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The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

1900.

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Rio de Janeiro.
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Adelaide.
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Brisbane.
p. Royal Society of Queensland.
Melbourne.
p. Observatory.
p. Royal Society of Victoria.
AB. University Library.
Sydney.
p. Australian Museum.
p. Geological Survey.
p. Linnean Society of New South Wales.
AB. Royal Society of New South Wales.
AB. University Library.

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p. Jugoslavenska Akademija Znanosti i Umjetnosti.
p. Societas Historico-Naturalis Croatica.

Austria (continued).
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Gratz.
AB. Naturwissenschaftlicher Verein für Steiermark.
Innsbruck.
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Prague.
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AB. Kaiserliche Akademie der Wissenschaften.
p. K.K. Geographische Gesellschaft.
AB. K.K. Geologische Reichsanstalt.
B. K.K. Naturhistorisches Hof-Museum.
B. K.K. Zoologisch-Botanische Gesellschaft.
p. Oesterreichische Gesellschaft für Meteorologie.
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Brussels.
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AB. Académie Royale des Sciences.
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p. Société Malacologique de Belgique.
Ghent.
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Belgium (continued).

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Colombo.

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England and Wales (continued).

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Aberdeen.

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Edinburgh.

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Scotland (continued).

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  - *p.* Royal Scottish Society of Arts.
  - AB Royal Society.
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  - *p.* Philosophical Society.

Servia.

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  - AB. Real Academia de Ciencias.

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  - *p.* Naturforschende Gesellschaft.
  - *p.* Sternwarte.

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  - *p.* Royal Society of Tasmania.

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- Baltimore.
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  - *p.* University of California.
- Boston.
  - AB. American Academy of Sciences.
  - B. Boston Society of Natural History.
  - A. Technological Institute.
- Brooklyn.
  - AB. Brooklyn Library.
- Cambridge.
  - AB. Harvard University.
  - B. Museum of Comparative Zoology.
- Chapel Hill (N.C.).
  - *p.* Elisha Mitchell Scientific Society.
- Charleston.
  - *p.* Elliott Society of Science and Art of South Carolina.
- Chicago.
  - AB. Academy of Sciences.
  - *p.* Astrophysical Journal.
  - *p.* Field Columbian Museum.
- Davenport (Iowa).
  - *p.* Academy of Natural Sciences.
- Granville (Ohio).
  - *p.* Journal of Comparative Neurology.
- Ithaca (N.Y.).
  - A. Journal of Physical Chemistry.
  - *p.* Physical Review (Cornell University).
- Lawrence.
  - *p.* Kansas University.
- Madison.
  - *p.* Wisconsin Academy of Sciences.
- Mount Hamilton (California).
  - A. Lick Observatory.
- New Haven (Conn.).
  - AB. American Journal of Science.
  - AB. Connecticut Academy of Arts and Sciences.
- New York.
  - *p.* American Geographical Society.
  - A. American Mathematical Society.
  - *p.* American Museum of Natural History.
  - AB. Columbia College Library.
  - *p.* New York Academy of Sciences.
  - *p.* New York Medical Journal.

United States (continued).

Philadelphia.
- AB. Academy of Natural Sciences.
- AB. American Philosophical Society.
- *p.* Franklin Institute.
- *p.* Wagner Free Institute of Science.

Rochester (N.Y.).
- *p.* Academy of Science.

St. Louis.
- *p.* Academy of Science.

Salem (Mass.).
- *p.* American Association for the Advancement of Science.
- AB. Essex Institute.

San Francisco.
- AB. California Academy of Sciences.

United States (continued).

Washington.
- AB. Patent Office.
- AB. Smithsonian Institution.
- AB. United States Coast Survey.
- B. United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.
- AB. United States Naval Observatory.
- *p.* United States Department of Agriculture.
- A. United States Department of Agriculture (Weather Bureau).

West Point (N.Y.)
- AB. United States Military Academy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1900, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to Professor MARCELLIN BERTHELOT, For. Mem.R.S., for his brilliant services to Chemical Science.

The RUMFORD MEDAL to Professor ANTOINE HENRI BECQUEREL, for his discoveries in Radiation proceeding from Uranium.

A ROYAL MEDAL to Major PERCY ALEXANDER MACMAHON, F.R.S., for the number and range of his contributions to Mathematical Science.

A ROYAL MEDAL to Professor ALFRED NEWTON, F.R.S., for his eminent contributions to the science of Ornithology and the Geographical Distribution of Animals.

The DAVY MEDAL to Professor GUGLIELMO KOERNER, for his brilliant investigations on the Position Theory of the Aromatic Compounds.

The DARWIN MEDAL to Professor ERNST HAECKEL, for his long-continued and highly-important work in Zoology, all of which has been inspired by the spirit of Darwinism.

The BAKERIAN LECTURE for the year 1900, "On the Specific Heat of Metals and the Relation of Specific Heat to Atomic Weight," was delivered by Professor W. A. TILDEN, F.R.S., on March 8, 1900.

The CROONIAN LECTURE for the year 1900, "On Immunity with Special Reference to Cell Life," was delivered by Professor Dr. PAUL EHRLICH, on March 22, 1900.

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PHILOSOPHICAL TRANSACTIONS.

I. *Mathematical Contributions to the Theory of Evolution.*—VII. *On the Correlation of Characters not Quantitatively Measurable.*

By KARL PEARSON, *F.R.S.*

(*From the Department of Applied Mathematics, University College, London.*)

Received February 7,—Read March 1, 1900.

NOTE.

In August, 1899, I presented a memoir to the Royal Society on the inheritance of coat-colour in the horse and of eye-colour in man, which was read November, 1899, and ultimately ordered to be published in the 'Phil. Trans.' Before that memoir was printed, Mr. YULE'S valuable memoir on Association was read, and, further, Mr. LESLIE BRAMLEY-MOORE showed me that the theory of my memoir as given in § 6 of the present memoir led to somewhat divergent results according to the methods of proportioning adopted. We therefore undertook a new investigation of the theory of the whole subject, which is embodied in the present memoir. The data involved in the paper on coat-colour in horses and eye-colour in man have all been recalculated, and that paper is nearly ready for presentation.* But it seemed best to separate the purely theoretical considerations from their application to special cases of inheritance, and accordingly the old memoir now reappears in two sections. The theory discussed in this paper was, further, the basis of a paper on the Law of Reversion with special reference to the Inheritance of Coat-colour in Basset Hounds recently communicated to the Society, and about to appear in the 'Proceedings.'†

While I am responsible for the general outlines of the present paper, the rough draft of it was taken up and carried on in leisure moments by Mr. LESLIE BRAMLEY-MOORE, Mr. L. N. G. FILON, M.A., and Miss ALICE LEE, D.Sc. Mr. BRAMLEY-MOORE discovered the u-functions; Mr. FILON proved most of their general properties and the convergency of the series; I alone am responsible for sections 4, 5, and 6. Mr. LESLIE BRAMLEY-MOORE sent me, without proof, on the eve of his departure for the Cape, the general expansion for z on p. 26. I am responsible for the present proof and its applications. To Dr. ALICE LEE we owe most of the illustrations and the table on p. 17. Thus the work is essentially a joint memoir in which we have equal part, and the use of the first personal pronoun is due to the fact that the material had to be put together and thrown into form by one of our number.—K. P.

CONTENTS.

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* Since ordered to be printed in the 'Phil. Trans.'

† Read January 25, 1900. 'Roy. Soc. Proc.,' vol. 66, p. 140.

§ (1.) *On a General Theorem in Normal Correlation.*

Let the frequency surface

$$z = \frac{\mathrm{N}}{2\pi\sqrt{(1-r^2)}\sigma_1\sigma_2}\, e^{-\frac{1}{2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}-\frac{2rxy}{\sigma_1\sigma_2}\right)},$$

where N = total number of observations,

σ_1, σ_2 = standard deviations of organs x and y,

r = correlation of x and y,

be divided into four parts by two planes at right angles to the axes of x and y at distances h' and k' from the origin. The total volumes or frequencies in these parts will be represented by a, b, c, and d in the manner indicated in the accompanying plan :—

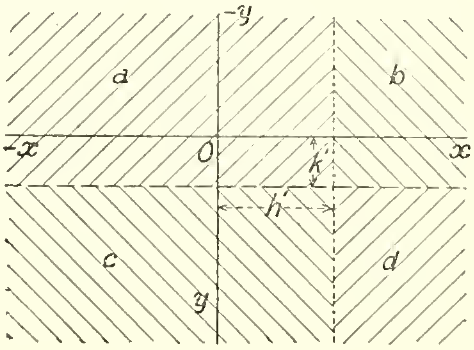

Table of Frequencies

a	b	a+b
c	d	c+d
a+c	b+d	N

Then clearly

$$d = \frac{\mathrm{N}}{2\pi\sqrt{(1-r^2)}\sigma_1\sigma_2}\int_{h'}^{\infty}\int_{k'}^{\infty} e^{-\frac{1}{2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}-\frac{2rxy}{\sigma_1\sigma_2}\right)}\,dx\,dy$$

$$= \frac{\mathrm{N}}{2\pi\sqrt{(1-r^2)}}\int_{h}^{\infty}\int_{k}^{\infty} e^{-\frac{1}{2}\frac{1}{1-r^2}(x^2+y^2-2rxy)}\,dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)},$$

if $h = h'/\sigma_1$ and $k = k'/\sigma_2$.

Further,

$$b + d = \frac{\mathrm{N}}{\sqrt{2\pi}\,\sigma_1}\int_{h'}^{\infty} e^{-\frac{1}{2}\frac{x^2}{\sigma_1^2}}\,dx,$$

$$= \frac{\mathrm{N}}{\sqrt{2\pi}}\int_{h}^{\infty} e^{-\frac{1}{2}x^2}\,dx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (\text{ii.}),$$

and

$$c + d = \frac{\mathrm{N}}{\sqrt{2\pi}}\int_{k}^{\infty} e^{-\frac{1}{2}y^2}\,dy \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (\text{iii.}),$$

$$\frac{(a + c) - (b + d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}}\int_{0}^{h} e^{-\frac{1}{2}x^2}\,dx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (\text{iv.}),$$

$$\frac{(a + b) - (c + d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}}\int_{0}^{k} e^{-\frac{1}{2}y^2}\,dy \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (\text{v.}).$$

Thus, when a, b, c, and d are known, h and k can be found by the ordinary table of the probability integral, say that of Mr. SHEPPARD ('Phil. Trans.,' A, vol. 192, p. 167, Table VI.*). The limits accordingly of the integral for d in (i.) are known.

Now consider the expression

$$\frac{1}{\sqrt{1 - r^2}}\,e^{-\frac{1}{2}\frac{1}{1 - r^2}(x^2 + y^2 - 2rxy)} = \mathrm{U}, \text{ say}, \quad . \; . \; . \; . \; . \; . \; . \; . \; (\text{vi.}),$$

and let us expand it in powers of r. Then, if the expansion be

$$\mathrm{U} = e^{-\frac{1}{2}(x^2 + y^2)}\left(u_0 + \frac{u_1 r}{\underline{\lvert 1}} + \frac{u_2 r^2}{\underline{\lvert 2}} + \ldots + \frac{u_n r^n}{\underline{\lvert n}} + \ldots\right) \quad . \; . \; . \; . \; (\text{vii.}),$$

we shall have

$$u_n = e^{\frac{1}{2}(x^2 + y^2)}\left(\frac{d^n\mathrm{U}}{dr^n}\right)_{r=0} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (\text{viii.}).$$

Taking logarithmic differentials, we get at once

$$(1 - r^2)^2\frac{d\mathrm{U}}{dr} = \{xy + r(1 - x^2 - y^2) + r^2xy - r^3\}\mathrm{U}.$$

Differentiating n times by LEIBNITZ's theorem, and putting $r = 0$, we have, after some reductions

$$\begin{aligned} u_{n+1} &= n(2n - 1 - x^2 - y^2)u_{n-1} \\ &\quad - n(n - 1)(n - 2)^2 u_{n-3} \\ &\quad + xy\{u_n + n(n - 1)u_{n-2}\} \quad . \; . \; . \; . \; . \; . \; . \; . \; (\text{ix.}). \end{aligned}$$

Hence we find

$$\left.\begin{aligned} u_0 &= 1 \\ u_1 &= xy \\ u_2 &= (x^2 - 1)(y^2 - 1) \\ u_3 &= x(x^2 - 3)\,y(y^2 - 3) \\ u_4 &= (x^4 - 6x^2 + 3)(y^4 - 6y^2 + 3) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; (\text{x.})$$

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* See, however, foot-note, p. 5.

Thus the following laws are indicated :—

$$u_n = v_n \times w_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xi.}),$$

where

$$v_n = xv_{n-1} - (n-1)v_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xii.}),$$

$$w_n = yw_{n-1} - (n-1)w_{n-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xiii.}).$$

We shall now show that these laws hold good by induction. Assume

$$u_{n+1} = v_{n+1}w_{n+1} = (xv_n - nv_{n-1})(yw_n - nw_{n-1}).$$

Thus

$$u_{n+1} = xyu_n + n^2u_{n-1} - n(yw_nv_{n-1} + xv_nw_{n-1}).$$

But by (ix.), substituting for u_{n-3} from (xi.) and (xiii.),

$$\begin{aligned} u_{n+1} = xy\{v_nw_n + n(n-1)v_{n-2}w_{n-2}\} &+ n(2n-1 - x^2 - y^2)v_{n-1}w_{n-1} \\ &- n(n-1)v_{n-1}w_{n-1} - xyn(n-1)v_{n-2}w_{n-2} \\ &+ n(n-1)(yv_{n-1}w_{n-2} + xv_{n-2}w_{n-1}). \\ = xyv_nw_n &+ n^2v_{n-1}w_{n-1} - n(x^2+y^2)v_{n-1}w_{n-1} \\ &+ n(n-1)(yv_{n-1}w_{n-2} + xv_{n-2}w_{n-1}) \\ = xyv_nw_n &+ n^2v_{n-1}w_{n-1} - n\{yv_{n-1}(yu_{n-1} - \overline{n-1}\,w_{n-2}) \\ &\qquad + xw_{n-1}(xv_{n-1} - \overline{n-1}\,v_{n-2}\} \\ = xyv_nw_n &+ n^2v_{n-1}w_{n-1} - n(yv_{n-1}w_n + xw_{n-1}v_n) \\ = v_{n+1}w_{n+1}&, \text{ as we have seen above.} \end{aligned}$$

Thus, if the theorem holds for u_n, it holds for u_{n+1}. Accordingly

$$\text{U} = e^{-\frac{1}{2}(x^2+y^2)}\left(1 + \frac{v_1w_1}{\underline{|1}}r + \frac{v_2w_2}{\underline{|2}}r^2 + \ldots + \frac{v_nw_n}{\underline{|n}}r^n + \ldots\right) \quad . \quad . \quad . \quad (\text{xiv.}),$$

where the v's and w's are given by (x.), (xii.), and (xiii.).

It is thus clear that $\frac{1}{2\pi}\int_h^\infty \int_k^\infty \text{U}\, dx\, dy$ consists of a series of which the general term is

$$\frac{1}{\underline{|n}}\text{V}_n\text{W}_n r^n$$

where

$$\text{V}_n = \frac{1}{\sqrt{2\pi}}\int_h^\infty e^{-\frac{1}{2}x^2}v_n dx$$

$$\text{W}_n = \frac{1}{\sqrt{2\pi}}\int_k^\infty e^{-\frac{1}{2}y^2}w_n dy.$$

It remains to find these integrals.

The general form of v_n is given by

$$v_n = x^n - \frac{n(n-1)}{2\,\underline{|1}}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2\,\underline{|2}}x^{n-4} - \&\text{c.} \quad . \quad . \quad (\text{xv.}).$$

For this obviously gives (x.). Assume it true for v_{n-1} and v_{n-2}, then

$$\begin{aligned} xv_{n-1} - (n-1)v_{n-2} &= x^n - \frac{(n-1)(n-2)}{2\,\underline{|1}}x^{n-2} + \frac{(n-1)(n-2)(n-3)(n-4)}{2^2\,\underline{|2}}x^{n-4} - \dots \\ &\quad - (n-1)x^{n-2} + \frac{(n-1)(n-2)(n-3)}{2\,\underline{|1}}x^{n-4} - \dots \\ &= x^n - \frac{n(n-1)}{2\,\underline{|1}}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2\,\underline{|2}}x^{n-4} - \dots \\ &= v_n. \end{aligned}$$

Thus the expression (xv.) is shown to hold by induction, the general terms being

$$\begin{aligned} &(-1)^r\frac{(n-1)(n-2)\dots(n-2r+1)}{2^{r-1}\,\underline{|r-1}}\left(\frac{n-2r}{2r}+1\right)x^{n-2r} \\ &\qquad\qquad = (-1)^r\frac{n(n-1)(n-2)\dots(n-2r+1)}{2^r\,\underline{|r}}x^{n-2r}, \end{aligned}$$

or the general term in u_n.

We notice at once that

$$\frac{dv_n}{dx} = nv_{n-1} \quad \dots\dots\dots\dots \quad \text{(xvi.)}.$$

Thus, by (xii.)

$$v_n = xv_{n-1} - \frac{dv_{n-1}}{dx}.$$

Multiply by $e^{-\frac{1}{2}x^2}$ and integrate

$$\int e^{-\frac{1}{2}x^2}v_n\,dx = \int xe^{-\frac{1}{2}x^2}v_{n-1}dx - \int e^{-\frac{1}{2}x^2}\frac{dv_{n-1}}{dx}\,dx.$$

Integrating the latter integral by parts, we have

$$\int v_n e^{-\frac{1}{2}x^2}dx = -e^{-\frac{1}{2}x^2}v_{n-1},$$

or

$$\mathrm{V}_n = \frac{1}{\sqrt{2\pi}}\int_h^\infty v_n e^{-\frac{1}{2}x^2}dx = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}h^2}(v_{n-1})_{x=h}.$$

Now $\dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}h^2}$ can be found from any table of the ordinates of the normal curve, *e.g.*, Mr. SHEPPARD'S, 'Phil. Trans.,' A, vol. 192, p. 153, Table I.* We shall accordingly put

$$\mathrm{H} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}h^2}, \qquad \mathrm{K} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}k^2} \quad \dots\dots \quad \text{(xvii.)},$$

and look upon H and K as known quantities.

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* For our present purposes the differences of Mr. SHEPPARD'S tables are occasionally too large, but the following series give very close results :—

Let

$$\chi_1 = \sqrt{\frac{\pi}{2}}\,\frac{(a+c)-(b+d)}{\mathrm{N}} = \int_0^h e^{-\frac{1}{2}x^2}dx \text{ by (iv.)},$$

$$\chi_2 = \sqrt{\frac{\pi}{2}}\,\frac{(a+b)-(c+d)}{\mathrm{N}} = \int_0^k e^{-\frac{1}{2}y^2}dy \text{ by (v.)}.$$

Further, let us write $(v_{n-1})_{x=h}$ as $\bar{v}_{n-1}$, and similarly $(w_{n-1})_{y=k}$ as $\bar{w}_{n-1}$. Thus

$$\text{V}_n = \text{H} \,.\, \bar{v}_{n-1}, \qquad \text{W}_n = \text{K} \,.\, \bar{w}_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xviii.)}.$$

We have then from (i.)

$$\frac{d}{\text{N}} = \frac{1}{2\pi}\int_h^\infty \int_k^\infty \text{U}\,dx\,dy$$

$$= \frac{1}{2\pi}\int_h^\infty \int_k^\infty e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy + \overset{\infty}{\underset{1}{\text{S}}}\left(\frac{r^n}{\underline{|n}}\text{HK}\,\bar{v}_{n-1}\bar{w}_{n-1}\right)$$

$$= \frac{(b+d)(c+d)}{\text{N}^2} + \overset{\infty}{\underset{1}{\text{S}}}\left(\frac{r^n}{\underline{|n}}\text{HK}\,\bar{v}_{n-1}\bar{w}_{n-1}\right)$$

by (ii.) and (iii.).

Or, remembering that $\text{N} = a + b + c + d$, we can write this

$$\frac{ad - bc}{\text{N}^2\text{HK}} = \overset{\infty}{\underset{1}{\text{S}}}\left(\frac{r^n}{\underline{|n}}\bar{v}_{n-1}\bar{w}_{n-1}\right)$$

$$= r + \frac{r^2}{2}hk + \frac{r^3}{6}(h^2 - 1)(k^2 - 1) + \frac{r^4}{24}h(h^2 - 3)k(k^2 - 3)$$

$$+ \frac{r^5}{120}(h^4 - 6h^2 + 3)(k^4 - 6k^2 + 3)$$

$$+ \frac{r^6}{720}h(h^4 - 10h^2 + 15)k(k^4 - 10k^2 + 15)$$

$$+ \frac{r^7}{5040}(h^6 - 15h^4 + 45h^2 - 15)(k^6 - 15k^4 + 45k^2 - 15)$$

$$+ \frac{r^8}{40320}h(h^6 - 21h^4 + 105h^2 - 105)k(k^6 - 21k^4 + 105k - 105) + \text{, \&c.}$$

$$. \quad . \quad . \quad . \quad . \quad \text{(xix.)}.$$

Then

$$h = \chi_1 + \frac{1}{\underline{|3}}\chi_1^3 + \frac{7}{\underline{|5}}\chi_1^5 + \frac{127}{\underline{|7}}\chi_1^7 + \ldots$$

$$\frac{1}{\text{H}} = \sqrt{2\pi}\left(1 + \frac{1}{\underline{|2}}\chi_1^2 + \frac{7}{\underline{|4}}\chi_1^4 + \frac{127}{\underline{|6}}\chi_1^6 + \ldots\right),$$

and

$$k = \chi_2 + \frac{1}{\underline{|3}}\chi_2^3 + \frac{7}{\underline{|5}}\chi_2^5 + \frac{127}{\underline{|7}}\chi_2^7 + \ldots$$

$$\frac{1}{\text{K}} = \sqrt{2\pi}\left(1 + \frac{1}{\underline{|2}}\chi_2^2 + \frac{7}{\underline{|4}}\chi_2^4 + \frac{127}{\underline{|6}}\chi_2^6 + \ldots\right).$$

These follow from the considerations that if

$$\chi_1 = \sqrt{2\pi}\,\phi_1, \qquad \chi_2 = \sqrt{2\pi}\,\phi_2,$$

$$\frac{d\phi_1}{dh} = \text{H}, \qquad \frac{d\phi_2}{dk} = \text{K},$$

$$\frac{d\text{H}}{d\phi_1} = -h, \qquad \frac{d\text{K}}{d\phi_2} = -k,$$

whence it is easy to find the successive differentials of h with regard to ϕ_1 and k with regard to ϕ_2, and then obtain the above results by MACLAURIN'S theorem. There is, of course, no difficulty in calculating H and K from (xvii.) directly. That method was adopted in the numerical illustrations.

Here the left-hand side is known, and since h and k are known, we can find the coefficients of any number of powers of r so soon as the first two have been found, from (xii.) and (xiii.).

Accordingly the correlation can be found if we have only made a grouping of our frequencies into the four divisions, a, b, c, and d.

If h and k be zero, we have from (xvii.) and (iv.)

$$\mathrm{H} = \mathrm{K} = \frac{1}{\sqrt{2\pi}}$$

$$a + c = b + d = \tfrac{1}{2}\mathrm{N}.$$

The right-hand side of (xix.) is now

$$r + \frac{1}{\underline{|3}}r^3 + \quad . \quad . \quad .$$

or equal to $\sin^{-1} r$.

Hence
$$r = \sin 2\pi\frac{(ad - bc)}{\mathrm{N}^2}$$
$$= \cos\pi\frac{b}{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xx.}),$$

which agrees with a result of Mr. SHEPPARD'S, 'Phil. Trans.,' A, vol. 192, p. 141. We have accordingly reached a generalised form of his result for *any* class-index whatever. Clearly, also, r being known, we can at once calculate the frequency of pairs of organs with deviations as great as or greater than h and k.

§ (2.) *Other Series for the Determination of r.*

For many purposes the series (xix.) is sufficiently convergent to give r for given h and k with but few approximations, but we will now turn to other developments.

We have by (vii.)

$$\int_0^r \mathrm{U}\,dr = e^{-\frac{1}{2}(x^2+y^2)}\left(u_0 r + \frac{u_1 r^2}{\underline{|2}} + \ . \ . \ . + u_n\frac{r^{n+1}}{\underline{|n+1}} + \ . \ . \ .\right).$$

Put $x = h$, $y = k$, and write for brevity

$$\epsilon = \frac{ad - bc}{\mathrm{N}^2\mathrm{HK}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxi.}).$$

It follows at once from (xix.) that

$$\epsilon = e^{\frac{1}{2}(h^2+k^2)}\int_0^r \mathrm{U}\,dr$$
$$= e^{\frac{1}{2}(h^2+k^2)}\int_0^r \frac{1}{\sqrt{1-r^2}}\,e^{-\frac{1}{2}\frac{1}{1-r^2}(h^2+k^2-2rhk)}\,dr$$

$$\left.\begin{aligned} &= e^{\frac{1}{2}k^2}\int_0^\theta e^{-\frac{1}{2}(h\tan\theta\,-\,k\sec\theta)^2}\,d\theta \\ &= e^{\frac{1}{2}h^2}\int_0^\theta e^{-\frac{1}{2}(k\tan\theta\,-\,h\sec\theta)^2}\,d\theta \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad \text{(xxii.)},$$

if $r = \sin\theta$.

Now either of the quantities under the sign of integration in (xxii.) can be expanded in powers of θ by MACLAURIN'S theorem. Thus let

$$\begin{aligned} \chi &= e^{-\frac{1}{2}(k\tan\theta\,-\,h\sec\theta)^2} \\ &= \chi_0 + \left(\frac{d\chi}{d\theta}\right)_0\theta + \left(\frac{d^2\chi}{d\theta^2}\right)_0\frac{\theta^2}{\underline{|2}} + \ldots + \left(\frac{d^n\chi}{d\theta^n}\right)_0\frac{\theta^n}{\underline{|n}} + \ldots \end{aligned}$$

Then

$$\epsilon = e^{\frac{1}{2}h^2}\left(\chi_0\theta + \left(\frac{d\chi}{d\theta}\right)_0\frac{\theta^2}{\underline{|2}} + \left(\frac{d^2\chi}{d\theta^2}\right)_0\frac{\theta^3}{\underline{|3}} + \ldots + \left(\frac{d^n\chi}{d\theta^n}\right)_0\frac{\theta^{n+1}}{\underline{|n+1}} + \ldots\right),$$

and it remains to find $\left(\dfrac{d^n\chi}{d\theta^n}\right)_0$.

Now $$\log\chi = -\tfrac{1}{2}(k\tan\theta - h\sec\theta)^2.$$
Hence

$$\cos^3\theta\,\frac{d\chi}{d\theta} = -\chi[(h^2+k^2)\sin\theta - hk(\tfrac{3}{2} - \tfrac{1}{2}\cos 2\theta)].$$

Differentiating $\overline{n-1}$ times by LEIBNITZ'S theorem, and putting $\theta = 0$,

$$\begin{aligned} 4\left(\frac{d^n\chi}{d\theta^n}\right)_0 &- 4hk\left(\frac{d^{n-1}\chi}{d\theta^{n-1}}\right) + \ldots \\ &+ \frac{(n-1)\ldots(n-r+1)}{\underline{|r-1}}\left[\cos\frac{r\pi}{2}\left\{\frac{n-r}{r}(3+3^r) - 4(h^2+k^2)\right\}\right. \\ &+ \left.\sin\frac{r\pi}{2}\,2^r hk\right]\left(\frac{d^{n-r}\chi}{d\theta^{n-r}}\right)_0 + \ldots = 0 \quad . \; . \; . \; . \; . \; . \; . \qquad \text{(xxiii.)}. \end{aligned}$$

Clearly $\chi_0 = e^{-\frac{1}{2}h^2}$, then we rapidly find

$$\begin{aligned} \left(\frac{d\chi}{d\theta}\right)_0 &= hk\,e^{-\frac{1}{2}h^2} \\ \left(\frac{d^2\chi}{d\theta^2}\right)_0 &= -(h^2+k^2-h^2k^2)e^{-\frac{1}{2}h^2} \\ \left(\frac{d^3\chi}{d\theta^3}\right)_0 &= hk\,\{h^2k^2 - 3(h^2+k^2) + 5\}. \end{aligned}$$

Or, finally

$$\epsilon = \theta + \tfrac{1}{2}hk\theta^2 - (h^2+k^2-h^2k^2)\frac{\theta^3}{6} + hk\{h^2k^2 - 3(h^2+k^2) + 5\}\frac{\theta^4}{24} + \ldots \qquad \text{(xxiv.)},$$

where more terms if required can be found by (xxiii.). If θ be fairly small, θ^5 will be negligible. Or if h and k be small, the lowest term in the next factor will be $h^2 + k^2$, and this into $\theta^5/\underline{|5}$ is generally quite insensible. Very often two or three terms on the right-hand side of (xxiv.) give quite close enough values of θ, and accordingly of $r = \sin\theta$. (xxiv.) is clearly somewhat more convergent than (xix.) if h and k are, as usually happens, less than unity.

Returning now to (xix.), let us write it

$$\epsilon = f(r, h, k).$$

This is the equation that must be solved for r. Suppose r_0 a root of this when we retain only few terms on the right, say a root of the quadratic

$$\epsilon = r + \tfrac{1}{2}hkr^2.$$

Then if $r = r_0 + \rho$,

$$\epsilon = f(r_0, h, k) + \rho f'(r_0, h, k) + \frac{1}{\underline{|2}}\rho^2 f''(r_0, h, k) + \&c.$$

Hence $\rho = \dfrac{\epsilon - f(r_0hk)}{f'(r_0hk)}$ to a third approximation

$$= -\frac{\frac{1}{6}(h^2-1)(k^2-1)r_0^{\,3}}{\dfrac{1}{\sqrt{1-r_0^{\,2}}}\,e^{-\frac{1}{2}\frac{1}{1-r_0^{\,2}}(h^2+k^2-2r_0hk)}} \text{ nearly } \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxv.}),$$

which gives us a value of ρ which, substituted in ρ^2 in the above equation, introduces only terms of the 6th order in r_0.

Another integral expression for ϵ of Equation (xxi.) may here be noticed :

$$\epsilon = e^{\frac{1}{2}(h^2+k^2)}\int_0^r \frac{dr}{\sqrt{1-r^2}}\, e^{-\frac{1}{2}\frac{1}{1-r^2}(h^2+k^2-2rhk)}.$$

Put $h = \dfrac{1}{\sqrt{2}}(\beta+\gamma)$, $k = \dfrac{1}{\sqrt{2}}(\beta-\gamma)$.

Hence

$$\epsilon = e^{\frac{1}{2}(\beta^2+\gamma^2)}\int_0^r \frac{dr}{\sqrt{1-r^2}}\, e^{-\frac{1}{2}\left(\frac{\beta^2}{1+r}+\frac{\gamma^2}{1-r}\right)}$$

$$= e^{\frac{1}{4}(\beta^2+\gamma^2)}\int_0^r \frac{dr}{\sqrt{1-r^2}}\, e^{-\frac{1}{4}\left(\beta^2\frac{1-r}{1+r}+\gamma^2\frac{1+r}{1-r}\right)}.$$

Let $\tan 2\phi = \dfrac{1-r}{1+r}$, or, $r = \cos 2\,\phi$.

Therefore

$$\epsilon = 2e^{\frac{1}{4}(\beta^2+\gamma^2)}\int_\phi^{45^\circ} e^{-\frac{1}{4}(\beta^2\tan^2\phi+\gamma^2\cot^2\phi)}\,d\phi,$$

$$= 2e^{\frac{1}{4}(\beta^2+\gamma^2)}\int_1^v e^{-\frac{1}{4}\left(\frac{\beta^2}{v^2}+\gamma^2v^2\right)}\frac{dv}{1+v^2}$$

where $v = \cot\phi$ and is > 1.

It seems possible that interesting developments for ϵ might be deduced from this integral expression.

§ (3.) *To show that the Series for r is Convergent if $r < 1$, whatever be the Values of h and k.*

Write the series in the form of p. 6, *i.e.* :—

$$\epsilon = \overset{n}{\underset{1}{\mathrm{S}}} \frac{r^n}{\underline{|n}} \bar{v}_{n-1} \bar{w}_{n-1}.$$

Now

$$\left.\begin{aligned} \bar{v}_{n+1} &= h\bar{v}_n - n\bar{v}_{n-1} \\ \bar{w}_{n+1} &= k\bar{w}_n - n\bar{w}_{n-1} \end{aligned}\right\} \text{by (xii.) and (xiii.).}$$

From these we deduce

$$\bar{v}_{n+1} = \{h^2 - (2n - 1)\} \bar{v}_{n-1} - (n - 1)(n - 2)\bar{v}_{n-3}$$

$$\bar{w}_{n+1} = \{k^2 - (2n - 1)\} \bar{w}_{n-1} - (n - 1)(n - 2)\bar{w}_{n-3}$$

Now let $s_n = \bar{v}_{n-1} r^{\frac{1}{2}n} \big/ \{\underline{|n}\}^{\frac{1}{2}}, \quad t_n = \bar{w}_{n-1} r^{\frac{1}{2}n} \big/ \{\underline{|n}\}^{\frac{1}{2}}.$

Then we find

$$s_{n+2} = \frac{h^2 - (2n - 1)}{\sqrt{(n + 1)(n + 2)}} s_n r - \sqrt{\frac{(n - 1)(n - 2)^2}{n(n + 1)(n + 2)}} s_{n-2} r^2,$$

$$t_{n+2} = \frac{k^2 - (2n - 1)}{\sqrt{(n + 1)(n + 2)}} t_n r - \sqrt{\frac{(n - 1)(n - 2)}{n(n + 1)(n + 2)}} t_{n-2} r^2.$$

Thus, when n is large, we find the ratio of successive terms s_{n+2}/s_n or t_{n+2}/t_n is given by ρ, where

$$\rho = -2r - r^2/\rho \text{ or, } \rho = -r.$$

The ultimate ratio of $s_{n+2}\, t_{n+2}$ to $s_n\, t_n$ is accordingly given by r^2, but this is the ratio of alternate terms of the original series. The original series thus breaks up into two series, one of odd and one of even powers of r. Both these series are absolutely convergent whatever h and k be, having an ultimate convergence ratio of r^2

§ (4.) *To find the Probable Error of the Correlation Coefficient as Determined by the Method of this Memoir.*

Given a division of the total frequency N into a, b, c, d groups, where $a + b + c + d = \mathrm{N}$, then the probable error of any one of them, say a, is ·67449 σ_a, where*

$$\sigma_a = \sqrt{\frac{a(\mathrm{N} - a)}{\mathrm{N}}} \quad \ldots\ldots\ldots\ldots \quad \text{(xxvi.)}.$$

Let $b + d = n_1$, $c + d = n_2$, then

---

* The standard deviation of an event which happens np times and fails nq times in n trials is well known to be $\sqrt{npq}$. The probable errors here dealt with are throughout, of course, those arising from different samples of the same general population.

$$\sigma_{n_1} = \sqrt{\frac{n_1(\mathrm{N} - n_1)}{\mathrm{N}}} \qquad \sigma_{n_2} = \sqrt{\frac{n_2(\mathrm{N} - n_2)}{\mathrm{N}}} \quad . \quad . \quad . \quad (\text{xxvii.}).$$

To obtain r_{cd} we have, if $\delta\eta$ denotes an error in any quantity η,

$$\delta c + \delta d = \delta n_2,$$

$$\therefore \ \sigma_c^2 + \sigma_d^2 + 2\sigma_c\sigma_d r_{cd} = \sigma_{n_2}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxviii.}),$$

by squaring, summing for all possible variations in c and d, and dividing by the total number of variations.

Hence, substituting the values of the standard deviations as found above, we deduce

$$\sigma_c\sigma_d r_{cd} = -cd/\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxix.}).$$

In a similar manner

$$\delta n_1 \delta d = \delta b\, \delta d + (\delta d)^2,$$

$$\sigma_d \sigma_{n_1} r_{dn_1} = \sigma_b \sigma_d r_{bd} + \sigma_d^2$$

$$\sigma_d \sigma_{n_1} r_{dn_1} = d\ (a + c)/\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxx.}).$$

and

$$\sigma_d \sigma_{n_2} r_{dn_2} = d\ (a + b)/\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxi.}),$$

Now

$$n_1 = \frac{\mathrm{N}}{\sqrt{2\pi}} \int_h^\infty e^{-\frac{1}{2}x^2}\, dx,$$

$$\therefore \ \delta n_1 = -\frac{\mathrm{N}}{\sqrt{2\pi}}\, e^{-\frac{1}{2}h^2}\, \delta h = -\mathrm{NH}\delta h.$$

Thus

$$\sigma_{n_1} = \mathrm{NH}\,\sigma_h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxii.}),$$

and similarly

$$\sigma_{n_2} = \mathrm{NK}\,\sigma_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxiii.}).$$

Hence the probable error of h

$$= \frac{\cdot 67449}{\mathrm{H}\sqrt{\mathrm{N}}} \sqrt{\frac{(b + d)(a + c)}{\mathrm{N}^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxiv.}),$$

and of k

$$= \frac{\cdot 67449}{\mathrm{K}\sqrt{\mathrm{N}}} \sqrt{\frac{(c + d)(a + b)}{\mathrm{N}^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxv.}).$$

They can be found at once, therefore, when H and K have been found from an ordinate table of the exponential curve, and a, b, c, d are given. We have thus the probable error of the means as found from any double grouping of observations.

Next, noting that

$$\delta n_1\, \delta n_2 = \mathrm{N}^2\mathrm{HK}\, \delta h\, \delta k,$$

we have

$$\sigma_{n_1}\sigma_{n_2} r_{n_1 n_2} = \mathrm{N}^2\mathrm{HK}\, \sigma_h \sigma_k r_{hk},$$

or

$$r_{n_1 n_2} = r_{hk}.$$

But

$$\delta n_1 \delta n_2 = (\delta b + \delta d)(\delta c + \delta d),$$

$$\sigma_{n_1}\sigma_{n_2} r_{n_1 n_2} = \sigma_b \sigma_c r_{bc} + \sigma_b \sigma_d r_{bcd} + \sigma_c \sigma_d r_{cd} + \sigma_d^2,$$

$$= \frac{ad - bc}{\mathrm{N}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxvi.}),$$

therefore

$$\sigma_h \sigma_k r_{hk} = \frac{ad - bc}{\mathrm{NHK}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxvii.}).$$

$$r_{hk} = \frac{ad - bc}{\sqrt{(b + d)(a + c)(c + d)(a + b)}} \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxviii.}).$$

This is an important result; it expresses the correlation between errors in the position of the means of the two characters under consideration. But if the probabilities were independent there could be no such correlation. Thus r_{hk} might be taken as a measure of divergence from independent variation. We shall return to this point later.

Since $\delta n_1 = - \mathrm{HN}\delta h$, we have $\delta n_1 \delta d = - \mathrm{HN}\delta d \delta h$, whence we easily deduce

$$r_{dn_1} = - r_{dh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxix.}).$$

Similarly

$$r_{dn_2} = - r_{dk} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xl.}).$$

Now d is a function of r, h, and k. Hence if $d = f(r, h, k)$,

$$\delta d = \frac{df}{dr}\,\delta r + \frac{df}{dh}\,\delta h + \frac{df}{dk}\,\delta k$$

$$= \gamma_0 \delta r + \gamma_1 \delta h + \gamma_2 \delta k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xli.}).$$

Whence transposing, squaring, summing, and dividing by the total number of observations, we find

$$\gamma_0^2 \sigma_r^2 = \sigma_d^2 + \gamma_1^2 \sigma_h^2 + \gamma_2^2 \sigma_k^2 - 2\gamma_1 \sigma_d \sigma_h r_{dh} - 2\gamma_2 \sigma_d \sigma_k r_{dk}$$
$$+ 2\gamma_1 \gamma_2 \sigma_h \sigma_k r_{hk}$$
$$= \sigma_d^2 + \left(\frac{\gamma_1}{\mathrm{HN}}\right)^2 \sigma_{n_1}^2 + \left(\frac{\gamma_2}{\mathrm{KN}}\right)^2 \sigma_{n_2}^2 + 2\left(\frac{\gamma_1}{\mathrm{HN}}\right) \sigma_d \sigma_{n_1} r_{dn_1}$$
$$+ 2\left(\frac{\gamma_2}{\mathrm{HN}}\right) \sigma_d \sigma_{n_2} r_{dn_2} + \frac{2\gamma_1 \gamma_2}{\mathrm{N^2HK}} \sigma_{n_1} \sigma_{n_2} r_{n_1 n_2} \quad . \quad . \quad . \quad . \quad . \quad (\text{xlii.}).$$

Substituting the values of the standard deviations and correlations as found above, we have

$$\sigma_r^2 = \frac{1}{\mathrm{N}\gamma_0^2}\left\{ d(a + b + c) + \left(\frac{\gamma_2}{\mathrm{HN}}\right)^2 (a + b)(d + c) + \left(\frac{\gamma_1}{\mathrm{KN}}\right)^2 (a + c)(d + b) \right.$$
$$\left. + \frac{2\gamma_1\gamma_2}{\mathrm{NHK}} (ad - bc) + \frac{2\gamma_2}{\mathrm{HN}} d(b + a) + \frac{2\gamma_1}{\mathrm{KN}} d(c + a) \right\} \quad . \quad . \quad . \quad (\text{xliii.}).$$

It remains now to determine γ_0, γ_1, and γ_2.

By Equation (i.)

$$d = f(r, h, k) = \frac{\mathrm{N}}{2\pi\sqrt{1-r^2}} \int_h^\infty \int_k^\infty e^{-\frac{1}{2}\frac{x^2+y^2-2xy}{1-r^2}}\,dx\,dy,$$

$$\gamma_1 = \frac{df}{dh} = -\frac{\mathrm{N}}{2\pi\sqrt{1-r^2}} \int_k^\infty e^{-\frac{1}{2(1-r^2)}(h^2+y^2-2rhy)}\,dy$$

$$= -\frac{\mathrm{N}}{2\pi\sqrt{1-r^2}} e^{-\frac{1}{2}h^2} \int_k^\infty e^{-\frac{1}{2}\frac{(y-rh)^2}{1-r^2}}\,dy$$

$$= -\mathrm{H}\frac{\mathrm{N}}{\sqrt{2\pi}} \int_{\beta_2}^\infty e^{-\frac{1}{2}z^2}\,dz \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(xliv.)},$$

where

$$\beta_2 = \frac{k - rh}{\sqrt{1-r^2}}.$$

Thus

$$\gamma_1/(\mathrm{NH}) = -\left(\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}z^2}\,dz - \frac{1}{\sqrt{2\pi}}\int_0^{\beta_2} e^{-\frac{1}{2}z^2}\,dz\right)$$

$$= \psi_2 - \tfrac{1}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(xlv.)}.$$

Similarly

$$\gamma_2/(\mathrm{NK}) = \psi_1 - \tfrac{1}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(xlvi.)}.$$

Here

$$\psi_1 = \frac{1}{\sqrt{2\pi}}\int_0^{\beta_1} e^{-\frac{1}{2}z^2}\,dz, \qquad \psi_2 = \frac{1}{\sqrt{2\pi}}\int_0^{\beta_2} e^{-\frac{1}{2}z^2}\,dz \quad . \; . \; . \; . \; . \; . \quad \text{(xlvii.)},$$

where

$$\beta_1 = \frac{h - rk}{\sqrt{1-r^2}}, \qquad \beta_2 = \frac{k - rh}{\sqrt{1-r^2}} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(xlviii.)},$$

and thus ψ_1 and ψ_2 can be found at once from the tables when β_1 and β_2 are found from the known values of r, h, k.

Lastly, we have from Equation (xxi.)

$$\frac{ad - bc}{\mathrm{N}^2\mathrm{HK}} = e^{\frac{1}{2}(h^2+k^2)}\int_0^r \mathrm{U}\,dr,$$

or

$$\frac{d}{\mathrm{N}} = \frac{(d+b)(d+c)}{\mathrm{N}^2} + \frac{1}{2\pi}\int_0^r \mathrm{U}\,dr.$$

Thus*

$$\gamma_0 = df/dr = \frac{1}{2\pi}\mathrm{NU},$$

$$\gamma_0/\mathrm{N} = \chi_0,$$

where

$$\chi_0 = \frac{1}{2\pi}\frac{1}{\sqrt{1-r^2}}\,e^{-\frac{1}{2(1-r^2)}(h^2+k^2-2rhk)} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(xlix.)}$$

a value which can again be found as soon as r, h, k are known. $\gamma_0 = \chi_0\mathrm{N}$ is clearly the ordinate of the frequency surface corresponding to $x = h$, $y = k$.

Substituting in Equation (xliii.) we have, after some reductions,

---

* By Equations (ii.) and (iii.), $d + b$ and $d + c$ are independent of r.

$$\text{Probable error of } r = {\cdot}67449\sigma_r$$

$$= \frac{{\cdot}67449}{\sqrt{\mathrm{N}}\chi_0}\left\{\frac{(a+d)(c+b)}{4\mathrm{N}^2} + {\psi_2}^2\frac{(a+c)(d+b)}{\mathrm{N}^2} + {\psi_1}^2\frac{(a+b)(d+c)}{\mathrm{N}^2}\right.$$

$$\left. + 2\psi_1\psi_2\frac{ad-bc}{\mathrm{N}^2} - \psi_2\frac{ab-cd}{\mathrm{N}^2} - \psi_1\frac{ac-bd}{\mathrm{N}_2}\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{l.}),$$

where χ_0, ψ_1, and ψ_2 are readily found from Equations (xlix.), (xlvii.), and (xlviii.). Thus the probable error of r can be fairly readily found. It must be noted in using this formula, that a is the quadrant in which the mean falls, so that h and k are both *positive* (see fig., p. 2). In other words, we have supposed $a + c > b + d$ and $a + b > c + d$. Our lettering must always be arranged so as to suit this result before we apply the above formula.

§ (5.) *To Find a Physical Meaning for the Series in r, or for the ϵ of Equation* (xxi.).

Return to the original distribution $\begin{array}{c|c} a & b \\ \hline c & d \end{array}$ of p. 2. If the probabilities of the two characters or organs were quite independent, we should expect the distribution

$$\begin{array}{c|c} \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} & \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} \\ \hline \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} & \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} \end{array}$$

Now re-arranging our actual data we may put it thus :

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array} = \begin{array}{c|c} \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} + \dfrac{ad-bc}{\mathrm{N}} & \mathrm{N}\dfrac{a+b}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} - \dfrac{ad-bc}{\mathrm{N}} \\ \hline \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{a+c}{\mathrm{N}} - \dfrac{ad-bc}{\mathrm{N}} & \mathrm{N}\dfrac{c+d}{\mathrm{N}}\dfrac{b+d}{\mathrm{N}} + \dfrac{ad-bc}{\mathrm{N}} \end{array}$$

Accordingly correlation denotes that $\frac{ad-bc}{\mathrm{N}}$ has been transferred from each of the second and fourth compartments, and the same amount added to each of the first and third compartments. If $\eta = (ad - bc)/\mathrm{N}^2$, then η is the *transfer per unit of the total frequency*. The magnitude of this transfer is clearly a measure of the divergence of the statistics from independent variation. It is physically quite as significant as the correlation coefficient itself, and of course much easier to determine. It must vanish with the correlation coefficient. We see from (xxi.) that

$$\eta = \epsilon \times \mathrm{HK},$$

or we have an interpretation for the series in r of (xix.).

Now, obviously any function of η, just like η itself, would serve as a measure of the divergence from perfectly independent variation. It is convenient to choose a function which shall lie arithmetically between 0 and 1.

Now consider what happens in the case of perfect correlation, *i.e.*, all the observations fall into a straight line. Hence if $ad > bc$, either b or c is zero, for a straight line cannot cut all four compartments, and a and d are obviously positive. Thus c and b can only be zero if $\eta = (c + d)(a + c)/\mathrm{N}^2$ or $(a + b)(b + d)/\mathrm{N}^2$. In order that b should be zero, it is needful that h and k, as given by (iv.) and (v.), should be positive or $a + c > b + d$, $a + b > c + d$, and the mean fall under the 45° line through the vertical and horizontal lines dividing the table into four compartments, *i.e.*, $h > k$. These conditions would be satisfied if $ad > bc$ and $a > d$, $c > b$. Now suppose our four-compartment table arranged so that

$$ad > bc, \quad a > d, \quad c > b,$$

and consider the function

$$\mathrm{Q}_1 = \sin \frac{\pi}{2} \frac{\eta}{(a + b)(b + d)/\mathrm{N}^2} \quad \text{(li.)},$$

or

$$\mathrm{Q}_1 = \sin \frac{\pi}{2} \frac{ad - bc}{(a + b)(b + d)} \quad \text{(lii.)}.$$

This function vanishes if $\eta = 0$, and it further = unity if $b = 0$. Thus it agrees at the limits 0 and 1 with the value of the correlation coefficient. Again, when h and k are both zero, $a = d$, $b = c$, and $\mathrm{Q}_1 = \sin \frac{\pi}{2} \frac{a - b}{a + b}$, is thus r by (xx.). Hence we have found a function which vanishes with r and equals unity with r, while it is also equal to r if the divisions of the table be taken through the medians.

Now, I take it that these are very good conditions to make for any function of a, b, c, d which is to vanish with the "transfer," and to serve as a measure of the degree of dependent variability, or what Mr. YULE has termed the degree of "association." Mr. YULE has selected for his coefficient of association the expression

$$\mathrm{Q}_2 = \frac{ad - bc}{ad + bc} \quad \text{(liii.)}.$$

This vanishes with the transfer, equals unity if b or c be zero, and minus unity if a or d be zero. The latter is, of course, unnecessary if we agree to arrange a, b, c, d so that ad is always greater than bc. Now it is clear that Q_2 possesses a great advantage over Q_1 in rapidity of calculation, but the coefficient of correlation is also a coefficient which measures the association, and it is a great advantage to select one which agrees to the closest extent with the correlation, for then it enables us to determine other important features of the system.

If we do not make all the above conditions, we easily obtain a number of coefficients which would vanish with the transfer. Thus for example the correlation r_{hk} of Equation (xxxviii.) is such an expression.* It has the advantage of a symmetrical form, and has a concise physical meaning. It does not, however, become unity when

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* In fact (xxxvii.) gives us $\epsilon = \sigma_h \sigma_k r_{hk}$.

either, but not both, b and c vanish, nor does it, unless we multiply it by $\pi/2$ and take its sine, equal the coefficient of correlation when $a = d$ and $b = c$.

Again, we might deduce a fairly simple approximation to the coefficient of correlation from the Equation (xxiv.) for θ, using only its first few terms. Thus we find

$$\text{Sin } 2\pi \frac{ad - bc}{\mathrm{N}^2\{1 - \frac{1}{2}(\chi_1^2 + \chi_2^2)\} + \pi\chi_1\chi_2(ad - bc)} \quad . \quad . \quad . \quad . \quad . \quad \text{(liv.)},$$

where

$$\chi_1 = \sqrt{\frac{\pi}{2}}\frac{(a + c) - (b + d)}{\mathrm{N}},$$

$$\chi_2 = \sqrt{\frac{\pi}{2}}\frac{(a + b) - (c + d)}{\mathrm{N}},$$

as an expression which vanishes with the transfer, and will be fairly close to the coefficient of correlation. It is not, however, exactly unity when either b or c is zero. But without entering into a discussion of such expressions, we can write several down which fully satisfy the three conditions :—

(i.) Vanishing with the transfer.
(ii.) Being equal to unity if b or $c = 0$.
(iii.) Being equal to the correlation for median divisions.

Such are, for example :—

$$\mathrm{Q}_3 = \sin\frac{\pi}{2}\frac{\sqrt{ad} - \sqrt{bc}}{\sqrt{ad} - \sqrt{bc}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lv.)},$$

$$\mathrm{Q}_4 = \sin\frac{\pi}{2}\frac{1}{1 + \dfrac{2bc\,\mathrm{N}}{(ad - bc)(b + c)}},\ ad > bc \quad . \quad . \quad . \quad \text{(lvi.)},$$

$$\mathrm{Q}_5 = \sin\frac{\pi}{2}\frac{1}{\sqrt{1 + \kappa^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lvii.)},$$

where

$$\kappa^2 = \frac{4abcd\,\mathrm{N}^2}{(ad - bc)^2(a + d)(b + c)}.$$

Only by actual examination of the numerical results has it seemed possible to pick out the most efficient of these coefficients. Q_1 was found of little service. The following table gives the values of Q_2, Q_3, Q_4, and Q_5 in the case of fifteen series selected to cover a fairly wide range of values :—

No.	r.	h.	k.	Q_2.	Q_3.	Q_4.	Q_5.
1	·5939 ± ·0247	− ·0873	− ·4163	·7067	·6054	·6168	·6100
2	·5557 ± ·0261	− ·4189	− ·4163	·6688	·5657	·5405	·5570
3	·5529 ± ·0247	− ·0873	− ·0012	·6828	·5809	·5699	·5813
4	·5264 ± ·0264	+ ·2743	+ ·3537	·6345	·5331	·5200	·5283
5	·5213 ± ·0294	+ ·6413	+ ·6966	·6530	·5511	·4878	·5160
6	·5524 ± ·0307	+ 1·0234	+ ·3537	·7130	·6118	·6169	·6138
7	·5422 ± ·0288	+ ·6463	+ ·5828	·6693	·5673	·5136	·5452
8	·2222 ± ·0162	+ ·3190	+ ·3190	·2840	·2268	·2164	·2251
9	·3180 ± ·0361	+ ·1381	+ ·0696	·3959	·3185	·3176	·3183
10	·5954 ± ·0272	+ 1·5114	+ ·7414	·7860	·7100	·6099	·6803
11	·4708 ± ·0292	+ ·0865	− ·0054	·5692	·4712	·4720	·4715
12	·2335 ± ·0335	+ ·0405	+ ·0054	·2996	·2385	·2385	·2385
13	·2451 ± ·0205	+ ·2707	+ ·0873	·3103	·2473	·2456	·2470
14	·1002 ± ·0394	+ ·4557	+ ·1758	·1311	·1032	·0993	·1029
15	·6928 ± ·0164	+ ·5814	+ ·5814	·8032	·7108	·6699	·6897

Now an examination of this table shows that notwithstanding the extreme elegance and simplicity of Mr. YULE's coefficient of association Q_2, the coefficients Q_3, Q_4, and Q_5, which satisfy also his requirements, are much nearer to the values assumed by the correlation. I take this to be such great gain that it more than counterbalances the somewhat greater labour of calculation. If we except cases (6) and (10), in which h or k take a large value exceeding unity, we find that Q_3, Q_4, and Q_5 in the fifteen cases hardly differ by as much as the probable error from the value of the correlation. If we take the mean percentage error of the difference between the correlation and these coefficients, we find

Mean difference of Q_2 = 24·38 per cent.
" " Q_3 = 3·95 "
" " Q_4 = 2·94 "
" " Q_5 = 2·72 "

Thus although there is not much to choose between Q_4 and Q_5, we can take Q_5 as a good measure of the degree of independent variation.

The reader may ask: Why is it needful to seek for such a measure? Why cannot we always use the correlation as determined by the method of this paper? The answer is twofold. We want first to save the labour of calculating r for cases where the data are comparatively poor, and so reaching a fairly approximate result rapidly. But labour-saving is never a wholly satisfactory excuse for adopting an inferior method. The second and chief reason for seeking such a coefficient as Q lies in the fact that all our reasoning in this paper is based upon the normality of the frequency. We require to free ourselves from this assumption if possible, for the difficulty, as is exemplified in Illustration V. below, is to find material which actually obeys within the probable errors any such law. Now, by considering the coefficient of regression, $r\sigma_1/\sigma_2 = S(xy)/(N\sigma_1\sigma_2)$, as the slope of the line which best fits the series of points determined as the means of arrays of x for given values of y, we have once and for all freed ourselves from the difficulties attendant upon assuming normal frequency. We become indifferent to the deviations from that law, merely observing how closely or not our means of arrays fall on a line. When we are not given arrays but gross grouping under certain divisions, we have seen that the "transfer" is also a physical quantity of a significance independent of normality. We want accordingly to take a function which vanishes with the transfer, and does not diverge widely from the correlation in cases that we can test. Here the correlation is not taken as something peculiar to normal distributions, but something significant for all distributions whatever. Such a function of a suitable kind appears to be given by Q_5.

§ 6. *On the "Excess" and its Relation to Correlation and Relative Variability.*

There is another method of dealing with the correlation of characters for which we cannot directly discover a quantitative scale which deserves consideration. It is capable of fairly wide application, but, unlike the methods previously discussed, it requires the data to be collected in a special manner. It has the advantage of not applying only to the normal surface of frequency, but to any surface which can be converted into a surface of revolution by a slide and two stretches.

It is well known that not only the normal curve but the normal surface has a type form from which all others can be deduced by stretching or stretching and sliding. Thus in 1895 the Cambridge Instrument Company made for the instrument room at University College, London, a "biprojector," an instrument for giving arbitrary stretches in two directions at right angles to any curve. In this manner by the use of type-templates we were able to draw a variety of curves with arbitrary parameters, *e.g.*, all ellipses from one circle, parabolas from one parabola, normal curves from one normal curve template. Somewhat later Mr. G. U. YULE commenced a model of a normal frequency surface on the BRILL system of interlaced curves. This, by the variable amount of slide given to its two rectangular systems of normal curves, illustrated the changes from zero to perfect correlation. This model was exhibited at a College *soirée* in June, 1897. Geometrically this property has been taken by Mr. W. F. SHEPPARD as the basis of his valuable paper on correlation in the 'Phil. Trans.,' A, vol. 192, pp. 101–167. It is a slight addition to, and modification of, his results that I propose to consider in this section.

The equation to the normal frequency surface is, as we have seen in § 1,

$$z = \frac{\mathrm{N}}{2\pi\sigma_1\sigma_1\sqrt{1-r^2}} \text{ expt. } \left\{-\tfrac{1}{2}\left(\frac{x^2}{\sigma_1{}^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2{}^2}\right)\frac{1}{1-r^2}\right\}$$

Now write $x/(\sigma_1\sqrt{1-r^2}) = x'$, $y/\sigma_2 = y'$. This is merely giving the surface two uniform stretches (or squeezes) parallel to the coordinate axes. We have for the frequency of pairs lying between x, $x + \delta x$, and y, $\delta + \delta y$,

$$z\delta x\delta y = \frac{\mathrm{N}}{2\pi}\,\delta x'\delta y' \text{ expt.}\left\{-\tfrac{1}{2}\left(\left(x'-\frac{ry'}{\sqrt{1-r^2}}\right)^2+y'^2\right)\right\}.$$

Now give the area a uniform slide parallel to the axis of x defined by $r/\sqrt{1-r^2}$ at unit distance from that axis. This will not change the basal unit of area $\delta\alpha = \delta x'\delta y'$, and analytically we may write

$$\mathrm{X} = x' - y'r/\sqrt{1-r^2}, \qquad \mathrm{Y} = y', \qquad \mathrm{R}^2 = \mathrm{X}^2 + \mathrm{Y}^2.$$

Whence we find

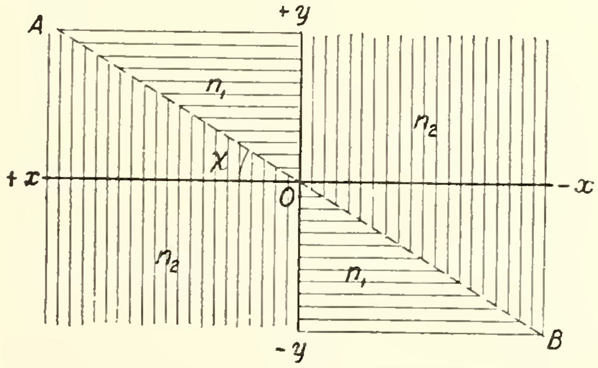
$$z\delta x\delta y = \frac{\mathrm{N}}{2\pi}\,\delta\alpha \text{ expt. } (-\tfrac{1}{2}\mathrm{R}^2).$$

This is the mechanical changing of the YULE-BRILL model analytically represented. The surface is now one of revolution, and the proof would have been precisely the same if we had written in the above results any function f, instead of the exponential.* It is easy to see that any volume cut off by two planes through the axis of the surface is to the whole volume as the angle between the two planes is to four right angles. Further the corresponding volumes of this surface and the original surface are to each other as unity to the product of the two stretches. Lastly, any plane through the z-axis of the original solid remains a plane through the z-axis after the two stretches and the slide. These points have all been dealt with by Mr. SHEPPARD (p. 101 *et seq.*, *loc. cit.*). I will here adopt his notation $r = \cos \mathrm{D}$, and term with him D the *divergence*. Thus cot D is (in the language of the theory of strain) the slide, and D is the angle between the strained positions of the original x and y directions. Now consider any plane which makes an angle χ with the plane of xz before strain. Then, since the contour lines of the correlation surface are ellipses, the volumes of the surface upon the like shaded opposite angles of the plan diagram below will be equal; and if they be n_1 and n_2, then $n_1 + n_2 = \frac{1}{2}\mathrm{N}$. If n_1' and n_2' be the volumes after strain, then by what precedes we shall have

$$n_1 = \sigma_1\sigma_2\sqrt{1-r^2} \times n_1', \qquad n_2 = \sigma_1\sigma_2\sqrt{1-r^2} \times n_2',$$

and

$$(n_2 - n_1)/(n_1 + n_2) = (n_2' - n_1')/(n_1' + n_2').$$



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* The generalisation is not so great as might at first appear, for I have convinced myself that this property of conversion into a surface of revolution by stretches and slides does not hold for actual cases of markedly skew correlation.

Now n_1' and n_2' will be as the angles between the strained positions of the planes bounding n_1 and n_2. Ox does not change its direction. Oy is turned through an angle $\pi/2 - \mathrm{D}$ clockwise, and χ becomes χ'', say. Hence

$$n_1' : n_2' :: \frac{\pi}{2} - \chi'' + \frac{\pi}{2} - \mathrm{D} : \frac{\pi}{2} + \chi'' - \frac{\pi}{2} + \mathrm{D}.$$

or

$$(n_2' - n_1')/(n_2' + n_1') = \frac{2}{\pi}(\chi'' + \mathrm{D}) - 1.$$

Let us write $\mathrm{E}_1 = 2(n_2 - n_1)$ and term it the *excess* for the y-character for the line AB. Then we easily find:

$$\tan\left(\frac{\mathrm{E}_1}{\mathrm{N}}\frac{\pi}{2} + \frac{\pi}{2}\right) = \tan(\chi'' + \mathrm{D}) = \frac{\cot\chi'' + \cot\mathrm{D}}{\cot\chi''\cot\mathrm{D} - 1} \quad . \quad . \quad . \quad . \quad \text{(lviii.)}.$$

It remains to determine $\tan\chi''$ and substitute. The stretches alter $\tan\chi$ into $\tan\chi'$, such that

$$\tan\chi' = \frac{\sigma_1\sqrt{1-r^2}}{\sigma_2}\tan\chi.$$

Further, by the slide

$$\cot\chi'' = \cot\chi' - \cot\mathrm{D} = \frac{\sigma_2}{\sigma_1\sqrt{1-r^2}}\cot\chi - \cot\mathrm{D}.$$

Hence we have by (lviii.) above

$$-\cot\left(\frac{\mathrm{E}_1}{\mathrm{N}}\frac{\pi}{2}\right) = \frac{\sigma_2}{\sigma_1\sqrt{1-r^2}}\cot\chi\bigg/\left(\frac{\sigma_2}{\sigma_1\sqrt{1-r^2}}\cot\chi\cot\mathrm{D} - \cot^2\mathrm{D} - 1\right),$$

or,

$$-\tan\left(\frac{\mathrm{E}_1}{\mathrm{N}}\frac{\pi}{2}\right) = \cot\mathrm{D} - \frac{\sigma_1}{\sigma_2}\frac{\tan\chi}{\sin\mathrm{D}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lix.)}.$$

Now the excess E_1 is the difference of the frequencies in the sum of the strips of the volume made by planes parallel to the plane yz on the two sides of the plane ABz (defined by χ), taken without regard to sign. For on one side of the mean yy this is $n_2 - n_1$, and on the other $-(n_1 - n_2)$. Hence we have this definition of E_1, the *column excess* for any line through the mean of a correlation table: *Add up the frequencies above and below the line in each column and take their differences without regard to sign, and their sum is the column excess.*

If we are dealing with an actual correlation table and not with a method of collecting statistics, then care must be taken to properly proportion the frequencies in the column in which the mean occurs, and also in the groups which are crossed by the line. It is the difficulty of doing this satisfactorily, especially if the grouping, as in eye and coat colour, is large and somewhat rough, that hinders the effective use of the method, if the statistics have not been collected *ad hoc.*

Now let E_2 be the *row excess* for the line AB, defined in like manner, then we have in the same way

$$-\tan\left(\frac{E_2}{N}\frac{\pi}{2}\right) = \cot D - \frac{\sigma_2}{\sigma_1}\frac{\cot\chi}{\sin D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{lix.}^{\text{bis}}).$$

Now eliminate σ_2/σ_1 between (lix.) and (lix.$^{\text{bis}}$) ; then

$$\left(\tan\left(\frac{E_1}{N}\frac{\pi}{2}\right) + \cot D\right)\left(\tan\left(\frac{E_2}{N}\frac{\pi}{2}\right) + \cot D\right) = \frac{1}{\sin^2 D}.$$

Whence we deduce

$$\cot D = \cot\frac{E_1 + E_2}{N}\frac{\pi}{2},$$

and, therefore,

$$r = \cos D = \cos\frac{E_1 + E_2}{N}\frac{\pi}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{lx.}).$$

Substituting for D in (lix.) we find further

$$\frac{\sigma_1}{\sigma_2} = \cot\chi\cos\left(\frac{E_2}{N}\frac{\pi}{2}\right)\bigg/\cos\left(\frac{E_1}{N}\frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{lxi.}).$$

Thus Equations (lx.) and (lxi.) give the coefficient of correlation and the relative variability of the two characters. The latter is, I believe, quite new, the former novel in form.

If we call m_1 the frequency in the angle χ (AOx of the figure above), then it is easy to see that $E_1 = 2(n_2 - n_1) = N - 4n_1$, and similarly $E_2 = N - 4m_1$. Thus $(E_1 + E_2)/N = 2(N - 2(n_1 + m_1))/N$. But $n_1 + m_1$ is the frequency in the first quadrant. This Mr. SHEPPARD terms P, while that in the second he terms R. We have thus $(E_1 + E_2)/N = 2R/(R + P)$, or

$$r = \cos\frac{R}{R + P}\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{lxii.}),$$

i.e., Mr. SHEPPARD's fundamental result* (' Phil. Trans.,' A, vol. 192, p. 141).

We can, of course, get Mr. SHEPPARD's result directly if we put $\chi = 0$, when we have at once $E_1 = 2(R - P)$, $E_2 = N = 2(R + P)$, and the result follows.

Equation (lxi.) may also be written in the form

$$\frac{\sigma_1}{\sigma_2} = \cot\chi\sin\left(\frac{m_1}{N}2\pi\right)\bigg/\sin\left(\frac{n_1}{N}2\pi\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (\text{lxiii.}).$$

If we put $\chi = 0$, then m_1 becomes zero, and the right-hand side of (lxiii.) is indeterminate. If we proceed, however, to the limit by evaluating the frequency in an indefinitely thin wedge of angle χ, we reach merely the identity $\sigma_1/\sigma_2 = \sigma_1/\sigma_2$. Hence there is no result corresponding to (lxi.) to be obtained by taking Mr. SHEPPARD's case of $\chi = 0$.

The following are the values of the probable errors of the quantities involved :—

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* In the actual classification of data (lx.) and (lxii.) suggest quite different processes. We can apply (lx.) where (lxii.) is difficult or impossible, *e.g.*, correlation in shading of birds' eggs from the same clutch.

$$\text{Probable error of } E_1 = \cdot 67449\sqrt{N(1 - E_1^2/N^2)} \quad . \quad . \quad . \quad . \quad (\text{lxiv.}).$$

$$\text{,,} \qquad \text{,,} \qquad E_2 = \cdot 67449\sqrt{N(1 - E_2^2/N^2)} \quad . \quad . \quad . \quad . \quad (\text{lxv.}).$$

$$\text{Correlation between errors in } E_1 \text{ and } E_2 = -\sqrt{\frac{(1 - E_1/N)(1 - E_2/N)}{(1 + E_1/N)(1 + E_2/N)}} \quad . \quad (\text{lxvi.}).$$

$$\text{Probable error in } r = \frac{\cdot 67449 \sin D\sqrt{D(\pi - D)}}{\sqrt{N}} \quad . \quad . \quad . \quad . \quad (\text{lxvii.}),$$

where $D = \frac{E_1 + E_2}{N}\frac{\pi}{2}$ (*cf.* SHEPPARD, *loc. cit.*, p. 148).

Probable error in ratio $\sigma_1/\sigma_2 =$

$$\frac{\cdot 67449}{\sqrt{N}}\frac{\sigma_1}{\sigma_2}\frac{\pi}{2}\left\{\left(1 - \frac{E_1^2}{N^2}\right)\tan^2\left(\frac{E_1}{N}\frac{\pi}{2}\right) + \left(1 - \frac{E_2^2}{N_2}\right)\tan^2\left(\frac{E_2}{N}\frac{\pi}{2}\right)\right.$$
$$\left. + 2\left(1 - \frac{E_1}{N}\right)\left(1 - \frac{E_2}{N}\right)\tan\left(\frac{E_1}{N}\frac{\pi}{2}\right)\tan\left(\frac{E_2}{N}\frac{\pi}{2}\right)\right\}^{\frac{1}{2}} \quad . \quad . \quad (\text{lxviii.}).$$

The application of the method here discussed to statistics without quantitative scale can now be indicated. If the characters we are dealing with have the same scale, although it be unknown, then, if the quantitative *order* be maintained, *i.e.*, individuals arranged in order of lightness or darkness of coat or eye-colour, the diagonal line on the table at 45° will remain unchanged, however we may suppose parts of the scale to be distorted, for the distortion will be the same at corresponding points of both axes. Further, if we suppose the mean of the two characters to be the same, this 45° line will pass through that mean, and will serve for the line AB of the above investigation. In this case we must take $\tan\chi = 1$, and consequently (lxi.) becomes

$$\sigma_1/\sigma_2 = \cos\left(\frac{E_2}{N}\frac{\pi}{2}\right)\Big/\cos\left(\frac{E_1}{N}\frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxix.}).$$

We can even, when the mean is a considerable way off the 45° line, get, in some cases, good results. Thus, the correlation in stature of husband and wife worked out by the ordinary product moment process is ·2872. But in this case $E_1 = 382{\cdot}062$ $E_2 = 806{\cdot}425$, and this gives the correlation ·2994. On the other hand, the actual ratio of variabilities is 1·12, while (lxix.) makes it 2·76! This arises from the fact that the errors in E_1 and E_2, due to the mean being off the 45° line, tend to cancel in $E_1 + E_2$, but tend in directly opposite directions in the ratio of the cosines. Similarly the correlation between father and son works out ·5666, which may be compared with the values given in Illustration V. below, ranging from ·5198 to ·5939. Again, correlation in eye-colour between husband and wife came out by the excess process ·0986, and by the process given earlier in the present Memoir ·1002. But all these are favourable examples, and many others gave much worse results. We ought really only to apply it to find σ_1/σ_2 when the means are on the 45° line, as in the correlation of the same character in brethren, and even in this case the statistics ought to be collected *ad hoc, i.e.*, we ought to make a very full quantitative order, and then notice for each individual case the number above and below the type. For example, suppose we had a diagram of some twenty-five to thirty eye tints in order (*e.g.*, like BERTRAND's), then we take any individual, note his tint, and observe how many relatives of a particular class—brethren or cousins, say—have lighter and how many darker eyes; the difference of the two would be the excess for this individual. The same plan would be possible with horses' coat-colour and other characters. After trying the plan of the excesses on the data at my disposal for horses' coat-colour and human eye-colour (which were not collected *ad hoc*), I abandoned it for the earlier method of this Memoir; for, the classification being in large groups, the proportioning of the excess (as well as the differences in the means) introduced too great errors for such investigations.

§ 7. *On a Generalisation of the Fundamental Theorem of the Present Memoir.*

If we measure deviations in units of standard deviations, we may take for the equation to the correlation surface for n variables

$$z = \frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}n}\sqrt{\mathrm{R}}}e^{-\frac{1}{2}\left\{\mathrm{S}_1\left(\frac{\mathrm{R}_{ss}}{\mathrm{R}}x_s^2\right) + 2\mathrm{S}_2\left(\frac{\mathrm{R}_{ss'}x_s x_{s'}}{\mathrm{R}}\right)\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(lxx.)},$$

where

$$\mathrm{R} = \left|\begin{array}{cccccc}
1 & r_{12} & r_{13} & \cdots & \cdots & r_{1n} \\
r_{21} & 1 & r_{23} & \cdots & \cdots & r_{2n} \\
r_{31} & r_{32} & 1 & \cdots & \cdots & r_{3n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
r_{n-1,\,1} & r_{n-1,\,2} & r_{n-1,\,3} & \cdots & 1 & r_{n-1,\,n} \\
r_{n,\,1} & r_{n,\,2} & r_{n,\,3} & \cdots & r_{n-1,\,n} & 1
\end{array}\right|$$

and R_{pq} is the minor obtained by striking out the pth row and qth column. r_{pq} is, of course, the correlation between the pth and qth variables, and equals r_{qp}. S_1 denotes a summation for s from 1 to n, and S_2 a summation of every possible pair out of the n quantities 1 to n.

Now take the logarithmic differential of z with regard to r_{pq}. We find

$$\frac{1}{z}\frac{dz}{dr_{pq}} = -\frac{1}{2\mathrm{R}}\frac{d\mathrm{R}}{dr_{pq}} - \tfrac{1}{2}\mathrm{S}_1\left\{\frac{d}{dr_{pq}}\left(\frac{\mathrm{R}_{ss}}{\mathrm{R}}\right)x_s^2\right\} - \mathrm{S}_2\left\{\frac{d}{dr_{pq}}\left(\frac{\mathrm{R}_{ss'}}{\mathrm{R}}\right)x_s x_{s'}\right\}$$

$$= -\frac{\mathrm{R}_{pq}}{\mathrm{R}} + \mathrm{S}_1\left(\frac{\mathrm{R}_{ps}\,\mathrm{R}_{qs}}{\mathrm{R}^2}x_s^2\right) + \mathrm{S}_2\left(\frac{\mathrm{R}_{ps}\,\mathrm{R}_{qs'} + \mathrm{R}_{ps'}\,\mathrm{R}_{qs}}{\mathrm{R}^2}x_s x_{s'}\right)$$

For $$d\mathrm{R}/dr_{pq} = 2\mathrm{R}_{pq}$$

and, generally, whether s is or is not $= s'$, or these are or are not $= p$ and q, we have

$$\frac{d}{dr_{pq}}\left(\frac{R_{ss'}}{R}\right) = -\frac{R_{ps}R_{qs'} + R_{ps'}R_{qs}}{R^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; (\text{lxxi.}).$$

This follows thus :

$$\frac{d}{dr_{pq}}\left(\frac{R_{ss'}}{R}\right) = \frac{1}{R}\frac{dR_{ss'}}{dr_{pq}} - \frac{R_{ss'}}{R^2}\frac{dR}{dr_{pq}} = \frac{1}{R}\frac{dR_{ss'}}{dr_{pq}} - \frac{2R_{ss'}R_{pq}}{R^2},$$

or we have to show that

$$\begin{aligned}\frac{dR_{ss'}}{dr_{pq}} &= \frac{2R_{ss'}R_{pq} - R_{ps}R_{qs'} - R_{ps'}R_{qs}}{R} \\ &= \frac{R_{ss'}R_{pq} - R_{ps}R_{qs'}}{R} + \frac{R_{ss'}R_{pq} - R_{ps'}R_{qs}}{R} \\ &= {}_{pq}R_{ss'} + {}_{qp}R_{ss'}\end{aligned}$$

where ${}_{pq}R_{ss'}$ is the minor corresponding to the term r_{pq} in $R_{ss'}$, and ${}_{qp}R_{ss'}$ the minor corresponding to the term r_{qp}.* But this last result is obvious because $R_{ss'}$ only contains r_{pq} in two places, *i.e.*, as r_{pq} and r_{qp}.

Putting $s = s'$, we have the other identity required above, *i.e.*,

$$\frac{d}{dr_{pq}}\left(\frac{R_{ss}}{R}\right) = -\frac{2R_{ps}R_{qs}}{R^2}.$$

Returning now to the value for $\frac{1}{z}\frac{dz}{dr_{pq}}$ on the previous page, we see that the two sum terms may be expressed as a product, or we may put

$$\frac{1}{z}\frac{dz}{dr_{pq}} = -\frac{R_{pq}}{R} + S_1\left(\frac{R_{ps}}{R}x_s\right) \times S_1\left(\frac{R_{qs}}{R}x_s\right).$$

Now write

$$z = \frac{N}{(2\pi)^{\frac{1}{2}n}\sqrt{R}}e^{-\phi}.$$

Then $$\frac{d\phi}{dx_p} = S_1\left(\frac{R_{ps}}{R}x_s\right), \quad \frac{d\phi}{dx_q} = S_1\left(\frac{R_{qs}}{R}x_s\right) \text{ and } \frac{d^2\phi}{dx_p\,dx_q} = \frac{R_{pq}}{R}.$$

Hence $$\frac{1}{z}\frac{dz}{dr_{pq}} = -\frac{d^2\phi}{dx_p\,dx_q} + \frac{d\phi}{dx_p}\frac{d\phi}{dx_q}.$$

Now differentiate log z with regard to x_p. Then

$$\frac{dz}{dx_p} = -z\frac{d\phi}{dx_p}$$

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[*] See also SCOTT, 'Theory of Determinants,' p. 59.

$$\frac{d^2z}{dx_p dx_q} = -z\frac{d^2\phi}{dx_p dx_q} - \frac{dz}{dx_q}\frac{d\phi}{dx_p}$$

$$\frac{1}{z}\frac{d^2z}{dx_p dx_q} = -\frac{d^2\phi}{dx_p dx_q} + \frac{d\phi}{dx_p}\frac{d\phi}{dx_q}$$

Thus finally

$$\frac{dz}{dr_{pq}} = \frac{d^2z}{dx_p dx_q} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(lxxii.)}.$$

In other words, the operator d/dr_{pq} acting on z can always be replaced by the operator $d^2/dx_p dx_q$.

Let $d/d\rho_{pq}$ denote the effect of applying the operator d/dr_{pq} to z, and putting r_{pq} zero after *all* differentiations have been performed, then the effect of this operator will be the same as if we used $d^2/dx_p dx_q$ on z, putting r_{pq} zero *before* differentiation. Generally, let F be any series of operations like d/dr_{pq}, then we see that

$$\mathrm{F}\left(\frac{d}{dr_{pq}}, \frac{d}{dr_{p'q'}}, \frac{d}{dr_{p''q''}} \cdots\cdots\right) z$$

$$= \mathrm{F}\left(\frac{d^2}{dx_p dx_q}, \frac{d^2}{dx_{p'} dx_{q'}}, \frac{d^2}{dx_{p''} dx_{q''}}, \cdots\cdots\right) \frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}n}} e^{-\frac{1}{2}\mathrm{S}_1(x_s^2)}.$$

Now let F be the function which gives the operation of expanding z by MACLAURIN'S theorem in powers of the correlation coefficients, *i.e.*,

$$\mathrm{F} = e^{\mathrm{S}_2\left(r_{ss'}\frac{d}{d\rho_{ss'}}\right)},$$

then

$$z = e^{\mathrm{S}_2\left(r_{ss'}\frac{d}{d\rho_{ss'}}\right.} z = \frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}n}} e^{\mathrm{S}_2\left(r_{ss'}\frac{d}{dx_s dx_{s'}}\right)} e^{-\frac{1}{2}\mathrm{S}_1(x_s^2)}.$$

This is the generalised form of result (xiv.) reached above.

Now let

$$z_0 = \frac{\mathrm{N}}{(2\pi)^{\frac{1}{2}n}} e^{-\frac{1}{2}\mathrm{S}_1(x_s^2)},$$

then z_0 is the ordinate of a frequency surface of the nth order, in which the distribution of the n variables is absolutely independent. We have accordingly the extremely interesting geometrical interpretation that the operator

$$e^{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)},$$

applied to a surface of frequency for n independent variables converts it into a surface of frequency for n dependent variables, the correlation between the sth and s'th variables being $r_{ss'}$.*

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* I should like to suggest to the pure mathematician the interest which a study of such operators would have, and in particular of the generalised form of projection in hyperspace indicated by them.

Expanding, we have

$$z = z_0 + \mathrm{S}_2\left(r_{rs'} \frac{d^2}{dx_s dx_{s'}}\right) z_0 + \frac{1}{\underline{2}}\left\{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^2 z_0$$

$$+ \ldots + \frac{1}{\underline{|m}}\left\{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^m z_0 + \ldots \ldots \quad \text{(lxxiii.).}$$

Our next stage is to evaluate the operation

$$\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)^m z_0.$$

Let us put

$${}_s v_1 = x_s, \qquad {}_s v_2 = x_s^2 - 1, \qquad {}_s v_3 = x_s(x_s^2 - 3),$$

and ${}_s v_p$ = the pth function of x_s as defined by (xv.).

Let ϵ_s be a symbol such that $\epsilon_s{}^p$ represents ${}_s v_p$. Then we shall show that

$$\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)^m z_0 = z_0\left\{\mathrm{S}_r\left(r_{ss'}\epsilon_s\epsilon_{s'}\right)\right\}^m \quad \ldots \ldots \quad \text{(lxxiv.).}$$

We shall prove this by induction.

By (xii.)

$${}_s v_{m+1} = x_s\, {}_s v_m - m\, {}_s v_{m-1},$$

or

$$\epsilon_s{}^{m+1} = x_s\, \epsilon_s{}^m - m\, \epsilon_s{}^{m-1},$$

and by (xvi.)

$$\frac{d_s v_m}{dx_s} = m\, {}_s v_{m-1}, \qquad \text{or} \qquad \frac{d\epsilon_s{}^m}{dx_s} = m\, \epsilon_s{}^{m-1}.$$

Now, let $\chi(\epsilon_s)$ be any function of ϵ_s

$$= \mathrm{S}(\mathrm{A}_q \epsilon_s{}^q),$$

if we suppose it can be expanded in powers of ϵ_s.

Then

$$\begin{aligned}
\frac{d}{dx_s}\chi(\epsilon_s) &= \mathrm{S}\left(\mathrm{A}_q \frac{d}{dx_s}(\epsilon_s{}^q)\right) \\
&= \mathrm{S}(\mathrm{A}_q\, q\epsilon_s{}^{q-1}) \\
&= \mathrm{S}(\mathrm{A}_q(x_s\, \epsilon_s{}^q - \epsilon_s{}^{q+1})) \\
&= x_s \mathrm{S}(\mathrm{A}_q\, \epsilon_s{}^q) - \epsilon_s\, \mathrm{S}(\mathrm{A}_q\, \epsilon_s{}^q) \\
&= (x_s - \epsilon_s)\,\chi(\epsilon_s) \quad \ldots \ldots \ldots \quad \text{(lxxv.).}
\end{aligned}$$

Similarly

$$\frac{d^2}{dx_s dx_{s'}}\chi(\epsilon_s, \epsilon_{s'}) = (x_s - \epsilon_s)(x_{s'} - \epsilon_{s'})\,\chi(\epsilon_s, \epsilon_{s'}) \quad \ldots \ldots \quad \text{(lxxvi.).}$$

Now suppose that

$$\left\{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\} z_0 = z_0\{\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^m,$$

then

$$\left\{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^{m+1} z_0 = \mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right) z_0 \mathrm{U},$$

where U stands for $\{\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^m$.

Hence, remembering that $dz_0/dx_s = -z_0 x_{s'}$,

$$\begin{aligned}\left\{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^{m+1} z_0 &= z_0\mathrm{S}_2(r_{ss'}x_s x_{s'})\mathrm{U} + z_0\mathrm{S}_2\left(r_{ss'}\frac{d^2\mathrm{U}}{dx_s dx_{s'}}\right)\\ &\quad - z_0\mathrm{S}_2\left(r_{ss'}\left(x_s\frac{d\mathrm{U}}{dx_{s'}} + x_{s'}\frac{d\mathrm{U}}{dx_s}\right)\right)\\ &= z_0\mathrm{S}_2(r_{ss'}x_s x_{s'})\mathrm{U} + z_0\mathrm{S}_2\{r_{ss'}(x_s - \epsilon_s)(x_{s'} - \epsilon_{s'})\}\mathrm{U}\\ &\quad - z_0\mathrm{S}_2\{r_{ss'}\{x_{s'}(x_{s'} - \epsilon_{s'}) + x_{s'}(x_s - \epsilon_s)\}\mathrm{U}\\ &= z_0\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\mathrm{U}\\ &= z_0\{\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^{m+1},\end{aligned}$$

which had to be proved.

But it is easy to show by simple differentiation that

$$\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right) z_0 = z_0\ \mathrm{S}_2(r_{ss'}\ {}_s v_1\ {}_{s'}v_1) = z_0\ \mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})$$

$$\begin{aligned}\left\{\mathrm{S}_2\left(r_{ss'}\frac{d^2}{dx_s dx_{s'}}\right)\right\}^2 z_0 &= z_0\ \mathrm{S}_2(r^2{}_{ss'}\ {}_s v_2\ {}_{s'}v_2 + 2r_{pq}r_{ss'}\ {}_p v_1\ {}_q v_1\ {}_s v_1\ {}_{s'}v_1 + 2r_{ps}r_{ss'}\ {}_s v_2\ {}_p v_1\ {}_{s'}v_1)\\ &= z_0\ \{\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxvii.)}.\end{aligned}$$

Hence the theorem is generally true.

Thus we conclude that

$$\begin{aligned}z = z_0\Bigg[1 + \mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'}) + \frac{1}{\underline{|2}}\left\{\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\right\}^2\\ + \ .\ .\ .\ + \frac{1}{\underline{|m}}\left\{\mathrm{S}_2(r_{ss'}\epsilon_s\epsilon_{s'})\right\}^m + \ .\ .\ .\ .\Bigg] \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxviii.)}.\end{aligned}$$

It is quite straightforward, if laborious, to write down the expansion for any number of variables.

Now let Q be the total frequency of complices of variables with x_1 lying between h_1 and ∞, x_2 between h_2 and ∞, . . . x_s between h_s and ∞, . . . x_n between h_n and ∞; and let Q_0 be the frequency of such complices if there were no correlations. Then

$$\mathrm{Q} = \int_{h_1}^{\infty}\int_{h_2}^{\infty} .\ .\ . \int_{h_3}^{\infty} .\ .\ . \int_{h_n}^{\infty} z\, dx_1\, dx_2\ .\ .\ .\ dx_s\ .\ .\ .\ dx_n$$

$$\mathrm{Q}_0 = \int_{h_1}^{\infty}\int_{h_2}^{\infty} .\ .\ . \int_{h_s}^{\infty} .\ .\ . \int_{h_n}^{\infty} z_0\, dx_1\, dx_2\ .\ .\ .\ dx_s\ .\ .\ .\ dx_n$$

Now let

$$\beta_s \mathrm{H}_s = \frac{1}{\sqrt{2\pi}}\int_{h_s}^{\infty} e^{-\frac{1}{2}x_s^2} dx_s$$

where

$$\mathrm{H}_s = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h_s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxix.)}.$$

We have
$$\mathrm{Q}_0 = \mathrm{N}\beta_1\beta_2 \;.\;.\; \beta_s \;.\;.\; \beta_n \,\mathrm{H}_1\, \mathrm{H}_2 \;.\;.\; \mathrm{H}_s \;.\;.\; \mathrm{H}_n.$$

But by (xviii.)

$$\frac{1}{\sqrt{2\pi}} \int_{h_s}^{\infty} {}_sv_p \, e^{-\frac{1}{2}x_s^2}\, dx_s = \mathrm{H}_s \, {}_s\bar{v}_{p-1} = \mathrm{H}_s\beta_s \frac{{}_s\bar{v}_{p-1}}{\beta_s},$$

where

$${}_s\bar{v}_{p-1} = \left[{}_sv_{p-1}\right]_{x_s = h_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxx.)},$$

and as above,

$$\beta_s = \int_{h_s}^{\infty} e^{-\frac{1}{2}x_s^2} dx_s \Big/ e^{-\frac{1}{2}h_s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxxi.)}.$$

Thus

$$\frac{1}{(\sqrt{2\pi})^n} \int_{h_1}^{\infty}\int_{h_2}^{\infty} \ldots \int_{h_s}^{\infty} \ldots \int_{h_n}^{\infty} {}_sv_p \, {}_{s'}v_{p'} \, {}_{s''}v_{p''} \ldots e^{-\frac{1}{2}(x_1^2 + x_2^2 + \ldots + x_s^2 + \ldots + x_n^2)} dx_1 dx_2 \ldots dx_s \ldots dx_n$$
$$= \mathrm{H}_1\mathrm{H}_2 \ldots \mathrm{H}_s \ldots \mathrm{H}_n\beta_1\beta_2 \ldots \beta_s \ldots \beta_n \frac{{}_s\bar{v}_{p-1}}{\beta_s} \frac{{}_{s'}\bar{v}_{p'-1}}{\beta_{s'}} \frac{{}_{s''}\bar{v}_{p''-1}}{\beta_{s''}} \ldots,$$

or

$$\int_{h_1}^{\infty}\int_{h_2}^{\infty} \ldots \int_{h_s}^{\infty} \ldots \int_{h_n}^{\infty} z_0\Pi({}_sv_p)\, dx_1 dx_2 \ldots dx_s \ldots dx_n = \mathrm{Q}_0\Pi\left(\frac{{}_s\bar{v}_{p-1}}{\beta_s}\right)$$
$$. \quad . \quad . \quad . \quad . \quad \text{(lxxxii.)}.$$

where Π denotes a product of ${}_sv_p$ for any number of v's with any s and p. The rule, therefore, is very simple. We must expand the value of z in v's as given by (lxxviii.) above, then the multiple integral of this will be obtained by lowering every v's right-hand subscript by unity (remembering that ${}_sv_0 = 1$), and further dividing by the β of the left-hand subscript. The general expression up to terms of the fourth order has been written down; it involves thirty-four sums, each represented by a type term All these would only occur in the case of the correlation of eight organs, or when we have to deal with twenty-eight coefficients of correlation. Such a number seems beyond our present power of arithmetical manipulation, so that I have not printed the general expressions. At the same time, the theory of multiple correlation is of such great importance for problems of evolution, in which over and over again we have three or four correlated characters to deal with,[*] that it seems desirable to place on record the expansion for these cases. I give four variables up to the fourth and three variables up to the fifth order terms. Afterwards I will consider special cases.

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[*] In my memoir on Prehistoric Stature I have dealt with five correlated organs, *i.e.*, ten coefficients. In some barometric investigations now in hand we propose to deal with at least fifteen coefficients, while Mr. BRAMLEY-MOORE, in the correlation of parts of the skeleton, has, in a memoir not yet published, dealt with between forty and fifty cases of four variables or six coefficients.

*Value of the Quadruple Integral in the Case of Four Variables.**

$$
\begin{aligned}
\frac{\mathrm{Q}-\mathrm{Q}_0}{\mathrm{Q}_0} = {} & \frac{r_{12}}{\beta_1\beta_2} + \frac{r_{13}}{\beta_1\beta_3} + \frac{r_{14}}{\beta_1\beta_4} + \frac{r_{23}}{\beta_2\beta_3} + \frac{r_{24}}{\beta_2\beta_4} + \frac{r_{34}}{\beta_3\beta_4} \\
& + \frac{1}{\underline{|2}}\left\{ \frac{r_{12}^{\;2}}{\beta_1\beta_2} v_1'v_1'' + \frac{r_{13}^{\;2}}{\beta_1\beta_3} v_1'v_1''' + \frac{r_{14}^{\;2}}{\beta_1\beta_4} v_1'v_1^{\text{iv}} + \frac{r_{23}^{\;2}}{\beta_2\beta_3} v_1''v_1''' + \frac{r_{24}^{\;2}}{\beta_2\beta_4} v_1''v_1^{\text{iv}} \right. \\
& + \frac{r_{34}^{\;2}}{\beta_3\beta_4} v_1'''v_1^{\text{iv}} + \frac{2r_{12}r_{13}}{\beta_1\beta_2\beta_3} v_1' + \frac{2r_{12}r_{14}}{\beta_1\beta_2\beta_4} v_1' + \frac{2r_{13}r_{14}}{\beta_1\beta_3\beta_4} v_1' \\
& + \frac{2r_{12}r_{23}}{\beta_1\beta_2\beta_3} v_1'' + \frac{2r_{13}r_{23}}{\beta_1\beta_2\beta_3} v_1''' + \frac{2r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} + \frac{2r_{12}r_{34}}{\beta_1\beta_2\beta_3\beta_4} \\
& + \frac{2r_{13}r_{34}}{\beta_1\beta_3\beta_4} \cdot v_1''' + \frac{2r_{14}r_{34}}{\beta_1\beta_3\beta_4} \cdot v_1^{\text{iv}} + \frac{2r_{23}r_{34}}{\beta_2\beta_3\beta_4} v_1''' \\
& + \frac{2r_{24}r_{34}}{\beta_2\beta_3\beta_4} v_1^{\text{iv}} + \frac{2r_{12}r_{24}}{\beta_1\beta_2\beta_4} v_1'' + \frac{2r_{13}r_{24}}{\beta_1\beta_2\beta_3\beta_4} \\
& \left. + \frac{2r_{14}r_{24}}{\beta_1\beta_2\beta_4} v_1^{\text{iv}} + \frac{2r_{23}r_{24}}{\beta_2\beta_3\beta_4} v_1'' \right\} \\
& + \frac{1}{\underline{|3}}\left\{ \frac{r_{12}^{\;3}}{\beta_1\beta_2} v_2'v_2'' + \frac{r_{13}^{\;3}}{\beta_1\beta_3} v_2'v_2''' + \frac{r_{14}^{\;3}}{\beta_1\beta_4} v_2'v_2^{\text{iv}} + \frac{r_{23}^{\;3}}{\beta_1\beta_3} v_2''v_2''' + \frac{r_{24}^{\;3}}{\beta_2\beta_4} v_2''v_2^{\text{iv}} \right. \\
& + \frac{r_{34}^{\;3}}{\beta_3\beta_4} v_2'''v_2^{\text{iv}} + \frac{3r_{12}^{\;2}\,r_{13}}{\beta_1\beta_2\beta_3} v_2'v_1'' + \frac{3r_{12}r_{13}^{\;2}}{\beta_1\beta_2\beta_3} v_2'v_1''' + \frac{3r_{12}^{\;2}\,r_{14}}{\beta_1\beta_2\beta_4} v_2'v_1''' \\
& + \frac{3r_{12}r_{14}^{\;2}}{\beta_2\beta_2\beta_4} v_2'v_1^{\text{iv}} + \frac{3r_{13}^{\;2}\,r_{14}}{\beta_1\beta_2\beta_4} v_2'v_1''' + \frac{3r_{13}r_{14}^{\;2}}{\beta_1\beta_3\beta_4} v_2'v_1^{\text{iv}} \\
& + \frac{3r_{12}^{\;2}\,r_{23}}{\beta_1\beta_2\beta_3} v_2''v_1' + \frac{3r_{12}r_{23}^{\;2}}{\beta_1\beta_2\beta_3} v_2''v_1''' + \frac{3r_{13}^{\;2}\,r_{23}}{\beta_1\beta_2\beta_3} v_2'''v_1' \\
& + \frac{3r_{13}r_{23}^{\;2}}{\beta_1\beta_2\beta_3} v_2'''v_1'' + \frac{3r_{14}^{\;2}\,r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1^{\text{iv}} + \frac{3r_{14}r_{23}^{\;2}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1''' \\
& + \frac{3r_{12}^{\;2}\,r_{24}}{\beta_1\beta_2\beta_4} v_2''v_1' + \frac{3r_{24}^{\;2}\,r_{12}}{\beta_1\beta_2\beta_4} v_2''v_1^{\text{iv}} + \frac{3r_{13}^{\;2}\,r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''' \\
& + \frac{3r_{13}r_{24}^{\;2}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1^{\text{iv}} + \frac{3r_{14}^{\;2}\,r_{24}}{\beta_1\beta_2\beta_4} v_2^{\text{iv}}v_1' + \frac{3r_{14}r_{24}^{\;2}}{\beta_1\beta_2\beta_4} v_2^{\text{iv}}v_1'' \\
& + \frac{3r_{23}^{\;2}\,r_{24}}{\beta_2\beta_3\beta_4} v_2''v_1''' + \frac{3r_{23}r_{24}^{\;2}}{\beta_2\beta_3\beta_4} v_2''v_1^{\text{iv}} + \frac{3r_{34}r_{12}^{\;2}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1'' \\
& + \frac{3r_{34}^{\;2}\,r_{12}}{\beta_1\beta_2\beta_3\beta_4} v_1'''v_1^{\text{iv}} + \frac{3r_{13}^{\;2}\,r_{34}}{\beta_1\beta_2\beta_4} v_2'''v_1' + \frac{3r_{13}r_{34}^{\;2}}{\beta_1\beta_3\beta_4} v_2'''v_1^{\text{iv}} \\
& + \frac{3r_{14}^{\;2}\,r_{34}}{\beta_1\beta_3\beta_4} v_2^{\text{iv}}v_1' + \frac{3r_{14}r_{34}^{\;2}}{\beta_1\beta_3\beta_4} v_2^{\text{iv}}v_1''' + \frac{3r_{23}^{\;2}\,r_{34}}{\beta_2\beta_3\beta_4} v_2'''v_1''
\end{aligned}
$$

---

* To simplify the notation, v_s', v_s'', v_s''' v_s^{iv} have been used for ${}_1\bar{v}_s$, ${}_2\bar{v}_s$, ${}_3\bar{v}_s$, ${}_4\bar{v}_s$.

$$+\frac{3r_{23}r_{34}^{2}}{\beta_2\beta_3\beta_4}v_2'''v_1^{\text{iv}}+\frac{3r_{24}^{2}r_{34}}{\beta_2\beta_3\beta_4}v_2^{\text{iv}}v_1''+\frac{3r_{24}r_{34}^{2}}{\beta_2\beta_3\beta_4}v_2^{\text{iv}}v_1'''$$

$$+\frac{6r_{12}r_{13}r_{14}}{\beta_1\beta_2\beta_3\beta_4}v_2'+\frac{6r_{12}r_{13}r_{23}}{\beta_1\beta_2\beta_3}v_1'v_1''v_1'''+\frac{6r_{12}r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1''$$

$$+\frac{6r_{13}r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1'''+\frac{6r_{12}r_{13}r_{24}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1''+\frac{6r_{12}r_{14}r_{24}}{\beta_1\beta_2\beta_4}v_1'v_1''v_1^{\text{iv}}$$

$$+\frac{6r_{12}r_{14}r_{24}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1^{\text{iv}}+\frac{6r_{12}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4}v_2''+\frac{6r_{13}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4}v_1''v_1'''$$

$$+\frac{6r_{14}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4}v_1''v_1^{\text{iv}}+\frac{6r_{12}r_{13}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1'''+\frac{6r_{12}r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1^{\text{iv}}$$

$$+\frac{6r_{12}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1'''+\frac{6r_{13}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_2'''+\frac{6r_{14}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_1'''v_1^{\text{iv}}$$

$$+\frac{6r_{12}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1^{\text{iv}}+\frac{6r_{13}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_1'''v_1^{\text{iv}}+\frac{6r_{14}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4}v_2^{\text{iv}}$$

$$+\frac{6r_{23}r_{24}r_{34}}{\beta_2\beta_3\beta_4}v_1''v_1'''v_1^{\text{iv}}+\frac{6r_{13}r_{14}r_{34}}{\beta_1\beta_3\beta_4}v_1\,v_1'''v_1^{\text{iv}}\Bigg\}$$

$$+\frac{1}{\underline{|4}}\Bigg\{\frac{r_{12}^{4}}{\beta_1\beta_2}v_3'v_3''+\frac{r_{13}^{4}}{\beta_1\beta_3}v_3'v_3'''+\frac{r_{14}^{4}}{\beta_1\beta_4}v_3'v_3^{\text{iv}}+\frac{r_{23}^{4}}{\beta_2\beta_3}v_3''v_3'''+\frac{r_{24}^{4}}{\beta_2\beta_4}v_3''v_3^{\text{iv}}+\frac{r_{34}^{4}}{\beta_3\beta_4}v_3'''v_3^{\text{iv}}$$

$$+\frac{4r_{12}^{3}r_{13}}{\beta_1\beta_2\beta_3}v_3\,v_2''+\frac{4r_{12}r_{13}^{3}}{\beta_1\beta_2\beta_3}v_3'v_2'''+\frac{4r_{12}^{3}r_{14}}{\beta_1\beta_2\beta_4}v_3'v_2''+\frac{4r_{12}r_{14}^{3}}{\beta_1\beta_2\beta_4}v_3'v_2^{\text{iv}}$$

$$+\frac{4r_{13}^{3}r_{14}}{\beta_1\beta_3\beta_4}v_3'v_3'''+\frac{4r_{13}r_{14}^{3}}{\beta_1\beta_3\beta_4}v_3'v_2^{\text{iv}}+\frac{4r_{12}^{3}r_{23}}{\beta_1\beta_2\beta_3}v_3''v_2'+\frac{4r_{12}r_{23}^{3}}{\beta_1\beta_2\beta_3}v_3''v_2'''$$

$$+\frac{4r_{13}^{3}r_{23}}{\beta_1\beta_2\beta_3}v_3'''v_2'+\frac{4r_{13}r_{23}^{3}}{\beta_1\beta_2\beta_3}v_3'''v_2''+\frac{4r_{14}^{3}r_{23}}{\beta_1\beta_2\beta_3\beta_4}v_2'v_2^{\text{iv}}+\frac{4r_{14}r_{23}^{3}}{\beta_1\beta_2\beta_3\beta_4}v_2''v_2'''$$

$$+\frac{4r_{12}^{3}r_{24}}{\beta_1\beta_2\beta_4}v_2'v_3''+\frac{4r_{12}r_{24}^{3}}{\beta_1\beta_2\beta_4}v_3''v_2^{\text{iv}}+\frac{4r_{13}^{3}r_{24}}{\beta_1\beta_2\beta_3\beta_4}v_2'v_2'''+\frac{4r_{13}r_{24}^{3}}{\beta_1\beta_2\beta_3\beta_4}v_2'\,v_2^{\text{iv}}$$

$$+\frac{4r_{14}^{3}r_{24}}{\beta_1\beta_2\beta_4}v_2'v_3^{\text{iv}}+\frac{4r_{14}r_{24}^{3}}{\beta_1\beta_2\beta_4}v_2''v_3^{\text{iv}}+\frac{4r_{23}^{3}r_{24}}{\beta_2\beta_3\beta_4}v_3''v_2'''+\frac{4r_{23}r_{24}^{3}}{\beta_2\beta_3\beta_4}v_3''v_2^{\text{iv}}$$

$$+\frac{4r_{12}^{3}r_{34}}{\beta_1\beta_3\beta_4}v_2'v_2''+\frac{4r_{12}r_{34}^{3}}{\beta_1\beta_2\beta_3\beta_4}v_2'''v_2^{\text{iv}}+\frac{4r_{13}^{3}r_{34}}{\beta_1\beta_3\beta_4}v_3'''v_2'+\frac{4r_{13}r_{34}^{3}}{\beta_1\beta_3\beta_4}v_3'''v_2^{\text{iv}}$$

$$+\frac{4r_{14}^{3}r_{34}}{\beta_1\beta_3\beta_4}v_2'v_3^{\text{iv}}+\frac{4r_{14}r_{34}^{3}}{\beta_1\beta_3\beta_4}v_2'''v_3^{\text{iv}}+\frac{4r_{23}^{3}r_{34}}{\beta_2\beta_3\beta_4}v_2''v_3''+\frac{4r_{23}r_{34}^{3}}{\beta_2\beta_3\beta_4}v_3'''v_2^{\text{iv}}$$

$$+\frac{4r_{24}^{3}r_{34}}{\beta_1\beta_3\beta_4}v_2''v_3^{\text{iv}}+\frac{4r_{24}r_{34}^{3}}{\beta_2\beta_3\beta_4}v_2'''v_3^{\text{iv}}+\frac{6r_{12}^{2}r_{13}^{2}}{\beta_1\beta_2\beta_3}v_3'v_1''v_1'''+\frac{6r_{12}^{2}r_{14}^{2}}{\beta_1\beta_2\beta_4}v_3'v_1''v_1^{\text{iv}}$$

$$+\frac{6r_{13}^{2}r_{14}^{2}}{\beta_1\beta_3\beta_4}v_3'v_1'''v_1^{\text{iv}}+\frac{6r_{12}^{2}r_{23}^{2}}{\beta_1\beta_2\beta_3}v_1'v_3''v_1'''+\frac{6r_{13}^{2}r_{23}^{2}}{\beta_1\beta_2\beta_3}v_1'v_1''v_3'''+\frac{6r_{14}^{2}r_{23}^{2}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1''v_1'''v_1^{\text{iv}}$$

$$+\frac{6r_{12}^{2}r_{24}^{2}}{\beta_1\beta_2\beta_4}v_1'v_3''v_1^{\text{iv}}+\frac{6r_{13}^{2}r_{24}^{2}}{\beta_1\beta_2\beta_3\beta_4}v_1'v_1''v_1'''v_1^{\text{iv}}+\frac{6r_{14}^{2}r_{24}^{2}}{\beta_1\beta_2\beta_4}v_1'v_1''v_3^{\text{iv}}+\frac{6r_{23}^{2}r_{24}^{2}}{\beta_2\beta_3\beta_4}v_3'\,v_1'''v_1^{\text{iv}}$$

$$+ \frac{6r_{12}^2 r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_1'''v_1^{\text{iv}} + \frac{6r_{13}^2 r_{34}^2}{\beta_1\beta_3\beta_4} v_1'v_3'''v_1^{\text{iv}} + \frac{6r_{14}^2 r_{34}^2}{\beta_1\beta_3\beta_4} v_1'v_1'''v_3^{\text{iv}} + \frac{6r_{23}^3 r_{34}^2}{\beta_2\beta_3\beta_4} v_1''v_3'''v_1^{\text{iv}}$$

$$+ \frac{6r_{24}^2 r_{34}^2}{\beta_2\beta_3\beta_4} v_1''v_1'''v_3^{\text{iv}} + \frac{12r_{12}^2 r_{13}r_{14}}{\beta_1\beta_2\beta_3\beta_4} v_3'v_1'' + \frac{12r_{12}r_{13}^2 r_{14}}{\beta_1\beta_2\beta_3\beta_4} v_3'v_1''' + \frac{12r_{12}r_{13}r_{14}^2}{\beta_1\beta_2\beta_3\beta_4} v_3'v_1^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{13}r_{23}}{\beta_1\beta_2\beta_3} v_2'v_2''v_1''' + \frac{12r_{12}r_{13}^2 r_{23}}{\beta_1\beta_2\beta_3} v_2'v_1''v_2''' + \frac{12r_{12}r_{13}r_{23}^2}{\beta_1\beta_2\beta_3} v_1'v_2''v_2'''$$

$$+ \frac{12r_{12}^2 r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2'' + \frac{12r_{12}r_{14}^2 r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1^{\text{iv}} + \frac{12r_{12}r_{14}r_{23}^2}{\beta^1\beta_2\beta_3\beta_4} v_1'v_2''v_1'$$

$$+ \frac{12r_{13}^2 r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2'' + \frac{12r_{13}r_{14}^2 r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1'''v^{\text{iv}} + \frac{12r_{13}r_{14}r_{23}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2'''v_1''$$

$$+ \frac{12r_{12}^2 r_{13}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2'' + \frac{12r_{12}r_{13}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1''' + \frac{12r_{12}r_{13}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_1^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{14}r_{24}}{\beta_1\beta_2\beta_4} v_2'v_2''v_1^{\text{iv}} + \frac{12r_{12}r_{14}^2 r_{24}}{\beta_1\beta_2\beta_4} v_2'v_1''v_2^{\text{iv}} + \frac{12r_{12}r_{14}r_{24}^2}{\beta_1\beta_2\beta_4} v_1'v_2''v_2^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{14}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1'''v_1^{\text{iv}} + \frac{12r_{13}r_{14}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2^{\text{iv}} + \frac{12r_{13}r_{14}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2^{\text{iv}}v_1''$$

$$+ \frac{12r_{12}^2 r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_3'' + \frac{12r_{12}r_{23}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_3''v_1''' + \frac{12r_{12}r_{23}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_3''v_1^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_2''' + \frac{12r_{13}r_{23}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2''' + \frac{12r_{13}r_{23}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_2''v_1'''v_1^{\text{iv}}$$

$$+ \frac{12r_{14}^2 r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1''v_2^{\text{iv}} + \frac{12r_{14}r_{23}^2 r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_1'''v_1^{\text{iv}} + \frac{12r_{14}r_{23}r_{24}^2}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{13}r_{34}}{\beta_1\beta_2\beta_3\beta_3} v_2'v_1''v_2''' + \frac{12r_{12}r_{13}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2''' + \frac{12r_{12}r_{13}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2'''v_1^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_1''v_1^{\text{iv}} + \frac{12r_{12}r_{14}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2'v_2^{\text{iv}} + \frac{12r_{12}r_{14}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1'''v_2^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{14}r_{34}}{\beta_1\beta_3\beta_4} v_2'v_2'''v_1^{\text{iv}} + \frac{12r_{13}r_{14}^2 r_{34}}{\beta_1\beta_3\beta_4} v_2'v_1'''v_2^{\text{iv}} + \frac{12r_{13}r_{14}r_{34}^2}{\beta_1\beta_3\beta_4} v_1'v_2'''v_2^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2''v_1''' + \frac{12r_{12}r_{23}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2''' + \frac{12r_{12}r_{23}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1''v_2'''v_1^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_3^{\text{iv}} + \frac{12r_{13}r_{23}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_3''' + \frac{12r_{13}r_{23}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_3'''v_1^{\text{iv}}$$

$$+ \frac{12r_{14}^2 r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_1'''v_2^{\text{iv}} + \frac{12r_{14}r_{23}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_2'''v_1^{\text{iv}} + \frac{12r_{14}r_{23}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_2'''v_2^{\text{iv}}$$

$$+ \frac{12r_{12}^2 r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2''v_1^{\text{iv}} + \frac{12r_{12}r_{24}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2''v_2^{\text{iv}} + \frac{12r_{12}r_{24}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1'''v_2^{\text{iv}}$$

$$+ \frac{12r_{13}^2 r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1'v_2'''v_1^{\text{iv}} + \frac{12r_{13}r_{24}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1''v_1'''v_2^{\text{iv}} + \frac{12r_{13}r_{24}r_{34}^2}{\beta_1\beta_2\beta_3\beta_4} v_2'''v_2^{\text{iv}}$$

$$+ \frac{12r_{14}{}^2 r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_3{}^{\text{iv}} + \frac{12r_{14}r_{24}{}^2 r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}''v_3{}^{\text{iv}} + \frac{12r_{14}r_{24}r_{34}{}^2}{\beta_1\beta_2\beta_3\beta_4} v_1{}'''v_3{}^{\text{iv}}$$

$$+ \frac{12r_{23}{}^2 r_{24}r_{34}}{\beta_2\beta_3\beta_4} v_2{}''v_2{}'''v_1{}^{\text{iv}} + \frac{12r_{23}r_{24}{}^2 r_{34}}{\beta_2\beta_3\beta_4} v_2{}''v_1{}'''v_2{}^{\text{iv}} + \frac{12r_{23}r_{24}r_{34}{}^2}{\beta_2\beta_3\beta_4} v_1{}''v_2{}'''v_2{}^{\text{iv}}$$

$$+ \frac{24r_{12}r_{13}r_{14}r_{23}}{\beta_1\beta_2\beta_3\beta_4} v_2{}'v_1{}''v_1{}''' + \frac{24r_{12}r_{13}r_{14}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_2{}'v_1{}''v_1{}^{\text{iv}} + \frac{24r_{12}r_{13}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_2{}''v_1{}'''$$

$$+ \frac{24r_{12}r_{14}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_2{}''v_1{}^{\text{iv}} + \frac{24r_{13}r_{14}r_{23}r_{24}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''v_1{}'''v_1{}^{\text{iv}} + \frac{24r_{12}r_{13}r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2{}'v_1{}'''v_1{}^{\text{iv}}$$

$$+ \frac{24r_{12}r_{13}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''v_2{}''' + \frac{24r_{12}r_{14}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''v_1{}'''v_1{}^{\text{iv}} + \frac{24r_{13}r_{14}r_{23}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_2{}'''v_1{}^{\text{iv}}$$

$$+ \frac{24r_{12}r_{13}r_{14}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2{}'v_1{}'''v_1{}^{\text{iv}} + \frac{24r_{12}r_{14}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}''v_2{}^{\text{iv}} + \frac{24r_{13}r_{14}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}'v_1{}'''v_2{}^{\text{iv}}$$

$$+ \frac{24r_{12}r_{23}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_2{}''v_1{}'''v_1{}^{\text{iv}} + \frac{24r_{13}r_{23}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}''v_2{}'''v_1{}^{\text{iv}} + \frac{24r_{14}r_{23}r_{24}r_{34}}{\beta_1\beta_2\beta_3\beta_4} v_1{}''v_1{}'''v_2{}^{\text{iv}}\bigg\} \quad \ldots\ldots \text{(lxxxiii.)}.$$

In the case of three variables, we must cancel in the above expression all terms involving β_4. Thus we shall have 3 instead of 6 first order terms, 6 instead of 21 second order terms, 10 instead of 56 third order terms, and 15 instead of 126 fourth order terms—a much more manageable series.

I give below the extra term necessary for calculating the value of $(Q - Q_0)/Q_0$ as far as the fifth order terms in the case of three variables.

Fifth Order Terms for Three Variables.

$$\frac{1}{\underline{|5}}\bigg\{\frac{r_{23}{}^5}{\beta_2\beta_3} v_4{}''v_4{}''' + \frac{r_{31}{}^5}{\beta_3\beta_1} v_4{}'''v_4{}' + \frac{r_{12}{}^5}{\beta_1\beta_2} v_4{}'v_4{}'' + \frac{5r_{31}{}^4r_{12}}{\beta_1\beta_2\beta_3} v_3{}'''v_4{}'$$

$$+ \frac{5r_{31}{}^4 r_{23}}{\beta_1\beta_2\beta_3} v_3{}'v_4{}''' + \frac{5r_{12}{}^4 r_{23}}{\beta_1\beta_2\beta_3} v_3{}'v_4{}'' + \frac{5r_{12}{}^4 r_{31}}{\beta_1\beta_2\beta_3} v_4{}'v_3{}''$$

$$+ \frac{5r_{23}{}^4 r_{31}}{\beta_1\beta_2\beta_3} v_3{}''v_4{}''' + \frac{5r_{23}{}^4 r_{12}}{\beta_1\beta_2\beta_3} v_4{}''v_3{}''' + \frac{10r_{31}{}^3 r_{12}{}^2}{\beta_1\beta_2\beta_3} v_4{}'v_1{}''v_2{}'''$$

$$+ \frac{10r_{31}{}^3 r_{23}{}^2}{\beta_1\beta_2\beta_3} v_2{}'v_1{}'\,v_4{}''' + \frac{10r_{12}{}^3 r_{23}{}^2}{\beta_1\beta_3\beta_2} v_2{}'v_4{}''v_1{}''' + \frac{10r_{12}{}^3 r_{31}{}^2}{\beta_1\beta_2\beta_3} v_4{}'v_2{}''v_1{}'''$$

$$+ \frac{10r_{23}{}^3 r_{31}{}^2}{\beta_1\beta_2\beta_3} v_1{}'v_2{}''v_4{}''' + \frac{10r_{23}{}^3 r_{12}{}^2}{\beta_1\beta_2\beta_3} v_1{}'v_4{}''v_2{}''' + \frac{20r_{23}{}^3 r_{31}r_{12}}{\beta_1\beta_2\beta_3} v_1{}'v_3{}''v_3{}'''$$

$$+ \frac{20r_{31}{}^3 r_{12}r_{23}}{\beta_1\beta_2\beta_3} v_3{}'v_1{}''v_3{}''' + \frac{20r_{12}{}^3 r_{23}r_{31}}{\beta_1\beta_2\beta_3} v_3{}'v_3{}''v_1{}''' + \frac{30r_{23}r_{31}{}^2 r_{12}{}^2}{\beta_1\beta_2\beta_3} v_3{}'v_3{}''v_2{}'''$$

$$+ \frac{30r_{31}r_{12}{}^2 r_{23}{}^2}{\beta_1\beta_2\beta_3} v_2{}'v_3{}''v_2{}''' + \frac{30r_{12}r_{23}{}^2 r_{31}{}^2}{\beta_1\beta_2\beta_3} v_2{}'v_2{}''v_3{}'''\bigg\}. \quad \ldots\ldots \text{(lxxxiv.)}.$$

A numerical illustration of these formulæ will be given in the latter part of this Memoir. It will, however, be clear that what we want are tables of $\log\left(\frac{{}_s\bar{v}_p}{\beta_s}\right)$, including $\log\left(\frac{{}_s\bar{v}_0}{\beta_s}\right)$ or $\log\left(\frac{1}{\beta_s}\right)$ for a series of values of h. Such tables would render the computation of $\frac{\mathrm{Q}-\mathrm{Q}_0}{\mathrm{Q}_0}$ fairly direct and rapid; they could be fairly easily calculated from existing tables for the ordinate and area of the normal curve, and I hope later to find some one willing to undertake them.

Meanwhile let us look at special cases. In the first place, suppose, in the case of three variables, that the division of the groups is taken at the mean, *i.e.*, $h_1 = h_2 = h_3 = 0$. Then we have

$$\beta_1 = \beta_2 = \beta_3 = \int_0^\infty e^{-\frac{1}{2}x^2}\,dx = \sqrt{\frac{\pi}{2}}\,.$$

$$v_1' = v_1'' = v_1''' = 0$$

$$v_2' = v_2'' = v_2''' = -1$$

$$v_3' = v_3'' = v_3''' = 0$$

$$v_4' = v_4'' = v_4''' = 3.$$

Hence we have

$$\begin{aligned}\mathrm{Q} = \int_0^\infty\!\int_0^\infty\!\int_0^\infty z\,dx_1\,dx_2\,dx_3 &= \mathrm{Q}_0\left\{1 + \frac{2}{\pi}(r_{12}+r_{13}+r_{23})\right\}\\ &\quad+ \frac{1}{|\underline{3}}\left\{\frac{2}{\pi}(r_{12}{}^3 + r_{13}{}^3 + r_{23}{}^3)\right\} + \frac{1}{|\underline{5}}\left\{\frac{2}{\pi}9\,(r_{23}{}^5 + r_{31}{}^5 + r_{12}{}^5)\right\} + \ldots\\ &= \mathrm{Q}_0\left\{1 + \frac{2}{\pi}(\sin^{-1}r_{12} + \sin^{-1}r_{13} + \sin^{-1}r_{23})\right\} \quad \ldots\ldots\ldots \quad \text{(lxxxv.)}.\end{aligned}$$

Let $r_{12} = \cos \mathrm{D}_{12}$, $r_{13} = \cos \mathrm{D}_{13}$, $r_{23} = \cos \mathrm{D}_{23}$, and let E be the spherical excess of the spherical triangle whose angles are the divergences D_{12}, D_{13}, D_{23}. Then we have

$$\frac{\mathrm{Q}-\mathrm{Q}_0}{\mathrm{Q}_0}\frac{\pi}{2} = \frac{3\pi}{2} - \mathrm{D}_{12} - \mathrm{D}_{13} - \mathrm{D}_{23} = \frac{\pi}{2} - \mathrm{E}.$$

Or:

$$\sin\frac{\mathrm{Q}-\mathrm{Q}_0}{\mathrm{Q}_0}\frac{\pi}{2} = \cos \mathrm{E} \quad \ldots\ldots\ldots \quad \text{(lxxxvi.)}.$$

Now take the case of four variables. Here we have

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \sqrt{\frac{\pi}{2}}$$

$$v_2' = v_2'' = v_2''' = v_2^{\mathrm{iv}} = 1$$

$$v_4' = v_4'' = v_4''' = v_4^{\mathrm{iv}} = 3,$$

and all the odd v's zero. Hence

$$\begin{aligned}\frac{Q-Q_0}{Q} &= \frac{2}{\pi}(r_{12}+r_{13}+r_{14}+r_{23}+r_{24}+r_{34}) + \left(\frac{2}{\pi}\right)^2(r_{14}r_{23}+r_{12}r_{34}+r_{13}r_{24}) \\ &+ \frac{2}{\pi}\frac{1}{\underline{|3}}(r_{12}{}^3+r_{13}{}^3+r_{14}{}^3+r_{23}{}^3+r_{24}{}^3+r_{34}{}^3) + \left(\frac{2}{\pi}\right)^2(r_{12}r_{13}r_{14}+r_{12}r_{23}r_{24} \\ &+ r_{13}r_{23}r_{34}+r_{14}r_{24}r_{34}) + \left(\frac{2}{\pi}\right)^2\frac{1}{\underline{|3}}(r_{14}{}^3r_{23}+r_{14}r_{23}{}^3+r_{13}r_{24}{}^3+r_{13}{}^3r_{24}+r_{12}{}^3r_{34} \\ &+ r_{12}r_{34}{}^3) + \left(\frac{2}{\pi}\right)^2\frac{1}{\underline{|2}}(r_{12}{}^2r_{14}r_{23}+r_{13}{}^2r_{14}r_{23}+r_{12}{}^2r_{13}r_{24}+r_{13}r_{14}{}^2r_{24}+r_{13}r_{23}{}^2r_{24} \\ &+ r_{14}r_{23}r_{24}{}^2+r_{12}r_{13}{}^2r_{34}+r_{12}r_{14}{}^2r_{34}+r_{12}r_{23}{}^2r_{34}+r_{14}r_{23}r_{34}{}^2+r_{12}r_{24}{}^2r_{34} \\ &+ r_{13}r_{24}r_{34}{}^2) + \dots\dots \qquad \text{(lxxxvii.)}.\end{aligned}$$

This is the correct value including terms of the fourth order, but to this order of approximation we can throw it into a much simpler form. Let $r_{ss'} = \sin\delta_{ss'}$, then

$$\begin{aligned}\frac{Q-Q_0}{Q_0}\frac{\pi}{2} &= \sin^{-1}r_{12}+\sin^{-1}r_{13}+\sin^{-1}r_{14}+\sin^{-1}r_{23}+\sin^{-1}r_{24}+\sin^{-1}r_{34} \\ &+ \frac{2}{\pi}(\sin^{-1}r_{12}\sin^{-1}r_{13}\sin^{-1}r_{14}+\sin^{-1}r_{12}\sin^{-1}r_{23}\sin^{-1}r_{24} \\ &+ \sin^{-1}r_{13}\sin^{-1}r_{23}\sin^{-1}r_{34}+\sin^{-1}r_{14}\sin^{-1}r_{24}\sin^{-1}r_{34}) \\ &+ \frac{2}{\pi}[\sin^{-1}r_{14}\sin^{-1}r_{23}\{(1-r_{12}{}^2)(1-r_{13}{}^2)(1-r_{24}{}^2)(1-r_{34}{}^2)\}^{-\frac{1}{2}} \\ &\quad + \sin^{-1}r_{12}\sin^{-1}r_{34}\{(1-r_{13}{}^2)(1-r_{14}{}^2)(1-r_{23}{}^2)(1-r_{24}{}^2)\}^{-\frac{1}{2}} \\ &\quad + \sin^{-1}r_{13}\sin^{-1}r_{24}\{(1-r_{12}{}^2)(1-r_{14}{}^2)(1-r_{23}{}^2)(1-r_{34}{}^2)\}^{-\frac{1}{2}}] \\ &= \delta_{12}+\delta_{13}+\delta_{14}+\delta_{23}+\delta_{24}+\delta_{34} \\ &+ \frac{2}{\pi}(\delta_{12}\delta_{13}\delta_{14}+\delta_{12}\delta_{23}\delta_{24}+\delta_{13}\delta_{23}\delta_{34}+\delta_{14}\delta_{24}\delta_{34}) \\ &+ \frac{2}{\pi}\left(\frac{\delta_{14}\delta_{23}\cos\delta_{14}\cos\delta_{23}+\delta_{12}\delta_{34}\cos\delta_{12}\cos\delta_{34}+\delta_{13}\delta_{24}\cos\delta_{13}\cos\delta_{24}}{\cos\delta_{12}\cos\delta_{13}\cos\delta_{14}\cos\delta_{23}\cos\delta_{24}\cos\delta_{34}}\right) \\ &\qquad \dots\dots \qquad \text{(lxxxviii.)}.\end{aligned}$$

We may write this
$$\sin\frac{Q-Q_0}{Q_0}\frac{\pi}{2} = \cos E' \qquad \dots\dots \qquad \text{(lxxxix.)}$$
where

$$\begin{aligned}E' &= \frac{\pi}{2}-\delta_{12}-\delta_{13}-\delta_{14}-\delta_{23}-\delta_{24}-\delta_{34} \\ &- \frac{2}{\pi}(\delta_{12}\delta_{13}\delta_{14}+\delta_{12}\delta_{23}\delta_{24}+\delta_{13}\delta_{23}\delta_{24}+\delta_{14}\delta_{24}\delta_{34}) \\ &- \frac{2}{\pi}\left(\frac{\delta_{14}\delta_{23}\cos\delta_{14}\cos\delta_{23}+\delta_{12}\delta_{34}\cos\delta_{12}\cos\delta_{34}+\delta_{13}\delta_{24}\cos\delta_{13}\cos\delta_{24}}{\cos\delta_{12}\cos\delta_{13}\cos\delta_{14}\cos\delta_{23}\cos\delta_{24}\cos\delta_{34}}\right)\end{aligned}$$

The expressions E and E′ of (lxxxvi.) and (lxxxix.) are of considerable interest, for they enable us to express the area of a spherical triangle in three-dimensioned space, and (up to the above degree of approximation) the volume of a "tetrahedron" on a "sphere" in hyperspace of four dimensions. In fact, the whole theory of hyperspace "spherical trigonometry" needs investigation in relation to the properties of multiple correlation.

In our illustrations (viii.) and (ix.) will be found examples of the above formulæ applied to important cases in triple and quadruple correlation in the theory of heredity. I consider that the formulæ above given will cover numerous novel applications, for many of which greater simplicity will be introduced owing to the choice of special values for the h's or for the correlation coefficients.

(8.) *Illustrations of the New Methods.*

Illustration I. Inheritance of Coat-colour in Horses.—The following represents the distribution of sires and fillies in 1050 cases of thoroughbred racehorses, the grouping being made into all coat-colour classed as "bay and darker," "chesnut and lighter":—

	Colour.	Sires.		
		Bay and darker.	Chesnut and lighter.	
Fillies.	Bay and darker . . .	631	125	756
	Chesnut and lighter .	147	147	294
		778	272	1050

a	b	$a + b$
c	d	$c + d$
$a + c$	$b + d$	N

Then we require the correlation between sire and filly in the matter of coat-colour, and also the probable error of its determination.

We have from (iv.) and (v.)

$$\alpha_1 = \frac{(a + c) - (b + d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}} \int_0^h e^{-\frac{1}{2}x^2} dx = \cdot 481{,}905,$$

$$\alpha_2 = \frac{(a + b) - (c + d)}{\mathrm{N}} = \sqrt{\frac{2}{\pi}} \int_0^k e^{-\frac{1}{2}y^2} dy = \cdot 440{,}000.$$

Hence from the probability integral tables

$$h = \cdot 64630, \qquad k = \cdot 58284.$$

We have then: log HK $= 1\cdot 037{,}3514$ by (xvii.), Thence $\epsilon = \dfrac{ad - bc}{\mathrm{N}^2\mathrm{HK}} = {}^{\cdot}619{,}068$ from (xxi.).

Calculating out the coefficients of the series in r in (xix.) we find

$$\cdot 619{,}068 = r + \cdot 188{,}345r^2 + \cdot 064{,}0814r^3 + \cdot 107{,}8220r^4 + \cdot 005{,}9986r^5 + \cdot 067{,}2682r^6 + \&c.$$

Neglecting powers of r above the second, we find by solving the quadratic and taking the positive root

$$r = \cdot 5600.$$

Solving by two approximations the sextic we finally determine

$$r = \cdot 5422,$$

correct, I think, to four places of figures.

Turning now to the probable error as given by Equation (l.), I find

$$h^2 + k^2 - 2rhk = \cdot 348{,}924,$$

and from (xlix.)

$$\log \chi_0 = \bar{1}\cdot 170{,}0947.$$

Further : $\dfrac{k - rh}{\sqrt{1 - r^2}} = \cdot 275{,}642\,, \qquad \dfrac{h - rk}{\sqrt{1 - r^2}} = \cdot 393{,}078.$

Hence from (xlvii.) and (xlviii.) we find

$$\psi_1 = \frac{1}{\sqrt{2\pi}}\int_0^{\cdot 393,078} e^{-\frac{1}{2}z^2dz}\,, \qquad \psi_2 = \frac{1}{\sqrt{2\pi}}\int_0^{\cdot 275,642} e^{-\frac{1}{2}z^2dz}\,,$$

and by means of the probability integral table

$$\psi_1 = \cdot 108{,}884, \qquad \psi_2 = \cdot 152{,}865.$$

By substituting in (l.), we find

$$\text{probable error of } r = \cdot 0288.$$

From (xxxiv.) and (xxxv.) we find

$$\text{p.e. of } h = \cdot 0282. \qquad \text{p.e. of } k = \cdot 0278.$$

Thus, finally, we may sum up our results

$$h = \cdot 6463 \pm \cdot 0282, \qquad k = \cdot 5828 \pm \cdot 0278,$$

$$r = \cdot 5422 \pm \cdot 0288.$$

The probable error of this r, if we had been able to find it from the product moment, would have been ·0147, or only about one-half its present value.

Illustration II.—Our analysis opens a large field suggested by the following problem :—*What is the chance that an exceptional man is born of an exceptional father?*

Of course much depends on how we define "exceptional," and any numerical measure of it must be quite arbitrary. As an illustration, let us take a man who possesses a character only possessed by one man in twenty as exceptional. For example, only one man in twenty is more than 6 feet 1·2 inches in height, and such a stature may be considered "exceptional." In a class of twenty students we generally find one of "exceptional" ability, and so on. Accordingly we have classed fathers and sons who possess characters only possessed by one man in twenty as exceptional. We first determine h and k, so that the tail of the frequency curve cut off is $\frac{1}{20}$ of its whole area. This gives us $h = k = 1{\cdot}64485$.

Next we determine $\mathrm{HK} = \dfrac{1}{2\pi} e^{-\frac{1}{2}(h^2+k^2)}$, and find $\log \mathrm{HK} = \bar{2}{\cdot}026{,}8228$.

Then we calculate the coefficients of the various powers of r in (xix.). We find

$$\log \tfrac{1}{2} hk = {\cdot}131{,}2225.$$

$$\log \tfrac{1}{6}(h^2 - 1)(k^2 - 1) = \bar{1}{\cdot}685{,}5683.$$

$$\log \frac{hk}{24}(h^2 - 3)(k^2 - 3) = \bar{3}{\cdot}990{,}1176.$$

$$\log \tfrac{1}{120}(h^4 - 6h^2 + 3)(k^4 - 6k^2 + 3) = \bar{1}{\cdot}464{,}4772.$$

$$\log \frac{hk}{720}(h^4 - 10h^2 + 15)(k^4 - 10k^2 + 15) = \bar{2}{\cdot}925{,}6367.$$

It remains to determine what value we shall give to r, the paternal correlation. It ranges from ·3 to ·5 for my own measurements as we turn from blended to exclusive inheritance. Taking these two extreme values we find

$$\frac{ad - bc}{\mathrm{N}^2} = {\cdot}0046344 \quad \text{or} \quad {\cdot}0096779.$$

But $\dfrac{ad-bc}{\mathrm{N}^2} = \dfrac{d}{\mathrm{N}} - \dfrac{(d+b)(d+c)}{\mathrm{N}^2}$, and the second term is the chance of exceptional fathers with exceptional sons, when variation is independent, *i.e.*, when there is no heredity, $= \frac{1}{20} \times \frac{1}{20} = {\cdot}0025$.

Thus $\qquad d/\mathrm{N} = {\cdot}007134 \quad \text{or} \quad {\cdot}012178\,;$

accordingly $\qquad b/\mathrm{N} = {\cdot}042866 \quad \text{or} \quad {\cdot}037822.$

Hence we conclude that of the 5 per cent. of exceptional men ·71 per cent. in the first case, and 1·22 per cent. in the second case, are born of exceptional fathers, and 4·29 per cent. in the first case and 3·78 per cent. in the second case of non-exceptional fathers. In other words, out of 1000 men of mark we may expect 142 in the first case, 244 in the second, to be born of exceptional parents, while 858 in the first and 756 in the second are born of undistinguished fathers. In the former case the odds are about 6 to 1, in the latter 3 to 1 against a distinguished son having a distinguished father. This result confirms what I have elsewhere stated, that we trust to the great mass of our population for the bulk of our distinguished men. On the other hand it does not invalidate what I have written on the importance of creating good stock, for a good stock means a bias largely above that due to an exceptional father alone.

In addition to this the $\frac{1}{20}$ of the population forming the exceptional fathers produce 142 or 244 exceptional sons to compare with the 858 or 756 exceptional sons produced by the $\frac{19}{20}$ of the population who are non-exceptional. That is to say that the *relative* production is as 142 to 45·2, or 244 to 39·8, *i.e.*, in the one case as more than 3 to 1, in the other case as more than 6 to 1. *In other words, exceptional fathers produce exceptional sons at a rate* 3 *to* 6 *times as great as non-exceptional fathers.* It is only because exceptional fathers are themselves so rare that we must trust for the bulk of our distinguished men to the non-exceptional class.

Illustration III. Heredity in Coat-colour of Hounds.—To find the correlation in coat-colour between Basset hounds which are half-brethren, say, offspring of the same dam.

Here the classification is simply into lemon and white (*lw*) and lemon, black and white or tricolour (*t*),

The following is the table for 4172 cases :—

Colour.	*t.*	*lw.*	Totals.
t.	1766	842	2608
lw.	842	722	1564
Totals	2608	1564	4172

Proceeding precisely in the same way as in the first illustration we find :

$$\alpha_1 = \alpha_2 = \cdot 25024$$
$$h = k = \cdot 318,957$$
$$\log \mathrm{KH} = \bar{1}\cdot 157,6378$$
$$\epsilon = \cdot 226,234.$$

It will be sufficient now to go to r^4. We have

$$\cdot 226,234 = r + \cdot 050,867\, r^2 + \cdot 134,480\, r^3 + \cdot 035,587\, r^4.$$

The quadratic gives $r = {\cdot}2237$. Using the NEWTONIAN method of approximating to the root we find

$$r = {\cdot}2222.$$

Summing up as before, after finding the probable errors, we have

$$h = k = {\cdot}3190 \pm {\cdot}0133,$$
$$r = {\cdot}2222 \pm {\cdot}0162.$$

Illustration IV. Inheritance of Eye-colour in Man.—To find the correlation in eye-colour between a maternal grandmother and her granddaughter. Here the classification is into eyes described as grey or lighter, and eyes described as dark grey or darker.*

	Tint.	Maternal grandmother.		Totals.
		Grey or lighter.	Dark grey or darker.	
Granddaughter.	Grey or lighter	254	136	390
	Dark grey or darker . .	156	193	349
	Totals	410	329	739

As before, we find

$$\alpha_1 = {\cdot}109{,}607, \qquad \alpha_2 = {\cdot}055{,}480,$$
$$h = {\cdot}138{,}105, \qquad k = {\cdot}069{,}593,$$
$$\log \mathrm{HK} = \bar{1}{\cdot}196{,}6267,$$
$$\epsilon = {\cdot}323{,}760.$$

Series for r up to r^4

$${\cdot}323{,}760 = r + {\cdot}004{,}806r^2 + {\cdot}162{\cdot}696r^3 + {\cdot}000{,}358r^4.$$

The quadratic gives $r = {\cdot}3233$, and the biquadratic

$$r = {\cdot}3180,$$

the value of the term in r^4 being ·000,00366, so that higher terms may be neglected.

Determining the probable errors as in Illustration I., we sum up :—

---

* According to Mr. GALTON'S classification, the first group contains eyes described as light blue, blue, dark blue, blue-green, grey; and the second eyes described as dark grey, hazel, light brown, brown, dark brown, verv dark brown, black.

$$h = \cdot 1381 \pm \cdot 0312,$$
$$k = \cdot 0696 \pm \cdot 0311,$$
$$r = \cdot 3180 \pm \cdot 0361.$$

Illustration V. Inheritance of Stature.—The following data have been found for the inheritance of stature between father and son from my Family Data cards, 1078 cases :—

Mean stature of father	67″·698
,, son	68″·661
Standard deviation of father . .	2″·7048
,, son . . .	2″·7321

Correlation $= \cdot 5198 \pm \cdot 0150$.

Now for purposes of comparison of methods the correlation has been determined for this material from various groupings of fathers and sons :—

(A.) *Fathers.*

Sons. Class.	Below 67″·5.	Above 67″·5.	Totals.
Below 67″·5 . .	269·25	95·75	365
Above 67″·5 . .	232·25	480·75	713
Totals . . .	501·5	576·5	1078

(B.) *Fathers.*

Sons. Class.	Below 66″·5.	Above 66″·5.	Totals.
Below 67″·5 . .	211·25	153·75	365
Above 67″·5 . .	152·75	560·25	713
Totals . . .	364	714	1078

(C.)

Fathers.

	Class.	Below 67″·5.	Above 67″·5.	Totals.
Sons.	Below 68·5″ . .	356·25	182·25	538·5
	Above 68·5″ . .	145·25	394·25	539·5
	Totals . . .	501·5	576·5	1078

(D.)

Fathers.

	Class.	Below 68″·5.	Above 68″·5.	Totals.
Sons.	Below 69″·5 . .	506	182	688
	Above 69″·5 . .	149·5	240·5	390
	Totals . . .	655·5	422·5	1078

(E.)

Fathers.

	Class.	Below 69″·5.	Above 69″·5.	Totals.
Sons.	Below 70″·5 . .	669	147	816
	Above 70″·5 . .	128	134	262
	Totals . . .	797	281	1078

(F.)

Fathers.

	Class.	Below 70″·5.	Above 70″·5.	Totals.
Sons.	Below 69″·5 . .	641·25	46·75	688
	Above 69″·5 . .	271·75	118·25	390
	Totals . . .	913	165	1078

TABLE of Results.

Classificatiom.	Correlation.	Mean of sons.	Mean of fathers.
		k.	*h*.
A	·5939 ± ·0247	68″:64 (– ·416,32)	67″·74 (– ·087,00)
B	·5557 ± ·0261	68″·64 (– ·416,32)	67″·63 (– ·418,86)
C	·5529 ± ·0247	68″·50 (– ·001,16)	67″·74 (– ·087,30)
D	·5264 ± ·0264	68″·53 (·353,71)	67″·77 (·274,30)
E	·5213 ± ·0294	68″·60 (·696,57)	67″·76 (·641,30)
F	·5524 ± ·0307	68″·53 (·353,71)	67″·73 (1·023,44)

Now these results are of quite peculiar interest. They show us :—

(i.) That the probable error of r, as found by the present method, increases with h and k. But the increase is not very rapid, so that the probable errors of the series range only between ·025 and ·031. Hence while it is an advantage, it is not a very great advantage, to take the divisions of the groups near the medians. It is an advantage which may be easily counterbalanced by some practical gain in the method of observation when the division is not close to the medians.

(ii.) While the probable error, as found from the present method of calculation, is 1·5 to 2 times the probable error as found from the product moment, it is by no means so large as to seriously weigh against the new process, if the old is unavailable. It is quite true that the results given by the present process for six arbitrary divisions differ very considerably among themselves. But a consideration of the probable errors shows that the differences are sensibly larger than the probable error of the differences, even in some case double ; hence it is not the method but the assumption of normal correlation for such distributions which is at fault. As we shall hardly get a better variable than stature to hypothesise normality for, we see the weakness of the position which assumes without qualification the generality of the GAUSSIAN law of frequency.

