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Florian Cajori.

PLANE AND SOLID GEOMETRY

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INTRODUCTION.

The study of Geometry is pursued with a threefold purpose.

1. To aid in the development of logical reasoning.

2. To stimulate the use of accurate and precise forms of expression.

3. To acquire facts and principles that may be of practical value in subsequent life.

The first two purposes are advocated because of their disciplinary importance; and when mathematics, because of its exactness, was the only science which furthered to a high degree these purposes, it was necessary for the student to devote a large part of his time to their study. But now other sciences, and even the languages and philosophy, claim disciplinary merit equal to that possessed by mathematics, although differing somewhat in the character of the training.

Hence it appears that the time has come when we can afford to hearken to the demands of the utilitarians, and give up those refinements in mathematics which have been retained for the mental discipline they bring about, but which are wholly lacking in practical application.

I have therefore, out of an experience as a computer and worker in applied mathematics, as well as a teacher, eliminated from this treatise all propositions that are not of practical value or needed in the demonstration of such propositions.

This exclusion leaves out about one-half of the matter usually included in our text-books on geometry. However, instructors will not entirely miss those familiar and interesting theorems which helped to swell the books they studied, — such theorems as fall below the practical standard are here given as exercises or as corollaries.

Until within the past two decades the verbiage of demonstrations was so elaborate that the student was tempted to memorize. The natural reaction resulted, and for a while our authors passed to the other extreme in symbolic expressions. While symbols and equational statements have the advantage of brevity, and convey information to the mind through its most receptive channel, — the eye, — still they discourage the use of language, and hence fail to develop by example and precept the employment of accurate and precise forms of expression.

I have therefore sought to use symbols and equations only in those cases where I could see no gain in spelling out their meaning.

Attention is called to the solution of problems. Ordinarily the problem is presumed to be solved, and then a demonstration is given to show that the solution was correct. This does not appear to me to be in the line of discovery. I have in all cases started with a statement of those known facts which plainly suggest the first step in the solution, then introduced the next step, giving the construction in connection with each stated fact, so that with the completed construction goes its own demonstration and the student sees the road along which he travelled, and understands from the beginning why he started upon it.

Great care has been exercised in the selection of exercises to follow each demonstration. They are intended to be variations upon the theorem demonstrated, or extensions of it, so that at least a portion of the required proof is suggested. At the end of each book will be found a larger collection of exercises, formulæ, and numerical examples.

In conclusion, I may state that no claim is made to originality in demonstration; I have employed those I deemed best; however, no statement is taken from another author unless it is the common property of several.

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GEOMETRY.

PRELIMINARY DEFINITIONS.

1. Space has extension in all directions, and so far as our experience can teach us it is limitless.

2. A material or physical body occupies a definite portion of space, and this space freed from the body is called a geometrical solid, which for brevity will be known as a *Solid*.

3. The limits, or boundaries, of a solid are *Surfaces*.

The limits, or boundaries, of a surface are *Lines*.

The intersection of two lines is a *Point*.

4. It is said a solid has three dimensions: *Length*, *Breadth*, and *Thickness*.

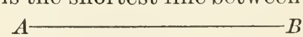
A surface has only two dimensions: length and breadth.

A line has only one dimension: length.

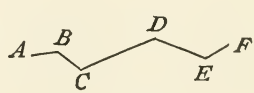
A point is without dimension, having simply position.

5. In drawings and diagrams material figures are employed for purposes of demonstration, but they are merely the representatives of mathematical figures.

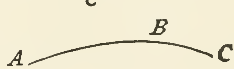
6. A *Straight Line*, or *Right Line*, is the shortest line between two points; as *AB*



7. A *Broken Line* is a line composed of different successive straight lines; as *ABCDEF*.



8. A *Curved Line*, or simply a *Curve*, is a line no portion of which is straight; as *ABC*.



9. A *Plane Surface*, or simply a *Plane*, is one such that the straight line which joins any two of its points lies entirely in the surface.

10. A *Curved Surface* is one no portion of which is plane.

11. A *Geometrical Figure* is any combination of points, lines, surfaces, or solids formed under specific conditions.

Plane Figures are formed by points and lines in a plane; *Rectilinear*, or *Right-lined Figures*, are formed of straight lines.

12. *Geometry* is that branch of mathematics which treats of the construction of figures, of their measurement, and of their properties.

Plane Geometry treats of plane figures.

Solid Geometry, sometimes called *Geometry of Space* and *Geometry of Three Dimensions*, treats of solids, of curved surfaces, and of all figures that are not represented on a plane.

13. A *Theorem* is a truth requiring demonstration.

14. A *Problem* is a question proposed for solution.

15. A *Postulate* assumes the possibility of the solution of some problem.

16. An *Axiom* is a truth assumed to be true, or a truth verified by intuition or our experience with material things.

17. A *Proposition* is a general term for theorem, axiom, problem, and postulate.

18. A *Demonstration* is the course of reasoning by which the truth of a theorem is established.

19. A *Corollary* is a conclusion which follows immediately from a theorem, but this conclusion may at times demand demonstration.

20. A *Lemma* is an auxiliary theorem required in the demonstration of a principal theorem.

21. A *Scholium* is a remark upon one or more propositions.

22. An *Hypothesis* is a supposition made either in the enunciation of a proposition or in the course of a demonstration.

23. A *Solution* of a problem is the method of construction which accomplishes the required end.

24. A *Construction* is the drawing of such lines and curves as may be required to prove the truth of a theorem, or to solve a problem.

25. The *Enunciation* of a theorem consists of two parts: the *Hypothesis*, or that which is assumed; and the *Conclusion*, or that which is asserted to follow therefrom.

POSTULATES.

26. 1. A straight line can be drawn between any two points.

2. A straight line can be produced indefinitely in either direction.

27. Given		Prove
Axiom (Assumed truth)	Proposition	Theorem That something is true
Postulate (Assumed possibility)		Problem That something can be done

AXIOMS.

28. 1. Things which are equal to the same thing are equal to each other.

2. If equals be multiplied or divided by equals, the results will be equal.

3. If equals be added to or subtracted from equals, the results will be equal.

4. If equals are added to or subtracted from unequals, the results will be unequal.

5. The whole is equal to the sum of its parts.

6. The whole is greater than any of its parts.

ABBREVIATIONS.

29. The following is a list of the symbols which will be used as abbreviations:

$+$, plus.	$>$, is greater than.
$-$, minus.	$<$, is less than.
$\times$, multiplied by.	$\therefore$, therefore.
$=$, equals.	$\angle$, angle.

In addition to these, the following may be used for writing demonstrations on the board or in exercise books, but no use is made of them in the present work.

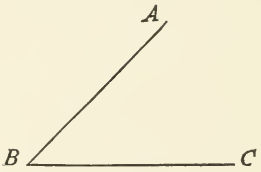
$\lfloor\!\underline{\mathrm{R}}$, right angle.	$\triangle\!\!\!\!\text{s}$, triangles.
$\angle\!\text{s}$, angles.	$\square$, parallelogram.
$\perp$, perpendicular.	$\boxed{\text{s}}$, parallelograms.
$\perp\!\text{s}$, perpendiculars.	$\odot$, circle.
$\parallel$, parallel.	Ⓢ, circles.
$\parallel\!\text{s}$, parallels.	$\frown$, arc.
$\triangle$, triangle.	

PLANE GEOMETRY

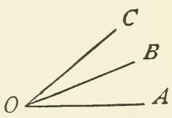
BOOK I.

RECTILINEAR FIGURES.

30. An *Angle* is the difference in direction of two lines that meet or might meet; if the two lines meet, the point of meeting is called the *Vertex*, and the lines are called its *Sides*. Thus, in the angle formed by AB and BC, B is the vertex, and AB and BC are the sides.



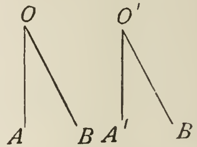
31. An isolated angle may be designated by the letter at its vertex, as "the angle O"; but when several angles are formed at the same point by different lines, as OA, OB, OC, we designate the angle intended by three letters; namely, by one letter on each of its sides, together with the one at its vertex, which must be written between the other two. Thus, with these lines there are formed three different angles, which are distinguished as AOB, BOC, and AOC.



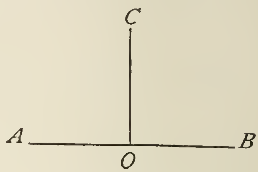
32. Two angles, such as AOB, BOC, which have the same vertex O and a common side OB between them, are called *Adjacent.*

33. The magnitude of an angle depends wholly upon the amount of divergence of its sides, and is independent of their length.

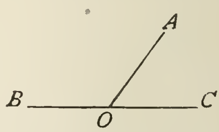
34. Two angles are *Equal* when one can be placed upon the other so that they shall coincide. Thus, the angles AOB and $A'O'B'$ are equal, if $A'O'B'$ can be superposed upon AOB so that while $O'A'$ coincides with OA, $O'B'$ shall also coincide with OB, or when the difference in the directions of the sides of one angle is the same as the difference in the directions of the sides of the other.



35. When one straight line meeting another makes the adjacent angles equal to each other, each of the angles is called a *Right Angle;* and the two lines thus meeting are said to be *perpendicular* to each other or at *right* angles to each other. Thus, if the adjacent angles AOC and BOC are equal to each other, each is a right angle, and the line CO is perpendicular to AB, and AB is perpendicular to CO. The point O is called the *Foot* of the perpendicular.

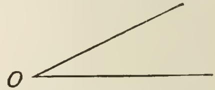


36. An *Oblique Angle* is formed by one straight line meeting another so as to make the adjacent angles *Unequal.*



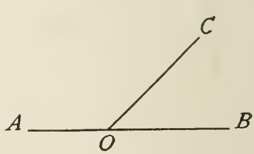
Oblique angles are subdivided into two classes, *Acute Angles* and *Obtuse Angles.*

37. An *Acute Angle* is less than a right angle, as the angle O.

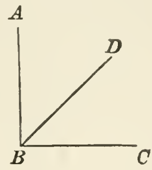


38. An *Obtuse Angle* is greater than a right angle, as the angle AOB (in 36).

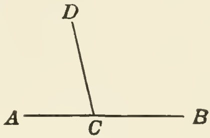
39. A *Straight Angle* has its sides extending in opposite directions so as to be in the same straight line. Thus, if OA, OB are in the same straight line, the angle formed by them is called a straight angle.



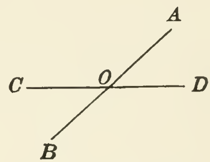
40. The *Complement* of an angle is the difference between a right angle and the given angle. Thus, ABD is the complement of the angle DBC; also DBC is the complement of the angle ABD.



41. The *Supplement* of an angle is the difference between a straight angle and the given angle. Thus, ACD is the supplement of the angle DCB; also DCB is the supplement of the angle ACD.



42. *Vertical Angles* are angles which have the same vertex, and their sides extending in opposite directions. Thus the angles AOD and COB are vertical angles, as also the angles AOC and DOB.



43. The magnitude of an angle is measured by finding the number of times which it contains another angle adopted arbitrarily as the unit of measurement.

The usual unit of measurement is the *Degree*, or the ninetieth part of a right angle. To express fractional parts of the unit, the degree is divided into sixty equal parts, called *Minutes*, and the minute into sixty equal parts, called *Seconds*.

Degrees, minutes, and seconds are denoted by the symbols $^\circ$, $'$, $''$, respectively; thus, $28^\circ\ 42'\ 36''$ stands for 28 degrees 42 minutes and 36 seconds.

EXERCISES.

1. How many degrees are there in the complement of 27°? of 65°? of $18^\circ\ 17'$? of $38^\circ\ 18'\ 35''$? of $\frac{3}{5}$ of a right angle?

2. How many degrees are there in the supplement of 68°? of $124^\circ\ 16'$? of $142^\circ\ 18'\ 46''$? of $\frac{4}{5}$ of a right angle?

3. How many degrees are there in an angle which is the complement of three times itself?

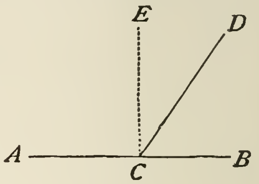
4. How many degrees are there in an angle whose supplement is two times its complement?

5. How many degrees are there in an angle if its complement and supplement are together equal to 120°?

Proposition I. Theorem.

44. *If a straight line meets another straight line, the sum of the adjacent angles is equal to two right angles.*

Let DC meet AB at C; then the sum of the angles DCA and DCB is equal to two right angles.



At C, let CE be drawn perpendicular to AB; then, by definition, the angles ECA and ECB are both right angles, and consequently their sum is equal to *two right angles.*

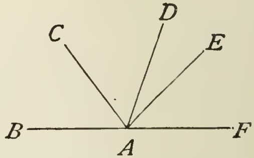
The angle DCA is equal to the sum of the angles ECA and ECD; hence,

$$\angle DCB + \angle DCA = (\angle DCB + \angle DCE) + \angle ECA$$
$$= \angle ECB + \angle ECA.$$

But by construction $\angle ECB$ and $\angle ECA$ are right angles, therefore

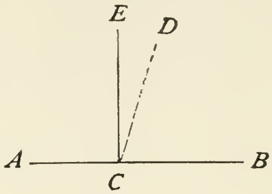
$$\angle DCB + \angle DCA = 2 \text{ right angles.}$$

45. Cor. 1. If one of the angles DCA, DCB, is a right angle, the other must also be a right angle.



46. Cor. 2. The sum of the angles BAC, CAD, DAE, EAF, formed about a given point on the same side of a straight line BF, is equal to two right angles. For their sum is equal to the sum of the angles EAB and EAF; which, from the proposition just demonstrated, is equal to two right angles.

47. Cor. 3. At a given point in a straight line, and on a given side of the line, only one perpendicular to that line can be erected. For if two could be erected, let them be EC and DC; then (by 35) $\angle ECB$ is a right angle, likewise $\angle DCB$ is a right angle, or (by 28)



$$\angle ECB = \angle DCB,$$

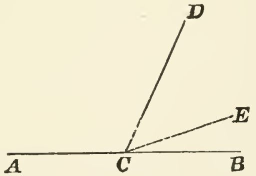
which is impossible, as $\angle DCB$ is a part of $\angle ECB$.

EXERCISE.

If in Cor. 2 the angles EAF, CAD, and BAC are equal, and each twice as large as the angle DAE, what will be the size of each angle in degrees?

Proposition II. Theorem.

48. Conversely,* *if the sum of two adjacent angles is equal to two right angles or to a straight angle, their exterior sides lie in the same straight line.*



Let the sum of the adjacent angles ACD and BCD be equal to two right angles.

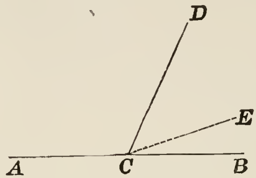
To prove that ACB is a straight line.

If ACB is not a straight line, let CE be in the same straight line with AC.

* Hereafter converse propositions will not be demonstrated, but given as corollaries or exercises. The method of demonstration, which in general will be identical to the direct proposition, or a proof that any other condition would not be true, will, however, be indicated.

Then (by 41) $\angle ECD$ is the supplement of $\angle ACD$.

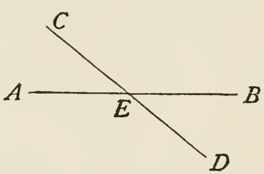
But by hypothesis $\angle BCD$ is the supplement of $\angle ACD$.



Therefore, since supplements of the same angle must be equal to one another, $\angle ECD$ must be equal to $\angle BCD$, which (by 28) is impossible except when CE coincides with CB, or CB is the only line that is a prolongation of AC.

Proposition III. Theorem.

49. *If two straight lines intersect each other, the vertical angles are equal.*



Let the straight lines AB and CD intersect at E.

To prove that $\angle AEC = \angle BED$.

By (44), $\angle AEC + \angle CEB =$ two right angles,

and $\angle BED + \angle CEB =$ two right angles.

Therefore, by (28),

$$\angle AEC + \angle CEB = \angle BED + \angle CEB.$$

Subtracting $\angle CEB = \angle CEB$,

we have $\angle AEC = \angle BED$. Q.E.D.

In the same way it may be proved that

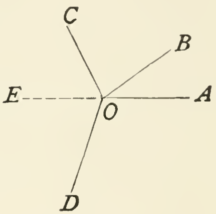
$$\angle AED = \angle CEB.$$

50. Cor. 1. *If two straight lines intersect each other, the four angles which they make at the point of intersection are together equal to four right angles.*

If one of the four angles is a right angle, the other three are right angles, and the lines are mutually perpendicular to each other.

51. Cor. 2. *If any number of straight lines meet at a point, the sum of all the angles having this vertex in common is equal to four right angles.*

The sum of all the angles AOB, BOC, COD, DOA, formed about a point, is equal to four right angles.



For if the line OA is produced to E, the sum of the angles AOB, BOC, and COE is equal to two right angles, and the same is true of the sum of the angles AOD and DOE.

Hence the sum of the angles AOB, BOC, COD, and DOA is equal to four right angles.

EXERCISES.

1. If in the above figure the angles AOB, BOC, and AOD are respectively 42°, 85°, and $\frac{7}{6}$ of a right angle, how many degrees are there in COD?

2. If in Prop. III. the angle CEA is $34° 21'$, how many degrees are there in each of the other angles?

3. If A's complement is equal to one-sixth of A's supplement, find A.

4. In Prop. III. the angle CEA is equal to one-fourth of angle CEB. How many degrees are there in each of the other angles?

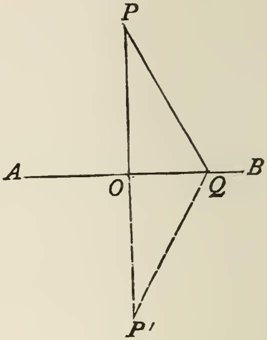
5. If in Cor. 2, angle COE is equal to angle EOD, show that angle COA is equal to angle AOD.

6. If the angles BOA, EOC, and COB are in the ratio of 2 : 3 : 5, how many degrees are there in each?

7. If the angles BOA, BOC, AOD, and COD are in the ratio of 1 : 2 : 3 : 4, how many degrees are there in each?

Proposition IV. Theorem.

52. *From a point outside a straight line only one perpendicular can be drawn to such straight line, and this perpendicular is the shortest distance from the point to the line.*



Let P be the point, AB the line, and PO a perpendicular.

1. To prove that PO is the only perpendicular that can be drawn, and that it is the shortest distance from P to AB.

Produce PO to P', making $OP' = OP$; then the angles POB and $P'OB$ are right angles.

If any other perpendicular can be drawn, suppose it be PQ, and join $P'Q$.

Revolve the figure OPQ about AB as an axis; then, since POQ is a right angle, OP will fall on OP', and the point P will coincide with P', and, Q remaining stationary, PQ will fall upon $P'Q$.

Therefore, if $\angle PQO$ be a right angle, $\angle P'QO$ must also be a right angle, and (from 48) the lines PQP' must be straight; this would give two straight lines joining P and P', which is impossible, or PQ cannot be perpendicular to AB.

Hence only one perpendicular can be drawn.

2. To prove that PO is the shortest distance from P to the line AB.

It was just shown that OP could be made to coincide with OP', and PQ with QP', or,

$$PO = P'O \text{ and } PQ = QP'.$$

Since PP' is a straight line, it is the shortest line that can be drawn from P to P', or $$PP' < PQP';$$

that is, $$PO + OP' < PQ + QP',$$

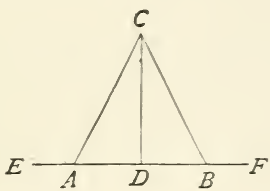
or $$2\,PO < 2\,PQ.$$

Dividing by 2, $$PO < PQ.$$

Hence the perpendicular is the shortest distance from a point to a straight line. Q.E.D.

PROPOSITION V. THEOREM.

53. *Two oblique lines drawn from a point to a straight line, cutting off equal distances from the foot of the perpendicular, are equal.*



Let the oblique lines CA and CB meet the line EF at equal distances from the foot of the perpendicular CD.

To prove that $CA = CB$.

Let the part CDA be revolved about CD until DE falls upon its prolongation, DF; then, since $AD = DB$ by hypothesis, and CD remains stationary, the point A will fall on B, and the line CA will fall on CB, and be equal to it.

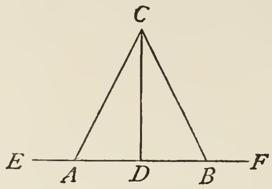
Hence $CA = CB$. Q.E.D.

54. COR. 1. Since D is the middle point of AB, DC a perpendicular, and C any point on this perpendicular, it is true that every point on the perpendicular bisector of a straight line is equally distant from the extremities of that line.

55. Cor. 2. When CAD was revolved, it was found that AD fell upon BD and AC upon BC.

$$\therefore \angle CAD = \angle CBD,$$

and, similarly, $\angle ACD = \angle DCB$;

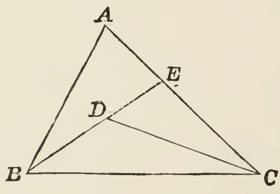


that is, the two equal oblique lines drawn from the same point in the perpendicular make equal angles with the perpendicular and also with the base.

56. Cor. 3. *If two points on a line are equally distant from the extremities of another line, the first line is a perpendicular bisector of the second.*

Proposition VI. Theorem.

57. *If two lines are drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but enveloped by them.*



Let AB and AC be drawn from the point A to the extremities of the line BC, and let DB and DC be two lines similarly drawn, but enveloped by AB and AC.

To prove that $AB + AC > DB + DC$.

Produce BD to meet AC at E.

Since BE is a straight line,

$$BE < BA + AE.$$

Add EC, $$BE + EC < BA + AE + EC$$
$$< BA + AC.$$

Since DC is a straight line,

$$DC < DE + EC.$$

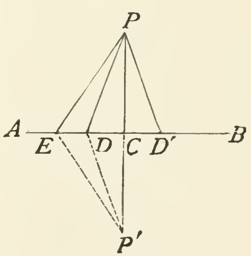
Add BD, $$BD + DC < (BD + DE) + EC$$
$$< BE + EC.$$

But $$BE + EC < BA + AC.$$
$$\therefore BD + DC < BA + AC.$$
$$\therefore AB + AC > DB + DC.$$ Q.E.D.

Proposition VII. Theorem.

58. *Of two oblique lines drawn from the same point to the same straight line, that which meets the line at the greater distance from the foot of the perpendicular is the greater.*

Let PC be perpendicular to AB, and PD and PE two oblique lines cutting off unequal distances from C.



To prove $PE > PD$.

Produce PC to P', making

$$CP' = CP;$$

join $P'D$ and $P'E$;

then (by 54) $P'D = PD$, and $P'E = PE$.

By (57) $$PE + P'E > PD + P'D,$$

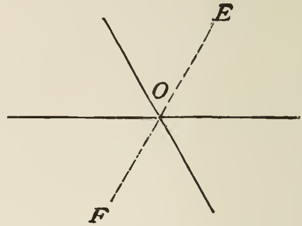
or $$2\,PE > 2\,PD.$$

Dividing by 2, $$PE > PD.$$ Q.E.D.

58 *a.* Cor. *Only two equal straight lines can be drawn from a point to a straight line.*

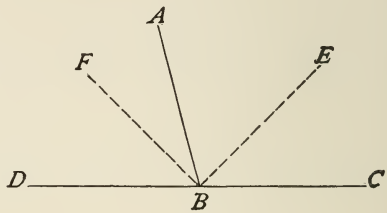
EXERCISES.

1. If the oblique lines are on opposite sides of the perpendicular, show that the theorem (58) is true.



2. Prove that the bisectors of two vertical angles are in the same straight line.

Suggestion. Show that the sum of the angles on one side of FE is equal to the sum of those on the other side.



3. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.

Suggestion. Show that $\angle EBF = \frac{1}{2}\, CBD = \frac{1}{2}$ a straight angle.

4. If the angle ABD is 86° 14′, how many degrees are there in each of the other angles formed at B?

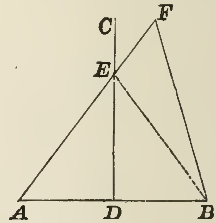
5. If the angle ABD is two-thirds of the angle ABC, how many degrees are there in each of the other angles formed at B?

6. If the straight line CD is the shortest line that can be drawn from C without the line AB to AB, show that CD is perpendicular to AB.

7. If a perpendicular is erected at the middle point of a line, any point without the perpendicular is unequally distant from the extremities of the line.

That is, $FA > FB$.

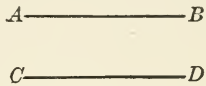
Suggestion. $EA = EB$; add EF to both sides of this equation.



8. If any point be taken within a triangle, show that the sum of the lines joining the point to the vertices is less than the sum of the sides of the triangle.

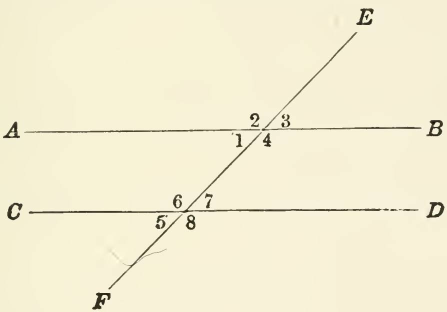
PARALLEL LINES.

59. Definition. Two straight lines are called *Parallel* when they lie in the same plane, and cannot meet nor approach each other, however far they may be produced; as *AB* and *CD*.



60. Axioms. 1. But one straight line can be drawn through a given point parallel to a given straight line.

2. Since parallel lines cannot approach each other, they are everywhere equally distant from each other.



If a straight line *EF* cut two other straight lines *AB* and *CD*, it makes with those lines eight angles, to which particular names are given.

The angles 1, 4, 6, 7 are called *Interior* angles.

The angles 2, 3, 5, 8 are called *Exterior* angles.

The pairs of angles 1 and 7, 4 and 6, are called *Alternate-interior* angles.

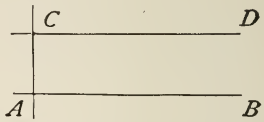
The pairs of angles 2 and 8, 3 and 5, are called *Alternate-exterior* angles.

The pairs of angles 1 and 5, 2 and 6, 4 and 8, 3 and 7, are called *Exterior-interior* angles.

The angles 2 and 6, 3 and 7, 4 and 8, 1 and 5, are called *Corresponding* angles.

Proposition VIII. Theorem.

61. *Two straight lines perpendicular to the same straight line are parallel.*



Let AB and CD be perpendicular to CA.

To prove AB and CD are parallel.

If they are not parallel, they will meet, and if they meet, there will be two lines from this point of meeting perpendicular to the same line, which (by 52) is impossible.

Therefore CD and AB, if perpendicular to AC, are parallel.

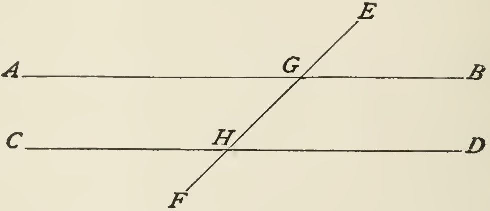
EXERCISES.

1. Prove that two straight lines parallel to the same straight line are parallel to each other.

2. Prove that a straight line perpendicular to one of two parallels is also perpendicular to the other.

Proposition IX. Theorem.

62. *If two parallel straight lines be cut by a third straight line, the alternate-interior angles are equal.*



Let AB and CD be two parallel straight lines cut by the line EF at G and H.

To prove $\angle AGH = \angle GHD$.

The lines AB and CD, being parallel, have the same direction.

The lines EG and GH, being in one and the same straight line, are similarly directed.

That is, the angles EGB and GHD have sides with the same direction; therefore the differences of their directions are equal,

or (by 30), $\angle EGB = \angle GHD$.

But (by 49), $\angle EGB = \angle AGH$.

Therefore (by 28), $\angle AGH = \angle GHD$. Q.E.D.

EXERCISES.

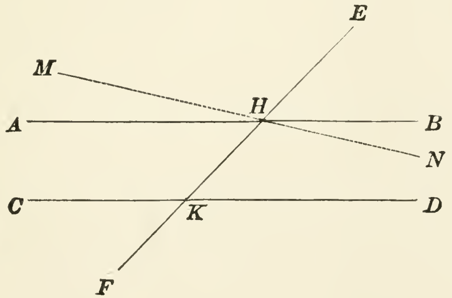
1. Prove that the alternate-exterior angles are equal,

or $\angle EGA = \angle DHF$, or $\angle EGB = \angle CHF$.

2. Prove that the sum of the two interior angles on the same side of the cutting line, or transversal, is equal to two right angles.

Suggestion. $\angle AGH = \angle GHD$; add $\angle GHC$.

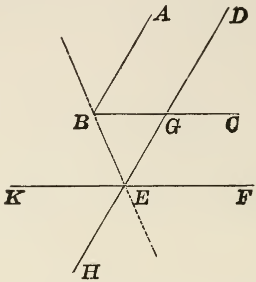
3. When two straight lines are cut by a third straight line, if the exterior-interior angles be equal, these two straight lines are parallel.



Suggestion. If AB is not parallel to CD, draw MN parallel to CD; then apply 62 and 28.

Proposition X. Theorem.

63. *Two angles whose sides are parallel each to each are either equal or supplementary.*



Let AB be parallel to DH and BC to KF.

To prove that the angle ABC is equal to DEF and supplementary to DEK.

Let BC and DE intersect at G.

1. Since EG and GD have the same direction, likewise EF and GC, the difference in the directions of EG and EF is the same as the difference in the directions of GD and GC; that is,

$$\angle GEF = \angle DGC.$$

Similarly, $\qquad \angle ABC = \angle DGC;$

(by 28) $\qquad \angle ABC = \angle DEF.$ Q.E.D.

2. $\angle GEF$ is the supplement of $\angle GEK$.

$\therefore$ $\angle ABC$, which is equal to $\angle GEF$, will be the supplement of $\angle GEK$. Q.E.D.

3. By (49) $\qquad \angle KEH = \angle GEF.$

But $\qquad \angle GEF = \angle ABC.$

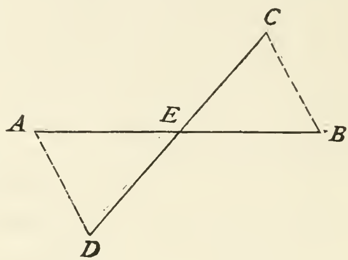
$\therefore$ by (28) $\qquad \angle KEH = \angle ABC.$

Scholium. Two parallels are said to be in the *same* direction, or in *opposite* directions, according as they lie on the same side or on opposite sides of the straight line joining their origins.

Thus AB and ED, and also BC and EF, are in the same direction because they lie on the same side of BE. But BA and EH, and also BC and EK, are in opposite directions.

63 *a.* The angles are *equal* when both pairs of parallel sides extend in the same direction, or in opposite directions.

The angles are *supplementary* when one pair of parallel sides extend in the same direction and the other pair in opposite directions.



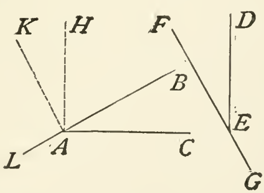
EXERCISE.

If AB and CD bisect each other at E, show that the straight lines CB and AD are parallel.

Suggestion. Apply CEB to DEA, and show that $\angle A = \angle B$.

Proposition XI. Theorem.

64. *Two angles having their sides perpendicular each to each, are either equal or supplementary.*



Let DE be perpendicular to AC, and FG to AB.

To prove that the angle BAC is equal to FED and supplementary to DEG.

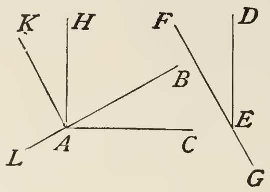
1. From A draw AH perpendicular to AC, and AK perpendicular to LB; then AH is parallel to DE, and AK to FG.

By (63) $\qquad\qquad \angle KAH = \angle FED.$

By construction the angles KAB and HAC are right angles, and are therefore equal; that is,

$$\angle KAB = \angle HAC$$

or $$\angle KAH + \angle HAB = \angle HAB + \angle BAC;$$



subtracting $\angle HAB$,

$$\angle KAH = \angle BAC.$$

But $$\angle KAH = \angle FED.$$

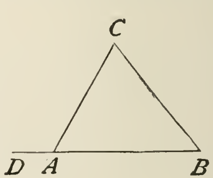
$\therefore$ (by 28) $$\angle BAC = \angle FED.$$

2. $\angle DEG$ is the supplement of $\angle FED$, and is therefore the supplement of the equal of $\angle FED$ or of $\angle BAC$.

TRIANGLES.

65. A *Triangle* is a plane figure bounded by three straight lines.

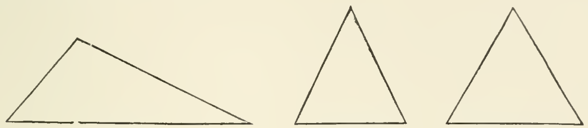
The three straight lines which bound a triangle are called its *Sides*. Thus AB, BC, CA, are the sides of the triangle ABC.



The *angles* of the triangle are the angles formed by the sides with each other; as BAC, ABC, ACB. The vertices of these angles are also called the *Vertices* of the triangle.

66. An *Exterior Angle* of a triangle is the angle formed between any side and the continuation of another side; as CAD.

The angles *BAC, ABC, BCA*, are called *Interior Angles* of the triangle. When we speak of the angles of a triangle, we mean the three interior angles.



SCALENE. ISOSCELES. EQUILATERAL.

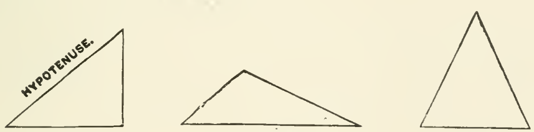
67. A *Scalene* triangle is one no two of whose sides are equal.

68. An *Isosceles* triangle is one two of whose sides are equal.

69. An *Equilateral* triangle is one three of whose sides are equal.

70. The *Base* of a triangle is the side on which the triangle is supposed to stand.

In an isosceles triangle, the side which is not one of the equal sides is considered the base.



RIGHT. OBTUSE. ACUTE.

71. A *Right* triangle is one which has one of the angles a right angle.

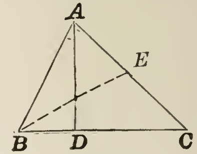
72. The side opposite the right angle is called the *Hypotenuse.*

73. An *Obtuse* triangle is one which has one of the angles an obtuse angle.

74. An *Acute* triangle is one which has all the angles acute.

75. An *Equiangular* triangle is one of which the three angles are equal.

76. When any side has been taken as the base, the opposite angle is called the *Vertical Angle* and its vertex is called the vertex of the triangle.



The *Altitude* of a triangle is the perpendicular drawn from the vertex to the base, produced if necessary.

Thus in the triangle ABC, BC is the base, BAC the vertical angle, and AD the altitude.

A *Medial* is a line that joins a vertex with the middle point of the opposite side, as BE.

77. Since a straight line is the shortest distance between two points (by 6), it follows that *either side of a triangle is less than the sum of the other two.*

78. By (77) $\quad BC < AB + AC.$

Transpose AB,

then $\quad BC - AB < AC;$

that is, *any side of a triangle is greater than the difference of the other two sides.*

PROPOSITION XII. THEOREM.

79. *The sum of the three angles of a triangle is equal to two right angles.*

Let ABC be any triangle.

To prove that $\angle A + \angle B + \angle BCA$ is equal to two right angles.

From C draw CE parallel to AB.

By (63) $\quad \angle ECF = \angle A.$

By (62) $\quad \angle BCE = \angle B.$

But $\quad \angle BCF = \angle BCE + ECF.$

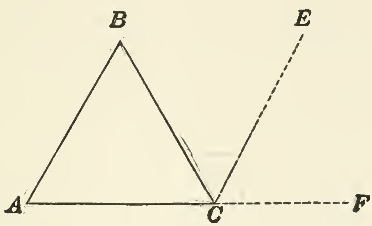
$$\therefore \angle BCF = \angle A + \angle B. \qquad (a)$$

Add the angle BCA, and we have

$$\angle BCF + \angle BCA = \angle A + \angle B + \angle BCA.$$

But, by (44), $\angle BCF + \angle BCA = 2$ right angles.

$$\therefore \angle A + \angle B + \angle BCA = 2 \text{ right angles.} \qquad \text{Q.E.D.}$$



80. Cor. 1. Equation (a) when expressed in words is: *the exterior angle of a triangle is equal to the sum of the two interior and opposite angles.*

81. Cor. 2. *If two angles of a triangle are given, or merely their sum, the third angle can be found by subtracting this sum from two right angles.*

82. Cor. 3. *If two triangles have two angles of the one equal to two angles of the other, the third angles are equal.*

83. Cor. 4. *A triangle can have but one right angle, or but one obtuse angle.*

84. Cor. 5. *In any right-angled triangle the two acute angles are complementary.*

85. Cor. 6. *Each angle of an equiangular triangle is two-thirds of a right angle.*

EXERCISES.

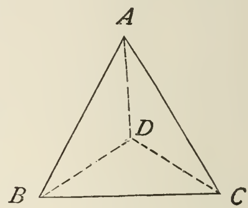
1. If one of the acute angles of a right triangle is $18° 24' 17''$, what is the value of the other acute angle?

2. If one angle of a triangle is $46° 17'$, and another is $\frac{4}{5}$ of a right angle, what is the value of the other angle?

3. If the angles of a triangle are in the proportion 1, 2, 3, what is the value of each angle?

4. How many degrees are there in each angle of an equiangular triangle?

5. If the unequal or vertical angle of an isosceles triangle is $46° 18'$, what will be the value of each of the angles at the base?

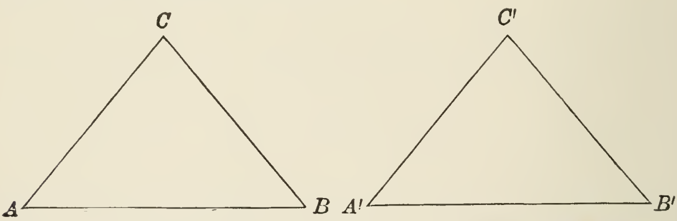


6. Show that the sum of the distances of any point in a triangle from the three angles is greater than half the sum of the three sides of the triangle.

$$DB + DA + DC > \tfrac{1}{2}(AB + BC + AC).$$

Proposition XIII. Theorem.

86. *Two triangles are equal each to each when two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*



In the triangles ABC and $A'B'C'$, let $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$.

To prove that the triangles are equal.

Place the triangle ABC upon the triangle $A'B'C'$ so that the side AB may fall upon $A'B'$, and since $AB = A'B'$, the point B will fall upon B'.

Since $\angle A = \angle A'$, the line AC will take the direction of $A'C'$, and these lines being equal, the point C will fall upon C'.

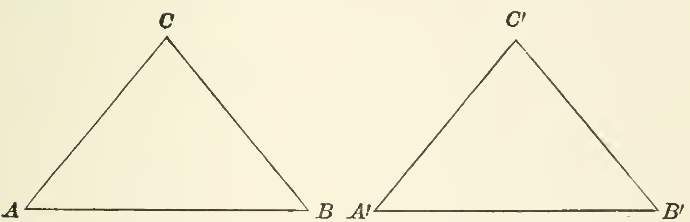
Therefore, as the points C and C', B and B' are coincident, the line joining $B'C'$ will coincide with the line joining BC, or the triangles will coincide throughout, and hence are equal. Q.E.D.

86 *a.* Two right triangles are equal if the two legs, or sides including the right angle, of the one are equal to the legs of the other.

87. Scholium. In equal figures, lines or angles similarly situated are called *Homologous.*

Proposition XIV. Theorem.

88. *Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.*



In the triangles ABC and $A'B'C'$, let $AB = A'B'$, $\angle A = \angle A'$, $\angle B = \angle B'$.

To prove that the triangles are equal.

Place the triangle ABC upon the triangle $A'B'C'$ so that AB may fall upon its equal $A'B'$.

Then, since $\angle A = \angle A'$, the line AC will take the direction of $A'C'$, and the point C will fall in the line $A'C'$.

Since $\angle B = \angle B'$, the line BC will take the direction of $B'C'$, and the point C will fall in the line $B'C'$.

$\therefore$ the point C, falling in the lines $A'C'$ and $B'C'$, it must be at the intersection of these lines, or at the point C'; that is, the two triangles coincide throughout and are equal. Q.E.D.

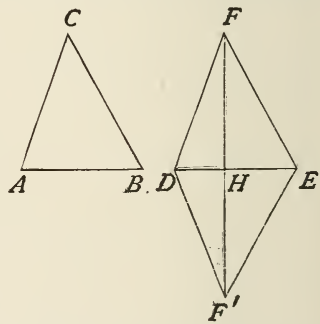
89. Cor. 1. *Two right-angled triangles are equal when the hypotenuse and an acute angle of the one are equal respectively to the hypotenuse and an acute angle of the other.*

90. Cor. 2. *Two right-angled triangles are equal when a side and an acute angle of the one are equal respectively to a side and homologous acute angle of the other.*

Proposition XV. Theorem.

91. *Two triangles are equal when the three sides of the one are equal respectively to the three sides of the other.*

In the triangles ABC and DEF, $AB = DE$, $AC = DF$, and $BC = EF$.



To prove that the two triangles are equal.

Apply the triangle ABC to DEF so that AB may coincide with DE but the vertex C fall on the opposite side of DE from F, that is, at F', and join FF'.

By hypothesis the points D and E are equally distant from F and F'; therefore (by 56) DH is perpendicular to FF' at its middle point, or the triangles DHF, DHF', FHE, and $F'HE$ are right triangles.

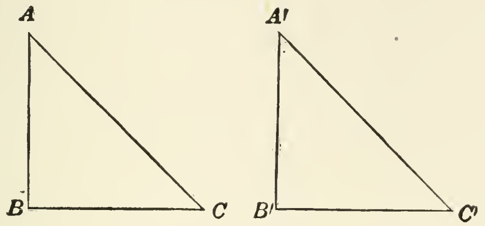
The right triangles DHF and DHF' have $DF' = DF$, $HF = HF'$, and DH common; therefore (by 86) they are equal,

or $$\angle FDH = \angle F'DH = \angle BAC.$$

This gives in ABC and DEF two sides and the included angle equal; therefore (by 86) the triangles are equal. Q.E.D.

Proposition XVI. Theorem.

92. *Two right triangles are equal when a side and the hypotenuse of the one are equal respectively to a side and the hypotenuse of the other.*



In the right triangles ABC and $A'B'C'$, let $AB = A'B'$, and $AC = A'C'$.

To prove that the triangles are equal.

Apply the triangle ABC to $A'B'C'$, so that BC will take the direction of $B'C'$.

Then, since $\angle B = \angle B'$, both being right angles, the side BA will take the direction of $B'A'$, and since $BA = B'A'$ the point A will fall on A'.

Since $AC = A'C'$ (by 53), they will cut off equal distances from the foot of the perpendicular; that is,

$$BC = B'C'.$$

Therefore the triangles ABC and $A'B'C'$ having three sides equal are (by 91) equal in all their parts. Q.E.D.

EXERCISES.

1. Prove that in any obtuse-angled triangle the sum of the acute angles is less than a right angle.

2. Prove that in any acute-angled triangle the sum of any two acute angles is greater than a right angle.

PROPOSITION XVII. THEOREM.

