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RELATIONS BETWEEN THE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH FOUR REGULAR SINGULAR POINTS

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RELATIONS

BETWEEN THE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH FOUR REGULAR SINGULAR POINTS

I. INTRODUCTION.

The most general type of linear differential equation of the second order with three regular singular points is the well-known hypergeometric equation. This classical equation is familiar to all mathematicians and the relations between its functions are of remarkable interest.

It is the purpose of this paper to study the relations between the solutions of an analogous equation with four regular singular points; the results of this study we can generalize to Klein's equation with n regular singular points, of which the hypergeometric function becomes a special case. This differential equation has been studied by both Karl Heun[1] and C. Franz[2], who obtained some special interesting properties about its series solutions. Here we manage to give a single way to derive the differential equation from that of Klein, and a general and idoneous method to get the 192 integrals, and we may also generalize to obtain the

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1 Heun, Zur Theioie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten, Mathem, Ann, Bd. 33 pg. 161 (1889).

2 Franz, Untersuchungen udie Lineare Homogene Differential gleichung 2. Ordnung der Fuchs'schen Lkasse mit drei im Endlichen gelegenon singularer Stellen, (1898).

$n!\,2^{n-1}$ integrals of Klein's equation. Since the equation has an arbitrary undetermined constant, this may be chosen with special value zero and supposing that a solution of this particular differential equation is known, we thus get its relation with the general solution, and consequently we can also apply to the most general Klein's equation.

II. Derivarion of the Differential Equation.

The most general linear homogeneous equation of the second order and of Fuchsian type, having n singularities in the finite part of the plane, say $a_1, a_2, \ldots a_n$, with exponents α_i, β_i respectively, and ∞ being an ordinary point, was given by Klein to be[1]

$$\frac{d^2y}{dx^2}+\left\{\sum_{i=1}^{n}\frac{1-\alpha_i-\beta_i}{x-a_i}\right\}\frac{dy}{dx}+\left\{\sum_{i=1}^{n}\frac{\alpha_i\beta_i}{(x-a_i)^2}+\sum_{i=1}^{n}\frac{D_i}{x-a_i}\right\}y=0 \cdots\cdots (1)$$

with the relations,

$$\sum_{i=1}^{n}(\alpha_i+\beta_i)=n-2,\quad \sum_{i=1}^{n}D_i=0,\quad \sum_{i=1}^{n}(a_iD_i+\alpha_i\beta_i)=0,\quad \sum_{i=1}^{n}(a_i^2D_i+2a_i\alpha_i\beta_i)$$

Taking $n=4$, and letting $a_1\to 0,\ a_2\to 1,\ a_3=a,\ a_4\to\infty,$

we have

$$\frac{d^2y}{dx^2}+\left(\frac{1-\alpha_1-\beta_1}{x}+\frac{1-\alpha_2-\beta_2}{x-1}+\frac{1-\alpha_3-\beta_3}{x-a}\right)\frac{dy}{dx}+\left[\frac{\alpha_1\beta_1}{x^2}+\frac{\alpha_2\beta_2}{(x-1)^2}+\frac{\alpha_3\beta_3}{(x-a)^2}+\frac{D_1}{x}+\frac{D_2}{x-1}+\frac{D_3}{x-a}\right]y=0\cdots(2)$$

with the relations,

$$\alpha_1+\beta_1+\alpha_2+\beta_2+\alpha_3+\beta_3+\alpha_4+\beta_4=2,\quad D_1+D_2+D_3=0,\quad D_2+aD_3+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=\pm\alpha_4\beta_4$$

Though the last relation has double signs, we can choose one, the other being obtainable by a simple transformation. Thus choosing the $+$ sign, and letting $aD_1=-\alpha_4\beta_4 q$, we obtain

$$\frac{d^2y}{dx^2}+\left(\frac{1-\alpha_1-\beta_1}{x}+\frac{1-\alpha_2-\beta_2}{x-1}+\frac{1-\alpha_3-\beta_3}{x-a}\right)\frac{dy}{dx}+\left[\frac{\alpha_1\beta_1}{x^2}+\frac{\alpha_2\beta_2}{(x-1)^2}+\frac{\alpha_3\beta_3}{(x-a)^2}+\frac{(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)x-\alpha_4\beta_4 q}{x(x-1)(x-a)}\right]y=0 \;\ldots\ldots (I)$$

---

1 Whittaker, Modern Analysis, (1920) p.209, and also, Forsyth, Theory of Differential Equations, (1902), Vol. IV. pp. 154-5.

Following the Rieman P-function (I) is defined by the scheme

$$P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & x \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{matrix}\right\}$$

For $\alpha_1=\alpha_2=\alpha_3=0,\ \beta_1=1-\gamma,\ \beta_2=1-\delta,\ \beta_3=\gamma+\delta-\alpha-\beta,\ \alpha_4=\alpha,\ \beta_4=\beta,$

we have Heun's equation[1]

$$x(x-1)(x-a)\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x^2-\{\alpha+\beta-\delta+1+(\gamma+\delta)a\}x+a\gamma\right]\frac{dy}{dx}+\alpha\beta(x-q)y=0 \ldots\ldots\ldots\ldots(I')$$

The scheme of Heun is

$$P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & x \\ 1-\gamma & 1-\delta & \gamma+\delta-\alpha-\beta & \beta & \end{matrix}\right\}$$

If a=1, q=1, the equation (I') becomes:

$$x(x-1)^2\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x^2-\{\alpha+\beta+\gamma+1\}x+\gamma\right]\frac{dy}{dx}+\alpha\beta(x-1)y=0,$$

Which simplified reduces to

$$x(x-1)\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x-\gamma\right]\frac{dy}{dx}+\alpha\beta y=0$$

which is the so-called hypergeometric equation and satisfied by

$$y=F(\alpha,\beta,\gamma,x).$$

and if a=o, q=o, The equation (I') degenerates into

$$x(x-1)\frac{d^2y}{dx^2}+\left[(\alpha+\beta+1)x-(\alpha+\beta-\delta+1)\right]\frac{dy}{dx}+\alpha\beta y=0,$$

---

1 Heun; Franz; Whittaker, pp. 576-7 Forsyth, pp. 158-9; Ince, Ordinary Differential Equations (1927) p. 394.

Hence it is satisfied by

$$y = F(\alpha, \beta, \alpha+\beta-\delta+1, x).$$

If $a \to \infty$, the equation degenerates also into the hypergeometric equation, a fact not pointed out by Heun and Franz. For from (2), we have

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{dy}{dx} + \left[\frac{D_1}{x} + \frac{D_2}{x-1} + \frac{D_3}{x-a}\right]y = 0$$

with the relations

$$\alpha+\beta+1 = \gamma+\delta+\epsilon, \quad D_1+D_2+D_3 = 0, \quad D_2 + aD_3 - \alpha\beta = 0$$

Let $a \to \infty$, then $D_3 = 0$ and $D_1 + D_2 = 0$. Hence we have

$$x(x-1)\frac{d^2y}{dx^2} + [(\gamma+\delta)x - \gamma]\frac{dy}{dx} - D_1 y = 0,$$

Let $D_1 = -(\delta-1)\gamma$, we have

$$x(x-1)\frac{d^2y}{dx^2} + [(\gamma+\delta)x-\gamma]\frac{dy}{dx} + (\delta-1)\gamma y = 0$$

which is satisfied by

$$y = F(\gamma, \delta-1, \gamma, x)$$

III. 192 solutions of the differential equation.

By a homographic transformation of the variable, the four points o, l, a, ∞, are interchanged, except that a may go into another a^1, among themselves. As is known, 24 such substitutions are possible, namely

x	$1-x$	$\frac{1}{x}$	$\frac{x-1}{x}$	$\frac{1}{1-x}$	$\frac{x}{x-1}$
$\frac{a}{x}$	$\frac{x-a}{x}$	$\frac{x}{a}$	$\frac{a-x}{a}$	$\frac{x}{x-a}$	$\frac{a}{a-x}$
$\frac{x-a}{x-1}$	$\frac{a-1}{x-1}$	$\frac{x-1}{x-a}$	$\frac{1-a}{x-a}$	$\frac{x-1}{a-1}$	$\frac{x-a}{1-a}$
$\frac{a(x-1)}{x-a}$	$\frac{(1-a)x}{x-a}$	$\frac{x-a}{a(x-1)}$	$\frac{(a-1)x}{a(x-1)}$	$\frac{x-a}{(1-a)x}$	$\frac{a(x-1)}{(a-1)x}$

The following table illustrates the results of the aforementioned substitutions :

$$\begin{array}{llll}
\{0\ \ 1\ \ a\ \ \infty\ \ x\} & \{\infty\ \ a\ \ 1\ \ 0\ \ \frac{a}{x}\} & \{a\ \ \infty\ \ 0\ \ 1\ \ \frac{x-a}{x-1}\} & \{1\ \ 0\ \ \infty\ \ a\ \ \frac{a(x-1)}{x-a}\} \\
\{1\ \ 0\ \ 1-a\ \ \infty\ \ 1-x\} & \{\infty\ \ 1-a\ \ 0\ \ 1\ \ \frac{x-a}{x}\} & \{1-a\ \ \infty\ \ 1\ \ 0\ \ \frac{a-1}{x-1}\} & \{0\ \ 1\ \ \infty\ \ 1-a\ \ \frac{(1-a)x}{x-a}\} \\
\{0\ \ \frac{1}{a}\ \ 1\ \ \infty\ \ \frac{x}{a}\} & \{\infty\ \ 1\ \ \frac{1}{a}\ \ 0\ \ \frac{1}{x}\} & \{1\ \ \infty\ \ 0\ \ \frac{1}{a}\ \ \frac{x-a}{a(x-1)}\} & \{\frac{1}{a}\ \ 0\ \ \infty\ \ 1\ \ \frac{x-1}{x-a}\} \\
\{1\ \ \frac{a-1}{a}\ \ 0\ \ \infty\ \ \frac{a-x}{a}\} & \{\infty\ \ 0\ \ \frac{a-1}{a}\ \ 1\ \ \frac{x-1}{x}\} & \{0\ \ \infty\ \ 1\ \ \frac{a-1}{a}\ \ \frac{(a-1)x}{a(x-1)}\} & \{\frac{a-1}{a}\ \ 1\ \ \infty\ \ 0\ \ \frac{1-a}{x-a}\} \\
\{\frac{1}{1-a}\ \ 0\ \ 1\ \ \infty\ \ \frac{x-1}{a-1}\} & \{\infty\ \ 1\ \ 0\ \ \frac{1}{1-a}\ \ \frac{x-a}{(1-a)x}\} & \{1\ \ \infty\ \ \frac{1}{1-a}\ \ 0\ \ \frac{1}{1-x}\} & \{0\ \ \frac{1}{1-a}\ \ \infty\ \ 1\ \ \frac{x}{x-a}\} \\
\{\frac{a}{a-1}\ \ 1\ \ 0\ \ \infty\ \ \frac{x-a}{1-a}\} & \{\infty\ \ 0\ \ 1\ \ \frac{a}{a-1}\ \ \frac{a(x-1)}{(a-1)x}\} & \{0\ \ \infty\ \ \frac{a}{a-1}\ \ 1\ \ \frac{x}{x-1}\} & \{1\ \ \frac{a}{a-1}\ \ \infty\ \ 0\ \ \frac{a}{a-x}\}
\end{array}$$

We proceed to find the 192 solutions. The solutions of the equation (I'), which is regular in the vicinity of x = o, and belongs to the exponent o is given by[1]

$$F(a,q;\alpha,\beta,\gamma,\delta,x) = 1+\alpha\beta\sum_{n=0}^{\infty}\frac{G_{n+1}(q)}{\underline{n+1}\,\gamma(\gamma+1)\cdots(\gamma+n)}\left(\frac{x}{a}\right)^{n+1},$$

---

1 Heun, Math. Ann XXXIII Beitrage zur Theorie der Lame' schen Functionen, and Franz.

Where $G_1(q)=q$, $G_2(q)=\alpha\beta q^2+\{(\alpha+\beta-\delta+1)+(\gamma+\delta)a\}q-a\gamma$,

$$G_{n+1}(q)=[n\{(\alpha+\beta-\delta+n)+(\gamma+\delta+n-1)a\}+\alpha\beta q]G_n(q)-(\alpha+n-1)(\beta+n-1)(\gamma+n-1)na\,G_{n-1}(q).$$

The series is absolutely convergent for $|x|<1$ if $(a)>1$;

and fonr $|x|<|a|$ if $(a)<1$. And when $(x)=1$ and $(a)>1$,

a sufficient condition[1] for absolute convergenee is that the real part of

$(\delta-2)$ shall be less than -1; when $(x)=a$, $|a|<1$,

the real part of $(\alpha+\beta-\gamma-\delta-1)$ shall be less than -1.

Moreover $F(a,q;\alpha,\beta,\gamma,\delta,1)$ has a definite value, if real part of

$(\delta-2)<0$ and $|a|>1$; and also $F(a,q;\alpha,\beta,\gamma,\delta,a)$,

if real part of $(\alpha+\beta-\gamma-\delta-1)<0$ and $(a)<1$.[2]

1. *Making the Fuchsian substitutions*[3]

$$y=x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}u$$

in equation (I) we have

$$\frac{d^2u}{dx^2}+\left(\frac{1+\alpha_1-\beta_1}{x}+\frac{1+\alpha_2-\beta_2}{x-1}+\frac{1+\alpha_3-\beta_3}{x-a}\right)\frac{du}{dx}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)x-(\alpha_1+\alpha_2-\alpha_1\beta_2-\alpha_2\beta_1)a-(\alpha_1+\alpha_3-\alpha_1\beta_3-\alpha_3\beta_1)-\alpha_4\beta_4 q}{x(x-1)(x-a)}u=0$$

Let $q_1=+\dfrac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+(\alpha_1+\alpha_2+\alpha_3+\beta_4)+\alpha_4\beta_4 q}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_3)}$

Hence the above equatiun becomes

$$\frac{d^2u}{dx^2}+\left(\frac{1+\alpha_1-\beta_1}{x}+\frac{1+\alpha_2-\beta_2}{x-1}+\frac{1+\alpha_3-\beta_3}{x-a}\right)\frac{du}{dx}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)(x-q_1)}{x(x-1)(x-a)}u=0 \quad\cdots\cdots\ (\mathrm{II}_1')$$

whose scheme is

$$P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & x \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{matrix}\right\}$$

and a particular solution of (I) we can easily see is

$$y=x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}F(a,q_1;\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\alpha_3+\beta_4,1+\alpha_1-\beta_1,1+\alpha_2-\beta_2,x),$$

provided $\beta_1-\alpha_1$ is not a negative integer. For simplicity, we shall, throughout this discussion,

---

1 Weierstrass, Abhandlungen aus der Functionenlehre p. 220. The condition is also necessary. Cf. Bromwich, Infinite Series, pp. 202-4.

2 Heun and Franz.

3 A formula given in L. Heffter, Linearen differential gleichungen (1894), pp. 224-6, also T. Craig, Linear differential equations (1889), Vol. 1, pp. 154-6.

assume none of the exponent differences $\beta_i-\alpha_i$, $(i=1,2,3,4)$ is zero or an integer, as in this exceptional case the general solution of the differential equation may involve logarithmic terms. *The formulae in the exceptional case can be found in Franz's work.* Now if in the above expression α_j be interchanged with β_j, $(j=1,2,3)$ singly, doubly, or triply while α_4 and β_4 remain fixed, it must still satisfy the differential equation (I), since the latter is unaffected by this change. We thus obtain altogether eight expressions. Moreover, if, in equation (II'₁), we set $t=1-x$, then we have

$$\frac{d^2u}{dt^2}+\left(\frac{1+\alpha_2-\beta_2}{t}+\frac{1+\alpha_1-\beta_1}{t-1}+\frac{1+\alpha_3-\beta_3}{t-\overline{1-a}}\right)\frac{du}{dt}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\{t-\{1-q_1\}\}}{t(t-1)(t-\overline{1-a})}u=0 \cdots (II'_2)$$

with scheme

$$P\left\{\begin{matrix} 1 & 0 & 1-a & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 1-x \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{matrix}\right\}$$

and a particular solution of (I) is

$$y=x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}F(1-a,\ 1-q_1;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_1-\beta_1,\ 1-x).$$

We thus obtain eight new expressions. Similarly, we set

$$t=\frac{x}{a},\quad \frac{a-x}{a},\quad \frac{x-1}{a-1},\quad \frac{x-a}{1-a}$$

respectively, we have the differential equations with the corresponding schemes and solutions as follows:

$$\frac{d^2u}{dt^2}+\left(\frac{1+\alpha_1-\beta_1}{t}+\frac{1+\alpha_2-\beta_2}{t-1}+\frac{1+\alpha_3-\beta_3}{t-\frac{1}{a}}\right)\frac{du}{dt}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)(t-\frac{q_1}{a})}{t(t-1)(t-\frac{1}{a})}u=0 \cdots\cdots (II'_3)$$

$$
P\left\{\begin{matrix} 0 & \frac{1}{a} & 1 & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{x}{a} \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{matrix}\right\}
$$

$$
y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\frac{1}{a}, \frac{q_1}{a}, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\alpha_3-\beta_3, \frac{x}{a}\right);
$$

$$
\frac{d^2u}{dt^2} + \left(\frac{1+\alpha_3-\beta_3}{t} + \frac{1+\alpha_1-\beta_1}{t-1} + \frac{1+\alpha_2-\beta_2}{t-\frac{a-1}{a}}\right)\frac{du}{dx} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\left(t-\frac{a-q_1}{a}\right)u}{t(t-1)\left(t-\frac{a-1}{a}\right)} = 0 \cdots\cdots (II'_4)
$$

$$
P\left\{\begin{matrix} 1 & \frac{a-1}{a} & 0 & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{a-x}{a} \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{matrix}\right\}
$$

$$
y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\frac{a-1}{a}, \frac{a-q_1}{a}; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_3-\beta_3, 1+\alpha_1-\beta_1, \frac{a-x}{a}\right);
$$

$$
\frac{d^2u}{dt^2} + \left(\frac{1+\alpha_2-\beta_2}{t} + \frac{1+\alpha_3-\beta_3}{t-1} + \frac{1+\alpha_1-\beta_1}{t-\frac{1}{1-a}}\right)\frac{du}{dt} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\left(t-\frac{q_1-1}{a-1}\right)u}{t(t-1)\left(t-\frac{1}{1-a}\right)} = 0 \cdots\cdots (II'_5)
$$

$$
P\left\{\begin{matrix} \frac{1}{1-a} & 0 & 1 & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{x-1}{a-1} \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{matrix}\right\}
$$

$$
y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\frac{1}{1-a}, \frac{q_1-1}{a-1}, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_2-\beta_2, 1+\alpha_3-\beta_3, \frac{x-1}{a-1}\right);
$$

$$
\frac{d^2u}{dt^2} + \left(\frac{1+\alpha_3-\beta_3}{t} + \frac{1+\alpha_2-\beta_2}{t-1} + \frac{1+\alpha_1-\beta_1}{t-\frac{a}{a-1}}\right)\frac{du}{dt} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)\left(t-\frac{q_1-a}{1-a}\right)u}{t(t-1)\left(t-\frac{a}{a-1}\right)} = 0 \cdots\cdots (II'_6)
$$

$$
P\left\{\begin{matrix} \frac{a}{a-1} & 1 & 0 & \infty & \\ 0 & 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \frac{x-a}{1-a} \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\beta_4 & \end{matrix}\right\}
$$

$$
y = x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\, F\left(\frac{a}{a-1}, \frac{q_1-a}{1-a}; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_3-\beta_3, 1+\alpha_2-\beta_2, \frac{x-a}{1-a}\right)
$$

We thus obtain $5 \times 8 = 40$ new expressions, which together with the original eight mkae forty-eight particular solutions of equation (I). The first set of forty-eight solutions may be written down as follows :

$$
\left.\begin{aligned}
y_1 &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(a, q_1; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\alpha_2-\beta_2, x) \\
y_2 &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(a, q_1'; \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\alpha_2+\alpha_3+\beta_4, 1+\beta_1-\alpha_1, 1+\alpha_2-\beta_2, x) \\
y_3 &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(a, q_1''; \alpha_1+\beta_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\beta_2-\alpha_2, x) \\
y_4 &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(a, q_1'''; \beta_1+\beta_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\beta_4, 1+\beta_1-\alpha_1, 1+\beta_2-\alpha_2, x) \\
y_5 &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(a, q_1^{iv}; \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\alpha_2+\beta_3+\beta_4, 1+\alpha_1-\beta_1, 1+\alpha_2-\beta_2, x) \\
y_6 &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(a, q_1^{v}; \alpha_1+\beta_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\alpha_1-\beta_1, 1+\beta_2-\alpha_2\ x) \\
y_7 &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(a, q_1^{vi}; \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\alpha_2+\beta_3+\beta_4, 1+\beta_1-\alpha_1, 1+\alpha_2-\beta_2\ x) \\
y_8 &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(a, q_1^{vii}; \beta_1+\beta_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_1-\alpha_1, 1+\beta_2-\alpha_2\ x)
\end{aligned}\right\} (\mathrm{II}_1')
$$

$$
\left.\begin{aligned}
y_9 &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(1-a, 1-q_1; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_2-\beta_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{10} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3} F(1-a, 1-q_1'; \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\alpha_2+\alpha_3+\beta_4, 1+\alpha_2-\beta_2, 1+\beta_1-\alpha_1, 1-x) \\
y_{11} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(1-a, 1-q_1''; \alpha_1+\beta_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\beta_2-\alpha_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{12} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3} F(1-a, 1-q_1'''; \beta_1+\beta_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\beta_4, 1+\beta_2-\alpha_2, 1+\beta_1-\alpha_1, 1-x) \\
y_{13} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{iv}; \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\alpha_2+\beta_3+\beta_4, 1+\alpha_2-\beta_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{14} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{v}; \alpha_1+\beta_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\beta_2-\alpha_2, 1+\alpha_1-\beta_1, 1-x) \\
y_{15} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{vi}; \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\alpha_2+\beta_3+\beta_4, 1+\alpha_2-\beta_2, 1+\beta_1-\alpha_1, 1-x) \\
y_{16} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3} F(1-a, 1-q_1^{vii}; \beta_1+\beta_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_2-\alpha_2, 1+\beta_1-\alpha_1, 1-x)
\end{aligned}\right\} (\mathrm{II}_2')
$$

$$\left.\begin{aligned}
y_{17} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{a},\tfrac{q_1}{a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{18} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{a},\tfrac{q_1'}{a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{19} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{a},\tfrac{q_1''}{a};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{20} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{a},\tfrac{q_1'''}{a};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_3-\beta_3,\ \tfrac{x}{a}\right)\\
y_{21} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{a},\tfrac{q_1^{iv}}{a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)\\
y_{22} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{a},\tfrac{q_1^{v}}{a};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)\\
y_{23} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{a},\tfrac{q_1^{vi}}{a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)\\
y_{24} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{a},\tfrac{q_1^{vii}}{a};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_3-\alpha_3,\ \tfrac{x}{a}\right)
\end{aligned}\right\}\ (\mathrm{II}_3')$$

$$\left.\begin{aligned}
y_{25} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1}{a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{26} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1'}{a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)\\
y_{27} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1''}{a};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{28} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1'''}{a};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)\\
y_{29} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1^{iv}}{a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{30} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1^{v}}{a};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_1-\beta_1,\ \tfrac{a-x}{a}\right)\\
y_{31} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1^{vi}}{a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)\\
y_{32} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{a-1}{a},\tfrac{a-q_1^{vii}}{a};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_1-\alpha_1,\ \tfrac{a-x}{a}\right)
\end{aligned}\right\}\ (\mathrm{II}_4')$$

$$
\left.\begin{aligned}
y_{33} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1-1}{a-1};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{34} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1'-1}{a-1};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{35} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1''-1}{a-1};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{36} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1'''-1}{a-1};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_3-\beta_3,\ \tfrac{x-1}{a-1}\right)\\
y_{37} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1^{iv}-1}{a-1};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)\\
y_{38} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1^{v}-1}{a-1};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)\\
y_{39} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1^{vi}-1}{a-1};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)\\
y_{40} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{1}{1-a},\tfrac{q_1^{vii}-1}{a-1};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_3-\alpha_3,\ \tfrac{x-1}{a-1}\right)
\end{aligned}\right\}(\mathrm{II}'_5)
$$

$$
\left.\begin{aligned}
y_{41} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1-a}{1-a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{42} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1'-a}{1-a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{43} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1''-a}{1-a};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)\\
y_{44} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\alpha_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1'''-a}{1-a};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)\\
y_{45} &= x^{\alpha_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1^{iv}-a}{1-a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{46} &= x^{\alpha_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1^{v}-a}{1-a};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)\\
y_{47} &= x^{\beta_1}(x-1)^{\alpha_2}(x-a)^{\beta_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1^{vi}-a}{1-a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{a-1}\right)\\
y_{48} &= x^{\beta_1}(x-1)^{\beta_2}(x-a)^{\beta_3}\,F\left(\tfrac{a}{a-1},\tfrac{q_1^{vii}-a}{1-a};\ \beta_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{a-1}\right)
\end{aligned}\right\}(\mathrm{II}'_6)
$$