(iii.) We cannot assert that the smaller the probable error the more nearly will the correlation, as given by the present process, agree with its value as found by the product moment. If we did we should discard ·5213, a very accordant result, in favour of ·5529, or even ·5939. The fact is that the higher the correlation the lower, *ceteris paribus*, the probable error, and this fact may obscure the really best result. Judging by the smallness of h and k and of the probable error, we should be inclined to select C or the value ·5529. This only differs from ·5198 by slightly more than the probable error of the difference (·033 as compared with ·029) ; but since both are found from the *same* statistics, and not from different samplings ot the same population, this forms sufficient evidence in itself of want of normality. The approximate character of all results based on the theory of normal frequency must be carefully borne in mind ; and all we ought to conclude from the present data for inheritance of stature from father to son would be that the correlation $= \cdot 55 \pm \cdot 015$, while the product moment method would tell us more definitely that its value was $\cdot 52 \pm \cdot 015$. There is no question that the latter method is the better, but this does not hinder the new method from being extremely serviceable; for many cases it is the only one available.

Illustration VI. Effectiveness of Vaccination.—To find the correlation between strength to resist small-pox and the degree of effective vaccination.

We have in the earlier illustrations chosen cases in which in all probability a scale of character might possibly, if with difficulty, be determined. In the present case, the relationship is a very important one, but a quantitative scale is hardly discoverable. Nevertheless, it is of great interest to consider what results flow from the application of our method. We may consider our two characters as strength to resist the ravages of small-pox and as degree of effective vaccination. No quantitative scales are here available; all the statistics provide are the number of recoveries and deaths from small-pox, and the absence or presence of a definite vaccination cicatrix. Taking the Metropolitan Asylums Board statistics for the epidemic of 1893, we have the table given below, where the cases of "no evidence" have been omitted. Proceeding in the usual manner we find

$$\alpha_1 = \cdot 86929 \qquad \alpha_2 = \cdot 54157$$
$$h = 1\cdot 51139 \qquad k = \cdot 74145$$
$$\epsilon = \cdot 782454.$$

Hence the equation for r is

$$\cdot 782,454 = r + \cdot 560,310 r^2 - \cdot 096,378 r^3 + \cdot 081,881 r^4 - \cdot 000,172 r^5 - \cdot 040,059 r^6$$

whence $r = \cdot 5954$.

Summing up we have, after calculating the probable errors,

$$h = 1\cdot 5114 \pm \cdot 0287,$$
$$k = \cdot 7414 \pm \cdot 0205,$$
$$r = \cdot 5954 \pm \cdot 0272.$$

Strength to resist Small-pox when incurred.

	Cicatrix.	Recoveries.	Deaths.	Total.
Degree of effective Vaccination.	Present	1562	42	1604
	Absent	383	94	477
	Total	1945	136	2081

We see accordingly that there is quite a large correlation between recovery and the presence of the cicatrix. The two things are about as closely related as a child to its "mid-parent." While the correlation is very substantial and indicates the protective character of vaccination, even after small-pox is incurred, it is, perhaps, smaller than some over-ardent supporters of vaccination would have led us to believe.

Illustration VII. Effectiveness of Antitoxin Treatment.—To measure quantitatively the effect of antitoxin in diphtheria cases.

In like manner we may find the correlation between recovery and the administration of antitoxin in diphtheria cases. The statistics here are, however, somewhat difficult to obtain in a form suited to our purpose. The treatment by antitoxin began in the Metropolitan Asylums Board hospitals in 1895, but the serum was then administered only in those cases which gave rise to anxiety. Hence we cannot correlate recovery and death with the cases treated or not treated in that year, for those who were likely to recover were not dosed. In the year 1896 the majority of the cases were, on the contrary, treated with antitoxin, and those not treated were the slight cases of very small risk ; hence, again, we are in great difficulties in drawing up a table.* Further, if we compare an antitoxin year with a non-antitoxin year, we ought to compare the cases treated with antitoxin in the former year with those which would probably have been treated with it in the latter year. Lastly, the dosage, nature of cases treated, and time of treatment have been modified by the experience gained, so that it seems impossible to club a number of years together, and so obtain a satisfactorily wide range of statistics. In 1897, practically all the laryngeal cases were treated with antitoxin. Hence the best we can do is to compare the laryngeal cases in two years, one before and one after the introduction of antitoxin. The numbers available are thus rather few, but will help us to form some idea of the correlation. I take the following data from p. 8 of the Metropolitan Asylums Board 'Report upon the Use of Antitoxic Serum for 1896' :—

Laryngeal cases.	Recoveries.	Deaths.	Totals.
With antitoxin, 1896	319	143	462
Without antitoxin, 1894 . . .	177	289	466
Totals	496	432	928

---

* When a new drug or process is introduced the medical profession are naturally anxious to give every patient the possible benefit of it, and patients of course rush to those who first adopt it. But if the real efficiency of the process or drug is to be measured this is very undesirable. No definite data by which to measure the effectiveness of the novelty are thus available.

Here I find $r = \cdot 4708 \pm \cdot 0292$.

A further table is of interest :—

Laryngeal cases.	Requiring tracheotomy.	Not requiring it.	Totals.
Without antitoxin, 1894 . . .	261	205	466
With antitoxin, 1896	188	274	462
Totals	449	479	928

In this case we have $r = \cdot 2385 \pm \cdot 0335$.

Lastly, I have drawn up a third table :—

Total Infantile Cases, Ages 0—5 years.

	Recovery.	Death.	Totals.
With antitoxin, 1896 . . .	912	434	1346
Without antitoxin, 1894 . .	615	556	1171
Totals	1527	990	2517

Here we have* $r = \cdot 2451 \pm \cdot 0205$.

The three coefficients are all sensible as compared with their probable errors, and that between the administration of antitoxin and recovery in laryngeal cases is substantial. But the relationship is by no means so great as in the case of vaccination, and if its magnitude justifies the use of antitoxin, even when balanced against other ills which may follow in its train, it does not justify the sweeping statements of its effectiveness which I have heard made by medical friends. It seems until wider statistics are forthcoming a case for cautiously feeling the way forward rather than for hasty generalisations.

Illustration VIII. Effect on Produce of Superior Stock.—To find the effect of superiority of stock on percentage goodness of produce.

To illustrate this and also the formula (lxxxiii.) for six correlation coefficients, we will investigate the effect of selecting sire, dam, and one grandsire on the produce when there

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* The values of r for all the three cases of this Illustration were determined with great ease from Equation (xxiv.).

is selective pairing of dam and sire. We will suppose grandsire, dam, and sire to be above the average, and investigate what proportion of the produce will be above the average. As numbers very like those actually occurring in the case of dogs, horses, and even men, we may take

Correlation of grandsire and offspring . = ·25
,, sire or dam and offspring = ·5 in both cases
,, sire and grandsire . . . = ·5
Selective mating for sire and dam. . . = ·2

We will suppose zero correlation between paternal grandsire and dam, although with selective mating this may actually exist.* We have then the following system :—

$$r_{14} = \cdot 25, \quad r_{24} = \cdot 5, \quad r_{34} = \cdot 5, \quad r_{23} = \cdot 2, \quad r_{12} = \cdot 5, \quad r_{13} = 0.$$

Hence, substituting these values in (lxxxvii.), we find—after some arithmetic :

$$(Q - Q_0)/Q_0 = 1\cdot 4851.$$

But Q_0 is the chance of produce above the average if there were no heredity between grandsire, sire, and dam, and no assortative mating.

Hence it equals $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\text{N} = \frac{\text{N}}{16}$ $\therefore$ $Q = \cdot 1553\,\text{N}$.

Or, of the produce ·5 N above the average, ·1553 N instead of ·0625 N are born of the superior stock owing to inheritance, &c. In other words, out of the ·5 N above the average, ·1553 N are produced by the stock in sire, dam, and grandsire above the average, or by ·1827 of the total stock.† The remaining ·8173 only produce ·3447 N, or the superior stock produces produce above the average at over twice the rate of the inferior stock. Absolutely, the inferior stock being seven times as numerous produces about seven-tenths of the superior offspring.

Illustration IX. Effect of Exceptional Parentage.—Chance of an exceptional man being born of exceptional parents.

Let us enlarge the example in Illustration II., and seek the proportion of exceptional men, defined as one in twenty, born of exceptional parents in a community with assortative mating.

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[*] A correlation, if there be substantial selective mating, may exist between a man and his mother-in-law. Its rumoured absence, if established scientifically, would not, however, prove the non-existence of selective mating, for A may be correlated with B and C, but these not correlated with each other.

[†] The proportion of pairs of parents associated with a grandsire above the average was found by putting ·5, ·2, and 0 for the three correlation coefficients in (lxxxv.). In comparing with Illustration II., the reader must remember we there dealt with an exceptional father, 1 in 20, here only with relatives above the average—a very less stringent selection.

Here we take for father and son $r_{12} = \cdot 5$, for mother and son $r_{13} = \cdot 5$, and for assortative mating, $r_{23} = \cdot 2$.

We have then to apply the general formulæ (lxxxiii.) and (lxxxiv.) for the case of three variables. We have

$$\begin{aligned} h_1 &= h_2 = h_3 = 1\cdot 64485 \\ \beta_1 &= \beta_2 = \beta_3 = \cdot 484{,}795 \\ v_1' &= v_1'' = v_1''' = 1\cdot 644{,}850 \\ v_2' &= v_2'' = v_2''' = 1\cdot 705{,}532 \\ v_3' &= v_3'' = v_3''' = -\ \cdot 484{,}356 \\ v_4' &= v_4'' = v_4''' = -\ 5\cdot 913{,}290 \end{aligned}$$

Whence, after some arithmetical reduction, we find

$$(Q - Q_0)/Q_0 = 20\cdot 0389.$$

But $Q_0 = \frac{1}{20} \times \frac{1}{20} \times \frac{1}{20} N = \frac{1}{8000} N$. Hence $Q = \cdot 00263\ N$.

We must now distinguish between the absolute and relative production of exceptional men by exceptional and non-exceptional parents. The exceptional pairs of parents are obtained by (xix.), whence we deduce, putting $r = \cdot 2$, $h = k = 1\cdot 64485$,

$$\frac{ad - bc}{N^2} = \frac{d}{N} - \frac{(d + b)(d + c)}{N^2} = \frac{d}{N} - \frac{1}{400} = \cdot 002745.$$

Whence the number of pairs of parents, both exceptional

$$= \cdot 005245\,N.$$

Thus, ·005245 N pairs of exceptional parents produce ·00263 N exceptional sons, and ·994755 N pairs of parents, non-exceptional in character, produce ·04737 N exceptional sons, *i.e.*, the remainder of the $\frac{1}{20}$ N. The rates of production are thus as ·5014 to ·0476. Or: *Pairs of exceptional parents produce exceptional sons at a rate more than ten times as great as pairs of non-exceptional parents.* At the same time, eighteen times as many exceptional sons are born to non-exceptional as to exceptional parents, for the latter form only about $\frac{1}{2}$ per cent. of the community.

The reader who will carefully investigate Illustrations II., VIII., and IX. will grasp fully why so many famous men are born of undistinguished parents, but will, at the same time, realise the overwhelming advantage of coming of a good stock.

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II. *Electrical Conductivity in Gases Traversed by Cathode Rays.*

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Communicated by Professor J. J. Thomson, *F.R.S.*

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Though it has been known that a gas becomes a conductor when traversed by cathode rays, yet the laws connecting this electrical conductivity have not hitherto been studied.

The theory has been put forward by J. J. Thomson and Rutherford[*] that when a gas becomes a conductor under a radiation, it does so in virtue of the production of positive and negative ions throughout its mass. This view has been established by their experiments on Röntgenised gases, and confirmed by those of Zeleny[†] on the same subject. The recent work of Rutherford on Uranium Radiation[‡] also affords another example of such a process in the gases traversed.

The object of the experiments which are described in this paper was to investigate the nature of the conductivity in different gases when cathode rays of definite strength passed through them, and to measure the number of ions produced. With this in view, I have worked with cathode rays produced, after the method of Lenard, outside the discharge tube, as these were found to be more easily dealt with than those inside.

The investigation is described under the following subdivisions :—

1. Form of tube adopted for the production of cathode rays.
2. Ionization by cathode rays.
3. Discharging action of cathode rays.
4. Ionization not due to Röntgen rays.
5. Discussion of methods for measuring the ionizations produced in different gases.
6. Description of apparatus used.
7. Explanation of the method adopted for comparing ionizations.
8. Ionization in different gases at the same pressure.
9. Ionization in air at different pressures.

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[*] 'Phil. Mag.,' November, 1896, p. 393.

[†] 'Phil. Mag.,' July, 1898, p. 120.

[‡] 'Phil. Mag.,' January, 1899, p. 109.

10. Ionization in a gas independent of its chemical composition.
11. Comparison of ionizations produced by cathode and by Röntgen rays.
12. Summary of results.

1. *Form of Tube adopted for the production of Cathode Rays.*

To produce the rays, a modified form of the tube devised by LENARD,[*] fig. 1, was used. The disc *a* which closed the end and carried the aluminium window formed the anode. To hold this disc in position, and to render the joint airtight, recourse was had to sealing-wax, which was allowed to set on the previously warmed glass and metal, after which the parts were made to unite by slightly melting the surfaces and pressing them together. By running round the joint with the pointed flame of a blowpipe, any air bubbles present were removed, and complete union was effected. Joints made in this way were found to hold for any time desired.

In making the aluminium window airtight, marine glue could be used, but the ordinary commercial soft wax was found to be more suitable. This was especially so when the experiments were in the tentative state and alterations were frequently necessary. The wax melted at a lower temperature than the glue, and besides being much more manageable than the latter, it was also less disagreeable to handle. A coating of it on the sealing wax also prevented cracking.

As shown in the figure, the anode was provided with a shoulder round the opening of the window. This was found very convenient when the action of the rays on the air in a partially exhausted receiver such as A was being examined. The receiver was provided with a similar but larger shoulder, and by slipping it over that on the

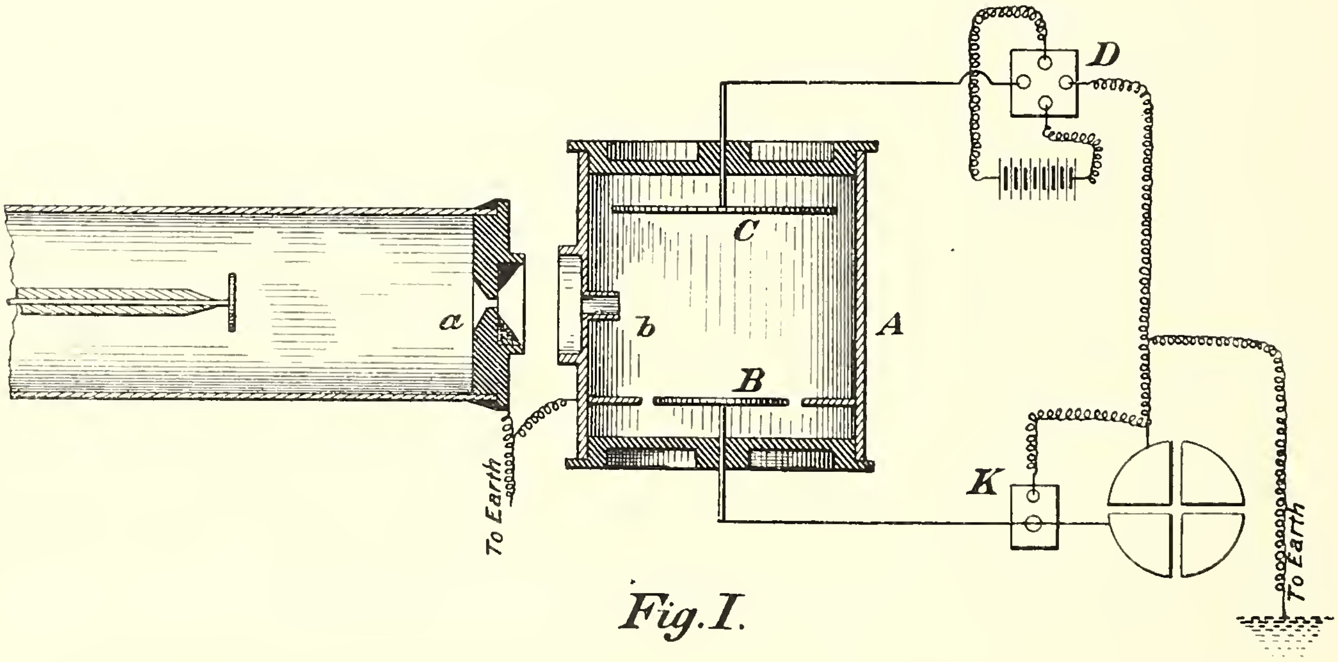

Fig. I.

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[*] 'Wied. Ann.,' vol. 51, p. 225 (1894).

anode and applying a coating of wax, an airtight connection could be readily made without interfering with that which secured the aluminium foil to the disc. This latter connection was effected by placing a thin coating of wax upon the brass disc and gently applying heat after the foil was laid upon it. All the space within the projecting shoulder was then covered with a thick coating of the wax, excepting the central portion of the aluminium.

In all the experiments with these tubes the anode was well earthed, as was also the positive terminal of the induction coil used to produce the discharge.

As regards the distance between the cathode and the anode, it was found best not to make it too small. Otherwise, the discharge would pass in the tube before the available maximum potential difference was reached. The velocity of the carriers has been shown by J. J. THOMSON[*] to vary with the potential difference between the electrodes, and as a consequence an intense radiation was more readily obtained when the distance between the anode and cathode was considerable.

In the case of tubes constructed with a short distance between the electrodes, the device adopted by McCLELLAND[†] of inserting an air gap in series with the tube very largely increased the intensity of the radiation.

The foil used by LENARD for the aluminium window was ·003 millim. in thickness. In practice it was exceedingly difficult to obtain such foil free from holes. Aluminium about three times as thick was, however, much better in this regard. The induction coil used in the experiments was, besides, very powerful, and, as a radiation sufficiently intense could be obtained with it, this thickness was used throughout the investigation.

2. *Ionization by Cathode Rays.*

It has been shown by LENARD[‡] that air, when traversed by cathode rays, acquires the property of discharging electrified conductors against which it may be blown, and that, further, it retains this property for some time after the rays producing it have been cut off.

According to the theory of Professor THOMSON, the air, when in this state, is ionized, and the discharging action is brought about by a motion of the ions in the gas to the charged conductor. Owing to the separation of the positive and negative ions, recombination can take place but gradually, and this readily explains why the discharging power is retained by the air for some time. In order to show that these positive and negative ions are produced in a gas traversed by the rays, the apparatus shown in fig. 1 was used.

The cathode rays issuing from the aluminium window *a* passed through a narrow tube, *b*, into an earth-connected metal chamber, A. B was a disc of brass supported

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[*] 'Phil. Mag.,' October, 1897, p. 315.

[†] 'Proc. Roy. Soc.,' vol. 61, No. 373, p. 227.

[‡] 'Wied. Ann.,' vol. 63, p. 253 (1897).

by an ebonite plug, and surrounded by a guard ring. A wire led from this electrode to one pair of quadrants of an electrometer, and the other pair was put to earth. Care was taken to screen off electrostatic induction by surrounding the wire and electrometer with earth-connected conductors. The second electrode, C, also supported by an ebonite plug, was connected by a commutator, D, to one of the terminals of a battery of small storage cells, the other terminal being connected to earth.

The tube, *b*, was made narrow, and penetrated a short distance into the chamber in order to confine the rays to a slender pencil, and to prevent their impinging upon the electrodes. By means of the key, K, the electrode, B, could be put to earth when necessary.

With such an apparatus, and no field initially between the electrodes, it was found on exciting the discharge tube and breaking the earth connection, K, that the electrometer gained a small negative charge, which did not go on increasing, but soon attained a limiting value.

On the assumption that the cathode rays produce positive and negative ions throughout the gas, the explanation of this is obvious. The cathode rays carried a negative charge into the gas, and set up a field which caused the negative ions to move to the walls of the chamber and to the electrode, B. The charge which the latter soon gained, however, set up a field of its own, and a state of equilibrium was reached when the conduction to the electrode was just equal to that proceeding from it. If, instead of there being no field initially between the electrodes, C was joined to the positive terminal of the battery, then the electrode, B, gained a positive charge when the tube was excited, and the rate at which its potential rose depended upon the capacity joined to B and the electrometer.

With C joined to the negative terminal of the battery, a similar charging took place, except that in this case the charge accumulated was a negative one.

This reversal in the sign of the charge collected may be shown with a field of a few volts a centimetre, and clearly points to the existence of positive and negative ions in the gas. Since the cathode rays themselves carry a negative charge, the presence of these carriers alone in the chamber would account for the negative charge obtained with a negative field. With a positive field, however, these carriers would be attracted to the electrode C, and it seems impossible to explain how the electrode B, under these circumstances, could receive a positive charge unless ions were produced by the rays.

3. *Discharging Action of Cathode Rays.*

In connection with the experiments of LENARD,[*] already referred to, cathode rays were allowed to fall upon a charged conductor surrounded with air at atmospheric pressure. This conductor consisted of a wire attached to a gold-leaf electroscope,

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[*] 'Wied. Ann.,' vol. 63, p. 253.

and was placed within a zinc box in which was a small opening covered with a film of aluminium, thin enough to allow the rays to pass through. The end of this wire was placed in front of the window and close to it, with the electroscope clear of the direct path of the rays. The box itself was connected to earth and set in position, with its window opposite that of the discharge tube.

Using this apparatus, LENARD found that positive and negative charges alike were completely dissipated by a single discharge through the tube when the aluminium windows were at any distance up to 4 centims. apart. At greater distances than this a similar but only partial discharging of both kinds of electricity occurred when the same amount of rays was used.

This loss of charge was no doubt brought about by means of the ionization in the air surrounding the conductor. The known behaviour of an ionized gas, however, would have led one to expect a somewhat different result, especially in regard to the effect obtained with short distances between the windows. When an insulated metal conductor is placed in air ionized by Röntgen rays, ZELENY* has shown that, owing to the greater velocity with which the negative ions diffuse, this conductor takes up a small negative charge, while the gas itself is left with a positive one. If then the ionizations in the two cases are of the same nature, one would have expected that in LENARD's experiments the wire and electroscope would not, under any circumstances, have been finally discharged completely, but would have been left with at least a small negative charge. When, further, it is remembered that the impinging cathode rays themselves carried a negative charge to the wire, this fact affords an additional reason for expecting such a result.

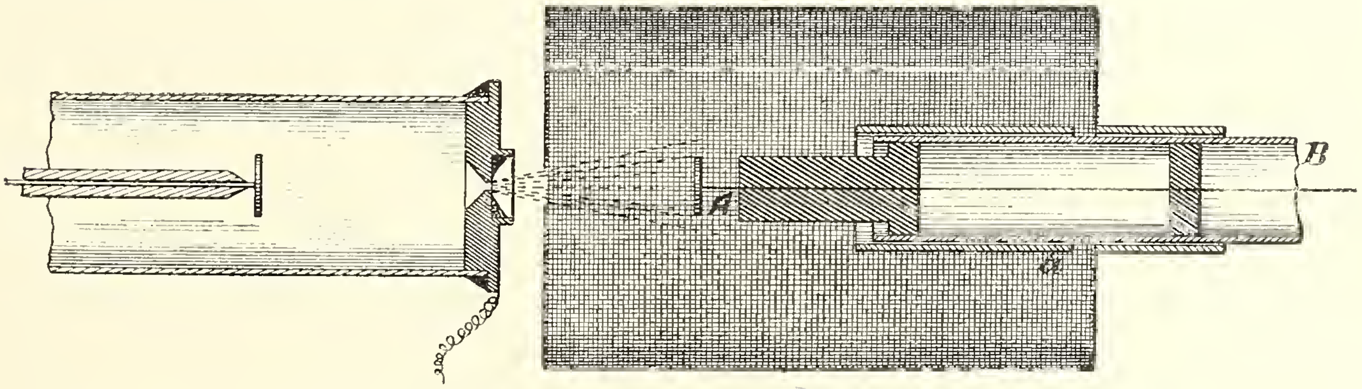

Fig. II.

Now the gold-leaf electroscope, as used by LENARD (EXNER's type), was not sensitive to small differences of potential, and it was consequently not a suitable instrument for the detection and measurement of effects of this kind. As the explanation of his results seemed, then, to be connected with this lack of sensibility in the measuring instrument, his experiments were repeated, and a quadrant electrometer was used in place of the electroscope.

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* 'Phil. Mag.,' July, 1898, p. 134.

The arrangement was that shown in fig. 2. A copper wire, terminated by a disc, A, of the same metal, was insulated by ebonite from an earth-connected copper tube B, through which it passed to the electrometer. To this tube there was fastened, as shown in the figure, a large, finely-meshed copper gauze which completely protected the disc from electrostatic induction. The tube B also carried a short concentric cylinder *a*, made of copper, which could be slid out when desired so as to surround the projecting end of the wire and the disc.

On placing this apparatus in front of the aluminium window so that the cathode rays fell on the disc, it was found that, although the rays caused a discharging of positive and of negative electricity, still in no case observed was a negative charge on the disc and wire ever completely dissipated.

Negative charges fell, however, to limiting values, represented in some cases by potentials of the order of ·25 volt, and then remained stationary. In the case of initial positive charges the discharging was not only complete but the disc also gained this limiting negative charge. A similar charging action was observed when there was no initial charge on the disc.

Here the disc was subjected to two influences, namely, the cathode rays carrying a negative charge to it and the ionized gas about it acting as a conductor and tending to discharge it. This limiting charge can, then, just as in the case already cited, be looked upon as representing a state of equilibrium in which the convection to the disc was just equal to the conduction away from it.

As the electric field produced by a given charge on the disc would vary with the distance between it and neighbouring conductors at a different potential, the conduction from the wire could consequently be increased or decreased according as an earth-connected conductor was brought close to the disc or removed farther from it. If then a means were devised of altering in this way the conduction without altering the intensity of the rays impinging on the disc, the value of this limiting charge could be subjected to definite variations.

The sliding cylinder *a* afforded a simple means of accomplishing this result. If when the tube was excited a stationary state was reached, with this cylinder shoved well back, and it was then brought forward over the wire and disc, the limiting negative charge at once dropped and assumed a steady but smaller value. In order to restore the charge to its original value it sufficed merely to slide the cylinder back to its former position.

Another simple verification of this view was afforded by the use of a blast of air. If when the rays were impinging on the disc a blast of air was directed towards it and at right angles to the rays, the limiting charge at once increased to another limiting value, and when the blast stopped it again dropped to its original amount.

As the velocity of the cathode rays has been estimated by J. J. THOMSON* to be of the order of 10^{10} centims. per second, it is clear that any ordinary blast could produce

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* 'Phil. Mag.,' October, 1897, p. 315.

very little effect on the motion of these carriers. On the other hand, the velocity of the ions in Röntgenised air has been found by Rutherford* to be about 1·6 centims. per second, under a field of a volt a centimetre, and consequently of the order of that of the blast. In the experiment described, the effect of the blast, therefore, was to decrease the conduction away from the electrode by removing the ionized gas; and as no change was made in the intensity of the rays impinging on the disc, this consequently produced an increase in the residual charge. This increase, however, did not go on indefinitely, but ceased when the field it set up was sufficient to neutralise the effect of the blast; hence the second stationary value for the charge.

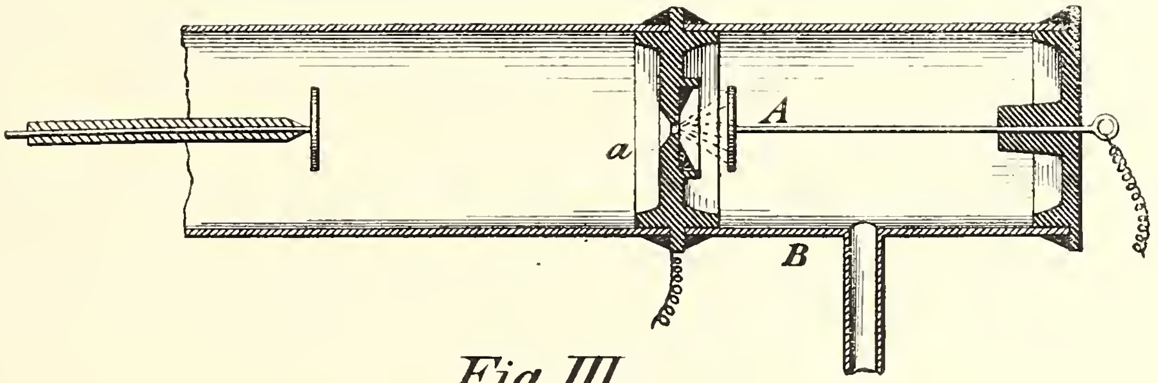

Fig. III.

Another means of increasing this limiting charge was afforded by the removal of the air surrounding the electrode. To show this the gauze cap was removed from the apparatus in fig. 2, and the metal tube surrounding the wire was brought forward and sealed to the anode of the discharge tube. The arrangement is shown in fig. 3.

With this apparatus it was found that, as the exhaustion proceeded in the chamber B, the negative charge received by the electrode A gradually increased, until finally, at a very high vacuum, a momentary discharge of the rays was sufficient to raise its potential beyond the range of the electrometer. This result, therefore, confirms the explanation already given of the discharging action of the rays. In a recent paper by Lenard† this charging action of the cathode rays in a high vacuum was described, but its connection with the ionized air surrounding the electrode was not brought out. From the experiments just described it is clear that, while this action is directly due to the fact that the cathode rays carry a negative charge, the extent of the effect obtained in all cases depends to a very great degree upon the opposing influence exerted by the ionized air surrounding the electrode upon which the rays fall.

4. *Ionization not due to Röntgen Rays.*

It has been thought by some that the ionization produced by cathode rays was due to Röntgen rays, which might possibly be sent out from the window at the same time. The results of experiment are, however, entirely opposed to this view.

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[*] 'Phil. Mag.,' November, 1897, p. 436.

[†] 'Wied. Ann.,' vol. 63, p. 253.

In order to investigate the point an apparatus similar to that shown in fig. 3 was adopted. Different thicknesses of aluminium foil were in turn used for the window, and the air in the chamber B was kept at a pressure low enough to absorb but little of any radiation coming from the window, and yet sufficiently high to afford considerable conductivity when ionized.

With different thicknesses of the foil down to ·04 millim., it was found that the electrode A did not gain any charge when the tube was excited. Further, if in these cases a charge, either positive or negative, was given independently to the electrode, this charge was maintained when the discharge passed in the tube, and no leak occurred. But when the window was made of foil ·008 millim. in thickness, the effect obtained was such as that already described in the last paragraph. Under these conditions the electrode A, if carrying initially a positive or a negative charge, finally assumed a stationary state, in which it carried a definite negative charge whose value, as has already been pointed out, depended upon the pressure of the air in the chamber B. As, then, no leak from the electrode occurred when the aluminium was ·04 millim. in thickness, it seems justifiable to conclude that if any Röntgen rays were present under these circumstances they were of an extremely weak character. If Röntgen rays of even very moderate intensity had entered the chamber, a leak would have taken place which could have been observed. In practice the aluminium foil used in my experiments was about ·008 millim. in thickness, and with this foil intense ionization was observed. From the known character of Röntgen rays, it was quite impossible for this great ionization to be produced by rays which could be absorbed by a layer of aluminium ·032 millim.—the difference in thickness of the two windows.

Again, an ordinary focus tube illustrates very well the fact that the Röntgen rays produced issue in a large measure from the face of the anticathode, upon which the cathode rays fall, while the radiation appearing to come from the opposite face is always very weak. The theory now generally accepted is that the Röntgen rays are electromagnetic pulses sent through the ether when the moving electrified particles which constitute the cathode rays are suddenly stopped. If then the Röntgen radiation sent out in the direction of propagation of the cathode rays, when these carriers were stopped by foil ·04 millim. in thickness, was at most but very feeble, it appears highly improbable that a strong radiation of this kind could be produced by those carriers that passed through the thinner foil without being stopped.

The conductivity produced in a gas by cathode rays is, moreover, far in excess of that excited by even the strongest Röntgen rays. In order to make a direct comparison, measurements were taken of the ionizations produced in the same chamber by both radiations, and the following illustration gives an indication of their respective efficiencies. By using the apparatus shown in fig. 1, it was found that, under the action of cathode rays with a saturating intensity of field, a capacity of

750 electrostatic units attached to the electrode B gained in 15 seconds a charge represented by 300 divisions on an arbitrary scale. A Röntgen ray focus tube giving out very strong rays was then used in place of that for producing the cathode rays, and was excited by an induction coil capable of giving a 50-centims. spark. Under these circumstances, with the same field, which was also in this case a saturating one, a capacity of 150 electrostatic units was charged in one minute to an amount represented by 20 on the same scale. This case, which is an extreme one, shows that the ionization by cathode rays was about 300 times that due to an intense Röntgen radiation. In the present investigation these latter rays, even if they did accompany the cathode rays, must have been very feeble, and could therefore only exert an ionizing influence which may be left out of consideration.

The known action of a magnetic field naturally suggested itself as a means of sifting out the cathode from any accompanying Röntgen rays. The intensity of the cathode rays, however, soon falls off owing to their rapid absorption by the air, and on this account it was necessary to place the chamber in which the ionization was measured close up to the discharge tube. Under these conditions it was found impossible to deflect the rays outside the tube without also deflecting those inside. This difficulty consequently rendered the test indecisive, and the method had to be abandoned.

5. *Discussion of Methods for Measuring the Ionizations produced in Different Gases.*

In the construction of Röntgen-ray bulbs, the disengagement of gas from the electrodes and the inside of the glass is facilitated by the application of heat to the tube. In the case of Lenard tubes, however, the joints are made of wax, and the final stage of exhaustion cannot be hastened by adopting this device. In practice a tube was kept attached to the mercury pump, and exhausted while the discharge was passing through it. After some hours of this procedure the coil was stopped, and the exhaustion was continued until only some traces of air were being taken over. On then exciting the tube, the vacuum was found to be sufficiently high for the cathode rays produced to penetrate the aluminium window. After running the coil for a short time, a small quantity of gas accumulated in the tube, and the pressure rose so high that the rays ceased to be propagated outside. After this air had been removed the vacuum again became good, and the original intensity of the rays was restored. As the ionizing power of the rays was very great, charges sufficiently large to be accurately measured were easily accumulated by exciting the tube only for short periods. By following this course quite satisfactory results were obtained and much loss of time was avoided.

On account of this running down of the discharge tube, it was impossible, in comparing the ionizations in two different gases, to use an apparatus with a single chamber, such as that shown in fig. 1. In order to obtain accurate results, it was necessary either to have a constant source of rays, or else to be able to ascertain the relative intensities of the rays used with the different gases.

One method which suggested itself was the use in series of two chambers, such as that shown in fig. 1. By inserting a thin aluminium membrane between them a different gas could be put in each chamber, and a single pencil of rays could be used to produce the ionization in both chambers. With this arrangement it was thought that the ionization obtained in the first chamber might perhaps bear a constant ratio to that produced in the second. But this relation was not found to hold, and further, as the cathode rays are rapidly absorbed, the amount of ionization obtained in the first chamber was so very much greater than that in the second, that even if the ratio had been fairly constant the method would not have been at all satisfactory.

This led to a trial of two receivers in parallel. Although the cathode rays on issuing from the window diverge very greatly, mechanical difficulties made it impracticable to receive part of the issuing rays in each chamber, and so recourse was had to the use of two windows. With a single large disc as cathode, a stream of rays was received in each of the chambers. The ratio of their intensities, however, as measured by the ionizations they produced, did not remain constant but varied quite irregularly. The explanation of this is probably found in a paper by A. A. C. SWINTON,[*] where he points out that the carriers are shot off in a hollow cone from the cathode, and that the dimensions of such a cone of rays vary with the degree of exhaustion in the tube. Besides, the aluminium windows were opposite to eccentric points on the cathode, and the ratio of the intensities of the two pencils was in this way greatly influenced by slight variations in the directions of the rays within the tube. A cathode formed of two small discs was then tried, and the results obtained were very satisfactory. The ratio of the discharges from the windows was in this case quite constant, and it was therefore possible to make measurements with confidence. The main difficulty of the investigation was in this way overcome, and the method was applied to obtain among other things a knowledge of—1, the absorption of the rays; 2, the ionizations produced by them in air at different pressures; and 3, the relative ionizations in different gases.

6. *Description of Apparatus used.*

A diagram of the apparatus is shown in fig. 4, and the way in which the connections were made is exhibited in fig. 5. The exciting tube was slightly over 3 centims. in diameter. The two discs of the cathode were each about a centimetre in diameter, and they were placed with their centres directly in front of the aluminium windows. That portion of the apparatus in which the ionizations were

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[*] 'Proc. Roy. Soc.,' vol. 61, p. 79 (1897).

measured consisted of two chambers, A and B, each made of brass and similar in form to that shown in fig. 1. The two electrodes C and D were held in position by ebonite plugs, which closed the ends of the receivers and at the same time served as insulators.

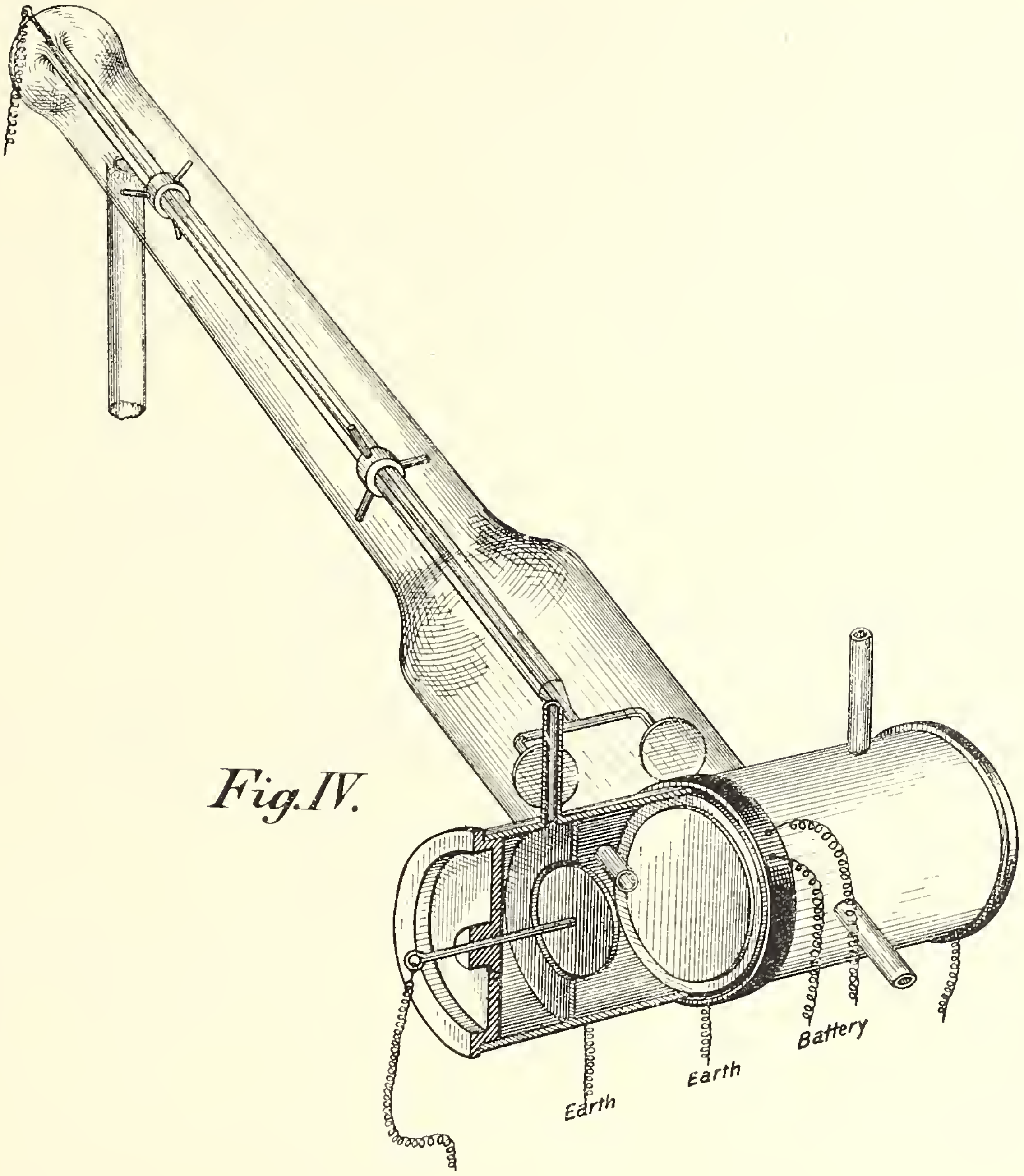

Fig. IV.

In each experiment the receivers themselves were well earthed, and also, initially, the electrodes C and D. As the electrostatic induction was very intense in the neighbourhood of the discharge tube, it was found necessary to take special precautions in regard to the earth connections. Wires of but very small resistance were used, and these were led to water mains and all the joints carefully soldered.

The two chambers were separated by a disc of ebonite, and to its faces were attached thin brass plates, *a* and *b*. By means of wires passing out through the ebonite to the battery of storage cells E, these plates could be charged to any desired potential, high or low. As the electrodes and the walls of the receiver were earthed, this afforded a means of setting up in each chamber a field which could be readily modified. The fields themselves, moreover, were quite distinct, each disc serving as a screen to cut off any action arising from the other.

Each of the chambers was provided with a projecting shoulder, which slid over a corresponding one on the anode surrounding the window opposite. By coating these joints with wax the chambers were then not only made airtight, but also were entirely separated from each other.

In the apparatus used, the diameter of the chambers A and B was about 3 centims., and the distance between each of the electrodes and its corresponding plate *a* or *b* about 1·6 centims. The diameter of the narrow cylinders which admitted the rays to the chambers was 3 millims., and the distance between the aluminium windows and points corresponding to the centres of the electric fields was about 2 centims.

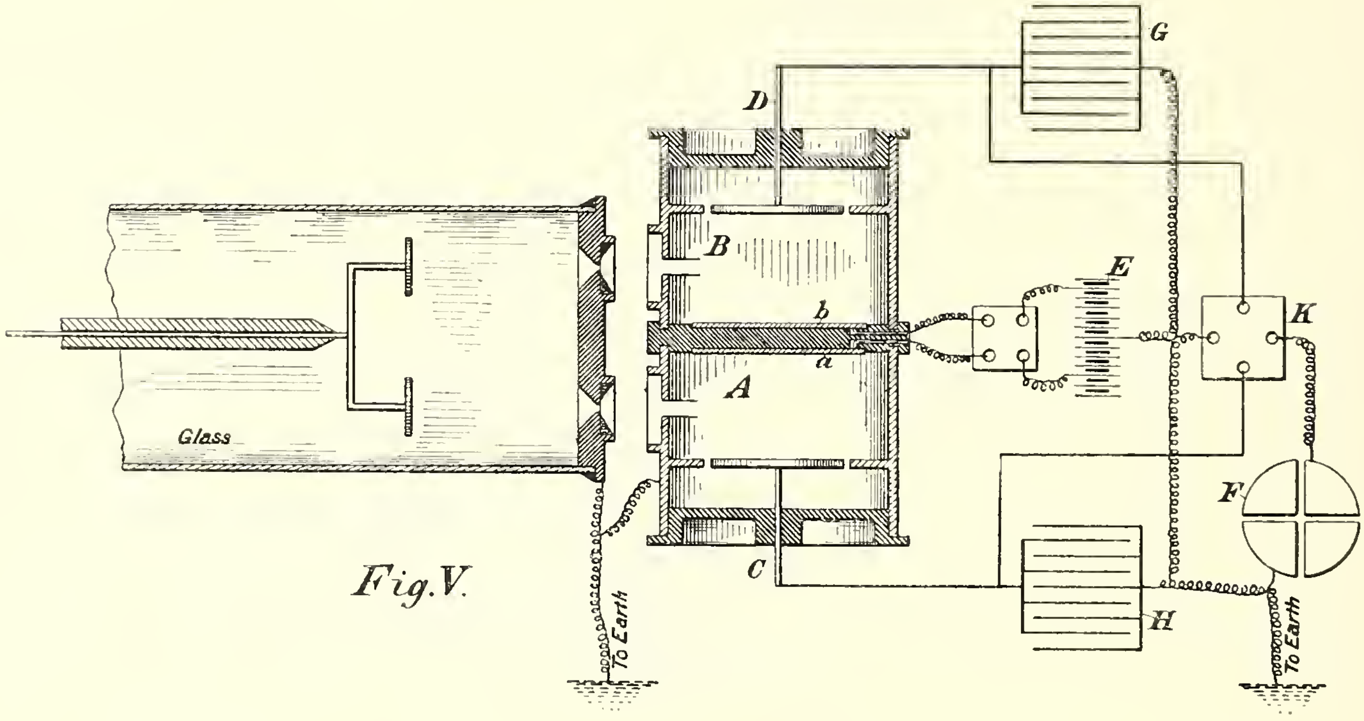

Fig. V.

Each of the electrodes was connected to an air condenser, whose capacity was about 600 electrostatic units. These condensers, G and H, were each made of two sets of parallel plates separated by small ebonite supports. The plates were made by coating both sides of a sheet of glass by a single sheet of tinfoil. In this way plates tolerably plane were obtained, and yet difficulties arising from electric absorption were avoided, the glass merely serving as a support for the foil plates.

The measurements were made with a quadrant electrometer F, and the tube was excited by a 50-centims. spark-length induction coil, whose positive terminal, together with the anode of the tube, was kept to earth. This coil was provided with an Apps interruptor, and besides being very powerful was also very efficient. It required a potential of only eight volts to excite it, and with the interruptor working slowly, this was sufficient to produce sparks of the maximum length in air at normal pressure. In practice, the interruptions were made at the rate of 20 to 25 per second.

7. *Explanation of the Method adopted for Comparing Ionizations.*

It is well known that, in conduction in Röntgenised gases, and in gases acted upon by uranium radiation, the current of electricity obtained does not increase in proportion to the electromotive force applied. The current, after reaching a certain critical value, becomes practically stationary, and increases but very little when very large increases are made in the electromotive forces. This maximum, or saturation current, was also found to characterise the conductivity produced by the passage of cathode rays through a gas. With Röntgen or uranium radiation, a field of 400 or 500 volts a centimetre has been found to give saturation in most simple gases; but with cathode rays it was necessary to apply fields of much stronger intensity.

As already stated, the distance between either of the electrodes C and D, fig. 5, and the dividing partition was about 1·6 centims. In order to ascertain the saturating electromotive force, the plate *b* was kept at a very high potential, while that of *a* was gradually increased from zero. At each stage the ratio of the currents obtained in the two chambers was noted, and it was not until a potential of about 900 volts was applied to *a* that an approximation to the saturation current was obtained in the chamber A. With a potential difference of 1200 volts the increase in the current was small, and an increase only slightly larger was obtained with a potential of 1600 volts, or 1000 volts a centimetre. This small increment in the current very probably arose from the influence of the field itself. It may be that in certain parts of the receiver the rays, acting in conjunction with the applied difference of potential, had not quite sufficient intensity to produce dissociation. An increase in the field under these circumstances would produce greater ionization, and consequently a larger current would be obtained. As this field of 1000 volts a centimetre practically produced saturation currents in both chambers, it was used throughout in measuring the ionizations. Sparking was prevented by using in the charging circuit liquid resistances, such as xylol.

An explanation of the saturation current is that the number of ions used up by the current in a given time is exactly equal to the number produced by the rays in the same time, or in other words, the ions are removed so rapidly by the applied field that recombination is practically eliminated. The saturation current is then a direct measure of the ionization produced, and in order to compare the ionizations in any two gases, it suffices to measure their saturation currents. In this investigation the saturating electromotive force was applied to the plates *a* and *b*, the discharge tube was then excited, and the currents obtained were used to charge up the condensers G and H. The discharge having been stopped, the potentials of the two condensers were then successively determined.

As the effective capacity of the electrometer was the same fraction of that of each of the two equal condensers, the deflection readings were direct measures of the charges obtained. The charging of both condensers proceeded for the same time, and consequently the electrometer deflections were also direct measures of the saturation currents, and therefore of the ionizations in the two chambers.

The method possessed the advantage of being independent of the time of charging and of the strengths of the rays coming from the two windows, provided only that the ratio of their intensities remained constant. In using the electrometer the needle was kept at a high potential, and one pair of quadrants always connected to earth. Though with this arrangement slow losses from the needle occurred, yet the short interval required for the two readings made the gradual change in the effective capacity of the electrometer inappreciable.

In practice, the electrometer was initially connected to one of the condensers, and the tube allowed to run until a suitable deflection was obtained. After noting this reading, the electrometer, having been put to earth, was then connected to the other condenser and the second reading taken. In this way the ratio of the ionizations in the two chambers was obtained.

From the experiments described in Section 2, it is clear that the signs of the charges obtained in the condensers depended on the signs of the charges given to *a* and *b* by the battery. In case these plates were positively charged, the charges collected were positive, and were due entirely to ionization. With a negative field, however, the negative charges obtained included not only negative ions produced by the rays, but also the negative carriers, constituting the rays, that were stopped in their motion by the gas. For this reason the positive field was always used, and consequently the charges obtained gave a measure of the number of ions produced in the gas by the passage of the rays.

8. *Ionization in different Gases at the Same Pressure.*

To compare the ionization in a selected gas with that in air at the same pressure, the saturating electromotive force was applied to the plates *a* and *b*, fig. 5. The two chambers A and B were first filled with air at atmospheric pressure, and a series of readings taken, the mean of which gave the ratio of the saturation currents in the two chambers. The air was then removed from A, and the gas to be tested introduced. A set of readings similarly taken gave a ratio for the saturation current obtained with the given gas in A, compared with that obtained with air in B. The combination of these results gave the ratio of the saturation current in A, when filled with the given gas, to that in the same chamber when filled with air. This ratio was, consequently, the ratio of the ionization produced in the selected gas to that produced in air at an equal pressure under the action of cathode rays entering the chamber with the same intensity in both cases.

The results obtained from this method for hydrogen, air, and carbon dioxide are given in the first column of Table I. In the second column are given the relative ionizations found by J. J. THOMSON[*] for these same gases when ionized by Röntgen rays of constant intensity.

TABLE I.

Name of gas.	Column I.	Column II.
	Ionization by cathode rays.	Ionization by Röntgen rays.
Hydrogen	2·65	·33
Air	1·00	1·00
Carbon dioxide . . .	·34	1·40

These numbers, it will be seen, present a very marked difference. In the one case the ionization decreased as the density of the gas traversed increased, while in the other a law directly the reverse of this was followed.

One explanation of this difference in the results is that the character of the ionization under cathode rays may be essentially different from that produced by Röntgen rays. Apart from these numbers, however, there seems to be but little ground for this view. Strong experimental evidence now exists to support the assumption that the cathode rays consist of small particles of matter carrying negative charges of electricity. We may therefore regard the ionization they produce as being due to their impinging on the molecules of a gas, and to the consequent breaking up of the latter. On this hypothesis it is not clear that the resulting ions should differ in character from those produced under the influence of Röntgen radiation.

It appeared rather that the true explanation was to be found in the varying absorbing powers of the different gases. LENARD,[†] who studied these rays by the fluorescence they excited, found that the absorption of cathode rays by gases at atmospheric pressure was considerable. He was also led by his experiments to propound the law, that while different gases at the same pressure absorbed the rays to different degrees, yet their absorption depended only upon the densities of the gases, and not upon their chemical composition.

In the apparatus here used, the distance traversed by the rays after they left the discharge tube until they reached the centre of the field where the ionization was

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[*] 'Proc. Camb. Phil. Soc.,' vol. 10, Part I., p. 12.

[†] 'Wied. Ann.,' vol. 56, p. 255 (1895).

measured, was about 2 centims. From LENARD's conclusions, it is obvious that in this distance the absorption of the rays by carbon dioxide would be greater than by air, and very much greater than by hydrogen. The effective intensities of the rays in the three gases at the same pressure would then be very different, and numbers such as those given in Column I. follow naturally under these circumstances, without assuming any difference in the character of the two ionizations.

9. *Ionization in Air at Different Pressures.*

In order to study more closely the influence of absorption, a number of experiments were carried out similar to that just described. The same apparatus was used, and the same method followed, but the ionizations, instead of being measured in different gases at the same pressure, were determined for the same gas at different pressures.

TABLE II.

Pressure.	Ionization measured.
millims.	
767	1·00
530	1·44
340	1·92
205	2·32
104	2·68
53	2·74
Between 40 and 45 millims. a sudden large increase was obtained in the ionization. This was found to be due to the action of the field itself in dissociating the gas.	

The results obtained with air are shown in Table II. The pressures are expressed in heights of columns of mercury at the same temperature. The ionizations given are relative, that corresponding to atmospheric pressure being taken as unity, and each value is the average of a large number of readings.

The results are also shown graphically in fig. 6, where the abscissæ represent pressures, and the ordinates corresponding relative ionizations.

The numbers show that as the pressure decreased the ionization obtained with a saturating electromotive force steadily increased, until a pressure of about 75 millims. of mercury was reached. This result, though surprising, can be readily explained by the great absorption of the rays at atmospheric pressure.

The rays had to travel at least 1·5 centims. from the window before they reached that part of the chamber from which the saturation current was obtained. For this reason their effective intensity was very largely determined by the pressure of the gas traversed.

While a diminution in the pressure would not affect the original intensity of a pencil of rays issuing from the window, it would, owing to a decrease in the absorption, increase the ionizing power of this pencil at the centre of the receiver. In this way, although the available amount of matter to be ionized was lessened by lowering the pressure, it could happen that the resultant ionization, as measured by the saturation current, would at first exhibit increasing values. This in all probability accounts for the numbers obtained in Table II.

Now from this point of view such a condition would only hold down to a stage when the two influences produced equal effects. The ionization would then be a maximum, and would afterwards fall off with diminishing pressures. Although the numbers obtained for the saturation current do not show definitely that a maximum value was obtained for the ionization, still there are indications from them, as the curve shown on fig. 6 illustrates, that the maximum value was reached at a pressure of about 75 millims. of mercury.

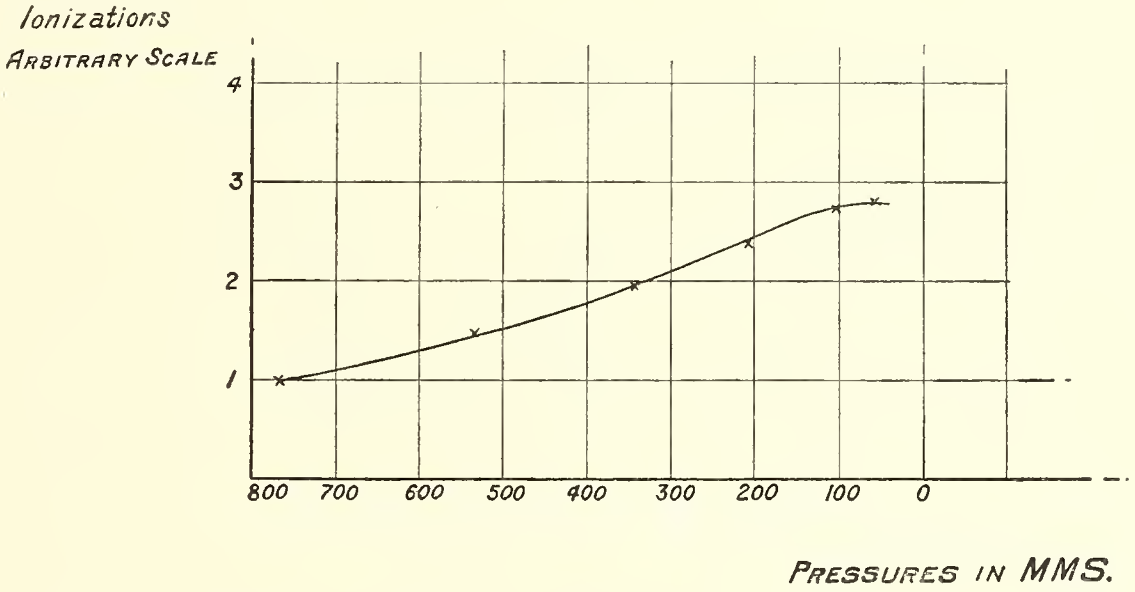

Fig. VI.

As indicated in Table II., the conditions of the experiment made it impossible to measure the ionization in air at pressures much below 50 millims. At about 40 millims. pressure a sudden large increase was obtained in the value of the saturation currrent, whch was found to be due to the influence exerted by the applied field in breaking down the gas. At these low pressures the electric intensity, which was 1000 volts a centimetre, was sufficient to dissociate the attenuated gas and to produce a discharge on its own account between the electrodes. This was shown by simply connecting the electrometer to one of the electrodes, C for example, and applying the potential difference without exciting the discharge tube. On then exhausting the chamber, the electrometer showed no leak until the critical pressure was reached, when it immediately began to charge up.

10. *Ionization in a Gas independent of its Chemical Composition.*

An important result in connection with these experiments is the agreement exhibited between the number given in Table I. for the ionization in hydrogen at atmospheric pressure and that given in Table II. for the ionization in air at a pressure of 53 millims.

Here two gases, hydrogen and air, were introduced in succession into the same measuring chamber and adjusted to the same density. Cathode rays of the same intensity were projected into this chamber in the two cases, and these rays, after traversing a certain length of the gas, reached a point where the ionization they there produced was measured. The values obtained show that under the circumstances the same number of ions was produced in both gases.

Since the rays issuing from the window were in both cases of the same intensity, it follows from LENARD's absorption law that the disposition of the rays, their actual intensities, and the quantities of them absorbed from point to point in the chamber, were precisely the same in both gases. Under these circumstances, therefore, the equal ionizations obtained in hydrogen and in air at the same density not only form a confirmation of LENARD's absorption law, but also show that where equal absorption occurs equal ionization is produced.

In the case of Röntgen radiation, RUTHERFORD* has made a determination of the relative absorbing powers of a number of gases. Taking I to denote the intensity of the rays on entering a particular gas, and $Ie^{-\lambda x}$ their intensity after traversing a length x, he has found that the values of the coefficient of absorption for the different gases practically represented the relative conductivities produced in these same gases by Röntgen rays. It is thus interesting to note that with cathode rays, just as with Röntgen rays, equal absorption gives equal ionization.

To test still further the accuracy of this conclusion a detailed examination was made of the ionization produced in a number of different gases. Throughout the experiments air in the chamber B, fig. 5, was taken as the standard. In some comparisons this air was kept at atmospheric pressure, while in others lower pressures were taken, the pressure selected being maintained through each complete determination. In making a comparison the chamber A was filled in turn with the two gases to be examined, and their pressures were adjusted so as to reduce them to the same density. Two ratios were in this way found for the ionizations in the chambers A and B, and as the influence of absorption was eliminated on account of the equal densities, these ratios represented the relative ionizations in the two gases under cathode rays of the same intensity.

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* 'Phil. Mag.,' April, 1897, p. 254.

These ratios were determined by taking the mean of a number of readings. Samples of the results obtained in five different comparisons are given in Tables III., IV., V., VI., and VII., the numbers being consecutive readings with each gas in A. They represent very well the working of the method. Although the variations were considerable, similar ones occurred in both sets of observations in each comparison, and as the number of readings taken was very large, any errors were in a great measure compensated.

Table III.—Oxygen and Air.

Air in both chambers at 746·7 millims.		Oxygen in A at 675·1 millims. Air in B at 746·7 millims.	
Ionization in A.	Ionization in B.	Ionization in A.	Ionization in B.
1·09	1·00	1·40	1·00
1·37	,,	1·28	,,
1·54	,,	1·39	,,
1·24	,,	1·20	,,
1·35	,,	1·54	,,
1·25	,,	1·41	,,
1·07	,,	1·20	,,
1·41	,,	1·26	,,
1·54	,,	1·41	,,
1·31	,,	1·33	,,
1·32	1·00	1·34	1·00

Table IV.—Nitrogen and Air.

Air in A at 734·3 millims. Air in B at 757 millims.		Nitrogen in A at 757 millims. Air in B at 757 millims.	
Ionization in A.	Ionization in B.	Ionization in A.	Ionization in B.
1·10	1·00	1·04	1·00
1·02	,,	1·12	,,
1·21	,,	1·03	,,
1·05	,,	1·34	,,
1·29	,,	1·03	,,
1·10	,,	1·06	,,
1·18	,,	1·15	,,
1·04	,,	1·12	,,
1·07	,,	1·08	,,
1·00	,,	1·06	,,
1·11	1·00	1·10	1·00

TABLE V.—Carbon Dioxide and Air.

Air in both chambers at 772·7 millims.		Carbon dioxide in A at 504·7 millims. Air in B at 772·7 millims.	
Ionization in A.	Ionization in B.	Ionization in A.	Ionization in B.
1·22	1·00	1·17	1·00
1·12	,,	1·16	,,
1·17	,,	1·23	,,
1·33	,,	1·31	,,
1·02	,,	1·37	,,
1·30	,,	1·00	,,
1·11	,,	1·21	,,
1·17	,,	1·24	,,
1·03	,,	1·31	,,
1·23	,,	1·00	,,
1·17	1·00	1·20	1·00

TABLE VI.—Hydrogen and Air.

Air in both chambers at 53·2 millims.		Hydrogen in A at 770·9 millims. Air in B at 53·2.	
Ionization in A.	Ionization in B.	Ionization in A.	Ionization in B.
1·58	1·00	1·52	1·00
1·77	,,	1·82	,,
1·64	,,	1·91	,,
1·41	,,	1·63	,,
1·62	,,	1·58	,,
1·63	,,	1·80	,,
1·79	,,	1·70	,,
1·73	,,	1·32	,,
1·85	,,	1·75	,,
1·81	,,	2·04	,,
1·68	1·00	1·71	1·00

Table VII.—Nitrous Oxide and Air.

Air in both chambers at 759 millims.		Nitrous oxide in A at 499 millims. Air in B at 759 millims.	
Ionization in A.	Ionization in B.	Ionization in A.	Ionization in B.
1·08	1·00	1·03	1·00
1·10	,,	1·21	,,
1·24	,,	1·15	,,
1·12	,,	1·07	,,
·99	,,	1·13	,,
1·08	,,	1·07	,,
1·11	,,	1·17	,,
1·23	,,	1·02	,,
1·07	,,	1·05	,,
1·12	,,	1·10	,,
1·11	1·00	1·10	1·00

Care was taken to insure the purity of the gases, and they were also well dried before being passed into the ionizing chamber.

The oxygen was prepared electrolytically, and was freed from ozone by being passed through a strong solution of potassium iodide and caustic potash.

The nitrogen was prepared by gently heating a mixture of ammonium chloride with a nearly saturated solution of sodium nitrite. The gas given off was passed through a U-tube containing strong caustic potash, and also through a second containing concentrated sulphuric acid. A Kipp apparatus was used for the preparation of carbon dioxide, which was made in the ordinary manner by allowing dilute hydrochloric acid to act on marble. In making hydrogen a Kipp apparatus was also used, dilute sulphuric acid being allowed to act on zinc. The gas was passed through a strong potassium permanganate solution, and then through a U-tube containing a strong solution of caustic potash.

The nitrous oxide was prepared by heating ammonium nitrate in a flask, and the gas was collected over water.

TABLE VIII.—Summary of Measurements.

Gases compared.		Pressures.	Ionizations.
		millims.	
Air	mean of 30 readings	746·7	1·31
Oxygen.	„ 30 „	675·1	1·32
Air	mean of 25 readings	734·3	1·11
Nitrogen	„ 25 „	757	1·09
Air	mean of 30 readings	772·7	1·20
Carbon dioxide . .	„ 30 „	505·4	1·18
Air	mean of 18 readings	53·2	1·70
Hydrogen	„ 18 „	770·9	1·79
Air	mean of 23 readings	759	1·09
Nitrous oxide . .	„ 23 „	499·3	1·10

A summary of complete sets of observations on the different gases is given in Table VIII. This statement includes the number of readings made in each case and the pressures at which these were taken. The ionizations quoted are the averages of the several sets of readings.

The close agreement exhibited by the numbers corresponding to each comparison fully bears out the conclusion deduced from the earlier experiments. It not only forms a striking corroboration of LENARD'S absorption law, but also shows that the ionization follows an analogous one, which may be stated thus :—When cathode rays of a given strength pass through a gas, the number of ions produced per second in 1 cub. centim. depends only upon the density of the gas, and is independent of its chemical composition.

The similarity in the laws of absorption and ionization, holding, as it does, with so many gases over such a wide range of pressures, is a clear indication that when cathode rays are absorbed to a certain extent, the positive and negative ions produced by these absorbed rays are of a definite amount, which bears a constant ratio to the quantity of the rays absorbed; that is to say, the absorption of a definite amount of radiant energy is always accompanied by the appearance of a fixed amount of potential energy in the form of free ions.

This granted, it follows that in order to ascertain the relative ionizations produced in any two gases by cathode rays of the same intensity, it is sufficient to determine the absorbing powers of the two gases for the same rays. In other words, the coefficients of ionization for a series of gases are fully determined when the coefficients of absorption for these same gases are known.

The existence of this general relation between absorption and ionization for both cathode and Röntgen rays is especially interesting when we remember that the two radiations are so very different in many respects.

In the one case, according to the generally-accepted view, the rays consist of small charged particles of matter moving with high velocities in space, while in the other they are supposed to consist of electromagnetic impulses propagated in the ether. With the one the dissociation is in all probability brought about by a series of impacts between the moving particles and the molecules of the gas; with the other it seems to be due to the direct action of the intense electric field forming the impulse. Again, while the absorption of cathode rays depends only upon the density of the medium traversed, the absorption of Röntgen rays, according to RUTHERFORD's results, does not seem to depend to any great extent upon the molecular weight of the gas. But while all these differences exist in the two radiations, with both of them it holds good that the same number of ions are always produced in a gas when the same amount of rays traversing it are absorbed.

11. *Comparison of Ionizations produced by Cathode and by Röntgen Rays.*

The method just described gives definite and conclusive information regarding the ionizations produced by cathode rays in gases of the same density; but where the gases are of different densities, it cannot be satisfactorily applied. As stated in Section IX., the rays, after entering the ionizing chamber, must travel some distance before reaching that part of the field from which the current is drawn. On this account, though rays entering the chamber may originally be of the same strength, still their effective intensities become at ordinary pressures quite different, when the gases traversed are not of the same density.

Also as it is impossible to define exactly the disposition of the electric field within the chamber, these effective intensities cannot be calculated with any degree of accuracy.

A difficulty arises, too, from the dispersion of the rays. As shown by LENARD, they issue from the window in a pencil whose form is greatly influenced by the density of the gas traversed. At very low pressures they pass through the aluminium window practically without deviation, but as the pressure increases, they spread out until finally they issue in all directions.

The conclusion arrived at in the last section, however, suggests a means of calculating the ionization which would be produced by rays of constant intensity in different gases at the same pressure.

LENARD,* who investigated the absorption powers of a number of gases at different

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* 'Wied. Ann.,' vol. 56, p. 258.

pressures, has shown that for any particular gas the coefficient of absorption varies directly as the pressure. In the case of air, taking I to denote the intensity of the rays issuing from the window of the discharge tube, and $Ie^{-\lambda x}$ their intensity at a distance x from the window, he found for λ the values given in Table IX.

TABLE IX.

Air pressure.	Coefficient of absorption.
millims.	
760	3·43
331	1·51
165	·661
83·7	·396
40·5	·235
19·3	·117
10·0	·0400
2·7	·0166
·78	·00416

These numbers, it will be seen, amply support LENARD'S conclusion. Similar tables, given by him for a number of gases, all exhibit the same relation between the values of λ and the corresponding pressures of the gas.

Now, if the values of the coefficient of absorption are taken to represent the relative ionizations produced in a gas, at a point where the pressure is varied but the intensity of the rays kept constant, it follows from LENARD'S numbers that the ionization in any particular gas would vary directly as the pressure to which it was subjected.

This result, which follows as a deduction from the preceding experiments, has also been found experimentally by PERRIN* to characterise the ionization produced by Röntgen rays. It is true that with Röntgen rays a number of experimenters have found quite different relations to hold between the ionization and the pressure; but in most cases they have vitiated their results either through omitting to use saturating electromotive forces, or through neglecting to arrange their experiments so as to eliminate the metal effect observed by PERRIN.

With uranium radiation also, RUTHERFORD† has found the ionization to be proportional to the pressure of the gas traversed.

The direct experimental verification of a law of this kind is always accompanied by a serious difficulty. The law has reference to the action of rays whose intensity is constant throughout the region ionized. With rays that are easily absorbed by gases at ordinary pressures, this condition can be realised either by the use of very thin layers of gas or by investigating the ionizations at very low pressures. Owing

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* 'Comptes Rendus,' vol. 123, p. 878.

† 'Phil. Mag.,' January, 1899, p. 136.

to mechanical difficulties, however, the former method is generally impracticable, while the action of the applied electric field in breaking down the insulation of the gas precludes the use of the latter artifice.

It is then open to measure the ionizations produced by rays traversing layers of gas of considerable thickness. But before any relation connecting ionizations and pressures can be deduced from such measurements, it is necessary to have definite information regarding the absorptive powers of the gases at different pressures, and to know exactly the form and dimensions of the region from which the ions are drawn.

Although the absorption laws for cathode rays have been fully developed by LENARD, and are quite definite and clear, it is scarcely possible to define even approximately the region in the ionizing chambers (fig. 5) from which the ions go to make up the saturation current.

On this account a direct verification of the proportionality law is not possible; but, as already pointed out, the results of the experiments described in Section X. strongly support the conclusion that, in the case of a gas subjected to increasing pressure, the ionizations produced by rays of constant intensity bear the same ratio to each other as the coefficients of absorption corresponding to these pressures.

If, then, the ionization in a gas varies with the pressure, it follows at once that if rays of the same intensity were allowed to traverse thin layers of different gases at a constant pressure, the ionizations produced would be directly proportional to the densities of these gases.

Take, for example, carbon dioxide and air. It has been shown that the ionization produced in carbon dioxide at a pressure of 504·7 millims. of mercury is the same as that produced in air at 772·7 millims. by rays of the same intensity.

According to the proportion law the ionization produced by these same rays in CO_2 at 772·7 would then be just 1·53 times that obtained at the lower pressure; that is, with rays of the same intensity the ionizations in carbon dioxide and in air would be to each other as 1·53 to 1 when these gases were subjected to the same pressure.

A similar conclusion may be deduced from a consideration of the other gases examined. Hence, on this view, the relative ionizations produced by rays of constant intensity in a series of gases subjected to the same pressure would be expressed by the numbers which under these circumstances give their relative densities.

These numbers are given for the gases examined in Column I., Table X., while in Column II. are given the values found by J. J. THOMSON[*] for the relative ionizations produced by Röntgen rays of constant intensity in the same gases.

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[*] 'Proc. Camb. Phil. Soc.,' vol. 10, Part I., p. 12.

TABLE X.

Gases examined.	Column I.	Column II.
	Densities (shown above to be proportional to ionization by cathode rays), air = 1.	Ionization by Röntgen rays. Ionization of air taken as unity.
Air	1·00	1·00
Oxygen	1·106	1·1
Nitrogen	·97	·89
Carbon dioxide	1·53	1·4
Hydrogen	·069	·33
Nitrous oxide	1·52	1·47

The numbers, with the exception of those for hydrogen, present an agreement which is very striking, and show that although the two forms of radiation are so very different, still the products of their action upon the gases cited are practically the same.

While the difference in the numbers for hydrogen is very large, there seems to be some doubt as to the proper value to be assigned to the conductivity produced by Röntgen rays in this gas. The conductivities under Röntgen rays in the gases named have been measured by a number of experimenters, and while their values for the other gases differ but little, a very wide divergence exists in their numbers for hydrogen. RUTHERFORD* gives the value ·5, while PERRIN† has obtained the number ·026 by a method entirely different from that of any of the others.

Though we have been thus led to conclude that the density of a gas should determine its conductivity under cathode rays, strong evidence exists against adopting any such general conclusion regarding the conductivity produced by Röntgen rays, notwithstanding the general agreement indicated above for the gases cited.

With such gases as HCl, Cl_2, SO_2, and H_2S, J. J. THOMSON, RUTHERFORD, and PERRIN have found the conductivities given in Table XI.

From an examination of these values and a comparison with those of Table X., it is evident that it is quite impossible to deduce any such relation between the densities of the gases and their conductivities under this radiation.

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* 'Phil. Mag.,' April, 1897, p. 254

† 'Thèse présentée à la Faculté des Sciences de Paris,' 1897, p. 46.

TABLE XI.—Conductivity under Röntgen Rays.

Gas.	Density.	Measured by		
		J. J. THOMSON.	RUTHERFORD.	PERRIN.
HCl	1·25	8·9	11	8
SO_2	2·23	6·4	4	6
Cl_2	2·45	17·4	18	—
H_2S	1·19	6·0	6	—

Although the laws of ionization and absorption for cathode rays are clearly defined by these results, it is difficult to apply them in practice to the direct calculation of the relative ionizations in any particular experiment.

Take, for example, the case of a pencil of parallel rays, 1 sq. centim. in cross section, traversing air at a pressure p.

Let q = the rate at which ions are produced in 1 cub. centim. of air at unit pressure by cathode rays of unit intensity

and λ_0 = the coefficient of absorption of air for unit pressure.

Consider then the ionization between two planes distant x and $x + dx$, from the source of the rays.

If I denotes the original intensity of the rays, $I.e^{-p\lambda_0 x}$ will represent their intensity at a distance x, and $p.q.I.e^{-p\lambda_0 x}dx$ will then represent the total number of ions produced between these two planes in one second.

Imagine now a saturating electric field applied at right angles to the rays and confined between the limits r and $r + d$.

The value of the total saturation current obtained with this field would then be represented by $\int_r^{r+d} p.q.I.e^{-p\lambda_0 x}dx$,

or

$$i = \frac{Iq}{\lambda_0}.e^{-\lambda r}(1 - e^{-\lambda d}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $p\lambda_0$ is replaced by the quantity λ, whose values for different pressures are given in Table IX.

If the air traversed be now subjected to diminishing pressures, the saturation current will assume different values and will reach a maximum when

$$\frac{di}{d\lambda} = 0\,,$$

i.e.,

$$(r + d)e^{-\lambda d} - r = 0\,,$$

or

$$e^{\lambda d} = \frac{r + d}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

An experiment somewhat analogous to this is described in Section IX. The apparatus used is shown in figs. 4 and 5. The diameters of the electrodes C and D were each about 1 centim. and, as already stated, the distance between the window and the centre of each of the chambers was about 2 centims.

By applying the equation (2) to this experiment, and taking $r = 1{\cdot}5$ centims. and $d = 1$ centim., it follows that the saturation current would be a maximum when

$$e^{\lambda} = 1{\cdot}66 \ . \ . \ .$$

or

$$\lambda = {\cdot}5.$$

From LENARD's values, Table IX., it will be seen that this value corresponds approximately to a pressure of about 120 millims. of mercury. The observed results, however, Table II. and fig. 6, indicate a maximum of about 75 centims. Further, the calculated values of the current from equation (1) exhibit a more rapid rise than that actually observed.

But the difference in the results is not surprising. The field within the receiver was far from uniform, being disturbed by the proximity of the walls of the chamber. The presence of the narrow tube through which the rays were conducted into the receiver also produced irregularities. On this account it was impossible to define, even approximately, the region from which the saturation current was drawn. Moreover, the actual paths of the rays, as LENARD has pointed out, are largely influenced by the pressure of the gas traversed. Even at best, then, the calculated results can scarcely be regarded as more than a rough approximation.

12. *Summary of Results.*

1. The conductivity impressed upon a gas by cathode rays is similar to that produced by Röntgen and uranium rays, and can be fully explained on the hypothesis that positive and negative ions are produced by the radiation throughout the volume of the gas traversed.

2. When cathode rays are allowed to fall upon insulated metallic conductors surrounded by air at atmospheric pressure,

   (*a.*) such conductors if initially uncharged gain a small limiting negative charge,
   (*b.*) positive charges are completely dissipated,
   (*c.*) negative charges drop to a small limiting value,
   (*d.*) the loss of charge is due to the action of the ionized air surrounding the conductor, and the value of the limiting negative charge is determined by the extent of the conduction in this air.

3. The ionization produced in a gas by rays coming from the aluminium window in a Lenard discharge tube is due to cathode rays and not to Röntgen rays.

4. Lenard's results obtained by fluoroscopic methods on the absorption of cathode rays are confirmed by a study of the ionization these rays produce in gases.

5. When cathode rays of a given strength are passed through a gas, the number of ions produced in 1 cub. centim. depends only upon the density of the gas, and is independent of its chemical composition.

6. With rays of constant intensity the ionization in any particular gas varies directly with the pressure to which it is subjected.

7. The relative ionizations produced by cathode rays of constant intensity in air, oxygen, nitrogen, carbon dioxide, hydrogen, and nitrous oxide, at the same pressure, are expressed by the numbers which represent their densities.

8. With cathode rays, just as with Röntgen rays, the number of ions produced in a gas bears a definite ratio to the amount of the radiant energy absorbed.

I gladly avail myself of this opportunity to record my grateful sense of the never failing encouragement and assistance received from Professor J. J. Thomson.

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PART III.—*On Eye-colour Inheritance in Man.*

NOTE.

This memoir was originally presented to the Society on August 5, 1899, and read on November 16, 1899. In working out by the same theory the coefficients of inheritance for Basset Hounds, Mr. LESLIE BRAMLEY-MOORE discovered that the method adopted was not exact enough in its process of proportioning. Accordingly, with the assistance of Mr. L. N. G. FILON, we immensely developed the theory, so that it was necessary to rewrite the theoretical part of the original memoir. This has been carried out in Part VII. of this series. The present memoir consists substantially of the portions of the original memoir relating to the inheritance of coat-colour in Horses and eye-colour in Man, with the numerical details and the resulting conclusions modified, so far as the extended theory rendered this necessary. In the very laborious work of reconstructing my original tables I have received the greatest possible assistance from Dr. ALICE LEE, and I now wish to associate her name with mine on the memoir.* The memoir was at my request returned to me for revision after it had been accepted for the 'Philosophical Transactions.'

PART I.—INTRODUCTORY.

(1.) A CERTAIN number of characters in living forms are capable of easy observation, and thus are in themselves suitable for observation, but they do not admit of an exact quantitative measurement, or only admit of this with very great labour. The object of the present paper is to illustrate a method by which the correlation of such characters may be effectively dealt with in a considerable number of cases. The conditions requisite are the following :—

(i.) The characters should admit of a quantitative order, although it may be impossible to give a numerical value to the character in any individual.

Thus it is impossible at present to give a quantitative value to a brown, a bay, or a roan horse, but it is not impossible to put them in order of relative darkness of shade. Or, again, we see that a blue eye is lighter than a hazel one, although we cannot *à priori* determine their relative positions numerically on a quantitative scale.

Even in the markings on the wings of butterflies or moths, where it might be indefinitely laborious to count the scales, some half dozen or dozen specimens may be taken to fix a quantitative order, and all other specimens may be grouped by inspection in the intervals so determined.

We can even go a stage further and group men or beasts into simply two categories—light and dark, tall and short, dolichocephalic and brachycephalic—and so we might ascertain by the method adopted whether there is, for example, correlation between complexion and stature, or stature and cephalic index.

(ii.) We assume that the characters are a function of some variable, which, if we

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* I have further to thank Mr. LESLIE BRAMLEY-MOORE, Mr. L. N. G. FILON, M.A., Mr. W. R. MACDONELL, M.A., LL.D. and Miss C. D. FAWCETT, B.Sc., for much help in the arithmetic, often for laborious calculations by processes and on tables, which were none the less of service if they were afterwards discarded for others. To Mr. BRAMLEY-MOORE I owe the extraction and part of the arithmetical reduction of the horse-colour tables.

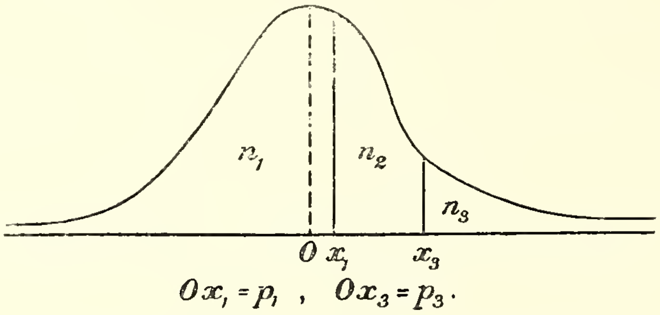
could determine a quantitative scale, would give a distribution obeying—at any rate to a first approximation—the normal law of frequency.

The whole of the theoretical investigations are given in a separate memoir, in which the method applied is illustrated by numerical examples taken from inheritance of eye-colour in man, of coat-colour in horses and dogs, and from other fields. We shall not therefore in this paper consider the processes involved, but we may make one or two remarks on the justification for their use. If we take a problem like that of coat-colour in horses, it is by no means difficult to construct an order of intensity of shade. The variable on which it depends may be the amount of a certain pigment in the hair, or the relative amounts of two pigments. Much the same applies to eye-colour. In both cases we may fail to obtain a true quantitative scale, but we may reasonably argue that, if we could find the quantity of pigment, we should be able to form a continuous curve of frequency. We make the assumption that this curve—to at any rate a first approximation—is a normal curve. Now if we take any line parallel to the axis of frequency and dividing the curve, we divide the total frequency into two classes, which, so long as there is a quantitative order of tint or colour, will have their relative frequency unchanged, however we, in our ignorance of the fundamental variable, distort its scale. For example, if we classify horses into bay and darker, chestnut and lighter, we have a division which is quite independent of the quantitative range we may give to black, brown, bay, chestnut, roan, grey, &c.

Precisely the same thing occurs with eye-colour; we classify into brown and darker, hazel and lighter, and the numbers in these classes will not change with the quantitative scale ultimately given to the various eye-tints. Our problem thus reduces to the following one: Given two classes of one variable, and two classes of a second variable correlated with it, deduce the value of the correlation. Classify sire and foal into bay and darker, chestnut and lighter; mother and daughter into brown and darker, hazel and lighter, and then find the correlation due to inheritance between the coat-colour or eye-colour of these pairs of relations. The method of doing this is given in Memoir VII. of this series. Its legitimacy depends on the assumptions (i.) and (ii.) made above, which may I think be looked upon as justifiable approximations to the truth.

Of course the probable error of the method is larger than we find it to be when correlation is determined from the product-moment. Its value varies with the inequality of the frequency in the two classes given by the arbitrary division. It will be least when we make that frequency as nearly equal as possible—a result which can often be approximately reached by a proper classification. In our present data the probable errors vary from about ·02 to ·04, values which by no means hinder us from drawing general conclusions, and which allow of quite satisfactory general results.

(2.) So far we have only spoken of the two classes, which are necessary if we merely want to determine the correlation. But if we wish to deal with relative variability we must have more than two classes. We have, in fact, in our tables preserved Mr. GALTON's eight eye-colour classes and the seventeen classes under which the coat-colour of thoroughbred horses is classified in WETHERBY's studbooks. Such a classification enables us at any rate approximately to ascertain relative variability, and, what is more, to reconstruct approximately the quantitative scale according to which the tints must be distributed in order that the frequency should be normal. For, in order to attain this result, we have to ascertain from a table of the areas of the normal curve the ratio of the length of the abscissa to the standard deviation which corresponds to any given increase of frequency. Let us suppose that three classes have been made—n_1, n_2, n_3, represented by the areas of the normal curve in the accompanying diagram so marked. Let p_1 and p_3 be the distances of the mean from the two boundaries of n_2. Here p_1 may be negative, or p_3 infinite, &c. Then if $h_1 = p_1/\sigma$, $h_3 = p_3/\sigma$, we find at once, if N = total frequency,



$$\frac{n_1 - n_2 - n_3}{\mathrm{N}} = \sqrt{\frac{2}{\pi}}\int_0^{h_1} e^{-\frac{1}{2}x^2}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}.$$

$$\frac{n_1 + n_2 - n_3}{\mathrm{N}} = \sqrt{\frac{2}{\pi}}\int_0^{h_3} e^{-\frac{1}{2}x^2}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}.$$

Now the integrals on the right are tabulated, and thus, since the left-hand side is a known numerical quantity, it follows that p_1/σ and p_3/σ, and accordingly the range $(p_3 - p_1)/\sigma$ of the class in terms of the standard deviation, are fully determined.

Thus, if ϵ be the range on the scale of tint or colour of the group of which the observed frequency is n_2, we have $\epsilon = p_3 - p_1$, and thus $\epsilon/\sigma = q$ say, is known. For a second series $\epsilon/\sigma' = q'$. Hence $\sigma/\sigma' = q'/q$, and accordingly the ratio of the variabilities of the two series is determined.

Again, the ratio $p_1/(p_3 - p_1)$ enables us to find the position of the mean in terms of the range on the scale occupied by the tint corresponding to the frequency n_2. As a rule we shall take this tint to be that in which the mean actually lies, in which case we shall have $p_1/(p_3 + p_1)$ as determining the ratio in which the mean divides the true quantitative range of this particular tint.

(3.) Let

$$\eta = p_1(p_3 - p_1) = h_1/(h_3 - h_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii.)},$$

$$\zeta = \sigma/\sigma' = (h_3' - h_1')/(h_3 - h_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}.$$

It remains to find the probable errors of these quantities.

Suppose Σ_x to be the standard deviation for the errors in a quantity x, and R_{xy} the correlation coefficient for errors in two quantities x and y.

Further let

$$H = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{v.}),$$

where subscripts and dashes may be attached to H to correspond to like distinguishing marks attached to h.

Since

$$\frac{2n_1 - N}{2N} = \frac{1}{\sqrt{2\pi}} \int_0^{h_1} e^{-\frac{1}{2}x^2} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vi.}),$$

we have at once $\delta n_1 = NH_1 \delta h_1$,

and $\Sigma_{h_1} = \Sigma_{n_1}/(NH_1)$ . . . . (vii.).

Similarly, $\delta n_3 = - NH_3 \delta h_3$, whence:

$$\Sigma_{h_3} = \Sigma_{n_3}/(NH_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{viii.}).$$

Further, we have $\Sigma_{h_1}\Sigma_{h_3}R_{h_1h_3} = - \Sigma_{n_1}\Sigma_{n_3}R_{n_1n_3}/(N^2H_1H_3)$ . . . . . (ix.);

but, as is shewn in Part VII., § 4,

$$\Sigma_{n_1}{}^2 = \frac{n_1(N - n_1)}{N}, \qquad \Sigma_{n_3}{}^2 = \frac{n_3(N - n_3)}{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{x.}),$$

$$\Sigma_{n_1}\Sigma_{n_3}R_{n_1n_3} = - \frac{n_1n_3}{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xi.}).$$

Thus we find

$$\text{Probable error of } h_1 = \cdot 67449 \Sigma_{h_1}$$

$$= \frac{\cdot 67449}{\sqrt{N}} \frac{1}{H_1} \sqrt{\frac{n_1(N - n_1)}{N^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xii.}).$$

$$\text{Probable error of } h_3 = \frac{\cdot 67449}{\sqrt{N}} \frac{1}{H_3} \sqrt{\frac{n_3(N - n_3)}{N^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xiii.}).$$

Correlation in errors in h_1 and h_3, or $R_{h_1h_3}$, is given by

$$\Sigma_{h_1}\Sigma_{h_3}R_{h_1h_3} = \frac{n_1n_3}{N^3H_1H_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xiv.}).$$

Let $u = h_3 - h_1$, $u' = h_3' - h_1'$ be the ratio to the respective standard deviations of the ranges corresponding to the groups n_2 and n_2'. Then

$$\Sigma_u^2 = \Sigma_{h_1}{}^2 + \Sigma_{h_3}{}^2 - 2\Sigma_{h_1}\Sigma_{h_3}R_{h_1h_3}$$

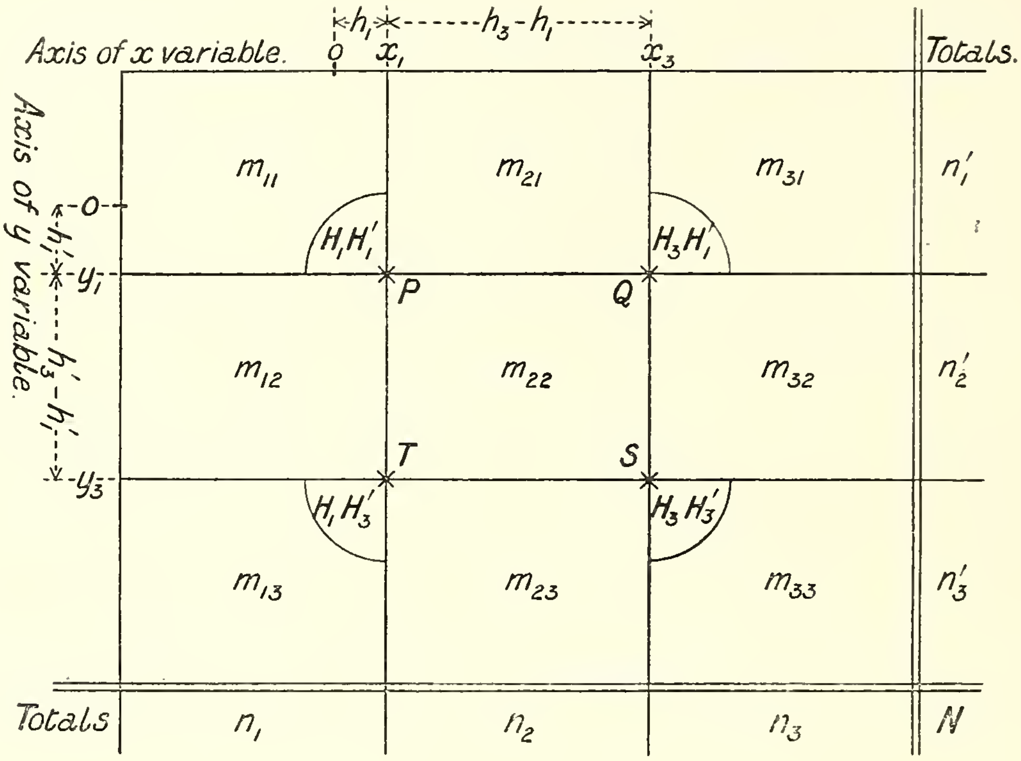
$$= \frac{1}{N^3}\left\{\frac{n_1(N - n_1)}{H_1{}^2} + \frac{n_3(N - n_3)}{H_3{}^2} - \frac{2n_1n_3}{H_1H_3}\right\};$$

whence, if ν be a proportional frequency $= n/N$, we readily find

$$\text{Probable error of } u = \frac{\cdot 67449}{\sqrt{N}} \left\{\frac{\nu_1}{H_1{}^2} + \frac{\nu_3}{H_3{}^2} - \left(\frac{\nu_1}{H_1} + \frac{\nu_3}{H_3}\right)^2\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad (\text{xv.}).$$

$$\text{Probable error of } u' = \frac{\cdot 67449}{\sqrt{N}} \left\{\frac{\nu_1'}{H_1'^2} + \frac{\nu_3'}{H_3'^2} - \left(\frac{\nu_1'}{H_1'} + \frac{\nu_3'}{H_3'}\right)^2\right\}^{\frac{1}{2}} . \quad . \quad . \quad (\text{xvi.}).$$

I now proceed to determine the correlation in the errors made in determining the ranges corresponding to any two classes of any two variables which are correlated. For this purpose let the frequency correlation table be dressed as follows, in the diagram below.



Here m_{ij} denotes the frequency of individuals common to the two classes n_i and n'_j. Let M_{ij} denote its "conjugate," or all the frequency which appears in neither n_i nor n'_j; then

$$N = M_{ij} + n_i + n'_j - m_{ij} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xvii.)}.$$

As before, we have

$$\Sigma_{n_i}^2 = \frac{n_i(N - n_i)}{N}, \qquad \Sigma_{n'_j}^2 = \frac{n'_j(N - n'_j)}{N} \quad . \quad . \quad . \quad . \qquad \text{(xviii.)}.$$

$$\Sigma_{M_{ij}}^2 = \frac{M_{ij}(N - M_{ij})}{N}, \qquad \Sigma_{m_{ij}}^2 = \frac{m_{ij}(N - m_{ij})}{N} \quad . \quad . \quad . \quad . \qquad \text{(xix.)}.$$

Further, since m_{ij} and M_{ij} are mutually exclusive, we have

$$\Sigma_{M_{ij}}\Sigma_{m_{ij}}R_{M_{ij}m_{ij}} = -\frac{M_{ij}m_{ij}}{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xx.)}.$$

From (xvii.) we have for small variations

$$\delta n_i + \delta n'_j = \delta m_{ij} - \delta M_{ij}.$$

Hence

$$2\Sigma_{n_i}\Sigma_{n'_j}R_{n_i n'_j} = \Sigma_{m_{ij}}^2 + \Sigma_{M_{ij}}^2 - \Sigma_{n_i}^2 - \Sigma_{n_{ij}}^2 - 2\Sigma_{m_{ij}}\Sigma_{M_{ij}}R_{M_{ij}m_{ij}} \quad . \quad . \quad . \qquad \text{(xxi)}.$$

Substituting the values given above we find, after some reductions,

$$\Sigma_{n_i}\Sigma_{n'_j}R_{n_i n'_j} = \frac{Nm_{ij} - n_i n'_j}{N} \quad \ldots\ldots\ldots \quad \text{(xxii.)}$$

This result, which is extremely simple in form, gives the correlation in errors made in determining the frequencies in any two classes whatever of any two correlated variables.

I next proceed to find the correlation between errors in u and u', the ratio of the ranges occupied by any two classes to their respective standard deviations.

We have

$$\delta n_2 + \delta n_3 = -NH_1\delta h_1 ;$$
$$\delta n_3 = -NH_3\delta h_3.$$

Hence $$\delta(h_3 - h_1) = \frac{\delta n_2}{NH_1} + \frac{\delta n_3}{N}\left(\frac{1}{H_1} - \frac{1}{H_3}\right).$$

Similarly $$\delta(h_3' - h_1') = \frac{\delta n'_2}{NH_1'} + \frac{\delta n_3'}{N}\left(\frac{1}{H_1'} - \frac{1}{H_3'}\right).$$

Multiply the first by the second, and summing as usual for all possible errors, we have, by using (xxii.)

$$\Sigma_u\Sigma_{u'}R_{uu'} = \frac{1}{N}\left\{\frac{Nm_{22} - n_2 n_2'}{N^2H_1H_1'} + \frac{Nm_{23} - n_2 n_3'}{N^2H_1}\left(\frac{1}{H_1'} - \frac{1}{H_3'}\right) + \frac{Nm_{32} - n_2' n_3}{N^2H_1'}\left(\frac{1}{H_1} - \frac{1}{H_3}\right)\right.$$
$$\left. + \left(\frac{Nm_{33} - n_3 n_3'}{N^2}\right)\left(\frac{1}{H_1} - \frac{1}{H_3}\right)\left(\frac{1}{H_1'} - \frac{1}{H_3'}\right)\right\}.$$

Collecting the like H's we find, after very considerable reductions,

$$\Sigma_u\Sigma_{u'}R_{uu'} = \frac{1}{N}\left\{\frac{Nm_{11} - n_1 n_1'}{N^2H_1H_1'} + \frac{Nm_{33} - n_3 n_3'}{N^2H_3H_3'} + \frac{Nm_{13} - n_1 n_3'}{N^2H_1H_3'} + \frac{Nm_{31} - n_3 n_1'}{N^2H_3H_1'}\right\}$$
$$\ldots\ldots \quad \text{(xxiii.)},$$

or, $$\Sigma_u\Sigma_{u'}R_{uu'} = \frac{1}{N}\left\{\frac{\mu_{11} - \nu_1\nu_1'}{H_1H_1'} + \frac{\mu_{33} - \nu_3\nu_3'}{H_3H_3'} + \frac{\mu_{13} - \nu_1\nu_3'}{H_1H_3'} + \frac{\mu_{31} - \nu_3\nu_1'}{H_3H_1'}\right\} \quad \ldots \quad (\text{xxiii.}^{\text{bis}}).$$

where $\mu_{ij} = m_{ij}/N$ = proportional frequency.

A glance at our diagram on the previous page of the correlation table divided into nine classes, shows at once the symmetrical formation of this result. By writing at the points P, Q, S, and T, the ordinate there of the normal surface, on the supposition of no correlation and N = 1, the construction of the result is still more clearly brought out.

We are now in a position to determine the probable errors of η and ζ. We have

$$\delta\eta = \frac{h_3\delta h_1 - h_1\delta h_3}{u^2}.$$

Hence

$$\Sigma_\eta^2 = \frac{1}{u^4}\left(h_3{}^2\Sigma_{h_1}{}^2 + h_1{}^2\Sigma_{h_3}{}^2 - 2h_3h_1\Sigma_{h_1}\Sigma_{h_3}\mathrm{R}_{h_1h_3}\right)$$

$$= \frac{h_1{}^2h_3{}^2}{u^4\mathrm{N}}\left\{\frac{n_1(\mathrm{N}-n_1)}{(h_1\mathrm{H}_1)^2\,\mathrm{N}^2} + \frac{n_3(\mathrm{N}-n_3)}{(h_3\mathrm{H}_3)^2\mathrm{N}^2} - \frac{2n_1n_3}{(h_1\mathrm{H}_1)(h_2\mathrm{H}_3)\mathrm{N}^2}\right\}.$$

Or, Probable error of η

$$= \frac{\cdot 67449}{\sqrt{\mathrm{N}}}\,\frac{h_1h_3}{u^2}\left\{\frac{\nu_1}{(h_1\mathrm{H}_1)^2} + \frac{\nu_3}{(h_3\mathrm{H}_3)^2} - \left(\frac{\nu_1}{h_1\mathrm{H}_1} + \frac{\nu_3}{h_3\mathrm{H}_3}\right)^2\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad \text{(xxiv.)}.$$

where u is the range $h_3 - h_1$, and ν_1 and ν_3 are the proportional frequencies, as before. Care must be taken, if the class n_2 cover, as it usually will in our present investigations, the mean, to put h_1 negative within the radical. In other words, for a class covering the mean we have

Probable error of η

$$= \frac{\cdot 67449}{\sqrt{\mathrm{N}}}\,\frac{h_1h_3}{(h_1+h_3)^2}\left\{\frac{\nu_1}{(h_1\mathrm{H}_1)^2} + \frac{\nu_3}{(h_3\mathrm{H}_3)^2} - \left(\frac{\nu_1}{h_1\mathrm{H}_1} - \frac{\nu_3}{h_3\mathrm{H}_3}\right)^2\right\}^{\frac{1}{2}} \quad . \quad . \quad \text{(xxv.)}.$$

Lastly we have

$$\zeta = u'/u\,,$$

or,

$$\delta\zeta = \frac{u\delta u' - u'\delta u}{u^2} = \frac{u'}{u}\left(\frac{\delta u'}{u'} - \frac{\delta u}{u}\right).$$

Hence

$$\Sigma_\zeta{}^2 = \frac{u'^2}{u^2}\left\{\frac{\Sigma_{u'}{}^2}{u'^2} + \frac{\Sigma_u{}^2}{u^2} - \frac{2\Sigma_u\Sigma_u\mathrm{R}_{uu'}}{uu'}\right\}.$$

Thus : Probable error of ζ

$$= \cdot 67449\zeta\left\{\frac{\Sigma_u{}^2}{u^2} + \frac{\Sigma_{u'}{}^2}{u'^2} - \frac{2\Sigma_u\Sigma_{u'}\mathrm{R}_{uu'}}{uu'}\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxvi.)},$$

where we have by (xv.), (xvi.), and (xxiii.$^{\text{bis}}$)

$$\Sigma_u{}^2 = \frac{1}{\mathrm{N}}\left\{\frac{\nu_1}{\mathrm{H}_1{}^2} + \frac{\nu_3}{\mathrm{H}_3{}^2} - \left(\frac{\nu_1}{\mathrm{H}_1} + \frac{\nu_3}{\mathrm{H}_3}\right)^2\right\},$$

$$\Sigma_{u'}{}^2 = \frac{1}{\mathrm{N}}\left\{\frac{\nu_1{}'}{\mathrm{H}_1{}'^2} + \frac{\nu_3{}'}{\mathrm{H}_3{}'^2} - \left(\frac{\nu_1{}'}{\mathrm{H}_1{}'} + \frac{\nu_3{}'}{\mathrm{H}_3{}'}\right)^2\right\},$$

$$\Sigma_u\Sigma_{u'}\mathrm{R}_{uu} = \frac{1}{\mathrm{N}}\left\{\frac{\mu_{11} - \nu_1\nu_1{}'}{\mathrm{H}_1\mathrm{H}_1{}'} + \frac{\mu_{33} - \nu_3\nu_3{}'}{\mathrm{H}_3\mathrm{H}_3{}'} + \frac{\mu_{13} - \nu_1\nu_3{}'}{\mathrm{H}_1\mathrm{H}_3{}'} + \frac{\mu_{31} - \nu_3\nu_1{}'}{\mathrm{H}_3\mathrm{H}_1{}'}\right\},$$

where, as before, μ's and ν's represent proportional frequencies.

In the following investigations on coat-colour and eye-colour inheritance I have not thought it needful to give in every one of the thirty-six relationships dealt with the probable errors of the means, ratio of variabilities, and the coefficients of inheritance (η, ζ, and r). The arithmetical labour would have been too great, for the expressions as given above are somewhat complex. It is, however, necessary to keep the approximate values of these probable errors in view, and, as our results classify themselves easily into groups for which our data, as well as the intensity of heredity, are approximately the same, one series of these errors has been found for each group.

(4.) If we have ground for our assumption that the variable at the basis of our tint classification can be so selected as to give a normal distribution, we may determine the relative lengths on the scale of that variable occupied by each tint or shade. Thus if σ_1 be the standard deviation of the variable for male eye-colour, σ_2 for female eye-colour, I measured the range on the scale in terms of σ_1 and σ_2 for Mr. GALTON'S eight eye-colour tints for 3000 cases of male and 3000 cases of female eye-colour. I found the spaces occupied on the unknown scale to be as follows :—

No.	Tint.	Range in terms of σ_2.	Range in terms of σ_1.
1	Light blue	∞	∞
2	Blue, dark blue . . .	1·39276	1·34918
3	Grey, blue-green . . .	·73468	·77596
4	Dark grey, hazel . . .	·40027	·41992
5	Light brown	·03893	·00856
6	Brown	·43679	·35895
7	Dark brown.	·84161	·64167
8	Very dark brown, black.	∞	∞

These results are not so regular as we might have hoped for, on the assumption that the ratio of σ_1/σ_2 would be the same from whatever part of the scale it be determined. The general conclusion, however, would be that σ_1 is slightly larger than σ_2, which is confirmed by other investigations. Actually a tint may be rather vaguely described, and where the data were obtained by untrained observers without the assistance of a plate of eye-colours, a good deal of rather rough classification is likely to have taken place. I do not think it would be safe to go further than stating that on the quantitative colour scale the tints as described must occupy spaces in about the following proportions :—

Light Blue.	Blue, Dark Blue.	Grey, Blue-Green.	Dark Grey, Hazel.	Light Brown.	Brown.	Dark Brown.	Very dark Brown, Black.
∞	1·37	·75	·41	·02	·40	·74	∞

Taking 2000 colts and 2000 fillies, the standard deviations being σ_1 and σ_2 respectively, I have worked out the coat-colour ranges in terms of σ_2 and σ_1 for each of the sixteen colours* occurring in the records. We have the following results :—

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* See p. 92, below.

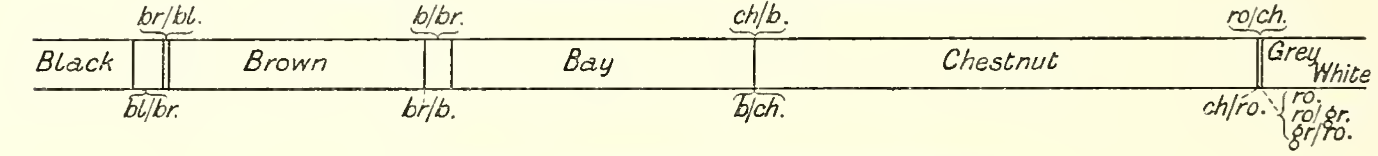
Tint.	Range in σ_2.	Range in σ_1.		Range in σ_2.	Range in σ_1.
1	∞	∞	9	·00000	·00000
2	·12683	·10768	10	1·96956	2·01658
3	·00000	·03313	11	·00000	·00000
4	·91747	1·11055	12	·02490	·00000
5	·00000	·00352	13	·00000	·00000
6	·11059	·10451	14	·00000	·00000
7	1·34684	1·27688	15	·00000	·00000
8	·00000	·00000	16	∞	∞

Here again it seems to me that the most we can safely do is to consider that on a suitable scale the relative lengths occupied by the classes of coat-colours recognised by thoroughbred horse breeders would be somewhat as follows:—

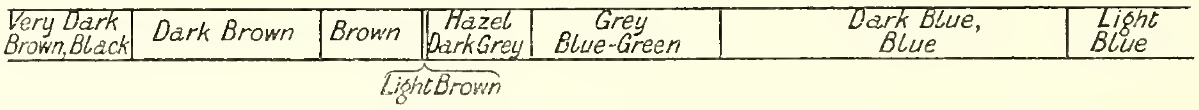
bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.
∞	·12	·02	1·01	·00	·11	1·31	·00	·00	1·99	·00	·01	·00	·00	·00	∞

The reader must carefully bear in mind that these represent scale-lengths occupied by the coat-colour and not the frequency of horses of these individual coat-colours. What we are to understand is this: that if eye-colour in man and coat-colour in horses were measured on such quantitative scales as we have given in skeleton, then the distribution of the frequency of the several colours would be very approximately normal. The actual skeleton scales are represented in the accompanying diagram, which puts them at once before the eye.

Normal Scale of Colour Ranges in Thoroughbred Horses.



Normal Scale of Eye Colour Ranges in Man.



(5.) It is necessary here to draw attention to a distinction of some importance in heredity, namely, that between *blended* and *exclusive* inheritance. In my treatment of the law of ancestral heredity,* it is assumed that we have to deal with a quantitatively measurable character, and that the ancestry contribute to the quantity of this character in certain proportions which on the average are fixed and follow certain definite numerical laws. Such an inheritance is *blended* inheritance. But another

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* 'Roy. Soc. Proc.,' vol. 62, p. 386.

type of inheritance is possible. We may have one progenitor, prepotent over all others and absorbing all their shares, who hands down to the offspring not a proportion of his character, but the whole of it without blend. If this progenitor is a parent we have *exclusive* inheritance, if a higher ancestor a case of reversion. I have dealt at some length with this type of inheritance under the title of the Law of Reversion in another paper.* We must consider in outline the main features of such inheritance, for the cases of eye-colour in man and coat-colour in the horse approximate more closely to the numerical values required by it, than to those indicated by the law of ancestral heredity. The chief feature of exclusive inheritance is the absolute prepotency of one parent with regard to some organ or character. It need not always be the parent of the same sex, or the same parent throughout the same family. Some offspring may take absolutely after one, others after another parent for this or that organ or character only. I believe Mr. GALTON first drew attention, in his 'Natural Inheritance' (p. 139), to this exclusive or, as he terms it, *alternative* heritage in eye-colour. In going through his data again I have been extremely impressed by it; even those cases in which children might be described as a blend, rare as they are, are quite possibly the result of reversion rather than blending. If we suppose exclusive inheritance to be absolute, and there to be no blending or reversion, it is not hard to determine the laws of inheritance. Supposing the population stable, one-half the offspring of parentages with one parent of given eye-colour would be identical with that parent in eye-colour, the other half would regress to the general population mean, *i.e.*, the mean eye-colour of all parents. Hence, taken as a whole, the regression of children on the parent would be ·5. In the case of the grandparent the regression would be ·25; of a great grandparent ·125, and so on. With an uncle a quarter of the offspring of his brother will be identical in eye-colour with him, the other three-quarters will regress to the population mean, thus the regression will be ·25. If we have n brethren in a family, and take all possible pairs of fraternal relations out of it, there will be $\frac{1}{2}n(n-1)$ such pairs; $\frac{1}{2}n$ brothers will have the same eye-colour that of one parent, the other $\frac{1}{2}n$ brother that of the other parent. Hence selecting any one brother, $\frac{1}{2}n - 1$ would have his eye-colour, and on the average $\frac{1}{2}n$ would have regressed to the mean of the general population. In other words, the coefficient of regression would be $(\frac{1}{2}n - 1)/(\frac{1}{2}n - 1 + \frac{1}{2}n) = (\frac{1}{2}n - 1)/(n - 1)$.

Accordingly $n = 3$ Regression $= \cdot 25$
$n = 4$,, $= \cdot 3333$
$n = 4 \cdot 7$,, $= \cdot 3649$
$n = 5$,, $= \cdot 375$
$n = 5 \cdot 3$,, $= \cdot 3833$
$n = 6$,, $= \cdot 4$
$n = \infty$,, $= \cdot 5$

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* 'Roy. Soc. Proc.,' vol. 66, pp. 140 *et seq.*

It will at first appear, therefore, that the fraternal regression with the size of families actually occurring will vary from ·35 to ·4.

To some extent these values would be modified by assortative mating, which actually exists in the case of eye-colour. The correlations between parent and offspring and between brothers would both be slightly increased. Thus if ρ be the coefficient of assortative mating, R_f the fraternal correlation with and r_f without assortative mating, and r the coefficient for parent and offspring,[*]

$$R_f = \frac{r_f + 2\rho r^2}{1 - 2\rho r^2}.$$

If we put $r_f = {\cdot}36$, $r = {\cdot}5$, $\rho = {\cdot}1$, we find

$$R_f = {\cdot}39.$$

Thus we see that the regression or correlation for fraternal inheritance in the case of exclusive inheritance could not, with the average size of families, be very far from ·4 of blended inheritance.

A further source which can modify immensely, however, the fraternal correlation is the prepotency of one or other parent, not universally, but within the individual family. In the extreme case all the offspring might be alike in each individual family. Thus fraternal correlation might be perfect although parental correlation were no greater than ·5. Hence, where for small families we get a fraternal correlation greater than ·4 to ·5, it is highly probable that there exists either a sex prepotency (in this case, one of the parental correlations will be considerably greater than the other) or an individual prepotency (in which case the parental correlations based on the average may be equal). We shall see that fraternal correlations occur greater than ·5 in our present investigations. I have dealt with these points in my Memoir on the 'Law of Reversion,'[†] and also in the second edition of the 'Grammar of Science.'[‡]

Another point also deserves notice, namely, that with the series ·5, ·25, ·125, &c., for the ancestral coefficients in the direct line, the theorems proved in my Memoir on Regression, Heredity, and Panmixia[§] for the series of coefficients $r, r^2, r^3 \ldots$ exactly hold. That is to say, if we have absolutely exclusive inheritance, the partial regression coefficients for direct ancestry are all zero except in the case of the parents. This it will be observed is not in agreement with Mr. GALTON's views as expressed in Chapter VIII. of the 'Natural Inheritance.' But I do not see how it is possible to treat exclusive inheritance on the hypothesis that the parental regression is about ·3.[||] Actual investigation shows that for this very character, *i.e.*, eye-colour, it is nearer ·5. If we take Table XIX. of Mr. GALTON's appendix, and make the following groups, both

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[*] This is shown in a paper not yet published on the influence of selection on correlation.

[†] 'Roy. Soc. Proc.,' vol. 66, pp. 140 *et seq.*.

[‡] "On Prepotency," p. 459; "On Exclusive Inheritance," p. 486.

[§] 'Phil. Trans.,' A, vol. 187, p. 302, etc.

[||] Mr. GALTON takes $\frac{1}{3}$ throughout his arithmetic.

parents light, one parent light and one medium, one light and one dark, we reach the following results :—

Parents' eye-colour.	Children, actual.		Light-eyed children, calculated.	
	Total.	Light-eyed.	Exclusive inheritance.	Ancestral law with knowledge of parents and grandparents.
Both light	355	334	355	321
Light and medium .	215	170	161	160
Light and dark . .	211	107	105	117

Here the exclusive inheritance leads us to misplace thirty-two and the ancestral law thirty-three children. The evidence, therefore, of the correctness of the latter is hardly greater than that of the former. Indeed, if the former were modified for reversion, it would very possibly give better results than the latter.

I am inclined accordingly to look upon eye-colour inheritance as an exclusive inheritance modified by reversion, and, to some extent, by assortative mating, rather than a mixture of exclusive inheritance with a slight amount of blending. In either case exclusive inheritance gives results like the above so closely in accord with the ancestral law that the latter might be supposed to hold. But, theoretically, I do not understand how the ancestral law is compatible with exclusive inheritance. Theoretically, we have values of parental, avuncular, and grand-parental correlation greater than the ancestral law would permit of, and these theoretical values are, on the whole, closer to observation, as we shall see in the sequel, than those given by the law of ancestral heredity. The following table gives the two systems :—

Table I.—Theoretical Values of the Regression Coefficients.

Relationship.	Blended inheritance, ancestral law.	Exclusive inheritance, absolute, no reversion.
Parent and offspring	·3	·5
Grandparent and offspring . . .	·15	·25
Great-grandparent and offspring .	·075	·125
Brethren	·4	·35 to ·5*
Uncle and nephew	·15	·25

Now, if exclusive inheritance be modified by reversion or assortative mating, or if blended inheritance be modified by "taxation,"† then we shall get values different

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* This varies with the size of the family.

† 'Roy. Soc. Proc.,' vol. 62, p. 402.

from the above, and possibly filling up the numerical gap between them. To this point I shall return when dealing with the observed values for eye-colour in man.

PART II.—ON COLOUR-INHERITANCE IN THOROUGHBRED RACEHORSES.

(6.) All the data were extracted from WEATHERBY'S stud-books, the colours being those of the horses as yearlings. My first step was to form an order, not a quantitative scale, of horse-colours. With this end in view, the recorded colours were examined, and, including the arabs, the following seventeen colours were at first found :—

1. Black (bl.).
2. Black or brown (bl./br.).
3. Brown or black (br./bl.).
4. Brown (br.).
5. Brown or bay (br./b.).
6. Bay or brown (b./br.).
7. Bay (b.).
8. Bay or chestnut (b./ch.).
9. Chestnut or bay (ch./b.).
10. Chestnut (ch.).
11. Chestnut or roan (ch./ro.).
12. Roan or chestnut (ro./ch.).
13. Roan (ro.).
14. Roan or grey (ro./g.).
15. Grey or roan (g./ro.).
16. Grey (g.).
17. White (w.).

Now, if we take the alternative colours to mean that the first alternative is the prominent element, we see that these colours in use among breeders admit of *only one arrangement from black to white.* That is to say, that a continuous shade-change is actually in use for the colour-nomenclature of thoroughbred horses.* Thus without any hypothesis as to the quantitative relative values of bay or roan, we have an order which serves for all our present purposes. Following this order, Appendix I., Tables I.—XII., for the colour correlation of related pairs of horses was compiled by Mr. LESLIE BRAMLEY-MOORE from the stud-books. When dealing with relationship in the ♀ line only, no weight has been given to fertility, as each mare has had only one foal attributed to it, or two in the case of fraternal correlation. In the case of the ♂ line, the colours of the older sires were harder to ascertain, and we did not obtain altogether more than 600 sire-colours. Thus one, two, or, in a few cases, three or four colts or fillies were taken from each sire.

I shall now discuss the results which may be drawn from these tables for the theory of heredity, first placing in a single table all the numerical constants calculated from the data in Tables I. to XII. of Appendix I.

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* Among the 6000–8000 horses dealt with only *four* were found with colours not entered in this scale, but these entries of bl./ch., br./ch., b./ro., in no way contradict the *order* of the scale, but merely show a rougher appreciation on the part of the nomenclator, or possibly printers' or editor's errors.

TABLE II.—Coat-colour Inheritance in Thoroughbred Horses.

Pair of relatives.		Division of bay range by the mean		Ratio of variabilities.		Coefficients of correlation.	Coefficients of regression.		Number of cases.
x.	y.	η_x.	η_y.	$\zeta = \sigma_x/\sigma_y$.	$\zeta = \sigma_y/\sigma_x$.	r_{xy}.	R_{xy}.	R_{yx}.	N.
Sire.	Colt. .	·6111	·5713	·8712	1·1478	·4913	·4280	·5639	1300
Sire.	Filly .	·6061	·5719	·8298	1·2051	·5422	·4499	·6534	1050
Dam	Colt. .	·5359	·6027	·9500	1·0526	·4862	·4619	·5118	1000
Dam	Filly .	·5565	·6051	·9036	1·1067	·5668	·5122	·6273	1000
Maternal grandsire	Colt. .	·6583	·5867	·7030	1·4225	·3590	·2524	·5107	1000
Maternal grandsire	Filly .	·6359	·6042	·7678	1·3024	·3116	·2392	·4058	1000
Colt . . . . . (Half	Colt. . siblings)	·5908	·5908	1	1	·3551	·3551	·3551	2000
Colt . . . . . (Whole	Colt. . siblings)	·5620	·5620	1	1	·6232	·6232	·6232	2000
Filly . . . . . (Half	Filly . siblings)	·5665	·5665	1	1	·4265	·4265	·4265	2000
Filly . . . . . (Whole	Filly . siblings)	·5684	·5684	1	1	·6928	·6928	·6928	2000
Filly . . . . . (Half	Colt. . siblings)	·5633	·5865	·9607	1·0409	·2834	·2723	·2950	1000
Filly . . . . . (Whole	Colt. . siblings)	·5410	·5711	·9555	1·0466	·5827	·5568	1·0466	1000

In this table $R_{xy} = r_{xy}\sigma_x/\sigma_y$, $R_{yx} = r_{xy}\sigma_y/\sigma_x$. Half-siblings* are those having the same dam, but different sires. Further, η is measured from the brown end of the bay range up to the mean.

(7.) *On the Mean Coat-Colour of Horses.*—If our theory be correct, this colour will not differ much from the median colour, and we notice at once a secular change going on. We have the following order:—

Maternal grandsire of colt . . . . . . η = ·6583
Maternal grandsire of filly . . . . . . = ·6359
Sire of colt . . . . = ·6111
Sire of filly . . . . = ·6061
Colt (mean value of seven series) . . . = ·5816

Dam of colts . . . . η = ·5359
Dam of fillies . . . . = ·5565
Fillies (mean value of seven series) . . = ·5753

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[*] I have introduced this expression in my paper on "The Law of Reversion," 'Roy. Soc. Proc.,' vol. 66, as a convenient expression for a pair of offspring from same parents whatever be their sex.

Now the colours of all the horses are returned when they are foals, so that there is no question of any variation of colour with age, yet we notice that—

(i.) The horse is lighter in colour than the mare.

(ii.) If we go back two generations (grandsire) the horse is lighter than if we only go back one generation (sire), and the sires are again lighter than their colts. In other words, there seems a progressive change towards a darker coat.

(iii.) On the other hand, the mares one generation back appear to be darker than their daughters.

(iv.) The average sire of colts is lighter than the average sire of fillies; the average dam of colts is darker than the average dam of fillies.

Now these conclusions seem to indicate that the older horse was lighter in coat, and the older mare darker in coat than either the colt or filly of to-day, and that there is a tendency in the thoroughbred racehorse of to-day to approach to an equality of colour in the two sexes, an equality which is a blend of the sensibly divergent sex-colour of the older generation.

Whether this secular change is due to the "breeding out" of the influence of light Arabian sires, or to a tendency in the past to select light-coloured sires and dark-coloured mares for breeding, or to the fact that such coloured sires and mares are the most fertile, *i.e.*, to an indirect effect of reproductive selection, is not so easy to determine. But what does appear certain is that the average thoroughbred is approaching to a blend between its male and female ancestry, which were sensibly divergent.*

(8.) *On the Relative Variability of Sex and Class in Horses.*—The following table gives the length of the bay range in terms of the standard deviation for each class. If ϵ represent this range, then in terms of the previous notation $\epsilon = u \times \sigma = u' \times \sigma'$, and from these values of u and u' the ratio, $\zeta = \sigma/\sigma'$ of Table II. was calculated.

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* Mean of dams and sires of colts = ·5735, *i.e.*, $\frac{1}{2}$(·6111 + ·5359).
Mean of dams and sires of fillies = ·5813, *i.e.*, $\frac{1}{2}$(·6061 + ·5565).

These are curiously enough almost exactly equal to the mean values ·5753 and ·5816 obtained for fillies and colts. This *inverse* relationship is too close to the probable errors of the quantities under investigation for real stress to be laid on it, but it may still turn out to be suggestive.

III.—Table of Bay Ranges.

Relative Pair.		Bay Range.		Probable Error of Median.	
x.	y.	$u \times \sigma_x$.	$u' \times \sigma_y$.	x.	y.
Sire	Colt	1·46943σ_x	1·28019σ_y	± ·0160	± ·0183
Sire	Filly	1·64075σ_x	1·36149σ_y	± ·0159	± ·0192
Dam	Colt	1·36645σ_x	1·29819σ_y	± ·0196	± ·0206
Dam	Filly	1·38165σ_x	1·24845σ_y	± ·0193	± ·0214
Maternal grandsire	Colt	1·69694σ_x	1·19293σ_y	± ·0158	± ·0224
Maternal grandsire	Filly	1·65021σ_x	1·26702σ_y	± ·0162	± ·0211
Colt (Half	Colt siblings)	1·23953σ_x	1·23953σ_y	± ·0153	± ·0153
Colt (Whole	Colt siblings)	1·27688σ_x	1·27688σ_y	± ·0148	± ·0148
Filly (Half	Filly siblings)	1·39619σ_x	1·39619σ_y	± ·0135	± ·0135
Filly (Whole	Filly siblings)	1·34684σ_x	1·34684σ_y	± ·0140	± ·0140
Filly (Half	Colt siblings)	1·33479σ_x	1·28229σ_y	± ·0202	± ·0208
Filly (Whole	Colt siblings)	1·41501σ_x	1·35207σ_y	± ·0189	± ·0198

To explain the last double column I note that Mr. SHEPPARD has shown (' Phil. Trans.,' A, vol. 192, p. 134) that the probable error of the median equals

$$\cdot 84535\ \sigma/\sqrt{\mathrm{N}}.$$

Hence in terms of the bay range we have

$$\frac{\text{probable error of median}}{\text{length of bay range}} = \cdot 84535/(u\sqrt{\mathrm{N}}).$$

I have found that this simple result gives a value close to but slightly larger than the probable error of the quantity η (p. 82), from which I have determined the position of the mean in the bay range. It is much easier to calculate, but of course not so exact, as we take no account of possible errors in the bay range itself or their correlation with errors in the median. I have accordingly tabulated its values in the last double column as a rough guide to the errors made in the numbers upon which the statements in the previous section depend. I shall return to the consideration of the probable errors below. Turning to columns 3 and 4 of our Table II., we can draw the following conclusions as to the variability of sex and class :—

(*a*.) *The Younger Generation is more Variable than the Old.*—Thus, foals are more variable than their sires, and, looking at the expressions in Table III. for the bay range, sires than grandsires. This is a rule I have now found true in a very great number of cases of inheritance. Parents are a fairly closely selected body of the general population, and so less variable than that population at large. This might appear pretty obvious in the case of thoroughbred horses when we are dealing with sires and grandsires. I have already pointed out that it was impossible to take 1000 to 1300 colts or fillies with as many independent sires, the fashion in sires is too marked; and of course the number of independent grandsires was still fewer.* But even in the case of dams, where we have taken as many independent dams as fillies, we see this reduction in variability in the older generation. As it also occurs with stature, &c., in man as well as with coat-colour in horses—in which latter case we expect artificial selection—it deserves special consideration. Without weighting with fertility, there exists a selection of the individuals destined to be parents in each generation. We have to ask whether the change in mean and variability from parent to offspring in each generation is secular or periodic, or to what extent it is partly one and partly the other. The importance of settling this point is very great; it concerns the stability of races. So far as my fairly numerous series of measurements yet go, I cannot say that a "stable population" has definitely shown itself; the characters of each race when measured for two generations seem to vary both in mean and standard deviation. Palæontologists tell us of species that have remained stable for thousands of years, but this is a judgment hitherto based on a *qualitative* appreciation. A quantitative comparison of the means, variabilities, and correlations of some living species in its present and its fossil representatives would be of the greatest interest and value. For myself, I must confess that my numerical investigations so far tend to impress me with the unstable character of most populations.

(*b*.) *There is fairly good evidence that the horse is more variable than the mare in coat-colour.* It would be idle to argue from the first four results of Table III. that the mare is more variable than the horse, in that these results show the dam to be more variable than the sire. For, as we have shown, the process of breeding and our method of extracting the data tend to produce a much more intense selection of sires than of dams. But if we compare the mean bay range in terms of the standard deviation of colts for our seven series of colts with that for the seven series of fillies in Table III., we find for the first 1·27458 σ_c and for the second 1·33854 σ_f. Hence we are justified in concluding that σ_c is greater than σ_f. In fact in only *one* case out of the seven does the series of fillies give a less variability than the corresponding series of colts, *i.e.*, colts corresponding to dams are somewhat less variable than fillies corresponding to dams. It must, however, be remembered that this conclusion is based upon the coat-colour of the animals recorded as yearling foals.† Thus, the coat-

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* For some account of the extent of in and in breeding in the thoroughbred horse, see my memoir on "Reproductive Selection," 'Phil. Trans.,' A, vol. 192, p. 257 *et seq.*

† The reader must always bear in mind that when we speak of the variability of colour in sire or dam, &c., it means the variability of this class when they were yearlings.

colour may change both in intensity and variability with age, much as variability in stature changes with children from birth to adult life.

(*c*.) As a more or less natural result of (*b*) it follows *that any group, male or female, having male relatives is more variable than the same group with female relatives.* Thus sires of colts are more variable than sires of fillies; fillies half-sisters to colts are more variable than fillies half-sisters to fillies, &c. But out of the nine cases provided by our data there are three exceptions to the rule, and perhaps not much stress can be laid on it, at any rate in the above form. It would appear that males, relatives of males, are sensibly more variable than males relatives of females. The bay ranges are 1·3926 σ' and 1·4447 σ respectively, which indicates that the average σ' is larger than σ. But if we treat the groups of females alone, we find for females with male relatives the bay range = 1·3694 σ, and for females with female relatives 1·3433 σ', which indicates that the latter are more variable. The difference is, however, not very sensible. Possibly the rule is simply that extremes tend to produce their own sex, but our data are not sufficient for really definite conclusions on the point.

In order that we may have a fair appreciation of the probable errors of the quantities involved and the weight that is to be laid upon their differences, I place here a table* of the probable errors of η, of $\zeta = \sigma_x/\sigma_y$ and of r_{xy} for typical cases.

IV.—Table of Probable Errors.

Relations.	η_x	η_y	ζ	u	u'	r_{xy}
Sire and Filly . .	·0143	·0170	·0243	·0363	·0330	·0288
Grandsire and Colt	·0143	·0199	·0237	·0385	·0319	·0333
Colt and Colt . . (Whole siblings)	·0186	·0186	—	·0328	·0328	·0259
Filly and Colt . . (Half siblings)	·0179	·0185	·0315	·0335	·0328	·0363

It will be seen from this table that the probable error in η is about 3 per cent., in ζ about 2 to 4 per cent., in u about 2 to 2·5 per cent., and the values of r about ·03, growing somewhat larger as r grows smaller. The probable errors are thus somewhat larger than those which we obtain by the old processes when the characters are capable of quantitative measurement, but they are not so large as to seriously affect the use of the new processes in biological investigations. As we have already indicated, the probable errors of the η's may be roughly judged by Mr. SHEPPARD's formula for the median (p. 95).

It will be seen that the differences in the η's and ζ's of Table II., or the u's of

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* I have to thank Mr. W. R. MACDONELL for friendly aid in the somewhat laborious arithmetic involved in calculating these probable errors.

Table III., are as a rule larger than the probable errors of the differences, sometimes several times larger. Yet in some cases they are not such large multiples of the probable errors of the differences that we can afford to lay great stress on the divergence of η or ζ or u in a pair of special cases. We must lay weight rather on the general tendency of the tables when all the series are taken together. Thus, while we may have small doubt about the correctness of (i.) of § 7 or (*b*) of § 8, we should look upon (iv.) of § 7 as an important suggestion which deserves serious consideration rather than a demonstrated law. The same again holds good for (*c*) of § 8. It is because of their suggestiveness that they are here included. That a differential fertility or an individualisation in the sex of offspring should be correlated with colour, would, if proved, be a fact of very considerable interest. It would again emphasise the important part which genetic selection plays in the modification of characters.* *À priori* it is not more unreasonable to expect coat-colour in horses than to suppose hair-colour in men to be correlated with fertility. But the fertility of man does seem to vary from the light to the dark races. The special feature of interest here, however, is that a different colour in the two sexes appears to influence the preponderance of one or other sex in the offspring. It would be an interesting inquiry to determine whether the sex-ratio in the offspring of "mixed marriages" varies when the races of the two parents are interchanged.

(9.) *On the Inheritance of Coat-colour in Thoroughbred Horses.*—(*a.*) *Direct Line. First Degree.*—Having regard to the probable errors—about ·03—in the values of the correlation coefficient r_{xy}, it seems quite reasonable to suppose that the mean parental correlation, actually ·5216, is practically ·5. It is quite impossible to imagine it the ·3 of Mr. GALTON's view of the Law of Ancestral Heredity. If we adopt the view of that law given in my paper on the Law of Ancestral Heredity,† and take the coefficient γ to be different from unity, then it is shown in my paper on the Law of Reversion‡ that we cannot on the theory of blended inheritance get parental correlation as high as ·5 without values of the fraternal correlation which are much higher than those hitherto observed, certainly much higher than, as we shall see later, we find in the case of coat-colour in horses. Coat-colour in horses does not thus appear to be at all in accord with Mr. GALTON's view of ancestral inheritance, or even with my generalised form of his theory. It does accord very well with what we might expect on the theory of exclusive inheritance as developed above, p. 91, on the assumption that there is no reversion.

Looking at the matter from the relative standpoint, we see that not much stress can be laid on the respective influences of the sire and dam on the colt, or of the sire and dam on the filly; but, on the other hand, the filly does appear to inherit more from

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* See the concluding remarks in the memoir on "Genetic (Reproductive) Selection," 'Phil. Trans.,' A, vol. 192, pp. 257—330.

† 'Roy. Soc. Proc.,' vol. 62, p. 386 *et seq.*

‡ 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

both parents than the colt does. There is certainly (judged from coat-colour) no preponderance of the sire's influence over the dam's such as breeders appear occasionally to imagine. The average influence of the dam on the offspring indeed appears to be slightly greater than that of the sire, but the difference is of the order of the probable error, and not of the overwhelming character exhibited in the case of Basset Hounds. There is indeed in the case of thoroughbred horses not the same chance of carelessness produced by a misalliance afterwards screened by the defaulter. There exists, however, a far greater premium—considering the great value of yearlings from fashionable sires—set upon dishonesty. Again it is possible that when stallions receive too many public or private mares, or are still used in their old age, that they may, without losing the power of fertilising, lose some of the power of transmitting their characters. The divergences, so far as the probable errors are concerned, are not such that we are forced out of our way to explain them. With the single exception of sire and colt we see that our table shows the universal prevalence of the rule that:

Relatives of the same sex are more closely correlated than relatives of the same grades of the opposite sex. Thus:—

A colt is more like his sire than his dam.
A filly is more like her dam than her sire.
A dam is more like her filly than her colt.
A grandsire is more like his grand-colt than his grand-filly.
A colt is more like his brother colt than his sister filly.
A filly is more like her sister filly than her brother colt.

the latter two cases being true for both whole and half siblings.

The solitary exception is that a sire is more like his filly than his colt.

If we were to assume it a rule that a filly in the matter of coat-colour has stronger inheritance all round than a colt, we should find it agree with our results for parental inheritance, and receive considerable support for the much stronger correlation of fillies than of colts, when either whole or half siblings. But it would not be in accordance with our results for grandparents, for which, however, we have only details for two out of the eight possible cases. On the whole, I think we must content ourselves with the statements that parental correlation is certainly about ·5, and that with high probability each sex is more closely correlated with its own sex of the same grade of relationship.

(*b.*) *Direct Line, Second Degree.*—My data here are unfortunately only for two cases out of the possible eight. I hope some day to finish the series, but the labour of ascertaining from the studbooks the coat-colour of 700 or 800 separate sires is very great. Indeed it is not easy to follow up the pedigree through the male line when the sire is not one of the famous few. On the other hand, it is much easier through the female line. For this reason the *maternal* grandsire was taken. Even in this case we had to seek back for each sire—the year of whose birth was unknown—until we found the record of his coat-colour given under the heading of his dam in the year of his birth.

The average of our two cases gives a coefficient of correlation = ·3353, the colt having a greater degree of resemblance to the grandsire than the filly. This value is substantially greater than the ·25 we might expect for exclusive inheritance, and more than double the value ·15, to be expected for the grandparental correlation with Mr. GALTON's unmodified law for blended inheritance. Of course the ·25 is to be expected as the mean of the *eight* grandparental series, and, as we shall see for eye-colour in man, these may vary very much in magnitude. But allowing for this, it seems quite impossible that the average value could be reduced to ·15. I take it therefore that the grandparental, like the parental, data point to a law of inheritance other than that which has been described in my paper on the Law of Ancestral Heredity. This peculiar strengthening of the grandparental heritage has already been noted by me in my paper on the Law of Reversion,* and the difficulties of dealing with it on the principle of reversion therein discussed. There may be some opinion among breeders as to the desirability of emphasising the dam's strain in the choice of a sire which leads to this result, but if so it is unknown to me, nor do I see how it would work without also emphasising the correlation of the dam and foal. The mean value of the correlation for the maternal grandfather and grandchildren for eye-colour in man is ·3343—a result in capital agreement with that for coat-colour in horses. In that case the average of the eight series, as we shall see later, is considerably above ·25, and we must, I think, suspend our judgment as to whether this could possibly in the case of horses be the true mean value. As to the value ·15 it seems quite out of the question.

As already remarked, the influence of the maternal grandsire (unlike that of the sire) is substantially greater on the colt than on the filly.

(*c.*) *Collateral Heredity, First Degree.*—Here we have more ample data to go upon, namely, a complete set of six tables of both whole and half siblings of both sexes.

We notice one or two remarkable features straight off. In the first place, in the case of both fillies and colts, the whole siblings of the *same* sex have not a correlation the double of that of the half siblings, but have a correlation very considerably less than this. *À priori* we might very reasonably expect the one to be the double of the other. This is what would happen in the case of blended inheritance on the hypothesis of equipotency of the parents. As the half siblings are on the dam's side, we might assert a considerable prepotency of the dam over the sire. This cannot indeed be the explanation of the divergence in the case of Basset Hounds, where the half siblings have a correlation considerably *less* than half that of whole siblings,†

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* 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

† 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

and yet the prepotency of the dam in coat-colour is very marked. But in the present case there is on the average only a slight, if indeed it be a real, prepotency of the dam. Further, if we turn to the correlation, no longer of siblings of the same sex, but of opposite sexes, we find the correlation of the whole siblings is approximately double that of the half siblings, as we should *à priori* have expected.

Taking averages on the assumption that the correlation for whole siblings should be double that for half siblings, we have the following results:—

Correlation between colts based on results for whole and half siblings .	·6667
Correlation between fillies based on results for whole and half siblings .	·7729
Correlation between filly and colt based on results for whole and half siblings	·5747
Mean correlation of siblings based upon all results for whole siblings .	·6329
Mean correlation of siblings based upon all results for half siblings . .	·7100
Mean correlation of siblings based upon results for both whole and half siblings	·6714

We can draw the following conclusions:—

(i.) In whatever manner we deduce the fraternal correlation it is very much larger than the ·4 for whole brethren, or the ·2 for half brethren, required by the unmodified Galtonian law. Such values, as we see above, could be deduced from the modified Galtonian law by taking γ greater than unity,* but this would involve values for the parental correlation sensibly less than those given by theory. We are again compelled to assert that the modified as well as the unmodified theory of blended inheritance, based on the Galtonian law, does not fit the facts. The above values, however, are quite compatible with the theory of exclusive inheritance on the assumption that there is an individual (not a sexual) prepotency from one pairing to another.

(ii.) In whatever way we consider it, it would appear that the average value of the fraternal correlation, as deduced from siblings with the same dam only, is greater than that deduced from siblings with both the same dam and the same sire.

I am not able to explain this in any way, for I cannot assert a very substantial prepotency of the dam. All I can say from the data at present available is that for horses and dogs there appears to be no simple numerical relation between the correlation of whole and half brethren.

(iii.) Offspring of the same sex are more alike than offspring of opposite sexes.

This appears to be generally true, so far as our data at present extend, and will be fairly manifest from the table below.

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* 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

TABLE V.—Collateral Heredity.

Pair.	Man.			Dog.	Horse.	
					Coat-colour.	
	Stature.[*]	Cephalic Index.[†]	Eye-colour.[‡]	Coat-colour.[§]	Whole Siblings.	Half Siblings.
Brother-Brother .	·3913	·3790	·5169	·5257	·6232	·3551
Sister-Sister . .	·4436	·4890	·4463		·6928	·4265
Brother-Sister .	·3754	·3400	·4615		·5827	·2834

It will be noted that, with the single exception of eye-colour in man, sister and sister are more alike than brother and brother.

The mean value of the fraternal correlation for stature is ·4034, and for cephalic index ·4027. These results are in excellent accordance with the ·4 required by the Galtonian theory of blended inheritance. The mean values for eye-colour in man, coat-colour in dogs, and coat-colour in horses are: ·4749, ·5257, and ·6329. These are quite incompatible with that theory. I venture accordingly to suggest that we have here cases of an inheritance which does not blend, and that it is an inheritance which is far more closely described by the numbers we have obtained on the theory before developed of exclusive inheritance than by the law of ancestral heredity.

Taking in conjunction with these results for collateral heredity, those for parental and grandparental inheritance, we see that coat-colour in horses diverges widely from the laws which have been found sufficient in the cases of stature and cephalic index in man. The latter characters are really based on a complex system of parts, while the determination of coat-colour may depend on a simple or even single factor in the plasmic mechanism. Here Mr. GALTON'S suggestion of an exclusive inheritance of separate parts ('Natural Inheritance,' p. 139) may enable us to understand why stature and cephalic index differ so widely in their laws of inheritance from coat- and eye-colours.

PART III.—ON THE INHERITANCE OF EYE-COLOUR IN MAN.

(10.) *On the Extraction and Reduction of the Data.*—The eye-colour data used in this memoir were most generously placed at my disposal by Mr. FRANCIS GALTON. They are contained in a manuscript wherein, by a simple notation, we can see at a

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[*] PEARSON, 'Phil. Trans.,' A, vol. 187, p. 253 *et seq.* See Note I. at the end of this paper.

[†] FAWCETT and PEARSON, 'Roy. Soc. Proc.,' vol. 62, p. 413 *et seq.*

[‡] Present memoir, p. 113 *et seq.*

[§] PEARSON, 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

glance the distribution in eye-colour of a whole family in its numerous male and female lines. Such complete details of the various direct and collateral relationships I have not hitherto come across, and from them I was able to form, in the course of some months of work, the twenty-four tables of correlation which are given in Appendix II. These tables, for the first time, give the whole eight series of grandparental and the whole eight series of avuncular relationships, besides such as we have deduced for other characters previously, *i.e.*, the four parental, the three fraternal relationships, and the table for assortative mating. The very great importance of this material will at once be obvious, and I cannot sufficiently express my gratitude to Mr. GALTON for allowing me to make free use of his valuable data.

At the same time we must pay due regard to the limitations of this material, which it is needful to enumerate, so that too great stress may not be laid on the irregularities and divergences which arise when we attempt to reduce the results to laws. These limitations are as follows :—

(*a.*) While the data of about 780 marriages are given in the record, they belong to less than 150 separate families. All our relationships, therefore, contain pairs weighted with the fertility of the individual families. Thus it was necessary to enter every child of a mother, every nephew of an uncle, and so forth. In the horse data we could take 1000 distinct mares and give to each one foal only. That is not possible in the present case.

(*b.*) The colour of eyes alters considerably with age. It is not clear that some of the eye-colours are not given for infants under twelve months, and certainly the eye-colours in the case of grandparents and others must have been taken in old, perhaps extreme old, age. In a large number of other cases of great grandfather, great great-grandfather, &c., great uncles, and so forth, the eye-colours must have been given from memory or taken from portraits—in neither alternative very trustworthy sources. Thus while the horse colour is always given for the yearling foal by the breeder, the eye-colour is given at very different ages, and comes through a variety of channels.

(*c.*) The personal equation in the statement of eye-colour, when the scale contains only a list of tint-names is, I think, very considerable. The issue for the collection of data of a plate of eye-colours like that of BERTRAND would be helpful, but we can hardly hope for a collection of family eye-colours so comprehensive as Mr. GALTON's to be again made for a long time to come.

These causes seem to me to account for a good deal of the irregularity which appears in the numerical reduction of the results, but they are not, I hold, sufficient to largely impair the very great value of Mr. GALTON's material.

