h)^2 : d^2+h^2 = a^2 : b^2$$

即　$(a-b)d^2 = (a+b)b^2 - 2b^2 h$

$$- (a-b)h^2$$

50. 在上題中，$a=48$ 尺，　$b=30$ 尺，　$d=48$ 尺

代入上式，則

$$18(48)^2 = 78(900) - 1800h - 18h^2$$

即　　　　$h^2 + 100h - 1596 = 0$

即　　　　$(h-14)(h+114) = 0$

$$\therefore \quad h = 14 \text{ 尺} \qquad （負值不合）$$

51. 設峭壁高 y 尺,塔高 x 尺

因　$\angle CBD = \angle CAD = \beta$

故 A, B, C, D 在一圓周上

$\therefore EC \cdot ED = EB \cdot EA$

即　$y(x+y) = ab$　　　(1)

設　$\tan \phi = y/a$

則　$\tan(\phi + \beta) = (x+y)/a$

即　　$\dfrac{y/a + \tan \beta}{1 - y \tan \beta/a} = \dfrac{x+y}{a} = \dfrac{b}{y}$

即　　$y^2 + (a+b)\tan \beta \cdot y - ab = 0$　　　(2)

$(1) - (2)$,　$xy = (a+b)\tan \beta \cdot y$

即　　　　$x = (a+b)\tan \beta$

52. 在上題中設 $\tan \beta = 1/10$,　$AE = 11$,　$BE = 9$,山高為 y

從上題(2)　　　　$y^2 + 2y - 99 = 0$

即　$(y-9)(y+11) = 0$　　　　$\therefore$　山高 $y = 9$

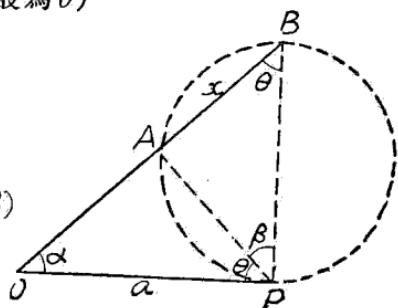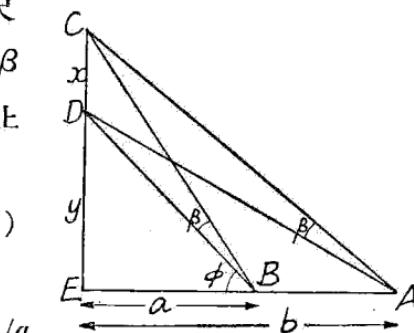**53.** 參考 297 頁例十七,知 OP 為圓 APB 之切線

故知　$\angle B = \angle OPA$　(設為 θ)

今　$\theta + \beta + \theta + \alpha = \pi$

$\therefore$　$\theta = \dfrac{\pi}{2} - \dfrac{1}{2}(\alpha + \beta)$

$\theta + \beta = \dfrac{\pi}{2} - \dfrac{1}{2}(\alpha - \beta)$

今　$\dfrac{x}{\sin \beta} = \dfrac{AP}{\sin \theta}$

$$\vartriangleleft \quad \frac{AP}{\sin \alpha} = \frac{a}{\sin(\theta+\beta)}$$

$$\therefore \quad \frac{x}{\sin \beta} = \frac{a \sin \alpha}{\sin \theta \sin(\theta+\beta)} = \frac{a \sin \alpha}{\cos \frac{1}{2}(\alpha+\beta)\cos \frac{1}{2}(\alpha-\beta)}$$
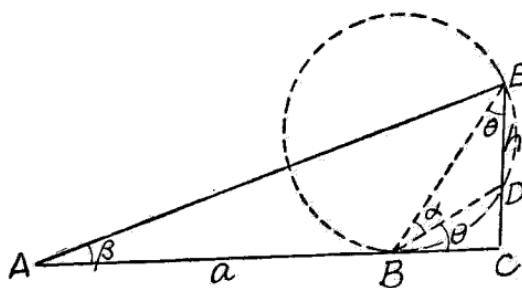
$$\therefore \quad x = a \sin \alpha \sin \beta \sec \tfrac{1}{2}(\alpha+\beta)\sec \tfrac{1}{2}(\alpha-\beta)$$

54. 從上題知 AB 切於圓 BDE

設　　$DE = h,$　　$\angle DBC = \theta = \angle BED$

則　$\angle AEB = \theta + \alpha - \beta,$　　又　$\theta + \alpha = 90° - \theta$

$$\therefore \quad \frac{BE}{\sin \beta} = \frac{a}{\sin(\theta+\alpha-\beta)}$$

$$\therefore \quad BE = \frac{a \sin \beta}{\sin(90°-\theta-\beta)} = \frac{a \sin \beta}{\cos(\theta+\beta)}$$

又　　　　$$\frac{h}{\sin \alpha} = \frac{BE}{\sin(\theta+\alpha)}$$

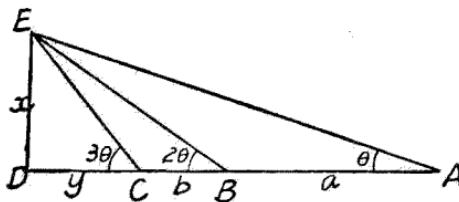$$\therefore \quad h = \frac{a \sin \alpha \sin \beta}{\sin(\theta+\alpha)\cos(\theta+\beta)}$$

$$= \frac{2a \sin \alpha \sin \beta}{\sin(2\theta+\alpha+\beta) + \sin(\alpha-\beta)}$$

$$= \frac{2a \sin \alpha \sin \beta}{\cos \beta + \sin(\alpha-\beta)}$$

55. 設 $DE=x,\quad DC=y,\qquad$ 則 $\tan 3\theta=x/y,$

$\tan 2\theta=x/(b+y),\qquad \tan\theta=x/(a+b+y)$

今 $$\tan 3\theta=\frac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$$

$$\therefore\quad \frac{x}{y}=\frac{\dfrac{x}{b+y}+\dfrac{x}{a+b+y}}{1-\dfrac{x}{b+y}\cdot\dfrac{x}{a+b+y}}=\frac{x(a+b+y)+x(b+y)}{(b+y)(a+b+y)-x^2}$$

即 $$(b+y)(a+b+y)-x^2=(a+2b+2y)y$$

即 $$x^2+y^2-b(a+b)=0 \qquad (1)$$

又 $$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$\therefore\quad \frac{x}{b+y}=\frac{\dfrac{2x}{a+b+y}}{1-\dfrac{x^2}{(a+b+y)^2}}=\frac{2x(a+b+y)}{(a+b+y)^2-x^2}$$

即 $$(a+b+y)^2-x^2=2(b+y)(a+b+y)$$

即 $$x^2+y^2+2by+(a+b)(b-a)=0 \qquad (2)$$

$(2)-(1),\qquad 2by+(a+b)(2b-a)=0$

$$y=\frac{(a+b)(a-2b)}{2b}$$

代入(1)得　$x^2 = b(a+b) - \dfrac{(a+b)^2(a-2b)^2}{4b^2}$

$$= \dfrac{(a+b)(4b^3 - a^3 + 3a^2b - 4b^3)}{4b^2}$$

$$= \dfrac{a^2(a+b)(3b-a)}{4b^2}$$

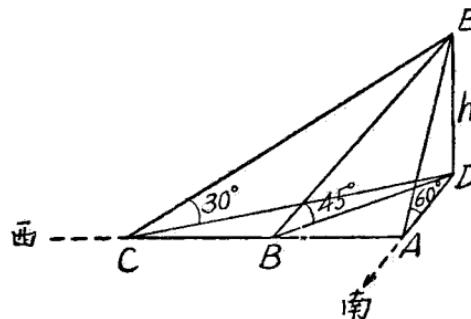$$\therefore \quad x = \dfrac{a\sqrt{(a+b)(3b-a)}}{2b}$$

今如　　　$a = 50, \quad b = 20$

則　　　　$x = \dfrac{50\sqrt{70(10)}}{2(20)} = \dfrac{25}{2}\sqrt{7} = 33.07$

56. 設 DE 爲塔高，且等於 h

則　$AD = h \cot 60° = h/\sqrt{3}, \qquad BD = h \cot 45° = h,$

$CD = h \cot 30° = \sqrt{3}\,h$

$\therefore \quad \angle DAB = \angle DAC = 90°$

$\therefore \quad \overline{BA}^2 + \overline{AD}^2 = \overline{BD}^2 \qquad$ 卽　$\overline{BA}^2 = h^2 - \dfrac{h^2}{3} = \dfrac{2h^2}{3}$

又　　　$\overline{CA}^2 = \overline{CD}^2 - \overline{AD}^2 = 3h^2 - \dfrac{h^2}{3} = \dfrac{8h^2}{3}$

$$\therefore \quad CA = 2\cdot\sqrt{\frac{2h^2}{3}} = 2BA = BA + CB$$

$$\therefore \quad BA = CB \qquad\qquad 故如題云.$$

57. 如在上題中 $AB = \frac{1}{2}$　$\therefore$　$\frac{2h^2}{3} = \frac{1}{4}$　$\therefore$　$h^2 = \frac{3}{8}$

$$\therefore \quad h = \frac{1}{4}\sqrt{6}\ 哩 = 440\sqrt{6}\ 碼$$

58. 見題 24.

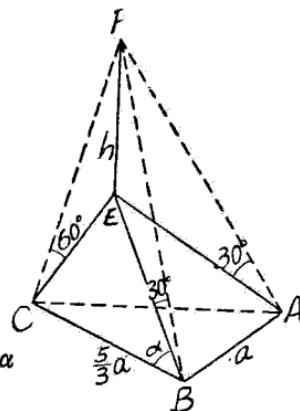**59.** 設塔高 EF 為 h, $\angle EBC = \alpha$

則　$\angle ABE = 90° - \alpha$

故　$CE = h\cot 60° = \dfrac{h}{\sqrt{3}}$,

$\quad AE = h\cot 30° = \sqrt{3}\,h$,

$\quad BE = \sqrt{3}\,h$

今 $\begin{cases} 3h^2 = a^2 + 3h^2 - 2\sqrt{3}\,ha\sin\alpha \\ \dfrac{h^2}{3} = \dfrac{25}{9}a^2 + 3h^2 - \dfrac{10a\sqrt{3}}{3}h\cos\alpha \end{cases}$

即 $\begin{cases} \sin\alpha = \dfrac{a}{2\sqrt{3}\,h} & (1) \\ \cos\alpha = \dfrac{24h^2 + 25a^2}{30\sqrt{3}\,ah} & (2) \end{cases}$

$$\therefore \quad \frac{a^2}{12h^2} + \frac{576h^4 + 1200a^2h^2 + 625a^4}{2700\,a^2h^2} = 1$$

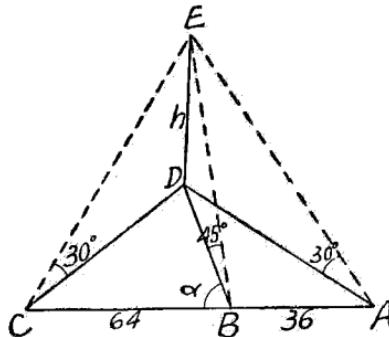即 $\qquad 288h^4 - 750a^2h^2 + 425a^4 = 0$

即 $\qquad (6h^2 - 5a^2)(48h^2 - 85a^2) = 0$

$$\therefore \quad h = \sqrt{\frac{5}{6}}\,a = \frac{a}{6}\sqrt{30} \quad 或 \quad h = \sqrt{\frac{85}{48}}\,a = \frac{a}{12}\sqrt{255}$$