Where

$$q_1 = \frac{(\alpha_1+\alpha_2-\alpha_1\beta_2-\alpha_2\beta_1)\,a+(\alpha_1+\alpha_3-\alpha_1\beta_3-\alpha_3\beta_1)+\alpha_4\beta_4 q}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\beta_4)}$$

$$q_1' = \frac{(\beta_1+\alpha_2-\beta_1\beta_2-\alpha_2\alpha_1)\,a+(\beta_1+\alpha_3-\beta_1\beta_3-\alpha_3\beta_1)+\alpha_4\beta_4 q}{(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\beta_4)}$$

$$q_1'' = \frac{(\alpha_1+\beta_2-\alpha_1\alpha_2-\beta_2\beta_1)\,a+(\alpha_1+\alpha_3-\alpha_1\beta_3-\alpha_3\beta_1)+\alpha_4\beta_4 q}{(\alpha_1+\beta_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\beta_4)}$$

$$q_1''' = \frac{(\beta_1+\beta_2-\beta_1\alpha_2-\beta_2\alpha_1)\,a+(\beta_1+\alpha_3-\beta_1\beta_3-\alpha_3\alpha_1)+\alpha_4\beta_4 q}{(\beta_1+\beta_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\alpha_3+\beta_4)}$$

$$q_1^{IV} = \frac{(\alpha_1+\alpha_2-\alpha_1\beta_2-\alpha_2\beta_1)\,a+(\alpha_1+\beta_3-\alpha_1\alpha_3-\beta_3\beta_1)+\alpha_4\beta_4 q}{(\alpha_1+\alpha_2+\beta_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\beta_4)}$$

$$q_1^{V} = \frac{(\alpha_1+\beta_2-\alpha_1\alpha_2-\beta_1\beta_2)\,a+(\alpha_1+\beta_3-\alpha_1\alpha_3-\beta_1\beta_3)+\alpha_4\beta_4 q}{(\alpha_1+\beta_2+\beta_3+\alpha_4)(\alpha_1+\beta_2+\beta_3+\beta_4)}$$

$$q_1^{VI} = \frac{(\beta_1+\alpha_2-\beta_1\beta_2-\alpha_1\alpha_2)\,a+(\beta_1+\beta_3-\beta_1\alpha_3-\alpha_1\beta_3)+\alpha_4\beta_4 q}{(\beta_1+\alpha_2+\beta_2+\alpha_4)(\beta_1+\alpha_2+\beta_3+\beta_4)}$$

$$q_1^{VII} = \frac{(\beta_1+\beta_2-\beta_1\alpha_2-\alpha_1\beta_2)\,a+(\beta_1+\beta_3-\beta_1\alpha_3-\alpha_1\beta_3)+\alpha_4\beta_4 q}{(\beta_1+\beta_2+\beta_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\beta_4)}$$

2. We certainly can obtain the other solutions from any one of equations (II'_1), but in doing so we have to change the dependent variable, and use Fuchsian substitutions. Wherefore we prefer to use equation (I). Letting $t = \frac{1}{x}$ [1], we have

$$\frac{d^2y}{dt^2} + \frac{(1-\alpha_4-\beta_4)(t-1)(t-\frac{1}{a}) + (1-\alpha_2-\beta_2)t(t-\frac{1}{a}) + (1-\alpha_3-\beta_3)t(t-1)}{t(t-1)(t-\frac{1}{a})}\frac{dy}{dt}$$

$$+\frac{\alpha_1\beta_1(t-1)^2(t-\frac{1}{a})^2 + \alpha_2\beta_2(t-\frac{1}{a})^2 + \alpha_3\beta_3\frac{1}{a^2}(t-1)^2 + \frac{1}{a}\left\{(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)-\alpha_4\beta_4 q t\right\}(t-1)(t-\frac{1}{a})}{t^2(t-1)^2(t-\frac{1}{a})^2}\, y = 0 \ldots\ldots (3)$$

with scheme

$$P\left\{\begin{matrix} \infty & 1 & \frac{1}{a} & 0 & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \frac{1}{x} \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{matrix}\right\}.$$

Using Fuchsian substitution

$$y = t^{\alpha_4}(t-1)^{\alpha_2}(t-\tfrac{1}{a})^{\alpha_3}u,$$

we have the differential equation

$$\frac{d^2u}{dt^2} + \left(\frac{1+\alpha_4-\beta_4}{t} + \frac{1+\alpha_2-\beta_2}{t-1} + \frac{1+\alpha_3-\beta_3}{t-\frac{1}{a}}\right)\frac{du}{dt} + \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)(t-q_2)u}{t(t-1)(t-\frac{1}{a})} = 0 \ldots\ldots (II''_1)$$

With scheme

$$P\left\{\begin{matrix} \infty & 1 & \frac{1}{a} & 0 & \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & 0 & \frac{1}{x} \\ \beta_1+\alpha_2+\alpha_3+\alpha_4 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\}$$

Where

$$q_2 = -\frac{\left\{(\alpha_3+\alpha_4)(\beta_3+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\} + \left\{(\alpha_2+\alpha_4)(\beta_2+\beta_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\right\}\frac{1}{a}}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)}$$

Hence a particular solution of (I) is

$$y = x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2} F\left(\tfrac{1}{a}, q_2, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\alpha_2+\alpha_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\alpha_2-\beta_2, \tfrac{1}{x}\right).$$

---

[1] Whittaker, Modern Analysis, gives a formula 10.4. p. 202

Moreover, we set $\quad \xi = \frac{a}{x},\ \frac{x-1}{x},\ \frac{x-a}{x},\ \frac{x-a}{(1-a)x},\ \frac{a(x-1)}{(a-1)x}$

hence $\quad t = \xi,\ 1-\xi,\ \frac{1-\xi}{a},\ \frac{1}{a}-\frac{1-a}{a}\xi,\ 1-\frac{a-1}{a}\xi$ respectively

By making substitutions in equation (II^1), we obtain

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_4-\beta_4}{\xi}+\frac{1+\alpha_3-\beta_3}{\xi-1}+\frac{1+\alpha_2-\beta_2}{\xi-a}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\xi-q_2 a)\,u}{} = 0 \ldots\ldots (II_2'')$$

$$P\left\{\begin{matrix} \infty & a & 1 & 0 & \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0\,. & 0 & \frac{a}{x} \\ \beta_1+\alpha_2+\alpha_3+\alpha_4 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\},$$

$$y = x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(a,\ aq_2;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_3-\beta_3,\ \frac{a}{x}\right)$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_2-\beta_2}{\xi}+\frac{1+\alpha_4-\beta_4}{\xi-1}+\frac{1+\alpha_3-\beta_3}{\xi-\frac{a-1}{a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\xi-\overline{1-q_2})\,u}{\xi(\xi-1)(\xi-\frac{a-1}{a})} = 0 \ldots\ldots (II_3'')$$

$$P\left\{\begin{matrix} \infty & 0 & \frac{a-1}{a} & 1 & \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & 0 & \frac{x-1}{x} \\ \beta_1+\alpha_2+\alpha_3+\alpha_4 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & . \end{matrix}\right\}$$

$$y = x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\frac{a-1}{a},\ 1-q_2;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_4-\beta_4,\ \frac{x-1}{x}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_3-\beta_3}{\xi}+\frac{1+\alpha_4-\beta_4}{\xi-1}+\frac{1+\alpha_2-\beta_2}{\xi-\overline{1-a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\xi-\overline{1-aq_2})\,u}{\xi(\xi-1)(\xi-\overline{1-a})} = 0 \ldots\ldots (II_4'')$$

$$y = x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(1-a,\ 1-aq_2;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_4-\beta_4,\ \frac{x-a}{x}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_3-\beta_3}{\xi}+\frac{1+\alpha_2-\beta_2}{\xi-1}+\frac{1+\alpha_4-\beta_4}{\xi-\frac{1}{1-a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\xi-\frac{1-q_2 a}{1-a})\,u}{\xi(\xi-1)(\xi-\frac{1}{1-a})} = 0 \ldots\ldots (II_5'')$$

$$
P\left\{\begin{matrix} \infty & 1 & 0 & \frac{1}{1-a} & \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & 0 & \frac{x-a}{(1-a)x} \\ \beta_1+\alpha_2+\alpha_3+\alpha_4 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\},
$$

$$
y = x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2} F\left(\frac{1}{1-a}, \frac{1-q_2 a}{1-a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\alpha_2+\alpha_3+\alpha_4, 1+\alpha_3-\beta_3, 1+\alpha_2-\beta_2, \frac{x-a}{(1-a)x}\right);
$$

$$
\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_2-\beta_2}{\xi}+\frac{1+\alpha_3-\beta_3}{\xi-1}+\frac{1+\alpha_4-\beta_4}{\xi-\frac{a}{a-1}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)\left(\xi-\frac{a(1-q_2)}{a-1}\right)u}{\xi(\xi-1)\left(\xi-\frac{a}{a-1}\right)} = 0 \ldots\ldots (II_6'')
$$

$$
P\left\{\begin{matrix} \infty & 0 & 1 & \frac{a}{a-1} & \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & 0 & \frac{a(x-1)}{(a-1)x} \\ \beta_1+\alpha_2+\alpha_3+\alpha_4 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\},
$$

$$
y = x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2} F\left(\frac{a}{a-1}, \frac{a(1-q_2)}{a-1};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\alpha_2+\alpha_3+\alpha_4, 1+\alpha_2-\beta_2, 1+\alpha_3-\beta_3, \frac{a(x-1)}{(a-1)x}\right).
$$

The second set of forty-eight solutions may be written down as follows:

$$\left.\begin{aligned}
y_{49} &= x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\tfrac{1}{a}, q_2;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_2-\beta_3,\ \tfrac{1}{x}\right)\\
y_{50} &= x^{-(\beta_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\tfrac{1}{a}, q_2';\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3,\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\alpha_2-\beta_2,\ \tfrac{1}{x}\right)\\
y_{51} &= x^{-(\alpha_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{1}{a}, q_2'';\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\beta_2-\alpha_2,\ \tfrac{1}{x}\right)\\
y_{52} &= x^{-(\beta_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{1}{a}, q_2''';\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\beta_2-\alpha_2,\ \tfrac{1}{x}\right)\\
y_{53} &= x^{-(\alpha_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{1}{a}, q_2^{iv};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_2-\beta_2,\ \tfrac{1}{x}\right)\\
y_{54} &= x^{-(\alpha_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{1}{a}, q_2^{v};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\beta_2-\alpha_2,\ \tfrac{1}{x}\right)\\
y_{55} &= x^{-(\beta_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{1}{a}, q_2^{vi};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\alpha_2-\beta_2,\ \tfrac{1}{x}\right)\\
y_{56} &= x^{-(\beta_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{1}{a}, q_2^{vii};\ \alpha_1+\beta_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\beta_2-\alpha_2,\ \tfrac{1}{x}\right)
\end{aligned}\right\}\ (\mathrm{II}_1'')$$

$$\left.\begin{aligned}
y_{57} &= x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(a, aq_2;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_3-\beta_3,\ \tfrac{a}{x}\right)\\
y_{58} &= x^{-(\beta_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(a, aq_2';\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\alpha_3-\beta_3,\ \tfrac{a}{x}\right)\\
y_{59} &= x^{-(\alpha_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(a, aq_2'';\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_3-\beta_3,\ \tfrac{a}{x}\right)\\
y_{60} &= x^{-(\beta_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(a, aq_2''';\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\alpha_3-\beta_3,\ \tfrac{a}{x}\right)\\
y_{61} &= x^{-(\alpha_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(a, aq_2^{iv};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\beta_3-\alpha_3,\ \tfrac{a}{x}\right)\\
y_{62} &= x^{-(\alpha_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(a, aq_2^{v};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\beta_3-\alpha_3,\ \tfrac{a}{x}\right)\\
y_{63} &= x^{-(\beta_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(a, aq_2^{vi};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\beta_3-\alpha_3,\ \tfrac{a}{x}\right)\\
y_{64} &= x^{-(\beta_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(a, aq_2^{vii};\ \alpha_1+\beta_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\beta_3-\alpha_3,\ \tfrac{a}{x}\right)
\end{aligned}\right\}\ (\mathrm{II}_2'')$$

$$
\left.\begin{aligned}
y_{65} &= x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\tfrac{a-1}{a},1-g_2;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_4-\beta_4,\ \tfrac{x-1}{x}\right)\\
y_{66} &= x^{-(\beta_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\tfrac{a-1}{a},1-g_2';\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_4-\alpha_4,\ \tfrac{x-1}{x}\right)\\
y_{67} &= x^{-(\alpha_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{a-1}{a},1-g_2'';\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_4-\beta_4,\ \tfrac{x-1}{x}\right)\\
y_{68} &= x^{-(\beta_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{a-1}{a},1-g_2''';\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_4-\alpha_4,\ \tfrac{x-1}{x}\right)\\
y_{69} &= x^{-(\alpha_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{a-1}{a},1-g_2^{iv};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_4-\beta_4,\ \tfrac{x-1}{x}\right)\\
y_{70} &= x^{-(\alpha_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{a-1}{a},1-g_2^{v};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_4-\beta_4,\ \tfrac{x-1}{x}\right)\\
y_{71} &= x^{-(\beta_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{a-1}{a},1-g_2^{vi};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_4-\alpha_4,\ \tfrac{x-1}{x}\right)\\
y_{72} &= x^{-(\beta_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{a-1}{a},1-g_2^{vii};\ \alpha_1+\beta_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_4-\alpha_4,\ \tfrac{x-1}{a}\right)
\end{aligned}\right\}\ (\mathrm{II}_3'')
$$

$$
\left.\begin{aligned}
y_{73} &= x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(1-a,1-ag_2;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_4-\beta_4,\ \tfrac{x-a}{x}\right)\\
y_{74} &= x^{-(\beta_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(1-a,1-ag_2';\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_4-\alpha_4,\ \tfrac{x-a}{x}\right)\\
y_{75} &= x^{-(\alpha_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(1-a,1-ag_2'';\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_4-\beta_4,\ \tfrac{x-a}{x}\right)\\
y_{76} &= x^{-(\beta_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(1-a,1-ag_2''';\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_4-\alpha_4,\ \tfrac{x-a}{x}\right)\\
y_{77} &= x^{-(\alpha_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(1-a,1-ag_2^{iv};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_4-\beta_4,\ \tfrac{x-a}{x}\right)\\
y_{78} &= x^{-(\alpha_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(1-a,1-ag_2^{v};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_4-\beta_4,\ \tfrac{x-a}{x}\right)\\
y_{79} &= x^{-(\beta_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(1-a,1-ag_2^{vi};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_4-\alpha_4,\ \tfrac{x-a}{x}\right)\\
y_{80} &= x^{-(\beta_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(1-a,1-ag_2^{vii};\ \alpha_1+\beta_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_4-\alpha_4,\ \tfrac{x-a}{x}\right)
\end{aligned}\right\}\ (\mathrm{II}_4'')
$$

$$
\left.\begin{aligned}
y_{81} &= x^{-(\alpha_2+\alpha_3+\alpha_4)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{(1-a)x}\right)\\
y_{82} &= x^{-(\beta_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{(1-a)x}\right)\\
y_{83} &= x^{-(\alpha_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{(1-a)x}\right)\\
y_{84} &= x^{-(\beta_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{(1-a)x}\right)\\
y_{85} &= x^{-(\alpha_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{(1-a)x}\right)\\
y_{86} &= x^{-(\alpha_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{(1-a)x}\right)\\
y_{87} &= x^{-(\beta_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_2-\beta_2,\ \tfrac{x-a}{(1-a)x}\right)\\
y_{88} &= x^{-(\beta_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{1}{1-a},\tfrac{1-g_2a}{1-a};\ \alpha_1+\beta_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_2-\alpha_2,\ \tfrac{x-a}{(1-a)x}\right)
\end{aligned}\right\}(\mathrm{II}''_5)
$$

$$
\left.\begin{aligned}
y_{91} &= x^{-(\alpha_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_3-\beta_3,\ \tfrac{a(x-1)}{(a-1)x}\right)\\
y_{90} &= x^{-(\beta_4+\alpha_3+\alpha_2)}(x-1)^{\alpha_3}(x-a)^{\alpha_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_3-\beta_3,\ \tfrac{a(x-1)}{(a-1)x}\right)\\
y_{91} &= x^{-(\alpha_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_3-\beta_3,\ \tfrac{a(x-1)}{(a-1)x}\right)\\
y_{92} &= x^{-(\beta_4+\alpha_3+\beta_2)}(x-1)^{\alpha_3}(x-a)^{\beta_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\alpha_3-\beta_3,\ \tfrac{a(x-1)}{(a-1)x}\right)\\
y_{93} &= x^{-(\alpha_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_3-\alpha_3,\ \tfrac{a(x-1)}{(a-1)x}\right)\\
y_{94} &= x^{-(\alpha_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_3-\alpha_3,\ \tfrac{a(x-1)}{(a-1)x}\right)\\
y_{95} &= x^{-(\beta_4+\beta_3+\alpha_2)}(x-1)^{\beta_3}(x-a)^{\alpha_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_2-\beta_2,\ 1+\beta_3-\alpha_3,\ \tfrac{a(x-1)}{(a-1)x}\right)\\
y_{96} &= x^{-(\beta_4+\beta_3+\beta_2)}(x-1)^{\beta_3}(x-a)^{\beta_2}F\left(\tfrac{a}{a-1},\tfrac{a(1-g_2)}{a-1};\ \alpha_1+\beta_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_3-\alpha_3,\ \tfrac{a(x-1)}{(a-1)x}\right)
\end{aligned}\right\}(\mathrm{II}''_6)
$$

Where

$$q_2 = -\frac{\{(\alpha_2+\alpha_4)(\beta_3+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}+\{(\alpha_2+\alpha_4)(\beta_2+\beta_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\alpha_3+\alpha_4)},$$

$$q_2' = -\frac{\{(\alpha_2+\beta_4)(\beta_3+\alpha_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}+\{(\alpha_2+\beta_4)(\beta_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\alpha_2+\alpha_3+\beta_4)(\beta_1+\alpha_2+\alpha_3+\beta_4)},$$

$$q_2'' = -\frac{\{(\beta_2+\alpha_4)(\beta_3+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}+\{(\beta_2+\alpha_4)(\alpha_2+\beta_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\beta_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\alpha_3+\alpha_4)},$$

$$q_2''' = -\frac{\{(\beta_2+\beta_4)(\beta_3+\alpha_4-1)-\alpha_1\beta_1+\alpha_2\beta_2\}+\{(\beta_2+\beta_4)(\alpha_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\beta_2+\alpha_3+\beta_4)(\beta_1+\beta_2+\alpha_3+\beta_4)},$$

$$q_2^{iv} = -\frac{\{(\alpha_2+\alpha_4)(\alpha_3+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}+\{(\alpha_2+\alpha_4)(\beta_2+\beta_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\alpha_2+\beta_3+\alpha_4)(\beta_1+\alpha_2+\beta_3+\alpha_4)},$$

$$q_2^{v} = -\frac{\{(\beta_2+\alpha_4)(\alpha_3+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}+\{(\beta_2+\alpha_4)(\alpha_2+\beta_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\beta_2+\beta_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\alpha_4)},$$

$$q_2^{vi} = -\frac{\{(\alpha_2+\beta_4)(\alpha_3+\alpha_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}+\{(\alpha_2+\beta_4)(\beta_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\alpha_2+\beta_3+\beta_4)(\beta_1+\alpha_2+\beta_3+\beta_4)},$$

$$q_2^{vii} = -\frac{\{(\beta_2+\beta_4)(\alpha_3+\alpha_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}+\{(\beta_2+\beta_4)(\alpha_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_3\beta_3-\alpha_4\beta_4 q\}\frac{1}{a}}{(\alpha_1+\beta_2+\beta_3+\beta_4)(\beta_1+\beta_2+\beta_3+\beta_4)}.$$

3. To find the particular solutions when ∞ lies in the 2nd column of the scheme, we set $Z = 1-X$ in (I) and obtain

$$\frac{d^2y}{dz^2}+\left[\frac{1-\alpha_2-\beta_2}{z}+\frac{1-\alpha_1-\beta_1}{z-1}+\frac{1-\alpha_3-\beta_3}{z-(1-a)}\right]\frac{dy}{dz}+\left[\frac{\alpha_2\beta_2}{z^2}+\frac{\alpha_1\beta_1}{(z-1)^2}+\frac{\alpha_3\beta_3}{(z-a)^2}+\frac{(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)z-(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3+\alpha_1 a q)}{z(z-1)(z-1-a)}\right]y=0 \qquad \cdots\cdots(4)$$