In tabulating the data, I have followed the scale of tints adopted by Mr. GALTON, and I have used the entire material available in the cases of the grandparental, avuncular, and marital relations. I nearly exhausted the data for the parental relationships, but in these tables, which were first prepared, I stopped short at 1000 for the sake of whole numbers. I found, however, that it did not make the arithmetic sensibly shorter, and I afterwards dropped this limitation. In the case of brethren I took 1500 of each case—I daresay I could have got 2000 out of the records. As the light-eyed brethren are entered *first* in Mr. GALTON'S MS., the First Brother in my unsymmetrical tables is always lighter-eyed than the Second Brother, hence the tables had to be rendered symmetrical by interchanging and adding rows and columns before we could reduce them. Thus the symmetrical tables have an apparent entry of 3000 pairs. Of course 1500 is the number used in finding the probable error of the correlation coefficient. The like difficulty does not occur in the brother-sister table, where indeed the difference of mean eye-colour for the two sexes would not allow of our making the table symmetrical. A comparison of the symmetrical with unsymmetrical tables for colts-colts and fillies-fillies, will show how little need there is for rendering the tables symmetrical when pairs are taken out of any similar class and tabulated without regard to the relative magnitude of the character in the two individuals of the pair, *i.e.*, WEATHERBY'S record places the individuals simply in order of birth and not of darkness or lightness of coat-colour.

Table VII. gives the value of the chief numerical constants deduced from the twenty-four eye-colour tables in Appendix II.*

(11.) *On the Mean Eye-colour having regard to Sex and Generation.*—In order to test the degree of weight to be given to our conclusions, I have drawn up a table of probable errors for four typical cases—cases by no means selected to give the lowest possible values. Further, in Table VIII. I have given the probable error in the position of the median as determined in terms of the grey, blue-green range by the modification of Mr. SHEPPARD'S formula (see p. 95). The grey, blue-green range of eye-colour is about one-fifth of the total observed range, so that the probable error in the position of the median varies from about ·4 to 1 per cent. of that range. This is not a large error, but, relative to the small variations of value with generation and sex, it is sensible, and we must not draw too fine conclusions on the basis of single inequalities.

TABLE VI.—Table of Probable Errors in Eye-colour Data.

Relations.	η_x	η_y	ζ	u	u'	r_{xy}
Mother and Son . .	·0253	·0188	·0431	·0267	·0256	·0283
Maternal Grandmother and Granddaughter .	·0348	·0350	·0767	·0276	·0314	·0361
Sister and Sister . .	·0244	·0244	—	·0216	·0216	·0234
Maternal Aunt and Nephew	·0230	·0186	·0414	·0255	·0250	·0302

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* The theoretical formulæ by aid of which these constants were determined, have been indicated in the earlier part of this memoir, and in Part VII. of the present series on Evolution. The actual work of reduction has been most laborious, but I trust that our results are free from serious error.

If we examine this table we see that the error in η amounts to from ·02 to ·025 when we have upwards of 1000 tabulated cases, but can amount to ·035 when we have as few as 700 to 750 tabulated cases. An examination of the values of η in Table VII. shows us that most of our differences with probable errors taken on this scale are very sensible. A comparison with Table VIII. shows us that the probable error of the median is always greater than the probable error of η, and accordingly the former, being much easier of calculation, may be taken as a maximum limit. The probable errors of ζ, *i.e.*, the ratio of σ_x to σ_y, are more considerable, amounting to about ·04 for our longer series, and even to ·077 in the case of the short series of grandmother and granddaughter, but in this case ζ actually takes its maximum value of 1·291, so that the error is under 6 per cent.; in the longer series it is under 5 per cent. Again, we see that in most cases our differences in the ratio of variabilities are quite sensible. It must be admitted, however, that the ratio of variabilities as based on the grey blue-green range of eye-colour is not as satisfactory as that based on the bay range of coat-colour in horses. In the latter case, one-half of the horses fall into the bay range, but only about a quarter of mankind fall into the grey blue-green range of eye-colour, and, further, the appreciation of eye-colour seems to me by no means so satisfactory as that of coat-colour in horses.

I have tried a further series of values for the ratio of the variabilities by measuring the ranges occupied not only by the tints grey blue-green, but by the whole range of tints 3, 4, 5, and 6 of Mr. GALTON's classification (see p. 87). Lastly, I have taken a third method of appreciating the relative variabilities, namely, by using the method of column and row excesses, E_1 and E_2, discussed in Part VII. of this series. While this method has the advantage of using all and not part of the observations to determine the ratio of σ_x to σ_y, and so naturally agrees better with the results based on the four than the one tint ranges, it suffers from the evil that these excesses can only be found by interpolation methods, which are not very satisfactory when our classes are, as in this case, so few and so disproportionate. On the whole, this investigation of relative variability is the least satisfactory part of our eye-colour inquiry, and I attribute this to two sources:—

(i.) The vagueness in appreciation of eye-colour when no colour scale accompanies the directions for observation (*cf.* p. 103, (*c*)).

(ii.) A possible deviation from true normality in the factor upon which eye-colour really depends (*cf.* p. 80, (ii) 80).

Lastly, we may note that the probable error in the correlation amounts in most cases to less than ·03, rising only somewhat above this value for grandparental inheritance, where our series are somewhat short—650 to 750 instead of 1000. Here again most of the divergences are quite sensible.

Allowing accordingly for the comparative largeness of our probable errors, we shall do best to base conclusions on the general average of series; to insist on general inequalities rather than on exact quantitative differences, and to emphasise the

TABLE VII.—Eye-colour Inheritance in Man.

Pair of relatives.		Division of grey blue-green range by mean.		Ratio of variabilities.		Coefficient of correlation.	Coefficients of regression.		Number of cases.
x.	y.	η_x.	η_y.	σ_x/σ_y.	σ_y/σ_x.	r_{xy}.	R_{xy}.	R_{yx}.	N.
Father	Son	·5418	·5922	1·0667	·9375	·5503	·5870	·5159	1000
Father	Daughter . . .	·5064	·7322	1·1256	·8885	·4370	·4919	·3883	1000
Mother	Son	·8085	·5937	·9363	1·0680	·4817	·4510	·5145	1000
Mother	Daughter . . .	·8495	·7727	1·0188	·9815	·5096	·5192	·5002	1000
Brother	Brother	·6378	·6378	1	1	·5169	·5169	·5169	1500
Sister	Sister	·8827	·8827	1	1	·4463	·4463	·4463	1500
Brother	Sister	·6643	·8051	·9130	1·0953	·4615	·4213	·5055	1500
Husband	Wife	·4346	·7632	·9209	1·0859	·1002	·9227	1·0881	774
Paternal Grandfather .	Grandson . . .	·4665	·7135	1·0098	·9903	·4213	·4254	·4172	765
Paternal Grandfather .	Granddaughter .	·4698	·8134	1·0384	·9629	·3802	·3948	·3661	681
Maternal Grandfather .	Grandson . . .	·2154	·7129	·9122	1·0963	·3717	·3391	·4075	771
Maternal Grandfather .	Granddaughter .	·3115	·8652	·9675	1·0336	·2969	·2872	·3069	687
Paternal Grandmother .	Grandson . . .	·8491	·6733	·8659	1·1549	·2722	·2357	·3144	741
Paternal Grandmother .	Granddaughter .	·9493	·8156	·9259	1·0800	·2205	·2042	·2381	717
Maternal Grandmother .	Grandson . . .	·9371	·7159	1·1887	·8412	·2523	·2999	·2122	756
Maternal Grandmother .	Granddaughter .	·7673	·9090	1·2909	·7746	·3180	·4105	·2463	739
Paternal Uncle . . .	Nephew	·7387	·6402	·8815	1·1345	·3204	·2824	·3635	1290
Paternal Uncle . . .	Niece	·8087	·9063	·9874	1·0128	·2844	·2808	·2880	1128
Maternal Uncle . . .	Nephew	·4850	·7728	1·0192	·9812	·3706	·3777	·3636	1242
Maternal Uncle . . .	Niece	·4556	·8816	1·0520	·9506	·1938	·2039	·1842	1134
Paternal Aunt . . .	Nephew	·6982	·6563	·9012	1·1096	·2837	·2557	·3148	1186
Paternal Aunt . . .	Niece	·7988	·7539	1·1078	·9027	·1992	·2207	·1798	1149
Maternal Aunt . . .	Nephew	·8179	·6839	·9812	1·0192	·2576	·2528	·2625	1145
Maternal Aunt . . .	Niece	·7210	·7982	·9117	1·0968	·2101	·1915	·2304	1058

TABLE VIII.—On the relative Variability in Eye-colour.

Pair of relatives.		Ratio of variabilities from "excess."		Range of tint 3 in terms of σ. Range $= u\sigma$.		Range of tints 3, 4, 5, 6 in terms of σ. Range $= v\sigma$.		Probable error of median.	
x.	y.	σ_x/σ_y	σ_y/σ_x	u_x.	u_y.	v_x.	v_y.	x.	y.
Father	Son	·9981	1·0019	·6746	·7196	1·4950	1·5110	·0396	·0371
Father	Daughter	1·0532	·9495	·6189	·6966	1·3413	1·6178	·0432	·0384
Mother	Son	·9959	1·0041	·7617	·7132	1·6653	1·4904	·0351	·0375
Mother	Daughter	1·0424	·9593	·7178	·7313	1·5961	1·6500	·0372	·0366
Brother	Brother	1	1	·7759	·7759	1·5634	1·5634	·0199	·0199
Sister	Sister	1	1	·7347	·7347	1·6107	1·6107	·0210	·0210
Brother	Sister	·9917	1·0084	·8010	·7313	1·6492	1·5916	·0273	·0298
Husband	Wife	·9383	1·0657	·8061	·7423	1·7113	1·6203	·0377	·0409
Paternal Grandfather	Grandson	·9536	1·0487	·7939	·8017	1·3992	1·6008	·0385	·0381
Paternal Grandfather	Granddaughter	·9923	1·0077	·7144	·7418	1·3730	1·6793	·0453	·0437
Maternal Grandfather	Grandson	·9667	1·0344	·8120	·7407	1·6221	1·5528	·0366	·0411
Maternal Grandfather	Granddaughter	1·0775	·9281	·7990	·7730	1·4199	1·7836	·0404	·0417
Paternal Grandmother	Grandson	1·0242	·9764	·8882	·7691	1·6016	1·5846	·0350	·0404
Paternal Grandmother	Granddaughter	1·1019	·9075	·7925	·7338	1·5615	1·7600	·0398	·0430
Maternal Grandmother	Grandson	1·0197	·9807	·6332	·7527	1·6379	1·5891	·0486	·0408
Maternal Grandmother	Granddaughter	1·1519	·8681	·5922	·7645	1·4447	1·7965	·0525	·0407
Paternal Uncle	Nephew	1·0179	·9825	·8242	·7265	1·4636	1·5268	·0286	·0324
Paternal Uncle	Niece	1·1728	·8527	·7695	·7598	1·4253	1·8304	·0327	·0331
Maternal Uncle	Nephew	·9984	1·0016	·7977	·8130	1·5620	1·5763	·0301	·0295
Maternal Uncle	Niece	1·0154	·9849	·7468	·7856	1·4396	1·8208	·0336	·0320
Paternal Aunt	Nephew	·9964	1·0036	·8208	·7397	1·6311	1·5929	·0299	·0332
Paternal Aunt	Niece	1·1804	·8472	·7126	·7894	1·4974	1·8097	·0350	·0316
Maternal Aunt	Nephew	·9332	1·0716	·8022	·7871	1·7337	1·5371	·0311	·0317
Maternal Aunt	Niece	1·0730	·9320	·8925	·8137	1·7202	1·7873	·0291	·0319

general tendency of a series rather than pick out single differences for special consideration. If we do this we shall still find that remarkable results flow from our Tables VII. and VIII., most of which seem hitherto to have escaped attention.

I return now to the special topic of the present section, the mean eye-colour, after this lengthy—if needful—digression on the probable error of the data given in our tables.

We may, I think, safely draw the following conclusions :—

(*a*.) *Man has a mean eye-colour very substantially lighter than that of woman.*

If we compare the mean eye-colour of father with mother, of son with daughter, of brother with sister, of grandfather with mother, of uncle with aunt, of grandson with granddaughter, of nephew with niece, we have the same result—man is distinctly lighter eyed than woman.

(*b*.) *There appears to be a secular change taking place in eye-colour, but this is more marked and definite in the man than in the woman.*

Thus we have the following mean values for η in classes, which must roughly represent successive generations :—

Grandfather . . .	·3658	·4449	Grandmother . .	·8757	·8523
Father	·5241		Mother	·8290	
Son	·5929	·6484	Daughter. . . .	·7524	·8016
Grandson. . . .	·7039		Granddaughter. .	·8508	

Another comparison may be made by noting that grandsons are darker than grandfathers, sons than fathers, nephews than uncles. Similarly, granddaughters are lighter than grandmothers, daughters than mothers, but nieces are not lighter than aunts, as we might have expected. Thus, while the records show a definite darkening of the eyes of men, there appears to be a certain but less sensible lightening of the eyes of women. Again, the younger generations are much closer in eye-colour than the older generations. The difference in eye-colour between grandsons and granddaughters, sons and daughters, nephews and nieces is only about 15 per cent. of the grey blue-green range, but for fathers and mothers it is 30 per cent., and for grandfathers and grandmothers 50 per cent.

When we realise that difference in eye-colour appears to be a sexual character, the true explanation of this secular change in eye-colour becomes still more obscure.

If the lighter eye-colour of the older generation be due to an effect of old age, why is it conspicuous only in the male and not in the female? Why is the mother sensibly darker than the daughter, but the father sensibly lighter than the son?

Further, supposing light eyes much commoner among our grandfathers than among their grandsons, and dark eyes among our grandmothers than among their granddaughters, we cannot attribute the great approach in eye-colour to a blending of the parental characters, for, as we shall see later, eye-colour does not seem to blend, it is rather an exclusive character. We should, therefore, be thrown back on prepotency of the mother—a conclusion possibly warranted by our results in the case of daughters, but not in the case of sons. Again, why was there such a marked difference in eye-colour between the men and women of three or four generations back?[*] And if it was a sexual character, why is it disappearing? Was it not, perhaps, a racial difference? Light and dark eyes are not unusually associated with distinct races, and it is just possible that the change in eye-colour is a product of reproductive selection; the old blue-eyed element of the population may be dwindling owing to the greater fertility of the women of dark-eyed race, and thus without any obvious struggle for existence and survival of the fitter, the blue-eyed race may be disappearing from England, as the Langobard type has so largely gone from Italy and the Frank from France.[†] It will not do to be dogmatic about these matters, but the more one measures characters in different generations, the less stable do races appear to be. We speak of the national characters of the Englishman or the Frenchman based upon our experience of how these races have acted in past history, but although there has been no great racial invasion nor struggle, can we safely assert the physical characters of the Englishman to-day do not differ substantially from those of the Englishman of the Commonwealth? It seems to me that the possibly continuous change of characters in a race, not subjected to very apparent internal or external struggle, is a problem of the highest interest to the anthropologist and ultimately to the statesman.

Whatever be the explanation of this secular change in eye-colour, it appears to correspond singularly enough to the secular change we have noted in the coat-colour of thoroughbred horses—in the older generation the sexes differ more widely than in the younger.

(c.) *The maternal male relative (grandfather and uncle) is substantially lighter-eyed than the paternal male relative (grandfather and uncle).*—I see no explanation for this curious result, but it seems worth while to specially note it, for there are curious anomalies in the inheritance through the various male and female lines which may find their complete explanation some day when more and possibly more trustworthy characters have been investigated.

(12.) *On the Variability of Eye-colour with reference to Sex and Class.*—The determination of the relative variability of not exactly measurable characters is, as we have already seen (p. 105), a somewhat delicate problem. It is more so in the case of eye-colour in man than of coat-colour in horses, for there is greater difference in the means, and accordingly the ratio of σ_x/σ_y, as found from the ratio of the "excesses" (p. 105), will be even less reliable.[‡] The class indices corresponding to the

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[*] Mr. GALTON'S records went back to great-grandfathers, many of whom accordingly appear in our data for grandfathers.

[†] See Note II. at the end of this paper.

[‡] The relative variability of all classes was worked out for thorough-bred horses by the "excess" method, and in only one case—that of dam and colt—did it differ from the bay range method in its determination of the class with the greater variability.

grey blue-green range are also not entirely satisfactory in their results, nor those taken for a still larger range covering tints 3, 4, 5, and 6, or blue-green, grey, hazel, light brown, and brown, which cover roughly about 1·5 to 1·6 times the standard deviation. We shall now consider the results of three methods of considering the relative variability, (α) from the excesses given in columns 1 and 2 of Table VIII.; (β) from the grey blue-green range given in columns 3 and 4 of Table VIII.; and (γ) from the range of tints 3 to 6 inclusive given in columns 5 and 6 of Table VIII. As we have already indicated, these methods are not likely to give the same relative magnitude numerically for the variabilities; we must content ourselves if they agree in making the ratio of σ_x to σ_y greater or less than unity. Now, in the twenty-two cases

α and β disagree in 10 cases.
β and γ disagree in 7 cases.
α and γ disagree in 5 cases.

Further, for the five cases in which α and γ disagree, those for father and son, paternal grandfather and grandson, maternal uncle and nephew, show so little difference of variability in the two sexes that both methods give sensibly the same results, *i.e.*, equality of variability. In the cases of the paternal grandfather and grandchildren, the two methods diverge rather markedly.

It will be of interest accordingly to work out the probable errors as given by the excess method for one, say the first of these cases. The theory is given in Part VII. of the present series. Here $E_1 = 275·165$, $E_2 = 309·013$, whence we find probable error of $E_1 = 17·273$, probable error of $E_2 = 16·925$, correlation between errors in E_1 and $E_2 = -·4424$, probable error in $\sigma_1/\sigma_2 = ·0394$.

Thus the probable error in the ratio of the variabilities is about 4 per cent., and of the order of the quantities by which we are distinguishing the relative size of σ_1 and σ_2.

Further, there is another source of error in evaluating E_1 and E_2 due to the method of interpolation used, and this would still further increase the probable error in σ_1/σ_2. We cannot therefore lay any very great stress on the manner in which the ratios of variabilities for the paternal grandfather and grandchildren have swung round from (α) to (γ).

A further examination shows us that in all five cases wherein γ differs from α it is in accord with β. I shall accordingly take γ as the standard criterion, but in those cases in which it has agreement with α, its conclusions must be given greater weight.

(a.) *On the Relative Variability of Sex in Eye-colour.*—The following male groups are more variable than the corresponding female groups:—

Sons of fathers than daughters of fathers.
Sons of mothers than daughters of mothers.
Brothers of brothers than sisters of sisters.

Grandsons (in four series) than granddaughters (in four same series).
Nephews (in four series) than nieces (in four same series).
Fathers (in two series) than mothers (in two series).
Grandfathers (in four series) than grandmothers (in four like series).
Uncles (in four series) than aunts (in four like series).

The following female groups are more variable than the corresponding male groups :—

Sisters of brothers than brothers of sisters.
Wives than husbands.

We have thus 21 series with male preponderance against only two with female preponderance of variability.

Again, the mean range of tints 3, 4, 5, 6 in 22 male series equals 1·5424 σ_x, and in 22 female series equals 1·6740 σ_y, or we have enough evidence to show that the ratio of male to female variability is about 1·08.*

This greater variability of the male in eye-colour is of considerable interest. It does not appear to be a result of sexual selection, for so far as our comparatively small series weighs, husbands are less variable than wives. That mothers are, however, less variable than fathers seems to indicate that dark-eyed women are more fertile† than light-eyed, for we must bear in mind that mothers have on the average a darker eye-colour than wives. We have thus again reached the same conclusion as before, namely, that a dark-eyed element in the population with a prepotent fertility is replacing the blue-eyed element.

The other female exception to the general rule of greater variability in the eye-colour of the male is that in mixed families the sisters appear to be more variable than their brothers, notwithstanding that brothers of brothers are more variable than sisters of sisters. In other words, so far as eye-colour is concerned an exceptional man is more likely to have brothers than sisters, but an exceptional woman also is more likely to have brothers than sisters. The inference is not very strong, as the excess method (α) makes brothers of sisters and sisters of brothers of sensibly equal variability; it rests therefore on (β) and (γ) only. Still it deserves fuller investigation.

(*b*.) Let A and B be two grades of relationship, of which A refers to the older generation, and A and B refer to either sex. Then the variability of all the A's

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* It is worth noting that the ratio of male to female variability in the coat-colour of horses is 1·05 (see p. 96). In both cases the female is darker, *i.e.*, has less of "colour"; thus if we could take a coefficient of variation ratio instead of standard deviation ratio as the test, we should find the difference of variability less, possibly even zero.

† For if mothers are to be less variable than wives, their distribution must be more compressed round the mean than that of wives; this denotes that fertility is correlated with eye-colour, and the darker eye-colour goes with the greater fertility. [See Note II. at end of memoir, however.]

who have female B's is invariably greater than the variability of all the A's who have male B's.

The law appears to be universal, at least it is absolutely true for all the 10 cases to which we can apply it. Thus the father of sons is less variable than the father of daughters, the maternal grandmother of grandsons less variable than the maternal grandmother of granddaughters, or the paternal uncle of nephews less variable than the paternal uncle of nieces. In other words, although women appear, in eye-colour, to be less variable than men, they spring from more variable stocks.

This law is a remarkable one, but in face of the evidence for it, it seems difficult to doubt its validity. Should it be true for more characters in man than eye-colour,* the conclusions to be drawn from it will be somewhat far-reaching, however difficult it may be to interpret its physiological significance.

(*c.*) *On the Relative Variability of Different Generations.*—We have already had occasion to refer to the general rule that the older generation will be found less variable than the younger, for it is in itself a selection, namely, of those able to survive and reproduce themselves. But this rule is obscured in the present case by several extraneous factors, thus :—

(i.) The male is sensibly more variable than the female, consequently it is quite possible that an elder male generation should appear more variable than a younger female generation.

(ii.) There appears to be a secular change in eye-colour going on. Thus while the grandparental population is a selection from the general population, the general population, at a given time, is a selection from that of an earlier period.

Thus, taking means in the cases of the grandparental and avuncular relationships, we have from (γ) the following results :—

The father is more variable than son and than daughter.
The mother is less variable than son and more than daughter.
The grandfather is more variable than grandson and than granddaughter.
The grandmother is less variable than grandson and more than granddaughter.
The uncle is more variable than nephew and more than niece.
The aunt is less variable than nephew and more than niece.

In other words, the older generation is always more variable than the younger, except when rule (a), that the male is more variable than the female, comes in to overturn this law. If we confine ourselves to comparisons of the same sex the rule is seen to be universal.

We are thus forced again to ask for an explanation of the decreasing variability of eye-colour, and can only seek it in that secular change we have several times had

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* Fathers of daughters are more variable in stature than fathers of sons ('Phil. Trans.,' A, vol. 187, p. 274). I propose to reinvestigate the question with regard to mothers from the material of my family measurement cards, which is far more extensive than the material I had at my disposal in 1895.

occasion to refer to. Mean and standard deviation of eye-colour appear to have changed sensibly during the few generations covered by Mr. GALTON'S eye data.

It is difficult to understand how any obscurity about the recording of eye-colours could lead to anything but chaos in the numerical results. It does not seem to me possible that such results as we have reached under (a), (b), and (c), namely, greater variability in the male, greater variability in the stock of the female, and secular change in variability, can be due to any process of recording. I am forced to the conclusion that they are peculiar to the character under investigation, and are not due to the manner of taking the record or of dealing with it arithmetically. I have purposely avoided drawing attention to small differences and forming any conclusions which did not depend on whole series of groups and substantial averages.

(13.) *On the Inheritance of Eye-colour.* (a.) *Assortative Mating.*—Before we enter on the problem of inheritance, it is as well to look at the substantial correlation obtained between the eye-colour in husband and wife. When in 1895 I reached the value ·0931 ± ·0473 for stature, I wrote, "we are justified in considering that there is a definite amount of assortative mating with regard to height going on in the middle classes."[*] Since then we have worked out the coefficients of correlation in stature, forearm, and span for 1000 husbands and wives (instead of 200) from my family data[†] cards. The results, which are very substantial, will be dealt with in another paper, and amply confirm my view that assortative mating is very real in the case of mankind. The value (·0931) cited above is in close agreement with the result now reached (·1002 ± ·0378) for eye-colour in the *same* material. The correlation between husband and wife for two very divergent characters is thus shown to be about ·1, or is 25 *per cent. greater than is required between first cousins*[‡] *by the law of ancestral heredity.*

This remarkable degree of likeness between husband and wife—the scientific demonstration that like seeks like—cannot be overlooked. It shows that sexual selection, at least as far as assortative mating is concerned, is a real factor of evolution, and that we must follow DARWIN rather than WALLACE in this matter.[§]

(b.) *Collateral Heredity. First Degree.*—I deal first with this form of heredity, as it offers least points for discussion. The values of the correlation ·5169 for brothers, and ·4463 for sisters and sisters are considerably less than what we have found for coat-colour in horses, but, like the value ·4615 for brothers and sisters, are substantially greater than ·4 to be expected from the unmodified Galtonian law. They could be reached by making γ greater than unity in my statement of the law of ancestral heredity.[‖] They could also be given by the law of exclusive inheritance

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[*] 'Phil. Trans.,' A, vol. 187, p. 273.

[†] See also 'Grammar of Science,' second edition, pp. 429–437.

[‡] 'Roy. Soc. Proc.,' vol. 62, p. 410.

[§] 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

[‖] I have considered possible explanations of this apparently large assortative mating (i.) in both stature (see p. 90) with a certain degree of prepotency in the individual pairing. As we have already noted, collateral inheritance of the first degree alone considered will not enable us to discriminate between blended and exclusive inheritance.

We note that the male in collateral inheritance predominates over the female, brothers being more alike than sisters in eye-colour, and brother and sister more alike than sister and sister. The mean value for inheritance in the same sex is, however, greater than the value for inheritance between opposite sexes (*cf.* p. 102).

(*c.*) *Collateral Heredity. Second Degree.*—A very cursory inspection of the coefficients of correlation for the eight series of avuncular relationships shows us that it is quite impossible that the mean value should be ·15 as required by the Galtonian Law. The average value of the avuncular correlation is ·2650, and of the regression of nephew and niece on uncle or aunt is ·2733. The probable error of the former result will not be more than ·02, and of the latter something greater, as the ratio of the variabilities is open to larger error. This mean value is accordingly, within the limits of errors of investigation, identical with the ·25 to be expected on the theory of exclusive inheritance. It is a value which appears to be quite impossible on the theory of blended inheritance even with my generalised form of the ancestral law.

We may draw several other important conclusions from our table of avuncular correlations:—

(i.) Nephews are more closely related to both uncles and aunts than nieces are. This is true in each individual case, whether it be judged by correlation or regression. The mean correlations for uncles and for aunts are as ·3081 to ·2219 respectively.

(ii.) Uncles are more closely related to nephews and nieces than aunts are. This is true for three out of the four individual cases; in the fourth case the difference is of the order of the probable error of the difference. The mean correlations ot nephews and nieces are as ·2923 and ·2377 respectively.

(iii.) Paternal uncles and aunts are more closely correlated with their nieces and nephews than maternal uncles and aunts. The mean values are as ·2719 to ·2580.

(iv.) Resemblance between individuals of the same sex is closer than between individuals of opposite sex. The mean values for the avuncular correlation in the same sex and in the opposite sex are respectively ·2751 and ·2549.

(v.) Uncles are more closely related to nephews than aunts to nieces (mean correlations as ·3455 to ·2046). In fact, generally, we see a very considerable preponderance of heredity in the male line so far as these avuncular relations for

and eye-colour, being characters of local races, or even families, and the husband seeking his wife in his own locality or kin; (ii.) in a possible correlation of homogamy and fertility. See 'Roy. Soc. Proc.,' vol. 66, p. 28. Neither seem very satisfactory. Consciously or unconsciously, man and woman appear to select their own type in eye-colour and stature, until they are apparently more alike than such close blood relations as first cousins! Until we know how far this correlation extends to other characters, it would, perhaps, be idle to draw conclusions as to its bearing on widely current views as to first cousin marriages.

eye-colour extend. It is noteworthy that while the two highest correlations are reached for nephew with paternal and with maternal uncles, nearly the two lowest are found for niece with paternal and with maternal aunts. Without laying special stress on each small difference, it must be admitted that the avuncular correlations vary in a remarkable manner with sex, and differ very widely from the practical equality of resemblance which we might *à priori* have expected to exist in this relationship.

(*d.*) *Direct Heredity. First Degree.*—Here we have a mean value of the paternal correlation = ·4947. This is in excellent agreement with the ·5 to be expected by our theory of exclusive inheritance; it is thus in practical agreement with the value of the parental correlation obtained for the inheritance of coat-colour in horses. It would not be inconsistent with a high value for γ in the theory of blended inheritance, but such a value of γ is rendered impossible by the values we have obtained for collateral heredity (see 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*).

We may draw the following special conclusions:—(i.) The son inherits more strongly from his parents than the daughter, the mean correlations are as ·5160 to ·4733; (ii.) The son inherits more strongly from his father than his mother, and the daughter more strongly from her mother than her father.

This is part of the general principle which we have seen to hold, namely: that change of sex weakens the intensity of heredity.

The correlation of father and daughter appears to be abnormally below the other three, but something of the same kind has been noted in certain stature data; as it is, the high correlation of father and son renders the mean paternal correlation with offspring (·4936) sensibly equal to the mean maternal correlation (·4956).

(*e.*) *Direct Heredity. Second Degree.*—If we take the mean value of the eight grandparental correlations, we find it equals ·3164, while the mean value of the regression of offspring on their grandparents is ·3136. These results are absolutely incompatible with the ·15 required by Mr. GALTON's unmodified theory, and they in fact put the theory of blended inheritance entirely out of court. At the same time, unlike the cases of parental, avuncular, and fraternal inheritance, they cannot be said to be in good agreement with the value ·25 required by the theory of exclusive inheritance. We have to admit that our grandparental data are shorter series than in the other cases, and that guesses as to grandparents' eye-colour, based on memory, miniatures, &c., were more likely to be made. Further, such guesses might easily be biased by a knowledge of the eye-colour of more recent members of the family. Still a reduction from ·32 to ·25 is a very large reduction, and we have to remember that for long series in the case of the thoroughbred horses, with no such guessing at colour as may occur with ancestors' eyes, we found ·3353 for the maternal grandsires, a result in excellent agreement with the ·3343 found for the maternal grandfathers in the present case. Thus while the theory of exclusive inheritance *without reversion* suffices to describe the quantitative values we have found for the parental, the avuncular and the fraternal correlation in the cases of both horse and man, it is yet in both these cases unsatisfactory so far as the grandparental inheritance is concerned. It may be imagined that if we allowed for reversion, we might emphasise the grandparental correlation beyond the value ·25 suggested by theory. But I have shown in my memoir on the "Law of Reversion," that with the parental correlation as high as ·5, we cannot hope to have the grandparental correlation even with reversion higher than ·25. (See 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*) Clearly the values obtained for grandparental correlation in this paper—the first I believe hitherto investigated—seem to present anomalies which our theory of blended inheritance totally fails to account for, and which may require some modification of our views on reversion before we can meet them on our theory of exclusive inheritance.

I note the following general results deduced from our values of the grandparental correlations :—

(i.) Grandsons are more closely correlated with both grandparents than granddaughters are. This is true for three out of the four cases; the exception, maternal grandmother, is covered by another rule (iv.). The mean correlation for grandparents and grandsons is ·3294, and for grandparents and granddaughters ·3039.

(ii.) Grandfathers are more closely correlated with grandchildren than grandmothers are. This is true in three out of the four cases, the fourth being again subject to rule (iv.). The mean correlations for grandfathers and grandmothers are ·3675 and ·2658 respectively.

(iii.) Paternal grandparents appear to be more closely correlated with their grandchildren than maternal grandparents, the average values of the two correlations being ·3236 and ·3097 respectively.

(iv.) Resemblance between individuals of the same sex is closer than between individuals of the opposite sex. The mean values for the grandparental and grandchild correlation in the same and the opposite sexes are ·3329 and ·3004 respectively.

(v.) Grandfathers are more closely related to grandsons than grandmothers to granddaughters, the mean correlations being as ·3965 and ·2693 respectively. It will be noted at once that these five rules are identical with those we have obtained for the avuncular correlations. So that there is small doubt that they are general rules relating to all grades of relationship for this character.

It seems to me probable that the correct form of (iii.) is : Paternal grandfathers are more highly correlated with grandchildren (·4006) than maternal grandfathers (·3343), and paternal grandmothers (·2468) less highly correlated than maternal grandmothers (·2851). I have not stated the rule in this form, because it is not confirmed by the corresponding results for uncles and aunts. Paternal uncles (·3024) are more closely correlated with nephews and nieces than maternal uncles (·2722), but paternal aunts (·2414) are slightly more instead of less correlated with nephews and nieces than maternal aunts (·2338). I consider, however, that the correlation of paternal aunt and nephew (·2837) in our series is abnormally high.

Now it will, I believe, be seen that the investigation of the eight avuncular and the eight grandparental relationships, here made for the first time,* enables us to draw far wider conclusions than when, as hitherto, only parental and fraternal correlations are dealt with. In making the subjoined general statements, however, I must emphasise the following limitations :—

(α.) The rules are deduced only from data for one character in one type of life.

(β.) This character appears to be undergoing a secular change, a change very possibly due to a correlation between eye-colour and fertility in woman. Thus such a change might not unlikely differentiate the male and female influences in heredity.

My conclusions, definitely true for eye-colour in man, and at the very least suggestive for investigations on other characters in other types of life, are :—

(i.) That the younger generation takes, as a whole, more after its male than its female ascendants and collaterals.

(ii.) That the younger generation is more highly correlated with an ascendant or collateral of the same than of the opposite sex.

(iii.) That the younger generation is more highly correlated with an ascendant or higher collateral reached by a line passing through one sex only than if the line changes sex.

Thus correlation is greater with a paternal uncle than with a maternal uncle, or with a maternal grandmother than a paternal grandmother.

(iv.) Males are more highly correlated with their ascendants and collaterals than females are.

The above rules apply to the *averages ;* individual exceptions will be generally found to arise from a conflict of rules. Thus (ii.) and (iii.) may in special cases come into conflict with (i.). When we have more data for a greater variety of characters, we shall see better the relative weight of these rules in cases where they conflict.

(*f.*) *Exclusive Inheritance in Eye-Colour.*—A cursory examination of the eye-colour records shows at once how rare is a blend of the parental tints. Even when such is recorded, it is by no means clear that we have not to deal with a medium tint which is really a case of reversion to a medium tinted ancestor. The failure of eye-colour to blend is, I think, well illustrated by what Mr. GALTON has termed cases of "particulate" inheritance. In the thousands of eye-colours I have been through, I noticed some half-dozen cases only in which the two eyes of the same individual were of different tint, or the iris of one pupil had streaks of a second tint upon it.†

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* I anticipate equally valuable results when characters are first correlated for the nine possible cousin series.

† In the same manner the occurrence of particulate inheritance in coat-colour in horses may be really an argument against the existence of blends. In the many volumes of the studbooks I have examined, the recorded instances of piebalds are vanishingly few in number.

If we allow that it is from the theory of exclusive inheritance that we must seek results in the present cases, we see that for parental, collateral, and avuncular relationships we get quite excellent results, but that the grandparental relationship is somewhat anomalous. *À priori* it might appear that reversion would aid us in increasing the correlation between offspring and remote ascendants. But, as I have shown elsewhere,* this superficial view of reversion forgets that the parents as well as the offspring revert, and if we increase the grandparental correlation above ·25, we at once reach difficulties in the values of the parental correlation, provided we adopt what appear to be reasonable assumptions as to reversion being a continuous and decreasing factor from stage to stage of ancestry. I am inclined accordingly to suspend judgment on the grandparental relationships, thinking that the smallness of the number of families dealt with in Mr. Galton's data (200) may have something to do with my peculiar results. Meanwhile I shall endeavour to get the remaining six grandparental tables for thoroughbred horses worked out, and see whether they confirm the high values already found for the two maternal grandsires and offspring, or give an average value much nearer ·25.

That the reader may see at a glance the general results hitherto obtained in this and other papers, I append the following table of inheritance :—

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* See my paper on "The Law of Reversion," 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.* Also 'The Grammar of Science,' second edition, 1900, pp. 486–96, "On Exclusive Inheritance."

TABLE IX.—Theoretical and Actual Results for Inheritance.

Relationship.	Theory.		Man.			Horse.	Hound.	Daphnia.
	Blended inheritance.[1]	Exclusive inheritance.[2]	Stature[3].	Head index.[4]	Eye-colour.[5]	Coat-colour.[6]	Coat-colour.[7]	Spine.[8]
Parental	·3000	·5000	·3355	·3348	·4947	·5216	·3507	[·3295]
Mid-parental . . .	·4242	—	·4745	·4735	—	—	—	·4660
Grandparental . .	·1500	·2500	—	—	·3166	·3353	·1340	[·1360]
G. Grandparental .	·0750	·1250	—	—	—	—	·0404	—
Avuncular . . .	·1500	·2500	—	—	·2650	—	—	—
Whole sibling . .	·4000	·4 to 1·0	·4034	·4025	·4749	·6329	·5170	·6934
Half sibling . . .	·2000	·2 to ·5	—	—	—	·3550	·1646	—

[1] Mr. GALTON'S unmodified hypothesis. See "Law of Ancestral Heredity," 'Roy. Soc. Proc.,' vol. 62, p. 397.

[2] Without any reversion. See "Law of Reversion," 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.* The values for the fraternal correlation depend on the degree of prepotency of either parent within the union.

[3] See 'Phil. Trans.,' A, vol. 187, p. 270.

[4] See 'Roy. Soc. Proc.,' vol. 62, p. 413. The paternal correlations, for reasons stated in the paper, are excluded from the result.

[5] See p. 113 *et seq.* of the present memoir.

[6] See p. 98 *et seq.* of the present memoir. The grandparental correlation is based on two cases only.

[7] See 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

[8] See 'Roy. Soc. Proc.,' vol. 65, p. 154. I have deduced the value for parents and grandparents from Dr. WARREN'S results for midparent and midgrandparent. The value for whole siblings I obtained from Dr. WARREN'S measurements, which he with great kindness placed at my disposal.

(14.) *Conclusions.*—The course of this investigation has not been without difficulties, and I am fully prepared to admit that more obscurity and greater probable errors are likely to arise when we deal with the inheritance of a character not directly measurable, than when we take that of a character to which we can at once apply a quantitative scale. But I contend that many of the characters, the inheritance of which it is most important to investigate, do not at present, and perhaps never will, admit of a quantitative measurement. We can arrange in order, we can classify, we can say more or less intense, but we cannot read off value on a scale. It is just such characters also, which the not highly trained observer can most easily appreciate and record. Hence we have been compelled to devise some method of dealing with them, and the present paper illustrates how the method invented can be applied to reach results of considerable interest and of substantial validity.

In order to illustrate the method, I chose two characters, coat-colour in horses and eye-colour in man, which seemed sufficiently diverse both as to origin and species.* The new method enabled me to reach results for half-brethren, grandparents and uncles and aunts, which had not yet been independently considered. The conclusions arrived at for eye-colour in man at no point conflict with those for coat-colour in horses, and both in the main accord with the theory of exclusive inheritance without reversion herein developed. We find—

(i.) No approach to a single value for the coefficient of inheritance for each grade of relationship; it varies widely with the sex, and the line through which the relationship is traced.

(ii.) No approach in average values to those which would be indicated by Mr. GALTON'S Law.

Nor does the modification of Mr. GALTON'S Law, which I have termed the Law of Ancestral Heredity, give better results. For, if we cause it to give the parental values, it then renders results inconsistent with the fraternal values.

(iii.) There is agreement with the theory of exclusive inheritance without reversion for the parental, avuncular and fraternal series; but there is some anomaly in the case of grandparental inheritance. This requires further investigation, and possibly a modification of our views on the nature of reversion.

We want a list formed of characters in various types of life, which are supposed to be exclusively inherited, and then experiments ought to be made and statistics collected with regard to these characters. It is in this field of exclusive inheritance that we must look for real light on the problem of reversion.

If we consider the three known forms of inheritance, the blended, the exclusive, and the particulate (which may possibly be combined in one individual, if we deal with different organs); if we consider further that these forms may possibly have to be supplemented by others not yet recognised (*e.g.*, reversional theories depending, say, on heterogamous unions), then it would appear that the time is hardly ripe even for provisional *mechanical* theories of heredity. What we require to know first is, the class of organs and the types of life for which one or other form of inheritance predominates. As variation in no wise depends on the existence of *two* germ-plasms, so biparental heredity can by no means be treated as the result of their simple quantitative mixture; the component parts of these germ-plasms corresponding to special characters and organs, must be able to act upon each other in a variety of qualitatively different ways. To adopt for a moment the language of DARWIN'S theory of pangenesis, the multiplying gemmules from an organ in the father must (i.) cross with gemmules from that organ in the mother, and the hybrid gemmules give rise to blended inheritance, (ii.) must without crossing multiply alongside the gemmules of the mother, and give rise to particulate inheritance, (iii.) must alone survive, or alone

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* Since supplemented by my investigations, based on Mr. GALTON'S data, for coat-colour in hounds, 'Roy. Soc. Proc.,' vol. 66, p. 140 *et seq.*

be destroyed in a struggle for existence with those of the mother, and give rise to exclusive inheritance. And all these three processes may be going on within the same germ-plasm mixture at the same time! Even without using the language of gemmules, processes analogous to the above must be supposed to take place. Thus a quantitative "mixture of germ-plasms" becomes a mere name, screening a whole range of mechanical processes; and very possibly a new one could be found for each new form of heredity as it occurs. Such processes like the old ones would still remain without demonstrable reality under the veil of "mixture of germ-plasms."

What I venture to think we require at present is not a hypothetical plasmic mechanics, but careful classifications of inheritance for the several grades of relationship, for a great variety of characters, and for many types of life. This will require not only the formation of records and extensive breeding experiments, but ultimately statistics and most laborious arithmetic. Till we know what class of characters blend, and what class of characters is mutually exclusive, we have not within our cognizance the veriest outlines of the phenomena which the inventors of plasmic mechanisms are in such haste to account for. Such inventors are like planetary theorists rushing to prescribe a law of attraction for planets, the very orbital forms of which they have not first ascertained and described. Without the observations of Tycho Brahé, followed by the arithmetic of Kepler, no Newton had been possible. The numerical laws for the intensity of inheritance must first be discovered from wide observation before plasmic mechanics can be anything but the purest hypothetical speculation.

Appendix I.

Tables of Colour Inheritance in Thoroughbred Racehorses, extracted by Mr. Leslie Bramley-Moore from Weatherby's Studbooks.

Table of Colours.

1 = black (bl.)
2 = black or brown (bl./br.).
3 = brown or black (br./bl.).
4 = brown (br.).
5 = brown or bay (br./b.).
6 = bay or brown (b./br.).
7 = bay (b.).
8 = bay or chestnut (b./ch.).
9 = chestnut or bay (ch./b.).
10 = chestnut (ch.).
11 = chestnut or roan (ch./ro.).
12 = roan or chestnut (ro./ch.).
13 = roan (ro.).
14 = roan or grey (ro./gr.).
15 = grey or roan (gr./ro.)
16 = grey (gr.).

Table I.—Direct Inheritance. First Generation.

Sires and Colts.

Sires.

Colts.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	3	0	0	7	0	0	12	0	0	7	0	0	0	0	0	0	29
2	bl./br.	0	0	0	2	0	1	4	0	0	0	0	0	0	0	0	0	7
3	br./bl.	0	0	0	1	0	0	3	0	0	0	0	0	0	0	0	0	4
4	br.	2	0	0	64	0	4	105	0	0	38	1	0	0	0	0	1	215
5	br./b.	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	3
6	b./br.	0	0	0	12	0	0	23	0	0	8	0	0	0	0	0	1	44
7	b.	1	2	1	89	0	14	389	0	0	122	0	0	0	0	0	1	619
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
10	ch.	1	2	0	34	0	0	147	0	0	185	0	0	0	0	0	3	372
11	ch./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	0	0	0	0	0	3	0	0	2	0	0	0	0	0	0	5
Totals...	...	7	4	1	209	0	19	691	0	0	362	1	0	0	0	0	6	1300

TABLE II.—Direct Inheritance. First Generation.

Sires and Fillies.

Sires.

Fillies.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	0	0	0	3	0	0	10	0	0	3	0	0	0	0	0	0	16
2	bl./br.	0	0	0	8	0	0	3	0	0	1	0	0	0	0	0	0	12
3	br./bl.	0	0	0	1	0	0	6	0	0	0	0	0	0	0	0	0	7
4	br.	2	1	1	42	0	6	76	0	0	23	0	0	0	0	0	0	151
5	br./b.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
6	b./br.	0	0	0	6	0	0	29	0	0	7	0	0	0	0	0	0	42
7	b.	2	1	0	57	0	12	364	0	0	91	0	0	0	0	0	0	527
8	b./ch.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	2
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	2	1	0	21	0	1	118	0	0	133	2	0	0	0	0	1	279
11	ch./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	3
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
16	gr.	0	0	0	1	0	0	0	0	0	4	0	0	0	0	0	3	8
Totals...	...	6	3	1	139	0	19	610	0	0	264	2	0	0	0	0	6	1050

TABLE III.—Direct Inheritance. First Generation.

Dams and Colts.

Dams.

Colts.

		1 bl.	2 bl./br.	3 br./bl.	4 br.	5 br./b.	6 b./br.	7 b.	8 b./ch.	9 ch./b.	10 ch.	11 ch./ro.	12 ro./ch.	13 ro	14 ro./gr.	15 gr./ro.	16 gr.	Totals.
1	bl.	5	1	0	3	0	0	3	0	0	4	0	0	0	0	0	0	16
2	bl./br.	1	0	0	2	0	0	4	0	0	0	0	0	0	0	0	0	7
3	br./bl.	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
4	br.	6	1	0	59	0	5	60	0	0	18	0	0	0	0	0	0	149
5	br./b.	0	0	0	3	0	0	1	0	0	1	0	0	0	0	0	0	5
6	b./br.	1	0	0	14	0	3	15	0	0	6	0	0	0	0	0	0	39
7	b.	9	1	0	76	0	6	295	0	1	86	0	1	1	0	0	4	480
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	6	0	0	27	0	2	127	0	0	133	0	0	2	0	0	3	300
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
16	gr.	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	2
Totals...	...	28	3	1	184	0	16	505	0	1	249	1	1	4	0	0	7	1000

TABLE IV.—Direct Inheritance. First Generation.

Dams and Fillies.

Dams.

Fillies.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	1	1	0	1	0	1	4	0	0	2	0	0	0	0	0	0	10
2	bl./br.	1	0	0	4	0	0	8	0	0	2	0	0	0	0	0	0	15
3	br./bl.	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
4	br.	3	0	0	52	0	2	67	0	0	18	0	0	1	0	0	0	143
5	br./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	b./br.	3	0	0	15	0	1	31	0	0	6	0	0	0	0	0	0	56
7	b.	13	1	0	74	0	6	288	1	0	79	0	0	1	0	0	1	464
8	b./ch.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	4	1	0	27	0	7	109	0	1	146	0	1	0	0	0	3	299
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
15	gr./ro.	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	2
16	gr.	0	0	0	1	0	0	1	0	0	2	0	0	0	0	0	3	7
Totals ...	...	25	3	0	175	0	18	509	1	1	256	0	1	4	0	0	7	1000

TABLE V.—Direct Inheritance. Second Generation.

Maternal Grandsires and Colts.

Grandsires.

Colts.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Totals.
		bl.	bl./br.	br./bl.	br.	br./b.	b./br.	b.	b./ch.	ch./b.	ch.	ch./ro.	ro./ch.	ro.	ro./gr.	gr./ro.	gr.	
1	bl.	0	0	0	1	0	1	12	0	0	4	0	0	0	0	0	0	18
2	bl./br.	0	0	0	2	0	0	11	0	0	0	0	0	0	0	0	0	13
3	br./bl.	0	0	0	1	0	0	2	0	0	1	0	0	0	0	0	0	4
4	br.	1	0	0	29	0	2	94	0	0	29	0	0	0	0	0	1	156
5	br./b.	0	0	0	2	0	0	1	0	0	0	0	0	0	0	0	0	3
6	b./br.	2	0	0	7	0	3	24	0	0	12	0	0	0	0	0	0	48
7	b.	0	1	0	51	0	8	286	0	0	100	0	0	0	0	0	1	447
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	0	2	0	17	0	1	153	0	0	128	2	0	0	0	0	1	304
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
14	ro./gr.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	0	0	0	0	1	2	0	0	1	0	0	0	0	0	1	5
Totals...	...	3	3	0	110	0	16	587	0	0	275	2	0	0	0	0	4	1000

Table VI.—Direct Inheritance. Second Generation.

Maternal Grandsires and Fillies.

Grandsires.

Fillies.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	0	0	0	5	0	0	16	0	0	8	0	0	0	0	0	1	30
2	bl./br.	0	0	0	2	0	0	6	0	0	1	0	0	0	0	0	0	9
3	br./bl.	0	0	0	0	0	0	1	0	0	2	0	0	0	0	0	0	3
4	br.	0	2	0	22	0	4	72	0	0	31	0	0	0	0	0	0	131
5	br./b.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
6	b./br.	0	1	0	6	0	3	31	0	0	7	0	0	0	0	0	0	48
7	b.	1	3	0	55	0	5	305	0	0	101	0	0	0	0	0	0	470
8	b./ch.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	0	1	0	34	0	3	141	0	0	116	1	0	0	0	0	1	297
11	ch./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	2
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
16	gr.	0	0	0	0	0	0	2	0	0	2	0	0	0	0	0	2	6
Totals...	...	1	7	0	124	0	15	579	0	0	269	1	0	0	0	0	4	1000

Table VII.—Collateral Inheritance. First Degree. (Half-Brothers.)

Colts and Colts. Same Mare but different Sires.

First Colt.

Second Colt.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	2	1	0	6	0	1	8	0	0	5	0	0	0	0	0	0	23
2	bl./br.	0	0	0	2	0	0	5	0	0	6	0	0	0	0	0	0	13
3	br./bl.	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	2
4	br.	4	3	0	44	0	10	72	0	0	34	0	0	0	0	1	0	168
5	br./b.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
6	b./br.	1	0	0	4	0	10	19	0	0	8	0	0	0	0	0	0	42
7	b.	5	3	2	62	1	19	242	0	0	119	0	0	0	0	0	1	454
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
10	ch.	2	2	0	24	0	5	119	0	0	134	0	0	0	0	0	0	286
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	2
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	0	0	1	0	0	2	0	0	3	0	0	0	0	0	2	8
Totals...	...	14	9	2	143	1	46	470	0	0	311	0	0	0	0	1	3	1000

TABLE VII$^{\text{bis}}$.—Collateral Heredity. First Degree. (Half-Brothers.)

Colts and Colts. Same Mare but different Sires.

Symmetrical Distribution.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	4	1	0	10	0	2	13	0	0	7	0	0	0	0	0	0	37
2	bl./br.	1	0	0	5	0	0	8	0	0	8	0	0	0	0	0	0	22
3	br./bl.	0	0	0	0	0	1	3	0	0	0	0	0	0	0	0	0	4
4	br.	10	5	0	88	0	14	134	0	0	58	0	0	0	0	1	1	311
5	br./b.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	2
6	b./br.	2	0	1	14	0	20	38	0	0	13	0	0	0	0	0	0	88
7	b.	13	8	3	134	1	38	484	0	1	238	0	0	1	0	0	3	924
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
10	ch.	7	8	0	58	1	13	238	0	0	268	0	0	1	0	0	3	597
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	2
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
16	gr.	0	0	0	1	0	0	3	0	0	3	0	0	0	0	0	4	11
Totals...	...	37	22	4	311	2	88	924	0	1	597	0	0	2	0	1	11	2000

TABLE VIII.—Collateral Heredity. First Degree. (Whole Brothers.)

Colts and Colts. Same Mare and Sire.

First Colt.

Second Colt.		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	3	0	0	7	0	0	8	0	0	0	0	0	0	0	0	0	18
2	bl/br.	0	0	1	3	0	0	0	0	0	0	0	0	0	0	0	0	4
3	br./bl.	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	2
4	br.	5	1	0	62	0	2	67	0	0	30	0	0	1	0	0	0	168
5	br./b.	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
6	b./br.	0	0	0	14	0	2	17	0	0	3	0	0	0	0	0	0	36
7	b.	9	4	1	74	0	18	290	0	0	81	0	0	0	0	0	1	478
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	2	2	0	27	0	4	89	0	0	164	0	0	1	0	0	0	289
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
14	ro./gr.	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2
Totals...	...	20	7	2	188	1	26	473	0	0	278	0	0	2	0	0	3	1000

TABLE VIII$^{\text{bis}}$.—Collateral Heredity. First Degree. (Whole Brothers.)

Colts and Colts. Same Mare and Sire.

Symmetrical Distribution.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	6	0	1	12	0	0	17	0	0	2	0	0	0	0	0	0	38
2	bl./br.	0	0	1	4	0	0	4	0	0	2	0	0	0	0	0	0	11
3	br./bl.	1	1	0	0	0	0	2	0	0	0	0	0	0	0	0	0	4
4	br.	12	4	0	124	0	16	141	0	0	57	0	0	1	1	0	0	356
5	br./b.	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	2
6	b./br.	0	0	0	16	0	4	35	0	0	7	0	0	0	0	0	0	62
7	b.	17	4	2	141	0	35	580	0	0	170	0	0	1	0	0	1	951
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	2	2	0	57	0	7	170	0	0	328	0	0	1	0	0	0	567
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	3
14	ro./gr.	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	4	5
Totals...	...	38	11	4	356	2	62	951	0	0	567	0	0	3	1	0	5	2000

TABLE IX.—Collateral Heredity. (Half Sisters.)

Fillies and Fillies. Same Mare but different Sire

First Filly.

Second Filly.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	2	0	0	4	0	1	13	0	0	4	0	0	0	0	0	0	24
2	bl./br.	0	0	0	1	0	0	3	0	0	3	0	0	0	0	0	1	8
3	br./bl.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
4	br.	4	1	0	42	0	5	64	0	0	25	0	0	0	0	0	0	141
5	br./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	b./br.	0	1	0	3	0	6	23	0	0	7	0	0	0	0	0	0	40
7	b.	9	3	1	61	0	21	313	0	0	103	0	0	0	0	0	2	513
8	b./ch.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	5	5	0	26	0	10	91	0	0	119	0	0	0	0	1	1	258
11	ch./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	2	0	0	2	0	0	0	0	0	0	4
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	2
16	gr.	0	0	0	3	0	0	2	0	0	1	0	0	0	0	0	1	7
Totals...	...	20	10	1	140	0	44	513	0	0	266	0	0	0	0	1	5	1000

TABLE IX$^{\text{bis}}$.—Collateral Heredity. First Degree. (Half-Sisters.)

Fillies and Fillies. Same Mare but different Sires.

Symmetrical Distribution.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	4	0	0	8	0	1	22	0	0	9	0	0	0	0	0	0	44
2	bl./br.	0	0	0	2	0	1	6	0	0	8	0	0	0	0	0	1	18
3	br./bl.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	2
4	br.	8	2	0	84	0	8	125	0	0	51	0	0	0	0	0	3	281
5	br./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	b./br.	1	1	0	8	0	12	44	0	0	17	0	0	0	0	1	0	84
7	b.	22	6	1	125	0	44	626	0	0	194	1	0	2	0	1	4	1026
8	b./ch.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	9	8	1	51	0	17	194	1	0	238	0	0	2	0	1	2	524
11	ch./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	2	0	0	2	0	0	0	0	0	0	4
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0	3
16	gr.	0	1	0	3	0	0	4	0	0	2	0	0	0	0	0	2	12
Totals...	...	44	18	2	281	0	84	1026	1	0	524	1	0	4	0	3	12	2000

Table X.—Collateral Heredity. First Degree. (Whole Sisters.)

Fillies and Fillies. Same Mare and same Sire.

First Filly.

Second Filly.

		1. bl.	2. bl./br.	3. br./bl.	4 br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	6	1	0	6	0	1	11	0	0	4	0	0	0	0	0	0	29
2	bl./br.	1	0	0	2	0	2	3	0	0	0	0	0	0	0	0	0	8
3	br./bl.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	br.	6	5	0	64	0	4	53	0	0	21	0	0	0	0	0	0	153
5	br./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	b./br.	1	1	0	5	0	4	18	0	0	3	0	0	0	0	0	0	32
7	b.	8	2	0	55	0	17	336	0	0	77	0	0	0	0	0	1	496
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	4	1	0	23	0	3	77	0	0	168	0	1	0	0	0	0	277
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	4	5
Totals...	...	26	10	0	155	0	31	499	0	0	273	0	1	0	0	0	5	1000

Table X^{bis}.—Collateral Heredity. First Degree. (Whole Sisters).

Fillies and Fillies. Same Mare and same Sire.

Symmetrical Distribution.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	12	2	0	12	0	2	19	0	0	8	0	0	0	0	0	0	55
2	bl./br.	2	0	0	7	0	3	5	0	0	1	0	0	0	0	0	0	18
3	br./bl.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	br.	12	7	0	128	0	9	108	0	0	44	0	0	0	0	0	0	308
5	br./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	b./br.	2	3	0	9	0	8	35	0	0	6	0	0	0	0	0	0	63
7	b.	19	5	0	108	0	35	672	0	0	154	0	0	0	0	0	2	995
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	8	1	0	44	0	6	154	0	0	336	0	1	0	0	0	0	550
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
13	ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	0	0	0	0	0	2	0	0	0	0	0	0	0	0	8	10
Totals...	...	55	18	0	308	0	63	995	0	0	550	0	1	0	0	0	10	2000

TABLE XI.—Collateral Heredity. First Degree. (Half Sister and Half Brother.)

Fillies and Colts. Same Mare but different Sires.

Fillies.

Colts.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	0	0	0	4	0	0	7	0	0	7	0	0	0	0	0	0	18
2	bl./br.	1	0	0	2	0	0	2	0	0	0	0	0	0	0	0	0	5
3	br./bl.	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
4	br.	3	0	0	34	0	8	71	0	0	32	0	0	1	0	1	0	150
5	br./b.	0	1	0	2	0	1	0	0	0	1	0	0	0	0	0	0	5
6	b./br.	0	0	0	13	0	10	18	0	0	5	0	0	0	0	0	1	47
7	b.	5	8	0	61	0	22	263	0	0	112	0	0	3	0	0	2	476
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	4	2	0	28	0	15	125	0	0	116	0	0	1	0	1	0	292
11	ch./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	2
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
16	gr.	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	3
Totals...	...	13	11	0	145	0	57	489	0	0	274	0	0	5	0	2	4	1000

TABLE XII.—Collateral Heredity, First Degree. (Whole Brother and Whole Sister.)

Fillies and Colts. Same Mare and Sire.

Fillies.

Colts.

		1. bl.	2. bl./br.	3. br./bl.	4. br.	5. br./b.	6. b./br.	7. b.	8. b./ch.	9. ch./b.	10. ch.	11. ch./ro.	12. ro./ch.	13. ro.	14. ro./gr.	15. gr./ro.	16. gr.	Totals.
1	bl.	6	0	0	5	0	0	6	0	0	3	0	0	0	0	0	0	20
2	bl./br.	0	2	0	0	0	0	1	0	0	1	0	0	0	0	0	0	4
3	br./bl.	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
4	br.	3	1	0	63	0	9	69	0	0	22	0	0	0	0	1	0	168
5	br./b.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
6	br./b.	1	0	0	5	0	3	11	0	0	6	0	0	0	0	0	0	26
7	b.	8	5	0	63	0	12	332	0	0	77	0	0	0	0	0	2	499
8	b./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	ch./b.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	ch.	1	1	0	26	0	4	97	0	0	141	0	0	1	0	0	0	271
11	ch./ro.	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
12	ro./ch.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	ro./gr.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	gr./ro.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	gr.	0	1	0	0	0	2	2	0	0	2	0	0	0	0	0	2	9
Totals ...	...	20	10	0	162	0	30	520	0	0	252	0	0	1	0	1	4	1000

APPENDIX II.

Tables of Eye-colour Inheritance in Man, extracted by KARL PEARSON from Mr. FRANCIS GALTON'S Family Records.

TABLE OF TINTS.

1 = light blue.
2 = blue, dark blue.
3 = blue-green, grey.
4 = dark grey, hazel.
5 = light brown.
6 = brown.
7 = dark brown.
8 = very dark brown, black.

This grouping is not quite in keeping with more recent divisions of eye-colour, but being that adopted by Mr. GALTON in his original collection of data, it could not be modified in accordance with present practice.

TABLES for the Direct Inheritance of Eye-colour. First Generation.

I.—Fathers and Sons. 1000 Cases.

Fathers.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Sons.	1	9	12	5	5	0	1	2	0	34
	2	10	163	65	36	1	7	15	4	301
	3	10	73	124	41	1	12	18	5	284
	4	4	21	34	55	0	11	11	1	137
	5	0	0	0	0	2	2	1	0	5
	6	1	26	12	19	1	19	16	6	100
	7	1	23	16	14	0	11	31	2	98
	8	1	4	8	10	0	1	7	10	41
	Totals	36	322	264	180	5	64	101	28	1000

II.—Fathers and Daughters. 1000 Cases.

Fathers.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Daughters.	1	4	9	11	4	0	1	2	1	32
	2	11	139	57	31	0	6	24	5	273
	3	9	73	111	38	1	15	19	3	269
	4	5	43	34	54	2	10	14	3	165
	5	0	1	3	0	0	3	1	0	8
	6	1	45	13	19	0	23	15	3	119
	7	2	27	10	12	0	7	41	6	105
	8	0	8	4	0	0	2	11	4	29
	Totals	32	345	243	158	3	67	127	25	1000

III.—Mothers and Sons. 1000 Cases.

Mothers.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Sons.	1	5	14	6	3	0	1	0	6	35
	2	12	119	83	29	8	20	21	9	301
	3	13	54	113	35	4	37	14	8	278
	4	3	21	26	54	1	17	6	6	134
	5	0	1	1	0	0	3	0	0	5
	6	1	9	26	10	1	30	24	3	104
	7	1	9	19	16	0	18	31	7	101
	8	0	7	15	4	1	3	5	7	42
	Totals	35	234	289	151	15	129	101	46	1000

IV.—Mothers and Daughters. 1000 Cases.

Mothers.

	Tints.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Daughters.	1	5	15	3	2	2	2	2	0	31
	2	7	99	67	29	2	15	23	13	255
	3	7	77	111	38	1	26	14	6	280
	4	5	22	34	46	2	27	21	7	164
	5	0	2	2	0	3	1	2	1	11
	6	0	13	27	20	1	35	17	7	120
	7	0	13	21	16	1	19	26	9	105
	8	1	5	7	2	1	4	12	2	34
	Totals.	25	246	272	153	13	129	117	45	1000

TABLES for the Collateral Inheritance of Eye-colour.

V^a.—Brothers and Brothers. 1500 Cases.

First Brother.

	First.	1.	2.	3.	4.	5.	6.	7	8.	Totals.
Second Brother.	1	8	2	3	4	0	0	0	0	19
	2	36	202	23	17	0	6	4	3	291
	3	16	182	209	26	0	4	2	2	441
	4	6	36	71	84	0	7	2	0	206
	5	0	3	2	1	0	1	0	0	7
	6	3	56	50	39	0	34	5	6	193
	7	6	37	76	48	1	36	36	2	242
	8	4	24	26	18	0	8	6	15	101
	Totals	79	542	460	237	1	96	55	30	1500

VI^a.—Sisters and Sisters. 1500 Cases.

First Sister.

Second Sister.

First.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
1	10	2	1	1	0	0	0	0	14
2	17	147	29	6	0	10	6	2	217
3	10	136	186	24	0	9	5	3	373
4	3	75	94	66	0	1	10	0	249
5	0	3	5	2	2	0	1	0	13
6	2	57	69	55	5	52	9	2	251
7	4	56	61	52	10	59	49	0	291
8	2	20	10	7	6	13	26	8	92
Totals	48	496	455	213	23	144	106	15	1500

V^b.—Brothers and Brothers. Symmetrical System.

Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
1	16	38	19	10	0	3	6	6	98
2	38	404	205	53	3	62	41	27	833
3	19	205	418	97	2	54	78	28	901
4	10	53	97	168	1	46	50	18	443
5	0	3	2	1	0	1	1	0	8
6	3	62	54	46	1	68	41	14	289
7	6	41	78	50	1	41	72	8	297
8	6	27	28	18	0	14	8	30	131
Totals	98	833	901	443	8	289	297	131	3000

VI^b.—Sisters and Sisters. Symmetrical System.

Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
1	20	19	11	4	0	2	4	2	62
2	19	294	165	81	3	67	62	22	713
3	11	165	372	118	5	78	66	13	828
4	4	81	118	132	2	56	62	7	462
5	0	3	5	2	4	5	11	6	36
6	2	67	78	56	5	104	68	15	395
7	4	62	66	62	11	68	98	26	397
8	2	22	13	7	6	15	26	16	107
Totals	62	713	828	462	36	395	397	107	3000

VII.—Brothers and Sisters. 1500 Cases.

Brothers.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Sisters.	1	5	9	18	4	0	0	1	0	37
	2	20	163	101	36	0	28	19	13	380
	3	9	98	193	50	0	37	17	14	418
	4	5	36	49	67	3	28	13	16	217
	5	0	2	5	1	1	2	2	3	16
	6	3	47	41	27	4	42	17	14	195
	7	4	34	49	22	3	27	30	19	178
	8	1	10	7	1	0	10	8	22	59
	Totals	47	399	463	208	11	174	107	91	1500

TABLE for Assortative Mating in Eye-colour.

VIII.—Husbands and Wives. 774 Cases.

Husbands.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Wives.	1	2	13	4	3	0	1	2	0	25
	2	6	87	42	26	0	16	13	6	196
	3	6	56	93	31	1	16	11	6	220
	4	4	32	35	18	1	15	6	1	112
	5	0	0	5	1	0	0	1	0	7
	6	2	38	27	10	1	12	10	1	101
	7	5	20	28	7	1	6	12	4	83
	8	2	8	8	2	0	2	4	4	30
	Totals	27	254	242	98	4	68	59	22	774

TABLES for the Direct Inheritance of Eye-colour. Second Generation.

IX.—Paternal Grandfather and Grandson. 765 Cases.

Paternal Grandfather.

	First.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Grandson.	1	4	10	3	0	0	1	3	0	21
	2	7	115	31	20	1	6	13	3	196
	3	5	64	109	21	0	10	22	4	235
	4	2	25	40	21	0	9	13	6	116
	5	0	0	0	1	0	1	0	0	2
	6	0	14	32	11	0	15	5	2	79
	7	4	16	16	9	1	11	21	2	80
	8	0	6	5	0	0	3	16	6	36
	Totals	22	250	236	83	2	56	93	23	765

X.—Paternal Grandfather and Granddaughter. 681 Cases.

Paternal Grandfather.

	First.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Granddaughter.	1	3	6	4	5	0	1	1	0	20
	2	2	94	32	10	2	6	16	4	166
	3	5	67	71	17	0	9	20	3	192
	4	4	35	33	26	1	10	9	3	121
	5	0	3	4	1	0	1	2	1	12
	6	1	16	21	11	0	15	6	4	74
	7	3	10	20	11	1	8	15	3	71
	8	0	2	5	1	1	1	10	5	25
	Totals	18	233	190	82	5	51	79	23	681

TABLE XI.—Maternal Grandfather and Grandson. 771 Cases.

Maternal Grandfather.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Grandson.	1	3	11	3	1	—	—	1	—	19
	2	8	113	46	22	1	13	5	3	211
	3	10	87	89	12	—	11	8	3	220
	4	4	33	35	22	—	15	6	2	117
	5	—	1	1	—	—	—	—	—	2
	6	2	25	25	7	—	14	7	4	84
	7	1	22	26	6	2	9	10	4	80
	8	—	12	12	6	—	4	3	1	38
	Totals	28	304	237	76	3	66	40	17	771

TABLE XII.—Maternal Grandfather and Granddaughter. 687 Cases.

Maternal Grandfather.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Granddaughter.	1	1	3	7	2	—	—	1	—	14
	2	8	84	35	11	—	13	6	2	159
	3	11	67	76	18	—	7	15	5	199
	4	7	41	40	14	—	15	11	5	133
	5	—	5	2	—	—	2	1	—	10
	6	4	21	32	1	—	16	5	2	81
	7	4	15	14	8	1	7	19	2	70
	8	—	5	5	1	—	1	5	4	21
	Totals	35	241	211	55	1	61	63	20	687

TABLE XIII.—Paternal Grandfather and Grandson. 741 Cases.

Paternal Grandmother.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Grandson.	1	1	2	7	1	1	2	3	—	17
	2	6	62	69	22	4	15	25	4	207
	3	4	31	95	22	1	25	33	9	220
	4	3	18	36	20	4	15	16	4	116
	5	—	—	—	1	—	—	1	1	3
	6	1	10	23	6	2	16	10	4	72
	7	3	15	10	4	1	14	13	11	71
	8	1	10	3	3	—	5	5	8	35
	Totals	19	148	243	79	13	92	106	41	741

TABLE XIV.—Paternal Grandmother and Granddaughter. 717 Cases.

Paternal Grandmother.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Granddaughter.	1	—	3	3	2	—	—	6	2	16
	2	7	53	56	14	2	13	28	8	181
	3	8	35	65	22	6	29	28	7	200
	4	4	29	36	20	3	16	23	8	139
	5	1	—	3	1	—	5	1	—	11
	6	—	9	29	2	4	27	8	3	82
	7	—	10	15	12	2	10	12	7	68
	8	1	2	1	2	1	4	4	5	20
	Totals	21	141	208	75	18	104	110	40	717

TABLE XV.—Maternal Grandmother and Grandson. 756 Cases.

Maternal Grandmother.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Grandson.	1	1	10	1	3	—	—	3	1	19
	2	10	68	53	23	—	24	13	13	204
	3	9	39	67	38	—	32	23	11	219
	4	3	34	19	30	—	19	8	4	117
	5	—	—	1	1	—	—	—	—	2
	6	—	20	11	10	—	24	18	1	84
	7	2	9	23	11	1	17	17	—	80
	8	—	4	6	5	—	6	3	7	31
	Totals	25	184	181	121	1	122	85	37	756

Table XVI.—Maternal Grandmother and Granddaughter. 739 Cases.

Maternal Grandmother.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Granddaughter.	1	2	16	—	—	—	—	—	—	18
	2	7	66	34	13	—	21	15	6	162
	3	12	62	55	25	1	27	23	5	210
	4	6	32	36	25	—	23	15	7	144
	5	—	1	3	2	—	3	3	—	12
	6	1	14	21	11	—	27	17	2	93
	7	—	19	17	7	—	16	17	3	76
	8	—	5	4	2	—	7	3	3	24
	Totals	28	212	170	85	1	124	93	26	739

Tables for the Collateral Inheritance of Eye-colour. Second Degree.

XVII.—Paternal Uncle and Nephew. 1290 Cases.

Paternal Uncle.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Total.
Nephew.	1	4	10	11	6	1	4	5	2	43
	2	11	136	98	40	0	26	48	12	371
	3	8	84	157	26	1	27	54	7	364
	4	0	29	69	36	1	19	27	12	193
	5	0	2	1	0	0	2	1	0	6
	6	1	31	35	7	1	30	19	3	127
	7	2	21	27	24	1	13	34	11	133
	8	0	11	7	6	0	10	8	11	53
	Total	26	324	405	145	5	131	196	58	1290

XVIII.—Paternal Uncle and Niece. 1128 Cases.

Paternal Uncle.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Total.
Niece.	1	2	10	6	2	0	1	6	2	29
	2	7	85	61	27	0	29	26	13	248
	3	5	82	126	29	1	26	43	7	319
	4	2	47	73	40	1	29	40	5	237
	5	1	8	1	4	0	1	5	4	24
	6	1	26	35	12	1	8	42	3	128
	7	1	20	25	19	0	22	26	7	120
	8	0	4	2	3	0	5	3	6	23
	Total	26	282	329	136	3	121	191	47	1128

Table XIX.—Maternal Uncle and Nephew. 1242 Cases.

Maternal Uncle.

	Tint.	1.	2.	3.	4.	5.	6.	7	8.	Totals.
Nephew.	1	1	8	13	3	0	3	4	1	33
	2	17	137	71	29	0	19	14	9	296
	3	10	128	153	26	0	29	34	3	383
	4	2	50	62	28	0	22	14	1	179
	5	0	1	0	0	0	0	1	0	2
	6	4	33	29	12	0	35	20	3	136
	7	1	33	40	11	0	26	27	2	140
	8	0	9	17	23	0	8	3	13	73
	Totals	35	399	385	132	0	142	117	32	1242

Table XX.—Maternal Uncle and Niece. 1434 Cases.

Maternal Uncle.

	Tint.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Niece.	1	2	15	9	2	0	2	1	0	31
	2	6	99	76	23	0	18	13	11	246
	3	18	100	108	23	0	37	36	10	332
	4	4	72	64	28	0	16	21	9	214
	5	0	14	2	3	0	8	5	0	32
	6	5	38	41	10	0	23	11	4	132
	7	1	27	25	7	0	19	14	3	96
	8	0	15	5	6	0	9	11	5	51
	Totals	36	380	330	102	0	132	112	42	1134

Table XXI.—Paternal Aunt and Nephew. 1186 Cases.

Paternal Aunt.

	Tints.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Nephew.	1	6	13	5	3	1	4	6	4	42
	2	19	113	83	45	—	36	29	5	330
	3	10	81	147	30	—	29	35	8	340
	4	8	28	66	38	—	18	22	11	191
	5	—	—	—	—	—	—	—	—	0
	6	3	23	35	12	1	35	10	5	124
	7	5	22	28	19	—	18	15	5	112
	8	1	4	9	8	—	6	4	13	47
	Totals	52	284	373	155	2	148	121	51	1186

TABLE XXII.—Paternal Aunt and Niece. 1149 Cases.

Paternal Aunt.

	Tints.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Niece.	1	2	11	2	3	—	2	11	2	33
	2	15	89	62	37	2	25	40	14	284
	3	12	93	119	40	3	41	26	12	346
	4	10	36	62	43	5	25	21	11	213
	5	—	5	7	—	—	1	3	—	16
	6	1	24	33	16	1	29	19	5	128
	7	2	20	28	12	2	10	22	4	100
	8	—	7	—	4	—	5	9	4	29
	Totals	42	285	313	155	13	138	151	52	1149

TABLE XXIII.—Maternal Aunt and Nephew. 1145 Cases.

Maternal Aunt.

	First.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Nephew.	1	4	8	7	3	—	2	3	1	28
	2	5	117	81	29	—	43	29	6	310
	3	1	73	132	38	—	57	43	3	347
	4	1	20	54	27	1	21	11	—	135
	5	—	—	3	2	—	1	—	—	6
	6	—	24	35	22	—	30	23	3	137
	7	—	26	29	20	1	26	25	8	135
	8	—	14	5	10	—	12	4	2	47
	Totals	11	282	346	151	2	192	138	23	1145

TABLE XXIV.—Maternal Aunt and Niece. 1058 Cases.

Maternal Aunt.

	First.	1.	2.	3.	4.	5.	6.	7.	8.	Totals.
Niece.	1	2	3	10	—	—	—	—	—	15
	1	5	87	86	31	—	23	14	12	258
	3	3	71	125	32	1	49	41	3	325
	4	—	39	51	31	1	33	19	6	180
	5	—	4	6	1	—	8	5	3	27
	6	1	25	47	10	—	24	9	2	118
	7	—	30	29	11	—	14	18	10	112
	8	—	5	4	5	—	2	4	3	23
	Totals	11	264	358	121	2	153	110	39	1058

[Notes added July 3, 1900.

Note I. *Inheritance of Temper and Artistic Instinct.*—In additional to the fraternal correlations given on p. 102, I have dealt with Mr. GALTON's statistics for the inheritance of good and bad temper given in his 'Natural Inheritance' (p. 235). The following gives the distribution of good and bad temper among 1,294 brethren, as deduced by Mr. YULE.

		First Brother.		
		Good Temper.	Bad Temper.	Totals.
Second Brother.	Good temper.	330	255	585
	Bad temper.	255	454	709
	Totals	585	709	1294

The correlation is ·3167.

A like table is that for artistic instinct in the direct line :—

		Parentage.		
		Artistic.	Non-artistic.	Totals.
Child.	Artistic.	296	173	469
	Non-artistic.	372	666	1038
	Totals	668	839	1507

In this case the correlation is ·4039.

The fraternal correlation is somewhat low. The exact significance of the parental correlation is also somewhat vague, as the parentage is classified as artistic when *one* or *both* parents are artistic. But the two tables are very suggestive, they indicate how the new method will enable us to deal quantitatively even with characters like temper and artistic instinct to which it is impossible to apply directly a quantitative scale. With the introduction of a third or medium class, I believe it will be possible to obtain excellent results for heredity from very simple observations, and I have in hand at the present time a large series of observations on collateral heredity based upon such simple classifications. The reader should further consult

Mr. G. U. YULE's remarks on the association of temper and of artistic instinct in his memoir on "Association," 'Phil. Trans.,' A, vol. 194, p. 290, 1900.

Note II. *On the Correlation of Fertility and Eye-Colour.*—In the course of the present paper I have frequently referred to a probable influence of reproductive selection as the source of the progressive change in eye-colour, *i.e.*, to a possibility that eye-colour is correlated with fertility. I saw from Mr. GALTON's tables that in many cases the whole family had not been recorded, probably the eye-colour of the dead or of absentees being unknown. It appeared to me accordingly that it would be impossible to deal directly with the problem of fertility. However, it has since occurred to me that there is nothing likely to give the missing members of families a bias towards one rather than another eye-colour, and that we may simply treat them as a purely random subtraction from the total results. Assuming this, Mr. L. N. FILON, M.A., has prepared for me tables of father's and mother's eye-colour and of the recorded number of their children. From these* I take first the following results, premising (i.) that I call "light eyed," persons with eye-colours 1, 2 and 3, and "dark eyed," persons with eye-colours 4, 5, 6, 7, 8, *i.e.*, drawing the line between light and dark grey; (ii.) that I take as small families those with 0, 1, 2, or 3, recorded children and as large those with 4 or more recorded children.

Father.

		Light Eyed.	Dark Eyed.	Totals.
Family.	Small.	313	141	454
	Large.	264	139	403
Totals		577	280	857

Mother

		Light Eyed.	Dark Eyed.	Totals.
Family.	Small.	253	202	455
	Large.	225	169	394
Totals		478	371	849

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[*] Correlation tables were prepared of the size of families 0 to 15, and of the eye-colours 1 to 8, but it does not seem needful to print them *in extenso*.

We have, accordingly, by the method of the present memoir :—

Correlation of size of family with darkness of eye-colour

$$= \quad \cdot0595, \text{ for fathers.}$$
$$= - \cdot0239, \text{ for mothers.}$$

The former is just sensible, the latter hardly sensible relative to the probable error. So far as they can be relied upon, they would denote that fathers have more children the darker eyed they are, and mothers more children the lighter eyed they are. This is in accordance with the result given in the memoir, that the modern generation is darker than its male and lighter than its female ancestry, but it is not the explanation given in the text, although it is probably the true one. If it be the true one, dark fathers and light mothers ought to present the most fertile unions, and it seemed desirable to test this directly. We have already seen that there exists an assortative mating in eye-colour, like tending to mate with like, the co-efficient of correlation being about ·1 ; hence if we were to correlate the eye-colour of mothers and fathers, *i.e.*, husbands and wives weighted with their fertility, we ought to find this result substantially reduced. The following is the table :—

Fathers.

		Light Eyed.	Dark Eyed.	Totals.
Mothers.	Light Eyed.	1183	612	1795
	Dark Eyed.	826	455	1281
	Totals	2009	1067	3076

We find $r = \cdot0239$, or the correlation has been reduced to a fifth of its previous value, and is now of the order of its probable error. To mark still further this increased fertility of heterogamous unions, I add two further tables, giving the mean number of recorded offspring for various classifications of parental eye-colour.

		Fathers.		
		Colours 1–3.	Colours 4–8.	Average of totals.
Mothers.	Colours 1–3.	3·83	4·57	4·07
	Colours 4–8.	3·82	3·73	3·79
	Average of totals	3·83	4·17	3·94

		Fathers.		
		Colours 1–2.	Colours 3–8.	Average of totals
Mothers.	Colours 1–2.	3·19	4·52	3·86
	Colours 3–8.	3·98	3·96	3·97
	Average of totals	3·68	4·09	3·94

The first table entirely confirms all the conclusions reached,—dark fathers and light mothers are most fertile absolutely and in union. The second table shows that it is the blue-green and grey rather than pure blue-eyed mothers who are most fertile. This supplementary investigation accordingly seems to support the view of the text of the memoir, namely that reproductive selection is the source of the secular change in eye-colour noted, only the prepotent fertility which is replacing the blue-eyed element is in the first place that of the dark-eyed male, and only in the second place due to mothers having eye-colours dark or light other than true blue. We seem accordingly in eye-colour to have reproductive selection working through heterogamy rather than through homogamy as in the case of stature.* The effect, however, is like,—the progressive elimination of one type of character.]

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* See 'Roy. Soc. Proc.,' vol. 66, p. 30, and vol. 66, p. 316 *et seq.*

INDEX SLIP.

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Equations, differential partial, of second order, two independent variables, one dependent.
Dixon, A. C. Phil. Trans., A, vol. 195, 1900, pp. 151–191.

IV. *On Simultaneous Partial Differential Equations.*

By A. C. DIXON, *Sc.D.*

Communicated by J. W. L. GLAISHER, *Sc.D.*

Received May 9—Read June 15, 1899.

CONTENTS.

§ 1. IN this paper, without touching on the question of the existence of integrals of systems of simultaneous partial differential equations, I have given a method by which the problem of finding their complete primitives may be attacked.

The cases discussed are two: that of a pair of equations of the first order in two dependent and two independent variables, and that of a single equation of the second order, with one dependent and two independent variables.

I follow, as far as possible, the analogy of the method of LAGRANGE and CHARPIT, and with this object introduce the conception of the "bidifferential" or differential element of the second order, which bears the same relation to a Jacobian taken with respect to two independent variables as a differential does to a differential coefficient.

The solutions considered are, in general, complete primitives, that is, such as contain arbitrary constants in such number that the result of their elimination is the system of equations proposed for solution. The existence of such primitives is sufficiently established (see the papers of FRAU VON KOWALEVSKY and Professor KÖNIGSBERGER, quoted hereafter); it will therefore be assumed, and the object of the investigation

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will be to find conditions that must be satisfied by the equations of the solution and to put these conditions in a convenient form for solution by inspection.

I should add that I am greatly indebted to the referees for their suggestions and for help in removing obscurities.

To the list of authorities given by Dr. FORSYTH ('Theory of Differential Equations,' Part I., pp. 299, 331), may be added the following:—

JULIUS KÖNIG. Math. Annalen, vol. 23, pp. 520, 521.
LEO KÖNIGSBERGER. Crelle, vol. 109, pp. 261–340.* Math. Annalen, vol. 41, pp. 260–285.† Math. Annalen, vol. 44, pp. 17–40.
ED. V. WEBER. München Ber., vol. 25, 423–442.
J. M'COWAN. Edinb. Math. Soc. Proc., vol. 10, 63–70.
HAMBURGER. Crelle, vol. 110, pp. 158–176.
C. BOURLET. Annales de l'École Normale (3), vol. 8.
RIQUIER. Comptes Rendus, vols. 114, 116, 119. Annales de l'École Normale (3), vol. 10.
LLOYD TANNER. Proc. Lond. Math. Soc., vols. 7–11.
J. BRILL. Quarterly Journal of Math., vol. 30, pp. 221–242.

Several of the above papers are only known to me through abstracts.

On Bidifferentials.

§ 2. The idea of a "complete differential" plays an important part in the theory of differential equations. In this paper I shall try to show the importance of an extension of the same idea to differential elements of higher orders, such as enter into multiple integrals.

An expression $\mathrm{X}dx + \mathrm{Y}dy$ is called a complete differential when X, Y are functions of the independent variables x, y, such that

$$\partial\mathrm{Y}/\partial x = \partial\mathrm{X}/\partial y.$$

If this is the case, then, under certain restrictions, the value of $\int(\mathrm{X}dx + \mathrm{Y}dy)$ depends only on the limiting values of the variables, and not on the intermediate ones by which these limits are connected, or, as generally expressed, on the path along which the integral is taken.

This depends on the theorem that

$$\int(\mathrm{X}dx + \mathrm{Y}dy) = \iint\left(\frac{\partial\mathrm{Y}}{\partial x} - \frac{\partial\mathrm{X}}{\partial y}\right)dxdy$$

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[*] For reasons stated below, I am not in agreement with the results given in the latter part of this paper.

[†] In this paper it should be noticed that the equations (52) on p. 266 are not more general than (46).

when the single integral is taken round the boundary of the area over which the double integral is to extend.

Further, X, Y are in this case the partial derivatives of a single function.

§ 3. Let us consider the double integral

$$\int\int(\mathrm{X}\,dy\,dz + \mathrm{Y}\,dz\,dx + \mathrm{Z}\,dx\,dy),$$

where X, Y, Z are functions of the independent variables x, y, z. It is known that this, taken over a closed surface under certain restrictions, is equal to the triple integral

$$\int\int\int(\partial\mathrm{X}/\partial x + \partial\mathrm{Y}/\partial y + \partial\mathrm{Z}/dz)\,dx\,dy\,dz$$

taken over the space enclosed by that surface.

Hence, if $\partial\mathrm{X}/\partial x + \partial\mathrm{Y}/\partial y + \partial\mathrm{Z}/\partial z = 0$ identically the double integral taken over a closed surface vanishes, and taken over two open surfaces with the same boundary has the same value; that is to say, the value of the double integral depends on the values of x, y, z at the boundary only, and not, under certain restrictions, on the form of the surface enclosed by the boundary.

By analogy we may call the element of the double integral a "complete double differential," or a "complete bidifferential" under these circumstances; the condition that $\mathrm{X}\,dy\,dz + \mathrm{Y}\,dz\,dx + \mathrm{Z}\,dx\,dy$ may be a complete bidifferential is thus

$$\partial\mathrm{X}/\partial x + \partial\mathrm{Y}/\partial y + \partial\mathrm{Z}/\partial z = 0.$$

§ 4. A complete bidifferential may be expressed as a single term, such as $du\,dv$. For let u, v be two independent solutions of the equation

$$\mathrm{X}\frac{\partial u}{\partial x} + \mathrm{Y}\frac{\partial u}{\partial y} + \mathrm{Z}\frac{\partial u}{\partial z} = 0,$$

so that $u = a$, $v = b$ are integrals of the system

$$dx/\mathrm{X} = dy/\mathrm{Y} = dz/\mathrm{Z};$$

then $$\mathrm{X} = \theta\frac{\partial(u, v)}{\partial(y, z)}, \quad \mathrm{Y} = \theta\frac{\partial(u, v)}{\partial(z, x)}, \quad \mathrm{Z} = \theta\frac{\partial(u, v)}{\partial(x, y)},$$

θ being some multiplier,

and $$\frac{\partial\mathrm{X}}{\partial x} + \frac{\partial\mathrm{Y}}{\partial y} + \frac{\partial\mathrm{Z}}{\partial z} = \frac{\partial(\theta, u, v)}{\partial(x, y, z)}.$$

Since the last vanishes identically θ is a function of u, v only; a function w of u, v may be found, such that $\partial w/\partial u = \theta$, and thus

$$\mathrm{X} = \frac{\partial(w, v)}{\partial(y, z)}, \quad \mathrm{Y} = \frac{\partial(w, v)}{\partial(z, x)}, \quad \mathrm{Z} = \frac{\partial(w, v)}{\partial(x, y)}.$$

Now in finding the value of the double integral taken over a part of any surface, it will be natural to suppose the co-ordinates of any point of such a surface to be functions of two parameters, say p, q, and to transform the integral into one taken with respect to these. The integral as transformed is

$$\iint\left\{\mathrm{X}\frac{\partial(y, z)}{\partial(p, q)} + \mathrm{Y}\frac{\partial(z, x),}{\partial(p, q)} + \mathrm{Z}\frac{\partial(x, y)}{\partial(p, q)}\right\}dp\,dq,$$

and the known values in terms of p, q are to be substituted for x, y, z and their derivatives.

The subject of integration is

$$\frac{\partial(w, v)}{\partial(y, z)}\,\frac{\partial(y, z)}{\partial(p, q)} + \frac{\partial(w, v)}{\partial(z, x)}\,\frac{\partial(z, x)}{\partial(p, q)} + \frac{\partial(w, v)}{\partial(x, y)}\,\frac{\partial(x, y)}{\partial(p, q)}, \quad \text{or}$$

$$\begin{vmatrix} \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p}, & \frac{\partial v}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial p} \\ \frac{\partial w}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial w}{dy}\frac{\partial y}{\partial q} + \frac{\partial w}{dz}\frac{\partial z}{\partial q}, & \frac{\partial v}{dx}\frac{\partial x}{\partial q} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial q} \end{vmatrix} \quad \text{or}$$

$$\frac{\partial(w, v)}{\partial(p, q)}.$$

The integral is therefore

$$\iint\frac{\partial(w, v)}{\partial(p, q)}\,dp\,dq, \qquad \text{or} \qquad \iint dw\,dv,$$

and if we take a single element we may write

$$\mathrm{X}\,dy\,dz + \mathrm{Y}\,dz\,dx + \mathrm{Z}\,dx\,dy = dw\,dv,$$

dropping the parameters p, q, since the values which x, y, z have in terms of them are immaterial.

This equation is meaningless unless the expression in terms of parameters is understood. The same is true of ordinary differentials. If when u is a function of x, y, z we write

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz,$$

we mean that if x, y, z are supposed to be any functions whatever of a single parameter p, then

$$\frac{du}{dp} = \frac{\partial u}{\partial x}\frac{dx}{dp} + \frac{\partial u}{\partial y}\frac{dy}{dp} + \frac{\partial u}{\partial z}\frac{dz}{dp}.$$

This equation being true quite independently of the expressions assumed for x, y, z in terms of p, we drop the denominator dp for convenience; but in modern works on the Differential Calculus it is quite understood that a differential by itself is meaningless apart from this or some equivalent convention.

§ 5. The functions w, v are not uniquely determined. They may be replaced by W, V, where W, V are functions of w, v, one of which, say W, is arbitrary, while V is only restricted by the condition

$$\frac{\partial(\mathrm{W}, \mathrm{V})}{\partial(w, v)} = 1.$$

The transformations of w, v which are allowable will thus form a group. For a single integral the operations of the corresponding group consist in the addition of different constants, that is, in varying the constant of integration; the theory of periodic functions is connected with discontinuous sub-groups of this. It is possible that an investigation of the discontinuous sub-groups of the group of transformations of two variables which leaves their bidifferential unchanged may lead to an extended theory of periodic functions of the two variables.

§ 6. The finding of the functions w, v may be considered as the indefinite integration of the bidifferential expression. It is simplified by Jacobi's theory of the last multiplier, which is here a constant.

Since
$$\mathrm{X} = \frac{\partial(w, v)}{\partial(y, z)}, \quad \mathrm{Y} = \frac{\partial(w, v)}{\partial(z, x)}$$

we have
$$\mathrm{X}dy - \mathrm{Y}dx = \frac{\partial v}{\partial z} dw - \frac{\partial w}{\partial z} dv\,;$$

and thus, on the supposition that v is constant,

$$dw = \frac{\mathrm{X}\,dy - \mathrm{Y}\,dx}{\frac{\partial v}{\partial z}} = \frac{\mathrm{Y}\,dz - \mathrm{Z}\,dy}{\frac{\partial v}{\partial x}} = \frac{\mathrm{Z}\,dx - \mathrm{X}\,dz}{\frac{\partial v}{xy}}$$

$$= \frac{(\mu\mathrm{Z} - \nu\mathrm{Y})\,dx + (\nu\mathrm{X} - \lambda\mathrm{Z})\,dy + (\lambda\mathrm{Y} - \mu\mathrm{X})\,dz}{\lambda\frac{\partial v}{\partial x} + \mu\frac{\partial v}{\partial y} + \nu\frac{\partial v}{\partial z}}.$$

Hence w may be found, if v is known, by integrating this last expression on the supposition that v is constant; λ, μ, ν may have any values and the constant of integration is to be replaced by an arbitrary function of v. Thus, when one of the functions w, v is known, the other is found by ordinary integration. The only restriction on the one found first is the equation

$$\mathrm{X}\frac{\partial v}{\partial x} + \mathrm{Y}\frac{\partial v}{\partial y} + \mathrm{Z}\frac{\partial v}{\partial z} = 0.$$

§ 7. Let us now suppose a greater number of independent variables. Let u be a function of $x_1, x_2 \ . \ . \ . \ x_n$.

We have the relation

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ . \ . \ . \ + \frac{\partial u}{\partial x_n} dx_n.$$

Here the differentials represent simultaneous infinitesimal increments, those of the independent variables being arbitrary. The equation may also be interpreted by supposing $x_1, x_2 \ldots x_n$ to depend in any manner on a single parameter p, when the equation

$$\frac{du}{dp} = \sum_{r=1}^{n} \frac{\partial u}{\partial x_r} \frac{dx_r}{dp}$$

holds whatever functions of the parameter we suppose $x_1 \ldots x_n$ to be.

To get the idea of a double differential we must suppose two sets of simultaneous infinitesimal increments; denote them by d, δ. The bidifferential of x, y is then $dx \,.\, \delta y - \delta x \,.\, dy$.* This vanishes if x, y are not functionally independent, just as dx vanishes if x is a constant. The analogy is very clearly shown if we say that dx vanishes when some function $\phi(x)$ vanishes, $dx\, dy$ vanishes when some function $\phi(x, y)$ vanishes.

If u, v are functions of n independent variables $x_1, x_2 \ldots x_n$, we have

$$du = \sum_{r=1}^{n} \frac{\partial u}{\partial x_r} dx_r, \qquad \delta u = \sum_{r=1}^{n} \frac{\partial u}{\partial x_r} \delta x_r$$

$$dv = \sum_{r=1}^{n} \frac{\partial v}{\partial x_r} dx_r, \qquad \delta v = \sum_{r=1}^{n} \frac{\partial v}{\partial x_r} \delta x_r, \text{ and hence}$$

$$du \,.\, \delta v - \delta u \,.\, dv = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\partial u}{\partial x_r} \frac{\partial v}{\partial x_s} (dx_r \,.\, \delta x_s - dx_s \,.\, \delta x_r),$$

or

$$du\, dv = \Sigma \frac{\partial(u, v)}{\partial(x_r, x_s)} dx_r\, dx_s$$

the summation being taken over all pairs of different suffixes r, s. Hence the expression for $du\, dv$ is formed by multiplying together

$$\sum_{r=1}^{n} \frac{\partial u}{\partial x_r} dx_r \quad \text{and} \quad \sum_{r=1}^{n} \frac{\partial v}{\partial x_r} dx_r$$

with the conventions

$$dx\, dy = -\, dy\, dx,$$
$$dx\, dx = 0.$$

We shall often use the notation $d(x, y)$ for $dx\, dy$.

§ 8. For the purpose of double integration of such an expression as $\sum_{r, s} X_{rs}\, d(x_r, x_s)$, in which the coefficients X are functions of $x_1 \ldots x_n$, it is natural to suppose $x_1 \ldots x_n$ expressed throughout the range of the integration in terms of two parameters, say p, q. The integral thus becomes

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* The dot is used here and throughout the paragraph to distinguish multiplication in the ordinary algebraic sense from multiplication according to the Grassmann conventions stated at the end of the paragraph.

$$\iint \sum_{r,s} X_{rs} \frac{d(x_r, x_s)}{d(p, q)} dp\, dq.$$

If $X_{rs} = \dfrac{\partial(u, v)}{\partial(x_r, x_s)}$ for all pairs of suffixes, the subject of integration in the last integral is $d(u, v)/d(p, q)$, so that the integral becomes $\iint du\, dv$. Its value will therefore only depend on the values of u, v, that is of $x_1, x_2 \ldots x_n$, at the boundary of the range of integration, and not on the form of the relations giving $x_1, x_2 \ldots$ in terms of p, q, which define the particular surface over which the integral is taken.

In this case we may write

$$\sum_{r,s} X_{rs}\, d(x_r, x_s) = d(u, v)$$

and call it a complete bidifferential. It is easily seen that the coefficients X satisfy the relations

$$X_{rs} X_{ij} + X_{ri} X_{js} + X_{rj} X_{si} = 0_1 \quad \ldots \ldots \quad (1)$$

$$\frac{\partial X_{rs}}{\partial x_i} + \frac{\partial X_{ir}}{\partial x_s} + \frac{\partial X_{si}}{\partial x_r} = 0 \quad \ldots \ldots \quad (2),$$

for all combinations of suffixes, where it is understood that the term $X_{rs} d(x_r, x_s)$ may be also written $X_{sr} d(x_s, x_r)$, so that

$$X_{rs} = - X_{rs}.$$

The conditions (2) are those which must be satisfied in order that the value of the double integral may depend only on the boundary. The difference of two values of the double integral, for which the same boundary is assumed, will be its value over a closed surface passing through the boundary curve, and this may be transformed into the triple integral

$$\iiint \sum_{r,s,i} \left(\frac{\partial X_{rs}}{\partial x_i} + \frac{\partial X_{ir}}{\partial x_s} + \frac{\partial X_{si}}{\partial x_r} \right) dx_i\, dx_r\, dx_s$$

taken through the volume of any solid bounded by this closed surface. Hence this integral must vanish for any solid. By taking an infinitesimal solid, for every point of which all but x_i, x_r, x_s are constant, we find the condition (2).

The conditions (2) would be satisfied by an expression which was the sum of two or more complete bidifferentials, but (1) in general would not.

§ 9. We next try to find whether these conditions are sufficient as well as necessary. Now all the coefficients X cannot vanish. Suppose that X_{12} does not, then we have from (1)

$$X_{rs} = \frac{X_{1r} X_{2s} - X_{1s} X_{2r}}{X_{12}} \quad (r, s = 3, 4 \quad \ldots \quad n)$$

and in virtue of these all the conditions (1) are satisfied.

Taking the values thus given for X_{rs}, X_{ir}, X_{si} we have

$$(rsi) \equiv \frac{\partial X_{rs}}{\partial x_i} + \frac{\partial X_{ir}}{\partial x_s} + \frac{\delta X_{si}}{\partial x_r}$$

$$= \frac{X_{2s}}{X_{12}}\left(\frac{\partial X_{1r}}{\partial x_i} + \frac{\partial X_{i1}}{\partial x_r}\right) + \frac{X_{1s}}{X_{12}}\left(\frac{\partial X_{r2}}{\partial x_i} + \frac{\partial X_{2i}}{\partial x_r}\right) + \frac{X_{1r}}{X_{12}}\left(\frac{\partial X_{2s}}{\partial x_i} + \frac{\partial X_{i2}}{\partial x_s}\right)$$

$$+ \frac{X_{2r}}{X_{12}}\left(\frac{\partial X_{s1}}{\partial x_i} + \frac{\partial X_{1i}}{\partial x_s}\right) + \frac{X_{1i}}{X_{12}}\left(\frac{\partial X_{s2}}{\partial x_r} + \frac{\partial X_{2r}}{\partial x_s}\right) + \frac{X_{2i}}{X_{12}}\left(\frac{\partial X_{1s}}{\partial x_r} + \frac{\partial X_{r1}}{\partial x_s}\right)$$

$$- \frac{X_{rs}}{X_{12}}\frac{\partial X_{12}}{\partial x_i} - \frac{X_{ir}}{X_{12}}\frac{\partial X_{12}}{\partial x_s} - \frac{X_{si}}{X_{12}}\frac{\partial X_{12}}{\partial x_r}$$

$$= \frac{X_{2s}}{X_{12}}(1ri) + \frac{X_{1s}}{X_{12}}(r2i) + \frac{X_{1r}}{X_{12}}(2si) + \frac{X_{2r}}{X_{12}}(s1i) + \frac{X_{1i}}{X_{12}}(s2r) + \frac{X_{2i}}{X_{12}}(1sr)$$

$$+ \frac{1}{X_{12}}\frac{\partial}{\partial x_1}(X_{ir}X_{2s} + X_{si}X_{2r} + X_{rs}X_{2i}) + \frac{1}{X_{12}}\frac{\partial}{\partial x_2}(X_{ri}X_{1s} + X_{is}X_{1r} + X_{sr}X_{1i})$$

$$+ \frac{X_{ir}}{X_{12}}(21s) + \frac{X_{si}}{X_{12}}(21r) + \frac{X_{rs}}{X_{12}}(21i).$$

Thus the conditions (2) are not independent, but all follow from those in which at least one of the suffixes 1, 2 enters. If they are satisfied then the equations

$$\sum_{r=2}^{n} X_{1r}\,dx_r = 0,\ \sum_{r=1}^{n} X_{2r}\,dx_r = 0$$

can be satisfied by two integrals of the form $u = a$, $v = b$; that is, these last equations will give x_1, x_2 as functions of the rest, such that

$$\frac{\partial x_1}{\partial x_r} = \frac{X_{2r}}{X_{12}},\ \frac{\partial x_2}{\partial x_r} = \frac{X_{r1}}{X_{12}}.$$

For the conditions necessary and sufficient* for this are the vanishing of such expressions as

$$\frac{\partial}{\partial x_s}\frac{X_{r1}}{X_{12}} + \frac{X_{2s}}{X_{12}}\frac{\partial}{\partial x_1}\frac{X_{r1}}{X_{12}} + \frac{X_{s1}}{X_{12}}\frac{\partial}{\partial x_2}\frac{X_{r1}}{X_{12}} - \frac{\partial}{\partial x_r}\frac{X_{s1}}{X_{12}} - \frac{X_{2r}}{X_{12}}\frac{\partial}{\partial x_1}\frac{X_{s1}}{X_{12}} - \frac{X_{r1}}{X_{12}}\frac{\partial}{\partial x_2}\frac{X_{s1}}{X_{12}}$$

in which 1, 2 may be interchanged and r, s are any two of the other suffixes. This expression may be written

$$\frac{1}{X_{12}}(r1s) - \frac{X_{r1}}{X_{12}^2}(12s) + \frac{X_{s1}}{X_{12}^2}(12r) + \frac{1}{X_{12}^2}\frac{\partial}{\partial x_1}(X_{12}X_{rs} + X_{r1}X_{2s} + X_{2r}X_{1s}),$$

so that it vanishes and the conditions of integrability of the equations $\sum_r X_{1r}dx_r = 0$, $\sum_r X_{2r}dx_r = 0$ are satisfied. If $u = a$, $v = b$ are the integrals, then, since $\sum_r X_{1r}dx_r$ does not contain the differential of x_1, we must have

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* For proof of this statement see FORSYTH, 'Theory,' part I., pp. 43–51.

$$\sum_r X_{1r}\, dx_r = \theta \left\{ \frac{\partial u}{\partial x_1}\, dv - \frac{\partial v}{\partial x_1}\, du \right\}$$

and $$X_{1r} = \theta \frac{\partial(u, v)}{\partial(x_1, x_r)}.$$

In like manner $$X_{2r} = \theta \frac{\partial(u, v)}{\partial(x_2, x_r)},$$ the multiplier θ being the same.

Hence $$X_{rs} = \theta \frac{\partial(u, v)}{\partial(x_r, x_s)}$$

and $$(rst) = \frac{\partial(\theta, u, v)}{\partial(x_r, x_s, x_t)}.$$

Since this vanishes for all combinations of suffixes, θ is a function of u, v, and if another function of them, w, is so chosen that

$$\partial w / \partial u = \theta,$$

we shall have $$\sum_{r, s} X_{rs}\, d(x_r, x_s) = \theta\, d(u, v) = d(w, v).$$

Linear Differential Equations.

§ 10. If $u = a$ is an integral of the linear partial differential equation

$$X_1 \frac{\partial x_{n+1}}{\partial x_1} + X_2 \frac{\partial x_{n+1}}{\partial x_2} + \ldots + X_n \frac{\partial x_{n+1}}{\partial x_n} = X_{n+1},$$

where $X_1, X_2 \ldots X_{n+1}$ are functions of $x_1, \ldots x_{n+1}$, then u satisfies the condition

$$\sum_{r=1}^{n+1} X_r \frac{\partial u}{\partial x_r} = 0,$$

and the complete differential du is a linear combination[*] of the determinants

$$\left\| \begin{matrix} dx_1, & dx_2, & dx_3 & \ldots & dx_n, & dx_{n+1} \\ X_1, & X_2, & X_3 & \ldots & X_n, & X_{n+1} \end{matrix} \right\|$$

the coefficients in the combination being usually functions of $x_1, \ldots x_{n+1}$.

If $u = a$ is a common solution of the above equation and of

$$X_1' \frac{\partial x_{n+1}}{\partial x_1} + \ldots + X_n' \frac{\partial x_{n+1}}{\partial x_n} = X'_{n+1},$$

then, in like manner, du is a linear combination of the determinants

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[*] This is generally expressed by saying that "$u = a$ is an integral of the equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_{n+1}}{X_{n+1}}."$$

For the sake of the analogy with the work of § 11, I prefer the phrase in the text, which expresses no more and no less than the one generally used.

$$\left| \begin{matrix} dx_1, & dx_2, & \ldots, & dx_{n+1}, \\ X_1, & X_2, & \ldots, & X_{n+1}, \\ X_1', & X_2', & \ldots, & X'_{n+1}, \end{matrix} \right\|$$

but in general, of course, it will not be possible to combine them so as to form a perfect differential.

§ 11. An analogous process of integration may be given for two simultaneous equations

$$\left.\begin{aligned} \sum_{i,j}\{A_{ij}(p_iq_j - p_jq_i)\} + \sum_i B_ip_i + \sum_i C_iq_i + E = 0 \\ \sum_{i,j}\{A'_{ij}(p_iq_j - p_jq_i)\} + \sum_i B'_ip_i + \sum_i C'_iq_i + E' = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (3),$$

in which the coefficients A, B, C, E, A′, B′, C′, E′ are functions of n independent variables, $x_1, x_2 \ldots x_n$, and two dependent y, z, and

$$p_i = \frac{\partial y}{\partial x_i}, \quad q_i = \frac{\partial z}{\partial x_i}.$$

To fix the ideas, take $n = 3$ and let x_4, x_5 stand for y, z respectively, A_{i4} for C_i, A_{i5} for $-B_i$, A_{45} for E, and make similar changes in the accented letters. Then, if $u = a$, $v = b$ are two equations constituting a solution,* a, b being arbitrary constants, we must have

$$\left.\begin{aligned} \sum_{i,j=1,2\,..\,5} A_{ij}\frac{\partial(u,v)}{\partial(x_i,x_j)} = 0 \\ \sum_{i,j} A'_{ij}\frac{\partial(u,v)}{\partial(x_i,x_j)} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (4);$$

for $$p_1\frac{\partial(u,v)}{\partial(y,z)} + \frac{\partial(u,v)}{\partial(x_1,z)} = 0, \text{ \&c., if } u = a, \ v = b,$$

and the values thus given for $p_1, q_1, p_2, q_2, p_3, q_3$ must satisfy the equations (3) identically, since a, b are supposed arbitrary. The equations to be solved are thus reduced to others which are linear and homogeneous in the Jacobians, and which do not contain the dependent variables.

The equations (4) give two of the Jacobians of u, v linearly in terms of the others; if we substitute for these two in the identity

$$d(u,v) = \sum_{i,j}\frac{\partial(u,v)}{\partial(x_i,x_j)}\,d(x_i,x_j),$$

we find that $d(u,v)$ is a linear combination of the determinants of the matrix of ten† columns.

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* This solution will not be a complete primitive unless a certain number of other arbitrary constants are involved as well as a, b, a supposition which is neither made nor excluded.

It may be well to point out that the solution here assumed consists of two equations, and not of one equation involving an arbitrary function; in fact, any solution whatever necessarily consists of two equations, and one point of the present method is that these are to be sought together, not successively.

† For n independent variables the number of columns in the matrix will be $\frac{1}{2}(n+1)(n+2)$, the number of rows being still three.

$$\begin{Vmatrix} d(x_1, x_2), & d(x_1, x_3), & \ldots & d(x_i, x_j) & \ldots & d(x_4, x_5) \\ A_{12}, & A_{13}, & \ldots & A_{ij} & \ldots & A_{45} \\ A'_{12}, & A'_{13}, & \ldots & A'_{ij} & \ldots & A'_{45} \end{Vmatrix}$$

There are thus eight bidifferential expressions, and the problem is to be solved by finding such multiples of these as, when added together, will form a complete bidifferential.

§ 12. As in the case of LAGRANGE'S linear equation, this will generally, in practice, be done by inspection, and the method will be useful for finding solutions in finite terms—when such exist. But in any case,[*] whether the inspection is successful or not, there can be no doubt of the existence of suitable multipliers, in infinite number. For it is certain that the equations (3) have—possibly among other solutions—an infinity of solutions, each involving two arbitrary constants at least, and any one of these may be written $u = a$, $v = b$, where a, b are the two constants ; u, v are functions of the variables, but may, of course, be implicit functions of great complexity. The functions u, v must satisfy the conditions (4), and it immediately follows that $d(u, v)$ must be a linear combination of the determinants of the matrix formed from (4) as above ; so that a corresponding system of multipliers must exist.

If the solution is not in finite terms it is not likely to be found by inspection, and it is quite probable that the best way to find it would be by solving the original equations (3) in series. By whatever means the solution is found, the corresponding system of multipliers is thereby determined.

If nine solutions of the form $u = a$, $v = b$ have been found, the nine bidifferentials $d(u_1, v_1)$, $d(u_2, v_2)$ . . . $d(u_9, v_9)$ must satisfy identically a linear relation, since they are all linear combinations of eight expressions only.

We shall say that one of the nine pairs of functions is a "bifunction" of the other eight pairs.

The following is, then, the definition of a bifunction. When the bidifferentials of any number of pairs of quantities are connected by an identical linear relation, with constant or variable coefficients, any one of these pairs is said to be a bifunction of the rest.

The word bifunction is simply used as an abbreviation—at least for the present. I am not without hope that at a future time it may be found to have some connotation.

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[*] If one of the dependent variables with its derivatives is altogether absent from the equations (3), or if it can be made to disappear by a change of the other dependent variable, the equations (3) will in general have no solution. This case will then be excluded; it is the only case in which the method of solution in series (as given, for instance, by FRAU VON KOWALEVSKY, 'Crelle,' vol. 80) cannot be used to prove that solutions actually exist.

Another case that may fairly be excluded is that in which all the derivatives of one of the dependent variables do not occur or may be made to disappear by a change of the other. Such a system is equivalent to a single partial differential equation with one dependent variable, since the one whose derivatives are absent may be eliminated.

It is, of course, evident that if u, v are functions of variables $x_1, x_2 \ldots$ then the pair u, v is a bifunction of all the pairs that can be formed from $x_1, x_2 \ldots$ Other examples will be found later on in the paper.

§ 13. Sometimes solutions exist for systems of partial differential equations in which the number of dependent variables is less than the number of equations.

If, for instance, with the system just considered we take a third equation of the same form, the coefficients being distinguished by two dashes, there may be solutions common to the three equations. If $u = a$, $v = b$ give such a solution, then it follows in like manner that $d(u, v)$ is a linear combination of the determinants of the following matrix :—

$$\left\|\begin{matrix} d(x_1, x_r) \ldots & d(x_i, x_j) \ldots \\ A_{12}, \ldots & A_{ij}, \ldots \\ A'_{12}, \ldots & A'_{ij}, \ldots \\ A''_{12}, \ldots & A''_{ij}, \ldots \end{matrix}\right\|$$

Similarly for a greater number of equations.

Application to other Differential Equations.

§ 14. There are two classes of equations whose solution depends on that of a pair of such linear homogeneous equations as we have just been considering ; they are, firstly, systems of two equations in two dependent and two independent variables, and, secondly, equations of the second order with one dependent variable and two independent. We shall consider them in order.

Firstly, let y, z be the dependent variables and x_1, x_2 the independent ; sometimes we shall write x_3 for y and x_4 for z. Let p_1, p_2 be the partial derivatives of y and q_1, q_2 those of z, and let the equations be

$$f_1(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0,$$

$$f_2(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0.$$

A complete primitive will consist of two equations connecting x_1, x_2, y, z and involving four arbitrary constants. By differentiation these equations yield four more involving p_1, p_2, q_1, q_2. As the two equations are supposed to be a complete primitive it must be possible to find expressions for the four arbitrary constants in terms of x_1, x_2, y, z, p_1, q_1, p_2, q_2 ; the elimination of the four constants must give $f_1 = 0$, $f_2 = 0$.

Let a_1, a_2, a_3, a_4 be the constants, and u_1, u_2, u_3, u_4 the expressions for them in terms of x_1, x_2, y, z, p_1, q_1, p_2, q_2. Suppose f_3, f_4, f_5, f_6 to stand for u_1, u_2, u_3, u_4 respectively. Then by differentiation we have for any value of the suffix i from 1 to 6,

$$\frac{\partial f_i}{\partial x_r} + p_r \frac{\partial f_i}{\partial y} + q_r \frac{\partial f_i}{\partial z} + \frac{\partial f_i}{\partial p_1}\frac{dp_1}{dx_r} + \frac{\partial f_i}{\partial p_2}\frac{dp_2}{dx_r} + \frac{\partial f_i}{\partial q_1}\frac{dq_1}{dx_r} + \frac{\partial f_i}{\partial q_2}\frac{dq_2}{dx_r} = 0 \; (r = 1, 2),$$

the letter d being used to denote differentiation with respect to x_1 or x_2 on the supposition that the other is constant, while ∂ indicates strictly partial differentiation.

Since $dp_2/dx_1 = dp_1/dx_2$, $dq_2/dx_1 = dq_1/dx_2$, we find by eliminating the derivatives of p_1, q_1, p_2, q_2, that

$$\begin{aligned} &\mathrm{J}(x_1, p_1, q_1, q_2) + p_1\mathrm{J}(y, p_1, q_1, q_2) + q_1\mathrm{J}(z, p_1, q_1, q_2) + \mathrm{J}(x_2, p_2, q_1, q_2) \\ &\qquad + p_2\mathrm{J}(y, p_2, q_1, q_2) + q_2\mathrm{J}(z, p_2, q_1, q_2) = 0, \end{aligned}$$

and
$$\begin{aligned} &\mathrm{J}(x_1, q_1, p_1, p_2) + p_1\mathrm{J}(y, q_1, p_1, p_2) + q_1\mathrm{J}(z, q_1, p_1, p_2) + \mathrm{J}(x_2, q_2, p_1, p_2) \\ &\qquad + p_2\mathrm{J}(y, q_2, p_1, p_2) + q_2\mathrm{J}(z, q_2, p_1, p_2) = 0 \end{aligned}$$

where J( ) denotes the Jacobian of any four of the functions $f_1, f_2, f_3, f_4, f_5, f_6$ with respect to the variables specified in the bracket. Of these equations there are thirty, but since they are given by the elimination of six quantities from twelve equations only six of the thirty can be independent.

§ 15. One pair of these auxiliary equations will contain Jacobians of f_1, f_2, f_3, f_4, and will in fact express the conditions that the equations

$$dy = p_1dx_1 + p_2dx_2$$

$$dz = q_1dx_1 + q_2dx_2$$

shall be integrable without restriction when p_1, p_2, q_1, q_2 have the values given by the equations $f_1 = 0 = f_2$, $f_3 = a_1$, $f_4 = a_2$.

Thus, if a pair of functions f_3, f_4 can be found satisfying these two auxiliary equations, the solution can be completed by solving a pair of simultaneous ordinary equations. (See MAYER's method, FORSYTH, 'Theory of Differential Equations,' pp. 59–62.)

The two auxiliary equations that f_3, f_4 must satisfy are linear and homogeneous in their Jacobians, the coefficients of the Jacobians not involving the functions f_3, f_4; the number of independent variables is apparently eight, but it may be taken as six, since two of the eight variables x_1, x_2, y, z, p_1, p_2, q_1, q_2 are given as functions of the other six by the relations $f_1 = 0$, $f_2 = 0$, and may be supposed eliminated from f_3, f_4, if that is desirable.

The columns of the matrix formed as at § 11 are the rows of the following array :—

$$\begin{array}{rlll} & d(x_1, x_2), & 0, & 0, \\ & d(x_1, y), & 0, & 0, \\ & d(x_1, z), & 0, & 0, \\ & d(x_2, y), & 0, & 0, \\ (5) & d(x_2, z), & 0, & 0, \end{array}$$

$$\begin{array}{llll}
 & d(y, z), & 0, & 0, \\
 & d(x_1, p_1), & \{q_1, q_2\}, & \{p_2, q_1\}, \\
 & d(x_2, p_1), & 0, & \{p_2, q_2\}, \\
 & d(y, p_1), & p_1\{q_1, q_2\} & p_1\{p_2, q_1\} + p_2\{p_2, q_2\}, \\
(10) & d(z, p_1), & q_1\{q_1, q_2\}, & q_1\{p_2, q_1\} + q_2\{p_2, q_2\}, \\
 & d(x_1, p_2), & 0, & \{q_1, p_1\}, \\
 & d(x_2, p_2), & \{q_1, q_2\}, & \{q_2, p_1\}, \\
 & d(y, p_2), & p_2\{q_1, q_2\}, & p_1\{q_1, p_1\} + p_2\{q_2, p_1\}, \\
 & d(z, p_2), & q_2\{q_1, q_2\}, & q_1\{q_1, p_1\} + q_2\{q_2, p_1\}, \\
(15) & d(x_1, q_1), & \{q_2, p_1\}, & \{p_1, p_2\}, \\
 & d(x_2, q_1), & \{q_2, p_2\}, & 0, \\
 & d(y_1, q_1), & p_1\{q_2, p_1\} + p_2\{q_2, p_2\}, & p_1\{p_1, p_2\}, \\
 & d(z, q_1), & q_1\{q_2, p_1\} + q_2\{q_2, p_2\}, & q_1\{p_1, p_2\}, \\
 & d(x_1, q_2), & \{p_1, q_1\}, & 0, \\
(20) & d(x_2, q_2), & \{p_2, q_1\}, & \{p_1, p_2\}, \\
 & d(y, q_2), & p_1\{p_1, q_1\} + p_2\{p_2, q_1\}, & p_2\{p_1, p_2\}, \\
 & d(z, q_2), & q_1\{p_1, q_1\} + q_2\{p_2, q_1\}, & q_2\{p_1, p_2\}, \\
 & d(p_1, p_2), & 0, & \{x_1, q_1\} + p_1\{y, q_1\} + q_1\{z, q_1\} + \{x_2, q_2\} + p_2\{y, q_2\} + q_2\{z, q_2\}, \\
 & d(p_1, q_1), & \{x_1, q_2\} + p_1\{y, q_2\} + q_1\{z, q_2\}, & -\{x_1, p_2\} - p_1\{y, p_2\} - q_1\{z, p_2\}, \\
(25) & d(p_1, q_2), & -\{x_1, q_1\} - p_1\{y, q_1\} - q_1\{z, q_1\}, & -\{x_2, p_2\} - p_2\{y, p_2\} - q_2\{z, p_2\}, \\
 & d(p_2, q_1), & \{x_2, q_2\} + p_2\{y, q_2\} + q_2\{z, q_2\}, & \{x_1, p_1\} + p_1\{y, p_1\} + q_1\{z, p_1\}, \\
 & d(p_2, q_2), & -\{x_2, q_1\} - p_2\{y, q_1\} - q_2\{z, q_1\}, & \{x_2, p_1\} + p_2\{y, p_1\} + q_2\{z, p_1\}, \\
 & d(q_1, q_2), & \{x_1, p_1\} + p_1\{y, p_1\} + q_1\{z, p_1\} + \{x_2, p_2\} + p_2\{y, p_2\} + q_2\{z, q_2\}, & 0,
\end{array} \qquad (5)$$

Here $\{p_1, q_1\}$, for instance, is written for $\partial(f_1, f_2)/\partial(p_1, q_1)$, and every fifth row is numbered.

§ 16. In order, then, to solve the equations $f_1 = 0, f_2 = 0$ we have to form such a linear combination of the determinants of this array as will be a complete bidifferential, say $d(f_3, f_4)$, f_3, f_4 being such functions that the equations $f_1 = 0 = f_2, f_3 = a_3, f_4 = a_4$ can be solved for p_1, q_1, p_2, q_2. The array contains twenty-eight rows, but thirteen of these are combinations of the other fifteen. For instance, multiply the first row by $\partial f_1/\partial x_2$, the second by $\partial f_1/\partial y$, the third by $\partial f_1/\partial z$, the seventh by $\partial f_1/\partial p_1$, the eleventh by $\partial f_1/\partial p_2$, the fifteenth by $\partial f_1/\partial q_1$, the nineteenth by $\partial f_1/\partial q_2$ and add; the resulting row is

$$d(x_1, f_1), \; 0, \; 0,$$

which vanishes. Other vanishing rows may be formed similarly by combining the rows of the array so as to have in the first column one of the following—

$$d(x_1, f_1), \; d(x_2, f_1), \; d(y, f_1), \; d(z, f_1), \; d(p_1, f_1), \; d(p_2, f_1), \; d(q_1, f_1), \; d(q_2, f_1),$$
$$d(x_1, f_2), \; d(x_2, f_2), \; d(y, f_2), \; d(z, f_2), \; d(p_1, f_2), \; d(p_2, f_2), \; d(q_1, f_2), \; d(q_2, f_2).$$

The coefficients in these combinations are partial derivatives of f_1 or f_2, thus, for instance,

$$d(p_1, f_2) = \frac{\partial f_2}{\partial x_1} d(p_1, x_1) + \frac{\partial f_2}{\partial x_2} d(p_1, x_2) + \frac{\partial f_2}{\partial y} d(p_1, y) + \frac{\partial f_2}{\partial z} d(p_1, z)$$
$$+ \frac{\partial f_2}{\partial q_1} d(p_1, q_1) + \frac{\partial f_2}{\partial p_2} d(p_1, p_2) + \frac{\partial f_2}{\partial q_2} d(p_1, q_2),$$

and so in other cases.

The number of these combinations is sixteen, but it is to be lowered by three, since $d(f_1, f_1)$ and $d(f_2, f_2)$ are identically zero and $d(f_1, f_2)$ can be formed by combining the sixteen in two ways, so that three linear combinations of the sixteen bidifferentials vanish identically.

Hence the array contains virtually only fifteen rows $(28 - 13)$ and as there are three columns, we have thirteen bidifferential expressions to combine. Any pair of the four functions x_1, x_2, y, z will satisfy the two auxiliary equations, as is clear either from the equations themselves or from an examination of the matrix; of course these solutions of the auxiliary equations will not give a complete primitive.

§ 17. If a complete primitive has been found it leads, as has been explained, to four equations

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4,$$

and any pair of these must satisfy the auxiliary equations. Thus twelve pairs of functions satisfying these are known, namely

$$x_i \text{ and } x_j \quad (i, j = 1, 2, 3, 4)$$
$$u_i \text{ and } u_j \quad (i, j = 1, 2, 3, 4).$$

These, however, are not all independent, but one pair is a bifunction of the other eleven.

For if

$$\left.\begin{aligned} \phi(x_1, x_2, x_3, x_4, a_1, a_2, a_3, a_4) &= 0 \\ \psi(x_1, x_2, x_3, x_4, a_1, a_2, a_3, a_4) &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

are the equations of the complete primitive, they must reduce to identities when u_1, u_2, u_3, u_4 are substituted for a_1, a_2, a_3, a_4 respectively.

Hence

$$\left.\begin{aligned} \phi(x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4) &= 0 \\ \psi(x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4) &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

identically, and

$$\sum_i \frac{\partial \phi}{\partial x_i} dx_i = - \sum_i \frac{\partial \phi}{\partial u_i} du_i,$$

$$\sum_i \frac{\partial \psi}{\partial x_i} dx_i = - \sum_i \frac{\partial \psi}{\partial u_i} du_i,$$

so that

$$\sum_{i,j} \frac{\partial(\phi, \psi)}{\partial(x_i, x_j)} d(x_i, x_j) = \sum_{i,j} \frac{\partial(\phi, \psi)}{\partial(u_i, u_j)} d(u_i, u_j) \quad . \quad . \quad . \quad . \quad (8),$$

and the bidifferentials of the twelve pairs of functions are connected by a linear relation.

§ 18. The method of CHARPIT for a single partial differential equation of the first order shows how all solutions may be deduced from one complete primitive, and it is a question of interest and importance whether there is any analogous method for simultaneous equations. Now it follows at once from the conditions for a complete bidifferential that a bifunction of the pairs that can be formed from m functions, say $u_1, u_2 \ldots\ldots u_m$, will be a pair of functions of $u_1 \ldots\ldots u_m$. In the present case a bifunction of the six pairs that can be formed with u_1, u_2, u_3, u_4 will be a pair of functions of these four, and the complete primitive to which it will lead will be the same as that given by u_1, u_2. For when a solution of the auxiliary equations is known it leads directly to one and only one complete primitive by the integration of the equations*

$$dy = p_1\, dx_1 + p_2\, dx_2$$
$$dz = q_1\, dx_1 + q_2\, dx_2\,;$$

also the complete primitive to which the equations $F_1(u_1, u_2, u_3, u_4) = \text{const.}$, $F_2(u_1, u_2, u_3, u_4) = \text{const.}$, will lead can be no other than is given by

$$u_1 = a_1, \quad u_2 = a_2, \quad u_3 = a_3, \quad u_4 = a_4.$$

It must not, however, be forgotten that the system $F_1 = \text{const.}$, $F_2 = \text{const.}$, $f_1 = 0$, $f_2 = 0$ may have a singular solution. If F_1, F_2 involve two other arbitrary constants this singular solution will involve four, and therefore in general be a complete primitive of the equations $f_1 = 0$, $f_2 = 0$. Moreover, all new complete primitives are included among those thus given.

For every solution implies six equations connecting $x_1, x_2, y, z, p_1, q_1, p_2, q_2$ (two of these six are of course $f_1 = 0$, $f_2 = 0$), and, therefore, by elimination of $x_1, x_2, y, z, p_1, q_1, p_2, q_2$, two equations or more connecting u_1, u_2, u_3, u_4, which are known in terms of these eight quantities. If u_1, u_2, u_3, u_4 are connected by four equations they are constants, and the solution is therefore included in the old complete primitive. Let us, then, suppose that u_1, u_2, u_3, u_4 are connected by two or by three equations,

$$F_\alpha(u_1, u_2, u_3, u_4) = 0 \qquad (\alpha = 1, 2 \text{ or } 1, 2, 3).$$

Now if p_1, p_2, q_1, q_2, are all expressed in terms of p, q, two of their number, and x_1, x_2, y, z, by means of the equations $f_1 = 0$, $f_2 = 0$, the expressions

$$dy - p_1 dx_1 - p_2 dx_2,\ dz - q_1 dx_1 - q_2 dx_2$$

must both be expressible in the form

$$A_1 du_1 + A_2 du_2 + A_3 du_3 + A_4 du_4,$$

---

* Otherwise thus—if in the auxiliary equations we suppose f_3 to have the known value u_1, they become a pair of linear equations for f_4, which must be satisfied by u_2, u_3, u_4; now two linear equations in six independent variables can only have four functionally independent solutions, and one of these is known, namely, u_1. (In exceptional cases the two linear equations for u_2, u_3, u_4 may be equivalent; for instance, suppose $f_1 = p_1 + q_1$, $u_1 = p_2 + q_2$, f_2 having any form.) Hence, except in special cases, the particular complete primitive is defined when one of the functions u_1, u_2, u_3, u_4, or more generally a combination of them, $F(u_1, u_2, u_3, u_4)$ is known. In the case supposed in the text two such combinations are known.

and since dp, dq are absent we must have in each case

$$\sum_{r=1}^{4} \mathrm{A}_r \frac{\partial u_r}{\partial p} = 0, \quad \sum_{r=1}^{4} \mathrm{A}_r \frac{\partial u_r}{\partial q} = 0.$$

Thus the equations

$$dy = p_1 dx_1 + p_2 dx_2, \quad dz = q_1 dx_1 + q_2 dx_2$$

become

$$\begin{Vmatrix} du_1, & du_2, & du_3, & du_4, \\ \dfrac{\partial u_1}{\partial p}, & \dfrac{\partial u_2}{\partial p}, & \dfrac{\partial u_3}{\partial p}, & \dfrac{\partial u_4}{\partial p} \\ \dfrac{\partial u_1}{\partial q}, & \dfrac{\partial u_2}{\partial q}, & \dfrac{\partial u_3}{\partial q}, & \dfrac{\partial u_4}{\partial q} \end{Vmatrix} = 0.$$

These two equations, connecting du_1, du_2, du_3, du_4, taken with the system

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} du_r = 0 \ (\alpha = 1, 2 \text{ or } 1, 2, 3),$$

show that if u_1, u_2, u_3, u_4 satisfy by themselves no other relations than $\mathrm{F}_\alpha = 0$ ($\alpha = 1, 2$ or $1, 2, 3$) we must have, as a consequence of the equations of the solution,

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u}{\partial p} = 0, \quad \sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial q} = 0.$$

If, then, there are two equations

$$\mathrm{F}_1 = 0, \quad \mathrm{F}_2 = 0,$$

the four equations

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial p} = 0, \quad \sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial q} = 0 \ (\alpha = 1, 2)$$

must reduce to two only. This will be the ordinary case, and we see that if the forms of F_1, F_2, have been found by any means, the solution is completed without integration; the process corresponds to CHARPIT's method of deducing all complete primitives from one, but it differs in that the functions F_1, F_2, are not arbitrary; they must, in fact, be so chosen that the four equations last written shall reduce to two, and the conditions for this are clearly very complicated in general, though in particular cases available forms for F_1, F_2 may be seen on inspection.

In the more uncommon case, when there are three equations

$$\mathrm{F}_1 = 0, \ \mathrm{F}_2 = 0, \ \mathrm{F}_3 = 0,$$

the six equations

$$\sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial p} = 0, \quad \sum_{r=1}^{4} \frac{\partial \mathrm{F}_\alpha}{\partial u_r} \frac{\partial u_r}{\partial q} = 0 \ (\alpha = 1, 2, 3),$$

must reduce to one only.

These two cases are further discussed, from a somewhat different point of view, in §§ 21—23.

It should not be forgotten that the form in which the new complete primitive has just appeared is not that in which complete primitives were discussed in § 14, since the equations are not here supposed to be solved for the arbitrary constants.

§ 19. In addition to the six pairs (u_i, u_j) of functions satisfying the auxiliary equations, we have also the six pairs (x_i, x_j); of these twelve, eleven are independent, the other being a bifunction of them. If we can find a bifunction of the eleven pairs which is not a bifunction of either set of six it will give a new complete primitive; whether every, or indeed any, other primitive is thus given is a matter for further inquiry.

Suppose $v_i = b_i$ $(i = 1, 2, 3, 4)$ to be a new complete primitive, then it gives six more pairs of functions satisfying the auxiliary equations, and thus we have in all eighteen pairs. The bidifferentials of these must be connected by $(18-13)$ five linear relations, one of which has been written (8); by means of the other four, an expression of either of the following forms—

$$\mathrm{A}d(v_1, v_2) + \mathrm{B}d(v_1, v_3) + \mathrm{C}d(v_1, v_4),$$
$$\mathrm{A}d(v_2, v_3) + \mathrm{B}d(v_3, v_1) + \mathrm{C}d(v_1, v_2),$$

can be found which will be equal to a linear combination of the twelve bidifferentials $d(x_i, x_j)$ and $d(u_i, u_j)$. It is natural to ask whether, conversely, any linear combination of these twelve which can be written in one of the above forms will lead to a complete primitive? In the first case this is not so, for if we take any function whatever, η, of six independent variables, $\xi_1, \xi_2 \ldots \xi_6$, we may choose the coefficients $\alpha_1, \ldots \alpha_6$, so that

$$\sum_{i=1}^{6} \alpha_i \, d(\eta, \xi_i)$$

shall be a linear combination of eleven* given bidifferentials; the expression $\Sigma \, \alpha_i \, d\xi_i$ may then be reduced to three terms, $\beta_1 \, d\zeta_1 + \beta_2 \, d\zeta_2 + \beta_3 \, d\zeta_3$, so that for an *arbitrary* function (η) a combination of the eleven given bidifferentials can be found of the form $\beta_1 \, d(\eta, \zeta_1) + \beta_2 \, d(\eta, \zeta_2) + \beta_3 \, d(\eta, \zeta_3)$, which is the same as $\mathrm{A}d(v_1, v_2) + \mathrm{B}d(v_1, v_3) + \mathrm{C}d(v_1, v_4)$. This argument does not apply to the second form

$$\mathrm{A}d(v_2, v_3) + \mathrm{B}d(v_3, v_1) + \mathrm{C}d(v_1, v_2),$$

and further investigation may show that any combination of the eleven that can be reduced to this form† will lead to a primitive.

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[*] Not of any lower number in general, since the most general bidifferential expression in this number of variables contains fifteen terms, while the expression just written vanishes identically if

$$\alpha_i = \partial\eta/\partial\xi_i,$$

so that there are virtually only five coefficients, of which one must be left arbitrary.

[†] The conditions necessary that a bidifferential expression may be reducible to this form include algebraic ones which are the same as for a complete bidifferential, since

$$\mathrm{A}d(v_2, v_3) + \mathrm{B}d(v_3, v_1) + \mathrm{C}d(v_1, v_2) = \frac{1}{\mathrm{A}}\left\{\mathrm{A}dv_2 - \mathrm{B}dv_1\right\}\left\{\mathrm{A}dv_3 - \mathrm{C}dv_1\right\}.$$

§ 20. Before we can claim in any sense to have found the general solution of the auxiliary equations, we must be in possession of thirteen pairs of functions satisfying them; we have only eleven when we know one complete primitive, and hence one more complete primitive, or even possibly two, must be found. An example (below, § 29) will show that one more is not always enough.

It is perhaps worth while to remark that any complete primitive defines the whole system of solutions, since it defines the differential equations.

§ 21. The question of finding new solutions when a complete primitive is known may be attacked by the method of varying the parameters. Take the equations (6) or (7) of § 17. The problem is then to find such variable values for u_1, u_2, u_3, u_4 as will satisfy the equations

$$\sum_{i=1}^{4} \frac{\partial \phi}{\partial u_i} du_i = 0, \quad \sum_{i=1}^{4} \frac{\partial \psi}{\partial u_i} du_i = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Since all variables are supposed functions of x_1, x_2, we may make one of two suppositions with respect to u_1, u_2, u_3, u_4; either they are connected by three relations and are all functions of the same variable, say t, which is of course a function of x_1, x_2, or they are only connected by two relations, so that two of them may be taken as functions of the other two.

Suppose first that they are all functions of the one variable t. Then, generally, the four equations (7), (9) will define x_1, x_2, x_3, x_4 also as functions of t, and hence this supposition is not admissible unless it is possible to choose the functions of t in such a way* that the four equations (7), (9) will be only equivalent to three. The

---

If these conditions are satisfied by an expression

$$\sum_{i,j=1,2\,.\,.\,6} \mathrm{A}_{ij} d(x_i, x_j),$$

it can be put in the form

$$\left(\sum_{i=1}^{6} \lambda_i \, dx_i\right)\left(\sum_{i=1}^{6} \mu_i \, dx_i\right),$$

and then it must further be possible to express

$$\sum_{i=1}^{6} \lambda_i \, dx_i \text{ and } \sum_{i=1}^{6} \mu_i \, dx_i$$

as linear combinations of *three* differentials, dv_1, dv_2, dv_3. The discussion of the conditions therefore belongs to the theory of the reduction of two such expressions, that is, of the extended PFAFF problem.

* It seems obvious that this will not generally be possible; but it may be well to give an example. Suppose the complete primitive to be

$$\left.\begin{aligned} y &= ax_1^2 + bx_2 + c, \\ z &= cx_1 + ex_2^2 + bx_1x_2^2 \end{aligned}\right\},$$

so that the differential equations are

$$\begin{aligned} y &= \tfrac{1}{2}p_1x_1 + p_2x_2 + q_1 - p_2x_2^2, \\ z &= q_1x_1 + \tfrac{1}{2}q_2x_2 - p_2x_1x_2^2, \end{aligned}$$

then the variations of the parameters a, b, c, e must satisfy the equations,

$$\begin{aligned} x_1^2da + x_2db + dc &= 0 \\ x_1dc + x_2^2de + x_1x_2^2db &= 0; \end{aligned}$$

number of conditions, which will be of the nature of ordinary differential equations, thus imposed on the four parameters must not be greater than three ; for if they are subjected to four conditions they are made invariable ; it may be, however, less than three. For instance, a complete primitive of the equations $p_1 = p_2$, $q_1 = q_2$ is given by

$$y = a(x_1 + x_2) + b, \; z = c(x_1 + x_2) + e\,;$$

the equations given by varying the parameters are

$$(x_1 + x_2)da + db = 0,$$
$$(x_1 + x_2)dc + de = 0,$$

which give the *single* differential equation connecting the parameters

$$da\,de = db\,dc.$$

We may then assume arbitrary forms for two parameters in terms of a third, and find the fourth by integration. Say, for instance,

$$b = \phi(a), \; c = \psi(a),$$

then

$$e = \int\phi'(a)\psi'(a)da,$$
$$x_1 + x_2 = -\phi'(a)\,;$$

thus we arrive at the known general solution

$$y = \chi(x_1 + x_2), \; z = \omega(x_1 + x_2).$$

---

whence, by elimination of x_1,

$$(x_2{}^2db + dc)^2(x_2db + dc) + x_2{}^4de^2da = 0.$$

This equation must fail to define x_2, so that b, c, and a or e must be constant ; thence it follows that all four parameters must be constant.

I lay stress on this, because it is not in agreement with the results of Professor KÖNIGSBERGER ('Crelle,' vol. 109, p. 318), and appears in fact to show that his method there given is faulty. Professor KÖNIGSBERGER assumes (p. 313 ; I take $m = 2$) that the most general integral of the equations

$$f_1(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0$$
$$f_2(x_1, x_2, y, z, p_1, p_2, q_1, q_2) = 0$$

has the form

$$y = \omega_1(x_1, x_2, \phi_1[\psi_1(x_1, x_2)], \phi_2[\psi_2(x_1, x_2)])$$
$$z = \omega_2(x_1, x_2, \phi_1[\psi_1(x_1, x_2)], \phi_2[\psi_2(x_1, x_2)]),$$

where ϕ_1, ϕ_2 denote arbitrary and ψ_1, ψ_2 definite functions. But suppose these equations solved for ϕ_1, ϕ_2 in the form

$$\phi_1[\psi_1(x_1, x_2)] = \chi_1(x_1, x_2, y, z)$$
$$\phi_2[\psi_2(x_1, x_2)] = \chi_2(x_1, x_2, y, z)$$

and the arbitrary functions eliminated by differentiation. The differential equations thus formed are of the first degree in p_1, p_2, q_1, q_2, and are not by any means of the general form assumed. The differential equations in the examples given by Professor KÖNIGSBERGER are, in fact, linear (see pp. 319, 328). The method appears to be founded on an interpretation of the last clause of § 2 (p. 290), which is not justified.

In the case of two equations of CLAIRAUT'S form

$$y = p_1x_1 + p_2x_2 + \phi(p_1, p_2, q_1, q_2),$$
$$z = q_1x_1 + q_2x_2 + \psi(p_1, p_2, q_1, q_2),$$

which will be more fully considered later, the number of differential relations among the parameters is *two*, so that one parameter may be taken as an arbitrary function of a second, and the other two found in terms of the second by solving two ordinary differential equations.

If the primitive* is

$$y = a\alpha + b\beta + c\gamma + e\delta$$
$$z = \mathrm{A}\alpha + \mathrm{B}\beta + \mathrm{C}\gamma + \mathrm{E}\delta,$$

where a, b, c, e are the parameters, A, B, C, E known functions of a, b, c, e, and α, β, γ, δ known functions of x_1, x_2, then the variations of the parameters must satisfy the relations

$$\alpha da + \beta db + \gamma dc + \delta de = 0,$$
$$\alpha d\mathrm{A} + \beta d\mathrm{B} + \gamma d\mathrm{C} + \delta d\mathrm{E} = 0,$$

and thus, in general, if a, b, c, e are all functions of one variable they are connected by *three* relations

$$d\mathrm{A}/da = d\mathrm{B}/db = d\mathrm{C}/dc = d\mathrm{E}/de.$$

The integral equivalent of these equations consists of three relations connecting a, b, c, e with three arbitrary constants, and by eliminating a, b, c, e we find a new solution of the original differential equations which is not a complete primitive, since it only contains three arbitrary constants.

These examples show that the number of conditions to be fulfilled by the parameters when all four are taken to be functions of one of them, may be one, two, or three; this number is to be made up to three by assuming arbitrary relations (two, one, or none, as the case may be).