60. 設塔高 $DE=h$ 碼， $\angle CBD=\alpha$

則　$AD=h \cot 30°=h\sqrt{3}$， $DC=h\sqrt{3}$， $BD=h$

今 $\begin{cases} 3h^2=h^2+36^2+72\,h \cos \alpha & (1) \\ 3h^2=h^2+64^2-128\,h \cos \alpha & (2) \end{cases}$

$16(1)+9(2)$，并化簡之得　　　$h^2=9\times128$

$\therefore \quad h=24\sqrt{2}$　　　　　即塔高 $24\sqrt{2}$ 碼

61. 設　$\tan \phi=a$，　　$\tan \theta=\beta$，　　$DE=h$

則　　$AD=h/a$，　　$BD=h/\beta$，　　$CD=h/a$
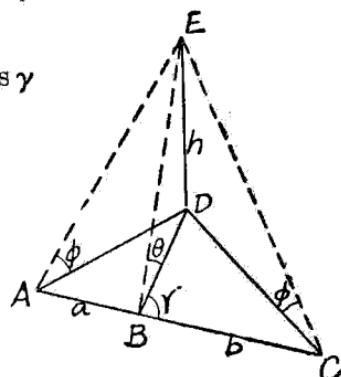
今 $\begin{cases} \dfrac{h^2}{a^2}=\dfrac{h^2}{\beta^2}+a^2+2a\left(\dfrac{h}{\beta}\right)\cos \gamma \\[2mm] \dfrac{h^2}{a^2}=\dfrac{h^2}{\beta^2}+b^2-2b\left(\dfrac{h}{\beta}\right)\cos \gamma \end{cases}$

$\therefore \quad (a+b)\dfrac{h^2}{a^2}=\dfrac{h^2}{\beta^2}(a+b)$
$\qquad\qquad\qquad +a^2b+b^2a$

即　$\dfrac{h^2}{a^2}-\dfrac{h^2}{\beta^2}=ab$

故　$h=a\beta\sqrt{\dfrac{ab}{\beta^2-a^2}}$

62. 見 298 頁例十九之圖,設 $\tan\alpha=\sqrt{2}$, $\tan\beta=\sqrt{3}+1$

則　$\cot\alpha=1/\sqrt{2}$, $\cot\beta=1/(\sqrt{3}+1)=(\sqrt{3}-1)/2$

設塔高爲 x 尺, 則因 $a=80$ 尺, 故從 299 頁

$$(\cot^2\alpha-\cot^2\beta)x^2+a\sqrt{3}\,x\cot\beta-a^2=0$$

即　$\left(\dfrac{1}{2}-\dfrac{2-\sqrt{3}}{2}\right)x^2+40\sqrt{3}(\sqrt{3}-1)x-6400=0$

即　$x^2+80\sqrt{3}\,x-6400(\sqrt{3}+1)=0$

即　$(x-80)[x+80(\sqrt{3}+1)]=0$　　　　$\therefore\ x=80$ 尺

63. 今因 A,B,C 對於桿高之仰

角相同

$\therefore\ AD=BD=CD=100\cot 60°$

$=100/\sqrt{3}$

故 D 爲 $\triangle ABC$ 之外心

又因各邊相等

故 D 亦爲其內心,重心

故　　　　$\angle ADB=120°$
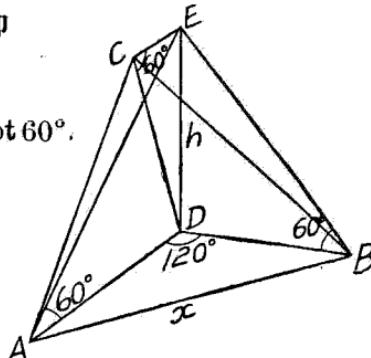
今　　　$\dfrac{AB}{\sin 120°}=\dfrac{AD}{\sin 30°}$
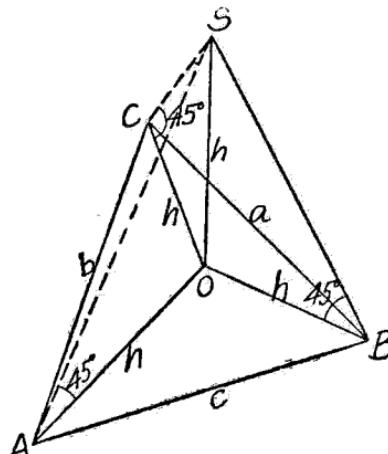
即　　　$x=\dfrac{\sqrt{3}}{2}\left(\dfrac{100}{\sqrt{3}}\right)2=100$ 尺

64. 設氣球高 $OS=h$, 則 $AO=BO=CO=h\tan 45°=h$

故 AO 爲 $\triangle ABC$ 之外接圓半徑

從公式 47 得　　　$\triangle=\dfrac{abc}{4R}$

$$\therefore \quad AO = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

65. 見299頁例二十之圖，今因 A, B, C, D 在同一圓周上

$$\therefore \quad \angle BDC = \angle BAC = \theta, \quad \angle BDA = \angle BCA = \theta$$

從 $\quad x^2\cot^2\gamma + x^2\cot^2\beta - 2x^2\cot\gamma\cot\beta\cos\theta$

$$= x^2\cot^2\alpha + x^2\cot^2\beta - 2x^2\cot\alpha\cot\beta\cos\theta$$

故 $\quad 2\cos\theta\cot\beta(\cot\alpha - \cot\gamma) = \cot^2\alpha - \cot^2\gamma$

即 $\quad\quad 2\cos\theta\cot\beta = \cot\alpha + \cot\gamma$

66. 設 $AB = x$, 今 $\angle DAC = \pi - (\alpha + \gamma)$
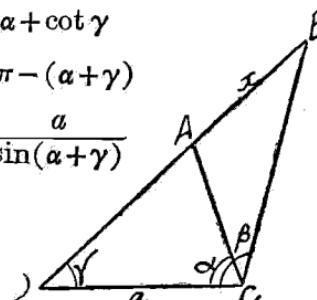
$$\therefore \quad \frac{AC}{\sin\gamma} = \frac{a}{\sin(\pi - \alpha - \gamma)} = \frac{a}{\sin(\alpha + \gamma)}$$

$$\therefore \quad AC = a\sin\gamma / \sin(\alpha + \gamma)$$

又 $\quad \angle ABC = \pi - (\beta + \gamma)$
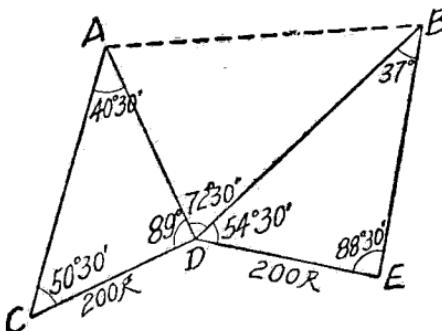
故在 $\triangle ABC$ 中

$$x = \frac{AC\sin(\beta - \alpha)}{\sin(\pi - \beta - \gamma)} = \frac{a\sin\gamma\sin(\beta - \alpha)}{\sin(\alpha + \gamma)\sin(\beta + \gamma)}$$

67 今 $\overline{BD}=\dfrac{200\sin 88°30'}{\sin 37°}$ 又 $\overline{AD}=\dfrac{200\sin 50°30'}{\sin 40°30'}$

∴ $\overline{AB}=\sqrt{\overline{BD}^2+\overline{AD}^2-2\cdot\overline{AD}\cdot\overline{BD}\cos 72°30'}$

今
$$\log 200 = 2.30103$$
$$\log \sin 88°30' = 9.99985 - 10$$
$$\text{colog} \sin 37° = 0.22054$$
$$\log \overline{BD} = 2.52142$$
$$2\log \overline{BD} = 5.04284$$
$$\therefore \ \overline{BD}^2 = 110367$$

$$\log 200 = 2.30103$$
$$\log \sin 50°30' = 9.88741 - 10$$
$$\text{colog} \sin 40°30' = 0.18746$$
$$\log \overline{AD} = 2.37590$$
$$2\log \overline{AD} = 4.75180$$
$$\overline{AD}^2 = 56468$$

$$\log 2 = 0.30103$$

$$\log \overline{AD} = 2.37590$$

$$\log \overline{BD} = 2.52142$$

$$\log \cos 72°30' = 9.47814 - 10$$

$$\log 2 \cdot \overline{AD} \cdot \overline{BD} \cos 72°30' = 4.67649$$

$$\therefore \quad 2\overline{AD} \cdot \overline{BD} \cos 72°30' = 47478$$

$$\therefore \quad \overline{AB} = \sqrt{119357}$$

今　$\log 119357 = 5.07685$,　$\frac{1}{2}\log 119357 = 2.53843$

$$\therefore \quad \overline{AB} = 345.48 \text{ 尺} \qquad \text{故 } AB \text{ 長爲 } 345 \text{ 尺.}$$

68.

今　$$\overline{AC} = \frac{200 \sin 83°}{\sin 42°29'}$$

$$\therefore \quad \overline{AD} = \sqrt{\overline{AC}^2 + \overline{CD}^2 - 2\overline{AC}\ \overline{CD}\ \cos 53°30'}$$

$$\therefore \quad \overline{AB} = \sqrt{\overline{AD}^2 + \overline{BD}^2 - 2\overline{AD} \cdot \overline{BD} \cdot \cos ADB}$$

今
$$\log 200 = 2.30103$$
$$\log \sin 83° = 9.99675 - 10$$
$$\text{colog} \sin 42°29' = 0.17045$$
$$\log \overline{AC} = 2.46823$$
$$2 \log \overline{AC} = 4.93646$$
$$\therefore \quad \overline{AC}^2 = 86390$$
$$\text{及} \quad \overline{CD}^2 = 40000$$

$$\log 2 = 0.30103$$
$$\log \overline{AC} = 2.46823$$
$$\log \overline{CD} = 2.30103$$
$$\log \cos 53°30' = 9.77439 - 10$$
$$\log \overline{2AC \cdot CD} \cos 53°30' = 4.84468$$
$$\therefore \quad 2\overline{AC} \cdot \overline{CD} \cos 53°30' = 69933$$

$$\therefore \quad \overline{AD} = \sqrt{56457}$$
$$\log 56457 = 4.75172$$
$$\log \overline{AD} = 2.37586$$

$$\sin ADC = \frac{\overline{AC} \sin 53°30'}{\overline{AD}}$$
$$\log \overline{AC} = \quad 2.46823$$
$$\log \sin 53°30' = \quad 9.90518 - 10$$
$$\text{colog} \overline{AD} = \quad 7.62414 - 10$$
$$\log \sin ADC = 19.99755 - 20$$
$$\therefore \quad \angle ADC = 83°55'$$

$$\therefore \quad \angle ADB = 72°30'$$

$$\log 2 = 0.30103$$

$$\log \overline{AD} = 2.37586 \qquad (見前)$$

$$\log \overline{BD} = 2.52142 \qquad (學者自求)$$

$$\log \cos 72°30' = 9.47814 - 10$$

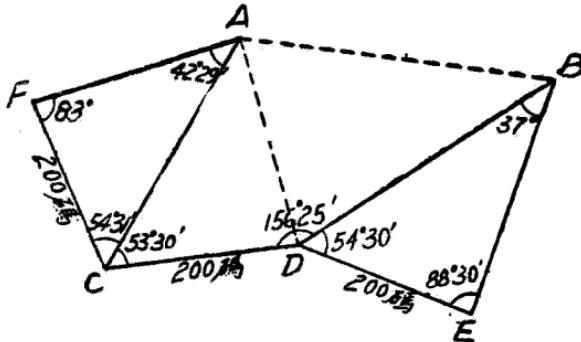$$\log 2\overline{AD} \cdot \overline{BD} \cos 72°30' = 4.67645$$

$$2\overline{AD} \cdot \overline{BD} \cos 72°30' = 47475$$

$$\therefore \quad \overline{AB} = \sqrt{56457 + 110367 - 47473} = \sqrt{119351}$$

