Astigmatism of Skew Pencils in Optical Systems Containing Toric Surfaces [1]

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A method is described for tracing skew rays through toric refracting surfaces and for determining the astigmatism of pencils around these rays.

1. Introduction

Toric refracting surfaces are used chiefly in two classes of optical systems, spectacle lenses, and anamorphotic photographic and projection objectives. In spectacles the toric surfaces are arranged to correct the axial astigmatism of the eye, and in anamorphotic systems two or more toric (possibly cylindrical) surfaces are included to give stigmatic imagery with different magnifications in different directions in the image plane.

When the axial pencil is refracted by a toric,[3] some astigmatism is introduced; if a pencil is incident obliquely, either centrally or eccentrically, astigmatism of different magnitude is introduced. It is therefore desirable in the case of both types of optical systems mentioned above to determine the astigmatism of obliquely incident pencils, in order to assess the imagery in the outer parts of the field. This is the problem which was studied in the work to be described.

The problem arose in connection with spectacle lenses and may be outlined as follows:

Figure 1 shows front and side views of a spectacle lens with toric surface; O is the center of the lens, C is the center of rotation of the eye, and P_1, P_2, P_3 are points of incidence of the chief rays of oblique pencils. Let the meridian and equatorial sections of the toric be, respectively, in the vertical and horizontal planes OP_1 and OP_3. Then the problem for chief rays passing through the axis and through P_1 and P_3 is trivial. The chief ray itself can be traced by the ordinary ray-tracing formulas for meridian rays and spherical surfaces, and the astigmatism can be determined by the usual methods for close sagittal and tangential fans, with the slight difference that the curvatures of the refracting surface in the two directions are not equal (see eq (8)).
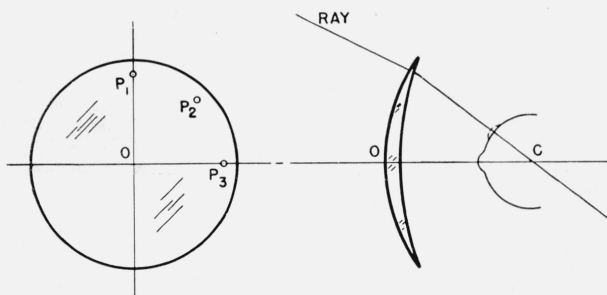

FIGURE 1.

If, however, the point of incidence of the chief ray does not lie in the meridian or equatorial section, as, for example, P_2 in figure 1, the problem becomes more complicated. In the first place, even if the incident chief ray intersects the axis of the system, the refracted ray in general does not, so that the chief ray must for complete generality be considered to be skew; thus the first part of the problem is to trace a skew chief ray through the toric.

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[3] In what follows, "toric surface" will be abbreviated to "toric."

Second, with oblique incidence as at any of the points P_1, P_2, or P_3, astigmatism is produced in the refracted pencil for two reasons: (a) because the angle of incidence is not zero, as in the ordinary case of spherical surfaces, and (b) because the refracting surface has two different principal curvatures at the point of incidence. In the case of P_2 the principal directions of the astigmatism due to (a) are parallel and normal to the plane of incidence of the chief ray, i. e., approximately parallel and perpendicular to line, OP_2; the principal directions of the astigmatism due to (b) are, however, approximately parallel to OP_1 and OP_3. Thus the resultant astigmatic pencil will have its focal lines in some intermediate directions, and the determination of these directions further complicates the calculation.[4] It follows that the formal distinction between sagittal and tangential directions can no longer be made; however, for the configurations of chief ray and toric that occur in spectacle lenses the rotation of the focal lines is unlikely to be above $10°$, so that there is little difficulty in relating the results for skew chief rays to those for rays in either principal section.

The results of such calculations would constitute a survey of the astigmatism of a toric spectacle lens at all obliquities and azimuths. They would help to answer questions such as the following: What is the best bending of a toric spectacle lens? What range of powers can be manufactured with a common back curvature without departing appreciably from the best bending? Is there any advantage in making both surfaces of a spectacle lens toric in shape? Such questions as these arise in the design of so-called best-form spectacle lenses. These have previously been designed on the basis of computations in the two principal sections only.[5][6] However, an illustration is given by Henker[6] showing the quality of an image produced by a pencil outside the principal sections of a typical sphero-toric lens of the Punktal best-form series; this illustration suggests that there is still room for improvement in the image. The work to be described makes it possible to compute the imagery by skew pencils and so determine whether any improvements can in fact be made.