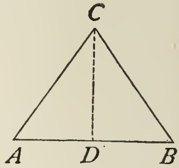
93. *In an isosceles triangle the angles opposite the equal sides are equal.*

Let ABC be an isosceles triangle in which AC and BC are the equal sides.

To prove that $\angle A = \angle B$.

Draw CD perpendicular to AB.

Then the triangles ADC and BDC having the side CD common and the hypotenuses equal, are (by 92) equal in all their parts, and $\angle A = \angle B$.



Q.E.D.

94. COR. 1. The equality of the triangles ADC and BDC also gives $AD = DB$.

Hence *the straight line which bisects the vertical angle of an isosceles triangle bisects the base at right angles.*

And, in general, if a straight line is drawn so as to satisfy any two of the following conditions,

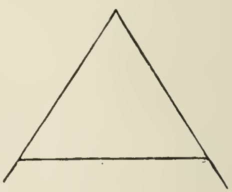
1. Passing through the vertex,
2. Bisecting the vertical angle,
3. Bisecting the base,
4. Perpendicular to the base,

it will also satisfy the remaining conditions.

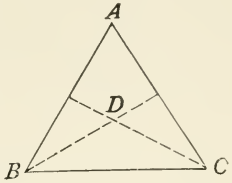
EXERCISES.

1. Show conversely, if two angles of a triangle are equal, the sides opposite are equal and the triangle is isosceles.

2. Show that if the equal sides of an isosceles triangle be produced, the angles formed with the base by the sides produced are equal.



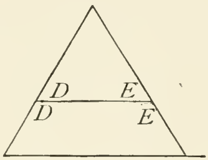
3. Show that if the perpendicular from the vertex to the base of a triangle bisects the base, the triangle is isosceles.



4. How many degrees are there in the exterior angle at each vertex of an equiangular triangle?

5. Show that the bisectors of the equal angles of an isosceles triangle form with the base another isosceles triangle; that is, DBC is isosceles.

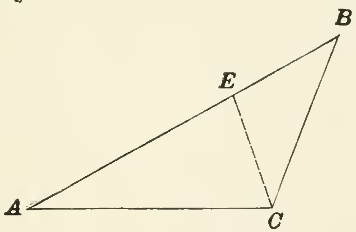
6. What are the relative values of the vertical angles D and A in the above?



7. Show that a straight line parallel to the base of an isosceles triangle makes equal angles with its sides, or $\angle D = \angle E$.

Proposition XVIII. Theorem.

95. *Of two sides of a triangle, that is the greater which is opposite the greater angle.*



In the triangle ABC let angle ACB be greater than angle B.

To prove that $AB > AC$.

From C draw CE, making $\angle ECB = \angle B$.

Then the triangle BEC is isosceles and the side $EB = EC$.

Add AE, $\qquad AE + EB = AE + EC$,

or $\qquad AB = AE + EC$.

But (by 77) $\qquad AE + EC > AC$.

$\therefore$ AB, which is equal to $AE + EC$, is greater than AC.

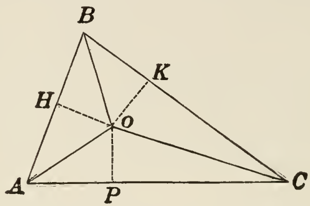
Q.E.D.

EXERCISE.

1. Show that of two angles of a triangle, that is the greater which is opposite the greater side.

Proposition XIX. Theorem.

96. *The three bisectors of the angles of a triangle meet in a point.*



Let AO and CO be the bisectors of the angles A and C of the triangle ABC.

To prove that the bisectors meet in a point.

Suppose AO and CO meet at O, and join BO.

Let fall the perpendiculars, OP, OH, and OK, forming the right triangles AOP, AOH, COP, and COK.

The triangles AOP and AOH are equal (by 89), having the hypotenuse AO common, and $\angle OAP = \angle OAH$; therefore

$$OP = OH.$$

For the same reason, the triangles OPC and OKC are equal,

or $$OP = OK,$$

or (by 28) $$OH = OK.$$

The two right triangles BOH and BOK have BO common, and $OH = OK$; therefore (by 92) they are equal; that is,

$$\angle HBO = \angle KBO,$$

or $$BO \text{ is a bisector of } \angle B.$$

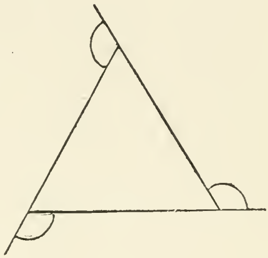
$\therefore$ the three lines meeting in O are bisectors of the angles.

97. Cor. Since $OP = OK = OH$, it is shown that the bisectors of angles are equally distant from their sides.

EXERCISES.

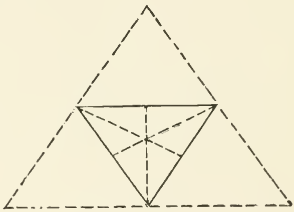
1. Show that the three perpendiculars erected at the middle points of the three sides of a triangle meet in a point, and this point is equally distant from the vertices.

SUG. See 53.



2. If an exterior angle is formed at each vertex of a triangle, their sum will be equal to four right angles.

3. Show that the perpendiculars from the vertices of a triangle to the opposite sides meet in a point.

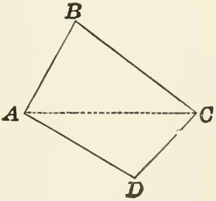


4. Show that every point unequally distant from the sides of an angle lies outside of the bisector of that angle.

QUADRILATERALS.

DEFINITIONS.

98. A *Quadrilateral* is a plane figure bounded by four straight lines; as *ABCD*.



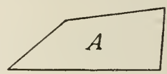
99. The bounding lines are called the *Sides* of the quadrilateral, and their points of intersection are called its *Vertices*

100. The *Angles* of the quadrilateral are the interior angles formed by the sides with each other.

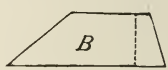
101. A *Diagonal* of a quadrilateral is a straight line joining two vertices not adjacent; as *AC*.

102. Quadrilaterals are divided into classes as follows:

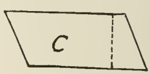
1st. The *Trapezium* (*A*), which has no two of its sides parallel.



2d. The *Trapezoid* (*B*), which has two sides parallel. The parallel sides are called the *Bases*, and the perpendicular distance between them the *Altitude* of the trapezoid.

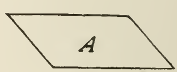


3d. The *Parallelogram* (*C*), which is bounded by two pairs of parallel sides.

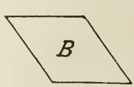


103. The side upon which a parallelogram is supposed to stand and the opposite side are called its lower and upper *Bases*. The perpendicular distance between the bases is the *Altitude*.

104. Parallelograms are divided into species as follows:

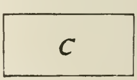


The *Rhomboid* (*A*), whose adjacent sides are not equal and whose angles are not right angles.



The *Rhombus* (*B*), whose sides are all equal.

The *Rectangle* (*C*), whose angles are all right angles.

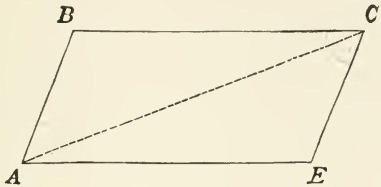


The *Square* (*D*), whose sides are all equal and whose angles are all equal.

105. The square is at once equilateral and equiangular.

Proposition XX. Theorem.

106. *In a parallelogram the opposite sides are equal, and the opposite angles are equal.*



Let the figure $ABCE$ be a parallelogram.

To prove that $AB = CE$, and $BC = AE$, and $\angle B = \angle E$, $\angle A = \angle C$.

Draw the diagonal AC.

Since AB and CE are parallel and AC cuts them,

(by 62), $\angle BAC = \angle ACE.$

Since AE and BC are parallel and AC cuts them,

(by 62), $\angle ACB = \angle CAE.$

Then the triangles ABC and ACE have the side AC common, and the two adjacent angles equal; they are, therefore (by 88), equal in all their parts,

or $AB = CE,\ BC = AE,$ and $\angle B = \angle E.$

Likewise, since $\angle BAC = \angle ACE,$

and $\angle CAE = \angle ACB.$

By addition, $\angle BAE = \angle BCE,$

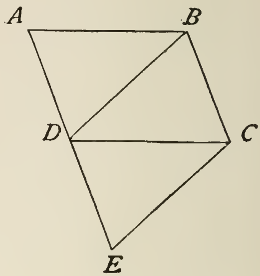
or $\angle A = \angle C.$ Q.E.D.

107. Cor. 1. *A diagonal of a parallelogram divides it into two equal triangles.*

108. Cor. 2. *Parallel lines included between parallels are equal.*

EXERCISES.

1. Show conversely, that if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.



2. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

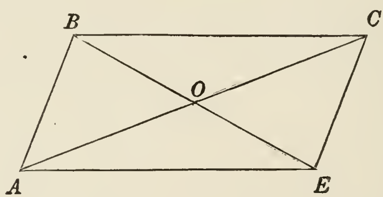
3. If one angle of a parallelogram is a right angle, the figure is a rectangle.

4. If two parallels are cut by a third straight line, the bisectors of the four interior angles form a rectangle. (See 58, Ex. 3.)

5. If CE is drawn parallel to BD, meeting AD produced, show that $BCED$ is a parallelogram and equal to the parallelogram $ABCD$.

Proposition XXI. Theorem.

109. *The diagonals of a parallelogram bisect each other.*



Let the figure $ABCE$ be a parallelogram, and let the diagonals AC and BE cut each other at O.

To prove that $AO = OC$ and $BO = OE$.

In the triangles BOC and AOE, $BC = AE$ (by 106), $\angle BCO = \angle OAE$, and $\angle OBC = \angle OEA$ (by 62); the triangles are therefore equal (by 88) in all their parts,

or $BO = OE$ and $AO = OC$. Q.E.D.

EXERCISES.

1. Show conversely, if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

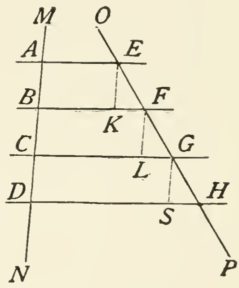
2. Show that the diagonals of a rhombus bisect each other at right angles.

3. Show that the diagonals of a rectangle are equal.

4. Show that two parallelograms are equal when two adjacent sides and the included angle of the one are equal to the two adjacent sides and the included angle of the other.

Proposition XXII. Theorem.

110. *If three or more parallels intercept equal lengths on any transversal, they intercept equal lengths on every transversal.*



Let AE, BF, CG, and DH be parallels, and MN and OP any two transversals.

To prove that if $AB = BC = CD$, $EF = FG = GH$.

Draw EK, FL, and GS parallel to MN.

Then, since EK and AB are parallels included between parallels, they are equal (by 108). Likewise, $FL = BC$, and $GS = CD$.

But $AB = BC = CD$ by hypothesis,

then (by 28) $$EK = FL = GS.$$

In the triangles EKF, FLG, and GSH the angles KEF, LFG, and SGH are equal (by 63); also the angles EFK, FGL, and GHS are equal (by 63).

Therefore these triangles are (by 88) equal in all their parts,

or $$EF = FG = GH.$$ Q.E.D.

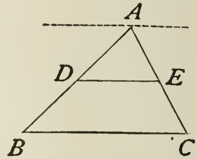
111. Cor. From the equality of the triangles EKF, FLG, and GSH, $KF = LG = SH$.

$HD - DS = SH$, but (by 108) $CG = DS$; therefore $HD - CG = SH$; likewise, $CG - BF = LG$, and $BF - AE = KF$.

Therefore *the intercepted part of each parallel will differ in length from the next intercept by the same amount.*

Proposition XXIII. Theorem.

112. *The straight line drawn through the middle point of a side of a triangle parallel to the base bisects the remaining side, and is equal to half the base.*



In the triangle ABC let E be the middle point of AC and DE parallel to BC.

To prove that D is the middle point of AB and that $DE = \frac{1}{2} BC$.

Through A draw a line parallel to DE, and it will be parallel to BC.

Then AB and AC are transversals cutting parallel lines; therefore (by 110), when $AE = EC$, $AD = DB$. Q.E.D.

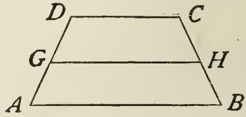
Likewise (by 111), $BC - DE = DE - AA = DE - 0 = DE$.

Transpose DE, then $\quad BC = 2\,DE$,

or $\quad DE = \frac{1}{2} BC$. Q.E.D.

Proposition XXIV. Theorem.

113. *The line drawn parallel to the bases through the middle point of one of the non-parallel sides of a trapezoid bisects the opposite side, and is equal to half of the parallel sides.*



Let $ABCD$ be a trapezoid, GH a line drawn from G, the middle point of AD parallel to AB.

To prove that $GH = \frac{1}{2}(AB + DC)$, and $HB = CH$.

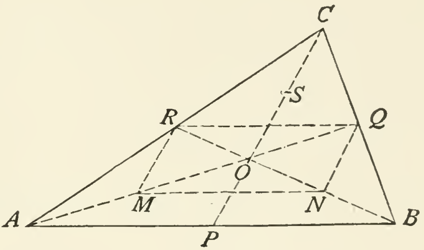
Since DA and CB are transversals, and DC, GH, and AB parallels (by 110), when $DG = GA$, $CH = HB$. Q.E.D.

Also (by 111) $AB - GH = GH - CD$, or $AB + CD = 2\,GH$; $\therefore\ GH = \frac{1}{2}(AB + CD)$.

Proposition XXV. Theorem.

114. *The three medial lines of a triangle meet in a point which is two-thirds of the way from each angle to the middle of the opposite side.*

Let ABC be a triangle; P, Q, R, the middle points of its respective sides; BR, AQ, two medial lines of the triangle; O, their point of intersection.



To prove that the third medial line CP passes through O, and that $CO = \frac{2}{3} CP$, $AO = \frac{2}{3} AQ$, and $BO = \frac{2}{3} BR$.

Bisect AO in M, and BO in N; join RM and QN and OC.

In the triangle AOC, M is the middle point of AO by construction, and R the middle point of AC by hypothesis; therefore (by 112) RM is parallel to CO and equal to one-half of CO.

In the triangle BOC, for the same reason, $NQ = \frac{1}{2} CO$ and is parallel to CO.

Therefore (by 28) RM is equal to and parallel with QN.

In the triangle ACB (by 112), $RQ = \frac{1}{2} AB$ and is parallel to it.

In the triangle AOB (by 112), $MN = \frac{1}{2} AB$ and is parallel to it.

Therefore $RQ = MN$ and is parallel to it.

Hence the figure $RMNQ$ is (by 102) a parallelogram.

Since $RMNQ$ is a parallelogram, (by 109) $OR = ON$, and $MO = OQ$.

But by construction $OM = AM$; therefore the three parts AM, MO, and OQ into which AQ is divided are equal.

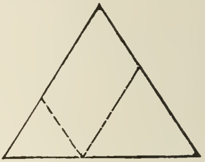
Therefore AO, which contains two of these parts, is two-thirds of the whole, or $AO = \frac{2}{3}AQ$, and likewise $BO = \frac{2}{3}BR$.

Q.E.D.

By taking CP and BR as medial lines intersecting at O, and joining S, the middle point of CO, with N, and with R, and drawing RP and NP, it can be shown in the same manner that $OC = \frac{2}{3}CP$, and that O is a point on all three medial lines.

EXERCISES.

1. The bisectors of the interior angles of a parallelogram form a rectangle.



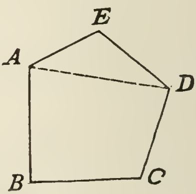
2. If the non-parallel sides of a trapezoid are equal, the angles which they make with the bases are equal.

3. If from any point in the base of an isosceles triangle parallels to the equal sides are drawn, the perimeter of the parallelogram thus formed is equal to the sum of the equal sides of the triangle.

Suggestion. See 112.

115. A *Polygon* is a plane figure bounded by straight lines; as *ABCDE*.

The straight lines are called the *Sides* of the polygon; and their sum is called the *Perimeter* of the polygon.



The *Angles* of the polygon are the angles formed by the adjacent sides with each other; and the vertices of these angles are also called the *Vertices* of the polygon.

116. The angles of the polygon within the polygon and included between its sides are called *Interior Angles.*

An *Exterior Angle* of a polygon is an angle between any side and the continuation of an adjacent side.

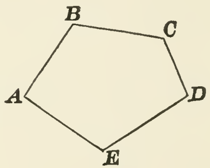
A *Diagonal* is a line joining any two vertices that are not adjacent, as AD.

117. Polygons are named from the number of their sides, as follows:

No. of Sides.	Designation.	No. of Sides.	Designation.
3	Triangle.	8	Octagon.
4	Quadrilateral.	9	Enneagon.
5	Pentagon.	10	Decagon.
6	Hexagon.	11	Hendecagon.
7	Heptagon.	12	Dodecagon, etc.

118. An *Equilateral* polygon is one all of whose sides are equal.

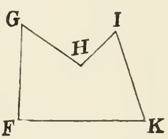
An *Equiangular* polygon is one all of whose angles are equal.



119. A polygon is called *Convex* when each of its angles is less than a straight angle; as *ABCDE.*

It is evident that in such a polygon no side, if produced, can enter the space enclosed by the perimeter.

120. A polygon is called *Concave* when at least one of its angles is greater than a straight angle; as *FGHIK*, in which the interior angle whose vertex is *H* is greater than a straight angle.

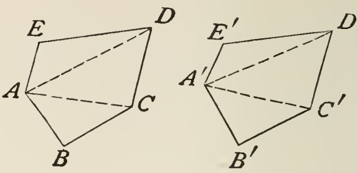


Such an angle is called *Reëntrant.*

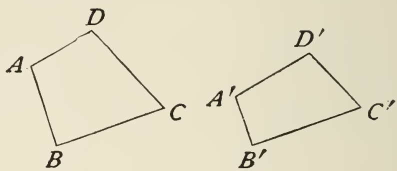
It is evident that in such a polygon at least two sides, if produced, will enter the space enclosed by the perimeter.

All polygons treated hereafter will be understood to be convex, unless the contrary is stated.

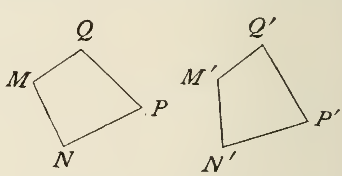
121. Two polygons, $ABCDE$, $A'B'C'D'E'$, are equal when they can be divided by diagonals into the same number of triangles, equal each to each, and similarly arranged; for the polygons can evidently be superposed, one upon the other, so as to coincide.



122. Two polygons are *mutually equiangular* when the angles of the one are respectively equal to the angles of the other, taken in the same order; as $ABCD$, $A'B'C'D'$, in which $A = A'$, $B = B'$, etc. The equal angles are called *Homologous Angles;* the sides containing equal angles, and similarly placed, are *Homologous Sides;* thus A and A' are homologous angles, AB and $A'B'$ are homologous sides, etc.



Two polygons are *mutually equilateral* when the sides of the one are respectively equal to the sides of the other, taken in the same order; as $MNPQ$, $M'N'P'Q'$, in which $MN = M'N'$, $NP = N'P'$, etc. The equal sides are homologous; and angles contained by equal sides similarly placed, are homologous; thus MN and $M'N'$ are homologous sides; M and M' are homologous angles.

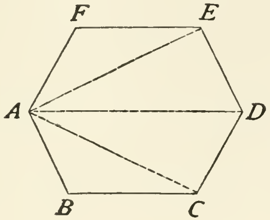


Two mutually equiangular polygons are not necessarily also mutually equilateral. Nor are two mutually equilateral polygons necessarily also mutually equiangular, except in the case of triangles (91).

If two polygons are mutually equilateral and also mutually equiangular, they are equal; for they can evidently be superposed, one upon the other, so as to coincide.

PROPOSITION XXVI. THEOREM.

123. *The sum of the interior angles of a polygon is equal to two right angles taken as many times as the polygon has sides less two.*



Let $ABCDEF$ be a polygon, and AE, AD, and AC diagonals.

These diagonals divide the polygon into triangles.

Since the first and last triangles involve *two* sides of the polygon, while each other triangle only involves one side of the polygon, there will always be two triangles less than the number of sides in the polygon.

The sum of the angles of the polygon will be equal to the sum of the angles of the triangles, but (by 79) the angles of each triangle are equal to two right angles; therefore, since the number of triangles is two less than the number of sides in the polygon, the angles of the polygon are equal to two right angles taken as many times, less two, as the figure has sides.

Q.E.D.

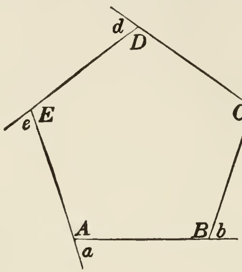
124. COR. *The sum of the angles of a quadrilateral is equal to four right angles; of a pentagon, six right angles; of a hexagon, eight right angles; etc.*

125. Scholium. If R denotes a right angle, and n the number of sides of the polygon, the sum of its angles is expressed by $2R \times (n-2)$, or $2nR - 4R$.

That is, *the sum of the angles of a polygon is equal to twice as many right angles as the figure has sides, less four right angles.*

Proposition XXVII. Theorem.

126. *The exterior angles of a polygon, made by producing each of its sides in succession, are together equal to four right angles.*



Let the figure $ABCDE$ be a polygon, having its sides produced in succession.

To prove that the sum of the angles, a, b, c, d, and e is equal to four right angles.

The sum of each exterior and its corresponding interior angle (by 44) is equal to two right angles.

That is, the sum of the interior and exterior angles is equal to twice as many right angles as the figure has sides.

But by (125) the interior angles are equal to twice as many right angles as the figure has sides, less four right angles.

Therefore the exterior angles alone are equal to four right angles. Q.E.D.

EXERCISES.

1. If one side of a regular hexagon is produced, show that the exterior angle is equal to the angle of an equilateral triangle.

2. The exterior angle of a regular polygon is 18° ; find the number of sides in the polygon.

3. The interior angle of a regular polygon is five-thirds of a right angle ; find the number of sides in the polygon.

4. How many degrees are there in each angle of a regular pentagon ? Of a regular hexagon ? Of a regular dodecagon ?

5. If two angles of a quadrilateral are supplementary, the other two angles are supplementary.

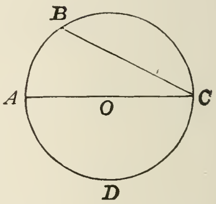
6. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

BOOK II.

THE CIRCLE.

DEFINITIONS.

127. A *Circle* is a plane figure bounded by a curve, all points of which are equally distant from a point within called the *Centre*.



The curve which bounds the circle is called the *Circumference*, and any portion of it is called an *Arc*.

128. A *Chord* is a straight line which joins any two points on the circumference, as BC.

When a chord passes through the centre, it has its greatest length, and is called the *Diameter*.

129. A *Radius* is a straight line drawn from the centre to the circumference, and since, by definition, this distance is the same for the same circle, all radii are equal; and each radius is one-half of the diameter.

130. An arc equal to one-half the circumference is called a *Semi-circumference*, and an arc equal to one-fourth of the circumference is called a *Quadrant*.

131. Two circles are *Equal* when they have equal radii, for they can evidently be applied one to the other so as to coincide throughout.

132. Two circles are *Concentric* when they have the same centre.

133. Postulate: the circumference of a circle can be described about any point as a centre and with any distance for a radius.

134. A *Segment* of a circle is a portion of a circle enclosed by an arc and its chord, as *AMB*, Fig. 1.

135. A *Semicircle* is a segment equal to one-half the circle, as *ADC*, Fig. 1.

136. A *Sector* of a circle is a portion of the circle enclosed by two radii and the arc which they intercept, as *ACB*, Fig. 2.

137. A *Tangent* is a straight line which touches the circumference, but does not intercept it, however far produced. The point in which the tangent touches the circumference is called the *Point of Contact*, or *Point of Tangency*.

138. Two *Circumferences* are tangent to each other when they are tangent to a straight line at the same point.

139. A *Secant* is a straight line which intersects the circumference in two points, as *AD*, Fig 3.

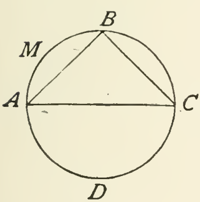


Fig. 1.

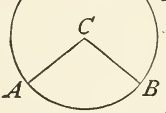


Fig. 2.

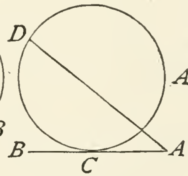


Fig. 3.

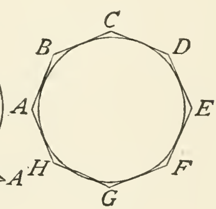


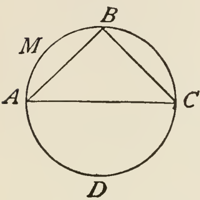
Fig. 4.

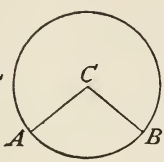
140. A straight line is *Inscribed* in a circle when its extremities lie in the circumference of the circle, as *AB*, Fig. 1.

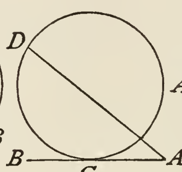
An angle is inscribed in a circle when its vertex is in the circumference, and its sides are chords of that circumference, as $\angle ABC$, Fig. 1.

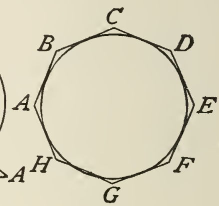
A polygon is inscribed in a circle when its sides are chords of the circle, as ABC, Fig. 1.

A circle is inscribed in a polygon when the circumference touches the sides of the polygon but does not intersect them, as in Fig. 4.





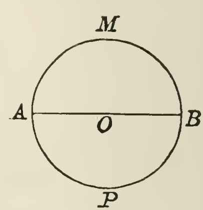




141. A polygon is *Circumscribed* about a circle when all the sides of the polygon are tangents to the circle, as in Fig. 4.

A circle is circumscribed about a polygon when the circumference passes through all the vertices of the polygon, as in Fig. 1.

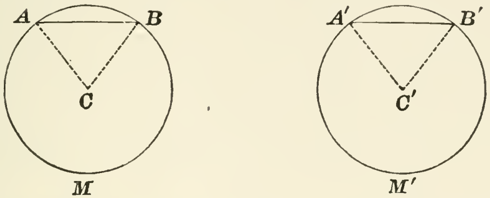
142. *Every diameter bisects the circle and its circumference.* For if we fold over the segment AMB on AB as an axis until it comes into the plane of APB, the arc AMB will coincide with the arc APB; because every point in each is equally distant from the centre O.



143. *A straight line cannot meet the circumference of a circle in more than two points.* For if it could meet it in three points, these three points would be equally distant from the centre (127). There would then be three equal straight lines drawn from the same point to the same straight line, which is impossible (58 a).

Proposition I. Theorem.

144. *In equal circles, or in the same circle, equal arcs are intercepted by equal central angles and have equal chords.*



Let ABM and $A'B'M'$ be two equal circles, in which $\angle C = \angle C'$.

To prove that the arc $AB =$ arc $A'B'$ and the chord AB $=$ chord $A'B'$.

1. Place the circle ABM upon $A'B'M'$ so that their centres may coincide, and A fall upon A'; then, since they are equal circles, they will coincide throughout.

Since $\angle C = \angle C'$, the radius CB will take the direction of $C'B'$, and, being radii of equal circles, B will fall upon B'.

Therefore the arc AB will coincide with arc $A'B'$ and be equal to it. Q.E.D.

2. The two triangles ACB and $A'C'B'$ have $AC = A'C'$ and $BC = B'C'$, being radii of equal circles (by 131), and $\angle C$ $= \angle C'$ by hypothesis.

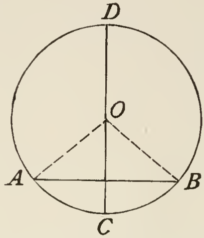
Therefore (by 86) the two triangles are equal in all their parts, or $AB = A'B'$. Q.E.D.

145. Cor. 1. Another form of statement is: *In equal circles, or in the same circle, equal central angles intercept equal arcs on the circumference.*

Cor. 2. Also the converse: *In equal circles, or in the same circle, equal chords subtend equal arcs and equal angles at the centre.*

Proposition II. Theorem.

146. *The diameter perpendicular to a chord bisects the chord and its subtended arcs.*



In the circle ADB, let the diameter CD be perpendicular to the chord AB.

To prove that DC bisects AB and its subtended arcs.

Let O be the centre of the circle, and join OA and OB.

Then since $OA = OB$, the triangle OAB is isosceles; and the line CD, passing through the vertex perpendicular to the base, bisects the base and also the vertical angle (94).

Hence $\angle AOC = \angle BOC$, and arc $AC =$ arc BC (144).

Subtracting the equal arcs AC and BC from the semicircumferences CAD and CBD, we have arc $AD =$ arc BD.

Therefore the diameter bisects the chord AB and its subtended arcs ACB and ADB.

147. Cor. *The perpendicular erected at the middle point of a chord passes through the centre of the circle.*

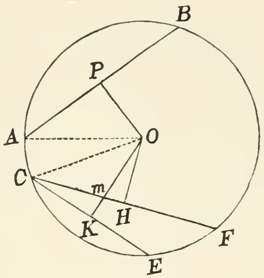
And in general, if a straight line is drawn so as to satisfy any two of the following conditions:

1. Passing through the centre,
2. Perpendicular to the chord,
3. Bisecting the chord,
4. Bisecting the less subtended arc,
5. Bisecting the greater subtended arc,

it will also satisfy the remaining conditions.

Proposition III. Theorem.

148. *In the same circle, or equal circles, equal chords are equally distant from the centre; and of two unequal chords the less is at the greater distance from the centre.*



In the circle $ABEC$ let the chord AB equal the chord CF, and the chord CE be less than the chord CF. Let OP, OH, and OK be perpendiculars drawn to these chords from the centre O.

To prove that $OP = OH$, and that $OK > OP$ or OH.

Join OA and OC.

1. The right triangles OAP and OCH have by hypothesis $AP = CH$, and $OA = OC$ being radii, therefore (by 92) the triangles are equal in all their parts, or $OP = OH$. Q.E.D.

2. Since Om is an oblique line, and OH a perpendicular,

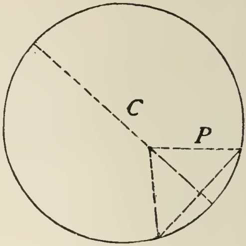
$$Om > OH, \text{ but } OK > Om,$$

therefore $$OK > OH \text{ or its equal } OP. \qquad \text{Q.E.D.}$$

149. Cor. *The conclusion is reached that the nearer the centre the greater the chord, therefore the greatest chord is at no distance from the centre, or passes through the centre, that is the diameter* (128).

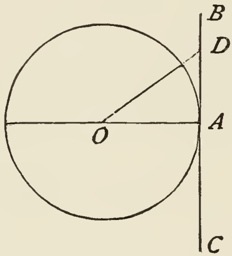
EXERCISE.

From a point within the circle, other than the centre, not more than two equal straight lines can be drawn to the circumference. (See 147.)



Proposition IV. Theorem.

150. *A straight line perpendicular to a radius at its extremity is a tangent to the circumference.*



Let BC be perpendicular to the radius OA at its extremity A.

To prove that BC is a tangent to the circumference.

Since OA is the shortest line that can be drawn from the point O to the line BC (by 52), any other line as OD will be longer than OA, or the point D will be at a distance from the centre greater than the radius, and hence is without the circle.

As D is any point other than A, A is the only point that is on the line and the circumference, therefore BC is a tangent to the circumference. Q.E.D.

151. Cor. *A perpendicular to a tangent at its point of contact passes through the centre of the circle.*

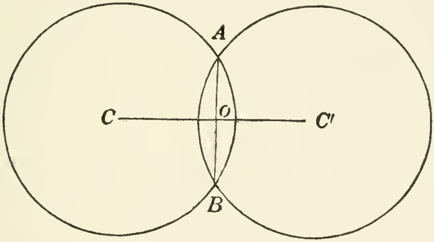
EXERCISES.

1. Show conversely, a tangent to the circumference is perpendicular to the radius drawn to the point of contact.

2. Prove that the tangents to a circle at the extremities of a diameter are parallel.

Proposition V. Theorem.

152. *If two circumferences intersect each other, the line which joins their centres is perpendicular to their common chord at its middle point.*



Let C and C' be the centres of two circumferences which intersect each other at A and B, and let the line CC' intersect their common chord at O.

To prove that CC' is a perpendicular bisector of AB.

Since A and B are points on both circles, they are equally distant from C and also from C' (by 127), the line CC' is perpendicular to AB at its middle point (by 56). Q.E.D.

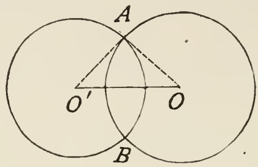
EXERCISES.

1. If two circumferences are tangent to each other, the straight line joining their centres passes through the point of contact.

Suggestion. Draw a common tangent.

Proposition VI. Theorem.

153. *If two circumferences intersect each other, the distance between their centres is less than the sum and greater than the difference of the radii.*



Let O and O' be two circles which intersect each other at A and B.

To prove that the distance between their centres is less than the sum of their radii.

Join AO' and AO.

Then, in the triangle $O'AO$,

$$OO' < AO' + AO \text{ (by 77).}$$ Q.E.D.

Also (by 78), $OO' > AO - AO'$.

154. Cor. 1. *If the distance of the centres of two circles is greater than the sum of their radii, they are wholly exterior to each other.*

155. Cor. 2. *If the distance of the centres of two circles is equal to the sum of the radii, they are tangent externally.*

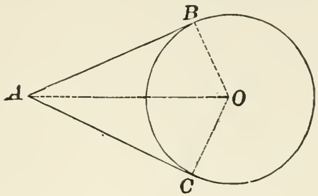
156. Cor. 3. *If the distance of the centres is less than the sum and greater than the difference of the radii, the circles intersect.*

157. Cor. 4. *If the distance of the centres is equal to the difference of the radii, the circles are tangent internally.*

158. Cor. 5. *If the distance of the centres is less than the difference of the radii, one circle is wholly within the other.*

Proposition VII. Theorem.

159. *The two tangents to a circumference from an outside point are equal.*



Let AB and AC be the tangents from the point A to the circumference whose centre is at O.

To prove that $AB = AC$.

Draw the radii OB and OC and join AO.

In the triangles ABO and AOC, $\angle C = \angle B$ (by 150), $BO = OC$, both being radii, and the side AO is common.

Therefore (by 92) they are equal in all their parts,

or $$AB = AC. \qquad \text{Q.E.D.}$$

160. Cor. The line OA bisects the angle BAC, the angle BOC, and the arc BC.

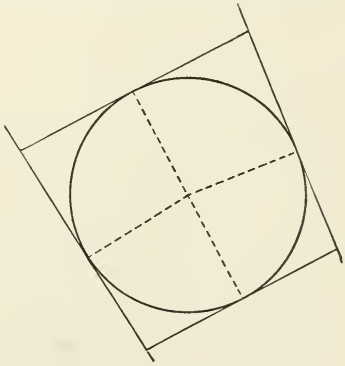
EXERCISES.

1. The straight line drawn from the centre of a circle to the point of intersection of two tangents bisects at right angles the chord joining their points of contact.

2. Show that the sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

3. The bisector of the angle between two tangents to a circumference passes through the centre.

4. If tangents are drawn to a circumference at the extremities of any pair of diameters, the figure thus formed is a rhombus.



Suggestion. See 151, Ex. 2.

ON MEASUREMENT.

161. *Ratio* is the relation with respect to magnitude which one quantity bears to another of the same kind, and is expressed by writing the first quantity as the numerator and the second as the denominator of a fraction.

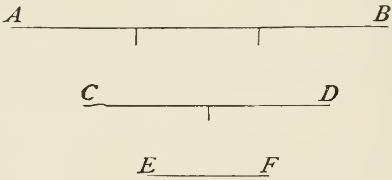
Thus the ratio of a to b is $\frac{a}{b}$; it is also expressed $a : b$.

The *numerical value* of a ratio is the quotient obtained by dividing the numerator by the denominator.

162. To *measure* a quantity is to find its ratio to another quantity of the same kind called the *unit of measure.*

163. The number which expresses how many times a quantity contains the unit, prefixed to the name of the unit, is called the *numerical measure* of that quantity; as 5 yards, etc.

164. Two quantities are *commensurable* when they have a common measure; that is, when there is some third quantity of the same kind which is contained an exact number of times in each.



Thus if EF is contained in AB 3 times and in CD 2 times, then AB and CD are commensurable, and EF is a common measure.

165. Two quantities are *incommensurable* when they have no common measure. The ratio of such quantities is called an *incommensurable* ratio. This ratio cannot be exactly expressed in figures; but its numerical value can be obtained approximately as near as we please.

Thus, suppose G and H are two lines whose ratio is $\sqrt{2}$. We cannot find any fraction which is *exactly* equal to $\sqrt{2}$ but by taking a sufficient number of decimals we may find $\sqrt{2}$ to any required degree of approximation.

Thus $\sqrt{2} = 1.4142135 \cdots,$

and therefore $\sqrt{2} > 1.414213$ and $< 1.414214.$

That is, the ratio of G to H lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these ratios by less than one-millionth. And since the decimals may be continued without end in extracting the square root of 2, it is evident that this ratio can be expressed as a fraction with an error *less than any assignable quantity.*

166. And in general, if the approximate numerical value of the ratio of two incommensurable quantities is desired within $\frac{1}{n}$, let the second quantity be divided into n equal parts, and suppose that one of these parts is contained between m and $m + 1$ times in the first quantity.

Then the numerical value of the ratio of the first quantity to the second is between $\frac{m}{n}$ and $\frac{m+1}{n}$; that is, the approximate numerical value of the ratio is $\frac{m}{n}$, correct within $\frac{1}{n}$.

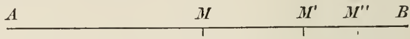
And since n can be taken as great as we please, $\frac{1}{n}$ is made correspondingly small, or until it becomes less than any assignable value, though it can never reach zero, or *absolute* nothing.

THE METHOD OF LIMITS.

167. A *Variable Quantity,* or simply a *Variable,* is a quantity, which under the conditions imposed upon it, may assume an indefinite number of values.

168. A *Constant* is a quantity which remains unchanged throughout the same discussion.

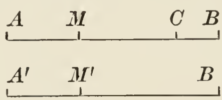
169. If a variable approaches nearer and nearer a constant so that the difference between it and the constant can become less than any quantity that may be assigned, then that constant is called the *Limit* of the variable.

170. Suppose a point 

to move from A toward B, under the conditions that the first second it shall move one-half the distance from A to B; that is, to M; the next second, one-half the remaining distance; that is, to M'; the next second, one-half the remaining distance; that is, to M''; and so on indefinitely.

Then it is evident that the moving point *may approach as near to B as we please, but will never arrive at B;* that is, the distance AB is the limit of the space passed over by the point.

171. Theorem. *If two variables are always equal and each approaches a limit, then the two limits are equal.*



Let AM and $A'M'$ be two equal variables which approach indefinitely the limits AB and $A'B'$ respectively.

To prove that $AB = A'B'$.

If possible, suppose $AB > A'B'$, and lay off $AC = A'B'$.

Then the variable AM may assume values between AC and AB, while the variable $A'M'$ is restricted (by 169) to values less than AC; which is contrary to the hypothesis that the variables should always be equal.

Hence AB cannot be $> A'B'$, and in like manner it may be proved that AB cannot be $< A'B'$; therefore $AB = A'B'$.

172. Cor. *If two variables are in a constant ratio, their limits are in the same ratio.*

Let x and y be two variables, so that $\frac{x}{y} = m$.

To prove that their limits have the same ratio.

Now let x approach the limit x', and y the limit y'.

Then since the variables x and my are always equal (by 171), their limits are equal; that is, $x' = my'$.

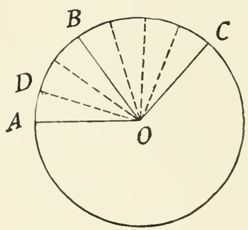
Therefore $\frac{x'}{y'} = m$.

MEASUREMENT OF ANGLES.

Proposition VIII. Theorem.

173. *In the same circle, or in equal circles, angles at the centre are in the same ratio as their intercepted arcs.*

Case I. When the arcs are commensurable.



In the circle O, let AOB and BOC be two angles at the centre intercepting the commensurable arcs AB and BC.

To prove that

$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Let AD be the common measure of the arcs AB and BC, and by applying it to the arcs it is found that AB contains it 3 times, and BC 4 times.

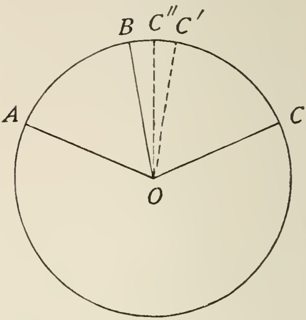
Therefore $$\frac{\text{arc } AB}{\text{arc } BC} = \frac{3}{4}.$$

If radii be drawn from the several points of division, they will divide the angle AOB into 3 parts which (by 145) are equal, and BOC into 4 parts which are equal.

Therefore $$\frac{\angle AOB}{\angle BOC} = \frac{3}{4}.$$

Hence (by 28) $$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$ Q.E.D.

Case II. When the arcs are incommensurable. If the arcs AB and BC are incommensurable, cut off a portion CC' which will have a common measure with AB.



Then (by 173)

$$\frac{\angle AOB}{\angle COC'} = \frac{\text{arc } AB}{\text{arc } CC'}.$$

By taking a smaller measure, an arc CC'' may be found which is commensurable with AB, which would give

$$\frac{\angle AOB}{\angle COC''} = \frac{\text{arc } AB}{\text{arc } CC''}.$$

Now CB is the limit of the arc, and $\angle COB$ is the limit of the angle, therefore since the ratio of the angles is equal to the ratio of the arcs at different stages of their variation, then (by 171) their limits will have the same ratio, that is,

$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}. \qquad \text{Q.E.D.}$$

174. Scholium. Since the angle at the centre of a circle and its intercepted arc increase and decrease in the same ratio, it is said that an angle at the centre is *measured* by its intercepted arc.

Proposition IX. Theorem.

175. *An inscribed angle is measured by one-half the arc intercepted between its sides.*

In the circle O, let BAC be an inscribed angle.

To prove that $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Draw the diameter AD and the radii OB and OC.

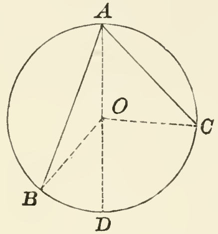
Since OB and OA are radii, the triangle OBA (by 68) is isosceles, and (by 93) $\angle B = \angle BAO$.

But $\angle BOD$ being an exterior angle, it is equal (by 80) to the sum of the interior and opposite angles, B and BAO; that is,

$$\angle BOD = \angle B + \angle BAO$$

$$= 2 \angle BAO,$$

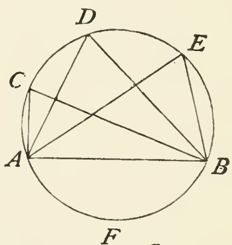
or $$\angle BAO = \tfrac{1}{2} \angle BOD.$$



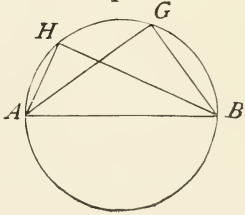
But $\angle BOD$ is measured by the arc BD (by 174).