今　$\log 119351 = 5.07682, \quad \log \overline{AB} = 2.53841$

$\therefore \quad \overline{AB} = 345.47$ 碼　　　　故 AB 長爲 345 碼。

69. 過 $\overline{BD}$ 作平面 $\perp VA$ 交 VA 於 K

則 $\angle BKD$ 即爲所求之角

今 $FK \perp VA$

故 $\triangle FKA$ 爲 $rt.\triangle$

又 $\angle VAF = \angle FAK$

$\therefore rt.\triangle VFA \backsim rt.\triangle FKA$

$\therefore FK : FA = VF : VA$

但 $FA = \dfrac{\sqrt{2}}{2}a,$

$VF = h, \quad VA = \sqrt{\dfrac{1}{2}a^2 + h^2} = \dfrac{\sqrt{2a^2 + 4h^2}}{2}$

$$\therefore \qquad FK = \frac{\frac{\sqrt{2}}{2}ah}{\frac{1}{2}\sqrt{2a^2+4h^2}} = \frac{ah}{\sqrt{a^2+2h^2}}$$

又 $\qquad BF = FA = \frac{\sqrt{2}}{2}a$

故在 $rt.\,\triangle KFB$ 中 ($\angle KFB = 90°$)
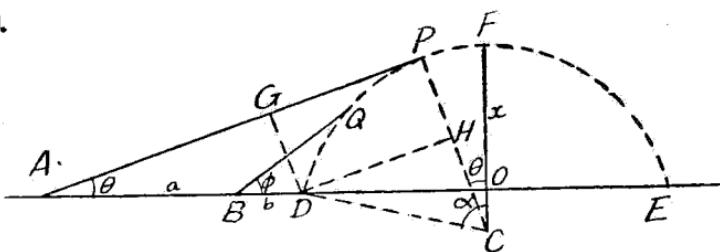
$$BK = \sqrt{\frac{a^2h^2}{a^2+2h^2}+\frac{1}{2}a^2} = \sqrt{\frac{4a^2h^2+a^4}{2a^2+4h^2}} = a\sqrt{\frac{a^2+4h^2}{2a^2+4h^2}}$$

$$\therefore \quad \sin BKF = \frac{BF}{BK} = \frac{\frac{\sqrt{2}}{2}a}{a\sqrt{\frac{a^2+4h^2}{2a^2+4h^2}}} = \sqrt{\frac{a^2+2h^2}{a^2+4h^2}}$$

$$\cos BKF = \sqrt{\frac{2h^2}{a^2+4h^2}} = h\sqrt{\frac{2}{a^2+4h^2}}$$

故 $\quad \sin 2BKF = \sin BKD = 2h\frac{\sqrt{2a^2+4h^2}}{a^2+4h^2}$

70.

如圖設測點 A, B 與 AP, BQ, D, E 同在一垂直平面

今 $AD = a$, $BD = b$, AP, BQ 均爲球山之切線

設 C 爲球心,半徑 $CF \perp DE$, 則 OF 爲所求之山高,設爲 x

今作 CP, 則 $CP \perp AP$, 又作 $DH \perp CP$, 則 $\angle PCF = \theta$

又連 CD, 設 $\angle DCF$ 爲 α, 則 $\angle DCP = \alpha - \theta$

$$\therefore \quad a \sin \theta = \overline{DG} = \overline{HP} = \overline{CP} - \overline{CH} = r - r \cos(\alpha - \theta)$$

$$= r[1 - \cos(\alpha - \theta)] = 2r \sin^2 \tfrac{1}{2}(\alpha - \theta)$$

$$\therefore \quad 2a \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta = 2r \sin^2 \tfrac{1}{2}(\alpha - \theta)$$

兩邊以 $2 \sin^2 \tfrac{1}{2}\theta$ 除之並開平方,則

$$\sqrt{a \cot \tfrac{1}{2}\theta} = \sqrt{r}\,(\sin \tfrac{1}{2}\alpha \cot \tfrac{1}{2}\theta - \cos \tfrac{1}{2}\alpha) \tag{1}$$

依同理 $\quad \sqrt{b \cot \tfrac{1}{2}\phi} = \sqrt{r}\,(\sin \tfrac{1}{2}\alpha \cot \tfrac{1}{2}\phi - \cos \tfrac{1}{2}\alpha) \tag{2}$

$(2)-(1)$, $\sqrt{b \cot \tfrac{1}{2}\phi} - \sqrt{a \cot \tfrac{1}{2}\theta}$

$$= \sqrt{r} \sin \tfrac{1}{2}\alpha (\cot \tfrac{1}{2}\phi - \cot \tfrac{1}{2}\theta)$$

$$\therefore \quad \sqrt{r} \sin \tfrac{1}{2}\alpha = \frac{\sqrt{b \cot \tfrac{1}{2}\phi} - \sqrt{a \cot \tfrac{1}{2}\theta}}{\cot \tfrac{1}{2}\phi - \cot \tfrac{1}{2}\theta} \tag{3}$$

但 $\quad x = \overline{OF} = \overline{CF} - \overline{CO} = r - r \cos \alpha = r(1 - \cos \alpha)$

$$= 2r \sin^2 \tfrac{1}{2}\alpha = 2(\sqrt{r} \sin \tfrac{1}{2}\alpha)^2 \tag{4}$$

以(3)代入(4),則得

$$x = 2\left(\frac{\sqrt{b \cot \tfrac{1}{2}\phi} - \sqrt{a \cot \tfrac{1}{2}\theta}}{\cot \tfrac{1}{2}\phi - \cot \tfrac{1}{2}\theta}\right)^2$$
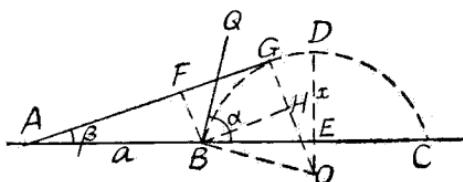
71. 設 $AB = a, BQ, AG$ 切球山, 而 $\quad GAB = \beta$, $\angle QBE = \alpha$ 半徑 $OD \perp BC$, 聯 OG, 作 $BH \perp OG$, 並設 DE 爲 x 又球山之半徑爲 r, 球心爲 O

則　$x = r - \overline{OE} = r - r\cos\alpha = r(1 - \cos\alpha)$

$\qquad = 2r\sin^2 \tfrac{1}{2}\alpha \qquad\qquad (1)$

$(\because\ OB \perp BQ,\ OD \perp BC\quad \therefore\ \angle BOD = \angle QBE = \alpha)$

又　$\overline{GH} = r - \overline{OH} = r - r\cos(\alpha - \beta)$

$\qquad = r[1 - \cos(\alpha - \beta)] = 2r\sin^2 \tfrac{1}{2}(\alpha - \beta)$

$(\because\ OG \perp AG,\ OD \perp BC\quad \therefore\ \angle GOD = \angle GAB = \beta$

$\qquad \therefore\ \angle BOG = \angle BOD - \angle GOD = \alpha - \beta)$

但　$\overline{GH} = \overline{BF} = a\sin\beta\quad \therefore\ a\sin\beta = 2r\sin^2 \tfrac{1}{2}(\alpha - \beta$

$\qquad \therefore\ 2r = a\sin\beta\csc^2 \tfrac{1}{2}(\alpha - \beta) \qquad (2)$

以(2)代入(1)，　則　$x = a\sin\beta\sin^2 \tfrac{1}{2}\alpha\csc^2 \tfrac{1}{2}(\alpha - \beta)$

72. 由 A 作切線 AC，由 B 作切線 BD

則　$\angle COA = \alpha,\ \angle DOB = \beta$

設半徑 $OC = x$，則　$\overset{\frown}{EF} = d$

$\qquad \therefore\ \angle EOF = d/x$

今　$AO = OC/\cos\alpha = x/\cos\alpha$

$\qquad BO = OD/\cos\beta = x/\cos\beta$
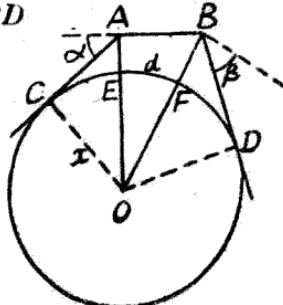
$\therefore\ \dfrac{AO}{BO} = \dfrac{\cos\beta}{\cos\alpha}$　　　　但　$\dfrac{AO}{BO} = \cos\dfrac{d}{x}$

$\therefore\ \dfrac{\cos\beta}{\cos\alpha} = \cos\dfrac{d}{x}$　　　$\therefore\ \dfrac{d}{x} = \cos^{-1}(\cos\beta\sec\alpha)$

$\therefore\ x = d/\cos^{-1}(\cos\beta\sec\alpha)$　　　（地球半徑）

73. 設 C 爲燈所在處，A,B 爲二測點

今　$EA = 64$ 呎，　$FB = 16$ 呎，　$OE = 4000$ 哩

船自 E 航至 F 費時 30 分鐘

因 $\widehat{EF}$ 比全圓周爲甚小

故　　$AB \doteqdot \widehat{EF}$

$$\overline{AC}^2 = \overline{OA}^2 - \overline{OC}^2$$

$$= (r+64)^2 - r^2 \doteqdot 128r$$

$\therefore$　$AC = 8\sqrt{2r}$　　　同理　　$BC = 4\sqrt{2r}$

$\therefore$　$AB = AC - BC = 4\sqrt{2r} = 4\sqrt{8000(5280)}$ 呎

故每時速　$\dfrac{6400\sqrt{66}}{6080} = 8.62$ 浬　（1 浬 $=6080.27$ 呎）

（參考 305 頁例二十七）
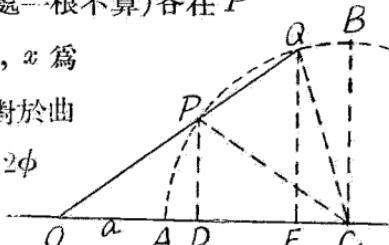
74. 設第 p 及 q 根電桿（A 處一根不算）各在 P

及 Q 點，C 爲圓心，x 爲

半徑，又連續兩桿對於曲

道中心之水平角爲 2ϕ

則　$2\phi = \dfrac{\pi}{2(n-1)}$

又　　　　$\angle PCO = 2p\phi$,　　$\angle QCO = 2q\phi$

$\therefore$　$\tan POD = \dfrac{\overline{PD}}{\overline{OD}} = \dfrac{\overline{QE}}{\overline{OE}}$

今　　　$\overline{PD} = \overline{PC} \sin 2p\phi = x \sin 2p\phi$

$\overline{OD} = \overline{OC} - \overline{DC} = a + x - x \cos 2p\phi$

$\overline{QE} = \overline{QC} \sin 2q\phi = x \sin 2q\phi$

$\overline{OE} = \overline{OC} - \overline{EC} = a + x - x \cos 2q\phi$
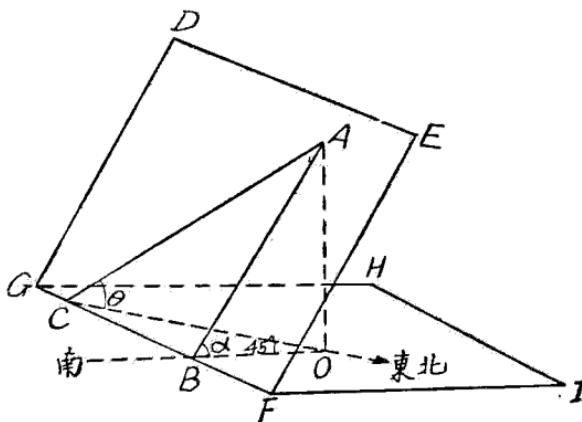
$$\therefore \qquad \frac{x \sin 2p\phi}{a+x-x\cos 2p\phi} = \frac{x \sin 2q\phi}{a+x-x\cos 2q\phi}$$