The scheme is

$$P\left\{\begin{array}{ccccc} 1 & 0 & 1-a & \infty & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & z\,(=1-x) \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{array}\right\}$$

Putting $t=\frac{1}{z}$ in (4), we have

$$\frac{d^2y}{dz^2}+\frac{(1+\alpha_2+\beta_2)(t-1)(t-\frac{1}{1-a})+(1-\alpha_1-\beta_1)(t-\frac{1}{1-a})+\frac{1}{1-a}(1-\alpha_3-\beta_3)(t-1)}{t(t-1)(t-\frac{1}{1-a})}\frac{dy}{dt}$$

$$+\frac{\alpha_2\beta_2(t-1)^2(t-\frac{1}{1-a})^2+\alpha_1\beta_1(t-\frac{1}{1-a})^2+\frac{1}{(1-a)^2}\alpha_3\beta_3(t-1)^2+\frac{1}{1-a}\{(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)-(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3-\alpha_1 a q)t\}(t-1)(t-\frac{1}{1-a})}{t^2(t-1)^2(t-\frac{1}{1-a})^2}y=0 \qquad \cdots\cdots(5)$$

The scheme is

$$P\left\{\begin{array}{ccccc} 1 & \infty & \frac{1}{1-a} & 0 & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & t\,(=\frac{1}{1-x}) \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{array}\right\}$$

Making the Fuchsian substitution

$$y=t^{\alpha_4}(t-1)^{\alpha_1}(t-\tfrac{1}{1-a})^{\alpha_3}u,$$

we obtain

$$\frac{d^2u}{dt^2}+\left(\frac{1+\alpha_4-\beta_4}{t}+\frac{1+\alpha_1-\beta_1}{t-1}+\frac{1+\alpha_3-\beta_3}{t-\frac{1}{1-a}}\right)\frac{du}{dt}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)(t-q_3)u}{t(t-1)(t-\frac{1}{1-a})} \cdots\cdots\cdots\cdots(II_1''')$$

With scheme

$$P\left\{\begin{array}{ccccc} 1 & \infty & \frac{1}{1-a} & 0 & \\ 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & t\,(=\frac{1}{1-x}) \\ \beta_1-\alpha_1 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{array}\right\}$$

where

$$q_3=\frac{(\alpha_1+\alpha_4-\alpha_4\beta_1-\alpha_1\beta_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)\frac{1}{1-a}-(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_3\beta_3+\alpha_2\beta_2)}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)}$$

And a particular solution of (I) is

$$y = (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(\tfrac{1}{1-a}, \gamma_3; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\alpha_1-\beta_1, \tfrac{1}{1-x}\right).$$

In equation (II_1'''), let

$$\xi = \frac{x}{x-1},\ \frac{x-a}{x-1},\ \frac{a-1}{x-1},\ \frac{x-a}{a(x-1)},\ \frac{(a-1)x}{a(x-1)}$$

Hence $$t = 1-\xi,\ \frac{\xi-1}{a-1},\ \frac{\xi}{1-a},\ \frac{a\xi-1}{a-1},\ \frac{a\xi-(a-1)}{1-a}$$

We have the following:

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_1-\beta_1}{\xi}+\frac{1+\alpha_4-\beta_4}{\xi-1}+\frac{1+\alpha_3-\beta_3}{\xi-\frac{a}{a-1}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)(\xi-\overline{1-\gamma_3})u}{\xi(\xi-1)\left(\xi-\frac{a}{a-1}\right)}=0 \text{ ------ } (\mathrm{II}_2''')$$

$$P\left\{\begin{matrix} 0 & \infty & \frac{a}{a-1} & 1 & \\ 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & \xi\left(=\frac{x}{x-1}\right) \\ \beta_1-\alpha_1 & \alpha_1+\beta_2+\alpha_3+\alpha_4 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\}$$

$$y = (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(\tfrac{a}{a-1}, 1-\gamma_3; \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_1-\beta_1, 1+\alpha_4-\beta_4, \tfrac{x}{x-1}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_3-\beta_3}{\xi}+\frac{1+\alpha_4-\beta_4}{\xi-1}+\frac{1+\alpha_1-\beta_1}{\xi-a}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)\{\xi-(1-\gamma_3\overline{a-1})\}u}{\xi(\xi-1)(\xi-a)}=0 \text{ ------- } (\mathrm{II}_3''')$$

$$P\left\{\begin{matrix} a & \infty & 0 & 1 & \\ 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & \xi\left(=\frac{x-a}{x-1}\right) \\ \beta_1-\alpha_1 & \alpha_1+\beta_2+\alpha_3+\alpha_4 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\},$$

$$y = (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(a, 1-\gamma_3(a-1); \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_3-\beta_3, 1+\alpha_4-\beta_4, \tfrac{x-a}{x-1}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_4-\beta_4}{\xi}+\frac{1+\alpha_3-\beta_3}{\xi-1}+\frac{1+\alpha_1-\beta_1}{\xi-\overline{1-a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)(\xi-q_3\,\overline{1-a})\,u}{\xi(\xi-1)(\xi-\overline{1-a})}=0\ldots\ldots(II_4^{\prime\prime\prime})$$

$$P\left\{\begin{matrix} 1-a, & \infty & 1 & 0 & \\ 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & \xi\left(=\frac{a-1}{x-1}\right) \\ \beta_1-\alpha_1 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\}$$

$$y=(x-1)^{-(\alpha_2+\alpha_3+\alpha_4)}x^{\alpha_1}(x-a)^{\alpha_4}F\left(1-a,\ q_3(1-a);\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_3-\beta_3,\ \frac{a-1}{x-1}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_3-\beta_3}{\xi}+\frac{1+\alpha_1-\beta_1}{\xi-1}+\frac{1+\alpha_4-\beta_4}{\xi-\frac{1}{a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)\left(\xi-\frac{(a-1)(q_3+1)}{a}\right)u}{\xi(\xi-1)\left(\xi-\frac{1}{a}\right)}=0\ldots\ldots(II_5^{\prime\prime\prime})$$

$$P\left\{\begin{matrix} 1 & \infty & 0 & \frac{1}{a} & \\ 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & \xi\left(=\frac{x-a}{a(x-1)}\right) \\ \beta_1-\alpha_1 & \alpha_1+\beta_2+\alpha_3+\alpha_4 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\}$$

$$y=(x-1)^{-(\alpha_4+\alpha_1+\alpha_3)}x^{\alpha_1}(x-a)^{\alpha_3}F\left(\frac{1}{a},\ \frac{(a-1)q_3+1}{a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_1-\beta_1,\ \frac{x-a}{a(x-1)}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_1-\beta_1}{\xi}+\frac{1+\alpha_3-\beta_3}{\xi-1}+\frac{1+\alpha_4-\beta_4}{\xi-\frac{a-1}{a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)\left(\xi-\frac{(1-a)(q_3-1)}{a}\right)u}{\xi(\xi-1)\left(\xi-\frac{a-1}{a}\right)}=0\ldots\ldots(II_6^{\prime\prime\prime})$$

$$P\left\{\begin{matrix} 0 & \infty & 1 & \frac{a-1}{a} & \\ 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & \xi\left(=\frac{(a-1)x}{a(x-1)}\right) \\ \beta_1-\alpha_1 & \alpha_1+\beta_2+\alpha_3+\alpha_4 & \beta_3-\alpha_3 & \beta_4-\alpha_4 & \end{matrix}\right\},$$

$$y=(x-1)^{-(\alpha_4+\alpha_1+\alpha_3)}x^{\alpha_1}(x-a)^{\alpha_3}F\left(\frac{a-1}{a},\ \frac{(1-a)(q_3-1)}{a},\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_3-\beta_3,\ \frac{(a-1)x}{a(x-1)}\right).$$

The third set of forty-eight solutions may be written down as follows:

$$\left.\begin{aligned}
y_{97} &= (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)}x^{\alpha_1}(x-a)^{\alpha_3}F\left(\tfrac{1}{1-a}, q_3;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\alpha_1-\beta_1, \tfrac{1}{1-x}\right)\\
y_{98} &= (x-1)^{-(\beta_4+\alpha_1+\alpha_3)}x^{\alpha_1}(x-a)^{\alpha_3}F\left(\tfrac{1}{1-a}, q_3';\ \alpha_1+\alpha_2+\alpha_3+\beta_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\beta_4-\alpha_4, 1+\alpha_1-\beta_1, \tfrac{1}{1-x}\right)\\
y_{99} &= (x-1)^{-(\alpha_4+\beta_1+\alpha_3)}x^{\beta_1}(x-a)^{\alpha_3}F\left(\tfrac{1}{1-a}, q_3'';\ \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\beta_1-\alpha_1, \tfrac{1}{1-x}\right)\\
y_{100} &= (x-1)^{-(\beta_4+\beta_1+\alpha_3)}x^{\beta_1}(x-a)^{\alpha_3}F\left(\tfrac{1}{1-a}, q_3''';\ \beta_1+\alpha_2+\alpha_3+\beta_4, \beta_1+\beta_2+\alpha_3-\beta_4, 1+\beta_4-\alpha_4, 1+\beta_1-\alpha_1, \tfrac{1}{1-x}\right)\\
y_{101} &= (x-1)^{-(\alpha_4+\alpha_1+\beta_3)}x^{\alpha_1}(x-a)^{\beta_3}F\left(\tfrac{1}{1-a}, q_3^{IV};\ \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\alpha_1-\beta_1, \tfrac{1}{1-x}\right)\\
y_{102} &= (x-1)^{-(\alpha_4+\beta_1+\beta_3)}x^{\beta_1}(x-a)^{\beta_3}F\left(\tfrac{1}{1-a}, q_3^{V};\ \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\beta_1-\alpha_1, \tfrac{1}{1-x}\right)\\
y_{103} &= (x-1)^{-(\beta_4+\alpha_1+\beta_3)}x^{\alpha_1}(x-a)^{\beta_3}F\left(\tfrac{1}{1-a}, q_3^{VI};\ \alpha_1+\alpha_2+\beta_3+\beta_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\beta_4-\alpha_4, 1+\alpha_1-\beta_1, \tfrac{1}{1-x}\right)\\
y_{104} &= (x-1)^{-(\beta_4+\beta_1+\beta_3)}x^{\beta_1}(x-a)^{\beta_3}F\left(\tfrac{1}{1-a}, q_3^{VII};\ \beta_1+\alpha_2+\beta_3+\beta_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_4-\alpha_4, 1+\beta_1-\alpha_1, \tfrac{1}{1-x}\right)
\end{aligned}\right\}\quad (\mathrm{II}_1''')$$

$$\left.\begin{aligned}
y_{105} &= (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)}x^{\alpha_1}(x-a)^{\alpha_3}F\left(\tfrac{a}{a-1}, 1-q_3;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_1-\beta_1, 1+\alpha_4-\beta_4, \tfrac{x}{x-1}\right)\\
y_{106} &= (x-1)^{-(\beta_4+\alpha_1+\alpha_3)}x^{\alpha_1}(x-a)^{\alpha_3}F\left(\tfrac{a}{a-1}, 1-q_3';\ \alpha_1+\alpha_2+\alpha_3+\beta_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\alpha_1-\beta_1, 1+\beta_4-\alpha_4, \tfrac{x}{x-1}\right)\\
y_{107} &= (x-1)^{-(\alpha_4+\beta_1+\alpha_3)}x^{\beta_1}(x-a)^{\alpha_3}F\left(\tfrac{a}{a-1}, 1-q_3'';\ \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\alpha_4, 1+\beta_1-\alpha_1, 1+\alpha_4-\beta_4, \tfrac{x}{x-1}\right)\\
y_{108} &= (x-1)^{-(\beta_4+\beta_1+\alpha_3)}x^{\beta_1}(x-a)^{\alpha_3}F\left(\tfrac{a}{a-1}, 1-q_3''';\ \beta_1+\alpha_2+\alpha_3+\beta_4, \beta_1+\beta_2+\alpha_3+\beta_4, 1+\beta_1-\alpha_1, 1+\beta_4-\alpha_4, \tfrac{x}{x-1}\right)\\
y_{109} &= (x-1)^{-(\alpha_4+\alpha_1+\beta_3)}x^{\alpha_1}(x-a)^{\beta_3}F\left(\tfrac{a}{a-1}, 1-q_3^{IV};\ \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\alpha_4, 1+\alpha_1-\beta_1, 1+\alpha_4-\beta_4, \tfrac{x}{x-1}\right)\\
y_{110} &= (x-1)^{-(\alpha_4+\beta_1+\beta_3)}x^{\beta_1}(x-a)^{\beta_3}F\left(\tfrac{a}{a-1}, 1-q_3^{V};\ \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\alpha_4, 1+\beta_1-\alpha_1, 1+\alpha_4-\beta_4, \tfrac{x}{x-1}\right)\\
y_{111} &= (x-1)^{-(\beta_4+\alpha_1+\beta_3)}x^{\alpha_1}(x-a)^{\beta_3}F\left(\tfrac{a}{a-1}, 1-q_3^{VI};\ \alpha_1+\alpha_2+\beta_3+\beta_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\alpha_1-\beta_1, 1+\beta_4-\alpha_4, \tfrac{x}{x-1}\right)\\
y_{112} &= (x-1)^{-(\beta_4+\beta_1+\beta_3)}x^{\beta_1}(x-a)^{\beta_3}F\left(\tfrac{a}{a-1}, 1-q_3^{VII};\ \beta_1+\alpha_2+\beta_3+\beta_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_1-\alpha_1, 1+\beta_4-\alpha_4, \tfrac{x}{x-1}\right)
\end{aligned}\right\}\quad (\mathrm{II}_2''')$$

$$
\left.\begin{aligned}
y_{113} &= (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(a, 1-q_3(a-1); \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_3-\beta_3, 1+\alpha_4-\beta_4, \tfrac{x-a}{x-1}\right)\\
y_{114} &= (x-1)^{-(\beta_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(a, 1-q_3'(a-1); \alpha_1+\alpha_2+\alpha_3+\beta_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\alpha_3-\beta_3, 1+\beta_4-\alpha_4, \tfrac{x-a}{x-1}\right)\\
y_{115} &= (x-1)^{-(\alpha_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(a, 1-q_3''(a-1); \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_3-\beta_3, 1+\alpha_4-\beta_4, \tfrac{x-a}{x-1}\right)\\
y_{116} &= (x-1)^{-(\beta_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(a, 1-q_3'''(a-1); \beta_1+\alpha_2+\alpha_3+\beta_4, \beta_1+\beta_2+\alpha_3+\beta_4, 1+\alpha_3-\beta_3, 1+\beta_4-\alpha_4, \tfrac{x-a}{x-1}\right)\\
y_{117} &= (x-1)^{-(\alpha_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\beta_3} F\left(a, 1-q_3^{iv}(a-1); \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\alpha_4, 1+\beta_3-\alpha_3, 1+\alpha_4-\beta_4, \tfrac{x-a}{x-1}\right)\\
y_{118} &= (x-1)^{-(\alpha_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\beta_3} F\left(a, 1-q_3^{v}(a-1); \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\alpha_4, 1+\beta_3-\alpha_3, 1+\alpha_4-\beta_4, \tfrac{x-a}{x-1}\right)\\
y_{119} &= (x-1)^{-(\beta_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\beta_3} F\left(a, 1-q_3^{vi}(a-1); \alpha_1+\alpha_2+\beta_3+\beta_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\beta_3-\alpha_3, 1+\beta_4-\alpha_4, \tfrac{x-a}{x-1}\right)\\
y_{120} &= (x-1)^{-(\beta_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\beta_3} F\left(a, 1-q_3^{vii}(a-1), \beta_1+\alpha_2+\beta_3+\beta_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_3-\alpha_3, 1+\beta_4-\alpha_4, \tfrac{x-a}{x-1}\right)
\end{aligned}\right\} \quad (\mathrm{II}_3'')
$$

$$
\left.\begin{aligned}
y_{121} &= (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(1-a, q_3(1-a); \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\alpha_3-\beta_3, \tfrac{a-1}{x-1}\right)\\
y_{122} &= (x-1)^{-(\beta_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(1-a, q_3'(1-a); \alpha_1+\alpha_2+\alpha_3+\beta_4, \alpha_1+\beta_2+\alpha_3+\beta_4, 1+\beta_4-\alpha_4, 1+\alpha_3-\beta_3, \tfrac{a-1}{x-1}\right)\\
y_{123} &= (x-1)^{-(\alpha_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(1-a, q_3''(1-a); \beta_1+\alpha_2+\alpha_3+\alpha_4, \beta_1+\beta_2+\alpha_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\alpha_3-\beta_3, \tfrac{a-1}{x-1}\right)\\
y_{124} &= (x-1)^{-(\beta_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(1-a, q_3'''(1-a); \beta_1+\alpha_2+\alpha_3+\beta_4, \beta_1+\beta_2+\alpha_3+\beta_4, 1+\beta_4-\alpha_4, 1+\alpha_3-\beta_3, \tfrac{a-1}{x-1}\right)\\
y_{125} &= (x-1)^{-(\alpha_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\beta_3} F\left(1-a, q_3^{iv}(1-a); \alpha_1+\alpha_2+\beta_3+\alpha_4, \alpha_1+\beta_2+\beta_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\beta_3-\alpha_3, \tfrac{a-1}{x-1}\right)\\
y_{126} &= (x-1)^{-(\alpha_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\beta_3} F\left(1-a, q_3^{v}(1-a); \beta_1+\alpha_2+\beta_3+\alpha_4, \beta_1+\beta_2+\beta_3+\alpha_4, 1+\alpha_4-\beta_4, 1+\beta_3-\alpha_3, \tfrac{a-1}{x-1}\right)\\
y_{127} &= (x-1)^{-(\beta_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\beta_3} F\left(1-a, q_3^{vi}(1-a); \alpha_1+\alpha_2+\beta_3+\beta_4, \alpha_1+\beta_2+\beta_3+\beta_4, 1+\beta_4-\alpha_4, 1+\beta_3-\alpha_3, \tfrac{a-1}{x-1}\right)\\
y_{128} &= (x-1)^{-(\beta_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\beta_3} F\left(1-a, q_3^{vii}(1-a), \beta_1+\alpha_2+\beta_3+\beta_4, \beta_1+\beta_2+\beta_3+\beta_4, 1+\beta_4-\alpha_4, 1+\beta_3-\alpha_3, \tfrac{a-1}{x-1}\right)
\end{aligned}\right\} \quad (\mathrm{II}_4''')
$$

$$
\left.\begin{aligned}
y_{129} &= (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3+1}{a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_1-\beta_1,\ \tfrac{x-a}{a(x-1)}\right)\\
y_{130} &= (x-1)^{-(\beta_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3'+1}{a};\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\alpha_1-\beta_1,\ \tfrac{x-a}{a(x-1)}\right)\\
y_{131} &= (x-1)^{-(\alpha_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3''+1}{a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_1-\alpha_1,\ \tfrac{x-a}{a(x-1)}\right)\\
y_{132} &= (x-1)^{-(\beta_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3'''+1}{a};\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_3-\beta_3,\ 1+\beta_1-\alpha_1,\ \tfrac{x-a}{a(x-1)}\right)\\
y_{133} &= (x-1)^{-(\alpha_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3^{iv}+1}{a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_1-\beta_1,\ \tfrac{x-a}{a(x-1)}\right)\\
y_{134} &= (x-1)^{-(\alpha_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\beta_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3^{v}+1}{a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_1-\alpha_1,\ \tfrac{x-a}{a(x-1)}\right)\\
y_{135} &= (x-1)^{-(\beta_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\beta_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3^{vi}+1}{a};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\alpha_1-\beta_1,\ \tfrac{x-a}{a(x-1)}\right)\\
y_{136} &= (x-1)^{-(\beta_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\beta_3} F\left(\tfrac{1}{a}, \tfrac{(a-1)q_3^{vii}+1}{a};\ \beta_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_3-\alpha_3,\ 1+\beta_1-\alpha_1,\ \tfrac{x-a}{a(x-1)}\right)
\end{aligned}\right\} (\mathrm{II}_5''')
$$

$$
\left.\begin{aligned}
y_{137} &= (x-1)^{-(\alpha_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3-1)}{a};\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_3-\beta_3,\ \tfrac{(a-1)x}{a(x-1)}\right)\\
y_{138} &= (x-1)^{-(\beta_4+\alpha_1+\alpha_3)} x^{\alpha_1}(x-a)^{\alpha_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3'-1)}{a};\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_3-\beta_3,\ \tfrac{(a-1)x}{a(x-1)}\right)\\
y_{139} &= (x-1)^{-(\alpha_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3''-1)}{a};\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_3-\beta_3,\ \tfrac{(a-1)x}{a(x-1)}\right)\\
y_{140} &= (x-1)^{(\beta_4+\beta_1+\alpha_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3'''-1)}{a};\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\alpha_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_3-\beta_3,\ \tfrac{(a-1)x}{a(x-1)}\right)\\
y_{141} &= (x-1)^{-(\alpha_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\beta_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3^{iv}-1)}{a};\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_3-\alpha_3,\ \tfrac{(a-1)x}{a(x-1)}\right)\\
y_{142} &= (x-1)^{-(\alpha_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\alpha_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3^{v}-1)}{a};\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_3-\alpha_3,\ \tfrac{(a-1)x}{a(x-1)}\right)\\
y_{143} &= (x-1)^{-(\beta_4+\alpha_1+\beta_3)} x^{\alpha_1}(x-a)^{\beta_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3^{vi}-1)}{a};\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_3-\alpha_3,\ \tfrac{(a-1)x}{a(x-1)}\right)\\
y_{144} &= (x-1)^{-(\beta_4+\beta_1+\beta_3)} x^{\beta_1}(x-a)^{\beta_3} F\left(\tfrac{a-1}{a}, \tfrac{(1-a)(q_3^{vii}-1)}{a};\ \beta_1+\alpha_2+\beta_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_3-\alpha_3,\ \tfrac{(a-1)x}{a(x-1)}\right)
\end{aligned}\right\} (\mathrm{II}_6'''
$$