Therefore $\angle BAD$ is measured by $\frac{1}{2}$ arc BD.

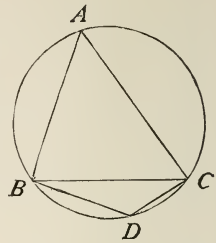
Likewise $\angle DAC$ is measured by $\frac{1}{2}$ arc DC, or $\angle BAC$ is measured by $\frac{1}{2}$ arc BDC. Q.E.D.



176. Cor. 1. *All angles inscribed in the same segment are equal;* for each is measured by one-half the same arc *AFB*.

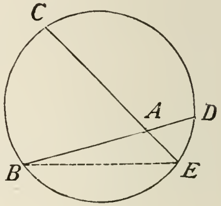


177. Cor. 2. *Every angle AHB, inscribed in a semicircle, is a right angle;* for it is measured by one-half a semi-circumference, or by a quadrant.



178. Cor. 3. *Every angle BAC, inscribed in a segment greater than a semicircle, is an acute angle;* for it is measured by one-half the arc BDC, which is less than a quadrant.

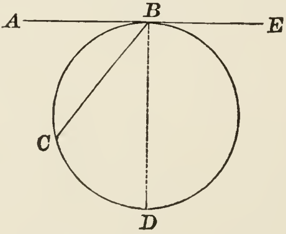
Every angle BDC, inscribed in a segment less than a semicircle, is an obtuse angle; for it is measured by one-half the arc BAC, which is greater than a quadrant.



An angle formed by two chords which intersect within a circle, is measured by one-half the sum of the arcs intercepted between its sides and between its sides produced. That is, $\angle CAB$ is measured by one-half $(BC + DE)$. See (80).

Proposition X. Theorem.

179. *An angle formed by a tangent and a chord is measured by one-half its intercepted arc.*



Let AE be tangent to the circumference BCD at B, and let BC be a chord.

To prove that $\angle ABC$ is measured by $\frac{1}{2}$ arc BC.

At B erect a perpendicular; then (by 151) it will be a diameter, and the angle ABD is a right angle.

Since a right angle is measured by a quadrant or one-half a semicircle, $\angle ABD$ is measured by $\frac{1}{2}$ arc BCD.

And (by 175) $\angle CBD$ is measured by $\frac{1}{2}$ arc CD.

But $\angle ABC = \angle ABD - \angle CBD$.

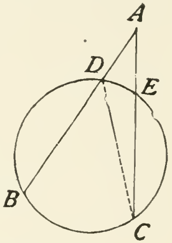
Therefore $\angle ABC$ is measured by $\frac{1}{2}$ arc $BCD - \frac{1}{2}$ arc CD,

or $\frac{1}{2}(BCD - CD) = \frac{1}{2}$ arc BC.

$\therefore \angle ABC$ is measured by $\frac{1}{2}$ arc BC. Q.E.D.

Proposition XI. Theorem.

180. *An angle formed by two secants, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.*



Let the angle BAC be formed by the secants AB and AC.

To prove that the angle BAC is measured by one-half the arc BC minus one-half the arc DE.

Join DC.

Then (by 80) $\angle BDC = \angle C + \angle DAC$ (or BAC).

By transposing, $\angle BDC - \angle C = \angle BAC$.

But $\angle BDC$ is measured (by 175) by $\frac{1}{2}$ arc BC,

and $\angle C$ is measured (by 175) by $\frac{1}{2}$ arc DE.

Therefore

$\angle BAC$ is measured by $\frac{1}{2}$ arc $BC - \frac{1}{2}$ arc DE.

EXERCISES.

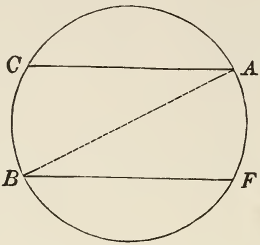
1. An angle formed by a tangent and a secant is measured by one-half the difference of the intercepted arcs.

2. The angle formed by two tangents is measured by one-half the difference of the intercepted arcs.

3. If a quadrilateral be inscribed in a circle, the sum of each pair of opposite angles is two right angles.

Proposition XII. Theorem.

181. *Two parallel lines intercept upon the circumference equal arcs.*



Let AC and BF be two parallel chords.

To prove that they intercept equal arcs; that is, arc $BC =$ arc AF.

Join AB.

$$\angle BAC = \angle ABF \text{ (by 62).}$$

But $\qquad \angle BAC$ is measured by $\frac{1}{2}$ arc BC (175),

and $\qquad \angle ABF$ is measured by $\frac{1}{2}$ arc AF.

Since the angles are equal, their measures are equal; that is,

$$\tfrac{1}{2} \text{ arc } BC = \tfrac{1}{2} \text{ arc } AF,$$

or (by 28) $\qquad$ arc $BC =$ arc AF. $\qquad$ Q.E.D.

EXERCISE.

1. Show that the above theorem is true if both lines are tangents, also when one is a chord and the other a tangent.

CONSTRUCTION.

Up to the present time it has been assumed that any needful line or combination of lines could be drawn, and the question has not arisen as to the possibility of drawing these lines with accuracy.

In order to show that any required combination of lines, angles, or parts of lines or angles fulfilled the required conditions, principles were needed long before they could be demonstrated.

Sufficient progress has now been made to render it possible to show that every assumed construction can be synthetically effected and proof furnished that each step is legitimate.

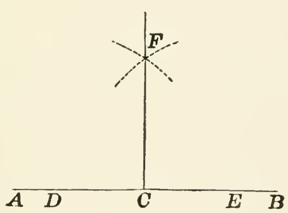
The only instruments that can be employed in Elementary Geometry are the ruler and compasses. The former is used for drawing or producing straight lines, and the compasses for describing circles and for the transference of distances.

The warrant for the use of these instruments is found in the three postulates already given (26).

PROBLEMS IN CONSTRUCTION.

Proposition XIII. Problem.

182. *At a given point in a straight line to erect a perpendicular to that line.*



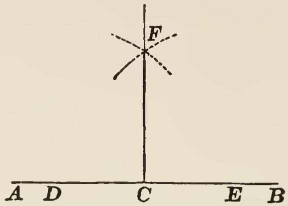
Let C be the given point in the line AB.

To erect a perpendicular to AB at C.

It is known (by 54) that every point that is equally distant from the extremities of a straight line is in the perpendicular bisector of that line.

Therefore it is simply necessary to make C the middle point of a portion of AB, by measuring off a distance CE, less than CB, and taking CD equal to CE.

To find another point equally distant from D and E, take any radius greater than DC and draw arcs, first with D as a

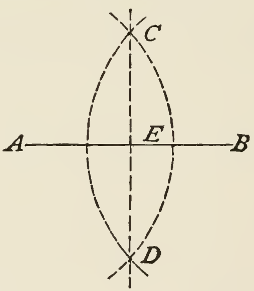


centre, then with E as a centre, and these arcs will intersect at some point, say F.

Draw FC, and it will be the perpendicular required, since C and F are equally distant from the points D and E.

PROPOSITION XIV. PROBLEM.

183. *To bisect a given finite straight line.*



Given, the line AB.

To bisect AB.

It is known (by 54) that every point that is equally distant from the extremities of a straight line is in the perpendicular bisector of that line.

Therefore it is necessary to find two or more points equally distant from A and B.

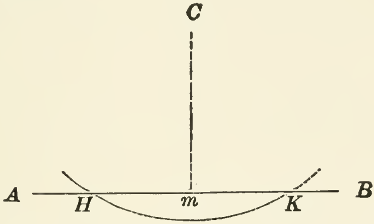
Since radii of equal circles are equal, it is suggested that A and B be made centres of circles of equal radii; then all points that are common to the two circles will be equally distant from A and B, and hence be on the bisector of AB.

With A as a centre and a radius manifestly greater than one-half of AB describe an arc, and with B as a centre describe an arc intersecting the former arc (by 156) at two points, say C and D.

Join CD, and it will be the bisector required.

Proposition XV. Problem.

184. *From a point without a straight line, to let fall a perpendicular upon that line.*



Let AB be a given straight line, and C a given point without the line.

To let fall a perpendicular from C to the line AB.

If a line through C is to be perpendicular to AB, it must have at least two points in it that are equally distant from two points in the line AB.

Let C be one of the former points; then, by drawing an arc of a circle with C as a centre and with a radius manifestly greater than the distance from C to the line AB, this arc will intersect AB in two points, say H and K.

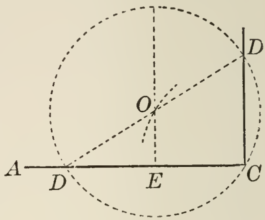
Therefore H and K are equally distant from C.

Another point equally distant from H and K will be on the bisector of HK, therefore bisect (by 183) HK, and let m be the point of bisection.

Therefore C and m are two points equally distant from H and K, and the line Cm is the perpendicular required.

EXERCISES.

1. From the extremity of a straight line to erect a perpendicular to that line.



Suggestion. Take any length CD, bisect it perpendicularly; take C as a centre, draw arc intersecting EO in O; with O as a centre and OC radius, describe circumference; draw DOD'; join $D'C$.

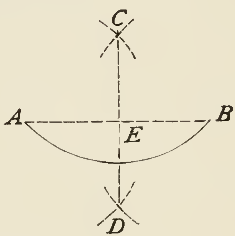
2. Divide a line into four equal parts.

3. Given the base and altitude of an isosceles triangle, to construct the triangle. See 94, 183.

4. Given the side of an equilateral triangle, to construct the triangle.

Proposition XVI. Problem.

185. *To bisect a given arc.*



Let AB be the given arc.

To bisect AB.

It is known (from 147) that the perpendicular bisector of a chord is also a bisector of the arc which it subtends.

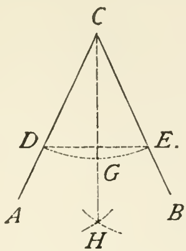
Therefore draw the chord AB and (by 183) bisect the chord, and the bisector CD will bisect the arc, say at E.

Proposition XVII. Problem.

186. *To bisect a given angle.*

Let ACB be the given angle.

To bisect $\angle ACB$.

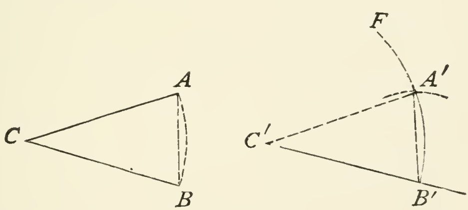


It is known (from 147) that the bisector of an arc also bisects the angle which it subtends.

Therefore draw the arc DE, and (by 185) bisect DE, say at G; then $\angle DCB$ is bisected by CG.

Proposition XVIII. Problem.

187. *At a given point in a given straight line to construct an angle equal to a given angle.*



Let C' be the given point in the given line $C'B'$, and C the given angle.

To construct at C' an angle equal to $\angle ACB$.

It is known (from 144) that in equal circles equal arcs subtend equal angles at the centre.

Therefore draw, with C as a centre and with C' as a centre, equal circles (or arcs); then, from B', measure off an arc equal to arc AB by taking B' as a centre, and with a radius equal to BA draw an arc intersecting arc $B'F$ at a point, say A', and join $A'C'$; then $\angle A'C'B' = \angle ACB$.

EXERCISES.

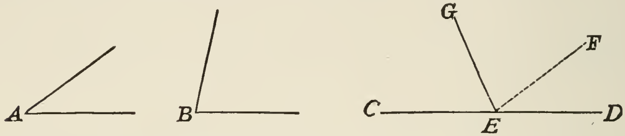
1. To construct a right triangle, given an acute angle and the base; given an acute angle and the hypotenuse.

2. To divide an angle into four equal parts.

3. Given an angle, to construct its complement; to construct its supplement. See 40, 41.

Proposition XIX. Problem.

188. *Given two angles of a triangle, to find the third.*



Let A and B be the given angles.

To find the third angle.

It is known (from 79) that the three angles of a triangle are equal to two right angles.

It is also known (from 46) that the sum of the angles around a point on one side of a straight line is equal to two right angles.

Therefore, if the two angles be added together so that their vertices may coincide and both fall on the same side of a straight line, then the remaining angle on that side will be the angle required.

Hence at a point, say E in the line CD, construct (by 187) an angle, say CEG equal to $\angle B$, and an angle FED equal to $\angle A$, then the remaining $\angle GEF$ will be the third angle required.

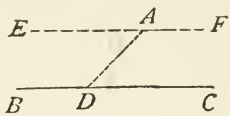
EXERCISES.

1. Given the base and vertical angle of an isosceles triangle, to construct the triangle.

2. Given the altitude and one of the equal angles of an isosceles triangle, to construct the triangle.

Proposition XX. Problem.

189. *Through a given point to draw a straight line parallel to a given straight line.*



Let BC be the line and A the point.

To draw through A a line parallel to BC.

It is known (from 62) that if one line intersect two other lines so as to make the interior and opposite angles equal, the lines will be parallel.

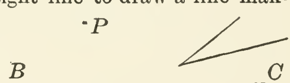
Therefore, draw a line from A to any point in BC, say D, making ADC one interior angle.

Then construct at A on the line DA an angle opposite $\angle ADC$ and (by 187) equal to it, that is, the angle DAE; then EF will be parallel to BC, and will pass through A as required.

EXERCISE.

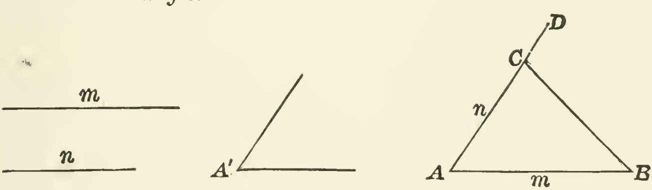
1. Through a given point without a straight line to draw a line making a given angle with that line.

Suggestion. Through P draw a line parallel to BC, then see 62.



Proposition XXI. Problem.

190. *Given two sides and the included angle of a triangle, to construct the triangle.*



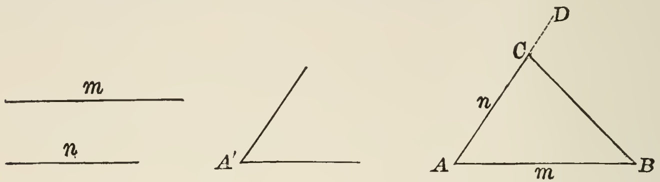
Let m and n be the given sides, and A' their included angle.

To construct the triangle.

Draw a line AB equal to m.

Construct (by 187) at A an angle BAC equal to $\angle A'$, and measure off on the side AD a part equal to n, and join CB.

Then ACB will be the triangle required, having two sides and the included angle given; no triangle differing from it could be constructed with these parts (86).

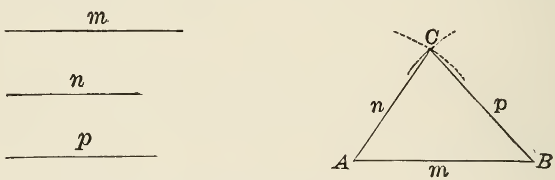


EXERCISES.

1. Given a side and two adjacent angles of a triangle, to construct the triangle.
2. Given a side and *any* two angles to construct the triangle.
3. Show when the problem (190) is impossible.

Proposition XXII. Problem.

191. *Given the three sides of a triangle, to construct the triangle.*



Let m, n, and p be the given sides.

To construct the triangle.

Lay off AB equal to m; then since the other vertex of the triangle must be at a distance n from A, it will lie on the circumference whose centre is at A and whose radius is n; therefore draw such a circle or a portion of it.

Likewise the same vertex must be at a distance of p from B; therefore it will lie on the circumference whose centre is B and radius p.

Draw such a circle, and where the two circles, or arcs, intersect will be the vertex C required.

Then the triangle ABC will have its sides equal to m, n, and p, and no triangle differing from it could have the same three sides (91).

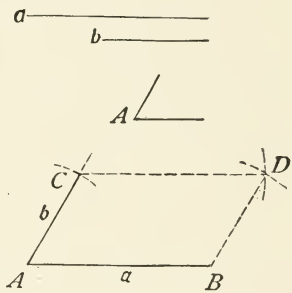
EXERCISES.

1. When is this problem impossible?

2. Two sides of a triangle and the angle opposite one of them being given, to construct the triangle.

Proposition XXIII. Problem.

192. *Given two adjacent sides and the included angle of a parallelogram, to construct the parallelogram.*



With the sides a, b, and the $\angle A$ the given angle, to construct the parallelogram.

Lay off AB equal to a, construct (by 187) the angle BAC equal to $\angle A$, and make $AC = b$.

Since the opposite sides of a parallelogram (by 106) are equal, the fourth vertex must be as far from C as B is from A; therefore, it will be on a circumference whose centre is at C and whose radius is equal to a. Likewise, this vertex will be on a circumference whose centre is at B and radius equal to b.

Hence if this vertex is on both the circumferences named, it will be at their intersection, say D.

Join DC and DB, and $ABDC$ will be the parallelogram required.

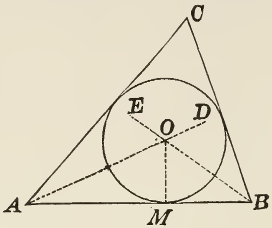
EXERCISES.

1. Construct a square upon a given straight line.

2. Given two diagonals of a parallelogram and their included angle to construct the parallelogram.

PROPOSITION XXIV. PROBLEM.

193. *To inscribe a circle in a given triangle.*



Let ABC be the given triangle.

To inscribe a circle in ABC.

It is known (from 96) that the point in which the bisectors of the angles of a triangle meet is equally distant from the three sides of the triangle.

Therefore, if this point be taken as a centre, and the distance from it to any one side be used as a radius, the circle so described will touch all three sides, or be inscribed in the triangle.

Hence bisect (by 186) any two of the angles of the triangle, and the point of intersection, say O, will be the centre, and the perpendicular OM the radius.

If the sides of a triangle are produced and the exterior angles are bisected, the intersections of the bisectors are the centres of three circles, each of which is tangent to one side of the triangle and the other two sides produced. These three circles are called *escribed* circles.

EXERCISES.

1. To draw an escribed circle.

2. Given the middle point of a chord in a given circle, to draw the chord.

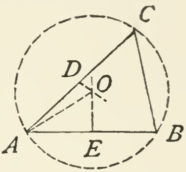
3. Construct an angle of 60°, one of 120°, and one of 45°.

Proposition XXV. Problem.

194. *To circumscribe a circle about a given triangle.*

Let ABC be the triangle.

To circumscribe a circle about ABC.



It is known (from 54) that every point that is equally distant from any two points is on the perpendicular bisector of the line joining these two points.

Therefore, any circle whose circumference passes through A and B must have its centre on the perpendicular bisector of AB.

Likewise, the circle whose circumference passes through A and C must have its centre on the perpendicular bisector of AC.

Therefore (by 183), bisect AB and AC; then the point in which the bisectors meet, say O, will be equally distant from A, B, and C, or will be the centre of the circumscribing circle.

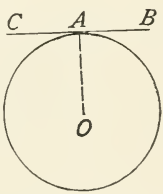
EXERCISES.

1. Through three points, not in a straight line, to draw a circle.
2. To circumscribe a circle about a given rectangle.

Proposition XXVI. Problem.

195. *At a given point in a given circumference, to draw a tangent to the circumference.*

Let O be the given circle and A the point on its circumference.



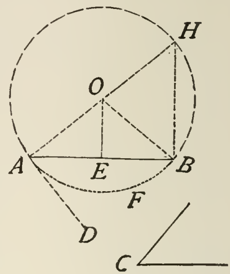
To draw a tangent through A.

It is known (from 150) that a line perpendicular to a radius at its extremity is a tangent to the circumference.

Therefore, draw the radius OA, and erect (by 182) a perpendicular to OA through A, and it, say CB, will be the tangent required.

EXERCISES.

1. On a given straight line, to describe a segment which shall contain a given angle.



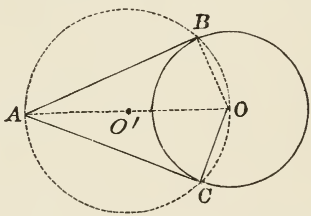
Suggestion. On AB construct (by 187) $\angle BAD = \angle C$; draw AH perpendicular to AD at D (by 184, Ex. 1, see 151); bisect AB (by 183), and O will be the centre (see 147) and AFB the required arc (see 179).

2. Through a given point inside of a circle other than the centre to draw a chord which is bisected at that point.

Suggestion. Find the centre, draw the diameter through the given point, and see 147.

Proposition XXVII. Problem.

196. *To draw a tangent to a given circle through a given point without the circumference.*



Let O be the centre of the given circle, and A the given point without the circumference.

To draw through A a tangent to the circumference.

It is known (from 150) that a radius and tangent drawn to its extremity are perpendicular to each other.

It is also known (from 177) that a diameter subtends a right angle.

Therefore the line joining the point through which the tangent is to pass and the centre of the given circle must subtend a right angle or be the diameter of an auxiliary circle.

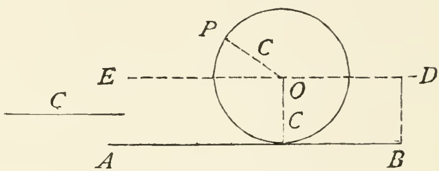
Hence draw AO, bisect it (by 183) at O', say, then describe a circle with $O'O$ as a radius and O' a centre, and connect A with the points where this circle intersects the given circle, say B and C, then AB and AC will be the tangents required, $\angle ABO$ and $\angle ACO$ being right angles.

EXERCISES.

1. To describe a circle tangent to a given straight line, having its centre at a given point.

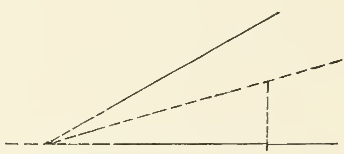
Suggestion. See 184 and 150.

2. Through a given point to describe a circle of given radius, tangent to a given straight line.



Suggestion. Erect DB $(=C)$ $\perp$ to AB at B, draw DE parallel to AB, with P as centre, and radius equal to C, cut ED in O, then O is the centre.

3. Show when the above problem is impossible.



4. To describe a circle touching two given straight lines, one of them at a given point.

Suggestion. See 160, 150.

5. To find the centre of a given arc or circle.

6. To draw a tangent common to two circles, C and c.

Suggestion. Draw two parallel radii CB and cb; draw Bb and continue it until it meets Cc produced in A. See 196.

BOOK III.

RATIO AND PROPORTION. SIMILAR FIGURES.

DEFINITIONS.

197. A *Proportion* is an equality of ratios. (See 161–163.)

That is, if the ratio of a to b is equal to the ratio of c to d, they form a proportion, which may be written

$$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}, \text{ or } a : b :: c : d,$$

and is read a is to b as c is to d.

198. The four terms of the two equal ratios are called the *Terms* of the proportion. The first and fourth terms are called the *Extremes*, and the second and third the *Means*. Thus, in the above proportion, a and d are the extremes, and b and c the means.

The first and third terms are called the *Antecedents*, and the second and fourth the *Consequents*. Thus, a and c are the antecedents, and b and d the consequents.

The fourth term is called a *Fourth Proportional* to the other three. Thus, in the above proportion, d is a fourth proportional to a, b, and c.

In the proportion $a : b = b : c$, c is a *third proportional* to a and b, and b is a *mean proportional* between a and c.

PROPOSITION I.

199. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Let $a : b = c : d.$

To prove $ad = bc.$

By definition (197), $\frac{a}{b} = \frac{c}{d}.$

Clearing of fractions, $ad = bc.$ Q.E.D.

200. COR. If $a : b = b : c,$

then (by 199) $b^2 = ac.$

$\therefore\ b = \sqrt{ac}.$ Q.E.D.

That is, *the mean proportional between two quantities is equal to the square root of their product.*

PROPOSITION II. THEOREM.

201. CONVERSELY, *if the product of two quantities is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Let $ad = bc.$

Dividing both members of the equation by bd,

$$\frac{ad}{bd} = \frac{bc}{bd}, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

That is, $a : b = c : d.$ Q.E.D.

PROPOSITION III. THEOREM.

202. *In any proportion the terms are in proportion by* **Alternation**; *that is, the first term is to the third as the second term is to the fourth.*

Let $a : b = c : d.$

Then (by 199) $ad = bc.$

Whence (by 201) $a : c = b : d.$ Q.E.D.

Proposition IV.

203. *If four quantities are in proportion, they are in proportion by* **Inversion**; *that is, the second term is to the first as the fourth term is to the third.*

Let $a : b = c : d.$

To prove $b : a = d : c.$

If $a : b = c : d,$

then (by 199) $bc = ad.$

Divide by ac, $\frac{bc}{ac} = \frac{ad}{ac};$

that is, $\frac{b}{a} = \frac{d}{c},$

or $b : a = d : c.$ Q.E.D.

Proposition V. Theorem.

204. *In any proportion the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Let $a : b = c : d.$

To prove $a + b : a :: c + d : c.$

If $a : b = c : d,$

then (by 199) $ad = bc.$

Adding both members of the equation to ac,

$$ac + ad = ac + bc,$$

or $a(c + d) = c(a + b).$

Therefore (by 201), $a + b : a :: c + d : c.$

Similarly, $a + b : b :: c + d : d.$ Q.E.D.

Proposition VI.

205. *If four quantities are in proportion, they are in proportion by* **Division**; *that is, the difference of the first and second is to the first as the difference of the third and fourth is to the third.*

Let $$a : b = c : d.$$

To prove $$a - b : a = c - d : c.$$

If $$a : b = c : d,$$

then (by 199) $$ad = bc.$$

Subtract both members of this equation from ac, then

$$ac - ad = ac - bc,$$

or $$a(c - d) = c(a - b).$$

Therefore (by 201), $a - b : a = c - d : c.$

Similarly, $$a - b : b = c - d : d.$$ Q.E.D.

Proposition VII.

206. *If four quantities are in proportion, they are in proportion by* **Composition** *and* **Division**; *that is, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

Let $$a : b = c : d.$$

To prove $$a + b : a - b = c + d : c - d.$$

(By 204), $$\frac{a+b}{b} = \frac{c+d}{d};$$

and (by 205) $$\frac{a-b}{b} = \frac{c-d}{d};$$

by division, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore\ a + b : a - b = c + d : c - d.$$ Q.E.D.

Proposition VIII.

207. The products of the corresponding terms of two or more proportions are proportional.

Let $a : b = c : d$, and $e : f = g : h$.

To prove $ae : bf = cg : dh$.

Writing the proportions in another form,

$$\frac{a}{b} = \frac{c}{d}, \text{ and } \frac{e}{f} = \frac{g}{h}.$$

Multiplying these equations member by member,

$$\frac{ae}{bf} = \frac{cg}{dh},$$

or $ae : bf = cg : dh$. Q.E.D.

208. Cor. If the corresponding terms of the proportions are equal; that is, if $e = a$, $f = b$, $g = c$, and $h = d$, the result of the preceding theorem becomes

$$a^2 : b^2 = c^2 : d^2.$$

And in general *in any proportion like powers of the terms are in proportion.*

Proposition IX. Theorem.

209. *In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b = c : d = e : f$.

To prove $a + c + e : b + d + f = a : b = c : d = e : f$.

Let r be the value of the equal ratios, that is,

$$\frac{a}{b} = r, \ \frac{c}{d} = r, \text{ and } \frac{e}{f} = r.$$

From these equations,

$$a = br, \quad c = dr, \quad e = fr,$$

or by addition,
$$a + c + e = br + dr + fr$$
$$= (b + d + f)r.$$

By dividing,
$$\frac{a + c + e}{b + d + f} = r.$$

But by hypothesis,
$$r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

Therefore
$$\frac{a + c + e}{b + d + f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

that is, $a + c + e : b + d + f = a : b = c : d = e : f.$ Q.E.D.

Proposition X. Theorem.

210. *In any proportion, if the antecedents are multiplied by any quantity, as also the consequents, the resulting terms will be in proportion.*

Let
$$a : b = c : d.$$

Then
$$\frac{a}{b} = \frac{c}{d}.$$

Multiplying both members of the equation by $\frac{m}{n}$,

$$\frac{ma}{nb} = \frac{mc}{nd}.$$

That is,
$$ma : nb = mc : nd.$$

In like manner,
$$\frac{a}{m} : \frac{b}{n} = \frac{c}{m} : \frac{d}{n}.$$
Q.E.D.

211. Scholium. Either m or n may be unity.

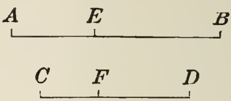
EXERCISES.

1. Show that equimultiples of two quantities are in the same ratio as the quantities themselves.

2. Show that if four quantities are in proportion, their like roots are in proportion.

PROPORTIONAL LINES.

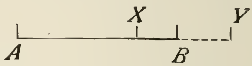
Two straight lines are said to be divided *proportionally* when their corresponding segments, or parts, are in the same ratio as the lines themselves.



Thus the lines AB and CD are divided proportionally at E and F if

$$AB : AE = CD : CF.$$

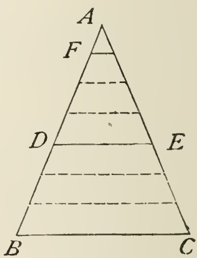
212. When a finite straight line, as AB, is cut at a point X between A and B, it is said to be *divided internally* at X, and the two parts AX and BX are called *segments*. But if the straight line AB is produced, and cut at a point Y beyond AB, it is said to be *divided externally* at Y, and the parts AY and BY are called *segments*. The given line is the *sum* of two *internal* segments, or the *difference* of two *external* segments.



When a straight line is divided internally and externally into segments having the same ratio, it is said to be divided *harmonically*.

Proposition XI. Theorem.

213. *A straight line parallel to one side of a triangle divides the other two sides proportionally.*



In the triangle ABC let DE be parallel to BC.

To prove $AD : DB = AE : EC$.

Case I. When AD and DB are commensurable.

Take AF, any common measure of AD and DB, and suppose it to be contained 4 times in AD and 3 times in DB.

Then $$\frac{AD}{DB} = \frac{4}{3}.$$

Through the several points of division of AB draw lines parallel to BC; then since these parallels cut off equal lengths on AB, they will (by 110) cut off equal lengths on AC.

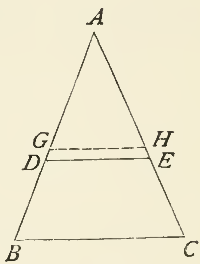
Therefore, AE will be divided into 4 equal parts and EC into 3; that is,

$$\frac{AE}{EC} = \frac{4}{3}.$$

Hence (by 28), $$\frac{AD}{DB} = \frac{AE}{EC},$$

or $$AD : DB = AE : EC.$$

Case II. When AD and DB are incommensurable.



In this case we know (170) that we may always find a line AG as nearly equal as we please to AD, and such that AG and DB are commensurable.

Draw GH parallel to BC; then

$$\frac{AG}{DB} = \frac{AH}{EC}. \qquad \text{(Case I.)}$$

As these two ratios are always equal while the common measure is indefinitely diminished, they will be equal as GH approaches DE.

Therefore, this quality of ratios will exist (by 172) as the limiting position DE is approached; that is,

$$\frac{AD}{DB} = \frac{AE}{EC}, \text{ or } AD : DB = AE : EC. \qquad \text{Q.E.D}$$

214. Cor. By composition (204),

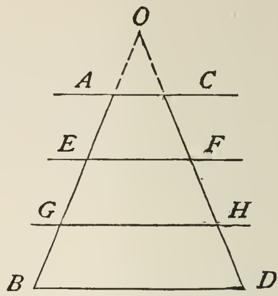
$$AD + DB : AD = AE + EC : AE,$$

or $$AB : AD = AC : AE.$$

Likewise (by 204), $$AB : DB = AC : EC,$$

and (by 202), $$AB : AC = DB : EC.$$

EXERCISES.



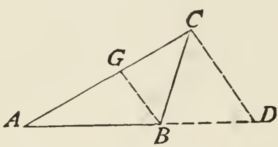
1. Conversely, if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

2. If two straight lines AB, CD are cut by any number of parallels, AC, EF, GH, BD, the corresponding intercepts are proportional.

Suggestion. See 214.

Proposition XII. Theorem.

215. *The bisector of any angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*



Let ABC be the triangle, and GB the bisector of the angle ABC.

To prove $AG : GC = AB : BC$.

Draw CD parallel to GB, and produce AB to D.

Then (by 63) $\angle BDC = \angle ABG$,

and (by 62) $\angle BCD = \angle GBC$.

But by construction, $\angle ABG = \angle GBC$;

therefore (by 28) $\angle BDC = \angle BCD$;

hence (by 93) the triangle BCD is isosceles, and

$$BC = BD.$$

It is known (from 213) that

$$AG : GC = AB : BD.$$

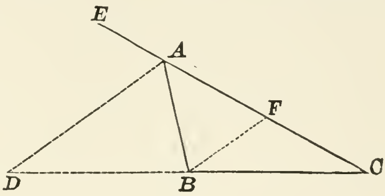
Substituting for BD its equal BC,

$$AG : GC = AB : BC. \qquad \text{Q.E.D.}$$

EXERCISES.

1. If a line divides one side of a triangle into segments that are proportional to the adjacent sides, it bisects the opposite angle.

2. The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.



3. If (in 215), $AB = 4$, $BC = 6$, and $CA = 9$, find AG and GC.

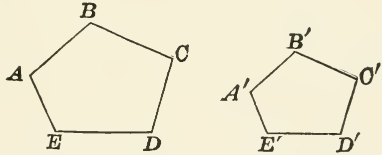
4. If $AB = 5$, $BC = 7$, and $CA = 8$, find AD and BD.

5. Bisectors of an interior and exterior angle at the vertex of a triangle divide the opposite side harmonically.

Suggestion. See 212.

SIMILAR POLYGONS.

216. Definitions. Two polygons are called *Similar* when they are mutually equiangular (122) and have their homologous sides proportional (122).



That is, the polygons $ABCDE$ and $A'B'C'D'E'$ are similar if:

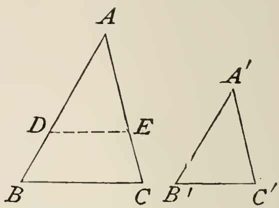
$$\angle A = \angle A', \ \angle B = \angle B', \ \angle C = \angle C', \text{ etc.,}$$

and
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

217. In two similar polygons, the ratio of any two homologous sides is called the *Ratio of Similitude* of the polygons.

Proposition XIII. Theorem.

218. *Triangles which are mutually equiangular are similar.*



Let ABC and $A'B'C'$ be two mutually equiangular triangles.

To prove that ABC and $A'B'C'$ are similar triangles.

Lay off on AB a distance equal to $A'B'$ and on AC make AE equal to $A'C'$.

The triangles ADE and $A'B'C'$ are (by 86) equal, having the included $\angle A$ equal to $\angle A'$, and the sides AD and AE equal to $A'B'$ and $A'C'$, by construction. Therefore $\angle ADE = \angle B'$, but by hypothesis, $\angle B = \angle B'$, hence $\angle ADE = \angle B$, therefore (by 63) DE is parallel to BC.

If DE is parallel to BC, we have (by 214)

$$AB : AD = AC : AE,$$

or substituting for AD its equal $A'B'$, and for AE, $A'C'$,

$$AB : A'B' = AC : A'C'.$$

Similarly, it can be shown, by laying off on BA a distance equal to $B'A'$, and on BC a distance equal to $B'C'$, that

$$BA : B'A' = BC : B'C'. \qquad \text{Q.E.D.}$$

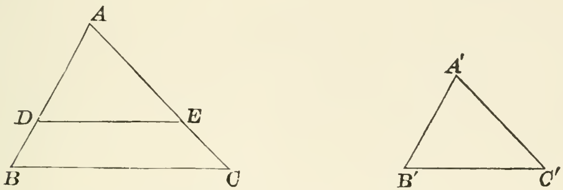
219. Cor. 1. *Two triangles are similar when two angles of the one are equal respectively to two angles of the other.* (See 82.)

220. Cor. 2. *A triangle is similar to any triangle cut off by a line parallel to one of its sides.*

221. Scholium. In similar triangles the homologous sides lie opposite the equal angles.

Proposition XIV. Theorem.

222. *Two triangles are similar when their homologous sides are proportional.*



In the triangles ABC and $A'B'C'$, let

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

To prove that the triangles are similar.

Take $AD = A'B'$ and $AE = A'C'$, and join DE.

Then from the given proportion we have

$$\frac{AB}{AD}=\frac{AC}{AE},$$

therefore (by converse of 214, Ex. 1) the line DE is parallel to BC, and the angles ADE and B having their sides parallel and similarly directed are (by 63) equal; likewise, $\angle AED = \angle C$.

Hence the triangles ADE and ABC are mutually equiangular and (by 218) are similar; that is,

$$\frac{AB}{AD}=\frac{BC}{DE}, \text{ or } \frac{AB}{A'B'}=\frac{BC}{DE}.$$

But, by hypothesis, $$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

These last two proportions agree term for term except the last in each, and these must therefore be equal, or $B'C' = DE$.

Hence the triangles ADE and $A'B'C'$ are mutually equilateral and therefore equal.

But the triangle ADE has been proved similar to ABC.

Hence the triangle $A'B'C'$ is similar to ABC. Q.E.D.

223. Scholium. Two polygons are similar when they are mutually equiangular *and* have their homologous sides proportional. But in the case of triangles we learn, from Propositions XIII. and XIV., that either of these conditions involves the other.

This, however, is not necessarily the case with polygons of more than three sides; for even with quadrilaterals, the angles can be changed without altering the sides, or the proportionality of the sides can be changed without altering the angles.

EXERCISES.

1. Two right triangles are similar when they have an acute angle of one equal to an acute angle of the other.

2. Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.

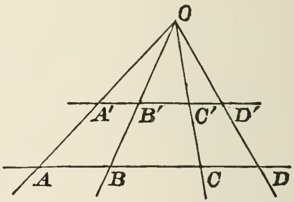
3. Two triangles are similar when the sides of one are parallel respectively to the sides of the other.

4. Two triangles are similar when the sides of one are perpendicular respectively to the sides of the other.

Suggestion. See 64.

5. The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.

6. If in any triangle a parallel be drawn to the base, all lines from the vertex will divide the base and its parallel proportionally.



Suggestion. See 218.

7. Two parallelograms are similar when they have an angle equal and the including sides proportional.

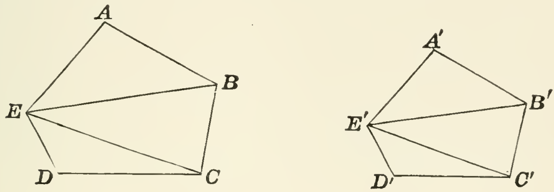
8. Two rectangles are similar when they have two adjacent sides proportional.

9. If two triangles stand upon the same base, and not between the same parallels, the figure formed by joining the middle points of their sides is a parallelogram.

10. If from any two diametrically opposite points on the circumference of a circle perpendiculars be drawn to a straight line outside the circle, the sum of these perpendiculars is constant.

Proposition XV. Theorem.

224. *Two polygons are similar when they are composed of the same number of triangles, similar each to each and similarly placed.*



Let $ABCDE$ and $A'B'C'D'E'$ be two polygons composed of the same number of similar triangles similarly placed.

To prove that the polygons are similar; that is, that they are mutually equiangular, and that their homologous sides are proportional.

Since the triangles AEB and $A'E'B'$ are similar by hypothesis, they are (by 223) mutually equiangular; that gives

$$\angle A = \angle A', \text{ and } \angle ABE = \angle A'B'E'.$$

Likewise, in the triangles EBC and $E'B'C'$,

$$\angle EBC = \angle E'B'C',$$

or by addition, $\angle ABE + \angle EBC = \angle A'B'E' + \angle E'B'C'$,

or $$\angle ABC = \angle A'B'C'.$$

In like manner,

$\angle BCD = \angle B'C'D'$, $\angle CDE = \angle C'D'E'$, and $\angle DEA = \angle D'E'A'$.

Since the triangles are similar, their homologous sides are proportional, which gives

$$\frac{AB}{A'B'} = \frac{BE}{B'E'}, \text{ and } \frac{BE}{B'E'} = \frac{BC}{B'C'},$$

or (by 28), $$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

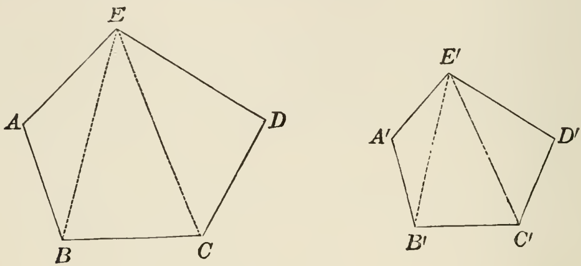
In like manner,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}. \qquad \text{Q.E.D.}$$

225. Cor. *Conversely, two similar polygons may be divided into the same number of triangles, similar each to each, and similarly placed.*

Proposition XVI. Theorem.

226. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



Let the two similar polygons be $ABCDE$ and $A'B'C'D'E'$, and let P and P' represent their perimeters.

To prove $P : P' :: AB : A'B'$.

Since the polygons are similar (by 223),

$$\frac{AE}{A'E'} = \frac{ED}{E'D'} = \frac{DC}{D'C'} = \frac{CB}{C'B'} = \frac{BA}{B'A'}.$$

The sum of the antecedents will have the same ratio to the sum of the consequents that any antecedent has to its consequent (by 209); that is,

$$\frac{AE + ED + DC + CB + BA}{A'E' + E'D' + D'C' + C'B' + B'A'} = \frac{AE}{A'E'},$$

or,

$$\frac{P}{P'} = \frac{AE}{A'E'}. \qquad \text{Q.E.D.}$$

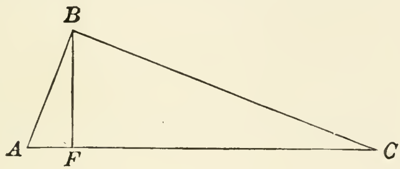
PROPOSITION XVII. THEOREM.

227. *If in a right triangle a perpendicular be drawn from the vertex of the right angle to the hypotenuse:*

I. *It divides the triangle into two right triangles which are similar to the whole triangle, and also to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Each side of the right triangle is a mean proportional between the hypotenuse and its adjacent segment.*



In the right triangle ABC, let BF be drawn from the vertex of the right angle B, perpendicular to the hypotenuse AC.

1. To prove that ABF is similar to BFC, and that each is similar to ABC.

The triangles ABF and ABC have the angle A common, and $\angle AFB = \angle ABC$, therefore their third angles (by 82) are equal.

Hence the triangles are mutually equiangular and (by 218) are similar.

Likewise, the triangles BFC and ABC are similar.

Therefore, if the triangles ABF and BFC are similar to ABC, they will be similar to one another.