Section 2 is a recapitulation of the properties of astigmatic pencils that are required in deriving the astigmatism equations. In section 3 the equations of a toric are given, and other properties are derived, in section 4 the ray-tracing equations for the skew chief ray are derived, and in section 5 the astigmatism formulas are derived. Section 6 deals with the forms in which the input data must be given. In section 7 the mode of presentation of the results is discussed, the results of a numerical example are given, and the possible uses of the computation are discussed.

2. Astigmatic Pencils

For the purposes of the present paper a pencil will be considered to be made up of a narrow bundle of rays originating from a point in the object and surrounding a chief, or principal, ray, together with all the corresponding wave fronts. The wave fronts are surfaces of constant optical path length from the object point, and the rays are normals to the wave fronts.

With a rectangular coordinate system with the x axis along the principal ray, the equation of the wave front passing through the origin will take the form

$$x = c_{20}y^2 + c_{11}yz + c_{02}z^2 + \text{terms of higher degree in } y \text{ and } z; \tag{1}$$

linear terms are absent because the yz plane must be tangent to the wave front at the origin. Also it is possible to rotate the y and z axes about the x axis, so that the coefficient of the term in yz vanishes when the equation takes the form

$$x = \tfrac{1}{2}(c_r y^2 + c_s z^2) + \text{higher-degree terms.} \tag{2}$$

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[4] This difficulty does not arise for points on the principal sections of the toric, such as P_1 and P_3, because then the directions of the focal lines are known by symmetry to be parallel and perpendicular to the principal sections.

[5] H. H. Emsley and W. Swaine, Ophthalmic lenses, 6th ed. (London, 1951).

[6] O. Henker, Introduction to the theory of spectacles (Translated by R. Kanthack) (Jena, 1924).

Effects depending on terms in y and z of higher degree than the second will be neglected. To this degree of approximation it is easily shown that all rays of the pencil; i. e., normals to the surface (2) intersect one line through the point $(1/c_r, 0, 0)$ parallel to the z axis and another line through the point $(1/c_s, 0, 0)$ parallel to the y axis, these being the two astigmatic focal lines. Thus this approximation takes account of astigmatism but not of aberrations such as spherical aberration and coma, which depend on higher powers of the aperture of the pencil. The coefficients have the significance that c_r is the curvature of the section of the wave front by the xy plane and c_s, that by the xz plane

Some results from differential geometry are now required. At any nonsingular point P of a surface the normal curvature (i. e., the curvature of the section of the surface by a plane through the normal) has a maximum and a minimum for two section planes which are at right angles to each other; furthermore, if these two planes, together with the tangent plane at P, are taken as coordinate planes, the equation of the surface in the neighborhood of P takes the form of eq (2), that is, the yz term is absent. The maximum and minimum normal curvatures are called the principal curvatures, and the corresponding section planes meet the tangent plane at P in lines that lie along the principal directions at P. Thus the principal directions are along the y and z axes for eq (2).

It is also known, by Euler's theorem, that if a normal section plane makes an angle α with the xy plane, the curvature in this section is

$$c(\alpha) = c_r \cos^2 \alpha + c_s \sin^2 \alpha. \tag{3}$$

From this it follows that these quantities are required to characterize an astigmatic pencil:

(a) The direction of the chief ray; this will be given by a unit vector, $\boldsymbol{Q}$, with components X, Y, and Z.

(b) The principal curvatures, c_r and c_s, of the wave front at a point on the chief ray.

(c) The principal directions on the wave front; these will be given by unit vectors $\boldsymbol{R}$ and $\boldsymbol{S}$, and from the above it follows that $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{S}$ are all perpendicular to each other.

3. Toric Surface

A toric surface is generated by the rotation of a circle (the generating circle, abbreviated to G. C.) about an axis in its plane but not passing through its center, as in figure 2, a and b. In optical applications the lens surface is, of course, only a cap-shaped portion of the whole surface, roughly centered on the point A. Such a surface is alined with a centered system of spherical surfaces by making the optical axis lie in the normal plane to the rotation axis, which contains the center of the G. C. The optical axis then meets the toric in the pole A.
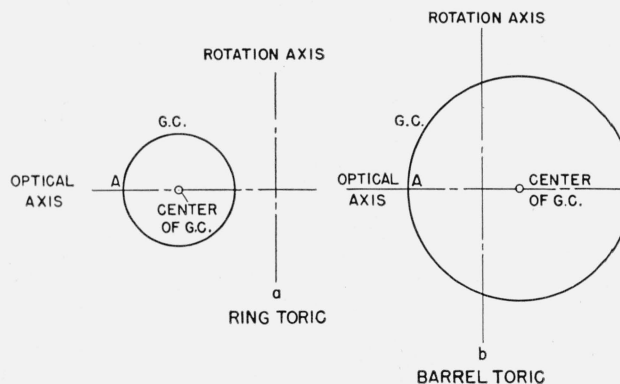

FIGURE 2.