Where

$$q_3 = \frac{(\alpha_1+\alpha_4-\alpha_4\beta_1-\alpha_1\beta_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\alpha_3+\alpha_4)},$$

$$q_3' = \frac{(\alpha_1+\beta_4-\beta_1\beta_4-\alpha_1\alpha_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\alpha_1+\alpha_2+\alpha_3+\beta_4)(\alpha_1+\beta_2+\alpha_3+\beta_4)},$$

$$q_3'' = \frac{(\beta_1+\alpha_4-\alpha_4\beta_1-\beta_1\beta_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\alpha_3+\alpha_4)},$$

$$q_3''' = \frac{(\beta_1+\alpha_4-\beta_1\beta_4-\beta_1\alpha_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\beta_1+\alpha_2+\alpha_3+\beta_4)(\beta_1+\beta_2+\alpha_3+\beta_4)},$$

$$q_4^{iv} = \frac{(\alpha_1+\alpha_4-\alpha_4\beta_1-\alpha_1\beta_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\alpha_1+\alpha_2+\beta_3+\alpha_4)(\alpha_1+\beta_2+\beta_3+\alpha_4)},$$

$$q_3^{v} = \frac{(\beta_1+\alpha_4-\alpha_4\alpha_1-\beta_1\beta_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\beta_1+\alpha_2+\beta_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\alpha_4)},$$

$$q_3^{vi} = \frac{(\alpha_1+\beta_4-\beta_1\beta_4-\alpha_1\alpha_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\alpha_1+\alpha_2+\beta_3+\beta_4)(\alpha_1+\beta_2+\beta_3+\beta_4)},$$

$$q^{vii} = \frac{(\beta_1+\beta_4-\alpha_1\beta_4-\beta_1\alpha_4-2\alpha_1\beta_1-\alpha_4\beta_4 q)^{\frac{1}{1-a}} - (\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2+\alpha_3\beta_3)}{(\beta_1+\alpha_2+\beta_3+\beta_4)(\beta_1+\beta_2+\beta_4)},$$

4. When ∞ lies in the third column of the scheme, we let

$Z = \frac{a-x}{a}$ in (I), and obtain

$$\frac{d^2y}{dz^2}+\left(\frac{1-\alpha_3-\beta_3}{z}+\frac{1-\alpha_1-\beta_1}{z-1}+\frac{1-\alpha_2-\beta_2}{z-\frac{a-1}{a}}\right)\frac{dy}{dz}+\left[\frac{\alpha_3\beta_3}{z^2}+\frac{\alpha_1\beta_1}{(z-1)^2}+\frac{\alpha_2\beta_2}{(z-\frac{a-1}{a})^2}+\frac{(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)z-(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3+\frac{1}{a}\alpha_4\beta_4 q)}{z(z-1)(z-\frac{a-1}{a})}\right]y=0 \qquad \text{-------}(6)$$

The scheme of (6) is

$$P\left\{\begin{array}{lllll} 1 & \frac{a-1}{a} & 0 & \infty & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & z\left(=\frac{a-x}{a}\right) \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{array}\right\}$$

Putting $t = \frac{1}{z}$ in (6) we have

$$\frac{d^2y}{dz^2}+\frac{(1+\alpha_3+\beta_3)(t-1)(t-\frac{a}{a-1})+(1-\alpha_1-\beta_1)(t-\frac{a}{a-1})+\frac{a}{a-1}(1-\alpha_2-\beta_2)(t-1)}{t(t-1)(t-\frac{a}{a-1})}\frac{dy}{dt}$$

$$+\frac{\alpha_3\beta_3(t-1)^2(t-\frac{a}{a-1})^2+\alpha_1\beta_1(t-\frac{a}{a-1})^2+(\frac{a}{a-1})^2\alpha_2\beta_2(t-1)^2+\frac{a}{a-1}\{(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)-(\alpha_4\beta_4-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3+\frac{1}{a}\alpha_4\beta_4 q)t\}(t-1)(t-\frac{a}{a-1})}{t^2(t-1)^2(t-\frac{a}{a-1})^2}y=0 \qquad \text{-------}(7)$$

The scheme of (7) is

$$P\left\{\begin{array}{lllll} 1 & \frac{a}{a-1} & \infty & 0 & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & t\left(=\frac{1}{z}\right) \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{array}\right\}$$

And letting $y = t^{\alpha_4}(t-1)^{\alpha_1}(t-\frac{a}{a-1})^{\alpha_2}u$, we obtain

$$\frac{d^2u}{dt^2}+\left(\frac{1+\alpha_4-\beta_4}{t}+\frac{1+\alpha_1-\beta_1}{t-1}+\frac{1+\alpha_2-\beta_2}{t-\frac{a}{a-1}}\right)\frac{du}{dt}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)(t-q_4)\,u}{t(t-1)(t-\frac{a}{a-1})}=0 \cdots\cdots (\mathrm{II}_1^{IV})$$

With scheme

$$P\left\{\begin{array}{lllll} 1 & \frac{a}{a-1} & \infty & 0 & \\ 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & t\left(=\frac{a}{a-x}\right) \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \alpha_1+\alpha_2+\beta_3+\alpha_4 & \beta_4-\alpha_4 & \end{array}\right\}$$

Where

$$q_4=-\frac{\frac{a}{a-1}\{\alpha_2(\beta_1-1)+\alpha_1(\beta_2-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\}+\{(\alpha_4+\alpha_2)(\beta_2+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\}}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)},$$

And a particular solution of (I) is

$$y=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\frac{a}{a-1}, q_4;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_1-\beta_1,\ \frac{a}{a-x}\right).$$

In equation (II_1^{IV}), let $\xi=\frac{x}{x-a},\ \frac{x-1}{x-a},\ \frac{1-a}{x-a},\ \frac{a(x-1)}{x-a},\ \frac{(1-a)x}{x-a}$,

Hence $t=1-\xi,\ \frac{a}{1-a}(\xi-1),\ \frac{a}{a-1}\xi,\ \frac{\xi-a}{1-a},\ \frac{(1-a)-\xi}{1-a}$ respectively.

We have the following:

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_1-\beta_1}{\xi}+\frac{1+\alpha_4-\beta_4}{\xi-1}+\frac{1+\alpha_2-\beta_2}{\xi-\frac{1}{1-a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)(\xi-1-q_4)u}{\xi(\xi-1)(\xi-\frac{1}{1-a})}=0 \cdots\cdots (II_2^{IV})$$

$$P\left\{\begin{matrix} 0 & \frac{1}{1-a} & \infty & 1 & \\ 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & \xi\left(=\frac{x}{x-a}\right) \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \alpha_1+\alpha_2+\beta_3+\alpha_4 & \beta_4-\alpha_4 & \end{matrix}\right\},$$

$$y=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\frac{1}{1-a}, 1-q_4;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_4-\beta_4,\ \frac{x}{x-a}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_2-\beta_2}{\xi}+\frac{1+\alpha_4-\beta_4}{\xi-1}+\frac{1+\alpha_1-\beta_1}{\xi-\frac{1}{a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)(\xi-[1+\frac{1-a}{a}q_4])u}{\xi(\xi-1)(\xi-\frac{1}{a})}=0 \cdots\cdots (II_3^{IV})$$

$$P\left\{\begin{matrix} \frac{1}{a} & 0 & \infty & 1 & \\ 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & \xi\left(=\frac{x-1}{x-a}\right) \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \alpha_1+\alpha_2+\beta_3+\alpha_4 & \beta_4-\alpha_4 & \end{matrix}\right\}$$

$$y=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\frac{1}{a}, 1+\frac{1-a}{a}q_4;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_2-\beta_2,\ 1+\alpha_4-\beta_4,\ \frac{x-1}{x-a}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_4-\beta_4}{\xi}+\frac{1+\alpha_2-\beta_2}{\xi-1}+\frac{1+\alpha_1-\beta_1}{\xi-\frac{a-1}{a}}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)\left(\xi-\frac{a-1}{a}q_4\right)u}{\xi(\xi-1)\left(\xi-\frac{a-1}{a}\right)}=0\;\cdots\cdots\;(II_4^{IV})$$

$$P\left\{\begin{matrix} \frac{a-1}{a} & 1 & \infty & 0 & \\ 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & \xi\left(=\frac{1-a}{x-a}\right) \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \alpha_1+\alpha_2+\beta_3+\alpha_4 & \beta_3-\alpha_3 & \end{matrix}\right\},$$

$$y=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\frac{a-1}{a},\frac{a-1}{a}q_4;\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_4-\beta_4,1+\alpha_2-\beta_2,\frac{1-a}{x-a}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_2-\beta_2}{\xi}+\frac{1+\alpha_1-\beta_1}{\xi-1}+\frac{1+\alpha_4-\beta_4}{\xi-a}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)(\xi-[a+\overline{1-a}q_4])u}{\xi(\xi-1)(\xi-a)}=0\;\cdots\cdots\;(II_5^{IV})$$

$$P\left\{\begin{matrix} 1 & 0 & \infty & a & \\ 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & \xi\left(=\frac{a(x-1)}{x-a}\right) \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \alpha_1+\alpha_2+\beta_3+\alpha_4 & \beta_3-\alpha_3 & \end{matrix}\right\};$$

$$y=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(a,a+(1-a)q_4;\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_1-\beta_1,\frac{a(x-1)}{x-a}\right);$$

$$\frac{d^2u}{d\xi^2}+\left(\frac{1+\alpha_1-\beta_1}{\xi}+\frac{1+\alpha_2-\beta_2}{\xi-1}+\frac{1+\alpha_4-\beta_4}{\xi-(1-a)}\right)\frac{du}{d\xi}+\frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)\{\xi-(1-a)(1-q_4)\}u}{\xi(\xi-1)(\xi-\overline{1-a})}=0\;\cdots\cdots\;(II_6^{IV})$$

$$P\left\{\begin{matrix} 0 & 1 & \infty & 1-a & \\ 0 & 0 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & 0 & \xi\left(=\frac{(1-a)x}{x-a}\right) \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \alpha_1+\alpha_2+\beta_3+\alpha_4 & \beta_4-\alpha_4 & \end{matrix}\right\},$$

$$y=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(1-a,(1-a)(1-q_4);\alpha_1+\alpha_2+\alpha_3+\alpha_4,1+\alpha_1-\beta_1,1+\alpha_2-\beta_2,\frac{(1-a)x}{x-a}\right)$$

The fourth set of forty-eight solutions, now, may be written down as follows :

$$
\left.\begin{aligned}
y_{145} &= (x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}\, x^{\alpha_1}(x-1)^{\alpha_2} F\left(\tfrac{a}{a-1},\, q_4;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_1-\beta_1,\ \tfrac{a}{a-x}\right)\\
y_{146} &= (x-a)^{-(\beta_4+\alpha_1+\alpha_2)}\, x^{\alpha_1}(x-1)^{\alpha_2} F\left(\tfrac{a}{a-1},\, q_4';\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\alpha_1-\beta_1,\ \tfrac{a}{a-x}\right)\\
y_{147} &= (x-a)^{-(\alpha_4+\beta_1+\alpha_2)}\, x^{\beta_1}(x-1)^{\alpha_2} F\left(\tfrac{a}{a-1},\, q_4'';\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\beta_1-\alpha_1,\ \tfrac{a}{a-x}\right)\\
y_{148} &= (x-a)^{-(\beta_4+\beta_1+\alpha_2)}\, x^{\beta_1}(x-1)^{\alpha_2} F\left(\tfrac{a}{a-1},\, q_4''';\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\beta_1-\alpha_1,\ \tfrac{a}{a-x}\right)\\
y_{149} &= (x-a)^{-(\alpha_4+\alpha_1+\beta_2)}\, x^{\alpha_1}(x-1)^{\beta_2} F\left(\tfrac{a}{a-1},\, q_4^{iv};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\alpha_1-\beta_1,\ \tfrac{a}{a-x}\right)\\
y_{150} &= (x-a)^{-(\alpha_4+\beta_1+\beta_2)}\, x^{\beta_1}(x-1)^{\beta_2} F\left(\tfrac{a}{a-1},\, q_4^{v};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\alpha_4-\beta_4,\ 1+\beta_1-\alpha_1,\ \tfrac{a}{a-x}\right)\\
y_{151} &= (x-a)^{-(\beta_4+\alpha_1+\beta_2)}\, x^{\alpha_1}(x-1)^{\beta_2} F\left(\tfrac{a}{a-1},\, q_4^{vi};\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\alpha_1-\beta_1,\ \tfrac{a}{a-x}\right)\\
y_{152} &= (x-a)^{-(\beta_4+\beta_1+\beta_2)}\, x^{\beta_1}(x-1)^{\beta_2} F\left(\tfrac{a}{a-1},\, q_4^{vii};\ \beta_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_4-\alpha_4,\ 1+\beta_1-\alpha_1,\ \tfrac{a}{a-x}\right)
\end{aligned}\right\}\ (\mathrm{II}_1^{\mathrm{IV}})
$$

$$
\left.\begin{aligned}
y_{153} &= (x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}\, x^{\alpha_1}(x-1)^{\alpha_2} F\left(\tfrac{1}{1-a},\, 1-q_4;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_4-\beta_4,\ \tfrac{x}{x-a}\right)\\
y_{154} &= (x-a)^{-(\beta_4+\alpha_1+\alpha_2)}\, x^{\alpha_1}(x-1)^{\alpha_2} F\left(\tfrac{1}{1-a},\, 1-q_4';\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_4-\alpha_4,\ \tfrac{x}{x-a}\right)\\
y_{155} &= (x-a)^{-(\alpha_4+\beta_1+\alpha_2)}\, x^{\beta_1}(x-1)^{\alpha_2} F\left(\tfrac{1}{1-a},\, 1-q_4'';\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_4-\beta_4,\ \tfrac{x}{x-a}\right)\\
y_{156} &= (x-a)^{-(\beta_4+\beta_1+\alpha_2)}\, x^{\beta_1}(x-1)^{\alpha_2} F\left(\tfrac{1}{1-a},\, 1-q_4''';\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_4-\alpha_4,\ \tfrac{x}{x-a}\right)\\
y_{157} &= (x-a)^{-(\alpha_4+\alpha_1+\beta_2)}\, x^{\alpha_1}(x-1)^{\beta_2} F\left(\tfrac{1}{1-a},\, 1-q_4^{iv};\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_4-\beta_4,\ \tfrac{x}{x-a}\right)\\
y_{158} &= (x-a)^{-(\alpha_4+\beta_1+\beta_2)}\, x^{\beta_1}(x-1)^{\beta_2} F\left(\tfrac{1}{1-a},\, 1-q_4^{v};\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_4-\beta_4,\ \tfrac{x}{x-a}\right)\\
y_{159} &= (x-a)^{-(\beta_4+\alpha_1+\beta_2)}\, x^{\alpha_1}(x-1)^{\beta_2} F\left(\tfrac{1}{1-a},\, 1-q_4^{vi};\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_4-\alpha_4,\ \tfrac{x}{x-a}\right)\\
y_{160} &= (x-a)^{-(\beta_4+\beta_1+\beta_2)}\, x^{\beta_1}(x-1)^{\beta_2} F\left(\tfrac{1}{1-a},\, 1-q_4^{vii};\ \beta_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_4-\alpha_4,\ \tfrac{x}{x-a}\right)
\end{aligned}\right\}\ (\mathrm{II}_2^{\mathrm{IV}})
$$

$$
\left.\begin{aligned}
y_{161}&=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g_4;\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_2-\beta_2,1+\alpha_4-\beta_4,\tfrac{x-1}{x-a}\right)\\
y_{162}&=(x-a)^{-(\beta_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g'_4;\alpha_1+\alpha_2+\alpha_3+\beta_4,\alpha_1+\alpha_2+\beta_3+\beta_4,1+\alpha_2-\beta_2,1+\beta_4-\alpha_4,\tfrac{x-1}{x-a}\right)\\
y_{163}&=(x-a)^{-(\alpha_4+\beta_1+\alpha_2)}x^{\beta_1}(x-1)^{\alpha_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g''_4;\beta_1+\alpha_2+\alpha_3+\alpha_4,\beta_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_2-\beta_2,1+\alpha_4-\beta_4,\tfrac{x-1}{x-a}\right)\\
y_{164}&=(x-a)^{-(\beta_4+\beta_1+\alpha_2)}x^{\beta_1}(x-1)^{\alpha_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g'''_4;\beta_1+\alpha_2+\alpha_3+\beta_4,\beta_1+\alpha_2+\beta_3+\beta_4,1+\alpha_2-\beta_2,1+\beta_4-\alpha_4,\tfrac{x-1}{x-a}\right)\\
y_{165}&=(x-a)^{-(\alpha_4+\alpha_1+\beta_2)}x^{\alpha_1}(x-1)^{\beta_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g^{IV}_4;\alpha_1+\beta_2+\alpha_3+\alpha_4,\alpha_1+\beta_2+\beta_3+\alpha_4,1+\beta_2-\alpha_2,1+\alpha_4-\beta_4,\tfrac{x-1}{x-a}\right)\\
y_{166}&=(x-a)^{-(\alpha_4+\beta_1+\beta_2)}x^{\beta_1}(x-1)^{\beta_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g^{V}_4;\beta_1+\beta_2+\alpha_3+\alpha_4,\beta_1+\beta_2+\beta_3+\alpha_4,1+\beta_2-\alpha_2,1+\alpha_4-\beta_4,\tfrac{x-1}{x-a}\right)\\
y_{167}&=(x-a)^{-(\beta_4+\alpha_1+\beta_2)}x^{\alpha_1}(x-1)^{\beta_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g^{VI}_4;\alpha_1+\beta_2+\alpha_3+\beta_4,\alpha_1+\beta_2+\beta_3+\beta_4,1+\beta_2-\alpha_2,1+\beta_4-\alpha_4,\tfrac{x-1}{x-a}\right)\\
y_{168}&=(x-a)^{-(\beta_4+\beta_1+\beta_2)}x^{\beta_1}(x-1)^{\beta_2}F\left(\tfrac{1}{a},1+\tfrac{1-a}{a}g^{VII}_4;\beta_1+\beta_2+\alpha_3+\beta_4,\beta_1+\beta_2+\beta_3+\beta_4,1+\beta_2-\alpha_2,1+\beta_4-\alpha_4,\tfrac{x-1}{x-a}\right)
\end{aligned}\right\}\ (\mathrm{II}_3^{IV})
$$

$$
\left.\begin{aligned}
y_{169}&=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g_4;\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_4-\beta_4,1+\alpha_2-\beta_2,\tfrac{1-a}{x-a}\right)\\
y_{170}&=(x-a)^{-(\beta_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g'_4;\alpha_1+\alpha_2+\alpha_3+\beta_4,\alpha_1+\alpha_2+\beta_3+\beta_4,1+\beta_4-\alpha_4,1+\alpha_2-\beta_2,\tfrac{1-a}{x-a}\right)\\
y_{171}&=(x-a)^{-(\alpha_4+\beta_1+\alpha_2)}x^{\beta_1}(x-1)^{\alpha_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g''_4;\beta_1+\alpha_2+\alpha_3+\alpha_4,\beta_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_4-\beta_4,1+\alpha_2-\beta_2,\tfrac{1-a}{x-a}\right)\\
y_{172}&=(x-a)^{-(\beta_4+\beta_1+\alpha_2)}x^{\beta_1}(x-1)^{\alpha_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g'''_4;\beta_1+\alpha_2+\alpha_3+\beta_4,\beta_1+\alpha_2+\beta_3+\beta_4,1+\beta_4-\alpha_4,1+\alpha_2-\beta_2,\tfrac{1-a}{x-a}\right)\\
y_{173}&=(x-a)^{-(\alpha_4+\alpha_1+\beta_2)}x^{\alpha_1}(x-1)^{\beta_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g^{IV}_4;\alpha_1+\beta_2+\alpha_3+\alpha_4,\alpha_1+\beta_2+\beta_3+\alpha_4,1+\alpha_4-\beta_4,1+\beta_2-\alpha_2,\tfrac{1-a}{x-a}\right)\\
y_{174}&=(x-a)^{-(\alpha_4+\beta_1+\beta_2)}x^{\beta_1}(x-1)^{\beta_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g^{V}_4;\beta_1+\beta_2+\alpha_3+\alpha_4,\beta_1+\beta_2+\beta_3+\alpha_4,1+\alpha_4-\beta_4,1+\beta_2-\alpha_2,\tfrac{1-a}{x-a}\right)\\
y_{175}&=(x-a)^{-(\beta_4+\alpha_1+\beta_2)}x^{\alpha_1}(x-1)^{\beta_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g^{VI}_4;\alpha_1+\beta_2+\alpha_3+\beta_4,\alpha_1+\beta_2+\beta_3+\beta_4,1+\beta_4-\alpha_4,1+\beta_2-\alpha_2,\tfrac{1-a}{x-a}\right)\\
y_{176}&=(x-a)^{-(\beta_4+\beta_1+\beta_2)}x^{\beta_1}(x-1)^{\beta_2}F\left(\tfrac{a-1}{a},\tfrac{a-1}{a}g^{VII}_4;\beta_1+\beta_2+\alpha_3+\beta_4,\beta_1+\beta_2+\beta_3+\beta_4,1+\beta_4-\alpha_4,1+\beta_2-\alpha_2,\tfrac{1-a}{x-a}\right)
\end{aligned}\right\}\ (\mathrm{II}_4^{IV})
$$

$$
\left.\begin{aligned}
y_{177}&=(x-a)^{-(\alpha_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(a,a+(1-a)g_4;\alpha_1+\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_2-\beta_2,1+\alpha_1-\beta_1,\tfrac{a(x-1)}{x-a}\right)\\
y_{178}&=(x-a)^{-(\beta_4+\alpha_1+\alpha_2)}x^{\alpha_1}(x-1)^{\alpha_2}F\left(a,a+(1-a)g'_4;\alpha_1+\alpha_2+\alpha_3+\beta_4,\alpha_1+\alpha_2+\beta_3+\beta_4,1+\alpha_2-\beta_2,1+\alpha_1-\beta_1,\tfrac{a(x-1)}{x-a}\right)\\
y_{179}&=(x-a)^{-(\alpha_4+\beta_1+\alpha_2)}x^{\beta_1}(x-1)^{\alpha_2}F\left(a,a+(1-a)g''_4;\beta_1+\alpha_2+\alpha_3+\alpha_4,\beta_1+\alpha_2+\beta_3+\alpha_4,1+\alpha_2-\beta_2,1+\beta_1-\alpha_1,\tfrac{a(x-1)}{x-a}\right)\\
y_{180}&=(x-a)^{-(\beta_4+\beta_1+\alpha_2)}x^{\beta_1}(x-1)^{\alpha_2}F\left(a,a+(1-a)g'''_4;\beta_1+\alpha_2+\alpha_3+\beta_4,\beta_1+\alpha_2+\beta_3+\beta_4,1+\alpha_2-\beta_2,1+\beta_1-\alpha_1,\tfrac{a(x-1)}{x-a}\right)\\
y_{181}&=(x-a)^{-(\alpha_4+\alpha_1+\beta_2)}x^{\alpha_1}(x-1)^{\beta_2}F\left(a,a+(1-a)g^{IV}_4;\alpha_1+\beta_2+\alpha_3+\alpha_4,\alpha_1+\beta_2+\beta_3+\alpha_4,1+\beta_2-\alpha_2,1+\alpha_1-\beta_1,\tfrac{a(x-1)}{x-a}\right)\\
y_{182}&=(x-a)^{-(\alpha_4+\beta_1+\beta_2)}x^{\beta_1}(x-1)^{\beta_2}F\left(a,a+(1-a)g^{V}_4;\beta_1+\beta_2+\alpha_3+\alpha_4,\beta_1+\beta_2+\beta_3+\alpha_4,1+\beta_2-\alpha_2,1+\beta_1-\alpha_1,\tfrac{a(x-1)}{x-a}\right)\\
y_{183}&=(x-a)^{-(\beta_4+\alpha_1+\beta_2)}x^{\alpha_1}(x-1)^{\beta_2}F\left(a,a+(1-a)g^{VI}_4;\alpha_1+\beta_2+\alpha_3+\beta_4,\alpha_1+\beta_2+\beta_3+\beta_4,1+\beta_2-\alpha_2,1+\alpha_1-\beta_1,\tfrac{a(x-1)}{x-a}\right)\\
y_{184}&=(x-a)^{-(\beta_4+\beta_1+\beta_2)}x^{\beta_1}(x-1)^{\beta_2}F\left(a,a+(1-a)g^{VII}_4;\beta_1+\beta_2+\alpha_3+\beta_4,\beta_1+\beta_2+\beta_3+\beta_4,1+\beta_2-\alpha_2,1+\beta_1-\alpha_1,\tfrac{a(x-1)}{x-a}\right)
\end{aligned}\right\}\ (\mathrm{II}_5^{IV})
$$