§ 22. Usually the parameters will not be functions of one variable only, and we may suppose two of them, u_3, u_4, to be functions of the other two, u_1, u_2.

The partial differential coefficients

$$\frac{du_3}{du_1},\ \frac{du_3}{du_2},\ \frac{du_4}{du_1},\ \frac{du_4}{du_2}$$

are then given by the equations (9), each of which is equivalent to two. The first, for instance, gives

$$\frac{\partial\phi}{\partial u_1} + \frac{\partial\phi}{\partial u_3}\frac{du_3}{du_1} + \frac{\partial\phi}{\partial u_4}\frac{du_4}{du_1} = 0,$$
$$\frac{\partial\phi}{\partial u_2} + \frac{\partial\phi}{\partial u_3}\frac{du_3}{du_2} + \frac{\partial\phi}{\partial u_4}\frac{du_4}{du_2} = 0.$$

The derivatives are thus given in terms of u_1, u_2, u_3, u_4, x_1, x_2, x_3, x_4, and the last

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* It is unnecessary to give the differential equations.

four may be eliminated by means of the relations (7); so that in the end we shall have two relations connecting u_1, u_2, u_3, u_4, and the derivatives; the problem is of the same form as the original one, to solve two simultaneous partial differential equations in two dependent and two independent variables.

Interchange of Variables and Parameters.

§ 23. A curious thing may be noticed at this point. If in the equations $\phi = 0$, $\psi = 0$, we treat x_1, x_2, x_3, x_4 as arbitrary constants and eliminate them by differentiation, we are led to the same differential equations connecting u_1, u_2, u_3, u_4 as were just now given by the variation of parameters. Thus two equations in two sets of four quantities will give two pairs of simultaneous partial differential equations by taking each set of the quantities in turn as variables and the other as arbitrary constants. The auxiliary equations, if expressed in terms of the eight quantities, will be the same in both cases; this gives a meaning to the six solutions of the form (x_i, x_j) which we found the auxiliary equations to have, for any one of the six will lead to the primitive $\phi = 0$, $\psi = 0$ of the second pair of differential equations, just as a solution (u_i, u_j) leads to this primitive for the first pair; any new solution of the auxiliary equations will in general lead to a new complete primitive for either pair, but an exception to this rule will arise when, for instance, the x differential equations have a complete primitive which gives three relations among u_1, u_2, u_3, u_4.

The array (5), transformed so that the variables are x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4, connected by the equations $\phi = 0$, $\psi = 0$, will have six rows of the form

$$d(x_i, x_j),\ 0,\ 0,$$

six of the form

$$d(u_i, u_j),\ 0,\ 0,$$

and in the other sixteen there will be

$$d(x_i, u_j) \text{ in the first column,}$$

in the second the minor of $\dfrac{\partial^2\phi}{\partial x_i \partial u_j}$ in the determinant:

$$\begin{vmatrix} \dfrac{\partial^2\phi}{\partial x_1\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_1\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_1\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_1\partial u_4}, & \dfrac{\partial\phi}{\partial x_1}, & \dfrac{\partial\psi}{\partial x_1} \\ \dfrac{\partial^2\phi}{\partial x_2\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_2\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_2\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_2\partial u_4}, & \dfrac{\partial\phi}{\partial x_2}, & \dfrac{\partial\psi}{\partial x_2} \\ \dfrac{\partial^2\phi}{\partial x_3\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_3\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_3\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_3\partial u_4}, & \dfrac{\partial\phi}{\partial x_3}, & \dfrac{\partial\psi}{\partial x_3} \\ \dfrac{\partial^2\phi}{\partial x_4\partial u_1}, & \dfrac{\partial^2\phi}{\partial x_4\partial u_2}, & \dfrac{\partial^2\phi}{\partial x_4\partial u_3}, & \dfrac{\partial^2\phi}{\partial x_4\partial u_4}, & \dfrac{\partial\phi}{\partial x_4}, & \dfrac{\partial\psi}{\partial x_4} \\ \dfrac{\partial\phi}{\partial u_1}, & \dfrac{\partial\phi}{\partial u_2}, & \dfrac{\partial\phi}{\partial u_3}, & \dfrac{\partial\phi}{\partial u_4}, & 0, & 0 \\ \dfrac{\partial\psi}{\partial u_1}, & \dfrac{\partial\psi}{\partial u_2}, & \dfrac{\partial\psi}{\partial u_3}, & \dfrac{\partial\psi}{\partial u_4}, & 0, & 0 \end{vmatrix} \quad \ldots\ldots\ldots \quad (10)$$

in the third the same expression with ϕ, ψ interchanged. The array is thus practically unchanged by interchanging the sets x and u, as should be the case.

§ 24. This transformation may be accomplished by taking the equations

$$\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_1} = 0 = \sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_2},$$

from which may be deduced

$$\frac{d}{dx_1}\left(\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_2}\right) = \frac{d}{dx_2}\left(\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_1}\right),$$

or

$$\sum_i\sum_j \frac{\partial^2\phi}{\partial u_i\partial u_j}\frac{du_i}{dx_2}\frac{du_j}{dx_1} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial x_1}\frac{du_i}{dx_2} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial y}p_1\frac{du_i}{dx_2} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial z}q_1\frac{du_i}{dx_2}$$

$$= \sum_i\sum_j \frac{\partial^2\phi}{\partial u_i\partial u_j}\frac{du_i}{dx_1}\frac{du_j}{dx_2} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial x_2}\frac{du_i}{dx_1} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial y}p_2\frac{du_i}{dx_1} + \sum_i \frac{\partial^2\phi}{\partial u_i\partial z}q_2\frac{du_i}{dx_1}.$$

Now p_1, q_1, p_2, q_2 are given by the relations

$$\frac{\partial\phi}{\partial x_1} + p_1\frac{\partial\phi}{\partial y} + q_1\frac{\partial\phi}{\partial z} = 0, \text{ \&c.},$$

and hence this equation may be written

$$\sum_i \left\{\frac{du_i}{dx_2}\frac{\partial\left(\phi, \psi, \frac{\partial\phi}{\partial u_i}\right)}{\partial(x_1, y, z)}\right\} = \sum_i \left\{\frac{du_i}{dx_1}\frac{\partial\left(\phi, \psi, \frac{\partial\phi}{du_i}\right)}{d(x_2, y, z)}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11);$$

in this ϕ, ψ may be interchanged so as to give another equation.

Now, suppose $\theta = a_1$, $\chi = a_2$ to be two of the four equations connecting u_1, u_2, u_3, u_4 with x_1, x_2, which yield a new complete primitive, and that y, z have been eliminated from θ, χ by means of the equations $\phi = 0$, $\psi = 0$, then the derivatives $\frac{du_1}{dx_1}$, $\frac{du_2}{dx_1}$, &c., are given by the following relations:—

$$\sum_i \frac{\partial\phi}{\partial u_i}\frac{du_i}{dx_1} = 0,$$

$$\sum_i \frac{\partial\psi}{\partial u_i}\frac{du_i}{dx_1} = 0,$$

$$\sum_i \frac{\partial\theta}{\partial u_i}\frac{du_i}{dx_1} + \frac{\partial\theta}{\partial x_1} = 0,$$

$$\sum_i \frac{\partial\chi}{\partial u_i}\frac{du_i}{dx_1} + \frac{\partial\chi}{\partial x_1} = 0,$$

and similarly for the derivatives with respect to x_2.

Substituting the values hence found for these derivatives in the equation (11), we have an equation linear in the Jacobians of the form

$$\frac{\partial(\theta, \chi)}{\partial(x_i, u_j)} \; (i = 1, 2; j = 1, 2, 3, 4),$$

the coefficient of the Jacobian written being the minor of $\frac{\partial^2\phi}{\partial x_i \partial u_j}$ in the determinant (10). Hence the constituents in the second column of the transformed array are as stated, and those of the third are found in like manner. It is not, of course, necessary that these columns should be the same as would be found by actual substitution of the values of p_1, p_2, q_1, q_2 in the columns of the original array; a linear transformation is allowable, with constant or variable coefficients.

The above process gives fifteen independent rows of the array; the others are deduced from the consideration that y, z are known in terms of x_1, x_2, u_1, u_2, u_3, u_4 from the equations $\phi = 0$, $\psi = 0$.

Examples.

§ 25. I. As a first example of the method of solution, take the equations

$$\alpha_1 = \alpha_2,\ \beta_1 = \beta_2,$$

where α_1, β_1 denote known functions of x_1, p_1, q_1 and α_2, β_2 known functions of x_2, p_2, q_2.

In the array (5) multiply the seventh row by $\frac{\partial(\alpha_1, \beta_1)}{\partial(x_1, p_1)}$, the fifteenth by $\frac{\partial(\alpha_1, \beta_1)}{\partial(x_1, q_1)}$, the twenty-fourth by $\frac{\partial(\alpha_1, \beta_1)}{\partial(p_1, q_1)}$, and add. The result in the first column is $d(\alpha_1, \beta_1)$, in the second, by virtue of the particular forms of f_1 and f_2,

$$\{x_1, p_1\}\{q_1, q_2\} + \{x_1, q_1\}\{q_2, p_1\} + \{p_1, q_1\}\{x_1, q_2\} \text{ or } 0,$$

and in the third,

$$\{x_1, p_1\}\{p_2, q_1\} + \{x_1, q_1\}\{p_1, p_2\} - \{p_1, q_1\}\{x_1, p_2\} \text{ or } 0.$$

Hence α_1, β_1 are two functions satisfying the auxiliary equations, and a solution is given by finding p_1, q_1, p_2, q_2 from the equations

$$\alpha_1 = \alpha_2 = a,$$
$$\beta_1 = \beta_2 = b,$$

and integrating. Two constants will be introduced by integration, so that the result is a complete primitive.

§ 26. II. Take, secondly, the equations

$$y = p_1 x_1 + \mathrm{F}(x_2, p_1, q_1, p_2, q_2),$$
$$z = q_1 x_1 + \mathrm{G}(x_2, p_1, q_1, p_2, q_2).$$

Here the twenty-fourth row is

$$d(p_1, q_1),\ 0,\ 0,$$

so that p_1, q_1 are two functions satisfying the auxiliary equations, and the integral is to be found by putting $p_1 = a$, $q_1 = b$. Thus we have

$$y - ax_1 = \mathrm{F}(x_2, a, b, p_2, q_2),$$
$$z - bx_1 = \mathrm{G}(x_2, a, b, p_2, q_2),$$

or

$$\eta = \mathrm{F}(\xi, a, b, \eta', \zeta'),$$
$$\zeta = \mathrm{G}(\xi, a, b, \eta', \zeta'),$$

where

$$\xi = x_2,\ \eta = y - ax_1,\ \zeta = z - bx_1,$$
$$\eta' = d\eta/d\xi,\ \zeta' = d\zeta/d\xi.$$

These are ordinary differential equations, the solution of which will involve two new arbitrary constants and so constitute a complete primitive of the original equations.

§ 27. III. The equations

$$y = p_1x_1 + p_2x_2 + \phi(p_1, p_2, q_1, q_2),$$
$$z = q_1x_1 + q_2x_2 + \psi(p_1, p_2, q_1, q_2),$$

are of special interest, because more complete primitives than one can be found. The obvious solution is $p_1 = a_1$, $p_2 = a_2$, $q_1 = b_1$, $q_2 = b_2$,

$$y = a_1x_1 + a_2x_2 + \phi(a_1, a_2, b_1, b_2),$$
$$z = b_1x_1 + b_2x_2 + \psi(a_1, a_2, b_1, b_2).$$

Suppose a_1, a_2, b_1, b_2 to be variable, but functions of one variable only—say a_1, then their variations must satisfy the relations

$$x_1\,da_1 + x_2\,da_2 + d\phi = 0,$$
$$x_1\,db_1 + x_2\,db_2 + d\psi = 0.$$

These define x_1, x_2, and, therefore, also y, z as functions of a_1, unless the determinants of the matrix

$$\left\| \begin{matrix} da_1, & da_2, & d\phi \\ db_1, & db_2, & d\psi \end{matrix} \right\|$$

vanish; it is necessary, then, that these determinants should vanish. Thus a_1, b_1, a_2, b_2 are connected by two ordinary differential equations. We may assume any third relation connecting them at will; suppose $b_1 = \mathrm{F}(a_1)$, F denoting an arbitrary function. Then by integration we may suppose a_2, b_2 found in terms of a_1.

Also a_1 is connected with x_1, x_2 by the relation

$$x_1 + x_2\frac{da_2}{da_1} + \frac{d\phi}{da_1} = 0,$$

so that a_1, b_1, a_2, b_2 are all known in terms of x_1, x_2, and by substitution the values of y, z are found.

§ 28. The solution may be verified. We have taken $b_1 = F(a_1)$, a known but arbitrary function of a_1, and a_2, b_2 other functions of a_1, such that

$$\frac{db_2}{da_1} = F'(a_1)\frac{da_2}{da_1}$$

$$\left[\frac{\partial\psi}{\partial a_1} + \frac{\partial\psi}{\partial b_1}F'(a_1) + \frac{\partial\psi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\psi}{db_2}\frac{db_2}{da_1}\right] = F'(a_1)\left[\frac{\partial\phi}{\partial a_1} + \frac{\partial\phi}{\partial b_1}F'(a_1) + \frac{\partial\phi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\phi}{\partial b_2}\frac{db_2}{da_1}\right].$$

Then we have the further relations

$$x_1 + x_2\frac{da_2}{da_1} + \frac{\partial\phi}{\partial a_1} + \frac{\partial\phi}{db_1}F'(a_1) + \frac{\partial\phi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\phi}{\partial b_2}\frac{db_2}{da_1} = 0,$$

$$x_1 F'(a_1) + x_2\frac{db_2}{da_1} + \frac{\partial\psi}{\partial a_1} + \frac{\partial\psi}{\partial b_1}F'(a_1) + \frac{\partial\psi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\psi}{db_2}\frac{db_2}{da_1} = 0,$$

which are of course not distinct. Also

$$y = a_1 x_1 + a_2 x_2 + \phi(a_1, a_2, b_1, b_2),$$

so that

$$p_1 = a_1 + \frac{da_1}{dx_1}\left[x_1 + x_2\frac{da_2}{da_1} + \frac{\partial\phi}{\partial a_1} + \frac{\partial\phi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\phi}{\partial b_1}\frac{db_1}{da_1} + \frac{\partial\phi}{\partial b_2}\frac{db_2}{da_1}\right] = a_1,$$

and in like manner $p_2 = a_2$.

Again $z = b_1 x_1 + b_2 x_2 + \psi(a_1, a_2, b_1, b_2)$, and

$$q_1 = b_1 + \frac{da_1}{dx_1}\left[x_1\frac{db_1}{da_1} + x_2\frac{db_2}{da_1} + \frac{\partial\psi}{\partial a_1} + \frac{\partial\psi}{\partial a_2}\frac{da_2}{da_1} + \frac{\partial\psi}{\partial b_1}\frac{db_1}{da_1} + \frac{\partial\psi}{db_2}\frac{db_2}{da_1}\right] = b_1,$$

and similarly $q_2 = b_2$.

Hence the original differential equations are actually satisfied. If the arbitrary relation assumed—which may if convenient involve more than two of the parameters—contains two arbitrary constants, the new solution will generally be a complete primitive, since two more constants are introduced by integration.*†

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* The ordinary equations to be integrated may have a singular solution with one arbitrary constant, or with none: if the arbitrary function has been chosen so as to involve three or four arbitrary constants, the whole number being thus raised to four, the solution so given may quite well be a complete primitive, and, in general, will be so.

† The above investigation in a modified form shows how to find integrals of a system of three equations

$$\left.\begin{aligned} f_1(u, v, p_1, p_2, q_1, q_2) &= 0, \\ f_2(u, v, p_1, p_2, q_1, q_2) &= 0, \\ f_3(u, v, p_1, p_2, q_1, q_2) &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $u = p_1 x_1 + p_2 x_2 - y$, $v = q_1 x_1 + q_2 x_2 - z$.

One solution is to take u, v, p_1, p_2, q_1, q_2 as constants connected by the three relations (12); if they are not constants we have

$$du = x_1\,dp_1 + x_2\,dp_2,$$
$$dv = x_1\,dq_1 + x_2\,dq_2.$$

§ 29. Let us now consider the new solutions of the auxiliary equations, given by the new complete primitive. The old solutions are the six pairs of the form x_i, x_j and the six of the form u_i, u_j, where $u_1 = p_1$, $u_2 = p_2$, $u_3 = q_1$, $u_4 = q_2$. The bi-differentials of these twelve satisfy the relation

$$d(y, z) - p_1 d(x_1, z) - p_2 d(x_2, z) - q_1 d(y, x_1) - q_2 d(y, x_2) + (p_1 q_2 - p_2 q_1) d(x_1, x_2)$$

$$= \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_1}, & x_2 + \frac{\partial\phi}{\partial p_2} \\ \frac{\partial\psi}{\partial p_1} & \frac{\partial\psi}{\partial p_2} \end{vmatrix} d(p_1, p_2) + \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_1}, & \frac{\partial\phi}{\partial q_1} \\ \frac{\partial\psi}{\partial p_1}, & x_1 + \frac{\partial\psi}{\partial q_1} \end{vmatrix} d(p_1, q_1)$$

$$+ \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_1}, & \frac{\partial\phi}{\partial q_2} \\ \frac{\partial\psi}{\partial p_1}, & x_2 + \frac{\partial\psi}{\partial q_2} \end{vmatrix} d(p_1, q_2) + \begin{vmatrix} x_1 + \frac{\partial\phi}{\partial p_2}, & \frac{\partial\phi}{\partial q_1} \\ \frac{\partial\psi}{\partial p_2}, & x_1 + \frac{\partial\psi}{\partial q_1} \end{vmatrix} d(p_2, q_1)$$

$$+ \begin{vmatrix} x_2 + \frac{\partial\phi}{\partial p_2}, & \frac{\partial\phi}{\partial q_2} \\ \frac{\partial\psi}{\partial p_2}, & x_2 + \frac{\partial\psi}{\partial q_2} \end{vmatrix} d(p_2, q_2) + \begin{vmatrix} \frac{\partial\phi}{\partial q_1}, & \frac{\partial\phi}{\partial q_2} \\ x_1 + \frac{\partial\psi}{\partial q_1}, & x_2 + \frac{\partial\psi}{\partial q_2} \end{vmatrix} d(q_1, q_2)$$

In the auxiliary equations we may take $x_1, x_2, p_1, p_2, q_1, q_2$ as independent variables, since y, z are given explicitly in terms of these six.

From (12) follow three more relations connecting the six differentials $du, dv, dp_1, dp_2, dq_1, dq_2$, so that their ratios are determinate, and therefore u, v, p_1, q_1, p_2, q_2 can only be functions of one variable. The two equations last written will then, generally, give x_1, x_2 in terms of this variable, which may not be. Hence we must have

$$du = \lambda dv, dp_1 = \lambda dq_1, dp_2 = \lambda dq_2,$$

and since $df_1 = 0$, $df_2 = 0$, $df_3 = 0$, and $du, dv, dp_1, dq_1, dp_2, dq_2$ do not vanish, λ must satisfy the equation:

$$\begin{vmatrix} \lambda\frac{df_1}{du} + \frac{df_1}{dv}, & \lambda\frac{df_1}{dp_1} + \frac{df_1}{dq_1}, & \lambda\frac{df_1}{dp_2} + \frac{df_1}{dq_2} \\ \lambda\frac{df_2}{du} + \frac{df_2}{dv}, & \lambda\frac{df_2}{dp_1} + \frac{df_2}{dq_1}, & \lambda\frac{df_2}{dp_2} + \frac{df_2}{dq_2} \\ \lambda\frac{df_3}{du} + \frac{df_3}{dv}, & \lambda\frac{df_3}{dp_1} + \frac{df_3}{dq_1}, & \lambda\frac{df_3}{dp_2} + \frac{df_3}{dq_2} \end{vmatrix} = 0.$$

If λ satisfies this equation the differential relations $du = \lambda dv$, $dp_1 = \lambda dq_1$, $dp_2 = \lambda dq_2$ reduce to two only, since u, v, p_1, q_1, p_2, q_2 are connected by the equations

$$f_1 = 0, f_2 = 0, f_3 = 0.$$

By integrating these two we find two more relations involving two arbitrary constants. Hence we may suppose v, p_1, p_2, q_1, q_2 expressed in terms of u, and find a solution by eliminating u from the following:—

$$u = p_1 x_1 + p_2 x_2 - y,$$
$$v = q_1 x_1 + q_2 x_2 - z,$$
$$1 = x_1 dp_1/du + x_2 dp_2/du.$$

Then

$$d(x_1, y) = p_2 d(x_1, x_2) + \left(x_1 + \frac{\partial\phi}{\partial p_1}\right) d(x_1, p_1) + \left(x_2 + \frac{\partial\phi}{\partial p_2}\right) d(x_1, p_2) + \frac{\partial\phi}{\partial q_1} d(x_1, q_1) + \frac{\partial\phi}{\partial q_2} d(x_1, q_2),$$

and similar expressions may be found for $d(x_2, y)$, $d(x_1, z)$, $d(x_2, z)$ in terms of the bidifferentials of the pairs of independent variables.

Let c_1, c_2, c_3, c_4 be the constants of integration in a new complete primitive found by the method of §§ 27–8. Let λ be the common value of the ratios dp_1/dq_1, dp_2/dq_2, $d\phi/d\psi$. Then, after integrating the equations $dp_1/dq_1 = dp_2/dq_2 = d\phi/d\psi$ $(= \lambda)$ by help of an assumed relation connecting, say, p_1, q_1, p_2, q_2, c_1, c_2 we have four relations among

$$p_1, p_2, q_1, q_2, \lambda, c_1, c_2, c_3\, c_4,$$

and we may therefore suppose p_1, p_2, q_1, q_2 expressed in terms of λ, c_1, c_2, c_3, c_4, unless λ is a constant, and therefore itself a function of c_1, c_2, c_3, c_4. Then

$$dp_1 - \lambda\, dq_1,\ dp_2 - \lambda\, dq_2,\ d\phi - \lambda\, d\psi$$

will be linear combinations of dc_1, dc_2, dc_3, dc_4, and so will some such expression as

$$\alpha\, dp_1 + \beta\, d\lambda,$$

where α vanishes if λ is one of the constants or a function of them. Conversely, dc_1, dc_2, dc_3, dc_4 will be linear combinations of

$$dp_1 - \lambda\, dq_1,\ dp_2 - \lambda\, dq_2,\ d\phi - \lambda\, d\psi,\ \alpha\, dp_1 + \beta\, d\lambda,$$

and the bidifferentials of c_1, c_2, c_3, c_4 in pairs will be linear combinations of the six following expressions :—

$$\begin{aligned}
&d(p_1, p_2) - \lambda d(q_1, p_2) - \lambda d(p_1, q_2) + \lambda^2 d(q_1, q_2),\\
&d(p_1, \phi) - \lambda d(q_1, \phi) - \lambda d(p_1, \psi) + \lambda^2 d(q_1, \psi),\\
&d(p_2, \phi) - \lambda d(q_2, \phi) - \lambda d(p_2, \psi) + \lambda^2 d(q_2, \psi),\\
&\beta d(p_1, \lambda) - \alpha\lambda d(q_1, p_1) - \beta\lambda d(q_1, \lambda),\\
&\alpha d(p_2, p_1) + \beta d(p_2, \lambda) - \alpha\lambda d(q_2, p_1) - \beta\lambda d(q_2, \lambda),\\
&\alpha d(\phi, p_1) + \beta d(\phi, \lambda) - \alpha\lambda d(\psi, p_1) - \beta\lambda d(\psi, \lambda).
\end{aligned}$$

These are combinations of the bidifferentials of p_1, p_2, q_1, q_2, in pairs, with the expressions

$$\begin{aligned}
&d(p_1, \lambda) - \lambda d(q_1, \lambda),\\
&d(p_2, \lambda) - \lambda d(q_2, \lambda),\\
&d(\phi, \lambda) - \lambda d(\psi, \lambda).
\end{aligned}$$

Now λ is a definite function of x_1, x_2, p_1, p_2, q_1, q_2, given by eliminating the differentials from the equations

$$dp_1 = \lambda dq_1,\ dp_2 = \lambda dq_2,\ d\phi = \lambda d\psi,$$
$$x_1\, dq_1 + x_2\, dq_2 + d\psi = 0.$$

By means of the first two, the third becomes

$$\left\{\lambda\frac{\partial\phi}{\partial p_1} + \frac{\partial\phi}{\partial q_1} - \lambda^2\frac{\partial\psi}{\partial p_1} - \lambda\frac{\partial\psi}{\partial q_1}\right\}dq_1 + \left\{\lambda\frac{\partial\phi}{\partial p_2} + \frac{\partial\phi}{\partial q_2} - \lambda^2\frac{\partial\psi}{\partial p_2} - \lambda\frac{\partial\psi}{\partial q_2}\right\}dq_2 = 0,$$

and the fourth

$$\left\{x_1 + \lambda\frac{\partial\psi}{\partial p_1} + \frac{\partial\psi}{\partial q_1}\right\}dq_1 + \left\{x_2 + \lambda\frac{\partial\psi}{\partial p_2} + \frac{\partial\psi}{\partial q_2}\right\}dq_2 = 0$$

The result of elimination is therefore

$$x_1\left(\lambda\frac{\partial\phi}{\partial p_2} + \frac{\partial\phi}{\partial q_2} - \lambda^2\frac{\partial\psi}{\partial p_2} - \lambda\frac{\partial\psi}{\partial q_2}\right) - x_2\left(\lambda\frac{\partial\phi}{\partial p_1} + \frac{\partial\phi}{\partial q_1} - \lambda^2\frac{\partial\psi}{\partial p_1} - \lambda\frac{\partial\psi}{\partial q_1}\right)$$
$$+\left(\lambda\frac{\partial\psi}{\partial p_1} + \frac{\partial\psi}{\partial q_1}\right)\left(\lambda\frac{\partial\phi}{\partial p_2} + \frac{\partial\phi}{\partial q_2}\right) - \left(\lambda\frac{\partial\psi}{\partial p_2} + \frac{\partial\psi}{\partial q_2}\right)\left(\lambda\frac{\partial\phi}{\partial p_1} + \frac{\partial\phi}{\partial q_1}\right) = 0.$$

This shows the form of λ as a function of x_1, x_2, p_1, p_2, q_1, q_2, not involving c_1, c_2, c_3, c_4. Now this choice of λ makes it possible to choose coefficients A, B, C, E, F, G, such that

$$x_1\, dp_1 + x_2\, dp_2 + d\phi = \mathrm{A}\,(dp_1 - \lambda dq_1) + \mathrm{B}\,(dp_2 - \lambda dq_2) + \mathrm{C}\,(d\phi - \lambda d\psi),$$
$$x_1\, dq_1 + x_2\, dq_2 + d\psi = \mathrm{E}\,(dp_1 - \lambda dq_1) + \mathrm{F}\,(dp_2 - \lambda dq_2) + \mathrm{G}\,(d\phi - \lambda d\psi).$$

Thus

$$\mathrm{A}\{d\,(p_1, \lambda) - \lambda d\,(q_1, \lambda)\} + \mathrm{B}\,\{d\,(p_2, \lambda) - \lambda d\,(q_2, \lambda)\}$$
$$+ \mathrm{C}\{d\,(\phi, \lambda) - \lambda d\,(\psi, \lambda)\} = x_1 d\,(p_1, \lambda) + x_2 d\,(p_2, \lambda) + d\,(\phi, \lambda)$$
$$= \text{multiples of bidifferentials of } p_1,\ p_2,\ q_1,\ q_2$$
$$+ \frac{\partial\lambda}{\partial x_1}\left\{x_1\, d\,(p_1, x_1) + x_2 d\,(p_2, x_1) + d\,(\phi, x_1)\right\}$$
$$+ \frac{\partial\lambda}{\partial x_2}\left\{x_1\, d\,(p_1, x_2) + x_2 d\,(p_2, x_2) + d\,(\phi, x_2)\right\}$$
$$= \frac{\partial\lambda}{\partial x_1}\left\{d\,(y, x_1) - p_2 d\,(x_2, x_1)\right\} + \frac{\partial\lambda}{\partial x_2}\left\{d\,(y, x_2) - p_1 d\,(x_1, x_2)\right\}$$
$$+ \text{multiples of bidifferentials of } p_1,\ p_2,\ q_1,\ q_2.$$

In like manner

$$\mathrm{E}\,\{d\,(p_1,\lambda)-\lambda d\,(q_1,\lambda)\}+\mathrm{F}\,\{d\,(p_2,\lambda)-\lambda d\,(q_2,\lambda)\}$$

$$+\,\mathrm{G}\,\{d\,(\phi_1\lambda)-\lambda d\,(\psi,\lambda)\}=\frac{\partial\lambda}{\partial x_1}\{d\,(z,x_1)-q_2d\,(x_2,x_1)\}+\frac{\partial\lambda}{\partial x_2}\{d\,(z,x_2)-q_1d\,(x_1,x_2)\}$$

$$+\text{ multiples of bidifferentials of } p_1,\, p_2,\, q_1,\, q_2.$$

Hence the three expressions

$$d(p_1,\lambda)-\lambda d(q_1,\lambda),$$
$$d(p_2,\lambda)-\lambda d(q_2,\lambda),$$
$$d(\phi,\lambda)-\lambda d(\psi,\lambda),$$

are all reduced to the same, save for a factor, by adding or subtracting multiples of the bidifferentials of x_1, x_2, x_3, x_4 and of u_1, u_2, u_3, u_4; the same is therefore true of the bidifferentials of c_1, c_2, c_3, c_4. Hence all the new complete primitives found by the method of §§ 27–8 only add one to the eleven known "bifunctionally" independent pairs of functions satisfying the auxiliary equations; one more pair, leading to a fresh complete primitive, is yet to be found.

§ 30. These results may be used to construct examples of bifunctions. For instance, the equations

$$y=p_1x_1+p_2x_2+q_1,$$
$$z=q_1x_1+q_2x_2+p_2,$$

lead to the following case among others :—

In the equations

$$\frac{dp_1}{dq_1}=\frac{dp_2}{dq_2}=\frac{dq_1}{dp_2}=\lambda,$$

put $q_2=\lambda+a$, $dq_2=d\lambda$, and integrate.

Thus $\qquad 2p_2=\lambda^2+b,\quad 3q_1=\lambda^3+c,\quad 4p_1=\lambda^4+e,$

and the arbitrary constants a, b, c, e in the new solution are respectively equal to

$$q_2-\lambda,\ 2p_2-\lambda^2,\ 3q_1-\lambda^3,\ 4p_1-\lambda^4,\ \text{where } \lambda^2x_1+x_2+\lambda=0.$$

Now from § 29 it follows that $d\,(c,e)$ can be expressed in terms of $d\,(a,b)$, the bidifferentials of x_1, x_2, y, z and those of p_1, p_2, q_1, q_2.

For convenience, let us write

u, v, w, x, y, z for x_1, λ, p_1, p_2, q_1, q_2 respectively; then

for x_2 we must put $-v(1+uv)$,

for y ,, ,, $wu-xv(1+uv)+y$,

for z ,, ,, $yu-zv(1+uv)+x$,

so that the eight original variables connected by two equations are now expressed in terms of six.

Thus $d\,(3y - v^3, 4w - v^4)$ can be expressed in terms of $d\,(z - v, 2x - v^2)$, the six bifferentials of w, x, y, z and those of

$$u, \quad -v(1 + uv), \quad wu - xv(1 + uv) + y, \quad yu - zv(1 + uv) + x,$$

that is, of

$$u, \quad v, \quad wu - xv(1 + uv) + y, \quad yu - zv(1 + uv) + x.$$

There is no difficulty in finding the relation. It is

$$\begin{aligned}
&u^2d(3y - v^3, 4w - v^4) - 6v^2(1 + uv)^2d(z - v, 2x - v^2) \\
&- 12u^2d(y, w) + 12v^2(1 + uv)^2d(z, x) \\
&+ 12v^2\{(1 + uv)(y - zv^2) - u(w - xv^2)\}d(v, u) \\
&- 12v^2(1 + uv)d(v, yu - zv - uzv^2 + x) \\
&+ 12uv^2d(v, wu - xv - uxv^2 + y) = 0.
\end{aligned}$$

Here then we have an identical linear relation connecting the bidifferentials of seven pairs of functions of six variables. Any one of the seven pairs is accordingly by definition a bifunction of the other six.

Second Application.

§ 31. Take now a differential equation of the second order,

$$f(x, y, z, p, q, r, s, t) = 0,$$

where p, q are the first and r, s, t the second partial derivatives of z with respect to x, y.

A complete primitive will consist of a single equation in x, y, z involving five arbitrary constants, say a_1, a_2, a_3, a_4, a_5. If we form the first and second derivatives of this equation we shall have, in all, six equations from which a_1, a_2, a_3, a_4, a_5 can be found in terms of x, y, z, p, q, r, s, t, and the original differential equation will be the result of eliminating a_1, a_2, a_3, a_4, a_5. Let u_1, u_2, u_3, u_4, u_5 represent the expressions found for a_1, a_2, a_3, a_4, a_5 respectively, in terms of x, y, z, p, q, r, s, t.

Then from the equations

$$f = 0, \quad u_1 = a_1, \quad u_2 = a_2,$$

by differentiating, we can form six equations which will involve the third derivatives of z; by eliminating these we deduce the following two differential equations to be satisfied by u_1, u_2:—

$$\begin{aligned}&J(x, r, t) + pJ(z, r, t) + rJ(p, r, t) + sJ(q, r, t) + J(y, s, t) + qJ(z, s, t)\\ &\qquad + sJ(p, s, t) + tJ(q, s, t) = 0, \quad \text{and}\end{aligned}$$

$$\begin{aligned}&J(x, s, r) + pJ(z, s, r) + rJ(p, s, r) + sJ(q, s, r) + J(y, t, r) + qJ(z, t, r) + sJ(p, t, r)\\ &\qquad + tJ(q, t, r) = 0.\end{aligned}$$

Here J ( ) denotes the Jacobian of f, u_1, u_2 with respect to the variables specified. These equations express the conditions which are necessary and sufficient in order that

$$\begin{aligned}dz &= pdx + qdy,\\ dp &= rdx + sdy,\\ dq &= sdx + tdy\end{aligned}$$

may be integrable without restriction, when r, s, t are given in terms of x, y, z, p, q, by the equations

$$f = 0, \; u_1 = a_1, u_2 = a_2\,;$$

the conditions must of course be satisfied by any three of the six functions u_1, u_2, u_3, u_4, u_5, f. We thus have forty equations, of which only eight can be algebraically independent.

§ 32. The conditions to be satisfied by u_1, u_2 are linear and homogeneous in their Jacobians with respect to the eight variables x, y, z, p, q, r, s, t; of these, one is given in terms of the rest by the equation $f = 0$, and may, if convenient, be supposed not to occur in u_1, u_2: hence the auxiliary equations in this case have seven independent variables and the dependent variables do not occur explicitly: to find a solution we are therefore to form a complete bidifferential, which shall be a linear combination of the determinants of the following array :—

$$\begin{array}{rlcc}
 & d(r, s), & 0 & - X - pZ - rP - sQ \\
 & d(r, t), & X + pZ + rP + sQ, & - Y - qZ - sP - tQ \\
 & d(s, t), & Y + qZ + sP + tQ, & 0 \\
 & d(p, r), & rT, & - rS - sT \\
(5) & d(p, s), & sT, & rR \\
 & d(p, t), & - rR - sS & sR \\
 & d(q, r), & sT, & - sS - tT \\
 & d(q, s), & tT, & sR \\
 & d(q, t), & - sR - tS, & tR \\
(10) & d(z, r), & pT, & - pS - qT \\
 & d(z, s), & qT, & pR \\
 & d(z, t), & - pR - qS, & qR \\
 & d(x, r), & T, & - S \\
 & d(x, s), & 0, & R
\end{array}$$

(15)	$d(x, t)$,	$-\mathrm{R}$	0
	$d(y, r)$,	0,	$-\mathrm{T}$
	$d(y, s)$,	T,	0
	$d(y, t)$,	$-\mathrm{S}$,	R
	$d(x, p)$,	0,	0
(20)	$d(y, p)$,	0,	0
	$d(z, p)$,	0,	0
	$d(x, q)$,	0,	0
	$d(y, q)$,	0,	0
	$d(z, q)$,	0,	0
(25)	$d(p, q)$,	0,	0
	$d(x, z)$,	0,	0
	$d(y, z)$,	0,	0
	$d(x, y)$,	0,	0

X, P . . . are written for $\partial f/\partial x, \partial f/\partial p$ . . .

Of these twenty-eight rows, only twenty-one are independent. For instance, multiply the 1st, 2nd, 4th, 7th, 10th, 13th, 16th by $-\mathrm{S}$, $-\mathrm{T}$, P, Q, Z, X, Y respectively and add; the resulting row is

$$d(f, r), \quad 0, \quad 0,$$

which vanishes since $f = 0$ by hypothesis.

Suppose $d\,(u_1, u_2)$ to be the complete bidifferential formed from the determinants of the array, then to complete the solution we have to find r, s, t from the equations

$$f = 0, \quad u_1 = a_1, \quad u_2 = a_2,$$

and integrate the equations

$$dz = pdx + qdy, \quad dp = rdx + sdy, \quad dq = sdx + tdy.$$

It will amount to the same thing if we treat u_1 as known in the auxiliary equations. They must be satisfied if u_3, u_4, u_5 are substituted in turn for u_2. Now two homogeneous linear partial differential equations in seven independent variables can at most have five common solutions, and here one of these, u_1, is known; the other four may be taken as u_2, u_3, u_4, u_5.

§ 33. Any two of the five functions x, y, z, p, q will satisfy the auxiliary equations, but as we have to solve for r, s, t, these solutions will not serve our purpose. They are ten in number, and ten more will be given by taking in pairs the expressions u_1, u_2, u_3, u_4, u_5 given by any complete primitive. These twenty are not all bifunctionally independent, for since there are three relations[*] among the ten expressions

$$x, y, z, p, q, u_1, u_2, u_3, u_4, u_5,$$

---

[*] Compare § 34, p. 184.

three linear relations can be formed connecting the twenty bidifferentials; one is formed from each pair of equations as at § 17 (8). Hence seventeen bifunctionally independent solutions of the auxiliary equations are known when we have one complete primitive. The full number is nineteen $\left(\frac{7.6}{1.2} - 2\right)$, and in order to know all we must have one, or possibly two (see § 41, p. 190), more complete primitives.

§ 34. New solutions found by varying the parameters may be divided into two classes, according as the parameters are or are not all functions of one variable; solutions of the former class only occur in exceptional cases, and the principles of § 21 apply to them with slight modification.

Let the three equations connecting

$$x,\ y,\ z,\ p,\ q,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5$$

be $$\phi_i(x,\ y,\ z,\ p,\ q,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5) = 0\ (i = 1, 2, 3);$$

(the forms ϕ_1, ϕ_2, ϕ_3 are not unrestricted, but must be such that the following relations hold identically

$$\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(x, p, q)} + p\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(z, p, q)} = 0,$$

$$\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(y, p, q)} + q\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(z, p, q)} = 0;$$

or we may take ϕ_1 as not involving p, q and ϕ_2 as $p\,\partial\phi_1/\partial z + \partial\phi_1/\partial x$
ϕ_3 as $q\,\partial\phi_1/\partial z + \partial\phi_1/\partial y$),

then the variations of the parameters must satisfy the three equations

$$\sum_{r=1}^{r=5} \frac{\partial\phi_i}{\partial u_r}\, du_r = 0\ (i = 1,\ 2,\ 3),$$

in order that the same relations may subsist among x, y, z, p, q, r, s, t and the parameters, as held when the parameters were constant.

If the parameters are functions of one variable, their forms must be so chosen that the three equations last written reduce to one only, otherwise we shall have five relations connecting x, y, z, p, q with this single variable.

§ 35. If the parameters are not functions of one variable, only the equations

$$\sum_{r=1}^{r=5} \frac{\partial\phi_i}{\partial u_r}\, du_r = 0$$

are equivalent to six, and determine the partial derivatives of u_3, u_4, u_5 with respect to u_1, u_2 in terms of the five parameters and x, y, z, p, q. By help of the relations $\phi_i = 0$ we may suppose x, y, z, p, q eliminated and thus arrive at a system of four partial differential equations connecting u_1, u_2, u_3, u_4, u_5.

The original system may also be taken to consist of four equations connecting five variables x, y, z, p, q, namely:

$$dz/dx = p,\ dz/dy = q,\ dp/dy = dq/dx$$
$$f(x, y, z, p, q, dp/dx, dp/dy, dq/dy) = 0,$$

and so the method of variation of parameters does not lead to any simplification of the problem in general.

§ 36. The interchange of variables and parameters is again possible; it is, perhaps, made clearer by taking three equations of perfectly general form,

$$\phi_i\,(x_1, x_2, x_3, x_4, x_5, u_1, u_2\ u_3, u_4, u_5) = 0\ (i = 1, 2, 3),$$

connecting two sets, each of five quantities.

Whichever set we suppose constant and eliminated by differentiation, we are led to a system of four partial differential equations connecting the quantities of the other set, two of the five being taken as independent variables. A new solution of either of these systems of differential equations will in general yield a new solution of the other.

Suppose, for instance, that we have a new solution of the u equations; this gives u_3, u_4, u_5, say, in terms of u_1, u_2. Then the six equations included in

$$\sum_{r=1}^{r=5} \frac{\partial\phi_i}{\partial u_r}\,du_r = 0\ (i = 1, 2, 3)$$

give two relations among $x_1, \ldots u_1, u_2$, since the four differential equations, which are consequences of these six, are supposed satisfied; by the help of these two, u_1, u_2, may be eliminated from the three relations $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$, and thus three relations are given connecting x_1, x_2, x_3, x_4, x_5; these three will constitute a solution of the x system of differential equations.

§ 37. In this more general case there will not seemingly, as a rule, be any more solutions for either system of differential equations. For the derivatives, say, of x_3, x_4, x_5 with respect to x_1, x_2 are given in terms of these five variables and two others, say u_1, u_2. The forms we may assign to u_1, u_2 are then restricted by three differential equations derived from the three conditions

$$\frac{d^2x_r}{dx_1\,dx_2} = \frac{d^2x_r}{dx_2\,dx_1}\ (r = 3, 4, 5),$$

and thus, generally speaking, no forms of u_1, u_2 will be suitable. In some cases the conditions are not inconsistent, and we may form an array by the method of § 11 such that if $d(\theta,\chi)$ is a combination of its determinants, then $\theta = a$, $\chi = b$, $\phi_1 = 0$,

$\phi_2 = 0$, $\phi_3 = 0$ will give suitable values for u_1, u_2. This array will have four columns and forty-five rows, ten such as

$$d(x_i, x_j),\ 0,\ 0,\ 0,$$

ten such as

$$d(u_i, u_j),\ 0,\ 0,\ 0,$$

and twenty-five of the following type. In the first column there is $d(x_i, u_j)$, in the $(r+1)$th the minor of $\partial^2\phi_r/\partial x_i \partial u_j$, in the determinant

$$\begin{vmatrix}
\dfrac{\partial^2\phi_r}{\partial x_1\,\partial u_1}, & \dfrac{\partial^2\phi_r}{\partial x_2\,\partial u_1}, & \dfrac{\partial^2\phi_r}{\partial x_3\,\partial u_1}, & \dfrac{\partial^2\phi_r}{\partial x_4\,\partial u_1}, & \dfrac{\partial^2\phi_r}{\partial x_5\,\partial u_1} & \dfrac{\partial\phi_1}{\partial u_1} & \dfrac{\partial\phi_2}{\partial u_1} & \dfrac{\partial\phi_3}{\partial u_1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \dfrac{\partial\phi_1}{\partial u_2} & \dfrac{\partial\phi_2}{\partial u_2} & \dfrac{\partial\phi_3}{\partial u_2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \dfrac{\partial\phi_1}{\partial u_3} & \dfrac{\partial\phi_2}{\partial u_3} & \dfrac{\partial\phi_3}{\partial u_3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \dfrac{\partial\phi_1}{\partial u_4} & \dfrac{\partial\phi_2}{\partial u_4} & \dfrac{\partial\phi_3}{\partial u_4} \\
\dfrac{\partial^2\phi_r}{\partial x_1\,\partial u_5} & \cdot & \cdot & \cdot & \dfrac{\partial^2\phi_r}{\partial x_5\,\partial u_5} & \dfrac{\partial\phi_1}{\partial u_5} & \dfrac{\partial\phi_2}{\partial u_5} & \dfrac{\partial\phi_3}{\partial u_5} \\
\dfrac{\partial\phi_1}{\partial x_1} & \dfrac{\partial\phi_1}{\partial x_2} & \dfrac{\partial\phi_1}{\partial x_3} & \dfrac{\partial\phi_1}{\partial x_4} & \dfrac{\partial\phi_1}{\partial x_5} & 0 & 0 & 0 \\
\dfrac{\partial\phi_2}{\partial x_1} & \dfrac{\partial\phi_2}{\partial x_2} & \dfrac{\partial\phi_2}{\partial x_3} & \dfrac{\partial\phi_2}{\partial x_4} & \dfrac{\partial\phi_2}{\partial x_5} & 0 & 0 & 0 \\
\dfrac{\partial\phi_3}{\partial x_1} & \dfrac{\partial\phi_3}{\partial x_2} & \dfrac{\partial\phi_3}{\partial x_3} & \dfrac{\partial\phi_3}{\partial x_4} & \dfrac{\partial\phi_3}{\partial x_5} & 0 & 0 & 0
\end{vmatrix}$$

for

$$r = 1,\ 2,\ 3.$$

This interchange of variables and parameters may take place whenever their numbers are equal, the differential equations being of the first degree.

Examples.

§ 38. I. As an example of the method of solution take the equation $\alpha = \beta$, where α is a function of r, s, $p - sy$, x and β a function of s, t, $q - sx$, y.

In the array (§ 32) multiply the first row by $\partial\alpha/\partial r$, the fifth by $\partial\alpha/\partial p$, the fourteenth by $\partial\alpha/\partial x$, the seventeenth by $-s\,\partial\alpha/\partial p$, and add; the resulting row is

$$d(\alpha, s),\ 0,\ 0.$$

Hence we take $\alpha = \beta = a$, $s = b$,

$$z = bxy + \mathrm{X} + \mathrm{Y},$$

X being a function of x only and Y a function of y only. Then $\alpha = a$ is a relation connecting x, $d\mathrm{X}/dx$, $d^2\mathrm{X}/dx^2$, and $\beta = a$ is a relation connecting y, $d\mathrm{Y}/dy$, $d^2\mathrm{Y}/dy^2$, and by solving these for X, Y respectively we shall have the complete primitive.

§ 39. II. As a second example take the equation

$$F(r, s, t, p - sy, q - ty, z - qy + \tfrac{1}{2}ty^2, x) = 0.$$

Here the third row of the array is

$$d(s, t),\ 0,\ 0,$$

so that the functions s, t satisfy the auxiliary equations. Put, then, $s = a$, $t = b$; thus

$$q = ax + by + c$$
$$z = axy + \tfrac{1}{2}by^2 + cy + X,$$

the last term being a function of x only. The differential equation thus becomes

$$F(d^2X/dx^2,\ a,\ b,\ dX/dx,\ ax + c,\ X,\ x) = 0,$$

an ordinary equation of the second order giving X in terms of x and two more arbitrary constants; hence the finding of a complete primitive is reduced to the solution of the equation last written.

§ 40. III. If the equation is of the particular form $F(r, s, t, p - rx - sy, q - sx - ty, z - px - qy + \frac{1}{2}rx^2 + sxy + \frac{1}{2}ty^2) = 0$, the first three rows of the array are

$$\begin{array}{ccc} d(r, s) & 0 & 0 \\ d(r, t) & 0 & 0 \\ d(s, t) & 0 & 0. \end{array}$$

Hence any two of the three functions r, s, t will satisfy the auxiliary equations, and a complete primitive is given by putting

$$r = a, \quad s = h, \quad t = b.$$

Hence $p = ax + hy + g$, $q = hx + by + f$

$$z = c + gx + fy + \tfrac{1}{2}(ax^2 + 2hxy + by^2),$$

where a, b, c, f, g, h are constants satisfying the relation

$$F(a, h, b, g, f, c) = 0.$$

This is a case in which other solutions are readily given by supposing the parameters variable and functions of one variable only, say a. The variations must satisfy the conditions

$$x^2da + 2xy\,dh + y^2db + 2x\,dg + 2y\,df + 2dc = 0,$$
$$x\,da + y\,dh + dg = 0,\ x\,dh + y\,db + df = 0,$$

whence follows $xdg+ydf+2dc = 0$, a simpler relation that may be taken instead of the first of the three.

These equations will define x, y in terms of the single variable a, unless all the first minors of

$$\begin{vmatrix} da & dh & dg \\ dh & db & df \\ dg & df & 2dc \end{vmatrix} \text{ vanish.}$$

We thus have three ordinary differential equations connecting a, b, c, f, g, h; they are connected also by the relation $\mathrm{F}(a, h, b, g, f, c) = 0$, and the fifth relation among them may be chosen arbitrarily, so that we may put $h = \phi(a)$, an arbitrary function.

Then we have

$$db/da = \{\phi'(a)\}^2, \quad df/da = \phi'(a)\, dg/da,$$

$$2dc/da = (dg/da)^2,$$

$$\mathrm{F}(a, \phi(a), b, g, f, c) = 0$$

as the equations determining b, g, f, c in terms of a. These are to be integrated, and then a is to be eliminated from the equations

$$x + y\, dh/da + dg/da = 0,$$

$$z = c + gx + fy + \tfrac{1}{2}(ax^2 + 2hxy + by^2).$$

The result of elimination will be a solution of the differential equation. Three constants are introduced by integration, and thus, if the function ϕ involves two constants, the new solution will generally be a complete primitive.

§ 41. The new complete primitive gives new solutions of the auxiliary equations which we shall now examine. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ be the new set of parameters. Then a, h, g, b, c, f are connected with these parameters by five equations, one of which is the original equation $\mathrm{F} = 0$. These five relations are such, that if

$$dh = \lambda da, \quad dg = \mu da,$$

then $$db = \lambda^2 da, \quad 2dc = \mu^2 da, \quad df = \lambda\mu da;$$

of these five, the first two define λ, μ in terms of a, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, and the others must then follow from the five equations that give h, g, b, c, f in terms of a and the same new constants. Thus, in general, we may suppose a, h, g, b, c, f, μ, expressed in terms of λ, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and the expressions will be such that

$$dh - \lambda da, \quad dg - \mu da, \quad db - \lambda^2 da, \quad 2dc - \mu^2 da, \quad df - \lambda\mu da$$

involve only the differentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$. One of these five is expressible in terms of the other four, since

$$\frac{\partial F}{\partial a}da + \frac{\partial F}{\partial h}dh + \frac{\partial F}{\partial b}db + \frac{\partial F}{\partial g}dg + \frac{\partial F}{\partial f}df + \frac{\partial F}{\partial c}dc = 0,$$

while one of the relations connecting $\lambda, \mu, a, b, \ldots$ is

$$\frac{\partial F}{\partial a} + \lambda\frac{\partial F}{\partial h} + \mu\frac{\partial F}{\partial g} + \lambda^2\frac{\partial F}{\partial b} + \lambda\mu\frac{\partial F}{\partial f} + \tfrac{1}{2}\mu^2\frac{\partial F}{\partial c} = 0.$$

Some expression such as $\nu d\lambda - \rho da$ will also involve the differentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ only. Hence the differentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ will be linear combinations of $\nu d\lambda - \rho da$, $dh - \lambda da$, $dg - \mu da$, $db - \lambda^2 da$, $df - \lambda\mu da$, $2dc - \mu^2 da$, of which the last five satisfy a linear relation.

Thus the bidifferentials of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ in pairs will be linear combinations of the bidifferentials of a, b, c, f, g, h (only five of the six need be used) in pairs, and of the expressions

$$d(h, \lambda) - \lambda d(a, \lambda),\ d(g, \lambda) - \mu d(a, \lambda),\ d(b, \lambda) - \lambda^2 d(a, \lambda),$$
$$d(f, \lambda) - \lambda\mu d(a, \lambda),\ 2d(c, \lambda) - \mu^2 d(a, \lambda),$$

of which last five, only four are independent.

Now λ, μ are connected not only by the equation

$$\frac{\partial F}{\partial a} + \lambda\frac{\partial F}{\partial h} + \mu\frac{\partial F}{\partial g} + \lambda^2\frac{\partial F}{\partial b} + \lambda\mu\frac{\partial F}{\partial f} + \tfrac{1}{2}\mu^2\frac{\partial F}{\partial c} = 0,$$

but also by the equation

$$x + \lambda y + \mu = 0,$$

so that they are definite functions of x, y, a, b, f, g, h.

Again
$$p = ax + hy + g,$$
$$d(p, x) - hd(y, x) = xd(a, x) + yd(h, x) + d(g, x),$$
$$d(p, y) - ad(x, y) = xd(a, y) + yd(h, y) + d(g, y).$$

Thus
$$y\{d(h, \lambda) - \lambda d(a, \lambda)\} + \{d(g, \lambda) - \mu d(a, \lambda)\}$$
$$= xd(a, \lambda) + yd(h, \lambda) + d(g, \lambda)$$
$$= \frac{\partial\lambda}{\partial x}[d(p, x) - hd(y, x)] + \frac{\partial\lambda}{\partial y}[d(p, y) - ad(x, y)],$$

+ multiples of bidifferentials of a, b, c, f, g, h.

In like manner

$$\begin{aligned} y\{d(b,\lambda)-\lambda^2 d(a,\lambda)\}+\{d(f,\lambda)-\lambda\mu d(a,\lambda)\}+x\{d(h,\lambda)-\lambda d(a,\lambda)\} \\ = xd(h,\lambda)+yd(b,\lambda)+d(f,\lambda) \\ = \frac{\partial\lambda}{\partial x}[d(q,x)-bd(y,x)]+\frac{\partial\lambda}{\partial y}[d(q,y)-hd(x,y)], \end{aligned}$$

+ multiples of bidifferentials of a, b, c, f, g, h.

Lastly,

$$\begin{aligned} 2xy\{d(h,\lambda)-\lambda d(a,\lambda)\}+y^2\{d(b,\lambda)-\lambda^2 d(a,\lambda)\}+2x\{d(g,\lambda)-\mu d(a,\lambda)\} \\ +2y\{d(f,\lambda)-\lambda\mu d(a,\lambda)\}+\{2d(c,\lambda)-\mu^2 d(a,\lambda)\} \\ = x^2 d(a,\lambda)+2xyd(h,\lambda)+y^2 d(b,\lambda)+2xd(g,\lambda)+2yd(f,\lambda)+2d(c,\lambda) \\ = 2\{d(z,x)-(hx+by+f)d(y,x)\}\partial\lambda/dx \\ +2\{d(z,y)-(ax+hy+g)d(x,y)\}\,\partial\lambda/dy, \end{aligned}$$

+ multiples of bidifferentials of a, b, c, f, g, h.

Hence, in all, nine combinations of the ten bidifferentials of α_1, α_2, α_3, α_4, α_5 can be expressed in terms of the bidifferentials of x, y, z, p, q and of a, b, c, f, g, h; that is, in terms of the bidifferentials of the seventeen known independent pairs of functions satisfying the auxiliary equations: thus the new complete primitive adds only one to the number of these known bifunctionally independent pairs, and one more must be added in order to give the full number.

This theory enables us again to construct examples of bifunctions of a number of known pairs which may reach eighteen.

§ 42. The foregoing investigation may be modified so as to give singular solutions of a pair of differential equations of the form in question, say

$$\begin{aligned} F_1(r,\ s,\ t,\ \bar{p},\ \bar{q},\ \bar{z}) = 0, \\ F_2(r,\ s,\ t,\ \bar{p},\ \bar{q},\ \bar{z}) = 0, \end{aligned}$$

where

$$\begin{aligned} \bar{p} &= p - rx - sy, \\ \bar{q} &= q - sx - ty, \\ \bar{z} &= z - \tfrac{1}{2}(p+\bar{p})x - \tfrac{1}{2}(q+\bar{q})y. \end{aligned}$$

A complete primitive would be given by supposing r, s, t, $\bar{p}$, $\bar{q}$, $\bar{z}$ constants connected by the above equations. Another solution would be given by solving the total differential equations found by supposing the relations

$$x\,dr + y\ \ ds + d\bar{p} = 0,$$
$$x\,ds + y\ \ dt + d\bar{q} = 0,$$
$$x\,d\bar{p} + y\ \ d\bar{q} + 2d\bar{z} = 0,$$

to reduce to the same relation linear in x and y. That is, we must solve the system

$$ds = \lambda dr, \quad dt = \lambda^2 dr, \quad d\bar{p} = \mu dr,$$

where λ, μ are given in terms of $\bar{p}$, $\bar{q}$, $\bar{z}$, r, s, t by the relations

$$\frac{\partial F_1}{\partial r} + \lambda \frac{\partial F_1}{\partial s} + \lambda^2 \frac{\partial F_1}{\partial t} + \mu \frac{\partial F_1}{\partial \bar{p}} + \lambda\mu \frac{\partial F_1}{\partial \bar{q}} + \tfrac{1}{2}\mu^2 \frac{\partial F_1}{\partial \bar{z}} = 0,$$

$$\frac{\partial F_2}{\partial r} + \lambda \frac{\partial F_2}{\partial s} + \lambda^2 \frac{\partial F_2}{\partial t} + \mu \frac{\partial F_2}{\partial \bar{p}} + \lambda\mu \frac{\partial F_2}{\partial \bar{q}} + \tfrac{1}{2}\mu^2 \frac{\partial F_2}{\partial \bar{z}} = 0,$$

and $\bar{q}$, $\bar{z}$ in terms of $\bar{p}$, r, s, t by the relations $F_1 = 0$, $F_2 = 0$.

The complete primitive of these ordinary equations will involve three arbitrary constants, and there may be singular solutions with a lower number; none of these will therefore constitute a complete primitive of the partial differential system

$$F_1 = 0, \ F_2 = 0.$$

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V. *The Velocity of the Ions produced in Gases by Röntgen Rays.*

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Communicated by Professor J. J. THOMSON, *F.R.S.*

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§ 1. INTRODUCTION.

THE electrical conductivity which is imparted to gases by their exposure to Röntgen rays has been explained by J. J. THOMSON and E. RUTHERFORD* on the hypothesis of a formation of oppositely charged carriers throughout the volume of the gas. The motion of these carriers or ions when in an electric field constitutes the observed conductivity, and the recovery of the insulating property of a gas after an exposure to the rays is due partly to the recombination of the oppositely charged ions and partly to their impact with the boundaries.

An estimate of the sum of the velocities with which the positive and negative ions move in air when in a unit electric field was first obtained by J. J. THOMSON and E. RUTHERFORD, and later E. RUTHERFORD,† by the same indirect method, determined the sum of the velocities of the ions in a number of gases. This method involved the determination of the rate of recombination of the ions, the saturation current obtained through the gas by the use of a strong electric field, and the current obtained with some small non-saturating electric force. E. RUTHERFORD also describes an experiment in which the velocities of the two ions in air were obtained separately by a direct method, and found to be approximately equal. The writer‡ has since shown that in general the two velocities are not equal, and for those gases for which the ratio of the two velocities was determined the negative ion moved the faster in nearly all cases.

The values of the velocities of the ions have recently been applied by J. J. THOMSON§ and J. S. TOWNSEND|| in the determination of important physical quantities, and it seemed desirable that a redetermination of the values of the velocities be

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* J. J. THOMSON, and E. RUTHERFORD, 'Phil. Mag.,' November, 1896

† E. RUTHERFORD, 'Phil. Mag.,' November, 1897

‡ J. ZELENY, 'Phil. Mag.,' July, 1898.

§ J. J. THOMSON, 'Phil. Mag.,' December, 1898.

|| J. S. TOWNSEND, 'Phil. Trans.,' A, vol. 193, 1899.

undertaken, partly because of advances in our understanding of some of the intricacies of the conduction, and partly because it seemed desirable that a satisfactory direct method be devised whereby the velocities of the two ions could be determined separately, and in which the experimental conditions could be subjected to a number of variations sufficient to ensure freedom from serious errors.

In undertaking this, an attempt was first made to use a modification of the method employed by the writer in the determination of the ratio of the two ionic velocities, which is described in a previous paper. The ions were made to go against a stream of gas in a tube by means of an electric field, and their velocity was compared to that of the gas stream. The presence of the gauzes necessary for the production of the electric field was found, however, to disturb the gas stream sufficiently to produce a turbulent motion in it and so prevented the attainment of absolute results.

The method which was then developed, and the one with which all of the results of this paper were obtained, also consisted in directly comparing the ionic velocity with that of a stream of gas, but avoided the difficulty of the above by having the electric field at right angles to the gas stream.

§ 2. The Method Used for Determining the Velocity.

A stream of gas is passed between two concentric cylinders which are kept at different potentials, and which at one place are traversed by a beam of Röntgen rays. The ions which are produced between the two cylinders by the rays are carried along by the stream of gas and at the same time, under the influence of the electric force, they move at right angles to the axis of the tubes. The resultant paths of the ions are inclined by an amount depending upon the relative value of the velocity of the gas stream to that of the ions.

Let CC′ in fig. 1 represent a section of a portion of the outer cylinder, and DB that of the inner one, and let *dd* represent a narrow beam of rays traversing the two cylinders at right angles to their common axis. When the two cylinders are at

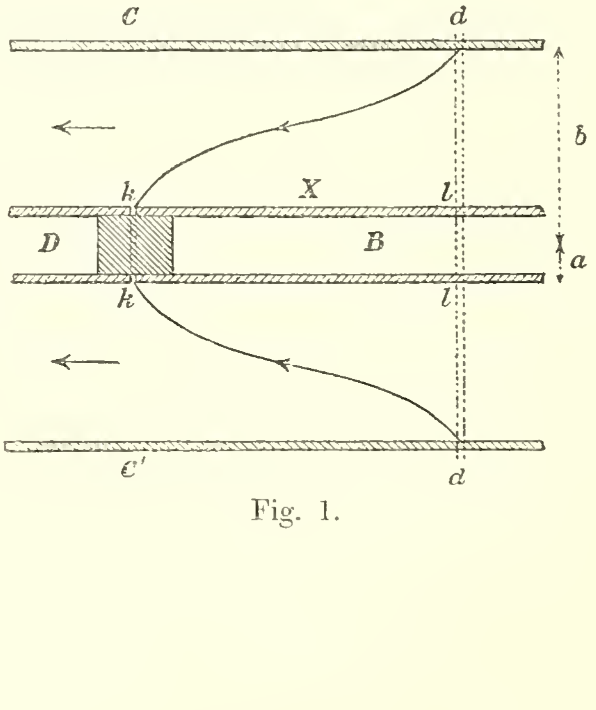

Fig. 1.

different potentials and the gas between them is at rest, an ion starting from the point d at the inner surface of the outer cylinder will move directly across to l under the electric force. But when a stream of gas is passing between the cylinders from right to left, the ion will also be carried along by the stream, and so follow a path somewhat like that represented by the curve dk, finally reaching the inner cylinder at some point, k, which can be determined. The paths of the ions are not straight lines, because the electric intensity and the velocity of the gas stream vary from point to point between the cylinders and according to a different law for each. The distance X that the ions have been carried along the tube by the gas stream while they are crossing between the two cylinders under the electric force is a measure of the relative velocities of the gas and of the ions, and so may be used in determining the velocity with which the ions move in a given electric field.

Let the outer cylinder be kept at a potential of A volts and the inner one at zero potential.

Let b be the inner radius of the outer cylinder and a the outer radius of the inner cylinder.

Then the potential at any point between the cylinders at a distance r from the common axis of the two cylinders is

$$\mathrm{R} = \frac{\mathrm{A} \log_e r/a}{\log_e b/a} \quad \ldots \ldots \ldots \ldots \quad (1),$$

and the electric intensity at this point is

$$\frac{d\mathrm{R}}{dr} = \frac{\mathrm{A}}{r \log_e b/a} \quad \ldots \ldots \ldots \ldots \quad (2).$$

If we let v represent the velocity with which an ion moves when in an electric field whose intensity is 1 volt per centim., and assume that its velocity is proportional to the strength of the field, then at a point whose electric intensity is represented by equation (2), the radial velocity of the ion will be

$$\mathrm{V} = \frac{\mathrm{A}v}{r \log_e b/a} \quad \ldots \ldots \ldots \ldots \quad (3).$$

The ion being carried by the moving gas also has a motion along the tubes. The velocity of the gas stream at any point depends upon its distance from the axis of the cylinders, which will be called the x axis.

Suppose that at the distance r from this axis the gas velocity is u.

The motion of the ion is represented by

$$\frac{dx}{dr} = \frac{u}{\mathrm{V}} \quad \ldots \ldots \ldots \ldots \quad (4),$$

and substituting the value of V from (3),

$$dx = \frac{ur \log_e b/a}{\mathrm{A}v} dr \quad \ldots \ldots \ldots \ldots \quad (5).$$

The distance X travelled by the ion in the direction of the x axis while it is traversing the whole distance between the cylinders, *i.e.*, from $r = b$ to $r = a$, is

$$X = \frac{\log_e b/a}{\mathrm{A}v} \int_a^b u\, r\, dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Now the average velocity of the gas stream as measured by the quotient of the total volume of gas emitted in a second by the area of the cross section is

$$U = \frac{2}{b^2 - a^2} \int_a^b u\, r\, dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

From (6) and (7)

$$X = \frac{U(b^2 - a^2)}{2\mathrm{A}v} \log_e \frac{b}{a} \quad . \quad . \quad . \quad (8), \qquad \text{and} \qquad v = \frac{U(b^2 - a^2)}{2\mathrm{A}X} \log_e \frac{b}{a} \quad . \quad . \quad . \quad (9).$$

This gives the value of the ionic velocity in a unit field in terms of quantities which can be experimentally determined.

The time required for the ions to pass from one cylinder to the other is

$$T = \int_a^b \frac{dr}{V} = \frac{\log_e b/a}{\mathrm{A}v} \int_a^b r\, dr = \frac{(b^2 - a^2)}{2\mathrm{A}v} \log_e \frac{b}{a} = \frac{X}{U} \quad . \quad . \quad . \quad . \quad (10).$$

The equations above apply to ions starting from the inner surface of the outer cylinder and moving inward to the inner cylinder. In practice it is not possible to limit the production of the ions by the Röntgen rays to the inner surface of the outer cylinder, so a narrow beam of rays is passed at right angles through the cylinders, as is represented by *dd* of fig. 1. Of the ions of this layer which move inward under the influence of the electric force, those that start from the circumference at *d* are carried the farthest by the gas stream before they reach the inner cylinder. Under these conditions the equations obtained can be applied by determining the point along the inner cylinder farthest from the beam of rays that is still reached by ions. For obtaining this point, the inner cylinder DB is divided at *k* into two parts, insulated from each other, the part B to the right being connected to earth, while the part D, to the left of the division at *k*, is connected to a pair of the quadrants of an electrometer.

If a definite stream of gas is maintained between the two cylinders, then while the potential of the outer tube CC′ is above a certain value, all of the ions from the volume *dd* which move inward will reach DB to the right of the juncture *k*, and so the electrometer reading will not change. By gradually diminishing the potential of CC′ a value is finally reached such that the ions starting from the outer edge *d* reach DB just to the left of *k*, as will be indicated by a changing electrometer reading. The value of the voltage A in equation (9) is thus determined, and the value of X, which corresponds to it, is the distance from the beam of rays to the juncture *k*. In getting X the corrections which must be made for the width of the beam of the rays and for the width of the juncture k will be considered later. The apparatus as used will now be described.

§ 3. The Apparatus.

The main parts of the apparatus are represented in fig. 2, where the lower part of the figure is a vertical section, while the electrical connections in the upper part are viewed from above.

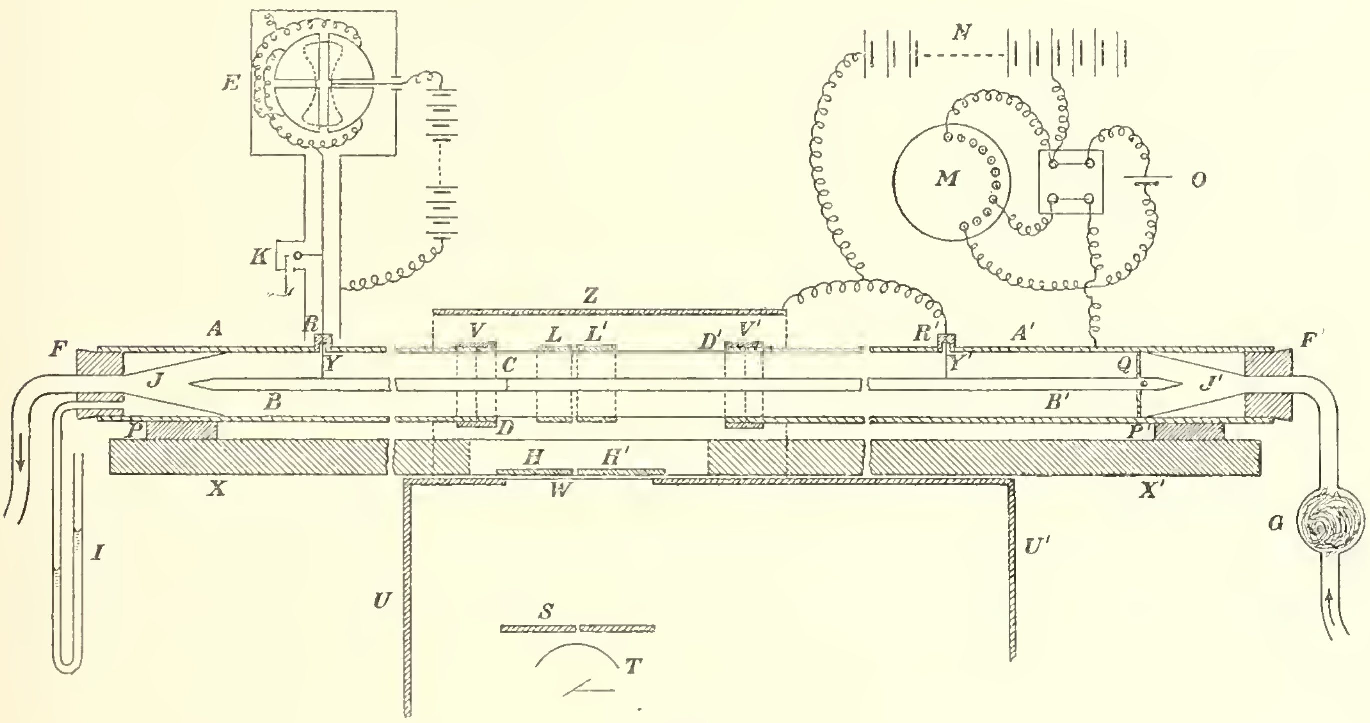

Fig. 2.

The outer cylinder, AA′, had an internal diameter of 5·11 centims., and a total length of 142 centims. For convenience the length is shortened in the figure by the omission of two sections. The part to the left of V, 41 centims. long, and the part to the right of V′, 81 centims. long, were made of strong brass tubing. The portion DD′ between these was 20 centims. long, and consisted of an aluminium tube, which was of the same internal diameter as the brass cylinders. Brass collars over the ends of the aluminium tube fitted into the external collars V and V′ soldered to the brass cylinders, and so formed close-fitting joints that were made gas-tight by sealing them on the outside. The whole cylinder was supported on a board, XX′, and insulated by means of four paraffin blocks, two of which are represented by P and P′.

The inner cylinder, BB′, was an aluminium tube 1 centim. in diameter, closed at its ends by conical pieces. At C the cylinder was divided so that the two portions were held one-half of a millimetre apart and insulated, by means of an ebonite plug. At the end, B′, the tube was supported and kept central by means of two small ebonite rods, Q. The tube was further supported by the two stiff brass wires, Y and Y′, which lead through the ebonite plugs, R and R′, in the outer cylinder, and served to make electrical connections. The part B′ was joined to earth, while the part B was connected to a pair of the quadrants of the electrometer, E. Great care was taken to adjust the position of the central cylinder so as to be accurately concentric with the outer one.

The ends of the outer cylinder were fitted with the large rubber stoppers F and F′. Through these passed the gas inlet and outlet tubes, whose ends were the elongated funnels J and J′. These funnels, together with the cone endings of the inner cylinder, made the lines of gas motion change less abruptly on entering and leaving the apparatus, and so aided in having the gas maintain a steady motion in DD′, where the observations were taken. At the left end, F, a rubber tube led to a gas bag of about 150 litres capacity. The manometer, I, measured the pressure of the gas in the apparatus. The right end, F′, was connected to the glass wool chamber, G, which served to remove dust and any stray electrification from the gas. A rubber tube then led to a drying or moistening apparatus, to be described later, which was connected to a large gasometer of the ordinary type. The pressure of the gas in the gasometer was measured by means of a manometer, and a scale was also attached to the gasometer for measuring its rate of descent during an experiment. The average velocity of the gas stream in the apparatus was determined from the volume emitted by the gasometer in a second, and from the area of the cross section between the two cylinders. To prevent the gas in the gasometer from getting moist too rapidly in those cases where dry gases were used, the surface of the water was covered with a layer of oil, such as is used for air pumps, because of its very low vapour pressure.

The board, XX′, with the attached cylinders was placed on the top of a lead-covered box, UU′, so that DD′, the aluminium portion of the outer tube, was above the aluminium window, W, in the box.

The box contained the Crookes' tube and the induction coil for operating it. The form of tube used was that which the writer has previously employed for similar work.* This form was more satisfactory than any of the others tried, and gave the best results when emitting weak rays, and when an interval of rest of at least three or four minutes was allowed between the periods of use, which did not exceed thirty seconds. A 6-inch Apps' coil was used with a hammer interrupter, which could be made to run with sufficient uniformity with an easy running weak ray tube. The source of the rays, T, was more than 20 centims. from the axis of the cylinders.

The narrow vertical beam of rays which was sent up through the cylinders was regulated by adjusting the position of the tube, T, and of the lead plate, S, with its narrow slit, and of the two lead rings, L and L′, which fitted over the cylinder, DD′. This adjustment was first made by geometrical arrangement, and then tested and completed with the aid of a fluorescent screen placed over the apparatus. The lead strips, H and H′, served to restrict the window, W, and the lead cover, Z, prevented any rays or ionized gas from reaching the outside air of the room.

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* J. ZELENY, 'Phil. Mag.,' July, 1898, p. 126.

The quadrant electrometer, E, used for making the measurements was a small bicellular one, the needle of which was suspended by a quartz fibre, and charged through the liquid below by means of a battery of 160 small storage cells. One pair of its quadrants was joined by a wire to the part BC of the inner cylinder. Both the electrometer and the connecting wire were surrounded by an earthed metal case.

The key, K, permitted the insulated quadrants to be connected to earth at any time.

The capacity of the two quadrants and the part of the inner cylinder connected to them, together with the connecting wire, was about 53 centims. The sensibility of the electrometer was about 500 divisions per volt, with the scale at a distance of 130 centims. The potential of the outer cylinder AA′ was maintained at any desired value by means of the battery of storage cells, N ; the arrangement of the extra cell, O, and the divided megohm, M, permitting the addition of a fractional part of a cell's voltage.

By opening a stop-cock on the gasometer the gas was made to pass from the gasometer, through the apparatus, into the gas bag on the other side, at a rate which was regulated by the weights on the gasometer. It could then be forced back into the gasometer and used again.

A large volume of gas is required for carrying out an experiment, and the method is therefore limited to a small number of gases that can be obtained in such quantities, and that do not act upon the materials of the apparatus.

§ 4. CORRECTIONS AND PRECAUTIONS OBSERVED IN THE EXPERIMENTS.

1. It is essential for these experiments that in its motion down that part of the tube where the observations are being taken, the different portions of the gas should move in paths parallel to the axis of the tube, *i.e.*, that the motion be uniform, and not turbulent with vortices. This condition depends upon the velocity of the gas stream.

O. REYNOLDS has shown* that for motion in a cylindrical tube a fluid when started in a turbulent state will tend to assume a uniform motion with the parts moving parallel to the axis when for the fluid the average velocity is less than a critical value,

$$V = \frac{\mu}{B\rho D},$$

where μ is the viscosity of the fluid relative to that of water at 0°, ρ is its density, D is the diameter of the cylinder, and B is a constant.

The value of B obtained was about 280 when D and V were measured in metres.

Applying this constant to the gases used, for a cylinder of the diameter of the

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* O. REYNOLDS, 'Phil. Trans.,' A, 1883.

outer one in the apparatus, we obtain for the value of the critical velocity for air about 55 centims. per second, and for hydrogen about 390 centims. per second. It is evident that in the apparatus used where there are two concentric cylinders, the maximum velocity consistent with a uniform motion must be considerably larger than if the gas were flowing through the outer cylinder alone. Nevertheless the largest value of the velocity used in any experiment was 25 centims. per second for air and 44 centims. per second for hydrogen. As these values are well within the limits given above for a cylindrical tube whose radius is equal to that of the outer one here used, the conditions for a stable motion are fulfilled. The entrance of the gas through a funnel-shaped aperture and its subsequent passage for a considerable distance through a uniform section allowed the motion to come to a permanent state before it reached the place where the observations were taken.

An experiment which was tried showed that by blowing a stream of air down a large glass tube and with a velocity greater than that used in these experiments, the gas assumed a motion parallel to the axis after it had traversed but a short length of the tube, as was made visible by the presence in the air of irregularly distributed ammonium chloride particles.

2. The volume of the gas emitted per second by the gasometer varied a little for different elevations of the gasometer, but there was a considerable range where it was quite constant, and this range only was used in making experiments, the rate of descent being determined in addition during each observation. Guide wheels prevented the tilting of the gasometer during its descent, and the readings on the attached scale could therefore be relied upon. The pressure of the gas was determined by a manometer attached to the gasometer, and the pressure in the apparatus was similarly obtained. The volume of the gas emitted by the gasometer per second was then reduced to the pressure in the apparatus, and dividing by the flow area in the tubes, the required value of U in equation (9) was obtained.

3. In order to understand more clearly the manner in which the values of A and X of equation (9) were determined, let us consider the following case. In fig. 3, CC′ represents a longitudinal section of the outer cylinder. DB is the

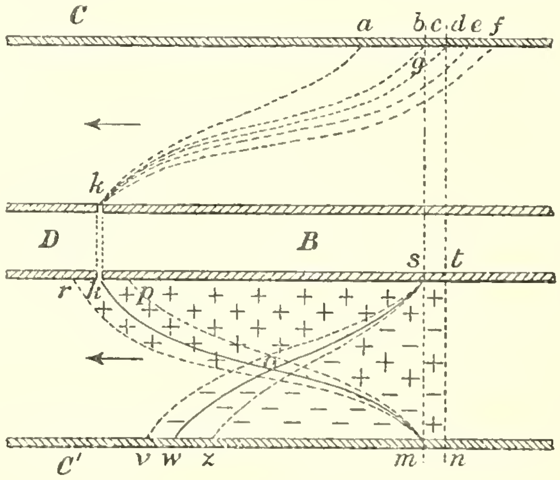

Fig. 3.

inner cylinder having the insulated juncture at *k*, the part D of the cylinder being connected to an electrometer.

The gas stream is supposed to flow from right to left in the figure, and *bdmn* is the beam of rays. DB being at zero potential, suppose that when the potential of CC′ is at a certain value the ions going towards DB move in paths parallel to the line *ak* in the upper half of the figure. An ion starting from any point to the left of *ak* would reach the part D and so influence the electrometer, but as all of the ions start from the beam of rays to the right of *ak*, all of them reach B. If the potential of CC′ is diminished so that the inclination of the ionic paths becomes *bk*, ions from the outermost rim of *bdmn* will just begin to reach the part D. By a certain decrement in the potential of CC′ the paths of the ions can be made parallel to *dk*, so that ions will reach D from a volume whose section is represented by the triangle *bdg*, the width of the beam of rays being *bd*. By a decrement in the potential of CC′ equal to the last one, the volume from which ions reach D is increased by a volume whose section is seen from the figure to be nearly a parallelogram of about twice the area of the triangle *bdg*. Another equal decrement in the potential increases the volume by almost the same amount as the last. As the potential is diminished further, the rate of increase of volume gradually diminishes. So if we represent the potentials used by abscissas and the volumes from which ions reach D by corresponding ordinates, we obtain a curve, fig. 4,

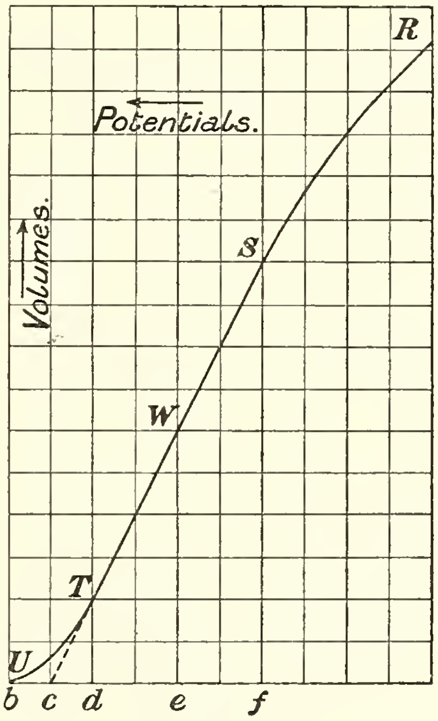

Fig. 4.

whose inclination to the axis of abscissas, as the potentials are increased, at first gradually increases (RS of fig. 4), then assumes a constant value (ST) and finally diminishes (TU) as the curve ends in the axis of abscissas. The point U corresponds to the inclination of the paths of the ions represented by *bk* of fig. 3, T corresponds to *dk*, and W to *ek*. As the paths change from *ek* to *dk*, the diminution in the number of ions reaching D is equal to about twice the number that are getting to D in the latter case ($We = 3\ Td$). If, therefore, the rate of diminution remained unchanged until ions just ceased to reach D, the change in potential required for this would be just a half of the change from *ek* to *dk* or from *dk* to *bk*. Thus in the curve it is seen that by prolonging WT it reaches the axis at *c*, half way between *b* and *d*. This corresponds to a potential which would be required for an ion starting from *c* (fig. 3) the middle point of the beam of rays *bd*, in order to have it just reach the juncture *k* in the inner cylinder. It is evident that the points T and U are not very sharply defined on an experimental curve, and hence cannot be determined as accurately as the point *c*, and so in practice the potential A of formula (9) has always been determined in this latter way. Evidently the value of X which is to be used with this value of A has to be measured from the middle of the beam of the rays where they cross the inner cylinder to the middle of the juncture *k*, as all ions reaching the middle point are drawn to D. The width of this juncture was only ·05 centim. The width of the beam of rays was used as small as possible, and in most cases was ·2 centim., this being a small part of the total distance X.

4. In considering the distribution of the ions between the two cylinders while the conduction is going on, it is seen from the lower part of fig. 3 that supposing the external tube to be positive, the negative ions starting from *s* will describe a path somewhat like *sw*, so that all of the negative ions will be confined to the space *wmnts*. Similarly the positive ions starting from *m* will describe the path *mk*, and all of the positive ions will be confined to the space *kmnts*. In the space where these two overlap, *i.e.*, *omnts*, both kinds of ions will be present and recombination will take place, the number of ions per cubic centim. diminishing, therefore, as we go from *sm* to *o*.

The space *owm* will be occupied by negative ions alone, and *oks* by positive ions alone. *wm* will usually be shorter than *ks*, because as a rule the negative ions travel the faster in the same electric field.

5. Of the ions starting from *m* towards *k* all will not follow the path *mk*, but some, due to the motions assigned to them by the kinetic theory of gases, will diffuse to either side so that the distribution, along the path, of the ions which started from *m* will lie between the two dotted lines *mr* and *mp*. This effect will produce a distortion in such a curve as that shown in fig. 4, and to bring all of the ions to the part B of the inner cylinder will require a greater force than would be necessary if there were no diffusion. The effect of this disturbance upon the value of the ionic velocity obtained in the manner described is to give a result that is too small because the potential A obtained is too large. Moreover the amount of the diffusion depends upon the time required for the ions to travel between the two

cylinders so that if we obtain values of the ionic velocity, in the manner already described, these will be the larger and nearer to the true value the smaller the time that is required for the passage of the ions across.

If this time were zero, then evidently all diffusion effects would disappear.

6. The free charges that exist in the gas, where the ions of one sign predominate, tend to spread on account of the mutual repulsion of the charged carriers.

This produces an effect similar to that of the diffusion just described. It increases with the time required for the ions to pass between the cylinders, but is less the smaller the density of the free charges, *i.e.*, the weaker the Röntgen rays used and the narrower the beam of the rays.

7. The presence of these free charges in the gas also has an influence upon the intensity of the electrostatic field between the two cylinders. To diminish this effect a sensitive electrometer was used in making the observations, as this allowed the employment of a weak radiation so that the charges in the gas were of a small density.

While it is not possible to make an exact calculation of the magnitude of this effect because of the unsymmetrical distribution of the ions, an approximation to it can still be obtained. Knowing the capacity of the receiving system and the charge received in a given time, and knowing the approximate velocity of the ions in the electric field and the approximate space occupied by the free charges, the density of these charges can be obtained roughly, and their effect upon the electrostatic field can be computed.

Computations of this kind made from the observations used for final results showed that the largest value of this correction made a diminution in the electrostatic field of less than 1 per cent. In some experiments where a large inner cylinder was used the intensity of the electric field employed was less, the ions moved slower, and the density of the free charges was therefore larger and in some instances the above correction was perhaps nearly 2 per cent. In all cases an increase in the strength of the field itself diminishes the percentage value of the correction, while the simultaneous diminution in the density of the free charges reduces it still further.

8. The motion of these free charges through the gas also produces a motion of the gas itself, as the writer has previously shown.* The amount of this is, however, very small compared to the velocity of the ions, so that it cannot have an appreciable disturbing effect upon the results of these experiments.

9. In conduction produced by Röntgen rays there is a noticeable fall of potential at the electrodes which diminishes the electric intensity in the intermediate space. As determined by the writer,† for conduction in air between two plates 1·2 centims. apart, this amounted to about 2 per cent. of the total potential difference for the

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* J. Zeleny, 'Proc. Camb. Phil. Soc.,' vol. 10, Pt. I., p. 13.

† J. Zeleny, 'Proc. Camb. Phil. Soc.,' vol. 10, Pt. I., p. 21.

strength of rays used. For the much weaker radiation and the greater distance here used the correction does not perhaps exceed 1 per cent. For gases other than air the effect has not been determined, and has been assumed to be no greater than with air.

10. J. PERRIN[*] has shown that when the Röntgen rays impinge upon a metal surface the ionization in the gas near it is increased by an amount depending upon the nature of the metal and upon the state of its surface. M. G. SAGNAC[†] and P. LANGEVIN have shown since that this is due to a secondary radiation started at the metal surface by the Röntgen rays. It is possible that the ions so produced are of a different nature from those produced by the direct rays, but in the absence of any evidence to that effect the much more probable case is assumed that the two kinds are identical.

The effect of the secondary rays, therefore, is to produce an uneven distribution of the ions in the space exposed to the direct rays, and also to widen the ionized area near the metal surfaces. This makes more difficult the accurate determination of the potential A in equation (9), the tendency being to get it too large. J. PERRIN found that the surface effect was by far the least for aluminium, what he calls the coefficient being ·0 for aluminium in air as compared to ·9 for gold in air. The effect is also very much dependent upon the cleanliness of the surface. It is thus seen that in the apparatus used this effect was made as small as possible by using unpolished aluminium as the material for those parts of the cylinders upon which the rays impinged. That the secondary rays did not produce an appreciable amount of ionization at a short distance to the side of the beam of the direct rays was shown by passing these rays near to the insulated juncture in the inner cylinder while the gas in the tubes was at rest. No conductivity was observed to that part of the inner cylinder which was not exposed to the direct rays.

Further experiments tried for the effect of the secondary rays by coating the inside of the aluminium cylinder on the apparatus with tin-foil will be described later among the observations for dry air.

11. W. C. RÖNTGEN[‡] has shown that the air itself where it is exposed to the rays acts as a source of a weak secondary radiation. The writer is not aware of any experiments showing any conductivity produced by this radiation, but the experiment referred to in the last section, where a beam of rays near the juncture of the inner cylinder produced no appreciable conductivity on the other side, shows that in these experiments the effect may be disregarded.

12. When D (fig. 3), the part of the inner cylinder joined to the electrometer, takes up a charge in the progress of an observation, the electric field in the vicinity of the juncture becomes slightly distorted, tending to lessen the number of ions

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[*] J. PERRIN, 'Comptes Rendus,' vol. 124, p. 455.

[†] M. G. SAGNAC, 'Journal de Physique,' 1899, p. 65.

[‡] W. C. RÖNTGEN, 'Wied. Ann.,' vol. 64, p. 18.

reaching D. As for each reading this effect starts from zero, the only influence of this upon a series of readings with different potentials is to diminish their values by small amounts nearly proportional to their size, thus having practically no effect upon the result obtained by projecting the curve as in fig. 4.

13. The velocity of the ions is evidently dependent upon the pressure of the gas. In these experiments the variations in the pressure were but small, being due mainly to the variations of the barometer. No experiments have been carried out on the effect of pressure upon the velocity of the ions produced by Röntgen rays, but E. RUTHERFORD* has shown that for the conduction produced by ultra-violet light the velocities of the ions in air are inversely as the pressure of the gas. This result will be used in these experiments to reduce all of the values of the velocities to the same pressure of 76 centims. of mercury.

14. The effect of temperature upon the ionic velocity is not known, so that corrections for temperature could not be made. The temperature was, however, taken in all cases, so that if necessary the correction can be applied later on.

15. In considering the various corrections above, it is seen that the effect of many of them is diminished or made negligible by using a narrow beam of weak rays, and by using unpolished aluminium for that part of the cylinders where the rays impinge. Those corrections which depend upon the time required for the ions to cross between the two cylinders could be made very small by sufficiently reducing the value of this time, but we are limited in doing so by the increase that is produced in the difficulty of measuring one of the required quantities. Resort must be had to finding the values of the ionic velocities for different times of crossing, and from these deriving the final results.

An estimated correction of 2 per cent. will be made for those effects considered above, especially (7) and (9), which tend to make the result too small by an undetermined but small amount.

§ 5. CHANGES MADE IN EXPERIMENTAL CONDITIONS.

The apparatus used permits of several changes in the experimental conditions, which are a test of the accuracy of the method, and allow us to draw conclusions about the effects of some of the corrections previously noted.

1. The velocity of the gas stream was varied by changing the weights on the gasometer. This necessitated a proportionate change in the value of the potential A of equation (9). The paths described by the ions are the same, but the time required for their passage between the two cylinders is changed. There are also changes in the amount of recombination of the ions and in the diffusion effect. The density of the free charges is changed, and so their effect upon the electric intensity is altered, and the spreading due to the mutual repulsion of the ions is also different.

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* E. RUTHERFORD, 'Proc. Camb. Phil. Soc.,' vol. 9, Pt. VIII., p. 414.

2. The distance of the beam of rays from the insulated juncture in the inner cylinder was also changed. This likewise necessitated a change in the value of the potential A, but in the opposite sense. The paths of the ions are now quite different, and changes are also produced in all of the quantities mentioned in the preceding case.

3. The intensity of the Röntgen rays was also varied. This produced alterations in the density of the free charges in the gas, and consequently in their effect upon the electric field between the cylinders and in the mutual repulsion of the ions. The amount of the recombination of the ions is also affected as well as the fall of potential at the electrodes.

4. By changing the diameter of the internal cylinder complete changes are produced in the configuration of the forces, and of the motions of the ions. All the other changes can also be tried in conjunction with this one.

5. The material of the inner surface of the outer cylinder was also altered to note the influence upon the result of increased ionization at the metal surface.

6. In trying to find the effect of any of these changes upon the observed velocity the greatest difficulty met with is due to the smallness of the effects, and their consequent masking by the irregularities of individual observations caused by the difficulty of maintaining a uniform radiation for a length of time sufficient to cover a number of readings. Individual observations taken under the same conditions may vary among themselves by a number of per cent., so a small change in the result cannot be detected unless a large number of observations is made.

§ 6. METHOD OF CONDUCTING THE EXPERIMENTS.

The following procedure was followed in taking readings with the apparatus. The Crookes' tube and the lead slits were accurately adjusted, so that the beam of rays occupied the desired position, and the distance X of equation (9) was carefully measured. The cylinder AA′ was connected to a chosen potential on the battery N. The electrometer quadrants, joined to the part B of the inner cylinder, were then disconnected from earth by means of the key K, and the zero reading was observed on the scale. The reading on the gasometer scale was also taken. At a definite time, observed on a chronometer, the valve at the gasometer was opened, so that the gas began to flow through the apparatus. After a short period, usually 10 seconds, sufficient to produce a steady state of flow in the apparatus, the primary of the induction coil was closed and the rays thus started. The rays were allowed to run for 30 seconds, and the primary of the coil was then broken, and the valve of the gasometer was also closed at a definite time. The electrometer reading was now taken, and the deflection produced was obtained. The key K was then closed, and the quadrants of the electrometer were connected to earth. From the reading on the gasometer scale the volume emitted was obtained, and with the aid of the pressure readings which were taken the average velocity of the gas stream in the apparatus could be calculated.

An interval of about three minutes was allowed as a rest for the tube, as this made it much more constant over a large number of readings. In the mean time, if necessary, gas was forced back from the gas bag into the gasometer. Guided by the previous electrometer deflection the potential of the outer cylinder was now changed, and the whole process repeated. In this way a number of readings were taken, such that the electrometer deflections ranged from some value down to near zero. These were taken in such an order that at first, say, a descending series of readings was obtained, and then immediately afterwards an ascending series. In this manner it is possible to detect any uniform changes which are taking place in the intensity of the rays, for in that case the two series of points would lie on curves of different inclinations.

It was seen in § 5 that the time of passage of the ions from one cylinder to the other could be varied by changing the velocity of the gas stream, and also by changing the distance X. Both of these were employed in practice, and it was found that the values of the velocity obtained diminished as the time increased; but they were practically the same for two different values of X if the velocity of the gas stream was changed in the same ratio, *i.e.*, if the time of passage of the ions was the same.

J. S. TOWNSEND* has recently observed that the rate of diffusion of the ions depends upon the moisture in the gas. In these experiments the gases were used both dry and saturated with aqueous vapour, and it was found that the velocity was different in the two cases.

For saturating a gas with aqueous vapour it was forced, in passing between the gasometer and the apparatus, to bubble through a water bottle and then to pass through a long horizontal tube half filled with water. After the gas had been passed several times back and forth between the gasometer and the gas bag, and before any readings were taken, the water bottle was cut out so as to avoid any unsteadiness in the pressure due to the bubbling.

For drying a gas the above arrangement was replaced by one in which the gas had to pass through a long, horizontal glass tube, partly filled with concentrated sulphuric acid, and then through a large volume of calcium chloride. In order to allow a sufficiently rapid stream with the small pressures used the calcium chloride was placed in a large, wide bottle, the gas entering above and leaving by a protected funnel-shaped tube near the bottom. It thus had to traverse a considerable length of calcium chloride, and on account of the large area of the bottle the velocity through it was small.

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* J. S. TOWNSEND, 'Phil. Trans.,' A, vol. 193.

§ 7. MOIST AIR.

The following is an example of a set of readings taken for the positive ions in air saturated with aqueous vapour.

Letters refer to corresponding quantities in formula (9).

Temperature = 14·5° C. X = 2·60 centims. a = ·50 centims. b = 2·555 centims.
Width of beam of rays = ·20 centim. Barometer = 75·4 centims.
Excess pressure inside gasometer = 1·56 centims. of mercury.
,, ,, in apparatus = ·59 centim. of mercury.
20 cells = 42·6 volts.

TABLE I.—Moist Air. Positive Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
+10	145	6·77
+12	105·5	6·79
+14	68·5	6·78
+16	29·5	6·72
+18	12	6·70
+19	7	6·83
+17	19	6·81
+15	52·5	6·78
+13	87	6·77
+11	128	6·76

In the middle of the observations the gasometer was refilled from the gas bag. The sectional area of the gasometer was 2904 sq. centims., and the area between the two cylinders was 19·73 sq. centims., so the average rate of descent of the gasometer above indicates an average velocity in the apparatus of 25·2 centims. per second, when corrected for the difference in pressure between the gasometer and the apparatus.

The voltages and their corresponding deflections are represented graphically in curve I. of fig. 5. The set of readings here given, and most of those which are to follow as examples, have been selected from among the best obtained.

It is seen that the curve at first approaches the axis of abscissas in nearly a straight line, but becomes convex when near to it. Had readings been taken for voltages smaller than those used, that part of the curve would have been concave to the axis of abscissas.

It has been explained in § 4 (3), why there is a nearly straight portion in the curve, while the width of the beam of rays and the various causes tending to spread the ions make the lower end of the curve approach the axis at a less rapid rate. It was also shown that the point on the axis of abscissas, obtained by prolonging the straight portion of the curve, would correspond to the voltage required to make an

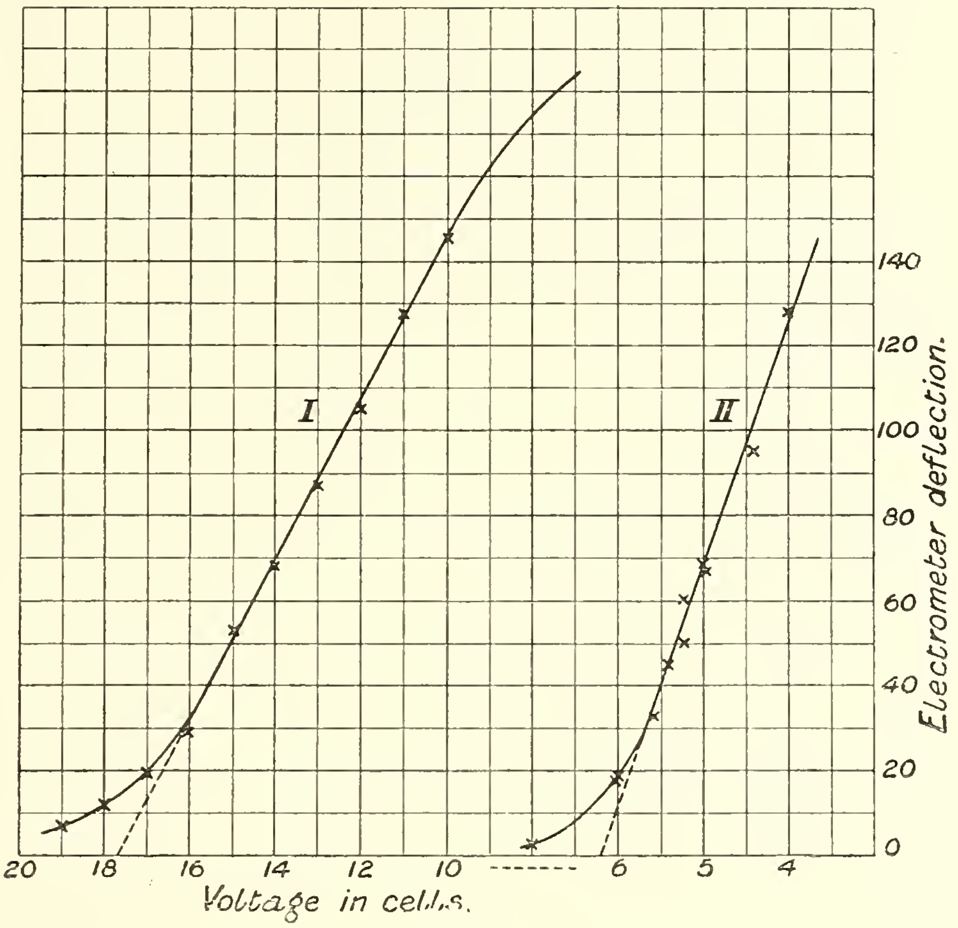

Fig. 5.

ion, starting from the surface of the outer cylinder in the middle of the beam of rays, just reach the middle of the juncture in the inner cylinder. But with diffusion and the other causes acting to produce a spreading of the ions, it is evident that the inclination of the straight part itself is affected and the result changed. Corrections for this error can only be made in conjunction with those of some other effects, and that by experiment, by producing alterations in the amount of these effects, by changes in the time of passage of the ions across the space between the cylinders.

The velocity obtained by the use of the voltage determined by the continuation of the straight part of the curve, as shown in the figure, will be called the ionic velocity for that determination, it being understood that it is not implied thereby that the velocity changes with the time, but that this is only a step towards the final result. From the above curve, A is seen to be 17·7 cells, which is equal to 37·7 volts.

Using equation (9),

$$v = \left\{\frac{b^2 - a^2}{2} \log_e \frac{b}{a}\right\} \frac{\mathrm{U}}{\mathrm{AX}} = 5{\cdot}118 \, \frac{25{\cdot}2}{2{\cdot}60 \times 37{\cdot}7} = 1{\cdot}315 \text{ centims. per second.}$$

The pressure in the apparatus is 76 centims. of mercury

From equation (10)

$$T = \frac{X}{U} = \frac{2{\cdot}6}{25{\cdot}2} = {\cdot}10 \text{ second.}$$

The following is a set of readings taken for negative ions in moist air. Unless otherwise mentioned the values of a, b, and the width of the beam of rays will hereafter be taken the same as in the previous example.

Temperature = 14·4° C. X = 6·42 centims. Barometer = 74·7 centims.
Excess pressure inside gasometer = 1·54 centims. of mercury.
,, ,, in apparatus = ·59 centim. of mercury.
8 cells = 16·5 volts.

Table II.—Moist Air. Negative Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
– 4	128	6·05
– 5	68·5	5·95
– 5·4	45	5·94
– 6	17·5	5·92
– 5·6	32·5	5·89
– 5·2	50	5·90
– 6	18	6·02
– 5·4	44·5	5·99
– 5	67·5	5·96
– 4·4	95	5·90
– 7·	2	5·90

The results are represented in Curve II. of fig. 5.

U = 22·1 centims. per second.
A = 12·7 volts.

$$v = 5{\cdot}118 \frac{22{\cdot}1}{6{\cdot}42 \times 12{\cdot}7} = 1{\cdot}39 \text{ centims. per second.}$$

The pressure in the apparatus = 75·3 centims.
The velocity reduced to 76 centims. pressure = 1·38 centims. per second.

$$T = \frac{6{\cdot}42}{22{\cdot}1} = {\cdot}29 \text{ second.}$$

The following is a summary of the results obtained for moist air for both the positive and negative ions. Each result was obtained from a series of observations as indicated by the above examples. The results are reduced to 76 centims. pressure.

Letters refer to quantities in equations (9) and (10).

Table III.—Moist Air. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					° C.			
1	4·33	20·6	+19·2	·21	15·3	76·2	—	1·28
2	4·18	10·85	+11·1	·39	15	77·6	—	1·225
3	4·18	11·1	−10·1	·38	15	77·6	1·37	—
4	4·18	10·73	− 9·8	·39	14·3	76·5	1·35	—
5	4·18	10·96	+11·35	·38	14·3	76·5	—	1·19
6	4·18	25·0	+23·4	·17	14·3	76·8	—	1·32
7	4·18	25·0	−21·2	·17	14·3	76·8	1·46	—
8	2·68	11·3	−15·65	·24	14·6	75·9	1·38	—
9	2·68	11·3	+17·5	·24	14·6	75·9	—	1·23
10	2·68	22·0	+32·15	·12	14	76·1	—	1·32
11	2·68	22·1	−29·1	·12	14	76·1	1·46	—
12	8·41	11·33	+ 6·26	·75	14·4	76·0	—	1·10
13	8·41	11·23	+ 6·03	·75	14·4	76·0	—	1·13
14	8·41	10·78	− 5·2	·78	14·4	76·0	1·26	—
15	8·41	11·33	− 5·41	·74	14·4	76·0	1·27	—
16	8·41	11·8	− 5·51	·72	14·4	76·0	1·30	—
17	8·41	24·8	−10·95	·34	14·2	76·2	1·39	—
18	8·41	24·73	+12·2	·34	14·2	76·2	—	1·24
19	8·41	24·8	+12·5	·34	14·2	76·2	—	1·22
20	8·41	24·8	−10·7	·34	14·2	76·2	1·42	—
21	6·42	10·67	− 6·57	·60	13·5	75·8	1·30	—
22	6·42	10·7	+ 7·26	·60	13·5	75·8	—	1·175
23	6·42	22·06	+14·0	·29	14·4	74·8	—	1·245
24	6·42	22·06	+14·15	·29	14·4	74·8	—	1·23
25	6·42	22·1	−12·7	·29	14·4	74·7	1·38	—
26	2·60	11·1	−15·7	·29	14·7	74·7	1·375	—
27	2·60	11·1	+17·0	·24	14·7	74·7	—	1·27
28	2·60	25·2	+37·7	·10	14·5	75·4	—	1·315
29	2·60	25·2	−34·1	·10	14·5	75·4	1·455	—
30	2·60	12·6	−17·8	·21	15	75·5	1·39	—
31	2·60	10·9	−15·7	·24	15	75·5	1·37	—
32	2·60	13·1	−18·3	·20	16	75·7	1·41	—
33	2·60	13·15	−17·8	·20	16	75·7	1·45	—

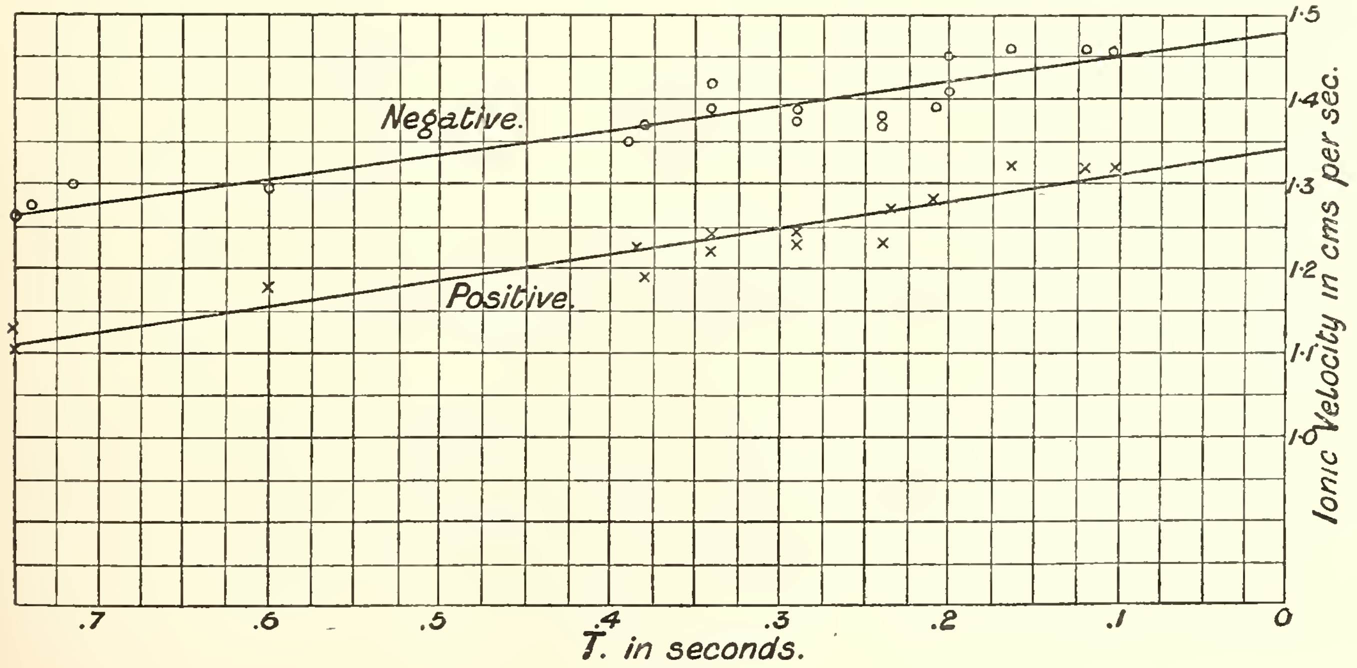

Fig. 6.

The results are represented in fig. 6, where the velocities are represented as ordinates and the corresponding values of T as abscissas. It is seen that the velocities decrease with T, and nearly in a linear manner. Considerable variations are observed among the individual results, but it is believed that they are not greater than is to be expected from the nature and difficulties of the experiments.

In § 4 it was seen that some of the corrections which act to give too small a value for the velocity diminish with T and disappear for $T = 0$. By drawing lines through the points in fig. 6 and prolonging them to the axis of ordinates where $T = 0$, we obtain the most probable values of the velocities. This gives for the negative ions 1·48 centims. per second, and for the positive ions 1·34 centims. per second.

In § 4 (15) it was stated that a correction of 2 per cent. would be made for disturbances not corrected by the above method. This gives for the final results for moist air the velocity in an electric field of 1 volt per centim. for the negative ions = 1·51 centims. per second, and for the positive ions = 1·37 centims. per second at a temperature of about 14° C., and a pressure of 76 centims. of mercury.

§ 8. DRY AIR.

The following set of readings was taken for the positive ions in dry air :—

Temperature = 13·8° C. X = 2·60 centims. Barometer = 76·1 centims.
Excess pressure in gasometer = 1·6 centims. of mercury.
,, ,, apparatus = ·45 ,, ,,
14 cells = 29·0 volts.

TABLE IV.—Dry Air. Positive Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions	Centims.
+ 8	117	4·29
+10	60	4·28
+12	22	4·26
+14	7	4·26
+11	40	4·22
+ 9	94	4·25
+ 7	153	4·25
+10	62	4·21
+ 8	123	4·23

These results are represented graphically in Curve I. of fig. 7.

The corrected value of U is 15·9 centims. per second.

$A = 24·8$ volts.

So $v = 5·118 \dfrac{15·9}{2·60 \times 24·8} = 1·26$ centims. per second, and when reduced to 76 centims. pressure this becomes 1·27 centims. per second.

$T = \dfrac{2·60}{15·9} = ·16$ second.

The following set of readings was taken for the negative ions in dry air :—

Temperature = 15·8° C. X = 2·60 centims. Barometer = 76 centims.
Excess pressure in gasometer = 1·0 centim.
,, ,, ,, apparatus = ·13 centim.
7 cells = 14·5 volts.

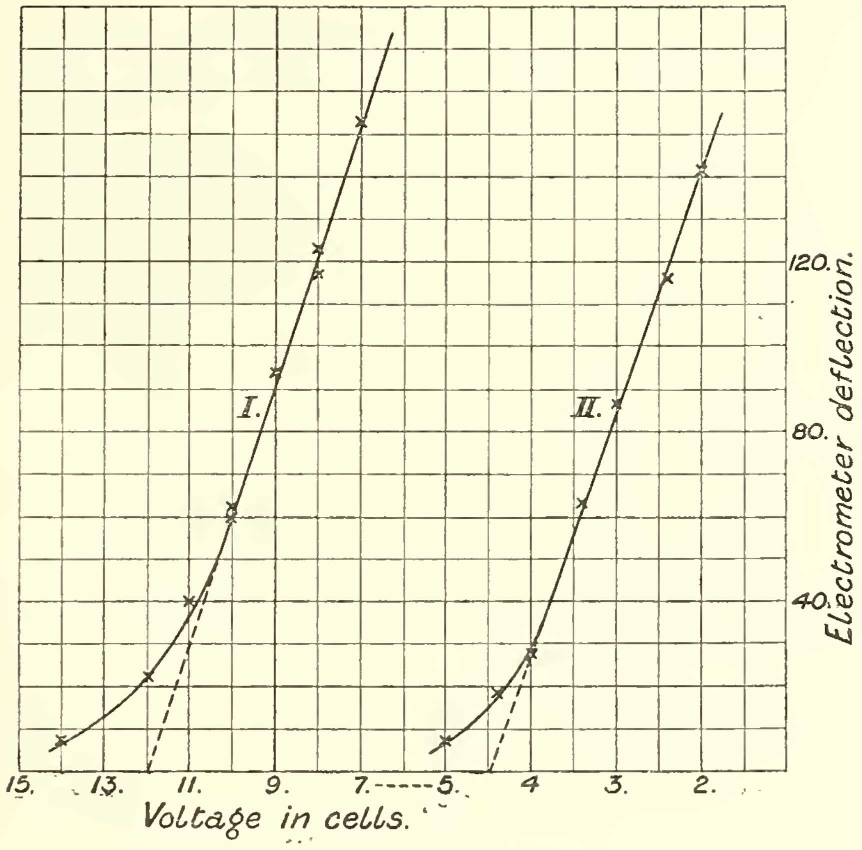

Fig. 7.

TABLE V.—Dry Air. Negative Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
− 2	141·5	2·07
− 3	87	2·05
− 4	28·5	2·08
− 5	6·5	2·05
− 4·4	18	2·08
− 3·4	63	2·05
− 2·4	116	2·05

The results are shown graphically in Curve II. of fig. 7.

U corrected for pressure = 7·64 centims. per second.
A = 9·21 volts.

$$v = 5{\cdot}118 \frac{7{\cdot}64}{2{\cdot}60 \times 9{\cdot}21} = 1{\cdot}63 \text{ centims. per second.}$$

$$T = -\frac{2{\cdot}60}{7{\cdot}64} = {\cdot}34 \text{ second.}$$

A summary of the results obtained for dry air for both the positive and the negative ions is given in Table VI.

TABLE VI.—Dry Air. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					° C.			
1	2·60	7·64	− 9·21	·34	15·8	76·2	1·63	—
2	2·60	7·64	+12·0	·34	15·8	76·2	—	1·25
3	2·60	7·64	+12·2	·34	16·3	75·8	—	1·23
4	2·60	7·57	− 9·31	·34	16·3	75·8	1·60	—
5	2·60	7·28	− 9·15	·36	12·2	76·6	1·58	—
6	2·60	7·23	+12·15	·36	12·2	76·6	—	1·18
7	2·60	15·8	−18·3	·16	13·8	76·6	1·71	—
8	2·60	15·9	+24·8	·16	13·8	76·6	—	1·27
9	2·60	15·9	−18.1	·16	13·8	76·6	1·74	—
10	2·60	15·9	−18·3	·16	13·8	76·6	1·725	—
11	2·60	16·6	−18·65	·16	14·6	75·8	1·75	—
12	2·60	16·3	−18·22	·16	14	76·8	1·78	—
13	2·60	15·5	+24·8	·17	12·5	76·7	—	1·25
14	2·60	15·5	+23·7	·17	10·7	77·6	—	1·31
15	2·60	15·6	−18·1	·17	10·7	77·6	1·72	—
16	5·15	8·65	+ 7·76	·60	11·4	77·3	—	1·13
17	5·15	8·62	− 5·9	·60	11·4	77·3	1·47	—
18	5·15	15·6	− 9·31	·33	11·4	77·6	1·67	—
19	5·15	15·7	+12·6	·33	11·4	77·6	—	1·26
20	5·15	8·58	+ 7·64	·60	11·7	77·5	—	1·14
21	5·15	8·58	− 5·88	·60	11·7	77·5	1·48	—

At No. 7 the drying apparatus was changed, and at No. 12 the Crookes' tube was replaced by a new one.

The results are represented in fig. 8, excluding the points marked by squares.

The final values thus obtained for dry air when the 2 per cent. correction mentioned in § 4 (15) has been added, give the velocity of the negative ions = 1·87 centims. per second, and of the positive ions = 1·36 centims. per second.

The temperature varied several degrees between the different observations, but was on the average about 13° C.

Most of the tests to which the method used in these experiments was subjected by changes of experimental conditions, were tried with dry air. Among these was tried the effect of changes in the intensity of the rays. By interposing aluminium plates the rays were diminished so that the conductivities produced by them changed in the ratio of three to one, but no noticeable change in the result could be observed. During the course of all of the experiments the rays were not of the same intensity, for the Crookes' tube had to be replaced several times, but in all cases without any marked effect upon the values obtained. It must be said, however, that rays of great intensity were never employed, the aim being always to have them as weak as possible for reasons previously stated.

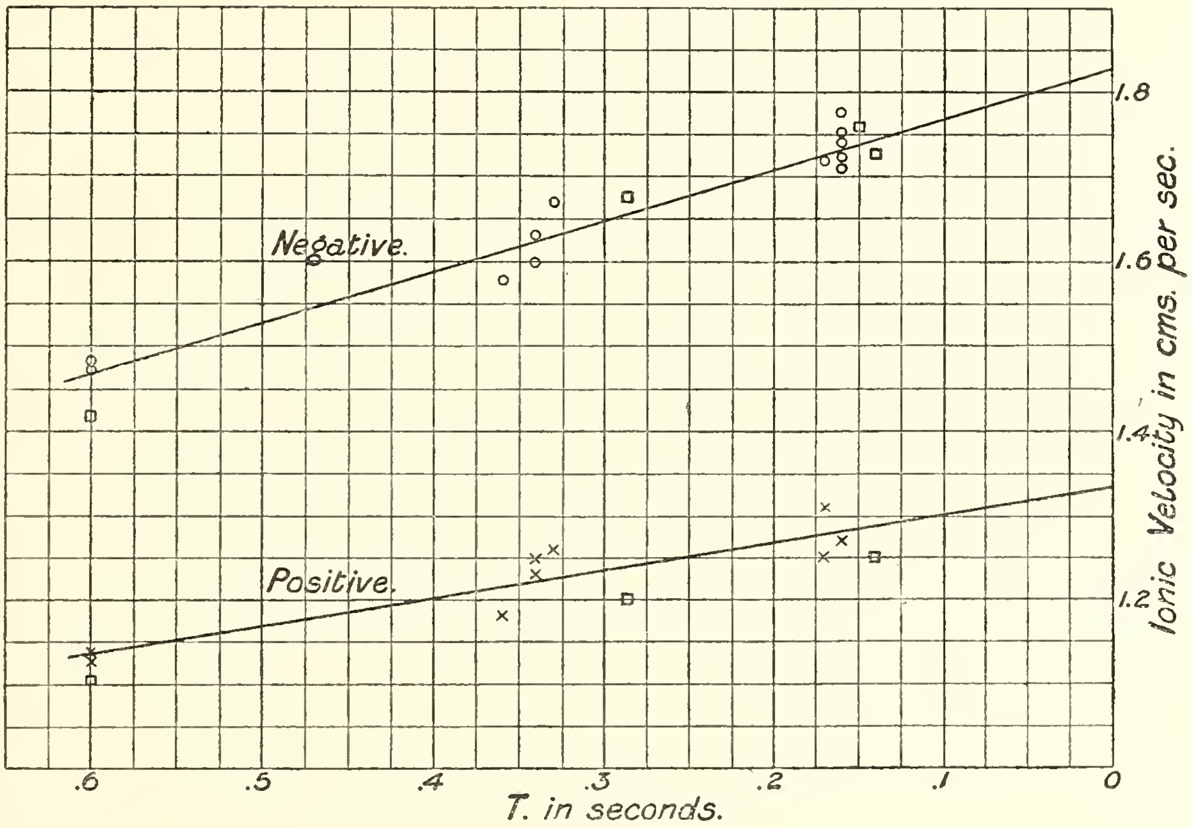

Fig. 8.

The most severe test to which the method was subjected was a change in the dimensions of the inner cylinder. In the above experiments the diameter of this cylinder was 1 centim., and it was now exchanged for one having a diameter of 2·8 centims. The distance between the inner and the outer cylinders was thus diminished to nearly one-half of its former value. The electric field between the two became much more uniform, and the gas velocities for different points of a cross-section now varied in a different manner. In order to keep the other quantities the same, the small distance between the two cylinders necessitated the use of voltages only about one-quarter as large as those used in the former arrangement. This increased the difficulty of the measurements and also some of the corrections which must be applied to get the final result. The density of the free charges in the gas was greater because the ions moved slower, being in a weaker field, and the same fall of potential at the electrodes was a larger percentage of the total voltage. The width of the beam of rays used was ·3 centim.

The following is a summary of the results obtained :—

TABLE VII.—Dry Air. Summary for Large Inner Cylinder.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					° C.			
1	5·4	10·7	−1·90	·50	11·8	75·9	1·43	—
2	5·4	12·6	−2·23	·43	12·4	75·9	1·43	—
3	5·4	12·7	+2·84	·43	12·4	75·9	—	1·13
4	5·4	10·4	+2·35	·52	12·4	75·9	—	1·10
5	3·77	11·55	+3·75	·33	15	75·8	—	1·12
6	3·77	11·55	−2·70	·33	15	75·8	1·555	—
7	3·77	11·65	−2·80	·33	15	75·8	1·51	—
8	5·11	13·9	−2·64	·37	15·4	76·8	1·43	—
9	5·11	13·95	+3·39	·37	15·4	76·8	—	1·115
10	3·26	13·8	−4·06	·24	15·4	76·8	1·57	—
11	6·35	8·83	−1·48	·72	16	77·2	1·315	—
12	6·35	8·83	+1·80	·72	16	77·2	—	1·08
13	6·35	14·05	+2·80	·45	15·8	76·0	—	1·09
14	6·35	14·2	−2·20	·45	15·8	76·0	1·40	—
15	2·63	14·9	−4·7	·18	15·8	76·0	1·65	—
16	2·63	7·7	+3·5	·34	16·3	75·9	—	1·14
17	2·63	7·04	−2·5	·37	16·3	75·9	1·465	—
18	2·63	13·9	−4·32	·19	16·3	75·9	1·67	—
19	2·63	14·06	+5·94	·19	16·3	75·9	—	1·23

The results are represented in fig. 9.

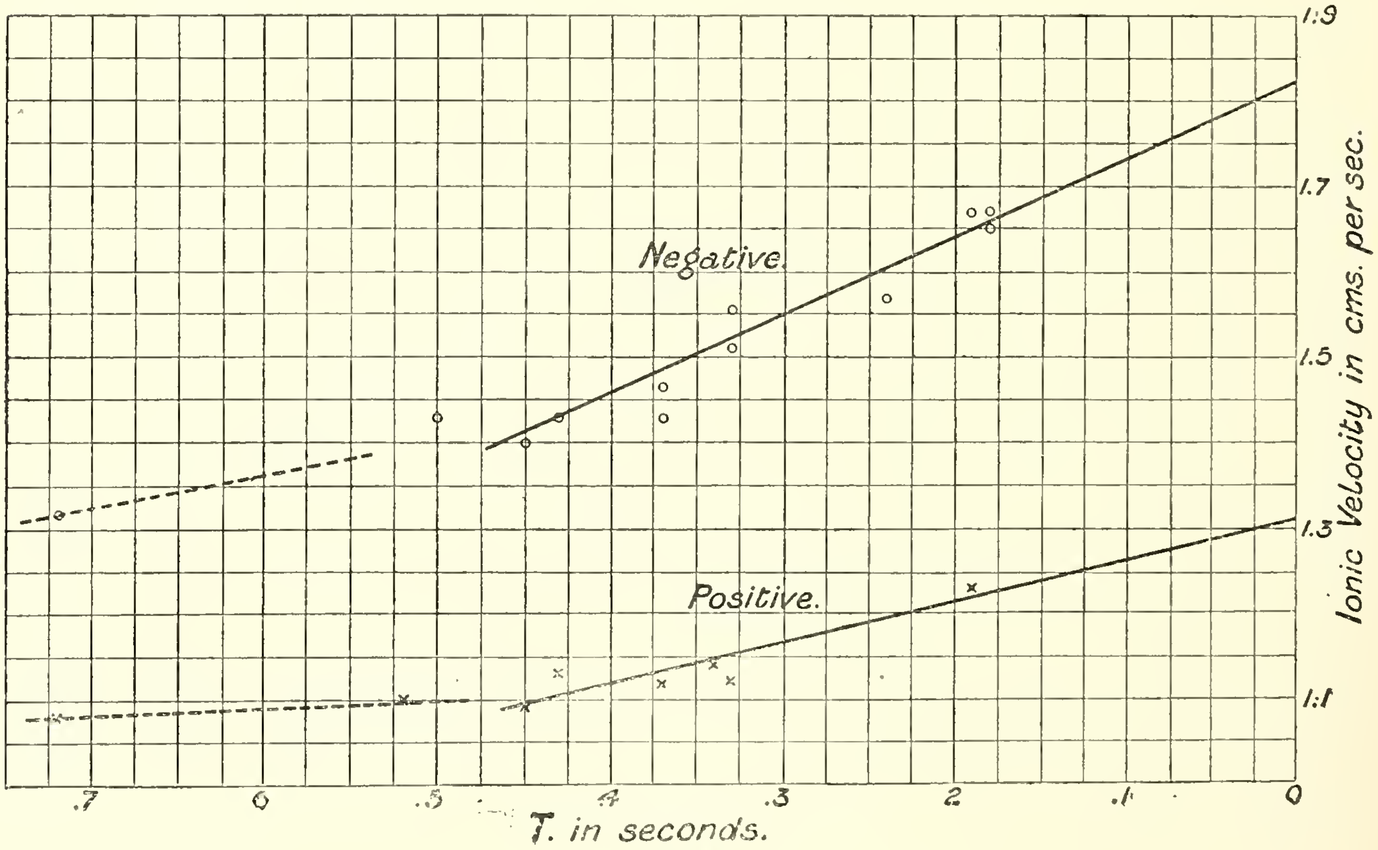

Fig. 9.

It is seen that in this case the values of the velocity change less rapidly as the values of T become large ; but for the smaller values of T the change is more rapid than it was when the smaller inner cylinder was used. The points on the curves are not advantageously distributed, and so do not allow of a very accurate projection of the lines to T = 0 ; but from those drawn it is seen that the results are but slightly smaller than those obtained with the smaller inner cylinder. This is considered a good agreement even if it is left out of account that an additive correction is still to be made.

An alteration which was tried to test the effect of surface ionization was a change in the material of the inner surface of the outer cylinder. The aluminium part DD′ (fig. 2) of the outer cylinder was coated on its inner surface with a layer of tin-foil. The rays in penetrating the cylinder now had a tin instead of an aluminium surface in contact with the air. J. PERRIN has shown (see § 4 (10)) that what he calls the coefficient of the increased ionization at a metal surface is for tin in air ·6 as against ·0 for aluminium in air. The effect varies with the state of the surface. In these experiments the aluminium surface used was an ordinary unpolished surface, while the tin surface used was that of bright tin-foil. It was thought that if an increase of the ionization near the metal surface has any marked effect upon the value of the velocity obtained, the difference should be observed by this new arrangement.

The results obtained are given in Table VIII., dry air being used as before. The smaller inner cylinder having a diameter of 1 centim. was used.

TABLE VIII.—Dry Air. Summary for Tin Surface.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					°C.			
1	5·22	8·56	- 6·06	·6	13·9	77·3	1·42	—
2	5·22	8·61	+ 7·82	·6	13·9	77·3	—	1·105
3	5·22	18·2	+15·2	·29	14·4	77·4	—	1·20
4	5·22	18·2	-10·8	·29	14·4	77·4	1·68	—
5	2·62	17·3	-19·6	·15	?	77·2	1·76	—
6	2·62	18·5	-21·2	·14	16·5	77·2	1·73	—
7	2·62	18·5	+29·4	·14	16·5	77·2	—	1·25

The points are plotted as squares on the curves in fig. 8, which represent corresponding values when the aluminium surface was used. It is seen that the points for the negative ions agree very well with the curve. The points for the positive ions are 2 to 3 per cent. below the values for the aluminium surface. Taking both results into consideration it was concluded, if the addition of a tin surface changed the values of the velocities by but such a small amount, that originally when the aluminium surface was used, the effect of the surface ionization could not have been sufficient to produce any marked error in the results.

The surface ionization also varies with the nature of the gas, but the values obtained by J. PERRIN for aluminium with the gases used in these experiments were in all cases much less than for tin in air.

§ 9. OXYGEN.

The gas used in these experiments was the commercial oxygen obtained from a cylinder, which contained about 5 per cent. of impurities, mostly nitrogen. Since the size and nature of the apparatus prevented the employment of the most pure gases, it seemed advisable to use the cylinder gas. The density was changed but little by the presence of the impurities, and, so far as known, the velocity should therefore be but slightly affected. The drying of the gas and its saturation with aqueous vapour were carried out in the same manner as with air. The following set of readings was taken for the negative ions in oxygen saturated with aqueous vapour:—

Temperature = 17·3° C. X = 5·01 centims. Barometer = 76·4 centims.
Excess pressure in gasometer = 1·54 centims.
,, ,, apparatus = ·55 centim.
8 cells = 16·3 volts.

TABLE IX.—Moist Oxygen. Negative Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
− 3	194·5	5·06
− 4	157	5·11
− 5	106·5	5·02
− 6	41	5·11
− 6·6	17	5·07
− 5·6	67	5·08
− 5·2	93·5	5·09
− 4·6	125·5	5·03

The results are shown in Curve I. of fig. 10.

The corrected value of U = 18·83 centims. per second.

A = 13·55 volts.

$v = 5{\cdot}118 \dfrac{18{\cdot}83}{5{\cdot}01 \times 13{\cdot}55} = 1{\cdot}413$ centims. per second, and when reduced to 76 centims. pressure this becomes 1·43 centims. per second.

$$T = \frac{5{\cdot}01}{18{\cdot}83} = {\cdot}27 \text{ second.}$$

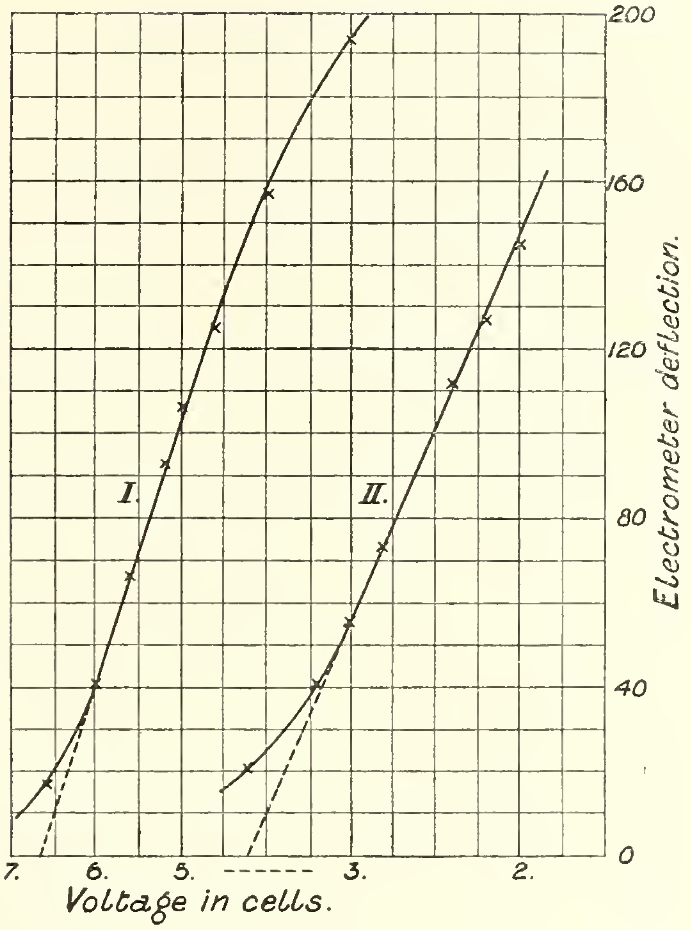

Fig. 10.

The following is a set of readings taken for the positive ions in oxygen saturated with aqueous vapour :—

Temperature = 15·6° C. X = 5·01 centims. Barometer = 76·5 centims.
Excess pressure in gasometer = ·44 centims.
,, ,, apparatus = ·16 ,,
6 cells = 12·35 volts.

TABLE X.—Moist Oxygen. Positive Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
+2	145	2·30
+3	55·5	2·31
+3·6	20·5	2·29
+3·2	41	2·29
+2·8	73·5	2·27
+2·4	112	2·25
+2·2	127	2·25

The results are represented by Curve II. of fig. 10.

The corrected value of U = 8·42 centims. per second.

A = 7·42 volts.

$v = 5{\cdot}118 \dfrac{8{\cdot}42}{5{\cdot}01 \times 7{\cdot}42} = 1{\cdot}16$ centims. per second, and when reduced to 76 centims. pressure this becomes 1·17 centims. per second.

$$\mathrm{T} = \frac{5{\cdot}01}{8{\cdot}42} = {\cdot}6 \text{ second.}$$

A summary of the results thus obtained for moist oxygen for both the positive and the negative ions is given in Table XI.

TABLE XI.—Moist Oxygen. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					°C.			
1	6·89	9·41	+ 6·04	·73	15·4	76·8	—	1·17
2	6·89	9·38	− 5·38	·73	15·4	76·8	1·31	—
3	6·89	16·8	− 8·80	·41	15·5	77·3	1·44	—
4	6·89	16·8	+ 10·5	·41	15·5	77·3	—	1·21
5	5·01	8·42	+ 7·42	·6	15·6	76·7	—	1·17
6	5·01	8·68	− 6·69	·58	15·6	76·7	1·34	—
7	5·01	18·8	− 13·55	·27	17·3	76·9	1·43	—
8	5·01	18·6	+ 15·7	·27	17·3	76·9	—	1·23
9	3·03	20·5	+ 27·3	·15	15·4	76·9	—	1·29
10	3·03	20·6	− 24·25	·15	15·4	76·9	1·46	—

The results are shown graphically in fig. 11.

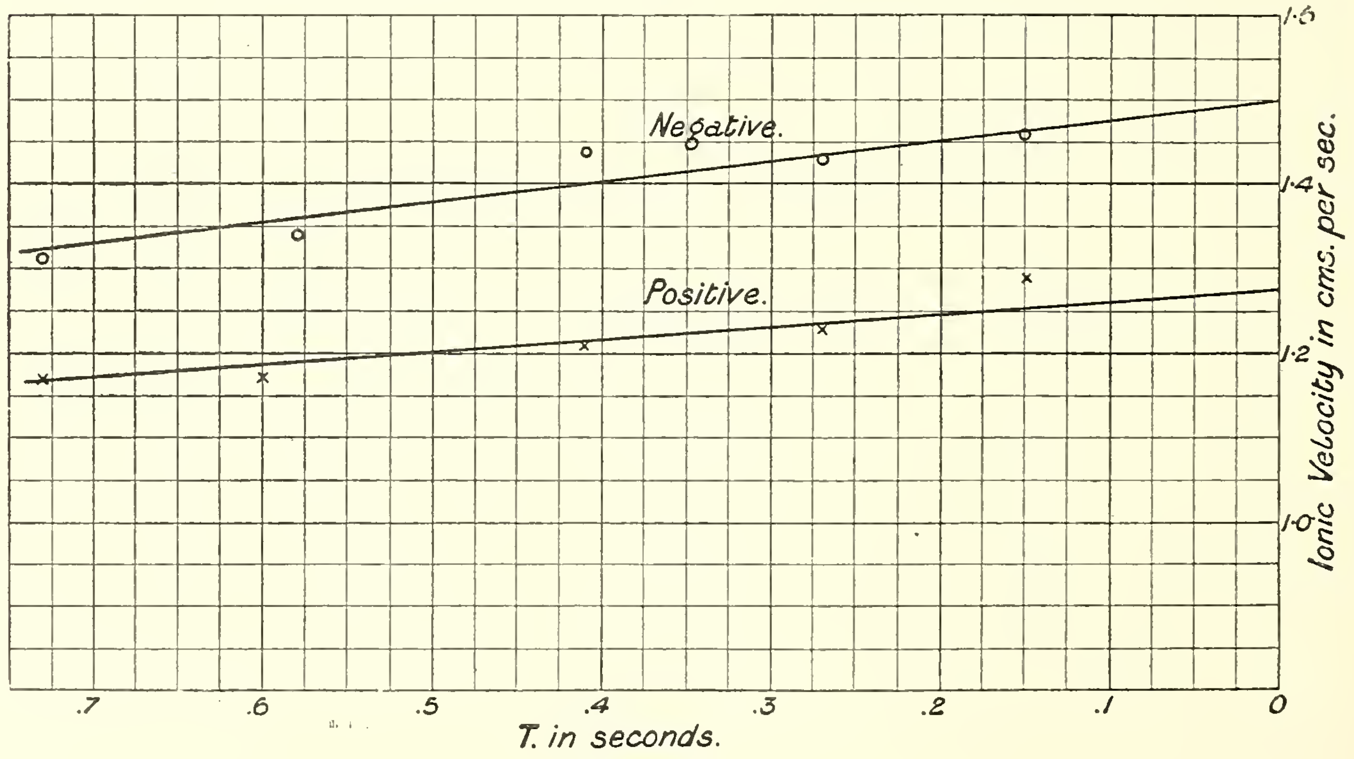

Fig. 11.

The correction mentioned in § 4 (15), which is to be applied to the values indicated at T = 0 has in this case been reduced to 1 per cent., 1 per cent. being allowed for an increase in the velocity due to a diminution of density caused by the impurities in the gas. The corrected value thus obtained for the velocity in moist oxygen is for the negative ions = 1·52 centims. per second, and for the positive ions = 1·29 centims. per second, at a pressure of 76 centims. and at a temperature of about 16° C.

The following is a summary of the results obtained for the positive and negative ions in dry oxygen :—

Table XII.—Dry Oxygen. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature. °C.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
1	2·73	13·9	+20·2	·20	20·3	77·3	—	1·31
2	2·73	13·8	−15·37	·20	20·3	77·3	1·71	—
3	2·73	13·3	+19·5	·21	19·4	77·4	—	1·31
4	2·73	13·3	−14·7	·21	19·4	77·4	1·72	—
5	2·73	13·3	+19·5	·21	19·4	77·4	—	1·30
6	3·89	16·65	−13·6	·23	15	78	1·66	—
7	3·89	17·7	+17·7	·22	15·6	78	—	1·35
8	3·89	16·9	−13·4	·23	15·6	78	1·71	—
9	3·89	7·97	− 6·82	·49	16	76·8	1·56	—
10	3·89	8·46	+ 9·45	·46	16	76·8	—	1·19
11	6·89	8·92	+ 5·64	·77	15·2	77·1	—	1·19
12	6·89	9·07	− 4·53	·76	15·2	77·1	1·51	—
13	6·89	17·35	− 8·05	·40	15·8	77·1	1·63	—
14	6·89	17·4	+10·33	·40	15·8	77·1	—	1·27

The results are represented graphically in fig. 12.

When, as in the case of moist oxygen, a 1 per cent. additive correction is applied to the values indicated in the figure by T = 0, the final result for the velocity in dry oxygen is for the negative ions = 1·80 centims. per second, and for the positive ions = 1·36 centims. per second for a pressure of 76 centims. and a temperature of about 17° C.

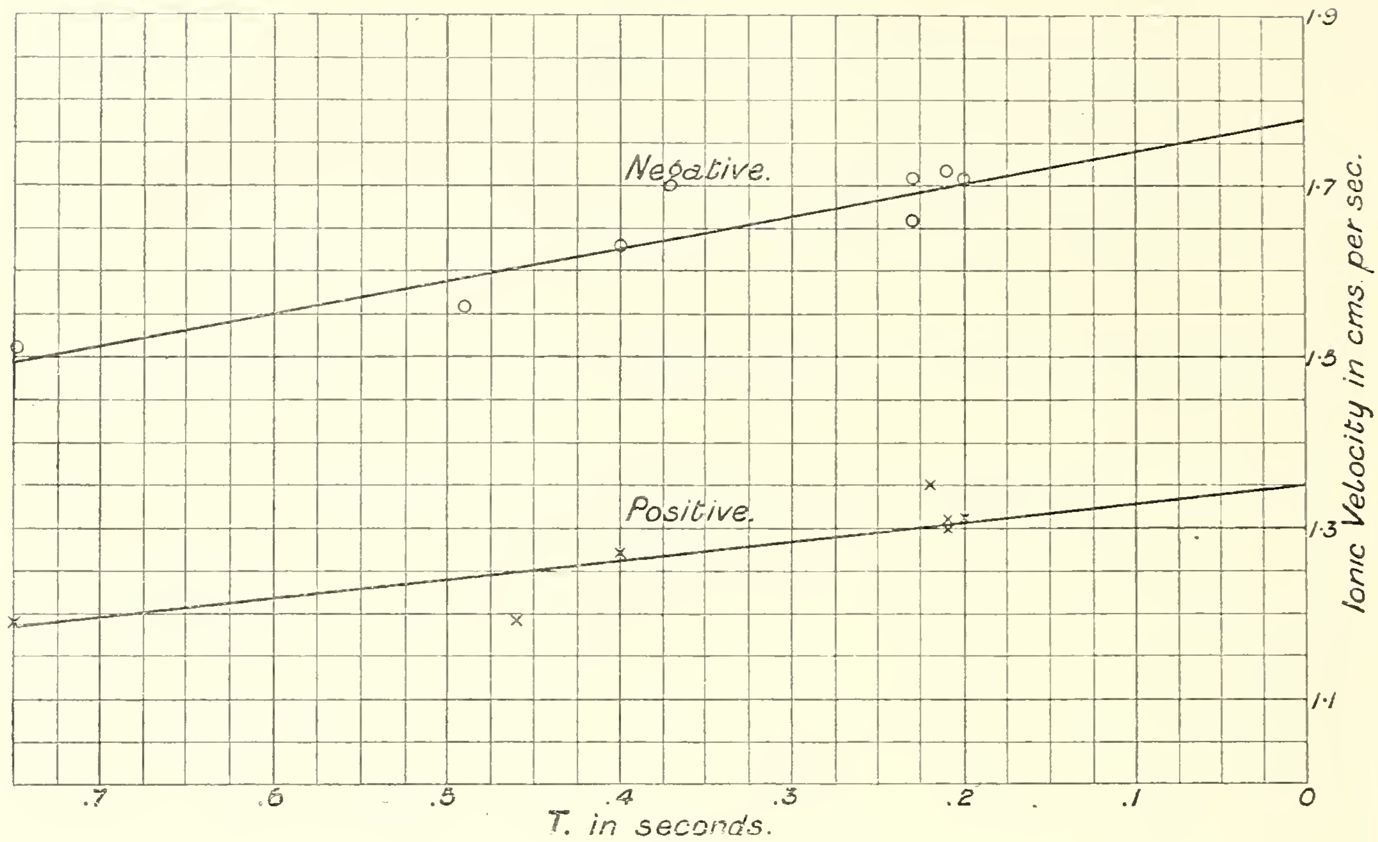

Fig. 12.