The toric refracting surface has different powers (in the optical sense) in different sections through the optical axis: the powers are determined by the radius of the G. C. for the section in the plane of the rotation axis and by the distance from A to the rotation axis for that perpendicular to it; these are known as the meridian and equatorial sections, respectively.

It can be seen that for given powers in the principal sections there are two possible forms of toric surface: the radius of the G. C. can be either less than or greater than the distance from A to the rotation axis, the two forms being known, respectively, as ring- and barrel-shaped torics. The ring form seems to be used almost exclusively for spectacles in commercial practice, but to the best of the author's knowledge, this is merely because more rings than barrels of given curvatures can be surfaced in one operation, not for any optical advantage. The possibility of improvements from the use of the barrel shape seems not to be considered in the literature on "best-form" series.

3.1. Equations of the Toric

One can write down the equation of the toric surface by taking rectangular coordinate axes with the origin at the pole A of the toric refracting surface (fig. 3), the x axis along the optical axis, the y axis parallel to the rotation axis, and the z axis in the equatorial plane. Let r_y and r_z be the radii of curvature in the xy and xz sections, respectively. Take angular parameters, θ and ϕ, as in the figure, so that θ is the longitude and ϕ the latitude of a point on the toric. Then the parametric equations of the toric are

$$\left. \begin{aligned} x &= r_z - (r_z - r_y + r_y \cos \phi) \cos \theta \\ y &= r_y \sin \phi \\ z &= (r_z - r_y + r_y \cos \phi) \sin \theta \end{aligned} \right\} \tag{4}$$

It can be seen that eq (4) represent a ring- or a barrel-shaped toric according as r_y is less than or greater than r_z in absolute magnitude,[7] in both cases the rotation axis is parallel to the y axis. When the rotation axis is to be parallel to the z axis the equations are obtained be interchanging y and z throughout, including subscripts, in eq (4); this is true also for the equations in the remainder of section 3.
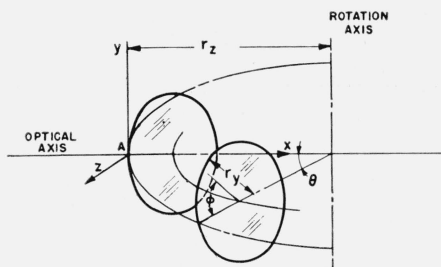

FIGURE 3.

An explicit equation can be obtained by eliminating θ and ϕ in eq (4) and solving for x:

$$x = r_z \pm \{ [r_z - r_y \pm (r_y^2 - y^2)^{1/2}]^2 - z^2 \}^{1/2}. \tag{5}$$

The ambiguous signs are due to the fact that there are four (possibly complex) values of x for any given y and z, since the toric is a quartic surface; the required solution is that which tends to zero as y and z tend to zero, so that the correct signs are, respectively, minus and plus. When these are inserted the result is unsuitable for numerical computation when y and

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[7] Provided r_y and r_z have the same sign, which is generally the case in spectacle lenses.

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z are small compared to r_y and r_z, as is generally the case. This difficulty can be circumvented by transforming the equation by means of conjugate surds in the usual way and using curvatures instead of radii, giving

$$x = \frac{F}{G\{G + [G^2 - c_z F]^{1/2}\}},\tag{6}$$

where

$$F = c_z z^2 G^2 + 2 c_y y^2 G - c_y^2 c_z y^4, \quad G = 1 + (1 - c_y^2 y^2)^{1/2}.$$

In this form the equation of the toric is suitable for numerical computation with all relevant ranges of the variables.

3.2. Principal Directions and Curvatures

In considering the refraction of an astigmatic pencil through a toric, it is necessary to know the shape of the toric in the neighborhood of the point of incidence. Thus, as in section 2, the following are required:

(a) The direction of the normal at the point of incidence; this will be given by a unit vector, U, with components, λ, μ, and ν.

(b) The principal curvatures, c_v and c_w, at the point of incidence.