即 $\quad \dfrac{\sin 2p\phi}{\sin 2q\phi} = \dfrac{a+x(1-\cos 2p\phi)}{a+x(1-\cos 2q\phi)} = \dfrac{a+2x\sin^2 p\phi}{a+2x\sin^2 q\phi}$

即 $\quad 2x(\sin^2 p\phi \sin 2q\phi - \sin^2 q\phi \sin 2p\phi)$

$$= a(\sin 2p\phi - \sin 2q\phi)$$

$$= 2a \sin(p-q)\phi \cos(p+q)\phi$$

$$\therefore \quad x = \frac{a \sin(p-q)\phi \cos(p+q)\phi}{2 \sin p\phi \sin q\phi (\sin p\phi \cos q\phi - \cos p\phi \sin q\phi)}$$

$$= \frac{a \sin(p-q)\phi \cos(p+q)\phi}{2 \sin p\phi \sin q\phi \sin(p-q)\phi}$$

$$= \frac{a \cos(p+q)\phi}{2 \sin p\phi \sin q\phi}$$

75. 設平面 $DEFG$ 代表山之傾斜面，平面 $HIFG$ 代表地平
面，作 $AB \perp GF$，$AO \perp$ 平面 $HIFG$，連 OB，則
$OB \perp GF$（三垂線定理），又作 AC，使成束北向，連
OC，則由題意知，設 $\angle COB = \beta$，則得 $\beta = 45°$，

設　$\angle ACO = \theta$, 則 $\sin \theta$ 卽爲所求者,設 $\angle ABO = \alpha$.

則　$\sin \alpha = 1:5$

今　$\triangle AOB$, $\triangle AOC$, $\triangle OBC$ 均爲直角三角形

故　$\overline{AO} = \overline{CO} \tan \theta = \overline{BO} \tan \alpha = \overline{CO} \cos \beta \tan \alpha$

$\therefore$　$\tan \theta = \cos \beta \tan \alpha$　　　（參考 302 頁例二十四）

但　$\sin \alpha = \dfrac{1}{5}$　　　　　　　$\therefore$　$\tan \alpha = \dfrac{1}{2\sqrt{6}}$

又　$\cos \beta = \cos 45° = \dfrac{\sqrt{2}}{2}$　　$\therefore$　$\tan \theta = \dfrac{1}{4\sqrt{3}}$

卽　$\sin \theta = \dfrac{1}{7}$　　　　　故如題云.

76. 由 302 頁例二十四得　$\tan \theta = \cos \beta \tan \alpha$

　a.　今　　　　　$\alpha = 15°$,　$\beta = 45°$

　　　$\therefore$　$\tan \theta = \cos 45° \tan 15°$

　　　　$\log \cos \beta = 9.84949 - 10$

　　　　$\underline{\log \tan \alpha = 9.42805 - 10}$

　　　　$\log \tan \theta = 9.27754 - 10$

　　　　　$\therefore$　$\theta = 10°44'$

　b.　今　　　　$\theta = 5°$,　$\alpha = 15°$

　　　$\therefore$　$\cos \beta = \dfrac{\tan \theta}{\tan \alpha} = \dfrac{\tan 5°}{\tan 15°}$

　　　　$\log \tan \theta = 8.94195 - 10$

　　　　$\underline{\text{colog} \tan \alpha = 0.57195}$

　　　　$\log \cos \beta = 9.51390 - 10$

　　　　　$\therefore$　$\beta = N 19°3' E$

習 題 二 十 四 (329—332 頁)

1. $(1)^2 + (2)^2$,　　　　$x^2 + y^2 = \rho^2$

2. 從(1)　　　$m \sin \phi = 1 - \cos \phi$　　　(3)

　　從(2)　　　$n \sin \phi = 1 + \cos \phi$　　　(4)

　　(3),(4)相乘得　　　$mn = 1$

3. 自(1)　　　　$\cot^2 t = \dfrac{x^2}{y^2}$　　　(3)

　　自(2)　　　　$\csc^2 t = \dfrac{(a-x)^2}{y^2}$　　　(4)

　　(4)-(3),　　　$\dfrac{a^2 - 2ax}{y^2} = 1$

　　即　　　　$y^2 = a^2 - 2ax$　　（拋物線）

4. 從(2)　　　$\cos\theta = \dfrac{a-y}{b}$　　　$\therefore$　　$\theta = \cos^{-1}\dfrac{a-y}{b}$

　　及　　　　$\sin\theta = \dfrac{\sqrt{b^2 - (a-y)^2}}{b}$

　　代入(1)得　　$x = a\cos^{-1}\dfrac{a-y}{b} - \sqrt{b^2 - (a-y)^2}$

5. 今　　$2\tan 2\psi = m+n$,　　　$2\sin 2\psi = m-n$

　　即　　$\cot 2\psi = \dfrac{2}{m+n}$,　　　$\csc 2\psi = \dfrac{2}{m-n}$

　　故得　　　$\dfrac{4}{(m-n)^2} - \dfrac{4}{(m+n)^2} = 1$

　　即　　　　$16mn = (m^2 - n^2)^2$

6. 由(2) $\sin\theta = \dfrac{n}{\sqrt{m^2+n^2}}$, $\cos\theta = \dfrac{m}{\sqrt{m^2+n^2}}$

代入(1) $\dfrac{ax}{m} - \dfrac{by}{n} = \dfrac{c^2}{\sqrt{m^2+n^2}}$

7. 即 $x\sin 2\phi = 1 - \cos 2\phi$, $y\sin 2\phi = 1 + \cos 2\phi$

相乘得 $xy = 1$ (等邊雙曲線)

8. 從(1) $a = \dfrac{1}{\sin\theta} - \sin\theta = \dfrac{\cos^2\theta}{\sin\theta}$ (3)

從(2) $b = \dfrac{1}{\cos\theta} - \cos\theta = \dfrac{\sin^2\theta}{\cos\theta}$ (4)

即 $a^2b = \cos^3\theta$ $ab^2 = \sin^3\theta$

$\therefore$ $(a^2b)^{\frac{2}{3}} + (ab^2)^{\frac{2}{3}} = 1$

9. 從(1), $\dfrac{x}{a} = 2\cos\dfrac{3\theta}{2}\cos\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2}\left(4\cos^2\dfrac{\theta}{2} - 3\right)$ (3)

$(1)^2 + (2)^2$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2(1 + \cos\theta) = 4\cos^2\dfrac{\theta}{2}$ (4)

(4)中 $\cos^2\dfrac{\theta}{2}$代入(3)得

$$\dfrac{2x}{a} = \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 3\right)$$

別解: $(1)^2 + (2)^2$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2 + 2\cos\theta$

即 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 1 + 2\cos\theta$ (5)

又從(1) $\dfrac{x}{a} + 1 = \cos\theta + 2\cos^2\theta = \cos\theta(1 + 2\cos\theta)$

從(2) $\dfrac{y}{b} = \sin\theta(1 + 2\cos\theta)$

$$\therefore \quad \left(\frac{x}{a}+1\right)^2+\frac{y^2}{b^2}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2$$

10. $(1)^2+(2)^2,\qquad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\qquad$ （橢圓）

11. 卽　$a\sec\theta=y+x\tan\theta,\quad b\sec\theta=-y\tan\theta+x$

上兩式自乘相加　$(a^2+b^2)\sec^2\theta=(x^2+y^2)(1+\tan^2x)$

卽　　　　　　　　$x^2+y^2=a^2+b^2\qquad$ （圓）

12. 從(1)，$x+y=3-(1-2\sin^2 2\theta)=2+2\sin^2 2\theta\qquad$ (3)

從(3)+(2)，　$x=(1+\sin 2\theta)^2$

$\qquad\qquad\therefore\quad x^{\frac{1}{2}}=1+\sin 2\theta\qquad\qquad\qquad$ (4)

從(3)−(2)，　$y=(1-\sin 2\theta)^2$

$\qquad\qquad\therefore\quad y^{\frac{1}{2}}=1-\sin 2\theta\qquad\qquad\qquad$ (5)

$\qquad\qquad\therefore\quad x^{\frac{1}{2}}+y^{\frac{1}{2}}=2\qquad$ （拋物線）

別解：用他法消去則得　$(x-y)^2=8(x+y-2)$

卽　　　　　　　$(x+y-4)^2=4xy$

卽　　　　　　　$x+y-4=-2\sqrt{xy}\quad$ （取負號）

卽　　　　　　　$x+y+2\sqrt{xy}=4$

卽　　　　　　　$x^{\frac{1}{2}}+y^{\frac{1}{2}}=2$

13. 從(1)，　$x=\cot\phi+\dfrac{1}{\cot\phi}=\dfrac{1+\cot^2\phi}{\cot\phi}=\dfrac{\csc^2\phi}{\cot\phi}$

從(2)，　$y=\csc\phi-\dfrac{1}{\csc\phi}=\dfrac{\csc^2\phi-1}{\csc\phi}=\dfrac{\cot^2\phi}{\csc\phi}$

$\qquad x^2y=\csc^3\phi,\qquad xy^2=\cot^3\phi\qquad$ （參考 320 頁例五）

故　$(x^2y)^{\frac{2}{3}}-(xy^2)^{\frac{2}{3}}=1\qquad$ 卽　$x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}}=1$

14. 從(1)移項得　$x \sec^2\theta = a \tan^3\theta$　　∴　$x = a\left(\dfrac{\tan^3\theta}{\sec^2\theta}\right)$

同理從(2)　　$y \tan^2\theta = a \sec^3\theta$　　∴　$y = a\left(\dfrac{\sec^3\theta}{\tan^2\theta}\right)$

∴　　　　$x^2 y^3 = a^5 \sec^5\theta, \quad x^3 y^2 = a^5 \tan^5\theta$

∴　$x^{\frac{4}{5}} y^{\frac{6}{5}} - x^{\frac{6}{5}} y^{\frac{4}{5}} = a^2$

15　從(1)　　$x = a\cos\theta(4\cos^2\theta - 3) = a\cos 3\theta$

從(2)　　$y = b\sin\theta(3 - 4\sin^2\theta) = b\sin 3\theta$

故　　　　$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$　　　　（橢圓）

16.　今　$\begin{cases} x = \cos(\theta - a) = \cos\theta\cos\alpha + \sin\theta\sin\alpha \\ y = \sin(\theta - \beta) = \sin\theta\cos\beta - \cos\theta\sin\beta \end{cases}$

即　$\begin{cases} \sin\alpha\cdot\sin\theta + \cos\alpha\cdot\cos\theta - x = 0 \\ \cos\beta\cdot\sin\theta - \sin\beta\cdot\cos\theta - y = 0 \end{cases}$

∴　$\dfrac{\sin\theta}{-y\cos\alpha - x\sin\beta} = \dfrac{\cos\theta}{-x\cos\beta + y\sin\alpha}$

$= \dfrac{1}{-\cos(\alpha - \beta)}$

故　$(x\sin\beta + y\cos\alpha)^2 + (x\cos\beta - y\sin\alpha)^2$

$= \cos^2(\alpha - \beta)$

∴　　$x^2 - 2xy\sin(\alpha - \beta) + y^2 = \cos^2(\alpha - \beta)$

今因　$\triangle = 4\sin^2(\alpha - \beta) - 4 \lessgtr 0$，故此為橢圓或拋物線.