2. To prove that $AF : BF = BF : FC$.

Since the triangles ABF and BFC are similar, their homologous sides are (by 223) proportional, that gives

$$AF : BF = BF : FC, \text{ or (by 199) } \overline{BF}^2 = AF \times FC.$$

3. To prove that

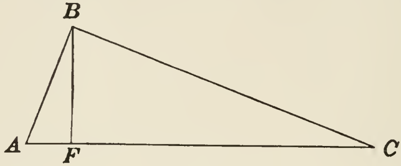
$$AC : BC = BC : FC, \text{ or (by 199) } \overline{BC}^2 = AC \times FC.$$

Since the triangles ABC and FBC are similar, their homologous sides (by 223) are proportional, which gives

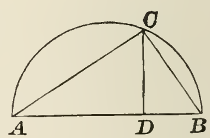
$$AC : BC = BC : FC.$$

In a similar manner, it can be shown that

$$AC : AB = AB : AF, \text{ or (by 199) } \overline{AB}^2 = AC \times AF. \quad \text{Q.E.D.}$$



228. COR. Since an angle inscribed in a semicircle is a right angle (177), it follows that



I. *The perpendicular from any point in the circumference of a circle to a diameter is a mean proportional between the segments of the diameter.*

II. *The chord drawn from the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.*

EXERCISES.

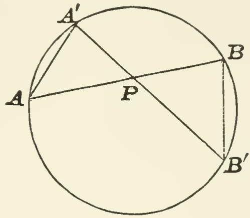
1. The squares on the two sides of the right triangle have the same ratio as the adjacent segments of the hypotenuse.

2. The square on the hypotenuse has the same ratio to the square on either side as the hypotenuse has to the segment adjacent to that side.

3. Two isosceles triangles are similar when their vertical angles are equal.

PROPOSITION XVIII. THEOREM.

229. *If any two chords are drawn through a fixed point in a circle, the product of the segments of one is equal to the product of the segments of the other.*



Let AB and $A'B'$ be any two chords of the circle ABB' passing through the point P.

To prove that $$AP \times BP = A'P \times B'P.$$

Join AA' and BB'.

In the two triangles APA' and BPB', the vertical angles $A'PA$ and BPB' are (by 49) equal, $\angle B'$ and $\angle A$ are equal, both being measured by one-half of the same arc $A'B$, and $\angle B = \angle A'$ for the same reason.

The triangles are therefore equiangular and (by 218) are similar, which gives

$$A'P : PB = AP : PB', \text{ or (by 199) } A'P \times PB' = AP \times PB.$$

230. When four quantities, such as the sides about two angles, are so related that a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first, the sides are said to be *reciprocally proportional.* Therefore

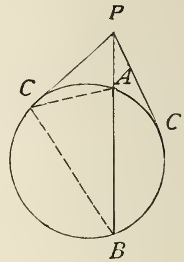
231. COR. 1. *If two chords cut each other in a circle, their segments are reciprocally proportional.*

232. COR. 2. *If through a fixed point within a circle any number of chords are drawn, the products of their segments are all equal.*

Proposition XIX. Theorem.

233. *If from a point without a circle a tangent and a secant be drawn, the tangent is a mean proportional between the whole secant and the external segment.*

Let PC and PB be a tangent and a secant drawn from the point P to the circle CAB.



To prove that $PB : PC = PC : PA$.

Join CA and CB.

In the two triangles PCA and PCB the angle P is common, and $\angle PCA = \angle PBC$, being measured by one-half of the same arc CA; then (by 82), $\angle PAC = \angle PCB$.

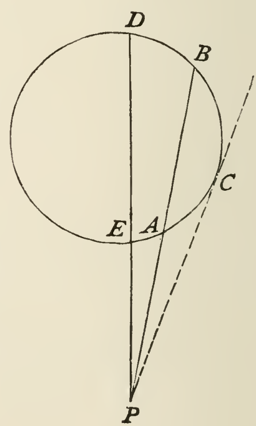
Therefore, the triangles are equiangular and are (by 218) similar, which gives

$$PB : PC = PC : PA; \text{ or (by 199), } \overline{PC}^2 = PB \times PA. \quad \text{Q.E.D.}$$

234. Cor. $\overline{PC'}^2 = BP \times PA$; therefore (by 28),

$$\overline{PC}^2 = \overline{PC'}^2, \text{ or } PC = PC'.$$

EXERCISES.



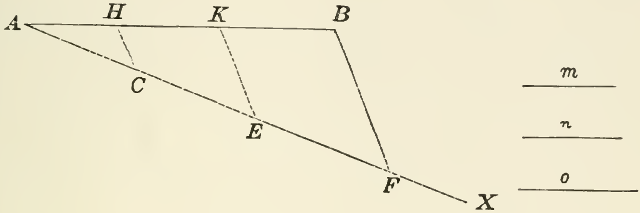
1. If from a point without a circle two secants be drawn, the product of one secant and its external segment is equal to the product of the other and its external segment.

2. If from a point without a circle any number of secants are drawn, the products of the whole secants and their external segments are all equal.

Suggestion. Draw a tangent PC and apply 233.

Proposition XX. Problem.

235. *To divide a given straight line into parts proportional to any number of given lines.*



Let AB, m, n, and o be given straight lines.

It is required to divide AB into parts proportional to the given lines m, n, and o.

It is known (from 213) that lines drawn parallel to the base of a triangle divide the other two sides proportionally.

Therefore, form with AB a triangle by drawing an indefinite straight line from A; measure off on this line a part equal to m, say AC; then n, say CE, and o, say EF, and join BF, thus forming the triangle AFB.

Through the points E and C draw lines parallel to FB, meeting AB in K and H; then

$$AH : HK : KB = AC : CE : EF = m : n : o.$$

236. Cor. 1. By making $AC = CE = EF$, the line AB will be divided equally.

237. Cor. 2. By making $AC = m$, $AH = n$, and $CE = o$, we would have (by 213), $m : n : : o : HK$; that is, HK would be a fourth proportional to m, n, and o.

EXERCISE.

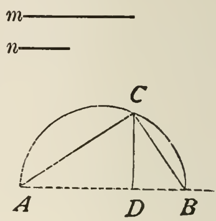
To divide a given straight line into three segments, A, B, and C, such that A and B shall be in the ratio of two given straight lines m and n, and B and C shall be in the ratio of two other straight lines o and p.

Proposition XXI. Problem.

238. *To find a mean proportional between two given straight lines.*

Let m and n be the two lines.

To find a mean proportional to them.



It is known (from 228) that the perpendicular from the circumference to the diameter is a mean proportional between the segments of the diameter.

Therefore, if a diameter be made of m and n and a perpendicular erected at their point of union, the portion included between the diameter and the circumference will be the mean proportional required; that is, lay off $AD = m$ and $DB = n$. Describe upon AB as a diameter a circle, erect (by 182) a perpendicular at D, and CD will be a mean proportional, or

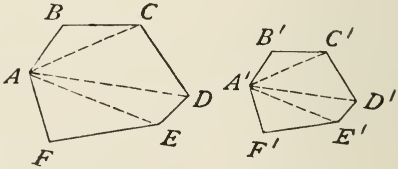
$$AD : CD = CD : DB, \quad \text{or} \quad m : CD = CD : n.$$

239. The mean proportional between two lines is often called the *geometric* mean, while half their sum is called the *arithmetic* mean.

Proposition XXII. Problem.

240. *On a given straight line, to construct a polygon similar to a given polygon.*

Let $ABCDEF$ be a polygon, and $A'B'$ be the straight line.



To construct on $A'B'$ a polygon similar to A–F.

It is known (from 224) that two polygons are similar when they are composed of the same number of similar triangles similarly placed; therefore, divide A–F into triangles by drawing the diagonals AC, AD, and AE.

It is known (from 218) that triangles are similar when they are equiangular.

Therefore, construct (by 187) on $A'B'$ a triangle equiangular with ABC, say $A'B'C'$; then on $A'C'$ a triangle equiangular with ACD, say $A'C'D'$. Likewise on $A'D'$, $A'D'E'$, and on $A'E'$, $A'E'F'$; then will A'–F' be the polygon required.

BOOK IV.

AREAS OF POLYGONS.

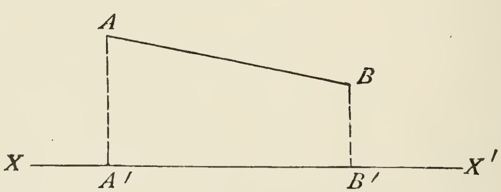
DEFINITIONS.

241. The *Area* of a surface is the numerical value of the ratio of this surface to another surface, called the *Unit of Surface*, or *Superficial Unit.*

242. The unit of surface is the square whose side is some *Unit of Length*, as an inch, a foot, a metre, etc., and the area is expressed as so many square inches, square feet, square metres, etc.

243. Two surfaces are equivalent when their areas are equal.

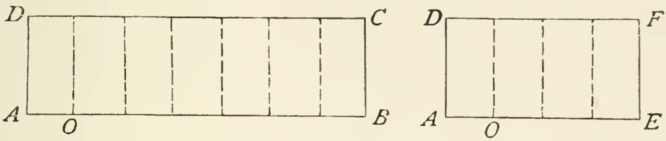
244. The projection of a point upon a straight line is the foot of the perpendicular let fall from the point upon the line. Thus A' is the projection of A.



The projection of a limited straight line upon another straight line, is the portion of the latter included between the projection of the terminal points of the former. Thus $A'B'$ is the projection of AB on XX'.

Proposition I. Theorem.

245. *Two rectangles* having equal altitudes are to each other as their bases.*



Let the two rectangles be AC and AF, having the same altitude AD.

To prove that $$\frac{ABCD}{AEFD} = \frac{AB}{AE}.$$

Case I. *When the bases are commensurable.*

Let AO be a common measure of AB and AE, and upon application it is found to be contained 7 times in AB and 4 times in AE.

At each point of division along AB erect perpendiculars, and likewise on AE. This will divide the first rectangle into 7 rectangles and the second into 4.

These small rectangles are equal, since having all parts the same they can be applied one to the other and will coincide throughout, thus

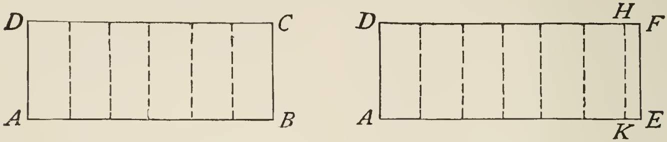
$$\frac{ABCD}{AEFD} = \frac{7}{4},$$

but $$\frac{AB}{AE} = \frac{7}{4}.$$

Therefore (by 28) $$\frac{ABCD}{AEFD} = \frac{AB}{AE}.$$

* By rectangles is meant the *area* of the rectangles.

Case II. *When AB and AE are incommensurable.*



In this case we find a portion of AE, say AK, which is commensurable with AB, erect the perpendicular KH; then (from first case),

$$\frac{ABCD}{AKHD} = \frac{AB}{AK}.$$

By diminishing the common measure a larger portion of AE can be found which will be commensurable with AB, but the above equality of ratios will exist.

The limit of $AKHD$ is $AEFD$, and the limit of AK is AE.

Therefore (by 172) $$\frac{ABCD}{AEFD} = \frac{AB}{AE}.$$ Q.E.D.

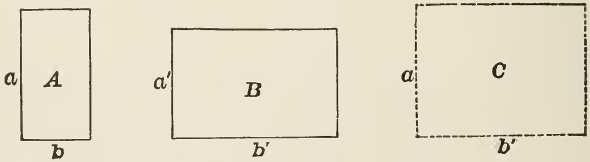
246. Cor. Since either side of a rectangle may be taken as the base, it follows that

Two rectangles having equal bases are to each other as their altitudes.

Proposition II. Theorem.

247. *Any two rectangles are to each other as the products of their bases by their altitudes.*

Note. By the *product* of two lines is to be understood the product of their *numerical measures* when referred to a common unit (§ 242).



Let A and B be any two rectangles having the altitudes a and a', and the bases b and b', respectively.

To prove that $$\frac{B}{A} = \frac{a' \times b'}{a \times b}.$$

Construct a rectangle C, with a base equal to the base of B and altitude equal to that of A.

Then (by 246), comparing B and C,

$$\frac{B}{C} = \frac{a'}{a}.$$

Likewise (by 245), comparing C and A,

$$\frac{C}{A} = \frac{b'}{b}.$$

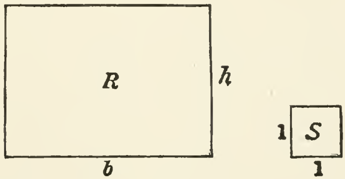
Multiplying these proportions (by 207),

$$\frac{B \times C}{C \times A} = \frac{a' \times b'}{a \times b},$$

or $$\frac{B}{A} = \frac{a' \times b'}{a \times b}.$$ Q.E.D.

PROPOSITION III. THEOREM.

248. *The area of a rectangle is equal to the product of its base and altitude.*



Let R be the rectangle, b the base, and h the altitude; and let S be a square whose side is the linear unit.

To prove the area of $R = h \times b$.

It is known (from 247) that two rectangles are to each other as the products of their bases by their altitudes; therefore,

$$\frac{R}{S} = \frac{h \times b}{1 \times 1}, = h \times b,$$

but S is the unit of area;

hence $$R = h \times b.$$

249. Cor. If $h = b$, then $R = b \times b = b^2$.

But when the base and altitude of a rectangle are equal, the figure (by 105) is a square, hence the area of a square is equal to the square of one of its sides.

250. Scholium. The statement of this proposition is an abbreviation of the following:

The number of units of area in a rectangular figure is equal to the product of the number of linear units in its base by the number of linear units in its altitude.

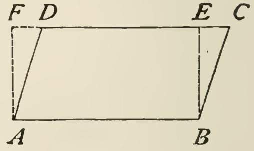
Proposition IV. Theorem.

251. *The area of a parallelogram is equal to the product of its base and altitude.*

Let $ABCD$ be a parallelogram, and EB its altitude.

To prove that the area of

$$ABCD = AB \times EB.$$



Erect the perpendiculars AF and BE and produce CD to F, forming the rectangle $ABEF$.

In the right triangles ADF and BCE the sides AD and BC are (by 106) equal, and AF and BE are (by 108) equal; therefore, the triangles are equal.

If from the entire figure $ABCF$ the triangle ADF be subtracted, the parallelogram $ABCD$ remains; and if from the same figure the equal triangle BEC be subtracted, the rectangle $ABEF$ remains.

Therefore (by 28)

$$ABCD = ABEF.$$

But (by 248) $\quad ABEF = AB \times EB.$

Hence $\quad ABCD = AB \times EB.$ Q.E.D.

252. Cor. 1. *Parallelograms having equal bases and equal altitudes are equivalent, because they are all equivalent to the same rectangle.*

253. Cor. 2. *Any two parallelograms are to each other as the products of their bases by their altitudes; therefore, parallelograms having equal bases are to each other as their altitudes, and parallelograms of equal altitudes are to each other as their bases.*

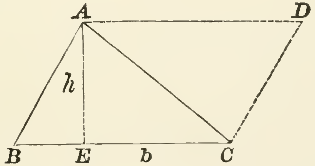
EXERCISES.

1. Show that the diagonals of a parallelogram divide it into four equivalent triangles.

2. Show that the area of a rhombus is equal to one-half the product of its diagonals.

Proposition V. Theorem.

254. *The area of a triangle is equal to one-half the product of its base and altitude.*



Let ABC be a triangle, having its altitude equal to h and its base equal to b.

To prove that $\quad$ area $ABC = \frac{1}{2} h \times b.$

Draw the lines AD and CD parallel to BC and AB.

Then $ABCD$ is a parallelogram having its altitude equal to h and its base equal to b.

It is known (from 107) that the diagonal of a parallelogram divides it into two equal triangles; therefore

$$ABC = \tfrac{1}{2} ABCD.$$

But (by 251) $\qquad ABCD = h \times b.$

Therefore $\qquad ABC = \tfrac{1}{2} h \times b.$ Q.E.D.

255. Cor. 1. *Two triangles having equal bases and equal altitudes are equivalent.*

256. Cor. 2. *Two triangles having equal altitudes are to each other as their bases; two triangles having equal bases are to each other as their altitudes; and any two triangles are to each other as the products of their bases by their altitudes.*

257. Cor. 3. *A triangle is equivalent to one-half of a parallelogram having the same base and altitude.*

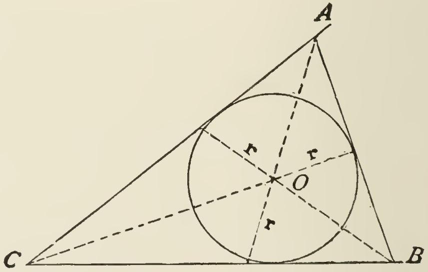
EXERCISES.

1. The area of a rectangle is 6912 square inches and its base is 2 yards. What is its perimeter in feet?

2. If the base and altitude of a triangle are 18 and 12, what is the length of the side of an equivalent square?

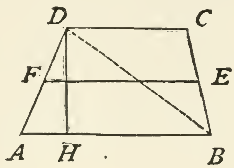
3. Show that the area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.

Suggestion. Join the centre with each vertex, and find the area of OBC, OBA, and OAC.



PROPOSITION VI. THEOREM.

258. *The area of a trapezoid is equal to the product of the half sum of its parallel sides by its altitude.*



Let $ABCD$ be a trapezoid, with AB and CD its parallel sides and DH the altitude.

To prove that the area of

$$ABCD = \tfrac{1}{2}(AB + CD) \times DH.$$

Join DB, making of the trapezoid two triangles.

It is known (from 254) that the area of

$$ADB = \tfrac{1}{2}\, AB \times DH,$$

and area of $$DCB = \tfrac{1}{2}\, DC \times DH.$$

Hence by adding

$$ADB + DCB = \tfrac{1}{2}\, AB \times DH + \tfrac{1}{2}\, DC \times DH,$$

or $$ABCD = \tfrac{1}{2}(AB + DC) \times DH.$$ Q.E.D.

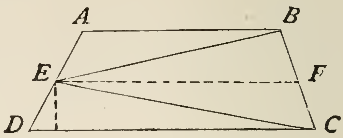
259. Since (by 113) the median line $FE = \tfrac{1}{2}(AB + DC)$, then the area of a trapezoid is equal to the product of the median joining the middle points of the non-parallel sides by the altitude.

$$\therefore \text{ area } ABCD = FE \times DH.$$

260. Occasionally the area of an irregular polygon is found by dividing the figure into trapezoids and triangles, and finding the area of each and taking their sum.

EXERCISE.

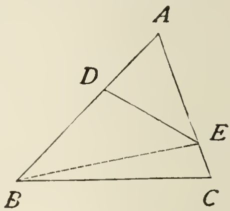
1. In a trapezoid the straight lines, drawn from the middle point of one of the non-parallel sides to the ends of the opposite side, form with that side a triangle equal to half the trapezoid.



Suggestion. Compare area of ABE, BEF and FEC, EDC.

Proposition VII. Theorem.

261. *The areas of two triangles having an angle of the one equal to an angle of the other, are to each other as the products of the sides including the equal angles.*



Let ABC and ADE be two triangles, having $\angle A$ common.

To prove that $\dfrac{ABC}{ADE}=\dfrac{AB\times AC}{AD\times AE}$.

Join BE, then the two triangles ABC and ABE having their bases in the same line AC and their vertices in the same point EB will have the same altitude, hence (by 256)

$$\frac{ABC}{ABE}=\frac{AC}{AE}.$$

Likewise the triangles ABE and ADE having their bases in the same line (AB), and their vertices in the same point (E), will have the same altitude, hence

$$\frac{ABE}{ADE}=\frac{AB}{AD}.$$

Multiplying these ratios (by 207),

$$\frac{ABC\times ABE}{ABE\times ADE}=\frac{AC\times AB}{AE\times AD},$$

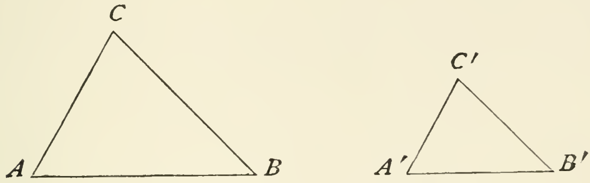
or

$$\frac{ABC}{ADE}=\frac{AC\times AB}{AE\times AD}.$$

Q.E.D.

Proposition VIII. Theorem.

262. *Two similar triangles are to each other as the squares of their homologous sides.*



Let AC and $A'C'$ be homologous sides of the similar triangles ABC and $A'B'C'$.

To prove that $$\frac{ABC}{A'B'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2}.$$

The triangles being similar, they are (by 223) equiangular, therefore (by 261)

$$\frac{ABC}{A'B'C'} = \frac{AC \times BC}{A'C' \times B'C'}.$$

But as the triangles are similar, we have (by 222)

$$\frac{BC}{B'C'} = \frac{AC}{A'C'}.$$

Therefore, substituting this equal ratio for $\frac{BC}{B'C'}$ in the above proportion, it gives

$$\frac{ABC}{A'B'C'} = \frac{AC \times AC}{A'C' \times A'C'} = \frac{\overline{AC}^2}{\overline{A'C'}^2}. \qquad \text{Q.E.D.}$$

263. Scholium. Two similar triangles are to each other as the squares of any two homologous lines.

264. Cor. *Two similar polygons are to each other as the squares of their homologous sides.*

Similar polygons (by 224) can be divided into the same number of similar triangles, and (by 209) the sum of the triangles of one polygon will be to the sum of the triangles of the other as any one triangle of the former is to a corresponding triangle of the latter.

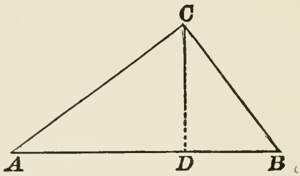
But these triangles are to each other (by 262) as the squares of their homologous sides.

Therefore the sums of the triangles or polygons are to each other as the squares of their homologous sides.

Proposition IX. The Pythagorean Theorem.*

265. *In any right triangle the square described upon the hypotenuse is equivalent to the sum of the square described upon the other two sides.*

First Method.



Let ABC be a right triangle.

To prove that the square described upon the hypotenuse AB is equivalent to the sum of the squares described upon the sides AC and BC.

Draw CD perpendicular to AB.

Then (by 227) $\overline{AC}^2 = AB \times AD,$

and $\overline{BC}^2 = AB \times BD.$

Adding, we have

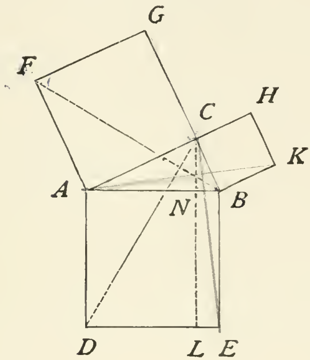
$$\overline{AC}^2 + \overline{BC}^2 = AB \times (AD + BD)$$
$$= AB \times AB$$
$$= \overline{AB}^2.$$

* This proposition is called *the Pythagorean Proposition* because it is said to have been first given by Pythagoras (born about 600 B.C.).

But $\overline{AC}^2$, $\overline{BC}^2$, and $\overline{AB}^2$ are the areas of the squares described upon the sides AC, BC, and AB (by 249).

Hence the square described upon AB is equivalent to the sum of the squares described upon AC and BC.

Second Method.



Construct upon AC, the square $ACGF$, upon CB, $CBKH$, and upon AB, $ABED$.

Draw CL perpendicular to DE, and join FB and CD.

The triangle FAB is one-half the square $ACGF$, having the same base AF and the same altitude AC.

The triangle $DAC = \frac{1}{2} ADLN$, having the same base AD and the same altitude AN.

The triangles FAB and CAD are equal, having the sides $FA = AC$, being sides of the same square, and for the same reason $AB = AD$. The included angles $FAB = \angle CAD$, both being equal to a right angle plus the common angle CAB. These two triangles are therefore (by 86) equal.

As these triangles are equal,

$$\tfrac{1}{2} ACGF = \tfrac{1}{2} ADLN,$$

or

$$ACGF = ADLN.$$

In a similar manner by joining AK and CE, it can be shown that

$$CHKB = NLEB,$$

or by addition,

$$ACGF + CHKB = ADLN + NLEB$$
$$= ABED,$$

or
$$\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2.$$

266. Cor. 1. From the last equation, by transposition,

$$\overline{AC}^2 = \overline{AB}^2 - \overline{CB}^2,$$

and
$$\overline{CB}^2 = \overline{AB}^2 - \overline{AC}^2,$$

or
$$AC = \sqrt{\overline{AB}^2 - \overline{CB}^2},$$

and
$$CB = \sqrt{\overline{AB}^2 - \overline{AC}^2}.$$

EXERCISES.

1. Show that the diagonal and side of a square are incommensurable.

2. Find the length of the diagonal of a rectangle whose area is 96 and whose altitude is 8.

3. Show that if similar polygons be similarly drawn on the sides of a right triangle, the polygon on the hypotenuse is equal to the sum of the polygons on the other sides.

Proposition X. Theorem.

267. *In any triangle, the square on the side opposite an acute angle is equivalent to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon that side.*

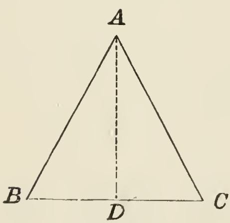


Fig. 1.

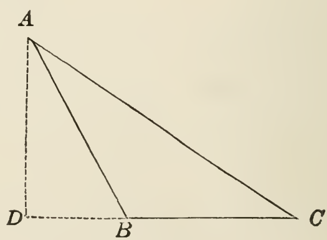


Fig. 2.

Let C be an acute angle of the triangle ABC, and DC the projection of AC upon BC.

To prove that

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times DC.$$

There will be two possible cases; in the first the projection of A will be within the base of the triangle (Fig. 1); in the second it will be on the base produced (Fig. 2).

In the first case,

$$DB = BC - DC;$$

in the second,

$$DB = DC - BC.$$

Squaring in either case,

$$\overline{DB}^2 = \overline{DC}^2 + \overline{BC}^2 - 2\,DC \times BC.$$

Add $\overline{AD}^2$ to both sides of the equation; then

$$\overline{AD}^2 + \overline{DB}^2 = \overline{AD}^2 + \overline{DC}^2 + \overline{BC}^2 - 2\,DC \times BC.$$

But (by 265),

$$\overline{AD}^2 + \overline{DB}^2 = \overline{AB}^2, \text{ and } \overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2;$$

therefore $\quad \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2\,DC \times BC.$

268. Cor. Another proof, using algebraic processes, is:

By (265) $\quad \overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2.$

But (by 266) $\quad \overline{AD}^2 = \overline{AC}^2 - \overline{DC}^2.$

Also $\quad BD = DC - BC, \quad$ (Fig. 2),

which gives, by substitution,

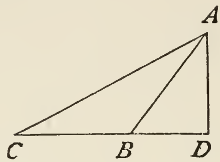
$$\overline{AB}^2 = \overline{AC}^2 - \overline{DC}^2 + (DC - BC)^2$$
$$= \overline{AC}^2 - \overline{DC}^2 + \overline{DC}^2 - 2\,DC \times BC + \overline{BC}^2;$$

or by cancellation,

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2\,DC \times BC. \qquad \text{Q.E.D.}$$

Proposition XI. Theorem.

269. *In an obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of these sides by the projection of the other side upon it.*



Let ABC be a triangle with obtuse angle ABC.

To prove that

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,BC \times BD.$$

$$CD = BD + BC.$$

Squaring, $$\overline{CD}^2 = \overline{BD}^2 + \overline{BC}^2 + 2\,BC \times BD.$$

Add $\overline{AD}^2$ to both sides of the equation,

$$\overline{AD}^2 + \overline{CD}^2 = \overline{AD}^2 + \overline{BD}^2 + \overline{BC}^2 + 2\,BC \times BD.$$

But (by 265)

$$\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2, \text{ and } \overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2.$$

Making these substitutions,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,BC \times BD.$$ Q.E.D.

270. Cor. *From the three preceding theorems, it follows that the square of the side of a triangle is less than, equal to, or greater than, the sum of the squares of the other two sides, according as the angle opposite this side is acute, right, or obtuse.*

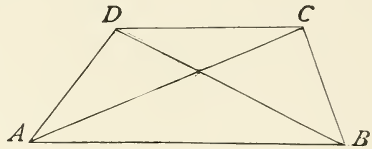
EXERCISES.

1. Prove the above by the method of 268.

2. Show that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on the four sides.

SUGGESTION. Apply 263 and 264.

3. The sum of the squares upon the diagonals of a trapezoid is equal to the sum of the squares upon the non-parallel sides plus twice the rectangle of the parallel sides.

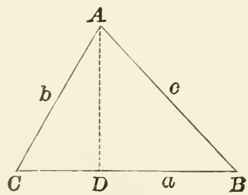


CONCLUSION. $AC^2 + BD^2 = AD^2 + BC^2 + 2\,AB \times CD.$

SUGGESTION. Take the triangles ACB and BCD, and apply 267 and 269.

PROPOSITION XII. PROBLEM.

271. *To find the area of a triangle when its three sides are given.*



Let a, b, and c denote the three sides of the triangle ABC, and draw AD perpendicular to BC.

Then if C is an acute angle, we have (by 267),

$$c^2 = a^2 + b^2 - 2\,a \times CD,$$

or
$$CD = \frac{a^2 + b^2 - c^2}{2\,a}.$$

Now (by 266) $\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2$

$$= b^2 - \left(\frac{a^2+b^2-c^2}{2\,a}\right)^2$$

$$= b^2 - \frac{(a^2+b^2-c^2)^2}{4\,a^2}$$

$$= \frac{4\,a^2b^2 - (a^2+b^2-c^2)^2}{4\,a^2}.$$

The second member of this equation is the difference of two squares, and hence can be factored; that is,

$$\overline{AD}^2 = \frac{[2\,ab + (a^2+b^2-c^2)]\,[2\,ab - (a^2+b^2-c^2)]}{4\,a^2}$$

$$= \frac{(2\,ab + a^2 + b^2 - c^2)(2\,ab - a^2 - b^2 + c^2)}{4\,a^2}$$

$$= \frac{[(a+b)^2 - c^2]\,[c^2 - (a-b)^2]}{4\,a^2}$$

$$= \frac{[(a+b-c)(a+b+c)]\,[c-(a-b)]\,[c+a-b]}{4\,a^2}$$

$$= \frac{(a+b-c)(a+b+c)(c-a+b)(c+a-b)}{4\,a^2}.$$

Let $\qquad a+b+c = 2\,s.$

Subtract $\qquad 2\,c = 2\,c,$

then $\qquad a+b-c = 2\,s - 2\,c = 2\,(s-c).$

Similarly, $\qquad a+c-b = 2\,(s-b),$

and $\qquad -a+c+b = 2\,(s-a).$

Substituting these values in the above equation,

$$\overline{AD}^2 = \frac{2\,s \cdot 2\,(s-a)\,2\,(s-b)\,2\,(s-c)}{4\,a^2}$$

$$= 4 \cdot \frac{s(s-a)(s-b)(s-c)}{a^2},$$

or

$$AD = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

But (by 254), Area of $ABC = \frac{1}{2} BC \times AD$

$$= \frac{1}{2} a \times AD$$

$$= \frac{1}{2} a \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

In which s is one-half of the perimeter.

EXERCISES.

1. If the sides of a triangle are 13, 14, 15, find the area.

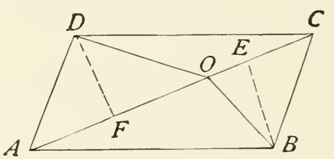
2. In the above, find the radius of the inscribed circle (see 257, Ex. 3).

3. The area of a rhombus is 24 and its side is 5; find the lengths of the diagonals.

4. If the sides of an isosceles triangle are a, a, and b, show that its area is $\frac{b}{4}\sqrt{4a^2 - b^2}$.

5. If from any point on the diagonal of a parallelogram lines be drawn to the opposite angles, the parallelogram will be divided into two pairs of equal triangles.

Area $OAD =$ area OAB.
Area $OCD =$ area OCB.



Suggestion. $OAD = \frac{1}{2} OA \times DF$, $OAB = \frac{1}{2} OA \times BE$; hence show that $DF = EB$.

6. Show that two quadrilaterals are equivalent when they have the following parts of the one respectively equal to the corresponding parts in the other:

I. Four sides and one diagonal.
II. Four sides and one angle.
III. Two adjacent sides and three angles.
IV. Three sides and the two included angles.

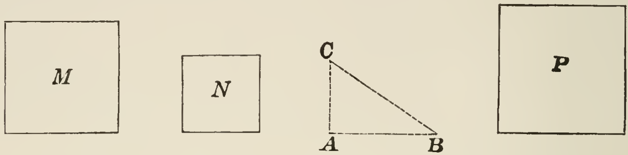
7. Construct a square having a given diagonal.

8. Show that the sum of the squares of the sides of a triangle is equal to double the square of the bisector of the base together with double the square of half the base.

PROBLEMS IN CONSTRUCTION.

Proposition XIII. Problem.

272. *To construct a square equivalent to the sum of two given squares.*



Let M and N be the given squares.

To construct a square equivalent to their sum.

It is known (from 265) that the square upon the hypotenuse is equivalent to the sum of the squares on the other two sides.

Therefore construct a right triangle whose base will be equal to a side of M, say AB, and whose altitude will be equal to a side of N, say AC, then the hypotenuse, say BC, will be a side of the square required.

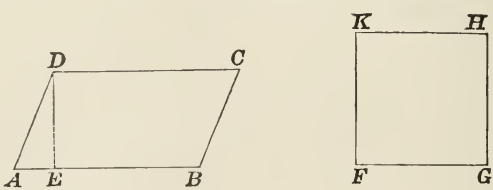
EXERCISES.

1. To construct a square equivalent to the sum of any number of squares.

2. To construct a square equivalent to the difference of two given squares.

Proposition XIV. Problem.

273. *To construct a square equivalent to a given parallelogram.*



Let $ABCD$ be the given parallelogram.

To construct a square equivalent to $ABCD$.

It is known (from 251) that the area of the parallelogram $ABCD = AB \times DE$, therefore any square to be equivalent to $ABCD$ must have such a side that its square must be equal to $AB \times DE$.

It is known (from 200) that when the square of one quantity is equal to the product of two other quantities the former is said to be a mean proportional to the other two.

Hence find (by 238) a mean proportional to AB and DE, say FG, then the square on FG will be the required square equivalent to $ABCD$.

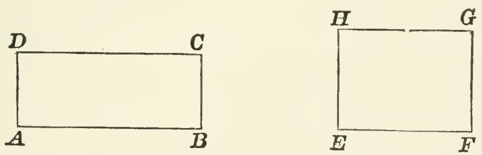
EXERCISES.

1. To construct a square equivalent to a given triangle.

2. To construct a square equivalent to the sum of two given triangles.

PROPOSITION XV. PROBLEM.

274. *Upon a given straight line, to construct a rectangle equivalent to a given rectangle.*



Let $ABCD$ be the given rectangle, and EF the given line.

To construct upon EF as a base a rectangle equivalent to $ABCD$.

The area of the given rectangle is $AB \times DA$, therefore if EF is the base of the required rectangle, its altitude must be such a value that when multiplied by EF the product will be equal to $AB \times AD$; that is, a fourth proportional to EF, AB, and DA.

Therefore find (by 237) a fourth proportional to EF, AB, DA; suppose it is HE, that is

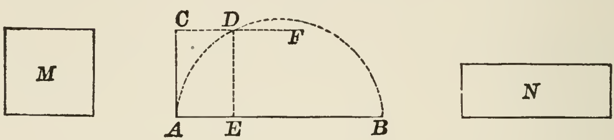
$$EF : AB = DA : HE, \text{ or}$$

$$EF \times HE = AB \times DA,$$

or
$$\text{area } HEFG = \text{area } ABCD.$$

Proposition XVI. Problem.

275. *To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.*



Let M be the given square and AB the given line.

To construct a rectangle equivalent to M, having the sum of its base and altitude equal to AB.

It is known (from 228) that the perpendicular let fall from any point in the circumference upon the diameter is a mean proportional between the segments into which it divides the diameter.

Hence we take AB as the diameter (183) and find a point on the circumference which is as far from the diameter as the side of the square.

To do this, erect at A (by 184, Ex. 1) a perpendicular to AB, equal to a side of M, say AC, through C, draw a line parallel to AB (by 189), say CF; then where CF intersects the circumference, say D, let fall the perpendicular DE.

We know (by 228)
$$DE^2 = AE \times EB,$$

but
$$DE^2 = M.$$

Construct a rectangle N whose base and altitude are EB and AE, then

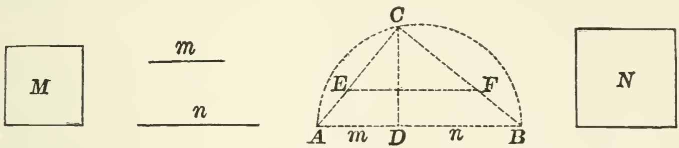
$$N = AE \times EB,$$

or

$$N = M \text{ and } AE + EB = AB.$$

Proposition XVII. Problem.

276. *To construct a square having a given ratio to a given square.*



Let M be the given square, and let the given ratio be that of the lines m and n.

To construct a square which shall have to M the ratio $m : n$.

It is known (from 228) that the perpendicular let fall from any point in the circumference upon the diameter divides it into segments which have the same ratio as the squares of the chords drawn from the same point to the two extremities of the same diameter.

Hence we lay off on a straight line $DA = m$, and $DB = n$, and on AB erect (by 183) a semicircumference.

At D erect (by 182) the perpendicular DC, and join CA and CB.

Then (by 228), $\overline{CA}^2 : \overline{CB}^2 = m : n$.

But neither CA nor CB is a side of the square M, that is, we must take a part of CA, say CE, that is equal to a side of M, and find some quantity that has the same ratio to CE that CB has to CA.

It is known (from 213) that a line drawn through E parallel to AB will divide CB into parts having the same ratio as the parts into which E divides CA.

Therefore draw (by 189) EF parallel to AB;

then $$CA : CB = CE : CF,$$

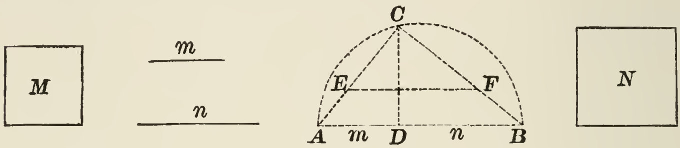
or (by 208) $$\overline{CA}^2 : \overline{CB}^2 = \overline{CE}^2 : \overline{CF}^2;$$

but $$\overline{CA}^2 : \overline{CB}^2 = m : n,$$

therefore $$\overline{CE}^2 : \overline{CF}^2 = m : n,$$

or CF is the side of the square required.

Cor. If a side of M is greater than CA, extend CA and CB, and proceed in the same manner.



EXERCISES.

1. To construct a square equivalent to the sum of a given triangle and a given parallelogram.

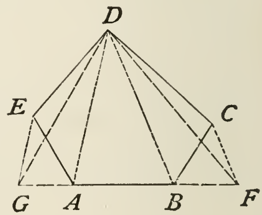
2. To construct an isosceles triangle equivalent to a given triangle, its altitude being given.

Proposition XVIII. Problem.

277. *To construct a triangle equivalent to a given polygon.*

Let $ABCDE$ be the given polygon.

To construct a triangle equivalent to $ABCDE$.



If it is possible to construct one polygon equivalent to another but with one side less, then a continuation of this operation would eventually result in a triangle.

One side of the polygon, say AB, can be extended without increasing the number of sides, then draw DA, in the effort to find upon DA and AB produced a triangle equivalent to DEA introducing *one* line in the place of *two*, that is DE and EA.

If DA is regarded as the base, then the required triangle must have (by 255) an altitude equal to the distance from E to DA; and since (by 60) parallel lines are everywhere equally distant, the vertex of the required triangle must lie on the parallel to DA drawn through E. Again, if AB produced is to be a side of the triangle, the vertex must also be on AB produced or at G.

Draw DG, and the triangle DGA will be the equivalent of DEA.

Add to $DABC$ the triangle DEA, and we have the original polygon; add to the same figure the equal triangle DGA, and we have the polygon $DGBC$; therefore $DGBC = DEABC$, and has one side less.

Draw CF parallel to DB and draw DF; then the triangle DGF will be equivalent to the polygon $ABCDE$.

EXERCISES.

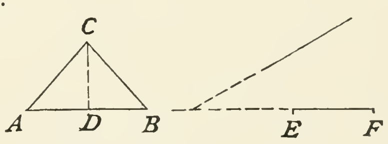
1. To draw a square equivalent to a given polygon.

2. To construct a square equal to two given polygons.

3. Two similar polygons being given, to construct a similar polygon equal to their sum.

Suggestion. See 240 and 272.

4. On a given straight line construct a triangle equal to a given triangle and having its vertex on a given straight line not parallel to the base.



Suggestion. Find (by 237) a fourth proportional to EF, AB, and $\frac{1}{2}CD$, and it will be the required altitude; then see 196, Ex. 2.

5. When is the last problem impossible?

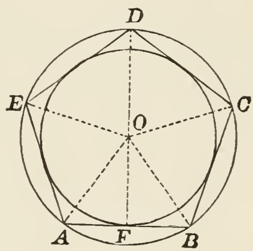
BOOK V.

REGULAR POLYGONS AND CIRCLES.

278. A *Regular Polygon* is a polygon which is equilateral and equiangular.

Proposition I. Theorem.

279. *A circle may be circumscribed about, or inscribed within, any regular polygon.*



Let $ABCDE$ be a regular polygon.

1. To prove that a circle may be circumscribed about it.

Let A, B, and C be any three vertices, and through them pass (by 194) a circle; let its centre be at O. Join OA, OB, OC, OD, and OE.

Since the polygon is equiangular,

$$\angle ABC = \angle BCD,$$

and since $OB = OC$, $\quad \angle OBC = \angle OCB.$

Subtracting these equal angles,

$$\angle ABC - \angle OBC = \angle BCD - \angle OCB,$$

or $$\angle OBA = \angle OCD;$$

therefore the triangles OCD and ABO have $OC = OB$ being radii, $AB = CD$ sides of the regular polygon, and $\angle OBA = \angle OCD$. They are therefore equal, and $OD = OA$.

Hence the circle passing through A, B, and C, also passes through D.

In the same manner it can be shown to pass through E.

2. To prove that a circle may be inscribed in $ABCDE$.

Since AB, BC, CD, DE, and EA are equal chords, they are (by 148) equally distant from the centre O.

Hence if a circle be described with O as a centre, and a radius equal to the perpendicular distance from O to one of the sides, the circumference will touch all the sides of the polygon. Q.E.D.

280. The *Centre* of a regular polygon is the common centre O of the circumscribed and inscribed circles.

281. The *Radius* of a regular polygon is the radius OA of the circumscribed circle.

282. The *Apothem* of a regular polygon is the radius OF of the inscribed circle.

283. The *Angle at the centre* is the angle included by the radii drawn to the extremities of any side.

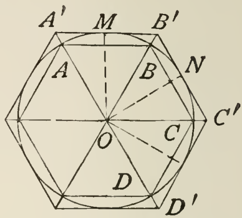
284. Cor. 1. *Each angle at the centre of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

Since the triangles OAB, OBC, OCD, etc., are equal, the angles AOB, BOC, COD, etc., are equal.

Therefore each angle is equal to four right angles (by 51) divided by the number of sides.