§ 10. Carbonic Acid.

The gas used was taken from a cylinder of liquid carbonic acid. The small amount of impurities in this does not produce any marked change in the density of the gas, and is assumed to be without noticeable effect upon the ionic velocities. As examples of the readings taken, the following two sets are given for carbonic acid gas saturated with aqueous vapour :—

Temperature = 16·3° C. X = 3·02 centims. Barometer = 75·4 centims.
Excess pressure in gasometer = ·44 centim.
,, ,, ,, apparatus = ·21 ,,
10 cells = 20·6 volts.

Table XIII.—Moist Carbonic Acid. Negative Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
– 6	136	2·34
– 7	106·5	2·33
– 8	76	2·33
– 9	41·5	2.30
– 10	17·5	2·33
– 9·4	28	2·29
– 8·4	57	2·28
– 7·4	91·5	2·27

The results are shown in Curve I. of fig. 13.

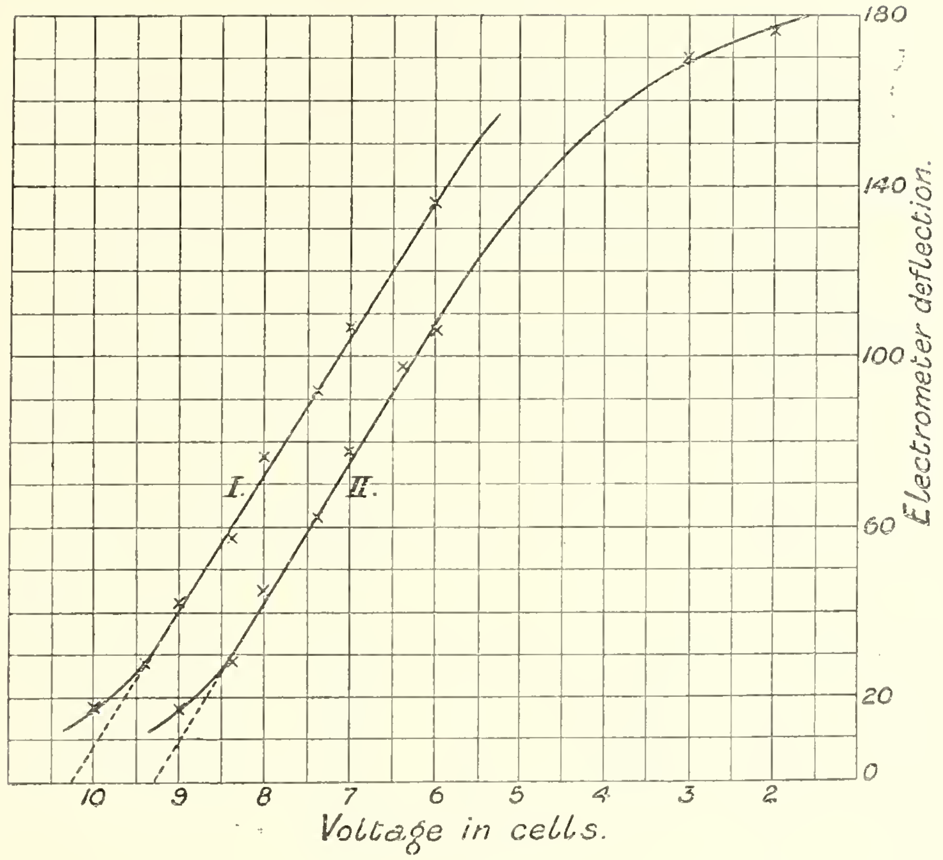

Fig. 13.

The corrected value of U = 8·52 centims. per second.

A = 21·1 volts.

$v = 5{\cdot}118 \dfrac{8{\cdot}52}{3{\cdot}02 \times 21{\cdot}1} = {\cdot}683$ centim. per second, which, reduced to 76 centims. pressure, becomes ·679 centim. per second.

$T = \dfrac{3{\cdot}02}{8{\cdot}52} = {\cdot}36$ second.

TABLE XIV.—Moist Carbonic Acid. Positive Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
+2	176·5	2·32
+3	170	2·31
+6	106	2·33
+7	77·5	2·32
+8	45	2·32
+9	17	2·32
+8·4	28	2·26
+7·4	62	2·25
+6·4	97·5	2·27

These results are represented in curve II. of fig. 13.

The corrected value of U = 8·48 centims. per second.

$$A = 19{\cdot}16 \text{ volts.}$$

$$v = 5{\cdot}118\,\frac{8{\cdot}48}{3{\cdot}02 \times 19{\cdot}16} = {\cdot}749$$ centim. per second, which reduced to 76 centims. pressure becomes ·745 centim. per second.

$$T = \frac{3{\cdot}02}{8.48} = {\cdot}36 \text{ second.}$$

A summary of the results for moist carbonic acid is given in Table XV.

Table XV.—Moist Carbonic Acid. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					° C.			
1	3·02	8·48	+19·16	·36	16·3	75·6	—	·745
2	3·02	8·52	−21·1	·36	16·3	75·6	·679	—
3	3·02	16·6	−39·2	·18	16·6	76·2	·717	—
4	3·02	16·75	+35·9	·18	16·6	76·2	—	·791
5	6·07	9·74	−12·3	·62	16·9	75	·658	—
6	6.07	9·97	+11	·61	16·9	75	—	·755
7	6·07	19·4	+21·5	·31	17·1	75·5	—	·755
8	6·07	18·1	−22·14	·33	17·1	75·5	·685	—
9	6·07	8·63	+ 9·95	·70	17·1	75	—	·722
10	6·07	13·05	+14·1	·47	17·6	75·1	—	·772
11	6·07	12·9	−15·85	·47	17·6	75·1	·678	—

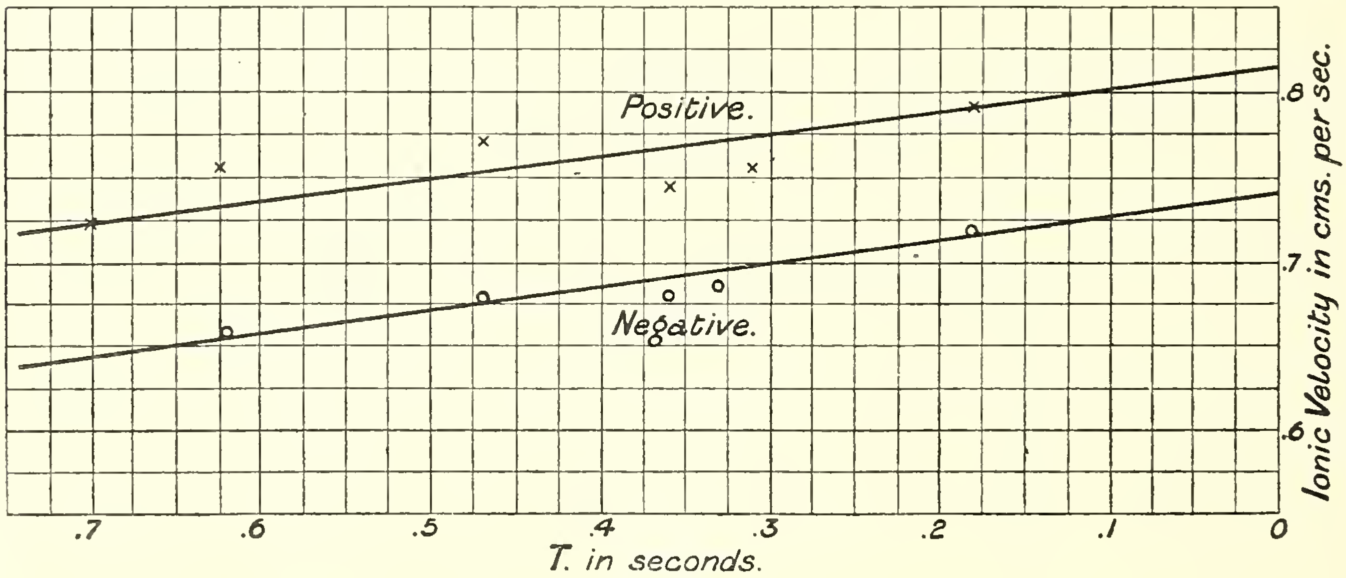

Fig. 14.

The results are represented in fig. 14, from which it is seen that the values corresponding to T = 0, when corrected similarly to those of moist oxygen, give as the velocity in moist carbonic acid, for the negative ions, ·75 centim. per second, and for the positive ions, ·825 centim. per second, for a pressure of 76 centims. and a temperature of about 17° C.

A summary of the results obtained for dry carbonic acid is given in Table XVI.

TABLE XVI.—Dry Carbonic Acid. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature. ° C.	Gas pressure.	Ionic Velocity.	
							Negative.	Positive.
1	6·07	8·61	− 9·15	·71	17·5	75·4	·787	—
2	6·07	8·63	+ 9·59	·71	17·5	75·4	—	·752
3	6·07	16·7	+19·07	·36	17·5	75·8	—	·737
4	6·07	17·1	−18·05	·36	17·5	75·8	·796	—
5	3·08	17·3	−36·2	·18	18·3	75·8	·793	—
6	3·08	17·3	+38·4	·18	18·3	75·8	—	·747
7	3·08	8·25	+18·96	·37	17·2	75·7	—	·725
8	3·08	8·53	−18·18	·36	17·2	75·7	·781	—
9	6·01	8·63	− 9·53	·70	17·3	75·7	·770	—
10	6·01	8·53	+ 9·64	·71	17·3	75·7	—	·753
11	6·01	12·8	+14·76	·47	17·5	75·9	—	·738
12	6·01	12·85	−14·04	·47	17·5	75·9	·777	—

The results are represented in fig. 15.

The velocities appear to vary but little with T.

The values for the positive velocity being comparatively large for the highest value of T, make it difficult to draw the line through the positive points, and the inclination of the one through the negative points has been used as a guide for drawing the one shown. The value thus found, when corrected, gives the velocity in dry carbonic acid for the negative ions as ·81 centim. per second, and for the positive ions as ·76 centim. per second for a pressure of 76 centims. and a temperature of 17·5° C.

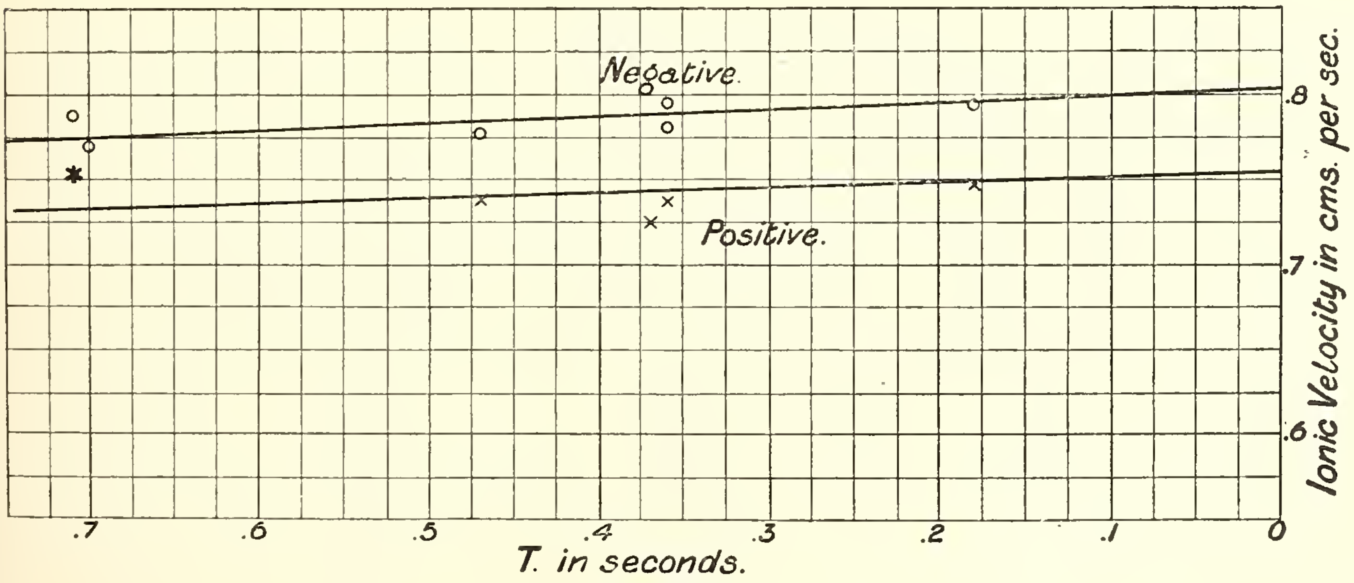

Fig. 15.

§ 11. HYDROGEN.

The gas was prepared from pure zinc and hydrochloric acid, and bubbled through three bottles of strong caustic potash and potassium permanganate to free it from the acid and other impurities. Great difficulty was experienced in maintaining the gas sufficiently pure on standing, because of the large surface of rubber exposed in the gas bag and in the connecting tubes of the apparatus. The density of hydrogen being so small compared to that of air, a small amount of the latter produces a large change in the density of the gas, and it was found that the ionic velocities were greatly affected thereby.

The following plan was finally adopted as the most practicable under the circumstances :—

The forenoon of a day was spent in the preparation of fresh hydrogen, this length of time being required to generate the large quantity necessary for use and for washing the more impure hydrogen out of the apparatus. Beginning early in the afternoon, readings were taken as rapidly as possible until after midnight, thus giving about eleven hours of continuous observations. The density of the gas was then determined by weighing a 600 cubic centims. flask filled first with dry air and then with dry gas from the gasometer. Since 1 per cent. of air in the gas made a difference of over 6 milligrams in the weight, this permitted a sufficiently accurate determination of the amount of the air impurity. A test was made by the eudiometer method, which showed that the impurity was practically all air.

The width of the beam of rays used was ·3 centim., as the conductivity was much less with the hydrogen than in the other cases.

The following is a set of readings taken for the negative ions in dry hydrogen :—

Temperature = 20° C. X = 2·95 centims. Barometer = 76·15 centims.
Excess pressure in gasometer = ·90 centim.
,, ,, apparatus = ·36 centim.
5 cells = 10·5 volts.

TABLE XVII.—Dry Hydrogen. Negative Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
− 2·6	12·5	9·59
− 3	9·7	9·52
− 3·4	6·5	9·31
− 3·8	3·8	9·44
− 3·6	6	9·46
− 3·2	8·8	9·50
− 2·8	11·2	9·38
− 2·4	14·2	9·40

The results are shown graphically in curve II. of fig. 16.

The corrected value of $U = 35{\cdot}0$ centims. per second.

$$A = 9{\cdot}04 \text{ volts.}$$

$v = 5{\cdot}118 \dfrac{35}{2{\cdot}95 \times 9{\cdot}04} = 6{\cdot}72$ centims. per second, which reduced to 76 centims. pressure becomes 6·76 centims. per second.

$$T = \frac{2{\cdot}95}{35} = {\cdot}084 \text{ second.}$$

The gas in this case contained 3·4 per cent. of air.

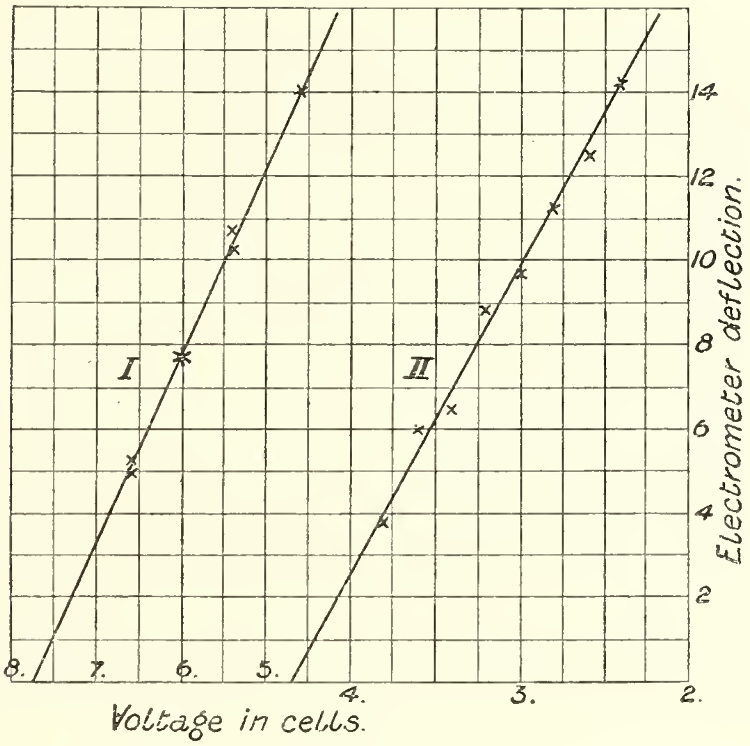

Fig. 16.

The following set of readings was taken for the positive ions in hydrogen saturated with aqueous vapour :—

Temperature = 20° C. X = 2·95 centims. Barometer = 76·7 centims.
Excess pressure in gasometer = ·78 centim.
,, ,, apparatus = ·35 centim.
9 cells = 18·5 volts.

TABLE XVIII.—Moist Hydrogen. Positive Ions.

Voltage of outer cylinder.	Electrometer deflection in 30 seconds.	Descent of gasometer in 40 seconds.
Cells.	Divisions.	Centims.
+6·6	5·25	10·38
+6	7·75	10·23
+5·4	10·25	10·18
+4·6	14	10·38
+5·4	10·75	10·25
+6	7·75	10·18
+6·6	5	10·16

The results are shown in curve I. of fig. 16.

The corrected value of $U = 43{\cdot}3$ centims. per second.

$A = 15{\cdot}9$ volts.

$v = 5{\cdot}118 \dfrac{43{\cdot}3}{2{\cdot}95 \times 15{\cdot}9} = 4{\cdot}73$ centims. per second, which reduced to 76 centims. pressure, $= 4{\cdot}80$ centims. per second,

$T = \dfrac{2{\cdot}95}{43{\cdot}3} = {\cdot}068$ second.

Besides the water vapour, the gas in this case contained 1·5 per cent. of air.

A summary of the results obtained with dry hydrogen containing 3·4 per cent. of air is given in Table XIX. On account of the smaller electrometer readings the

TABLE XIX.—Dry Hydrogen. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					° C.			
1	2·95	21·2	+ 6·65	·14	21·4	76·3	—	5·56
2	2·95	21·6	+ 7·73	·14	21·4	76·3	—	5·15
3	2·95	21·8	+ 7·14	·14	21·4	76·3	—	5·33
4	2·95	21·9	+ 7·37	·14	21·4	76·3	—	5·18
5	2·95	21·6	− 5·79	·14	21·4	76·3	6·52	—
6	2·95	21·4	− 5·75	·14	21·4	76·3	6·49	—
7	2·95	21·4	− 5·85	·14	21·4	76·3	6·39	—
8	2·95	21·4	+ 7·14	·14	21·4	76·3	—	5·24
9	2·95	35·4	+10·35	·083	20	76·5	—	5·97
10	2·95	35·3	+10·79	·083	20	76·5	—	5·71
11	2·95	35·0	− 9·03	·084	20	76·5	6·77	—
12	2·95	35·0	− 9·24	·084	20	76·5	6·60	—
13	2·95	34·9	− 8·90	·084	20	76·5	6·84	—
14	2·95	34·7	+11·10	·085	20	76·5	—	5·45
15	2·95	25·7	+ 8·08	·115	20	76·4	—	5·55
16	2·95	25·7	+ 7·90	·115	20	76·4	—	5·67

determinations for hydrogen are less accurate, and so results were obtained for small values of T, only because of their greater importance, and in order to expedite the readings.

The results are represented by I. and II. of fig. 17.

The lines projected to T = 0 indicate for the uncorrected velocity of the negative ions 7·3 centims. per second, and for the positive ions 6·2 centims. per second when under a pressure of 76 centims., and at a temperature of about 20° C. These values are for dry hydrogen containing 3·4 per cent. of air. The correction for the presence of the air can be found approximately by finding the value of the velocity in a gas having a larger percentage of air.

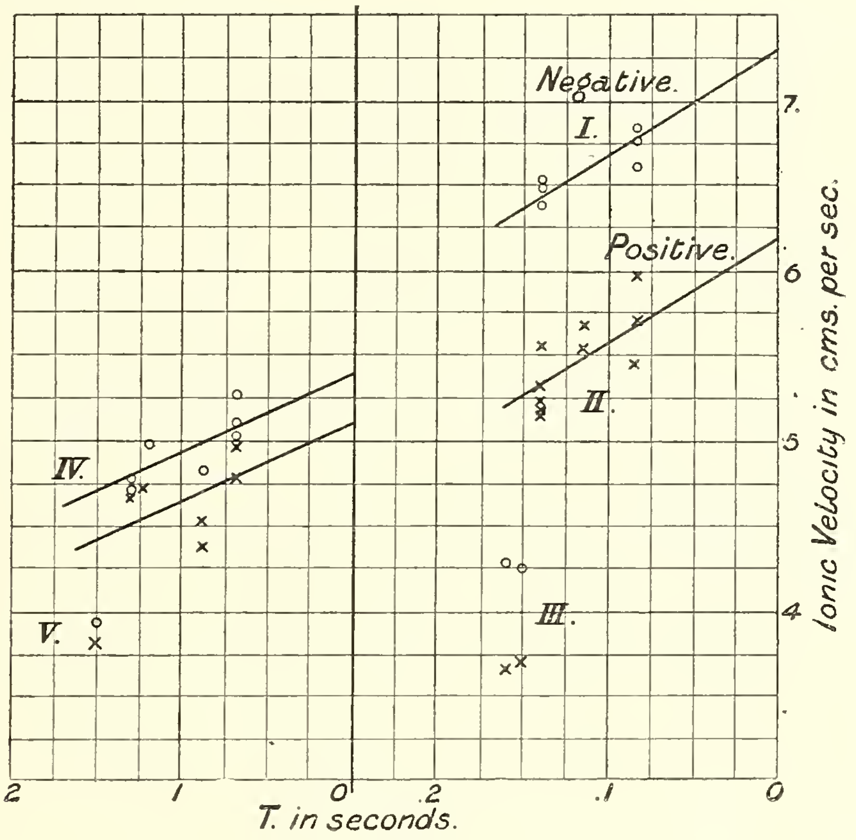

Fig. 17.

The following are a number of results obtained with dry hydrogen which contained 14·4 per cent. of air :—

TABLE XX.—Dry Hydrogen with 14·4 per cent. of Air.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					°C.			
1	2·95	20·4	+ 9·62	·15	21·2	76·3	—	3·70
2	2·95	20·2	− 8·28	·15	21·2	76·3	4·25	—
3	2·95	18·6	− 7·59	·16	22	76·4	4·27	—
4	2·95	18·6	+ 8·90	·16	22	76·4	—	3·65

These results are represented at III. in fig. 17.

By finding the difference between these points and the values in the curves above them corresponding to the same value of T, the diminution in the velocity is obtained that is produced by the addition of 14·4 — 3·4 = 11 per cent. of air. Assuming that up to this point the diminution in the ionic velocity is proportional to the amount of air present in the gas, the velocity in pure hydrogen is found by adding to the value obtained when 3·4 per cent. of air was present $\frac{3 \cdot 4}{11}$ part of the diminution observed as due to 11 per cent. of air. From the above results this correction is found to be ·65 for the negative ions and ·50 for the positive ions. Disregarding any minor corrections, the final result for pure dry hydrogen is thus found to be 7·95 centims. per second for the negative ions, and 6·70 centims. per second for the positive ions at a pressure of 76 centims., and at a temperature of about 20° C.

A summary of the results obtained with hydrogen saturated with aqueous vapour, and containing 1·5 per cent. of air, is given in Table XXI.

TABLE XXI.—Moist Hydrogen. Summary of Results.

Reference number.	X.	U.	A.	T.	Temperature.	Gas pressure.	Ionic velocity.	
							Negative.	Positive.
					° C.			
1	2·95	43·9	– 14·6	·067	20	77·1	5·26	—
2	2·95	43·7	– 15·0	·067	20	77·1	5·10	—
3	2·95	43·7	+ 15·5	·067	20	77·1	—	4·97
4	2·95	43·3	+ 15·9	·068	20	77·1	—	4·80
5	2·95	43·1	– 15·0	·069	20	77·1	5·03	—
6	2·95	23·8	– 8·43	·12	19·8	76·9	4·98	—
7	2·95	23·5	– 8·76	·13	19·8	76·9	4·72	—
8	2·95	23·4	– 8·64	·13	19·8	76·9	4·77	—
9	2·95	23·3	+ 8·77	·13	19·8	76·9	—	4·68
10	2·95	23·3	+ 8·69	·13	19·8	76·9	—	4·73
11	2·95	34·2	+ 13·3	·087	19·8	77	—	4·53
12	2·95	34·1	+ 13·8	·087	19·8	77	—	4·37
13	2·95	34	– 12·4	·087	19·8	77	4·82	—
14	2·98	19·8	– 8·74	·15	20·4	76·7	3·93	—
15	2·98	19·7	+ 8·97	·15	20·7	76·9	—	3·82

The results 1 to 13 are represented by IV. of fig. 17.

The results 14 and 15 were obtained with moist hydrogen containing 8 per cent. of air. These two were selected out of a number of results of which they represent about the average values. They are shown by V. of fig. 17, and by means of them the correction for the air present in the above experiments was made in the same manner as with dry hydrogen. The points IV. in the figure are so scattered that the inclination of the lines drawn through them had to be estimated mainly by comparison with those for dry hydrogen, remembering that with the smaller velocities here obtained the inclination would be somewhat less. The final values thus obtained for hydrogen saturated with aqueous vapour when corrected for the air present give for the velocity of the negative ions 5·6 centims. per second, and for the positive ions 5·3 centims. per second at a pressure of 76 centims., and a temperature of 20° C.

§ 12. Remarks on the Experiments.

The changes in the values obtained for the velocity with changes of T are observed to be greater for those cases where the ionic velocities are higher. With dry and moist carbonic acid, however, the inclination of the curves is somewhat different for nearly equal values of the velocities. In some instances, where the set of points for either the positive or the negative ions did not allow of a sufficiently accurate estimate of the inclination of the line to be drawn through them, the line through the other set of points was used as a guide.

The presence of water vapour diminished the velocity of the negative ions in all of the gases, while in carbonic acid the velocity of the positive ions was at the same time considerably increased. It seems most probable that these changes are due to some effect upon the size of the ions, and it is possible that a few molecules of the aqueous vapour collect upon the negative ions. It is interesting to note in this connection the recent results of C. T. R. Wilson,* showing that in supersaturated air the water condenses more readily upon the negatively charged ions.

While in most cases the readings indicate a greater accuracy, it is believed that the maximum error in any determination is less than five per cent. For convenience, all of the values obtained are here collected in one table, the results being given in centims. per second both for a field of one volt per centim. and for a field of one electrostatic unit per centim.

Table XXII.—Ionic Velocities.

Gas.	Velocities in centims. per second in a field of 1 volt per centim.		Velocities in centims. per second in a field of 1 E.S.U. per centim.		Ratio of Negative to Positive.	Temperature.
	Positive ions.	Negative ions.	Positive ions.	Negative ions.		° C.
Air, dry	1·36	1·87	408	561	1·375	13·5
,, moist	1·37	1·51	411	453	1·10	14
Oxygen, dry	1·36	1·80	408	540	1·32	17
,, moist	1·29	1·52	387	456	1·18	16
Carbonic acid, dry	·76	·81	228	243	1·07	17·5
,, ,, moist	·82	·75	246	225	·915	17
Hydrogen, dry	6·70	7·95	2010	2385	1·19	20
,, moist	5·30	5·60	1590	1680	1·05	20

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* C. T. R. Wilson, 'Phil. Trans.,' A, vol. 193, p. 289, 1899.

It is seen that the value of the velocity is greater for the negative ions in all cases except for moist carbonic acid. In comparing the values for the different gases, the temperature at which the observations were taken must be taken into consideration.

At the time the writer[*] determined the ratio of these velocities, the influence of moisture being unknown, the gases used were not dried, and so the values obtained were between those given above for the dry and the moist gases. Of the gases used in the former experiments, which were not used in these, the ammonia gas used had been passed through two long tubes of calcium oxide, the acetylene gas had been passed through a long tube of calcium carbide and the nitrogen monoxide was used directly from a cylinder.

The results obtained by E. RUTHERFORD[†] for the sum of the velocities of the ions produced by Röntgen rays are for:—

Air = 3·2 centims. per second.
Oxygen = 2·8 centims. per second.
Carbonic acid = 2·15 centims. per second.
Hydrogen = 10·4 centims. per second.

It is not stated whether the gases were dried, but the value for air agrees with that given above for the sum of the velocities in dry air, while the values for oxygen and hydrogen agree with the values for the moist gases. The value for carbonic acid is nearly 40 per cent. larger than that obtained here. It is of interest to compare the velocities of the ions produced by Röntgen rays with those of the ions produced by the action of ultra-violet light and in the discharge from points, as they show a close similarity.

For conduction produced by ultra-violet light, E. RUTHERFORD[‡] obtained with dry gases for the velocity of the negative ions in—

Air = 1·4 centims. per second.
Hydrogen = 3·9 centims. per second.
Carbonic acid = ·78 centim. per second.

The value for carbonic acid is quite near to that obtained above, but the other two are considerably smaller.

A. P. CHATTOCK[§] obtained for the velocities of the ions in dry air in the case of discharge from points—

413 centims. per second for the positive ions, and
540 ,, ,, ,, negative ions for a field of one electrostatic unit.

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[*] J. ZELENY, 'Phil. Mag.,' July, 1898.

[†] E. RUTHERFORD, 'Phil. Mag.,' November, 1897.

[‡] E. RUTHERFORD, 'Proc. Camb. Phil. Soc.,' vol. 9, Pt. VIII.

[§] A. P. CHATTOCK, 'Phil. Mag.,' November, 1899.

These values are nearly the same as those obtained above for the ions produced by Röntgen rays.

J. S. TOWNSEND* has shown that from the ionic velocity in a gas and the coefficient of diffusion of the ions in the gas, the value of Ne can be obtained, N being the number of molecules in a cubic centim. of the gas, and e the charge carried by an ion. By comparing this value with that obtained from the electrolysis of liquids, the relation between the charges on the ions in the two cases can be determined.

Using the values of the ionic velocities (v) given in Table XXII., and the corresponding coefficients of diffusion (K) from the tables given by J. S. TOWNSEND, the values of Ne are obtained from the equation $\mathrm{Ne} = \frac{3 \times 10^8 v}{\mathrm{K}}$ for the positive and the negative ions in both dry and moist gases.

The results are given in the following table:—

TABLE XXIII.—Values of Ne × 10^{-10}.

Gas.	Moist gas.		Dry gas.	
	Positive ions.	Negative ions.	Positive ions.	Negative ions.
Air	1·28	1·29	1·46	1·31
Oxygen	1·34	1·27	1·63	1·36
Hydrogen	1·24	1·18	1·63	1·25
Carbonic acid	1·01	·87	·99	·93

The corresponding value of Ne obtained for hydrogen from the electrolysis of liquids is 1·23 × 10^{10} at a pressure of 76 centims. of mercury, and a temperature of 15° C.

The values of Ne in the table for the positive and the negative ions in moist air, oxygen and hydrogen are perhaps in sufficient agreement to justify the statement that the charges carried by the positive and negative ions are the same, and that the value is also the same for the three gases, and corresponds to the charge carried by the hydrogen ion in the electrolysis of liquids.

The values of Ne for the negative ions in the same three gases when dry are not far from those in the moist gases, but the results for the positive ions are considerably larger. It seems very improbable, however, that the charges carried by the ions are different in the moist and dry gases, since most likely the moisture does not influence the act of the ionization itself, but either affects the ions after they are formed during the production of clusters of molecules around them, or changes the resistance

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* J. S. TOWNSEND, 'Phil. Trans.,' A, vol. 193, p. 152.

to their motion. So if the charges are equal in the moist gases, they should be equal in the dry gases also.

The values of Ne for carbonic acid are all less than that obtained for hydrogen by electrolysis, and so indicate a smaller charge on the ions; but from analogy with liquids we should expect that if the charges vary at all, it would be in the ratio of one to two or more, unless it is possible to have a charge smaller than that carried by hydrogen in electrolysis.

The writer cannot account for the differences in the values of Ne by supposing them due to errors in the ionic velocities obtained, since that would mean the presence in the experiments of some error which in some cases influenced the results for the positive ions alône, in other cases had an effect upon the values of both of the ions, and in still other cases was without effect.

The experiments described in this paper were performed at the Cavendish Laboratory, Cambridge, and I desire to express here my thanks to Professor J. J. THOMSON for the encouragement and valuable suggestions given in the course of the investigation.

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Rambaut, Arthur A. Phil. Trans., A, vol. 195, 1900, pp. 235–258.

VI. *Underground Temperature at Oxford in the Year* 1899, *as determined by five Platinum-resistance Thermometers.*

By ARTHUR A. RAMBAUT, *M.A., D.Sc., Radcliffe Observer.*

Communicated by E. H. GRIFFITHS, *F.R.S.*

Received May 17,—Read June 21, 1900.

[PLATES 1, 2.]

Description of the Apparatus and Mode of Reduction of the Observations of Earth Temperatures.

THE instruments with which the earth-temperatures given in this paper were observed, were five platinum-resistance thermometers of the Callendar and Griffiths pattern,* made by the Cambridge Scientific Instrument Company. These were purchased by the late Mr. STONE, and were placed in position under his direction shortly before his death.

The method of platinum thermometry seemed to be particularly suitable for this class of work, on account of the immunity it enjoys from certain errors attending the use of the long-stemmed mercurial or spirit thermometers ordinarily employed for underground temperatures.

A higher degree of accuracy might, therefore, reasonably be expected, and the discussion which follows of the first complete year's observations at the Radcliffe Observatory shows, I think, that this anticipation has been justified. Some discrepancies between theory and observation no doubt appear, but they are of a character which seems to indicate a difference between the assumptions on which the theory is based and the conditions actually prevailing in the stratum of gravel in which the thermometers are buried, rather than thermometric errors affecting the observations themselves.

The thermometers are inserted in undisturbed gravel, the first four lying one under the other, in a vertical plane beneath the grass of the south lawn, and within a few feet of the Stevenson screen in which the dry and wet bulb, and the maximum and minimum, thermometers are suspended.

In order that the thermometers might lie in practically unbroken ground, the following method of placing them was adopted. A pit was dug at the edge of the

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* See the Cambridge Scientific Instrument Company's "Descriptive List of Instruments," page 20.

grass about 5 feet long by 4 feet wide. One edge of the pit coincided with the edge of the grass plot, and the corresponding side of the pit was made as nearly vertical as possible. Into this vertical face four iron tubes were driven horizontally, the tubes being formed with spikes at their ends to facilitate this operation. The tubes are 4 feet long, and into them the thermometers were inserted with the leads attached, the mouths of the tubes were sealed up with tow and red lead, and the pit filled in.

The first four thermometers were placed at depths of (approximately) 6 inches, 1 foot 6 inches, 3 feet 6 inches, and 6 feet respectively ; but Mr. STONE soon saw the advisability of placing another at a lower level, and intended to have gone to a depth of 20 feet. But as water was met with at a depth of 10 feet 6 inches, he decided to place it just above the water level, at a depth of 10 feet.

This thermometer was buried, not directly under the four earlier ones, but in a separate pit at the other side of the Stevenson screen. This was apparently done to avoid disturbing the leads of the thermometers which were already in position, but it would have been rather more satisfactory if all had been placed in the same vertical plane.

It is also, perhaps, to be regretted that one or two similar thermometers were not buried to considerably greater depths. The presence of water, however, complicated matters and introduced conditions different from those which prevailed in the dry gravel above. It is not, for example, to be supposed that the thermal conductivity or the diffusivity of permanently water-logged gravel would be the same as that of the drier material above it. Hence it would appear necessary to put at least two thermometers below the permanent water-level in order to study the flow of heat under such circumstances. Besides, it is highly probable that the gravel stratum is not very much thicker than 10 feet. Excavations in the neighbourhood show that the blue Oxford clay is likely to be met with at any depth below 12 feet from the surface, and in this, of course, the thermal conditions would be likely to prove wholly different from those in the gravel.

The actual depths of the various thermometers as measured in October, 1898 (when the pits were standing open to enable us to re-standardise the thermometers) were as follows :—

Thermometer .	1	2	3	4	5
Depth	6½ in.	1 ft. 6 in.	3 ft. 6½ in.	5 ft. 8½ in.	9 ft. 11½ in.

These thermometers, with the Callendar and Griffiths resistance box, which could be connected with each thermometer through a switchboard, had been set up as I have stated, shortly before Mr. STONE's death.

On my appointment to the post of Radcliffe Observer, I took an early opportunity of examining the apparatus, and partly with a view of familiarising myself with all its details, I proceeded to determine the comparative values of the coils, and to re-standardise a spare thermometer which was kept in the observing room for general purposes.

This examination led to the discovery of discrepancies in the readings of the apparatus which troubled me for a long time, and which necessitated a large number of experiments extending at intervals over the greater part of a year before they were traced to their sources and eliminated.

In this part of the work I have to acknowledge the very generous help and advice of Mr. E. H. GRIFFITHS, F.R.S., who was kind enough to come to Oxford on more than one occasion to place his experience at our disposal, and who, at one stage of the investigation, took the resistance box and spare thermometer to Cambridge to subject them to a prolonged examination in his own laboratory.

These discrepancies, though serious in view of the accuracy which we had reason to expect from the apparatus, were still small quantities confined within one or two tenths of a centigrade degree. They were, for the most part, traced eventually to uncertainties in the contacts at the switchboard, and a want of perfect insulation in the older leads. These consisted of four india-rubber covered wires which, in the underground portion, passed through leaden pipes, but within the observing room were without the leaden covering. It was found that these were very susceptible to damp, and that the insulation fell away very rapidly when there was much moisture in the air, thus giving rise to very puzzling and troublesome discrepancies.

In September, 1898, the switchboard was improved and new composition cable leads substituted, which extended without interruption from the thermometers right up to the switchboard. Since these changes were effected the discrepancies have ceased to appear, except on one occasion (viz., October 27, 1899), when it was found that the short flexible lead from the switchboard to the resistance box was thoroughly damp. On lighting a fire in the observing room to dry the covering of this lead, the irregularities disappeared.

Since that date up till the end of March of this year (1900) I have kept a gas light burning continuously in the room, to prevent the deposition of moisture, and have experienced no further trouble of the sort.

The resistance box is in its general design similar to that described by Mr. GRIFFITHS,* but simplified to suit the particular class of work for which it was intended.

It is provided with three principal coils A, B, C, whose nominal values are 20, 40 and 80 box units respectively, a box unit being about 0·01 ohm. There are two additional coils, one for the calibration of the bridge wire, and another, which we have called the "concealed coil," whose value is about 240 box units, which was inserted for convenience to balance approximately the resistance of the thermometers at 0° C. when the coil A was also in the circuit, so that the reading of the bridge wire under these circumstances might be as nearly zero as possible.

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* 'Nature,' November 14, 1895.

On account of the differential character of the equation (a), (p. 242), the value of this coil does not concern us except in computing the correction for the temperature of the box, and then an approximate value only is required.

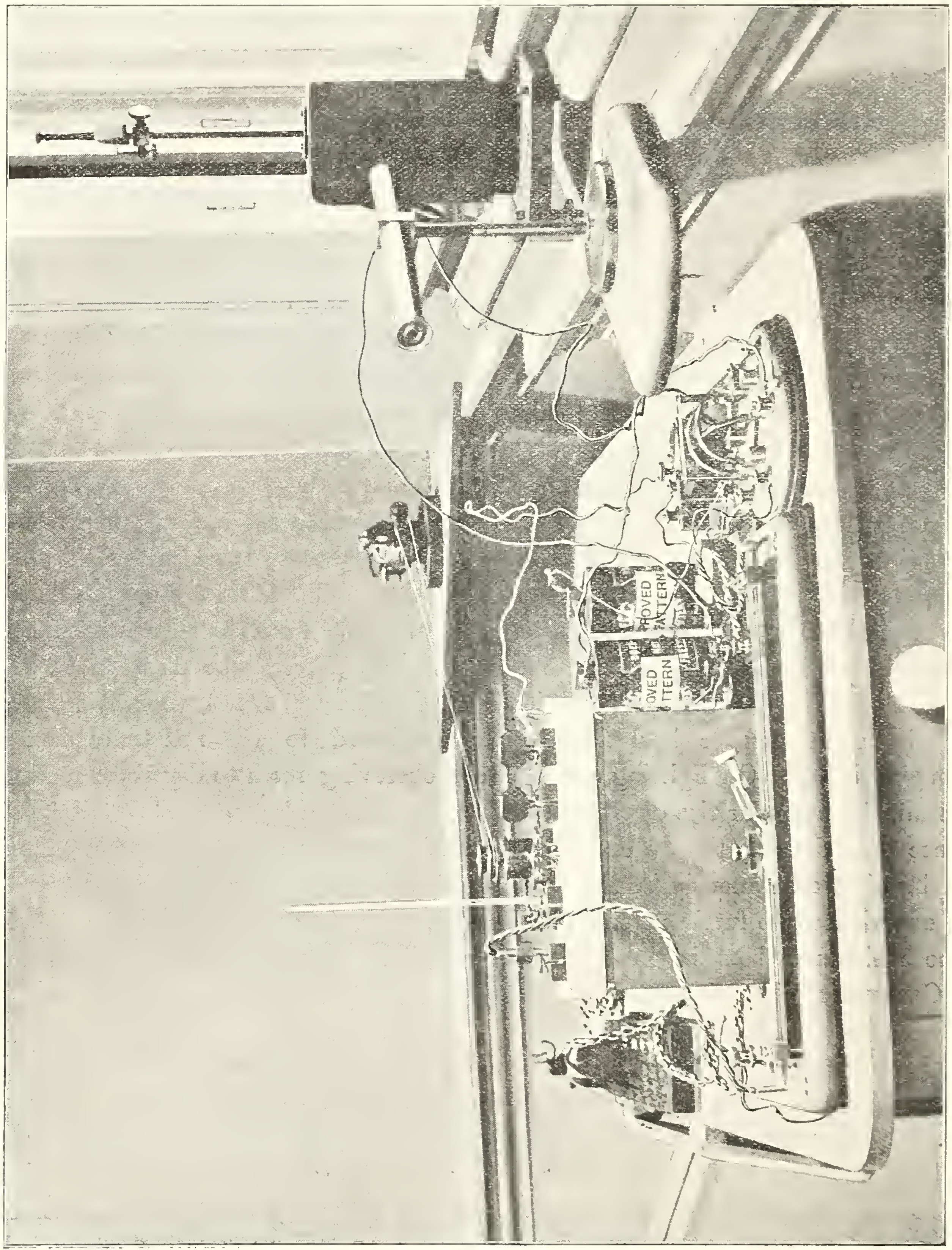

Fig. 1.—Resistance Box and Galvanometer at the Radcliffe Observatory, Oxford.

The apparatus is provided with a slow motion contact-maker of Mr. Horace Darwin's pattern,[*] and Griffiths' thermo-electric key.[†]

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[*] 'Nature, November 14, 1895.

[†] 'Phil. Trans.,' A, vol. 184 (1893), pp. 397–8.

The galvanometer microscope is placed on a window ledge to the right of the resistance box, in such a position that the observer can manipulate the commutator for reversing the direction of the current without removing his eye from the eye-piece.

The general arrangement of the apparatus is shown in fig. 1. To the right is the galvanometer and microscope; underneath in front is the commutator, and behind it the contact key. On the extreme left is the switchboard, and in the corner of the room is seen a small electric motor for stirring the oil in which the resistance coils are immersed.

In the standardisation of the apparatus the method described by Mr. GRIFFITHS in 'Nature,' of November 14, 1895, was in the main followed. The temperature coefficient was determined by Mr. GRIFFITHS when the apparatus was under examination in his own laboratory at Cambridge. Two separate determinations made in 1898 (July 27 and August 8) gave the following results :—

Date.	Range of Temp.	Temp. Coeff.
July 27 . .	9°·18	0·000242
August 8 . .	12 ·51	0·000240

In the reduction of the observations, the value 0·00024 has been adopted. The accuracy of this value has been borne out by subsequent observations in several different ways. Thus, for example, the invariable steadiness in the changes of No. 5, whatever might be the temperature of the box, indicated a high degree of accuracy in the adopted value of the constant.

For the determination of the coil values and the unit of the bridge wire scale, the following observations were made at the Radcliffe Observatory :—

	C − B − A = 20·051	B − A = 19·851	A = 19·603
	20·046	19·849	·600
	20·043	19·848	·601
	—	19·853	·602
	—	19·847	—
Means . .	20·047	19·850	19·601

From these we obtain, as in Mr. GRIFFITHS' paper referred to above,

$$\left.\begin{array}{l} C = 80{\cdot}158 \\ B = 39{\cdot}979 \\ A = 19{\cdot}863 \end{array}\right\}\text{mean box units.}$$

and one scale division of the bridge wire is equal to

1·0134 mean box units.

We thus get the following table giving the correction for the particular arrangement of coils in use, in which the corrections have each been diminished by 20 simply for the convenience of having the nominal readings of the box approximately equal to the temperature of the thermometers :—

TABLE I.—Correction for Coils.

Plugs in	Correction.	Coils in Circuit.
A, B, C	− 20·000	None
B, C	− 0·137	A
A, C	+ 19·979	B
C	+ 39·842	A, B
A, B	+ 60·158	C
B	+ 80·021	A, C
A	+100·137	B, C
None	+120·000	A, B, C

The correction to the bridge wire scale is also deduced from the following table, the correction being always of the same sign as the reading of the scale :—

TABLE II.—Bridge Wire Table.

R.	Corr.	R.	Corr.	R.	Corr.
0	0·000	6	0·080	12	0·161
1	·013	7	·094	13	·174
2	·027	8	·107	14	·188
3	·040	9	·121	15	·201
4	·054	10	·134	16	·214
5	·067	11	·147	17	·228
6	·080	12	·161	18	·241

We have next to correct for the temperature of the coils and bridge wire. This temperature is read from a mercury thermometer which stands in the oil in which the coils are immersed, its stem protruding through the marble slab which forms the top of the box. The oil in the box was stirred before each series of readings by means of a small electric motor. When the observations were prolonged for any considerable time, as in the process of standardising, the oil was stirred at frequent intervals.

All observations have been reduced to a standard temperature of 14° C., as being about the mean temperature of the observing room throughout the year.

If R_θ denotes the observed resistance at any temperature θ, and R_{14} the corresponding resistance for a temperature of the coils of 14° C., they are connected by the relation,

$$R_{14} = R_\theta\{1 + k(\theta - 14^\circ)\}.$$

or adopting the temperature coefficient, 0·00024, as determined by Mr. GRIFFITHS,

$$R_{14} - R_\theta = R_\theta \times 0·00024 \times (\theta - 14°).$$

In this expression R_θ is the total resistance in the circuit, and since this includes the resistance of the "concealed coil," we require to know approximately the value of that coil in the right-hand side of the equation. This is, perhaps, most easily determined from the observations of the thermometers themselves at 100° C. and 0° C., combined with the constant value found by Mr. GRIFFITHS for the ratio of the corresponding resistances R_1 and R_0.

Thus, if X be the value of this coil, r_0 that of the other coils in use and the bridge wire when the thermometer is packed in melting ice, and r_1 that of the coils and bridge wire when it is immersed in steam, reduced to mean box units at 14° C., then the total resistances in the two cases are, $X + r_1$ and $X + r_0$, and if we take the ratio of these resistances to be 1·3872,[*] as found by Mr. GRIFFITHS for the wire used in the construction of this instrument, then $\frac{X + r_1}{X + r_0} = 1·3872$, and therefore $X = \frac{r_1 - 1·3872 r_0}{0·3872}$.

The values of X found in this way from the observations made on October 4, 5, and 6, 1898, for the purpose of standardising the thermometers, are as follows :—

Thermometer.	X.
1	240·65
2	·65
3	·77
4	·77
5	·60
A	·65
Mean	240·68

For any arrangement of coils (Y) and any bridge wire reading (R) we have therefore the total resistance in the circuit, X + Y + R, and the coefficient of $(\theta - 14°)$ in the correction for temperature is

$$(X + Y + R) \times 0·00024.$$

We thus find the following table for the two different arrangements which have been used in the observations :—

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[*] 'Nature,' November 14, 1895, p. 45.

TABLE III.—Coefficient of $(\theta - 14^\circ)$.

Plugs in	B + C	A + C	Plugs in	B + C	A + C	Plugs in	B + C	A + C
Bridge wire reading.			Bridge wire reading.			Bridge wire reading.		
−18	0·0581	0·0629	−5	0·0612	0·0660	+8	0·0643	0·0691
17	83	31	4	14	62	9	46	94
16	86	34	3	17	65	10	48	96
15	88	36	2	19	67	11	50	·0698
14	90	38	−1	22	70	12	53	·0701
13	93	41	0	24	72	13	55	03
12	95	43	+1	26	74	14	58	06
11	·0598	46	2	29	77	15	60	08
10	·0600	48	3	31	79	16	62	10
9	02	50	4	34	82	17	65	13
8	05	53	5	36	84	+18	·0667	·0715
7	07	55	6	38	86			
−6	·0610	·0658	+7	·0641	·0689			

The temperature on the platinum scale corresponding to a resistance, R, is deduced from CALLENDAR'S formula,

$$pt = 100\ (\mathrm{R} - \mathrm{R}_0)/(\mathrm{R}_1 - \mathrm{R}_0)^* \quad . \ . \ . \ . \ . \ . \ . \ . \quad (a.)$$

where R_1 is the resistance at 100° C., and R_0 at 0° C., each of them being reduced to mean box units at the standard temperature (14°).

From a very careful series of observations made at the Radcliffe Observatory on October 4 and 6, 1898, the following separate values of the zero points were obtained, each value being the mean of several closely accordant settings :—

Zero Points of the Thermometers.

Thermometer .	1.	2.	3.	4.	5.
1898, October 4	0·306	0·420	0·481	0·323	—
„ 6	·306	·432	·496	·338	0·239
„ 6	·305	·434	·498	·333	·239
Adopted means	0·31	0·43	0·49	0·33	0·24

The temperature of steam was observed on October 4 and 5, with the following results for the several thermometers :—

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* 'Phil. Trans.,' A, vol. 178, p. 195.

Temperature of Steam.

Thermometer .	1.	2.	3.	4.	5.
1898, October 4	—	—	101·610	101·389	—
,, 5	101·286	101·474	·589	·367	101·179
Adopted values	101·29	101·47	101·60	101·38	101·18

To those who have standardised naked platinum thermometers the discrepancies in the separate results for the b. p.'s of Nos. 3 and 4 may appear large. It should, however, be pointed out that it was necessary to standardise these instruments while they were sealed up in strong brass tubes with heavy leaden-covered leads attached; hence it was impossible to eliminate altogether the effects of conduction along these tubes, but for reasons given on p. 244 it was not considered necessary to take further precaution against the small errors arising from this cause. In determining the zero points the thermometers were placed in a trough 3 feet long, and any error arising from this cause was very much diminished.

One of the most important considerations in connection with this subject is the degree of permanence in the fundamental points as determined at considerable intervals of time. It was the occurrence of discrepancies between the values which I found in my first observations and those determined about a year and a half previously at the time that the instruments were set up, which induced me to have all the thermometers exhumed and to make a thorough re-examination of the whole apparatus. This examination led eventually to my discarding the original leads, the insulation of which was found to fall off very much when they became damp.

Another series of discrepancies was traced to an uncertainty in the contacts at the switchboard. In the original form the four steel prongs in which the fourfold lead from the resistance box terminates, were inserted into mercury cups into which also were led the four brass strips to which the thermometer leads were soldered. By having the steel prongs amalgamated, and adding springs to keep each prong firmly pressed against the brass strips immersed in the mercury, a great improvement was experienced; and, since this change was effected, we have had no trouble from the same cause. It has been the habit, too, to make the observations from time to time with the four steel prongs in both positions, which affords a very satisfactory check on the character of the contacts.

Taking advantage of a visit from Mr. GRIFFITHS on October 6, 1899, I had thermometer No. 1 (6 inches) dug up, and we examined its zero point after exactly a year's continuous observations. Determined in the same way as in the previous year, the zero point of this thermometer was found to agree with the earlier value to less than 0°·005 C., the actual values being

In 1898	0·306
and in 1899	0·302

During the observations of 1898, the temperature of the box ranged from 15°·81 to 16°·05, whilst in 1899 it stood at 10°·36. The close agreement of these two results, therefore, taken at such an interval of time and at temperatures differing so considerably—by about one-sixth of the whole range with which we are concerned—affords a further confirmation of the accuracy of the adopted value of the temperature coefficient and the general consistency of the apparatus.

It is clear that in a series of observations, such as we are at present considering, in which the highest temperature does not exceed 25° C., any change in the zero point will have considerably more influence on our results than a corresponding variation in the boiling point. Also the researches of Professor CALLENDAR, Mr. GRIFFITHS, Messrs. HEYCOCK and NEVILLE, and others, show that the change in the fundamental interval, even when the instrument is exposed to enormous variations of temperature, is very slight.

Our experience with a sixth thermometer (A), which is kept in the observing room, points in the same direction. In this instrument the fundamental interval remained practically unaltered after the lapse of a year, the actual values found for it being

In 1898 . . . . . . 101·067
and in 1899 . . . . . . 101·059

On substituting the values found above for R_0 and R_1 in formula (a) we obtain the following expressions for the separate thermometers, giving the temperature on the platinum scale corresponding to any reading R :—

$$\begin{array}{rl}
\text{Thermometer 1.} & pt = (R - 0{\cdot}31)/1{\cdot}0098 = R - (0{\cdot}0097\,R + 0{\cdot}31) \\
2. & pt = (R - 0{\cdot}43)/1{\cdot}0104 = R - (0{\cdot}0103\,R + 0{\cdot}43) \\
3. & pt = (R - 0{\cdot}49)/1{\cdot}0111 = R - (0{\cdot}0110\,R + 0{\cdot}48) \\
4. & pt = (R - 0{\cdot}33)/1{\cdot}0105 = R - (0{\cdot}0104\,R + 0{\cdot}33) \\
5. & pt = (R - 0{\cdot}24)/1{\cdot}0094 = R - (0{\cdot}0093\,R + 0{\cdot}24)
\end{array}$$

The expressions in brackets on the right-hand side are the corrections which must be applied to R to obtain the temperature on the platinum scale. They may be tabulated in a very simple form for each thermometer, so that the platinum temperature can be at once deduced from the reading of the resistance.

TABLE IV.—Reduction to Platinum Scale.

Thermometer .	1.	2.	3.	4.	5.
R = −10 . .	−0·21	−0·33	−0·37	−0·23	−0·15
0 . .	−0·31	−0·43	−0·48	−0·33	−0·24
+10 . .	−0·41	−0·53	−0·59	−0·43	−0·33
+20 . .	−0·50	−0·64	−0·70	−0·54	−0·43
+30 . .	−0·60	−0·74	−0·81	−0·64	−0·52

A complete determination of temperature on the platinum scale by means of one of the sunken thermometers is, therefore, reduced to the following simple steps :—

(1) The balancing of the galvanometer and reading of the bridge wire scale (R) and the temperature (θ) of the box.

(2) To R is to be added the correction for the particular arrangement of coils used, from Table I.

(3) The correction to reduce the bridge wire reading to mean box units, from Table II.

(4) The reduction to standard temperature (14°). The quantity taken from Table III. multiplied by ($\theta - 14$) gives this correction.

(5) The correction from Table IV.

It only remains to reduce the temperature thus expressed from the platinum to the air scale.

The relation connecting these two, established by Professor CALLENDAR,* is

$$d = t - pt = \delta\left\{\left(\frac{t}{100}\right)^2 - \frac{t}{100}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

in which pt is the platinum temperature, t the temperature on the air scale, and δ a constant.

For a completely independent standardisation it would be necessary to determine the resistance at some third known temperature in order to obtain the value of δ, but the experiments of CALLENDAR and GRIFFITHS have shown that although the value of δ varies from one specimen of platinum to another, it is a constant for any particular sample of wire. References to the original papers bearing on this point are given in Mr. GRIFFITHS' article in 'Nature' cited above.

The value of δ for the particular wire used in the Oxford instrument was determined at Cambridge to be 1·512.† If it were intended to employ the Oxford apparatus for the determination of temperatures over a very wide range, it would doubtless have been advisable to make an independent determination of the value of this constant. Since, however, the range −15° C. to +25° C. will cover all the variations of earth temperatures with which alone we are here concerned, and since within that range the correction does not amount to as much as 0·3, an error of even 0·050 (which is quite inadmissible) in the value of δ would not affect our results.

Writing $pt + d$ for t in equation (b), and remarking that since $d/100$ is less than 0·003, its square may be neglected, we find

$$d = \delta(\tau^2 - \tau)/\{1 + (1 - 2\tau)\delta/100\}$$

τ being written for $pt/100$.

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[*] 'Phil. Trans.,' A, 1887.

[†] *Cf.* the Report of the Committee of the British Association for improving the Construction of Practical Standards for use in Electrical Measurements. Bradford, 1900. [September 16, 1900.]

We thus obtain the following table for the correction from the platinum to the air scale, for every degree of the former from $-15°$ to $+25°$.

TABLE V.—Reduction from the Platinum to the Air Scale.

pt.	Corr. to air.	*pt.*	Corr. to air.	*pt.*	Corr. to air.	*pt.*	Corr. to air.
°	°	°	°	°	°	°	°
−15	+0·256	−5	+0·078	+ 5	−0·071	+15	−0·191
14	·237	4	·062	6	·084	16	·201
13	·218	3	·046	7	·097	17	·211
12	·199	2	·030	8	·110	18	·221
11	·181	−1	+ ·015	9	·122	19	·231
10	·163	0	·000	10	·134	20	·240
9	·146	+1	− ·015	11	·146	21	·249
8	·128	2	·029	12	·158	22	·257
7	·111	3	·043	13	·169	23	·266
6	·095	4	·057	14	·180	24	·274
− 5	+ ·078	+5	− ·071	+15	− ·191	+25	− ·281

The reduction of the observations is thus of a very simple character; but it may be still further simplified, and the chance of arithmetical errors occurring in individual cases greatly reduced, if not wholly removed, by the preparation of a table giving the total correction to the bridge wire reading for each arrangement of coils, the arguments in each case being the bridge wire reading and the temperature of the box. Of course, if a great variety of coils were in use, such tables would attain dimensions out of proportion to their usefulness, as it would be necessary to construct a special table for each separate combination of coils. But in observations of underground temperature, the range of the readings of any particular thermometer is comparatively limited, so that only two different arrangements of the plugs are necessary, coil A serving for about eight months of the year, and coil B coming into use for about four months in summer.

It is thus a simple matter to compute tables that will cover all cases. Tables of this sort were prepared for each of the earth thermometers, from which the correction to the bridge wire reading, to reduce to the corresponding temperature on the air scale, could be obtained at one step by simple interpolation.

For some little time before and after the epochs at which the change from one coil to the other is made (about the end of May and the end of September) it is possible to read the thermometers with either arrangement, and thus a check can be imposed upon the general performance of the apparatus.

In 1899 the changes were made on June 1 and September 26, and the following readings of the 10-foot thermometer, in which the changes are very slow and regular, afford very satisfactory evidence of the consistency of the readings.

May 28	9·55° C.	Coil A.	Sept. 22	14·42° C.	Coil B.
29	9·59		23	14·41	
30	9·63		24	14·41	
31	9·66		25	14·40	
June 1	9·70	Coil B.	26	14·39	Coil A.
2	9·75		27	14·37	
3	9·80		28	14·35	
4	9·82		29	14·34	

Discussion of the Observations.

The first step in the discussion of the observations is to group them into monthly means, and thence to deduce the harmonic expressions which will represent the readings of each thermometer throughout the year.*

In this part of the work I have adopted the Fahrenheit scale, as the observations had already been reduced to this scale for comparison with our other meteorological results, and as the observations of the same kind at Greenwich† and Edinburgh‡ discussed by Professor EVERETT are expressed in the same scale, there seemed to be a distinct advantage in retaining it.

On account of the inequality in the lengths of the calendar months I have discarded them altogether, and, as far as possible, have divided the year into twelve portions which are alternately thirty and thirty-one days in length. As the observations are taken only once a day, it is of course necessary to have an integer number of days in each division, but the following scheme makes the differences in their lengths as small as possible, and with one exception, that of January, alternately thirty and thirty-one days. In Leap Year this exception would be removed by intercalating the extra day in January, instead of February.

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* Professor W. THOMSON, "On the Reduction of Observations of Underground Temperature," 'Trans. Roy. Soc. Edin.,' vol. 22, p. 409.

† 'Greenwich Observations,' 1860 (cxciii.).

‡ Professor EVERETT, 'Trans. Roy. Soc. Edin.,' vol. 22, p. 429.

Divisions of the Year.

Ordinary Years.			Leap Year.	
Division.		No. of days.		No. of days.
I.	Jan. 1 to Jan. 30 (inclusive)	30	Jan. 1 to Jan. 31	31
II.	Jan. 31 to Mar. 1 ,,	30	Feb. 1 ,, Mar. 1	30
III.	Mar. 2 ,, April 1 ,,	31		31
IV.	April 2 ,, May 1 ,,	30		30
V.	May 2 ,, June 1 ,,	31		31
VI.	June 2 ,, July 1 ,,	30		30
VII.	July 2 ,, Aug. 1 ,,	31		31
VIII.	Aug. 2 ,, Aug. 31 ,,	30		30
IX.	Sep. 1 ,, Oct. 1 ,,	31		31
X.	Oct. 2 ,, Oct. 31 ,,	30		30
XI.	Nov. 1 ,, Dec. 1 ,,	31		31
XII.	Dec. 1 ,, Dec. 31 ,,	30		30

For the sake of convenience, I have retained below the usual names of the months for these twelve divisions of the year. A good deal might, I think, be said in favour of adopting these intervals for all meteorological returns where the means of daily observations are taken.

The monthly means so obtained for the five thermometers are given in the following table:—

Mean Monthly Temperature of the Ground at the Radcliffe Observatory, Oxford, 1899.

Thermometer. .	1	2	3	4	5
Depth	6½ in.	1 ft. 6 in.	3 ft. 6½ in.	5 ft. 8½ in.	9 ft. 11½ in.
January . . .	40°·47	42°·07	44°·68	46°·80	49°·97
February . . .	40·09	41·34	43·25	45·08	48·33
March	41·34	41·91	43·21	44·74	47·42
April	48·77	47·66	46·61	46·40	47·37
May	54·86	52·95	50·96	49·54	48·53
June	66·73	62·89	58·29	54·73	50·88
July	69·43	65·93	62·15	58·74	53·85
August . . .	69·23	67·79	64·88	61·66	56·39
September . .	59·34	61·12	62·03	61·19	57·80
October . . .	48·99	51·14	54·29	56·12	56·71
November . .	46·73	48·43	50·99	52·69	54·48
December . .	38·14	41·08	45·35	48·58	52·33

The observations were taken within a short time of noon on every day throughout the year without exception. They were for the most part made by Mr.

McClellan in the ordinary routine, but during his vacation, or on Sundays, Mr. Wickham or Mr. Robinson took his place. All three are observers of skill and experience, and, as the results seem to show, the observations are of a remarkable degree of precision.

These means were deduced from the observations as directly obtained without any modification or correction, other than those taken from the tables referred to above, except on one date—October 27—when, as I have mentioned, the short flexible lead was found to be affected by the dampness of the air in the observing room.

This was indicated by a sudden change of about 0°·13 F. in the reading of the 10 feet thermometer, which, under ordinary circumstances changes so slowly and steadily, that its reading on any day might be predicted with certainty to within one-twentieth of a degree from the readings of two or three days preceding. On drying the lead, however, the abnormal readings disappeared by the next day, and the subsequent readings of this thermometer were found to lie along the same curve as before the discrepancy had arisen.

As the dampness of the lead disturbed only the reading of the resistance box and in no way affected the thermometers themselves, we were, therefore, able to take an interpolated value for the reading of No. 5 as a standard of comparison, and the difference between this and the actually-observed readings, viz.: 0°·13, was added as a correction to all observations made on that day.

This particular case illustrates very well the protection which the readings of a deep sunk thermometer afford against sudden changes occurring unobserved in the apparatus.

The monthly means are graphically represented in fig. 2; the daily readings for two periods of two months each are exhibited in Plates 1 and 2. These two periods have been selected, as the first includes the minimum and the second the maximum of the 10 feet thermometer, and both illustrate very well the steadiness of the changes in the indications of this instrument, and exhibit also the manner in which both the amplitude of a wave is diminished, and its phase retarded in passing from one thermometer to the one below it.

In fig. 3 are given the mean monthly temperature gradients beneath the surface, deduced from the same figures.

The harmonic expression to represent the temperature of any particular thermometer throughout the year will be

$$\theta = a_0 + a_1 \cos \lambda t + a_2 \cos 2\lambda t + \&c. \\ + b_1 \sin \lambda t + b_2 \sin 2\lambda t + \&c. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (c)$$

or

$$\theta = a_0 + \mathrm{P}_1 \sin (\lambda t + \mathrm{E}_1) + \mathrm{P}_2 \sin (2\lambda t + \mathrm{E}_2) + \&c. \ . \ . \ . \ . \ . \quad (d)$$

where t denotes the time represented as the fraction of a year, and λ is equal to 2π.

Fig. 2.—Mean Monthly Temperature of the Ground, 1899.

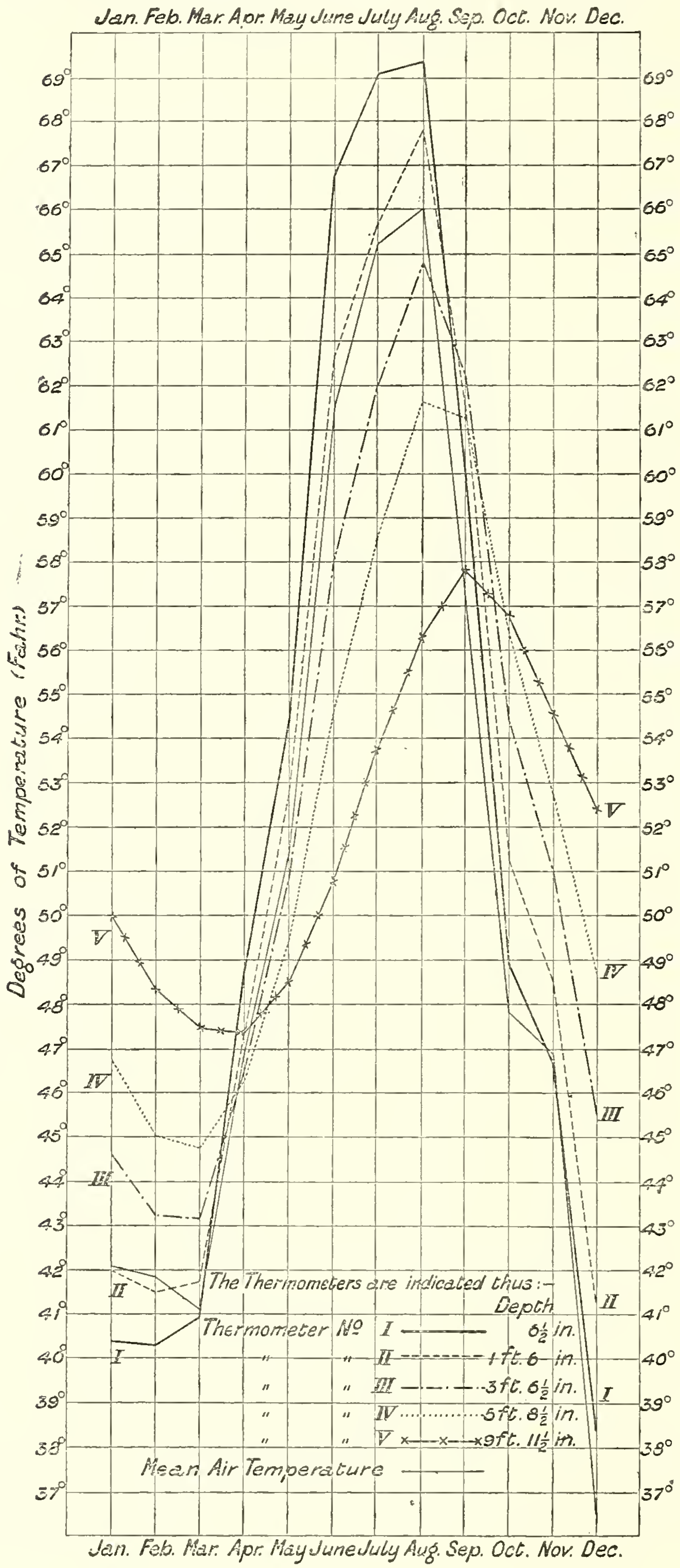
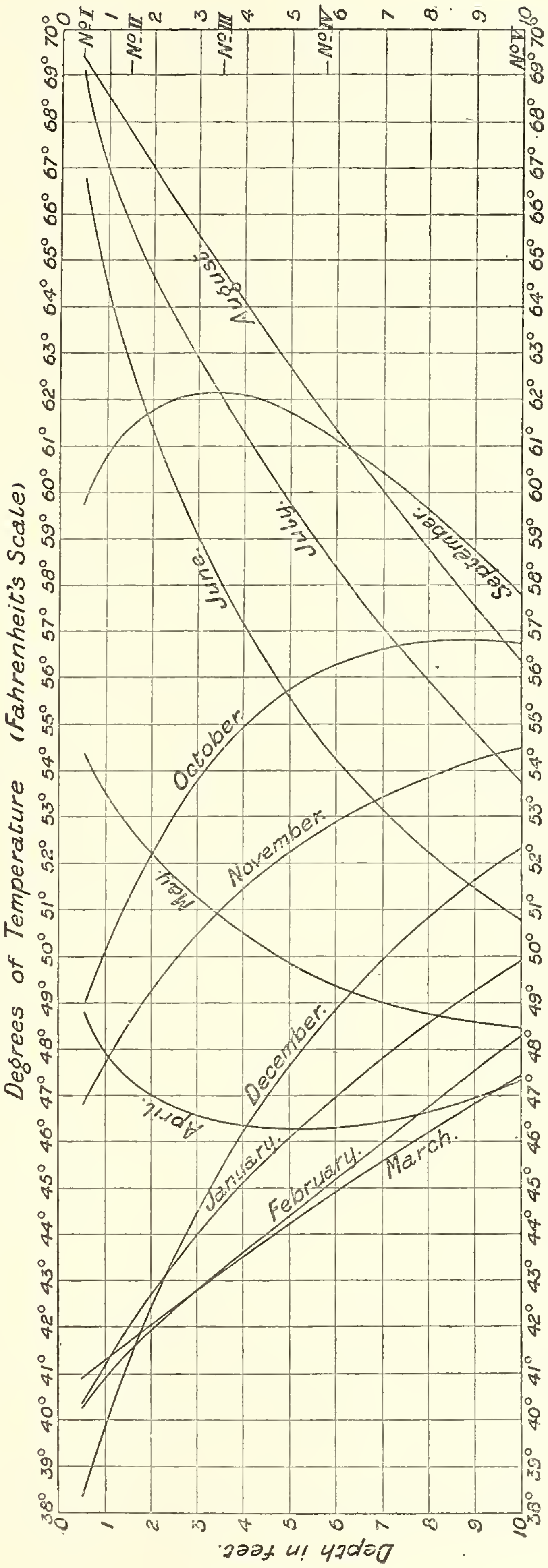


Fig. 3.—Mean Monthly Temperature Gradients.



NOTE :- The positions of the Thermometers are indicated by the Roman Numerals at the right of the diagram.

From the monthly means given above, we deduce the following :—

VALUES of the Coefficients.

No.	a_0	a_1	b_1	a_2	b_2	P_1	E_1	P_2	E_2
	°	°	°	°	°	°	° ′	°	° ′
5	52·005	− 1·970	−4·706	−0·068	+0·537	5·102	202 42·8	0·541	352 45·1
4	52·189	− 6·379	−5·318	+0·661	+1·029	8·305	230 10·9	1·223	32 42·1
3	52·224	− 9·467	−4·725	+1·370	+1·123	10·581	243 28·6	1·771	50 39·5
2	52·013	−12·932	−3·128	+2·130	+0·976	13·305	256 24·3	2·343	65 23·3
1	52·010	−15·337	−1·507	+3·017	+0·511	15·411	264 23·2	3·060	80 23·2
Air	50·396	−12·776	−1·621	+2·847	+1·410	12·878	262 46·1	3·177	63 39·2

In this table I have added, for comparison, the constants of the Fourier series representing the mean temperature of the air in the Stevenson screen at a height of 4 feet above the surface, deduced, as has been the custom for many years at this Observatory, from observations of a standard mercury thermometer at 8 A.M., noon, and 8 P.M.

Some interesting results appear at once from this table. As was to be expected the mean temperatures (denoted by a_0) of the soil at all depth exceeds that of the air, although the differences are less than those deduced by Professor EVERETT from the Greenwich results for thirteen years.* It will also be noticed that the annual range of temperatures for Nos. 1 and 2 exceeds that of the air. This is only true of the monthly means. The range of the mean *diurnal* air temperature is considerably greater than that observed for any of the underground thermometers. The maxima and minima are given in the following table :—

Thermometer.	Maxima.	Minima.
	°	°
Air	75·4	21·1
1	75·29	33·48
2	70·03	36·97
3	65·28	41·45
4	62·28	44·38
5	57·96	47·08

If the actual maximum and minimum for the air had been taken instead of the daily mean, the range would have been even greater.

I have not taken into account the terms depending on $3t$. For No. 5 this term is

$$0^{\circ}{\cdot}256 \sin (3\lambda t + 117^{\circ}\ 28'),$$

and in the case of the other thermometers they attain slightly larger dimensions;

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* 'Greenwich Observations,' 1860 (cxciii.).

but as we are here dealing with the observations of a single year, it would be unsafe to attach much importance to results deducible from the smaller terms.

The accuracy with which the observations are represented by the first three terms of the formulæ is shown by the following table, which contains the differences between the mean monthly temperatures as computed and those actually observed :—

Comparison of Computed and Observed Mean Monthly Temperatures, C—O.

Thermometer.	1	2	3	4	5
January . . .	−0°·78	−0°·86	−0°·55	−0°·32	0°·00
February . . .	−0·17	−0·18	+0·07	+0·15	+0·05
March	+0·65	+0·71	+0·48	+0·22	+0·02
April	−1·28	−0·70	−0·48	−0·19	0·00
May	+1·56	+0·91	+0·25	+0·01	−0·05
June	−1·13	−1·02	−0·52	−0·24	−0·02
July	+0·93	+1·15	+0·91	+0·49	+0·06
August . . .	−1·23	−1·10	−0·44	−0·07	+0·10
September . .	+0·58	−0·15	−0·69	−0·65	−0·23
October . . .	+1·51	+1·87	+1·29	+0·73	+0·07
November . .	−3·03	−2·14	−1·07	−0·31	+0·18
December . .	+2·41	+1·52	+0·75	+0·18	−0·18

It thus appears that while the three deeper thermometers are fairly well represented by the formulæ, there are considerable differences in the two upper ones. This, especially in the case of No. 1, is largely due to the diurnal variations which make themselves felt to a depth of about 3 feet.

The surface of the ground in the neighbourhood of the spot where the thermometers are sunk, being approximately level and the gravel being, as far as we know, of a fairly uniform character for a considerable distance in all directions, the flow of heat will be represented by FOURIER'S equation

$$\kappa \, d^2\theta/dx^2 = d\theta/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

in which κ denotes the diffusivity of the gravel, and x denotes the depth of a thermometer below the surface.

A solution of this is

$$\theta = \Sigma \mathrm{A}_n e^{-\alpha_n x} \sin (n\lambda t + \beta_n x + \gamma)$$

if $$\alpha_n^2 - \beta_n^2 = 0 \quad \text{and} \quad 2\alpha_n\beta_n\kappa = -n\lambda.$$

Hence $$\alpha_n = \sqrt{\frac{n\pi}{\kappa}} = -\beta_n.$$

Comparing this expression with the series (d) given on page 249 we have

$$\mathrm{P}_n = \mathrm{A}_n e^{-\alpha_n x} \quad \text{and} \quad \mathrm{E}_n = \beta_n x + \gamma.$$

For a second thermometer we have similarly

$$P_n' = A_n e^{-\alpha_n x'} \quad \text{and} \quad E_n' = \beta_n x' + \gamma,$$

and, therefore,

$$\left.\begin{aligned} \sqrt{\frac{n\pi}{\kappa}} &= \alpha_n = \frac{\log P_n - \log P_n'}{x' - x} \\ \text{and} \qquad \sqrt{\frac{n\pi}{\kappa}} &= -\beta_n = \frac{E_n - E_n'}{x' - x} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (f).$$

Thus from each wave as observed at any pair of thermometers, we obtain two determinations of the value of κ, one from the diminution of amplitude, and the other from the retardation of phase.

In computing the value of $\sqrt{\pi/\kappa}$ I have expressed the depths in Paris feet, in order to bring out the results in terms of the same units as those employed by Professor EVERETT in his Reduction of the Greenwich Observations,* and those of Lord KELVIN already referred to. The results are given in the following tables :—

Values of $\sqrt{\pi/\kappa}$ deduced from the Annual Wave.

Thermometers compared.	From diminution of amplitude.	From retardation of phase.
No. 5 and No. 4	0·1222	0·1202
„ 5 „ 3	·1212	·1181
„ 5 „ 2	·1208	·1181
„ 4 „ 3	·1191	·1141
„ 4 „ 2	·1193	·1159
„ 3 „ 2	·1196	·1178
Means . . .	0·1204	0·1174
Mean of both . . . 0·1189		

The values deduced from the half-yearly wave in a similar way, are as follows :—

---

* 'Greenwich Observations,' 1860 (cxciii).

Values of $\sqrt{\pi/\kappa}$ deduced from the Half-yearly Wave.

Thermometers compared.	From diminution of amplitude.	From retardation of phase.
No. 5 and No. 4	0·1445	0·1236
„ 5 „ „ 3	·1393	·1187
„ 5 „ „ 2	·1306	·1129
„ 4 „ „ 3	·1287	·1092
„ 4 „ „ 2	·1164	·1022
„ 3 „ „ 2	·1033	·0949
Means . . .	0·1271	0·1102
Mean of both . . . 0·1187		

I have omitted the results derived from the readings of No. 1, as they seem too much affected by short period variations to afford reliable results. This thermometer, too, is buried in a surface soil which is of quite a different character from the sandy gravel containing the other thermometers.

The values of $\sqrt{\pi/\kappa}$ deduced from the annual wave are, of course, much more trustworthy than those obtained from the half-yearly wave, and the larger discrepancies in the individual results from the latter are not surprising. It is, however, satisfactory to note how these corroborate the others, showing, for instance, larger values resulting from the comparison of Nos. 5 and 4 than from that of any other pair. This may possibly be due to a smaller value of κ for the stratum of gravel about 4 feet thick which separates these two thermometers, than for the higher strata, or to the fact of No. 5 being buried at some distance (9 feet 6 inches) from the vertical plane containing the other three.

Unfortunately, when the pits were open, no very critical examination of the character of the gravels at different depths was made; but it is proposed to repair this omission when next the thermometers are dug up.

The excessively close agreement of the mean values of $\sqrt{\pi/\kappa}$ derived from the annual and half-yearly waves is very remarkable (especially in view of the fact that the results are deduced from the observations of a single year), and seems to indicate a high degree of precision in the observations.

The systematically larger values found from the diminution of amplitude, as compared with those deduced from the retardation of phase must be traced to some other cause, and may possibly be due to the proximity of the Observatory building, the south front of which is situated at a distance of 36 feet from the thermometers. The temperature of the ground beneath the buildings would in all probability be different from that at an equal depth beneath the exposed surface. There would, therefore, be a transfer of heat which would render equation (e) no longer strictly applicable.

A solution on the hypothesis that the heat which is conducted in this way may be represented by $\mu\theta$ leads to the two values

$$\sqrt{\pi/\kappa} = 0{\cdot}1189 \text{ from the annual wave,}$$
$$\text{and } \sqrt{\pi/\kappa} = 0{\cdot}1184 \text{ from the half-yearly wave,}$$

a very satisfactory agreement; but the values found for μ/κ on the same hypothesis, viz.:—

$$\mu/\kappa = 0{\cdot}0007 \text{ from the annual wave,}$$
$$\text{and } \mu/\kappa = 0{\cdot}0080 \text{ from the half-yearly wave,}$$

differ too much to admit of any confidence in this additional term as representing the exchange of heat.

We cannot, however, be much in error in taking

$$0{\cdot}1188$$

as the value of $\sqrt{\pi/\kappa}$ for the gravel in which the thermometers are sunk.

The value of the same quantity as found by Professor EVERETT from the Greenwich Observations was

$$0{\cdot}09175,$$

and for the three stations at Edinburgh, from Professor FORBES' observations, Lord KELVIN obtained,

Calton Hill, trap rock	0·1154,
Experimental Garden, sand	·1098,
Craigleith Quarry, sandstone . . .	·06744.

From the equations (f) we find from each thermometer a value of the amplitude and of the retardation of phase of each wave at the surface.

Denoting the amplitude of the annual wave at the surface by P_{01}, we have

$$\log P_{01} = \log P_1 + \sqrt{\frac{\pi}{\kappa}} \,.\, x.\text{*}$$

Substituting the values of P_1 and x for each of the thermometers 2, 3, 4, and 5, we get four separate values of P_{01}. These are

Thermometer.	P_{01}.
	°
5 . . .	15·48
4 . . .	·70
3 . . .	·70
2 . . .	·72
Mean =	15·65 = amplitude of annual wave at surface.

---

* *Cf.* QUETELET, 'Annales de l'Observatoire Royal de Bruxelles,' tome iv., 1845, p. 110.

Similarly, for the half-yearly wave, we find

Thermometer.	P_{02}.
	°
5 . . .	2·60
4 . . .	3·01
3 . . .	3·09
2 . . .	2·97
	2·92 = amplitude of half-yearly wave at surface.

Putting these values in the first of equations (f) we can determine the value of x, corresponding to a given value of $P_{\prime\prime}$, or the depth at which the amplitude of the wave is reduced to any given value, on the hypothesis that the conditions prevailing above 10 feet remain unchanged at greater depths.

Although theoretically there is no invariable layer so long as equations (f) are applicable, still we may consider that an annual variation of 0°·02 F. is less than can be certainly detected. The stratum, therefore, at which the amplitude of the annual wave is reduced to 0°·01 may to all intents and purposes be considered as invariable.

For this depth we have M log $P_1 = -2$ (M being the modulus of common logarithms), and therefore

$$x = (2 + \log P_{01})/M\sqrt{\pi/\kappa}.$$

Thus we find as the depth at which the amplitude of the annual wave is reduced to 0°·01,

$$x_1 = 61·21 \text{ French feet} = 66·0 \text{ English feet},$$

and similarly for the half-yearly wave,

$$x_2 = 33·78 \text{ French feet} = 36·0 \text{ English feet}.$$

The depths at which the annual and half-yearly waves are reduced to an amplitude of 0°·1 F. are found in a similar way to be 45·3 and 21·4 English feet respectively.

This paper deals with the observations of a single year, and the results accordingly exhibit discrepancies between theory and observations which, although they are less than might have been *à priori* expected, are greater than one would like to see. These discrepancies are due partly to the fact that the temperature variations are not strictly of a periodic character as the theory supposes, and as such they might be expected to be diminished in the mean of a number of years, and partly to irregularities, physical and formal, in the surface.

The other source of irregularity considered by Lord KELVIN in his paper, referred to above, namely, thermometric errors arising from the uncertainty as to the temperature of the liquid in the long stems of the thermometers used in Professor

Forbes' observations, does not in this case apply, and if other errors peculiar to the platinum thermometers exist, they seem to be confined within much smaller limits.

In fact this mode of thermometry seems especially suited to the investigation of underground temperatures on account of its freedom from this source of error, and the convenience with which the observations may be made.

PLATE 1.

DAILY READINGS OF UNDERGROUND THERMOMETERS. March and April, 1899.

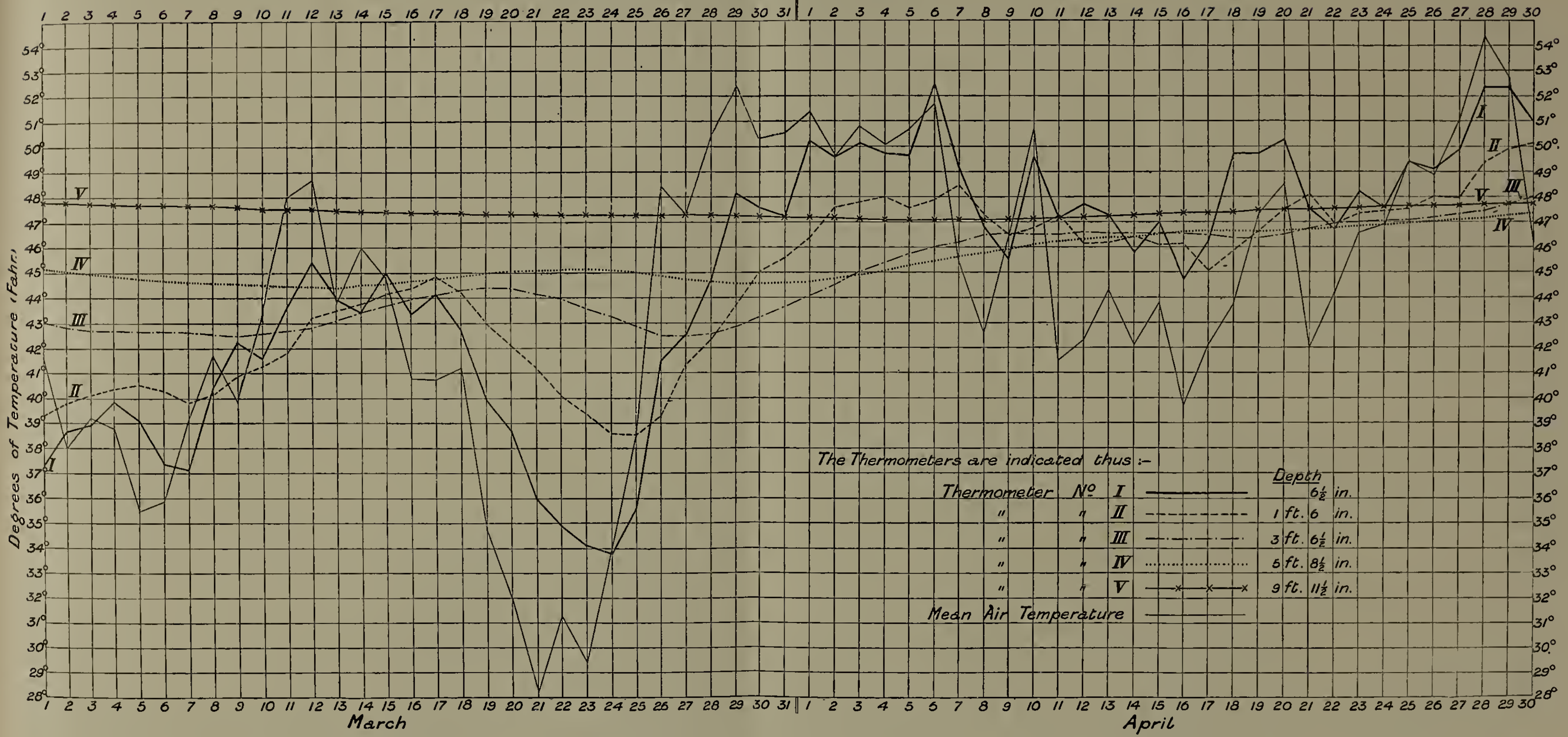
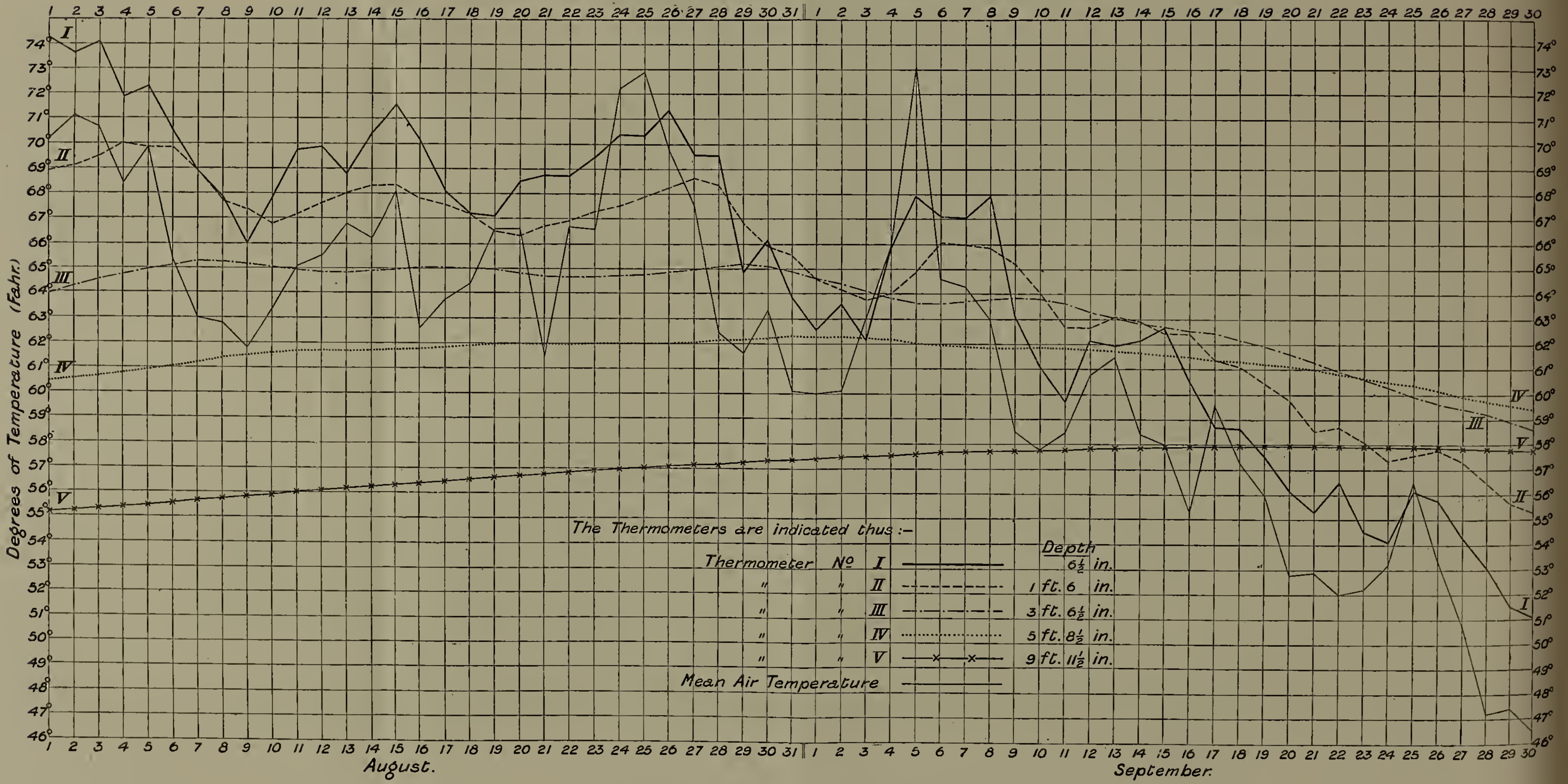


PLATE 2.

Daily Readings of Underground Thermometers. August and September, 1899.



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VII. *The Diffusion of Ions produced in Air by the Action of a Radio-active Substance, Ultra-violet Light and Point Discharges.*

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A GENERAL method of finding the rate of diffusion of ions into a gas has been described in a previous paper,* and an account was there given of the results obtained with ions produced by Röntgen rays. The present paper gives the results obtained with ions produced by a radio-active substance, by point discharges, and by ultra-violet light. The principle of the method consists in calculating the rate of diffusion from observations on the loss of conductivity of a gas as it passes along metal tubing.

The experiments were arranged so that the loss due to diffusion should be much greater than the loss due to other causes. In order to ensure this, there are two effects which must be considered in fixing the dimensions of the tubing: the recombination which occurs when there are both positive and negative ions present in the gas; and the effect due to the mutual repulsion of the ions which takes place when most of the ions are charged with electricity of the same sign. It is therefore necessary either to correct for these sources of error or to arrange the conditions of the experiments so that the loss of conductivity due to these causes is negligible.

The present paper is divided into five sections. The first section contains an investigation of the relative importance of the processes of diffusion, recombination, and mutual repulsion in causing loss of conductivity. The descriptions of apparatus, and the results of the experiments made on ions produced by a radio-active substance, by ultra-violet light, and by point discharges, are given in Sections II., III., and IV. respectively. The conclusions to be drawn from the experiments are discussed in Section V.

SECTION I.

In the previous paper we have shown that when a number of ions, A, are uniformly distributed throughout a gas, B, which is entering metal tubing, the ratio R, of the

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* JOHN S. TOWNSEND, 'Phil. Trans.,' A, vol. 193, 1899, p. 129.

number of ions which come through the tube without touching the sides to the number which enter is—

$$R = \cdot 193 E^{\frac{-7 \cdot 31 KZ}{2a^2 V}} + \cdot 0243 E^{\frac{-44 \cdot 5 KZ}{2a^2 V}} + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where K is the coefficient of diffusion of the ions into the gas, B, Z the length of the tube, a its radius, and V the mean velocity of the gas, B, along the tube. This value of R is obtained on the supposition that the velocity, W, of the gas, B, at a distance r from the axis of the tube is given by the formula $W = 2V(a^2 - r^2)/a^2$.

Let R_1 and R_2 be the values of R equation (1) corresponding to lengths Z_1 and Z_2, then $y = R_1/R_2$ is the ratio of the number of ions which come through tubes of lengths Z_1 and Z_2, the same uniform distribution entering each.

The ratio y can be easily determined experimentally, and the value of K can be found from the formula—

$$y = R_1/R_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

if the loss of conductivity arises principally from diffusion.

Before we proceed to estimate the loss due to recombination or mutual repulsion, we may here point out the advantages of having a small density of ionisation, and of using tubing of small bore.

The ratio y, equation (2), is independent of the number of ions used. It shows that the proportion of ions which are lost in passing through the tube is the same for large and small conductivities. When reduction in conductivity is caused by recombination or mutual repulsion, this law no longer holds. In both these cases the absolute loss is proportional to the square of the number present, so that the proportion of ions lost is proportional to the density of ionisation.

The ratio, y, is a function of $KZ_1/2a^2V$, when Z_1/Z_2 is constant. Let c be a value of $KZ_1/2a^2V$ which gives the value of y between $\cdot 3$ and $\cdot 6$ that can be found accurately by experiment. If the mean time, Z_1/V, during which the gas is in the longer tube be given, the radius can be selected so as to make $KZ_1/2a^2V = C$. In dealing with the corrections to be applied to y, we need only to consider the effect of recombination or mutual repulsion in the longer tube. In the apparatus which was used, $Z_1 = 4$ centims., $Z_2 = \cdot 5$ centim. The processes which give rise to errors take place during the time Z_1/V, and the rate at which they destroy the conductivity is independent of the radius of the tubing. The errors therefore arising from either of the processes which we are considering can be reduced to any desired extent by reducing the density of ionisation, or reducing the bore of the tubing.

Recombination.

Among the methods of producing conductivity with which we are dealing, it is only in the case of ionisation produced by the radio-active substance that both positive and negative ions appear simultaneously in the gas. From the results of the experiments on diffusion, we are led to conclude that the ions thus produced resemble very closely those produced by Röntgen rays, and carry the same charge. We will therefore assume that the laws governing the recombination will not be much different in the two cases. The method of finding the correction for recombination has been explained in the previous paper.* It was there shown that for small conductivities the loss due to recombination was about 4 per cent. of the loss due to diffusion to the sides.

The time, Z_1/V, in the experiments made with Röntgen rays was about $\frac{1}{10}$th of a second, the radius of the tubing being 1·5 millims. A new apparatus was made with finer tubing (a = ·5 millim.), so that without altering KZ_1/a^2V, the value of Z_1/V is reduced to $\frac{1}{90}$. The number, N, of ions which recombine is similarly reduced from N to N/9.

The radio-active substance was contained in a sealed glass tube, which cut down the radiation proceeding from it so as to produce densities of ionisation less than the smallest that was used in the experiments made with Röntgen rays. We may therefore assume that in the present experiments the process of recombination does not affect the value of y to the extent of ·5 per cent.

Mutual Repulsion.

When a gas contains ions of one sign (as in the case of ions produced by the action of ultra-violet light on a metal plate, or by a point discharge), the electrostatic field arising from the electric density is sometimes sufficient to exert a considerable force. It would be difficult to find the exact amount that this effect contributes to the loss of ions in a tube while diffusion is taking place, but it is easy to find an upper limit to the error it introduces.

Let us consider the case of a charged gas in a metal tube losing its electrification owing to the motion of the ions along lines of force from the axis to the surface. If we suppose that no diffusion is taking place, it is easy to show† that the density of electrification at any point is given by the formula—

$$\rho = \frac{\rho_0}{1 + 4\pi u\rho_0 t},$$

ρ_0 being the initial density, supposed uniform, u the velocity of an ion when acted on by unit electrostatic force, and t the time during which the density falls from ρ_0 to ρ.

The proportion of ions lost, $\frac{\rho_0 - \rho}{\rho_0}$, is practically $4\pi\rho_0 ut$ when the loss is small.

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* *Loc. cit., supra*, p. 144.

† JOHN S. TOWNSEND, 'Phil. Mag.,' June, 1898.

If $t = \frac{1}{90}$, $\rho_0 = 10^{-4}$ electrostatic unit, and $u = 450$ centims. per second, then $\frac{\rho_0 - \rho}{\rho_0}$ becomes ·006. This fraction is bigger than the correction to be applied to y, as can be seen by taking a simple example. Consider a case where 50 per cent. of ions are lost in $\frac{1}{90}$th of a second when diffusion is taking place in a gas having a mean electric density 10^{-4} electrostatic unit. There are less ions per cub. centim. in the gas near the surface than in the gas near the centre, so that the effect of mutual repulsion must be much less than the above estimate, which is made on the supposition that the distribution is uniform. We may safely consider that when $\rho_0 t$ is less than 10^{-6}, the effect of the charge does not introduce an error of 1 per cent.

SECTION II.—*Ions produced by a Radio-active Substance.*

The apparatus which was used for experiments with a radio-active substance is shown in fig. 1. It consists of a large brass tube, A (60 centims. long, 3·5 centims. in diameter), and two smaller tubes, B_1 and B_2 (16 centims. long), which fitted lightly into A. The tubes B had brass electrodes, E, supported by brass rods, F, which passed through ebonite plugs, G, in the tubes B.

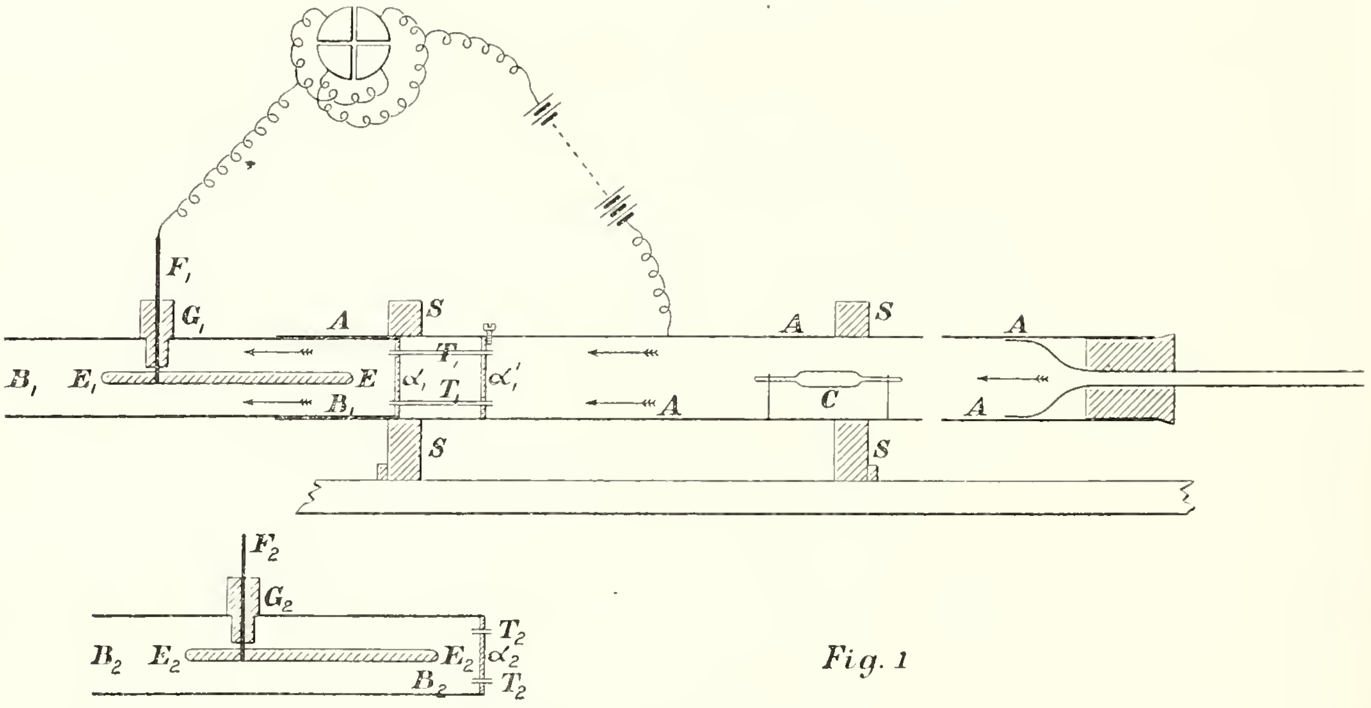


Fig. 1

The fine tubes in which the diffusion takes place consisted of two sets of twenty-four each. The longer set T_1 (4 centims. long and 1 millim. internal diameter) passed through holes in two brass discs, α_1 and α'_1, which fitted exactly into the large tube A. The tubes T_1, two of which are represented in the figure, projected 2 millims. from the discs at either end. The set of short tubes T_2 were passed through holes in a disc α_2, and projected 2 millims. on each side of the disc, so that the conducting air should enter each set under exactly the same conditions. The holes in the discs through which the tubes T passed were arranged on a circle of 2 centims. diameter, concentric with the boundary of the disc. When either set of tubes was pushed into A, a stream of air down the large tube divides itself equally among the twenty-four channels provided by the tubes T. The disc α_1 was soldered to the front of the tube B_1, and the disc α_2 to the front of the tube B_2.

When working with air at atmospheric pressure the stream of air was obtained by putting weights on the movable cylinder of a gasometer. For experiments with dry air the delivery tube of the gasometer was connected to wide tubes of calcium chloride, and a tube tightly packed with glass-wool was put between the drying tubes and the tube leading to A, so that particles of dust should not be carried into the diffusion apparatus. When it was desired to make experiments with moist air, the calcium chloride tubes were removed, and long tubes half filled with water were substituted. The velocity of the air along the tubes T could be varied by changing the weights on the gasometer.

The radio-active substance was obtained from E. de HAËN (Chemische Fabrik, List vor Hannover), and the preparation labelled "Radio-active Substance A" was used. A tube, C, of thin glass, containing some of the radio-active substance, was held by means of wire supports inside the tube A as shown in the figure. The radiation given out by the active substance was much more intense than uranium rays, and after passing through the glass tube was strong enough to ionize the surrounding air. The tube C was sealed in order to prevent any moisture from coming into contact with the radio-active substance, which was deliquescent. The tube A was fixed rigidly by ebonite supports, S, to the top of a heavy box, so that the tube C should not get shaken when the tubes B_1 and B_2 are fixed in position.

The tube A was connected to one terminal of a battery of forty lead cells, the other terminal being to earth. The rod F was connected to one pair of quadrants of an electrometer, the other pair of quadrants and the case of the electrometer being connected to earth. The rod F and the wire connecting it to the electrometer were surrounded by metal screens, so that external electric charges should not give any deflection on the electrometer scale.

Since the tube A is in metallic connection with all the parts of the diffusion apparatus except E, there is no electric force acting on a stream of gas until it comes out of the tubes T into the space between E and B. The air takes about one second to pass the electrode, and a difference of potential of a few volts between E and B would, under ordinary conditions, suffice to collect all the ions of one sign on E, but owing to the turbulent motion of the gas as it escapes from the tubes T, a much greater potential difference (80 volts) was used. The potential of the electrode during an experiment never exceeded 1 or 2 volts. It was found under similar conditions of ionisation and velocity of air that the electrometer deflection was not altered by charging A to 40 volts instead of 80. We therefore conclude that all the ions of one sign are collected on E, so that the electrometer deflection is proportional to the number of ions that come through the tubes T.

Methods of Conducting the Experiments with Air at Atmospheric Pressure.

The tube B was moved into A until the disc α_1' came up to a small screw that projected into A, and the rod F_1 was connected to the insulated quadrants of the electrometer. The tube B_2 was connected to the end of B_1 by means of a piece of tubing the same size as A. Before any observations were made the stream of air was allowed to pass for one minute through the apparatus in order to blow out the ions that accumulate in the air in A. The quadrants (connected to F_1) were then insulated, and the air, being ionised as it passes the tube C, carries some of the ions with it through the tube T_1. The electrometer deflection, n_1 divisions per minute, was then observed.

The positions of the two tubes B_1 and B_2 were then interchanged. The resistance to the passage of the air through the apparatus was unaltered, so that the stream of gas that passed through was the same as when the deflection n_1 was being observed. The rod F_2 was connected to the quadrants of the electrometer and the same observations were made, the electrometer deflections being n_2 divisions per minute. These experiments were repeated several times, and it was found that the numbers n_1 and n_2 were constant, showing that the rays emanating from the tube C did not vary with the time to any appreciable extent.

When the quadrants were insulated and the air inside A at rest, a small deflection (usually one division per minute) was obtained arising from the imperfect insulation of the plug G. This deflection has to be subtracted from the deflections obtained when the stream of air is passing along A. The leak across G gives rise to no inaccuracy, as it is perfectly constant and was easily determined. In the tables of observations the corrected values of n_1 and n_2 are given.

When the tube A is charged positively the deflections n_1 and n_2 refer to positive ions; similar numbers for negative ions were obtained by changing the sign of the potential of A.

Experiments at Lower Pressures.

In order to make experiments with air at pressures lower than the atmospheric pressure, it was found necessary to make a slight change in the diffusion apparatus. The experiments were made with the ions produced by the radio-active substance, as it is a source of very constant radiation. A flat ring was soldered to the end of the tube A, and similar rings were soldered outside the tubes B_1 and B_2. The ring on the tube B_1 was 8 centims. from the disc α_1', and the ring on the tube B_2 8 centims. from the disc α_2. The diffusion apparatus was made sufficiently air-tight by greasing the ring on A, and pressing the ring on B against it.

The arrangement of the apparatus for obtaining a stream of air at low pressure is shown in fig. 2. Rubber stoppers with glass tubes leading from them were fitted into the open ends of the tubes B_1 and B_2. A short brass tube L was soldered near the end of the tube A, and was connected to the manometer M, so that the pressure of the air in A could be found. The air from the room was admitted to the apparatus through capillary tubes K, having first passed through a tube of glass-wool G, to prevent the admission of any dust which might alter the resistance of the tubing. The capillary tubing was connected to the drying tubes, and the rest of the connections were made with wide tubing. The tube u leading from B was connected to two large vessels W_1 and W_2, which were exhausted by means of a water-pump.

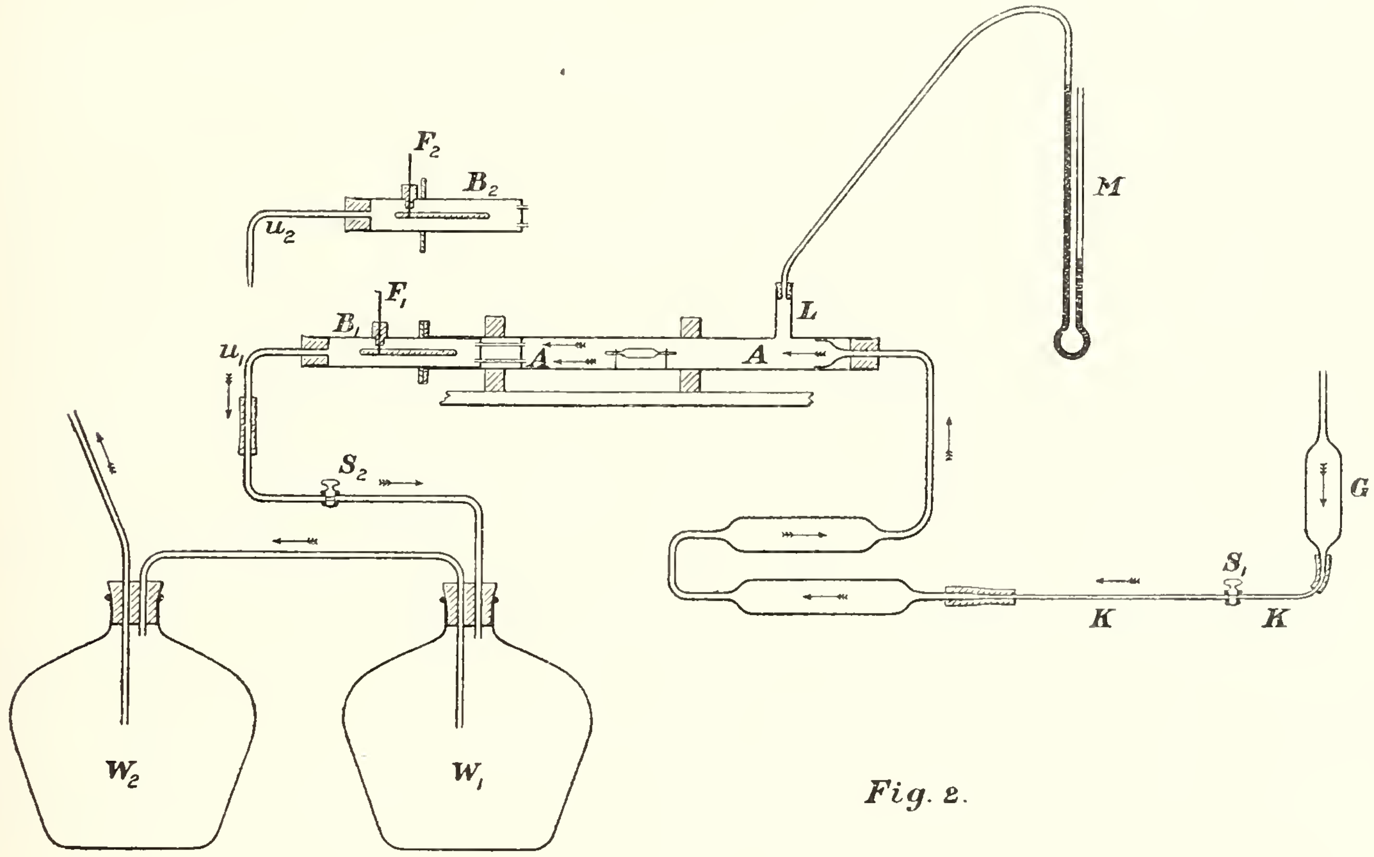


Fig. 2.

In order to obtain a stream of air through A at a given pressure P, and with a velocity between suitable limits, the stopcock S_1 was closed and the whole apparatus was exhausted until the pressure was a few millims. below the pressure P. The tube G was connected to the delivery tube of a gasometer, the movable cylinder of which was adjusted so as to be on the point of moving downwards when the gasometer was open to the air. The vessel W_2 was then connected with the water-pump, and S_1 was turned on for a few minutes. The velocities V (through the tubing T) which were desirable necessitated a larger supply through the apparatus than could be taken out by the water-pump, so that the pressure, as shown by the manometer, gradually rose at the rate of about 3 millims. per minute. The stopcock S_1 was turned off when the pressure was as much above P as it was below P at the beginning of the experiment. The velocity V in the tubes T can be accurately found by observing the volume of air that escaped from the gasometer and the time during which the stopcock S_1 was open. If the velocity so found was too big or too small, the length of the capillary tubing was changed so as to bring the velocity V within the required limits. Any alteration in the tubing leading into the diffusion apparatus necessitates a fresh determination of V before experiments on the conductivity are made.

The tube G is connected to the gasometer only for the purpose of determining V. Since the pressure inside the gasometer is the same as the atmospheric pressure, the air from the room may be allowed to enter G directly during experiments on the conductivity.

The electrical arrangements were the same as have been already described.

Method of Conducting the Experiments with Air at Low Pressures.

The apparatus is arranged as shown in fig. 2, and the pressure is reduced until it is a little lower than the pressure P at which the experiment is to be made. The stopcock S_1 is turned on and the air is allowed to run through the apparatus for about a minute before the quadrants, to which F_1 is connected, are insulated. When the manometer shows that the pressure is about 2 millims. lower than P, the quadrants are insulated and the deflection n_1 divisions per minute on the electrometer scale is observed. The observations are continued until the pressure is about 2 millims. above P, and the mean taken. The difference between the deflections in the first and last half minute was scarcely perceptible.

Having determined n_1, the stopcock S_2 was closed, and the air is admittted into the diffusion apparatus through S_1, the tube B_1 was then removed and B_2 put in its place. Before making observations with F_2 connected to the electrometer, it is necessary to test whether the joint between the two discs is air-tight. For this purpose S_1 was closed and S_2 opened so as to let some of the air from the diffusion apparatus into W_1. The stopcock S_2 was then closed and the manometer was observed. It was found that the air did not get into the apparatus at one-thousandth the rate at which it entered when S_1 was open.

The determination of n_2 is then made in the same way as n_1. Particular care was taken in all cases to make the observations over the same part of the electrometer scale.

Generally the deflection n_2 was obtained from two half-minute observations. Since n_1 is much smaller than n_2, its value was taken as the mean of a number of observations made while the spot of light was passing the same part of the scale as was used for the determination of n_2.

The following tables give the results of experiments at different pressures. Tables I. and II. refer to positive and negative ions respectively in dry air. Tables III. and IV. are the corresponding observations for moist air. The numbers in the columns n_1 and n_2 are the deflections obtained per minute; P is the pressure of the air in A in millims. of mercury ; V is the mean velocity in the tubing T_1 in centims. per second ; Θ is the temperature of the air during the experiment :—

TABLE I.—Positive Ions in Dry Air.

n_1.	n_2.	V.	P.	θ.
88·2	153	344	772	19
56·1	105·3	387	550	13
33·9	73·9	420	400	16
14·8	41·3	410	300	13
11·5	34·5	582	200	12

TABLE II.—Negative Ions in Dry Air.

n_1.	n_2.	V.	P.	θ.
63·4	138·6	344	772	19
43·0	93·8	387	550	13
24·8	68·0	420	400	16
10·6	39·9	410	300	13
7·6	31·5	582	200	12

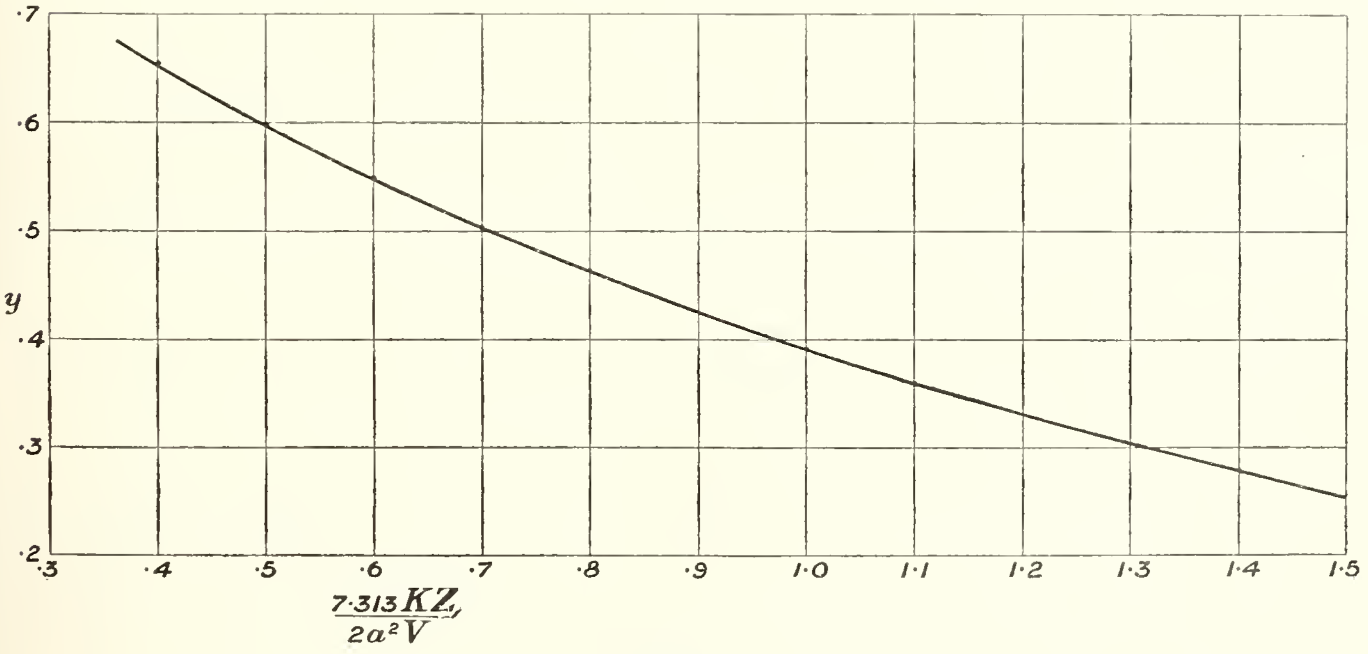
TABLE III.—Positive Ions in Moist Air.

n_1.	n_2.	V.	P.	θ.
81·3	145·8	368	772	18
24·7	58·5	430	400	11
9·5	31·2	609	200	9·5

TABLE IV.—Negative Ions in Moist Air.

n_1.	n_2.	V.	P.	θ.
71·1	135	368	772	18
21·0	56·3	430	440	11
7·6	27·1	609	200	9·5

The connection between the ratio y $(= n_1/n_2)$ and the coefficient of diffusion can be shown graphically by means of a curve representing equation 2, Section I. The



accompanying curve has for co-ordinates y and $\frac{7 \cdot 31\, KZ_1}{2a^2V}$, and from it we can deduce the values of K corresponding to the different pressures. These values are given in the following tables:—

TABLE V.—Positive Ions, Dry Air.

P.	K.	P × K.	θ.
772	·0317	24·5	19
550	·0420	23·1	13
400	·0578	23·1	16
300	·078	23·4	13
200	·118	23·6	12

TABLE VI.—Negative Ions, Dry Air.

P.	K.	P × K.	θ.
772	·0429	33	19
550	·0542	29·8	13
400	·078	31·2	16
300	·103	30·9	13
200	·155	31·0	12

TABLE VII.—Positive Ions, Moist Air.

P.	K.	P × K.	θ.
772	·0364	28·0	18
400	·0668	26·7	11
200	·134	26·8	9·5

TABLE VIII.—Negative Ions, Moist Air.

P.	K.	P × K.	θ.
772	·0409	31·5	18
400	·0771	30·8	11
200	·147	29·4	9·5

These tables show that in each case the rate of diffusion of ions into a gas is inversely proportional to the pressure of the gas.

The coefficients of diffusion at 772 millims. show a discrepancy from this law which is somewhat greater than the probable error of the experiments, but we should not expect closer agreement between the products P × K unless the temperature of the air was the same in each case. It will be noticed that the experiments at 772 millims. pressure were made when the temperature of the air was higher than the temperature during the other experiments.

SECTION III.—*Ions produced by the Action of Ultra-violet Light on a Metal Surface.*

The apparatus which is described in Section II. can, with slight alterations, be used for ions produced by various methods. In order to distribute ions produced by ultra-violet light in the stream of air passing along the tube A, the changes shown in fig. 3 were made. The windows W_1 and W_2 were cut in the long tube, and two short pieces, S_1 and S_2 were saddled on to it, each of them surrounding one of the windows. A quartz-plate, Q, was fixed to the end of S_1 by means of sealing wax, which made the joint air-tight, and a piece of wire gauze, having the same curvature as A, was placed in the window W_1, completely filling it. A piece of zinc, Z, of the same shape and size as the piece of brass which was cut out of the window W_2, was fixed to a brass rod, R, which passed through the ebonite disc D. The disc fitted tightly into the end of S_2, and the joint was made air-tight. The zinc did not touch the tube A, so that its potential relative to A could be varied as desired. When ultra-violet light passes through the quartz and the gauze, it falls on the zinc, and negative ions are produced at the surface of the metal. Some of these ions can be sent into a stream of air passing along A by lowering the potential of the zinc relative to A. For this purpose a small battery, H, was insulated and its positive terminal was connected to A and its negative terminal to R.

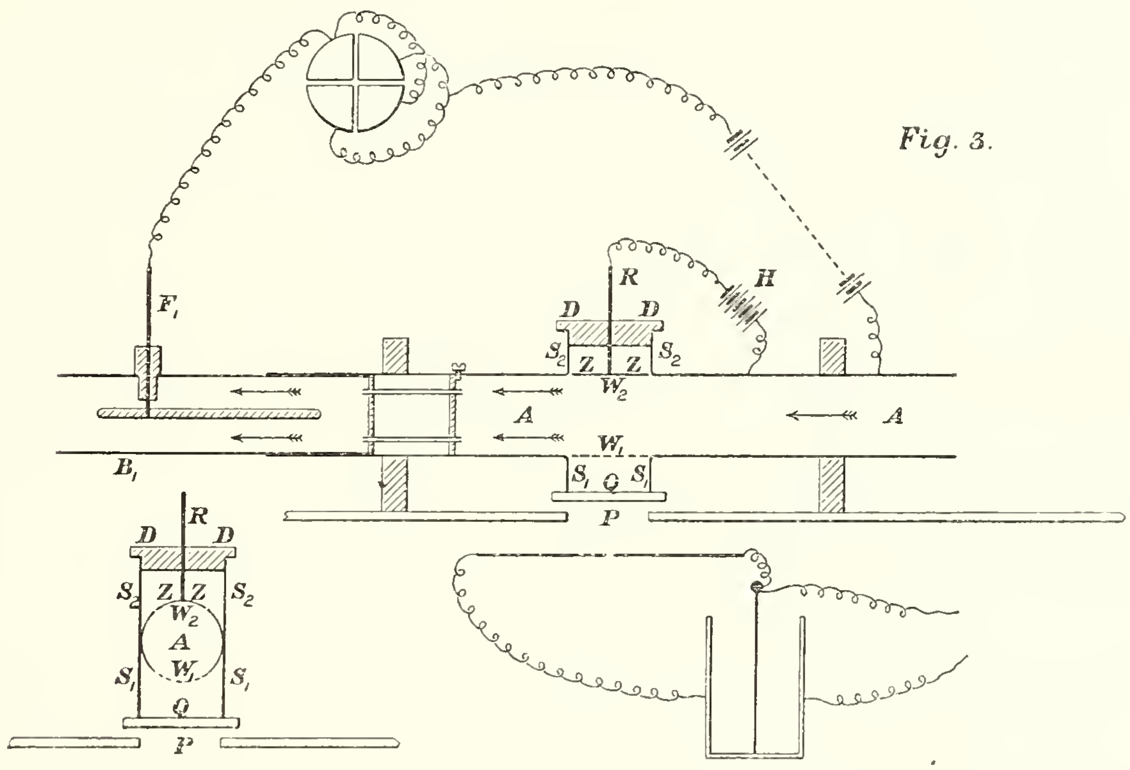


Fig. 3.

A spark between two aluminium wires was used as the source of ultra-violet light. The apparatus for producing the spark was contained inside a box covered with lead, with a small opening at P_1 through which the light from the spark fell on the quartz-plate Q. One of the terminals of the secondary of a Ruhmkorff coil was connected to the outer coating of a Leyden jar, and the other terminal to the inner coating. The jar was charged by the coil, and the discharge took place across the spark-gap formed by the two aluminium wires. The air in the neighbourhood of the spark becomes positively electrified, so that it was found necessary to pack wool round the tube S_1 to prevent the electrified air from coming into the neighbourhood of the rod F and the wire connecting it to the electrometer. When this precaution was taken it was found that the electrometer gave no deflection when the coil was turned on for several minutes.

Method of Conducting Experiments with Ultra-Violet Light.

The small battery of Clark cells (H, fig. 3) was insulated, and its terminals were connected to R and A so as to make the potential of R about 6 volts less than A. The tube A was connected to the negative terminal of a battery of 40 lead cells, the other terminal being to earth. The tubes B and the electrometer connections were arranged in the same way as when experimenting with the radio-active substance. The observations were made in a slightly different manner.

The quadrants connected to F were insulated and the stream of air allowed to run through the apparatus. About fifteen seconds were allowed for the stream of air to become constant; the coil working the spark gap was then turned on for a fixed time (twenty seconds generally). The electrometer deflection can be read when the spot of light becomes steady, which is an advantage of this method. When the coil was not working the electrometer reading did not vary more than ·5 division per minute, and was not affected by letting the stream of air pass through the apparatus.

The deflections n_2 (obtained in a similar manner when the ionized air passes through the short tubes T_2) were about twice as big as the deflections n_1, so that the latter observations were made twice in order to cover the same part of the scale.

The deflections obtained when the rod R was connected directly to A were about $\frac{1}{10}$th of the deflection obtained when the zinc plate was 6 volts negative compared with A.

The potential of A was changed to 80 volts positive, and the zinc plate made positive with respect to A, and no deflection was obtained on allowing the coil to run for two minutes.

The following tables give the deflections obtained with different velocities V through the tubes T. The numbers n_1 and n_2 are deflections per minute, and the coefficients of diffusion K were deduced from the curve, Section II. The air in these experiments was at atmospheric pressure H. The temperature only varied from 16° to 18° during the experiments, so that the rates of diffusion may be taken as corresponding to a temperature of 17° centigrade:—

TABLE IX.—Dry Air.

H.	n_1.	n_2.	V.	K.
761	32·8	65·1	356	·0427
748	44·0	76·2	475	·0438
761	55·7	109·3	377	·0440

TABLE X.—Moist Air.

H.	n_1.	n_2.	V.	K.
762	45·0	81·1	368	·0368
772	47·0	90·0	337	·0380

The density of ionization depends greatly on the state of the zinc surface, which has to be polished from time to time in order that the ionization should not be too small. In the above experiments the greatest density of electrification occurs in the third experiment with dry gas. The mean time, t, spent by any portion of the gas in the tubes T_1 is $\frac{4}{377}$ second. The total volume of gas that passed through the apparatus per minute was 4260 c.c.

On standardising the electrometer, it was found that each scale division corresponded to a charge of ·0044 electrostatic unit. The mean density of the electrification ρ was therefore $8 \cdot 5 \times 10^{-5}$ electrostatic unit per c.c. We have shown in Section I. that the product $\rho \times t$ must be less than 10^{-6} in order that the loss of ions due to self-repulsion should be less than 1 per cent. of the loss due to diffusion to the sides. In the present case the product $\rho \times t$ is $\cdot 9 \times 10^{-6}$, so that no correction need be made for the loss due to self-repulsion.

SECTION IV.—*Ions produced by the Point Discharge.*

In order to make the apparatus suitable for experimenting with the point discharge, the changes shown in fig. 4 were made. Two circular holes (1·6 centims. in diameter) were made in the tube A, and two tubes, Q and R, of the same diameter as

Fig. 4.

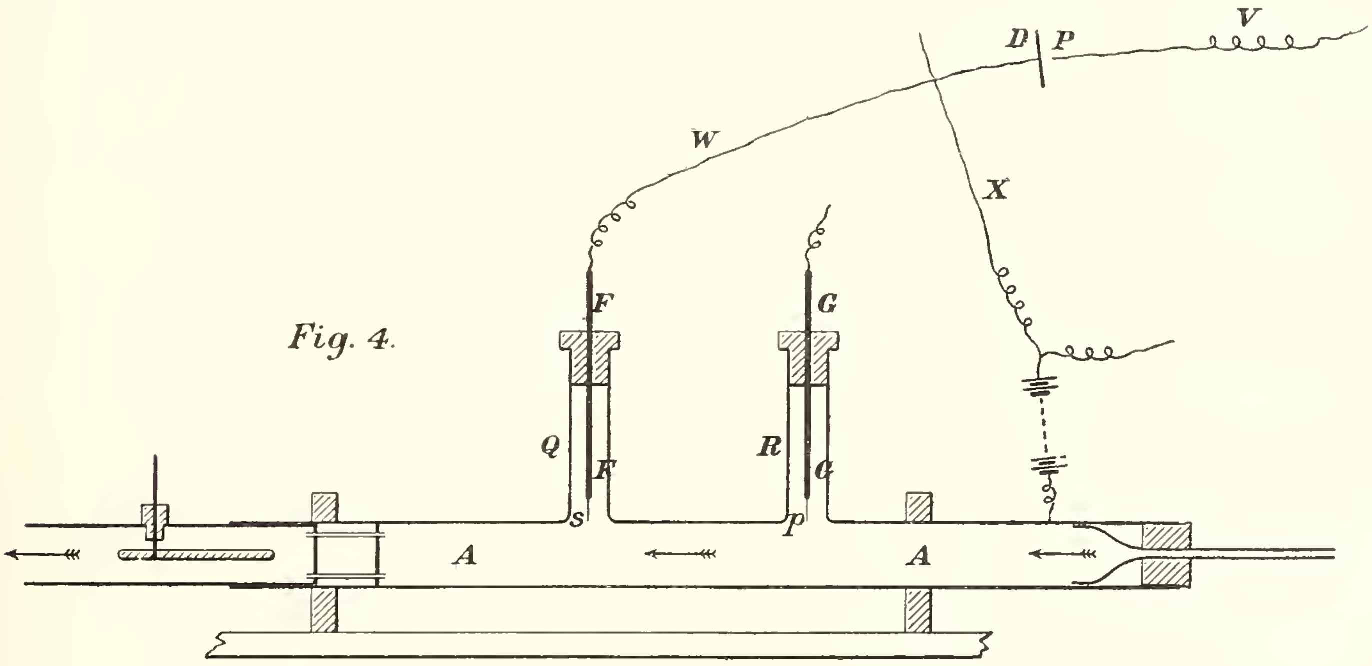
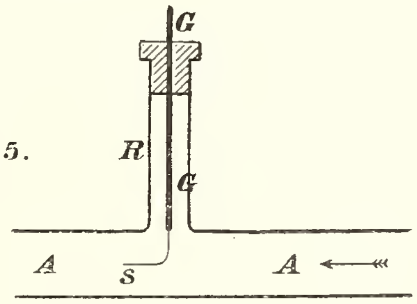


Fig. 5.



the holes were soldered to A in the positions shown in the figure. Ebonite plugs were fitted tightly into Q and R so that no air should escape by the side tubes. The brass rods F and G passed through holes in the plugs, and could be moved up and down so as to bring the metal points which were soldered to their ends to any desired position in the tubes. The point S at the end of F was a steel needle, and the point *p* was a short platinum wire.

The potential of the point was raised by means of a Wimshurst machine driven at a constant speed by an electric motor. One of the conductors of the machine was connected to earth and the other to the wire V, which terminated in a point P at a distance of about 5 millims. from the metal disc D. The disc was connected to one of the rods F or G by the insulated wire W. When the Wimshurst machine was working no discharge took place from the point inside the diffusion apparatus while the earth-connected wire X rested on W. When the earth-connected wire was raised off W, a discharge immediately took place from the point at the end of the rod F. A constant discharge could therefore be obtained inside the tube for any desired time, independent of the initial and final variations of the machine.

Most of the electricity discharged from the point inside A goes to earth through the battery, and only a small fraction is carried by the stream of air along the tube A.

Owing to the charges carried about in the air of the room (emanating partly from P and partly from points of the machine), it was found that gauze screens were not sufficient protection for the wire leading from F to the insulated quadrants; it was found necessary to cover the screens with tinfoil. When this precaution was taken the electrometer showed no deflection when the Wimshurst had been working for several minutes.

The experiments were conducted practically in the same manner as the experiments on ultra-violet light. The following tables give the results of the experiments, the numbers in the columns n_1, n_2, V and K having the same signification as in the previous tables :—

TABLE XI.—Positive Ions in Dry Air.

Expt.	H.	n_1.	n_2.	V.	K.
1	766	162	262	324	·0263
2	760	81	129	334	·0251
3	761	101	150	378	·0245
4	754	89·4	142	329	·0247
5	753	180·6	299	324	·0257
6	767	52·4	77·4	342	·0216

Experiment 1.—Made with steel point in the tube Q, the point being at the aperture in the tube A.

Experiments 2 and 3.—Made under same conditions as experiment 1, except that the tube R was used in order that the gas should have a smaller electrification when it reaches the tube T.

Experiment 4.—Same as experiments 2 and 3, with a platinum point substituted for the steel point.

Experiment 5.—The point was held in the tube A in the position shown in fig. 5.

Experiment 6.—Same as experiment 1, except that the point was drawn up the side tube Q 5 centims. from the aperture in A.

Experiments 1 and 5 are the only ones in which the effect of self-repulsion may contribute appreciably to the loss of ions in the tube T_1, so that the values of K deduced from these experiments may be a little too big.

The difference of about 5 per cent. which occurs between the values obtained in experiments 1 and 5 and the values obtained in experiments 2, 3, and 4, is probably due to this effect and not to any difference in the ions.

The ions which get into the stream of air in A from a point some distance up the side tube are larger than the others, as Experiment 6 shows that they diffuse more slowly.

TABLE XII.—Negative Ions in Dry Air.

Expt.	H.	n_1.	n_2.	V.	K.
1	768	72	138	337	·0382
2	766	86·2	165	326	·0367
3	758	78·5	150	323	·0368
4	767	91·2	160	342	·0324

Experiment 1.—Made with a steel point in the tube Q, the point being at the aperture in the tube A.

Experiment 2.—Same as Experiment 1, with a platinum point substituted for the steel needle.

Experiment 3.—The point was held in the tube A, as shown in fig. 5.

Experiment 4.—Same as Experiment 1, except that the point was drawn up the tube Q 2 centims. from the aperture in A.

The first three experiments give practically the same values for the coefficient of diffusion, but the fourth experiment shows that larger ions are produced when the point discharge takes place in the narrow tube Q.

TABLE XIII.—Positive Ions in Moist Air.

Expt.	H.	n_1.	n_2.	V.	K.
1	765	44·1	74·7	318	·0277
2	750	71·0	112	396	·0291
3	750	60	100	323	·0271

Experiment 1.—Made with a steel point in the tube R, the point being at the aperture in A.

Experiment 2.—Made with a platinum point in the tube Q (in order to get larger deflections).

Experiment 3.—Steel point in tube Q 3 centims. from the aperture in A.

The effect of drawing the point up the tube has not so much effect in this case as when the air is dry.

TABLE XIV.—Negative Ions in Moist Air.

Expt.	H.	n_1.	n_2.	V.	K.
1	774	56·4	113·6	321	·0395
2	750	66·7	121·5	396	·0399
3	774	45·8	97·4	321	·0376

These three experiments were made with the discharge from the wire in the side tube Q. In Experiments 1 and 2 the point was at the centre of the aperture in A. steel and platinum points being used in the two cases respectively. The third experiment was made with a steel point drawn up the tube Q 2 centims. from the aperture in the tube A.

Considering the experiments made with the point at the junction of Q and A, it will be seen that the electrification (n_2/V in arbitrary units) is greatest for those ions which move the slowest. This arises partly on account of the loss of charge in the tube A before the gas reaches the tubes T. If equal electrifications were produced at the source, we would expect less of the small ions than of the large ions to reach the tubes T with the gas.

Section V.—*Effect of Pressure.*

The theory of the interdiffusion of gases shows that the coefficient of diffusion is inversely proportional to the total pressure of the two diffusing gases. This law has been confirmed by the experiments of Loschmidt and others.* The results given in Section II. show that the law can be extended to the case where one of the gases consists of ions. The pressure that the ions exert is so small that it does not contribute to the total pressure by an amount which could be measured. The total pressure in this case is the pressure of the gas into which the ions are diffusing, and we see that between the pressures 772 and 200 (millims. of mercury) the rate of diffusion is inversely proportional to the pressure.

We conclude from this that the size of an ion does not change when the pressure varies between these limits.

Ions produced by Various Methods.

The experiments on diffusion show that the ions produced by Röntgen rays, radio-active substances, and ultra-violet light are nearly of the same size, and subject to the same changes arising from the presence of moisture. The following table of coefficients of diffusion of ions into air shows that there are differences in the various cases which are greater than what might arise from experimental errors.

The ions produced by the point discharge are larger than those produced by the other methods, since their rate of diffusion is much slower, except in the case of negative ions in moist air.

Coefficients of Diffusion of Ions produced in Air by different Methods.

Method.	Dry air.		Moist Air.	
	Positive ions.	Negative ions.	Positive ions.	Negative ions.
Röntgen rays	·028	·043	·032	·035
Radio-active substance	·032	·043	·036	·041
Ultra-violet light	—	·043	—	·037
Point discharge	·0247 ·0216	·037 ·032	·028 ·027	·039 ·037

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* Meyer, 'The Kinetic Theory of Gases,' Chap. VIII.

We shall first examine the relative values of the charges on the ions in the different cases. For this purpose a knowledge of the velocities of the ions through air, when acted on by a known electric force, is required.

If N is the number of molecules per c.c. of a gas at atmospheric pressure (10^6 dynes per square centim.), e the charge on the ion of the gas in electrostatic units, u the velocity of the ion in a field of 1 volt. per centim., and K the coefficient of diffusion of ions into the gas; then—

$$N \times e = \frac{3 \times 10^8 u}{K}.$$

If E is the charge on a hydrogen ion in a liquid electrolyte, then $N \times E = 1{\cdot}22 \times 10^{10}$, E being expressed in electrostatic units.

From these formulæ we see that when u and K are known, we can find e in terms of E.

It has already been shown in this manner, in the case of Röntgen rays, that the charge carried by the positive and negative ions is in all cases very nearly the same as the charge E, the differences not being greater than possible experimental errors. In order to make a similar investigation for ions produced by other methods, a complete knowledge of the velocities u would be necessary, but it is only in a few cases that these velocities have been found.

The value of u for ions produced by ultra-violet light given by Professor RUTHERFORD is 1·5 centims. per second in air at atmospheric pressure.* If we take the mean of the two values of K which we have found for dry and moist air, we find that in this case $Ne = 1{\cdot}12 \times 10^{10}$. The effect of moisture on the velocity under an electric force has not been examined, so that we cannot expect to obtain a nearer agreement with the electrolytic value of NE.

The velocities of the ions produced by point discharges have been investigated by Professor CHATTOCK.† The values given for the positive and negative ions are 1·37 and 1·80 (centims. per second in a field of 1 volt per centim.) respectively. Other values were also obtained smaller than these, which would agree better with the observed coefficients of diffusion, but Professor CHATTOCK is of opinion that the low values he obtained for the velocities are less reliable, owing to experimental errors.

The arrangement of the apparatus for the determination of the velocities was such that the point was not surrounded by a narrow tube, so that the larger values obtained for the coefficient of diffusion may be taken in conjunction with the above values of the velocities. The numbers so obtained for Ne are $1{\cdot}66 \times 10^{10}$ and $1{\cdot}46 \times 10^{10}$ for the positive and negative ions respectively.