17.　即　　$x\cos\theta + y\sin\theta = 1, \quad x'\cos\theta + y'\sin\theta = 1$

$$\therefore \quad \cos\theta = \frac{\begin{vmatrix} 1 & y \\ 1 & y \end{vmatrix}}{\begin{vmatrix} x & y \\ x' & y' \end{vmatrix}} = \frac{y'-y}{xy'-x'y},$$

$$\sin\theta = \frac{\begin{vmatrix} x & 1 \\ x' & 1 \end{vmatrix}}{\begin{vmatrix} x & y \\ x' & y' \end{vmatrix}} = \frac{x-x'}{xy'-x'y}$$

$$\therefore \quad \left(\frac{y'-y}{xy'-x'y}\right) + \left(\frac{x-x'}{xy'-x'y}\right)^2 = 1$$

即　　　$(x-x')^2 + (y'-y)^2 = (xy'-x'y)^2$

18. 從(1),(2)　$a = \sin\theta(1+2\cos\theta)$,

$\qquad\qquad b+1 = \cos\theta(1+2\cos\theta)$

$\qquad\therefore \quad a^2 + (b+1)^2 = (1+2\cos\theta)^2$ 　　　(3)

$(1)^2 + (2)^2$, 　$2+2\cos\theta = a^2+b^2$

$\qquad\therefore \quad 1+2\cos\theta = a^2+b^2-1$ 　　　　　(4)

從(3),(4)　$a^2+(b+1)^2 = (a^2+b^2-1)^2$

即　　　　$(a^2+b^2)^2 - 3(a^2+b^2) - 2b = 0$

19. $(1)^2$, 　$x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta = x^2 + y^2$

即　　$x^2(1-\sin^2\theta) + 2xy\sin\theta\cos\theta + y^2(1-\cos^2\theta) = 0$

即　　$x^2\cos^2\theta + 2xy\sin\theta\cos\theta + y^2\sin^2\theta = 0$

即　$(x\cos\theta + y\sin\theta)^2 = 0$ 　即　$x\cos\theta + y\sin\theta = 0$

$\therefore \quad \tan^2\theta = \frac{x^2}{y^2}$ 　$\sin^2\theta = \frac{x^2}{x^2+y^2}$, 　$\cos^2\theta = \frac{y}{x^2+y}$

代入(2)化簡得　　$x^2/a^2 + y^2/b^2 = 1$ 　　（橢圓）

20. 從(1)
$$x^{\frac{1}{m}} = a^{\frac{1}{m}} \cos\theta \cos\phi \tag{4}$$

從(2)
$$y^{\frac{1}{m}} = b^{\frac{1}{m}} \cos\theta \sin\phi \tag{5}$$

從(3)
$$z^{\frac{1}{m}} = c^{\frac{1}{m}} \sin\theta \tag{6}$$

$(4)^2 + (5)^2 + (6)^2$, $\left(\dfrac{x}{a}\right)^{\frac{2}{m}} + \left(\dfrac{y}{b}\right)^{\frac{2}{m}} + \left(\dfrac{z}{c}\right)^{\frac{2}{m}} = 1$

(參考 322 頁例九)

21. 舊版之題同 321 頁例七，今則改爲

$$\begin{cases} x = r\cos\theta + r\theta\sin\theta & (1) \\ y = r\sin\theta - r\theta\cos\theta & (2) \end{cases}$$

$(1)^2 + (2)^2$, $x^2 + y^2 = r^2 + r^2\theta^2$ ∴ $\theta = \sqrt{x^2 + y^2 - r^2}/r$

代入(1)得　$x = r\cos(\sqrt{x^2 + y^2 - r^2}/r)$

$$+ \sqrt{x^2 + y^2 - r^2}\sin(\sqrt{x^2 + y^2 - r^2}/r)$$

22. 從(2)　$a\sec\theta + b\tan\theta = y\sec\theta\tan\theta$ (3)

$(1)+(3)$, $(a+b)(\tan\theta+\sec\theta) = y\sec\theta\tan\theta + x$ (4)

$(1)-(3)$, $(a-b)(\tan\theta-\sec\theta) = x - y\sec\theta\tan\theta$ (5)

$(3)^2 - (1)^2$, $a^2 - b^2 = y^2\sec^2\theta\tan^2\theta - x^2$

∴ $\sec^2\theta\tan^2\theta = \dfrac{a^2 - b^2 + x^2}{y^2}$ (6)

又從(4),(5) $(\tan\theta + \sec\theta)^2 - (\tan\theta - \sec\theta)^2$

$$= \left(\frac{y\sec\theta\tan\theta + x}{a+b}\right)^2 - \left(\frac{x - y\sec\theta\tan\theta}{a-b}\right)^2$$

卽　$(a^2-b^2)^2\tan\theta\sec\theta = -ab(y^2\tan^2\theta\sec^2\theta + x^2)$

$$+ xy(a^2+b^2)\tan\theta\sec\theta$$

以(6)之值代入且移項而平方之得

$$a^2b^2(a^2-b^2+2x^2)^2$$

$$=[xy(a^2+b^2)-(a^2-b^2)^2]^2\tan^2\theta\sec^2\theta$$

即　$a^2b^2y\;(a^2-b^2+2x^2)^2$

$$=[xy(a^2+b^2)-(a^2-b^2)^2]^2(a^2-b^2+x^2)\qquad(7)$$

註：若從(1)·(3) 得

$$ab(\tan^2\theta+\sec^2\theta)+(a^2+b^2)\tan\theta\sec\theta$$

$$=xy\tan\theta\sec\theta$$

即　$(a^2+b^2-xy)^2\tan^2\theta\sec^2\theta=a^2b^2(1+4\tan^2\theta\sec^2\theta)$

即　$(a^2-b^2+x^2)[(a^2+b^2-xy)^2-4a^2b^2]=a^2b^2y^2\qquad(8)$

(8)與(7)相差爲(7)中一因式 a^2-b^2 未曾括出．

23. $(1)^2+(2)^2,\; x^2+y^2=a^2+b^2+2ab\cos\theta\qquad(3)$

從(1)　　　　$x+b=\cos\theta(a+2b\cos\theta)$

從(2)　　　　$y=\sin\theta(a+2b\cos\theta)$

$\therefore\;(x+b)^2+y^2=(a+2b\cos\theta)^2$

即　　　$(x+b)^2+y^2=\left(\dfrac{x^2+y^2-b^2}{a}\right)^2$

故　　　$(x^2+y^2-b^2)^2=a^2[(x+b)^2+y^2]$

24.　　　$x=a\cos\theta\cos2\theta+2a\sin\theta\sin2\theta$

$$=a\cos\theta(1-2\sin^2\theta)+4a\sin^2\theta\cos\theta$$

$$=a\cos\theta(1+2\sin^2\theta)$$

$$y=2a\cos\theta\sin2\theta-a\sin\theta\cos2\theta$$

$$=4a\sin\theta\cos^2\theta-a\sin\theta(2\cos^2\theta-1)$$

$$= a \sin \theta (1 + 2 \cos^2 \theta)$$

$$\therefore \quad x + y = a(\cos \theta + \sin \theta) + 2a \sin \theta \cos \theta (\cos \theta + \sin \theta)$$

$$= a(\cos \theta + \sin \theta)(1 + 2 \sin \theta \cos \theta)$$

$$= a(\cos \theta + \sin \theta)^3 \qquad \because \quad 1 = \cos^2 \theta + \sin^2 \theta$$

同理 $\qquad x - y = a(\cos \theta - \sin \theta)^3$

$$\therefore \quad (x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = a^{\frac{2}{3}}[(\cos \theta + \sin \theta)^2$$
$$+ (\cos \theta - \sin \theta)^2] = 2a^{\frac{2}{3}}$$

25. **即**
$$\begin{cases} x \cos \theta \cos \alpha - x \sin \theta \sin \alpha + y \sin \theta \cos \alpha \\ \qquad + y \cos \theta \sin \alpha - a \sin 2\theta = 0 \\ x \sin \theta \cos \alpha + x \cos \theta \sin \alpha - y \cos \theta \cos \alpha \\ \qquad + y \sin \theta \sin \alpha + 2a \cos 2\theta = 0 \end{cases}$$

設 $\quad A = x \cos \alpha + y \sin \alpha, \qquad B = x \sin \alpha - y \cos \alpha$

則為 $\begin{cases} A \cos \theta - B \sin \theta = a \sin 2\theta \\ A \sin \theta + B \cos \theta = -2a \cos 2\theta \end{cases}$

用十字法得 $\quad A = a(\cos \theta \sin 2\theta) - 2a(\cos 2\theta \sin \theta)$

$$= a \sin \theta - a \sin \theta \cos 2\theta$$

$$= a \sin \theta (2 \sin^2 \theta) = 2a \sin^3 \theta$$

又 $\qquad B = -a \cos \theta - a \cos 2\theta \cos \theta = -2a \cos^3 \theta$

$$\therefore \quad A^{\frac{2}{3}} + B^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$$

即 $\quad (x \cos \alpha + y \sin \alpha)^{\frac{2}{3}} + (x \sin \alpha - y \cos \alpha)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$

26. **今** $\begin{cases} x + y = (\sin \theta + \cos \theta)(1 + \sin 2\theta) = (\sin \theta + \cos \theta)^3 \\ x - y = (\sin \theta - \cos \theta)(1 - \sin 2\theta) = (\sin \theta - \cos \theta)^3 \end{cases}$

$$\therefore \quad (x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2$$

27.
$$a = 4(\cos^3\theta - \sin^3\theta) - 3(\cos\theta - \sin\theta)$$
$$= (\cos\theta - \sin\theta)(4 + 4\cos\theta\sin\theta - 3)$$
$$= b(1 + 4\cos\theta\sin\theta) \qquad (3)$$

又從(1) $\qquad b^2 = 1 - 2\cos\theta\sin\theta \qquad (4)$

比較(3),(4) $\quad \therefore \quad 2(1-b^2)b = a-b$

即 $\qquad\qquad 2b^3 - 3b + a = 0$

28. 從(1) $\quad \dfrac{a}{b} = \dfrac{\tan(\theta+\alpha) + \tan(\theta-\alpha)}{\tan(\theta+\alpha) - \tan(\theta-\alpha)} = \dfrac{\sin 2\theta}{\sin 2\alpha}$

故 $\qquad\qquad \sin^2 2\theta = \dfrac{a^2}{b^2}\sin^2 2\alpha \qquad (1)$

從(2) $\qquad \cos 2\theta = \dfrac{c - a\cos 2\alpha}{b}$

$\qquad\qquad \therefore \quad \cos^2 2\theta = \dfrac{(c - a\cos 2\alpha)^2}{b^2} \qquad (2)$

$\therefore \quad \dfrac{a^2\sin^2 2\alpha}{b^2} + \dfrac{c^2 - 2ac\cos 2\alpha + a^2\cos^2 2\alpha}{b^2} = 1$

即 $\qquad\qquad a^2 - 2ac\cos 2\alpha + c^2 = b^2$

29. $(1) + (2)$, $\;2(ax+by) = (\sin\theta + \cos\theta)(1 + 2\sin\theta\cos\theta)$
$$= (\sin\theta + \cos\theta)^3$$