285. Cor. 2. If a regular inscribed polygon is given, the tangents at the vertices of the given polygon form a regular circumscribed polygon of the same number of sides.

286. Cor. 3. If a regular inscribed polygon $ABCD \cdots$ is given, the tangents at the middle points M, N, P, etc., of the arcs AB, BC, CD, etc., form a regular circumscribed polygon whose sides are parallel to those of the inscribed polygon, and whose vertices A, B', C', etc., lie on the radii OAA', OBB' prolonged, etc.



For the sides AB, $A'B'$ are parallel, being perpendicular to OM.

Since $B'M = B'N$ (by 234), the right triangles MOB' and NOB' are (by 92) equal, hence the point B is on the bisector OB of the angle MON.

Likewise C and C', D and D', are on the same line.

287. Cor. 4. If the chords AM, MB, BN, etc., be drawn, the chords form a regular inscribed polygon of double the number of sides of $ABCD \cdots$.

288. Cor. 5. If through the points A, B, C, etc., tangents are drawn intersecting the tangents $A'B'$, $B'C'$, etc., a regular circumscribed polygon is formed of double the number of sides of $A'B'C'D' \cdots$.

289. Cor. 6. *If the circumference of a circle is divided into any number of equal arcs, their chords form a regular polygon inscribed in the circle.*

Since (by 145) equal arcs are subtended by equal chords, if the arcs are equal the chords will be equal.

And (by 175) each angle will be measured by one-half of the circumference excepting the two arcs subtended by the two equal chords forming the sides of the angle, hence each angle will have the same measure.

Therefore the polygon will be equiangular and equilateral, and hence regular.

EXERCISES.

1. Show that the interior angle of a regular polygon is the supplement of the angle at the centre

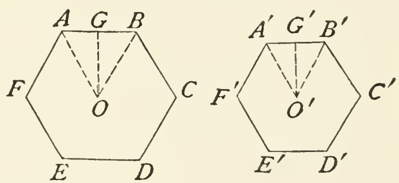
2. Show that the radius drawn to any vertex of a regular polygon bisects the angle at that vertex.

3. Show that if the circumference of a circle be divided into any number of equal arcs, the tangents at the points of division form a regular polygon circumscribed about the circle.

Proposition II. Theorem.

290. *Regular polygons of the same number of sides are similar.*

Let $ABCDEF$ and $A'B'C'D'E'F'$ be two regular polygons of the same number of sides.



To prove that they are similar.

The sum of the angles of the one polygon is (by 123) equal to the sum of the angles of the other; then since the number of angles are the same in both, each angle of the one will be equal to the corresponding angle of the other. Therefore they are mutually equiangular.

The polygons being regular,

$$AB = BC = CD, \text{ etc.},$$

and $$A'B' = B'C' = C'D', \text{ etc.}$$

Dividing the former equation by the latter, we have

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

Therefore the polygons are (by 223) similar. Q.E.D.

291. Cor. 1. Taking the above proportion by composition (204),

$$\frac{AB+BC+CD+\text{etc.}}{A'B'+B'C'+C'D'+\text{etc.}}=\frac{AB}{A'B'},$$

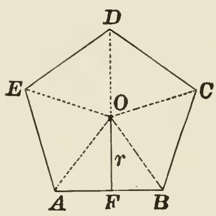
or
$$\frac{\text{Perimeter of } AB-F}{\text{Perimeter of } A'B'-F'}=\frac{AB}{A'B'};$$

that is,

The perimeters of regular polygons of the same number of sides are to each other as any two homologous sides or lines.

Proposition III. Theorem.

292. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Let r denote the apothem OF, and P the perimeter of the regular polygon $ABCDE$.

To prove that area $ABCDE=\frac{1}{2}P\times r$.

By drawing the radii OA, OB, OC, etc., the polygon may be divided into a series of triangles, OAB, OBC, etc., whose common altitude is r.

Then (by 254) area $OAB=\frac{1}{2}AB\times r$,

area $OBC=\frac{1}{2}BC\times r$, etc.

Adding, we have

area OAB + area OBC + etc. $=\frac{1}{2}(AB+BC+\text{etc.})\times r$.

That is, area $ABCDE=\frac{1}{2}P\times r$. Q.E.D.

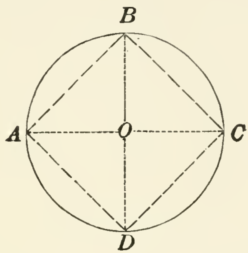
EXERCISE.

1. The apothem of a regular pentagon is 6, and a side is 4; find the perimeter and area of a regular pentagon whose apothem is 8.

SUGGESTION. See 291.

PROPOSITION IV. PROBLEM.

293. *To inscribe a square in a given circle.*



Let O be the centre of the given circle.

To inscribe a square within it.

Since the inscribed figure is to be a regular quadrilateral, each angle at the centre will be one-fourth of four right angles, or one right angle.

Therefore draw any diameter, say AC, and another perpendicular thereto, say BD, and join the ends of these diameters, and the figure inscribed will be the square required.

294. COR. 1. If tangents be drawn to the circle at the points A, B, C, D, the figure so formed will be a circumscribed square.

295. COR. 2. To inscribe and circumscribe regular polygons of 8 sides, bisect the arcs AB, BC, CD, DA, and proceed as before.

By repeating this process, regular inscribed and circumscribed polygons of 16, 32,$\cdots$, and, in general, of 2^n sides, may be drawn.

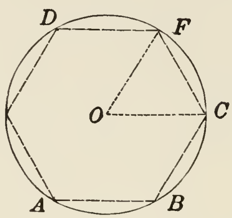
EXERCISES.

1. Show that the side of an inscribed square $= r\sqrt{2}$.

2. Find the ratio of the areas of an inscribed and a circumscribed circle.

Proposition V. Problem.

296. *To inscribe in a given circle a regular hexagon.*



Let O be the centre of the given circle.

To inscribe therein a regular hexagon.

Each central angle of an inscribed hexagon will be one-sixth of four right angles, or one-third of two right angles, leaving (from 79) two-thirds of two right angles for the two base angles of a triangle if we imagine lines drawn from the centre to each vertex.

But these lines being radii are equal, forming an isosceles triangle.

Therefore the angles at the base are equal or each will be one-third of two right angles, hence all the angles of the triangle are equal and the triangle is equilateral or the base is equal to the radius.

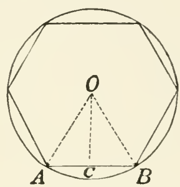
Therefore apply the radius six times to the circumference, join the points of division, and the inscribed figure is the hexagon required.

297. Cor. 1. By joining the alternate vertices, A, C, D, an equilateral triangle is inscribed in a circle.

298. Cor. 2. By bisecting the arcs AB, BC, etc., a regular polygon of 12 sides may be inscribed in a circle; and, by continuing the process, regular polygons of 24, 48, etc., sides may be inscribed.

Proposition VI. Theorem.

299. *If the number of sides of a regular inscribed polygon be increased indefinitely, the apothem will be an increasing variable whose limit is the radius of the circle.*



In the right triangle OCA, let OA be denoted by R, OC by r, and AC by b.

To prove that R is the limit of r.

In the triangle OAC, since one side of a triangle is (by 78) greater than the difference between the other two, we have

$$R - r < b.$$

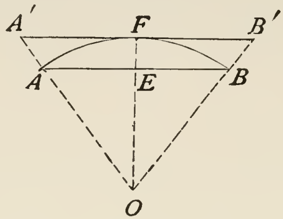
Now by increasing the number of sides each side diminishes in length, and hence the half side, b, can by increasing the number of sides indefinitely be made less than any assignable quantity, or the difference between R and r can be made less than any assignable quantity, hence R is the limit of r. Q.E.D.

Proposition VII. Theorem.

300. *If the number of sides of a regular inscribed, and of a similar regular circumscribed, polygon, is indefinitely increased,*

I. *The perimeter of each polygon approaches the circumference of the circle as a limit.*

II. *Their areas approach the area of the circle as a limit.*



1. Let AB be one side of a regular inscribed polygon, $A'B'$ a corresponding side of a regular circumscribed polygon of the same number of sides, and O the centre of the circle. Call the perimeter of the inscribed polygon P, and of the circumscribed, P'.

To prove that the limit of P and P' is the circumference of the circle.

It is evident that the inscribed polygon can never pass without the circle, nor can the circumscribed polygon come within; therefore, however near they may approach one another, they will be still nearer the circle.

Since the polygons are regular, they are (by 290) similar, and (by 291) we have $\frac{P}{P'} = \frac{OE}{OF}$, but (by 299) the difference between OE and OF, when the number of sides is indefinitely increased, approaches 0, therefore the difference between P and P' will approach 0.

That is, the perimeters of the inscribed and circumscribed polygons approach one another; hence each will approach the circle more nearly.

2. Let S and S' represent the areas of the inscribed and circumscribed polygons.

$$\frac{S'}{S} = \frac{\overline{OF}^2}{\overline{OE}^2} = \left(\frac{OF}{OE}\right)^2.$$

But since (by 299) OF approaches OE, $\frac{OF}{OE}$ approaches 1, or $\left(\frac{OF}{OE}\right)^2$ approaches 1.

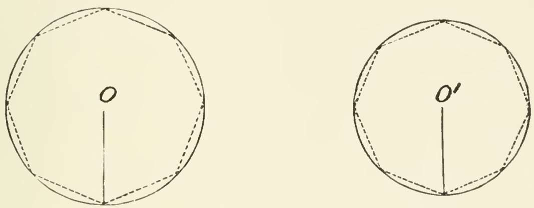
Hence $\frac{S'}{S}$ approaches 1, or S' approaches S.

But the area of the circle being intermediate, each polygon will approach the area of the circle more nearly. Q.E.D

301. Definition. In circles of different radii, *Similar Arcs Segments,* or *Sectors* are those which correspond to equal central angles.

Proposition VIII. Theorem.

302. *The circumferences of circles have the same ratio as their radii.*



Let C and C' be the circumferences, R and R' the radii of the two circles O and O'.

To prove $C : C' :: R : R'$.

Inscribe in the circle two regular polygons of the same number of sides, whose perimeters we shall call P and P'.

Then by (291)
$$\frac{P}{P'} = \frac{R}{R'}.$$

Or
$$P \times R' = P' \times R.$$

Suppose the number of sides to be indefinitely increased, the two to continually have the same number of sides.

Then $P \times R'$ will approach the limit $C \times R'$, and $P' \times R$ will approach the limit $C' \times R$.

Therefore (by 171) $C \times R' = C' \times R$, or $\frac{C}{C'} = \frac{R}{R'}$. Q.E.D.

303. Cor. By multiplying the last member by 2 we have
$$\frac{C}{C'} = \frac{2\,R}{2\,R'}, \text{ or } C : C' = 2\,R : 2\,R'.$$

Taking this by alternation (202), it becomes

$$C : 2\,R = C' : 2\,R', \text{ or } \frac{C}{2\,R} = \frac{C'}{2\,R'}.$$

That is, the ratio of the circumference of a circle to its diameter is a constant.

This constant is denoted by the Greek letter π,

or $$\frac{C}{2\,R} = \pi, \text{ or } C = 2\,\pi\,R.$$

EXERCISES.

1. Show that the side of an inscribed equilateral triangle $= r\sqrt{3}$.

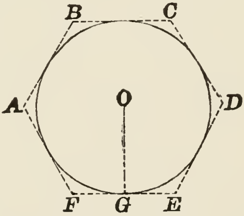
2. Show that the apothem of a regular inscribed hexagon $= \frac{r}{2}\sqrt{3}$.

3. Find the area of a square inscribed in a circle whose radius is 6.

4. Show that the area of a regular inscribed hexagon is a mean proportional between the areas of an inscribed, and of a circumscribed, equilateral triangle.

Proposition IX. Theorem.

304. *The area of a circle is equal to one-half the product of its circumference and radius.* (Compare 292.)



Let R denote the radius, C the circumference, and S the area of the circle.

To prove that $S = \frac{1}{2}\,C \times R.$

Circumscribe about the circle a regular polygon; let P denote its perimeter and P' its area.

Then (by 292), $P' = \frac{1}{2}\,P \times OG.$

But when the number of sides of the polygon increases indefinitely, P approaches C (by 300), OG remains R, and P' approaches S.

Therefore $$S = \tfrac{1}{2} R \times C.$$ Q.E.D.

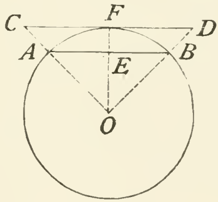
305. Cor. 1. $$C = 2\pi R \text{ (by 303)}.$$

Therefore $$S = \tfrac{1}{2} R \times 2\pi R$$
$$= \pi R^2.$$

306. Cor. 2. *Since a sector bears the same ratio to the circle that its arc bears to the circumference, the area of a sector is equal to one-half the product of its arc by its radius.*

Proposition X. Problem.

307. *Given the radius and the side of a regular inscribed polygon, to compute the side of a similar circumscribed polygon.*



Let AB be a side of the inscribed polygon, and $OF = R$, the radius of the circle.

To compute CD, a side of the similar circumscribed polygon.

Draw CO and DO; they will (by 286) intersect AB in A and B.

The triangles CFO and AEO are corresponding parts of similar polygons, and hence are similar.

Hence $$\frac{CF}{AE} = \frac{OF}{OE},$$

Multiplying by AE, $CF = \frac{OF \times AE}{OE} = \frac{R \times AE}{OE}$,

or $CD = \frac{R \times AB}{OE}$.

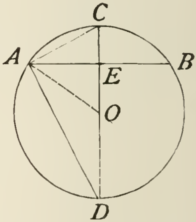
In the right triangle OAE (by 266)

$$OE = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{R^2 - \frac{\overline{AB}^2}{4}} = \tfrac{1}{2}\sqrt{4R^2 - \overline{AB}^2}.$$

Therefore $CD = \frac{2R \times AB}{\sqrt{4R^2 - \overline{AB}^2}}$.

PROPOSITION XI. PROBLEM.

308. *Given the radius and the side of a regular inscribed polygon, to compute the side of the regular inscribed polygon of double the number of sides.*



Given AB, a side of the regular inscribed polygon, and $OC = R$, the radius of the circle.

To compute AC, a side of an inscribed polygon of double the number of sides.

Draw OC bisecting arc ACB; then since it bisects the arc ACB, it will bisect AB at right angles (by 147).

Produce CO to D and draw AD; then since $\angle CAD$ is inscribed in a semicircle (by 177), it is a right angle.

Then (by 228) AC is a mean proportional between CD and CE, or

$$\overline{AC}^2 = CD \times CE = CD(CO - EO) = 2R(R - EO)$$
$$= R(2R - 2EO).$$

But in the right triangle (by 266),

$$EO = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{R^2 - \frac{\overline{AB}^2}{4}} = \tfrac{1}{2}\sqrt{4R^2 - \overline{AB}^2},$$

or $$2EO = \sqrt{4R^2 - \overline{AB}^2}.$$

Substituting this in the value for $\overline{AC}^2$, we have

$$\overline{AC}^2 = R(2\,R - \sqrt{4\,R^2 - \overline{AB}^2}),$$

or

$$AC = \sqrt{R(2\,R - \sqrt{4\,R^2 - \overline{AB}^2})}.$$

PROPOSITION XII. PROBLEM.

309. *To compute the ratio of the circumference of a circle to its diameter.*

It is known (from 303) that

$$C = 2\,\pi R.$$

If, therefore, we take a circle whose radius is unity, we have

$$C = 2\,\pi, \text{ or } \pi = \tfrac{1}{2}\,C;$$

that is, $\pi =$ a semicircumference of unit radius.

Hence the semiperimeter of each inscribed polygon is an approximate value of π, and the semiperimeter of each circumscribed polygon is also an approximate value of π. Therefore, if by constantly increasing the number of sides of these polygons the approximate values of π become practically identical, we know that as the circle lies between the inscribed and circumscribed polygons, this coincident value for π can be taken as the semicircumference of the circle of unit radius.

If we begin with the square we know that the side is the hypotenuse of an isosceles right triangle, the two equal sides being radii; hence

$$AB \text{ (in 308)} = \sqrt{2} = 1.4142136,$$

or

$$\text{semiperimeter} = 2.8284272.$$

Then each side of the circumscribed square is the diameter or twice the radius $= 2$, and the semiperimeter will be 4.

From the final equation in Prob. XI. it is easy to compute the semiperimeter of a polygon of 8 sides; then from the final equation in Prob. X. can be computed the semiperimeter of a circumscribed polygon of 8 sides; and so on.

In the following table are given the semiperimeters of inscribed and circumscribed polygons:

Number of Sides.	Inscribed.	Circumscribed.
4	2.8284271	4.0000000
8	3.0616675	3.3137085
16	3.1214452	3.1825927
32	3.1365485	3.1517249
64	3.1403312	3.1414184
128	3.1412773	3.1422236
256	3.1415138	3.1417504
512	3.1415729	3.1416321
1024	3.1415877	3.1416025
2048	3.1415914	3.1415951
4096	3.1415923	3.1415933
8192	3.1415926	3.1415928

The figures in face type show the approximation.

310. Scholium. By the aid of simpler methods the value of π has been computed to more than eight hundred places of decimals.

The first twenty figures of the result are

$$\pi = 3.14159\ 26535\ 89793\ 238,$$

$$\frac{1}{\pi} = 0.31830\ 98861\ 83790\ 6715,$$

$$\log \pi = 0.49714\ 98726\ 94133\ 85435.$$

For all practical purposes it is sufficient to take $\pi = 3.1416$.

EXERCISES.

1. If the radius of a circle is 4, find its circumference and area.

2. If the circumference of a circle is 30, find its radius and area.

3. If the diameter of a circle is 26, find the length of an arc of 72°.

4. If the radius of a circle is 12, find the area of a sector whose central angle is 80°.

5. If the apothem of a regular hexagon is 4, find the area of the circumscribing circle.

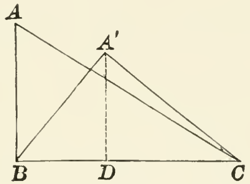
MAXIMA AND MINIMA.

311. Of quantities of the same kind, the one which is the greatest is called the *Maximum*, and the least is called the *Minimum.*

312. *Isoperimetric* figures are those which have equal perimeters.

Proposition XIII. Theorem.

313. *Of all triangles formed with two given sides, that in which these sides include a right angle is the maximum.*



Let ABC and $A'BC$ be two triangles having the sides AB and BC equal to the sides $A'B$ and BC respectively, and let the angle ABC be a right angle.

To prove that

$$\text{area } ABC > \text{area } A'BC.$$

Draw $A'D$ perpendicular to BC.

Then since (by 52) the oblique line $A'B$ is greater than the perpendicular $A'D$, we have

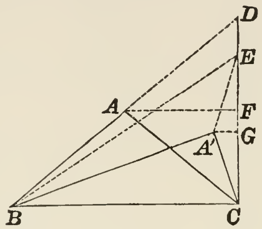
$$AB > A'D.$$

But AB and $A'D$ are the altitudes of the triangles ABC and $A'BC$, and as they have the same base, that triangle is the greater which has the greater altitude, or

$$\text{area } ABC > \text{area } A'BC. \qquad \text{Q. E. D.}$$

PROPOSITION XIV. THEOREM.

314. *Of isoperimetric triangles having the same base, that which is isosceles is the maximum.*



Let ABC and $A'BC$ be two isoperimetric triangles having the same base BC, and let the triangle ABC be isosceles.

To prove that area $ABC >$ area $A'BC$.

Produce AB to D, making $AD = AB$, and draw CD.

Since B, C, and D are equally distant from A, a circle with A as a centre could be drawn through B, C, and D, of which BD would be the diameter.

The angle BCD would therefore (by 177) be a right angle.

Draw AF and $A'G$ parallel to BC, take $A'E$ equal to $A'C$, and draw BE.

Since the triangles ABC and $A'BC$ are isoperimetric,

$$AB + AC = A'B + A'C = A'B + A'E.$$

But $$AC = AB = AD,$$

or $$AB + AC = BD,$$

hence $$A'B + A'E = BD.$$

But (by 6) $$A'B + A'E > BE;$$

that is, $$BD > BE.$$

Therefore (by 58) $$CD > CE.$$

Since the triangles CAD and $CA'E$ are isosceles by construction, and AF and $A'G$ perpendiculars upon their bases,

$$CF = \tfrac{1}{2}\,CD, \text{ and } CG = \tfrac{1}{2}\,CE.$$

But as CD is greater than CE, $CF > CG$.

CF is the altitude of the triangle BAC, and CG of $BA'C$, as these triangles have the same base, the one which has the greater altitude is the greater, or

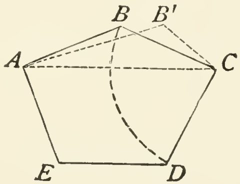
$$\text{area } ABC > A'BC. \qquad \text{Q.E.D.}$$

315. Cor. *Of all the triangles of the same perimeter, that which is equilateral is the maximum.*

For the maximum triangle having a given perimeter must be isosceles whichever side is taken as the base.

Proposition XV. Theorem.

316. *Of isoperimetric polygons having the same number of sides, that which is equilateral is the maximum.*



Let $ABCDE$ be an equilateral polygon.

To prove that it is greater than any other isoperimetric polygon of the same number of sides.

If not greater, suppose $AB'CDE$ is greater.

Draw AC.

Then ABC being an isosceles triangle, we know (by 314) area $ABC > AB'C$.

Add area $ACDE$ to this inequality,

$$ACDE + ABC > ACDE + AB'C,$$

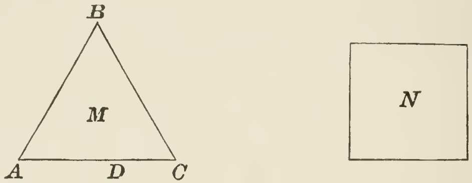
or

$$ABCDE > AB'CDE.$$

That is, AB and BC cannot be unequal without decreasing the area, and in like manner it can be shown that $BC = CD = DE$, etc., or the polygon is equilateral. Q.E.D.

Proposition XVI. Theorem.

317. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



Let M be an equilateral triangle, and N an isoperimetric square.

To prove that area $N >$ area M.

Let D be any point in the side AC of the triangle.

Then the triangle M may be regarded as an irregular quadrilateral, having the four sides AB, BC, CD, and DA; the angle at D being equal to two right angles.

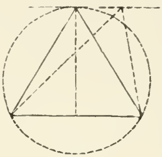
Hence, since the two quadrilaterals are isoperimetric,

$$\text{area } N > \text{area } M. \quad (316)$$

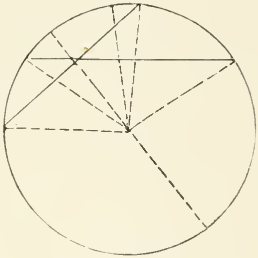
In like manner, it may be proved that the area of a regular pentagon is greater than that of an isoperimetric square; that the area of a regular hexagon is greater than that of an isoperimetric regular pentagon; and so on. Q.E.D.

318. Cor. *Since a circle may be regarded as a regular polygon of an infinite number of sides, it follows that the circle is the maximum of all isoperimetric plane figures.*

EXERCISES.



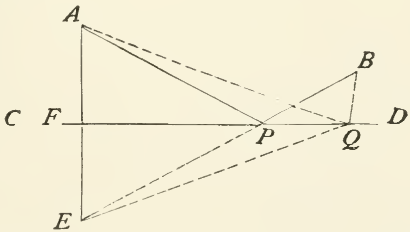
1. Of all triangles of given base and area, the isosceles is that which has the greatest vertical angle.



2. The shortest chord which can be drawn through a given point within a circle is the perpendicular to the diameter which passes through that point.

Proposition XVII. Theorem.

319. *The sum of the distances from two fixed points on the same side of a straight line to the same point in that line is a minimum when the lines joining the fixed points with the same point are equally inclined to the given line.*



Let CD be the straight line, A and B the fixed points, P such a point in CD that $\angle APC = \angle BPD$, and Q any other point in CD.

To prove that $AP + PB < AQ + BQ$.

Let fall the perpendicular AF, and continue it until it meet BP produced, say in E, and join QA and QE.

Since BE and CD are intersecting lines (by 49),

$$\angle BPD = \angle FPE.$$

By hypothesis, $\angle APF = \angle BPD$;

therefore $\angle APF = \angle FPE$.

The triangles APF and FPE are right triangles by construction, and having the side FP common, are (by 90) equal in all their parts; that is

$$AF = FE, \text{ and } AP = EP.$$

Then (by 53) $AQ = EQ$.

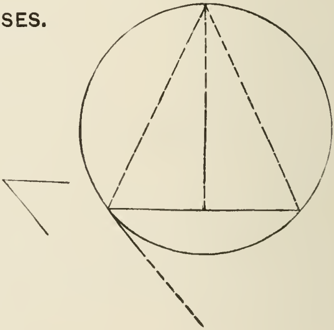
But (by 6) $EB < EQ + QB$,

or $EP + PB < AQ + QB$.

Therefore $AP + PB < AQ + QB$. Q.E.D.

Note. If CD is a reflecting surface, a ray of light in order to go from A to B by reflection, pursues the shortest path when the angle of incidence (APF) is equal to the angle of reflection (BPD). This is the physical law, thus furnishing one illustration of the economy in nature.

EXERCISES.



1. Given the base and the vertical angle of a triangle; to construct it so that its area may be a maximum.

Suggestion. See 195, Ex. 1.

2. Show that the greatest rectangle which can be inscribed in a circle is a square.

SOLID GEOMETRY.

BOOK VI.

PLANES AND SOLID ANGLES.

DEFINITIONS.

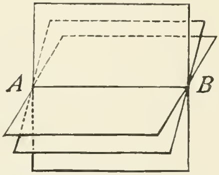
320. A *plane* is (by 9) a surface such that a straight line which joins any two of its points will lie wholly in the surface.

321. A plane is of unlimited extent in its length and breadth; but to represent a plane in a diagram it is necessary to take only a definite portion, and usually it is represented by a parallelogram which is supposed to lie in the plane.

322. A plane is said to be *determined* by any combination of lines or points when it is the only plane which contains these lines or points.

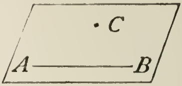
323. *Any number of planes may be passed through any given straight line.*

For if a plane is passed through any given straight line AB, the plane may be turned about AB as an axis, and made to occupy an infinite number of positions, each of which will be a different plane passing through AB.



From this it can be seen that a single straight line does not determine a plane.

324. *But a plane is determined by a straight line and a point without that line.*



For, if the plane containing the straight line AB turn about this line as an axis until it contains the given point C, the plane is evidently determined, for if turned in any other position it will not contain C.

325. *A plane is determined by a straight line and a point without that line, by two intersecting straight lines, or by two parallel lines.*

Since one straight line and a point without that line determine (by 324) the plane, it will be necessary to take only three points, two in one of the lines and the third in the other line, the first two giving the required line and the third the required point.

326. A straight line is *perpendicular to a plane* when it is perpendicular to every straight line of the plane which passes through its *foot*, that is, the point where it meets the plane.

Conversely, the plane is perpendicular to the line.

327. A straight line is said to be *parallel to a plane* when they cannot meet, however far they may be produced.

328. Two planes are said to be *parallel to each other* when they cannot meet, however far they may be produced.

329. The *projection of a point* on a plane is the foot of the perpendicular let fall from the point to the plane.

330. The *projection of a line* on a plane is the line through the projections of all its points.

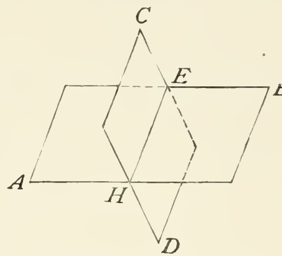
331. The *angle which a line makes with a plane* is the angle which it makes with its projection on the plane.

332. By the *distance* of a point from a plane is meant the *shortest* distance from the point to the plane.

Proposition I. Theorem.

333. *If two planes cut each other, their common intersection is a straight line.*

Let AB, CD be two planes which cut each other.



To prove their common intersection is a straight line.

Let H and E be two points in the intersection. Join them by the straight line HE.

By definition this straight line lies wholly in the plane AB, likewise H and E being points on CD the line HE must lie wholly in the plane CD.

Therefore HE being common to both planes, it must be their intersection. Q.E.D.

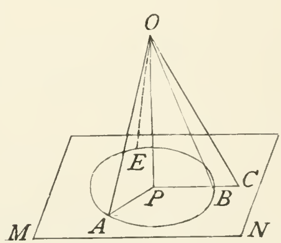
Proposition II. Theorem.

334. *If oblique lines are drawn from a point to a plane:*

(1) *Two oblique lines meeting the plane at equal distances from the foot of the perpendicular are equal.*

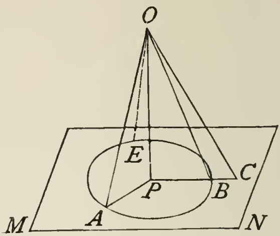
(2) *Of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer.*

Let OP be perpendicular to the plane MN, and $PA = PB$, but $PC > PA$.



To prove that $OA = OB$, but that $OC > OA$.

In the two right triangles OPA and OPB, the side OP is common and $AP = PB$ by hypothesis; therefore (by 86) the triangles are equal in all their parts, that is $OA = OB$.



Since OC meets the line PB produced at a point further from the point P than does OB, OC is (by 58) greater that OB.

But $\quad OB = OA,$
therefore $\quad OC > OA.$ Q.E.D.

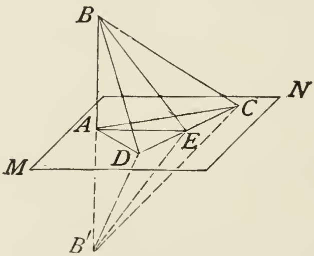
335. Cor. 1. *The perpendicular is the shortest distance from a point to a plane; therefore, by the distance of a point from a plane is meant the perpendicular distance from the point to the plane.*

Since OP is less than OA, OB, and OC, it is the shortest distance from the point to the plane.

336. Cor. 2. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines, the greater meets the plane at the greater distance from the foot of the perpendicular.*

Proposition III. Theorem.

337. *If a straight line is perpendicular to each of two straight lines at their point of intersection it is perpendicular to the plane of those lines.*



Let BA be perpendicular to AD and AC two intersecting lines in the plane MN, and let EA be any other line in MN passing through the point of intersection of AD and AC.

To prove that BA is perpendicular to EA and hence perpendicular to MN.

Make AC equal to AD, draw DC, and produce BA to B', making $AB' = AB$, and join B with D, E, and C.

Since D and C are equally distant from A, $BD = BC$ (by 334).

Since DA is a perpendicular bisector of BB', (by 54) $DB' = BD$, likewise $B'C = BC$, hence $B'DC$ is an isosceles triangle.

The triangles BDC and $B'DC$ have $BD = B'D$, $BC = B'C$, and the side DC common; they are therefore equal (by 91) in all their parts.

Therefore if the triangle $B'DC$ were applied to BDC they would coincide in all their parts, and the point E being fixed, the line BE would fall upon $B'E$ and be equal to it.

Hence E being equally distant from B and B', it is on the perpendicular bisector of BB', or EA is perpendicular to BA, or BA is perpendicular to AE.

As AE is any line in MN, BA is perpendicular to MN. Q.E.D.

338. Cor. 1. *Conversely, all the perpendiculars to a straight line at the same point lie in a plane perpendicular to the line.*

339. Cor. 2. *At a given point in a plane, only one perpendicular to the plane can be erected.*

340. Cor. 3. *From a point without a plane only one perpendicular can be drawn to the plane.*

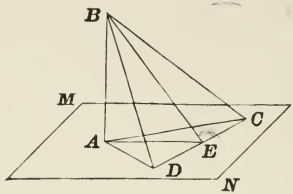
341. Cor. 4. *At a given point in a straight line one plane, and only one, can be drawn perpendicular to the line.*

342. Cor. 5. *If a right angle be turned round one of its arms as an axis, the other arm will generate a plane.*

343. Cor. 6. *Through a given point without a straight line one plane, and only one, can be drawn perpendicular to the line.*

Proposition IV. Theorem.

344. *If through the foot of a perpendicular to a plane a line is drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*



Let AB be a perpendicular to the plane MN.

Draw AE perpendicular to any line CD in the plane MN, and join the point E to any point B in the perpendicular.

To prove that BE is perpendicular to CD.

Take $EC = ED$, and draw AD, AC, BD, and BC.

Since A is on the perpendicular bisector of CD, $AD = AC$ (by 54).

Hence the oblique lines BD and BC meet the plane MN at points equally distant from the foot of the perpendicular and are (by 334) equal.

Therefore BDC is an isosceles triangle, and the line BE bisecting the base, by construction, will be (by 94) perpendicular to the base; that is, $\angle BEC$ is a right angle. Q.E.D.

EXERCISES.

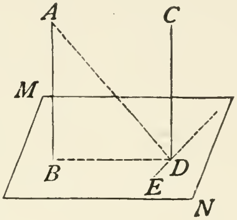
1. If a plane bisects a straight line at right angles, every point in the plane is equally distant from the extremities of the line.

2. Given a plane MN and two points A and B on the same side of the plane, find upon the plane a point C so that the sum of the distances AC and BC shall be a minimum.

3. If the points are on opposite sides, find C when the difference of the distances is a minimum.

Proposition V. Theorem.

345. *Two straight lines perpendicular to the same plane are parallel.*



Let AB and CD be two straight lines perpendicular to the plane MN.

To prove that AB and CD are parallel.

In the plane MN draw BD and AD and erect DE perpendicular to BD.

Since DC is perpendicular to the plane MN it is (by 326) perpendicular to DE.

Again, since BD is perpendicular to ED, by construction, AD is perpendicular (by 344) to ED.

Therefore ED is perpendicular to AD, BD, and CD; hence these lines all lie in one plane.

Consequently AB and CD are two lines in one plane perpendicular to the same line BD, therefore (by 61) they are parallel to one another. Q.E.D.

346. Cor. 1. *If one of two parallels is perpendicular to a plane, the other is also.*

347. Cor. 2. *Two straight lines that are parallel to a third straight line are parallel to each other.*

EXERCISES.

1. Two planes that have three points not in the same straight line in common coincide.

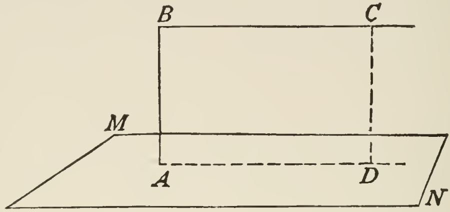
2. At a given point in a plane, erect a perpendicular to the plane.

3. From a point without a plane, let fall a perpendicular to the plane.

4. From a point without a plane draw a number of equal oblique lines to the plane.

Proposition VI. Theorem.

348. *If a straight line and a plane be perpendicular to the same straight line, they are parallel.*



Let the straight line BC and the plane MN be perpendicular to the straight line AB.

To prove that BC is parallel to MN. Pass a plane through BC and A meeting MN in the line AD, then from any point C in the line BC let fall the perpendicular CD to the plane, and draw AD.

Since CD is perpendicular to MN it will (by 326) be perpendicular to AD.

But BA is perpendicular to AD, therefore (by 345) BA is parallel to CD, and likewise BC and AD being perpendicular to BA, they will be parallel.

Therefore $BADC$ is a parallelogram and $CD = BA$, or the line BC is everywhere equally distant from MN, hence is parallel to MN. Q.E.D.

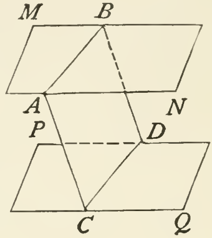
349. Cor. 1. *If two planes be perpendicular to the same straight line, they are parallel.*

350. Cor. 2. *Two parallel planes are everywhere equally distant.* *And,* conversely, *two planes that are everywhere equally distant are parallel.*

351. Cor. 3. *If two intersecting straight lines are each parallel to a given plane, the plane of these lines is parallel to the given plane.*

Proposition VII. Theorem.

352. *The intersections of two parallel planes by a third plane are parallel lines.*



Let MN and PQ be two parallel planes intersected by the plane AD in AB and CD.

To prove that AB and CD are parallel.

The lines AB and CD cannot meet since they lie in planes that are parallel, they themselves by hypothesis being in the same plane AD.

Therefore AB and CD are parallel. Q.E.D.

EXERCISES.

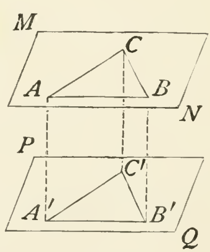
1. If a straight line is parallel to a line in a plane, it is parallel to the plane.

2. Parallel lines between parallel planes are equal.

Suggestion. See 108.

Proposition VIII. Theorem.

353. *If two angles not in the same plane have their sides respectively parallel and lying in the same direction, they are equal and their planes are parallel.*



Let the angles C and C' lie in the planes MN and PQ respectively, having their sides AC and $A'C'$ parallel, and also CB and $C'B$ parallel and in the same direction.

To prove that $\angle C = \angle C'$, and that MN and PQ are parallel.

1. Take $A'C' = AC$ and $C'B' = CB$, and draw AA', BB', and CC'.

Since AC and $A'C'$ are equal by construction and parallel by hypothesis, the figure $ACC'A'$ is a parallelogram; that is, AA' is equal and parallel to CC'.

For a similar reason, BB' and CC' are equal and parallel.

Since AA' and BB' are both equal and parallel to CC', they are equal and parallel to each other, or $ABB'A'$ is a parallelogram, and hence $AB = A'B'$.

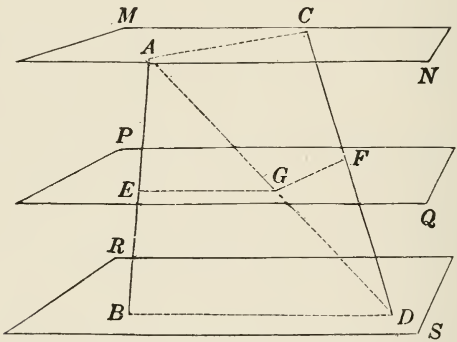
Therefore the triangles ACB and $A'C'B'$ have their sides respectively equal, and hence their angles are equal, or $\angle C = \angle C'$.

2. Since $AA' = BB' = CC'$, the two planes are equally distant, and hence parallel. Q.E.D.

354. Cor. *If two angles have their sides parallel, they are equal or supplemental.*

Proposition IX. Theorem.

355. *If two straight lines be intersected by three parallel planes their corresponding segments are proportional.*



Let AB and CD be intersected by the parallel planes MN, PQ, RS, in the points A, E, B, and C, F, D.

We are to prove $\dfrac{AE}{EB} = \dfrac{CF}{FD}$.

Draw AD, cutting the plane PQ in G.

Join the points E, G, and F, G.

In the triangle ABD, EG, being in the plane PQ parallel to RS, will be parallel to BD.

Therefore (by 213) $$\frac{AE}{EB}=\frac{AG}{GD}.$$

Likewise in the triangle DAC, GF is parallel to AC, and hence

$$\frac{CF}{FD}=\frac{AG}{GD}.$$

Hence (by 28) $$\frac{AE}{EB}=\frac{CF}{FD}.$$ Q.E.D.

356. Cor. *Any number of straight lines cut by parallel planes are divided into proportional segments.*

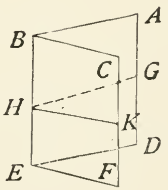
DIEDRAL ANGLES.

Definitions.

357. When two planes intersect they are said to form with each other a *Diedral* Angle.

The line of intersection is called the *Edge*. The planes are the *Faces*.

Thus in the diedral angle formed by the planes BD and BF, BE is the edge and BD and BF are the faces.



358. A diedral angle may be designated by two letters on its edge; or, if several diedral angles have a common edge, by four letters, one in each face and two on the edge, the letters on the edge being named between the other two.

Thus the diedral angle in the figure may be designated either as BE or $A–BE–C$.

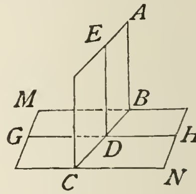
359. If a point is taken in the edge of the diedral angle, and two straight lines are drawn through this point, one in each face, and each perpendicular to the edge, the angle formed by these two lines is called the *Plane Angle* of the diedral angle, as $\angle GHK$.

360. Two diedral angles are equal if their plane angles are equal, or when their faces may be made to coincide.

361. The *magnitude* of a diedral angle depends solely on the amount of divergence of its faces, and is entirely independent of their extent.

362. Two diedral angles are *adjacent* when they have a common edge and a common face between them.

363. When the adjacent diedral angles which a plane forms with another plane on opposite sides are equal, each of these angles is called a *right diedral angle;* and the first plane is said to be *perpendicular* to the other.



Thus if the adjacent diedral angles $ABCM$, $ABCN$ are equal, each of these is a right diedral angle, and the planes AC and MN are perpendicular to each other.

Through a given line in a plane only one plane can be passed perpendicular to the given plane.

364. If the diedral angle is a right angle, the plane angle is also a right angle: therefore if two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.

365. The following principles are also true:

1. If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to that plane.

2. If two planes are perpendicular to each other, a straight line through any point of their intersection perpendicular to one of the planes will lie in the other.

3. If two planes are perpendicular to each other, a straight line from any point of one plane perpendicular to the other will lie in the first plane.

366. Vertical diedral angles are those which have a common edge, and the faces of one are prolongations of the faces of the other.

367. Diedral angles are *acute, obtuse, complementary, supplementary,* under the same conditions that hold for plane angles.

368. The demonstrations of many properties of diedral angles are the same as the demonstrations of analogous properties of plane angles.

For example:

1. Vertical diedral angles are equal.

2. Diedral angles whose faces are respectively parallel or perpendicular are either equal or supplementary.

3. Every point in the bisecting plane of a diedral angle is equally distant from the faces of the angle.

4. If a plane meets another, the sum of the adjacent diedral angles formed is equal to two right diedral angles; and conversely.

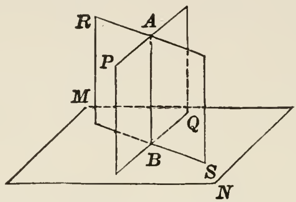
5. If two parallel planes are cut by a third plane, the alternate-interior diedral angles are equal, the alternate-exterior angles are equal, any diedral angle is equal to its corresponding angle, and the sum of the interior diedral angles on the same side of the secant plane is equal to two right diedral angles; and conversely.

6. Two diedral angles whose faces are parallel each to each are either equal or together equal to two right diedral angles.

7. Diedral angles are to each other as their plane angles; hence the plane angle may be taken as the measure of the diedral angle.

Proposition X. Theorem.

369. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*



Let the planes PQ and RS be perpendicular to MN.

To prove that their intersection AB is perpendicular to MN.

Let a perpendicular be erected to the plane MN at B.

Since B is a point in the plane RS, as RS is perpendicular to MN, the perpendicular BA will lie (by 365) in RS.

For the same reason BA will lie in PQ.

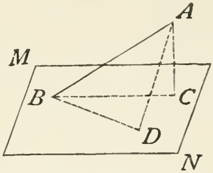
Therefore as BA lies in both planes, it must be in the intersection of those planes, or the intersection BA is perpendicular to MN. Q.E.D.