These results would seem to show that some of the ions carried a double charge. but we cannot attach much importance to the above numbers, since the coefficients of

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* E. RUTHERFORD, 'Cambridge Philosophical Society Proc.,' vol. 9, Part VIII., 1898.

† A. P. CHATTOCK, 'Phil. Mag.,' Nov., 1899.

diffusion show plainly that the sizes of the ions produced by point discharges vary with the arrangement of the apparatus in the neighbourhood of the point. What is necessary in order to come to a definite conclusion with regard to the charges is to examine the coefficients of diffusion and the velocities of ions produced under similar circumstances. This would be possible with the apparatus I have used for the determination of the coefficients of diffusion, and I hope to be able to make observations on the velocities which will lead to an accurate determination of Ne.

This method of obtaining the charge on an ion in a gas in terms of the charge on an ion in a liquid electrolyte is of some importance, as it enables us to obtain evidence of the atomic nature of electricity.

The results show that there is a similarity between the minimum subdivisions of electric charge in liquids and in gases.

The methods hitherto used for the determination of the charge in absolute units apply to ions produced in moist gases, and since all the determinations depend upon the rate at which a cloud falls, great accuracy cannot be expected. The results show that the charge on the carrier is of the same order for ions obtained by various methods. The charges have been obtained by Professor J. J. THOMSON for ions produced by Röntgen rays,* and by ultra-violet light;† the values are nearly the same, being between 6×10^{-10} and 7×10^{-10} electrostatic unit. These values do not differ very much from the value 5×10^{-10} which I obtained for the charge on the carrier in the charged gases given off by electrolysis.‡

If we consider that the charge is the same in all cases, we must assume that the mass surrounding the ion varies in order to account for the differences observed in the coefficients of diffusion. MCCLELLAND,§ by examining the velocities of the ions produced by an arc and by glowing wires, found that the mass attached to the ion depends to a great measure on circumstances connected with the ionization. The velocities undergo large changes for small differences of temperature of the wire, showing that the mass which collects round an ion is very variable. We would not expect that Röntgen rays or radio-active substances would have an effect upon the air, which would alter its tendency to collect round a charged ion, but it is possible that ionization is produced in different ways by different kinds of rays, so that the masses are not identical. With point discharges in air there are actions taking place which would tend to make the carrier increase in size. Thus the oxides of nitrogen which are formed might condense round the charge and lower the rate of diffusion by increasing the mass of the ion.

It is uncertain whether ultra-violet light has any effect on dry air, but WILSON‖

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* J. J. THOMSON, 'Phil. Mag.,' Dec., 1898.

† J. J. THOMSON, 'Phil. Mag.,' Dec., 1899.

‡ J. S. TOWNSEND, 'Phil. Mag.,' Feb., 1898.

§ J. A. MCCLELLAND, 'Camb. Phil. Soc. Proc.,' vol. 10, Part VI.

‖ C. T. R. WILSON, 'Phil. Trans.,' A, vol. 192, 1899.

has shown that if the light acts on moist air for some minutes, a cloud begins to appear. This effect must be very small in the present experiments, but it may account for the difference between the rates of diffusion of ions produced by Röntgen rays and ultra-violet light.

In conclusion, I must express my thanks to Professor THOMSON for many valuable suggestions in connection with this research.

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VIII. *The Crystalline Structure of Metals.* (Second Paper.)

By J. A. Ewing, *F.R.S., Professor of Mechanism and Applied Mechanics in the University of Cambridge, and* Walter Rosenhain, *B.A., St. John's College, Cambridge,* 1851 *Exhibition Research Scholar, University of Melbourne.*

Received July 12, 1900—Read, in Abstract, May 31, 1900.

[Plates 3—13.]

The investigations described in the present paper deal principally with the phenomena of annealing. They form a continuation of the research described in the Bakerian Lecture for 1899 ('Phil. Trans.,' A, vol. 193, 1900, pp. 353–377). In iron, steel, and brass these phenomena have been studied with the aid of the microscope by various workers, among whom Arnold, Charpy, Stead, and Roberts-Austen should be particularly mentioned. As a result of their labours it is well known that annealing is accompanied by a re-arrangement of the crystalline grains of the metal.

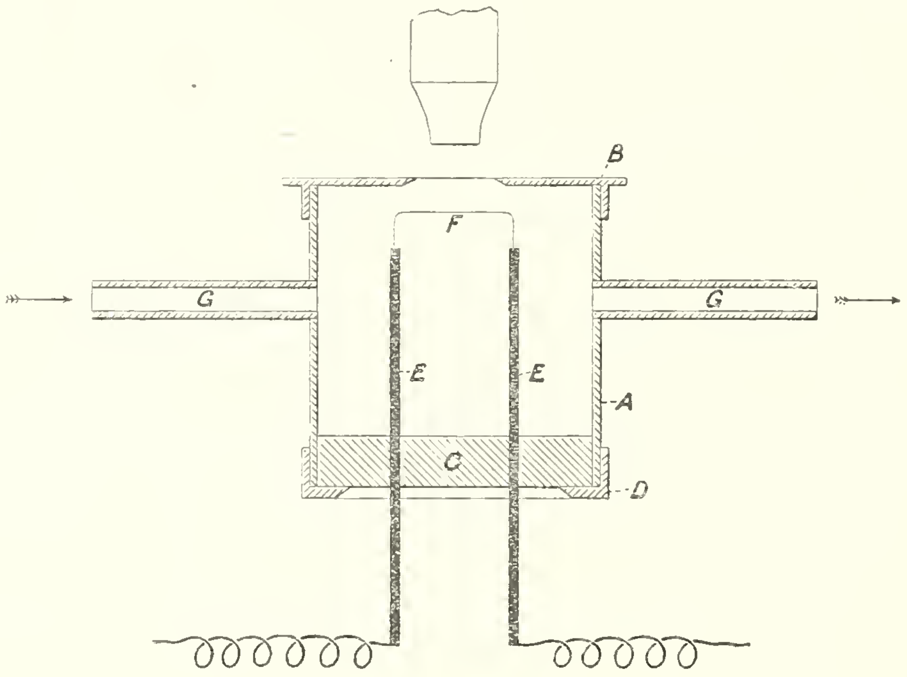
Thus, when a piece of iron is strained in tension its crystalline grains become elongated in the direction of tension; but when the specimen has been subsequently annealed by being heated to a bright red, all signs of such elongation disappear from the crystalline pattern revealed by the microscope. In fact it is not generally possible to find any definite connection between the crystalline pattern seen in the same specimen before and after annealing. In general, the pattern seen after annealing resembles that found in a similar specimen before it has been strained, but the scale and character of the pattern produced depend very much on the details of the annealing process, particularly upon the temperature applied, the time of its application, and the rate of cooling. Arnold and Stead have shown that prolonged annealing tends to produce large crystals in iron and steel. But even short exposure to a suitable temperature is well known to produce complete re-crystallisation, and it has been suggested that these changes occur at critical points corresponding to the "arrest-points" in the cooling of the metal. These arrest-points indicate evolutions of heat, and it is natural to suppose that they are evidences of re-arrangement of the structure of the metal.

At the outset of the present research we hoped to observe this change taking place under the microscope. Although the experimental difficulties of keeping a specimen under microscopic observation while it was being heated were successfully overcome, the attempt to watch the re-crystallisation of iron failed.

10.12.1900.

Our first expedient was to surround the specimen with an atmosphere of pure dry hydrogen gas, and to examine the surface through a glass or mica window in the containing vessel. The hydrogen was prepared in the usual way by the action of pure dilute sulphuric acid on pure granulated zinc, and the gas was dried and purified by bubbling through several wash-bottles containing pure sulphuric acid, caustic potash, and permanganate of potash solutions respectively. As a final precaution, it was passed through a long tube of glazed porcelain kept at a red heat and packed with clean pieces of sheet-iron. It was a specimen of similar iron that was to be experimented upon, and we hoped that after passing over this large quantity of hot iron, the hydrogen would be free from any impurities capable of tarnishing the polished surface of our specimen. The specimen itself was a strip of sheet-iron, of the pure kind used for transformer plates, measuring 3 centims. in length and 3 to 4 millims. in width. It was bent to a ⊓ shape, and the upper flat surface was polished and etched. The containing vessel is shown in section in fig. 1. It consisted of a brass cylinder, A, with a screw-cap, B, provided with a window of thin glass or mica. The bottom of the cylinder was formed by a plug of slate, C, fitting against a shoulder, and held in place by a screw-collar, D. Through this slate plug passed two stout copper wires. EE; the upper ends of these rods were split to receive the two legs of the specimen F, whose flat, polished surface thus came immediately beneath the window in the cap. At GG tubes entered the vessel; these allowed the passage of the current of hydrogen.

Fig. 1.



All the joints were very carefully fitted, and made as nearly air-tight as possible; leakage, however, was not an important matter as the hydrogen was kept at a pressure slightly higher than that of the atmosphere. On leaving the apparatus the hydrogen was burnt as a small jet.

The specimens were observed through the window by means of "vertical" illumination coming through the objective itself; as we were content with moderate magnifications (up to 100 diameters) an objective of long focus could be used.

In spite of the precautions taken, we did not succeed in keeping the polished surface untarnished after a red heat had been reached; but in the course of our observations an interesting phenomenon was observed. On beginning an observation, when the specimen was still cold, the "ferrite" grains could be clearly distinguished. If the temperature of the specimen were then slowly raised by gradually turning on an alternating electric current, the first visible change was a dimming of the image, which resulted in its becoming completely blotted out. This we supposed to be due to condensation of moisture on some part of the optical system, but we could not locate it. On raising the temperature further, the image of the crystals reappeared very vividly; at this point, by shutting off the reflected light, the metal could be seen to be just vividly red. On heating still further, the pattern was rapidly and totally obliterated by the tarnishing of the surface; the metal was now dull-red. The red then brightened, but suddenly dark spots appeared, and spread rapidly over the entire field of view. The speed at which they spread could, however, be regulated by suitably adjusting the heating current. The spots appeared well in the middle of the specimen, and the apparent darkening could only be pushed to the extreme edges by using a considerably higher temperature. On allowing the specimen to cool, no change was visible, either on passing through this range of temperature or at any other period; nor could the phenomenon be made to recur until the specimen had been cooled below redness; but, if this was done, the phenomenon would recur indefinitely in the same specimen. It seems probable that the action which causes this appearance occurs in the metal itself and not merely in the surface film of tarnish, as details in this film remain entirely unaffected by it, thus giving the observer the impression that he is looking at an action taking place beneath a thin and partially transparent film. On repeating these observations with the specimen maintained in other atmospheres than hydrogen, no such phenomenon was observed; and this leads us to suppose that the phenomenon is a result of chemical reaction between the hydrogen and the iron. From its occurrence just above redness it would seem to correspond to the arrest-point, about 487° C., discovered by Sir W. ROBERTS-AUSTEN ('Alloys Research Report,' Inst. Mechan. Engin., 1899).

Believing that this action of the hydrogen caused the surface of our specimens to tarnish on heating, we sought to overcome this difficulty by observing the surface of a specimen heated in vacuo. The heating was again done electrically, either by passing the current directly through the specimen, or else by placing the specimen in the centre of a coil of fine platinum wire wound on a piece of terra-cotta. In both cases the specimen was placed in a test-tube, the electrodes passing through a sealed cork at the end; the tube could be exhausted by means of a Fleuss pump. The specimen was placed with its polished and etched surface as near to the top of the tube as possible, and a 22 millims. objective was used. When the specimen became really hot, the radiation would have endangered the objective, and this difficulty was met by keeping a blast of air directed partly upon the lower end of the objective and partly upon the surface of the test-tube. The apparatus was thus kept quite cool without in any way disturbing the distinctness of the image.

We found that with the best vacuum we could obtain, even with the addition of phosphoric anhydride drying tubes, our specimens always tarnished if kept at a red heat for any length of time. We are inclined to ascribe this to the action of gases occluded in the metal and given out on heating. The tarnishing, however, was not very rapid, and a specimen could safely be heated up to bright red once before it became noticeable. This enabled us to make observations which very clearly showed that, even with the greatest experimental refinement, we could not hope to observe the process of re-crystallisation microscopically. We have found that if a strained, polished and etched specimen, showing distinctly elongated crystals, be gradually heated in vacuo no change whatever is visible in the outlines of the crystals; on removing the specimen from the apparatus it is sometimes found coated with a thin blue tarnish, and on examination with the microscope it is seen that this tarnish has not obliterated the original crystals, but in fact differentiates them by various colours on different crystals. If this tarnish be now polished off, as may easily be done by the use of rouge alone, and the specimen be re-etched, an entirely new set of crystals is revealed. This was readily established by drawing the outlines of the original crystals at a marked spot on the specimen before and after the light re-polishing. Simple re-polishing and etching a similar specimen, without annealing, produces only very small changes in the pattern. We may therefore conclude that in the above experiment the iron did re-crystallise when at a red heat, but that this re-crystallisation did not immediately affect the pattern on the surface. The explanation is obvious when we realise that the pattern seen in the microscope is due to the differences of level and texture in the surface of the specimen which have been produced by the action of the acid used in etching. This pattern, although in its origin dependent upon the actual crystalline structure of the metal at the time when the surface was etched, remains as a mere mask, beneath which re-orientation of crystalline elements may go on without affecting its appearance. The etched pattern is in this respect in the same position as a scratch or mechanical marking of the surface, and the persistence of the etched pattern after annealing is simply an extension into more minute detail of the well-known fact that the external shape of a piece of metal is not in general affected by annealing, although the crystalline structure is entirely changed.

These considerations show that we cannot expect to see the process of re-crystallisation in any metal where etching, staining, or relief polishing is needed to differentiate the constituents. We therefore abandoned the attempt to observe this process in iron, but our attention was directed to the study of similar processes of annealing or re-crystallisation as seen in more fusible metals, particularly lead.

The question, What is the temperature required to produced re-crystallisation in lead ? was raised by the appearance we observed in specimens of plumbers' sheet-lead that had been etched with dilute nitric acid. When thus treated, ordinary sheet-lead exhibits a remarkably brilliant crystalline structure on such a large scale that no previous polishing is required. The etched surface shows all the appearances due to the selective effect of oblique light on etched crystalline surfaces ; when the specimen is turned the light catches the various crystalline grains in turn, the colour and brightness being uniform over each grain, but different on different grains. Fig. 2, Plate 3, is a photograph of such a surface magnified two diameters : these surfaces tarnish immediately on drying, and must therefore be observed and photographed while wet with the acid solution.

An examination of this etched surface reveals a peculiarity in the configuration of the crystalline grains ; these are seen to have many remarkably straight boundaries meeting at sharp angles, several sets of parallel boundaries being frequently observed. These features, which strongly resemble what we had previously observed in wrought copper, are to be ascribed to the frequent occurrence of twin crystals. In our earlier observations on twin crystals their presence had always been readily detected by the configuration of slip-bands produced in them by slightly straining the specimen after polishing. An instance of twinning observed in sheet lead by this method has been given by us in a previous paper ('Phil. Trans.,' A, vol. 193, 1900, Plate 26, fig. 40). In the present instance this method of detecting twins is not available, as the roughness of the surface and the great depth of etching employed make it impossible to study the slip-bands. But the presence of twin lamellæ nevertheless becomes evident under slightly higher magnification with oblique light. Fig. 3, Plate 3, is a photograph of crystals with twin lamellæ magnified 40 diameters. The figure illustrates the selective effect of oblique illumination, which has picked out a few isolated crystals, lighting these brilliantly while neighbouring ones remain almost dark. Within the area of the brightly-illuminated grains, a number of dark patches are seen, and these show the straight boundaries occurring in parallel sets which are characteristic of twinning. In this instance there are three distinct parallel sets of dark bands, and the fact that they are twin lamellæ becomes apparent when the stage carrying the specimen is rotated, thus altering the incidence of the light. As the specimen is turned, the grains that were bright become dark, but presently some of the patches that were previously dark shine out brilliantly, all the bands which are parallel to one another flashing out simultaneously. Fig. 4 is a photograph of the same field as fig. 3 after a rotation of about 30°, and illustrates this appearance. But those parallel bands which catch the light simultaneously are evidently portions of the crystal in all of which the orientation of the elements has been changed by the same amount ; in other words, they are twin lamellæ.

Further evidence of the crystalline structure of sheet-lead is found when a slightly higher magnification (100 diameters) is applied. Many of the crystals are then seen to be marked with geometrical figures, most frequently hexagons, but sometimes triangles and rectangles. These figures are similar, and similarly oriented, over the entire area of any one crystal, but they are generally different in shape and orientation on adjacent crystals. In this respect they resemble the geometrical etched pits in iron and geometrical air-bubbles in cadmium described in our previous paper cited above. Their occurrence in *cast* lead that has been exposed to dilute nitric acid has been described by Professor MIERS and Mr. A. DICK in the 'Journal of the Mineralogical Society,' April, 1899. Professor MIERS measured their crystallographic angles, but does not connect them definitely with the crystallisation of the solid metal —and Mr. DICK's view is that they are promiscuously deposited crystals due to electrolytic transfer of lead from the upper to the lower surface of the specimen. Careful microscopic examination shows that in some instances this is the case in sheet lead. Some of the geometrical figures can be seen to be the outlines of deposited, or at all events, projecting crystals; but the remarkable fact remains that even here these crystals are deposited with their elements in the same orientation as those of the crystal upon which they are being deposited. In many cases, however, geometrical markings are found which form the outlines of pits instead of projections. Sometimes these pits occur on the same grain as the deposited crystals, and both then have similar outline and orientation.

Fig. 5, Plate 4, is a photograph of such geometrical markings on etched sheet-lead, seen under oblique light and magnified 100 diameters.

The observations here described form a full demonstration of the marked crystalline structure of sheet-lead, but the origin of this structure is not immediately apparent. Sheet-lead is produced by rolling out cast ingots of the metal without the aid of heat. In the sheets we have, therefore, metal which has been subjected to a very great amount of plastic deformation. The view of plastic strain enunciated in our former paper ('Phil. Trans.,' 1900, Series A, vol. 193, pp. 353–375) would lead us to suppose that the structure of such violently deformed metal would still be entirely crystalline, but its crystals would show peculiarities of shape corresponding with the nature of the plastic deformation imposed upon the mass of the metal. Thus, in a specimen which had been rolled from a solid lump into a sheet, we should expect to find the crystals thin and flat, but of considerable area parallel to the surface of the sheet. Sheet-lead, however, shows no such feature; on a transverse section of the sheet no flattening of the crystals is seen This led us to suppose that some process of annealing or re-crystallisation had been at work in the metal subsequent to its manufacture, and, as the lead had not been subjected to any thermal treatment, it seemed that re-crystallisation must have occurred in it at ordinary atmospheric temperatures. We therefore directed our experiments to test this supposition, and generally to investigate the phenomena of annealing or re-crystallisation in lead.

The first step was to determine the effect of very severe strain on the crystalline structure of lead. In such a soft, ductile metal, plastic deformation may be carried to so great an extent that the adaptability of the individual crystals to change their shape by means of slips on cleavage planes may be insufficient. Careful observation of the crystalline structure of a piece of lead under severe compression confirms this view. Up to a certain point the crystals are gradually flattened out in proportion to the flattening of the whole specimen, but when the "flow" becomes considerable it is found that the crystals, already very thin and flat, are driven into and through one another, this process resulting in a grain or structure which is small, but still entirely crystalline. The action is analogous to what occurs in the fracture of a more brittle metal, with this difference, that in a more brittle metal, when "slip" has gone so far as to extend right across a crystal, the new surfaces thereby brought into contact do not unite or "weld," and fracture results; in lead the freshly exposed surfaces do weld or re-unite under the pressure, a fact which is associated with the possession, on the part of the metal, of great ductility. Fig. 6 is a micro-photograph showing the crystalline structure of ordinary cast lead magnified 12 diameters, while fig. 7 shows the much more minute structure of freshly and severely-strained lead magnified 30 diameters. In this and in the following experiments with lead, the process of straining was carried out by squeezing a cast block of the metal in a compression-testing machine, letting it expand laterally until the block, originally about 1 inch high and $\frac{5}{8}$ inch diameter was flattened into a plate about $\frac{1}{8}$ inch thick.

In order to investigate the changes in the crystalline structure of such strained lead, we adopted the method of taking a series of photographs of a marked area of each specimen at intervals of time during which the metal was exposed either to the ordinary temperature of the room or was subjected to special thermal treatment. Before each photograph was taken the surface was thoroughly re-etched; our experiments on the annealing of iron had convinced us of the necessity of this proceeding, and our observations on lead specimens have confirmed the previous experience. In no case did thermal treatment in any way produce a visible change in the surface pattern until further etching had been resorted to, and fairly deep etching is required to obliterate the old pattern entirely. This applies more particularly to the channels which are etched out at the inter-crystalline boundaries—these may often be seen forming a network on the newly-formed pattern, but quite independent of the new structure.

The method of etching consisted of alternate applications of concentrated and very dilute nitric acid. In some cases, where very deep etching was required, an electrolytic method was used. One of the great advantages of dealing with a metal like lead arises from the large size of its crystals; by enabling us to use deep etching it allows all fine polishing to be dispensed with, and it becomes possible to obtain micro-photographs at low magnifications, and under oblique light, which exhibit clearly the various changes in crystalline structure.

The low power required is a further advantage in view of the fact that an etched lead surface must be kept wet with an acid solution while it is being examined and photographed, as such a surface becomes dull and useless as soon as it dries.

To identify the areas which were to be kept under observation, we marked them by scratches or furrows made with a steel point. Unsymmetrical patterns of marking were used for this purpose to facilitate replacing the specimen in its former position and orientation. It is an essential precaution in taking micro-photographs under oblique illumination which are to serve for purposes of comparison, that the orientation of the specimen and the angle of incidence of the light should be constant throughout the entire series. Our micro-photographic arrangements enabled us to comply with these conditions to a considerable degree of accuracy.

The first series of these micro-photographs (figs. 8 to 13 inclusive, Plates 5 and 6) was taken from a specimen of lead which, after being severely strained, was simply kept during six months at the temperature of an ordinary room without any special thermal treatment whatever, being carefully re-etched and photographed at intervals. The first of the series (fig. 8) was taken immediately after the specimen had been strained. The magnification (12 diameters) is too low to clearly resolve the minute crystalline structure which existed at this stage. The identification marks are very clear at this stage, but they become less clear after each successive etching.

Fig. 9 represents the same surface, re-etched after six days; a small amount of change is visible, and this becomes more marked in figs. 10, 11, 12, and 13, which represent the state of the specimen after one, two, four, and six months respectively. Comparing figs. 8 and 13, the great change in the crystalline structure is strikingly evident.

Measurement of the largest crystals seen in fig. 13 gives some rough idea of the rate at which these crystals have grown. In one case the dimensions are as follows :—

Length of crystal in freshly-strained specimen = ·0025″
,, ,, specimen after 1 month = ·0083″
,, ,, ,, ,, 4 months = ·0100″
,, ,, ,, ,, 6 ,, = ·0115″

From this it would appear that the rate of growth is greatest in the first month after the specimen has been strained; but a close examination of such specimens reveals a feature which considerably affects this question. It is seen that the crystals do not grow by the steady accretion of layers all over their surface, but that they throw out arms or branches, which invade neighbouring crystals, thus forming a skeleton crystal somewhat similar to those that are often noticed in the crystallisation of fused substances. The more or less dim and mottled appearance of the larger crystals in figs. 8 and 9, and the much greater brightness and uniformity shown by the same crystals in figs. 10, 11, and 12, is undoubtedly due to the formation of these skeletons, and their subsequent filling in.

The figures given above must therefore be taken as no more than an extremely rough indication of the order of magnitude with which we are concerned in these questions of growth at ordinary temperatures.

Taking a specimen of plumbers' sheet-lead showing fairly large crystals as an example of metal in which air-temperature annealing has continued for a long time, we next tried the experiment of applying a higher temperature, so as to determine whether any further change would occur. Our observations show that the metal in this state is very sensitive to moderately high temperatures, three minutes' exposure of ordinary sheet lead to a temperature of 200° C. being sufficient to produce a great change in the crystalline pattern. If the specimen be kept at 200° C. for a long time there is a continued change, but it becomes very slow, and ultimately a state is reached when further change is imperceptible.

Figs. 14 to 19, Plates 6 and 7, inclusive are a series of micro-photographs of a specimen treated in this way. Fig. 14 shows the appearance of a typical specimen of sheet-lead, and fig. 15 shows the change produced by 30 minutes at 200° C. Except in one or two isolated instances it is very difficult to trace any connection between the pattern now seen and the original. Fig. 16 shows the same surface, re-etched after a further 30 minutes at 200° C. There has been further change, but not to anything like the same extent as in the first half-hour. The change is most marked in a large crystal on the left-hand side of the figure ; in fig. 15 it shows a mottled or skeleton appearance, and this becomes filled in in fig. 16, while there is a considerable increase in the length and thickness of the two tongue-like projections that start on the right-hand side of the twin band.

A further development of this crystal is seen in fig. 17, which was taken after 48 hours' further annealing at 200° C. Here another twin band has become evident, and a further filling in and straightening of the boundaries has taken place. This process has gone slightly further in fig. 18, taken after four days' further annealing. This photograph may be taken as representing the final state of this specimen, as very prolonged annealing produced no further considerable change. This specimen, however, showed another interesting feature, which we have often observed in other cases. The marked area chosen for photography occupied the centre of the surface of the specimen, which measured approximately $\frac{3}{4}$ inch square by $\frac{1}{8}$ inch thick. Somewhat to our annoyance, this marked area did not show by any means the best development of crystalline growth. In this case, as in many others, we found that the largest and most rapidly growing crystals were formed at or near the edges of the specimen. Fig. 19, taken at the same stage as fig. 18, and with the same magnification, but outside the marked area, shows the remarkable development of crystals in some parts of the specimen.

In comparing the members of such a series of photographs, one consideration must be borne in mind, the great difference produced in the appearance of the surface by a minute change in the angle of incidence of the light. In spite of the utmost precaution to keep the incidence constant, a small change will sometimes cause the entire disappearance of a crystal whose facets happen to be near the critical angle. But, provided any portion of a crystal is shown bright in the photograph, any changes in its outline or surface are certainly genuine, for light that catches the facets on one part of the crystal must necessarily catch all similarly oriented facets also. Where, therefore, the same crystal appears in successive members of a series, the photographs may be taken to represent its life-history accurately.

Another feature of these photographs requires explanation. In figs. 18 and 19, and more or less in most of the others, an irregular network of dark lines or channels is seen to cover the surface, and these lines obviously have little or no relation to the crystals. As a matter of fact, these lines are only found in specimens that have been etched, and then annealed and re-etched. In such a case as figs. 3 and 4, where only one etching has been applied, similar lines are seen to follow the inter-crystalline boundaries. In that case they are simply "corrosion channels" caused by the action of the acid at the crystalline junctions. These channels are of comparatively great depth, and when such a specimen is allowed to re-crystallise, and is then re-etched to develop the new crystalline structure, this second etching is rarely carried deep enough to remove the old etched corrosion channels, and these remain as a network over the surface, simply indicating the positions in which crystalline boundaries formerly existed before the structure became changed. This view is borne out by the observation that deeper etching will eventually remove all trace of this network. In many instances these traces of the former crystalline boundaries are very useful as indicating the nature and extent of the growth that has taken place. On the other hand, the deep etching that would be required to remove these lines was considered undesirable, as very deep etching might of itself alter the apparent shape of the crystals. By observing the effect of successive deep etchings on the pattern of a stable specimen we satisfied ourselves that it would require exceedingly vigorous and prolonged etching to cause any visible change of pattern, and that, in the specimens illustrated in this paper, the changes cannot possibly be ascribed to that cause.

Our next experiment was to expose a freshly-crushed specimen of lead to a temperature of 200° C. for a long time, and to watch the changes produced in its crystalline structure. The annealing ovens* used for this purpose consisted of wooden boxes, lined with asbestos cloth, and heated by means of one or more incandescent electric lamps. The temperature could be regulated by regulating the current in the lamps, and, more conveniently, by adjusting the lids of the boxes so as to admit more or less air. Variations of 10° C. in the temperature were rare.

Figs. 20 to 27 inclusive, Plates 8, 9, and 10, illustrate our observations on one specimen treated in this way. Fig. 20 represents a marked area on the etched surface

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* These annealing ovens were devised and used by Mr. S. R. Roget in his researches on effects of prolonged heating on the magnetic qualities of iron. 'Proc. Roy. Soc.,' May 12 and Dec. 8, 1898.

of a freshly-squeezed specimen; the following photographs were taken after the specimen had been exposed to 200° C. for the time shown in the table :—

Number.	Days.	Hours.
Fig. 21 . . .	—	17·5
,, 22 . . .	1	17·5
,, 23 . . .	2	17·5
,, 24 . . .	5	16
,, 25 . . .	39	20
,, 26 . . .	39	20

Fig. 20 shows the small structure characteristic of freshly-strained lead, with one or two larger crystals that have persisted from the original crystallisation. In fig. 21 these are seen to have grown considerably, and a general change of pattern is observed; the most striking feature is the large skeleton crystal that has developed in the lower left-hand corner of the marked area. This skeleton is seen to grow and fill in continuously in figs. 22 and 23. Figs. 24 and 25 were taken under a different angle of illumination in order to show another large crystal which gave signs of vigorous growth. In fig. 24 it is still somewhat skeletal, but in fig. 25, while it has become much more consolidated, all its outlying arms have disappeared, having been absorbed by a well-defined crystal, part of which is seen as a dark arm in fig. 25. Fig. 26 is a photograph of the specimen at the same stage as fig. 25, but at a lower magnification (8 diameters), and so illuminated as to bring the new crystal, whose dark arm is seen in fig. 25, into brightness. This new crystal is seen to be enormously larger than its neighbours, and from its position relatively to the marked area it can be recognised as the same crystal whose early stages are seen in figs. 22 and 23. This large crystal is an excellent example of what may be called an aggressive individual crystal. Another example, also at 8 diameters, is shown in fig. 27, Plate 10. In both figures the edge of the specimen can be seen, and the photographs illustrate the observation that the largest crystals are generally near the edges of the specimen. It should be added that these large crystals are not mere surface layers, but extend well into and across the thickness of the specimen, and can be readily identified on the opposite surface when, as in this case, the specimen is a plate about one-eighth of an inch thick.

The occurrence of such large crystals in the annealed metal is apparently in no way dependent upon the size of the crystals in the original state before straining; the large crystal in fig. 27 was developed in a specimen whose original crystals were very small.

A remarkable feature, well illustrated by the photographs (figs. 26 and 27) of these large crystals, is the frequent occurrence of twin crystals, both as inclusions in the body of the crystal and at the boundaries. In fig. 26 three distinct sets of straight parallel boundaries are visible, and their "twin" character is evident when they are examined in the manner described above.

In our previous paper we have already given the observation that twin crystals are rarely, if ever, found in a *cast* metal, but that they are frequently developed by strain, and that they are very commonly found in metal that has been annealed after severe straining. Our present observations on lead entirely agree with these statements; but it may now be possible to suggest an explanation for the frequent occurrence of twins in annealed lead. When the metal solidifies from the liquid state, it does so by the formation of skeleton crystals, starting from a great number of centres, and the arms of these skeletons continue to grow until arrested by meeting with other growths.

From these arms other arms again shoot out, and so on until the entire metal is solidified; but each crystalline element as it settles into place on any of these arms must assume the proper orientation to enable it to fit in, and in the process of filling space by means of such a system of many meeting and interlacing arms, the formation of a twin would be almost impossible. But when the metal re-crystallises after severe strain, it does so by the growth of skeleton arms that must often start from a cleavage plane of an actual solid crystal, and probably the new elements deposited upon such a plane would find it as easy to assume the twin orientation as the normal. The idea that twin crystals are formed in annealed metal by growth starting from cleavage planes which have been sheared across in the process of compression is suggested by the very straight boundaries observed as a characteristic feature of twin lamellæ.

Having observed the comparatively rapid growth of crystals in strained lead at 200° C., we extended our experiments to both higher and lower temperatures. Our observations were recorded photographically as before, but the general character of the results is so similar to those already illustrated that it is unnecessary to reproduce these photographs. Experiments were made at temperatures of 100° C., 150° C., 250° C., and 300° C.

In the absence of a reliable method of measuring the rate of growth of crystals, the following statements must be taken as based upon a rough general estimate. With this reservation, our observations over this range of temperatures may be summed up as follows :—

(1) In lead which has been severely strained, re-crystallisation goes on at all temperatures from that of an ordinary room up to the melting-point.

(2) The higher the temperature the more rapid are the changes in the crystalline structure.

(3) The rate of change varies with different specimens, probably depending upon the nature and quantity of impurities present, and upon the severity of the strain to which the metal has been subjected.

(4) The size of the crystals ultimately produced does not vary appreciably between 100° C. and 300° C. In lead annealed at the temperature of the air, very large crystals have not yet been obtained, but this is probably only a question of time.

(5) The phenomena of growth of crystals occur in lead only when the metal has previously been subjected to severe plastic strain. The structure of a cast specimen remains unaltered at temperatures which cause a strained specimen to show rapid change. By casting in a mould arranged to cause rapid cooling, specimens of lead can be obtained having a minute crystalline structure, whose scale is not very different from that of severely crushed lead; such a specimen was exposed to 200° C. for nearly seven days, but no visible change of structure occurred. A piece of this specimen was then strained by severe crushing, and on further exposure to 200° C. vigorous growth took place.

(6) The rate at which a lead specimen is cooled from temperatures of 200° C. to 300° C. down to the ordinary air-temperature has no visible effect on the structure. Even "quenching" in liquid air has no visible effect; quenching in water, cooling in air, and slow cooling in the oven, were all tried on a number of specimens without in any way affecting them.

(7) Lead is mechanically hardened to a small extent by severe strain, and the subsequent effect of annealing in restoring softness is correspondingly small. In one of the experiments a specimen of lead was crushed under a given load in the testing-machine, and the load was left on until no further creeping occurred. The specimen was then annealed and again placed under the same load, when a distinct amount of further crushing was seen to take place.

Some of the experiments described above as having been made with lead were extended to certain other metals that lend themselves to similar treatment; those used were tin, cadmium and zinc.

The crystalline structure of tin is well shown when a surface of a cast ingot of the metal is etched with strong hydrochloric acid. These crystals are generally large, but a much more striking display is obtained on etching the surface of commercial tin-plate. Even before etching, the inter-crystalline boundaries may be seen on the surface, where they are marked by fine grooves or channels. The presence of these channels is readily accounted for by the method of manufacture, during which these plates are drawn out of a bath of melted tin, and allowed to drain. As the plate is drawn out, the layer of tin adhering to it crystallises, but any fusible impurity present in the tin would remain fluid slightly longer, and, being forced by the crystallising tin into the inter-crystalline junctions, the still fluid impurities will drain off, thus leaving a minute channel.

The appearance of the etched surface of commercial tin-plate is shown in fig. 28, Plate 10, which is a photograph at one-half the natural size. In this photograph the outlines of the large crystals can be clearly seen, but it also illustrates another and peculiar feature of etched tin-plate. In all cases of an etched crystalline metal viewed by oblique light we have always observed that, under a given incidence of light, certain crystals were bright while others were dark, and that the illumination was uniform over the entire area of each crystal. In the etched tin-plate this is not the case, the brightness shades off across individual crystals, sometimes uniformly, so as to give the crystal a concave appearance; at other times in patches, giving a mottled effect. On closer inspection, it appears that such variations in brightness occur only in one direction in each crystal; in that direction the brightness may be made to move across the crystal by a slight oscillation of the plate, but another direction can generally be found where the crystal appears uniformly bright and is quite free from the concave appearance.

It is well known that the usual selective effect of oblique illumination on etched crystalline surfaces is due to the fact that etching develops on the surfaces of crystals a great number of minute but similarly oriented facets or pits, and the uniformity of illumination over any area is a direct result of the constancy of orientation of these pits over that area. In the specimens of tin-plate here described we would therefore expect to find that the orientation of the etched pits or facets was not strictly constant over the entire area of each grain.

By examining the etched surface of tin-plate under the higher powers of the microscope the etched pits or facets can be clearly resolved. Their appearance is illustrated at a magnification of 100 diameters in figs. 29 and 30. Under this power their orientation is apparently quite constant, and even under a power of 1000 diameters the departure from strict constancy is only very slight. That it does exist could only be seen by comparing the outlines of apparently similar pits at opposite ends of a large crystal; even there the change in the angles of the figure or of the orientation of one of the sides, as compared by means of a cross wire in the eye-piece, was too small to be measured, but in certain cases the character of one of the outlines of the pit altered slightly, being a fine line at one end of the crystal, and a narrow wedge-shaped dark area at the other. This appearance indicates a change in the slope of that side of the pit, and thus a change of orientation of the internal facets of the pit. This change of orientation seems therefore to occur principally—if not entirely—in the vertical plane, *i.e.*, in the plane of least thickness of the layer of tin.

Under these circumstances it seems that the term "crystal" should only be used with some qualification in speaking of the patches developed by etching the surface of commercial tin-plate. In a crystal which is undistorted by elastic strain constancy of orientation of the elements is an essential characteristic of a true crystal. Some light is thrown on this matter by the fact that we have not observed this "concave" appearance in samples of solid tin, but only on commercial tin-plate where we are dealing with a very thin layer of tin adhering to sheet-iron. Even in tin-plate this effect may be made to disappear by re-melting the tin over a flame and allowing it to cool slowly and at rest. The appearance produced on re-etching such a specimen is shown in fig. 31 (to the same scale as fig. 28). Here, although the crystals are marked with a multitude of straight lines, suggesting a great number of twin crystals, there is little or no trace of the "concave" effect seen in fig. 28.

In view of these observations we are inclined to attribute the "concave" appearance to a distortion or bending of true crystalline plates; such distortion would probably arise from differences in the coefficients of expansion of iron and tin brought into action by a suitable rate of cooling. Considering the extreme thinness of the layer of tin, the amount of distortion might well be purely elastic and insufficient to produce slip in the crystals of tin.

Fig. 32, taken from a different specimen, but to the same scale as fig. 31, illustrates the change in the dimensions and arrangements of the tin crystals that can be effected by changing the rate of solidification; the crystals in fig. 31 were formed in a slowly cooled specimen; those in fig. 32 by quenching the specimen in water while the surface layer of tin was still melted. By means of local quenching and re-melting a great variety of patterns can be obtained; such processes have long been in commercial use in the manufacture of what is called "*moirée metallique*."

It is important to notice that the small crystals of tin which are obtained by quenching the melted metal in water do not show any growth when the metal is exposed for long periods to temperatures short of the melting-point. Even a temperature just short of fusion does not make them grow or re-arrange themselves. A solid block of tin may, however, be reduced to a minutely crystalline structure by severe compression, and in specimens so treated we have observed re-crystallisation to occur at 150° C.

We also made some experiments on the re-crystallisation of cadmium at moderate temperatures. This metal also can be strained by compression until its crystalline structure becomes minute through interpenetration of the original larger crystals. Fig. 33, Plate 12, is a low-power (12 diameters) photograph of an etched and marked area on the surface of a freshly-strained piece of cadmium. Fig. 34 shows the same area, re-etched, after 24 hours' exposure to 200° C. It now shows a well-defined crystalline structure. Fig. 35 shows the same area again, after six days' further exposure to 200° C., and a very considerable increase in the size of the crystals is visible. In this case, although the gradual growth of some of the crystals is very strikingly shown, many of the features that we have observed in the case of lead are entirely absent. In the cadmium we can see no invading branches and no aggressive individuals, nor does there seem to be any considerable amount of twinning.

Experiments similar to those just described were also made on specimens of zinc, with the result that specimens of zinc strained by compression at ordinary temperatures were found to re-crystallise on exposure to 200° C. Some results obtained with sheet-zinc, such as that used for electric batteries, were particularly interesting. It is a well known fact that the mechanical properties of zinc are widely different at different temperatures, particularly that the metal is soft and ductile at temperatures slightly above 100° C., and that it is generally worked at that temperature, while it is known to become very brittle at and above 200° C. Commercial sheet-zinc, rolled at temperatures above 100° C., remains fairly soft and flexible at ordinary temperatures, and its crystalline structure is too minute to be seen in specimens etched without previous polishing.

If a specimen of such sheet-zinc be exposed to a temperature of 200° C. for about half-an-hour, it shows on etching with strong hydrochloric acid a brilliantly crystalline texture as illustrated in fig. 36, Plate 13, magnified 8 diameters. The metal is then much harder, but at the same time brittle ; when bent it emits a "cry" like that of tin, and finally breaks with a crystalline fracture which zig-zags along the cleavage-planes of the various crystals through which it passes.

Many of the phenomena described above as occurring in such metals as lead, tin, zinc, and cadmium, have close and well-known analogues in iron and steel. In those metals, however, the temperatures involved are much higher, and the experimental difficulties are consequently greater ; we have consequently been able to make many observations on the re-crystallisation of lead which it has not yet been possible to make in the case of iron or steel. The question, therefore, suggests itself, how far the analogy between the various metals holds true. One view which is held by many persons—perhaps unconsciously—may be called the theory of "corresponding temperatures" ; the central idea of this view is that the properties and behaviour of metals are to a great extent a function of their distance from their own melting points. Thus a phenomenon seen in lead at 200° C. is strictly analogous to a similar phenomenon seen in iron at 800° C. To a certain extent this view is justified, but it may easily be carried too far. Thus, while lead at 10° C. is soft and ductile like iron at 800° C., yet gold at 10° C. is also soft and ductile, while it is further from its melting-point than iron at the same temperature. Again, one of the most characteristic features of lead is the extreme straightness of its "slip-bands," while those of iron are characteristically curved and irregular. We have not had an opportunity of examining the slip-bands produced by straining red-hot iron, but we have made the converse experiment of straining lead at the temperature of liquid air, with the result that the slip-bands developed were as characteristically straight and regular as ever, thus showing that the straightness of the slip-bands, which is so closely associated with softness and ductility, is a characteristic of the metal rather than a function of the temperature. In order to thoroughly test the same question in relation to annealing, a long series of experiments to determine whether or not gradual re-crystallisation goes on in strained iron at ordinary temperatures and a similar series on lead at much lower temperatures would be required. In the case of glass-hard steels it is well known that a certain amount of annealing takes place at 100° C., and even at atmospheric temperatures ; but in the case of steel, where a distinct chemical change is involved, the nature of the annealing action may differ very considerably from that in an approximately pure metal. Certain phenomena in the recovery of elasticity after over-strain in both iron and steel (MUIR, J., "Recovery of Iron and Steel from Over-strain," 'Phil. Trans.,' 1899) also point to the fact that changes of internal structure may occur in iron at very moderate temperatures ; further evidence in the same direction is afforded by the effect produced (ROGET, S. R., 'Proc. Roy. Soc.,' 1898) on the magnetic properties of soft iron by roasting at moderate temperatures. On the other hand, it has been very widely believed that annealing or re-crystallisation, particularly in iron and steel, are "critical" phenomena which can only occur at or above certain definite temperatures. ARNOLD has gone so far as to make a careful determination of such an "annealing point." Various of the "arrest-points" in the cooling of iron and steel have also been regarded as representing critical points in respect to annealing, but the connection between the two is by no means fully proved. Having found phenomena of annealing or re-crystallisation in other metals, it therefore became interesting to inquire whether any corresponding arrest-points could be found in the cooling of these metals. We investigated the matter by means of a pyrometric arrangement consisting of two thermo-electric junctions, a very sensitive D'Arsonval galvanometer, and a potentiometer somewhat similar to that used by Sir W. ROBERTS-AUSTEN; the deflections of the galvanometer were, however, observed by means of a telescope and scale, instead of being photographically recorded. It may be, therefore, that either from this cause, or from insufficient sensitiveness of the whole arrangement, some minute arrest-points were overlooked; but between the melting-points and the ordinary temperature of the air no trace of an arrest-point was observed in the three metals tried, *i.e.*, lead, tin and cadmium. Such arrest-points, if they exist at all, may be found at much lower temperatures than those to which our experiments were carried.

Having thus failed to correlate the phenomena of re-crystallisation in lead, &c., with any definite point at which heat is evolved during the cooling of the metal, and having reason to believe that even in iron the arrest-points are not necessarily intimately connected with annealing, we look for a theoretical explanation of these actions in another direction. The theory of re-crystallisation which we shall now advance as a working hypothesis for the explanation of the phenomena described in this paper ascribes an important part in the action to the impurities present even in "pure" metals. The impurities which we believe to be of importance are those which are capable of forming eutectic alloys, or fusible compounds, with the metal itself; they would therefore be mainly metals, particularly the more fusible metals, such as bismuth, tin, cadmium, mercury, sodium, or even rarer metals, such as gallium. It is well known that when a metal containing a small proportion of such impurities crystallises, the impurities are, for the most part, segregated in the inter-crystalline boundaries. The crystals themselves form at a temperature when the eutectic alloys are still quite fluid, and the growing crystals gradually push the remaining eutectic into the boundaries. Where the quantity of impurities present is sufficiently great, this eutectic can be seen under the microscope forming an inter-crystalline cement. Mild steel, where the "pearlite" plays the part of a eutectic, is a good example of such a structure; other examples can be found in the gold-aluminium alloys illustrated by Messrs. HEYCOCK and NEVILLE.* Where the quantity of eutectics present is very small, the meshes of inter-crystalline cement

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* "Gold-Aluminium Alloys," HEYCOCK and NEVILLE, 'Phil. Trans.,' A, vol. 194, plates 4, 5.

cease to be visible, but the presence of the impurity makes itself felt by the formation of deep grooves or channels along the inter-crystalline boundaries on etching. Even a very minute amount of impurity would suffice to form a thin but practically continuous film of eutectic in the crystalline boundaries.

The close analogy between alloys and salt solutions has been so fully worked out that it is barely an assumption to say that a eutectic alloy in contact with crystals of one of its constituents will behave in much the same way as a saturated solution of a salt in contact with crystals of that salt would do. Our view, then, is that there is constant diffusion from the surface of the crystal into the eutectic film and equally constant re-deposition of metal upon the crystal from the eutectic film. If there are several crystals in contact with the same eutectic, then there will be, under some conditions, a state of dynamic equilibrium between them, the amount dissolved from each being exactly counterbalanced by the amount deposited upon it; if, however, there is any difference in the solubility of various crystals, in other words, if there is any difference in their "solution pressure" in respect to the eutectic, then the less soluble will grow at the expense of the more soluble. In the case of salt crystals in an ordinary solution it is well known that the large crystals gradually absorb the small ones, a transformation generally explained on the ground that a system tends to assume a position of minimum potential energy. In the case of metallic crystals in a solid metal we have, however, this special condition, that the eutectic (or solution) exists as a mere thin film in contact with only one face of one crystal on either side. If, therefore, these two crystal faces differ in solubility or "solution pressure" in the eutectic, gradual transfer of the metal by diffusion through the eutectic film from one crystal to the other would result, and the one crystal would grow at the expense of the other. This action might go on while the eutectic was solid—Sir W. C. ROBERTS-AUSTEN has demonstrated that diffusion does occur in solid metals,* but at extremely slow rates, unless the metals are near their melting-points. The metal constituting the eutectic films, being much nearer its melting-point than the rest of the mass, would thus be favourable to comparatively rapid diffusion, but the rate of such diffusion and, consequently, the rate of growth of crystals, would be enormously increased by heating the metal to a temperature above the melting-point of the eutectic in question.

The theory† which we suggest to explain crystal growth in a solid metal depends upon the existence of a difference in the solubility of the two crystal faces in contact with a eutectic film. The only difference between these two faces is, apparently, in the orientation of the crystalline elements; but this very difference is sufficient to produce a difference in the rate of solution of such surfaces in an acid. A polished surface of metal when attacked by an acid is not attacked at a uniform

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* Bakerian Lecture, "Diffusion of Metals," by W. C. ROBERTS-AUSTEN, 'Phil. Trans.,' A, 1896, vol. 187, pp. 383–416; "Diffusion of Gold into Lead," 'Proc Roy. Soc.,' May 5th, 1900.

† It is proper to say that the credit for this theory belongs to Mr. ROSENHAIN.—J. A. E.

rate all over, but at different rates over different crystals, in such a way as to produce marked differences of level between adjacent crystals. Another phenomenon, seen best in etching lead in dilute nitric acid, is also of interest in this connection ; it has already been described in the present paper (see p. 284). We there have a case of lead dissolved from one crystal and deposited upon another crystal in its proper orientation.

In view of these facts, we think it must be admitted that different crystal faces, having a different orientation of their elements, differ in solubility in the same solvent. To inquire into the cause of this difference is a further step in speculation which is, perhaps, hardly necessary in this connection. Such differential actions may, however, most probably be attributed to differences of electrical potential in the surfaces involved. If we accept this view of the matter, then the diffusion across films of eutectic becomes a case of electrolysis. Now, while diffusion in metals and alloys is a proved fact, the possibility of electrolysis in an alloy has not yet been demonstrated experimentally.* On the other hand, the close analogy with salt solutions leads one to expect that alloys could be electrolysed, and those who have experimented in the matter are not by any means certain that greater experimental resources will not enable them to electrolyse alloys.

There is at least one fact in the phenomena of re-crystallisation which the solution theory, apart from electrolysis, does not cover, while the electrolytic theory explains it very readily. We refer to the fact that only strained crystals will grow, while unstrained crystals show no tendency to change even at higher temperatures. The explanation, on the electrolytic theory, is that in the unstrained state the crystals are surrounded by practically continuous films of eutectic, and that electrolysis only becomes possible when severe distortion has broken through these films in places, allowing the actual crystals to come into contact ; the electrolytic circuit would then be for each pair of crystals, from one crystal to the other by direct contact and back through the eutectic film.

Our view of the phenomena of re-crystallisation in solid metals may be summed up thus :—We believe that the action is one of solution and diffusion of the pure metal constituting the crystals into the fusible and mobile eutectic forming the inter-crystalline cement. This diffusion results in the growth of one crystal at the expense of the other, owing to differences in solubility of the crystal faces on opposite sides of the eutectic film, and it seems probable that this phenomenon of directed diffusion is really a form of electrolysis.

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* It has, however, been shown by GARNIER ('Comptes Rendus de l'Académie des Sciences,' vol. 116, 1893, pp. 144–9), that the diffusion of carbon into iron is affected by the action of an electric current. He interposed a layer of carbon between two iron electrodes enclosed in a fire-clay tube ; the whole was heated to 1000° C., and a current of 55 ampères was passed for three hours, when the anode was found to be unchanged, while the cathode had undergone considerable cementation. This action in the interior of the iron is practically electrolysis of the carbon-iron eutectic.

There is one deduction from this solution theory of re-crystallisation that lends itself to experimental investigation. If eutectics play an essential part in re-crystallisation, these phenomena should disappear in the total absence of impurities capable of forming eutectics. In a perfectly pure metal, re-crystallisation by annealing after severe strain should not occur; but it is almost hopeless to obtain a specimen of such purity as to justify the conclusion experimentally in this form. The degree of purity required can be roughly gauged from the fact that 0·1 per cent. of carbon in iron is easily visible as "pearlite" under the microscope; so that, probably, an impurity of one part in one million would vitiate the experiment.

But our conclusion can be narrowed down to more practicable limits by putting it in this way:—That if the presence of a eutectic is essential to crystalline growth, then a crystalline boundary free from eutectic should be a barrier to all such growth. This condition can be approximately realised by means of a weld between two clean-cut surfaces of a metal. Lead lends itself particularly well to such experiments, as it welds readily under pressure without the aid of heat. In this way we have obtained a striking experimental verification of the conclusion deduced from the above theory.

The welds were made in various ways; generally two lead discs about $1\frac{1}{2}$ inch in diameter had their surfaces scraped clean with a clean, sharp knife, the two surfaces being put into contact immediately after scraping. They were then subjected to a pressure of 5 tons steadily applied in a testing-machine; in some cases pressures up to 50 tons were used. The behaviour of the specimens under pressure depended upon the previous preparation of the lead discs; as a rule, and in order to obtain the metal in a condition where its crystals would grow rapidly, these discs were prepared by crushing a cast cylinder an inch long by $\frac{5}{8}$ inch diameter. In other cases the discs were obtained by casting, and were then only strained when the welding pressure came upon them; in these cases there was considerable "flow" while the two discs were in contact, but their ultimate behaviour was the same in all cases.

The welded discs so obtained were found to be firmly united and could be sawn into sections as desired; they were cut up into sections suitable for microscopic examination, sometimes before, but generally after "annealing." The annealing was done by exposing the specimens to a temperature of 200° C. for a considerable time—varying from 24 hours to over a month, and the crystals in all cases grew vigorously. Sections at right angles to the plane of the weld were then cut smooth and etched for examination. On the freshly-cut surface the line of the weld could never be distinguished. The etching had to be carried to a considerable depth, because we found that the cut surface was covered by a thin layer of very minute crystals—evidently the result of the violent strains set up by the cutting-tool.

Microscopic examination, generally at 80 diameters, of these etched sections showed that, although the crystals on either side of the weld had grown vigorously, none of them crossed the line of the weld, which was clearly visible as an inter-crystalline boundary continuous along the whole specimen; in many cases aggressive individual crystals had grown up to the line of the weld and there ended quite abruptly. It must not, however, be supposed that this weld line was mechanically weak; it proved on trial to be quite as difficult to cut or tear the metal along the weld as in any other direction. The weld therefore behaved as a true inter-crystalline boundary, differing only in the absence of eutectic, and therefore forming a barrier to crystalline growth. Fig. 37 shows the appearance of such a weld in section after annealing and etching, at a magnification of 30 diameters. The line AB is the weld. As these experiments were made on commercial lead, we were prepared to find that, as a mere matter of probability, some eutectic would have occasionally found its way into the welding surfaces, but this seems to have happened only very rarely. We examined some forty specimens, and only in two instances did we see a slight amount of crystalline growth crossing the line of the weld. We think that we are justified in attributing these rare exceptions to the accidental presence of impurity.

We then went a step further. If we have in a welding surface an inter-crystalline junction which acts as a barrier to crystalline growth owing to the absence of eutectic, then if a suitable eutectic be supplied, growth should occur there as elsewhere.

Our first experiment was to interpose a thin but continuous layer of lead-bismuth eutectic between the lead discs in welding; the specimen was then annealed for several days at 200° C.—well above the melting-point of the eutectic—but on examination it was found that the layer of eutectic had persisted as such, and allowed no growth to cross it. But in this case the film of eutectic introduced at the weld was continuous, and the conditions were therefore analogous to those which hold at the boundaries of unstrained crystals, where, as we have pointed out, growth does not occur. To make the experiment conclusive it was necessary to have a discontinuous film of eutectic at the weld. We accordingly tried another experiment, introducing only a few small flakes of the same alloy, and after annealing we found that crystalline growths had crossed the line of the weld in many places. This experiment was repeated many times, various impurities being used, such as the lead-tin-cadmium-bismuth eutectic, pure tin, cadmium, bismuth, and mercury. All these gave the same result, showing considerable growth across the weld after prolonged annealing at 200° C., but the amount of growth observed varied very much.

Fig. 38 shows a typical example of crystals that have grown across the weld; the line of the weld is still faintly indicated by a discontinuous line, CD, probably representing an included impurity of a non-metallic character, around which the crystals have grown much as they grow around the slag in wrought-iron.

In order to remove all doubt as to the action of the impurities which were introduced, and particularly to obviate the possible contention that their action was either purely mechanical or else of the nature of that of the "dirt" more or less requisite in many chemical actions, certain further experiments on welds in lead were made. In these, the matter introduced at the weld was—

(1) Flakes of lead.
(2) Clean iron filings.
(3) Clean sand.

In all three cases no growth across the weld took place, which confirms the view that the presence of a more fusible eutectic in an inter-crystalline boundary is essential to crystalline growth across that boundary. We think, therefore, that we are justified in regarding the results of these experiments as strong confirmation of the solution theory of crystalline growth in annealing.

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Fig. 6. A typical view of the structure seen in etched cast-lead under oblique light and magnified 12 diameters. (Plate 4.)

Fig. 7. A typical view of the structure of freshly crushed lead seen under oblique light and magnified 30 diameters. (Plate 4.)

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Fig. 37. Section through a cold weld in lead, using clean surfaces, after prolonged annealing; the line of the weld is seen at AB. Oblique light, magnification 30 diameters. (Plate 13.)

Fig. 38. Section through a cold weld, using eutectic in the weld, after prolonged annealing. The line of the weld is at CD. Oblique light, magnification 30 diameters. (Plate 13.)

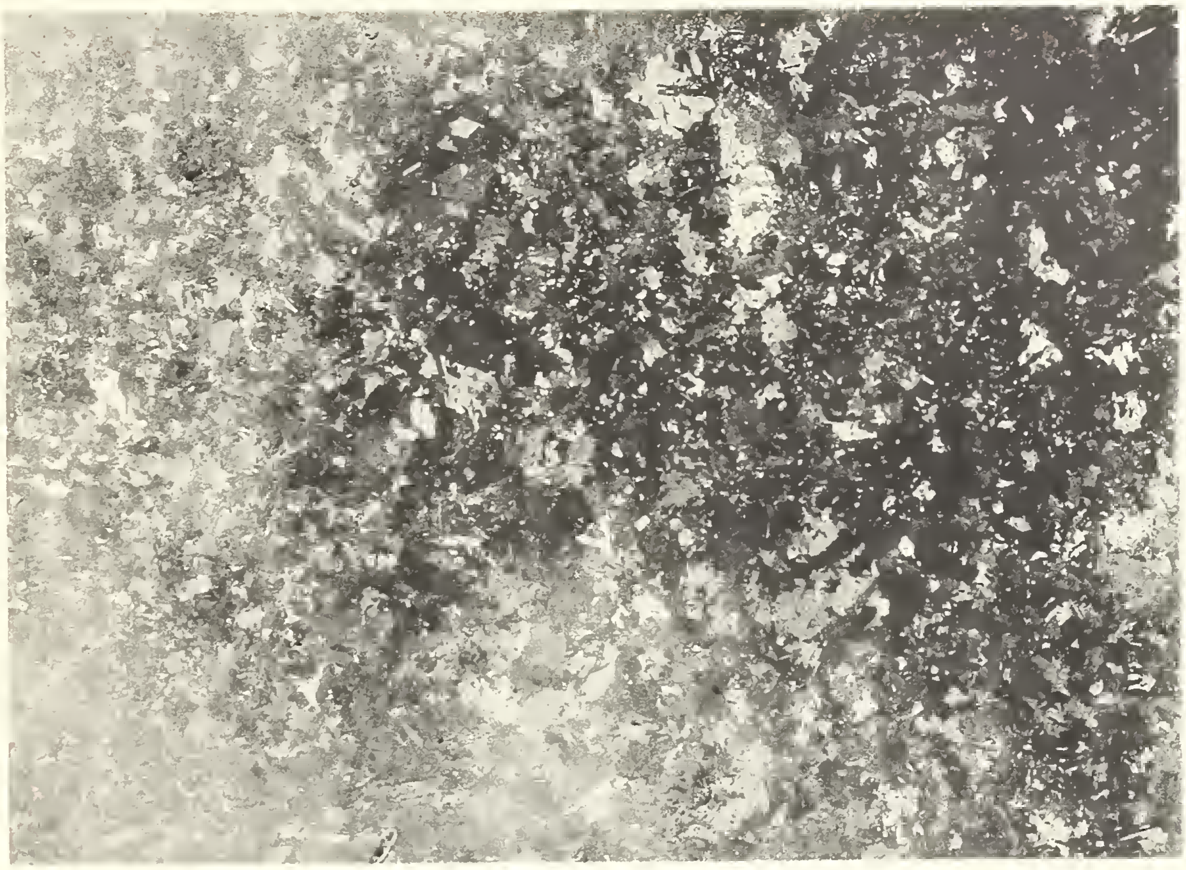

Fig. 2.—Etched Sheet lead. Two diameters.

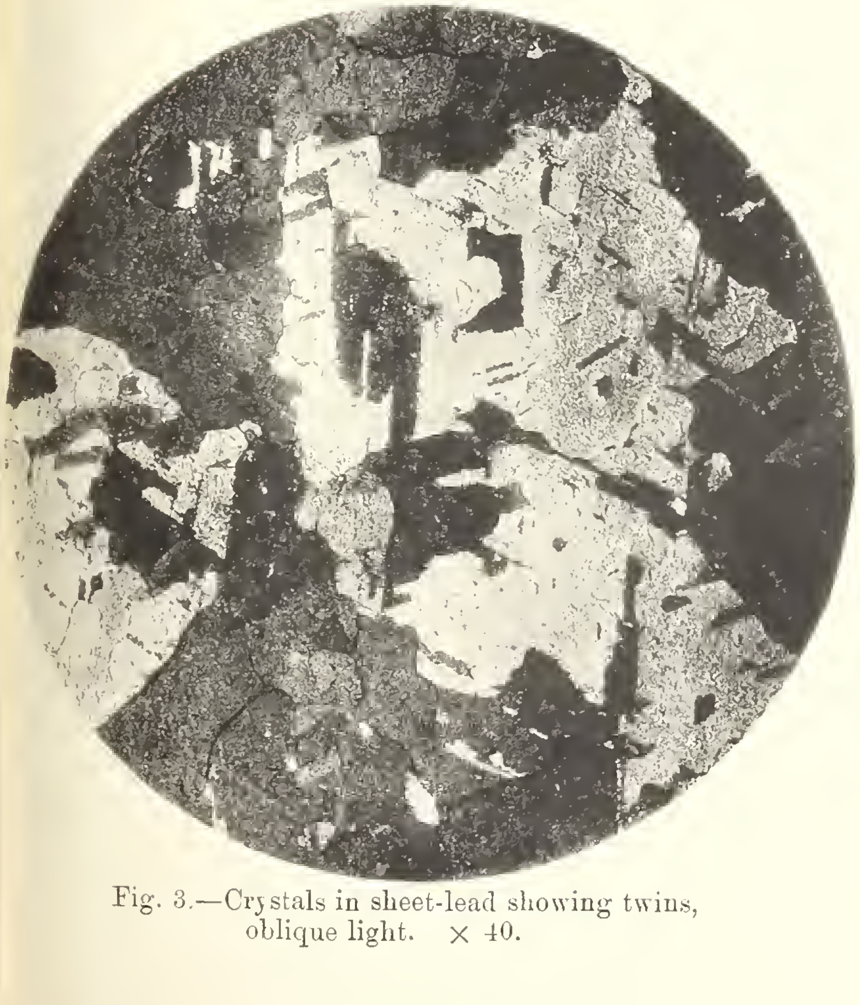

Fig. 3.—Crystals in sheet-lead showing twins, oblique light. × 40.

Fig. 4.—Same field as Fig. 3 with direction of illumination rotated through 30°. × 40.

Fig. 5.—Etched sheet-lead. × 100.

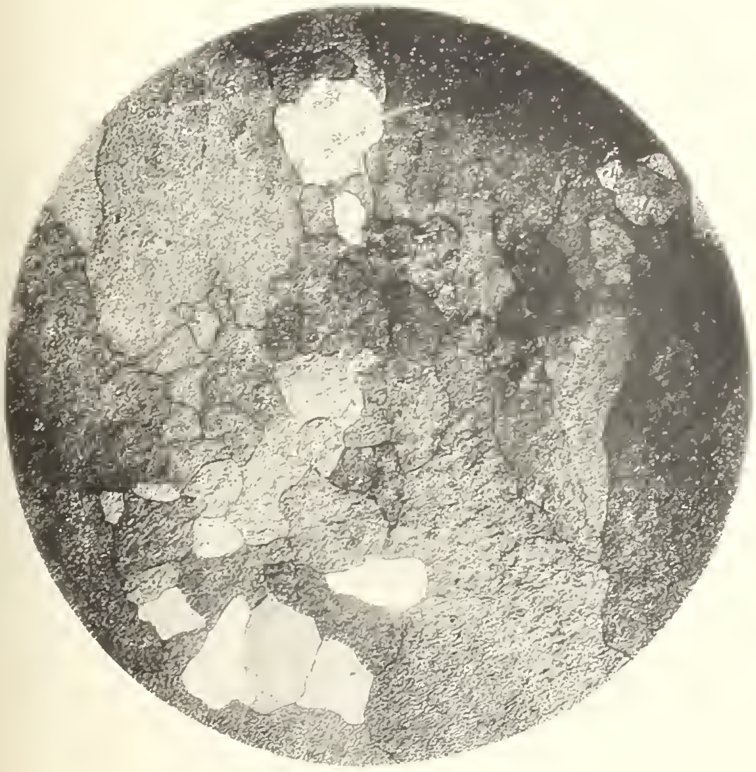

Fig. 6.—Cast lead. × 12.

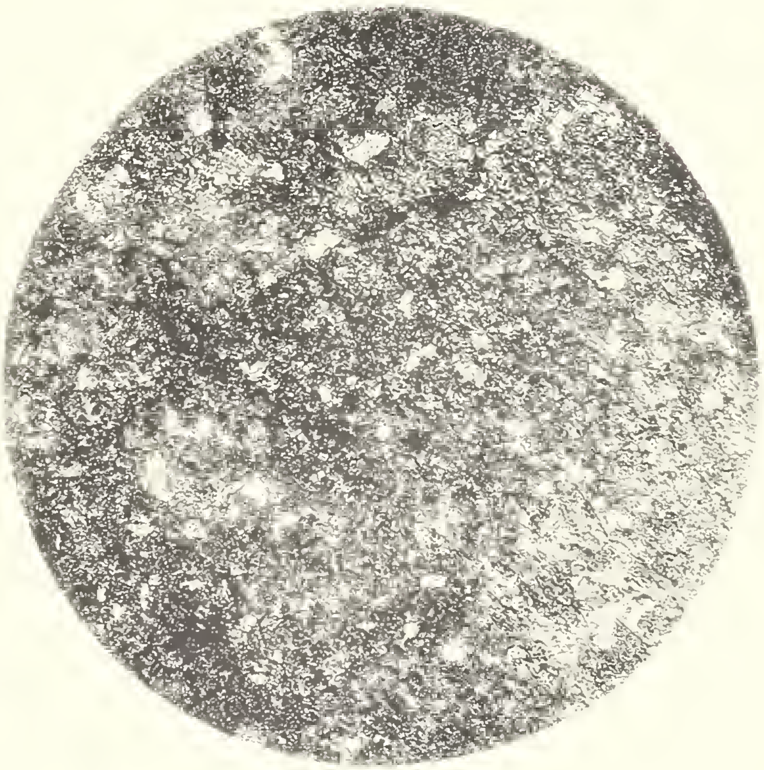

Fig. 7.—Freshly crushed lead. × 30.

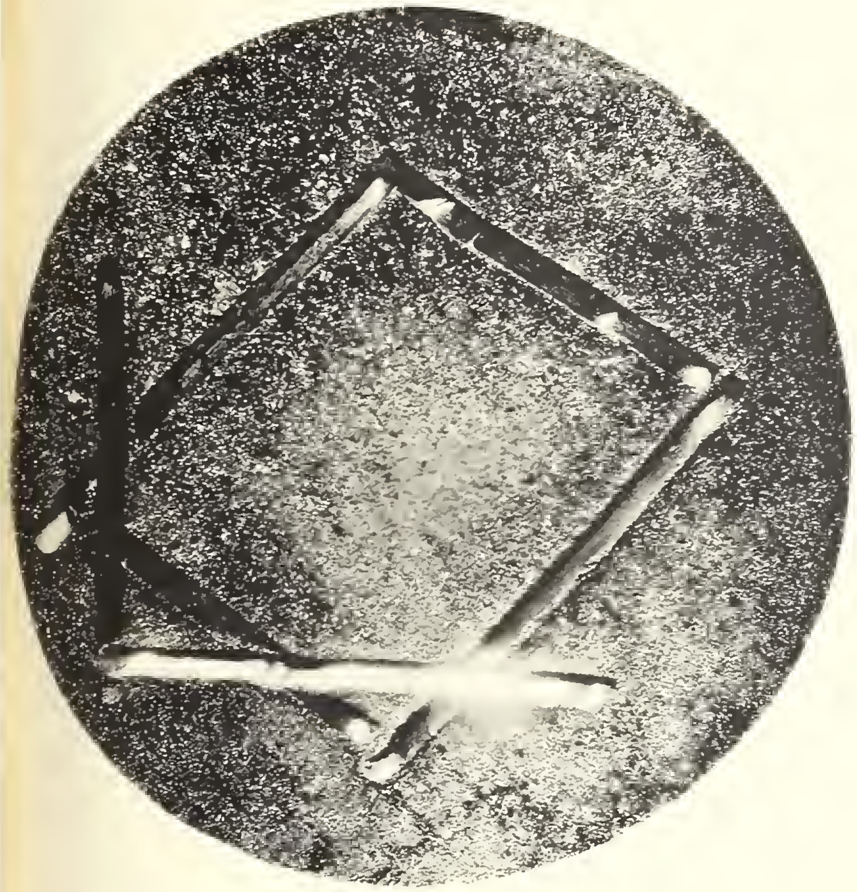

Fig. 8.—Lead. × 12. Freshly crushed.

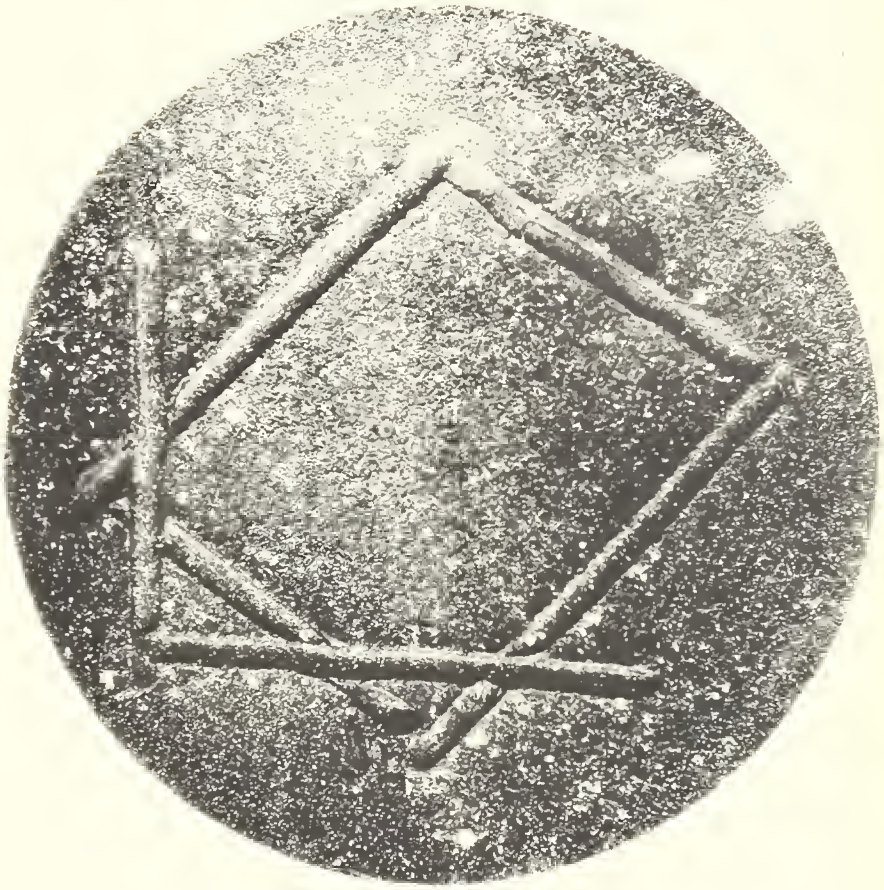

Fig. 9.—Same after six days.

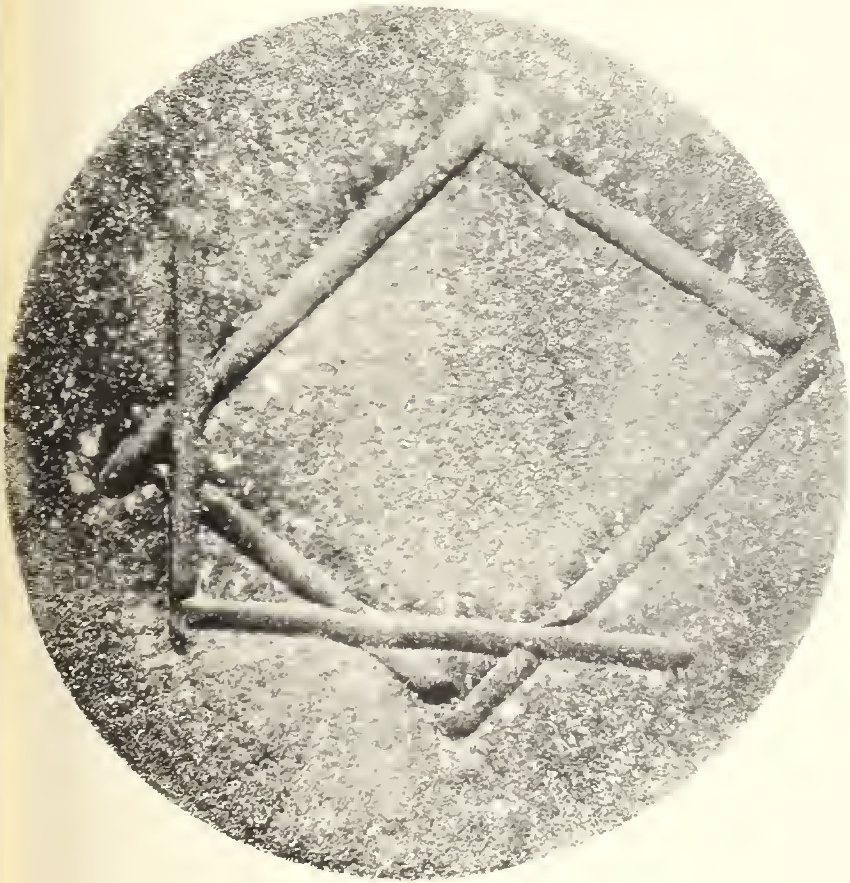

Fig. 10.—Same after one month.

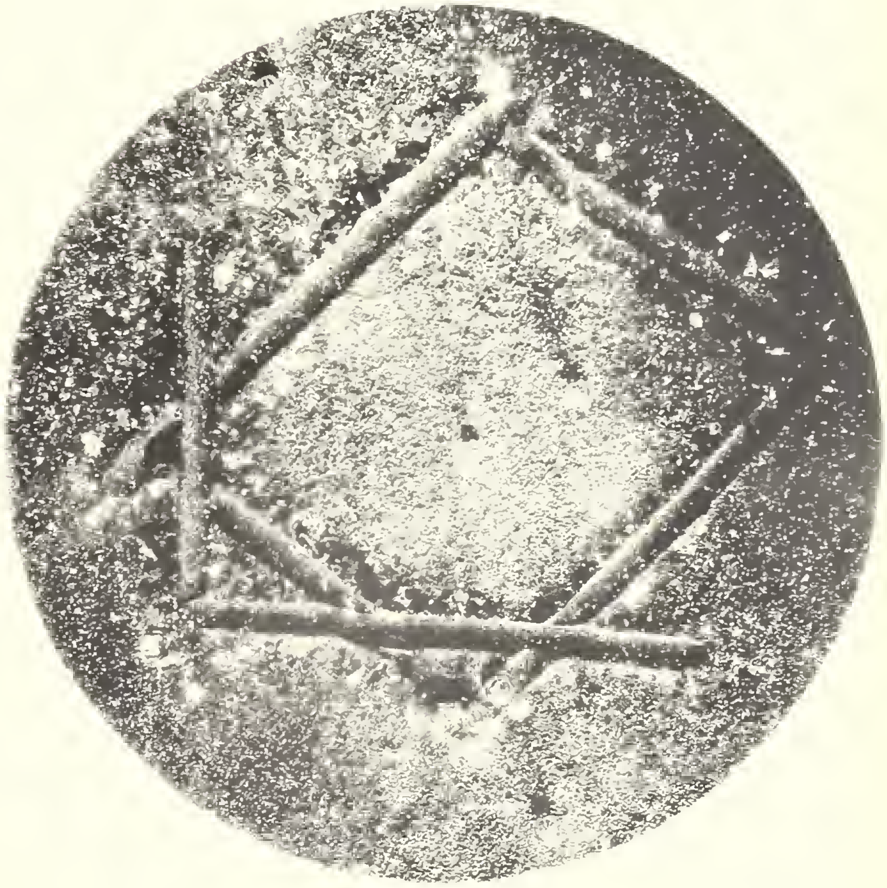

Fig. 11.—Same after two months.

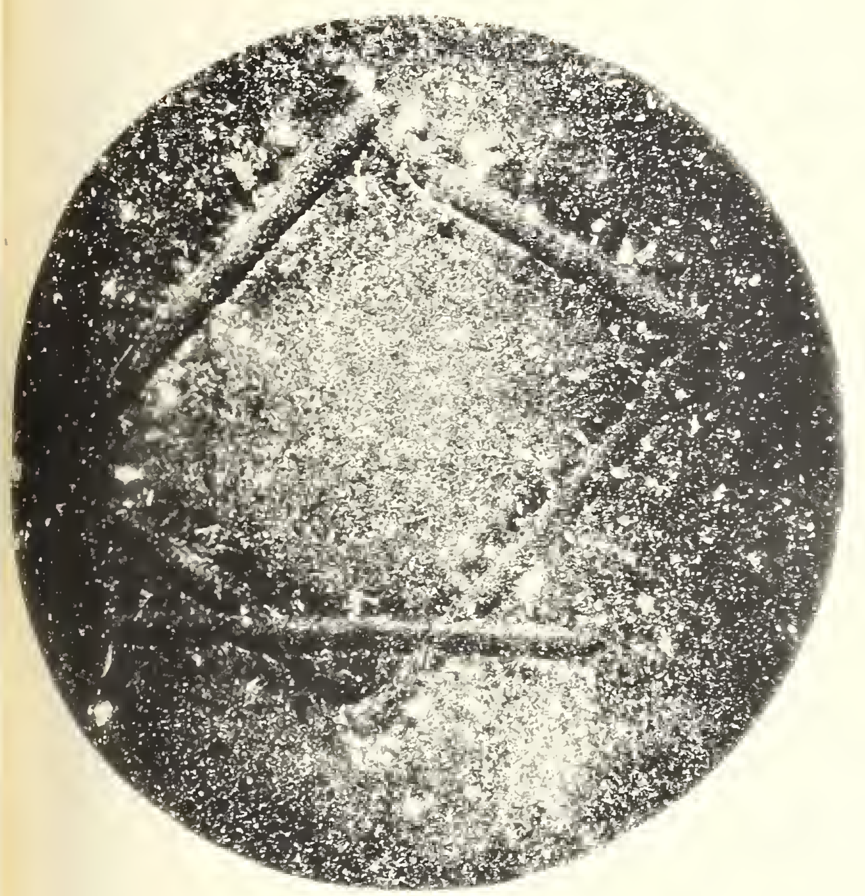

Fig. 12.—Same after four months.

Fig 13.—Same after six months.

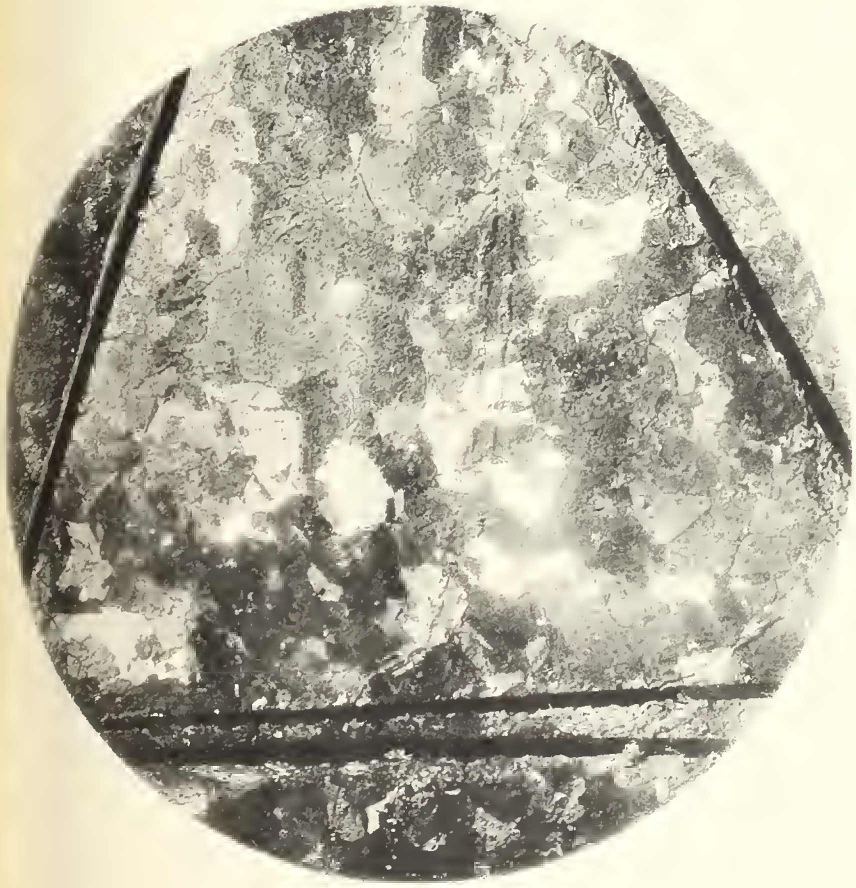

Fig. 14.—Sheet lead. × 12.

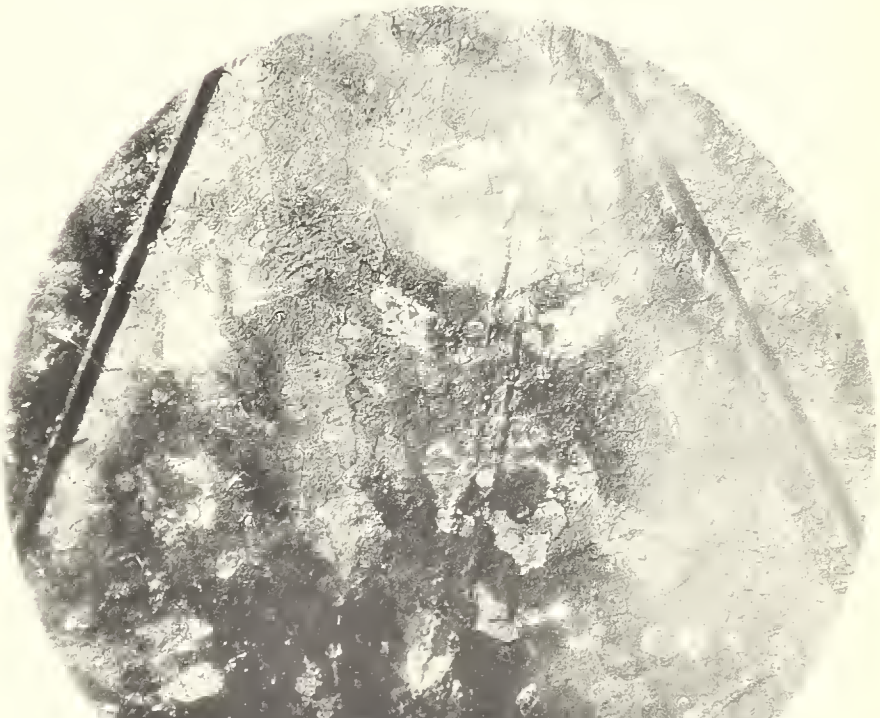

Fig. 15.—Same sheet lead. × 12. After 30 minutes at 200° C.

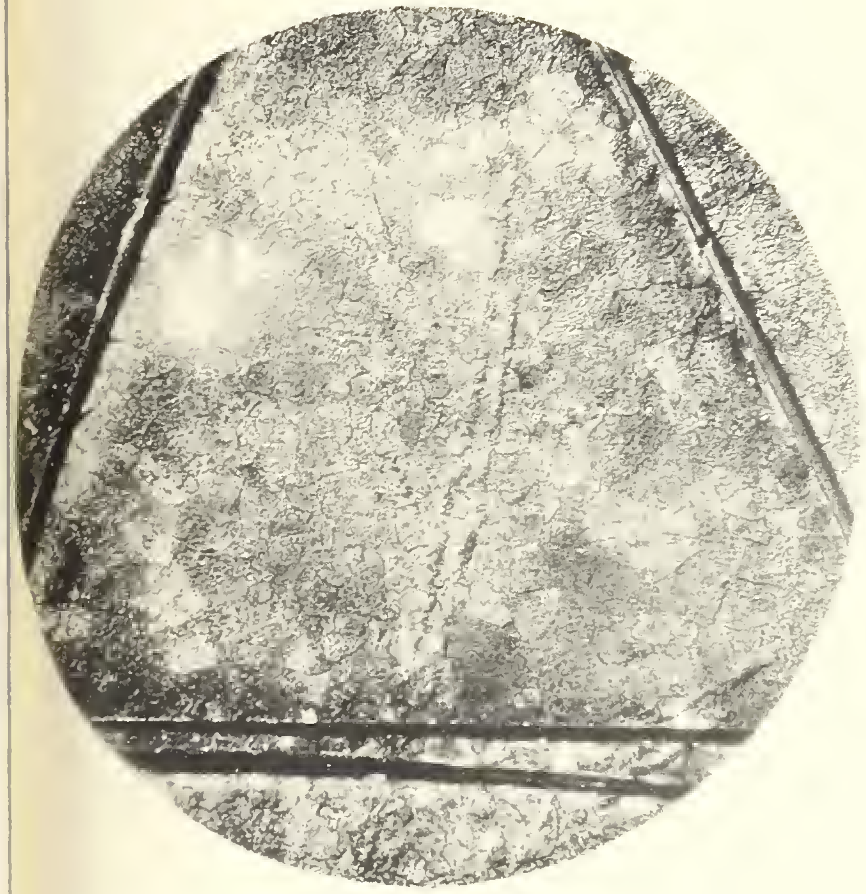

Fig 16.—Same after 30 minutes further at 200° C.

Fig. 17.—Same after 48 hours further at 200° C.

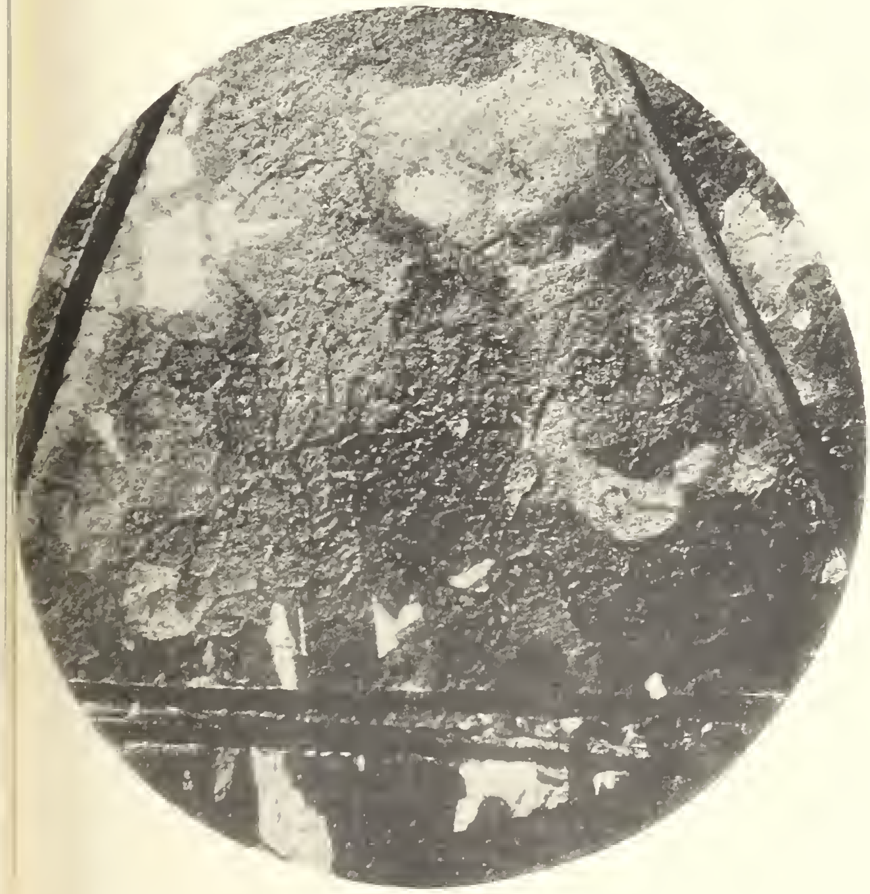

Fig 18.—Same after 4 days further at 200° C.

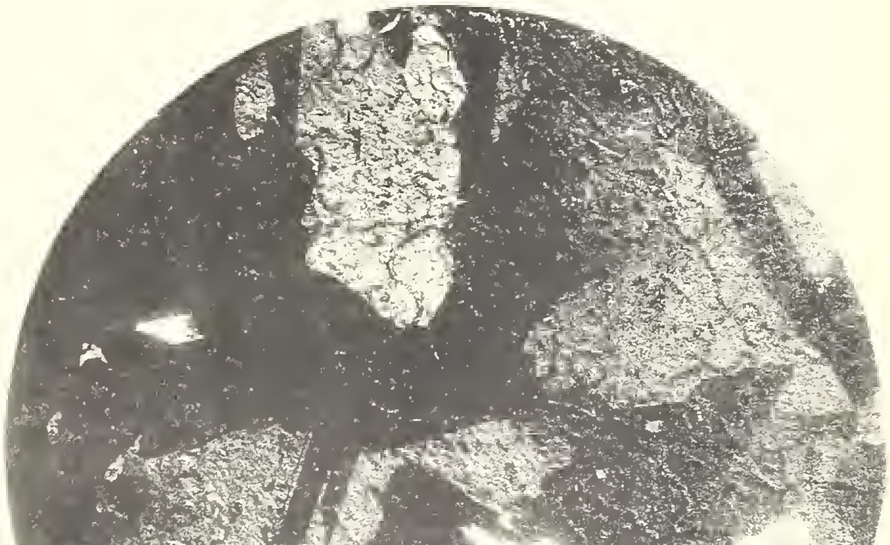

Fig. 19.—Sheet lead. × 12. Another part of the same specimen as Fig. 18.

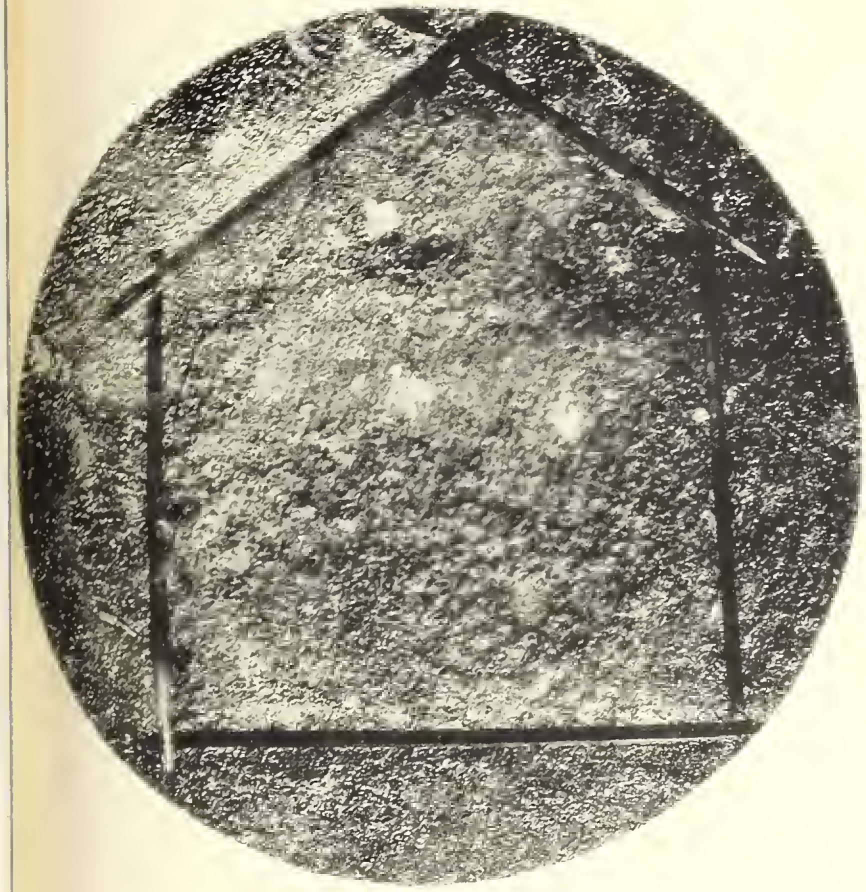

Fig. 20.—Freshly crushed lead. × 12.

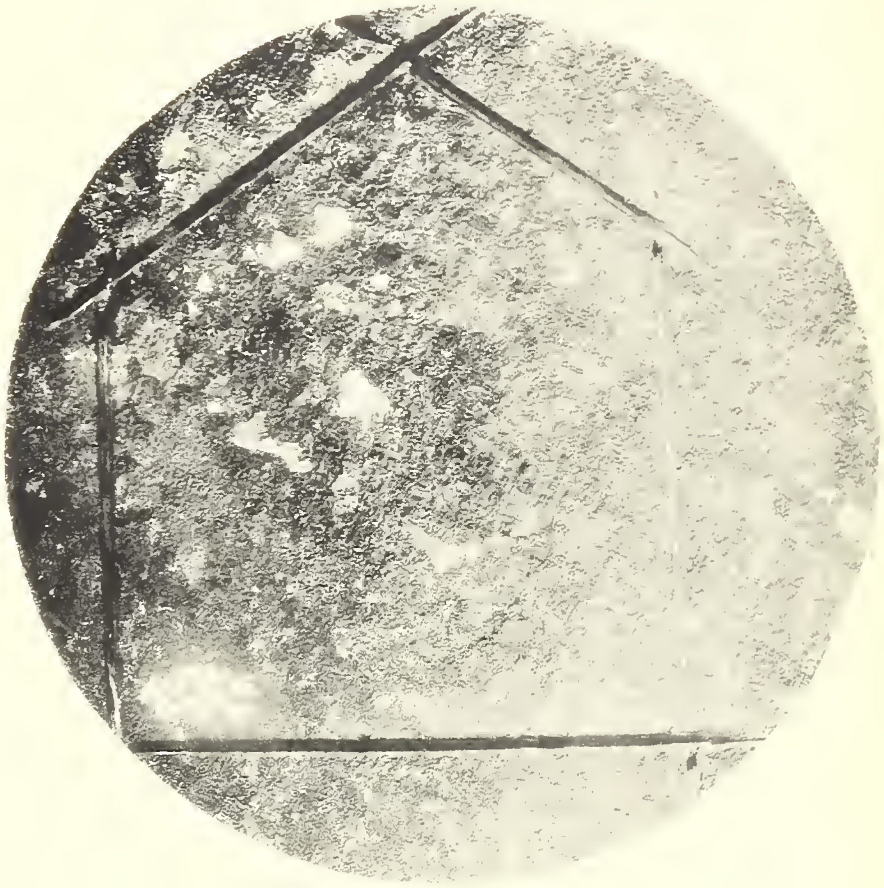

Fig. 21.—Same after 17½ hours at 200° C.

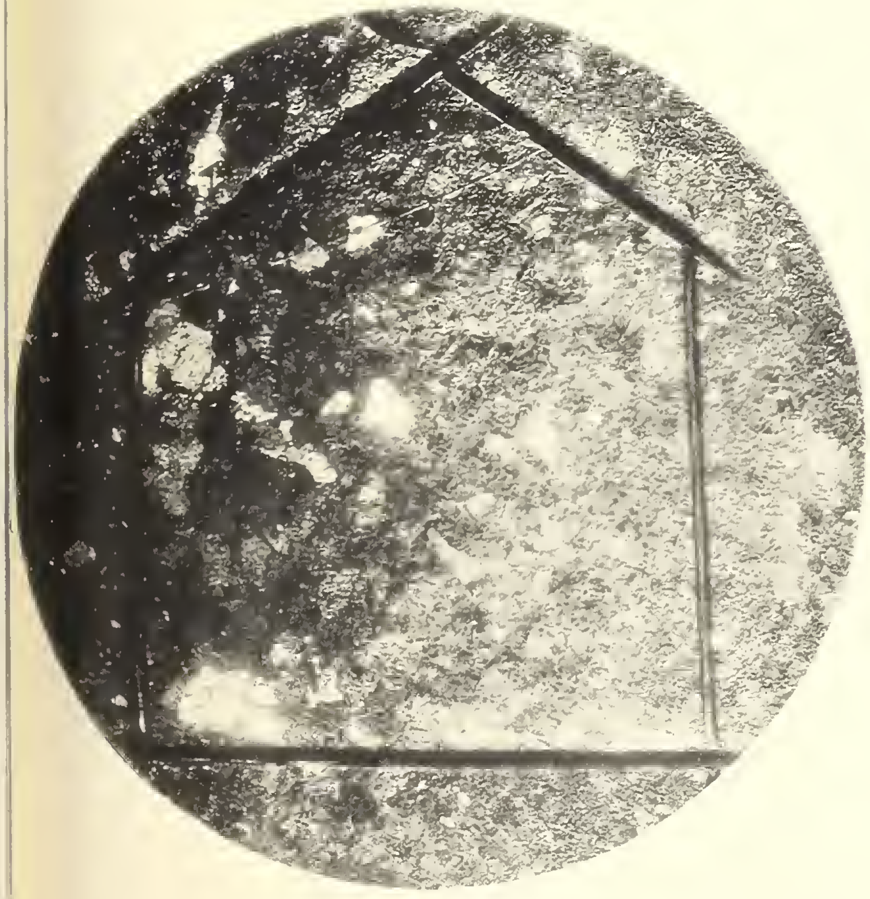

Fig. 22.—Same after 1 day 17½ hours at 200° C.

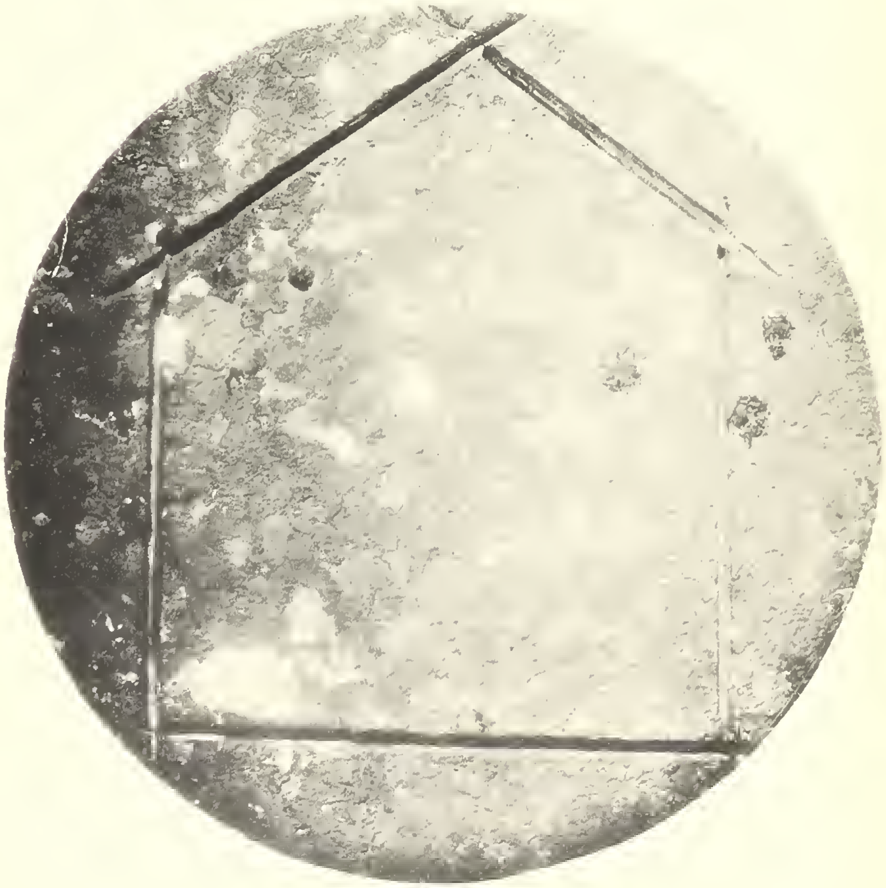

Fig. 23.—Same after 2 days 17½ hours at 200° C.

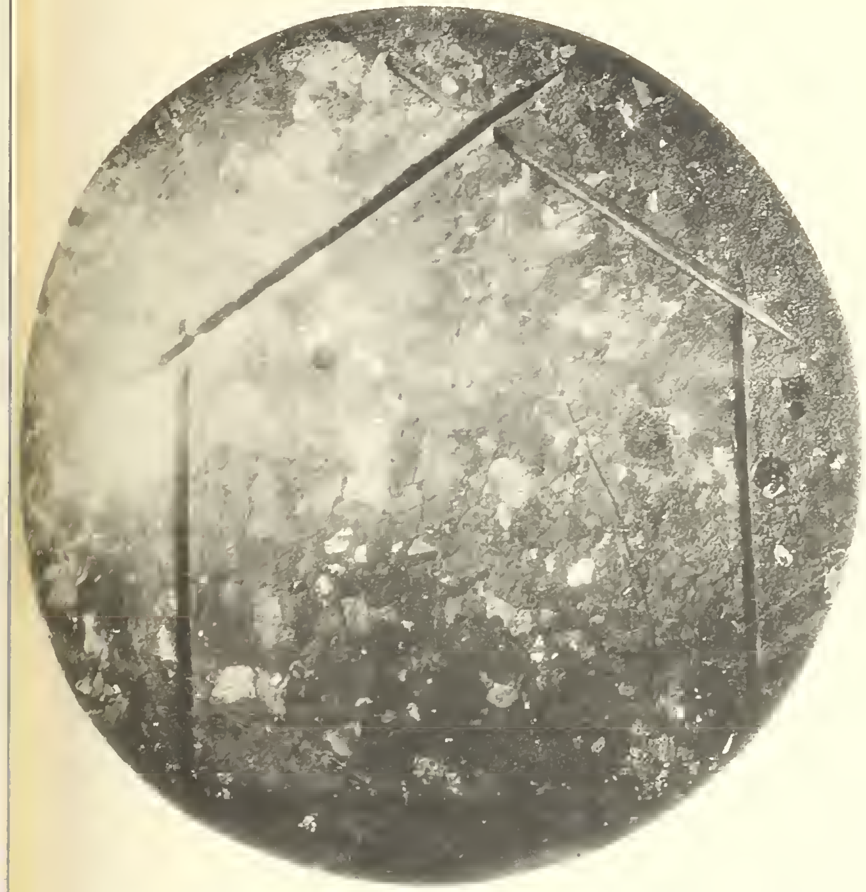

Fig. 24.—Same after 5 days 16 hours at 200° C. with incidence of light altered.

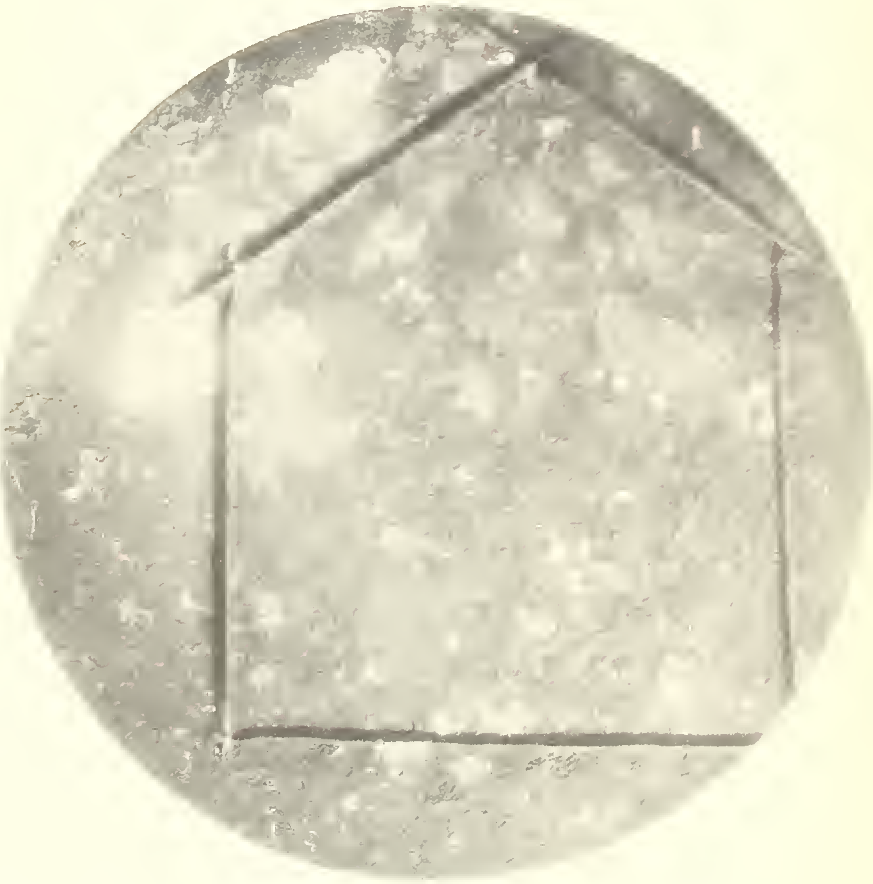

Fig. 25.—Same after 40 days at 200° C.

Fig. 26.—Same specimen as Fig. 25 (after 40 days at 200° C.) with incidence of light altered. × 8.

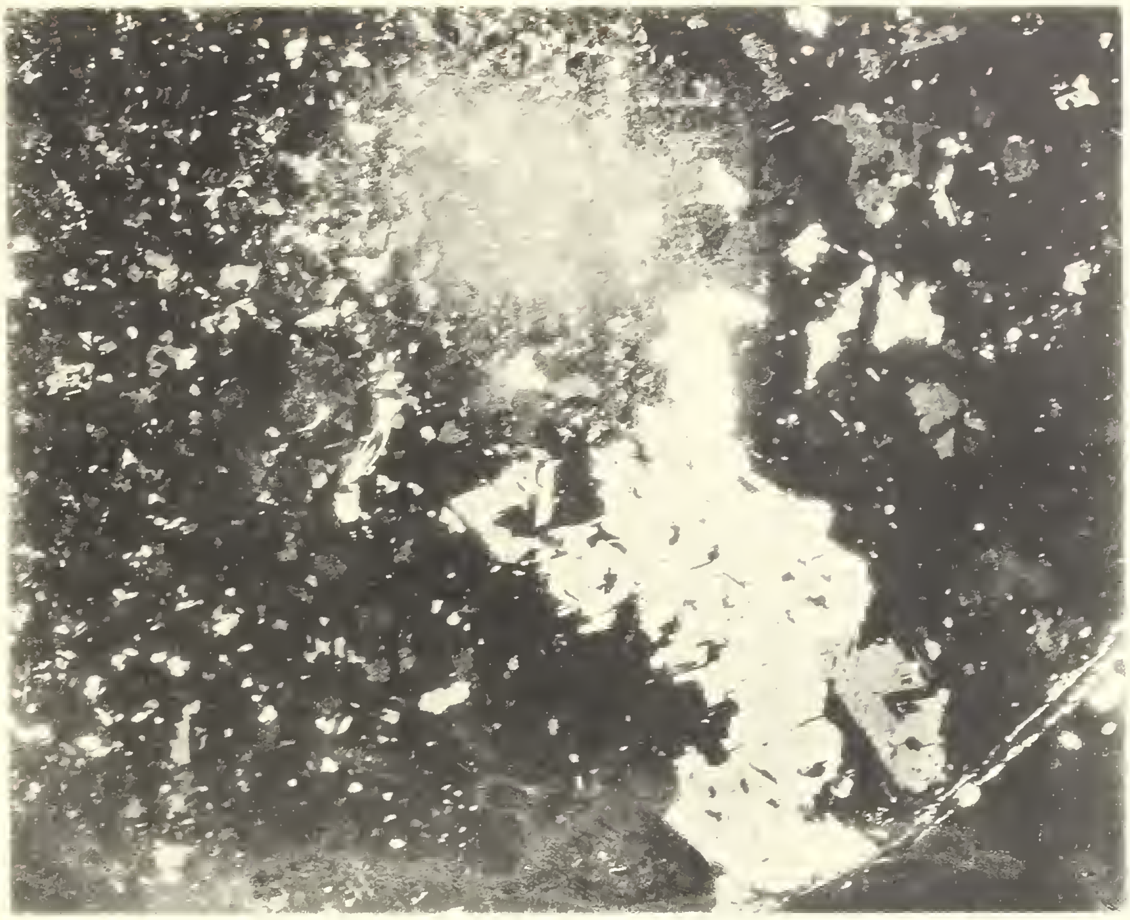

Fig. 27.—Crushed lead, after prolonged annealing. × 8.

Fig. 28.—Tin-plate, etched. × $\frac{1}{2}$.

Fig. 29.—Tin-plate, etched. × 100.

Fig. 30.—Tin-plate, etched. × 100.

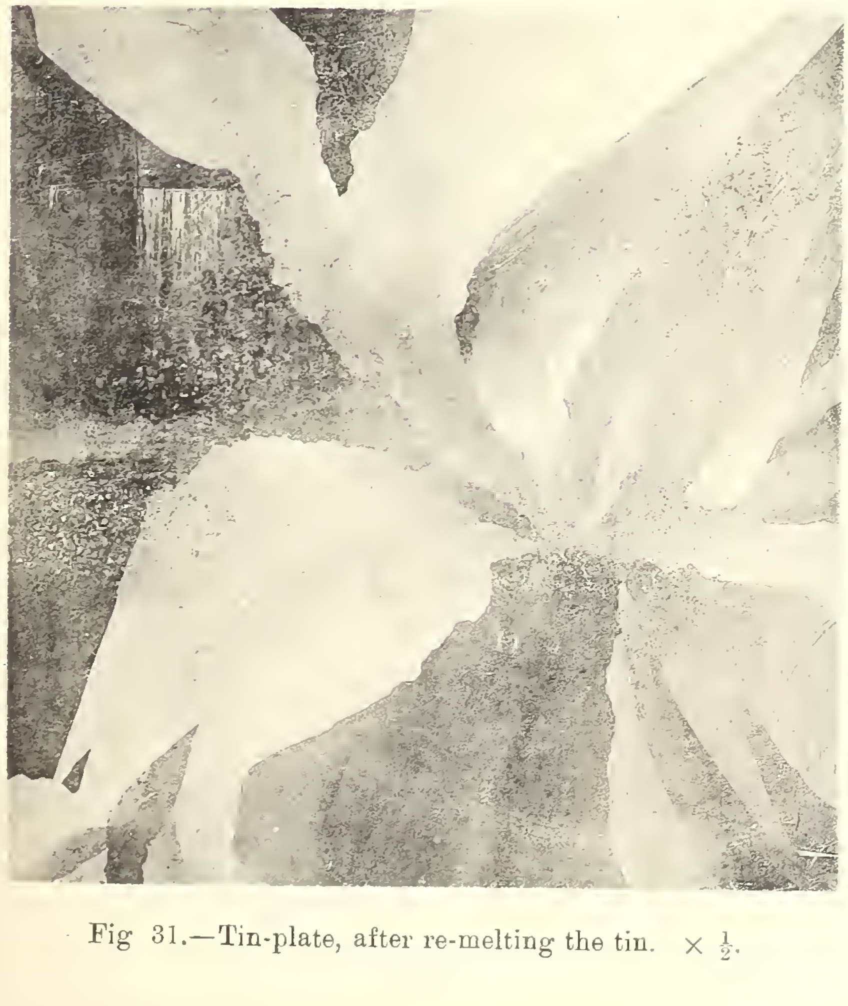

Fig 31.—Tin-plate, after re-melting the tin. × $\frac{1}{2}$.

Fig. 32.—Tin-plate, after re-melting the tin and cooling quickly. × $\frac{1}{2}$.

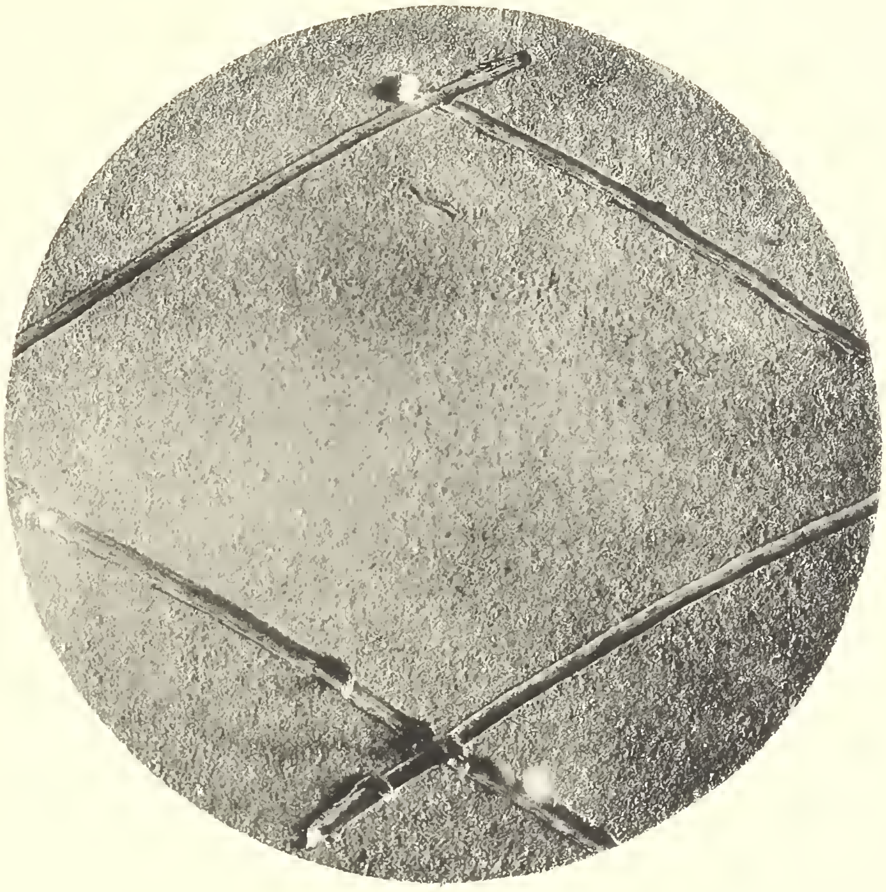

Fig. 33.—Cadmium, freshly strained. × 12.

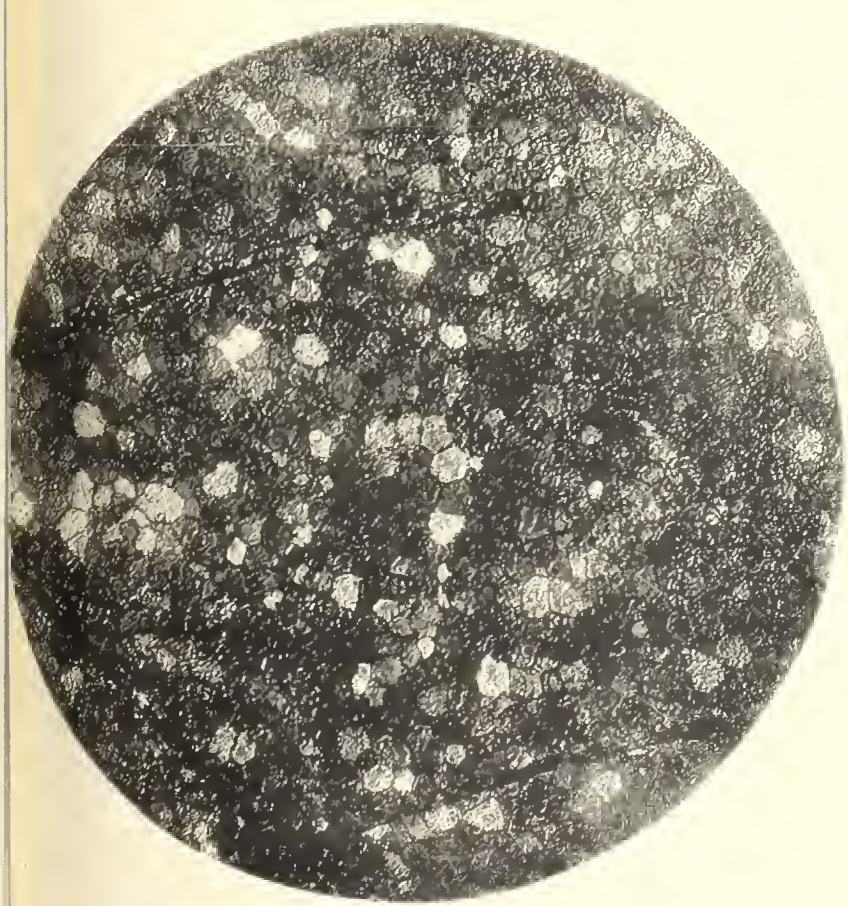

Fig. 34.—Same after 24 hours at 200° C.

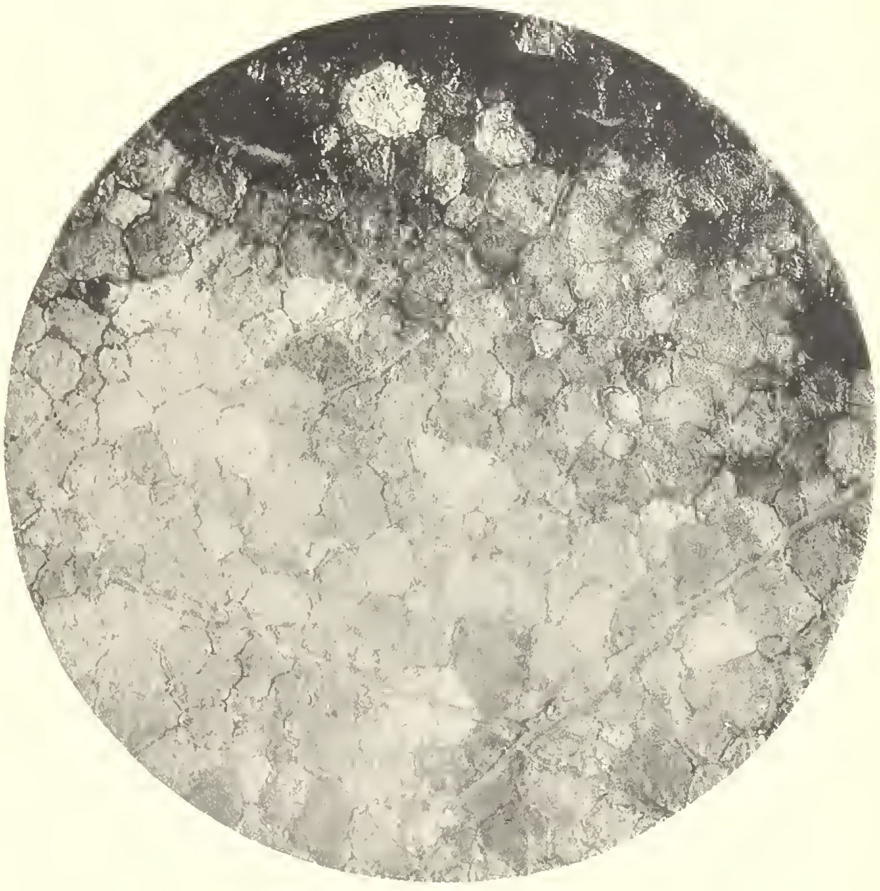

Fig. 35.—Same after seven days at 200° C.

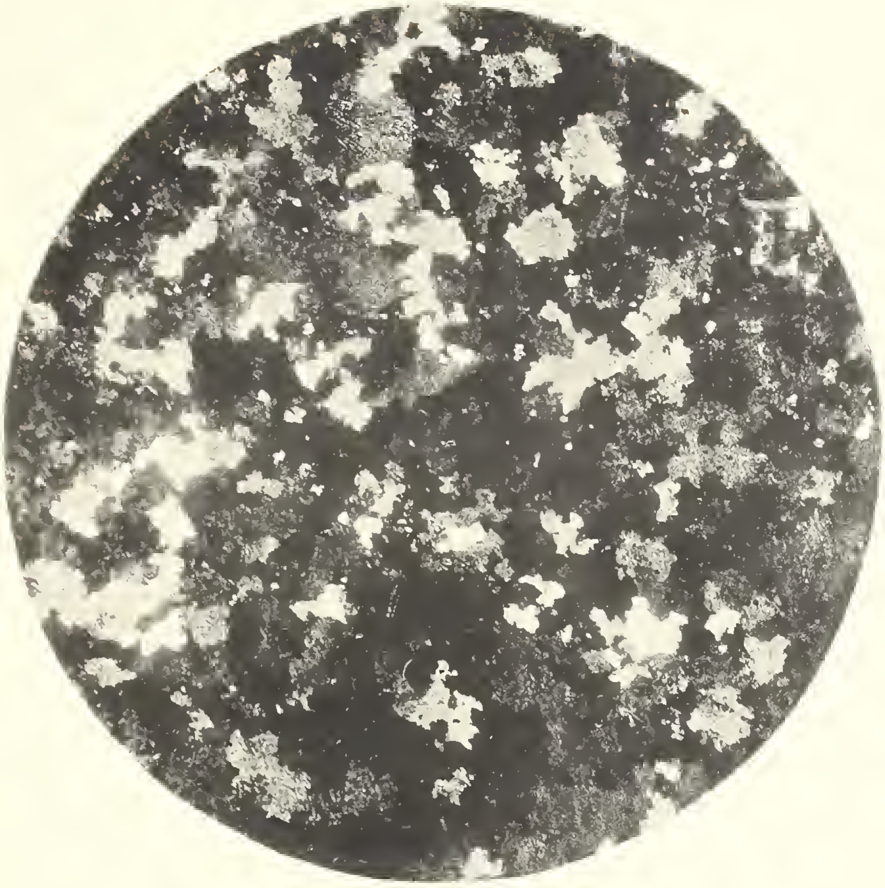

Fig. 36.—Zinc, after heating at 200° C. × 8.

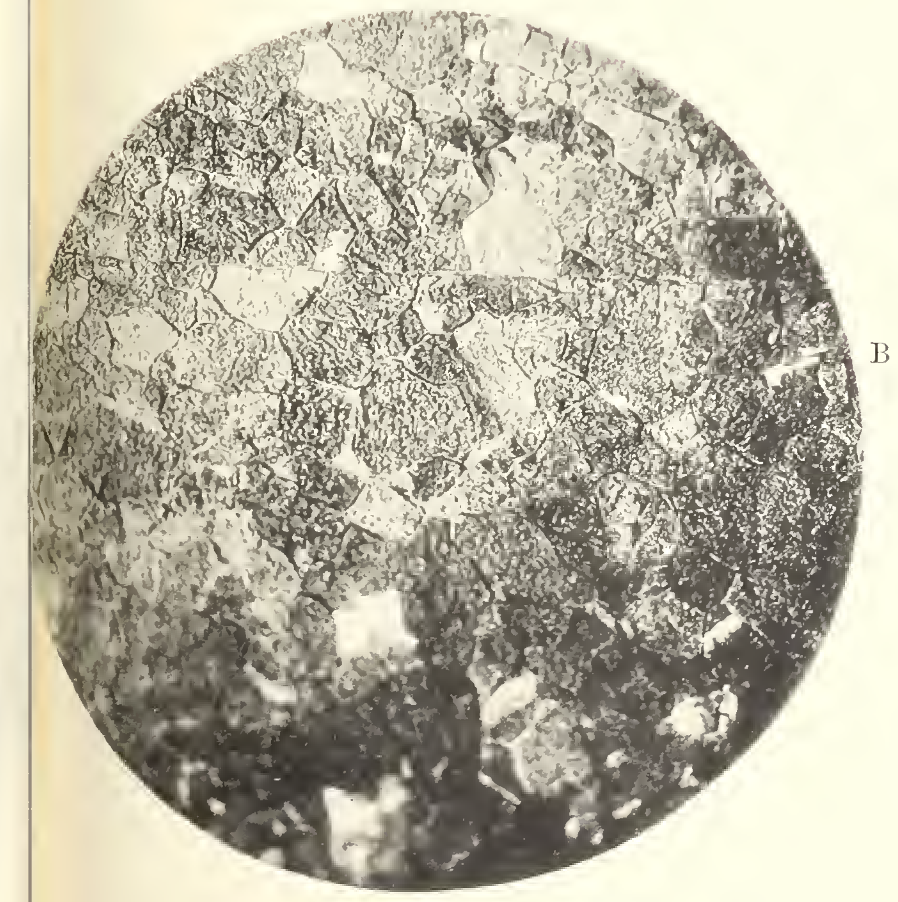

Fig. 37.—Weld in Lead. × 30

Fig. 38.—Weld in Lead, with eutectic. × 30.

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IX. *Lines of Induction in a Magnetic Field.*

By Professor H. S. Hele-Shaw, *LL.D.*, *F.R.S.*, *and* Alfred Hay, *B.Sc.*

Received June 13,—Read June 21, 1900.

[Plates 14-21.]

The following paper, which is partly experimental and partly mathematical, has arisen from the discovery that two-dimensional cases of magnetic lines of force could apparently be represented by the flow of a viscous liquid.* The original experiments upon which this assumption was made, showed that the stream lines which were obtained by the method in question, gave results very similar to those which had been calculated and plotted for the cases of an elliptical and circular cylinder. In order to ascertain definitely that the stream lines under these circumstances actually gave the exact position and direction of the corresponding magnetic lines of force, a result which, if verified, could be used for many practical investigations—it was necessary to undertake a long research dealing with the various points involved, a research which has proved extremely laborious, extending without intermission over a period of nearly two years.

In the first place it was necessary to devise some method by which a thin sheet of transparent or semi-transparent medium could be obtained of any required thickness, and on which, when placed between two sheets of glass, the required section of the body to be investigated could be formed.

Next it was necessary to determine the laws connecting the thickness of the thin film of liquid with the quantity flowing through it in a given time, so that the relative differences of thickness corresponding to the differences of permeability of the substances in a magnetic field could be ascertained.

Lastly, a mathematical investigation was undertaken of some cases suitably selected so as to afford, when plotted out, as severe a test as possible for ascertaining if the experimental method really determined for any case, accurately, the position and character of the lines of force in a magnetic field.

It may at once be stated that after overcoming in succession a very large number of difficulties, the case selected, viz., that of an ellipse with the major axis parallel

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* "Stream-line Motion of a Viscous Film": 'British Association Report' (Section A), Bristol Meeting, 1898.

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to the lines of force in the undisturbed magnetic field, showed unmistakably absolute agreement between the results obtained by calculation and experiment.

On account of the great practical importance of magnetic phenomena, various methods have from time to time been devised for the study of magnetic fields, and by means of such methods the approximate distribution of the lines of magnetic induction in various cases may be investigated. One of the earliest of these methods was that in which the lines are approximately mapped out by means of iron filings. This method, which has been known for a long time (LA HIRE mentions it as far back as 1717), was very largely used by FARADAY in his study of magnetic fields. It must, however, be regarded as a very rough-and-ready method, and does not enable us to trace out the exact shape of the lines with any degree of accuracy. Another method, which when carefully applied is capable of giving good results, is that in which a magnetic needle is moved from point to point of the field, the consecutive positions of the two ends of the needle being marked, and a line drawn through the points so determined. This method gives the shape of the lines very accurately if the length of the needle is small in comparison with the radius of curvature of the lines. But although in a diagram so obtained the relative intensity of the field at various points may be roughly estimated by noting the convergence or divergence of the lines, yet it is impossible to indicate this variation of intensity by the distance apart of consecutive lines, since there is no means of ascertaining how far apart the consecutive lines should be drawn. A third method, which we owe to the genius of FARADAY, and which is one of very great practical importance, is that of a search-coil connected to a ballistic galvanometer. In applying this method, the search-coil is either jerked out of a given position in the field, or turned through a small angle, or else the field is suddenly removed, or reversed, while the search-coil remains stationary. This method, however, notwithstanding its great importance as a method for ascertaining the magnetic flux through a given area in the field, can hardly be regarded as a method for delineating lines of induction. The same remark is applicable to two other methods; that in which the field intensity is ascertained by measuring the resistance of a bismuth spiral; and that in which a conductor conveying a known current is placed in the field, and the pull on the conductor is measured.

The experimental method which the above investigation proves to be accurate, is applicable to two-dimensional problems only. In so far as this is the case, its scope may appear restricted. But when we consider the fact that in the practical applications of magnetism, the bulk of the phenomena with which we have to deal are of the two-dimensional order, it will be recognised that it offers a solution of problems whose interest and importance are by no means inconsiderable. The magnetic field in and around the armature of a dynamo or alternator, and that in and around a cylindrical case which is used for purposes of magnetic shielding, are examples of important practical problems in two dimensions.

One very important feature which characterises the method, is the fact that it enables us to obtain the exact shape of the lines of induction not only in air, but *in the para-magnetic or dia-magnetic material itself.* This cannot be accomplished by any of the older methods. The method has therefore been applied to determine the exact form of the lines of induction for a number of cases of mathematical interest, several of which have been also plotted by calculation. Inasmuch as the laws connecting the relative thicknesses of two liquid films with the corresponding values of the permeability, have by the above investigation been determined, it is possible to investigate any practical case which may be of interest, and some examples are given of the application of the method in cases of interest to electricians.

The mathematical portion of the work which has been necessary, has not—as far as the authors are aware—hitherto been available in any published form, and although, as will be seen from the historical statement, parts of the problem have been dealt with by various writers, it is hoped that the account given will be found of use in the further investigation of this subject.

The scheme of the paper is as follows:—

I. (*a*) The experimental determination of the relation between the thickness of film and the corresponding rate of flow.
  (*b*) Statement of results and comparison with theory.
  (*c*) Description of appliances used in obtaining the stream-line diagrams.

II. (*a*) Account of the test case worked out mathematically and plotted, and its comparison with the actual photograph obtained by experiment.
  (*b*) Examples of various cases of mathematical interest, and also some of practical importance to the electrical engineer.

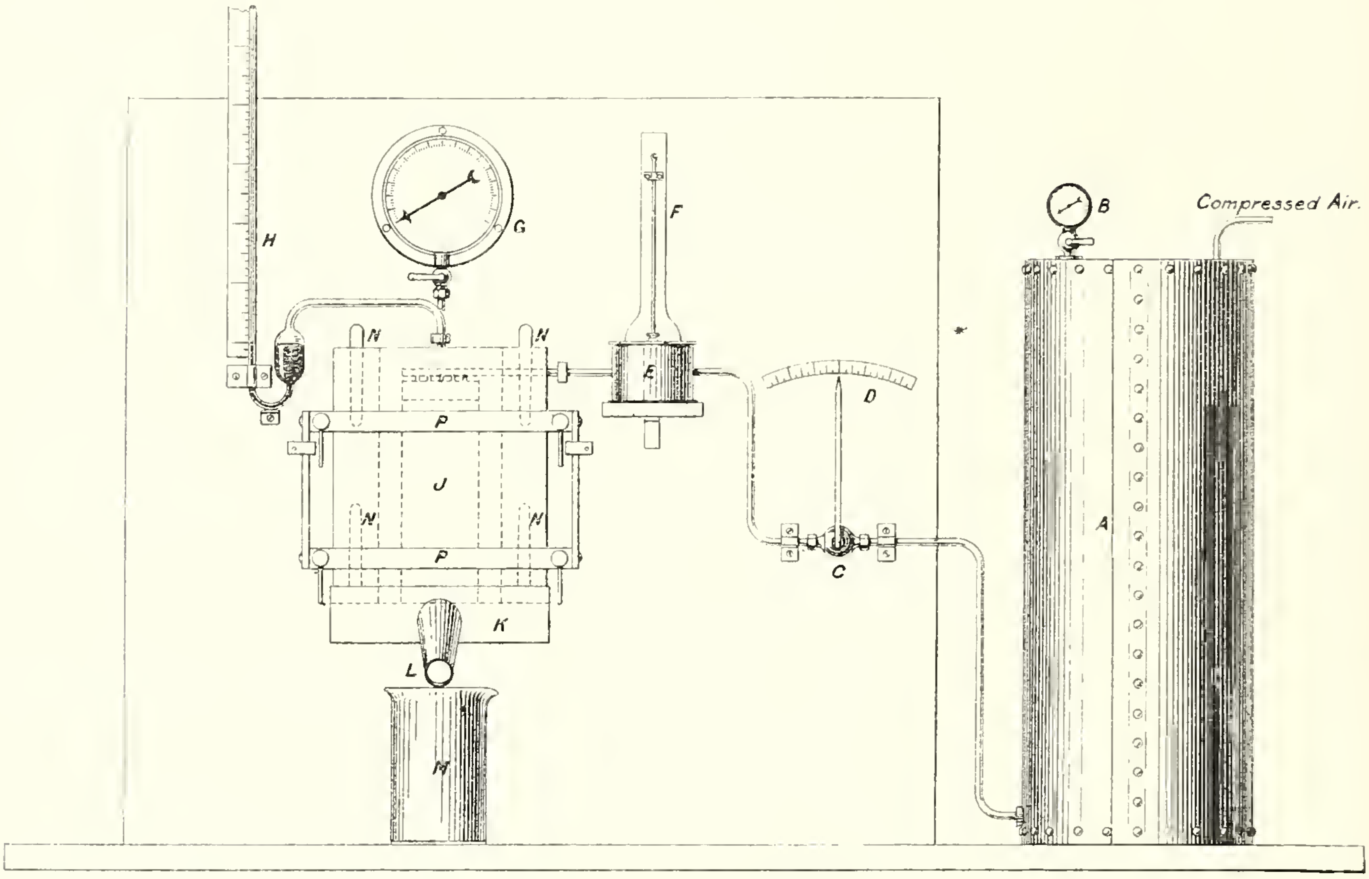
III. (*a*) Brief history of mathematical investigation of the subject, and
  (*b*) A general method of dealing with elliptic cylinders and con-focal elliptic shells.

SECTION I.

(*a*) The liquid is contained in the cylinder A, fig. 1, from which it is forced by means of compressed air, the pressure being determined by the pressure gauge B. It then passes through the cock C (fine adjustment being secured by the lever D which travels over a scale) to the thermometer box, E, where its temperature is recorded upon the thermometer F, and from thence to the slide J.

The slide consists of two sheets of plate-glass fixed within the frame P; a well is cut in the lower sheet into which the liquid is introduced, a channel being formed between the plates by using tin-foil as a border. The correct thickness is obtained by screwing down the plates upon the thickness gauges N, N, N, N, placed at the four corners of the slide. The pressure in the well, when low, is found by means of the barometer tube H: when high, by the pressure gauge G.

Fig. 1.



Apparatus for Determining the Viscosity of a Fluid by the method of a Thin Film.

The liquid is caught in the box K from which it passes through the spout L, into the vessel M, where it is collected. When viscous, the liquid was allowed to flow for a fixed time into M, and then weighed; when more mobile, it was collected into a larger vessel of known capacity, and the time observed which it took to fill the vessel up to a notch accurately made for the purpose.

The following tables give the results of the measurements:—

EXPERIMENTS on Flow of Thin Film between Parallel Plates.

(1) Glycerine at 20 lbs. pressure.

A = Amount in grammes flowing through in 10 minutes.
B = Time for 1000 grammes to flow.
C = Velocity in centims. per minute.

Thickness.	Temperature.	A.		B.	C.
inch.	° C.			minutes.	
·004	19·2 19·4	19·33 18·66	19	527	24
·006	17 17 17	60 64 61	61·6	162	51·8
·008	17 17 17·2	110 112 108	110	91	69·5
·009	17 17 17	170 168 166	168	59·5	94·2
·010	16·6 16·6 16·4	195 193 192	193·3	51·8	97·5
·011	17 17 17 16·4	271 269 268 267	268·75	37·3	125·5
·013	17 16·8 16·8	441 420 428	429	23·4	166·5
.014	17 17 16·6	556 559 556	557	18	210
·016	17·6 18	862 860	861	11·6	271
·017	19·6 19·6 19·5	1090 1076 1070	1078·6	9·28	320
·018	17·6	1174 1170	1172	8·04	329
·020	17·6 17·6	1640 1658	1649	6·05	416

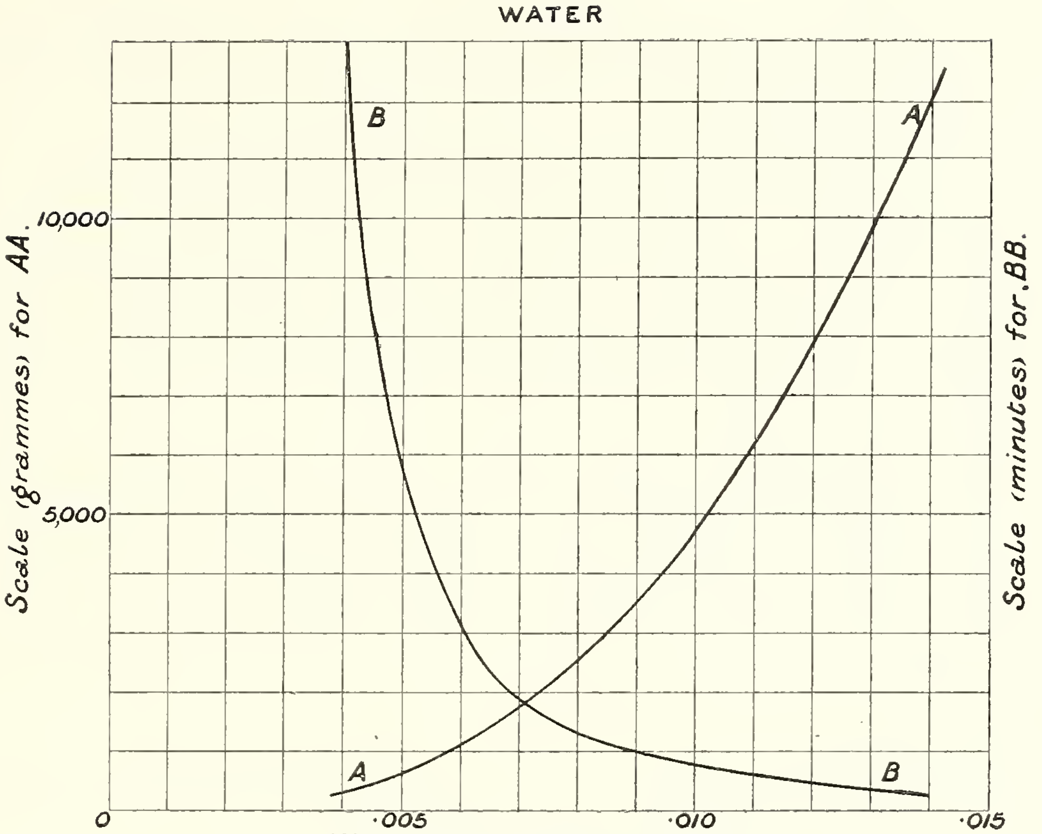
EXPERIMENTS on Flow of Thin Film between Parallel Plates.

(2) Water at 21 lbs. pressure.

A = Amount in grammes flowing through in 10 minutes.
B = Time for 3850 grammes to flow.
C = Velocity in centims. per minute.

Thickness.	Temperature.	A.	B.	C.
inch.	° C.		minutes.	
·004	16·6 14·2 14·3 13·8	308 307 305·5 305·5 } 306·5	126	475
·005	15 15 15·8 15·8 15·8	630 667 665 660 661 } 656	58·7	810
·006	16 16 16	1232 1226 1230 } 1229	31·3	1270
·007	17·5 16·8 15	2050 2058 2040 } 2049	19·2	1817
·008	13·6 13·5 13·5	3040 3048 3040 } 3043·6	12·65	2350
·009	15·2 15·4 15·6	3760	10·28 10·28 10·25 } 10·26	2600
·010	13 12·9 12·8	4470	8·65 8·58 8·58 } 8·6	2760
·011	16·1 16·3 15·2 15 14·6 14·5	6370	6·03 6 6·03 6·03 6·11 6·086 } 6·048	3580
·012	12·4 12·4 12·5 12·4	7050	5·38 5·48 5·5 5·48 } 5·46	3630
·014	14·8 14·8 14·6 14·3	12050	3·166 3·186 3·166 3·23 } 3·187	5325

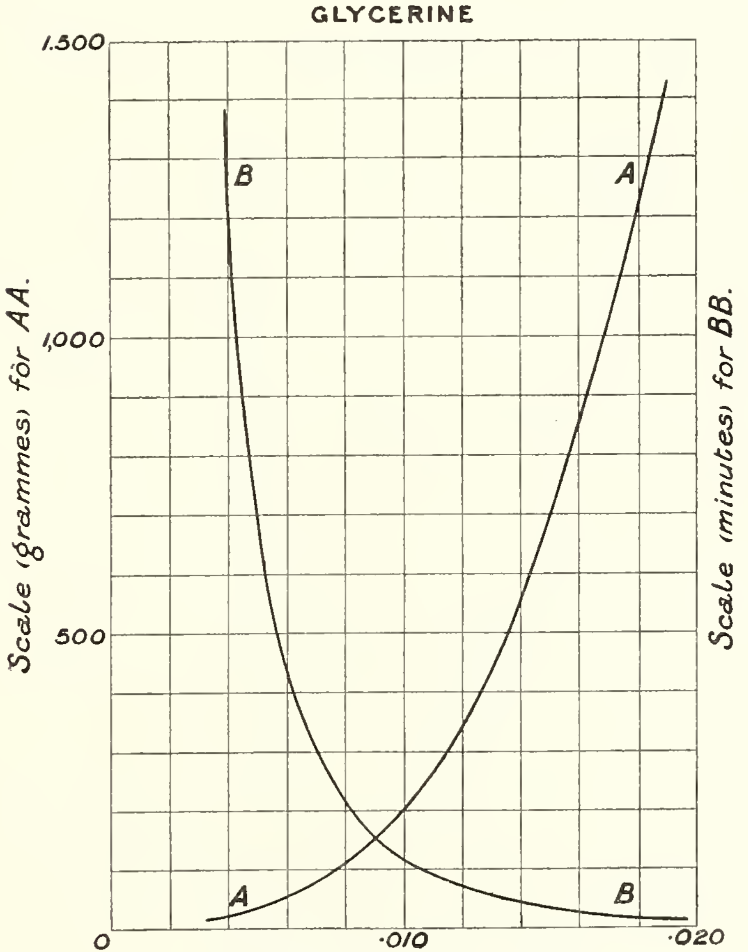
Fig. 2.