$(1) - (2)$, $\;2(bx+ay) = (\cos\theta - \sin\theta)(1 - 2\sin\theta\cos\theta)$
$$= (\cos\theta - \sin\theta)^3$$

$\therefore \quad 2^{\frac{2}{3}}(ax+by)^{\frac{2}{3}} + 2^{\frac{2}{3}}(bx+ay)^{\frac{2}{3}} = 2$

即 $\qquad (ax+by)^{\frac{2}{3}} + (bx+ay)^{\frac{2}{3}} = 2^{\frac{1}{3}}$

30. 從(1) $\qquad x = \csc\theta\tan^3\theta(\cot^2\theta + 2) + \sec\theta$

$$= \csc\theta\tan\theta + 2\csc\theta\tan^3\theta + \sec\theta$$

$$= \sec\theta + 2\sec\theta(\sec^2\theta - 1) + \sec\theta = 2\sec^3\theta$$

從(2)　　$y = \tan^3\theta(\cot^2\theta + 2) - \tan\theta$

$$= \tan\theta + 2\tan^3\theta - \tan\theta = 2\tan^3\theta$$

$$\therefore\quad x^{\frac{2}{3}} - y^{\frac{2}{3}} = 2^{\frac{2}{3}}$$

31. 從(1)　　　　$p^2\sin^2\theta = (\sin^2\theta + 1)^2$

即　　　　　$p^2(1 - \cos^2\theta) = (2 - \cos^2\theta)^2$

即　　$\cos^4\theta + (p^2 - 4)\cos^2\theta + 4 - p^2 = 0$　　　　　(3)

從(2)　　$q^2\cos^2\theta = \cos^4\theta + 2\cos^2\theta + 1$

即　　$\cos^4\theta + (2 - q^2)\cos^2\theta + 1 = 0$　　　　　(4)

自(3),(4)　　$\dfrac{\cos^4\theta}{p^2 - 4 - (p^2 - 4)(q^2 - 2)} = \dfrac{\cos^2\theta}{4 - p^2 - 1}$

$$= \dfrac{1}{2 - q^2 - p^2 + 4}$$

故　$(p^2 - 4)(-3 + q^2)(-6 + p^2 + q^2) = (p^2 - 3)^2$　　　(5)

又如(1)從 $\sin^4\theta$, $\sin^2\theta$ 着手則

$$(q^2 - 4)(p^2 - 3)(p^2 + q^2 - 6) = (q^2 - 3)^2 \tag{6}$$

(5)(6)實爲一式,因均等於

$$p^2 q^2(p^2 + q^2) - 4(p^4 + q^4) + 36(p^2 + q^2)$$

$$- 81 - 13 p^2 q^2 = 0 \ \text{也}.$$

32. 用十字法照 $1/a$, $1/b$, $1/c$, 解之得

$$\frac{1/a}{\dfrac{\cos 2\theta}{2} - \cos^2\theta} = \frac{1/b}{\cos 2\theta\cos\theta - \dfrac{\cos 3\theta}{2}} = \frac{1/c}{\cos 3\theta\cos\theta - \cos^2 2\theta}$$

即 $\qquad \dfrac{1/a}{-\dfrac{1}{2}} = \dfrac{1/b}{\dfrac{\cos\theta}{2}} = \dfrac{1/c}{\dfrac{\cos 2\theta - 1}{2}}$

即 $\qquad -\dfrac{1}{a} = \dfrac{1}{b\cos\theta} = -\dfrac{1}{2c\sin^2\theta}$

$\therefore \quad \sin^2\theta = \dfrac{a}{2c}, \quad \cos^2\theta = \dfrac{a^2}{b^2}$

$\therefore \quad \dfrac{a}{2c} + \dfrac{a^2}{b^2} = 1 \qquad$ 即 $\qquad ab^2 + 2c(a^2 - b^2) = 0$

33. $\quad \dfrac{\sin(\theta-\alpha)}{\sin(\theta-\beta)} = \dfrac{a}{b} \quad (1), \qquad \dfrac{\cos(\theta-\alpha)}{\cos(\theta-\beta)} = \dfrac{a'}{b'} \quad (2)$

從(1) $\quad b(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$\qquad\qquad = a(\sin\theta\cos\beta - \cos\theta\sin\beta)$

即 $\quad b(\tan\theta\cos\alpha - \sin\alpha) = a(\tan\theta\cos\beta - \sin\beta)$

$\therefore \quad \tan\theta = \dfrac{a\sin\beta - b\sin\alpha}{a\cos\beta - b\cos\alpha} \qquad (3)$

同理從(2) $\quad \tan\theta = \dfrac{-(a'\cos\beta - b'\cos\alpha)}{a'\sin\beta - b'\sin\alpha} \qquad (4)$

從(3),(4) $\quad (a\sin\beta - b\sin\alpha)(a'\sin\beta - b'\sin\alpha)$

$\qquad\qquad + (a\cos\beta - b\cos\alpha)(a'\cos\beta - b'\cos\alpha) = 0$

$\therefore \quad \cos(\alpha-\beta) = (aa' + bb')/(ab' + a'b)$

或從 $\qquad \dfrac{\sin 2(\theta-\alpha)}{\sin 2(\theta-\beta)} = \dfrac{aa'}{bb'}$

即 $\quad \dfrac{2\sin(2\theta - \alpha - \beta)\cos(\alpha - \beta)}{\sin 2(\theta-\beta)} = \dfrac{aa' + bb'}{bb'} \qquad (3)$

又(1)+(2), $\quad \dfrac{\sin(2\theta - \alpha - \beta)}{\sin(\theta-\beta)\cos(\theta-\beta)} = \dfrac{ab' + a'b}{bb'} \qquad (4)$

$(3) \div (4), \quad \cos(\alpha - \beta) = (aa' + bb')/(ab' + a'b)$

34. 從(1)　$\sin \theta \cos \alpha + \cos \theta \sin \alpha = 2a \sin \theta \cos \theta$

即　　$\tan \theta \cos \alpha - 2a \sin \theta + \sin \alpha = 0$　　　(3)

同理從(2)　$\tan \theta \cos \beta - 2a \sin \theta + \sin \beta = 0$　　(4)

$\therefore \quad \dfrac{\tan \theta}{2a(\sin \alpha - \sin \beta)} = \dfrac{\sin \theta}{\sin(-\beta + \alpha)} = \dfrac{1}{2a(\cos \beta - \cos \alpha)}$

即　$\dfrac{\tan \theta}{2a \cos \frac{1}{2}(\alpha + \beta)} = \dfrac{\sin \theta}{\cos \frac{1}{2}(\alpha - \beta)} = \dfrac{1}{2a \sin \frac{1}{2}(\alpha + \beta)}$

即　$\cot \theta = \tan \frac{1}{2}(\alpha + \beta), \qquad \csc \theta = \dfrac{2a \sin \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}$

$\therefore \quad 1 + \tan^2 \frac{1}{2}(\alpha + \beta) = \dfrac{4a^2 \sin^2 \frac{1}{2}(\alpha + \beta)}{\cos^2 \frac{1}{2}(\alpha - \beta)}$

即　$\dfrac{1}{\cos^2 \frac{1}{2}(\alpha + \beta)} = \dfrac{4a^2 \sin^2 \frac{1}{2}(\alpha + \beta)}{\cos^2 \frac{1}{2}(\alpha - \beta)}$

$\therefore \quad \cos^2 \frac{1}{2}(\alpha - \beta) = 4a^2 \sin^2 \frac{1}{2}(\alpha + \beta) \cos^2 \frac{1}{2}(\alpha + \beta)$

$\qquad\qquad = a^2 \sin^2(\alpha + \beta)$

35. $(1) + (2), \; a(\tan \theta + \cot \theta) + b(\cot 2\theta - \tan 2\theta) = 2c$

即　　$\dfrac{a}{\sin \theta \cos \theta} + \dfrac{b \cos 4\theta}{\cos 2\theta \sin 2\theta} = 2c$

即　　$\dfrac{a}{\sin 2\theta} + \dfrac{b \cos 4\theta}{\sin 4\theta} = c$

即　　$a \csc 2\theta + b \cot 4\theta = c$　　　　(3)

$(2) - (1), \; a(\cot \theta - \tan \theta) = b(\cot 2\theta + \tan 2\theta)$

即　　　$2a \cot 2\theta = b(\cot 2\theta + \tan 2\theta)$

即　　　$b \tan 2\theta = (2a - b)\cot 2\theta$

$$\therefore \quad \tan^2 2\theta = \frac{2a-b}{b}$$

則 $\quad \cot^2 2\theta = \frac{b}{2a-b}, \quad \csc^2 2\theta = \frac{2a}{2a-b}$

又 $\quad \cot 4\theta = \frac{\cot^2 2\theta - 1}{2 \cot 2\theta} = \frac{b-a}{\sqrt{b(2a-b)}}$

代入(3) $\quad a\sqrt{\frac{2a}{2a-b}} + (b-a)\sqrt{\frac{b}{2a-b}} = c$

$$\therefore \quad c\sqrt{2a-b} = a\sqrt{2a} - (a-b)\sqrt{b}$$

36. 從(2) $\quad \tan^2 \frac{\theta}{2} = \tan^{\frac{2}{3}}\alpha \qquad$ 即 $\quad \frac{1-\cos\theta}{1+\cos\theta} = \frac{\sin^{\frac{2}{3}}\alpha}{\cos^{\frac{2}{3}}\alpha}$

$$\therefore \quad \frac{\cos\theta}{1} = \frac{\cos^{\frac{2}{3}}\alpha - \sin^{\frac{2}{3}}\alpha}{\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha}$$

即 $\quad \frac{m^2-1}{3} = \frac{(\cos^{\frac{2}{3}}\alpha - \sin^{\frac{2}{3}}\alpha)^2}{(\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha)^2}$

即 $\quad m^2(\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha)^2 = 4(\cos^{\frac{4}{3}}\alpha - \cos^{\frac{2}{3}}\alpha\sin^{\frac{2}{3}}\alpha + \sin^{\frac{4}{3}}\alpha)$

即 $\quad m^2(\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha)^3 = 4[(\cos^{\frac{2}{3}}\alpha)^3 + (\sin^{\frac{2}{3}}\alpha)^3] = 4$

$$\therefore \quad \cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha = (2/m)^{\frac{2}{3}}$$

或自(2) $\quad \tan^2 \frac{\theta}{2} = \tan^{\frac{2}{3}}\alpha = \frac{1-\cos\theta}{1+\cos\theta}$

$$\therefore \quad \frac{\sin^{\frac{2}{3}}\alpha}{1-\cos\theta} = \frac{\cos^{\frac{2}{3}}\alpha}{1+\cos\theta} = \frac{\sin^{\frac{2}{3}}\alpha + \cos^{\frac{2}{3}}\alpha}{2} \qquad (3)$$

又(3)³, $\quad \frac{\sin^2\alpha}{(1-\cos\theta)^3} = \frac{\cos^2\alpha}{(1+\cos\theta)^3} = \frac{1}{2+6\cos^2\theta} = \frac{1}{2m^2}$

即　$\dfrac{\sin^{\frac{2}{3}}\alpha}{1-\cos\theta}=\left(\dfrac{1}{2m^2}\right)^{\frac{1}{3}}$　　∴　$\sin^{\frac{2}{3}}\alpha+\cos^{\frac{2}{3}}\alpha=\left(\dfrac{2}{m}\right)^{\frac{2}{3}}$