370. Cor. 1. If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to the third plane.

371. Cor. 2. If the planes PQ and RS include a right diedral angle, the three planes PQ, RS, MN, are perpendicular to one another; the intersection of any two of these planes is perpendicular to the third plane; and the three intersections are perpendicular to one another.

Proposition XI. Theorem.

372. *The acute angle between a straight line and its projection on a plane is the least angle which the line makes with any line of the plane.*



Let BC be the projection of AB on the plane MN, and BD be another line in the same plane passing through B.

To prove $\angle ABC < \angle ABD$.

Take $BD = BC$, and join AD and AC.

Since AC is the perpendicular (by 244) to the plane, it is shorter (by 335) than any oblique line from A to the plane, or $AC < AD$.

In the two triangles ABC and ABD, the side AB is common, the side $BD = BC$ by construction, but $AD > AC$.

Therefore (by 95) the greater angle lies opposite the greater third side, or

$$\angle ABC < ABD. \qquad \text{Q.E.D.}$$

EXERCISE.

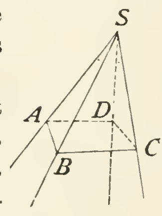
Show that a straight line makes equal angles with parallel planes.

POLYEDRAL ANGLES.

Definitions.

373. When three or more planes meeting in a point separate a portion of space from the rest of space, they form or include a *Polyedral Angle.*

The common point in which the planes meet is the *Vertex* of the angle, the intersections of the planes are the *Edges,* the portion of the planes between the edges are the *Faces,* and the plane angles formed by the edges are the *Face-angles.*



Thus, the point S is the vertex, the straight lines SA, SB, etc., are the edges, the planes SAB, SBC, etc., are the faces, and the angles ASB, BSC, etc., are the face-angles of the polyedral angle $S\text{–}ABCD$.

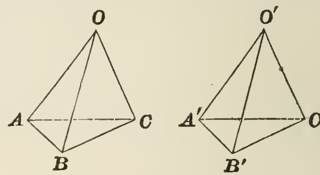
374. The edges of a polyedral angle may be produced indefinitely; but to represent the angle clearly, the edges and faces are supposed to be cut off by a plane, as in the figure above. The intersection of the faces with this plane forms a polygon, as $ABCD$, which is called the *Base* of the polyedral angle.

375. In a polyedral angle, each pair of adjacent faces forms a diedral angle, and each pair of adjacent edges forms a face-angle. There are as many edges as faces, and therefore as many diedral angles as faces.

376. The *magnitude* of a polyedral angle depends only upon the *relative position* of its faces, and is independent of their extent. Thus, by the face SAB is not meant the triangle SAB, but the indefinite plane between the edges SA, SB produced indefinitely.

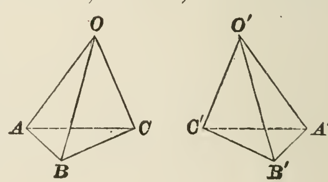
377. Two polyedral angles are *equal*, when the face and diedral angles of the one are respectively equal to the face and diedral angles of the other, and arranged in the *same order*.

Thus if the face angles AOB, BOC, and COA are equal respectively to the face angles $A'O'B'$, $B'O'C'$, and $C'O'A'$, and the diedral angles OA, OB, and OC to the diedral angles $O'A'$, $O'B'$, and $O'C'$, the triedral angles $O–ABC$ and $O'–A'B'C'$ are equal, for they can evidently be applied to each other so that their faces shall coincide.



378. Two polyedral angles are *symmetrical* when the face and diedral angles of one are equal to those of the other, each to each, but *arranged in reverse order.*

Thus if the face angles AOB, BOC, and COA are equal respectively to the face angles $A'O'B'$, $B'O'C'$, and $C'O'A'$, and the diedral angles OA, OB, and OC to the diedral angles $O'A'$, $O'B'$, and $O'C'$, the triedral angles $O–ABC$ and $O'–A'B'C'$ are symmetrical, for their equal parts are arranged in the reverse order.

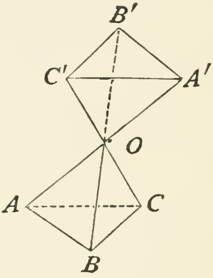


379. A polyedral angle of three faces is called a *Triedral* angle, one of four faces a *Quadraedral,* etc.

380. A triedral angle is called *Isosceles* if it has two of its face-angles equal; and *Equilateral* if three of its face-angles are equal.

381. Triedral angles are *Rectangular*, *Bi-rectangular*, or *Tri-rectangular*, according as they have one, two, or three right diedral angles.

382. A polyedral angle is *Convex*, if the polygon formed by the intersections of a plane with all its faces be a convex polygon.

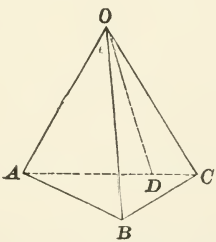


383. *Opposite* or *Vertical* polyedral angles are those in which the edges of the one are prolongations of the edges of the other.

Such angles are symmetrical, as $O\text{–}ABC$ and $O\text{–}A'B'C'$.

Proposition XII. Theorem.

384. *The sum of any two face-angles of a triedral angle is greater than the third.*

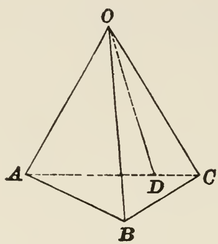


If the angles are equal, it is evident that the sum of any two will be greater than the third.

If unequal, let $\angle AOC$ be greater than $\angle AOB$ or $\angle BOC$ in the triedral $\angle O\text{–}ABC$.

In the plane AOC draw the line OD making $\angle AOD = \angle AOB$; draw AC cutting OD in D and pass a plane through AC so that it may cut off OB equal to OD.

Then the triangles OAD and OAB will be equal (by 86), having two sides and the included angle equal by construction, which gives $AD = AB$.



In the triangle ABC, $AB + BC > AC$ (by 77); subtracting the equals $AB = AD$, we have $BC > DC$.

In the triangles BOC and DOC, $OB = OD$, and the side OC is common, but the third side BC is greater than DC, therefore (by 95) $\angle BOC > \angle DOC$.

Add the equal angles, $\angle AOB = \angle AOD$,

and $$\angle AOB + \angle BOC > \angle AOD + \angle DOC,$$

or $$\angle AOB + \angle BOC > \angle AOC.$$ Q.E.D.

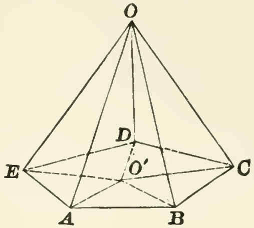
EXERCISES.

1. If a plane intersects all the faces of a triedral angle, what kind of a plane figure is formed by the lines of intersection? What in the case of a tetraedral angle?

2. The sides of an isosceles triangle are everywhere equally inclined to any plane passing through its base.

PROPOSITION XIII. THEOREM.

385. *The sum of the face-angles of any convex polyedral angle is less than four right angles.*



Let O–$ABCDE$ be a convex polyedral angle.

To prove that the sum of the face angles AOB, BOC, etc., is less than four right angles.

Pass the plane $ABCDE$ intersecting the edges in A, B, C, D, and E, and let O' be any point in this plane.

Join O' with A, B, C, D, and E.

Since the sum of any two face-angles at a triedral angle is greater than the third (by 384),

$$\angle OAB + \angle OAE > \angle EAB,$$

also $$\angle OBA + \angle OBC > \angle ABC, \text{ etc.}$$

That is, the sum of the base angles whose vertex is O is greater than the sum of the base angles whose vertex is O'.

But the sum of all the angles of the triangles whose vertex is O must be equal to the sum of all the angles of the triangles whose vertex is O', since the number of triangles in each case is the same, and (by 79) the value of the angles of each triangle is identical.

Therefore the angles at the vertex of the triangles, having the common vertex O, is less than the vertex angles at O', or less than four right angles. Q.E.D.

BOOK VII.

POLYEDRONS, CYLINDERS, AND CONES.

GENERAL DEFINITIONS.

386. A *Polyedron* is a solid bounded by planes. The *Faces* are the bounding planes, the *Edges* are the intersections of its faces, and the *Vertices* are the intersections of its edges.

387. The *Diagonal* of a polyedron is a straight line joining any two non-adjacent vertices not in the same plane.

388. A polyedron of four faces is called a *Tetraedron;* of six faces, a *Hexaedron;* of eight faces, an *Octaedron;* of twelve faces, a *Dodecaedron;* of twenty faces, an *Icosaedron.*

389. A polyedron is called *Convex* when the section made by any plane is a convex polygon.

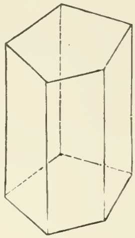
All polyedrons treated hereafter will be understood to be convex.

390. The *Volume* of a solid is the number which expresses its ratio to some other solid taken as a unit of volume. The *Unit of Volume* is a cube whose edge is a linear unit.

391. Two solids are *Equivalent* when their volumes are equal.

Prisms and Parallelopipeds.

392. A *Prism* is a polyedron two of whose faces are equal and parallel polygons, and the other faces are parallelograms.

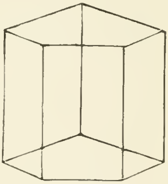


The equal and parallel polygons are called the *Bases* of the prism; the parallelograms are the *Lateral Faces;* the lateral faces taken together form the *Lateral* or *Convex Surface;* and the intersections of the lateral faces are the *Lateral Edges.*

The lateral edges are parallel and equal, and the area of the lateral surface is called the *Lateral Area.*

393. The *Altitude* of a prism is the perpendicular distance between its bases.

394. Prisms are *Triangular, Quadrangular, Pentangular,* etc., according as their bases are triangles, quadrangles, pentagons, etc.



RIGHT PRISM.

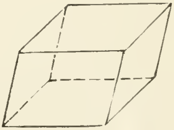
395. A *Right Prism* is a prism whose lateral edges are perpendicular to its bases.

396. An *Oblique Prism* is a prism whose lateral edges are oblique to its bases.

397. A *Regular Prism* is a right prism whose bases are regular polygons, and hence its lateral faces are equal rectangles.

398. A *Truncated Prism* is a portion of a prism included between either base and a section inclined to the base and cutting all the lateral edges.

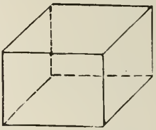
399. A *Right Section* of a prism is a section perpendicular to its lateral edges.



400. A *Parallelopiped* is a prism whose bases are parallelograms; therefore all the faces are parallelograms, and the opposite faces are equal and parallel.

401. A *Right Parallelopiped* is one whose lateral edges are perpendicular to its bases; that is, the lateral faces are rectangles.

402. A *Rectangular Parallelopiped* is a right parallelopiped whose bases are rectangles; that is, all the faces are rectangles.



Such a solid is sometimes called a *cuboid.* It is contained between three pairs of parallel planes.

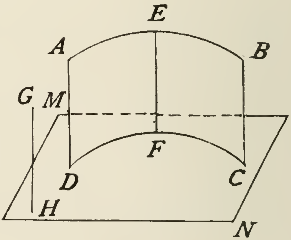
The *Dimensions* of a rectangular parallelopiped are the three edges which meet at any vertex.

403. A *Cube* is a rectangular parallelopiped whose six faces are all squares, and edges consequently equal.

404. *Similar Polyedrons* are those which are bounded by the same number of similar polygons, similarly placed.

Parts which are similarly placed, whether faces, edges, or angles, are called *Homologous.*

405. A *Cylindrical Surface* is a curved surface traced by a straight line, so moving as to intersect a given curve and always be parallel to a given straight line not in the curve.



Thus if the line EF moves so as to continually intersect the curve DC, and always be parallel to GH, the surface AC is a cylindrical surface.

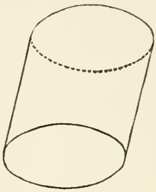
406. The moving line EF is the *Generatrix*, the fixed curve DC the *Directrix*, and EF in any of its positions is an *Element* of the surface.

407. A *General Cylinder* is a solid bounded by a cylindrical surface and two parallel planes called *Bases.*

408. The *Lateral Surface* is the curved surface.

408 *a*. A plane which contains an element of the cylinder and does not cut the surface is called a *tangent plane*, and the element contained by the tangent plane is the element of contact.

409. The *Altitude* of a cylinder is the perpendicular distance between the bases or the planes of the bases.



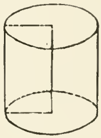
410. The *Right Cylinder* is the cylinder whose element is perpendicular to its base.

If the base is distorted so as to be no longer regular, the cylinder is still a *right* though not a regular cylinder.

411. A *Circular Cylinder* is one whose directrix is a circle.

Note. Hereafter the term *cylinder* is used for circular cylinder.

412. A right cylinder may be conceived as formed by the revolution of a rectangle about one of its sides.



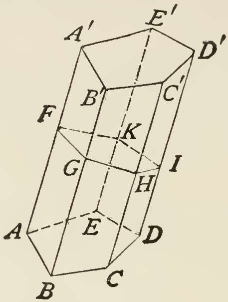
Similar cylinders of revolution are generated by similar rectangles.

413. Since the base of a cylinder is a polygon of an infinite number of sides, the cylinder itself may be regarded as a prism of an infinite number of faces; that is, a cylinder is only a prism under this condition of infinite faces.

414. Hence the cylinder will have the properties of a prism, and all demonstrations for prisms will include cylinders when so stated in the theorem or in the corollary.

PROPOSITION I. THEOREM.

415. *The lateral area of a prism is equal to the product of the perimeter of a right section by a lateral edge.*



Let AD' be a prism, and $FGHIK$ a right section.

To prove that the lateral area $= AA'(FG + GH + HI$, etc.).

Since a right section is perpendicular to the lateral edges, FG, GH, HI, etc., are altitudes of the parallelograms which form the faces of the prism. Hence,

$$\text{area of } A'ABB' = AA' \times FG \text{ (by 251)},$$

and $$\text{area of } B'BCC' = BB' \times GH, \text{ etc.}$$

But (by 392) the lateral edges are equal; that is,

$$AA' = BB' = CC', \text{ etc.}$$

Therefore the total lateral surface will be

$$AA' \times FG + AA' \times GH + AA' \times HI + \text{ etc.},$$

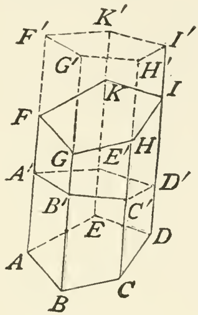
or lateral surface $AD = AA'(FG + GH + HI + \text{ etc})$. Q.E.D.

416. Cor. 1. *The lateral area of a right prism is equal to the product of the perimeter of its base by its altitude.*

417. Cor. 2. *The lateral area of a cylinder is equal to the perimeter of a right section of the cylinder multiplied by an element.*

Proposition II. Theorem.

418. *An oblique prism is equivalent to a right prism having for its base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism.*



Let $ABCDE\text{–}I$ be an oblique prism, and $A'B'C'D'E'$ a right section of it.

Produce $A'F$ to F', making $A'F' = AF$, likewise $B'G' = BG$, $C'H' = CH$, $D'I' = DI$, $E'K' = EK$; then will $F'\text{–}I'$ be a plane (by 350) parallel to $A'\text{–}D'$, which is a right section; hence $A'B'C'D'E'\text{–}I'$ will be a right prism.

To prove that prism $AI =$ prism $A'I'$.

The truncated prisms $F\text{–}I'$ and $A\text{–}D'$ are equivalent, since the faces $FGG'F'$ and $ABB'A'$ are equal by construction, likewise $G'GHH'$ and $B'BCC'$, and so with each pair of faces. The diedral angles are equal, being formed by a continuation of the same faces; that is,

$$\angle A'A = \angle FF', \ \angle B'B = \angle G'G, \text{ etc.}$$

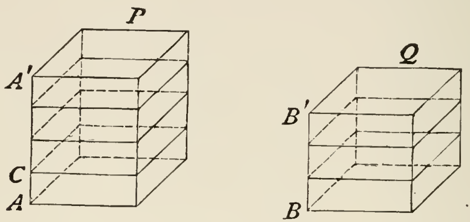
Therefore the space occupied by $A\text{–}D'$ could be exactly filled by $F\text{–}I'$, or *vice versa;* that is, the prisms are equivalent.

Hence if from the entire solid $A\text{–}I'$ we subtract the solid $A\text{–}D'$, we have left the right prism $A'\text{–}I'$, and if from the same prism we subtract the equal prism $F\text{–}I'$, we have left the oblique prism $A\text{–}I$.

Therefore prism $A\text{–}I =$ right prism $A'\text{–}I'$. Q.E.D.

Proposition III. Theorem.

419. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



Let P and Q be two rectangular parallelopipeds having equal bases, and let their altitudes AA' and BB' be commensurable.

To prove that $$\frac{P}{Q}=\frac{AA'}{BB'}.$$

Let AC be a common measure of AA' and BB', and suppose it to be contained 4 times in AA' and 3 times in BB'.

Then, $$\frac{AA'}{BB'}=\frac{4}{3}. \tag{1}$$

At the several points of division of AA' and BB' pass planes perpendicular to these lines.

Then the parallelopiped P will be divided into 4 equal parts, of which the parallelopiped Q will contain 3.

Therefore, $$\frac{P}{Q}=\frac{4}{3}. \tag{2}$$

From (1) and (2), we have

$$\frac{P}{Q}=\frac{AA'}{BB'}.$$ Q.E.D.

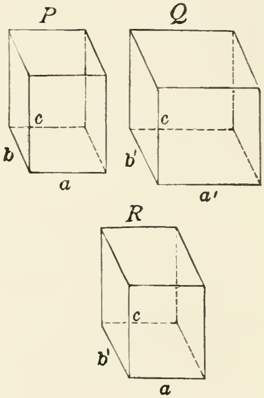
When the altitudes are incommensurable, the demonstration follows the method pursued in section 173.

420. Scholium. This theorem may also be expressed as follows:

Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.

Proposition IV. Theorem.

421. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let P and Q be two rectangular parallelopipeds having the common altitude c and the rectangles ab and $a'b'$ for bases.

To prove that $\dfrac{P}{Q} = \dfrac{ab}{a'b'}$.

Construct a third parallelopiped R which shall have a, b', and c for its dimensions.

Then since P and R have by construction two dimensions in common, we have (from 420)

$$\frac{P}{R} = \frac{b}{b'}.$$

For the same reason

$$\frac{R}{Q} = \frac{a}{a'}.$$

Hence by multiplication

$$\frac{P}{Q} = \frac{ab}{a'b'}.$$

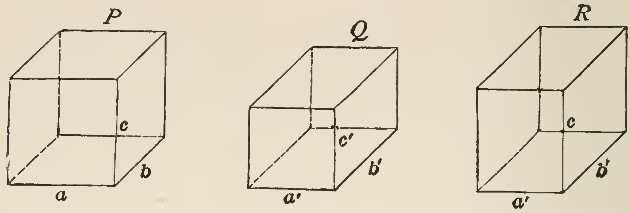
Q.E.D.

422. Scholium. The theorem may also be expressed:

Two rectangular parallelopipeds having one dimension in common are to each other as the products of their other two dimensions.

Proposition V. Theorem.

423. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let P and Q be two rectangular parallelopipeds having the dimensions a, b, c, and a', b', c', respectively.

To prove that $$\frac{P}{Q}=\frac{a \times b \times c}{a' \times b' \times c'}.$$

Construct a third parallelopiped having the dimensions a', b', and c.

Then since P and R have the dimension c in common, we have (by 422)

$$\frac{P}{R}=\frac{a \times b}{a' \times b'}.$$

Again, R and Q have the two dimensions a' and b' common, hence (by 420) we have

$$\frac{R}{Q}=\frac{c}{c'}.$$

Multiplying these equal ratios, it gives

$$\frac{P}{Q}=\frac{a \times b \times c}{a' \times b' \times c'}. \qquad \text{Q.E.D.}$$

424. Cor. 1. If $a'=b'=c'=1$, then Q will be the unit of volume, and the above proportion becomes $P=a \times b \times c$, or the product of its three dimensions.

425. Cor. 2. Since $a \times b$ gives the area of the base (from 248), we have the volume of a rectangular parallelopiped equal to the product of its base by its altitude.

426. Cor. 3. If $a=b=c$, then (from 424) $P=a \times a \times a=a^3$; that is, the volume of a cube (403) is the cube of its edge.

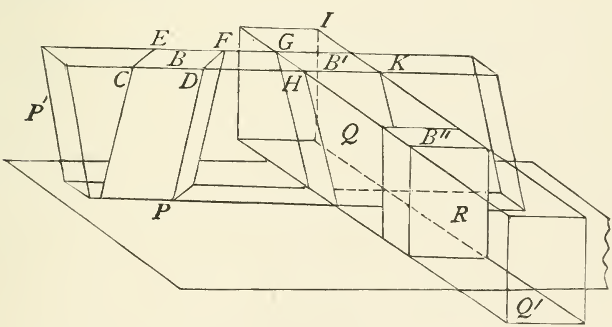
EXERCISES.

1. Show that the diagonals of a parallelopiped bisect each other.

2. Show that the square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three edges meeting at any vertex.

Proposition VI. Theorem.

427. *The volume of any parallelopiped is equal to the product of its base and altitude.*



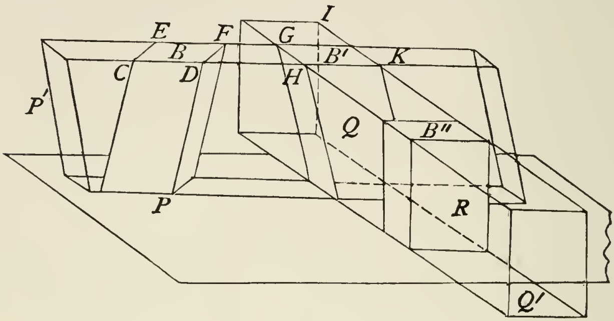
Let P be any parallelopiped with the base B and altitude h.

To prove that $\quad$ vol. $P = B \times h$.

Extend the lines CD and EF and also the corresponding lines of the base, and construct thereon the indefinite prism P'. Cut from this the right prism Q, whose altitude HK is equal to the lateral edge CD.

Then (from 418) the oblique prism $P =$ right prism Q.

Extend the lines GH and IK and also the corresponding lines of the base of Q, and construct thereon the indefinite prism Q'. From this cut the rectangular prism R, having its altitude equal to the lateral edge of Q and base B''.



Then (from 423) $\qquad Q = R$.

But it was shown that $P = Q$, therefore $P = R$.

Now R is a rectangular parallelopiped since its faces are perpendicular to each other, that gives $R = B'' \times h$.

But the parallelograms B and B' are equal, being between the same parallels; likewise $B' = B''$ for the same reason, therefore $B = B''$.

Hence $\qquad R = B \times h.$ $\qquad$ Q.E.D.

EXERCISES.

1. Show that in any parallelopiped the sum of the squares of the twelve edges is equal to the sum of the squares of its four diagonals.

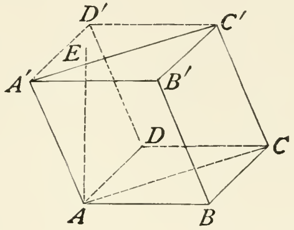
2. Find the length of the diagonal of a rectangular parallelopiped whose dimensions are 3, 4, and 5.

3. Find the volume of a rectangular parallelopiped whose surface is 932 and whose base is 4 by 12.

4. Find the side of a cube which contains as much as a rectangular parallelopiped 16 feet long, 4 feet wide, and 3 feet high.

Proposition VII. Theorem.

428. *The volume of a triangular prism is equal to the product of its base and altitude.*



Let AE be the altitude of the triangular prism $ABC\text{–}C'$.

To prove that

$$\text{volume } ABC\text{–}C' = ABC \times AE.$$

Construct the parallelopiped $ABCD\text{–}D'$ having its edges equal and parallel to AB, BC, and BB'.

Since the diagonal AC divides the parallelogram $ABCD$ into two equal parts, the two prisms $ABC\text{–}C'$ and $ADC\text{–}D'$, each being equivalent to a right prism of the same altitude and equal right section are equivalent.

But the parallelopiped (by 427) is equal to the product of its base by its altitude.

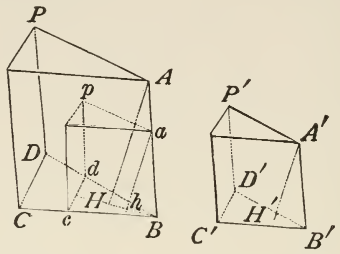
Therefore the half parallelopiped is equal to the product of the half base by its altitude; that is,

$$\text{volume } ABC\text{–}C' = ABC \times AE. \qquad \text{Q.E.D.}$$

429. Cor. *Since any prism can be divided into triangular prisms by diagonal planes, each prism being equal to the product of its base by its altitude, it follows that the volume of any prism is equal to the product of its base and altitude.*

Proposition VIII. Theorem.

430. *Similar triangular prisms are to each other as the cubes of their homologous edges.*



Let $CBD\text{–}P$ and $C'B'D'\text{–}P'$ be two similar triangular prisms, and let BC and $B'C'$ be any two homologous edges.

To prove that

$$CBD\text{–}P : C'B'D'\text{–}P' = \overline{BC}^3 : \overline{B'C'}^3.$$

Since the homologous triedral angles B and B' are equal, and the faces which bound them are (by 404) similar, these triedral angles may be applied, one to the other, so that the angle $C'B'D'$ will coincide with CBD, with the edge $B'A'$ on BA.

In this case the prism $C'B'D'\text{–}P'$ will take the position of $cBd\text{–}p$.

From A draw AH perpendicular to the common base of the prisms; then the plane BAH is (by 365) perpendicular to the plane of the base.

From a draw ah likewise in the plane BAH perpendicular to the intersector BH, and it will (by 364) be perpendicular to the plane of the base.

Since the bases BCD and Bcd are similar (by 262),

$$CBD : cBd = \overline{CB}^2 : \overline{cB}^2 \quad . \quad . \quad . \quad . \quad . \quad (a)$$

In the similar triangles ABH and aBh (by 218),

$$AH : ah = AB : aB.$$

In the similar parallelograms AC and ac (by 223),

$$AB : aB = BC : Bc;$$

therefore (by 28)

$$AH : ah = CB : cB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Multiplying (a) by (b), we have

$$CBD \times AH : cBd \times ah = \overline{CB}^3 : \overline{cB}^3.$$

But (by 428) $CBD \times AH$ is the volume of $CBD\text{–}P$, and $cBd \times ah$ is the volume of $C'B'D'\text{–}P'$, and $cB = C'B'$.

Therefore $\quad CBD\text{–}P : C'B'D'\text{–}P' = \overline{CB}^3 : \overline{C'B'}^3$, Q.E.D.

431. Cor. 1. *Any two similar prisms are to each other as the cubes of their homologous edges.*

For, since the prisms are similar, their bases are similar polygons (by 404*); and these similar polygons may each be divided into the same number of similar triangles, similarly placed (by* 121*); therefore, each prism may be divided into the same number of triangular prisms, having their faces similar and like placed; consequently, the triangular prisms are similar (by* 404*). But these triangular prisms are to each other as the cubes of their homologous edges, and being like parts of the polygonal prisms, the polygonal prisms themselves are to each other as the cubes of their homologous edges.*

432. Cor. 2. *Similar prisms are to each other as the cubes of their altitudes, or as the cubes of any other homologous lines.*

433. Cor. 3. *Since the cylinder is the limit of a prism of infinite number of sides, it follows that:*

The volume of a cylinder is equal to the product of its base and altitude.

434. Cor. 4. *The volumes of two prisms (cylinders) are to each other as the product of their bases and altitudes: prisms* *(cylinders) having equivalent bases are to each other as their altitudes: prisms (cylinders) having equal altitudes are to each other as their bases: prisms (cylinders) having equivalent bases and equal altitudes are equivalent.*

EXERCISES.

Find the lateral area and volume, when right:

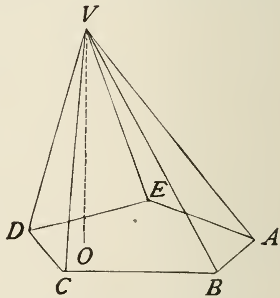
1. Of a triangular prism, each side of whose base is 3, and whose altitude is 8.

2. Of a regular hexagonal prism, each side of whose base is 2, and whose altitude is 12.

3. Of a triangular prism whose altitude is 18 and the sides of the base are 6, 8, and 10.

PYRAMIDS.

435. A *Pyramid* is a polyedron, one of whose faces is a polygon, and whose other faces are triangles having a common vertex without the base and the sides of the polygon for bases.



436. The polygon $ABCDE$ is the *Base* of the pyramid, the point V the *Vertex*, VBC, VCD, etc., the *Lateral*, or *Convex Surface*, VC, VB, etc., the *Lateral* edges, and the area of the lateral surface is called the *Lateral Area*.

437. The *Altitude* of a pyramid is the perpendicular distance from the vertex to the plane of the base.

438. A pyramid is called *Triangular, Quadrangular, Pentagonal*, etc., according as its base is a triangle, quadrilateral, pentagon, etc.

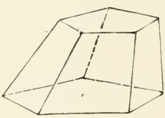
439. A triangular pyramid has but four faces, and is called a *Tetraedron;* any one of its faces can be taken for its base.

440. A *Regular Pyramid* is one whose base is a regular polygon, the centre of which coincides with the foot of the perpendicular let fall upon it from the vertex. The lateral edges of a regular pyramid are (by 334) equal, hence the lateral faces are equal isosceles triangles.

441. The *Slant Height* of a regular pyramid is the altitude of any one of its lateral faces; that is, the straight line drawn from the vertex of the pyramid to the middle point of any side of the base.

442. A *Truncated Pyramid* is the portion of a pyramid included between its base and a plane cutting all the lateral edges.

443. A *Frustum* of a pyramid is a truncated pyramid whose bases are parallel.

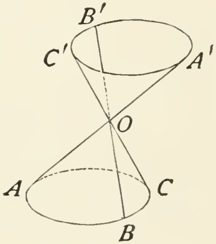


The *Altitude* of a frustum is the perpendicular distance between the planes of its bases.

444. The lateral faces of a frustum of a regular pyramid are equal trapezoids.

The *Slant Height* of a frustum of a regular pyramid is the altitude of any one of its lateral faces.

445. A *Conical Surface* is traced by a straight line so moving that it always intersects a given curve and passes through a given point.



Thus the straight line BB' continually intersects the curve ABC and passes through the point O, tracing the conical surface ABC–O–$A'B'C'$.

446. The straight line BB' is the *Generatrix*, the curve ABC the *Directrix*, O the *Vertex*, and O–ABC, O–$A'B'C'$ are the two *Nappes*, and OB is an *Element*.

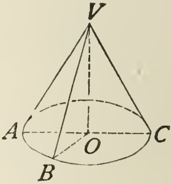
447. A *Cone* is a solid bounded by a conical surface and a plane which cuts all of the elements of the surface, as O–ABC.

448. This plane is called the *Base*, and the perpendicular from the vertex to the plane of the base is the *Altitude*.

449. A *Circular Cone* is one whose base is a circle.

NOTE. Hereafter *Cone* will be used for circular cone.

450. A *Right Cone* is a cone in which the perpendicular let fall from the vertex meets the base in its centre; it is also called a cone of revolution, since it can be formed by revolving a right triangle about one of its shorter sides, as V–ABC.

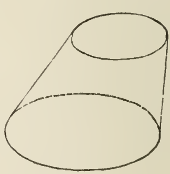


451. Since the cone has a circular base which is the limit of a polygonal base, a cone may be regarded as a pyramid of an infinite number of faces, hence the cone will have, in general, the properties of a pyramid, and all demonstrations for pyramids will include cones when so stated in the theorem or in the corollary.

452. A *Truncated Cone* is the portion of a cone included between its base and another plane cutting all its elements.

453. A *Frustum* of a cone is a truncated cone whose cutting planes or bases are parallel.

The *Altitude* of a frustum is the perpendicular distance between the planes of its bases.

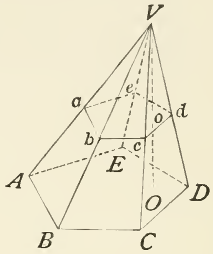


Proposition IX. Theorem.

454. *If a pyramid is cut by a plane parallel to its base:*

(1) *The edges and the altitude are divided proportionally.*

(2) *The section is a polygon similar to the base.*



Let $V–ABCDE$ be a pyramid cut by the plane $abcde$ parallel to the base.

1. To prove

$$\frac{Va}{VA}=\frac{Vb}{VB}\cdots=\frac{Vo}{VO}.$$

Suppose a plane to pass through V parallel also to the base; then (by 355)

$$\frac{Va}{VA}=\frac{Vb}{VB}=\cdots\frac{Vo}{VO}.$$

2. To prove that the section $abcde$ is similar to the base $ABCDE$.

Since ab is parallel to AB and bc parallel to BC, then (by 353) $\angle abc=\angle ABC$; likewise $\angle bcd=\angle BCD$, etc.

Again, ab and AB being parallel, we have (by 218)

$$\frac{ab}{AB}=\frac{Vb}{VB},$$

also $$\frac{bc}{BC}=\frac{Vb}{VB};$$

hence $$\frac{ab}{AB}=\frac{bc}{BC};$$

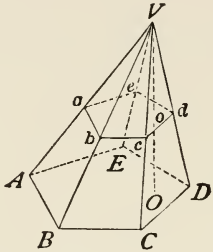
similarly $$\frac{bc}{BC}=\frac{cd}{CD}, \text{ etc.}$$

Therefore the polygons $abcde$ and $ABCDE$ are mutually equiangular and have their homologous sides proportional; hence (by 223) they are similar. Q.E.D

455. Cor. 1. Since *abcde* and *ABCDE* are similar polygons, we have (from 264)

$$\frac{abcde}{ABCDE}=\frac{\overline{ab}^2}{\overline{AB}^2}=\frac{\overline{Vb}^2}{\overline{VB}^2}=\frac{\overline{Vo}^2}{\overline{VO}^2};\text{ that is,}$$

The areas of parallel sections of a pyramid are proportional to the squares of their oblique or vertical distances from the vertex.

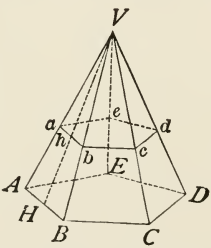


456. Cor. 2. *In two pyramids of equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.*

457. Cor. 3. *The section of a circular cone made by a plane parallel to the base is a circle.*

Proposition X. Theorem.

458. *The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.*



Let $V\text{–}ABCDE$ be a regular pyramid, and VH the slant height.

To prove that

$$\text{lateral area } V\text{–}ABCDE = (AB + BC + \text{etc.}) \times \tfrac{1}{2}\, VH.$$

The lateral area of the pyramid is equal to the sum of the areas of the triangles VAB, VBC, etc.

But (by 254) area $VAB = \frac{1}{2}\, AB \times VH$;

likewise area of $VBC = \frac{1}{2}\, BC \times VH$, etc.

Therefore

$$\text{lateral area } V\text{–}ABCDE = \tfrac{1}{2}\, AB \times VH + \tfrac{1}{2}\, BC \times VH + \text{etc.},$$

or $$= \tfrac{1}{2}\,(AB + BC + \text{etc.}) \times VH. \qquad \text{Q.E.D.}$$

459. Cor. 1. *The lateral area of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases multiplied by its slant height.*

460. Cor. 2. *The lateral area of a cone of revolution is equal to the circumference of its base multiplied by one-half its slant height.*

461. Cor. 3. *The lateral area of a frustum of a cone of revolution is equal to one-half the sum of the circumferences of its bases multiplied by its slant height.*

If R and R' denote the radii of the lower and upper bases of the frustum of a cone of revolution, and L the slant height, then

$$\text{lateral area} = \tfrac{1}{2}\,[2\,\pi R + 2\,\pi R'] \times L$$
$$= \pi\,(R + R') \times L.$$

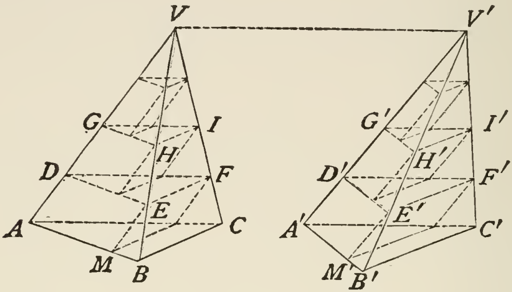
EXERCISES.

1. The radius of the lower base of the frustum of a cone of revolution is 12, the radius of the upper base is 6 and the altitude 8. Find the lateral area.

2. In the above what is the lateral area of the cone that was cut off to form this frustum?

Proposition XI. Theorem.

462. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Let $V\text{-}ABC$ and $V'\text{-}A'B'C'$ be two triangular pyramids having equivalent bases ABC and $A'B'C'$ and a common altitude.

To prove vol. $V\text{-}ABC =$ vol. $V'\text{-}A'B'C'$.

Divide the common altitude into any number of equal parts, and through these points of division pass planes parallel to the plane of the bases, say DEF, $D'E'F'$; GHI, $G'H'I'$; etc.

Upon DEF construct the prism $DEF\text{-}M$, and on $D'E'F'$ the prism $D'E'F'\text{-}M'$. These prisms are (by 434) equivalent.

Likewise, prism upon GHI is equivalent to the prism upon $G'H'I'$, and so on.

Therefore the sum of the prisms in $V\text{-}ABC$ is equivalent to the sum of the prisms in $V'\text{-}A'B'C'$.

Now let the number of divisions be indefinitely increased, then the sum of the prisms in $V\text{-}ABC$ will approach the pyramid $V\text{-}ABC$ as its limit, and the sum of the prisms in $V'\text{-}A'B'C'$ will approach the pyramid $V'\text{-}A'B'C'$ as its limit.

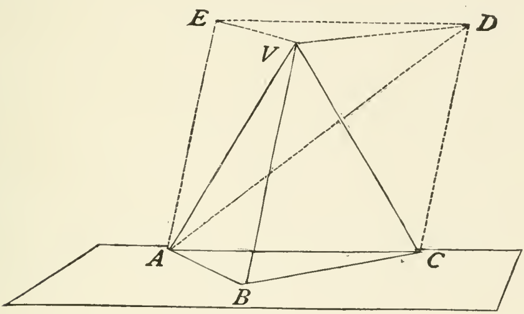
Therefore, since the sums of these prisms are always equivalent, their limits are equivalent, or

$$\text{vol. } V\text{-}ABC = \text{vol. } V'\text{-}A'B'C'. \qquad \text{Q.E.D.}$$

463. Since any pyramid can be divided into triangular pyramids by passing planes through the vertex and the diagonals of the base, it follows that *any two pyramids of equal altitudes and equivalent bases are equivalent.*

Proposition XII. Theorem.

464. *The volume of a triangular pyramid is equal to one-third of the product of its base and altitude.*



Let $V\text{–}ABC$ be a triangular pyramid with h for its altitude and ABC its base.

To prove that vol. $V\text{–}ABC = \frac{1}{3} h \times ABC$.

Upon the base ABC construct the prism $ABC\text{–}D$, having its lateral edges parallel to VB, and its altitude equal to h, or that of the pyramid.

Draw AD, the diagonal, and it will divide (by 107) the parallelogram $EACD$ into two equal triangles, and through AD and V conceive a plane to pass.

Then the prism will be divided into three triangular pyramids, $V\text{–}ABC$, $A\text{–}VED$, and $A\text{–}VCD$.

$V\text{–}ABC = A\text{–}VED$, having equivalent bases and equal altitudes.

$V\text{–}ABC$ can be regarded as having A for its vertex and

VBC for its base; then $A\text{–}VBC = A\text{–}VCD$, having a common vertex and equal bases (by 107).

Therefore the three triangular pyramids are equivalent, and each, say $V\text{–}ABC$, will be one-third of the triangular prism.

But the volume of the prism is (by 428) equal to the product of its base by its altitude, then the volume of $V\text{–}ABC = \frac{1}{3} h \times ABC$. Q.E.D.

465. Cor. 1. *Since any pyramid can be divided into triangular pyramids by passing planes through the vertex and the diagonals of its base, it follows that the volume of any pyramid is equal to one-third the product of its base and altitude.*

466. Cor. 2. *The volume of any cone is equal to the product of one-third of its base by its altitude.*

467. Cor. 3. *The volumes of two pyramids (cones) are to each other as the product of their bases and altitudes: having equivalent bases they are to each other as their altitudes: having the same altitude they are to each other as their bases: having equivalent bases and equal altitudes they are equivalent.*

468. Cor. 4. *If a triangle and a rectangle having the same base and equal altitudes be revolved about the common base as an axis, the volume generated by the triangle will be one-third that generated by the rectangle.*

Proposition XIII. Theorem.

469. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and for their bases the lower base, the upper base, and a mean proportional between the bases, of the frustum.*

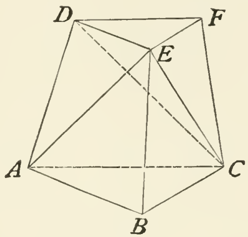
Let AF be a frustum of a triangular pyramid.

Denote the area of the lower base by B, the area of the upper base by b, and the altitude by h.

To prove that

$$\text{vol. } AF = \tfrac{1}{3}h \times B + \tfrac{1}{3}h \times b + \tfrac{1}{3}h \times \sqrt{B \times b} \quad (200)$$
$$= \tfrac{1}{3}h \times (B + b + \sqrt{B \times b}).$$

Pass a plane through the points A, C, and E, and another through the points C, D, and E, dividing the frustum into three triangular pyramids, $E\text{–}ABC$, $C\text{–}DEF$, and $E\text{–}ACD$.



Let these pyramids be denoted by P, Q, and R, respectively.

The pyramid P has for its altitude the altitude h of the frustum, and for its base the lower base B of the frustum.

Hence (by 464) $\quad P = \tfrac{1}{3}h \times B. \quad (1)$

And the pyramid Q has for its altitude the altitude of the frustum, and for its base the upper base b of the frustum.

Hence $\quad Q = \tfrac{1}{3}h \times b. \quad (2)$

Now the pyramids $E\text{–}ABC$ and $E\text{–}ACD$ may be regarded as having the common vertex C, and their bases AEB and AED in the same plane.

Then they have the same altitude, and are to each other (by 467) as their bases.

But the triangles AEB and AED have for their common altitude the altitude of the trapezoid $ABED$, and are to each other as their bases AB and DE (by 256).

Therefore $\quad \dfrac{P}{R} = \dfrac{AEB}{AED} = \dfrac{AB}{DE}. \quad (3)$

Again, the pyramids $E\text{–}ACD$ and $E\text{–}CDF$ have the common vertex E, and their bases ACD and CDF in the same plane.

Then they have the same altitude, and are to each other as their bases.

But the triangles ACD and CDF have for their common altitude the altitude of the trapezoid $ACFD$, and are to each other as their bases AC and DF.