$$
\left.\begin{aligned}
y_{185} &= (x-a)^{-(\alpha_4+\alpha_1+\alpha_2)} x^{\alpha_1}(x-1)^{\alpha_2} F\left(1-a,(1-a)(1-q_4);\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_2+\beta_3+\alpha_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_2-\beta_2,\ \frac{(1-a)x}{x-a}\right)\\
y_{186} &= (x-a)^{-(\beta_4+\alpha_1+\alpha_2)} x^{\alpha_1}(x-1)^{\alpha_2} F\left(1-a,(1-a)(1-q_4');\ \alpha_1+\alpha_2+\alpha_3+\beta_4,\ \alpha_1+\alpha_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\alpha_2-\beta_2,\ \frac{(1-a)x}{x-a}\right)\\
y_{187} &= (x-a)^{-(\alpha_4+\beta_1+\alpha_2)} x^{\beta_1}(x-1)^{\alpha_2} F\left(1-a,(1-a)(1-q_4'');\ \beta_1+\alpha_2+\alpha_3+\alpha_4,\ \beta_1+\alpha_2+\beta_3+\alpha_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_2-\beta_2,\ \frac{(1-a)x}{x-a}\right)\\
y_{188} &= (x-a)^{-(\beta_4+\beta_1+\alpha_2)} x^{\beta_1}(x-1)^{\alpha_2} F\left(1-a,(1-a)(1-q_4''');\ \beta_1+\alpha_2+\alpha_3+\beta_4,\ \beta_1+\alpha_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\alpha_2-\beta_2,\ \frac{(1-a)x}{x-a}\right)\\
y_{189} &= (x-a)^{-(\alpha_4+\alpha_1+\beta_2)} x^{\alpha_1}(x-1)^{\beta_2} F\left(1-a,(1-a)(1-q_4^{iv});\ \alpha_1+\beta_2+\alpha_3+\alpha_4,\ \alpha_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_2-\alpha_2,\ 1+\beta_2-\alpha_2,\ \frac{(1-a)x}{x-a}\right)\\
y_{190} &= (x-a)^{-(\alpha_4+\beta_1+\beta_2)} x^{\beta_1}(x-1)^{\beta_2} F\left(1-a,(1-a)(1-q_4^{v});\ \beta_1+\beta_2+\alpha_3+\alpha_4,\ \beta_1+\beta_2+\beta_3+\alpha_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_2-\alpha_2,\ \frac{(1-a)x}{x-a}\right)\\
y_{191} &= (x-a)^{-(\beta_4+\alpha_1+\beta_2)} x^{\alpha_1}(x-1)^{\beta_2} F\left(1-a,(1-a)(1-q_4^{vi});\ \alpha_1+\beta_2+\alpha_3+\beta_4,\ \alpha_1+\beta_2+\beta_3+\beta_4,\ 1+\alpha_1-\beta_1,\ 1+\beta_2-\alpha_2,\ \frac{(1-a)x}{x-a}\right)\\
y_{192} &= (x-a)^{-(\beta_4+\beta_1+\beta_2)} x^{\beta_1}(x-1)^{\beta_2} F\left(1-a,(1-a)(1-q_4^{vii});\ \beta_1+\beta_2+\alpha_3+\beta_4,\ \beta_1+\beta_2+\beta_3+\beta_4,\ 1+\beta_1-\alpha_1,\ 1+\beta_2-\alpha_2,\ \frac{(1-a)x}{x-a}\right)
\end{aligned}\right\}(\mathrm{II}^{\mathrm{iv}}_{l})
$$

Where

$$q_4 = -\frac{\frac{a}{a-1}\left\{\alpha_4(\beta_1-1)+\alpha_1(\beta_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\alpha_4+\alpha_2)(\beta_2+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\beta_3+\alpha_4)},$$

$$q_4' = -\frac{\frac{a}{a-1}\left\{\beta_4(\beta_1-1)+\alpha_1(\alpha_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\beta_4+\alpha_2)(\beta_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\alpha_1+\alpha_2+\alpha_3+\beta_4)(\alpha_1+\alpha_2+\beta_3+\beta_4)},$$

$$q_4'' = -\frac{\frac{a}{a-1}\left\{\alpha_4(\alpha_1-1)+\beta_1(\beta_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\alpha_4+\alpha_2)(\alpha_2+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\beta_1+\alpha_2+\alpha_3+\alpha_4)(\beta_1+\alpha_2+\beta_3+\alpha_4)},$$

$$q_4''' = -\frac{\frac{a}{a-1}\left\{\beta_4(\alpha_1-1)+\beta_1(\alpha_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\beta_4+\alpha_2)(\beta_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\beta_1+\alpha_2+\alpha_3+\beta_4)(\beta_1+\alpha_2+\beta_3+\beta_4)},$$

$$q_4^{iv} = -\frac{\frac{a}{a-1}\left\{\alpha_4(\beta_1-1)+\alpha_1(\beta_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\alpha_4+\beta_2)(\alpha_2+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\alpha_1+\beta_2+\alpha_3+\alpha_4)(\alpha_1+\beta_2+\beta_3+\alpha_4)}.$$

$$q_4^{v} = -\frac{\frac{a}{a-1}\left\{\alpha_4(\alpha_1-1)+\beta_1(\beta_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\alpha_4+\beta_2)(\alpha_2+\beta_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\beta_1+\beta_2+\alpha_3+\alpha_4)(\beta_1+\beta_2+\beta_3+\alpha_4)},$$

$$q_4^{vi} = -\frac{\frac{a}{a-1}\left\{\beta_4(\beta_1-1)+\alpha_1(\alpha_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\beta_4+\beta_2)(\alpha_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\alpha_1+\beta_2+\alpha_3+\beta_4)(\alpha_1+\beta_2+\beta_3+\beta_4)},$$

$$q_4^{vii} = -\frac{\frac{a}{a-1}\left\{\beta_4(\alpha_1-1)+\beta_1(\alpha_4-1)+2\alpha_1\beta_1+\frac{1}{a}\alpha_4\beta_4 q\right\}+\left\{(\beta_4+\beta_2)(\alpha_2+\alpha_4-1)-\alpha_1\beta_1-\alpha_2\beta_2\right\}}{(\beta_1+\beta_2+\alpha_3+\beta_4)(\beta_1+\beta_2+\beta_3+\beta_4)},$$

For $\beta_1 = 1-\gamma,\ \beta_2 = 1-\delta,\ \beta_3 = \gamma+\delta-\alpha-\beta,\ \alpha_1 = \alpha_2 = \alpha_3 = 0,\ \alpha_4 = \alpha,\ \beta_4 = \beta,$

we obtain the 192 solutions of (I^1) as follows :

$$y_1 = F(a, q_1;\ \alpha, \beta, \gamma, \delta, x)$$
$$y_2 = x^{1-\gamma} F(a, q_1';\ 1-\gamma+\alpha, 1-\gamma+\beta, 2-\gamma, \delta, x)$$
$$y_3 = (x-1)^{1-\gamma} F(a, q_1'';\ \alpha-\delta+1, \beta-\delta+1, \gamma, 2-\delta, x)$$
$$y_4 = x^{1-\gamma}(x-1)^{1-\delta} F(a, q_1''';\ \alpha-\gamma-\delta+2, \beta-\gamma-\delta+2, 2-\gamma, 2-\delta, x)$$
$$y_5 = (x-a)^{\gamma+\delta-\alpha-\beta} F(a, q_1^{iv};\ \gamma+\delta-\beta, \gamma+\delta-\alpha, \gamma, \delta, x)$$
$$y_6 = (x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F(a, q_1^{v};\ \gamma-\beta+1, \gamma-\alpha+1, \gamma, 2-\delta, x)$$
$$y_7 = x^{1-\gamma}(x-a)^{\gamma+\delta-\alpha-\beta} F(a, q_1^{vi};\ \delta-\beta+1, \beta-\gamma-\delta+2, 2-\gamma, \delta, x)$$
$$y_8 = x^{1-\gamma}(x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F(a\, q_1^{vii};\ 2-\beta, 2-\alpha, 2-\gamma, 2-\delta, x)$$

$$y_9 = F(1-a, 1-q_1;\ \alpha, \beta, \delta, \gamma, 1-x)$$
$$y_{10} = x^{1-\gamma} F(1-a, 1-q_1';\ \alpha-\gamma+1, \beta-\gamma+1, \delta, \gamma, 1-x)$$
$$y_{11} = (x-1)^{1-\delta} F(1-a, 1-q_1'';\ \alpha-\delta+1, \beta-\delta+1, 2-\delta, \gamma, 1-x)$$
$$y_{12} = x^{1-\gamma}(x-1)^{1-\delta} F(1-a, 1-q_1''';\ \alpha-\gamma-\delta+2, \beta-\gamma-\delta+2, \delta, \gamma, 1-x)$$
$$y_{13} = (x-a)^{\gamma+\delta-\alpha-\beta} F(1-a, 1-q_1^{iv};\ \gamma+\delta-\beta, \gamma+\delta-\alpha, \delta, \gamma, 1-x)$$
$$y_{14} = (x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F(1-a, 1-q_1^{v};\ \gamma-\beta+1, \gamma-\alpha+1, 2-\delta, \gamma, 1-x)$$
$$y_{15} = x^{1-\gamma}(x-a)^{\gamma+\delta-\alpha-\beta} F(1-a, 1-q_1^{vi};\ \delta-\beta+1, \delta-\alpha+1, \delta, 2-\gamma, 1-x)$$
$$y_{16} = x^{1-\gamma}(x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F(1-a, 1-q_1^{vii};\ 2-\beta, 2-\alpha, 2-\delta, 2-\gamma, 1-x)$$

$$y_{17} = F\left(\tfrac{1}{a}, \tfrac{q_1}{a};\ \alpha, \beta, \gamma, \alpha+\beta-\gamma-\delta+1, \tfrac{x}{a}\right)$$

$$y_{18} = x^{1-\gamma} F\left(\tfrac{1}{a}, \tfrac{q_1'}{a};\ \alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, \alpha+\beta-\gamma-\delta+1, \tfrac{x}{a}\right)$$

$$y_{19} = (x-1)^{1-\delta} F\left(\tfrac{1}{a}, \tfrac{q_1''}{a};\ \alpha-\delta+1, \beta-\delta+1, \gamma, \alpha+\beta-\gamma-\delta+1, \tfrac{x}{a}\right)$$

$$y_{20} = x^{1-\gamma}(x-1)^{1-\delta} F\left(\tfrac{1}{a}, \tfrac{q_1'''}{a};\ \alpha-\gamma-\delta+2, \beta-\gamma-\delta+2, 2-\gamma, \alpha+\beta-\gamma-\delta+1, \tfrac{x}{a}\right)$$

$$y_{21} = (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{1}{a}, \tfrac{q_1^{iv}}{a};\ \gamma+\delta-\alpha, \gamma, \gamma+\delta-\alpha-\beta+1, \tfrac{x}{a}\right)$$

$$y_{22} = (x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{1}{a}, \tfrac{q_1^{v}}{a};\ \gamma-\beta+1, \gamma-\alpha+1, \gamma, \gamma+\delta-\alpha-\beta+1, \tfrac{x}{a}\right)$$

$$y_{23} = x^{1-\gamma}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{1}{a}, \tfrac{q_1^{vi}}{a};\ \delta-\beta+1, \delta-\alpha+1, 2-\gamma, \gamma+\delta-\alpha-\beta+1, \tfrac{x}{a}\right)$$

$$y_{24} = x^{1-\gamma}(x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{1}{a}, \tfrac{q_1^{vii}}{a}, 2-\beta, 2-\alpha, 2-\gamma, \gamma+\delta-\alpha-\beta+1, \tfrac{x}{a}\right)$$

$$y_{25} = F\left(\tfrac{a-1}{a}, \tfrac{a-q_1}{a};\ \alpha, \beta, \alpha+\beta-\gamma-\delta+1, \gamma, \tfrac{a-x}{a}\right)$$

$$y_{26} = x^{1-\gamma} F\left(\tfrac{a-1}{a}, \tfrac{a-q_1'}{a};\ \alpha-\gamma+1, \beta-\gamma+1, \alpha+\beta-\gamma-\delta+1, 2-\gamma, \tfrac{a-x}{a}\right)$$

$$y_{27} = (x-1)^{1-\delta} F\left(\tfrac{a-1}{a}, \tfrac{a-q_1''}{a};\ \alpha-\delta+1, \beta-\delta+1, \alpha+\beta-\gamma-\delta+1, \gamma, \tfrac{a-x}{a}\right)$$

$$y_{28} = x^{1-\gamma}(x-1)^{1-\delta} F\left(\tfrac{a-1}{a}, \tfrac{a-q_1'''}{a};\ \alpha-\gamma-\delta+2, \beta-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, 2-\gamma, \tfrac{a-x}{a}\right)$$

$$y_{29} = (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{iv}}{a};\ \gamma+\delta-\beta, \gamma+\delta-\alpha, \gamma+\delta-\alpha, \gamma+\delta-\gamma-\delta+1, \gamma, \tfrac{a-x}{a}\right)$$

$$y_{30} = (x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{v}}{a};\ \gamma-\beta+1, \gamma-\alpha+1, \gamma+\delta-\alpha-\beta+1, \gamma, \tfrac{a-x}{a}\right)$$

$$y_{31} = x^{1-\gamma}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{vi}}{a};\ \delta-\beta+1, \delta-\alpha+1, \gamma+\delta-\alpha-\beta+1, 2-\gamma, \tfrac{a-x}{a}\right)$$

$$y_{32} = x^{1-\gamma}(x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{a-1}{a}, \tfrac{a-q_1^{vii}}{a};\ 2-\beta, 2-\beta, 2-\alpha, \gamma+\delta-\alpha-\beta+1, 2-\gamma, \tfrac{a-x}{a}\right)$$

$$y_{33} = F\left(\frac{1}{1-a}, \frac{q_1-1}{a-1};\ \alpha, \beta, \delta, \alpha+\beta-\gamma-\delta+1, \frac{x-1}{a-1}\right)$$
$$y_{34} = x^{1-\gamma} F\left(\frac{1}{1-a}, \frac{q_1'-1}{a-1}; \alpha-\gamma+1, \beta-\gamma+1, \delta, \alpha+\beta-\gamma-\delta+1, \frac{x-a}{a-1}\right)$$
$$y_{35} = (x-1)^{1-\delta} F\left(\frac{1}{1-a}, \frac{q_1''-1}{a-1}; \alpha-\delta+1, 2-\delta, \alpha+\beta-\gamma-\delta+1, \frac{x-1}{a-1}\right)$$
$$y_{36} = x^{1-\gamma}(x-1)^{1-\delta} F\left(\frac{1}{1-a}, \frac{q_1'''-1}{a-1}; \alpha-\gamma-\delta+2, \beta-\gamma-\delta+2, 2-\delta, \alpha+\beta-\gamma-\delta+1, \frac{x-1}{a-1}\right)$$
$$y_{37} = (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, \frac{q_1^{iv}-1}{a-1}; \gamma+\delta-\beta, \gamma+\delta-\alpha, \delta, \gamma+\delta-\alpha-\beta+1, \frac{x-1}{a-1}\right)$$
$$y_{38} = (x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, \frac{q_1^{v}-1}{a-1}; \gamma-\beta+1, \gamma-\alpha+1, 2-\delta, \gamma+\delta-\alpha-\beta+1, \frac{x-1}{a-1}\right)$$
$$y_{39} = x^{1-\gamma}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, \frac{q_1^{vi}-1}{a-1}; \delta-\beta+1, \delta-\alpha+1, \delta, \gamma+\delta-\alpha-\beta+1, \frac{x-a}{a-1}\right)$$
$$y_{40} = x^{1-\gamma}(x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, \frac{q_1^{vii}-1}{a-1};\ 2-\beta, 2-\alpha, 2-\delta, \gamma+\delta-\alpha-\beta+1, \frac{x-1}{a-1}\right).$$

$$y_{41} = F\left(\frac{a}{a-1}, \frac{q_1-a}{1-a};\ \alpha, \beta, \alpha+\beta-\gamma-\delta+1, \delta, \frac{x-a}{1-a}\right)$$
$$y_{42} = x^{1-\gamma} F\left(\frac{a}{a-1}, \frac{q_1'-a}{1-a}; \alpha-\gamma+1, \beta-\gamma+1, \alpha+\beta-\gamma-\delta+1, \delta, \frac{x-a}{1-a}\right)$$
$$y_{43} = (x-1)^{1-\delta} F\left(\frac{a}{a-1}, \frac{q_1''-a}{1-a}; \alpha-\delta+1, \alpha+\beta-\gamma-\delta+1, 2-\delta, \frac{x-a}{1-a}\right)$$
$$y_{44} = x^{1-\gamma}(x-1)^{1-\delta} F\left(\frac{a}{a-1}, \frac{q_1'''-a}{1-a}; \alpha-\gamma-\delta+2, \beta-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, 2-\delta, \frac{x-a}{1-a}\right)$$
$$y_{45} = (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, \frac{q_1^{iv}-a}{1-a};\ \gamma+\delta-\beta, \gamma+\delta-\alpha, \gamma+\delta-\alpha-\beta+1, \delta, \frac{x-a}{1-a}\right)$$
$$y_{46} = (x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, \frac{q_1^{v}-a}{1-a};\ \gamma-\beta+1, \gamma-\alpha+1, \gamma+\delta-\alpha-\beta+1, 2-\delta, \frac{x-a}{1-a}\right)$$
$$y_{47} = x^{1-\gamma}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, \frac{q_1^{vi}-a}{1-a};\ \delta-\beta+1, \delta-\alpha+1, \gamma+\delta-\alpha-\beta+1, \delta, \frac{x-a}{1-a}\right)$$
$$y_{48} = x^{1-\gamma}(x-1)^{1-\delta}(x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, \frac{q_1^{vii}-a}{1-a}, 2-\beta, 2-\alpha, \gamma+\delta-\alpha-\beta+1, 2-\delta, \frac{x-a}{1-a}\right)$$

$$y_{49} = x^{-\alpha} F\left(\tfrac{1}{a}, q_2; \alpha, \alpha-\gamma+1, \alpha-\beta+1, \delta, \tfrac{1}{x}\right)$$

$$y_{50} = x^{-\beta} F\left(\tfrac{1}{a}, q_2'; \beta, \beta-\gamma+1, \beta-\alpha+1, \delta, \tfrac{1}{x}\right)$$

$$y_{51} = x^{-(\alpha-\delta+1)}(x-a)^{1-\delta} F\left(\tfrac{1}{a}, q_2''; \alpha-\delta+1, \alpha-\gamma-\delta+2, 2-\beta+1, 2-\delta, \tfrac{1}{x}\right)$$

$$y_{52} = x^{-(\beta-\delta+1)}(x-a)^{1-\delta} F\left(\tfrac{1}{a}, q_2'''; \beta-\delta+1, \beta-\gamma-\delta+2, \beta-\alpha+1, 2-\delta, \tfrac{1}{x}\right)$$

$$y_{53} = x^{-(\gamma+\delta-\beta)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{1}{a}, q_2^{iv}; \gamma+\delta-\beta, \delta-\beta+1, \alpha-\beta+1, \delta, \tfrac{1}{x}\right)$$

$$y_{54} = x^{-(\gamma-\beta+1)}(x-1)^{\gamma+\delta-\alpha-\beta}(x-a)^{1-\delta} F\left(\tfrac{1}{a}, q_2^{v}; \gamma-\beta+1, 2-\beta, \alpha-\beta+1, 2-\delta, \tfrac{1}{x}\right)$$

$$y_{55} = x^{-(\gamma+\delta-\alpha)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(\tfrac{1}{a}, q_2^{vi}; \gamma+\delta-\alpha, \delta-\gamma+1, \beta-\alpha, \delta, \tfrac{1}{x}\right)$$

$$y_{56} = x^{-(\gamma-\alpha+1)}(x-1)^{\gamma+\delta-\alpha-\beta}(x-a)^{1-\delta} F\left(\tfrac{1}{a}, q_2^{vii}; \gamma-\alpha+1, 2-\alpha, \beta-\alpha+1, 2-\delta, \tfrac{1}{x}\right)$$

$$y_{57} = x^{-\alpha} F\left(a, aq_2; \alpha, \alpha-\gamma+1, \alpha-\beta+1, \alpha+\beta-\gamma-\delta+1, \tfrac{a}{x}\right)$$

$$y_{58} = x^{-\beta} F\left(a, aq_2'; \beta, \beta-\gamma+1, \beta-\alpha+1, \alpha+\beta-\gamma-\delta+1, \tfrac{a}{x}\right)$$

$$y_{59} = x^{-(\alpha-\delta+1)} F\left(a, aq_2''; \alpha-\delta+1, \alpha-\gamma-\delta+2, \alpha-\beta+1, \alpha+\beta-\gamma-\delta+1, \tfrac{a}{x}\right)$$

$$y_{60} = x^{-(\beta-\delta+1)}(x-a)^{1-\delta} F\left(a, aq_2'''; \beta-\delta+1, \beta-\gamma-\delta+2, \beta-\alpha+1, \alpha+\beta-\gamma-\delta+1, \tfrac{a}{x}\right)$$