37.　$(1)+(2)$,　$\sqrt{2}\,x(\sin\theta+\cos\theta)$

$$=\dfrac{a}{\sqrt{2}}+\dfrac{a}{\sqrt{2}}(\sin 2\theta+\cos 2\theta)$$

即　$2x(\sin\theta+\cos\theta)=a+a(\sin 2\theta+\cos 2\theta)$

$$=a(1+2\sin\theta\cos\theta)+a\cos 2\theta$$

$$=a(\sin\theta+\cos\theta)^2+a(\cos^2\theta-\sin^2\theta)$$

即　$2x=a(\sin\theta+\cos\theta)+a(\cos\theta-\sin\theta)$

即　$x=a\cos\theta$　　　　　　(3)

$(1)-(2)$,　　$\sqrt{2}\,y(\sin\theta-\cos\theta)$

$$=\dfrac{a}{\sqrt{2}}-\dfrac{a}{\sqrt{2}}(\sin 2\theta+\cos 2\theta)$$

即　$2y(\sin\theta-\cos\theta)=a(1-\sin 2\theta)-a\cos 2\theta$

即　$2y=a(\sin\theta-\cos\theta)+a(\cos\theta+\sin\theta)$

∴　　$y=a\sin\theta$　　　　　　(4)

$(3)^2+(4)^2$,　　　$x^2+y^2=a^2$

38.　$(1)+(2)$,　$(a+b)(\sin 3\theta+\cos 3\theta)$

$$=2c\cos\left(\theta+\dfrac{\pi}{12}\right)\cos\dfrac{\pi}{12}$$

即　　　$\dfrac{1}{\sqrt{2}}(a+b)(\sin 3\theta+\cos 3\theta)$

$$=\sqrt{2}\cdot c\left(\dfrac{\sqrt{3}}{2\sqrt{2}}+\dfrac{1}{2}\right)\cos\left(\theta+\dfrac{\pi}{12}\right)$$

$$(a+b)\sin\left(3\theta+\frac{\pi}{4}\right)=c\left(\frac{\sqrt{3}+1}{2}\right)\cos\left(\theta+\frac{\pi}{12}\right)$$

設　$\theta+\frac{\pi}{12}=\phi,\quad 3\theta+\frac{\pi}{4}=3\phi,\quad h=\frac{c(\sqrt{3}+1)}{2(a+b)}$

則　　　　　　　$\sin 3\phi = h\cos\phi$　　　　　　(3)

$(1)-(2),\quad (a-b)(\cos 3\theta-\sin 3\theta)$

$$=2c\sin\left(\theta+\frac{\pi}{12}\right)\sin\frac{\pi}{12}$$

$$(a-b)\cos\left(3\theta+\frac{\pi}{4}\right)=c\left(\frac{\sqrt{3}-1}{2}\right)\sin\left(\theta+\frac{\pi}{12}\right)$$

設　$k=\frac{c(\sqrt{3}-1)}{2(a-b)}$　　　則　$\cos 3\phi = k\sin\phi$　　(4)

$(3)\div(4),\quad h\cos^2\phi(4\cos^2\phi-3)$

$$=k\sin^2\phi(3-4\sin^2\phi)$$　　　　　(5)

又 $(3)^2+(4)^2,\quad 1=h^2\cos^2\phi+k^2\sin^2\phi$

即　　　　　$1-k^2=(h^2-k^2)\cos^2\phi$

或　　　　　$1-h^2=(k^2-h^2)\sin^2\phi$　　　　(6)

以(6)代入(5)　　$h\left(\frac{1-k^2}{h^2-k^2}\right)\left[4\left(\frac{1-k^2}{h^2-k^2}\right)-3\right]$

$$=k\left(\frac{1-h^2}{k^2-h^2}\right)\left[3-4\left(\frac{1-h^2}{k^2-h^2}\right)\right]$$

$$h(1-k^2)(4-k^2-3h^2)=k(1-h^2)(3k^2+h^2-4)$$

化簡之可得　$(h-k)^2(3-hk)=4(1-hk)^2$

39.　∵　$\theta=\frac{\pi}{2}+\phi$　　　∴　$\sin\theta=\cos\phi$ 及 $\cos\theta=-\sin\phi$

故原兩式爲
$$\begin{cases} \dfrac{ax}{\sin\phi} + \dfrac{by}{\cos\phi} + (a^2-b^2)=0 \\[2mm] \dfrac{by}{\sin\phi} - \dfrac{ax}{\cos\phi} + (a^2-b^2)=0 \end{cases}$$

$$\therefore\ \frac{\dfrac{1}{\sin\phi}}{(a^2-b^2)(by+ax)} = \frac{\dfrac{1}{\cos\phi}}{(a^2-b^2)(by-ax)} = -\frac{1}{a^2x^2+b^2y^2}$$

$$\therefore\ \left[\frac{a^2x^2+b^2y^2}{(a^2-b^2)(by+ax)}\right]^2 + \left[\frac{a^2x^2+b^2y^2}{(a^2-b^2)(by-ax)}\right]^2 = 1$$

$$\left[\frac{a^2x^2+b^2y^2}{a^2-b^2}\right]^2\left[\frac{1}{(by+ax)^2} + \frac{1}{(by-ax)^2}\right] = 1$$

即　$2(a^2x^2+b^2y^2)^3 = (a^2-b^2)^2(a^2x^2-b^2y^2)^2$

40. 從(2)　　$\tan\theta + \tan\phi = b\tan\theta\tan\phi$

以(1)代入　　$\tan\theta\tan\phi = a/b$

今　$\tan\alpha = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \dfrac{a}{1-\dfrac{a}{b}} = \dfrac{ab}{b-a}$

故　　　　$ab + (a-b)\tan\alpha = 0$

41. $(1)^2+(2)^2,\quad 2+2\cos(\theta-\phi)=a^2+b^2$

故　　　　$a^2+b^2 = 2(1+\cos\alpha)$

42. 從(1)　$\dfrac{\dfrac{x}{a}}{\sin\phi-\sin\theta} = \dfrac{\dfrac{y}{b}}{\cos\theta-\cos\phi} = \dfrac{1}{\sin(\phi-\theta)}$

$$\therefore\quad \frac{x}{a} = \frac{\cos\frac12(\theta+\phi)}{\cos\frac12(\phi-\theta)},\qquad \frac{y}{b} = \frac{\sin\frac12(\theta+\phi)}{\cos\frac12(\phi-\theta)}$$

$$\therefore\quad \frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{2}{1+\cos(\theta-\phi)} = \frac{2}{1+\cos2\alpha}$$

故　　　　　　　$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2}{1 + \cos 2\alpha}$

43. $(3)^2 - (2)^2,\ \ 2 + \dfrac{2}{\sin\theta\sin\phi} - \dfrac{2\cos\theta\cos\phi}{\sin\theta\sin\phi} = c^2 - b^2$

$$\therefore\ \ \frac{2 - 2\cos(\theta + \phi)}{\sin\theta\sin\phi} = c^2 - b^2$$

$$2[1 - \cos(\theta + \phi)] = (c^2 - b^2)\sin\theta\sin\phi \qquad (4)$$

從(3)　　　　$\sin\theta + \sin\phi = c\sin\theta\sin\phi \qquad (5)$

從(2)　　　　$\sin(\theta + \phi) = b\sin\theta\sin\phi \qquad (6)$

$(4)/(5),\quad 2c\sin\dfrac{\theta + \phi}{2} = (c^2 - b^2)\cos\dfrac{\theta - \phi}{2} \qquad (7)$

$(5)/(6),\quad b\cos\dfrac{\theta - \phi}{2} = c\cos\dfrac{\theta + \phi}{2} \qquad (8)$

$4(8)^2 + (7)^2,\ 4c^2 = [4b^2 + (b^2 - c^2)^2]\cos^2\dfrac{\theta - \phi}{2} \qquad (9)$

又從(1)　　　$2\cos\dfrac{\theta + \phi}{2}\cos\dfrac{\theta - \phi}{2} = a$

以(8)代入　　$2b\cos^2\dfrac{\theta - \phi}{2} = ac \qquad (10)$

從(9),(10)　$8bc^2 = ac[4b^2 + (b^2 - c^2)^2]$

即　　　　　　$8bc = a[4b^2 + (b^2 - c^2)^2]$

44. $(1)^2 + (2)^2.\quad a^2 + b^2 + 2ab\cos(\theta + \phi) = c^2$

即　　　　　　$\cos(\theta + \phi) = \dfrac{c^2 - a^2 - b^2}{2ab}$

即　　　　　　$\sec^2(\theta + \phi) = \dfrac{4a^2b^2}{(c^2 - a^2 - b^2)^2}$

從(3)　　　　$\tan^2(\theta + \phi) = x^2/y^2$

則　　　　$\dfrac{x^2}{y^2} = \dfrac{4a^2b^2}{(c^2-a^2-b^2)^2} - 1$

$$= \dfrac{(2ab-c^2+a^2+b^2)(2ab+c^2-a^2-b^2)}{(c^2-a^2-b^2)^2}$$

今分子 $= [(a+b)^2-c^2][c^2-(a-b)^2]$

$$= (a+b+c)(a+b-c)(c-a+b)(c+a-b)$$

$$= 16s(s-a)(s-b)(s-c)$$

（設　$a+b+c = 2s$）

代入上式，去分母又開平方，則得．

$$x(c^2-a^2-b^2) = 4y\sqrt{s(s-a)(s-b)(s-c)}$$

或從 $(1),(2)$ 消去 θ 得　$\cos\phi = \dfrac{b^2+c^2-a^2}{2bc}$

則　　　　　　　$\cos\theta = \dfrac{c^2+a^2-b^2}{2ca}$

設法化入 (3) 亦可求得，惟較此爲繁耳．

45. $(1)+(2)$,　$a+b = m\cos^2\phi + n\sin^2\phi$

即　$(a+b)(\sin^2\phi+\cos^2\phi) = m\cos^2\phi + n\sin^2\phi$

$\therefore$　$\tan^2\phi = (m-a-b)/(a+b-n)$　　　　　(4)

$m(2)+n(1)$,　$(ma+nb)\sin^2\theta + (mb+na)\cos^2\theta$

$$= mn(\sin^2\theta+\cos^2\theta)$$

$\therefore$　$\tan^2\theta = (mn-na-mb)/(ma+nb-mn)$

代入 (3)　$\dfrac{m(mn-na-mb)}{ma+nb-mn} - \dfrac{n(m-a-b)}{a+b-n} = 0$

化簡得　　$(a+b)[(a+b)(m+n)-2mn] = 0$

$\therefore$　　　　$a+b = 0$ 或 $a+b = 2mn/(m+n)$

46 $(1)^3$, $\qquad (\cos\theta+\cos\phi)(\cos^2\theta+\cos^2\phi$

$$+2\cos\theta\cos\phi)=a^3 \qquad (4)$$

自(3) $\quad 4(\cos^3\theta+\cos^3\phi)-3(\cos\theta+\cos\phi)=c$

$\therefore \qquad (\cos\theta+\cos\phi)(4\cos^2\theta-4\cos\theta\cos\phi$

$$+4\cos^2\phi-3)=c \qquad (5)$$

從(4),(5) $\quad 2a^3+c=(\cos\theta+\cos\phi)(6\cos^2\theta$

$$+6\cos^2\phi-3)$$

$$=3a(2\cos^2\theta+2\cos^2\phi-2+1)$$

$$=3a(\cos 2\theta+\cos 2\phi+1)$$

即 $\qquad 2a^3+c=3a(b+1) \qquad$ 〔以(2)值代入〕

47. $(1)\cdot(2)$, $\quad \sin 2\theta+\sin 2\phi+2\sin(\theta+\phi)=2ab$

即 $\qquad \sin(\theta+\phi)[\cos(\theta-\phi)+1]=ab \qquad (4)$

$(1)^2+(2)^2$, $\quad 2+2\cos(\theta-\phi)=a^2+b^2$

$$\therefore \quad \cos(\theta-\phi)=\frac{a^2+b^2-2}{2}$$

代入(4) $\qquad \sin(\theta+\phi)=\frac{2ab}{a^2+b^2} \qquad (5)$

$(2)^2-(1)^2$, $\quad \cos 2\phi+\cos 2\theta+2\cos(\phi+\theta)=b^2-a^2$

即 $\qquad 2\cos(\theta+\phi)[\cos(\theta-\phi)+1]=b^2-a^2$

$$\therefore \quad \cos(\theta+\phi)=\frac{b^2-a^2}{a^2+b^2}$$

今 $\quad \cos(\theta+\phi)+\cos(\theta-\phi)=\frac{b^2-a^2}{a^2+b^2}+\frac{a^2+b^2-2}{2}$