Therefore $$\frac{R}{Q}=\frac{ACD}{CDF}=\frac{AC}{DF}. \tag{4}$$

Now since the section DEF is similar to the base ABC (by 454),

$$\frac{AC}{DF}=\frac{AB}{DE}.$$

Whence from (3) and (4) (by 28),

$$\frac{R}{Q}=\frac{P}{R}, \text{ or } R^2=P\times Q.$$

Substituting in this equation the values of P and Q from (1) and (2),

$$R^2=(\tfrac{1}{3}h)^2\times(B\times b).$$

Whence, $$R=\tfrac{1}{3}h\times\sqrt{B\times b}.$$

Therefore $$\text{vol. } AF=P+Q+R$$
$$=\tfrac{1}{3}h\times(B+b+\sqrt{B\times b}). \quad \text{Q.E.D.}$$

470. Cor. 1. By the same reasoning as in 465 we may conclude that: *A frustum of any pyramid is equivalent to the sum of three pyramids, having the same altitude as the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases, of the frustum.*

471. Cor. 2. *The volume of a frustum of any cone is equal to the sum of the volumes of three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base,* *the upper base, and a mean proportional between the bases of the frustum.*

If R and R' denote the radii of the lower and upper bases of the frustum, and h the altitude, then $B = \pi R^2$, and $b = \pi R'^2$, hence $\sqrt{B \times b} = \pi RR'$.

Therefore $\quad$ vol. $= \frac{1}{3} h \times \pi [R^2 + R'^2 + RR']$.

EXERCISES

Find the lateral edge, lateral area, and volume:

1. Of a regular triangular pyramid, each side of whose base is 6, and whose altitude is 5.

2. Of a regular quadrangular pyramid, each side of whose base is 16, and whose altitude is 18.

3. Of a regular hexagonal pyramid, each side of whose base is 2, and whose altitude is 14.

4. Of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 3, and whose altitude is 6.

5. What is the volume of a frustum of a regular triangular pyramid, the sides of whose bases are 8 and 6, and whose lateral edge is 7?

6. Show that the number of plane angles at the vertices of a polyedron is an even number.

7. The sum of the face-angles of any polyedron is equal to four right angles taken as many times as the polyedron has vertices less two.

8. The base of a pyramid is regular, if its faces are equal isosceles triangles.

9. Find the difference between the volume of the frustum of a pyramid and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

Regular Polyedrons.

472. *A Regular Polyedron* is one whose faces are all equal regular polygons, and whose polyedral angles are all equal.

Proposition XIV. Theorem.

473. *There can be only five regular convex polyedrons.*

Proof. At least three faces are necessary to form a polyedral angle, and the sum of its face-angles must be less than 360° (by 385).

1. Because the angle of an equilateral triangle is 60°, each convex polyedral angle may have 3, 4, or 5 equilateral triangles. It cannot have 6 faces, because the sum of 6 such angles is 360°, reaching the limit. Therefore no more than three regular convex polyedrons can be formed with equilateral triangles; the *tetraedron, octaedron,* and *icosaedron.*

2. Because the angle of a square is 90°, each convex polyedral angle may have 3 squares. It cannot have 4 squares, because the sum of 4 such angles is 360°. Therefore only one regular convex polyedron can be formed with squares; the *hexaedron,* or *cube.*

3. Because the angle of a regular pentagon is 108°, each convex polyedral angle may have 3 regular pentagons. It cannot have 4 faces, because the sum of 4 such angles is 432°. Therefore only one regular convex polyedron can be formed of regular pentagons; the *dodecaedron.*

Because the angle of a regular hexagon is 120°, and the angle of every regular polygon of more than 6 sides is yet greater than 120°, therefore there can be no regular convex polyedron formed of regular hexagons or of any regular polygons of more than 6 sides.

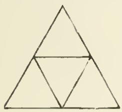
Therefore there can be only five regular convex polyedrons.

Q.E.D.

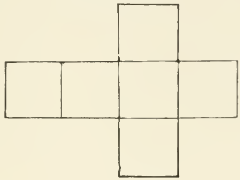
474. Scholium. Models of the regular polyedrons may be easily constructed as follows:

Draw the following diagrams on cardboard, and cut them out. Then cut halfway through the board in the dividing lines, and bring the edges together so as to form the respective polyedrons.

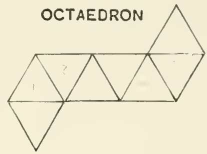
TETRAEDRON



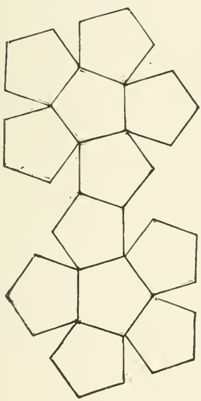
HEXAEDRON



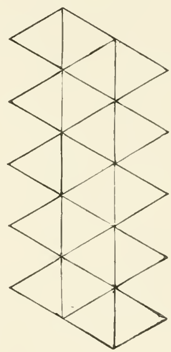
OCTAEDRON



DODECAEDRON



ICOSAEDRON

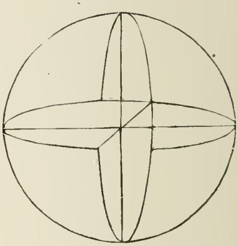


BOOK VIII.

THE SPHERE.

DEFINITIONS.

475. A *Sphere* is a solid bounded by a surface, all points of which are equally distant from a point within called the *centre.* A sphere may be generated by the revolution of a semicircle about its diameter as an axis.



476. A *Radius* of a sphere is the distance from its centre to any point in the surface. All the radii of a sphere are equal.

477. A *Diameter* of a sphere is any straight line passing through the centre and having its extremities in the surface of the sphere. All the diameters of a sphere are equal, since each is equal to twice the radius.

478. A *Section* of a sphere is a plane figure whose boundary is the intersection of its plane with the surface of the sphere.

479. Every section of a sphere made by a plane is a circle (see 334).

When the plane passes through the centre, the section is called a *Great Circle.*

480. Every great circle plane bisects the sphere.

481. Any two great circles bisect each other.

482. An *Axis* of a circle of a sphere is the diameter of the sphere *perpendicular* to the circle; and the extremities of the axis are the *Poles* of the circle.

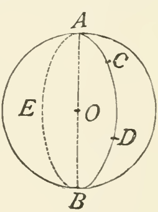
483. All points in the circumference of a circle of a sphere are equally distant from each of its poles, the distance being measured along the arcs of a great circle (see 334).

484. A straight line or a plane is said to be *tangent* to a sphere when it has but one point in common with the surface of the sphere.

The common point is called the *Point of Contact*, or *Point of Tangency.*

485. A plane perpendicular to a radius at its extremity is tangent to the sphere. (See 150.)

486. A great circle can be passed through any two points on a sphere, since a plane can be made to pass through these points and the centre, thus intersecting the surface of the sphere in a great circle.



By distance between two points is meant the shorter arc of the great circle passing through them, as CD.

Proposition I. Theorem.

487. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points in the arc of a great circle, it is the pole of that arc.*

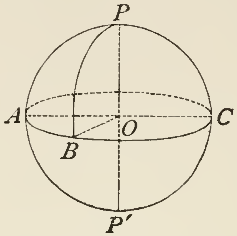
Let the point P be a quadrant's distance from each of the points A and B; that is, the arc joining P and A is one-fourth of the arc of a great circle.

To prove that P is the pole of the arc AB.

Let O be the centre of the sphere, and draw OA, OB, and OP.

Then since PA and PB are quadrants, the angles POA and POB are right angles.

Therefore PO (by 337) is perpendicular to the plane AOB; hence P is the pole of the arc AB. Q.E.D.



488. Cor. *The polar distance of a great circle is a quadrant.*

489. Scholium. The term *quadrant* in Spherical Geometry usually signifies a quadrant of a great circle.

SPHERICAL ANGLES AND POLYGONS.

Definitions.

490. The *Angle* between two intersecting arcs of circles on the surface of a sphere is the diedral angle between the planes of these circles.

A *Spherical Angle* is the angle between two intersecting arcs of *great circles* on the surface of a sphere.

491. A *Spherical Polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles.

The bounding arcs are the *Sides* of the polygon; the points of intersection of the sides are the *Vertices* of the polygon, and the angles which the sides make with each other are the *Angles* of the polygon.

A *Diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

492. A *Spherical Triangle* is a spherical polygon of three sides.

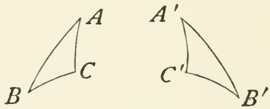
A spherical triangle is *Right* or *Oblique*, *Scalene*, *Isosceles*, or *Equilateral*, in the same cases as a plane triangle.

493. A *Spherical Pyramid* is a portion of the sphere bounded by a spherical polygon and the planes of the sides of the polygon. The centre of the sphere is the *Vertex* of the pyramid, and the spherical polygon is its *Base.*

494. Since the sides of a spherical polygon are arcs, they are usually expressed in *Degrees, Minutes,* and *Seconds.*

495. Two spherical polygons are *Equal* if they can be applied one to the other so as to coincide.

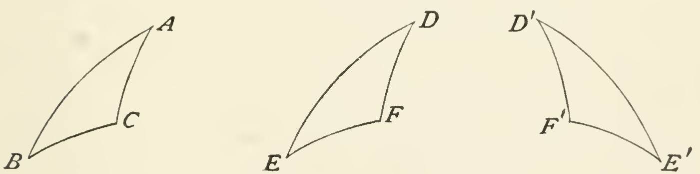
496. Two spherical polygons are *Symmetrical* when the sides and angles of the one are respectively equal to the sides and angles of the other, but taken in the reverse order.



Thus the spherical triangles ABC and $A'B'C'$ are symmetrical if the sides AB, BC, and CA are equal to $A'B$, $B'C'$, and $C'A'$, respectively, and the angles A, B, and C to the angles A', B', and C'.

497. Two spherical triangles on the same sphere or on equal spheres are equal (or symmetrical), under the same conditions as plane triangles, viz.:

a. When they have two sides and the included angle equal.
b. When they have two angles and the included side equal.
c. When they have three sides equal.

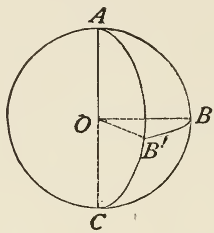


If the equal parts are in the same order as in ABC and DEF, equality is shown by superposition as in Plane Geometry.

If the parts are in the reverse order as ABC and $D'E'F'$, construct a triangle, DEF, symmetrical to $D'E'F'$, and then it can be shown that ABC and DEF are equal by superposition.

Proposition II. Theorem.

498. *A spherical angle is measured by the arc of a great circle described with its vertex as a pole, included between its sides produced if necessary.*



Let ABC and $AB'C$ be two intersecting arcs of great circles on the sphere AC, and O the centre of the sphere.

Pass the plane OBB' perpendicular to AC at O, intersecting the planes ABC and $AB'C$ in the radii OB and OB', and the sphere in the great circle BB'.

To prove that the spherical angle BAB' is measured by the arc BB'.

Since (by 359) BOB' is a plane angle, it is (by 368) the measure of the diedral angle $BACB'$.

But (by 174) the arc BB' is the measure of $\angle BOB'$.

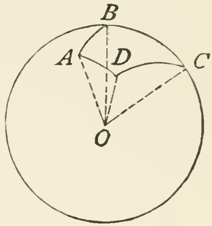
Therefore the spherical angle BAB' is measured by the arc BB'. Q.E.D.

499. Cor. 1. *A spherical angle is equal to the diedral angle between the planes of the two circles.*

500. Cor. 2. *If two arcs of great circles cut each other, their vertical angles are equal.*

501. Cor. 3. *The angles of a spherical polygon are equal to the diedral angles between the planes of the sides of the polygon.*

502. Because the planes of all great circles pass through the centre of the sphere, therefore the planes of the sides of a spherical polygon form a polyedral angle at the centre O whose face-angles AOB, BOC, etc., are measured by the sides AB, BC, etc., of the polygon, and whose diedral angles OA, OB, etc., are equal to the angles A, B, etc., of the spherical polygon ABC, etc.



We may therefore speak of *all* the parts of a spherical polygon as *Angles*, meaning thereby the face-angles, and the diedral angles between the faces, of the polyedral angle whose vertex is the centre of the sphere, and whose base is the spherical polygon.

503. Scholium. Since the sides and angles of a spherical polygon are measured by the face and diedral angles of the polyedral angle corresponding to the polygon, *we may, from any property of polyedral angles, infer an analogous property of spherical polygons.*

504. *Each side of a spherical triangle is less than the sum of the other two.* (384)

505. *Any side of a spherical polygon is less than the sum of the other sides.*

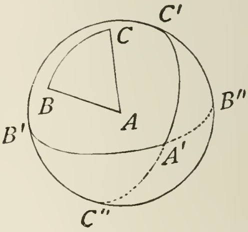
506. *The sum of the sides of a spherical polygon is less than 360°.* (385)

507. *Two mutually equilateral triangles on equal spheres are mutually equiangular, and are equal, or symmetrical and equivalent.*

508. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*

509. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base is perpendicular to the base, and bisects the vertical angle.*

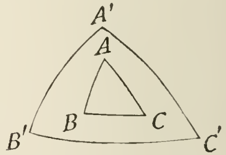
510. If with the vertices of a spherical triangle as poles arcs of great circles are described, a spherical triangle is formed which is called the *Polar Triangle* of the first.



Thus, if A, B, and C are the poles of the arcs $B'C'$, $C'A'$, and $A'B'$, then $A'B'C'$ is the polar triangle of ABC.

Proposition III. Theorem.

511. *If the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.*



Let $A'B'C'$ be the polar triangle of ABC.

To prove that ABC is the polar triangle of $A'B'C'$.

Since B is the pole of the arc $A'C'$, it is a quadrant's distance from A'. Also, since C is the pole of the arc $A'B'$, it is a quadrant's distance from A'.

Therefore A' is a quadrant's distance from B and C, hence from the arc BC, or is the pole of the arc BC.

Similarly, B' can be shown to be the pole of the arc AC, and C' the pole of AB.

Hence ABC is the polar triangle of $A'B'C'$. Q.E.D.

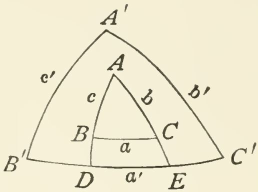
Proposition IV. Theorem.

512. *In two polar triangles, each angle of one is the supplement of the side opposite to it in the other.*

Let ABC and $A'B'C'$ be a pair of polar triangles in which A, B, C, A', B', and C' are the angles, and a, b, c, a', b', and c' the sides.

To prove that

$$A = 180° - a', \qquad A' = 180° - a,$$
$$B = 180° - b', \qquad B' = 180° - b,$$
$$C = 180° - c', \qquad C' = 180° - c.$$



Produce the arc AB until it meets $B'C'$ in D, and AC until it meet $B'C'$ in E.

Since B' is the pole of AC, it will be a quadrant's distance from E, or $B'E = 90°$; likewise, $C'D = 90°$.

Hence $$B'E + C'D = 180°,$$

or $$B'D + DE + C'D = 180°;$$

that is, $$B'C' + DE = 180°.$$

But (by 498) DE is the measure of $\angle A$,

therefore $$\angle A + a' = 180°,$$

or $$\angle A = 180° - a'.$$

The other relations may be proved in a similar manner. Q.E.D.

513. Scholium. Two spherical polygons are mutually equilateral or mutually equiangular when the sides or angles of one are equal respectively to the sides or angles of the other, whether taken in the same or in the reverse order.

514. Cor. *If two spherical triangles are mutually equiangular, their polar triangles are mutually equilateral.*

Since (by 512), any two homologous sides in the polar triangles are supplements of equal angles in the original triangles, hence they are equal.

515. If two spherical triangles are mutually equilateral, their polar triangles are mutually equiangular.

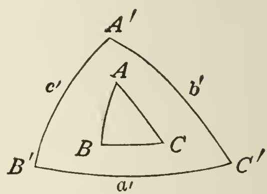
PROPOSITION V. THEOREM.

516. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

Let ABC be any spherical triangle.

To prove that

$$A+B+C>180°<540°.$$



Let $A'B'C'$ be the polar triangle, then (by 512)

$A+a'=180°, B+b'=180°, C+c'=180°,$

or $$A+B+C+a'+b'+c'=540°.$$

But (by 506)

$$a'+b'+c'<360°, \text{ and } a'+b'+c'>0.$$

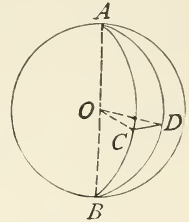
Therefore, by subtraction,

$$A+B+C>180°<540°. \qquad \text{Q.E.D.}$$

517. SCHOLIUM. The amount by which the three angles of a spherical triangle exceeds 180° is called the *Spherical Excess.*

518. COR. *A spherical triangle may have two, or even three, right angles; also two, or even three, obtuse angles.*

519. If a spherical triangle has two right angles, it is called a *Bi-rectangular Triangle;* and if a spherical triangle has three right angles, it is called a *Tri-rectangular Triangle.* Its surface is one-eighth part of the surface of the sphere.



520. A *Lune* is a portion of the surface of a sphere included between two semi-circumferences of great circles; as *ACBD.*

The *Angle* of a lune is the angle between the semi-circumferences which form its sides; as the angle *CAD*, or the angle *COD.*

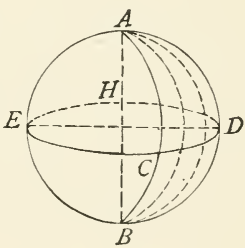
521. On the same, or on equal, spheres, lunes of equal angles are equal, as they are evidently superposable.

522. A *Spherical Wedge*, or *Ungula*, is the part of a sphere bounded by a lune and the planes of its sides; as *AOBCD.*

The diameter *AB* is called the *Edge* of the ungula, and the lune *ACBD* is called its *Base.*

Proposition VI. Theorem.

523. *The area of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*



Let *ACBD* be a lune, and *ECDH* the great circle whose poles are *A* and *B*; let *L* be the area of the lune, and *S* the surface of the sphere, and *A* the angle *CAD*, or the angle of the lune.

To prove that

$$\frac{L}{S}=\frac{A}{360°}, \text{ or } \frac{L}{S}=\frac{\text{arc } CD}{ECDH},$$

since angle *A* is measured by arc *CD*, and *ECDH* is the circumference. Apply a common measure to *CD* and *ECDH*, and suppose it is contained *n* times in *CD* and *m* times in *ECDH*, then $\dfrac{CD}{ECDH}=\dfrac{n}{m}\cdot$

Through these points of division pass great circles; then the lune $ACBD$ will contain n equal lunes, and the entire sphere m equal lunes,

or $$\frac{L}{S}=\frac{n}{m}.$$

Therefore $$\frac{L}{S}=\frac{\text{arc } CD}{ECDH}=\frac{A}{360^\circ}.$$ Q.E.D.

The student can supply the proof for the case when CD and $ECDH$ are incommensurable.

524. Cor. Let A denote the numerical measure of the angle of a lune referred to a right angle as the unit, and T the area of the tri-rectangular triangle.

Then since the surface of the sphere is expressed by $8\,T$, we have (by 523)

$$\frac{L}{8\,T}=\frac{A}{4}, \text{ or } L=2\,A\times T.$$

That is, *if the unit of measurement for angles is the right angle, the area of a lune is equal to twice its angle multiplied by the area of the tri-rectangular triangle.*

For example, if $A=60^\circ=\frac{2}{3}$ of a right angle, its area would be $\frac{4}{3}$ of the area of the tri-rectangular triangle. Then if the surface of the sphere were 120 square inches, the area of the tri-rectangular triangle is 15 square inches, $\frac{4}{3}$ of which is 20 square inches or the area of the lune.

EXERCISES.

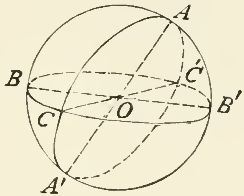
1. Show that in a spherical triangle, each side is greater than the difference between the other two.

2. If the radius of the sphere is 12, what is the linear length of the sides of the triangle whose angular measures are 40°, 60°, and 80°?

3. Find the area of a lune when the angle is 135°, and the surface of the sphere 300 square inches.

PROPOSITION VII. THEOREM.

525. *If two arcs of great circles BAB' and CAC' intersect each other on the surface of a hemisphere, the sum of the opposite triangles ABC and $AB'C'$ is equivalent to a lune whose angle is equal to the angle BAC included between the given arcs.*



Draw the diameters AA', BB', CC'.

Since $A'BA$ is a semicircle, it is equal to BAB'; subtract the portion BA, and we have arc $A'B = AB'$; likewise arc $A'C = AC'$, and $BC = B'C'$, both being measures of the equal vertical angles.

Therefore $\qquad A'BC = AB'C'$.

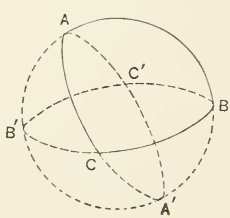
Adding BAC, we have

$$A'BC + BAC = BAC + AB'C',$$

or $\qquad$ lune $ABA'C = BAC + AB'C'$. Q.E.D.

PROPOSITION VIII. THEOREM.

526. *The area of a spherical triangle is equal to its spherical excess.*



Let A, B, C be the numerical measures of the angles of the spherical triangle ABC; let the right angle be the unit of angular measure, and the tri-rectangular triangle T be the unit of areas.

To prove that

$$\text{Area } ABC = (A + B + C - 2) \times T.$$

Continue any side, say AB, so as to complete the great circle, and produce the other sides until they meet this circle in B' and A'.

$$\text{Area } ABC + A'BC = \text{lune } ABA'C = 2\,A \times T;$$

likewise $\quad ABC + AB'C = \text{lune } ABCB' = 2\,B \times T,$

and $\quad ABC + A'B'C = \text{lune } ACBC' = 2\,C \times T.$

By addition,

$$3\,ABC + A'BC + AB'C + A'B'C = (2\,A + 2\,B + 2\,C) \times T.$$

But $ABC + A'BC + AB'C + A'B'C =$ the hemisphere $= 4\,T.$

Therefore $\quad 2\,ABC + 4\,T = (2\,A + 2\,B + 2\,C) \times T,$

$$ABC + 2\,T = (A + B + C) \times T,$$

or $\quad ABC = (A + B + C - 2) \times T.$ Q.E.D.

The greater the excess of $A + B + C$ over 2 right angles, the greater will be the area.

527. Cor. *The area of a spherical polygon is equal to its spherical excess.*

THE SPHERE.

528. A *Zone* is a portion of the surface of a sphere included between parallel planes.

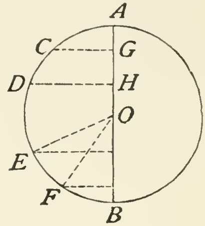
The circumference of the circles which bound the zone are called the *Bases*, and the distance between their planes the *Altitude*.

One of the bases may be a tangent plane.

529. A *Spherical Segment* is a portion of the volume of the sphere included between two parallel planes; the planes are the *Bases*, and their distance apart is the *Altitude*.

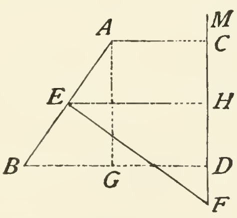
529 *a*. A *Spherical Sector* is the portion of a sphere generated by the revolution of a circular sector about the diameter of its circle as an axis.

530. Let the sphere be generated by the revolution of the semicircle $ACDEFB$ about its diameter AB as an axis; and let CG and DH be drawn perpendicular to the axis. The arc CD generates a zone whose altitude is GH, and the figure $CDHG$ generates a spherical segment whose altitude is GH. The circumferences generated by the points C and D are the bases of the zone, and the circles generated by CG and DH are the bases of the segment.



PROPOSITION IX. THEOREM.

531. *The area generated by the revolution of a straight line about an axis in its plane is equal to the projection of the line on the axis, multiplied by the circumference of a circle whose radius is the length of the perpendicular erected at the middle point of the line and terminating in the axis.*



Let AB be the straight line revolving about the axis FM, CD its projection on FM, and EF the perpendicular erected at the middle point of AB, terminating in the axis.

To prove that

$$\text{area generated by } AB = CD \times 2\pi \times EF.$$

Draw AG parallel to CD, and EH perpendicular to CD.

The area generated by AB is the lateral surface of a frustum of a cone of revolution, with AC and BD as radii of the upper and lower bases.

Therefore (by 461)

$$\text{area } AB = AB \times 2\pi \times EH.$$

The triangles are (by 64 and 218) similar, hence

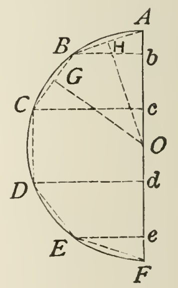
$$\frac{AB}{AG} = \frac{EF}{EH}, \text{ or } AB \times EH = AG \times EF = CD \times EF.$$

Substituting this value for $AB \times EH$, we have

$$\text{area } AB = CD \times 2\pi \times EF. \qquad \text{Q.E.D}$$

Proposition X. Theorem.

532. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*



Let the sphere be generated by the revolution of the semicircle $ABDF$ about the diameter AF, let O be the centre, R the radius, and denote the surface of the sphere by S.

To prove $S = AF \times 2\pi R.$

Inscribe in the semicircle a regular semi-polygon $ABCDEF$ of any number of sides.

Draw Bb, Cc, Dd, etc., perpendicular to AF, and OH perpendicular to BA; then (from 146) AB is bisected in H.

Then (from 531) area $AB = Ab \times 2\pi \times OH.$

Likewise area $BC = bc \times 2\pi \times OG$, etc.

But (by 148) $OG = OH.$

Therefore area generated by $ABC = Ac \times 2\pi \times OH.$

Now the sum of the projections of the sides of the semi-polygon make up the diameter AF, hence the

$$\text{area generated by } ABCDEF = AF \times 2\pi \times OH.$$

Now let the number of sides of the inscribed semi-polygon be indefinitely increased.

The semi-perimeter will approach the semi-circumference as its limit, and OH will approach the radius R as its limit.

Therefore the surface of revolution will approach the surface of the sphere as its limit; hence

$$S = AF \times 2\,\pi \times R.$$ Q.E.D.

533. Cor. 1. Since $AF = 2\,R$,

$$S = 2\,R \times 2\,\pi R = 4\,\pi R^2.$$

Therefore, *the area of the surface of a sphere is equal to the area of four great circles of that sphere.*

534. Cor. 2. *The areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

535. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

Proposition XI. Theorem.

536. *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*

Let V denote the volume of a sphere, S the area of its surface, and R its radius.

To prove that $V = S \times \frac{1}{3}\,R.$

Conceive any polyedron as circumscribed about the sphere; then if the number of faces be indefinitely increased, each face will be diminished, and the limit of the surface of the polyedron is the surface of the sphere, and the limit of the volume of the polyedron is the volume of the sphere.

Let each vertex of the polyedron be joined to the centre of the sphere, then the entire polyedron will be made up of pyramids.

The volume of the entire polyedron is equal to the sum of the volumes of the pyramids; that is, the sum of the bases multiplied by one-third of the common altitude, or the radius of the sphere.

Since this is true whatever the number of faces may be, the limiting volume will be equal to the limiting surface multiplied by one-third of the radius, or

$$V = S \times \tfrac{1}{3} R.$$

537. Cor. 1. Since (by 533)

$$S = 4\,\pi R^2,$$

$$V = \tfrac{4}{3}\,\pi R^3,$$

and $$V = \tfrac{1}{6}\,\pi D^3.$$

538. Cor. 2. *The volumes of two spheres are to each other as the cubes of their radii.*

539. Cor. 3. *The volume of a spherical sector is equal to the area of the zone which forms its base multiplied by one-third the radius of the sphere.*

For a spherical sector, like the entire sphere, may be conceived as consisting of an indefinitely great number of pyramids whose bases make up its base, and whose common altitude is the radius of the sphere.

540. Cor. 4. The volume of the cylinder circumscribed about a sphere $= 2\,\pi R^3$.

Therefore, *the volume of a sphere is equal to two-thirds the volume of the circumscribing cylinder.*

EXERCISES.

1. Find the surface and volume of a sphere whose radius is 12.
2. Find the diameter and surface of a sphere whose volume is 896.

3. If the radii of the bases of a spherical segment are 4 and 6 and the altitude is 5, find the radius of the sphere.

4. Find the number of cubic feet in a log 18 feet long and $6\frac{1}{2}$ feet in diameter.

5. Find the number of gallons in a cistern 6 feet in diameter and 10 feet deep, if 231 cu. in. make a gallon.

6. Find the weight of an iron shell 4 inches in diameter, the iron being $1\frac{1}{2}$ in. thick, and weighing $\frac{1}{4}$ of a pound to the cubic inch.

7. Show that the surface of a sphere is equal to the lateral surface of its circumscribing cylinder.

541. FORMULÆ.

Notation.

$S =$ surface (or area).	$r =$ radius of inscribed circle.
$V =$ volume.	$R =$ radius of circle (general).
$h =$ altitude.	$R' =$ radius of upper base.
$b =$ lower base (linear).	$D =$ diameter.
$b' =$ upper base (linear).	$L =$ slant height.
$s = \frac{1}{2}(a + b + c)$.	$C =$ circumference.
$P =$ perimeter.	$B =$ area of base.
$P' =$ perimeter of upper base.	$B' =$ area of upper base.

Parallelogram;	$S = h \times b.$	(251)
Triangle;	$S = \frac{1}{2} h \times b.$	(254)
	$= \sqrt{s(s-a)(s-b)(s-c)}.$	(271)
Trapezoid;	$S = \frac{1}{2} h [b + b'].$	(258)
Polygon;	$S = \frac{1}{2} P \times r.$	(292)
Circle;	$C = 2\pi \times R = \pi \times D.$	(303)
	$S = \pi \times R^2.$	(305)
Sector;	$S = \frac{1}{2} \text{arc} \times R.$	(306)

	$\pi = 3.141592635 \cdots$	(310)
	$\log \pi = 0.4971498726 \cdots$	
Right Prism;	lateral $S = P \times h.$	(416)
	$V = B \times h.$	(428)
Cylinder;	lateral $S = 2\pi \times R \times h.$	(417)
	$V = \pi \times R^2 \times h.$	(433)
Parallelopiped;	$V = B \times h.$	(425)
Pyramid;	lateral $S = \frac{1}{2} P \times L.$	(458)
	$V = \frac{1}{3} B \times h.$	(465)
Cone;	lateral $S = \pi \times R \times L.$	(460)
	$V = \frac{1}{3} \pi \times R^2 \times h.$	(466)
Frustum of a pyramid;	lateral $S = \frac{1}{2}(P + P') \times L.$	(459)
	$V = \frac{1}{3} h[B + B' + \sqrt{B \times B'}].$	(469)
Frustum of a cone;	lateral $S = \pi(R + R') \times L.$	(461)
	$V = \frac{1}{3}\pi \times h[R^2 + R'^2 + R \times R'].$	(471)
Sphere;	$S = 4\pi \times R^2.$	(533)
	$V = \frac{1}{6}\pi \times D^3.$	(537)
Zone;	$S = 2\pi \times R \times h.$	(535)

APPENDIX.

ADDITIONAL EXERCISES ON BOOK I.

1. Each exterior angle of an equilateral triangle equals how many times each interior angle?

2. From a point without a line, show that only two oblique lines can be drawn so as to make equal angles with the given line.

3. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected, show that the bisectors meet at right angles.

4. How many sides has a polygon, the sum of whose interior angles is equal to the sum of its exterior angles?

5. The sum of the three medial lines of a triangle is less than the perimeter, and greater than half the perimeter of the triangle.

6. How many sides has a polygon, the sum of whose interior angles is double that of its exterior angles?

7. If BC, the base of an isosceles triangle ABC, is produced to any point D, show that AD is greater than either of the equal sides.

8. Prove that any point not in the bisector of an angle is unequally distant from its sides.

9. The diagonals of a rhombus are perpendicular to each other and bisect the angles of the rhombus.

10. The angles a, b, c, d are such that $a + b + c + d =$ a straight angle, and $a = 2b = 4c = 8d$. How many degrees in a, b, c, d?

11. If an angle at the base of an isosceles triangle is n times the vertical angle, what fraction is the latter of a straight angle?

12. The straight line AE which bisects the angle exterior to the vertical angle of an isosceles triangle ABC, is parallel to the base BC.

13. The lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.

14. If both diagonals of a parallelogram are drawn, of the four triangles thus formed those opposite are equal.

15. In a triangle ABC, if AC is not greater than AB, show that any straight line drawn through the vertex A and terminated by the base BC is less than AB.

16. The lines joining the middle points of the sides of a rhombus, taken in order, include a rectangle.

17. The lines joining the middle points of the sides of any quadrilateral, taken in order, enclose a parallelogram.

18. In any right triangle, the straight line drawn from the vertex to the middle point of the hypotenuse is equal to half the hypotenuse.

19. If a diagonal of a quadrilateral bisects two angles, the quadrilateral has two pairs of equal sides.

20. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shorter side.

21. The perimeter of a quadrilateral is less than twice the sum of its two diagonals.

22. If, in a quadrilateral, two opposite sides are equal, and the two angles which a third side makes with the equal sides are equal, then the other two angles are equal also.

23. The straight lines joining the middle points of the opposite sides of any quadrilateral bisect each other.

24. Any two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the third side.

25. If ABC is an equilateral triangle, and if BD and CD bisect the angles B and C, the lines DE, DF, parallel to AB, AC, respectively, divide BC into three equal parts.

26. If from a *variable point* in the base of an isosceles triangle parallels to the sides are drawn, a parallelogram is formed whose perimeter is *constant*.

27. The diagonals of a square or rhombus bisect each other at right angles, and bisect the angles whose vertices they join.

28. If BE bisects the angle B of a triangle ABC, and CE bisects the exterior angle ACD, the angle E is equal to one-half the angle A.

29. The sum of the perpendiculars dropped from any point within an equilateral triangle to any one of the three sides is constant, and equal to the altitude.

30. The median to any side of a triangle is less than the half-sum of the other two sides, but greater than half of the difference between their sum and the third side.

31. If the bisectors of two angles of an equilateral triangle meet, and from the point of meeting lines be drawn parallel to any two sides, these lines will trisect the third side.

32. The sum of four lines drawn to the vertices of a quadrilateral from any point except the intersection of the diagonals, is greater than the sum of the diagonals.

33. In a quadrilateral, the sum of either pair of opposite sides is less than the sum of its two diagonals.

34. The interior angle of a regular polygon is five-thirds of a right angle. Find the number of sides in the polygon.

35. The exterior angle of a regular polygon is one-fifth of a right angle. Find the number of sides in the polygon.

36. If one side of a regular hexagon is produced, show that the exterior angle is equal to the angle of an equilateral triangle.

37. How many braces would it take to stiffen a three-sided plane figure? Four-sided? Five-sided?

38. If from any point equidistant from two parallels two transversals are drawn, they will cut off equal segments of the parallels.

39. If a quadrilateral have two of its opposite sides parallel, and the other two equal but not parallel, any two of its opposite angles are together equal to two right angles.

40. The sum of the perpendiculars from any point in the interior of an equilateral triangle is equal to the distance of any vertex from the opposite side.

41. A line is drawn terminated by two parallel lines; through its middle point any line is drawn and terminated by the parallel lines. Show that the second line is bisected at the middle point of the first.

ADDITIONAL EXERCISES ON BOOK II.

1. If an isosceles triangle be constructed on any chord of a circle, its vertex will be in a diameter, or a diameter produced.

2. The perimeter of an inscribed equilateral triangle is equal to half the perimeter of the circumscribed equilateral triangle.

3. If two equal chords intersect, their segments are severally equal.

4. The perpendiculars from the angles upon the opposite sides of the triangle are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.

5. A straight line will cut a circle, or lie entirely without it, according as its distance from the centre is less than, or greater than, the radius of the circle.

6. The bisectors of the angle contained by the opposite sides (produced) of an inscribed quadrilateral, intersect at right angles.

7. A, B, and C are three points on the circumference of a circle, the bisectors of the angles A, B, and C meet at D, and AD produced meets the circle in E; prove that $ED = EC$.

8. If a variable tangent meets two parallel tangents it subtends a right angle at the centre.

9. If through any point in a radius two chords are drawn, making equal oblique angles with it, these chords are equal.

10. Two circles are tangent internally at P, and a chord AB of the larger circle touches the smaller at C. Prove that PC bisects the angle APB.

11. All chords of a circle which touch an interior concentric circle are equal, and are bisected at the points of contact.

12. If a straight line cuts two concentric circles, the parts of it intercepted between the two circumferences are equal.

13. The angle formed by two tangents drawn to a circle from the same point, is supplementary to that formed by the radii to the points of contact.

14. In two concentric circles any chord of the outer circle, which touches the inner, is bisected at the point of contact.

15. If through the points of intersection of two circumferences, parallels be drawn to meet the circumferences, these parallels will be equal.

16. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the straight line joining the ends of the diameters passes through the other point of intersection.

17. If a circle is inscribed in a trapezoid that has equal angles at the base, each nonparallel side is equal to half the sum of the parallel sides.

18. If two circles touch externally at P, the straight line joining the extremities of two parallel diameters, towards opposite parts, passes through P.

19. OC is drawn from the centre O of a circle perpendicular to a chord AB. Prove that the tangents at A, B, intersect in OC produced.

20. Prove that two of the straight lines which join the ends of two equal chords are parallel, and the other two are equal.

21. If two pairs of opposite sides of a hexagon inscribed in a circle are parallel, the third pair of opposite sides are parallel.

22. To construct a triangle having given the two exterior angles and the included side.

23. To divide a right angle into three equal parts.

24. Construct an isosceles triangle having its sides each double the length of the base.

25. Through a given point to draw a line making a given angle with a given line.

26. Construct a right triangle, having given an arm and the altitude from the right angle upon the hypotenuse.

27. To draw a line through a given point, so that it shall form with the sides of a given angle an isosceles triangle.

28. From a given point in a given line to draw a line making an angle supplemental to a given angle.

29. Through a given point P within a given angle to draw a straight line, terminated by the sides of the angle, which shall be bisected at P.

30. Through a given point to draw a line which shall make equal angles with the two sides of a given angle.

31. Construct an equilateral triangle having a given altitude AB.

32. To construct an isosceles triangle, having given the base and the opposite angle.

33. On a given straight line as hypotenuse, construct a right triangle having one of its acute angles double the other.

34. The bisectors of the angles of a circumscribed quadrilateral pass through a common point.

35. Draw a straight line equally distant from three given points.

36. Find the bisector of the angle that would be formed by two given lines, without producing the lines.

37. In any side of a triangle, find the point which is equidistant from the other two sides.

38. Through two given points to draw the two lines forming, with a given line, an equilateral triangle.

39. From two given points to draw lines making equal angles with a given line, the points being on (1) the same side of the given line, (2) opposite sides of the given line.

40. In any side of a triangle, find the point from which the lines drawn parallel to the other two sides are equal.

41. To draw a tangent to a given circle so that it shall be parallel to a given straight line.

42. With a given point as centre, describe a circle which shall be divided by a given straight line into segments containing given angles.

43. To describe a circumference passing through a given point, and touching a given line, or a given circle, in a given point.

44. To draw a tangent to a given circle, perpendicular to a given line.

45. Draw a line parallel to a given line, so that the segment intercepted between two other given lines may equal a given segment.

46. To draw a tangent to a given circle which shall be parallel to a given straight line.

47. Describe a circle of given radius to touch two given lines. Show that the solution is, in general, impossible if the lines are parallel, but that otherwise there are four solutions.

ADDITIONAL EXERCISES ON BOOK III.

1. Any parallelogram that can be circumscribed about a circle must be equilateral.

2. Any parallelogram that can be inscribed in a circle will have the intersection of its diagonals at the centre of the circle.

3. The bisector of an angle formed by a tangent and a chord bisects the intercepted arc.

4. Give the construction for cutting off two-sevenths of a given straight line.

5. Any two altitudes of a triangle are inversely proportional to the corresponding bases.

6. In any isosceles triangle, the square of one of the equal sides is equal to the square of any straight line drawn from the vertex to the base plus the product of the segments of the base.

7. The difference of the squares of two sides of any triangle is equal to the difference of the squares of the projections of these sides on the third side.

8. If any two chords cut within the circle, at right angles, the sum of the squares on their segments equals the square on the diameter.

9. If a straight line AB is divided at C and D so that $AB \times AD = \overline{AC}^2$, and if from A any straight line AE is drawn equal to AC, then EC bisects the angle DEB.

10. The intersection of the diagonals of an equiangular quadrilateral is the centre of the circumscribed circle.

11. If two circles intersect in P, and the tangents at P to the two circles meet the circles again at Q and R, prove that $PQ : PR$ in the same ratio of the radii of the circles.

12. The tangents to two intersecting circles drawn from any point in their common chord produced, are equal.

13. If any two circles touch each other, either internally or externally, any two straight lines drawn from the point of contact will be cut proportionally by the circumferences.

14. If the diagonals of an inscribed quadrilateral bisect each other, what kind of a quadrilateral is it?

15. The diagonals of a trapezoid cut each other in the same ratio.

16. Describe a circumference passing through two given points and having its centre in a given straight line. When is this impossible?

17. If two circles touch each other, secants drawn through their point of contact and terminating in the two circumferences are divided proportionately at the point of contact.

18. The intersection of the diagonals of an equilateral quadrilateral is the centre of the inscribed circle.

19. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionately by the circumference of the smaller circle.

20. The bisectors of any angle of an inscribed quadrilateral, and the opposite exterior angle, meet on the circumference.

21. $ABCD$ is a quadrilateral having two of its sides, AB, CD, parallel. AF, CG are drawn parallel to each other, meeting BC, AD respectively in F, G. Prove that BG is parallel to DF.

22. If the line of centres of two circles meets the circumferences at the points A, B, C, D, and meets the common exterior tangent at P, then $PA \times PD = PB \times PC$.

23. Find a point equidistant from three given points. When is the problem impossible?

24. A tree casts a shadow 90 ft. long, when a vertical rod 6 ft. high casts a shadow 5 ft. long. How high is the tree?

25. The sides of a triangle are 5, 6, 7. In a similar triangle the side homologous to 7 is equal to 35. Find the other two sides.

26. Two circles touch in C, a point D is taken outside them such that the radii, AC, BC, subtend equal angles at D, and DE, DF are tangents to the circles. If EF cuts DG in G, prove that $DE : DF = EG : GF$.

27. The bisectors of the angles formed by producing the opposite sides of an inscribed quadrilateral to meet, are perpendicular to each other.

28. How long must a ladder be to reach a window 24 ft. high, if the lower end of the ladder is 10 ft. from the side of the house?

29. If, in a right triangle, the altitude upon the hypotenuse divides it in extreme and mean ratio, the lesser arm is equal to the farther segment.

30. The base of a triangle is given and is bisected by the centre of a given circle. If the vertex be at any point of the circumference, show that the sum of the squares on the two sides of the triangle is constant.

31. The altitudes of a triangle are inversely proportional to the sides upon which they are drawn.

32. If the diagonals of an inscribed quadrilateral are perpendicular to each other, the line through their intersection perpendicular to any side bisects the opposite side.

33. Through a given point between two given straight lines, draw a straight line which shall be terminated by the given lines and divided by the point in a given ratio.

34. The distance from the centre of a circle to a chord 10 in. long is 12 in. Find the distance from the centre to a chord 24 in. long.

35. The square on the base of an isosceles triangle is equal to twice the product of either side by the part of that side intercepted between the perpendicular let fall on the side from the opposite angle and the end of the base.