$$y_{61} = x^{-(\gamma+\delta+\beta)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(a, aq_2^{iv}; \gamma+\delta-\beta, \delta-\beta+1, \alpha-\beta+1, \gamma+\delta-\alpha-\beta+1, \tfrac{a}{x}\right)$$

$$y_{62} = x^{-(\gamma-\beta+1)}(x-1)^{\gamma+\delta-\alpha-\beta}(x-a)^{1-\delta} F\left(a, aq_2^{v}; \gamma-\beta+1, 2-\beta, \alpha-\beta+1, \gamma+\delta-\alpha-\beta+1, \tfrac{a}{x}\right)$$

$$y_{63} = x^{-(\gamma+\delta-\alpha)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(a, aq_2^{vi}; \gamma+\delta-\alpha, \delta-\alpha+1, \beta-\alpha+1, \gamma+\delta-\alpha-\beta+1, \tfrac{a}{x}\right)$$

$$y_{64} = x^{-(\gamma-\alpha+1)}(x-1)^{\gamma+\delta-\alpha-\beta}(x-a)^{1-\delta} F\left(a, aq_2^{vii}; \gamma-\alpha+1, 2-\alpha, \beta-\alpha+1, \gamma+\delta-\alpha-\beta+1, \tfrac{a}{x}\right)$$

$$y_{65} = x^{-\alpha} F\left(\frac{a-1}{a}, 1-q_z; \alpha, \alpha-\gamma+1, \delta, \alpha-\beta+1, \frac{x-1}{a}\right)$$

$$y_{66} = x^{-\beta} F\left(\frac{a-1}{a}, 1-q_z'; \beta, \beta-\gamma+1, \delta, \beta-\alpha+1, \frac{x-1}{x}\right)$$

$$y_{67} = x^{-(\alpha-\delta+1)} (x-a)^{1-\delta} F\left(\frac{a-1}{a}, 1-q_z''; \alpha-\delta+1, \alpha-\gamma-\delta+2, 2-\delta, \alpha-\beta+1, \frac{x-1}{x}\right)$$

$$y_{68} = x^{-(\beta-\delta+1)} (x-a)^{1-\delta} F\left(\frac{a-1}{a}, 1-q_z'''; \beta-\delta+1, \beta-\gamma-\delta+2, 2-\delta, \beta-\alpha+1, \frac{x-1}{x}\right)$$

$$y_{69} = x^{-(\gamma+\delta-\beta)} (x-1)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a-1}{a}, 1-q_z^{iv}; \gamma+\delta-\beta, \delta-\beta+1, \delta, \alpha+\beta+1, \frac{x-1}{x}\right)$$

$$y_{70} = x^{-(\gamma-\beta+1)} (x-1)^{\gamma+\delta-\alpha-\beta} (x-a)^{1-\delta} F\left(\frac{a-1}{a}, 1-q_z^{v}; \gamma-\beta+1, 2-\beta, \delta, \alpha-\beta+1, \frac{x-1}{x}\right)$$

$$y_{71} = x^{-(\gamma+\delta-\alpha)} (x-1)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a-1}{a}, 1-q_z^{vi}; \gamma+\delta-\beta, \delta-\beta+1, \delta, \beta-\alpha+1, \frac{x-1}{x}\right)$$

$$y_{72} = x^{-(\gamma-\alpha+1)} (x-1)^{\gamma+\delta-\alpha-\beta} (x-a)^{1-\delta} F\left(\frac{a-1}{a}, 1-q_z^{vii}; \gamma-\alpha+1, 2-\delta, \beta-\alpha+1, \frac{x-1}{x}\right)$$

$$y_{73} = x^{-\alpha} F\left(1-a, 1-aq_z; \alpha, \alpha-\gamma+1, \alpha+\beta-\gamma-\delta+1, \alpha-\beta+1, \frac{x-a}{x}\right)$$

$$y_{74} = x^{-\beta} F\left(1-a, 1-aq_z'; \beta-\gamma+1, \alpha+\beta-\gamma-\delta+1, \beta-\alpha+1, \frac{x-a}{x}\right)$$

$$y_{75} = x^{-(\alpha-\delta+1)} (x-a)^{1-\delta} F\left(1-a, 1-aq_z''; \alpha-\delta+1, \alpha-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, \alpha-\beta+1, \frac{x-a}{x}\right)$$

$$y_{76} = x^{-(\beta-\delta+1)} (x-a)^{1-\delta} F\left(1-a, 1-aq_z'''; \beta-\delta+1, \beta-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, \beta-\alpha+1, \frac{x-a}{x}\right)$$

$$y_{77} = x^{-(\gamma+\delta-\beta)} (x-1)^{\gamma+\delta-\alpha-\beta} F\left(1-a, 1-aq_z^{iv}; \gamma+\delta-\beta, \delta-\beta+1, \gamma+\delta-\alpha-\beta+1, \alpha-\beta+1, \frac{x-a}{x}\right)$$

$$y_{78} = x^{-(\gamma-\beta+1)} (x-1)^{\gamma+\delta-\alpha-\beta} (x-a)^{1-\delta} F\left(1-a, 1-aq_z^{v}; \gamma-\beta+1, \gamma+\delta-\alpha-\beta+1, \alpha-\beta+1, \frac{x-a}{x}\right)$$

$$y_{79} = x^{-(\gamma+\delta-\alpha)} (x-1)^{\gamma+\delta-\alpha-\beta} F\left(1-a, 1-aq_z^{vi}; \gamma+\delta-\alpha, \delta-\alpha+1, \gamma+\delta-\alpha-\beta+1, \beta-\alpha+1, \frac{x-a}{x}\right)$$

$$y_{80} = x^{-(\gamma-\alpha+1)} (x-1)^{\gamma+\delta-\alpha-\beta} (x-a)^{1-\delta} F\left(1-a, 1-aq_z^{vii}; \gamma-\alpha+1, \gamma+\delta-\alpha-\beta+1, \beta-\alpha+\frac{x-a}{x}\right)$$

$$y_{81} = x^{-\alpha} F\left(\frac{1}{1-a}, \frac{1-q_2 a}{1-a}; \alpha, \alpha-\gamma+1, \alpha+\beta-\gamma-\delta+1, \delta, \frac{x-a}{(1-a)x}\right)$$
$$y_{82} = x^{-\beta} F\left(\frac{1}{1-a}, \frac{1-q_2' a}{1-a}; \beta, \beta-\gamma+1, \alpha+\beta-\gamma-\delta+1, \delta, \frac{x-a}{(1-a)x}\right)$$
$$y_{83} = x^{-(\alpha-\delta+1)}(x-a)^{1-\delta} F\left(\frac{1}{1-a}, \frac{1-q_2'' a}{1-a}; \alpha-\delta+1, \alpha-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, 2-\delta, \frac{x-a}{(1-a)x}\right)$$
$$y_{84} = x^{-(\beta-\delta+1)}(x-a)^{1-\delta} F\left(\frac{1}{1-a}, \frac{1-q_2''' a}{1-a}; \beta-\delta+1, \beta-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, 2-\delta, \frac{x-a}{(1-a)x}\right)$$
$$y_{85} = x^{-(\gamma+\delta-\beta)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, \frac{1-q_2^{iv} a}{1-a}; \gamma+\delta-\beta, \delta-\beta+1, \gamma+\delta-\alpha-\beta+1, \delta, \frac{x-a}{(1-a)x}\right)$$
$$y_{86} = x^{-(\gamma-\beta+1)}(x-1)^{\gamma+\delta-\delta-\beta}(x-a)^{1-\delta} F\left(\frac{1}{1-a}, \frac{1-q_2^{v} a}{1-a}; \gamma-\beta+1, 2-\beta, \gamma+\delta-\alpha-\beta+1, 2-\delta, \frac{x-a}{(1-a)x}\right)$$
$$y_{87} = x^{-(\gamma+\delta-\alpha)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, \frac{1-q_2^{vi} a}{1-a}; \gamma+\delta-\alpha, \delta-\alpha+1, \gamma+\delta-\alpha-\beta+1, \delta, \frac{x-a}{(1-a)x}\right)$$
$$y_{88} = x^{-(\gamma-\alpha+1)}(x-1)^{\gamma+\delta-\alpha-\beta}(x-a)^{1-\delta} F\left(\frac{1}{1-a}, \frac{1-q_2^{vii} a}{1-a}; \gamma-\alpha+1, \gamma+\delta-\alpha-\beta+1, 2-\delta, \frac{x-a}{(1-a)x}\right)$$

$$y_{89} = x^{-\alpha} F\left(\frac{a}{a-1}, \frac{a(1-q_2)}{a-1}; \alpha, \alpha-\gamma+1, \delta, \alpha+\beta-\gamma-\delta+1, \frac{a(x-1)}{(a-1)x}\right)$$
$$y_{90} = x^{-\beta} F\left(\frac{a}{a-1}, \frac{a(1-q_2')}{a-1}; \beta, \beta-\gamma+1, \delta, \alpha+\beta-\gamma-\delta+1, \frac{a(x-1)}{(a-1)x}\right)$$
$$y_{91} = x^{-(\alpha-\delta+1)}(x-a)^{1-\delta} F\left(\frac{a}{a-1}, \frac{a(1-q_2'')}{a-1}; \alpha-\delta+1, \alpha-\gamma-\delta+2, 2-\delta, \alpha+\beta-\gamma-\delta+1, \frac{a(x-1)}{(a-1)x}\right)$$
$$y_{92} = x^{-(\beta-\delta+1)}(x-a)^{1-\delta} F\left(\frac{a}{a-1}, \frac{a(1-q_2''')}{a-1}; \beta-\delta+1, \beta-\gamma-\delta+2, 2-\delta, \alpha+\beta-\gamma-\delta+1, \frac{a(x-1)}{(a-1)x}\right)$$
$$y_{93} = x^{-(\gamma+\delta-\beta)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, \frac{a(1-q_2^{iv})}{a-1}; \gamma+\delta-\beta, \delta-\beta+1, \delta, \gamma+\delta-\gamma-\delta+1, \frac{a(x-1)}{(a-1)x}\right)$$
$$y_{94} = x^{-(\gamma-\beta+1)}(x-1)^{\gamma+\delta-\alpha-\beta}(x-a)^{1-\delta} F\left(\frac{a}{a-1}, \frac{a(1-q_2^{v})}{a-1}; \gamma-\beta+1, 2-\beta, 2-\delta, \gamma+\delta-\alpha-\beta+1, \frac{a(x-1)}{(a-1)x}\right)$$
$$y_{95} = x^{-(\gamma+\delta-\alpha)}(x-1)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, \frac{a(1-q_2^{vi})}{a-1}; \gamma+\delta-\alpha, \delta-\alpha+1, \delta, \gamma+\delta-\alpha+\beta+1, \frac{a(x-1)}{(a-1)x}\right)$$
$$y_{96} = x^{-(\gamma-\alpha+1)}(x-1)^{\gamma+\delta-\alpha-\beta}(x-a)^{1-\delta} F\left(\frac{a}{a-1}, \frac{a(1-q_2^{vii})}{a-1}; 2-\alpha, 2-\delta, \gamma+\delta-\alpha-\beta+1, \frac{a(x-1)}{(a-1)x}\right)$$

$$y_{97} = (x-1)^{-\alpha} F\left(\frac{1}{1-a}, g_3; \alpha, \alpha-\delta+1, \alpha-\beta+1, \gamma, \frac{1}{1-x}\right)$$
$$y_{98} = (x-1)^{-\beta} F\left(\frac{1}{1-a}, g_3'; \beta, \beta-\delta+1, \beta-\alpha+1, \gamma, \frac{1}{1-x}\right)$$
$$y_{99} = (x-1)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{1}{1-a}, g_3''; \alpha-\gamma+1, \alpha-\gamma-\delta+2, \alpha-\beta+1, 2-\gamma, \frac{1}{1-x}\right)$$
$$y_{100} = (x-1)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(\frac{1}{1-a}, g_3'''; \beta-\gamma+1, \beta-\gamma-\delta+2, \beta-\alpha+1, 2-\gamma, \frac{1}{1-x}\right)$$
$$y_{101} = (x-1)^{-(\gamma+\delta-\beta)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, g_3^{IV}; \gamma+\delta-\beta, \gamma-\beta+1, \alpha-\beta+1, \gamma, \frac{1}{1-x}\right)$$
$$y_{102} = (x-1)^{-(\delta-\beta+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, g_3^{V}; 2-\beta, \alpha-\beta+1, 2-\gamma, \frac{1}{1-x}\right)$$
$$y_{103} = (x-1)^{-(\gamma+\delta-\alpha)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, g_3^{VI}; \gamma+\delta-\alpha, \gamma-\alpha+1, \beta-\alpha+1, \gamma, \frac{1}{1-x}\right)$$
$$y_{104} = (x-1)^{-(\delta-\alpha+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{1-a}, g_3^{VII}; \delta-\alpha+1, 2-\alpha\ \beta-\alpha+1, 2-\gamma, \frac{1}{1-x}\right)$$

$$y_{105} = (x-1)^{-\alpha} F\left(a, 1-g_3; \alpha, \alpha-\delta+1, \gamma, \alpha-\beta+1, \frac{x}{x-1}\right)$$
$$y_{106} = (x-1)^{-\beta} F\left(a, 1-g_3'; \beta, \beta-\delta+1, \gamma, \beta-\alpha+1, \frac{x}{x-1}\right)$$
$$y_{107} = (x-1)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{a}{a-1}, 1-g_3''; \alpha-\gamma+1, \alpha-\gamma-\delta+2, 2-\gamma, \alpha-\beta+1, \frac{x}{x-1}\right)$$
$$y_{108} = (x-1)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(\frac{a}{a-1}, 1-g_3'''; \beta-\gamma+1, \beta-\gamma-\delta+2, 2-\gamma, \beta-\alpha+1, \frac{x}{x-1}\right)$$
$$y_{109} = (x-1)^{-(\gamma+\delta-\beta)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, 1-g_3^{IV}; \gamma+\delta-\beta, \gamma-\beta+1, \gamma, \alpha-\beta+1, \frac{x}{x-1}\right)$$
$$y_{110} = (x-1)^{-(\delta-\beta+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, 1-g_3^{V}; \delta-\beta+1, 2-\beta, 2-\gamma, \alpha-\beta+1, \frac{x}{x-1}\right)$$
$$y_{111} = (x-1)^{-(\gamma+\delta-\alpha)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, 1-g_3^{VI}; \gamma-\alpha+1, \gamma, \beta-\alpha+1, \frac{x}{x-1}\right)$$
$$y_{112} = (x-1)^{-(\delta-\alpha+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a}{a-1}, 1-g_3^{VII}; \delta-\alpha+1, 2-\alpha, 2-\gamma, \beta-\alpha+1, \frac{x}{x-1}\right)$$

$$y_{113} = (x-1)^{-\alpha} F\left(a, 1-q_3(a-1); \alpha, \alpha-\delta+1, \alpha+\beta-\gamma-\delta+1, \alpha-\beta+1, \frac{x-a}{x-1}\right)$$
$$y_{114} = (x-1)^{-\beta} F\left(a, 1-q_3'(a-1); \beta, \beta-\delta+1, \alpha+\beta-\gamma-\delta+1, \beta-\alpha+1, \frac{x-a}{x-1}\right)$$
$$y_{115} = (x-1)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(a, 1-q_3''(a-1); \alpha-\gamma+1, \alpha-\gamma-\delta+2\ \alpha+\beta-\gamma-\delta+1, \alpha-\beta+1, \frac{x-a}{x-1}\right)$$
$$y_{116} = (x-1)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(a, 1-q_3'''(a-1); \beta-\gamma+1, \beta-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, \beta-\alpha+1, \frac{x-a}{x-1}\right)$$
$$y_{117} = (x-1)^{-(\gamma+\delta-\beta)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(a, 1-q_3^{iv}(a-1); \gamma+\delta-\beta\ \gamma-\beta+1, \gamma+\delta-\alpha-\beta+1, \alpha-\beta+1, \frac{x-a}{x-1}\right)$$
$$y_{118} = (x-1)^{-(\delta-\beta+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(a, 1-q_3^{v}(a-1); \delta-\beta+1, 2-\beta, \gamma+\delta-\alpha-\beta+1, \alpha-\beta+1, \frac{x-a}{x-1}\right)$$
$$y_{119} = (x-1)^{-(\gamma+\delta-\alpha)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(a, 1-q_3^{vi}(a-1); \gamma+\delta-\alpha; \gamma-\alpha+1, \gamma+\delta-\alpha-\beta+1, \beta-\alpha+1, \frac{x-a}{x-1}\right)$$
$$y_{120} = (x-1)^{-(\delta-\alpha+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(a, 1-q_3^{vii}(a-1); \delta-\alpha+1, 2-\alpha, \gamma+\delta-\alpha-\beta+1, \beta-\alpha+1, \frac{x-a}{x-1}\right)$$

$$y_{121} = (x-1)^{-\alpha} F\left(1-a, g_3(1-a); \alpha, \alpha-\delta+1, \alpha-\beta+1, \alpha+\beta-\gamma-\delta+1, \frac{a-1}{x-1}\right)$$
$$y_{122} = (x-1)^{-\beta} F\left(1-a, g_3'(1-a); \beta, \beta-\delta+1, \alpha-\beta+1, \gamma+\beta-\gamma-\delta+1, \frac{a-1}{x-1}\right)$$
$$y_{123} = (x-1)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(1-a, g_3''(1-a); \alpha-\gamma+1, \alpha-\gamma-\delta+2\ \alpha-\beta+1, \alpha+\beta-\gamma-\delta+1, \frac{a-1}{x-1}\right)$$
$$y_{124} = (x-1)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(1-a, g_3'''(1-a); \beta-\gamma+1, \beta-\gamma-\delta+2, \beta-\alpha+1, \alpha+\beta-\gamma-\delta+1, \frac{a-1}{x-1}\right)$$
$$y_{125} = (x-1)^{-(\gamma+\delta-\beta)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(1-a, g_3^{iv}(1-a); \gamma+\delta-\beta, \alpha-\beta+1, \gamma+\delta-\alpha-\beta+1, \frac{a-1}{x-1}\right)$$
$$y_{126} = (x-1)^{-(\delta-\beta+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(1-a, g_3^{v}(1-a); \delta-\beta+1, 2-\beta, \alpha-\beta+1, \gamma+\delta-\alpha-\beta+1, \frac{a-1}{x-1}\right)$$
$$y_{127} = (x-1)^{-(\gamma+\delta-\alpha)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(1-a, g_3^{vi}(1-a); \gamma+\delta-\alpha, \gamma-\alpha+1, \beta-\alpha+1, \gamma+\delta-\alpha-\beta+1, \frac{a-1}{x-1}\right)$$
$$y_{128} = (x-1)^{-(\delta-\alpha+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(1-a, g_3^{vii}(1-a); \delta-\alpha+1, 2-\alpha, \beta-\alpha+1, \gamma+\delta-\alpha-\beta+1, \frac{a-1}{x-1}\right)$$

$$y_{129} = (x-1)^{-\alpha} F\left(\frac{1}{a}, \frac{(a-1)q_3+1}{a}; \alpha, \alpha-\delta+1, \alpha+\beta-\gamma-\delta+1, \gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{130} = (x-1)^{-\beta} F\left(\frac{1}{a}, \frac{(a-1)q_3'+1}{a}; \beta, \beta-\delta+1, \alpha+\beta-\gamma-\delta+1, \gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{131} = (x-1)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{1}{a}, \frac{(a-1)q_3''+1}{a}; \alpha-\gamma+1, \alpha-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, 2-\gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{132} = (x-1)^{-(\gamma+\delta-\alpha)} x^{1-\gamma} F\left(\frac{1}{a}, \frac{(a-1)q_3'''+1}{a}; \beta-\gamma+1, \beta-\gamma-\delta+2, \alpha+\beta-\gamma-\delta+1, 2-\gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{133} = (x-1)^{-(\gamma+\delta-\beta)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{a}, \frac{(a-1)q_3^{iv}+1}{a}; \gamma+\delta-\beta, \gamma-\beta+1, \gamma+\delta-\alpha-\beta+1, \gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{134} = (x-1)^{-(\delta-\beta+1)} x^{1-\gamma} (x-1)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{a}, \frac{(a-1)q_3^{v}+1}{a}; \delta-\beta+1, 2-\beta, \gamma+\delta-\alpha-\beta+1, 2-\gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{135} = (x-1)^{-(\gamma+\delta-\alpha)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{a}, \frac{(a-1)q_3^{vi}+1}{a}; \gamma+\delta-\alpha, \gamma-\alpha+1, \gamma+\delta-\alpha-\beta+1, \gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{136} = (x-1)^{-(\delta-\alpha+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{1}{a}, \frac{(a-1)q_3^{vii}+1}{a}; \delta-\alpha+1, \beta-\gamma-\delta+2, \gamma+\delta-\alpha-\beta+1, 2-\gamma, \frac{x-a}{a(x-1)}\right)$$

$$y_{137} = (x-1)^{-\alpha} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3-1)}{a}; \alpha, \alpha-\delta+1, \gamma, \alpha+\beta-\gamma-\delta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{138} = (x-1)^{-\beta} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3'-1)}{a}; \beta, \beta-\delta+1, \gamma, \alpha+\beta-\gamma-\delta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{139} = (x-1)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3''-1)}{a}; \alpha-\gamma+1, \alpha-\gamma-\delta+2, 2-\gamma, \alpha+\beta-\gamma-\delta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{140} = (x-1)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3'''-1)}{a}; \beta-\gamma+1, \beta-\gamma-\delta+2, 2-\gamma, \alpha+\beta-\gamma-\delta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{141} = (x-1)^{-(\gamma+\delta-\beta)} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3^{iv}-1)}{a}; \gamma+\delta-\beta, \gamma-\beta+1, \gamma, \gamma+\delta-\alpha-\beta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{142} = (x-1)^{-(\delta-\beta+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3^{v}-1)}{a}; \delta-\beta+1, 2-\beta, 2-\gamma, \gamma+\delta-\alpha-\beta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{143} = (x-1)^{-(\gamma+\delta-\alpha)} (x-a)^{\gamma-\delta-\alpha-\beta} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3^{vi}-1)}{a}; \gamma+\delta-\alpha, \gamma-\alpha+1, \gamma, \gamma+\delta-\alpha-\beta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{144} = (x-1)^{-(\delta-\alpha+1)} x^{1-\gamma} (x-a)^{\gamma+\delta-\alpha-\beta} F\left(\frac{a-1}{a}, \frac{(1-a)(q_3^{vii}-1)}{a}; \delta-\alpha+1, 2-\alpha, 2-\gamma, \gamma+\delta-\alpha-\beta+1, \frac{(a-1)x}{a(x-1)}\right)$$