Thickness of film in inches.

AA. Quantity for given time.
BB. Time for given quantity.

Fig. 3.



Thickness of film in inches.

AA. Quantity for given time.
BB. Time for given quantity.

From these tables the curves shown in figs. 2 and 3 have been drawn, and these curves were actually used to determine the relation between different depths of well and corresponding permeabilities.

The authors are indebted to Mr. J. C. W. HUMFREY, B.Sc., Victoria University Scholar in Engineering, for valuable assistance in carrying out the above experiments.

(*b*) After the results embodied in the curves, figs. 2 and 3, had been obtained, we thought it would be interesting to see in how far they were in agreement with the deductions from theory. Let

t = thickness of liquid film.
s = tangential stress (per unit area) due to viscosity at a distance x from the middle layer.
v = velocity of flow at distance x from middle layer.
η = coefficient of viscosity.

Then

$$s = \eta\, dv/dx.$$

At a point distant $x = \delta x$ from the middle layer the tangential stress will be

$$s' = \eta \frac{dv}{dx} + \eta \frac{d^2v}{dx^2} \delta x.$$

Hence if we consider a layer of unit length, unit width and thickness δx, the drag to which this elementary layer is subject on account of viscosity is given by

$$\eta \frac{d^2v}{dx^2} \delta x.$$

Now, since we suppose that the flow is steady, the elementary layer considered is not undergoing any acceleration. Hence the backward drag due to viscous resistance must be balanced by the forward push due to the difference of pressure over the two ends of the elementary layer.

If dp/dy stand for the gradient of pressure, then

$$\eta \frac{d^2v}{dx^2} \delta x = - \frac{dp}{dy} \delta x,$$

or

$$\eta\, d^2v/dx^2 = - dp/dy = - f, \text{ say,}$$

where f is the fall of pressure per unit length of the liquid layer.

Integrating this equation once, we get

$$\eta\, dv/dx = - fx + C,$$

where C is a constant. Since when $x = 0$, $dv/dx = 0$ (the velocity in the middle being at its maximum), $C = 0$. Integrating a second time, we get

$$\eta v = - \tfrac{1}{2} f x^2 + C'.$$

In order to find C', we notice that $v = 0$ when $x = \frac{1}{2}t$, the velocity at the boundary vanishing. This gives

$$v = \frac{f}{2\eta}\left(\frac{t^2}{4} - x^2\right).$$

The volume flowing through per second per unit width of the layer of thickness t is

$$q = \int_{-\frac{1}{2}t}^{+\frac{1}{2}t} v\,dx = 2\int_0^{\frac{1}{2}t} v\,dx$$

$$= \frac{ft^3}{12\eta}.$$

The rate of flow of a viscous liquid in a thin layer between parallel plane walls is thus, for a given fall of pressure along the layer, seen to be proportional to the cube of the thickness of the layer.

An examination of the curves obtained experimentally shows a satisfactory agreement with this theoretical law.

The data contained in the curves also enable us to calculate approximately the coefficient of viscosity. The apparatus is not, however, well adapted for exact measurements, since, in addition to the great difficulty in measuring accurately the thickness of the liquid film, it is doubtful if the glass plates are either sufficiently true or sufficiently rigid (notwithstanding their great thickness) for refined measurements. Since the formula

$$\eta = \frac{ft^3}{12q}$$

for the coefficient of viscosity involves the cube of the thickness of the liquid layer, a comparatively small error in the determination of t will give rise to a large error in η. Taking the water curve (fig. 2), we find for the coefficient of viscosity in C.G.S. units (using the point on the curve corresponding to a thickness of ·012″) the value ·0092, a value which is considerably too low, and which may be accounted for either by an error in estimating the thickness, or by slight irregularities in the containing walls, or slight bulging of the walls in the centre—all of which causes combined might easily account for the error.*

The glycerine curve (fig. 3) gives a value of the viscosity equal to 2·5; the density of the glycerine being 1·23. This value is in fair agreement with the results of previous experimenters.†

As, however, our object was not to carry out careful absolute measurements, but to find relative values which would enable us to apply the method to two-dimensional problems in magnetic induction, and, as the curves obtained (figs. 2 and 3) were

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* It must also be remembered that the water used in these experiments was ordinary tap water, which had not been freed from dissolved gases.—July 18, 1900.

† See p. 131 of ARCHBUTT and MOUNTFORD's 'Lubrication and Lubricants.'

found to be in agreement with the theoretical law, we did not pursue the subject of viscosity measurements any further.

(*c*) A description of the general method of providing for the flow of colour bands in a thin film has already been given in a paper read before the Institution of Naval Architects.* The apparatus, however, for doing this has been considerably improved and simplified, and shows in figs. 4, 5, and 6, the combination of glass plates which has been devised in connection with the present paper. It will be seen that instead of having three parallel plates of glass, only two are used, in one of which—made very thick—two wells are sunk, used for the clear and coloured liquids respectively. The figures show the construction of the apparatus, and it may be remarked that with this contrivance it is very easy to use a border of paper, cut from a template, and to employ the opposite plate of glass, or the cover-plate, for the purpose of providing the required well or depression through which the liquid flows. The depth of this well is made to correspond to the required magnetic permeability, the outline of the well having the form of a cross-section of the body under investigation. How this well or depression was obtained must now be described, as the difficulties experienced in this case proved much greater than when a solid obstacle was required, as in the paper above mentioned.

In order to solve the problem, it was necessary to obtain a transparent material that could be worked to any shape, and be cemented to the glass cover-plate of the slide to form the recess or well, corresponding in shape to the paramagnetic body for which the lines of induction were required. Originally, for simple circular forms, a sheet of glass such as is used for microscopical work, was cemented to the cover plate, but this was not satisfactory, as the glass could not readily be obtained of the exact thickness required. Further, the thickness of the layer of cement between the plates rendered accuracy very difficult to secure, and any figures other than circles could not easily be cut in glass. Celluloid had the advantage that it could be worked to any shape; but again the difficulty arose that it could not be obtained of the thicknesses required for different resistances, and moreover could not be cemented to the cover plate so as to lie perfectly flat. Various papers and tracing cloths were tried, but they all more or less altered in thickness when moistened, and it was ultimately found that any material requiring to be cemented to the glass could not be used unless it was capable of being brought to a true surface afterwards, as no cement was found capable of holding sufficiently for the object in view. Plates of white metal were thought of, but the labour involved in producing them, and the difficulty of obtaining photographs of the stream-lines, led to the abandonment of this idea also.

The method finally adopted was to cast a layer of paraffin wax on the cover-plate of glass, and although this was not transparent enough to enable the stream-lines

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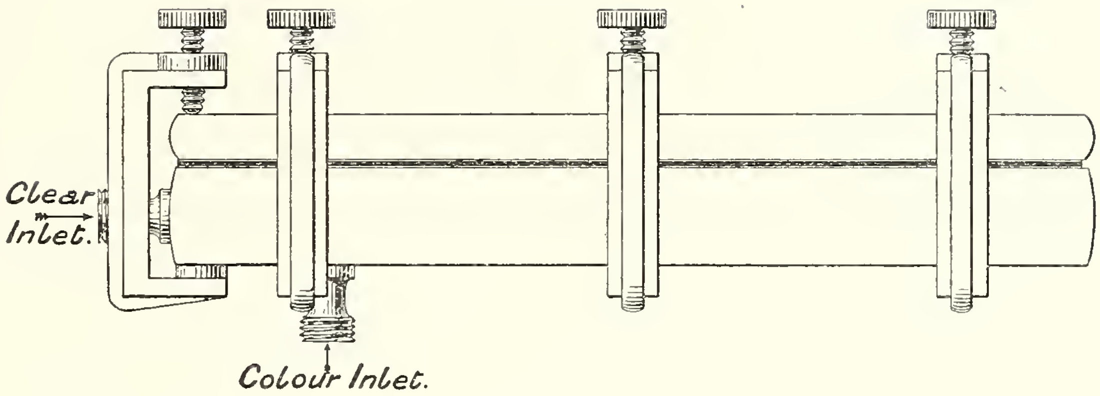
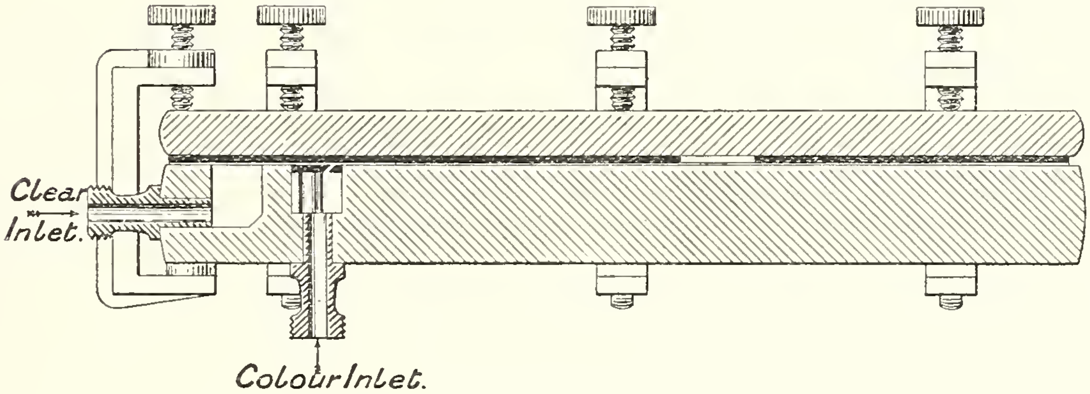
* "Investigation of the Nature of Surface Resistance of Water and of Stream-line Motion under Certain Experimental Conditions." 'Trans. Inst. Naval Architects,' vol. 40, 1898.

Fig. 4.



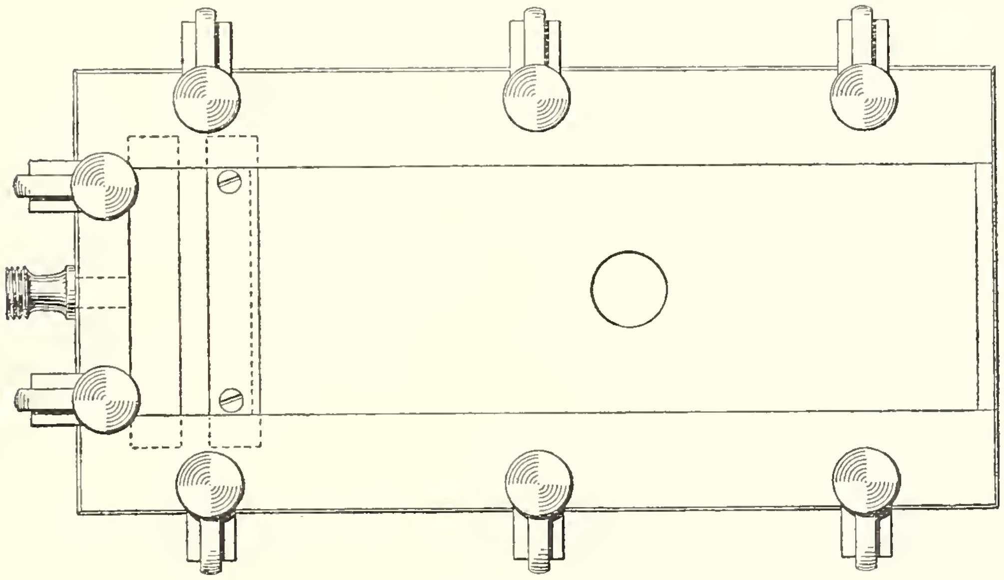
Side view.

Fig. 5.



Sectional view.

Fig. 6.

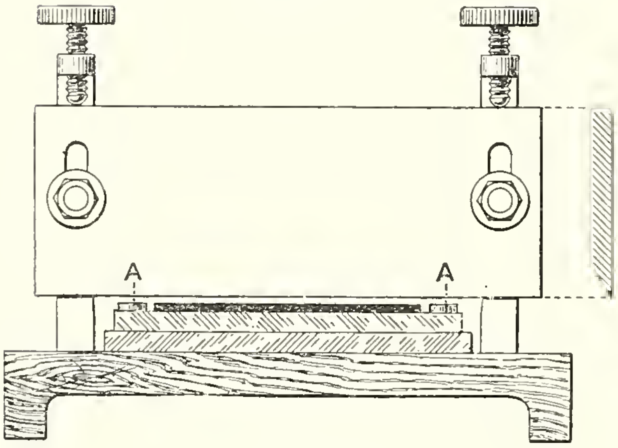


Front view.

Arrangement of Glass Plates and Clips.

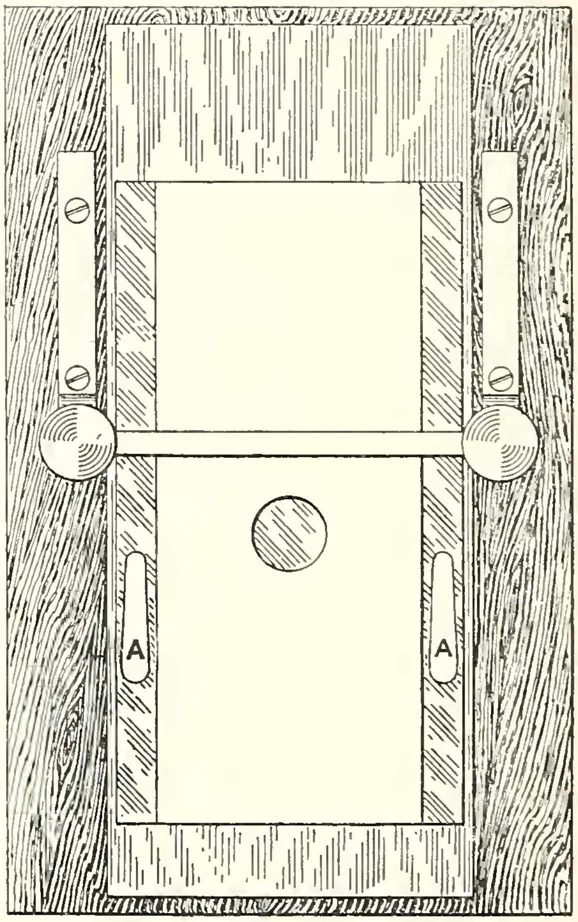
to be projected on the screen, it allowed, when illuminated from behind, sufficient light to pass to enable the stream-lines to be photographed. At first the difficulties of using paraffin wax were numerous, for on casting the wax and allowing it to cool, it became covered with star-like cracks beneath the surface, which were evidently caused by unequal heating and cooling, and imprisoned air bubbles. Casting in a vacuum was also tried, but without success. The difficulty was at length overcome by a systematic method of covering the plates, which was as follows: The plate on which the layer of wax was required, was placed in an iron tray, the wax was shredded on to the plate, and the whole heated gradually and uniformly, a plate of iron 1 inch thick being inserted between the tray and the Bunsen burner used for heating. By this means the plate could be covered to any required thickness in a satisfactory manner. In order to reduce the layer of wax to the thickness required for the experiment, the simple planing machine illustrated in figs. 7 and 8, was used:

Fig. 7.



End elevation.

Fig. 8.



Plan.

Planing Instrument for Paraffin Surface.

the gauges (AA) for obtaining the required thickness being shown in position. The milled-headed screws enabled the cutter to be adjusted so that the desired thickness of wax could be accurately obtained of any dimensions above ·004 inch to within ·001 inch with the greatest ease. The method of illuminating the slide had, however, to be abandoned, as the heat of the arc lamp melted the wax, thus at once putting an end to the experiment. Instead of this, the slide was taken to a place where it could be illuminated by strong daylight, and the exposure given to the sensitive photographic plate was of course correspondingly increased. Zinc templates of the shapes required were made, and by placing the template on the wax, the outline was cut with a sharp knife, and the particular well thus formed. In cases where the flow in the well was very slow, air bubbles became imprisoned, and could only be removed after great difficulty; in fact, in one or two cases—as will be seen in the photographs, figs. 31 and 32 (Plate 19)—they could not be removed at all.

Section II.

(*a*) In order to test the applicability of the stream-line method to the solution of two-dimensional magnetic problems, it was decided to work out mathematically the case of an infinite cylinder of elliptic section placed in an originally uniform magnetic field with its major axis along the field, to plot the lines of magnetic induction corresponding to a permeability of 100, and to compare the diagram so obtained with a stream-line diagram. The theoretical diagram is given in fig. 9 (Plate 14) and the corresponding stream-line diagram in fig. 10. In making the comparison, a greatly enlarged photograph of fig. 10 was prepared, and fig. 9,* which was actually drawn to a much larger scale, was then superposed on it: the coincidence of the lines in the two diagrams satisfactorily established the soundness of the stream-line method. It may be noted, however, that slight local divergences along the elliptic boundary are clearly observable. Instead of the sharp refraction of the lines as they enter the elliptic cylinder in the theoretical diagram, we have in fig. 10 a slight curvature at the ends of the otherwise perfectly straight lines crossing the ellipse. This feature is noticeable, to a greater or smaller extent, in all the stream-line diagrams accompanying the present paper. It is more marked in those cases where the difference between the thicknesses of the two liquid layers, *i.e.*, the permeability in the corresponding magnetic problem, is greater. It is clear that the presence of the highly permeable cylinder disturbs the originally uniform distribution of the lines, and in order to effect a satisfactory comparison between the theoretical and the experimental diagrams, it became necessary to assign to the liquid layer an external boundary whose shape corresponded to that of a particular stream-line in the theoretical diagram. The boundary of the liquid in fig. 10 is clearly shown by the dark shadows on either side of the diagram, and the shape of their boundaries is the same as that of the outer stream-lines in fig. 9. The method by which the solution was obtained for the theoretical case will be found fully explained in the mathematical appendix to the present paper.

(*b*) In most of the succeeding diagrams the boundaries of the liquid layer are straight lines. Interpreted magnetically, this means that the diagram gives the

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* For details see Mathematical Section of this paper.

solution not for a single cylinder of the given cross-section placed in a magnetic field, but of a whole row or grating of such cylinders, the distance apart of any two neighbouring cylinders being equal to the breadth of the liquid film.

Fig. 11 gives the solution for an elliptic cylinder similar to that in fig. 10, but having a permeability of 20. In fig. 12 we again have a cylinder of the same cross-section, but of permeability equal to 1000. On account of the great depth of the elliptic wall, the lines crossing the ellipse appear to be almost entirely obliterated; notwithstanding this fact, they emerge on the other side, preserving their identity and not mixing with the body of the liquid. The distortion of the lines close to and on the inner side of the elliptic boundary is in this case very strongly marked, and well illustrates one of the difficulties encountered in attempting to imitate the effects of highly permeable bodies.

An interesting feature, well-known as a result of theoretical deductions, is clearly brought out by a comparison of the three diagrams, viz., the gradual decrease in the angle made by the external lines with the normal to the ellipse as the permeability is increased. Thus in fig. 11 this angle of incidence of the lines is quite large at certain points of the elliptic boundary; it is greatly reduced in fig. 10, and in fig. 12, for which the permeability is 1000, it is practically zero.

It will be seen that the refraction of the lines in fig. 11 is very sharp—the "weir effect" being extremely feeble. No trouble was experienced on this account in any of the diagrams so long as the permeability did not exceed about 100.

Figs. 13 and 14 (Plate 15) are a set relating to circular cylinders of permeability 2 and 100 respectively. They illustrate clearly the well-known theoretical result that, on account of the shape of the cylinder, large changes of permeability produce only relatively slight changes in the magnetic induction through the cylinder.

In figs. 15—20 we have a set of diagrams which will be found interesting in connection with the important question of magnetic shielding—a subject which has recently attracted a good deal of attention. Fig. 15 is a diagram corresponding to the case of a hollow circular cylindric shield surrounding a solid cylinder, both cylinders having a permeability of 100. Fig. 16 relates to a hollow cylinder whose cross-section is bounded by two confocal ellipses. As shown in the mathematical appendix, the field produced inside such a cylinder is uniform if the original impressed field is uniform. Fig. 17 (Plate 16) represents the effect produced by a double cylindrical shield; it is one of the earliest diagrams obtained by us, and the permeability is so extremely low that, even with the double shield, the field in the innermost space is comparable with the undisturbed field. It would correspond to a highly-saturated double cylindric shield in a field of very great intensity. Figs. 18, 19, and 20 have a more practical interest, the permeability being 100 in each case. In fig. 18 we have a double cylindric shield, with a solid cylinder of iron placed concentrically in the innermost space; the powerful shielding effect on the central cylinder is sufficiently evident. Fig. 20 shows the effect produced by a triple concentric shield. It is interesting to compare this latter stream-line diagram with the theoretical diagram of fig. 19; the agreement in the shape and general distribution of the lines between the two cases is very striking.

Fig. 21 (Plate 17) is a theoretical diagram for an elliptic cylinder of permeability 100, and the ratio of whose axes is 2 : 1, placed in a uniform field with the major axis of the elliptic section of the cylinder making an angle of 45° with the impressed field. Fig. 22 gives a stream-line diagram for a cylinder of the same permeability, but the ratio of whose axes is 3 : 1. Fig. 23 is a similar diagram for a very thin elliptic plate, and fig. 24 relates to the hollow cylinder of fig. 16, but here turned through an angle of 45°. It will be noticed that fig. 24 confirms the theoretical result that the field in the interior of a hollow elliptic cylinder bounded by two confocal surfaces is uniform if the impressed field be uniform.

We have hitherto dealt with cases which may be treated theoretically as well as experimentally. But the number of such cases is very limited, and the vast majority of two-dimensional magnetic problems are beyond the powers of analysis. It is in such cases that the stream-line method employed by us becomes a powerful weapon of research. Figs. 25–28 relate to cylinders of rectangular section, figs. 25 and 26 giving the field distributions for cylinders of square section placed with one of their diagonals at 45° to the field and parallel to it respectively. In fig. 25, where the width of the cylinder remains constant along the direction of the impressed field, we notice that the lines inside the cylinder are concave outwards; in fig. 26, on the other hand, where the cylinder tapers rapidly as we proceed along the field, the curvature of the lines presents a convexity outwards. This suggests that an intermediate form between the two might be found for which the lines exhibit neither convexity nor concavity, *i.e.*, are straight; and, as a matter of fact, we know of one such intermediate form—a circular cylinder. Fig. 26 closely corresponds to the theoretical case which Dr. C. H. LEES has recently succeeded in working out analytically in connection with a problem in heat conduction.* Figs. 27 and 28 are intended to illustrate the effect of increasing the length of one of the sides of the rectangular section while keeping the other constant. We know from theoretical considerations that this has the effect of reducing the de-magnetising factor, and thus increasing the flux through the cylinder. This point is very clearly brought out by a comparison of figs. 25, 27, and 28.

Figs. 29 and 30 (Plate 19) show the magnetic fields corresponding to a cylinder of triangular section in two different positions.

Figs. 31 and 32—a solid circular cylinder inside a hollow square one, and a solid square one inside a hollow circular one—are interesting in connection with the problem of magnetic shielding. Both these diagrams are slightly disfigured by air-bubbles. These latter are extremely difficult to get rid of, once they are allowed to reach a part of the field where the thickness of the liquid film varies.

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* 'Phil. Mag.,' February, 1900, p. 225.

Fig. 33 (Plate 20) is a field diagram for a hollow square cylinder; the shielding effect is seen to be very powerful.

Fig. 35 gives approximately the field distribution between the tapered pole-pieces of an electro-magnet; and fig. 36 is intended to illustrate the pull of an electro-magnet on an armature.

The next few diagrams relate to cases of practical interest and importance, and serve to show how the stream-line method may be made to yield results of great interest to the electrical engineer. Figs. 37 and 38 (Plate 21) show an ordinary SIEMENS shuttle-wound armature—such as is used in connection with telephone call apparatus—in two positions. The direction of the torque acting on the armature in its second position may be at once inferred from an inspection of the diagram. It is also interesting to note the leakage lines outside the armature.

Figs. 34 and 39 relate to a toothed-core armature. Fig. 34 shows the symmetrical field distribution obtained when the teeth are well under cover of the pole-pieces. The induction in the air-gap consists of alternate maxima and minima, and the lines are only slightly curved at the level of the teeth. Below this level, they curve round sharply to enter the flanks of the teeth. The permeability of the core is assumed to be 100.

It is evident that a diagram of this description furnishes a means of calculating the line-integral of magnetic intensity along a line drawn from the polar surface to the base of a tooth, provided the total flux per tooth and the permeability are known.

In fig. 39 we have the same armature, but now shown emerging from under the polar surface. The fringe of the magnetic field is clearly exhibited. The distribution of the lines in the core below the level of the bases of the teeth is not correct. In an actual armature, the lines in this region, after passing into the core, would turn sharply to the left, proceeding towards the neighbouring pole-piece. In the stream-line diagram, on account of the straight-line boundary on the left, the lines are forced to go on in a downward direction. Yet it is curious to note the strong twist towards the left which the lines exhibit immediately after leaving the teeth, and which indicates the direction in which the flow would naturally continue if not subject to the artificial constraint just mentioned.

An objection which might be raised in connection with some of the diagrams is the fact that in those cases where the induction varies from point to point, the permeability in any actual magnetic substance is also variable. In so far as this is the case, of course the stream-line diagrams do not afford a rigid solution of the problem. But it must be remembered that there are many cases in which the magnetic intensity is either so weak or so strong that the permeability does not vary greatly within certain limits of the induction. For all such cases, the solution obtained by the stream-line method is a very close approximation. And even in cases where considerable variations of permeability occur, the stream-line method affords, at any rate, the first rough approximation to the solution of the problem.

SECTION III.—*Mathematical Appendix.*

Brief History and General Solution for Elliptic Cylinders and Confocal Elliptic Shells.

(*a*) The foundation of the modern mathematical theory of magnetic induction was laid by POISSON, between 1821 and 1838. He was the first to work out in detail the solution for a solid or hollow sphere of paramagnetic material placed in a field of uniform intensity.* Subsequently, he extended his investigations to the case of an ellipsoid placed in a uniform field.† GREEN, at a later date, gave an approximate solution for a cylinder of finite length placed with its axis along a uniform field. In 1848, J. NEUMANN attacked the problem of an ellipsoid of revolution placed in any given field.‡ In 1854, KIRCHHOFF succeeded in solving the same problem for a circular cylinder of infinite length. In 1881, A. G. GREENHILL considered the case of a hollow ellipsoid.§

Lord KELVIN was the first to publish, in 1872,|| diagrams of lines of induction for spheres of paramagnetic and diamagnetic material placed in a uniform field. On account of their frequent reproduction (they figure in almost every text-book on the subject), these diagrams are now very well known. In MAXWELL'S great treatise are to be found a number of two-dimensional diagrams. These include the following cases:—(1) two circular cylinders rigidly magnetised transversely, and placed with their magnetic axes at right angles to each other (vol. 2, fig. 14); (2) a circular cylinder permanently magnetised in a transverse direction, and placed in a uniform field, so that the direction of magnetisation of the cylinder is coincident with that of the field; (3) a cylinder of diamagnetic material in a uniform field (vol. 2, fig. 15); (4) a permanently magnetised cylinder in a uniform field whose direction is at right angles to the direction of magnetisation of the cylinder (vol. 2, fig. 16); (5) a uniform field disturbed by a current in an infinitely long straight cylindrical conductor normal to the direction of the field (vol. 2, fig. 17).

In 1882, STEFAN¶ published a long paper dealing with the induced magnetisation of an infinitely long hollow circular cylinder. He obtains a solution by assuming the magnetic potential V to be of the form $(\mathrm{A}r + \mathrm{B}/r) \cos \phi$, where r is the distance of the point considered from the axis of the cylinder, and ϕ the angle between r and the direction of the impressed field; the constants A and B assuming different values for the regions within the cylinder, in its substance, and outside the cylinder respectively. The function $\mathrm{V} = \mathrm{A}x + \mathrm{B}x/r^2$ is an integral of the equation $\frac{\partial^2 \mathrm{V}}{\partial x^2} + \frac{\partial^2 \mathrm{V}}{\partial y^2} = 0$, and it is the real part of the function $\mathrm{A}z + \mathrm{B}/z$ of the complex

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* 'Maxwell,' vol. 2, p. 59.

† *Ibid.*, p. 66.

‡ 'Crelle,' Bd. 37 (1848).

§ 'Journal de Physique' (1881).

|| 'Reprint of Papers on Electrostatics and Magnetism,' pp. 493–495.

¶ 'Wien. Ber.,' 85, Part 2, p. 613.

variable $z = x + iy$. The real factor of the imaginary part of this function is $U = Ay - By/r^2$, and this gives the system of lines of induction.

STEFAN further deals with the case of a hollow cylinder along whose axis are placed two wires (infinitely close together), conveying currents in opposite directions.

In 1894, RÜCKER* considered in detail the question of magnetic shielding by spherical shells, with special reference to the problem of maximum shielding effect for a given total weight of material. In the same year PERRY† contributed a brief paper to the Physical Society on the magnetic shielding of external space by a hollow iron cylinder enclosing two parallel conductors conveying equal currents in opposite directions in a diametrical plane of the cylinder, each conductor being at the same distance from the axis.

In 1897, fresh interest was given to the problem of magnetic shielding by the discussion which followed the reading of Mr. MORDEY's paper on "Dynamos," before the Institution of Electrical Engineers.‡ Towards the close of that year, DU BOIS§ commenced the publication of an elaborate series of articles on magnetic shielding, in which he briefly reviewed the history of the subject, and gave diagrams of magnetic fields for the case of hollow circular cylinders of varying thickness placed in a uniform field.||

Almost simultaneously with DU BOIS, SEARLE¶ published, in January 1898, a mathematical paper on the magnetic field due to a current in a wire placed parallel to the axis of a cylinder of iron. This paper is accompanied by some extremely interesting diagrams of magnetic fields.

The latest contribution to this subject is a paper by A. P. WILLS,** "On the Magnetic Shielding Effect of Trilamellar Spherical and Cylindrical Shells," which may be considered as an extension of DU BOIS' investigations to triple shields.

(*b*) The problem of the induced magnetisation due to a uniform impressed field in infinite cylinders of elliptic section or infinite cylindrical shells bounded by confocal elliptic surfaces, may be dealt with by the following method.

The general problem of magnetic induction in space of two dimensions may be regarded as consisting in the determination of a continuous potential function V, *i.e.*, a function satisfying LAPLACE's equation for two-dimensional space, $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$, which fulfils the condition of continuity of normal induction across any surface of separation between two media, viz.,

$$\mu_1 \frac{\partial V}{\partial n_1} + \mu_2 \frac{\partial V}{\partial n_2} = 0,$$

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* 'Phil. Mag.' [5], vol. 37, p. 95.

† 'Proc. Phys. Soc.,' vol. 13, p. 227.

‡ 'Journal,' vol. 26, p. 564.

§ 'The Electrician,' vol. 40.

|| *Ibid.*, vol. 40, p. 513.

¶ *Ibid.*, vol. 40, p. 453.

** 'Physical Review' [9], pp. 193–213, October, 1899 where μ_1 and μ_2 stand for the permeabilities of the two media, and n_1, n_2 for the normals drawn from any point of the surface *into* the corresponding media.

In dealing with elliptic cylinders and cylindrical shells, it is convenient to abandon the use of Cartesian rectangular co-ordinates, and to have recourse to the circular and hyperbolic functions.

We shall take

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as our ellipse of reference. Then

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

where λ is a variable parameter such that $\lambda > -b^2$, represents the family of ellipses confocal with (1), while

$$\frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} = 1,$$

where ν is a variable parameter such that $-b^2 > \nu > -a^2$, represents the family of hyperbolas confocal with (1).

If now we put

$$\frac{x}{\sqrt{a^2 + \lambda}} = \cos\theta, \qquad \frac{y}{\sqrt{b^2 + \lambda}} = \sin\theta,$$

$$\frac{x}{\sqrt{a^2 + \nu}} = \cosh u, \qquad \frac{y}{\sqrt{-(b^2 + \nu)}} = \sinh u.$$

then $u =$ constant is the equation to a certain ellipse, and $\theta =$ constant to a hyperbola, both curves being confocal with (1).

It is obvious that u (which may vary from 0 to ∞), and θ (which may vary from 0 to 2π), completely and uniquely determine the position of a point in the plane of the ellipse (1). We shall use u and θ as our co-ordinates.

In order to arrive at the form of the induced potential function, we make use of the following considerations. POISSON has shown that the magnetisation of a solid ellipsoid placed in a uniform field is uniform. Let one of the axes of the ellipsoid be made infinite. Then we pass from the three-dimensional case of the ellipsoid to the two-dimensional one of the elliptic cylinder, and we see that in this case also the magnetisation is uniform. Such a magnetisation might be supposed to be produced by imagining two solid cylinders of the imaginary magnetic matter, of volume-density ρ and of opposite sign, originally coincident, to be displaced relatively to each other through a small distance δs in the direction of magnetisation, such that $\rho\delta s$ is equal to the actual intensity of magnetisation of the cylinder. If then ρV stand for the potential at any point due to one of the solid cylinders of the imaginary magnetic matter, the combined potential of the pair of displaced cylinders is $\rho \frac{\partial V}{\partial s} \delta s$. But

$\rho \frac{\partial V}{\partial s}$ is the intensity-component in the direction of δs due to one of the cylinders. Thus the problem of finding the magnetic potential function is reduced to that of determining the intensity due to a solid cylinder of attracting or repelling matter. The components of intensity due to this latter cylinder may be shown to be

$$X = \frac{4\pi ab\rho}{\sqrt{a^2 - b^2}} e^{-u} \cos\theta,$$

$$Y = \frac{4\pi ab\rho}{\sqrt{a^2 - b^2}} e^{-u} \sin\theta.$$

We are thus led to adopt tentatively the somewhat more general forms

$$(A \cosh u - B \sinh u) \cos\theta$$

and

$$(C \cosh u - D \sinh u) \sin\theta,$$

where A, B, C, and D are constants, as the typical magnetic potential functions for magnetised elliptic cylinders and confocal elliptic cylindric shells.

In the above expressions, A, B, C and D will all be different for the space included between two confocal bounding surfaces; $A = B$ and $C = D$ for the external space extending to infinity. And $B = 0$, $C = 0$ for the space inside the innermost bounding surface, which includes the foci.

If we assume the permeability to be constant, then we need only consider the two standard cases of magnetisation along the two axes of the ellipse (1), as any intermediate direction may be obtained by a proper superposition of the two standard cases.

We shall in the first place consider the case of a solid cylinder whose bounding surface is the ellipse of reference (1), the impressed field H being along the major axis of the ellipse.

If V_o, V_i and V_e stand for the impressed potential, the induced potential inside, and that outside the cylinder respectively, then

$$V_o = -Hx = -H\sqrt{a^2 - b^2} \cos\theta \cosh u,$$

and we assume

$$V_i = Ax = A\sqrt{a^2 - b^2} \,.\, \cos\theta \cosh u,$$

$$V_e = B\sqrt{a^2 - b^2} \,.\, e^{-u} \cos\theta.$$

In order that these functions—which satisfy LAPLACE'S equation—may afford a solution of the problem, they must satisfy the conditions of (1) continuity of potential and (2) continuity of normal induction—two conditions which enable us to determine the constants A and B.

The first condition $\left(\text{when } u = \tanh^{-1}\frac{b}{a}\right)$ gives

$$aA - (a - b)B = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The second gives

$$\mu\frac{\partial(V_o + V_i)}{\partial n} = \frac{\partial(V_o + V_e)}{\partial n}, \quad \text{or} \quad (\mu - 1)\frac{\partial V_o}{\partial n} = \frac{\partial V_e}{\partial n} - \mu\frac{\partial V_i}{\partial n},$$

where μ is the permeability, and n is the outward-drawn normal at any point of the ellipse (1).

Now $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial u} \cdot \frac{du}{dn}$, so that the last equation becomes

$$(\mu - 1)\frac{\partial V_o}{\partial u} = \frac{\partial V_e}{\partial u} - \mu\frac{\partial V_i}{\partial u}.$$

Carrying out the differentiations, and then putting $u = \tanh^{-1}\frac{b}{a}$ we get for the second condition

$$\mu b\mathrm{A} + (a - b)\mathrm{B} = (\mu - 1)b\mathrm{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Solving (2) and (3) for A and B, we find

$$\mathrm{A} = \frac{\mu - 1}{\mu b + a} \cdot b\mathrm{H}$$

$$\mathrm{B} = \frac{\mu - 1}{\mu b + a} \cdot \frac{ab}{a - b} \cdot \mathrm{H}.$$

We thus have

$$V_o + V_i = -\frac{a + b}{\mu b + a} \cdot \mathrm{H}x$$

$$V_o + V_e = \frac{\sqrt{a^2 - b^2} \cdot \mathrm{H}}{(\mu b + a)(a - b)} \cdot \{(\mu b^2 - a^2)\cosh u \cos\theta - (\mu - 1)ab\sinh u\cos\theta\}$$

and the equipotential lines may at once be plotted from these equations.

The equation to the lines of induction inside the cylinder is obviously $y =$ constant.

In order to find the equation to the external lines, we have to determine the function which is conjugate to $V_o + V_e$. Since the function which is conjugate to $\cosh u \cos\theta$ is $\sinh u \sin\theta$, and that conjugate to $\sinh u \cos\theta$ is $\cosh u \sin\theta$, we have

$$\mathrm{K}\operatorname{cosec}\theta = \cosh u - \frac{\mu b^2 - a^2}{(\mu - 1)ab}\sinh u$$

for the equation to the lines of the external field, K being a constant which varies from one line to another.

If we next assume that the direction of the impressed field is along the minor axis of the ellipse, we similarly find for the equation to the lines of the external field

$$\mathrm{K}\sec\theta = \cosh u - \frac{(\mu - 1)ab}{\mu a^2 - b^2}\sinh u,$$

K being as before a parameter which varies from one line to another.

The treatment given above for the case of a solid elliptic cylinder may be extended to any number of hollow confocal elliptic cylinders of different permeabilities. For if (taking the direction of H to be along the major axis of the ellipse) we assume that the induced potential V_i in the innermost space is of the form

$$V_i = A\sqrt{a^2 - b^2} \,.\, \cosh u \cos\theta,$$

that the induced potential V_n in the substance of any one of the hollow confocal elliptic cylinders is

$$V_n = \sqrt{a^2 - b^2}(A_n \cosh u + B_n \sinh u) \cos\theta,$$

and that finally the induced potential in external space is

$$V_e = A_e \sqrt{a^2 - b^2} \,.\, e^{-u} \cos\theta,$$

then all the functions $V_i \ldots V_n \ldots V_e \ldots$ satisfy LAPLACE'S two-dimensional equation, and the last function V_e vanishes at an infinite distance. If, therefore, we can determine the various constants $A, \ldots A_n, \ldots A_e$, so as to satisfy (1) continuity of potential, and (2) continuity of normal induction, the only possible solution of the problem will have been obtained.

Let us suppose that there are m hollow cylinders. The number of constants in the assumed expressions for the potential will in that case be $2m + 2$. There are $m + 1$ bounding surfaces, and at every such surface the two conditions of continuity of potential and continuity of normal induction must be fulfilled, thus giving two conditional equations for each boundary. There will therefore be $2(m+1)$ equations, and these will completely determine the values of the $2m + 2$ constants.

By way of further illustration, we shall work out the case of a hollow elliptic cylinder of iron placed in a uniform field. Let the internal bounding surface be the ellipse

$$x^2/a^2 + y^2/b^2 = 1;$$

or $u = \tanh^{-1}(b/a)$, and the external bounding surface the ellipse

$$\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1,$$

or $$u = u_1, \text{ say, where } u_1 = \tanh^{-1}\sqrt{\frac{b^2 + \lambda_1}{a^2 + \lambda_1}}.$$

Let the impressed potential V_o be

$$V_o = -H\sqrt{a^2 - b^2} \,.\, \cosh u \cos\theta,$$

corresponding to a direction of the field along the major axis of the cylinder.

We assume for the induced magnetic potential functions :—

$$V_i = A\sqrt{a^2 - b^2} \,.\, \cosh u \cos\theta,$$

$$V_1 = A\sqrt{a^2 - b^2}(A_1 \cosh u + B_1 \sinh \mathrm{u}) \cos\theta$$

and $$V_e = A_e\sqrt{a^2 - b^2} \,.\, e^{-u} \cos\theta.$$

When $u = \tanh^{-1}(b/a)$, $V_i = V_1$, and when $u = u_1$, $V_1 = V_e$; this gives

$$Aa = aA_1 + bB_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and
$$A_1 \cosh u_1 + B_1 \sinh u_1 = A_e \cosh u_1 - A_e \sinh u_1 \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Again, when $u = \tanh^{-1}(b/a)$, we must have

$$(\mu - 1)\frac{\partial V_o}{\partial u} = \frac{\partial V_i}{\partial u} - \mu\frac{\partial V_1}{\partial u};$$

and when $u = u_1$,

$$(\mu - 1)\frac{\delta V_o}{\delta u} = \frac{\delta V_e}{\delta u} - \mu\frac{\delta V_1}{\delta u}.$$

These two conditions give

$$-(\mu - 1)bH = bA - \mu bA_1 - \mu aB_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$\begin{aligned}-(\mu - 1)H\sinh u_1& \\ &= A_e \sinh u_1 - A_e \cosh u_1 - \mu A_1 \sinh u_1 - \mu B \cosh u_1 \quad . \quad . \quad . \quad (7)\end{aligned}$$

Solving equations (4), (5), (6), and (7) for A, A_1, B_1, and A_e, we get

$$A = \frac{(\mu a - b)(a \sinh u_1 - b \cosh u_1)}{(\mu a^2 - b^2)(\mu \sinh u_1 + \cosh u_1) - (\mu - 1)ab(\mu \cosh u_1 + \sinh u_1)} \cdot (\mu - 1)H,$$

$$A_1 = \frac{(\mu a^2 - b^2)\sinh u_1 - ab(\mu \cosh u_1 + \sinh u_1)}{(\mu a^2 - b^2)(\mu \sinh u_1 + \cosh u_1) - (\mu - 1)ab(\mu \cosh u_1 + \sinh u_1)} \cdot (\mu - 1)H,$$

$$B_1 = \frac{a}{b}(A - A_1),$$

$$A_e = \frac{A_1 \cosh u_1 + B_1 \sinh u_1}{e^{-u_1}}.$$

These expressions may be reduced to a somewhat simpler form by putting $\sqrt{a^2 + \lambda_1} = a_1$, $\sqrt{b^2 + \lambda_1} = b_1$, so that a_1, b_1 stand for the semi-axes of the external elliptic bounding surface. Remembering that $u_1 = \tanh^{-1}(b_1/a_1)$, we get

$$A = \frac{(\mu a - b)(\mu - 1)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot H$$

$$A_1 = \frac{\left\{\mu a - (a + b)\dfrac{bb_1}{ab_1 - a_1 b}\right\}(\mu - 1)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot H$$

$$B_1 = \frac{ab(a_1 + b_1)/(ab_1 - a_1 b)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot (\mu - 1)H$$

$$A_e = \frac{(\mu aa_1 + bb_1)/(a_1 - b_1)}{\mu^2 a + b + \mu(a + b)\dfrac{aa_1 - bb_1}{ab_1 - a_1 b}} \cdot (\mu - 1)H$$

The equations to the lines of induction are :—

For the central space inside the shell, $y =$ constant.

For the space occupied by the substance of the shell,

$$\mathrm{K} \operatorname{cosec} \theta = \cosh u - \frac{\mathrm{H} - \mathrm{A}_1}{\mathrm{B}_1} \sinh u,$$

and for the external space

$$\mathrm{K} \operatorname{cosec} \theta = \cosh u - \frac{\mathrm{A}_e - \mathrm{H}}{\mathrm{A}_e} \sinh u,$$

K being in each case a variable parameter.

If we suppose that the impressed field is along the minor axis of the ellipse, so that

$$\mathrm{V}_o = - \mathrm{H} \sqrt{a^2 - b^2} \sinh u \sin \theta,$$

then, assuming

$$\mathrm{V}_i = \mathrm{A}' \sqrt{a^2 - b^2} \, . \sinh u \sin \theta,$$

$$\mathrm{V}_1 = \sqrt{a^2 - b^2} \, (\mathrm{A}'_1 \cosh u + \mathrm{B}'_1 \sinh u) \sin \theta,$$

and

$$\mathrm{V}_e = \sqrt{a^2 - b^2} \, . \, \mathrm{A}'_e e^{-u} \sin \theta,$$

we find, proceeding as before

$$\mathrm{A}' = \frac{(\mu b - a)(b \cosh u_1 - a \sinh u_1)}{(\mu b^2 - a^2)(\mu \cosh u_1 + \sinh u_1) - (\mu - 1) ab (\mu \sinh u_1 + \cosh u_1)} \, . \, (\mu - 1) \mathrm{H},$$

$$\mathrm{A}_1' = \frac{abe^{u_1}}{(\mu b^2 - a^2)(\mu \cosh u_1 + \sinh u_1) - (\mu - 1) ab (\mu \sinh u_1 + \cosh u_1)} \, . \, (\mu - 1) \mathrm{H},$$

$$\mathrm{B}_1' = \mathrm{A}' - \frac{a}{b} \mathrm{A}_1',$$

$$\mathrm{A}_e' = \frac{\mathrm{A}_1' \cosh \mu_1 + \mathrm{B}_1' \sinh u_1}{e^{-u_1}} .$$

Or, in terms of a_1, b_1,

$$\left. \begin{aligned} \mathrm{A}' &= \frac{\mu b - a}{\Delta} \, . \, (\mu - 1) \mathrm{H}, \\ \mathrm{A}_1' &= \frac{ab(a_1 + b_1)/(a_1 b - ab_1)}{\Delta} \, . \, (\mu - 1) \mathrm{H}, \\ \mathrm{B}_1' &= \frac{\mu b - \dfrac{aa_1(a + b)}{a_1 b - ab_1}}{\Delta} \, . \, (\mu - 1) \mathrm{H}, \\ \mathrm{A}_e' &= \frac{(\mu bb_1 + aa_1)/(a_1 - b_1)}{\Delta} \, . \, (\mu - 1) \mathrm{H}, \end{aligned} \right\} \text{where } \Delta = \mu^2 b + a - \mu(a + b) \frac{aa_1 - bb_1}{a_1 b - ab_1}.$$

The equations to the lines of induction are :—

$$x = \text{constant, in the innermost space,}$$

$$\mathrm{K} \sec \theta = \sinh u + \frac{\mathrm{B}_1' - \mathrm{H}}{\mathrm{A}_1'} \cosh u, \text{ in the substance of the shell, and}$$

$$\mathrm{K} \sec \theta = \sinh u - \frac{\mathrm{A}_e' + \mathrm{H}}{\mathrm{A}_e'} \cosh \mu, \text{ in external space.}$$

In working out numerical examples with the view of plotting curves, it is convenient to use a table of hyperbolic sines and cosines, such as the one compiled by T. H. BLAKESLEY and published by the Physical Society of London. Corresponding values of u and θ having been found, it is an easy step to pass to Cartesian rectangular co-ordinates, which are more convenient for plotting the curves.

The method given above may, as already mentioned, be extended to any number of confocal elliptic cylindric shells. In general, however, when numerical data are available, it is best to substitute these at once in the equations instead of first trying to obtain a solution in general terms.

The case of concentric circular cylindric shells may be regarded as a limiting case of elliptic shells, the two axes of the ellipse becoming equal. If in the expressions obtained for the constants A, A_1, &c., in the case of a hollow elliptic shell we regard a as constant and b as variable, and then proceed to the limit $b = a$, we find that the values so obtained are in agreement with those deduced by DU BOIS for a circular cylindric shell in his articles on "Magnetic Shielding."* But although an interesting verification of the method for a special case is thus obtained, it is much simpler, when dealing with circular cylindric shells, to follow the method developed by STEFAN in the paper already referred to.†

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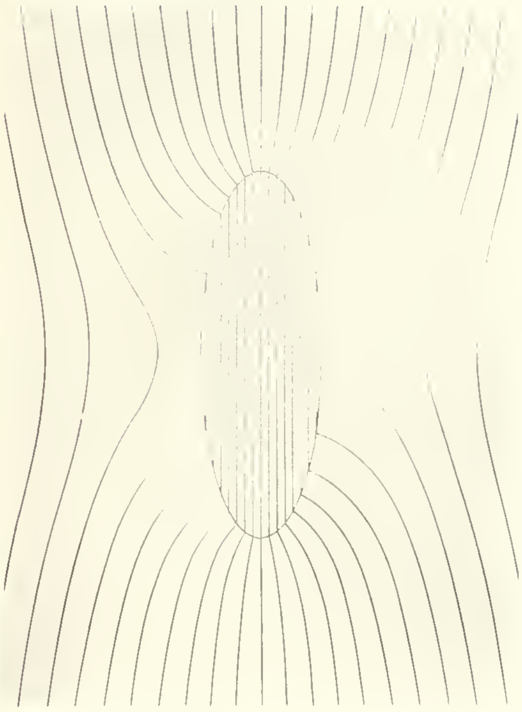
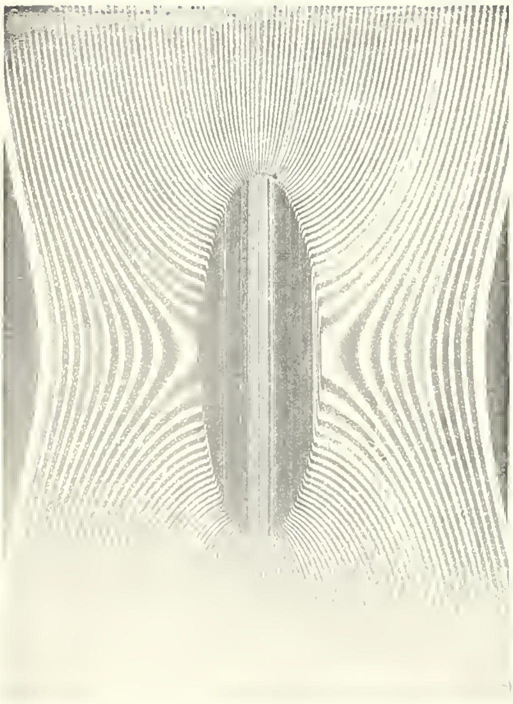
* 'The Electrician,' vol. 40.

† 'Wien. Akad. Sitz. Ber.,' vol. 85, Seetion 2, p. 613.



9

Theoretical Diagram for Infinite Elliptic Cylinder placed in uniform field. Ratio of axes 3 : 1. Permeability = 100.



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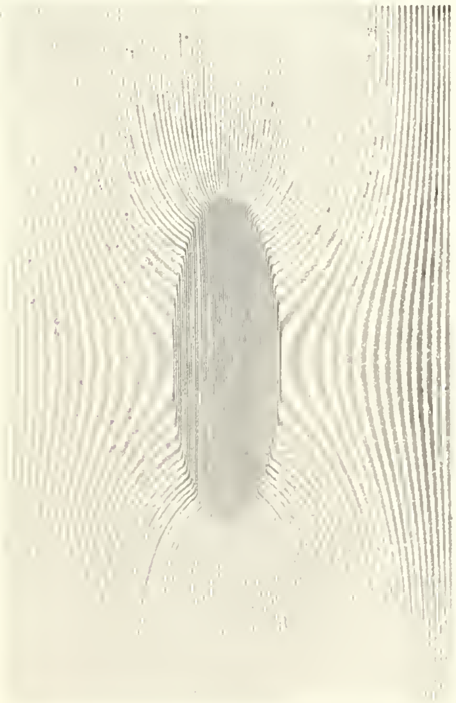
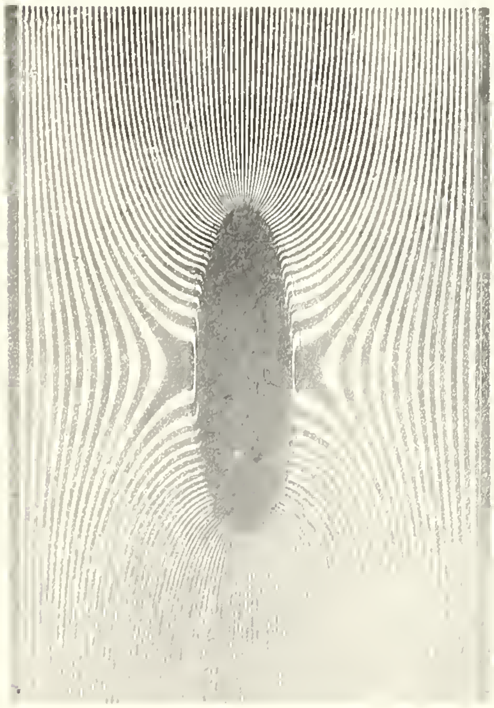
Stream-line Diagram corresponding to theoretical Diagram of Fig. 9.



11

Infinite Elliptic Cylinder in uniform field. Ratio of axes 3 : 1. Permeability = 20.



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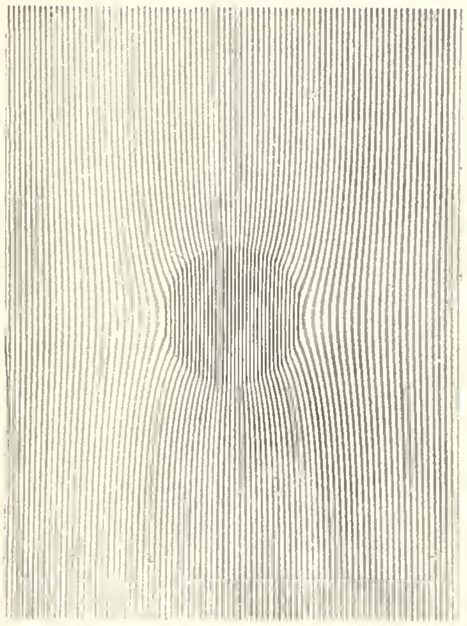
Infinite Elliptic Cylinder in uniform field. Ratio of axes 3 : 1. Permeability 1000.



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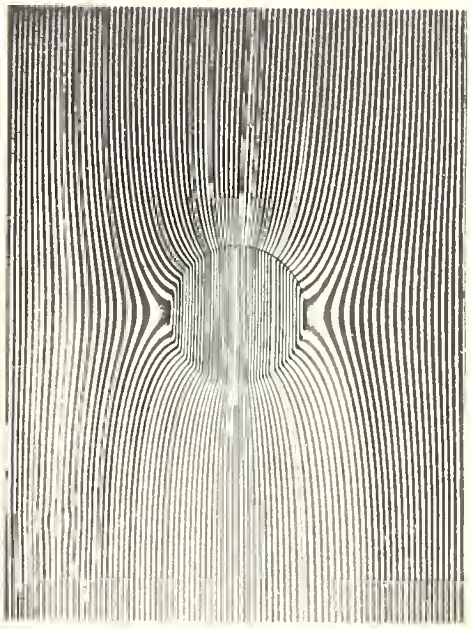
Infinite Circular Cylinder in uniform field.
Permeability=2.



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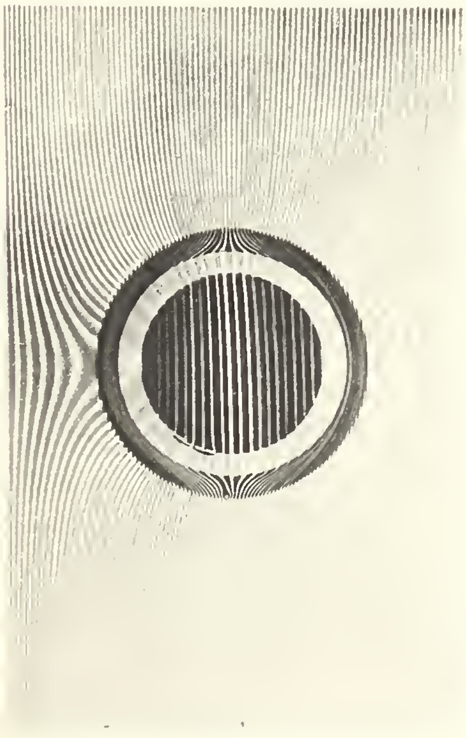
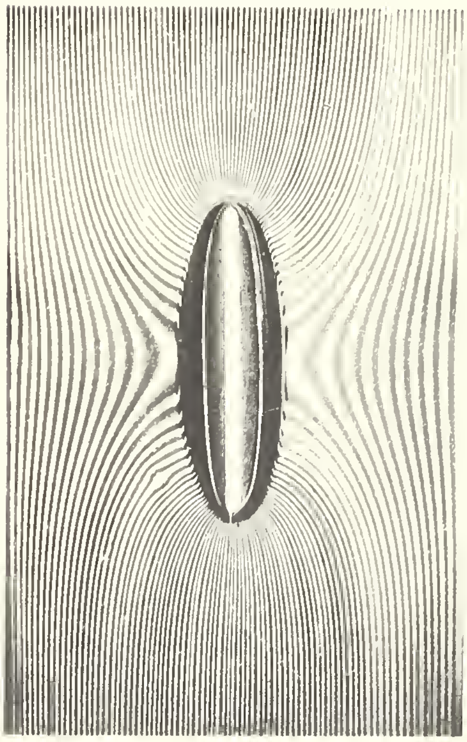
Infinite Circular Cylinder in uniform field.
Permeability=100.



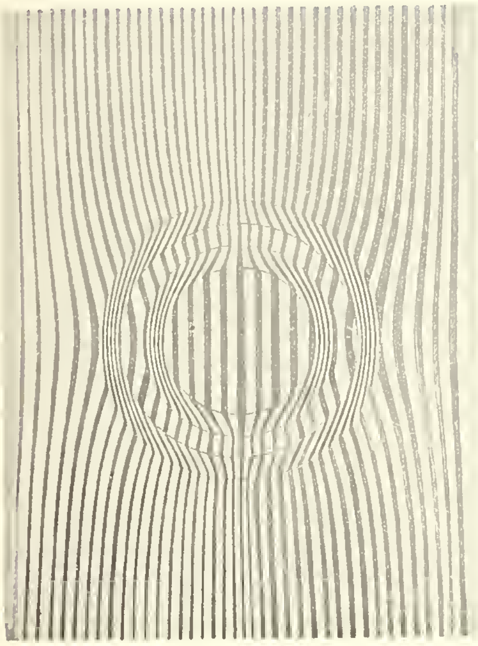
15

Solid Circular Cylinder shielded by hollow one. Permeability=100.



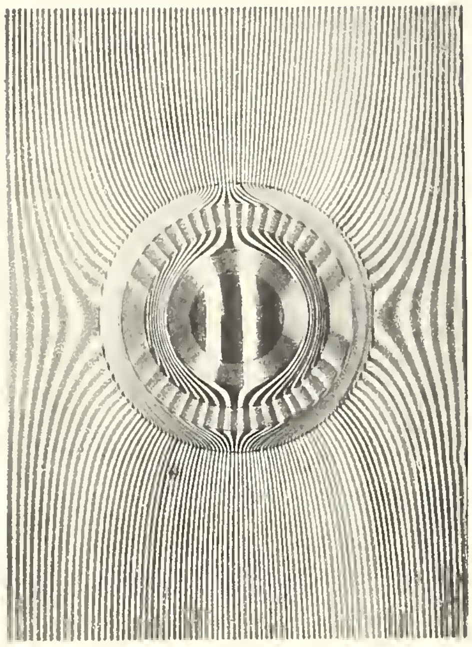
16

Infinite Hollow Elliptic Cylinder in uniform field. Permeability=100.



17

Double Cylindric Shield of very low permeability.



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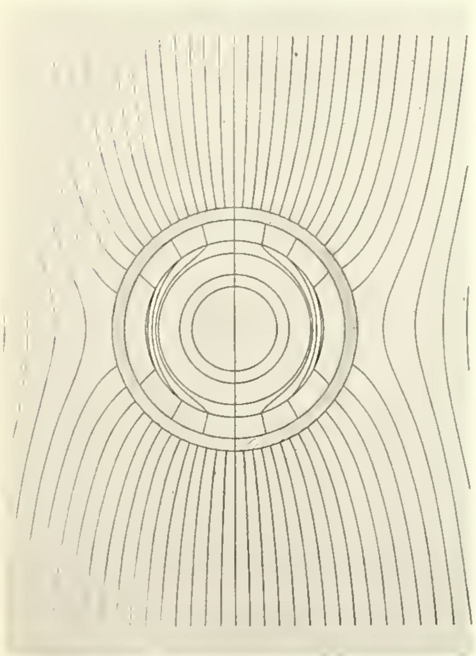
Double Cylindric Shield enclosing Solid Cylinder. Permeability = 100.



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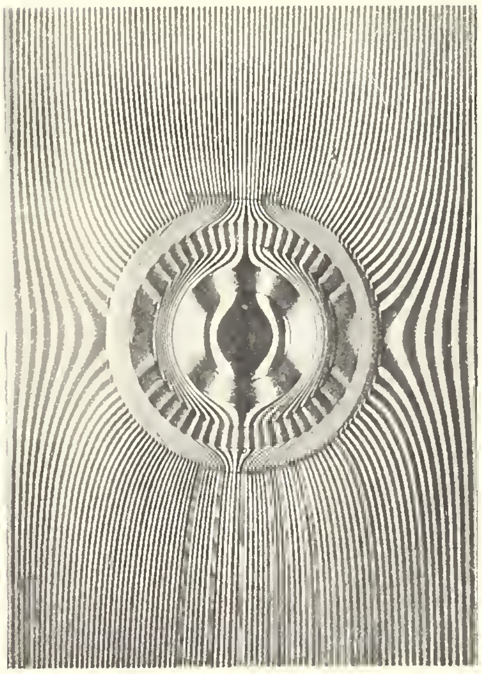
Triple Circular Cylindric Shield.
Theoretical Diagram.
Permeability = 100.



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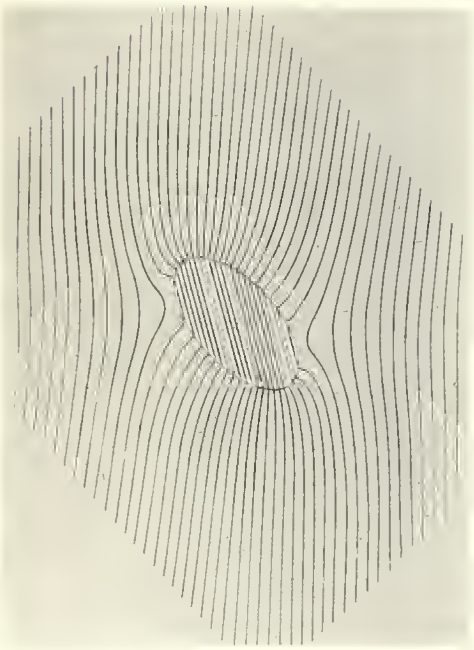
Triple Circular Cylindric Shield.
Stream-line Diagram.
Permeability = 100.



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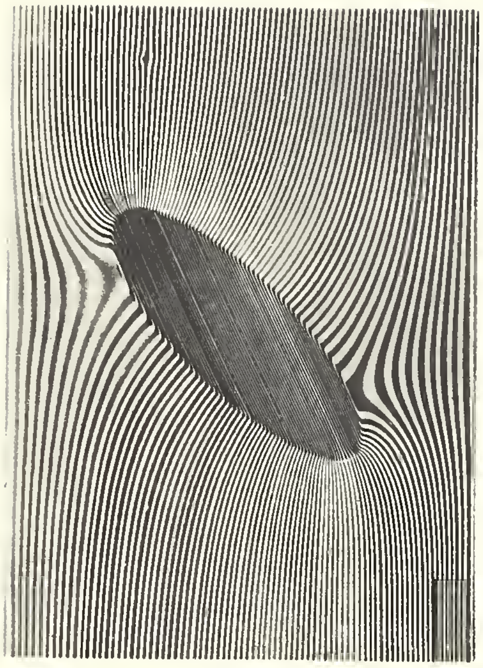
Elliptic Cylinder. Ratio of axes 2 : 1, major axis inclined at 45° to field. Theoretical Diagram. Permeability = 100.



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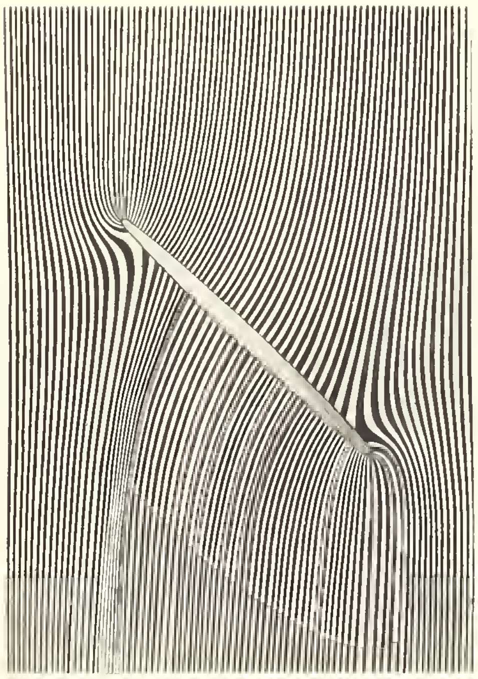
Elliptic Cylinder. Ratio of axes 3 : 1, major axis inclined at 45° to field. Stream-line Diagram. Permeability = 100.

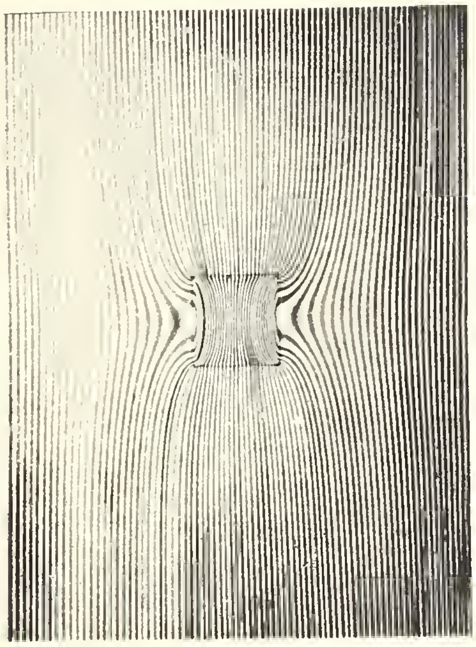


23

Flat Elliptic Plate, inclined 45° to field. Permeability = 100.

24

Hollow Elliptic Cylinder. Major axis inclined 45° to field. Permeability = 100.



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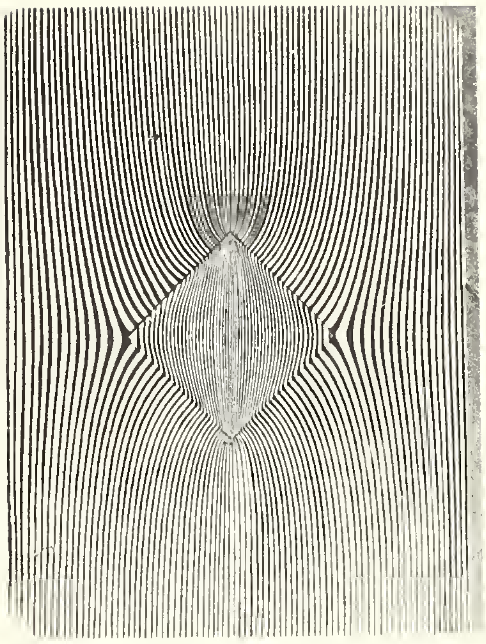
Infinite Cylinder of Square Section.
Permeability = 100.



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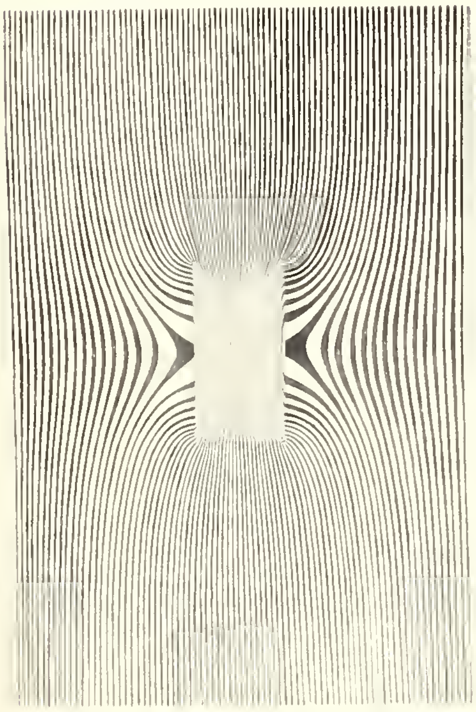
Infinite Cylinder of Square Section.
Permeability = 100.



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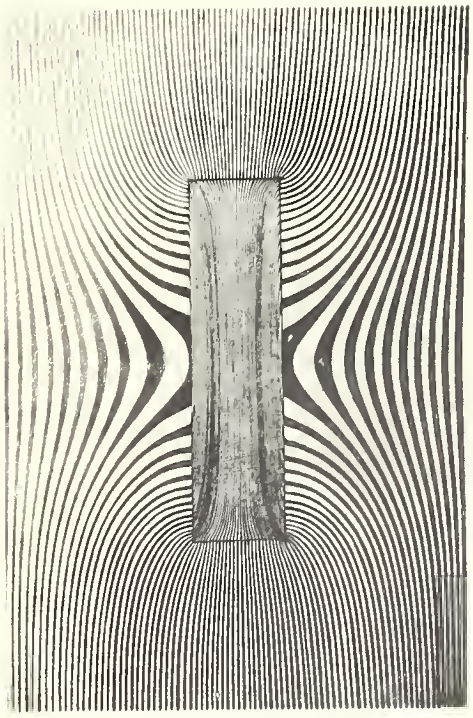
Infinite Cylinder of Rectangular Section.
Permeability = 100.



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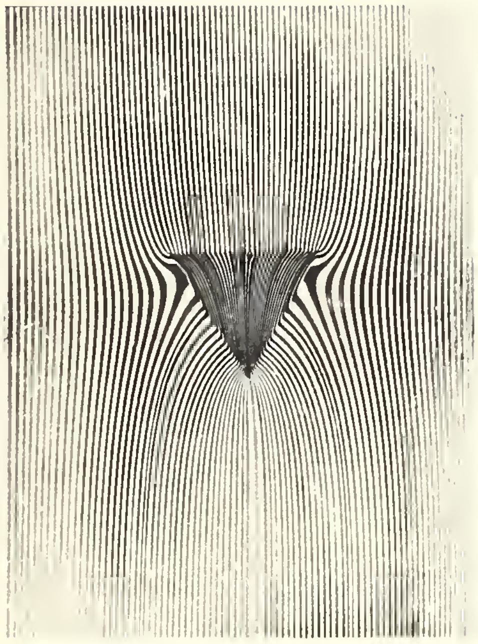
Infinite Cylinder of Rectangular Section.
Permeability = 100.



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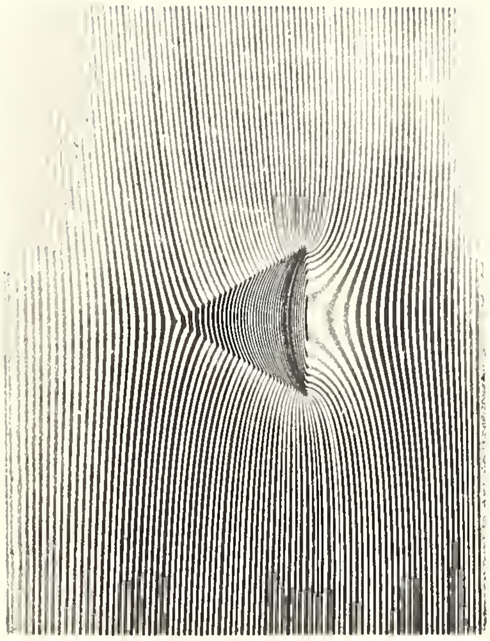
Infinite Cylinder of Triangular Section.
Permeability = 100.



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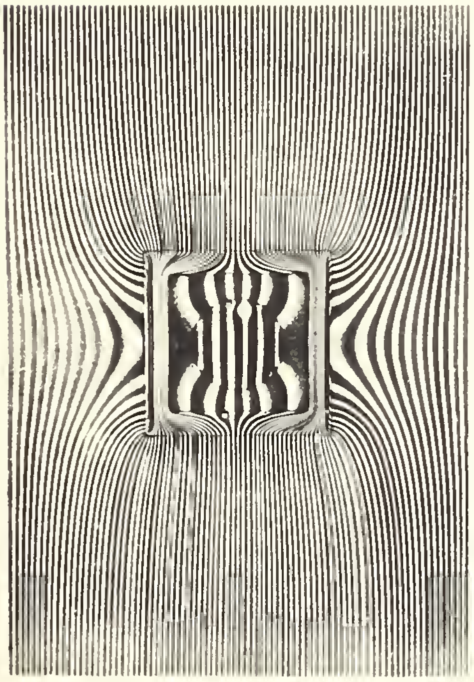
Infinite Cylinder of Triangular Section.
Permeability = 100.



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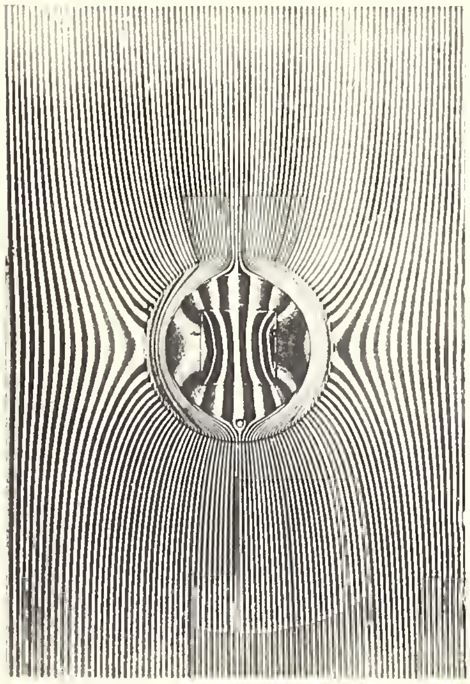
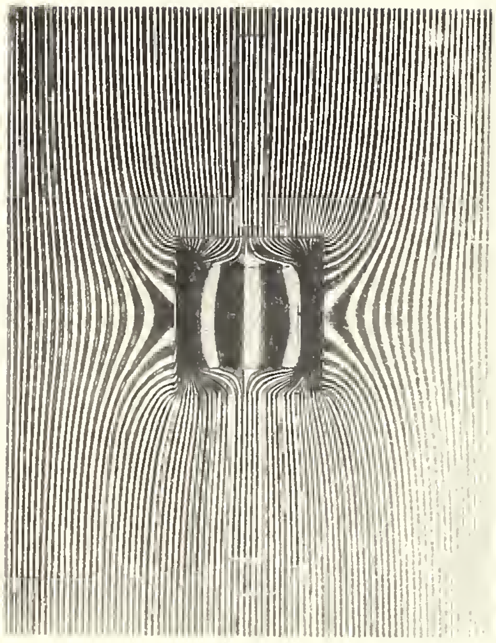
Hollow Square Cylinder enclosing Circular Solid one. Permeability = 100.



32

Hollow Circular Cylinder enclosing Solid Square one. Permeability = 100.



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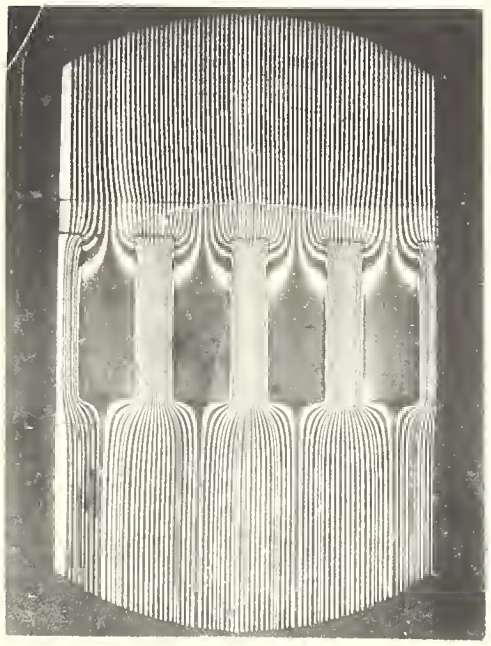
Hollow Square Shield. Permeability = 100.



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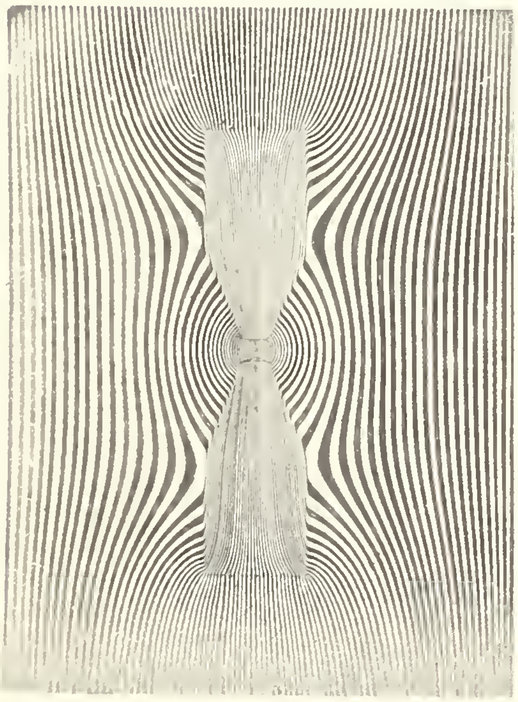
Induction in Air-gap and Teeth of Toothed-core Armature. Permeability = 100.



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Field between Tapered Pole-pieces of Electromagnet.

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Attraction of Armature by Pole-piece.

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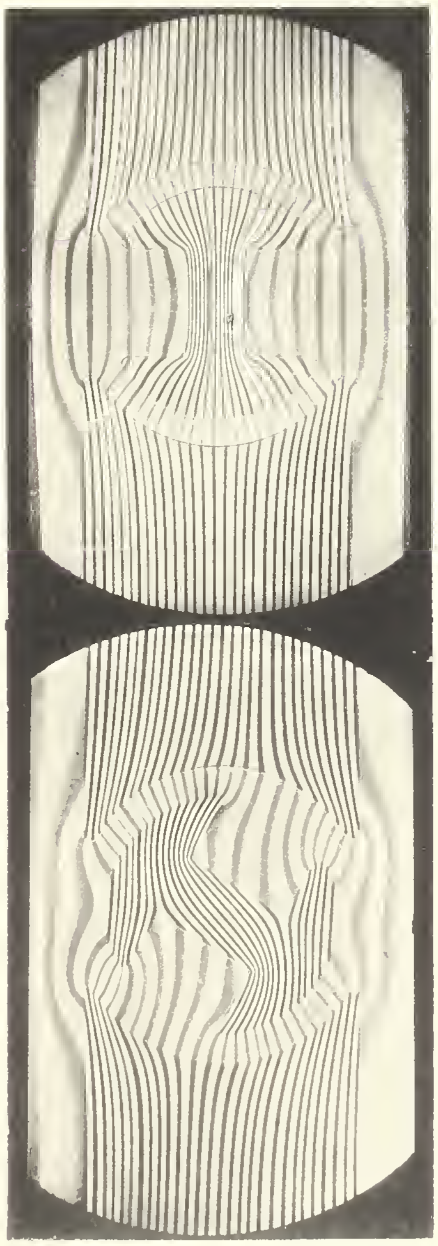

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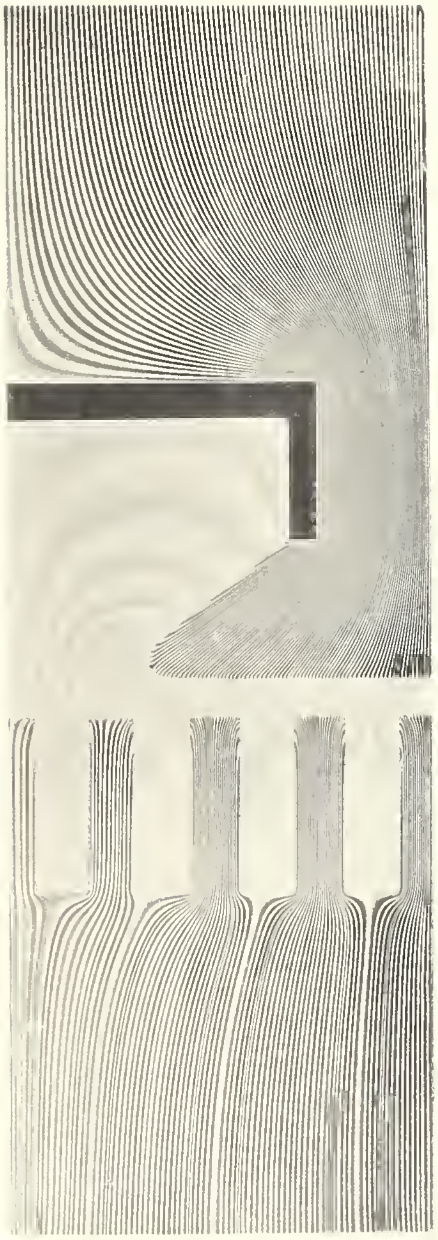
Siemens Shuttle-wound Armature
in two positions.



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Fringe of Magnetic Field near edge of
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X. *On the Application of Fourier's Double Integrals to Optical Problems.*

By CHARLES GODFREY, *B.A., Scholar of Trinity, Isaac Newton Student in the University of Cambridge.*

Communicated by Professor J. J. THOMSON, *F.R.S.*

Received June 12—Read June 15, 1899.

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INTRODUCTORY.

§ 1. THE object of the following work is to make some progress with the mathematical representation of the motions which go to compose natural light.

§ 2. It has always been recognised that interference phenomena forbid us to regard any natural radiation as consisting of an unending train of simple waves, such as may be represented by sine functions. At the same time, the equations of optics find their simplest solution in circular functions. It is desirable to enquire how far we may resolve a natural luminous motion with a sum of simple wave-trains by means of FOURIER'S "Theorem of Double Integrals." This procedure was first suggested by GOUY.[*]

§ 3. Doubts have often been entertained as to the permissibility of this process. Writers have been sceptical as to the physical meaning and independence of the simple waves thus introduced. In the following pages will be found an attempt at a strict justification of the method. It is based upon two principles (i.) that we are cognisant of light only by means of the integral effects produced by the light during an interval of time which depends upon the nature of the detector in use (the eye, a photographic plate, &c.); (ii.) that we are not concerned with simple wave-lengths, but rather with short ranges of wave-length, whose integrated energy we observe. The former principle is generally accepted; the latter has been put forward with great force by Lord RAYLEIGH.[†]

§ 4. In what follows we shall deal solely with plane and plane-polarised light.

§ 5. The matter at issue cannot be introduced better than by a quotation from GOUY[‡]:—

"On sait que la théorie ondulatoire, dans les explications qu'elle donne des phénomènes optiques, a pour objet immédiat le *mouvement simple*, dans lequel la vitesse vibratoire[||] d'un point quelconque est donnée par une équation de la forme

$$\nu = a \sin 2\pi\left(\frac{t}{\theta} + b\right)$$

---

[*] GOUY, 'Journ. de Physique,' ser. 2, vol. 5, p. 354. (1886.)

[†] Lord RAYLEIGH, 'Phil. Mag.,' vol. 27, 1889.

[‡] GOUY, 'J. de Ph.,' ser. 2, vol. 5, p. 354.

[||] It is clearly immaterial whether we speak of velocities and displacements of an elastic medium, or of electric and magnetic forces. The real objects of discussion are vectors, which can be interpreted in various ways.

t désignant le temps, a, b, et θ des constants. Cette équation définit une suite entièrement *illimitée* de vibrations pendulaires, d'une *régularité absolue*, dont la période est θ.

"Si ces conditions de regularité et de durée ne se trouvent pas rigoureusement réalisées, l'équation du mouvement est différente, et, par suite, un problème nouveau se trouve posé, la solution fournie par la théorie pour un mouvement simple n'étant plus applicable en général. S'il s'agit, par exemple, d'un phénomène d'interférence ou de diffraction, on voit immédiatement que toute irrégularité et toute interruption entraîne une perturbation dans l'effet produit, comme on l'a remarqué depuis longtemps. Il en sera encore de même, abstraction faite des interférences, toutes les fois que l'on aura à considérer des milieux doués de dispersion. En effet, le mouvement vibratoire dans les divers milieux devant toujours satisfaire aux équations différentielles des petits mouvements de ces milieux, il n'est nullement permis de faire subir une altération, quelle qu'elle soit, au mouvement vibratoire (i.) et de supposer en suite que ce mouvement se comportera, dans les milieux doués de dispersion, comme s'il n'était pas altéré. Ainsi, par exemple, on n'est pas en droit de supposer que le mouvement (i.) ne comprend qu'un nombre de vibrations limité, et qu'il se propagera dans les divers milieux comme s'il formait la suite entièrement illimitée que definit l'équation (i.).

"D'autre part, on a remarqué depuis longtemps qu'aucune source lumineuse ne peut produire une série de vibrations indéfinie et parfaitement régulière, ne fût-ce qu'en raison du renouvellement incessant des particules incandescentes. Ainsi aucun mouvement lumineux réel, même le moins complexe qu'on puisse supposer, ne rentre rigoureusement dans le cas du mouvement simple que considère la théorie ondulatoire.

"Cette difficulté est présentée dès l'origine de cette théorie. On y répond d'ordinaire en supposant que les sources lumineuses produisent des séries de vibrations régulières, mais troublées de temps à autre par des perturbations subites ou de courte durée. Si la série, entre deux perturbations, comprend un grand nombre de vibrations, on peut prévoir que l'effet moyen d'un pareil mouvement différera peu de celui d'un mouvement simple. Mais cet aperçu, reposant sur une hypothèse, ne peut servir de base à une étude rationnelle du problème qui nous occupe, et nous verrons que, pour les sources donnant des spectres continus, on peut se faire une idée beaucoup moins étroite au mouvement lumineux. De plus, cet aperçu ne nous apprend rien sur les effets des perturbations elles-mêmes, qui paraissent jouer un rôle important dans la constitution des spectres fournis par les vapeurs et les gaz incandescents."

§ 6. The general process to which GOUY alludes is the analysis of any disturbance whatever by means of FOURIER's theorem. He considers a function which is defined within a given interval of time; this is analysed into a sum of circular functions of time; the periods of the terms being the interval itself and all sub-multiples of it.

It seems unnecessary to restrict the function by defining it for a finite interval alone. We may at once discuss a motion given for all values of the time, from $-\infty$ to $+\infty$.

By the theorem of FOURIER's double integrals*

$$f(t) = \int_0^\infty (\mathrm{C}\cos ut + \mathrm{S}\sin ut)du$$

where $$\mathrm{C} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos uvdu, \quad \mathrm{S} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin uvdv.$$

The disturbance is thus analysed into a sum of elementary simple vibrations, of which

$$du(\mathrm{C}\cos ut + \mathrm{S}\sin ut)$$

is typical.

Each of these is a simple circular function; the results of the undulatory theory are directly applicable to it.

The periods of the elementary vibrations have all values from zero to infinity. Now, if the disturbance could be analysed as a number of simple circular functions with distinct periods, the separate elements would have meaning, as in the familiar harmonic analyses of tides, vibrations of musical instruments, &c. The question we have to answer is, have the simple elements meaning in the limit, when their number is infinite, and the sum becomes an integral?

§ 7. We will at once notice an obvious criticism; this was, in fact, offered by POINCARÉ,† soon after GOUY's article appeared. Each of the component vibrations $du(\mathrm{C}\cos ut + \mathrm{S}\sin ut)$ exists unchanged through all time. This is true whatever be the nature of the disturbance we are analysing. But this disturbance may, for instance, be zero, except within a certain definite interval of time. Take the case of a flash of light. Now a spectroscope, says M. POINCARÉ, will separate the component vibrations laterally; they may be examined separately. Hence a spectroscope will enable us to see the light for an infinite time before it is kindled, and for an infinite time after it is extinguished. The analysis must therefore be fallacious.

The answer to this objection is as follows. No spectroscope possesses infinite analysing power. A given point at the focus of the telescope will be illuminated by light of a whole range of periods. Or, to look at the matter from another point of

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* *On the Applicability of Fourier's Double Integral to Functions occurring in Physical Problems.*—In pure mathematics the applicability of FOURIER's theorems to functions is subject to certain limitations. These limitations exist when the functions possess infinite sets of discontinuities, or infinite sets of fluctuations, or infinities of certain types. Now, in concrete physical cases, we find neither infinities nor discontinuities. It is true that infinities and discontinuities may occur in functions commonly used to represent physical quantities. But the presence of such features is due to the abstract character of the method; a function more closely realising the properties of the physical quantity in question would be without infinite or discontinuous features.

† POINCARÉ, "Spectres Cannelés," 'C. R.,' 120, pp. 757–762, 1895; also see SCHUSTER, 'C. R.,' 120, pp. 987–989, 1895.

view, a perfectly monochromatic train of waves will, by virtue of diffraction, illuminate, not a point, but a small area of the focal plane. The different elements of the Fourier integral will not be distinguished separately; they will to some extent be superposed and recombine. The result will be, at each point of the focal plane, a disturbance not altogether different from the original motion in duration and character. We see then that the Fourier analysis may after all have meaning and application, and not lead to a paradox such as POINCARÉ anticipated.

It must be noticed that this recombination of the different elements of the integral is essentially connected with the phase-relation which exists between the said simple elements.

§ 8. We have seen that POINCARÉ'S objection will not prevent us from regarding the original ether-motion as mathematically equivalent to the Fourier integral. Whatever services the Fourier analysis can render us we may safely accept.

It will be found that the different simple elements of the Fourier integral cannot *in general* be said to have any independent physical existence. On the other hand, part of the following essay is an attempt to prove that in certain cases the different Fourier elements can be regarded as having such physical existence. A special case of this nature is that of a steady emission, such as the radiation of an incandescent gas. We shall inquire to what extent such radiations are equivalent to mixed light, presenting a continuous spectrum of composition determined by Fourier analysis.

The Fundamental Theorem.

§ 9. We will now introduce a theorem proved by Professor SCHUSTER.* A particular case was given by Lord RAYLEIGH.†

The theorem is as follows :—

$$\int_{-\infty}^{\infty} f(t)\phi(t)dt = \frac{1}{\pi}\int_{0}^{\infty}(A_1A_2 + B_1B_2)du,$$

where

$$A_1 = \int_{-\infty}^{+\infty} f(\lambda)\cos u\lambda d\lambda, \qquad B_1 = \int_{-\infty}^{+\infty} f(\lambda)\sin u\lambda d\lambda$$

$$A_2 = \int_{-\infty}^{+\infty} \phi(\lambda)\cos u\lambda d\lambda, \qquad B_2 = \int_{-\infty}^{+\infty} \phi(\lambda)\sin u\lambda d\lambda.$$

In other words, if

$$f(t) = \int_{0}^{\infty} R_1 \cos(ut + \psi_1)du$$

$$\psi(t) = \int_{0}^{\infty} R_2 \cos(ut + \psi_2)du$$

---

* Professor SCHUSTER, 'Phil. Mag.,' vol. 37, p. 533, 1894.

† Lord RAYLEIGH, 'Phil Mag.,' vol. 27, 1889.

where R_1, R_2, ψ_1, ψ_2 are functions of u,

then
$$\int_{-\infty}^{+\infty} f(t)\phi(t)dt = \pi\int_0^{\infty} R_1R_2 \cos(\psi_1 - \psi_2)du.$$

The integrated value of $f(t)\phi(t)$ depends therefore upon the distribution of energy in the separate elements, and upon the *difference of phases* of corresponding elements in the integrals.

§ 10. Professor SCHUSTER needs the theorem in order to prove that, in case of a bifurcated beam of light interfering with itself, "the amount of interference depends on the distribution of energy only, and not on any assumption respecting the regularity or irregularity of vibration."

The proposition has, however, very much wider consequences.

§ 11. For the present, the discussion will be confined to the particular case of *constant light*, *i.e.* light which does not present any *perceptible* fluctuations or other time-features.

§ 12. All our cognizance of radiation other than the long waves of HERTZ is by average effects. We average over a length of time great compared with the periods of vibration. This is true whatever be the means used to perceive and register the radiations, whether by direct visual perception, or by chemical effect, photographic or other, or by heating effect (bolometric), or by luminescence which the radiation excites. For Hertzian waves, on the other hand, the features of a single wave can be discovered.

In discussing the qualitative effect of constant light, with a view to discriminating between different wave-lengths, we are concerned solely with the integral effect over a certain interval of time.

It is doubtful how far we are at liberty to consider the molecule as a simple vibrator. But, in so far as this assumption is justified, we may prove that the observed effects of constant light will depend on nothing but the partition of energy among the different elements of the equivalent Fourier integral. The phases will of course determine whether the light shall be "constant" or no; but, if that condition is fulfilled, their further influence will not be perceived.

§ 13. *Energy.*—The whole energy of the light motion $f(t)$ depends upon $\int_{-\infty}^{+\infty} f^2(t)dt$

By the theorem of SCHUSTER and RAYLEIGH, this is equal to
$$\pi\int_0^{\infty} R^2 du$$

where
$$f(t) = \int_0^{\infty} R\cos(ut + \psi)du.$$

Interference with given path difference 2T.

This depends on

$$\int dt[\{f(t)+f(t+2\tau)\}^2 - f^2(t) - f^2(t+2\tau)] = 2\int f(t)f(t+2\tau)dt.$$

Now
$$f(t) = \int_0^\infty \text{R} \cos (ut + \psi)du$$

$$f(t + 2\tau) = \int_0^\infty \text{R} \cos (ut + 2u\tau + \psi)du,$$

$$\therefore \qquad \int_{-\infty}^{+\infty} f(t)f(t+2\tau)dt = \pi\int_0^\infty \text{R}^2 \cos 2u\tau du.$$

The phase ψ has disappeared.*

Influence of Light on a Vibrator.

The phenomena of refraction, dispersion and absorption, can be explained (subject to the above reservation) by considering the action of light waves on a vibrator.

The equation of motion of the vibrator is

$$\ddot{x} + 2k\dot{x} + p^2x = f(t) = \int_0^\infty \text{R} \cos (ut + \psi)du$$
$$= \text{real part of} \int_0^\infty \text{R}e^{i(ut+\psi)}du.$$

The "general" solution of this equation will be of the form $\text{A}e^{-kt} \cos (rt + \phi)$, where $r^2 = p^2 - k^2$ and A, ϕ are arbitrary. The above solution is to hold for all time from $-\infty$ to $+\infty$. We must therefore put $\text{A} = 0$, and the complete solution will be

$$x = \text{real part of} \int_0^\infty \frac{\text{R}e^{i(ut+\psi)}du}{p^2 - u^2 + 2kiu}\dagger$$

$$= \int_0^\infty \frac{\text{R}du}{(p^2-u^2)^2 + 4k^2u^2}\{(p^2 - u^2)\cos(ut+\psi) + 2ku \sin(ut+\psi)\}.$$

The average energy of the motion excited will depend upon

$$\int_0^\infty \frac{\text{R}^2du}{(p^2-u^2) + 4k^2u^2}.$$

Again, the work done by the light depends upon $\int f(t)\dot{x}dt$. Applying SCHUSTER'S

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* SCHUSTER, 'Phil. Mag.,' vol. 37, p. 533.

† We verify that this is a solution. This involves the process of differentiating inside the integral. Now, the condition that a Fourier integral shall admit of being so treated is, that the function represented shall be free from discontinuities and shall vanish at ± ∞. No mathematical discontinuities will occur in a physical problem; and, if necessary, the conditions at infinity may be satisfied by introducing into $f(t)$ a factor (such as $e^{-a^2t^2}$ where a is small) which shall ensure dying away at both extremities, and which at the same time will not affect the Fourier resolution in any marked degree.

theorem, we find that the terms connecting different periods drop out (a familiar property of ordinary harmonic analysis), and the *rate of absorption* is dependent on

$$\pi\int_0^\infty \frac{2ku^2\mathrm{R}^2du}{(p^2-u^2)^2+4k^2u^2}$$

The phase ψ has again disappeared.

Heating effects are directly dependent upon absorption. So again with *physiological effects.*

§ 14. In the case of *chemical* and *electrical* effects produced by light we probably have some kind of dissociation. This is perhaps true of *luminescence* also. It may be that we do not yet understand the mechanism of dissociation. But, if the dissociation arises from separation of ions as their light-excited vibrations become large, the vibrator analogy will apply here as well. Doubtless some molecules will split up sooner and others later; for the individual molecule the precise timing of its own vibrations with the phase of the incident light will be all-important. But on the average of a large number of molecules, the amount of dissociation will perhaps depend on the rate of absorption of energy by a vibrator typifying the average structure.

It is necessary to repeat that our assumption of a linear equation for the vibration of a molecule cannot be regarded as more than a first step towards a solution of a difficult problem. In the words of Sir GEORGE STOKES, "Linearity applies to the small disturbance of the single elastic medium—the ether—but it does not follow that linearity applies to all the effects produced in a complex system of molecules."

§ 15. Let us consider the application of the present treatment to the *spectroscopic analysis* of light.

The light emergent from the instrument in a given direction is compounded of different wave-lengths. The element

$$\mathrm{R}\cos(ut+\psi)du$$

of the integral will contribute a component

$$\mathrm{R}\phi(u)\cos(ut+\psi)du.$$

In this expression $\phi(u)$ depends upon the structure of the instrument, and the direction chosen, as well as upon u. The change of phase $\psi-\theta$ depends upon the same causes. We must note, however, that neither $\phi(u)$ nor $\psi-\theta$ depends upon ψ.

The emergent light will be

$$\int_0^\infty \mathrm{R}\phi(u)\cos(ut+\theta)du.$$

Now we have seen that the phase only enters as determining the constancy of the light. If the light which comes into the instrument is constant, so also is the emergent beam. The phase has no further part to play; hence, the *spectroscopic* *analysis of constant light depends solely upon the instrument and the distribution of energy among the elements of the Fourier integral.*

§ 16. A dispersive medium, apart from its possible selective absorption of the different wave-lengths, will always alter the relative phases of the different elements. The transmitted light will thus be altered. But the preceding work has shown that these phase-changes will not affect the sensible properties of the light.

§ 17. We have arrived at the conclusion that the different simple components of constant light are not only superposed, but also independent as regards all energy properties.

Radiations composed of a random Aggregate of Pulses.

§ 18. A constantly-recurring problem in optics is that of the composition of an irregular sequence of pulses of a given type.

The question occurs in dealing with the radiation of an incandescent gas. The pulse here consists of the train of waves given off by the molecule during its free path; after an encounter the train will be entirely changed, and practically independent of the former train.

Again, what is the total effect on radiation of the damping to which the vibrations of the molecules are subject? The question was raised by LOMMEL.[*] This author was content to analyse $e^{-kt}\sin(pt+\psi)$ as a Fourier integral, and assume that the different elements are independent. This, of course, will not be true for the simple pulse which LOMMEL considered. It is true that the motion $e^{-kt}\sin(pt+\psi)$ can be reconstructed by means of an infinite series of vibrators whose amplitudes follow the law of the Fourier expansion. But the phases of these vibrators will not be independent; they must be carefully adjusted to give the requisite effect.

§ 19. We shall find that, when we deal with an infinite and irregular succession of such pulses, the energy properties do completely specify the motion. The disturbing influence of phase will disappear; in the Fourier integral representing the complete motion, the phase will be a rapidly-fluctuating function of the wave-length, and all distinctive phase-properties will average out.

The omission to deal with a sequence of pulses has exposed LOMMEL's analysis to adverse criticism. It will be seen that a more complete treatment confirms the results which he obtained as regards the widening of spectrum lines through damping.

§ 20. Another case in point is that of Röntgen rays. These are satisfactorily covered by Professor J. J. THOMSON's theory of electric pulses. The pulses are of given type; each one may be analysed by FOURIER's theorem. We find a certain energy-wave-length curve; in dealing with an infinite succession of pulses the phase-

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[*] LOMMEL, 'Wied. Ann.,' 3, 251, 1878.

relations disappear, and we are left with the energy curve to completely specify the properties of the sequence of pulses. These statements will be justified below.

§ 21. The subject has been opened by Lord RAYLEIGH* in his paper on "The Complete Radiation at a given Temperature." He proposes to regard this as an irregular sequence of pulses of the type $e^{-c^2x^2}$.

Now

$$e^{-c^2x^2} = \frac{1}{c\pi^{\frac{1}{2}}}\int_0^\infty e^{-u^2/4c^2}\cos uxdu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the whole energy of the pulse

$$\int_{-\infty}^{+\infty} e^{-c^2x^2}dx = \frac{1}{c^2}\int_0^\infty e^{-u^2/2c^2}du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The intensity corresponding to the limits u and $u + du$ is therefore $c^{-2}e^{-u^2/2c^2}du$.

"If an infinite number of impulses, similar but not necessarily equal to (8), and of arbitrary sign, be distributed at random over the whole range from $-\infty$ to $+\infty$, the intensity of the resultant for an absolutely definite value of u would be indeterminate. Only the *probabilities* of various resultants could be assigned; and if the value of u were changed, by however little, the resultant would again be indeterminate. Within the smallest assignable range of u there is room for an infinite number of independent combinations. We are thus concerned only with an average, and the intensity of each component may be taken to be proportional to the total number of impulses (if equal) without regard to their phase-relations. In the aggregate vibrations, the law according to which the energy is distributed is still, for all practical purposes, that expressed by (22)."

§ 22. This important paragraph suggests the whole theory. But when we come to take a closer view of it, it will be found that there are certain questions which still remain to be solved.

Suppose, for instance, that the elementary pulse is confined to a certain small range of time, such as are the pulses in Professor THOMSON'S theory, how many of those pulses must be present in order to give the properties which Lord RAYLEIGH associates with an infinite succession? Again, we might suppose that different consequences would follow from different degrees of crowding among the pulses. They may be so close, on the average, that a great number of them are everywhere found overlapping; or, again, they may be so thinly scattered as to be, on the average, far apart in comparison with the space occupied by each. Experiment does not allow us to say which of these suppositions is correct; we may enquire what is the test which determines the applicability of Lord RAYLEIGH'S theorem to the aggregate.

§ 23. In order that the sequence of pulses may, for us, be equivalent to a spectrum,

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* 'Phil. Mag.,' 27, 1889.

an obvious condition is that the pulses shall not be so far apart as to be separately distinguishable. Photography can fix 10^{-7} second; hence there must be many pulses in 10^{-7} second. This is a condition certainly fulfilled by Röntgen rays. The coarser the means of observation which we use, the more thinly may the pulses be scattered. The results which we are about to investigate may be true, for a certain radiation, in the present state of experimental science; but will cease to be true for that particular kind of radiation when our instrumental means shall have been so improved as to enable us to distinguish structure in that radiation.

It shall be shown that this is the only condition necessary in order that a random sequence of similar pulses may be equivalent to radiation of a spectral composition given by the analysis of a single pulse.

§ 24. We have already proved that we are concerned simply with an integral effect over a time T of the order of the shortest observable interval. If we are content to view the radiation with the eye, or to use a slow photographic plate, T may be taken as great as we please. If, on the other hand, we are investigating the radiation with the shortest possible exposure, T may be reduced as far as our experimental skill will allow.

Let us examine the Fourier composition of a numerous sequence of random similar pulses.

Suppose the pulse to be

$$f(t) = \int_0^\infty \phi(u)\cos(ut+\psi)du$$

the angle ψ being a definite function of u. We are to examine the Fourier integral, equivalent to

$$f(t-\tau_1)+f(t-\tau_2)+f(t-\tau_3)+\ .\ .\ .\ +f(t-\tau_n)$$

where $\tau_1,\ \tau_2,\ \tau_3,\ .\ .\ .\ \tau_n$ define a large number of points of time distributed at random in an interval T. The breadth of the pulse is to be small compared with T.

The resultant integral is

$$\int_0^\infty \phi(u)\{\cos\overline{ut-u\tau_1+\psi}+\cos\overline{ut-u\tau_2+\psi}\ .\ .\ .\ +\cos\overline{ut-u\tau_n+\psi}\}du.$$

Consider the quantity

$$\phi(u)\{\cos\overline{ut+\psi-u\tau_1}+\cos\overline{ut+\psi-u\tau_2}+\ .\ .\ .\ +\cos\overline{ut+\psi-u\tau_n}\}\quad .\ \ .\quad \text{A}.$$

First, suppose that the time of vibration, $2\pi/u$, is small compared with T or $\tau_n-\tau_1$. Draw from an origin lines of length $\phi(u)$ making with the prime vector angles $\psi-u\tau_1$, $\psi-u\tau_2$, &c. Then the bounding lines of the angles $\psi-u\tau_1$, $\psi-u\tau_2$, . . .

$\psi - u\tau_n$ lie at random all round an origin. The phase of the resultant will be arbitrary, while the mean value of its modulus is

$$n^{\frac{1}{2}} \,.\, \phi(u).^{*}$$

Now this is the amplitude of the compound harmonic motion A.

If we pass to a frequency $u + du$, where Tdu is small, the new phases will differ but slightly from the old ; but if Tdu is finite or great, the new phases will differ finitely from the old, and the resultant for $u + du$ will have no apparent connection as regards phase with the resultant for u.

Let us consider what happens when we observe this radiation. First, we can only observe the content of a certain interval of time, which we have taken to be T ; we receive into our apparatus the total energy of all the pulses in T. Now SCHUSTER's theorem expresses the connection between the total energy of a radiation, and its expression as a Fourier integral (see p. 335). If the Fourier expression for the resultant of the n pulses in the present case is

$$\int_0^\infty \mathrm{R}\cos(ut + \theta)du,$$

the total energy is

$$\pi\int_0^\infty \mathrm{R}^2 du.$$

We have just seen that R is not a definite function of u, but partakes of the random character of the sequence of pulses. At this point we make use of RAYLEIGH's principle ; that we are not concerned with particular wave-lengths, but rather with the average energy over small ranges of wave-length. Bearing in mind the average value of R, we see that, for practical purposes, we have a spectrum whose intensity of energy for period $2\pi/u$ is

$$n\phi^2(u).$$

§ 25. The fluctuations in the energy-wave-length curve will be less rapid as we descend to the longer waves of the spectrum. As we have just seen, when the time of vibration is small compared with T, the fluctuations are so crowded as to be indistinguishable ; the eye, or sensitive plate, will take the meancurve $n\phi^2(u)$. But if the time of vibration is large compared with T the range of angle included in the set

$$\psi - u\tau_1,\ \psi - u\tau_2,\ .\ .\ .\ \psi - u\tau_n$$

will be but small ; the resultant will possess a phase intermediate between the extreme values, and a modulus of almost $n\phi(u)$. As we pass continuously to quicker vibrations, the modulus will diminish from $n\phi(u)$ to zero, and so on ; the divergences of the energy curve from $\phi^2(u)$ being no longer rapid but on a broad and theoretically distinguishable scale.

It will be found, however, that for short pulses the amount of energy in the slow

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[*] Lord RAYLEIGH, 'Phil. Mag.,' Aug., 1880 ; or 'Theory of Sound,' ed. 1894, p. 40.

waves is inconsiderable. The prepotent part of the energy resides in those quicker waves for which the energy curve is of the normal form $n\phi^2(u)$. (Compare the magnetic pulses of Professor THOMSON, treated in the next chapter).

§ 26. To recapitulate, the pulses of the sequence will not be separately distinguishable; their effect depends upon the integral of energy over an interval of time comparable with T; the phase of the Fourier element will have no further effect; and all the observable properties of the sequence will be bound up with the energy function $\phi^2(u)$.

§ 27. Hitherto it has been assumed that the interval T comprises a large number n of complete pulses, these being for the moment supposed not to be of infinite breadth. In general the boundaries of the interval T will find themselves *in* a pulse: there will be a number of incomplete pulses near each end. But these are few compared with the whole number n; they will not perceptibly affect the aggregate.

The spectrum is independent of n as regards composition, if n is large. The total intensity, however, varies as n; thus a variation in the crowding of the pulses causes a corresponding variation of the brightness of the spectrum—a result which might have been expected.

§ 28. Let us consider how these results are affected when the individual pulses are of infinite breadth. Suppose that we examine the type which Lord RAYLEIGH suggested,

$$f(t) = e^{-c^2t^2}.$$

The displacement becomes comparatively small when the distance from the centre of the pulse is great compared with $1/c$. In fact, we are tempted to regard these pulses as practically equivalent to pulses of finite breadth $1/c$ or thereabouts.

Suppose that the least observable interval T comprises a large number of central points of pulses. Suppose also that T is great compared with $1/c$. The interval will contain a contribution from each of the infinite succession of pulses. But, since $\int_b^\infty e^{-x^2}dx$ is small when b is great, only those pulses which contribute finite displacements will affect the aggregate content of T. Now the centres of these will lie either in the interval, or at a distance from its extremities of order $1/c$. As $1/c$ is small compared with T we shall practically be concerned only with the large number of pulses which lie almost entirely within T. We are, therefore, justified in regarding $1/c$ as the effective order of breadth of these strictly infinite pulses.

§ 29. If the pulses (supposed finite) are crowded, so that they overlap largely, we shall not find the characteristic spectrum unless the time-interval T which we are investigating is large compared with the breadth of a pulse. If T cuts into many pulses, but is not large compared with the breadth of each, we shall lose sight of individual pulses, it is true; but the energy function will be largely affected by the incomplete pulses. In other words, if we can shorten the exposure till it is comparable with the duration of a pulse, the spectrum observed will begin to show deviations from the normal spectrum as taken with a much larger exposure, or as observed by the eye.

§ 30. The extension to aggregates of pulses which are not all similar is obvious. Suppose for instance that we have a sequence of pulses of constant displacement, the lengths of the pulses varying, while at the same time the proportions of different lengths are given. The pulses of lengths between x and $x + dx$ are, say, $f(x)\,dx$, of the whole. They may be taken as equal pulses; suppose that they give an energy function $\phi^2(u, x)$. Then the whole energy of the mixture is

$$\int_0^\infty \int_0^\infty f(x)\phi^2(u,x)\,dx\,du.$$

Röntgen Rays and Ordinary Light.

§ 31. Professor THOMSON[*] explains Röntgen radiation by supposing it to consist of a succession of electro-magnetic pulses. Each pulse is practically a pulse of constant magnetic force, lasting for a short time. The thickness of a pulse is comparable with the diameters of the particles composing the cathode stream. Lord RAYLEIGH has pointed out[†] that these pulses may be regarded as simple waves of short wave-length. He did not explicitly consider the properties of a *succession* of pulses. Perhaps on account of this insufficiency of statement, Professor THOMSON[‡] has not fully accepted the above-mentioned view. He has held that the Fourier analysis of a pulse has no physical meaning. Now this is a valid objection to the identification of the single pulse with ordinary light of any composition whatever. The different elements of the integral will possess definite phase-relation; they are in no sense independent.

On the other hand, it has been proved in the course of the present essay that the succession of pulses will actually be equivalent to a spectrum of definite composition. The Thomson pulses will certainly possess the property of being brief in comparison with the shortest observable interval of time; there will be a great number of them in such an interval; it follows that, as the instrument averages over small ranges of wave-length, phase properties will be lost; the processes of time- and wave-length averaging will efface all distinction between the succession of pulses and that mixture of light which is determined by the analysis of the single pulse.

§ 32. We proceed to consider the energy-distribution in the scale of wave-length.

We must express as a Fourier integral a function of x which is zero from $-\infty$ to $-d$; E from $-d$ to $+d$; zero from $+d$ to $+\infty$.
We find

$$\phi(x) = \frac{2\mathrm{E}}{\pi}\int_0^\infty \frac{\sin ud}{u}\cos ux\,du.$$

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[*] Professor THOMSON, 'Phil. Mag.,' February, 1898.

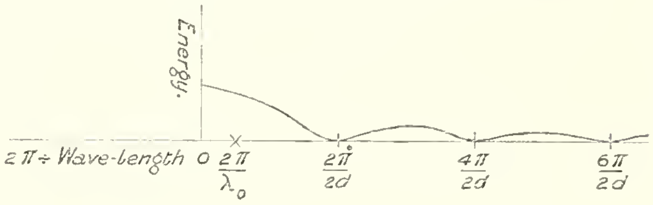
[†] Lord RAYLEIGH, 'Nature,' April 28, 1898.

[‡] Professor J. J. THOMSON, 'Nature,' May 5, 1898.

Accordingly the distribution of energy in the spectrum given by the succession of such pulses is that shown in the curve

$$y = \frac{\sin^2 ud}{u^2}.$$

Fig. 1.—Energy Curve of Röntgen Rays.



If we take the particles of the cathode stream to be at least as great as molecules, $2d$, the thickness of pulse, is small compared with the wave-length of visible light (see THOMSON'S paper); d/λ_0 may be taken as $\frac{1}{1000}$, where λ_0 is the wave-length of yellow light, say. In the scale of fig. 1 $2\pi/\lambda_0$ is very near to 0. It appears that long waves have the greatest amplitude; practically the same amplitude is maintained onwards through the visible spectrum, and in fact till we approach to wave-lengths comparable with the diameter of the molecules.

§ 33. The measure with which we are concerned, however, is not amplitude, but integral energy through ranges of wave-length. Considering this, we see at once that the short waves are all-important. The total energy of the pulse is of order E^2d. The energy contained in waves of length from infinity to λ_0 is of order $E^2d. (d/\lambda_0)$. Remembering that the visible spectrum includes an octave, we may say, roughly, that $\frac{1}{1000}$ of the energy of the radiation will reside in the visible part of the spectrum; and, of the rest, practically the whole in waves of length comparable with the diameter of the molecules. It is noteworthy that waves of length equal to the thickness of the pulse or sub-multiples thereof will be excluded from the spectrum.

§ 34. Inequalities in the thickness of the pulses will slightly modify the features of the equivalent spectrum. Such inequalities will arise partly from the fact that different pulses arrive in slightly different directions; they come from different parts of the glass (an effect diminishing with distance). Furthermore, the particles are not stopped at a single impact in the molecules of the glass.

It also appears that, if Röntgen rays can be made powerful enough, they will affect the eye as ordinary white light.

§ 35. Professor THOMSON'S magnetic pulses are all negative. A mixture of negative and positive pulses will present the same features except in so far as the long-waves are concerned. If the negative and positive are present in equal quantities, the amplitude of the infinite wave will vanish.

§ 36. It is to be remarked that Professor THOMSON'S magnetic pulses differ in one important respect from the thin pulses by which Sir GEORGE STOKES* has sought to

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* WILDE Lecture, 'Proc. Manchester Phil. Soc.,' 1897.

explain Röntgen rays. In the former, we have the magnetic force great and negative throughout the pulse; in the latter, positive and negative are to be so balanced that the force integrated through the thickness of the pulse shall vanish. On this property, together with that of the thinness of the pulses, Sir GEORGE STOKES[*] bases his proof that there will be no sensible diffraction.

The mathematical consequence of this property will be zero amplitude for the infinite wave-length. Practically this means that the energy in the visible spectrum is very much smaller than for the Thomson pulse. It will be of order $E^2d \,.\, (d/\lambda_0)^3$, instead of $E^2d \,.\, d/\lambda_0$; a proportion of 10^{-9} of the whole, instead of 10^{-3}. Now diffraction depends chiefly on waves whose lengths are of this order; very much shorter waves will not be diffracted, but will penetrate matter; and in any case would give much smaller diffraction patterns. Pulses of both the proposed forms will be sensibly free from diffractive properties; those of STOKES in a much higher degree than those of THOMSON.

Radiation of an Incandescent Gas.

§ 37. As an example of the composition of a large number of independent pulses of uniform type, we will take the case of radiation from an incandescent gas. We will suppose the mass of gas to be at a great distance, and to have no visible diameter; we shall thus be enabled to consider the radiation as composed of plane-waves travelling at right angles to their own wave fronts. Furthermore, the amount of gas is to be so small that the emission is not sensibly affected by absorption. The gas is to consist of molecules all having the same period of free vibration.

§ 38. The light received by the spectator will not be homogeneous. One reason for this is the Doppler effect.[†] The velocities of the molecules in the line of sight will alter the period of the light received. Another cause will doubtless be the altered vibrations of two molecules when very near to one another. This will perhaps become important at high pressures, but we will not further consider it at present.

§ 39. Lastly, we have to take into account the fact that the train of single waves emitted by each vibrating molecule is not infinite in length, but has a definite beginning and ending. The effect of this cause is investigated below. The Doppler effect is included in the same piece of analysis. We shall arrive at the remarkable result that the limiting width of the spectrum line when the pressure is indefinitely diminished is *less* by some 10 per cent. than the width calculated by Lord RAYLEIGH, who took into account nothing but the Doppler effect.

§ 40. The vibration of a molecule will be altered by collision with another. The velocity will also be altered, in both magnitude and direction. The vibration *received by the spectator* from this molecule will therefore be suddenly and fortuitously altered in period, amplitude and phase. The total radiation received

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[*] WILDE Lecture, 'Proc. Manchester Phil. Soc.,' 1897.

[†] Lord RAYLEIGH, 'Phil. Mag.,' vol. 27, April, 1889.

from the whole gas will consist of a great number of finite trains superposed. We must consider those trains as practically independent. It is true that each individual train is connected in one respect, namely, instant of beginning or ending, with two others, the trains emitted by the same molecule before and after. But this element of regularity will be overwhelmed by the independence of the different molecules. All we have to do is, to find the Fourier integral equivalent to a finite train of waves, to find the distribution of energy in the scale of frequency, and to sum up the energy for all possible trains.

Fourier Analysis of a Train of m Complete Sine Waves.

§ 41. The general theorem is

$$\pi f(x) = \int_0^{\infty} \int_{-\infty}^{+\infty} \cos \omega(\lambda - x) f(\lambda) d\omega \, d\lambda$$

In the present case $f(x) = 0$, except from 0 to $2\pi m/\kappa$, within which limits $f(x) = \cos \kappa x$.

Thus

$$\pi f(x) = \int_0^{\infty} \int_0^{2\pi m/\kappa} \cos \omega(\lambda - x) \cos \kappa\lambda \, d\omega \, d\lambda$$

$$= \tfrac{1}{2}\int_0^{\infty} \int_0^{2\pi m/\kappa} \{\cos \overline{(\omega + \kappa)\lambda - \omega x} + \cos \overline{(\omega - \kappa)\lambda - \omega x}\} d\omega \, d\lambda$$

$$= \int_0^{\infty} d\omega \left\{ \frac{\cos \omega \left(x - \frac{\pi m}{\kappa}\right) \sin \frac{\pi m \omega}{\kappa}}{\omega + \kappa} + \frac{\cos \omega \left(x - \frac{\pi m}{\kappa}\right) \sin \frac{\pi m \omega}{\kappa}}{\omega - \kappa} \right\}$$

If we consider the quantities $\dfrac{\sin \frac{\pi m \omega}{\kappa}}{\omega + \kappa}$, $\dfrac{\sin \frac{\pi m \omega}{\kappa}}{\omega - \kappa}$, we see that the latter attains to a maximum value $\pi m/\kappa$ at $\omega = \kappa$, and that the former is small in comparison since m is generally a considerable number.

We shall be concerned only with the values of ω near to κ; accordingly the first term shall be neglected. We shall then have a distribution of energy,

$$\int_0^{\infty} d\omega \frac{\sin^2 \pi m \omega/\kappa}{(\omega - \kappa)^2},$$

neglecting numerical coefficients which do not alter the distribution.

Let $\omega - \kappa = 2\pi n$; then at a distance n from the maximum (n being reciprocal wave-length) we have energy proportional to

$$\frac{\sin^2 \frac{\pi m}{\kappa}(\omega - \kappa)}{n^2} = \frac{\sin^2 \pi r n}{n^2}$$

where r is the length of the train.

For this single train, then, we have energy falling off from a maximum at κ according to the law

$$\frac{\sin^2 \pi r n}{n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

(the ordinary law for the diffraction pattern of an edge), where n is the distance from the brightest part measured in the scale of reciprocal wave-length.

Summation for all Molecules having Definite Velocities both Athwart and in Line of Sight.

§ 42. For these we have a definite position κ of maximum brightness, and definite resultant velocity v. We have to integrate for the different lengths of train.

Now Tait* has shown that, of all atoms moving with velocity v, a fraction $e^{-f\rho}$ penetrates unchecked to distance ρ, where

$$f = 4\pi n s^2 \ . \ \frac{h^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} \left(\frac{1}{4h^2 v} e^{-hv^2} + \overline{\frac{1}{4h^2 v^2} + \frac{1}{2h}} \int_0^v e^{-hv^2} dv \right).$$

We may write this function of v as follows :—

$$\begin{aligned} f &= \pi^{\frac{1}{2}} n s^2 \left[\frac{e^{-\mathrm{P}^2}}{\mathrm{P}} + \left(\frac{1}{\mathrm{P}^2} + 2 \right) \int_0^{\mathrm{P}} e^{-\mathrm{P}^2} d\mathrm{P} \right] \\ &= c^2 f(\mathrm{P}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.),} \end{aligned}$$

where

$$\mathrm{P} = v h^{\frac{1}{2}},$$

$$f(\mathrm{P}) = \frac{e^{-\mathrm{P}^2}}{\mathrm{P}} + \left(\frac{1}{\mathrm{P}^2} + 2 \right) \int_0^{\mathrm{P}} e^{-\mathrm{P}^2} d\mathrm{P},$$

$$c^2 = \pi^{\frac{1}{2}} n s^2,$$

n = number of atoms in unit volume,

s = diameter of atom.

From this we see that, of molecules moving with velocity v, a fraction

$$f e^{-f\rho} d\rho$$

have free paths between ρ and $\rho + d\rho$.

Now, such a molecule will emit an undisturbed train of waves of length between r and $r + dr$, where $r = \frac{\mathrm{V}}{v}\rho$, and V is the velocity of light.

Hence, of all molecules moving with velocity v, a fraction $\frac{vf}{\mathrm{V}} e^{-\frac{vfr}{\mathrm{V}}} dr$, will give free paths between r and $r + dr$.

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* Tait, 'Edinb. Trans.,' vol. 33 p. 72.

Returning to the expression for the energy of a single train of length r (i.), we see that with the aggregates of molecules now under consideration (definite thwart and line-of-sight velocities) we have for n a proportion of energy

$$\frac{fr}{n^2\mathrm{V}}\int_0^\infty e^{-\frac{vf}{\mathrm{V}}.r}\sin^2\pi nr\,.\,dr$$

$$= \frac{fv}{2n^2\mathrm{V}}\int_0^\infty e^{-\frac{vf}{\mathrm{V}}r}(1-\cos 2\pi nr)\,dr$$

$$= \frac{fv}{2n^2\mathrm{V}}\left|\left(-\frac{\mathrm{V}}{vf}-\frac{-\frac{vf}{\mathrm{V}}\cos 2\pi nr+2\pi n\sin 2\pi nr}{\left(\frac{vf}{\mathrm{V}}\right)^2+4\pi^2n^2}\right)e^{-\frac{vfr}{\mathrm{V}}}\right|_0^\infty$$

$$= \tfrac{1}{2}\,\frac{1}{n^2+\left(\frac{vf}{\mathrm{V}}\right)^2}\,.\;.\quad\text{(iii.)}$$

We next integrate for a definite velocity p in the line of sight, and all possible velocities q athwart.

The proportion of molecules with thwart velocities between q and $q+dq$ is $qe^{-hq^2}dq$. Hence, omitting the $\frac{1}{2}$ from (iii.) (it does not affect the distribution of energy), we have

$$\int_0^\infty qe^{-hq^2}dq\,.\,\frac{1}{n^2+\left(\frac{vf}{\mathrm{V}}\right)^2}\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\quad\text{(iv.)}$$

where $v^2 = p^2 + q^2$.

Lastly, we introduce all possible velocities in line of sight.

Here the Doppler effect enters; the mid-point of the spectrum (iv.) will be different for different p's. Let x be the distance from the centre of the final spectrum line (measured, as before, in reciprocal wave-lengths), we have

$$\int_{-\infty}^{+\infty} e^{-hp^2}\int_0^\infty qe^{-hq^2}\frac{1}{\left(x-\frac{p}{\lambda\mathrm{V}}\right)^2+\left(\frac{vf}{2\pi\mathrm{V}}\right)^2}\,dp\,dq\;.\;.\;.\;.\;.\;.\;.\;.\quad\text{(v.)}$$

This integral, regarded as a function of x, gives the distribution of light in the spectrum.

To make further progress, we will change the variables of integration from p and q to p and v, where $v^2 = p^2 + q^2$. We must remember that f is a function of v.

We have

$$\int_0^\infty e^{-hp^2}dp \int_p^\infty ve^{-hq^2}\frac{1}{\left(x-\frac{p}{\lambda \mathrm{V}}\right)^2+\left(\frac{vf}{2\pi \mathrm{V}}\right)^2}dv$$

$$+\int_{-\infty}^0 e^{-hp^2}dp \int_{-p}^\infty ve^{-hq^2}\frac{1}{\left(x+\frac{p}{\lambda \mathrm{V}}\right)^2+\left(\frac{vf}{2\pi \mathrm{V}}\right)^2}dv$$

$$=\int_0^\infty\int_p^\infty e^{-hv^2}\,.\,v\,dp\,dv\left\{\frac{1}{\left(x-\frac{p}{\lambda \mathrm{V}}\right)^2+\left(\frac{vf}{2\pi \mathrm{V}}\right)^2}+\frac{1}{\left(x+\frac{p}{\lambda \mathrm{V}}\right)^2+\left(\frac{vf}{2\pi \mathrm{V}}\right)^2}\right\},$$

or, changing the order of integration,

$$\int_0^\infty\int_0^v dv\,dp\,.\,ve^{-hv^2}\left\{\frac{1}{\left(x-\frac{p}{\lambda \mathrm{V}}\right)^2+\left(\frac{vf}{2\pi \mathrm{V}}\right)^2}+\frac{1}{\left(x+\frac{p}{\lambda \mathrm{V}}\right)^2+\left(\frac{vf}{2\pi \mathrm{V}}\right)^2}\right\} \quad . \quad . \text{(vi.)}$$

Visibility Curve.

§ 43. Professor MICHELSON* has shown that, although the breadths of elementary spectrum lines cannot in general be examined directly, yet the application of his interference method enables one to obtain much more detailed information. The light is made to interfere with itself, at a relative retardation u of the two half streams. Interference bands are produced and their "visibility" estimated for different values of u, the path-difference. From the visibility-curve thus constructed we can work backwards to the breadth of the spectrum line, and find out something about the distribution of light in this breadth.

MICHELSON has shown that, if $\phi(x)$ represent the intensity of light for position x in the spectrum, and

$$\mathrm{C}=\int_{-\infty}^{+\infty}\phi(x)\cos 2\pi ux\,dx$$

$$\mathrm{S}=\int_{-\infty}^{+\infty}\phi(x)\sin 2\pi ux\,dx$$

$$\mathrm{Q}=\int_{-\infty}^{+\infty}\phi(x)dx$$

and

$$\mathrm{V}^2=\frac{\mathrm{C}^2+\mathrm{S}^2}{\mathrm{Q}^2},$$

then V is the visibility-function, in terms of u, the path-difference.

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[*] MICHELSON, 'Phil. Mag.,' vols. 31 and 34.

In the present case S = 0, and

$$C = \int_{-\infty}^{+\infty} \int_0^{\infty} \int_0^{v} \cos 2\pi ux \, dx \, . \, e^{-hv^2} . \, v \, dv \, dp \Sigma \frac{1}{\left(x \pm \frac{p}{\lambda V}\right)^2 + \left(\frac{vf}{2\pi V}\right)^2}$$

Now

$$\int_{-\infty}^{+\infty} \cos 2\pi ux \frac{dx}{\left(x - \frac{p}{\lambda V}\right)^2 + \left(\frac{vf}{2\pi V}\right)^2}$$

$$= \cos \frac{2\pi up}{\lambda V} \int_{-\infty}^{+\infty} \frac{\cos 2\pi ux \, dx}{x^2 + \left(\frac{vf}{2\pi V}\right)^2} = \cos \frac{2\pi up}{\lambda V} \, . \, \frac{2\pi^2 V}{vf} \, . \, c^{-\frac{vfu}{V}} .$$

We may omit certain constant factors, and write

$$C = \int_0^{\infty} \int_0^{v} \cos \frac{2\pi up}{\lambda V} \, . \, \frac{1}{vf} . \, e^{-\frac{vfu}{V}} e^{-hv^2} . \, v \, dv \, dp$$

$$= \frac{\lambda V}{2\pi u} \int_0^{\infty} \sin \frac{2\pi uv}{\lambda V} \, . \, \frac{du}{f} \, . \, e^{-\left(hv^2 + \frac{vfu}{V}\right)} .$$

We may still further simplify this by introducing the notation of page 347. Omitting unnecessary factors,

$$C = \frac{1}{u} \int_0^{\infty} \sin \frac{u \kappa P}{V h^{\frac{1}{2}}} \cdot \frac{dP}{f(P)} e^{-P^2} e^{-\frac{\pi^{\frac{1}{2}} n s^2}{V h^{\frac{1}{2}}} \, . \, u P f(P)}$$

$$= \frac{1}{u} \int_0^{\infty} \sin 2buP \, . \, \frac{dP}{f(P)} \, . \, e^{-P^2 - a^2 u P f(P)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{vii.}),$$

where

$$2b = \frac{\kappa}{V h^{\frac{1}{2}}}$$

$$a^2 = \frac{\pi^{\frac{1}{2}} n s^2}{V h^{\frac{1}{2}}}$$

$$P = v h^{\frac{1}{2}}$$

$$f(P) = \frac{e^{-P^2}}{P} + \left(\frac{1}{P^2} + 2\right) \int_0^{P} e^{-P^2} dP.$$

To deduce V (the visibility function) from C, all we have to do is to put $u = 0$ in C, and divide C by the quantity thus formed.

Limiting Case of Zero Pressure.

§ 44. If the pressure is very small, n, the number of molecules in unit volume becomes small, and with it a^2. We are thus reduced to

$$C_0 = \frac{1}{u} \int_0^{\infty} \sin 2buP \, . \, \frac{e^{-P^2} dP}{f(P)}$$

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* TAIT, 'Edinb. Trans.,' vol. 33, p. 95.

§ 45. This expression as it stands is quite intractable. We are enabled, however, to make progress by means of tables which Tait* gives in his paper on the kinetic theory. Our subject of integration is

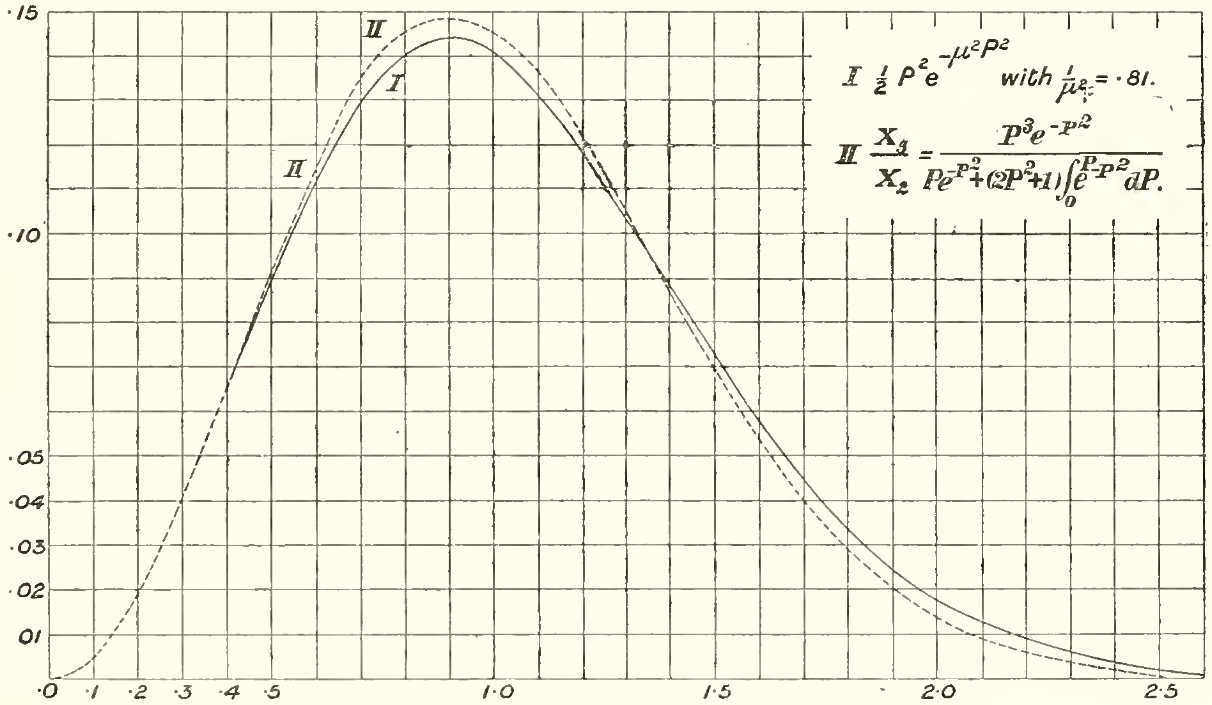
$$\frac{e^{-\mathrm{P}^2}\sin 2bu\mathrm{P}}{f(\mathrm{P})} = \frac{\sin 2bu\mathrm{P}}{\mathrm{P}} \cdot \frac{\mathrm{P}^3 e^{-\mathrm{P}^2}}{\mathrm{P}e^{-\mathrm{P}^2} + (2\mathrm{P}^2+1)\int_0^{\mathrm{P}} e^{-\mathrm{P}^2} d\mathrm{P}}$$

$$= \frac{\sin 2bu\mathrm{P}}{\mathrm{P}} \cdot \frac{\mathrm{X}_3}{\mathrm{X}_2} \text{ in Tait's notation.}$$

On marking out the graph of $\mathrm{X}_3/\mathrm{X}_2$ by means of the values in column 6 of Tait's table, it becomes obvious that the general outline of the function is very near to that of

$$\tfrac{1}{2}\mathrm{P}^2 e^{-\mu^2\mathrm{P}^2}.$$

Fig. 2.



To make the agreement as good as possible we choose μ so that the two curves may attain their maximum for the same value of P.

Now the latter function has its maximum at $\mathrm{P} = \dfrac{1}{\mu}$; graphically we see that we must choose $\dfrac{1}{\mu} = \cdot 9025$ (about); $\dfrac{1}{\mu^2} = \cdot 81$ . . .

In the accompanying figure, I. (continuous curve) is $\mathrm{X}_3/\mathrm{X}_2$; II. (dotted curve) is $\frac{1}{2}\mathrm{P}^2 e^{-\mathrm{P}^2/\cdot 81}$. The agreement is so good that the error in the integral through using II. instead of I. will be only about 1 per cent. But this is of the order of the errors in the observed visibilities, with which we propose to compare our results.

Making this substitution, we have

$$C_0 = \frac{1}{2u}\int_0^\infty \frac{\sin 2buP}{P} \,.\, P^2 e^{-\mu^2P^2} dP = \frac{b\pi^{\frac{1}{2}}}{4\mu^3} e^{-\frac{b^2u^2}{\mu^2}}$$

$$\therefore \quad V_0 = e^{-\frac{b^2u^2}{\mu^2}} = e^{-\frac{\kappa^2u^2}{4\mu^2V^2\lambda}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(viii.)}$$

§ 46. The result is now in a form which we can interpret. Lord RAYLEIGH* has worked out the width of the spectrum lines, taking into account the Doppler effect alone. The visibility function deduced from his work in accordance with MICHELSON'S definition of "visibility"† is

$$e^{-\kappa^2u/4V^2h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ix.)}$$

The two functions (viii.) and (ix.) are of the same form, and differ only in the presence of μ^2 in (viii.). Now MICHELSON'S experiments gave visibility-curves agreeing in general character with (ix.); they would therefore agree equally well with the function (viii.) which has been found above.

Numerical Estimates.

§ 47. MICHELSON has investigated the lines of several gases with his interferometer. He compares the "half-widths" (value of u for which visibility is $\frac{1}{2}$ of maximum) of the visibility-curves with the values deduced by Lord RAYLEIGH from DOPPLER'S principle. Unfortunately MICHELSON misquotes RAYLEIGH'S result, and has dropped a 2; in RAYLEIGH'S formula the path-difference is 2Δ; MICHELSON has taken it to be Δ. The last column of figures in page 294 of MICHELSON'S paper‡ should be all doubled. This would give the observed "half-widths" in every case much less than the calculated values. Furthermore, using the function which has been obtained in the present paper,

$$V = 2^{-\frac{\kappa^2u^2}{4l\mu^2V^2h}}$$

where l is the Napierian log. of 2.

Hence for half-width

$$\frac{\kappa u}{2\mu\sqrt{l}Vh^{\frac{1}{2}}} = 1$$

$$\therefore \quad u = \frac{2\mu V\lambda^{\frac{1}{2}}\sqrt{l}}{\kappa}.$$

But $v = \frac{2}{\pi^{\frac{1}{2}}h^{\frac{1}{2}}}$; v being the average velocity of the molecules.

Hence
$$u = \lambda \,.\, \frac{2\mu\sqrt{l}}{\pi^{\frac{3}{2}}} \,.\, \frac{V}{v}.$$

$$\therefore \quad u/\lambda = \cdot 33V/v, \text{ instead of MICHELSON'S } \cdot 15V/v.$$

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* 'Phil. Mag.,' vol. 27, p. 304, 1889.

† MICHELSON, 'Phil. Mag.,' vols. 31 and 34.

‡ 'Phil. Mag.,' vol. 34, 1892.

Lord RAYLEIGH's expression for the half-width is

$$u = \lambda \, . \, \frac{2\sqrt{l}}{\pi^{\frac{3}{2}}} \, . \, \frac{\mathrm{V}}{v}.$$

Now $\mu = 10/9$; hence the theory developed in the present paper necessitates a further addition of about 10 per cent. to the values calculated from Lord RAYLEIGH's formula. But the effect of this is quite obscured by the above-mentioned necessity of doubling the figures which MICHELSON deduces from theory. The conclusion of his paper should be that Lord RAYLEIGH's theory accounts for *a certain fraction of the observed widths of spectral lines.* The fraction varies from $\frac{2}{3}$ to $\frac{1}{4}$ for the different substances examined.

If the distribution of energy in the spectral line be given by

$$\phi(x) = 2^{-x^2/\delta_0{}^2} = e^{-\frac{lx^2}{\delta_0{}^2}},$$

then
$$\mathrm{C} = \int \phi(x) \cos 2\pi ux \, dx = \frac{\pi^{\frac{1}{2}}\delta_0}{\sqrt{l}} \, . \, e^{-\frac{\pi^2 u^2 \delta_0{}^2}{l}}$$

and
$$\mathrm{V} \doteq e^{-\frac{\pi^2 u^2 \delta^2}{l}} = 2^{-\frac{\pi^2 u^2 \delta_0{}^2}{l^2}}.$$

If we construct the curve representing the energy of the spectrum on a scale of reciprocal wave-lengths, the "half-width" in this curve will be δ_0. Hence the "half-widths" in the energy-curve and the visibility-curve are connected by the relation

$$n_0 = \frac{l}{\pi} \, . \, \frac{1}{\delta_0}.$$

§ 48. Our result that 10 per cent. ought to be added to the visibility half-width means that the theoretical width of the spectrum line should be diminished by a similar percentage.

It might have been expected that, with pressure small and collisions comparatively few, the modifying effect of the curtailment of free trains would disappear, the free paths being now on the average long. We might have expected that the formula for the width of the lines would converge to that given by Lord RAYLEIGH, in which the Doppler effect alone is considered. If the above reasoning is valid, there is no such convergence of the two theories when the pressure is indefinitely reduced; the results derived from them differing by some 10 per cent. It is noteworthy that the present theory leads us to expect narrower lines than does Lord RAYLEIGH. This result is certainly paradoxical, and calls for further justification.