即 $\quad \cos\theta\cos\phi=[(a^2+b^2)^2-4a^2]/4(a^2+b^2) \qquad (6)$

從(3) $\sin(\theta+\phi)=c\cos\theta\cos\phi$

以(5),(6)代入,得 $\dfrac{2ab}{a^2+b^2}=c\dfrac{(a^2+b^2)^2-4a^2}{4(a^2+b^2)}$

卽 $\dfrac{8ab}{c}=(a^2+b^2)^2-4a^2$

48. (1)−(2), $\dfrac{x}{a}\cdot 2\sin\dfrac{\theta+\phi}{2}\sin\dfrac{\phi-\theta}{2}$

$$+\dfrac{y}{b}\cdot 2\sin\dfrac{\theta-\phi}{2}\cos\dfrac{\theta+\phi}{2}=0$$

卽 $\dfrac{x}{a}\sin\dfrac{\theta+\phi}{2}-\dfrac{y}{b}\cos\dfrac{\theta+\phi}{2}=0$

∴ $\dfrac{\sin\dfrac{\theta+\phi}{2}}{ay}=\dfrac{\cos\dfrac{\theta+\phi}{2}}{bx}=\dfrac{1}{\sqrt{a^2y^2+b^2x^2}}$ (4)

從(1),(2)得 $\sin\theta\left(1-\dfrac{x}{a}\cos\phi\right)=\sin\phi\left(1-\dfrac{x}{a}\cos\theta\right)$

卽 $\dfrac{x}{a}\sin(\phi-\theta)=\sin\phi-\sin\theta$

$$=2\sin\dfrac{\phi-\theta}{2}\cos\dfrac{\phi+\theta}{2}$$

∴ $\cos\dfrac{\theta-\phi}{2}=\dfrac{a}{x}\cos\dfrac{\theta+\phi}{2}=\dfrac{ab}{\sqrt{b^2x^2+a^2y^2}}$ (5)

從(3) $a^2\left(\cos\dfrac{\theta-\phi}{2}-\cos\dfrac{\theta+\phi}{2}\right)$

$$+b^2\left(\cos\dfrac{\theta-\phi}{2}+\cos\dfrac{\theta+\phi}{2}\right)=2c^2$$

卽 $(a^2+b^2)\cos\dfrac{\theta-\phi}{2}-(a^2-b^2)\cos\dfrac{\theta+\phi}{2}=2c^2$

以(4),(5)之值代入，化簡之得

$$b[x(b^2-a^2)+a(a^2+b^2)]=2c^3\sqrt{b^2x^2+a^2y^2}$$

或法：從(3), 設 $\tan\dfrac{\theta}{2}=t$　$\tan\dfrac{\phi}{2}=t'$, 則為

$$(a^2tt'+b^2)^2=c^4\sec^2\dfrac{\theta}{2}\sec^2\dfrac{\phi}{2}$$

$$=c^4(1+t^2)(1+t'^2)$$

$$=c^4[1+(t+t')^2-2tt'+t^2t'^2] \qquad (4)$$

今　$\sin\theta=\dfrac{2t}{1+t^2}$, $\cos\theta=\dfrac{1-t^2}{1+t^2}$

代入(1),　$b(a+x)t^2-2ayt+b(a-x)=0$

同理可得　$b(a+x)t'^2-2ayt'+b(a-x)=0$

故 t,t' 為下式 T 之根

$$b(a+x)T^2-2ayT+b(a-x)=0$$

∴　$t+t'=2ay/b(a+x)$, $tt'=(a-x)/(a+x)$

代入(4),　$b^2[x(b^2-a^2)+a(a^2+b^2)]^2=4c^4(b^2x^2+a^2y^2)$

再法：從(3),設 $r=\cos\theta$, $r'=\cos\phi$, 則

$$a^2\sqrt{(1-r)(1-r')}+b^2\sqrt{(1+r)(1+r')}=2c^2 \quad (5)$$

從(1),　　　　　$\dfrac{y}{b}\sin\theta=1-\dfrac{x}{a}\cos\theta$

即　　　　　$a^2y^2(1-r^2)=b^2(a-rx)^2$

即　　　$(b^2x^2+a^2y^2)r^2-2ab^2xr+a^2(b^2-y^2)=0$

同理從(2),　$(b^2x^2+a^2y^2)r'^2-2ab^2xr'+a^2(b^2-y^2)=0$

故 r,r' 為下式 A 之根

$$(b^2x^2 + a^2y^2)A^2 - 2ab^2xA + a^2(b^2 - y^2) = 0$$

$$\therefore \quad r + r' = \frac{2ab^2x}{b^2x^2 + a^2y^2}, \quad rr' = \frac{a^2(b^2 - y^2)}{b^2x^2 + a^2y^2}$$

今　$(1-r)(1-r') = 1 - (r + r') + rr$

$$= \frac{b^2(a-x)^2}{b^2x^2 + a^2y^2},$$

$$(1+r)(1+r') = 1 + (r + r') + rr'$$

$$= \frac{b^2(a+x)^2}{b^2x^2 + a^2y^2}$$

代入(5)得　$b^2[x(b^2 - a^2) + a(a^2 + b^2)]^2 = 4c^4(b^2x^2 + a^2y^2)$

49. $(3)^2,$　　$1 - \sin(\theta + \phi) = 2(x^2 - y^2)^2$　　　　(4)

$(1)^2 + (2)^2,$　$2 + 2\sin(\theta + \phi) = 16x^2y^2 + 4(x^2 - y^2)^2$

$$= 4(x^2 + y^2)^2$$

$$\therefore \quad 1 + \sin(\theta + \phi) = 2(x^2 + y^2)^{\frac{1}{2}} \tag{5}$$

$(4) + (5),$　　$2 = 2(x^2 - y^2)^2 + 2(x^2 + y^2)^2$

$$\therefore \quad x^4 + y^4 = \tfrac{1}{2}$$

50. 即

$$\begin{cases} x - y\cos\gamma - z\cos\beta = 0 \\ x\cos\gamma - y + z\cos\alpha = 0 \\ x\cos\beta + y\cos\alpha - z = 0 \end{cases}$$

今設三式中 x, y, z 有共同之解,則

$$\triangle = \begin{vmatrix} 1 & -\cos\gamma & -\cos\beta \\ \cos\gamma & -1 & \cos\alpha \\ \cos\beta & \cos\alpha & -1 \end{vmatrix} = 0$$

即　　$1 - 2\Pi\cos\alpha - \cos^2\alpha - \cos^2\beta - \cos^2\gamma = 0$

$$-2-2\Pi\cos\alpha+\Sigma\sin^2\alpha=0$$

$$\therefore\quad\Sigma\sin^2\alpha=2(1+\Pi\cos\alpha)$$

51. 即　　　　$bx\cos\theta+ay\sin\theta=ab$　　　　(1)

　　　　　　$bx\cos\phi+ay\sin\phi=ab$　　　　(2)

從(1)　　　$b^2x^2\cos^2\theta=a^2(b-y\sin\theta)^2$

化去 $\cos\theta$ 且合併之

　$(b^2x^2+a^2y^2)\sin^2\theta-2a^2by\sin\theta+b^2(a^2-x^2)=0$

同樣化(2)可得 $\sin\phi$ 之同一形式,故知 $\sin\theta$, $\sin\phi$ 爲

　$(b^2x^2+a^2y^2)\lambda^2-2a^2by\lambda+b^2(a^2-x^2)=0$ 之根.

故從根與係數關係得

　　　$\sin\theta+\sin\phi=2a^2by/(b^2x^2+a^2y^2)$　　　(4)

同理從(1),(2),各化去正弦可知 $\cos\theta$, $\cos\phi$ 爲

　$(b^2x^2+a^2y^2)\mu^2-2ab^2x\mu-a^2(b^2-y^2)=0$ 之根.

故　　$\cos\theta+\cos\phi=2ab^2x/(b^2x^2+a^2y^2)$　　　(5)

以(4),(5)代入(3)　$\dfrac{2a^2b^2}{b^2x^2+a^2y^2}+\dfrac{2a^2b^2}{b^2x^2+a^2y^2}=4$

即　　　　　$b^2x^2+a^2y^2=a^2b^2$

即　　　　　$x^2/a^2+y^2/b^2=1$　　　（橢圓）

52. 化第一式爲　$y\sin\theta=2a-x\cos\theta$　　　(1)

　　　　　　$y\sin\phi=2a-x\cos\phi$　　　(2)

從(1)　$y^2(1-\cos^2\theta)=4a^2-4ax\cos\theta+x^2\cos^2\theta$

即　$(x^2+y^2)\cos^2\theta-4ax\cos\theta+(4a^2-y^2)=0$

同理化(2)　$(x^2+y^2)\cos^2\phi-4ax\cos\phi+(4a^2-y^2)=0$

故知 $\cos\theta,\ \cos\phi$ 爲下式 λ 之根

$$(x^2+y^2)\lambda^2-4ax\lambda+(4a^2-y^2)=0$$

故　　　　$\cos\theta+\cos\phi=4ax/(x^2+y^2),$

$$\cos\theta\cos\phi=(4a^2-y^2)/(x^2+y^2)$$

化 (3)　　　　$4\cos^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\phi=1$

即　　　　$(1+\cos\theta)(1+\cos\phi)=1$

即　　　　$(\cos\theta+\cos\phi)+\cos\theta\cos\phi=0$

代入得　$\dfrac{4ax}{x^2+y^2}+\dfrac{4a^2-y^2}{x^2+y^2}=0$　　　　故　$y^2=4a(x+a)$

或從　$x\cos\theta+y\sin\theta=2a,\quad x\cos\phi+y\sin\phi=2a$

得　$\dfrac{x}{\sin\phi-\sin\theta}=\dfrac{y}{\cos\theta-\cos\phi}=\dfrac{a}{\sin(\phi-\theta)}$

即　$\dfrac{x}{\cos\tfrac{1}{2}(\theta+\phi)}=\dfrac{y}{\sin\tfrac{1}{2}(\theta+\phi)}=\dfrac{2a}{\cos\tfrac{1}{2}(\theta-\phi)}$

$$=\dfrac{x+2a}{\cos\tfrac{1}{2}(\theta+\phi)+\cos\tfrac{1}{2}(\theta-\phi)}$$

即　$\dfrac{x^2+y^2}{\cos^2\tfrac{1}{2}(\theta+\phi)+\sin^2\tfrac{1}{2}(\theta+\phi)}=\left(\dfrac{x+2a}{2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\phi}\right)^2$

即　　　　$\dfrac{x^2+y^2}{1}=\left(\dfrac{x+2a}{1}\right)^2$

即　　　　$y^2=4a(x+a)$　　　　（拋物線）