$$y_{145} = (x-a)^{-\alpha} F\left(\frac{a}{a-1}, q_4; \alpha, \gamma+\delta-\beta, \alpha-\beta+1, \gamma, \frac{a}{a-x}\right)$$

$$y_{146} = (x-a)^{-\beta} F\left(\frac{a}{a-1}, q_4'; \beta, \gamma+\delta-\alpha, \beta-\alpha+1, \gamma, \frac{a}{a-x}\right)$$

$$y_{147} = (x-a)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{a}{a-1}, q_4''; \alpha-\gamma+1, \delta-\beta+1, \alpha-\beta+1, 2-\gamma, \frac{a}{a-x}\right)$$

$$y_{148} = (x-a)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(\frac{a}{a-1}, q_4'''; \beta-\gamma+1, \delta-\alpha+1, \beta-\alpha+1, 2-\gamma, \frac{a}{a-x}\right)$$

$$y_{149} = (x-a)^{-(\alpha-\delta+1)} (x-1)^{1-\delta} F\left(\frac{a}{a-1}, q_4^{IV}; \alpha-\delta+1, \gamma-\beta+1, \alpha-\beta+1, \gamma, \frac{a}{a-x}\right)$$

$$y_{150} = (x-a)^{-(\alpha-\gamma-\delta-2)} x^{1-\gamma} (x-1)^{1-\delta} F\left(\frac{a}{a-1}, q_4^{V}; \alpha-\gamma-\delta+2, 2-\beta, \alpha-\beta+1, 2-\gamma, \frac{a}{a-x}\right)$$

$$y_{151} = (x-a)^{-(\beta-\delta+1)} (x-1)^{1-\delta} F\left(\frac{a}{a-1}, q_4^{VI}; \beta-\delta+1, \gamma-\alpha+1, \beta-\alpha+1, \gamma, \frac{a}{a-x}\right)$$

$$y_{152} = (x-a)^{-(\beta-\gamma-\delta+2)} x^{1-\delta} (x-1)^{1-\delta} F\left(\frac{a}{a-1}, q_4^{VII}; \beta-\gamma-\delta+2, 2-\alpha, \beta-\alpha+1, 2-\gamma, \frac{a}{a-x}\right)$$

$$y_{153} = (x-a)^{-\alpha} F\left(\frac{1}{1-a}, 1-q_4; \alpha, \gamma+\delta-\beta, \gamma, \alpha-\beta+1, \frac{x}{x-a}\right)$$

$$y_{154} = (x-a)^{-\beta} F\left(\frac{1}{1-a}, 1-q_4; \beta, \gamma+\delta-\alpha, \gamma, \beta-\alpha+1, \frac{x}{x-a}\right)$$

$$y_{155} = (x-a)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{1}{1-a}, 1-q_4''; \alpha-\gamma+1, \delta-\beta+1, 2-\gamma, \alpha-\beta+1, \frac{x}{x-a}\right)$$

$$y_{156} = (x-a)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(\frac{1}{1-a}, 1-q_4'''; \beta-\gamma+1, \delta-\alpha+1, 2-\gamma, \beta-\alpha+1, \frac{x}{x-a}\right)$$

$$y_{157} = (x-a)^{-(\alpha-\delta+1)} (x-1)^{1-\delta} F\left(\frac{1}{1-a}, 1-q_4^{IV}; \alpha-\delta+1, \gamma-\beta+1, \gamma, \alpha-\beta+1, \frac{x}{x-a}\right)$$

$$y_{158} = (x-a)^{-(\alpha-\gamma-\delta+2)} x^{1-\gamma} (x-1)^{1-\delta} F\left(\frac{1}{1-a}, 1-q_4^{V}; \alpha-\gamma-\delta+2, 2-\beta, 2-\gamma, \alpha-\beta+1, \frac{x}{x-a}\right)$$

$$y_{159} = (x-a)^{-(\beta-\delta+1)} (x-1)^{1-\delta} F\left(\frac{1}{1-a}, 1-q_4^{VI}; \beta-\delta+1, \gamma-\alpha+1, \gamma, \beta-\alpha+1, \frac{x}{x-a}\right)$$

$$y_{160} = (x-a)^{-(\beta-\gamma-\delta+2)} x^{1-\gamma} (x-1)^{1-\delta} F\left(\frac{1}{1-a}, 1-q_4^{VII}; \beta-\gamma-\delta+2, 2-\alpha, 2-\gamma, \beta-\alpha+1, \frac{x}{x-a}\right)$$

$$y_{161} = (x-a)^{-\alpha} F\left(\frac{1}{a}, 1+\frac{(1-a)}{a}q_x ; \alpha, \gamma+\delta-\beta, \delta, \alpha-\beta+1, \frac{x-1}{x-a}\right)$$
$$y_{162} = (x-a)^{-\beta} F\left(\frac{1}{a}, 1+\frac{(1-a)}{a}q'_x ; \beta, \gamma+\delta-\alpha, \delta, \beta-\alpha+1, \frac{x-1}{x-a}\right)$$
$$y_{163} = (x-a)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{1}{a}, 1+\frac{1-a}{a}q''_x ; \alpha-\gamma+1, \delta-\beta+1, \delta, \alpha-\beta+1, \frac{x-1}{x-a}\right)$$
$$y_{164} = (x-a)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(\frac{1}{a}, 1+\frac{1-a}{a}q'''_x ; \beta-\gamma+1, \delta-\alpha+1, \delta, \beta-\alpha+1, \frac{x-1}{x-a}\right)$$
$$y_{165} = (x-a)^{-(\alpha-\delta+1)} (x-1)^{1-\delta} F\left(\frac{1}{a}, 1+\frac{1-a}{a}q^{IV}_x ; \alpha-\delta+1, \gamma-\beta+1, 2-\delta, \alpha-\beta+1, \frac{x-1}{x-a}\right)$$
$$y_{166} = (x-a)^{-(\alpha-\gamma-\delta+2)} x^{1-\gamma}(x-1)^{1-\delta} F\left(\frac{1}{a}, 1+\frac{1-a}{a}q^{V}_x ; \alpha-\gamma-\delta+2, 2-\beta, 2-\delta, \alpha-\beta+1, \frac{x-1}{x-a}\right)$$
$$y_{167} = (x-a)^{-(\beta-\delta+1)} (x-1)^{1-\delta} F\left(\frac{1}{a}, 1+\frac{1-a}{a}q^{VI}_x ; \beta-\delta+1, \gamma-\alpha+1, 2-\delta, \beta-\alpha+1, \frac{x-1}{x-a}\right)$$
$$y_{168} = (x-a)^{-(\beta-\gamma-\delta+2)} x^{1-\gamma}(x-1)^{1-\delta} F\left(\frac{1}{a}, 1+\frac{1-a}{a}q^{VII}_x ; \beta-\gamma-\delta+2, 2-\alpha, 2-\delta, \beta-\alpha+1, \frac{x-1}{x-a}\right)$$

$$y_{169} = (x-a)^{-\alpha} F\left(\frac{a-1}{a}, q_x\frac{a-1}{a} ; \alpha, \gamma+\delta-\beta, \alpha-\beta+1, \delta, \frac{1-a}{x-a}\right)$$
$$y_{170} = (x-a)^{-\beta} F\left(\frac{a-1}{a}, \frac{a-1}{a}q'_x ; \beta, \gamma+\delta-\alpha, \beta-\alpha+1, \delta, \frac{1-a}{x-a}\right)$$
$$y_{171} = (x-a)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(\frac{a-1}{a}, \frac{a-1}{a}q''_x ; \alpha-\gamma+1, \delta-\beta+1, \alpha-\beta+1, \delta, \frac{1-a}{x-a}\right)$$
$$y_{172} = (x-a)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(\frac{a-1}{a}, \frac{a-1}{a}q'''_x ; \beta-\gamma+1, \delta-\alpha+1, \beta-\alpha+1, \delta, \frac{1-a}{x-a}\right)$$
$$y_{173} = (x-a)^{-(\alpha-\delta+1)} (x-1)^{1-\delta} F\left(\frac{a-1}{a}, \frac{a-1}{a}q^{IV}_x ; \alpha-\delta+1, \gamma-\beta+1, \alpha-\beta+1, 2\delta, \frac{1-a}{x-a}\right)$$
$$y_{174} = (x-a)^{-(\alpha-\gamma-\delta+2)} x^{1-\gamma}(x-1)^{1-\delta} F\left(\frac{a-1}{a}, \frac{a-1}{a}q^{V}_x ; \alpha-\gamma-\delta+2, 2-\beta, \alpha-\beta+1, 2-\delta, \frac{1-a}{x-a}\right)$$
$$y_{175} = (x-a)^{-(\beta-\delta+1)} (x-1)^{1-\delta} F\left(\frac{a-1}{a}, \frac{a-1}{a}q^{VI}_x ; \beta-\delta+1, \gamma+\delta-\alpha, \beta-\alpha+1, 2-\delta, \frac{1-a}{x-a}\right)$$
$$y_{176} = (x-a)^{-(\beta-\gamma-\delta+2)} x^{1-\gamma}(x-1)^{1-\delta} F\left(\frac{a-1}{a}, \frac{a-1}{a}q^{VII}_x ; \beta-\gamma-\delta+2, 2-\alpha, \beta-\alpha+1, 2-\delta, \frac{1-a}{x-a}\right)$$

$$y_{177} = (x-a)^{-\alpha} F\left(a, a+(1-a)q_\psi ; \alpha, \gamma+\delta-\beta, \delta, \gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{178} = (x-a)^{-\beta} F\left(a, a+(1-a)q'_\psi ; \beta, \gamma+\delta-\alpha, \delta, \gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{179} = (x-a)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(a, a+(1-a)q''_\psi ; \alpha-\gamma+1, \delta-\beta+1, \delta, 2-\gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{180} = (x-a)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(a, a+(1-a)q'''_\psi ; \beta-\gamma+1, \delta-\alpha+1, \delta, 2-\gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{181} = (x-a)^{-(\alpha-\delta+1)} (x-1)^{1-\delta} F\left(a, a+(1-a)q^{iv}_\psi ; \alpha-\delta+1, \gamma-\beta+1, 2-\delta, \gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{182} = (x-a)^{-(\alpha-\gamma-\delta+2)} x^{1-\gamma} (x-1)^{1-\delta} F\left(a, a+(1-a)q^{v}_\psi ; \alpha-\gamma-\delta+2, 2-\beta, 2-\delta, 2-\gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{183} = (x-a)^{-(\beta-\delta+1)} (x-1)^{1-\delta} F\left(a, a+(1-a)q^{vi}_\psi ; \beta-\delta+1, \gamma-\alpha+1, 2-\delta, \gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{184} = (x-a)^{-(\beta-\gamma-\delta+2)} x^{1-\gamma} (x-1)^{1-\delta} F\left(a, a+(1-a)q^{vii}_\psi ; \alpha-\gamma-\delta+2, 2-\alpha, 2-\delta, 2-\gamma, \frac{a(x-1)}{x-a}\right)$$

$$y_{185} = (x-a)^{-\alpha} F\left(1-a, (1-a)(1-q_\psi) ; \alpha, \gamma+\delta-\beta, \gamma, \delta, \frac{(1-a)x}{x-a}\right)$$

$$y_{186} = (x-a)^{-\beta} F\left(1-a, (1-a)(1-q'_\psi) ; \beta, \gamma+\delta-\alpha, \gamma, \delta, \frac{(1-a)x}{x-a}\right)$$

$$y_{187} = (x-a)^{-(\alpha-\gamma+1)} x^{1-\gamma} F\left(1-a, (1-a)(1-q''_\psi) ; \alpha-\gamma+1, \delta-\beta+1, 2-\gamma, \delta, \frac{(1-a)x}{x-a}\right)$$

$$y_{188} = (x-a)^{-(\beta-\gamma+1)} x^{1-\gamma} F\left(1-a, (1-a)(1-q'''_\psi) ; \beta-\gamma+1, \delta-\alpha+1, 2-\gamma, \delta, \frac{(1-a)x}{x-a}\right)$$

$$y_{189} = (x-a)^{-(\alpha-\delta+1)} (x-1)^{1-\delta} F\left(1-a, (1-a)(1-q^{iv}_\psi) ; \alpha-\delta+1, \gamma-\beta+1, \gamma, 2-\delta, \frac{(1-a)x}{x-a}\right)$$

$$y_{190} = (x-a)^{-(\alpha-\gamma-\delta+2)} x^{1-\gamma} (x-1)^{1-\delta} F\left(1-a, (1-a)(1-q^{v}_\psi) ; \alpha-\gamma-\delta+2, 2-\beta, 2-\gamma, 2-\delta, \frac{(1-a)x}{x-a}\right)$$

$$y_{191} = (x-a)^{-(\beta-\delta+1)} (x-1)^{1-\delta} F\left(1-a, (1-a)(1-q^{vi}_\psi) ; \beta-\delta+1, \gamma-\alpha+1, \gamma, 2-\delta, \frac{(1-a)x}{x-a}\right)$$

$$y_{192} = (x-a)^{-(\beta-\gamma-\delta+2)} x^{1-\gamma} (x-1)^{1-\delta} F\left(1-a, (1-a)(1-q^{vii}_\psi) ; \beta-\gamma-\delta+2, 2-\alpha, 2-\gamma, 2-\delta, \frac{(1-a)x}{x-a}\right)$$

Where

$$
\left\{
\begin{aligned}
q_1 &= q \\
q_1' &= \frac{(1-\gamma)\{a\delta+\alpha+\beta-\gamma-\delta+1\}+\alpha\beta q}{(\alpha-\gamma+1)(\beta-\gamma+1)} \\
q_1'' &= \frac{(1-\delta)a\gamma+\alpha\beta q}{(\alpha-\delta+1)(\beta-\gamma-\delta+2)} \\
q_1''' &= \frac{(2-\gamma-\delta)a+(1-\gamma)(\alpha+\beta-\gamma-\delta+1)+\alpha\beta q}{(\alpha-\delta+1)(\beta-\delta+1)} \\
q_1^{iv} &= \frac{\gamma(\gamma+\delta-\alpha-\beta)+\alpha\beta q}{(\gamma+\delta-\beta)(\gamma+\delta-\alpha)} \\
q_1^{v} &= \frac{(1-\delta)a\gamma+(\gamma+\delta-\alpha-\beta)\gamma+\alpha\beta q}{(\gamma-\beta+1)(\gamma-\alpha+1)} \\
q_1^{vi} &= \frac{(2-\gamma-\delta)a+(\delta-\alpha-\beta+1)+\alpha\beta q}{(\delta-\beta+1)(\delta-\alpha+1)} \\
q_1^{vii} &= \frac{(2-\gamma-\delta)a+(\delta-\alpha-\beta+1)+\alpha\beta q}{(2-\beta)(2-\alpha)}
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
q_2 &= -\frac{\alpha(\gamma+\delta-\alpha-1)+\{\alpha(\beta-\delta)-\alpha\beta q\}\frac{1}{a}}{\alpha(\alpha-\delta+1)} \\
q_2 &= -\frac{\beta(\gamma+\delta-\beta-1)+\{\beta(\alpha-\delta)-\alpha\beta q\}\frac{1}{a}}{\beta(\beta-\gamma+1)} \\
q_2'' &= -\frac{(\alpha-\delta+1)(\gamma+\delta-\alpha-1)+\{(\alpha-\delta+1)(\beta-1)-\alpha\beta q\}\frac{1}{a}}{(\alpha-\delta+1)(\alpha-\gamma-\delta+2)} \\
q_2''' &= -\frac{(\beta-\delta+1)(\gamma+\delta-\alpha-1)+\{(\alpha-\delta+1)(\beta-1)-\alpha\beta q\}\frac{1}{a}}{(\beta-\alpha+1)(\beta-\gamma-\delta+2)} \\
q_2^{iv} &= -\frac{\alpha(\beta-1)+\{\alpha(\beta-\delta)-\alpha\beta q\}\frac{1}{a}}{(\gamma+\delta-\beta)(\delta-\beta+1)} \\
q_2^{v} &= -\frac{(\alpha-\delta+1)(\beta-1)+\{(\alpha-\delta+1)(\beta-1)-\alpha\beta q\}\frac{1}{a}}{(\gamma-\beta+1)(2-\beta)} \\
q_2^{vi} &= -\frac{\beta(\alpha-1)+\{\beta(\alpha-\delta)-\alpha\beta q\}\frac{1}{a}}{(\gamma+\delta-\alpha)(\delta-\alpha+1)} \\
q_2^{vii} &= -\frac{(\beta-\delta+1)(\alpha-1)+\{(\beta-\delta+1)(\alpha-\ )-\alpha\beta q\}\frac{1}{a}}{(\gamma-\alpha+1)(2-\alpha}
\end{aligned}
\right.
$$

$$
\begin{cases}
q_3 = \dfrac{\alpha(\gamma-\beta q)\frac{1}{1-a}-\alpha\beta}{\alpha(\alpha-\delta+1)} \\[2ex]
q_3^{\prime} = \dfrac{\beta(\gamma-\alpha q)\frac{1}{1-a}-\alpha\beta}{} \\[2ex]
q_3^{\prime\prime} = \dfrac{\{(\beta-\gamma)(1-\alpha-\beta)+\alpha-\alpha\beta q\}\frac{1}{1-a}-\alpha\beta}{(\alpha-\gamma+1)(\alpha-\gamma-\delta+2)} \\[2ex]
q_3^{\prime\prime\prime} = \dfrac{\{(1-\gamma)(1-\alpha-\beta)+\alpha-\alpha\beta q\}\frac{1}{1-a}-\alpha\beta}{(\beta-\gamma+1)(\beta-\gamma-\delta+2)} \\[2ex]
q_3^{IV} = \dfrac{\alpha(\gamma-\beta q)\frac{1}{1-a}-\alpha\beta}{(\gamma+\delta-\beta)(\gamma-\beta+1)} \\[2ex]
q_3^{V} = \dfrac{\{(1-\gamma)(1-\beta)+\alpha-\alpha\beta q\}\frac{1}{1-a}-\alpha\beta}{(\delta-\beta+1)(2-\beta)} \\[2ex]
q_3^{VI} = \dfrac{(\beta\gamma-\alpha\beta q)\frac{1}{1-a}-\alpha\beta}{(\gamma+\delta-\alpha)(\gamma-\alpha+1)} \\[2ex]
q_3^{VII} = \dfrac{\{(1-\gamma)(1-\alpha)+\beta-\alpha\beta q\}\frac{1}{1-a}-\alpha\beta}{(\delta-\alpha+1)(2-\alpha)}
\end{cases}
$$

$$
\begin{cases}
q_4 = \dfrac{\frac{a}{a-1}\{\frac{1}{a}\alpha\beta q-\alpha\gamma\}+\alpha(\beta-\delta)}{\alpha(\gamma+\delta-\beta)} \\[2ex]
q_4^{\prime} = -\dfrac{\frac{a}{a-1}\{\frac{1}{a}\alpha\beta q-\beta\gamma\}+\beta(\alpha-\delta)}{\beta(\gamma+\delta-\alpha)} \\[2ex]
q_4^{\prime\prime} = -\dfrac{\frac{a}{a-1}\{(1-\gamma)(\beta-1)-\alpha+\frac{1}{a}\alpha\beta q\}+\alpha(\beta-\delta)}{(\alpha-\gamma+1)(\delta-\beta+1)} \\[2ex]
q_4^{\prime\prime\prime} = -\dfrac{\frac{a}{a-1}\{(1-\gamma)(\alpha-1)-\beta+\frac{1}{a}\alpha\beta q\}+\beta(\alpha-\delta)}{(\beta-\delta+1)(\delta-\alpha+1)} \\[2ex]
q_4^{IV} = -\dfrac{\frac{a}{a-1}\{\frac{1}{a}\alpha\beta q-\alpha\gamma\}+(\alpha-\delta+1)(\beta-1)}{(\alpha-\delta+1)(\gamma-\beta+1)} \\[2ex]
q_4^{V} = -\dfrac{\frac{a}{a-1}\{(1-\gamma)(\beta-1)-\alpha+\frac{1}{a}\alpha\beta q\}+(\alpha-\delta+1)(\beta-1)}{(\alpha-\gamma-\delta+2)(2-\beta)} \\[2ex]
q_4^{VI} = -\dfrac{\frac{a}{a-1}\{\frac{1}{a}\alpha\beta q-\beta\gamma\}+(\beta-\delta+1)(\alpha-1)}{(\beta-\delta+1)(\gamma-\alpha+1)} \\[2ex]
q_4^{VII} = -\dfrac{\frac{a}{a-1}\{(1-\gamma)(\alpha-1)-\beta+\frac{1}{a}\alpha\beta q\}+(\beta-\delta+1)(\alpha-1)}{(\beta-\gamma-\delta+2)(2-\alpha)}
\end{cases}
$$

IV. Relations between the Particular Solutions.

It has been shown that 192 expressions are solutions of the differential equation (I') and, from the general theory of linear differential equations of the second order, it follows that any solution is linearly expressible in terms of two independent solutions. Let us first divide the above 192 functions, according to the common domain of existence, into eight gronps of twenty-four each, and for clearness we may write in the following :

Exponent 0				Exponent $1-\gamma$			
y_1	y_3	y_5	y_6	y_2	y_4	y_7	y_8
y_{17}	y_{19}	y_{21}	y_{22}	y_{18}	y_{20}	y_{23}	y_{24}
y_{105}	y_{106}	y_{109}	y_{111}	y_{107}	y_{108}	y_{110}	y_{112}
y_{137}	y_{138}	y_{141}	y_{143}	y_{139}	y_{140}	y_{142}	y_{144}
y_{153}	y_{154}	y_{157}	y_{159}	y_{155}	y_{156}	y_{158}	y_{160}
y_{185}	y_{186}	y_{189}	y_{191}	y_{187}	y_{188}	y_{190}	y_{192}

There are convergent in the region of point $x = 0$;

Exponent 0			
y_{9}	y_{10}	y_{13}	y_{15}
y_{33}	y_{34}	y_{37}	y_{39}
y_{65}	y_{66}	y_{67}	y_{68}
y_{89}	y_{90}	y_{91}	y_{92}
y_{161}	y_{162}	y_{163}	y_{164}
y_{177}	y_{178}	y_{179}	y_{180}

Exponent $1-\delta$			
y_{11}	y_{12}	y_{14}	y_{16}
y_{35}	y_{36}	y_{38}	y_{40}
y_{69}	y_{70}	y_{71}	y_{72}
y_{93}	y_{94}	y_{95}	y_{96}
y_{165}	y_{166}	y_{167}	y_{168}
y_{181}	y_{182}	y_{183}	y_{184}