36. If two circles are tangent externally, a common exterior tangent is a mean proportional between their diameters.

37. The radius of a circle is 6 in. Through a point 10 in. from the centre tangents are drawn. Find the lengths of the tangents, and also of the chord joining the points of contact.

38. Upon a given portion AC of the diameter AB of a semicircle another semicircle is described. Draw a line through A so that the part intercepted between the semicircles may be of given length.

39. If three circles intersect each other, their three common chords pass through the same point.

40. Inscribe a square in a given segment of a circle.

41. From the end of a tangent 20 in. long a secant is drawn through the centre of the circle. If the exterior segment of this secant is 10 in., find the radius of the circle.

42. From a given point without a circle draw a secant divided by the circumference into a given ratio.

43. Divide any side of a triangle into two parts proportional to the other sides.

44. If a perpendicular is let fall from any point on the circumference, to any diameter, it is the mean proportional between the segments into which it divides that diameter.

45. In a chord produced, find a point such that the tangents drawn from it to the circle shall be equal to a given line.

46. The sides of a triangle are 4, 5, 6. Is the largest angle acute, right, or obtuse ?

47. From a given point on the circumference of a given circle draw two chords so as to be in a given ratio and to contain a given angle.

ADDITIONAL EXERCISES ON BOOK IV.

1. If two triangles are on equal bases and between the same parallels, then any line parallel to their bases, and cutting the triangles, will cut off equal triangles.

2. If the middle points of two adjacent sides of a parallelogram are joined, a triangle is formed which is equivalent to one-eighth of the entire parallelogram.

3. Two equilateral triangles have their areas in the ratio of 1:2. Find the ratio of their sides to the nearest 0.01.

4. The sides of a triangle are 9, 11, 14 inches. Is the triangle right-angled? obtuse-angled?

5. The square of the base of an isosceles triangle is equivalent to twice the rectangle contained by either of the arms and the projection of the base upon that side.

6. The perimeter of a rectangle is 144 ft., and the length is three times the altitude; find the area.

7. The area of a triangle is equal to the product of its three sides divided by four times the radius of the circumscribed circle; that is, denoting this radius by R,

$$S = \frac{abc}{4R}.$$

8. The area of a trapezoid is equal to the product of one of the legs and the distance from this leg to the middle point of the other leg.

9. Any quadrilateral is divided by its interior diagonals into four triangles which form a proportion.

10. The sides of a triangle are 4, 11, 13 units long. Is the angle opposite 13 right? obtuse? acute?

11. Three times the sum of the squares of the sides of a triangle is equal to four times the sums of the squares of the medians.

12. On a given straight line construct a triangle equal to a given triangle and having its vertex in a given straight line not parallel to the base.

13. What part of a parallelogram is the triangle cut off by a line drawn from one vertex to the middle point of one of the opposite sides?

14. If one angle of a triangle is one-third of a straight angle, show that the square on the opposite side equals the sum of the squares on the other two sides less their rectangle.

15. If any point within a parallelogram be joined with the vertices, the sums of the opposite pairs of triangles are equivalent.

16. Construct a parallelogram that shall be equal in area and perimeter to a given triangle.

17. If the side of one equilateral triangle is equal to the altitude of another, what is the ratio of their areas?

18. If two fixed parallel tangents are cut by a variable tangent, the rectangle of the segments of the latter is constant.

19. Of the four triangles formed by drawing the diagonals of a trapezoid, (1) those having as bases the non-parallel sides are equivalent; (2) those having as bases the parallel sides are as the squares of those sides.

20. A triangle is divided by each of its medians into two parts of equal area.

21. If two triangles have a common angle and equal areas, the sides containing the common angle are inversely proportional.

22. In AC, a diagonal of the parallelogram $ABCD$, any point H is taken, and HB, HD are drawn; show that the triangle BAH is equal in area to the triangle DAH.

23. The sum of the squares of the four segments of any two chords that intersect at right angles is constant.

24. If any point in one side of a triangle be joined to the middle points of the other sides, the area of the quadrilateral thus formed is one-half that of the triangle.

25. Find the ratio of a rectangle 18 yds. by $14\frac{1}{2}$ yds. to a square whose perimeter is 100 ft.

26. In a trapezoid the straight lines, drawn from the middle points of one of the non-parallel sides to the ends of the opposite sides, form with that side a triangle equal to half the trapezoid.

27. Show that the line joining the middle points of the two parallel sides of a trapezoid divides the area into two equal parts.

28. To transform a parallelogram into a parallelogram having one side equal to a given length.

29. Construct an isosceles triangle on the same base as a given triangle, and equivalent to it.

30. Construct a parallelogram having a given angle upon the same base as a given square, and equivalent to it.

31. To construct a triangle equal to a given parallelogram, and having one of its angles equal to a given angle.

32. Construct an isosceles triangle equal in area to a given triangle and having a given vertical angle.

33. To find a point within a triangle, such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.

34. Divide a given line into two segments, such that their squares shall be as $8:5$.

35. On a given line to construct a rectangle equal to a given rectangle.

36. With a given altitude to construct an isosceles triangle equal to a given triangle.

37. Divide a straight line into two parts, such that the sum of the squares on the parts may be equal to a given square.

38. To divide a given triangle into two equivalent parts by drawing a line perpendicular to one of the sides.

39. To construct a triangle, given its angles and its area.

40. Bisect a triangle by a line parallel to the base.

41. On the base of a given triangle to construct a rectangle equal to the given triangle.

42. Construct a square that shall be one third of a given square.

43. Given any triangle, to construct an isosceles triangle of the same area whose vertical angle is an angle of the given triangle.

44. To construct a square equal to half the sum of two given squares.

45. Construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.

46. Construct a triangle equivalent to a given triangle, and having one side equal to a given line.

47. Construct a triangle similar to a given triangle ABC which shall be to ABC in the ratio of AB to BC.

48. Construct a triangle equal to a given triangle and having one of its angles equal to an angle of the triangle, and the sides containing this angle in a given ratio.

49. Construct a square that shall be to a given triangle as 7 is to 6.

50. Bisect a triangle by a straight line drawn through a given point in one of its sides.

ADDITIONAL EXERCISES ON BOOK V.

1. What is the radius of the circle circumscribing the triangle whose sides are 3, 4, 5?

2. The area of the regular inscribed hexagon is half the area of the circumscribed equilateral triangle.

3. The apothem of a regular pentagon is 6 and a side is 4; find the perimeter and area of a regular pentagon whose apothem is 8.

4. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and the circumscribed equilateral triangles.

5. The area of an inscribed regular octagon is equal to that of a rectangle whose sides are equal to the sides of the inscribed and the circumscribed squares.

6. If the diagonals of an inscribed quadrilateral are perpendicular to each other, then the sum of the products of the two opposite sides equals twice the area of the quadrilateral.

7. The apothem of an inscribed equilateral triangle is equal to half the radius of the circle.

8. The radius of a circle is 8; find the apothem, perimeter, and area of the inscribed equilateral triangle.

9. If $a =$ the side of a regular pentagon inscribed in a circle whose radius is R, then,

$$a = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}.$$

10. If $a =$ the side of a regular octagon inscribed in a circle whose radius is R, then,

$$a = R\sqrt{2 - \sqrt{2}}.$$

11. Upon the six sides of a regular hexagon squares are constructed outwardly. Prove that the exterior vertices of these squares are the vertices of a regular dodecagon.

12. The area of an inscribed equilateral triangle is half that of a regular hexagon inscribed in the same circle.

13. If $a =$ the side of a regular dodecagon inscribed in a circle whose radius is R, then,

$$a = R\sqrt{2-\sqrt{3}}.$$

14. The radius of a circle is ten: find the perimeter and area of the regular inscribed octagon.

The radius of a circle is 4; find the area of the inscribed square.

15. The radius of an inscribed regular polygon is the mean proportional between its apothem and the radius of the similar circumscribed polygon.

16. What is the radius of that circle of which the number of square units of area equals the number of linear units of circumference?

17. The altitude of an equilateral triangle is to the radius of the circumscribing circle as 3 is to 2.

18. If $a =$ the side of a regular pentedecagon inscribed in a circle whose radius is R, then,

$$a = \frac{R}{4}\left(\sqrt{10+2\sqrt{5}}+\sqrt{3}-\sqrt{15}\right).$$

19. The chord of an arc is 24 in., and the height of the arc is 9 in. Find the diameter of the circle.

20. What is the radius of that circle of which the number of square units of area equals the number of linear units of radius?

21. The square inscribed in a semicircle is to that inscribed in a circle as 2 is to 5.

22. The area of the regular inscribed hexagon is equal to twice the area of the regular inscribed triangle.

23. The diagonals drawn from the vertex of a regular pentagon to the opposite vertices trisect that angle.

24. If $a =$ the side of a regular polygon in a circle whose radius is R, and $A =$ the side of the similar circumscribed polygon, then,

$$A = \frac{2\,aR}{\sqrt{(4\,R^2-a^2)}}, \quad a = \frac{2\,aR}{\sqrt{(4\,R^2+A^2)}}.$$

25. Find the area of a sector, if the radius of the circle is 28 ft., and the angle at the centre 45°.

26. Find the areas of circles with radii 5, 8, 21, 33, 47, 52. (In these computations, let $\pi = 3.1416$.)

27. The intersecting diagonals of a regular pentagon divide each other in extreme and mean ratio.

28. If $d =$ the diagonal of a regular pentagon inscribed in a circle whose radius is R, then,

$$d = \frac{R}{2}\sqrt{10 + 2\sqrt{5}}.$$

29. The square of the side of the inscribed equilateral triangle is three times the square of a side of the regular inscribed hexagon.

30. The perpendiculars from two vertices of a triangle upon the opposite sides divide each other into segments reciprocally proportional.

31. Find the areas of circles with diameters 2, 8, 11, 31, 42, 97.

32. Inscribe an equilateral triangle in a given square, so as to have a vertex of the triangle at a vertex of the square.

33. The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.

34. The Egyptians said: "Construct a square the side of which is $\frac{8}{9}$ of the diameter of a circle, and its area will equal that of the circle." From this compute their value of π.

35. Construct a square that shall be $\frac{2}{3}$ of a given square.

36. The square inscribed in a semicircle is equal to $\frac{2}{5}$ the square inscribed in the whole circle.

37. Lines drawn from one vertex of a parallelogram to the middle points of the opposite sides trisect one of the diagonals.

38. Of all triangles in a given circle, that which has the greatest perimeter is equilateral.

39. Construct a regular hexagon that shall be $\frac{4}{5}$ of a given regular hexagon.

40. The area of a circle is 40 ft.; find the side of the inscribed square.

41. Construct a square equivalent to the sum of a given triangle and a given parallelogram.

42. Find a point in a given straight line such that the tangents drawn from it to a given circle contain the maximum angles.

43. Find the angle subtended at the center of a circle by an arc 6 ft. long, if the radius is 8 ft. long.

44. To construct a triangle, given its angles and its area.

45. Through a point of intersection of two circumferences, draw the maximum line terminated by the two circumferences.

46. Find the length of the arc subtended by one side of a regular octagon inscribed in a circle whose radius is 10 ft.

47. To construct an equilateral triangle having a given area.

48. Of all triangles of a given base and area the isosceles has the greatest vertical angle.

49. Find the area of a circular sector, the chord of half the arc being 10 in. and the radius 25 in.

50. What is the area of the largest triangle that can be inscribed in a circle of radius 10?

51. To construct a triangle, given its base, the ratio of the other sides, and the angle included by them.

52. The radius of a circle is 5 ft. Find the radius of a circle 16 times as large.

53. Every equilateral polygon circumscribed about a circle is equiangular, if the number of sides be odd.

54. Given a square of area 1. Find the area of an isoperimetric (1) equilateral triangle, (2) regular hexagon, (3) circle.

55. Find the height of an arc, the chord of half the arc being 10 ft., and the radius 24 ft.

56. What is the only rectilinear polygon that is necessarily plane? Why?

57. Find the length of the arc subtended by one side of a regular dodecahedron in a circle whose radius is 12.5 ft.

58. Find the area of a segment whose height is 16 in., the radius of the circle being 20 in.

59. Find the area of a segment whose arc is 100, the radius being 24 ft.

60. If AB be a side of an equilateral triangle inscribed in a circle, and AD a side of the inscribed square, prove that three times the square on AD is equal to twice the square on AB.

MISCELLANEOUS QUESTIONS. BOOKS I.–V.

(PLANE GEOMETRY.)

1. The sum of the distances of any point in the base of an isosceles triangle from the equal sides is equal to the distance of either extremity of the base from the opposite side.

2. Prove that the square constructed on the difference of two straight lines is equal to the sum of the squares constructed on the lines, diminished by twice the rectangle of the lines.

3. Any chord of a circle is a mean proportional between its projection on the diameter from any one of the extremities, and the diameter itself.

4. To divide a circle into two segments so that the angle contained in one shall be double that contained in the other.

5. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base.

6. The bisectors of the external angles of a quadrilateral form a circumscribed quadrilateral, the sum of whose opposite angles is equal to two right angles.

7. The angles made with the base of an isosceles triangle by perpendiculars from its extremities on the equal sides are each equal to half the vertical angle.

8. Construct a triangle, having given the base, the vertical angle, and (1) the sum, or (2) the difference of the sides.

9. Given the base, one of the angles at the base, and the difference of the sides of a triangle, to construct the triangle.

10. Given two sides of a triangle and the straight line drawn from the extremity of one of them to the middle point of the other, to construct the triangle.

11. In the triangle ABC, the angle $A = 50^\circ$, the angle $B = 70^\circ$. What angle will the bisectors of these two angles make with each other?

12. How many sides has a polygon, the sum of whose interior angles is four times that of its exterior angles?

13. In a given circle to draw a chord equal and parallel to a given line.

14. From a given isosceles triangle cut off a trapezoid having for base that of the triangle, and having its other three sides equal.

15. Find the number of degrees in the arc whose length is equal to the radius of the circle.

16. A straight line touches a circle at A, and from any point P, in the tangent, PB is drawn meeting the circle at B so that $PB = PA$. Prove that PB touches the circle.

17. If one of the parallel sides of a trapezoid is double the other, the diagonals intersect one another in a point of trisection.

18. Find the side of a square equivalent to a circle whose radius is 40 ft.

19. Find the radius of the circle whose sector of 45° is .125 sq. in.

20. A circle is described passing through the ends of the base of a given triangle; prove that the straight line joining the points, in which it meets the sides or the sides produced, is parallel to a fixed straight line.

21. Two circles touch externally at A; the tangent at B to one of them cuts the other in C, D; prove that BC and BD subtend supplementary angles at A.

22. C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA; join AB and produce it to meet the circle again at D, and let the tangent at D meet CB produced at E: show that BDE is an isosceles triangle.

23. What is the width of a ring between two concentric circumferences whose lengths are 160 ft. and 80 ft.?

24. The circumference of a circle is 78.54 in.; find (1) its diameter, and (2) its area.

25. Find the point inside a given triangle at which the sides subtend equal angles.

26. The figure formed by the five diagonals of a regular pentagon is a regular pentagon.

27. With a given radius, describe a circle touching two given circles.

28. Describe a circumference passing through a given point and touching a given line in a given point.

29. Through one of the points of intersection of two given circles draw a secant forming chords that are in a given ratio.

30. Describe a circumference touching two parallel lines and passing through a given point.

31. If three circles touch one another externally in P, Q, R, and the chords PQ, PR of two of the circles be produced to meet the third circle again in ST, prove that ST is a diameter.

32. Two tangents are drawn to a circle at the opposite extremities of a diameter, and intercept from a third tangent a portion AB; if C be the centre of the circle, show that ACB is a right angle.

33. Through the vertices of a quadrilateral straight lines are drawn parallel to the diagonals; prove that the figure thus formed is a parallelogram which is double the quadrilateral.

34. Describe a circle which shall pass through two given points and touch a given straight line. Two solutions.

35. To divide one side of a given triangle into segments proportional to the adjacent sides.

36. To describe a circle which shall pass through two given points and touch a given circle.

37. Show that the sum of the perpendiculars from any point inside a regular hexagon to the six sides is equal to three times the diameter of the inscribed circle.

38. The three sides of a triangle are 9, 10, 17 in., respectively; find (1) its area and (2) the area of the inscribed circle.

39. In a given circle inscribe a triangle similar to a given triangle.

40. Through one of the points of intersection of two circumferences, draw a straight line, terminated by the circumferences, which shall have a given length.

41. Construct a parallelogram, having given (1) two diagonals and the angle between them, (2) one side, one diagonal, and the angle between the diagonals.

42. Describe a circle with given radius to touch a given line in a given point. How many such circles can be described?

43. Construct a triangle, having given a median and the two angles into which the angle is divided by that median.

44. Every equiangular polygon inscribed in a circle is equilateral if the number of sides be odd.

45. Prove that the rectangle of the sum and difference of two straight lines is equal to the difference of the squares constructed on the lines.

46. If the straight line joining the middle points of two opposite sides of any quadrilateral divide the area into two equal parts, show that the two bisected sides are parallel.

47. Describe a circle which shall touch a given straight line at a given point and pass through another given point not in the line.

48. The apothem of a regular hexagon is 12; find the area of the circumscribing circle.

49. The angle included between the internal bisector of one base angle of a triangle and the external bisector of the other base angle is equal to half the vertical angle.

50. If the exterior angles of a triangle are bisected, the three exterior triangles formed on the sides of the original triangle are equiangular.

51. The angle formed by the bisectors of any two consecutive angles of a quadrilateral is equal to the sum of the other two angles.

52. AB is the diameter and C the centre of a semicircle; show that O, the centre of any circle inscribed in the semicircle, is equidistant from C and from the tangent to the semicircle parallel to AB.

53. To find in one side of a given triangle a point whose distances from the other sides shall be to each other in a given ratio.

54. Through a point in a circle draw a chord that is bisected in that point, and show that it is the least chord through that point.

55. To draw through a point P, exterior to a given circle, a secant PAB so that $AP : BP = 2 : 3$.

56. Prove that the square constructed on the sum of two straight lines is equal to the sum of the squares upon each of the two straight lines plus twice the rectangle of the lines.

57. Having given the greater segment of a line divided in extreme and mean ratio, to construct the line.

58. Construct a right triangle, having given the hypotenuse and the perpendicular from the right angle on it.

59. The position and magnitude of two chords of a circle being given, describe the circle.

60. Construct a right triangle, having given the hypotenuse and the difference of the other sides.

61. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

62. BAC is a triangle having the angle B double the angle A. If BD bisects the angle B and meets AC at D, show that BD is equal to AD.

63. ABC, DEF are triangles having the angles A and D equal, and AB equal to DE. Prove that the triangles are to each other as AC is to DF.

64. Construct a parallelogram, having given:

Two adjacent sides and a diagonal.

A side and both diagonals.

65. With a given point as centre describe a circle which shall intersect a given circle at the ends of a diameter.

66. Through a given point within a given circle draw two equal chords which shall contain a given angle.

67. Through a given point inside the circle which is not the centre, draw a chord bisected at that point.

ADDITIONAL EXERCISES ON BOOK VI.

1. What is the reason that a three-legged chair is always stable on the floor while a four-legged one may not be?

2. Prove that parallel lines have their projections on the same plane in lines that are coincident or parallel.

3. Show that all the propositions in Plane Geometry which relate to triangles are true of triangles in space, however situated.

4. Show that those propositions are not true of polygons of more than three sides situated in any way in space.

5. To construct a plane containing a given line, and parallel to another given line.

6. If the projections of a number of points on a plane are in a straight line, these points are in one plane.

7. A plane can be passed perpendicular to only one edge or to two faces of a polyedral angle.

8. If each of the projections of the line AB upon two intersecting planes is a straight line, the line AB is a straight line.

9. The edge of a diedral angle is perpendicular to the plane of the measuring angle.

10. If a line makes equal angles with three lines in the same plane, it is perpendicular to that plane.

11. If a plane bisects a line perpendicularly, every point of the plane is equally distant from the extremities of the line.

12. Through a given point, to pass a plane perpendicular to a given straight line.

13. Through a given straight line, to pass a plane perpendicular to a given plane.

14. The faces of a diedral angle are perpendicular to the plane of the measuring angle.

15. If a plane be passed through one diagonal of a parallelogram, the perpendiculars to that plane from the extremities of the other diagonal are equal.

16. If four lines in space are parallel, in how many planes may they lie when taken two at a time?

17. To bisect a diedral angle.

18. Through a given line in a plane pass a plane making a given angle with that plane.

19. If two lines not in the same plane are intersected by the same line, how many planes may be determined by the three lines taken two and two?

20. Through a given point, to pass a plane parallel to a given plane.

21. If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

22. Through the edge of a given diedral angle pass a plane bisecting that angle.

23. To draw a straight line perpendicular to a given plane from a given point outside of it.

24. To draw a straight line perpendicular to a given plane from a given point in the plane.

25. To determine that point in a given straight line which is equidistant from two given points not in the same plane with the given line.

26. Two parallel planes intersecting two parallel lines cut off equal segments.

27. In a given plane find a point equidistant from three given points without the plane.

28. Parallel lines make equal angles with parallel planes.

29. A straight line makes equal angles with parallel planes.

30. If a line is parallel to each of two intersecting planes, it is parallel to their intersection.

31. If a line is parallel to each of two planes, the intersections which any plane passing through it makes with the planes are parallel.

32. To determine the point whose distances from the three faces of a given triedral angle are given. Is it unique?

33. If the projections of any line upon two intersecting planes are each of them straight lines, prove that the line itself is a straight line.

34. Two planes which are not parallel are cut by two parallel planes. Prove that the intersections of the first two with the last two contain equal angles.

35. Pass a plane through a given point parallel to a given plane.

ADDITIONAL EXERCISES ON BOOK VII.

1. The lateral surface of a pyramid is greater than the base.

2. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude and diameter of the cylinder.

3. An open cistern 6 ft. long and $4\frac{1}{2}$ ft. wide holds 108 cubic ft. of water. How many cubic feet of lead will it take to line the sides and bottom, if the lead is $\frac{1}{8}$ in. thick?

4. Find the surface and volume of a rectangular parallelopiped whose edges are 4, 5, and 6 ft.

5. What is the volume of a right prism whose altitude is 45 in. and whose base contains 3 sq. ft.?

6. Find the volume of a rectangular parallelopiped whose surface is 104 and whose base is 2 by 6.

7. How many square feet of lead will be required to line a cistern open at the top, which is 4 ft. 6 in. long, 2 ft. 8 in. wide, and contains 42 cu. ft.?

8. A brick has the dimensions, 25 cm., 12 cm., 6 cm., but on account of shrinkage in baking, the mould is 27.5 cm. long and proportionally wide and deep. What per cent does the volume of the brick decrease in baking?

9. The altitude of a pyramid is divided into five equal parts by planes parallel to the base. Find the ratios of the various frustums to one another and to the whole pyramid.

10. To find two straight lines in the ratio of the volumes of two given cubes.

11. To cut a cube by a plane so that the section shall be a regular hexagon.

12. The lateral areas of the two cylinders generated by revolving a rectangle successively about each of its containing sides are equal.

13. Find the volume of a cube the diagonal of whose face is $a\sqrt{2}$.

14. The dimensions of one rectangular parallelopiped are 2 ft., 15 ft., and 14 ft., respectively; those of another are 4 ft., 5 ft., and 13 ft., respectively. What is the ratio of the first solid to the second?

15. Find the length of the diagonal of a rectangular parallelopiped whose edges are 4, 5, and 6.

16. The base of a pyramid contains 169 sq. ft. A plane parallel to the base and 4 ft. from the vertex cuts a section containing 64 sq. ft.; find the height of the pyramid.

17. If a slant height of a cone of revolution is equal to the diameter of its base, its total area is to that of the inscribed sphere as 9 is to 4.

18. What should be the edge of a cubical box that shall contain 8 gallons dry measure?

19. Find the volume of a rectangular parallelopiped whose surface is 832 and whose base is 8 by 6.

20. The dimensions of a trunk are 5 ft., 4 ft., 3 ft. What are the dimensions of a similar trunk holding four times as much?

21. Find the difference between the volume of the frustum of a pyramid and the volume of a prism of the same altitude, whose base is a section of the frustum parallel to its bases and equidistant from them.

The difference may be expressed in the form

$$\frac{h}{12}(\sqrt{B} - \sqrt{b})^2,$$

if B and b are the areas of the bases, and h the altitude of the frustum.

22. A pyramid 24 ft. high has a square base measuring 16 ft. on a side. What will be the area of a section made by a plane parallel to the base and 4 ft. from the vertex?

23. An equilateral triangle revolves about one of its altitudes. What is the ratio of the lateral surface of the generated cone to that of the sphere generated by the circle inscribed in the triangle?

24. What should be the edge of a cube so that its entire surface shall be 2 sq. ft.?

25. The height of a regular hexagonal pyramid is 36 ft., and one side of the base is 6 ft. What are the dimensions of a similar pyramid whose volume is one-twentieth that of the first?

26. A man wishes to make a cubical cistern whose contents are 186,624 cu. in.; how many feet of inch boards will line it?

27. Find the height in feet of a pyramid when the volume is 26 cu. ft. 936 cu. in., and each side of its square base is 3 ft. 6 in.

28. The base of a pyramid is 18 sq. ft. and its altitude is 9 ft. What is the area of a section parallel to the base and 3 ft. from it?

29. A conical tent of slant height 10 ft. covers a circular area 10 ft. in diameter. Find the volume, and the area of canvas.

30. The lateral edge of a right prism is equal to the altitude.

31. The base edge of a regular pyramid with a square base is 40 ft., the lateral edge 101 ft.; find its volume in cubic feet.

32. The total area of the equilateral cylinder inscribed in a sphere is a mean proportional between the area of the sphere and the total area of the inscribed equilateral cone. The same is true of the volumes of these bodies.

33. The volumes of two similar cones are 54 cu. ft. and 432 cu. ft. The height of the first is 6 ft., what is the height of the other?

34. The base of a cone is equal to a great circle of a sphere, and the altitude is equal to a diameter of the sphere. What is the ratio of their volumes?

35. The perimeter of the base of a pyramid is 20 in.; its slant height is 9 in. What is the lateral surface?

36. Find the dimensions of a right circular cylinder fifteen-sixteenths as large as a similar cylinder whose height is 40 ft., and diameter 20 ft.

37. The lateral areas of right prisms of equal altitudes are as the perimeters of their bases.

38. The volume of a sphere is to that of the inscribed cube as π is to $2 \div \sqrt{3}$.

39. The height of a frustum of a right cone is two-fifths the height of the entire cone. Compare the volumes of the frustum and the entire cone.

40. The bases of two pyramids are 8.1 sq. ft. and 10 sq. ft., respectively; their altitudes are 10 ft. and 9 ft. respectively. What is their ratio?

41. Having the base edge a, and the total surface T, of a regular pyramid with a square base, find the volume V.

42. If the lateral surface of a right circular cylinder is a, and the volume is b, find the radius of the base and the height.

43. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

44. A sphere is to the circumscribed cube as π is to 6.

45. The bases of a frustum of a pyramid are 9 sq. ft. and $5\frac{1}{2}$ sq. ft. respectively, and its altitude is 6 ft. What is its volume?

46. The height of a right circular cone is equal to the diameter of its base; find the ratio of the area of the base to the lateral surface.

47. Any straight line drawn through the centre of a parallelopiped, terminating in a pair of faces, is bisected at the centre.

48. The bases of a frustum of a pyramid are 24 sq. in. and 8.3 sq. in. respectively. Its volume is 500 cu. in. What is its altitude?

49. Find the volume of a prism the area of whose base is 24 sq. in. and altitude 7 ft.

50. Every section of a prism, by a plane parallel to the lateral edges, is a parallelogram.

51. What length of canvas $\frac{3}{4}$ yd. wide is required to make a conical tent 14 ft. in diameter and 10 ft. high?

52. In a cube the square of a diagonal is three times the square of an edge.

53. How many square feet of tin will be required to make a funnel, if the diameters of the top and the bottom are to be 30 in. and 15 in. respectively, and the height 25 in.?

54. The four middle points of two pairs of opposite edges of a tetraedron are in one plane, and at the vertices of a parallelogram.

55. The section of a triangular pyramid made by a plane passed parallel to two opposite edges is a parallelogram.

56. The diameters of the bases of a frustum of a cone are 10 in. and 8 in. respectively, and its slant height is 14 in. Find its lateral area.

57. A right circular cylinder 6 ft. in diameter is equivalent to a right circular cone 7 ft. in diameter. If the height of the cone is 8 ft., what is the height of the cylinder?

58. Find the surface of a cubical cistern whose contents are 373,248 cu. in.

59. The frustum of a right circular cone is 14 ft. high, and has a volume of 924 cu. ft. Find the radii of its bases if their sum is 9 ft.

60. The plane which bisects a diedral angle of a tetraedron divides the opposite side into segments, which are proportional to the areas of the adjacent faces.

61. Find the area of a section in Problem 59 equidistant from the bases.

62. A Dutch windmill in the shape of a frustum of a right cone is 12 metres high. The outer diameters at the bottom and the top are 16 metres and 12 metres, the inner diameters 12 metres and 10 metres, respectively. How many cubic metres of stone were required to build it?

63. The volume of a truncated parallelopiped is equal to the product of a right section by one-fourth the sum of its four lateral edges.

64. The Pyramid of Cheops was originally 480.75 ft. high, and 764 ft. square at the base. What was its volume?

65. The volume of a cylinder of revolution is equal to the product of its lateral area by half its radius.

66. If a spherical shell have an exterior diameter of 14 in., what should be the thickness of the wall so that it may contain 696.9 cu. in.?

67. Find the depth of a cubical cistern which shall hold 2000 gallons, each gallon being 231 cu. in.

68. If an iron sphere, 12 in. in diameter, weighs n lbs., what will be the weight of an iron sphere whose diameter is 16 in.?

69. Find the depth of a cubical box which shall contain 100 bu. of grain, each bushel holding 2150.42 cu. in.

70. A cone of revolution whose vertical angle is 60°, is circumscribed about a sphere. Compare the area of the sphere and the lateral area of the cone. Compare their volumes.

71. The altitudes of two similar cones of revolution are as 11 to 9. What is the ratio of their total areas? Of their volumes?

ADDITIONAL EXERCISES ON BOOK VIII.

1. How many points on a spherical surface determine a small circle? How many, in general, determine a great circle?

2. The polar triangle of a trirectangular triangle is a trirectangular triangle coinciding with the triangle itself.

3. Any lune is to a trirectangular triangle as its angle is to half a right angle.

4. If the radius of a sphere is bisected at right angles by a plane, the two zones into which the surface of the sphere is divided are to each other as 3 : 1.

5. If the radii of two spheres are 6 in. and 4 in. respectively, and the distance between their centres is 5 in., what is the area of the circle of intersection of these spheres?

6. Find the diameter of a sphere whose volume is one cubic foot.

7. Find the area of a spherical triangle each of whose angles is 70°, on a sphere whose surface is 144 sq. in.

8. Find the radius of the circle determined, in a sphere of 3 in. diameter, by a plane 1 in. from the centre.

9. Find the area of a birectangular triangle whose vertical angle is 108° on a sphere whose surface is 400 sq. in.

10. A spherical triangle is to the surface of the sphere as the spherical excess is to eight right angles.

11. In any right spherical triangle, if one side be greater than a quadrant, there must be a second side greater than a quadrant.

12. Find the angles of an equilateral spherical triangle whose area is equal to that of a great circle.

13. Considering the moon as a circle of diameter 2160.6 mi., whose centre is 23,4820 mi. from the eye, what is the volume of the cone whose base is the full moon and whose vertex is the eye?

14. One sphere has twice the volume of another. Find the ratio of the radius of the first to the radius of the second.

15. The circumference of a hemispherical dome is 132 ft. How many square feet of lead are required to cover it?

16. Find the surface of a sphere whose volume is 2 cu. ft.

17. If the ball on the top of St. Paul's Cathedral in London is 6 ft. in diameter, what would it cost to gild it at 7 cents per square inch?

18. What is the ratio of the surface of a sphere to the *entire* surface of its hemisphere?

19. Find the ratios of the areas of two spherical triangles on the same sphere, the angles being 60°, 84°, 129°, and 80°, 110°, 114° respectively.

20. Prove that the areas of zones on equal spheres are proportional to their altitudes.

21. Find a circumference of a small circle of a sphere whose diameter is 20 in., the plane of the circle being 5 in. from the centre of the sphere.

22. The diameter of a sphere is 21 ft. Find the curved surface of a segment whose height is 6 ft.

23. Find the volume of a sphere inscribed in a cube whose volume is 1331 cu. in.

24. The altitude of the torrid zone is about 3200 mi. Find its area in square miles, assuming the earth to be a sphere with a radius of 4000 mi.

25. A spherical pyramid has for base a trirectangular triangle. What fraction is the pyramid of the sphere?

26. Find the ratio of a spherical surface to the cylindrical surface of the circumscribed cylinder.

27. The radii of two concentric spheres are 8 and 12 in.; a plane is drawn tangent to the interior sphere. Find the area of the section made in the outer sphere.

28. If an iron ball 4 in. in diameter weighs 9 lbs., what is the weight of a hollow iron shell 2 in. thick, whose external diameter is 20 in.?

29. The surface of a sphere is to be 800 sq. in. What radius should be taken?

30. What is the ratio of the entire surface of a cylinder circumscribed about a sphere to the entire surface of its hemisphere?

31. To construct on the spherical blackboard a spherical triangle, having a side 75°, and the adjacent angles 110° and 87°.

32. The radius of the base of the segment of a sphere is 16 in., and the radius of the sphere is 20 in. Find its volume.

33. Two spheres have radii of 8 in. and 7 in. respectively. What is the ratio of the surfaces of those spheres? Of their volumes?

34. A cone has for its base a great circle of a sphere, and for its vertex a pole of that circle. Find the ratio of the curved surfaces of the cone and hemisphere; of the entire surfaces.

35. To draw an arc of a great circle perpendicular to a spherical arc, from a given point without it.

36. The radius of the base of a segment of a sphere is 40 ft., and its height is 20 ft.; find its volume.

37. Find the altitude of a zone whose area is 100 sq. in., on a surface of a sphere of 13 in. radius.

38. The area of a zone of one base (the other base is zero) equals that of a circle whose radius is the chord of the generating arc.

39. At a given point in a great circle, to draw an arc of a great circle, making a given angle with the first.

40. The surface of a sphere is 81 sq. in. Find its volume.

41. In a right-angled spherical triangle, if one side is equal to a quadrant, so is another side.

42. The altitude of a prism is 9 ft. and the perimeter of the base 12 ft.; find the altitude and perimeter of a base of a similar prism one-third as great.

43. The volume of a sphere is 7 cu. ft. Find its diameter and surface.

44. The volume of a spherical sector is 36 cu. in.; the diameter of the sphere is 18 in. Find the area of the zone that forms the base of the sector.

45. The mean radii of the earth and moon are respectively 3956 mi.; 1080.3 mi. Show that their volumes are as 49 to 1, nearly.

46. If lines are drawn from any point in the surface of a sphere to the ends of a diameter, they will form with each other a right angle.

47. The volumes of two spheres are as 27 is to 64. Find the ratio (1) of their diameters; (2) of their surfaces.

48. The mean diameter of the planet Jupiter being 86,657 mi., find the ratio of its volume to that of the earth.

49. If two straight lines are tangent to a sphere at the same point, the plane of these lines is tangent to the sphere.

50. In a sphere whose radius is 5 in., find the altitude of a zone whose area shall be that of a great circle.

51. The sun's diameter is about 109 times the earth's. Find the ratio of their volumes.

52. Any lune is to a trirectangular triangle as its angle is to half a right angle.

53. What is the radius of that sphere whose number of square units of surface equals the number of cubic units of volume?

54. The largest possible cube is cut out of a sphere one foot in diameter. Find the length of an edge.

55. Given a sphere of radius 10. How far from the centre must the eye be in order to see one-fourth of its surface?

56. What is the radius of that sphere whose number of cubic units of volume equals the number of square units of area in one of its great circles?

57. Spherical polygons are to each other as their spherical excesses.

58. A cone, a sphere, and a cylinder have the same altitudes and diameters. Show that their volumes are in arithmetical progression.

59. If the angles of a spherical triangle are respectively 65°, 112°, and 85°, how many degrees are there in each side of its polar triangle?

60. A metre was originally intended to be 0.000 000 1 of a quadrant of the circumference of the earth. Assuming it to be such, and the earth to be a sphere, find (1) its radius in kilometers; (2) its volume in cubic kilometers.

61. Given the spherical triangle whose sides are respectively 80°, 90°, and 140°, find the angles of its polar triangle.

62. If the atmosphere extends to a height of 45 miles above the earth's surface, what is the ratio of its volume to the volume of the earth, assuming the latter to be a sphere with a diameter of 7912 mi.?

63. What part of the surface of a sphere is a lune whose angle is 45°? 54°? 80°?

MISCELLANEOUS EXERCISES. BOOKS VI.–VIII.

(SOLID.)

1. Find the lateral area of a right pentagonal pyramid whose slant height is 9 in., and each side of the base 6 in.

2. A pyramid 20 ft. high has a base containing 169 sq. ft. How far from the vertex must a plane be passed parallel to the base, so that the section may contain 100 sq. ft.?

3. A pyramid 16 ft. high has a square base 10 ft. on a side. Find the area of a section made by a plane parallel to the base and 6 ft. from the vertex.

4. The volume of the frustum of a regular hexagonal pyramid is 12 cu. ft., the sides of the bases are 2 ft. and 1 ft.; find the height of the frustum.

5. A regular pyramid 8 ft. high is transformed into a regular prism with an equivalent base; find the height of the prism.

6. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

7. The lateral area of a given cone of revolution is double the area of its base; find the ratio of its altitude to the radius of its base.

8. Find the volume of the frustum of a cone of revolution, the radii of the bases being 8 ft. and 4 ft. and the altitude 6 ft.

9. The projections of parallel straight lines on any plane are themselves parallel.

10. Construct an equilateral triangle equal to a given triangle.

11. Construct an isosceles triangle having each angle at the base double the third angle.

12. The total area of a cone of revolution is 500 sq. in.; its altitude is 10 in. What is the diameter of its base?

13. What is the lateral area and the total area of a frustum of a cone of revolution whose altitude is 30 in., and the diameters of whose bases are 9 in. and 21 in. respectively?

14. The diameters of the bases of a frustum of a cone of revolution are $7\frac{1}{2}$ in. and 12 in. respectively; its volume is 575 cu. in. What is its altitude?

15. If the altitude of a cylinder of revolution is equal to the diameter of its base, the volume is equal to the product of its total area by one-third of its radius.

16. Find the ratio of two rectangular parallelopipeds, if their dimensions are 4, 7, 9, and 8, 14, 18 respectively.

17. The volume of a sphere is one cubic foot. Find the surface of the circumscribing cylinder.

18. How far from the base must a cone, whose altitude is 64 in., be cut by a plane so that the frustum shall be equivalent to half the cone?

19. What should be the altitude of a cone of revolution whose base has a diameter of 15 in., so that the lateral area may be 20 square feet?

20. The altitude of a cone of revolution is four times the radius of its base; the lateral area is 1000 sq. in. Find the radius and altitude.

21. The total area of a cylinder of revolution is 800 sq. in.; its altitude is 16 in. What is the diameter of a base?

22. What should be the volume of a cylinder of revolution whose altitude is 20 in., so that the lateral area shall be 3 sq. ft.?

23. Two given straight lines do not intersect and are not parallel. Find a plane on which their projections will be parallel.

24. Describe a circle which shall touch a given circle and two given straight lines which themselves touch the given circle.

25. Pass a plane perpendicular to a given straight line through a given point not in that line.

26. In any triedral angle, the planes bisecting the three diedral angles all intersect in the same straight line.

27. Draw a straight line through a given point in space, so that it shall cut two given straight lines not in the same plane.

28. Find the dimensions of a cube whose surface is numerically equal to its contents.

29. The base of a regular pyramid is a hexagon whose side is 12 ft. Find the height of the pyramid if the lateral area is eight times the area of the base.

30. Find the volume of the frustum of a regular triangular pyramid, the sides of whose bases are 18 and 16, and whose lateral edge is 11.

31. Find the lateral area of a right pyramid whose slant height is 8 ft., and whose base is a regular octagon of which each side is 6 ft. long.

32. Find the volume of the frustum of a square pyramid, the sides of whose bases are 16 and 12 ft., and whose altitude is 24 ft.

33. The altitudes of two similar cylinders of revolution are as 6 to 5. What is the ratio of their total areas? Of their volumes?

34. Find the ratio of two rectangular parallelopipeds, if their altitudes are each 6 ft., and their bases 8 ft. by 4 ft., and 15 ft. by 10 ft. respectively.

35. In order that a cylindrical tank with a depth of 24 ft. may contain 2000 gal., what should be its diameter?

36. How many cubic inches of iron would be required to make that tank, its walls being one-fourth of an inch thick?

37. The diameter of a right circular cylinder is 12 ft., and its altitude 9 ft. What is the side of an equivalent cube?

38. A sphere 6 in. in diameter has a hole bored through its centre with a 2-inch auger; find the remaining volume.

39. How high above the earth must a person be raised in order that he may see one-fifth of its surface?

40. How much of the earth's surface would a man see if he were raised to the height of the diameter above it?

41. Find the volume of a spherical segment of one base whose altitude is 4 ft., the radius of the sphere being 10 ft.

42. Find the surface of a sphere inscribed in a cube whose surface is 216.

43. The volumes of two similar cones of revolution are to each other as 3 : 5; find the ratio of their lateral areas, and of their volumes.

44. The lateral area of a cone of revolution is 60π and its slant height is 16; find its volume.

45. Find the lateral area of the frustum of a cone of revolution, the radii of the bases being 42 and 12 in., and the altitude 36 in.

46. The two legs of a right triangle are a and b; find the area of the surface generated when the triangle revolves about its hypotenuse.

47. Find the volume of a regular icosaedron whose edges are each 20 ft.

48. The base of a cone is equal to a great circle of a sphere, and the altitude of the cone is equal to a diameter of the sphere; compare the volumes of the cone and the sphere.

49. The lateral area of a cylinder of revolution is $116\frac{2}{3}$ sq. ft., and the altitude is 14 ft.; find the diameter of its base.

50. Find the number of cubic feet in the trunk of a tree, 70 ft. long, the diameters of its ends being 9 and 7 ft.

51. The heights of two cylinders of revolution of equal volumes are as 9 : 16; the diameter of one is 6 ft. Find the diameter of the other.

52. The volume of a sphere is 113; find its diameter and its surface.

53. The volume of a sphere is $776\,\pi$; find its diameter and its surface.

54. Find the weight of an iron shell 4 in. in diameter, the iron being 1 in. thick, and weighing $\frac{1}{4}$ lb. to the cubic inch.

55. If an iron ball 8 in. in diameter weighs 72 lbs., find the weight of an iron shell 10 in. in diameter, the iron being 2 in. thick.

56. A sphere, 2 ft. in diameter, is cut by two parallel planes, one at 3 and the other at 9 in. from the centre; find the volume of the segment included between them.

57. If the angle of a lune is 50°, find its area on a sphere whose surface is 72 sq. in.