The modification in theory has been to substitute for mathematically homogeneous light proceeding from each molecule, a radiation giving a certain continuous spectrum. We could hardly have foreseen without a complete analysis that, for zero pressure, the integrated effect of all these spectra gives an intensity curve for the total spectral line *steeper* than before.

§ 49. The following considerations may tend to remove the doubts which this result may arouse.

The whole set of molecules with given velocities, thwart and in line-of-sight, and with given length of free path, will emit light of a certain spectral composition, namely, that given by the function $\frac{\sin^2 \pi rn}{n^2}$ (p. 346), r being the length of the train of waves emitted during the single free path. In theory, this will give a set of lines in the spectrum of the same pattern as the diffraction lines of a straight edge. Other sets of molecules with other velocities and path-lengths will give other sets of lines; the whole aggregate of lines overlapping and compounding to give such spectrum lines as actually exist, and were measured by MICHELSON.

§ 50. Now, by lowering the pressure of the gas, we may lengthen the average free path, and the average trains of waves emitted in a single flight, this lengthening being theoretically without limit. The effect of this will be to narrow the curve $\frac{\sin^2 \pi rn}{n^2}$, also without limit. We are tempted, when this happens, to substitute for the aggregate of curves $\frac{\sin^2 \pi rn}{n^2}$ the aggregate of their maximum ordinates. If we do this, and also assume that the average length of wave-train is the same for all different velocities in the gas, we shall, in effect, be following Lord RAYLEIGH'S procedure, and we shall obtain his expression for the width of the resultant line.

§ 51. On closer examination, it will be obvious that the molecules moving with greater velocities emit, on the average, *shorter* wave-trains. For, given the velocity, the mean free path is $1/f$ (p. 347), a function of v; while the corresponding train of waves has length V/vf. It is not difficult to verify that this function of n diminishes as v increases.

Now, the molecules with greater velocities in the line of sight have, on the average, greater resultant velocities. These, therefore, give shorter trains of waves, and smaller ordinates in the energy curve (the maximum of $\frac{\sin^2 \pi rn}{n^2}$ is $\pi^2 r^2$, for $n = 0$). But these molecules will provide light that goes towards the edges of the spectrum line. The energy-curve will accordingly be steeper, and the line narrower than would follow from the assumption that the mean free path is the same for two groups of molecules having two different velocities.

§ 52. Furthermore, it is not allowable to substitute for the component curves their maximum ordinates, however steep and narrow these curves may become. The maximum of $\frac{\sin^2 \pi rn}{n^2}$ is $\pi^2 r^2$. If we substitute these maximum ordinates and then form an energy-curve by summing them all into a smooth curve, each will contribute energy proportional to r^2. But, in reality, the total energy connected with $\frac{\sin^2 \pi rn}{n^2}$ is $\int_{-\infty}^{+\infty} \frac{\sin^2 \pi rn^2}{n^2}\, dn = \pi^2 r$. Thus, trains of length r ought to contribute total energy proportional to r, not to r^2. In fact, the taller the component curves are, the narrower they are. This consideration is overlooked if we allow ourselves to substitute the maxima of the curves for the curves themselves when these become narrow.

§ 53. This latter source of error, by which r^2 is substituted for r, will not affect the shape of the resultant energy-curve if the average length of train is taken to be independent of the velocity. But we have seen that greater velocities give shorter trains. And this error tends in the same direction as the other ; for it gives too great prominence to the longer trains, *i.e.*, to the smaller velocities, which velocities send light to the middle of the spectrum line. Hence the effect of the error is to make the resultant curve too steep.

Accordingly, the only accurate way of investigating the limiting width for zero pressure is to form the general energy function as on p. 349, and then to proceed to the limit by diminishing the number of molecules in unit volume.

Effect of Damping in the Widths of Spectrum Lines.

§ 54. It has been urged by LOMMEL[*] that, whereas the vibrations of an atom are undoubtedly damped by radiation, the light emitted by a simple gas will be to some extent continuous. The same idea has been recently developed by JAUMANN.[†]

Both of these writers rely on a Fourier analysis of the vibration

$$e^{-\kappa t}\sin(pt+\psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{x.})$$

Their results have not been accepted generally ; as has been pointed out in the course of the present paper, their procedure will have no physical meaning for the single train of waves with which they deal.

But we have also seen that a vast aggregate of independent emissions of this form will really give the result which LOMMEL suggests.

§ 55. The trains of waves emitted by the molecules of a gas cannot strictly be represented by (x.). As a matter of fact, such a motion as (x.) will never be allowed to go on indefinitely ; it will always be checked at a certain stage by a new collision. Nevertheless, if the radiation is so rapid that the vibration of a molecule has generally become insignificant before the next collision occurs, we shall not be making a serious error by allowing (x.) to represent the train of waves. We will proceed to investigate the effect on the assumption that the damping is so rapid as to allow this procedure.

§ 56. Both LOMMEL and JAUMANN make an erroneous application of FOURIER'S theorem. The analysis should be as follows :—

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[*] LOMMEL, 'Wied. Ann.,' vol. 3, p. 251, 1878.

[†] JAUMANN, 'Wied. Ann.,' vols. 53 and 54.

$$\mathrm{F}(t) = 0 \text{ for } t = -\infty \text{ to } t = 0$$
$$\text{and } \mathrm{F}(t) = e^{-\kappa t} \cos pt \text{ for } t = 0 \text{ to } t = +\infty.$$

(The phase ψ will not make any important difference in the energy-function.)

$$\pi \mathrm{F}(t) = \int_0^\infty \int_0^\infty e^{-\kappa\beta} \cos p\beta \cos u(\beta - t) du\, d\beta$$
$$= \tfrac{1}{2} \int_0^\infty du \int_0^\infty e^{-\kappa\beta} [\cos(\overline{p+u}\beta - ut) + \cos(\overline{p-u}\beta + ut)] d\beta$$
$$= \tfrac{1}{2} \int_0^\infty du \left\{ \frac{\kappa \cos ut + \overline{p+u} \sin ut}{\kappa^2 + (p+u)^2} + \frac{\kappa \cos ut - \overline{p-u} \sin ut}{\kappa^2 + (p-u)^2} \right\}.$$

Now κ/p is small since the damping is gradual ; accordingly, both $\frac{1}{\kappa^2 + (p+u)^2}$ and $\frac{1}{\kappa^2 + (p-u)^2}$ will be of order $1/p^2$ unless p is near to u. In that case, the latter of the two expressions will attain to the order $1/\kappa$. We are justified in approximating to the extent of neglecting the former expression.

§ 57. The energy of the train of waves will depend on

$$\int_0^\infty \frac{du}{\kappa^2 + (p-u)^2}.$$

This function will define the spectrum to which a vast concourse of such damped trains is equivalent.

Now this will be a widened line in the spectrum. The "half-width" will be of order κ in frequency. The half-widths which MICHELSON has observed for irresoluble lines are of order

$$10^{-5} \times p.$$

If κ is of this order, $\frac{\kappa \times 10^5}{p}$ is finite.

Now $10^5/p$ is comparable with the time of 10^5 vibrations. Again, there are on the average 10^5 vibrations in the free path. In time t the vibrations are reduced in the ratio $e^{-\kappa t} : 1$; if κt is finite, the reduction of the energy may be very noticeable.

§ 58. We are thus led to conclude that the vibrations of molecules may be very considerably damped in the course of their free paths and yet no widening of lines be produced beyond what is actually observed. It will be remembered that the kinetic theory, without damping, gave widths varying from one-quarter to two-thirds of the widths observed. It seems not impossible that, for small densities, the residual width is to be ascribed to radiative damping. When the density becomes considerable, the mutual effect of neighbouring molecules will doubtless become so important as to obscure both damping and Doppler effect

Character of the Æther Motions in nearly Homogeneous Light.

§ 59. It has been thought that the possibility of producing a large number of interference fringes from white light is an indication of a certain regularity in the æther motion corresponding to such light. This view has been abundantly refuted by Gouy, Rayleigh, and Schuster.* The fringes cannot be produced without the use of a spectral apparatus; and the number of the fringes is an index, not of the regularity of the white light, but of the resolving power of the spectroscope.

A large number of fringes can also be produced without a spectroscope, by using radiations which naturally possess a high degree of homogeneity. The number of these fringes is a test of the homogeneity; and in this case, it is also a test of the regularity of the æther motion.

§ 60. We have justified the representation of light by a Fourier integral of the form

$$\int_0^\infty \mathrm{R} \cos(ut + \psi)du,$$

where R, ψ are functions of u.

It can be shown that, for light of long duration, ψ will fluctuate rapidly in terms of u.

For approximately homogeneous light of mean frequency p, we will use the notation

$$\int \mathrm{R} \cos(\overline{p + u}t + \psi)du.$$

In this expression R will be insignificant, except for values of u small compared with p.

The above integral can be written

$$\cos pt \int \mathrm{R} \cos(ut + \psi)du - \sin pt \int \mathrm{R} \sin(ut + \psi)du.$$

The ranges of integration will practically be confined to a region on either side of zero, small compared with p.

Each of the integrals

$$\int \mathrm{R} \cos(ut + \psi)du, \qquad \int \mathrm{R} \sin(ut + \psi)du$$

is a function of t, whose variations are slow compared with those of $\cos pt$. The expression $\cos pt \int \mathrm{R} \cos(ut + \psi)\,du$ may be taken to denote a simple vibration of period $2\pi/p$, whose amplitude varies slowly. The zeros of the expression will be, effectively, the zeros of $\cos pt$. Similar statements will apply to

$$\sin pt \int \mathrm{R} \sin(ut + \psi)du.$$

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* Gouy, 'J. de Ph.,' series 2, vol. 5, p. 354; Rayleigh, Art. "Wave Theory," 'Encycl. Brit.'; 'Phil. Mag.,' vol. 27, p. 460, 1889; Schuster, 'Phil. Mag., vcl. 54, p. 509, 1894

Finally, the motion which we have analysed as

$$\cos pt \int \mathrm{R} \cos (ut + \psi) du - \sin pt \int \mathrm{R} \sin (ut + \psi) du$$

may be described as a simple vibration whose amplitude *and phase* vary slowly. We might equally well say that the amplitude and period vary slowly; the latter within narrow limits. In passing it is interesting to note that all the effects of white light would be produced by an approximately simple vibration whose period varies slowly, but within wide limits; the whole range of variation being traversed a great number of times in the course of the shortest observable interval of time The great gap between 10^{-15} second (the period of vibration) and 10^{-7} second (the shortest observable time) will give room for a rate of variation small compared with the one measure, and great compared with the other.

§ 61. Returning to the nearly homogeneous light, let $\pm s$ be the effective range of integration in

$$\int \mathrm{R} \cos (ut + \psi) du, \qquad \int \mathrm{R} \sin (ut + \psi) du.$$

This of course means that R becomes small outside the limits $\pm s$. The width of the spectrum line will be of order s. Now the two integrals just quoted will give irregularly sinuous time-curves, the average extent of a sinuosity being of order $1/s$. Thus, the varying simple vibration

$$\int_{p-s}^{p+s} \mathrm{R} \cos (ut + \psi) du$$

will have entirely changed in amplitude and phase after a time of order $1/s$. It will therefore be impossible to produce sensible interference with time-differences of more than $1/s$. This is another aspect of the fact that for lines of width s (measured in frequency), the maximum path-difference for interference is of order $1/s$.

§ 62. We can look at the same matter from yet another point of view. We may go back to the composition of the radiation from a gas. This we have seen to be built up of finite trains of waves. For the moment, let us omit the Doppler effect and take all the trains to be of the same period. In an interval during which only a small proportion of molecules collide, the amplitude and phase of the composite vibrations is but little altered. But after an interval comparable with the mean free time of molecules, most of the molecules will have obtained new and independent vibrations; the composite motion will be entirely altered in amplitude and phase. For the discharge tubes used by MICHELSON this time is of order 10^5 periods.

Again, isolating the Doppler effect, we deal with the superposition of infinite simple trains. We have already seen that the composite vibration will have a materially altered amplitude and phase after a time equal to the reciprocal of the range of frequency in the component trains. For MICHELSON'S experiments it happens that this time is again some 10^5 periods.

In the actual radiation the two causes coexist; the joint effect is of the same order; the radiation will attain independence after every 10^5 periods. In accordance with this fact, we have interference up to path-difference of some 10^5 wave-lengths; while the width of the lines corresponds to a fraction $1/10^5$ of the frequency of the light.

Effect of a Natural Light on a Vibration.

§ 63. We have already proved that the complete solution of

$$\ddot{x} + 2\kappa\dot{x} + p^2x = f(t) \equiv \int_0^\infty \mathrm{R}\cos(ut + \psi)du$$

is
$$x = \int_0^\infty \frac{\mathrm{R}du}{(p^2 - u^2)^2 + 4\kappa^2u^2}\{(p^2 - u^2)\cos(ut + \psi) + 2\kappa u\sin(ut + \psi)\},$$

$f(t)$ being such a function of time as can actually represent a natural radiation (*see* footnote, page 336).

Now we have already pointed out the hypothetical character of this treatment of the molecule as a simple vibrator. Nevertheless the method has a historical interest, and may be regarded as a foreshadowing of the truth; it may be worth while to sketch the result of applying our analysis.

The composition of the exciting light is

$$\int_0^\infty \mathrm{R}^2du,$$

of the light emitted by the molecule

$$\int_0^\infty \frac{\mathrm{R}^2du}{(p^2 - u^2)^2 + 4\kappa^2u^2},$$

while the rate of absorption is dependent on

$$\int_0^\infty \frac{2\kappa u^2\mathrm{R}^2du}{(p^2 - u^2)^2 + 4\kappa^2u^2}.$$

We will take the light to be constant. Its effect on the vibrator is seen to depend entirely upon its spectrum.

§ 64. This is equally true for nearly homogeneous light. The effect of the irregularities in the light may be deduced from the observed widening of the spectrum line.

Now SELLMEIER treated of this problem in the paper which laid the foundations of the modern theory of dispersion.* He recognised that no natural radiation is a perfect train of simple waves, and he investigated the effect of the irregularities upon a vibrator. The period of the light was to differ from that of the natural vibrations of the molecule. He came to the conclusion that the irregularities in the light would

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* SELLMEIER, 'Pogg. Ann.,' 1872, vol. 145, p. 520.

not arouse the natural vibrations of the molecule. He held that the motion of the molecule would be mainly in the mean period of the incident light. This conclusion we take to be erroneous.

§ 65. SELLMEIER does not adopt the Fourier method; he builds up the incident light by means of a great number of finite superimposed trains of simple waves, all having the same phase and period, but of amplitudes, durations, and positions so adjusted as to give the actual fluctuation of amplitude which is present in the resultant motion. In passing, it may be noticed that this arrangement will give no irregularity of *phase* in the light motion; whereas we have shown that such irregularity will generally be present. The defect may be remedied by removing the condition that the component trains shall be of the same phase; but this consideration will not alter SELLMEIER's reasoning in any essential.*

SELLMEIER supposes the motion of the vibrator to be free from all damping. We will show that in this case his conclusion ought to be that the natural periods of the vibrator become continually more and more prominent without limit.

§ 66. His reasoning is as follows. Each new train of waves, as it strikes the vibrator, arouses :—

i. a vibration in the period of the incident light (forced),
ii. a vibration in the period proper to the vibrator (natural).

The forced vibration (i) will be of the same phase as the exciting train, and of amplitude proportional to that of the said train. The natural vibration will generally have an amplitude of the same order; its phase will, however, be different.

§ 67. The motion of the vibrator at any time results from the superposition of all the vibrations previously started. It will be partly forced and partly "natural." The forced vibrations at any instant will clearly differ from the incident motion at that instant by a numerical factor only.

The natural vibration, on the other hand, is compounded of members, whose phases are practically fortuitous. This is easily seen as follows. The phase of natural vibration aroused by the beginning of a new train depends upon the point of time at which this event takes place. The difference of a fraction of a period in the position of this point of time produces a finite change in the phase of the natural vibration aroused. But, in building up the slowly varying vibration by means of simple trains, the instant at which each start is, to a few periods, immaterial.

§ 68. SELLMEIER concludes from this that the natural vibration will be insignificant compared with the forced vibration. This mistake arises from the fact that

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* It will be convenient to get rid of the endings of these trains. This we may do by supposing that a train, once started, is unending; and by introducing a train of equal amplitude and opposite phase, whose beginning is so fixed as to extinguish the former train at the right moment. By this arrangement we shall only have to deal with beginnings.

he really considers the effect of only a single waxing and waning of the incident light. It is true that, on the average, the amplitude of the sum of a large number of vectors of random phase is small compared with the sum of the amplitudes. At the same time the energy is, on the average, equal to the sum of the component energies. In the present case the right deduction is, that the energy of the natural vibration will vary with the number of the component vibrations; in other words, will vary as the time elapsed since the light began to act. It will become greater without limit. It is easily seen from SELLMEIER'S analysis that there is no tendency for the natural vibrations excited by successive fluctuation to counteract one another. As regards the forced vibrations, on the other hand, the phases are, so to speak, arranged so that there shall be no accumulation of energy.

§ 69. For a frictionless vibrator, then, common homogeneous light will give a continually-increasing motion in the natural mode; this feature being entirely due to the irregularities.

But if there were a friction, however small, the motion would be prevented from mounting up indefinitely. The vibrations started by the component trains would not persist; in fact we may expect to find that the natural vibration settles down to a definite state, depending on the damping and the nature of the irregularities.

§ 70. The whole of this matter becomes quite simple on the application of Fourier. Let us first try to solve

$$\ddot{x} + p^2x = \int_0^\infty \mathrm{R}\cos(ut + \psi)du.$$

We are tempted to take for solution

$$\int_0^\infty \frac{\mathrm{R}\cos(ut + \psi)}{p^2 - u^2}du.$$

This expression, however, has no definite value. The integrand involves an infinity at $u = p$; furthermore, the infinity is of such a nature that the integral

$$\int_0^{p-\epsilon} + \int_{p+\epsilon'}^\infty$$

depends upon ϵ/ϵ'.

The above integral, in fact, is not a solution of the equation. We are forced to include a frictional term in the equation. But this corresponds to the actual properties of the vibrator; we have shown that, but for damping, the natural vibration would continually increase; a state of things unknown among the observed effects of light.

§ 71. On page 336 it has been shown that light of composition $\int \mathrm{R}^2du$ will excite vibration of composition

$$\int \frac{\mathrm{R}^2du}{(p^2 - u^2)^2 + 4\kappa^2u^2}.$$

For nearly homogeneous light, of period $\frac{2\pi}{q}$, R is small if $\frac{q-u}{q}$ is finite. The integrand is therefore unimportant, except for values of u

i. near to q, where we may neglect κ, and use $\frac{1}{p^2-q^2}\int \mathrm{R}^2 du$,

ii. near to p, where the emission is practically $\frac{1}{4p^2}\int \frac{\mathrm{R}^2 du}{\kappa^2+(p-u)^2}$.

If the light emitted from the vibrator is analysed by a spectroscope, theoretically a spectrum of two lines should be revealed; the lines being at frequencies p and q. Let d be the half-width of the bright line in the incident light; then d will also be the half-width of the q line in the emitted light.

The total intensity of the q line in the emitted light is therefore of order $\mathrm{R}_q^2 d/p^4$.

The half-width of the p line will be κ; the total intensity of the p line is of order $\mathrm{R}_p^2/\kappa p^2$.

The ratio of these intensities is

$$\frac{\mathrm{E}q}{\mathrm{E}p} = \frac{\mathrm{R}_q^2}{\mathrm{R}_p^2}\cdot\frac{\kappa}{p}\cdot\frac{d}{p}.$$

Now $\mathrm{R}_q^2/\mathrm{R}_p^2$ is great, the incident light at p being by hypothesis invisible. On the other hand κ/p and d/p are both small. It therefore appears that, so far as the theory of the vibrator carries us, the natural vibration may be as prominent as the forced vibrator; the natural vibration varying inversely as the index of damping. Whether or no it is strong enough to be visible, depends upon the spectrum of the incident light and the constants of the vibrator.

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XI. *An Experimental Investigation into the Flow of Marble.*

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Communicated by Professor Callendar, *F.R.S.*

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[Plates 22–25.]

Contents.

I. Introduction.

That rocks under the conditions to which they are subjected in many parts of the earth's crust become bent and twisted in the most complicated manner is a fact which was recognised by the earliest geologists, and it needs but a glance at any of the accurate sections of contorted regions of the earth's crust which have been prepared in more recent years to show not only that in many cases even the hardest rocks have been folded, but that there has often been a marked transfer or "flow" of material from one place to another in the folds. While, however, these facts are undisputed, the manner in which this contortion, with its concomitant flowing, has taken place is a matter concerning which there has been much discussion and a wide divergence of opinion. Some authorities—among whom HEIM,[*] whose work in Alpine geology must command the admiration of all, may be mentioned—have held that while, in the upper portions of the earth's crust, rocks, when submitted to pressure, will break, giving rise to faults and overthrusts, the same rocks in the deeper portions of the earth's crust are unable to break up in this way, owing to the great weight of the superincumbent strata. The lines of fracture become smaller and greatly increase in number, the various minerals constituting the rock thus breaking down into grains, which, however, move around and past one another, the adjacent grains always remaining within the sphere of cohesion. The structure becomes cataclastic; the rock mass, acting as plastic bodies do, and flowing in the direction of least resistance, maintains its coherence while altering its shape. HEIM believes that there is a further stage in the process which he thus describes:—

"Wird die umformende Kraft endlich so gross dass sie anstatt an ein, paar tausend Stellen die Festigkeit durch Bruch aufheben zu können, dieselbe in jedem einzelnen Punkte überwindet, so wird das Spaltennetz unendlich fein und das Gesteinskorn zur Kleinheit eines Moleküles reducirt, d. h. die mechanische Bewegungseinheit ist nicht mehr ein Gesteinsbrocken sondern unendlich klein so dass die Bewegung eine continuirliche Umformung ohne Bruch wird."

Now, according to SPRING,[†] the property known as regelation is really due to a power which fragments of bodies have of uniting if brought within the range of the molecular forces, a property which, although possessed in a marked manner by ice, is also, as he has experimentally demonstrated, exhibited by many other bodies, and would probably be displayed by all if the required conditions could be attained. The "flow of rocks" would therefore, according to this view, be a manifestation of regelation on an enormous scale.

Other writers on this subject have maintained that rocks are absolutely destitute of plasticity in any proper sense of the term. Thus MALLET[‡] based his theory on the supposition that in the earth's crust rocks under pressure are shattered. PFAFF[§] has held that in the depths of the earth great pressure alone will tend rather to prevent molecular movement and thus keep the rocks rigid. Those holding such views attribute the deformation of rocks either to crushing with subsequent recementation of the fragments by mineral matter deposited from percolating waters as the movements proceed or after they are completed,[||] or to a continuous process of

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[*] 'Der Mechanismus der Gebirgsbildung,' p. 31; see also VAN HISE, C. R., "Metamorphism of Rocks and Rock Flowage," 'Bull. Geol. Soc. of America,' vol. 9, 1898.

[†] "Recherches sur la propriété que possèdent les corps de se souder sous l'action de la pression." 'Revue Universelle des Mines,' 1880.

[‡] 'Philosophical Transactions,' vol. 163, 1874.

[§] 'Der Mechanismus der Gebirgsbildung,' pp. 19–21.

[||] STAPFF, "Zur Mechanik der Schichtenfaltungen," 'Neues Jahrbuch für Mineralogie,' 1879, p. 792; REYER, 'Theoretische Geologie,' p. 443.

solution and redeposition of the minerals which make up the rock. The percolating waters, it is held, tend to dissolve material at those points where the pressure is greatest, and to redeposit it where the pressure is wholly or partially relieved; the movements thus being accompanied by a more or less complete recrystallisation of the whole rock. Moisture would thus be a necessary factor in all rock folding or contortion, and recrystallisation the essential feature of the phenomenon. The deformation of a body of dry rock would be impossible. The opinion that water is a very important, if not an absolutely essential, factor in the folding of rocks was held by MacCulloch, De la Beche, and a number of the earlier geologists; who based their opinions on the fact that rocks are often much softer while they still contain their quarry water than after they are thoroughly dry, a fact which has been emphasised by tests of the relative strength of wet and dry rocks recently carried out at the arsenal at Watertown, Mass.* It is a matter of great difficulty, and, in fact, in most cases it is quite impossible to decide with certainty upon the relative merits of these conflicting views from a study of the deformed rocks themselves. Had this been possible, the controversy would long since have been brought to a close. Heim, however, in his great work on the 'Mechanism of Mountain Making,'† published some twenty years since, refers to the very valuable results which might be looked for in elucidation of these questions from carefully conducted experiments upon the deformation of rocks under conditions as nearly as possible approximating those which obtain in the deeper parts of our earth's crust. He expresses grave doubts, however, as to the possibility of reproducing the conditions in question.‡

From the time of Sir James Hall§ experimental investigations have been undertaken at intervals, aiming more particularly at the reproduction of the forms exhibited by folded strata. Those by Daubrée,|| Reyer,¶ Cadell,** Favre,†† Obermeyer,‡‡ Forcheimer,§§ and Bailey Willis|||| may be especially mentioned. In these experi-

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* 'Report of the Tests of Metals and other Materials for Industrial Purposes made at Watertown Arsenal, Mass., during 1894,' Washington, Government Printing Office, 1895. Also subsequent Report of same series for 1895.

† 'Untersuchungen über den Mechanismus der Gebirgsbildung,' vol. 2, pp. 4, 84.

‡ "Zum Mechanismus der Gebirgsbildungen," 'Zeit. d. deutsch. Geol. Gesell.,' 1880. See also Baltzer, 'Der Glärnisch,' p. 52.

§ "On the Vertical Position and Convolutions of Certain Strata," 'Trans. Roy. Soc. Edin.,' vol. 7, 1815.

|| 'Etudes Synthétiques de Géologie Expérimentale,' Paris, 1879.

¶ 'Ursachen der Deformationen und der Gebirgsbildung,' Leipzig, 1892.

** "Experimental Researches in Mountain Building," 'Trans. Roy. Soc. Edin.,' vol. 35, 1888.

†† "The Formation of Mountains," 'Nature,' December 5, 1878.

‡‡ "Versuche über das Ausfluss plastischen Thones," 'Sitz. der Wiener Akad. Math.-Natur. Class,' 58, 1868.

§§ "Über Sanddruck und Bewegungs-Erscheinungen im Inneren trockenen Sandes," 'Inaugural Dissertation der Eberhard-Carls-Universität in Tübingen.' Aachen, 1883.

|||| 'Thirteenth Annual Report U.S. Geological Survey.'

ments comparatively low pressures, and materials, such as paper, wax, clay, &c.—much less resistant than the rocks themselves—were employed, so that while they have thrown much light upon the dynamics of mountain making, they have left the aspects of the subject referred to above, and especially dealt with in the present paper, untouched. In another series of interesting investigations, specimens of the rocks themselves have been submitted to the action of direct pressure or heat, the conditions being otherwise those which obtain at the earth's surface. HODGKINSON,[*] for instance, showed that if thin ribs of stone, 7 feet long and 1 inch thick, properly supported at the extremities, are submitted to transverse strain they undergo a permanent deformation, no matter how small the strain to which they are subjected may be. He does not, however, state what stone he employed. MIALL,[†] in more elaborate experiments of a similar character, measured the amount of permanent deformation produced in thin slabs of gypsum and limestone. He found that a more marked deformation without rupture could be obtained if these rocks were embedded in pitch before being submitted to the action of pressure. He could not, however, succeed in permanently deforming slates or sandstones to any noticeable extent. In the tests made at the Watertown Arsenal,[‡] by employing greater pressure, a slight though permanent "set" was given not only to marble but to sandstone, and the same effect was produced by the simple application of heat without pressure. REYER[§] has stated, as the result of experimental study, that while it is possible to slowly deform gypsum by the aid of low pressures continuously applied, the action is greatly accelerated if the material be kept moist.

A few other investigations, among which those of GÜMBEL and KICK are especially worthy of mention, bear more directly upon the question at issue. These have been designed with the object of reproducing, at least in some of their features, the conditions existing at great depths in the earth's crust, and in this way bringing about such rock deformation as there results. GÜMBEL[||] subjected little cylinders of orthoclase, quartz, Iceland spar and alabaster, enclosed in steel collars, and having an area of 1 centim. in cross section, and a height of between half a centim. and 1 centim., to pressures varying from 22,000 to 25,000 atmospheres in a powerful testing machine. The cylinders of orthoclase and quartz crushed to an incoherent powder. The cylinder of calcite, on the other hand, retained its coherence. It became perfectly opaque, and while still retaining its cleavage, is stated to have had a conchoidal fracture induced in it by the pressure. The cleavage faces showed their usual lustre,

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[*] 'Athenæum,' 1853, p. 1165: "Report of the 23rd Meeting of the British Association for the Advancement of Science."

[†] "Experiments on the Contortion of Mountain Limestone," 'Geological Magazine,' November, 1869; and a subsequent paper in the 'Popular Science Review.'

[‡] *Loc. cit.* See also MELLARD READE'S 'Origin of Mountain Ranges,' pp. 16 and 24.

[§] 'Theoretische Geologie,' p. 444.

[||] "Das Verhalten der Schichtgesteine in gebogenen Lagen," 'Sitzungsber. d. königl. Bayer. Akad. d. Wiss.'; Math. Phy. Classe, 1880, 4, 596–623.

and the surfaces of fracture also had a vitreous lustre. The calcite, however, was observed to have been forced into little depressions and cracks which existed in the collar, but in such cases the calcite occupying the depressions and cracks, as well as the portions of the cylinder adjacent to them, was reduced to a finely pulverulent condition, and did not show the cleavage possessed by other portions of the mass. The alabaster deported itself in a similar manner under pressure. GÜMBEL considered that these experiments proved an entire absence of plasticity on the part of the several minerals in question, except possibly in the case of the alabaster, and concluded from their results, fortified by extended observations in the field, that the folding of the older crystalline rocks had taken place before they had become hardened. He also believed that in cases where a folded rock shows distinctly under the microscope that it has been crushed, its coherence is due to a recementation of the crushed mass by subsequent infiltration of mineral matter. It was shown by ROSENBUSCH,* however, in reviewing GÜMBEL's work, that while in the case of the quartz and orthoclase the minerals had undoubtedly been crushed to powder; in the experiment with the calcite column it was by no means proved that deformation without rupture, or "flow," would not equally well account for the phenomenon observed. Unfortunately, no examination of the microscopical characters and optical properties of the several minerals, before and after they had been submitted to pressure, was made.

Somewhat similar experiments on limestone were carried out by PFAFF.† He enclosed a small column of lithographic limestone from Solenhofen in a steel block, except at the top where a piston of the same metal came down upon it. A very small hole was drilled through the side of the block to the limestone, and this was filled with wax. The marble was then submitted to a pressure amounting to 9970 atmospheres, which was continued for seven weeks. The wax was not displaced, and the limestone suffered no alteration. In another experiment, a specimen of the same limestone having a polished surface, was submitted to a pressure of 21,800 atmospheres, delivered by means of a small ring-shaped steel die. The limestone did not flow into the centre of the ring, and only a slight depression was left on the polished surface of the rock. From these experiments PFAFF drew the conclusion that pressure alone is incapable of inducing any plasticity in limestone.

KICK,‡ in his experiments on deformation, made use of many different materials, the only rock investigated being marble. One of his experiments reproduces very closely the conditions in the first of PFAFF's experiments just described; but the result obtained was entirely different. A stout casting was bored out to receive a piston, the hole being closed at the lower end. In the bottom of the hole a steel die,

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* 'Neues Jahrbuch für Mineralogie,' 1882, 1, 222.

† 'Der Mechanismus der Gebirgsbildung,' pp. 16–19.

‡ "Die Principien der mechanischen Technologie und die Festigkeitslehre," 'Zeit. des Vereines Deutscher Ingenieure,' Bd. 36, p. 919 (1892).

having some device standing out from its surface, was placed face upwards. On this was laid a circular disc of marble. Oil was then poured in to fill up all vacant spaces. The piston was then inserted, and by it pressure was brought to bear upon the marble, which pressure was gradually increased to 13,000 atmospheres. The oil, which could escape only through the very narrow space between the piston and the casting, served to maintain a considerable pressure on all parts of the apparatus and the marble to which it had access, while the raised portions of the die coming in contact with the marble were pressed against it with great force. It was found that a well-marked, although not very perfect, reproduction of the device upon the die was impressed upon the marble. He also placed a small marble sphere in a stout copper box, filling the space between the marble and the sides of the box with alum or sulphur, poured in while molten. A heavy cover was then placed upon the box, and the whole was squeezed down to a fraction of its former height by means of a powerful press. After compression the alum or sulphur was dissolved away, setting free the enclosed marble, which was found to have been considerably flattened in a direction at right angles to the pressure.* In another experiment he enclosed a marble cylinder in an iron tube, and having filled the intervening space with water, bent the whole transversely by the application of a high pressure. When the tube was sawn open, the marble was found to have acted "like a plastic body," without having "altered its original characters."† In connection with these experiments, however, it must be mentioned that the marble, as will be seen later, could not have preserved its original character in all respects, although it retained its coherence, and **Reyer** in referring to the experiment says that the marble was crushed, and "nur mässig zementirt."‡ It is doubtful in the case of the deformed spheres of the first-mentioned experiment in how far the deformation obtained is traceable to plastic flow. Three of these deformed spheres were presented by Professor **Kick** to the University of Zürich, and are preserved in the Geological Museum of the University. Two of them, each about 2 centims. in diameter, certainly show a decided flattening, such as might be produced by plastic flow; but the third, which is considerably larger, and is so flattened that the length of the smallest diameter is about two-thirds that of the greatest, shows in its surface a series of fine cracks crossing obliquely, as if the rock had undergone some sort of complicated shearing, and where cracked across in one place the interior is seen to present a shelly structure, resembling in appearance the successive coats of an onion.

Daubrée§ also obtained some very interesting results bearing on this subject, in

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* "Die Principien der mechanischen Technologie und die Festigkeitslehre," 'Zeit. des Vereines Deutscher Ingenieure,' Bd. 36, p. 919, 1892.

† 'Das Gesetz der proportionalen Widerstande,' p. 76.

‡ 'Theoretische Geologie,' p. 444.

§ "Recherches expérimentales sur le rôle possible des gaz à hautes températures," 'Bull. de la Société Géologique de France,' 3e série, tome 19, p. 340.

carrying out his experiments on the perforation of rocks by means of explosives. Little cylinders of marble, granite, and other rocks, having a diameter of about 2 centims., were cut in two vertically, the two halves tightly bound together again by means of wire, and inserted firmly in a tube connected with a chamber in which small charges of dynamite were exploded. The escaping gas completely perforated the rock along the line of contact of the two half cylinders. Under the influence of the explosive force the marble cylinders apparently became somewhat plastic, the wire binding them often leaving slight depressions on the surface when removed. The cylinder was also found to be somewhat shorter and stouter, the marble had lost its translucency and had become opaque. Under the microscope, he says, it was seen to have been crushed to powder, and then squeezed together again into a solid mass. The cylinders also had a concentric structure induced in them around the central perforation.

The several experimental investigations hitherto made on the flow of rocks (and it is believed that the summary given above outlines all the experimental work in which the rocks themselves were employed, the results of which have appeared up to the present time), while very interesting and instructive, are inconclusive and in certain cases apparently mutually contradictory. Neither has account been taken in any of these investigations of the rapidity with which the pressure was applied, of the temperature of the rock during compression, nor, except in a very few cases, of the duration of the pressure. Nor have we in any case an accurate description of the character of the rock before and after the experiment, or of the strength of the deformed rock, so that the actual nature of the effect produced by the pressure can be determined.

II. Conditions to be reproduced in Experimental Work.

It is generally agreed that three chief factors contribute to bringing about the conditions to which rocks are subjected in the deeper parts of the earth's crust, where folding with concomitant flowing is most marked. These are :—

1. Great pressure.
2. High temperature.
3. Percolating waters.

With regard to the first factor it must be noted, that mere cubic compression will not produce movements of the nature of flowing, although it may produce molecular rearrangement in the rock. A differential pressure is necessary to give movement to the mass. Heim* has stated the conditions of movement, so far as pressure affects them, as follows :—

"Plastische Umformung geschieht also nur, wenn allseitig ein Druck wirkt, der

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* 'Untersuchung über den Mechanismus der Gebirgsbildung,' Band II., s. 91.

jedenfalls grösser als die Festigkeit, aber auf verschiedenen Seiten nicht gleich gross ist, so dass Ausweichen seitlich zum Maximaldruck stattfinden kann. Ist das vorhandene Druckminimum kleiner als die Festigkeit, so tritt Zerbrechen und damit Ausquetschen, 'Umformung mit Bruch,' ein." In rock movement resulting from these several factors the additional factor of time may play an important part. Whether all these factors, or only certain of them, are actually necessary for the production of rock deformation is unknown, but can probably be determined by experiment. For by experiment the action of each may be studied separately, as well as in combination with the others.

In experimental work, therefore, the first condition to be reproduced is that of a differential pressure which, even in the direction of its minimum value, exceeds the elastic limit of the rock under investigation. The action of this pressure should then be studied when combined with heat, and then with heat in the presence of moisture. Finally, the effect of time or rapidity of motion should be investigated.

III. Deformation of Carrara Marble.

A. *Methods employed.*

In the present paper* a first contribution to such a study is presented, pure Carrara marble being the rock selected. At the outset the endeavour was made to submit this rock to the first of the three conditions above mentioned only—that is, to bring to bear upon it great pressure from all sides, a pressure, however, which should not be equally great in every direction, but which, while always exceeding the elastic limit of the rock, should be greater in one direction than in others, thus tending not merely to bring about cubic compression but to determine a flow of the material in one direction. For this purpose it was sought to enclose the marble in some material having a much higher elastic limit than the rock itself, and possessing at the same time a very considerable ductility, so that it would move without rupture when the pressure became sufficiently high. Under such conditions it was believed the marble could not break in the ordinary way, even when submitted to a pressure far above that which under ordinary conditions would be required to crush it, for it would be enclosed on all sides by a stronger substance, and the pressure being increased it would remain intact until the elastic limit of the enclosing material had been exceeded, when it would commence to move, acting as water or any other enclosed fluid might.

As it was proposed to extend the investigation eventually to granites, and possibly other rocks, a long series of experiments was made on various alloys in the endeavour to obtain a material which possessed a sufficiently high elastic limit combined with the necessary ductility to fulfil the requirements as enclosing material in all cases;

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* A preliminary notice of these experiments was read before Section C of the British Association for the Advancement of Science, at the Toronto Meeting in 1897, an abstract of which appears in the 'Proceedings' of the Association for that year.

but it was found that none possessed a sufficiently high elastic limit combined with the required ductility, except certain aluminium bronzes, which however it was difficult to obtain with constant composition and properties. Heavy tubes of wrought iron were then made on the plan adopted in the construction of ordnance by rolling a thin strip of Low Moor iron around a bar of soft iron and welding the strip to the bar as it was rolled around it. The core of soft iron composing the bar was then bored out, leaving a tube of welded Low Moor iron, the sides being about a quarter of an inch thick, and so constructed that the fibres of the iron ran around the tube instead of being parallel to its length. These were found to answer the requirements admirably. The following procedure was then adopted: Columns of the marble, 0·81 inch, or in some cases 1 inch, in diameter and 1·53 inch in length, were accurately turned and polished, by Messrs. VOIGT and HOCHGESANG, of Göttingen. The tube was then very accurately fitted around the marble. This was accomplished by giving a very slight taper to both the column and the interior of the tube, and so arranging it that the marble would only pass about half way into the tube when cold. The tube was then expanded by heating, so as to allow the marble to pass completely into it and leave about 1·25 inch of the tube free at either end. On allowing the tube to cool a perfect contact between the iron and marble was obtained, and it was no longer possible to withdraw the latter. This perfect fit was considered indispensable in order to prevent the limestone crumbling when pressure was applied, as it would have done had it not been supported at every point. In some experiments the tube was subsequently turned down, so as to be somewhat thinner immediately around the marble. Into either end of the tube containing the column an accurately fitting steel plug was then inserted, and by means of these the pressure was applied. The high pressure required was obtained from the city water mains by using a double hydraulic "intensifier"; the whole arrangement being shown in the accompanying photograph (Plate 22, fig. 1).

A cylinder containing a moveable piston, whose upper portion is cast of square shape so as to form a press plate, has another press plate mounted opposite to it by means of four strong steel columns. The small cylinder containing the marble with the two steel plugs is set up between the two press plates, the plugs being kept in axial alignment with each other by having their enlarged ends fitted into cylindrical holes bored in a small but massive casting (A), which acts as a guide to them when under pressure. The sliding piston in the large cylinder is 20 inches in diameter, and is kept tight by cup leather packing. The strong copper vessel (B) has its upper half filled with a heavy oil, and thence is led the only pressure connection to the cylinder (C), to which oil, but no water, is admitted, in order that corrosion and undue leakage may be averted. For moderate pressures the city mains are connected directly to the lower half of the copper vessel, but for high pressures to the larger end of the small intensifier (D), and a pipe then leads from the upper end of the same to the lower end of the copper vessel. In either case the pressure is kept steady for weeks at a time when necessary by means of a small spring relief valve (a) with an adjusting screw, so that the water from the mains is allowed to overflow at any desired pressure, which thus cannot be exceeded. A recording gauge (b) attached to cylinder (C) registers the history of the experiment throughout its course. The allowance to be made for the friction of the 20-inch diameter cup leather was carefully determined, so that a close estimate of the pressure to which the rock is subjected can be formed. This was done by observing the amounts of compression of a specimen of hard steel due to various loads applied by a Buckton testing machine, and then inferring the loads to which it was afterwards subjected in the actual press from the compressions to which these gave rise. The compressions were measured by means of a Martens' mirror extensometer reading to the 1/100,000th of an inch; and any possible difference in the Young's modulus of the steel in two successive loadings was got rid of, after the manner of BAUSCHINGER, by first alternately stretching and compressing it, and so reducing it to a "state of ease." In this way the cup leather friction was found to be approximately constant in quantity (*i.e.*, independent of the pressure) within the limits of pressure employed, and to amount to about 400 lbs. Thus, if p be the gauge-pressure in the 20-inch cylinder, and P the pressure per square inch on the rock, of area a, we shall have :—

$$\mathrm{P} = \frac{314p - 400}{a} \quad (a \text{ being in sq. inches}).$$

We may tabulate the values of P for the three sizes of rock cylinder employed, viz., 1 inch, 0·8 inch, and 0·4 inch diameter, corresponding to various values of p, from 50 lbs. to 300 lbs. per square inch in the cylinder (C). (The latter was the greatest pressure allowable in the 20-inch cast-iron cylinder as designed.)

p.	Rock 1 inch dia.		Rock ·8 inch dia.		Rock ·4 inch dia.	
	P., lbs./sq. in.	atm.	lbs./sq. in.	atm.	lbs./sq. in.	atm.
50	19,500	1330	31,300	2130	125,300	8530
100	39,500	2680	63,400	4320	253,500	17,200
200	79,500	5400	127,600	8700	510,300	34,700
300	119,500	8150	191,800	13,050	767,100	52,100

It having been ascertained that columns of the Carrara marble, 1 inch in diameter and 1·585 inch high, crushed at a pressure of from 11,430 to 12,026 lbs. to the square inch when free from any lateral support, the column enclosed in its wrought-iron tube in the manner above described was placed in the machine and pressure applied gradually, the exterior diameter of the tube being accurately measured at frequent intervals. No effect was noticeable until a pressure upon the marble, varying of course with the thickness of the enclosing tube, but ordinarily about 18,000 lbs. to the square inch, was reached, when the tube was found to slowly bulge. This bulge was symmetrical and confined to that portion of the tube surrounding the marble. This distension was allowed to increase until the tube showed signs of rupture, when the pressure was removed and the experiment concluded.

B. *Deformation of the Dry Rock at Ordinary Temperatures.*

Eight experiments were made in this manner on the dry rock at ordinary temperatures, the rate at which the pressure was applied differing in different cases, the consequent deformation in some cases being very slow and in others taking place more rapidly, the time occupied by the experiment being from 10 minutes to 64 days. The pressure was regularly increased so soon as the movement ceased, and in this way the rate of motion was kept as nearly constant as possible. The final amount of deformation was not in all cases equal, as some of the tubes showed signs of rupture sooner than others, thus requiring the experiment to be brought to a close.

Plate 23, fig. 1, shows one of the tubes enclosing a marble column before the pressure has been applied, and beside it the same column after the completion of the experiment. The deformation in this case was carried out very slowly, the time occupied by the experiment being 64 days.

After the bulging of the tube had been carried as far as possible, consistent with safety, the tube was removed from the press, the plugs taken out, and the tube was slit through longitudinally by means of a narrow cutter in a milling machine along two lines opposite one another. The marble within, however, was found to be still firm and compact and to hold the respective sides of the tube, now completely severed from one another, so firmly together that it was impossible without mechanical aids to tear them apart. By means of a steel wedge, driven in between them, however, they could be separated, but only at the cost of splitting the marble through longitudinally. Columns so split, with the portions of the tube adhering to them, are shown in Plate 23, figs. 3 and 4, the marble column in the former case having been reduced to one-half of its original height in 4 hours, while in the latter case the deformation occupied 17 days. The marble was in one or two instances detached from the tube without breaking it further, by striking the latter a smart blow on the back with a hammer, but usually it adhered so firmly that it could be released from the tube only by spreading the latter in a vice. The exterior surface of the marble where it had been in contact with the tube was smooth and conformed to the curve of the bulging iron, its surface reproducing perfectly all the fine tool marks on the latter.

Fig. 2 of Plate 23 shows the deformed marble, freed in this way from the tube shown in fig. 1 of the same plate, and beside it a marble column of the dimensions which it originally possessed, for purposes of comparison.

The deformed marble is uniform and compact, and seems to break with equal ease in all directions. It differs somewhat in appearance from the original rock in possessing a dead white colour, somewhat like chalk, the glistening cleavage surfaces of the calcite being no longer visible, and the difference being well brought out in certain cases owing to the fact that a certain portion of the original marble often remains unaltered and unaffected by the pressure. This, when present, has the form of two cones of obtuse angle, whose bases are the original ends of the column resting against the faces of the steel plugs, while the apices extend into the mass of the deformed marble and point toward one another. These cones, or rather parabolas of rotation, are developed, as is well known, in all cases where cubes of rock, Portland cement, or cast iron are crushed in a testing machine in the ordinary manner. In the present experiments they seldom constituted any large proportion of the whole mass, and in some cases are absent or but faintly indicated, but there is always in immediate contact with the ends of the steel plugs a thin cake at least of marble possessing the characters of the original rock.

In order to ascertain the strength of the deformed rock as compared with the original marble, and also whether, in the case of the former, the rate of deformation influenced the strength, three of the half columns, obtained by splitting the deformed columns as above described and freeing them from the collar, were selected and tested in compression by means of an Emery testing machine. The results are presented in the following table, the measurements being given in inches :—

	Original height.	Original diameter.	Greatest diameter after deformation.	Time of deformation.	Crushing load for deformed marble, lbs. per square inch.
Experiment A ...	1·594	1·000	1·407	64 days	5350
„ O ...	1·594	1·000	1·203	1½ hours	4000
„ P ...	1·505	1·000	1·388	10 minutes	2776

As already mentioned, columns of the marble of the original dimensions, namely, 1·5 inch high and 1 inch in diameter, were found to have a crushing strength of between 11,430 and 12,026 lbs. per square inch. These figures show that, making all due allowance for the difference in shape of the specimens tested, the marble, after deformation, while in some cases still possessing considerable strength, is much weaker than the original rock. They also tend to show that when the deformation is carried on slowly the resulting rock is stronger than when the deformation is rapid. The specimens of the deformed rock when tested, in all cases crushed in exactly the same manner as the columns of the original marble, namely, with the development of two cones whose bases are the end faces of the columns, and whose apices point toward one another, with the appearance, when the limit of strength is reached, of a series of inward curving cracks running from top to bottom of the specimen along which strips of the rock split away from the cones in question. These cones while in the deformed rock possibly influenced in their position to some extent by the cones produced in deforming the rock, do not result from them, since, as above mentioned, they are always observed in the case of the original marble as well.

Thin sections of the deformed column passing vertically through the unaltered cone and the deformed portion of the rock were readily made, and when examined under the microscope clearly showed the nature of the movement which had taken place. The deformed portion of the rock can be distinguished at once by its turbid appearance, differing in a marked manner from the clear transparent mosaic of the unaltered cone. In those cases where the deformation has been rapid, as in Experiment P of the above list, an anastomosing and complicated meshwork of curved and branching lines, which are especially turbid in appearance, are seen running through the rock. These, when magnified 500 diameters, are resolved into strings or bands of very small calcite granules. They mark lines along which shearing has taken place. The calcite individuals along these lines have broken down, and the fragments so produced have moved over and past one another and remained as a compact mass after the movement ceased. In these lines of granulated material are enclosed great numbers of irregular fragments and shreds of calcite crystals, bent and twisted, which have been carried along in the moving granulated mass as the shearing progressed. The structure is therefore cataclastic, and is identical with that seen in the felspars and many gneisses. A microphotograph of a thin section of the deformed marble showing this structure is seen in Plate 25, fig. 1. It is taken in ordinary light, and magnified 70 diameters. The original column in this case had a diameter of 1·067 inch, which was increased by the pressure to 1·356 inch along the line of greatest bulging. The deformation was carried out in 7 hours. The dark areas are the granulated portions of the rock in which the fragments of calcite individuals, often distinctly twinned, are seen to be embedded.

Between these lines of granulated material the marble shows movements of another sort. Most of the calcite individuals in these portions can be seen to have been squeezed against one another, and in many cases a distinct flattening of the grains has resulted with marked strain shadows, indicating that they have been bent or twisted. They show, moreover, a finely fibrous structure in most cases, which, when highly magnified, is seen to be due to an extremely minute polysynthetic twinning. The chalky aspect of the deformed rock is chiefly due to the destruction by this repeated twinning of the continuity of the cleveage surfaces of the calcite individuals, thus making the reflecting surfaces much smaller.

By this twinning the calcite individuals are enabled under the pressure to alter their shape somewhat, while the flattening of the grains is evidently due to movements along the gliding planes of the crystals. This, however, will be referred to again.

In these parts, therefore, the rock presents a continuous mosaic of somewhat flattened grains. A microphotograph of a portion of the rock showing this structure is seen in Plate 25, fig. 2. It was taken in ordinary light, and magnified 50 diameters. Every stage can be traced, however, from the mosaic of twinned and somewhat flattened grains to the areas of perfectly granulated material. Minute lines of granulated calcite first appear along directions of intense twisting in the mosaic, then these become more numerous, and finally the complete breaking down of the mosaic into finely granulated material, filled with twisted remnants of the calcite grains, can be seen. The question of time does not seem to play any important part in the character of the deformation. The structure of the marble deformed in 64 days is essentially the same in character as that which was deformed to the same extent in 10 minutes. In both cases the lines of cataclastic structure and the intervening areas composed of flattened grains are found. It seems probable, however, from a study of the thin sections, that very rapid deformation tends to render the former structure more pronounced and more abundant, and as the granulated calcite is apparently the weakest portion of the mass, in this way to make the rock which is rapidly deformed weaker, as it is shown to be by the tests. The fact that the twinning and other structures above described are not developed in the cones proves that they are not produced by statical pressure or cubic compression, but that they are developed only when actual movement takes place in the mass.

In one experiment, of which a photograph is given in Plate 23, fig. 6, under the pressure of the two pistons, the marble was deformed as above described, causing the enclosing tube to bulge in a marked manner, and the pressure being continued, the enclosed marble tore the wrought-iron tube apart, developing a ragged rent across the fibres of the iron in a vertical direction, and commenced to fall out of the rent in the form of a fine white powder. On removing the pressure and milling open the tube, the remaining marble was found to be still firm and compact, except in the vicinity of the rent, where it was pulverulent.

C. *Deformation of the Dry Rock at* 300° *C. and at* 400° *C.*

It was next sought to determine experimentally in what respect the second factor, namely, heat, would influence the result. A column of the same Carrara marble and of the same dimensions as those used in the former experiments was enclosed in a wrought-iron tube of the same construction as before. This, which is marked (A) in the accompanying figure (fig. 1), is surrounded by a cast-iron jacket (B), which is bored to receive it. The casting is so arranged that hot gases circulate in an annular channel (D) within it and outside of the wrought-iron cylinder (A), so that the marble is kept at a high temperature while the pressure is applied. The casting is as massive as possible so as to equalise the temperature of the interior and enable that of the enclosed rock to be inferred by a Callendar's platinum resistance thermometer (C), which is inserted at the side of the shell in the air space (E). The hot gases are excluded from this space by the wall (F); and the heat flows into the cylinder and rock rather by the ends that across the badly-conducting air space (E). The whole is well lagged with asbestos. The heat is supplied by means of a Bunsen flame. The temperature, which was observed thrice daily, was maintained as nearly as possible at 300° C., and was within a few degrees of this most of the time, the extreme limits of variation being 270° C. and 352° C. The marble was maintained at this temperature for 124 days, or four months, and was during this time deformed as slowly as possible and at as nearly as possible a constant rate. The dimensions of the column in inches are given in the following table :—

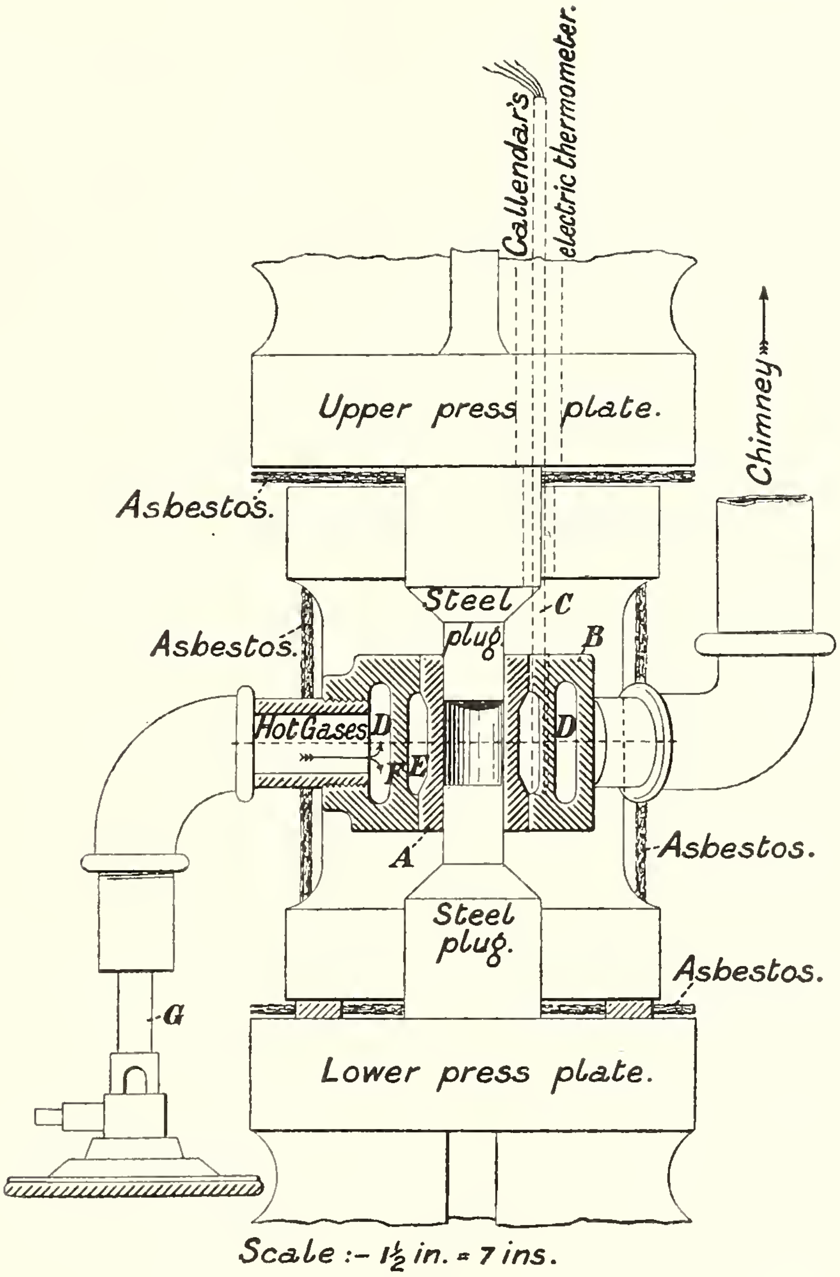

Fig. 1.

Experiment K.

Height before deformation.	Height after deformation.	Diameter before deformation.	Diameter after deformation.	Time of deformation.	Crushing load after deformation in lbs. per sq. inch.
1·53	1·355	1·002	1·110	124 days	10,652

The column was thus shortened by ·175 inch, or 11·4 per cent.

On removing the pistons and slitting the tube open the marble within was found to be so hard and compact that it was necessary to insert a steel wedge between the two halves of tube and drive it in by means of a hammer in order to split the marble so that the adhering portions of the tube might be removed. The rock broke with a clear, even fracture along a vertical plane passing through the centre.

Cones were not visible in this marble, the whole column (although not in so marked a manner as in the former experiments) presenting the dead white appearance characteristic of the deformed marble, although the ends of the column in contact with the plugs were a trifle less chalky in aspect than the rest of the rock. This difference was, however, by no means well marked, and little glistening cleavage faces could be seen throughout the whole mass of rock. One of the half-columns obtained by splitting the deformed marble was freed from the tube which still adhered firmly to it, in the usual manner. It separated as a single solid mass, which was quite smooth on the surface but stained with spots of a deep-brown colour where it had been in contact with the hot iron. The polish had, however, disappeared owing to the movements which had taken place over the surface, except on the ends and along a narrow zone at either end of the column where the lustre was still retained. The half-column was, of course, distinctly bulged. A photograph of bulged column, together with one of original size, is seen in Plate 23, fig. 5.

In order to determine the strength of this limestone after deformation, the half-column was then placed in an Emery testing machine and tested in compression. The pressure was gradually increased without developing any signs of distress until a load of 4200 lbs. had been reached when it suddenly crushed to fragments. Rude half-cones appeared to have sheared in at either end, which, however, were not coincident with the traces of cones of the original marble, and strips split off the sides longitudinally, precisely as in the case of the columns of the original marble when tested in a similar manner. Columns of the original marble, in all respects identical with those employed in the experiment, as has already been mentioned, crushed at a pressure of between 11,430 and 12,025 lbs. per square inch. The crushing load of the marble of the deformed half-column is equivalent to 10,652 lbs. per square inch. Although therefore the two cannot be compared with absolute exactness, owing to their difference in shape, and to the fact that but a single test of the deformed marble was made, it is clear that the deformed marble, if not quite as strong, is at least very nearly as strong as the original rock. Twenty thin sections were cut from a portion of the other half-column and examined under the microscope. The sections show that the deformed rock possesses a more or less distinct foliation except at the ends of the column, where practically no motion had taken place. Here scarcely any trace of foliation is visible. Cataclastic structure is absent, but almost every grain shows an exceedingly fine fibrous structure. When examined under a high power this fibrous structure resolves itself into an extremely narrow polysynthetic twinning —the whole grain consisting of slightly sinuous twin lamellæ, extinguishing in alternate sets. Each individual is usually twinned throughout, the lamellæ passing from end to end, although a single lamella often varies somewhat in width from place to place. The calcite grains which in the original rock are practically equidimensional, are now often distinctly flattened (fig. 2), some of them being three or even four times as long as they are wide. Some grains can be seen to have been bent around others adjacent to them, the twin lamellæ and the extinction curving with the twisted grain (fig. 3). In other twisted individuals the twin lamellæ only extend in to a certain distance from the margin of the grain, leaving a clear untwinned portion in the centre (fig. 4); and other crystals again show not only the fibrous structure due to twinning in one direction, but broader lamellæ crossing this obliquely. As the twinning in all cases is probably parallel to $-\frac{1}{2}R-$ this is due to the appearance of a set of twin lines parallel to a second face of the rhombohedron (fig. 5).

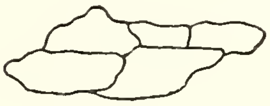

Fig. 2.

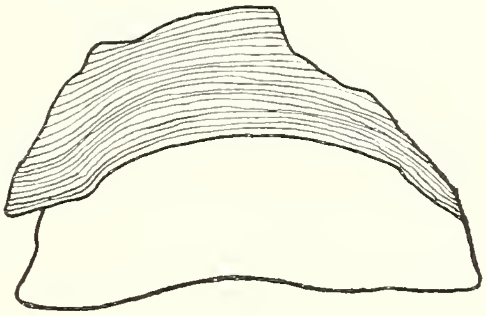

Fig. 3.

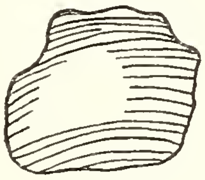

Fig. 4.

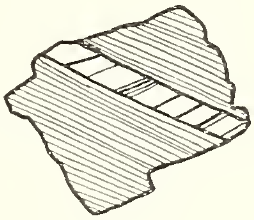

Fig. 5.

Cleavage is not developed by the crushing, for it is not seen in the majority of grains, even where they are most deformed. The proportion of grains which show it is no greater in the deformed than in the original limestone. What cleavage is visible was probably developed during the grinding of the section, as it is seen in places in sections of all marbles. There has been no breaking—the rock has not been crushed in the ordinary sense of the term. The movement has been brought about partly by twinning but chiefly by a deformation of the grains due to a slipping on their gliding planes. The structure is essentially that presented by those portions of the marble lying between the lines of granulated calcite in the case of the marble deformed at the ordinary temperature. In the accompanying plates, microphotographs of the marble before and after deformation at 300° C. are shown.

Plate 24, fig. 1, shows the appearance of a thin section of the original Carrara marble in ordinary light, magnified 50 diameters. The individuals are approximately equidimensional, and only three or four show twinning.

Plate 24, fig. 2, is the marble after having been slowly deformed at a temperature of 300° C., photographed between crossed nicols in polarised light and magnified 50 diameters. The individual grains can be seen to be distinctly flattened in a horizontal direction, giving a certain foliation to the rock. The fibrous appearance above referred to, as due to polysynthetic twinning, is also seen.

Plate 24, fig. 4, is a microphotograph of a few grains of calcite, the thinnest edge of another section of the same, taken between crossed nicols in polarised light and magnified 150 diameters. The polysynthetic twinning is well seen. Two sets of lamellæ cross and two of the bands represented in fig. 5 of the text are seen on the left. The lamellæ curve somewhat and vary more or less in width from place to place.

In the case of ice crystals a rise in temperature develops a greater ease of movement along their gliding planes, and this experiment seemed to show that the same is true of calcite. The individual grains are more plastic and accommodate themselves to the deforming forces by flowing around each other more readily rather than by breaking. The rock is therefore much stronger than when deformed at the ordinary temperature, the lines of cataclastic structure being apparently lines of weakness. As, however, the deformation in this experiment was carried on with extreme slowness, it was impossible to determine in how far this latter factor had influenced the result. Another trial was accordingly made in which the deformation was carried out quickly and at the same time at a much higher temperature. The amount of deformation induced in the marble was nearly the same as in the last case. The height of the column before compression was 1·552 inch, and after compression 1·352 inch; that is to say, the column was shortened by 12·9 per cent. The time occupied in the deformation, however, was only $8\frac{1}{4}$ hours, and the rock was maintained at a temperature of about 400° C.; the extremes of variation of the temperature during the experiment being 380° C. to 415° C. The temperature measurements were made by means of a special modification of the Le Chatelier pyrometer, calibrated by H. M. Tory, M.A., of McGill University. On slitting the tube in the usual manner and inserting the wedge to split the marble, the latter was found to offer more resistance than in any of the former experiments, and was finally pulled out of the separated halves of the tube without splitting in two as in all other cases; but unfortunately it was traversed by a few slight cracks developed in the process which rendered it impossible to test its strength. The surface where it had been in contact with the hot collar was brown in colour and maintained its polish, except around the central zone of maximum bulging, where the lustre had disappeared. There had been no disassociation of the calcium carbonate, as fragments of the powdered marble tested with moist turmeric paper gave no trace of an alkaline reaction. When sliced and examined under the microscope, the rock showed no trace of cataclastic structure, but the grains were seen to be distinctly flattened, giving to the rock a foliation which in some places was very pronounced. The calcite individuals showed the very narrow polysynthetic twinning producing the fibrous appearance before described. The twin lamellæ are in some cases twisted, the twisting being accompanied by strain shadows, which phenomenon, however, in this rock is neither very common nor very striking. The individuals are seen in many cases to have been squeezed into very irregular shapes, and in some cases to have been forced into wedge-shaped forms (fig. 6), quite different from those of the regular mosaic of the original rock. The individual grains have to all appearance acted as plastic bodies. A very pronounced movement along gliding planes, coinciding in direction with the course of twin lamellæ, is undoubted.

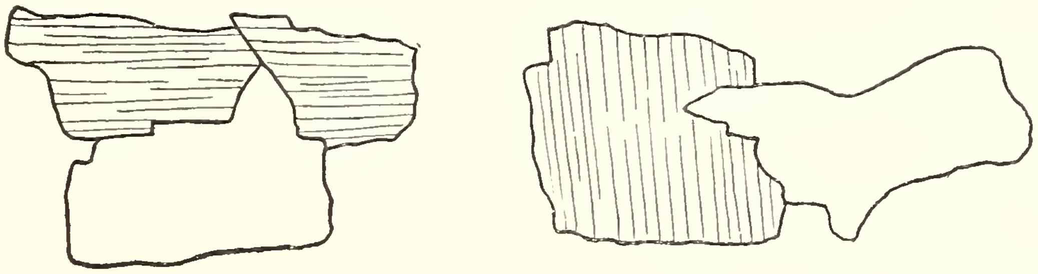

Fig. 6.

Apart from the evidence of this presented by the form of the calcite individuals, direct evidence can be seen in many cases in the step-like outline of the grains, as shown in fig. 6, the steps coinciding in direction with the twin lamellæ. In one instance, shown in fig. 7, a lamella was seen to have moved inward between

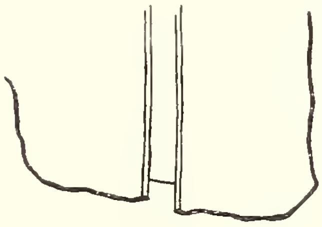

Fig. 7.

two other very narrow lamellæ on either side. This is of especial interest, as it is precisely this movement of individual lamellæ of measurable width over one another that gives rise to the phenomenon of the "flow" of metals as described in Section IV. Calcite, however, is apparently much more prone to twin during this deformation than metals are, although the greater difficulty of recognising twinning in metals—the latter being opaque—may have led to the frequency of this phenomenon in their case being underestimated.

The character of the movement in the case of quick deformation at a high temperature shows therefore that calcite has freer movement in its gliding planes at a high temperature and breaks less readily than when cold. A microphotograph of a section of this deformed marble (400° C.), taken in ordinary light and magnified 70 diameters, is seen in Plate 24, fig. 3. The evenness of movement and freedom from all fracturing or cataclastic action is well seen. The flattening of the grains is also distinctly shown and is especially noticeable if it be compared with the section of the original marble beside it, forming fig. 1 of the same Plate.

D. *Deformation of the Rock at 300° C. in the presence of Water.*

It was next sought to introduce the third factor above mentioned as possibly having an influence on rock deformation, namely, moisture.

For this purpose a modification of the apparatus employed in the experiments just described was used. A drawing of this is given in fig. 8. A hole was bored through the cast-iron jacket (B), as well as through the end of the wrought-iron cylinder (A) which contained the marble, so as to reach the surface of the steel piston at H, just above its contact with the marble column. Through this hole a stout copper pipe (K) was passed and having been screwed into A was brazed. Water was then forced through this tube by means of a hydraulic accumulator, similar in construction to that marked D in fig. 1 of Plate 22, while at the same time the required temperature was obtained by means of a gas flame as before. Even under the great hydraulic pressure employed, the water passed so slowly that the temperature could be easily maintained at 300° C.; the water making its way between the side of the steel piston and the tube (A) to the marble and passing through the latter and out of the lower steel piston by the hole (N) drilled through it. In order to prevent the water entering at H from passing upwards along the piston instead of downward into the marble and thus escaping, a heavy brass cap (P) was screwed on the end of the ring (Q), which in its turn was screwed into the jacket (B). The cap was turned with a projecting ring on its lower surface, while the upper surface of Q was slightly hollowed. In the space (M) thus intervening between the two, a ring of lead was placed, which, on screwing down the cap (P), was forced to occupy the whole space and to make a perfectly water-tight joint around the piston. This arrangement was repeated in the case of the lower steel piston.

The Callendar thermometer was inserted at C as before. In this way a column of the Carrara marble, enclosed in its iron tube, was slowly deformed while at a temperature of 300° C., but in the presence of water vapour under a pressure of 460 lbs. to the square inch. The deformation was carried on very slowly and at as uniform a rate as possible for a period of 54 days, or nearly 2 months. The following are the dimensions of the column stated in inches :—

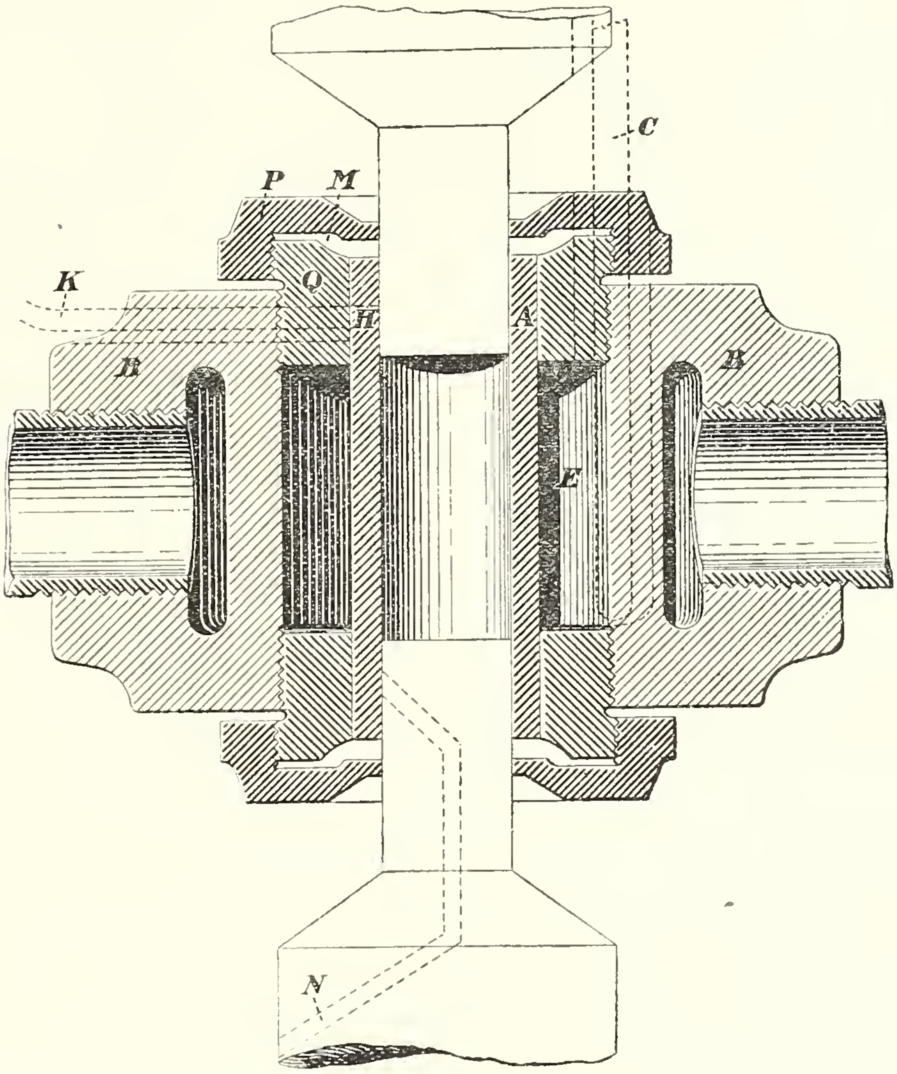

Fig. 8.

Experiment L.

Height before deformation.	Height after deformation.	Diameter before deformation.	Greatest diameter after deformation.	Time of deformation.	Crushing load after deformation.
1·513	1·127	·813	1·0205	54 days	Stronger than the original rock

The column was thus shortened by ·386 of an inch, equivalent to 25·51 per cent. On cutting through the tube the deformed marble was found, as in former experiments, to be so hard that a steel wedge had to be employed to split it. The ends of the column were found to be nearly black in colour from the deposition upon them of a thin ferruginous coating, derived apparently from the inner surface of the iron accumulator by means of which the water had been forced through the rock, as a similar deposit was found lining the tubes conveying the water both to and from the marble. This deposit, which is probably identical with that above referred to as coating the surface of the marble, was found upon examination to be composed of oxide of iron, a few little flakes of copper, carbonate of lime, and some material insoluble in acids, probably derived from the evaporation of the water on entering the heated portion of the apparatus.

In the thin ferruginous coating on the end of the column, and thus immediately beneath the face of the piston, a few minute flecks of metallic copper were also visible, showing that a little copper had been dissolved from the copper pipe carrying the water from the accumulator and redeposited on the surface of the marble. This coating although less pronounced was also visible around the sides of the column where it was in contact with the heated iron tube enclosing it. It penetrates into the marble for a short distance at one or two spots at the top and bottom of the column, but is not seen elsewhere in the inner part of the marble.

On splitting open the marble column, cones could be seen within it at either end, but they were not very sharply defined. The deformed portion of the marble, that is to say, the portion of the column not included in the cones, presented the same dead white or chalk-like appearance noted in former experiments. One of the halves of the deformed limestone column after being freed from the iron tube was tested in compression in an Emery testing machine. The pressure was raised gradually without developing any signs of distress in the marble until a load of 3090 lbs. had been reached, when a minute crack developed. The pressure was then gradually increased to 3240 lbs., when the column suddenly crushed. In breaking down it split from top to bottom, like a perfectly homogeneous body, and without reference to the above-mentioned cones. Two columns of the same marble employed in the experiment and of the same dimensions as the original column, when similarly tested, broke suddenly by shearing, under loads of 4870 lbs. and 5760 lbs. respectively. These figures, however, cannot be used for the purpose of comparing the strength of the limestone before and after compression, as in the experiment at present under consideration, the bulge given to the marble was so considerable that with material of equal strength the new form would certainly be considerably stronger than a half-column of the original dimensions. In order to make a direct comparison, however, a fragment of the original marble was cut into the form of a bulged half-column of the same dimensions as that produced in the present case by compression. This when tested in compression suddenly sheared to pieces when the pressure rose to 3050 lbs.

While, therefore, the averages of a number of trials would be required to establish the exact relative strength of the original marble and the marble after deformation, the results of the test just described show that the marble after deformation is not weaker, but actually somewhat stronger than the original rock.

A large number of thin sections of the deformed rock (some radial and some transverse) were prepared and examined. The rock shows the continuous mosaic before referred to with the exception of a little turbid line or band in each section, starting from the periphery of the top of the cylinder, and curving down toward the middle, following approximately the curve of the surface of one of the cones. Under the microscope this is seen to owe its appearance to the presence of a number of fine and very narrow reticulating lines, which appear to be lines of motion along which there has been a very minute granulation of the marble. Between these, and elsewhere throughout the column, there are no signs of granulation or cataclastic structure. This granulated material is so trivial in amount, that the deformation may be said to be due exclusively to movements on the gliding planes of the calcite, accompanied by polysynthetic twinning. It is thus identical in character with that seen in the case of the marble when deformed while dry, either at 300° C. or 400° C. The calcite individuals in the original rock are approximately equidimensional (none are more than twice as long in one direction as in the other), but in the deformed rock a very distinct foliation is often seen in the thin sections, owing to the flattening of the calcite grains, many individuals being three or even four times as long as they are wide. Some few of these flattened grains show strain shadows but no twinning, while the grains in their immediate vicinity show well-defined twinning, giving rise to the fibrous appearance before described. In some cases a grain will show strain shadows at one end, which will pass into a very narrow polysynthetic twinning at the other. The twin lamellæ in many grains are so narrow that even when magnified 1050 diameters, they are not very clearly resolved. The individual lamellæ in several sets which were measured, were found to have an average width of between ·0005 and ·0006 of a millim., and some were even narrower.

Where the iron stain has penetrated into the substance of the rock, it appears under the microscope as little lines of ferruginous material between the calcite grains, which latter are twinned and flattened in every way like those above described. There are no signs of solution and redeposition of calcium carbonate even in this iron-stained portion of the rock.

The presence of water, therefore, did not influence the character of the deformation. It is just possible, however, that there may have been a deposition of infinitesimal amounts of calcium carbonate along very minute cracks or fissures, thus contributing to maintain the strength of the rock. No signs of such deposition, however, are visible.

IV. Comparison of the Structures produced in Carrara Marble by Artificial Deformation with those produced by Deformation in the case of Metals.

Mügge,* whose researches in the movements set up by pressure in ice, and in various minerals and artificially prepared salts, are so extensive and so well known, in a paper read on January 14, 1899, presents the results of his investigations into the effect of pressure on metals and the nature of the movements resulting from it; and, in two papers read on March 16 and May 18 respectively of the same year, Ewing and Rosenhain† describe a series of investigations carried out by them on the same subject, and which cover practically the same ground and yield the same results. It is pointed out that all simple metals when examined under the microscope, are seen to be allotriomorphic aggregates of metallic crystals, the structure being precisely that of a block of marble.

When the metal is deformed by compression or tension, the effect being identical in both cases, the movement is found to be due to the distortion of each grain by slipping along gliding planes, with or without the accompaniment of twinning. This was observed in gold, silver, platinum, tin, copper, lead, cadmium, bismuth, antimony, nickel, iron, steel, and various alloys. It is in fact in this way that metals move or "flow" when submitted to pressure or impact.

Polysynthetic twinning was found to accompany the movement on gliding planes, in the case of most of the metals enumerated above, both phenomena often presenting themselves in the same grain.

Mügge shows that in the case of soft iron, gliding can take place along six planes, and that twinning is probably also developed by pressure. Ewing and Rosenhain, in their first paper, give three photographs of the same surface of soft iron showing the results of progressive deformation of the constituent crystalline grains under pressure, which photographs could not be distinguished from those of thin sections of the marble described in the present paper at corresponding stages of deformation. In the case of a specimen of Swedish iron, strained by a pull, the width of the lamellæ between the lines of slip was found to average 1/400 of a millim.

Messrs. Ewing and Rosenhain sum up the results of their experiments in the following words:—

"These experiments throw what appears to us new light on the character ot plastic strain in metals and other irregular crystalline aggregates. Plasticity is due to slip on the part of the crystals along cleavage or gliding surfaces. Each crystalline grain is deformed by numerous internal slips occurring at intervals

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* "Ueber neue Structurflächen an den Krystallen der gediegenen Metalle," 'Nachricht. der k. Gesell. der Wissen. zu Göttingen'; Math.-phys. Klasse. 1899. Heft 1.

† "Experiments in Micro-metallurgy: Effects of Strain (Preliminary Notice)," 'Roy. Soc. Proc., vol. 65; "The Crystalline Structure of Metals," Bakerian Lecture, 'Roy. Soc. Proc.,' vol. 65.

throughout its mass. In general these slips no doubt occur in three planes or possibly more, and the combination of the three allows the grain to accommodate itself to its envelope of neighbouring grains as the strain proceeds. The action is discontinuous; it is not a homogeneous shear but a series of finite slips, the portion of the crystal between one slip and the next behaving like a rigid solid. The process of slipping is one which takes time, and in this respect the aggregate effect is not easily distinguishable from the deformation of a viscous liquid.

"We infer from the experiments that 'flow' or non-elastic deformation in metals occurs through slip within each crystalline grain of portions of the crystal on one another along surfaces of cleavage or gliding surfaces. There is no need to suppose the portions which slip to be other than perfectly elastic. The slip when it occurs involves the expenditure of work in an irreversible manner. It is because the metal is an aggregate of irregular crystals that it is plastic as a whole, and is able to be deformed in any manner as a result of the slips occurring in individual crystals. Plasticity requires that each portion should be able to change its shape and its position. Each crystalline grain changes its shape through slips occurring within itself, and its position through slips occurring in other grains."

By a comparison of these results in the deformation of metals with those presented in the present paper in the deformation of marble, it will be seen that the agreement between the two is so close that the term "flow" is just as correctly applicable to the movement of heated marble in compression, under the conditions described, as it is of the movement which takes place in gold when a button of that metal is squeezed flat in a vice, or in iron when a billet is passed between a pair of rolls.

V. COMPARISON OF THE STRUCTURES PRODUCED IN CARRARA MARBLE BY ARTIFICIAL DEFORMATION WITH THOSE OBSERVED IN THE LIMESTONES AND MARBLES OF HIGHLY CONTORTED PORTIONS OF THE EARTH'S CRUST.

While the microscopic structure of the silicated rocks of the earth's crust has been made the subject of most exhaustive researches during the past half-century, comparatively little attention has been paid to the minute structure of limestones and marbles. The papers which have appeared on this subject deal chiefly with unaltered limestones, and there is but little mention made of structures resulting from pressure. PFAFF,* in his somewhat extended study of limestones and dolomites from many widely separated localities, in which he examined some 700 thin sections of these rocks, representing, however, chiefly unaltered strata of Mesozoic or Neozoic age, does not mention a single instance in which cataclastic structure was observed, and states that in only two instances was a flattening of the calcite grains seen, producing a species of foliation in the rock. Both of these were from the Alps, one of them a

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* "Einiges über Kalksteine und Dolomite," 'Sitzber. der Math.-phys. Classe der K. BAYER. Akad. der Wissen. zu München,' 1882. Heft 4, p. 566.

specimen of the well-known Lochseitenkalk from some locality not specified, and the other from the Fläscherberg, near Ragatz.

VOGT,* in his recent studies on marble, mentions cataclastic structure only in the marbles from a few localities along the contact zone in Velfjorden, where it was produced by dynamic action on the already altered limestones of the contact zone. He states that this structure often makes the marbles of this district so brittle that they are unfit for use, but mentions no case in which any foliation in the marble is produced by the flattening of the calcite individuals.

Cataclastic structure has been noted in a few instances in marbles from other parts of Europe, but it would seem to be very uncommon. The development of a foliation through the mechanical flattening of the calcite grains by dynamic action is, with the exception of the cases mentioned by PFAFF, so far as we are aware, unrecorded. HEIM refers to this structure in certain Swiss limestones, but regards the grains as broken fragments.

Twinning in the calcite of marbles is common, and has frequently been described. PFAFF states that it is rare except in the primitive limestones ("Urkalken"), where it is always present. ZIRKEL† states that, as a rule, the greater proportion of the calcite individuals in marbles are untwinned, but that when present twinning is for the most part undoubtedly due to pressure and has a "Gleitflächencharakter."

As, therefore, but very few dynamically altered limestones or marbles have been made the subject of a microscopical study sufficiently detailed to enable a comparison to be instituted between their structures and those seen in the artificially deformed limestones described in the present paper, a series of 42 limestones and marbles from highly folded or metamorphosed districts were selected and studied for the purpose of instituting such a comparison. The following is a list of the limestones and marbles selected, with the localities from which they were derived. The list is divided into three parts—the first comprising the rocks in which the effects of the movements, due to dynamic action, are distinctly visible, and either closely resemble or are identical with those seen in the artificially deformed marbles; the second embraces several mesozoic limestones from intensely folded portions of the Alps, whose structure is of doubtful origin; while the third includes those rocks in which evidence of movement under pressure is doubtful or absent. Of the 42, as will be seen, 15 exhibit the structures seen in the artificially deformed marbles described in this paper.

Limestones and Marbles showing the Structures of the Artificially Deformed Marbles.

1. Marble. Troviken, Norway.
2. ,, Tyrol, Austria.

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* "Der Marmor in Bezug auf seine Geologie, Structur und seine mechanischen Eigenschaften," 'Zeit. für prakt. Geol.,' Jan. und Feb., 1898.

† 'Lehrbuch der Petrographie,' Bd. 3, p. 447.

3. Marble. Andermatt, Switzerland.
4. ,, Schaftelen, ,,
5. Limestone. Bützistöckli, Switzerland.
6. ,, Flims, ,,
7. ,, Griesbach, Germany.
8. Marble. Carrara, Italy.
9. ,, Lot 12, Range V., Township of Burleigh, Canada.
10. ,, Lot 11, Range IV., Township of Burleigh, Canada.
11. ,, Lot 38, Range VIII., Township of Anstruther, Canada.
12. ,, Lot 29, Range XI., Township of Cardiff, Canada.
13. Limestone. Lot 28, Range XI., Township of Monmouth, Canada.
14. ,, Lot 27, Range XIV., Township of Monmouth, Canada.
15. Marble. Lachute, Province of Quebec, Canada.

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16. Limestone. Längis Grat, Switzerland.
17. ,, Lochseite, ,,
18. ,, Saasberg, ,,
19. ,, Färnigen, ,,
20. ,, Meienthal, ,,
21. ,, Haslithal, ,,

Limestones and Marbles not showing any distinct Pressure Structures.

22. Marble. Pentelicus, Greece.
23. ,, Hymettos, ,,
24. ,, Segelfor, Norway.
25. ,, Leifset, ,,
26. ,, Kvandal ,,
27. ,, Saxenvig, ,,
28. ,, Langesundfjord, Norway.
29. Limestone. Asker, Norway.
30. Marble. Carassiner Thal, Switzerland.
31. ,, Ascona, Switzerland.
32. ,, Lot 30, Range IX., Township of Methuen, Canada.
33. ,, Lot 8, Range IX., Township of Monmouth, Canada.
34. ,, Lot 9, Range XXIII., Township of Cardiff, Canada.
35. Dolomite. Lot 16, Range VI., Township of Cardiff, Canada.
36. ,, Lot 15, Range XI., Township of Wollaston, Canada.
37. Marble. Lot 15. Range XIII., Township of Galway, Canada.
38. ,, Lot 13, Range XIV., Township of Lutterworth, Canada.

39. Marble. Lot 14, Range III., Township of Lake, Canada.
40. Limestone. Lot 1, Range I., Township of Lake, Canada.
41. ,, Lot 16, Range XII., Township of Wollaston, Canada.
42. Marble. Lot 12, Range V., Township of Burleigh, Canada.

Limestones and Marbles showing the Structures of the Artificially Deformed Marbles.

1. *Marble. Troviken, Norway.*—This is a beautiful white marble from the contact zone in the Velfjorden. It is cited by VOGT as an example of a marble showing cataclastic structure, and is figured in his paper on marble before referred to. It is composed of large irregular-shaped individuals or fragments of calcite, embedded in a mass of smaller grains. In the hard specimens the cleavage surfaces of the large individuals can often be observed to be bent or curved in a striking manner. Under the microscope the large grains are seen to be in the act of breaking down into smaller grains. Almost every grain is twinned, and the great majority show strain shadows, which are often very marked. The structure is cataclastic, the smaller grains having been derived from the breaking down of larger ones, some of which survive in part as the remnants. There has not, however, been that rolling out and flattening of the grains seen in No. 13. The rock is stated by VOGT to owe its coarsely crystalline character to contact metamorphism, and its secondary cataclastic structure to subsequent dynamic action.

2. *Marble. Tyrol, Austria.*—A medium grained white saccharoidal marble of Liassic age, the precise locality of which it has been impossible to ascertain. The rock has undergone incipient deformation, and under the microscope presents an appearance similar to that seen in those artificially deformed marbles where the motion is due to twinning and gliding. The individuals of calcite with scarcely a single exception are twinned, often showing a double set of twin lines crossing one another. Many of the grains are bent or twisted along certain lines marked by deep strain shadows. The individual grains are approximately uniform in size and usually come together along smooth sweeping lines.

3. *Marble. Alte Kirke, Andermatt, Switzerland.*—This well-known marble, believed to be of Jurassic age, and which has, according to HEIM, been reduced to one-tenth of its original thickness by the enormous pressure to which it has been subjected during the folding of the Alps, is distinctly foliated, consisting of rude bands of larger and smaller grains of calcite. The foliation is chiefly due to the flattening of the calcite grains. Almost every grain is twinned and many show strain shadows. The sections also show little streaks or areas of much more finely crystalline calcite, containing a good deal of dark colouring matter, apparently a carbonaceous pigment. These are quite different in structure from the rest of the rock, and evidently represent the last remains of the original fine-grain limestone, from

the alteration of which the marble was produced. There are a few little strings of sericite between the calcite grains at intervals. While the movements in this rock probably took place chiefly before or during its recrystallisation from the finer grained limestone, the flattened character of the grains, accompanied as they are by twinning and strain shadows, indicate that there have been very considerable movements in the rock since its recrystallisation. The structure closely resembles that of the Carrara marble artificially deformed at 400° C. A microphotograph of a thin section of this rock is shown in Plate 22, fig. 2. It is taken between crossed Nicols in polarised light, and is magnified 70 diameters. A part of one of the areas of the finely crystalline aggregate above mentioned is seen at the margin of the field.

4. *Marble. Near Schaftelen, Switzerland.*—This occurrence, believed to be Upper Jurassic (Malm) in age, is crossed by the Susten Road near the village of Schaftelen. Like that at Andermatt, it has been caught up in the folding of the Alps and the rock has been greatly compressed. It is a pure white marble, consisting of a very fine-grained alabaster-like base, in which there are numerous remnants of large twisted calcite individuals which are almost entirely destroyed by cataclastic action. These have an irregular elongated form, with their longer axes generally parallel to one another. Under the microscope the rock is seen to possess a most perfect cataclastic structure. The large calcite remnants are traversed by narrow twin lines and show most pronounced twisting, with strain shadows and other accompanying pressure phenomena. Many of them are seen to be in the act of disappearing by being resolved into a mass of smaller grains like those making up the mass of the rock. The smaller grains, produced in the way described, are flattened in one plane, having the form of little disks or cakes of somewhat irregular outline, as can be seen by examining sections cut parallel with and at right angles to the foliation of the rock. They do not show twinning, but frequently show strain shadows. The fine-grained portion of the rock somewhat resembles No. 5. In this rock both cataclastic action and the flattening of the small grains resulting from the breaking down of the large ones, by what must be a movement on their gliding planes, is plainly seen; both of which are structures exhibited by artifically deformed marble. Although the deformation of the calcite in this case is undoubted and intense, the twinning lines are not nearly so numerous as in the artificially deformed rock. The conditions here have evidently been less favourable to twinning.

5. *Limestone. Bützistöckli, Switzerland.*—A limestone of Upper Jurassic age (Malm) from the Canton of Glarus and forming a portion of the Glarner Double Fold. As shown by HEIM, it has been greatly squeezed and rolled out by the pressure to which it has been subjected. It is greyish-blue in colour, has a slabby structure, and shows no signs of recrystallisation. Under the microscope it is seen to be so extremely fine in grain that an enlargement of 500 diameters is barely sufficient to resolve it. In structure it closely resembles the finely granulated calcite

in the little shear zones of the artificially deformed marble. There are, however, a few rather coarser-grained streaks in the section, and these are composed of calcite grains which show marked twinning, and which are being broken down by granulation into minute grains like those composing the mass of the rock. These latter are seen under a very high power to be distinctly flattened, while the pigment still remains as minute black dots scattered throughout the mass. The somewhat coarser-grained streaks evidently result from the rolling out of little veins of calcite formed in the rock during the earlier stages of its deformation, as shown by the fact that they cut obliquely across the foliation of the rock in many cases. They consequently are free from pigment, but have been greatly crushed by later movements, and now consist of small calcite fragments in a finely granulated groundmass, presenting a typical cataclastic structure. These fragments have precisely the same "fibrous" structure as that seen in the calcite of artificially deformed marbles. The fact that these later veins have not been recrystallised would seem to indicate that the finer grained groundmass of the rock is still intact in this respect, and that the flattening of the minute calcite grains has probably been produced by the pressure to which the rock has been subjected, as it is in the case of the Carrara marble in the experiments described in this paper.

6. *Limestone. Flims, Switzerland.*—A very fine-grained bluish Upper Jurassic limestone, showing structures similar to those described in No. 5.

7. *Limestone. Griesbach, Erzgebirge, Germany*—A light grey granular limestone or marble, rather fine in grain, with an indistinct banded appearance caused by the alternation of lighter and darker streaks or bands. Under the microscope the rock shows what is to all appearances a well-marked cataclastic structure. There are larger grains of irregular elongated form, with their longer axes lying in the same direction, and between them smaller grains which look as if they had been torn from the larger ones. Almost every grain, large or small, is highly twinned, often showing two sets of two lamellæ crossing one another. The twinning is usually in very narrow polysynthetic bands, often so narrow that the grains have a fibrous appearance, exactly like that in the artificially deformed limestones. Strain shadows are also common, but usually the grains are so highly twinned that the strain seems to have been relieved in this way. The larger grains are often as much as seven times as long as they are wide and are ragged in outline. The whole appearance of the rock indicates movement under great compression. The structures are exactly those seen in the deformed Carrara marble. The cataclastic structure, however, as in Nos. 10 and 11, has a more coarse-grained development than that produced artificially. The original rock was composed of larger individuals, and the granulated material is not so finely triturated. The other structure, which consists of the deformation and flattening of the component individuals of the rock by twinning and movement on their gliding planes, is exactly like that seen in the Carrara marble when deformed at 300° C. or 400° C. In thin sections the finer-grained portions of this

Griesbach limestone cannot be distinguished under the microscope from the Carrara marble deformed at 400° C., the structures being identical. A microphotograph of this rock is shown on Plate 22, fig. 3. It was taken between crossed Nicols in polarised light and is magnified 70 diameters. The rock also contains a few grains of quartz and muscovite which usually show marked strain shadows.

8. *Marble. Carrara, Italy.*—Carrara marble is usually free from any evidence of pressure or deformation, its normal character being that of the marble described and figured in the former part of this paper, and upon which the experiments in deformation were carried out. In this specimen, however, there is a suggestion of parallelism in the glistening cleavage surfaces of the broken rock, and under the microscope the calcite grains show a distinct tendency to assume a flattened form. A very large proportion of the grains are twinned, and strain shadows are seen in some cases. The appearance of the sections indicates that the flattening of the grains has been produced by movements along gliding planes under the influence of dynamic action. As in the case of No. 15, pressure acting subsequent to the recrystallisation of the rock has probably set up movements in certain parts of the mass, from one of which this specimen has been derived.

Nos. 9 to 15 are from the Grenville series, of the Laurentian system, of Canada. The first six are from the counties of Peterborough and Hastings, in the province of Ontario, in the district to the north of the lake ot that name, and the last is from a point about 40 miles west of Montreal, in the province of Quebec.

9. *Marble. Lot 12, Range V., Township of Burleigh, Ontario.*—This marble comes from the same great limestone belt as Nos. 10 and 11, although several miles distant from the locality from which the latter was obtained. The stratigraphical relations point to great movements along this line, a fact which is borne out by the structure of the limestones themselves. The limestone at this locality is coarsely crystalline, in some cases becoming very coarse, the constituent grains being as much as an inch in diameter. As in the case of the Carrara deposits, it is for the most part massive and free from any foliation, but along certain lines or bands it presents a very marked foliation, and cataclastic structure is distinctly seen in hand specimens, large and more or less lenticular and much twisted calcite remnants lying with their longer axes parallel to one another in a fine-grained base derived from their partial destruction. Under the microscope the evidence of this action is most striking. The large remnants are twinned and curved, showing marked strain shadows, and can in many cases be seen to be in the act of breaking down into smaller grains, especially about their margins. A microphotograph of the rock is shown in Plate 25, fig. 3. It is taken between crossed Nicols in polarised light, and is magnified 47 diameters. In the small grains constituting the base twinning and strain shadows are also frequently seen, and there is presented a distinct tendency to flattening in the same direction as that in which the longer axes of the large remnants lie. A number of twisted grains of quartz, showing very marked strain shadows, and in some cases even a marked granulation, are also present in the rock.

10 *Marble. Lot* 11, *Range IV., Township of Burleigh, Ontario.*—This is identical with No. 9, except that the base is very much finer in grain. The larger remnants are so highly twinned that they often present the fibrous appearance before referred to. They lie scattered about in the fine-grained base, and wedge-shaped tongues of the finer-grained material can often be observed penetrating them. The structure is identical with that seen in the Carrara marble when deformed at ordinary temperatures, that is, with a marked development of cataclastic structures rather than of movement on gliding planes. The whole in the case of the natural marble, however, is on a larger scale ; the original rock was more coarsely crystalline, and the resulting product was not so finely granulated.

11. *Marble. Lot* 38, *Range VIII., Township of Anstruther, Ontario.*—Practically identical with 10 in every respect. A microphotograph of a thin section of a highly granulated portion of this rock is shown on Plate 25, fig. 4. It is photographed between crossed Nicols in polarised light and is magnified 70 diameters.

12. *Marble. Lot* 29, *Range XI., Township of Cardiff, Ontario.*—A very fine-grained marble, through which are distributed occasional large twisted calcite remnants, which indicate that the rock in its present form has resulted from the granulation of a coarsely crystalline marble. The rock bears a very strong resemblance to No. 4, but the granulation is more advanced and the calcite remnants less numerous. The granulated portion of the rock is also identical with that of No. 10 ; in fact, No. 10, if more completely granulated, would be identical in character with this rock.

13. *Limestone. Lot* 28, *Range XI., Township of Monmouth, Ontario.*—At two places in this township (Nos. 13 and 14) the coarsely crystalline white limestone of the Laurentian contains somewhat irregular-shaped streaks or bands which are bluish-black in colour and very fine in grain. These are portions of the original limestone in a comparatively unaltered condition. In these bluish-black portions the calcite grains are very small, and have the dark carbonaceous colouring matter distributed all through their substance. An enlargement of 500 diameters is required for their study. With this power the rock is seen to be perfectly crystalline, the minute calcite individuals being fitted together along boundaries which are smooth or in some cases slightly crenulated. The grains are usually distinctly flattened, but this is not seen in all cases. Some of them are twinned, and many of them show strain shadows. The white marble with which this blue limestone is associated consists of a much more coarsely grained aggregate of calcite grains. These show the most marked evidence of motion, being very much twisted and flattened in the direction of the foliation of the rock, with twinning and very pronounced strain shadows. The carbonaceous pigment has been destroyed. Distributed in the usual more or less rounded forms through both the blue and the white varieties, but especially abundant in the latter, are grains of several other minerals—plagioclase, pyroxene, biotite, &c.—the results of metamorphic action. These generally show the effects of pressure, often in a striking manner.

The structure of this rock is due in part to cataclastic action, but chiefly to the deformation of the calcite grains by motion on their gliding planes. Evidently the original blue limestone was recrystallised throughout the greater part of its mass, with the development of numerous secondary minerals, and the whole was then subjected to dynamic action, which resulted in the movements described, which, while affecting both rocks, are most noticeable in the coarser-grained marble.

14. *Limestone. Lot 27, Range XIV., Township of Monmouth, Ontario.*—Identical in character with No. 13.

15. *Marble. Lachute, Province of Quebec.*—This rock has a very distinct foliated structure, the plane of foliation being emphasised by the presence of little graphite flakes, which have the appearance of being smeared along it. The rock has a marked cataclastic structure, and has clearly been derived from the squeezing of a coarse-grained marble. The deformation of the calcite grains, accompanied by strain shadows, is very marked in all but the smallest grains, which would probably show the phenomenon also if their surfaces were sufficiently large to render the shadows visible. Twinning is also common, although many grains which show a marked deformation are free from all traces of it. While, therefore, the structure is cataclastic, it is combined with a most marked development of deformation of the calcite grains by movement along their gliding planes.

Mesozoic Limestones from the Alps whose Structure is of Doubtful Origin.

The Jurassic limestones which have been caught up in the mighty foldings of the Alps, and which are found not only in the flanks of the mountain system but along certain lines in the deep synclinals of the chain, although extremely compressed and contorted, in many cases show but little signs of alteration, while elsewhere they have become converted into coarsely crystalline marbles. The marbles of Andermatt (No. 3) and of Schaftelen (No. 4) are believed to be of Mesozoic age, and to represent these limestones in their highly altered condition; while the limestones of Bützistöckli (No. 5) and Flims (No. 6) represent the rocks in a comparatively unaltered state. The former, as has been shown, present certain structures which are clearly attributable to deformation under pressure, but in some of the mesozoic limestones which have undoubtedly been extremely plicated and subjected to enormous internal movements, the evidence of these movements in the minute structure of the rock is by no means striking. To the latter class belongs the rocks of this division, Nos. 16 to 21. Their structure is in all cases essentially the same, and they are closely related to Nos. 5 and 6 described above. This structure is that which is the most important element in HEIM'S "Umformung ohne Bruch." The individual grains of calcite are flattened at right angles to the pressure, and in the direction of the movement of the rock. Whether, however, this flattening, inducing what HEIM terms "microclivage," has been brought about by pressure, the flattened grains flowing on their gliding planes and moving over one another and thus always adapting their shape to the space to be occupied, or whether the structure is in part due to recrystallisation, is not perfectly certain. HEIM holds the former view, and believes that microclivage and fluidal structure are essentially the same. "Es gibt in der That," he writes, "keine Grenze und keinen wirklichen mechanischen Unterschied zwischen beiden."* If this be the true explanation of the structure, these rocks are closely related to those of the class just described. We intend in a subsequent series of experiments, making use of fine-grained limestones, to endeavour to reproduce this structure also by artificial compression, and thus, if possible, to determine its origin.

16. *Limestone. Längis Grat, Switzerland.*—A fine-grained grey limestone from the Längis Grat, which rises above the Furka Road, opposite the Rhone Glacier, and which is believed to be a continuation of the same limestone as that which further east appears as the Andermatt marble (No. 3). It breaks up into long thin chip-like fragments, and where it disintegrates in damp places falls into a mass of needle-like calcite grains. It is indistinctly streaked in very narrow lines in lighter and darker shades. With the exception of a little carbonaceous matter and a few mica plates it consists altogether of calcite. In some places it holds belemnites. It has a very distinct foliated structure, due to the calcite grains being all flattened in one direction. The mass of the rock is made up of very small grains, but there are at intervals lines of similarly flattened grains of larger size. As shown by the study of longitudinal and transverse sections, the grains have the shape of short laths of irregular outline, resembling very closely in form the little leaves of quartz seen in certain gneisses, and are frequently as much as six times as long as they are wide. The larger grains are frequently twinned, but the smaller grains rarely show this structure. Strain shadows are not seen. It seems doubtful whether this structure is attributable to recrystallisation in the case of such a fine-grained limestone which still retains its organic pigment. It is not cataclastic, but may be due to the flattening of the calcite grains by gliding, under the influence of the great pressure to which the rock has been subjected.

17. *Limestone. Lochseite, Switzerland.*—The Malm limestone which is such a striking element in the succession in the Glarner Double Fold, and which derives its local name from Lochseite, near Schwanden, presents the same flattening of the constituent calcite grains as described in No. 16. The rock from Lochseite itself is very impure and extremely fine in grain, so that the structure is not well seen, but the elongation or flattening of the minute calcite grains composing the rock was observed in a number of slides of the Lochseiten-Kalk from various localities, which are preserved in the collections of the Geological Department of the University of Zürich.

18. *Limestone. Saasberg, Switzerland.*—The rock from the Saasberg, near the Bützistöckli, shows this structure excellently.

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* 'Untersuchungen über den Mechanismus der Gebirgsbildung,' Bd. 2, p. 56.

19. *Limestone. Färnigen, Switzerland.*—Near the eastern end of the Susten Pass a synclinal of Mesozoic rocks is pinched in by the folding of the Alps. Among these at Färnigen is a very fine-grained, slabby, blue limestone, which is of especial interest in that it contains numerous belemnites, which have been greatly elongated and in some cases torn apart. The rock shows no signs of recrystallisation, except that white calcite has been deposited between the fragments of the broken belemnites. Under the microscope it is seen to be composed of very minute elongated calcite grains, like those above described. There are also dotted all through the rock, groups of darker coloured grains of a rhombohedral carbonate, probably dolomite. These are untwinned and apparently uncrushed. The foliation of the rock, due to the flattening of the calcite grains, curves around them as it does around the garnets in a schist. There are also lines of more coarsely crystalline calcite, as described in No. 5, whose origin is identical in both cases.

20. *Limestone. Meienthal, Switzerland.*—Other specimens in the Zürich collection labelled simply "Meienthal," but probably also from near Färnigen, show exactly the same structure as described in No. 19. The little elongated calcite grains not having an identical orientation extinguish between crossed Nicols in different positions.

21. *Limestone. Haslithal, Switzerland.*—This rock, which is a typical example of Heim's 'Bruchlose gefaltete Malm Kalk,' shows the same flattening of the minute calcite individuals composing it. The little calcite veins referred to in No. 5 are here folded in with the rock and are more coarsely crystalline. Their presence shows that the rock was at first brittle and became shattered under the pressure, the fissures thus formed becoming filled and giving rise to calcite veins more coarsely crystalline in character than the rest of the rock. With the continuance of the pressure the rock became plastic and the veins were folded, the calcite grains composing them becoming flattened like those constituting the mass of the rock. The plane of the flattening or foliation of the grains cuts across the veins quite irrespective of their course, the position of the latter being marked by their lighter colour and coarser grain. The motion evidently took place in connection with the flattening of the calcite grains, and possibly, as above noted, by their movements over each other.

Limestones and Marbles not showing any Distinct Pressure Structures.

The limestones and marbles of this class (Nos. 22 to 42) do not here merit individual description. They do not present any undoubted evidence of movement under pressure. Their structure is that of a mosaic, apparently resulting in each case from the recrystallisation of a previously existing finer-grained limestone. This process, as described by Lepsius* in the Attic marbles, consists of the enlargement or growth of certain of the constituent grains at the expense of others until finally a coarse-grained mosaic is produced. Traces of this are seen in several of these rocks. Twinning is

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* 'Geologie von Attika; ein Beitrag zur Lehre vom Metamorphismus der Gesteine,' p. 186.

often present, but there has been no distinct alteration in the shape of the grains by pressure. The individual grains are, in some cases, very irregular in shape and often come together along more or less crenulated lines, and in No. 24 the peculiar intergrowth of separate calcite individuals described by VOGT[*] was observed. The structure of the limestones and marbles of this class is in fact quite different from those included in the first list, although they might readily give rise to such rocks as these, were they subjected to dynamic action under the required conditions.

It will thus have been seen that the deformed limestones and marbles met with in nature, present in many cases at least precisely the structures developed in marble by artificial deformation. Among these are to be especially noted, in the first place, cataclastic structure ; and, in the second place, the twisting, elongation, and flattening of the component calcite individuals either with or without the concomitant development of twinning and strain shadows, these latter phenomena being almost invariably seen in the larger individuals but less frequently observed in the very small grains, apparently on account of the very smallness of their surface. When a large, highly twinned and strained calcite individual is observed breaking down into a mass of smaller grains, it can be distinctly seen that each individual grain resulting from this granulation is so small that it is, in the great majority of cases, derived from a single twin lamella, and its surface is so limited that the strain shadow upon it would be scarcely noticeable.

While, therefore, recrystallisation undoubtedly plays an important, and in many cases probably a chief, part in the great movements which are observed to have taken place in the limestones of contorted districts, this process is by no means the only one by which such movements are brought about. Many limestones under pressure in the earth's crust *flow* precisely as metals do by deformation of the compressed grains and without the intervention of water or any other solvent.

VI. SUMMARY OF RESULTS.

1. By submitting limestone or marble to differential pressures exceeding the elastic limit of the rock and under the conditions described in this paper, permanent deformation can be produced.
2. This deformation, when carried out at ordinary temperatures, is due in part to a cataclastic structure and in part to twinning and gliding movements in the individual crystals composing the rock.
3. Both of these structures are seen in contorted limestones and marbles in nature.
4. When the deformation is carried out at 300° C., or, better, at 400° C., the cataclastic structure is not developed, and the whole movement is due to changes in the shape of the component calcite crystals, by twinning and gliding.
5. This latter movement is identical with that produced in metals by squeezing or

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[*] *Loc. cit.*, p. 13.

hammering, a movement which in metals as a general rule, as in marble, is facilitated by increase of temperature

6. There is therefore a flow of marble just as there is a flow of metals under suitable conditions of pressure.
7. The movement is also identical with that seen in glacial ice, although in the latter case the movement may not be entirely of this character.
8. In these experiments the presence of water was not observed to exert any influence.
9. It is believed, from the results of other experiments now being carried out but not yet completed, that similar movements can, to a certain extent at least, be induced in granite and other harder crystalline rocks, and that several structures developed in these rocks in nature in highly contorted regions can thus be reproduced.

Explanation of Plates.

PLATE 22.

Fig. 1. The machine used in the investigation. A marble column is in process of deformation. The experiment is being carried out in the absence of moisture and at the ordinary temperature. The small boiler on the extreme right does not belong to this machine.

Fig. 2. Thin section of the marble from Alte Kirke, Andermatt, Switzerland. The grains are slightly flattened in a horizontal direction, and are repeatedly twinned in almost every case. On the right there is a fine-grained aggregate which represents a remnant of the original fine-grained limestone, from the recrystallisation of which the marble was derived. The structure resembles that of Carrara marble artificially deformed at 300° or 400° C. Photographed between crossed Nicols. × 70 diameters.

Fig. 3. Thin section of the limestone or marble from Griesbach, in the Erzgebirge. The smaller grains have probably been derived from the breaking down of larger individuals, a portion of one of which is seen. All the grains show most pronounced polysynthetic twinning, two sets of lamellæ crossing one another being visible in most individuals. Movement on gliding planes is also pronounced, the structure being identical with that produced by the artificial deformation of Carrara marble at 300° C. or 400° C. Photographed between crossed Nicols. × 70 diameters.

PLATE 23.

Fig. 1. On the left the iron tube enclosing the marble of Experiment A is shown ready to be placed in the machine. On the right the same, after the marble had been slowly deformed during a period of 64 days. 13/14 of natural size.

Fig. 2. The deformed marble of Experiment A freed from the enclosing iron tube, and beside it a marble column of the dimensions which it originally possessed. 14/13 of natural size.

Fig. 3. Tube containing the deformed marble, milled open, and the marble split in two as described. The marble column in this case was reduced to one-half its original height in 4 hours. Natural size.

Fig. 4. Another experiment similar to that shown in fig. 3 ; the deformation, however, is less marked. The experiment in this case occupied 17 days. The cones were quite distinct in the original. 10/11 of natural size.

Fig. 5. Column of marble (Experiment K) deformed at 300° C. The experiment occupied 124 days. Beside it is a column of the original dimensions. Natural size (very nearly).

Fig. 6. In this case the pressure on the marble was continued so long and the deformation carried so far that the moving marble within tore the iron tube apart, as shown. This tube when opened is shown in fig. 3.

PLATE 24.

Fig. 1. Microphotograph of the Carrara marble used in the experiments. The rock as found in nature. The individual grains have very nearly the same diameter in every direction, although differing somewhat in size among themselves. Twinning is seen only in two or three grains, and in these is represented by a few broad lamellæ. Photographed in ordinary light. × 50 diameters.

Fig. 2. A microphotograph of the Carrara marble after having been slowly deformed during 124 days at a temperature of 300° C. The individual grains can be seen to be distinctly flattened in a horizontal direction, giving a certain foliation to the rock, and to possess the fibrous appearance referred to in the text as due to polysynthetic twinning. Photographed between crossed Nicols in polarised light. × 50 diameters.

Fig. 3. Microphotograph of the Carrara marble deformed at 400° C. A uniform mosaic of somewhat flattened grains, free from all fracturing or cataclastic action. Photographed in ordinary light. × 70 diameters.

Fig. 4. Microphotograph of a few grains of the calcite on the thinnest edge of a section of the deformed marble shown in fig. 2. The polysynthetic twinning is well seen. Two sets of twin lamellæ cross one another in the large grain, curving somewhat, and varying more or less in width from place to place. Photographed between crossed Nicols in polarised light. × 150 diameters.

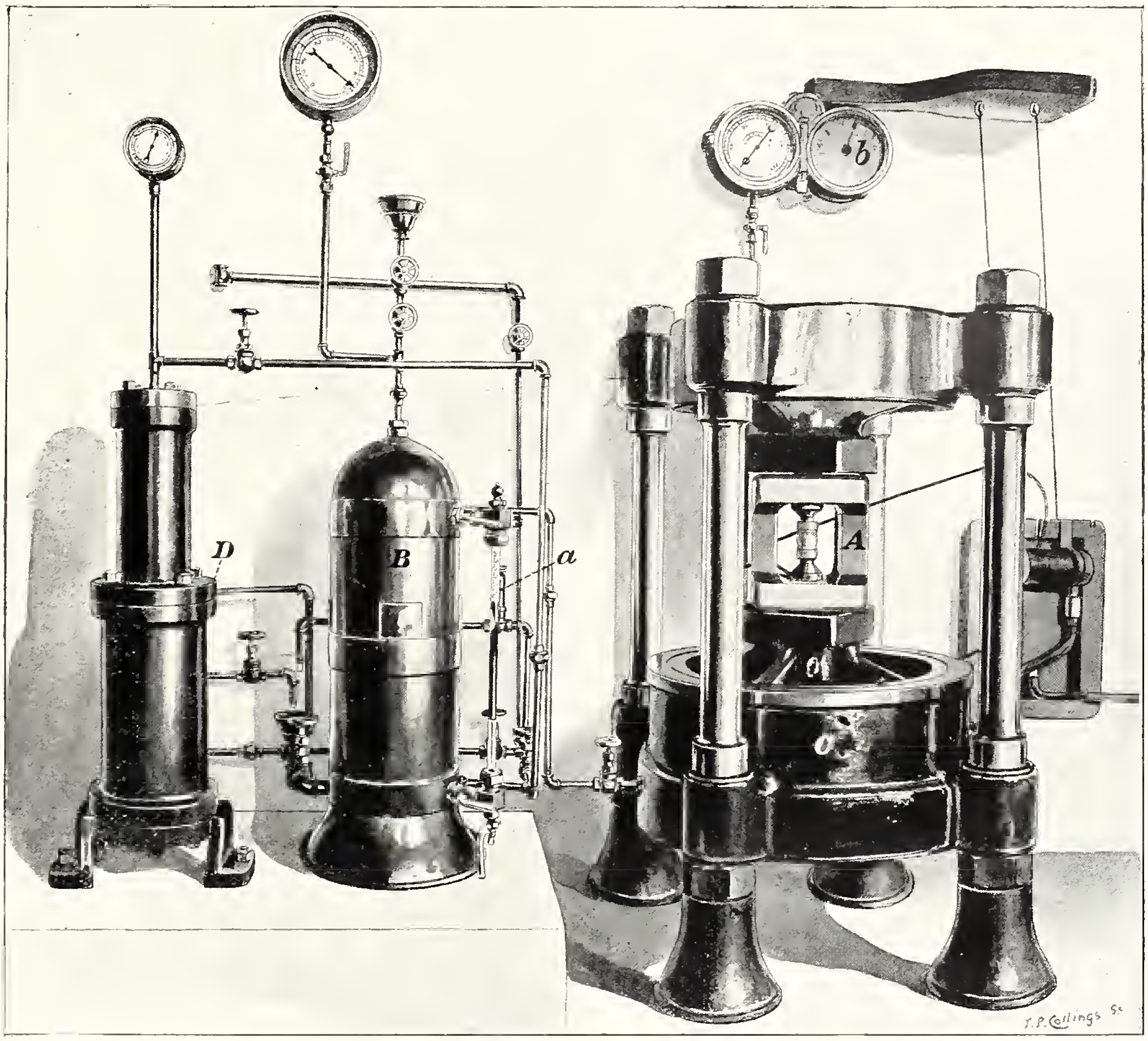
PLATE 25.

Fig. 1. Microphotograph of a thin section of the Carrara marble (shown in Plate 24, fig. 1) deformed at the ordinary temperature in 7 hours. The dark areas are the granulated portions of the rock. Irregularly shaped fragments of calcite individuals, often distinctly twinned, are seen scattered through it. Photographed in ordinary light. × 70 diameters.

Fig. 2. Microphotograph of a thin section of the same marble between the lines of granulated material. It presents a continuous mosaic of flattened grains. Photographed in ordinary light. × 50 diameters.

Fig. 3. Microphotograph of a thin section of the Laurentian marble from Lot 12, Range V., of the township of Burleigh, Ontario. Presents a cataclastic structure identical with that shown by the deformed marble of fig. 1 of this Plate, but on a larger scale. The original rock was much more coarsely crystalline, and the granulation has not been so minute. The twisting and twinning of the large remnants in process of granulation is well seen. Photographed between crossed Nicols in polarised light. × 47 diameters.

Fig. 4. Microphotograph of a thin section of the Laurentian marble from Lot 38, Range VIII., of the township of Anstruther, Ontario. The rock is identical in character with that shown in fig. 3, but the section represents a more thoroughly granulated portion. The structure is identical with that seen in the artificially deformed marble of fig. 1, but the granulation is not quite so minute. Photographed between crossed Nicols in polarised light × 70 diameters.



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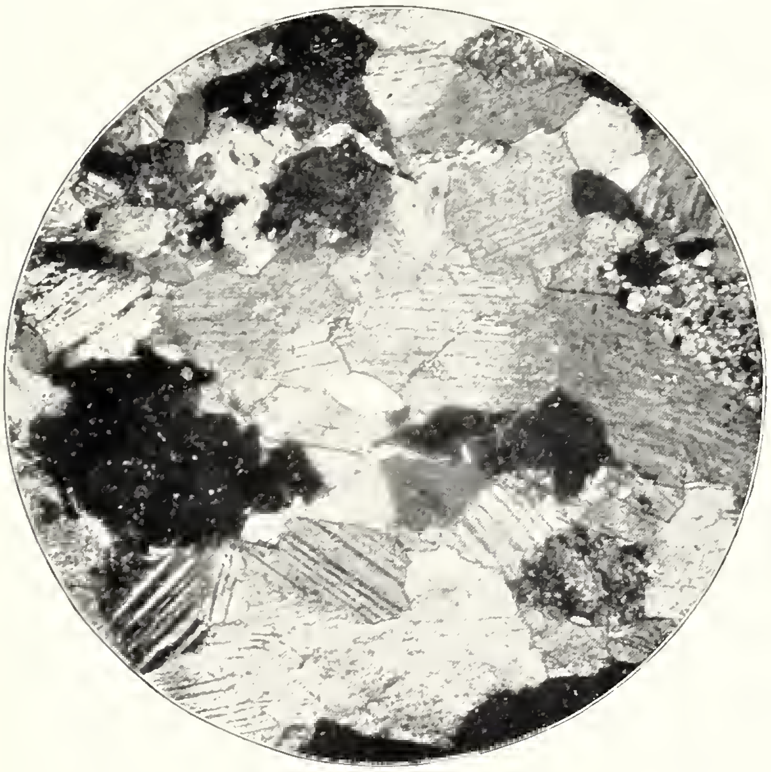

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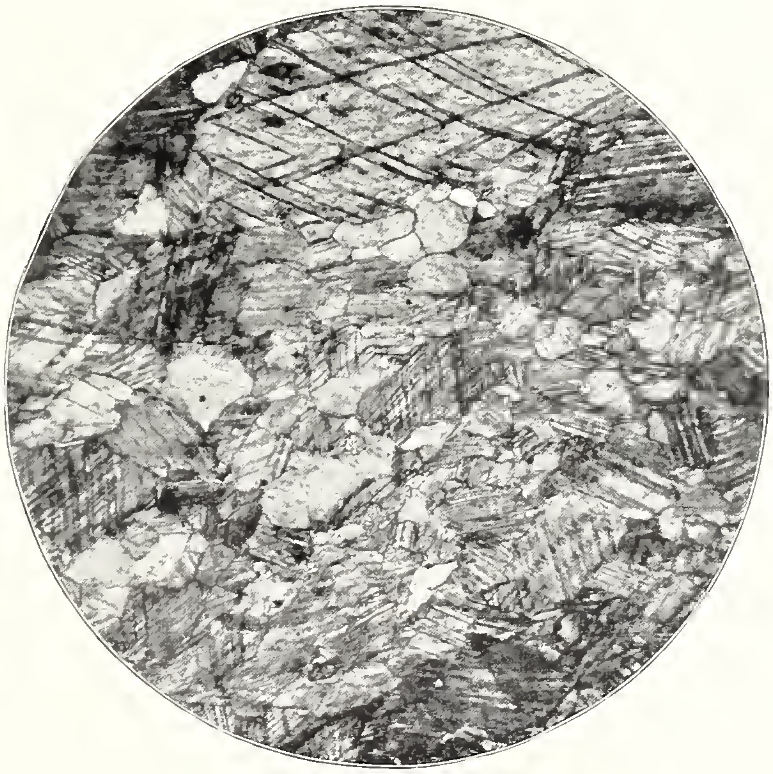

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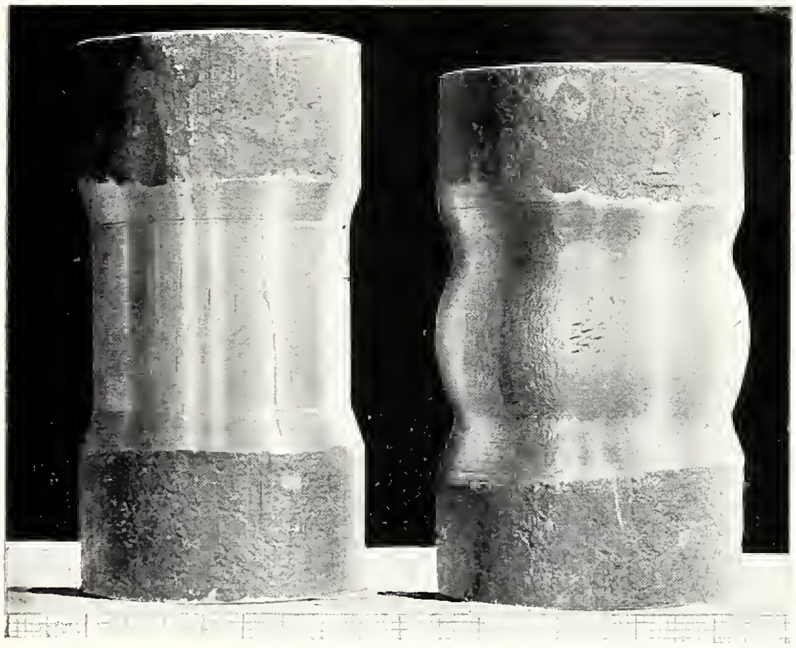

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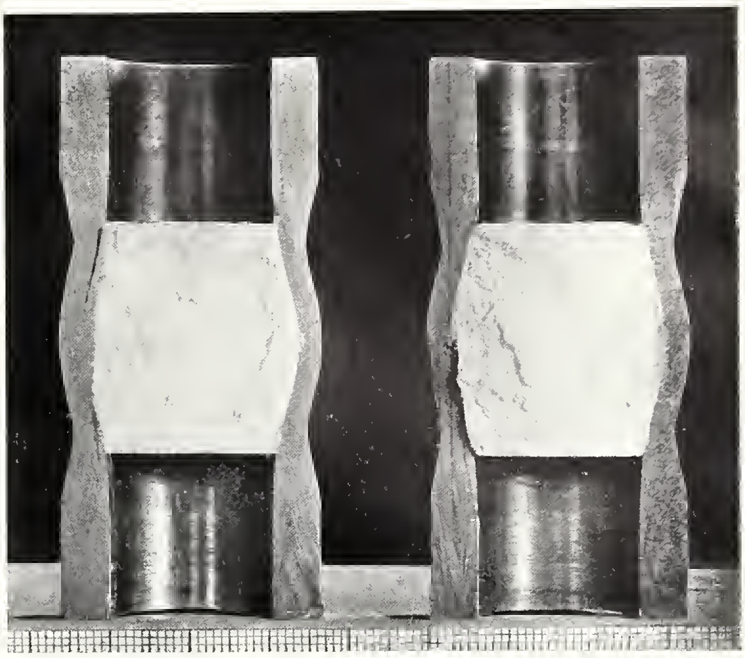

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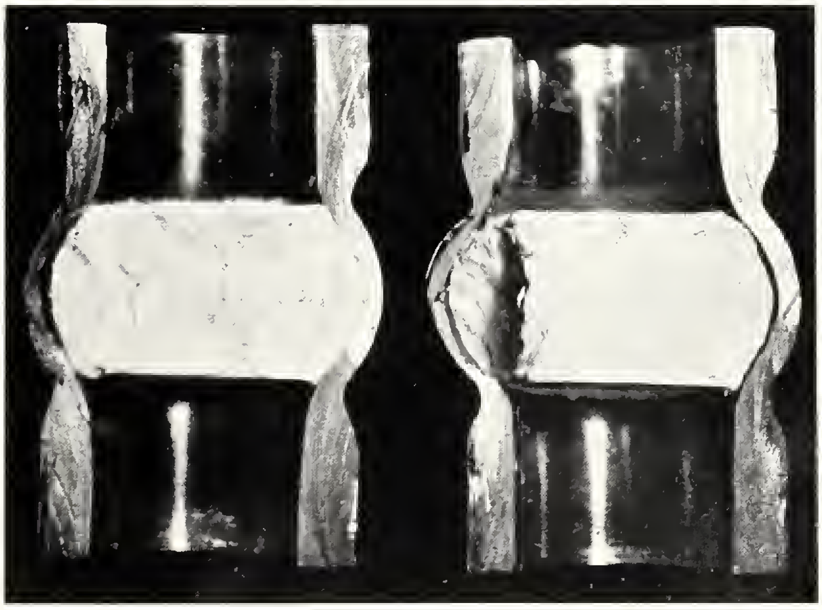

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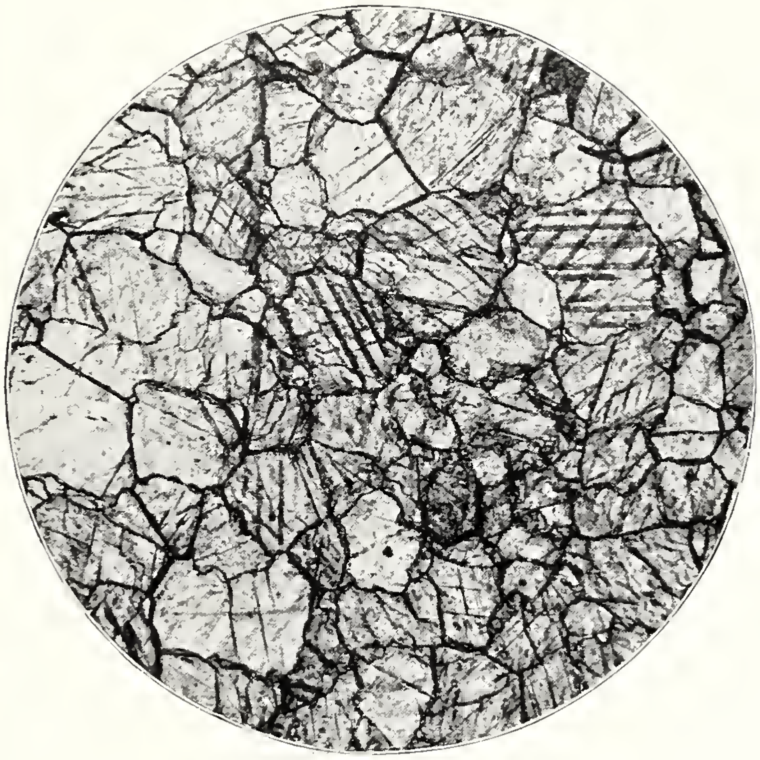

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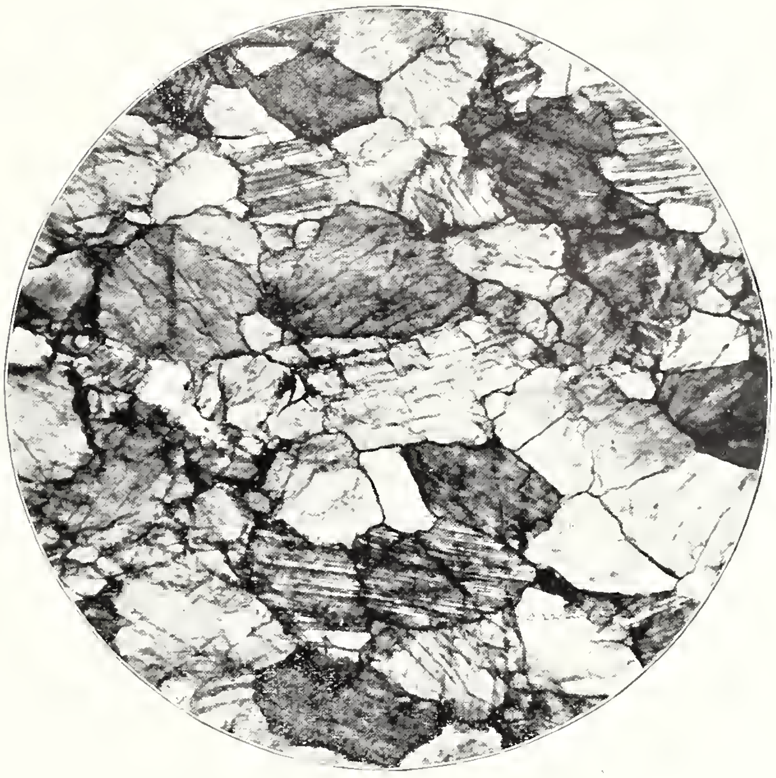

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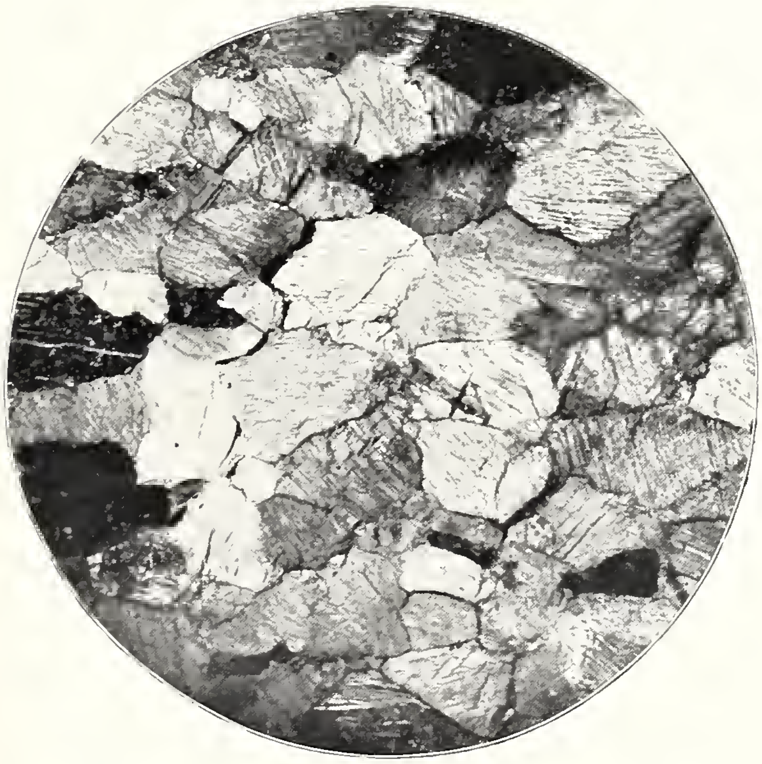

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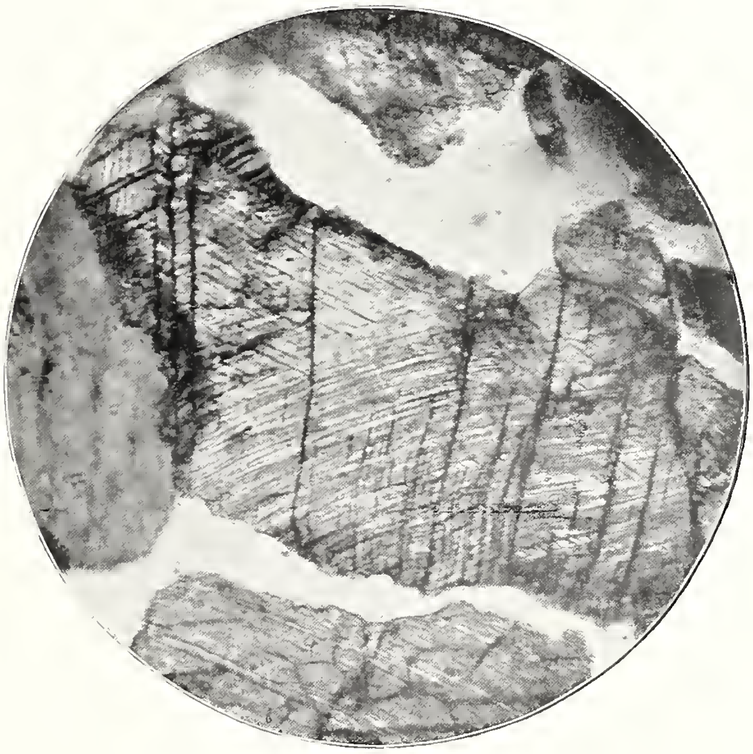

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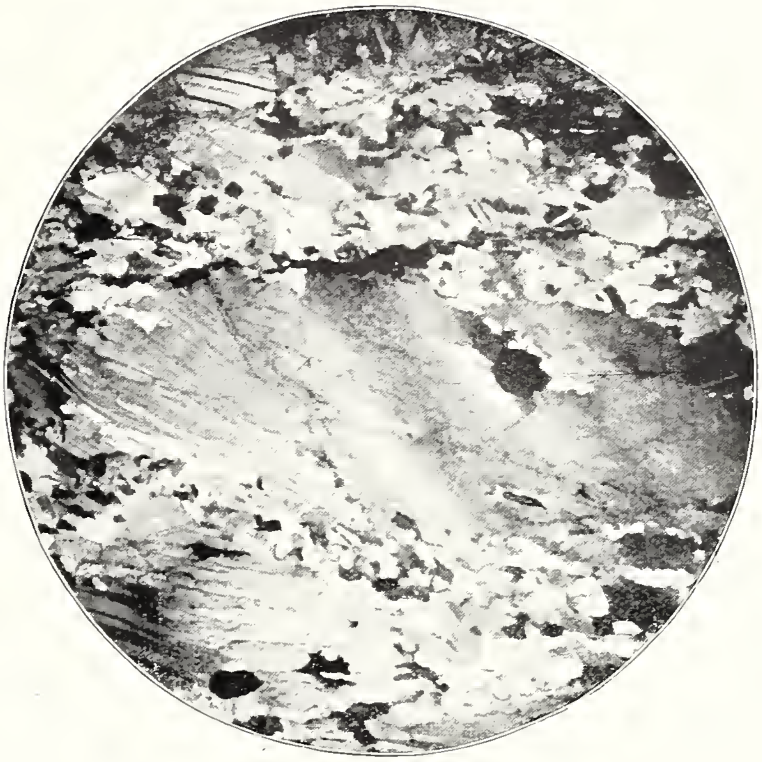

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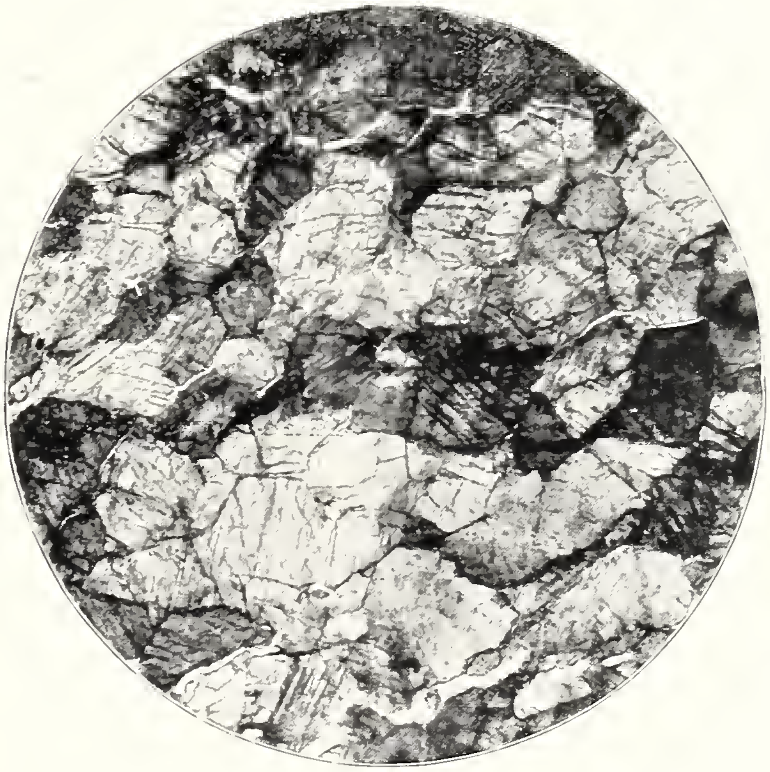

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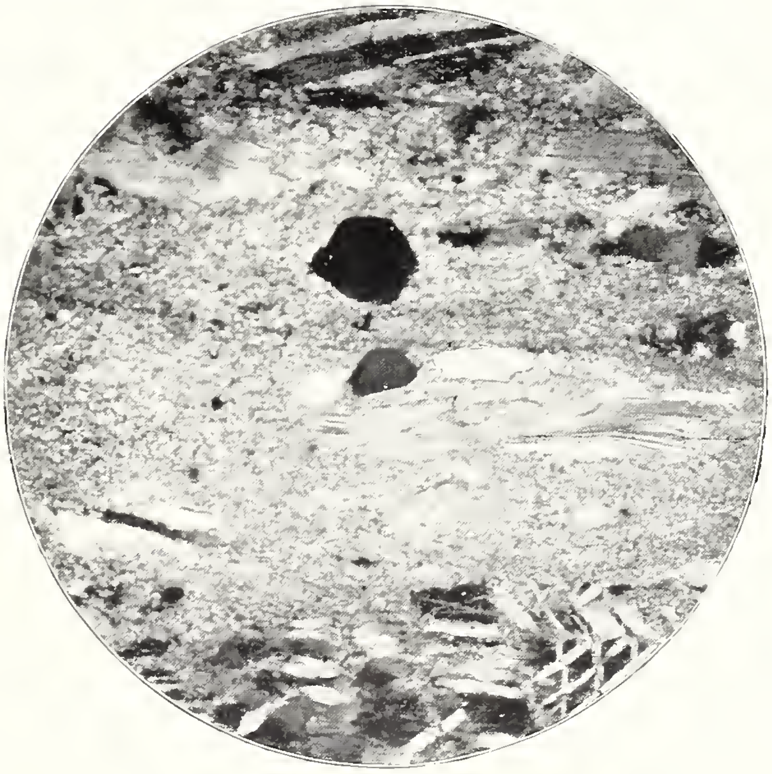

Fig. 4.

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TO THE

PHILOSOPHICAL TRANSACTIONS.

SERIES A, VOL. 195.

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Vol. 168 (extra)	3	0	0
1878. Part I...	1	16	0
—— Part II...	3	0	0
1879. Part I...	2	0	0
—— Part II...	1	12	0
1880. Part I...	2	5	0
—— Part II...	2	0	0
—— Part III...	1	1	0

	£	s.	d.
1881. Part I...	2	10	0
—— Part II...	1	10	0
—— Part III...	2	2	0
1882. Part I...	1	14	0
—— Part II...	2	0	0
—— Part III...	2	10	0
—— Part IV...	1	0	0
1883. Part I...	1	10	0
—— Part II...	2	10	0
—— Part III...	1	12	0
1884. Part I...	1	8	0
—— Part II...	1	16	0
1885. Part I...	2	10	0
—— Part II...	2	5	0
1886. Part I...	1	8	0
—— Part II...	1	15	0
1887. (A.)	1	3	0
—— (B.)	1	16	0
1888. (A.)	1	10	0
—— (B.)	2	17	6
1889. (A.)	1	18	0
—— (B.)	1	14	0
1890. (A.)	1	16	6
—— (B.)	1	5	0
1891. (A.)	2	2	0
—— (B.)	3	3	0
1892. (A.)	2	1	0
—— (B.)	2	2	0
1893. (A.)	3	14	0
—— (B.)	2	13	0
1894. (A.) Pt. I.	1	5	0
—— (A.) Pt. II.	1	10	0
—— (B.) Pt. I.	3	10	0
—— (B.) Pt. II.	1	11	6
1895. (A.) Pt. I.	1	14	0
—— (A.) Pt. II.	1	3	6
—— (B.) Pt. I.	1	15	0
—— (B.) Pt. II.	0	16	6
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—— (B) Vol. 187.	0	19	0
—— (A) Vol. 188.	1	9	0
—— (B) Vol. 188.	0	19	6
1897 (A) Vol. 189.	1	0	0
—— (B) Vol. 189.	0	17	6
—— (A) Vol. 190.	1	8	6
—— (B) Vol. 190.	2	5	0
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