These are convergent in the region of point x = 1;

Exponent 0			
y_{25}	y_{26}	y_{27}	y_{28}
y_{41}	y_{42}	y_{43}	y_{44}
y_{73}	y_{74}	y_{77}	y_{79}
y_{81}	y_{82}	y_{85}	y_{87}
y_{113}	y_{114}	y_{115}	y_{116}
y_{129}	y_{130}	y_{131}	y_{132}

Exponent $\gamma+\delta-\alpha-\beta$			
y_{29}	y_{30}	y_{31}	y_{32}
y_{45}	y_{46}	y_{47}	y_{48}
y_{75}	y_{76}	y_{78}	y_{80}
y_{83}	y_{84}	y_{86}	y_{88}
y_{117}	y_{118}	y_{119}	y_{120}
y_{133}	y_{134}	y_{135}	y_{136}

These are convergent in the region of point x = a;

Exponent α			
y_{49}	y_{5}	y_{55}	y_{56}
y_{57}	y_{59}	y_{63}	y_{64}
y_{97}	y_{99}	y_{103}	y_{104}
y_{121}	y_{123}	y_{127}	y_{128}
y_{145}	y_{147}	y_{149}	y_{150}
y_{169}	y_{171}	y_{173}	y_{174}

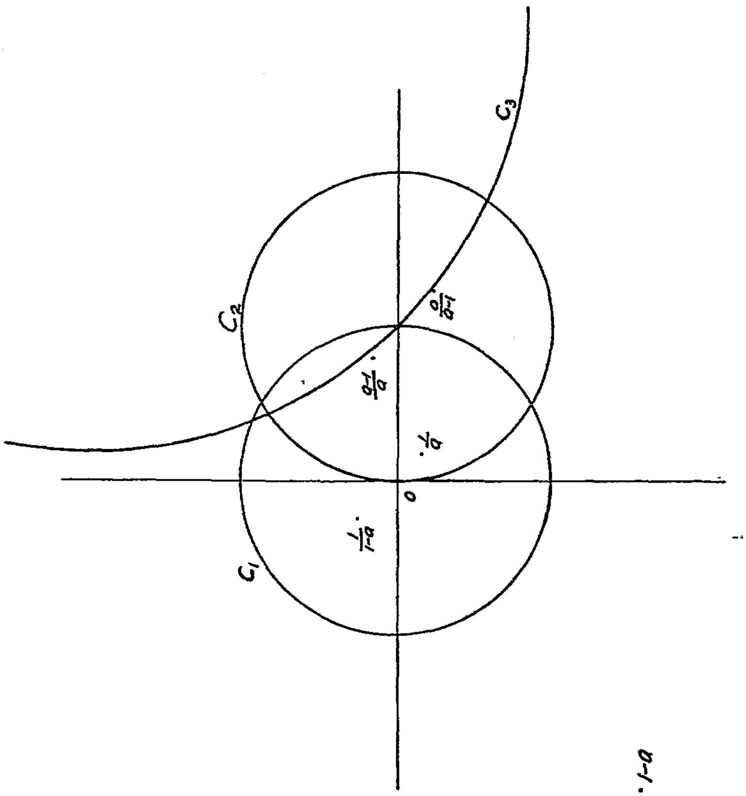
Exponent β			
y_{50}	y_{52}	y_{53}	y_{54}
y_{58}	y_{60}	y_{61}	y_{62}
y_{98}	y_{100}	y_{101}	y_{102}
y_{122}	y_{124}	y_{125}	y_{126}
y_{146}	y_{148}	y_{151}	y_{152}
y_{170}	y_{172}	y_{175}	y_{176}

These are convergent in the region of point $x = \infty$

The functions in the same group are not independent. If from each pair of coupled groups we choose any two, not both from the same group, we will form one fundamental system, and any other solution in the same region is connected with these two by a linear relation with constant coefficients.

To obtain the relations which connect the different pairs of groups is much more difficult. Let us give point a some definite value, for instance, say $a = 3+2i$, hence a moves in six positions during the homographic transformations of the independent variable, i.e., a, $1-a$, $\frac{1}{a}$, $\frac{1}{1-a}$, $\frac{a-1}{a}$, $\frac{1}{1-a}$ and $\frac{a}{a-1}$.

The following figure illustrates the positions of a for this particular value :



C_3
C_2
C_1
$\frac{a}{a-1}$
$\frac{a-1}{a}$
$\frac{1}{a}$
0
$\frac{1}{1-a}$
$1-a$

Let us consider the following four pairs of solutions:

y_1, y_2 ; y_9, y_{11} ; y_{41}, y_{45} ; y_{49}, y_{50} .

As to their regions of convergence, it is clear that for the pair (y_1, y_2) we have the circle (C_1) of radius unity whose centre is at the origin ; for the pair (y_9, y_{11}) the circle (C_2) whose centre is at $X = 1$ and whose radius is unity ; for the pair (y_{41}, y_{45}) the circle (C_3) whose centre is at $x = a$ and whose radius equals $(1-a)$; and for the pair (y_{49}, y_{50}) we have the entire plane outside of the circle (C_1).

We see now that the regions of $x = o$, $x = 1$ and $x = a$ havn an area in common, as have also the regions of $x = 1$, $x = a$ and $x = \infty$ It follows, therefore, that in the common area, say, to $x = o$, $x = 1$ and $x = a$ each one of the two pairs (y_9, y_{11}) and (y_{41}, y_{45}) are linearly expressible in terms of y_1 and y_2. If now, in this common area, we have expressed (y_9, y_{11}) and (y_{41}, y_{45}) *in terms of* y_1 and y_2, we can by a known process in the theory of functions obtain uniform and convergent developments for (y_9, y_{11}) and (y_{41}, y_{45}) in the whole regions of $x = o$. Similar remarks apply to the linear relations connecting the integrals (y_{49}, y_{50}) with (y_9, y_{11}) and (y_{41}, y_{45}), and consequently with (y_1, y_2). Thus we have the relations

$$\text{i)}\begin{cases} y_9 = A_1 y_1 + A_2 y_2, \\ y_{11} = A_3 y_1 + A_4 y_2 ; \end{cases} \quad \text{ii)}\begin{cases} y_{41} = B_1 y_1 + B_2 y_2, \\ y_{45} = B_3 y_1 + B_4 y_2 ; \end{cases} \quad \text{iii)}\begin{cases} y_{49} = C_1 y_1 + C_2 y_2, \\ y_{50} = C_3 y_1 + C_4 y_2 . \end{cases}$$

wherein the coefficients A_1, B_1, C_1, are constants. These constants are not all independent; there exist relations between them which will now be determined.

Let us take the first one of i), namely,

$$y_q = A_1 y_1 + A_2 y_2 .$$

To determine A_1 and A_2 the substitution of any two particular values of x will be sufficient; let then x = 1 and x = o, and suppose $1-\gamma$ a positive quantity so that $x^{1-\gamma}$ is zero when x = o; we have for these two cases

$$1 = A_1 F(a, q_1 ; \alpha, \beta, \gamma, \delta, 1) + A_2 F(a, q_1 ; \alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, \delta, 1),$$

$$F(1-a, 1-q_1' ; \alpha, \beta, \delta, \gamma, 1) = A_1$$

To evaluate A_1 and A_2 we must find the value of the series for argument unity. It is unfortunately, not like the hypergeometric series, we can easily express its value, when x = 1, in terms of gamma-functions[1]. But to determine the relations between them, we may proceed as the following way : The equation (I') in the form

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{1-\alpha+\beta-\gamma-\delta}{x-a}\right)\frac{dy}{dx} + \frac{\alpha\beta(x-q)}{x(x-1)(x-a)}\, y = 0 \text{------------} (I')$$

Is transformed by the substitution

$$y = e^{-\int I dx}\, v$$

Into $$v'' + I v = 0$$

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}P'$, and $P = \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{1+\alpha+\beta-\gamma-\delta}{x-a}$,

$$Q = \frac{\alpha\beta(x-q)}{x(x-a)(x-1)} .$$

---

[1] C. Jordan, *Cours D'Analyse*, Tome I, Nos. 379 et 382 (1909).

Letting $w = \frac{v}{\gamma} = \sum_{n=0}^{\infty} w_n q^n$, we have

$$\left(\sum_{n=0}^{\infty} w_n' q^n\right) + \left(\sum_{n=0}^{\infty} w_n q^n\right) + I = 0$$

Accordingly,

$$w_0' + w_0^2 + I_0 = 0 , \qquad I_0 = (I)_{q=0}$$

This is the well-known Riccati equation.

Suppose a particular solution

$$w_0 = \left(\frac{y_1}{y_1}\right)_{q=0}$$

is known.

Hence

$$w_0 = \frac{F'(a,(q)_0;\alpha,\beta,\gamma,\delta,x)}{F(a,(q)_0;\alpha,\beta,\gamma,\delta,x)} + \frac{1}{2}P$$

We also have

$$w_1' + 2w_0 w_1 = \frac{\alpha\beta}{x(x-1)(x-a)}$$
$$w_2' + 2w_0 w_2 + w_1^2 = 0,$$
$$w_3' + 2w_0 w_3 + 2w_1 w_2 = 0,$$
$$\cdots\cdots$$
$$w_n' + 2w_0 w_n + 2w_1 w_{n-1} + \cdots + 2w_{\frac{n-1}{2}} w_{\frac{n+1}{2}} = 0, \quad (n: odd)$$
$$\cdots\cdots .$$

since $\frac{v'}{v} = \frac{1}{2}P + \frac{y'}{y} = \sum_{n=0}^{\infty} w_n q^n ;$

Thus by comparing the coefficients of like terms, we have

$w_1 = \frac{\alpha\beta}{a\gamma}$, $w_2 = w_3 = \cdots\cdots = 0$, when x = 0.

Now, we have

$$w_1 = C_1 e^{-\int_0^x 2w_0\,dx} + \alpha\beta e^{-\int_0^x 2w_0\,dx} \int_0^x \frac{e^{\int_0^x 2w_0\,dx}}{x(x-1)(x-a)}\,dx$$

To show $C_1 = 0$, when x = 0, we proceed as follow :

Rewrite the above equation as

$$w_1 = C_1 e^{-\int_0^x 2w_0\,dx} + \alpha\beta\, e^{-\int_0^x 2w_0\,dx} \int_0^x \left(\frac{1}{ax} + \frac{1}{(1-a)(x-1)} + \frac{1}{a(a-1)(x-a)}\right) e^{\int_0^x 2w_0\,dx}\,dx,$$

Using Taylor's expansion, let

$$F(a,(q)_0;\alpha,\beta,\gamma,\delta,x)(x-1)^{\delta}(x-a)^{1+\alpha+\beta-\gamma-\delta}=\sum_{n=0}^{\infty}a_n x^n,\qquad a_0\neq 0\ ;$$

And

$$\frac{1}{(1-a)(x-1)}+\frac{1}{a(a-1)(x-a)}=\sum_{n=0}^{\infty}b_n x^n,\quad b_0\neq 0.$$

Thus

$$w_1=C_1 e^{-\int_0^x 2w_0\,dx}+\frac{\alpha\beta}{x^{\gamma}\sum_{n=0}^{\infty}a_n x^n}\int_0^x\frac{\sum_{n=0}^{\infty}a_n x^n}{a\,x^{1-\gamma}}\,dx$$

$$+\int_0^x x^{\gamma}\sum_{n=0}^{\infty}a_n x^n\sum_{n=0}^{\infty}b_n x^n\,dx$$

Integrating and putting $w_1=\frac{\alpha\beta}{a\gamma}$, $x=0$, we can easily see $C_1=0$.

Therefore we obtain

$$w_1=\alpha\beta\,e^{-\int_0^x 2w_0\,dx}\int_0^x\frac{e^{\int_0^x 2w_0\,dx}}{x(x-1)(x-a)}\,dx\ ;$$

Similarly, we have

$$w_2=-e^{-\int_0^x 2w_0\,dx}\int_0^x w_1^2\,e^{\int_0^x 2w_0\,dx}\,dx,$$

$$w_3=-2\,e^{-\int_0^x 2w_0\,dx}\int_0^x w_1w_2\,e^{\int_0^x 2w_0\,dx}\,dx,$$

- ,

$$w_n=-2\,e^{-\int_0^x 2w_0\,dx}\int_0^x\left(w_1w_{n-1}+w_2w_{n-2}+\cdots+w_{\frac{n-1}{2}}w_{\frac{n+1}{2}}\right)e^{\int_0^x 2w_0\,dx}\,dx,$$

- .

Hence, finally we obtain

$$F(a,q;\alpha,\beta,\gamma,\delta,x)=F(a,(q)_0;\alpha,\beta,\gamma,\delta,x)\,e^{\sum_{n=1}^{\infty}q^n\int_0^x w_n\,dx}$$

Thus we have, when $x=0$,

$$F(a,q;\alpha,\beta,\gamma,\delta,0)=F(a,(q)_0;\alpha,\beta,\gamma,\delta,0)=1\ ;$$

And when $x=1$,

$$F(a,q;\alpha,\beta,\gamma,\delta,1)=F(a,(q)_0;\alpha,\beta,\gamma,\delta,1)\,e^{\sum_{n=1}^{\infty}a_n^{(1)}q^n}$$

Let us write

$$y_q=A_1y_1+A_2y_2$$

As

$$y_q=F(1-a,1-q';\alpha,\beta,\delta,\gamma,1)\,y_1+A_2y_2,$$

or

$$\sum_{n=0}^{\infty} a_n^{(q)} q^n = F(1-a, 1-q'; \alpha, \beta, \delta, r, 1) F(a, (q)_0; \alpha, \beta, r, \delta, x)$$

$$\times \left[1 + \sum_{n=1}^{\infty} a_n^{(1)} q^n + \frac{\left(\sum_{n=1}^{\infty} a_n^{(1)} q^n\right)^2}{2!} + \ldots\ldots \right] + A_2 \sum_{n=0}^{\infty} a_n^{(2)} .$$

By comparing the coefficients of like terms and putting $x = 1$, we have

$$A_2 = \left[\frac{1 - F(1-a, (1-q')_{q=0}; \alpha, \beta, \delta, r, 1) F(a, (q)_0; \alpha, \beta, r, \delta, 1)}{a_0^{(2)}} \right]_{x=1.}$$

We may also proceed to find the relations which connect any two particular solutions with their differential coefficients, either in the same or different pairs.

It is known that, if Y_1 and Y_2 be two particular solutions of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

Then

$$Y_1 \frac{dY_2}{dx} - Y_2 \frac{dY_1}{dx} = C e^{-\int P dx},$$

Where C has a constant value which depends upon the set of particular solutions selected.

Choosing $Y_1 = y_1$ $Y_2 = y_2$, we can easily obtain

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = a^{r+\delta-\alpha-\beta-1}(1-r)x^{-r}(x-1)^{-\delta}(x-a)^{r+\delta-\alpha-\beta-1}$$

We have already shown that

$$y_9 = A_1 y_1 + A_2 y_2$$

in which A_1 and A_2 are constants. This gives on differentiation

$$\frac{dy_9}{dx} = A_1 \frac{dy_1}{dx} + A_2 \frac{dy_2}{dx},$$

And therefore

$$y_1 \frac{dy_9}{dx} - y_9 \frac{dy_1}{dx} = A_2 a^{r+\delta-\alpha-\beta-1}(1-r)x^{-r}(x-1)^{-\delta}(x-a)^{r+\delta-\alpha-\beta-1}$$

V. Generalizations applied to Klein's equation.

It may be generalized to the most general Klein s equation. Recalling that we set

$a_1 \to 0$, $a_n \to \infty$, the Klein's equation becomes

$$\frac{d^2y}{dx^2} + \left(\sum_{i=1}^{n-1} \frac{1-\alpha_i-\beta_i}{x-a_i} \right) \frac{dy}{dx} + \left\{ \sum_{i=1}^{n-1} \frac{\alpha_i \beta_i}{(x-a_i)^2} + \sum_{i=1}^{n-1} \frac{D_i}{x-a_i} \right\} y = 0,$$

Where

$$\sum_{i=1}^{n} (\alpha_i+\beta_i) = n-2, \quad \sum_{i=1}^{n-1} D_i = 0, \quad \alpha_n \beta_n = \sum_{i=1}^{n-1} (a_i D_i + \alpha_i \beta_i).$$

This equation may be also written

$$\frac{d^2y}{dx^2} + \frac{\sum_{i=1}^{n-1} (1-\alpha_i-\beta_i) \prod_{j(\neq i)=1}^{n-1} (x-a_j)}{\prod_{i=1}^{n-1} (x-a_i)} \frac{dy}{dx}$$

$$+ \frac{\sum_{i=1}^{n-1} \alpha_i \beta_i \prod_{j(\neq i)=1}^{n-1} (x-a_j)^2 + \sum_{i=1}^{n-1} D_i (x-a_i) \prod_{j(\neq i)=1}^{n-1} (x-a_j)^2}{\prod_{i=1}^{n-1} (x-a_i)^2} y = 0 \cdots\cdots (\bar{I})$$

With above relations. This equation has n-3 arbitrary undetermined constants[1] which will be denoted by q_i $(i=1,2,\cdots n-3)$. The scheme of $(\bar{I})$ is

$$P \left\{ \begin{matrix} 0 & a_2 & a_3 & \cdots\cdots & a_{n-1} & \infty & \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots\cdots & \alpha_{n-1} & \alpha_n & x \\ \beta_1 & \beta_2 & \beta_3 & \cdots\cdots & \beta_{n-1} & \beta_n & \end{matrix} \right\}$$

If we choose the exponents such that $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$,

$\alpha_n = \alpha_n'$, $\beta_1 = 1-\beta_1'$, $\beta_2 = 1-\beta_2'$, $\cdots$ $\beta_{n-2} = 1-\beta_{n-2}'$, $\beta_{n-1} = \sum_{i=1}^{n-2} \beta_i' - \alpha_n' - \beta_n'$, $\beta_n = \beta_n'$,

we have the simple equation

$$\frac{d^2y}{dx^2} + \left\{ \sum_{i=1}^{n-2} \frac{\beta_i'}{x-a_i} + \frac{1+\alpha_n'+\beta_n'-\sum_{i=1}^{n-2}\beta_i'}{x-a_{n-1}} \right\} \frac{dy}{dx} + \sum_{i=1}^{n-1} \frac{D_i}{x-a_i} y = 0 \quad \cdots\cdots (\bar{I}')$$

Where $\sum_{i=1}^{n-1} D_i = 0$, $\alpha_n' \beta_n' = \sum_{i=1}^{n-1} a_i D_i$. The scheme of $(\bar{I}')$ is

$$P \left\{ \begin{matrix} 0 & a_2 & a_3 & \cdots\cdots & a_{n-2} & a_{n-1} & \infty & \\ 0 & 0 & 0 & \cdots\cdots & 0 & 0 & \alpha_n' & x \\ 1-\beta_1' & 1-\beta_2' & 1-\beta_3' & \cdots\cdots & 1-\beta_{n-2}' & \sum_{i=1}^{n-2}\beta_i'-\alpha_n'-\beta_n' & \beta_n' & \end{matrix} \right\}$$

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1 MacRobert, Functions of a Complex Variable, p. 244 (1933)

Let the integral of this equation, which is regular in the vicinity of x = o and belong to the exponent o, be denoted by

$$F\left(a_2, a_3, \cdots a_{n-1}; q_1, q_2, \cdots q_{n-3}; \alpha_n', \beta_n', \beta_{n-1}', \cdots \beta_2', \beta_1'; x\right).$$

Now we apply to $(\bar{I})$ the Fuchsian substitution

$$y = \prod_{i=1}^{n-1} (x-a_i)^{\alpha_i} u,$$

And obtain

$$\prod_{i=1}^{n-1}(x-a_i)^2 \frac{d^2u}{dx^2} + \left[2\prod_{i=1}^{n-1}(x-a_i)^2 \sum_{i=1}^{n-1} \frac{\alpha_i}{x-a_i} + \prod_{i=1}^{n-1}(x-a_i)\right]\frac{du}{dx}$$

$$+\left[\prod_{i=1}^{n-1}(x-a_i)^2 \left\{\sum_{i=1}^{n-1} \frac{\alpha_i(\alpha_i-1)}{(x-a_i)^2} + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \frac{2\alpha_i \alpha_k}{(x-a_i)(x-a_k)}\right\} + \prod_{i=1}^{n-1}(x-a_i) P \sum_{i=1}^{n-1} \frac{\alpha_i}{x-a_i} + Q\right] u = 0$$

Where

$$P = \sum_{i=1}^{n-1} (1-\alpha_i-\beta_i) \prod_{j(\neq i)=1}^{n-1} (x-a_j), \qquad \text{-------- } (\bar{II}_1)$$

$$Q = \sum_{i=1}^{n-1} \alpha_i \beta_i \prod_{j(\neq i)=1}^{n-1} (x-a_j)^2 + \sum_{i=1}^{n-1} D_i (x-a_i) \prod_{j(\neq i)=1}^{n-1} (x-a_j)^2,$$

And the scheme is

$$P\left\{\begin{matrix} 0 & a_2 & a_3 & \cdots\cdots & a_{n-1} & \infty & \\ 0 & 0 & 0 & \cdots\cdots & 0 & \sum_{i=1}^{n} \alpha_i & x \\ \beta_1-\alpha_1 & \beta_2-\alpha_2 & \beta_3-\alpha_3 & \cdots\cdots & \beta_{n-1}-\alpha_{n-1} & \sum_{i=1}^{n-1} \alpha_i+\beta_n & \end{matrix}\right\}$$

Thus a particular solution of $(\bar{I})$ may be written down as

$$y = \prod_{i=1}^{n-1} (x-a_i)^{\alpha_i} F\left(a_2, a_3, \cdots a_{n-1}; q_1, q_2, \cdots q_{n-3}; \alpha_1, \alpha_2, \cdots \alpha_n; \beta_1, \beta_2, \cdots \beta_n; x\right).$$

By interchanging the exponents (α_i, β_i), we have 2^{n-1} particular integrals. Since the homographic transformations of the independent variable have $\underline{n}$ different forms, in doing so, we will obtain altogether n corresponding equations, namely $\bar{II}_1, \bar{II}_2, \cdots \bar{II}_n$, and consequently we will totally have $n!\, 2^{n-1}$ integrals.

Similarly, we may divide the $n!\, 2^{n-1}$ functions into 2n groups of $\underline{n-1}\, 2^{n-2}$ each, and write them into n couples.

Moreover, by a well-known theorem of power series, the radius of convergence reaches the nearest singular point, so we can find a common area in the finite part of the plane such that the function associated with the functions which are convergent at ∞ have linear relation with sonstant coefficients. If a particular solution of $(\bar{I}')$, in which all q_i, are zero, is known, we can in a similar way determine the relations between the constants.

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