(c) The principal directions at the point of incidence; these will be specified by unit vectors, V and W, and again the vectors, U, V, and W, form a trirectangular system. These quantities are easily determined for a current point (x, y, z); if the toric has principal curvatures, c_y and c_z, at the origin and if the rotation axis is parallel to the y axis, as in section 2.1, the results are

$$U = (\cos \phi \cos \theta, -\sin \phi, -\cos \phi \sin \theta),\tag{7}$$

$$\phi = \arcsin c_y y, \quad \theta = \arcsin \frac{G c_z z}{G - c_z c_y y^2},$$

$$\left.\begin{aligned} c_v &= c_y \\ c_w &= \frac{G c_z \cos \phi}{G - c_y c_z y^2} \end{aligned}\right\},\tag{8}$$

$$\left.\begin{aligned} V &= (\sin \phi \cos \theta, \;\; \cos \phi, \; -\sin \phi \sin \theta) \\ W &= (\sin \theta, \qquad\quad 0, \qquad\quad \cos \theta) \end{aligned}\right\}.\tag{9}$$

4. Tracing the Skew Chief Ray

Using the quantities defined in sections 2 and 3.2, the problem can be stated as follows:

Given Q and the coordinates of a point on the chief ray, the coordinates (x, y, z) of the point of incidence of the ray on the toric must be determined (transfer process); next U at the point of incidence must be found and Q', the direction of the refracted chief ray, determined (refraction process). The quantities c_r, c_s, R, and S will be known for the incident pencil and c_v, c_w, V, and W can be found, since the point of incidence is known; from these the quantities c_r', c_s', R', and S' for the refracted pencil must be determined. This last, the astigmatism calculation, will be described in section 5.

4.1. Transfer Process

The transfer problem is that of finding the intersection of a straight line with a toric. A direct approach would require the solution of an algebraic equation of the fourth degree; this would be a laborious process, so an iterative method was adopted. In addition to the general advantage of iterative methods, that mistakes in the early stages of the calculation are automatically eliminated at the cost of extra computing time, this method also selects the correct solution from the four possible solutions.

Let P_1 be a point on the ray near the required point of incidence.[8] A point P_1' on the toric is found with the same y and z coordinates and the tangent plane T_1 at P_1' is determined. The next approximation P_2 is taken at the intersection of the ray with T_1 and the cycle $P_2 \to P_2' \to T_2 \to P_3$ . . . repeated until the coordinates of P_n and P_{n+1} are the same to the required number of significant figures. These are then the coordinates of P, the point of incidence.

Let the coordinates of P_n be (x_n, y_n, z_n) and those of P_n' be (x_n', y_n, z_n); then x_n' is obtained by substituting y_n and z_n in eq (6). The equation of the plane T_n is $\lambda_n(x - x_n') + \mu_n(y - y_n) + \nu_n(z - z_n) = 0$,
where λ_n, μ_n, and νz are given by eq (7) so that x_{n+1} is given by

$$\left.\begin{aligned} x_{n+1} &= x_n + \delta_n \\ \delta_n &= \frac{\lambda_n X(x_n' - x_n)}{\lambda_n X + \mu_n Y + \nu_n Z} \end{aligned}\right\} . \tag{10}$$

Also,

$$\left.\begin{aligned} y_{n+1} &= y_n + \frac{Y}{X}\delta_n \\ z_{n+1} &= z_n + \frac{Z}{X}\delta_n \end{aligned}\right\} \tag{11}$$

This completes the iterative cycle. It can be shown to be a second-order iteration,[9] that is, δ_{n+1} is of order of magnitude δ_n^2, so that it is always rapidly convergent, provided the initial approximation is good enough.

The choice of the initial approximation depends on the available means of computation. If a high-speed automatic-computing engine is available, P_1 may be taken as the point where the ray meets the yz plane. Let (x_0, y_0, z_0) be the coordinates of the point P_0 where the ray leaves the previous refracting surface (or the entrance pupil);[10] then P_1 is given by

$$\left.\begin{aligned} x_1 &= 0, \\ y_1 &= y_0 - x_0 Y/X, \\ z_1 &= z_0 - x_0 Z/X \end{aligned}\right\} . \tag{12}$$

Starting from this approximation, some four or five iterations would probably be needed to reach a result correct to six significant figures.

If a desk computing machine is to be used, an appreciable saving in computing time can be effected by using an improved first estimate. It usually happens that chief rays directed at the same pupil have already been traced in the meridian and equatorial sections of the toric. Let the x coordinates of their points of incidence be x_y and x_z, respectively. The section of the toric by a plane through the x axis and making an angle α with the xy plane is approximately a circle, of curvature $c_y\cos^2\alpha + c_z\sin^2\alpha$, so that a chief ray in this plane will meet the toric at a point of which the x coordinate is approximately $x_y \cos^2\alpha + x_z \sin^2\alpha$. This P_1 is given by

$$\left.\begin{aligned} x_1 &= x_y \cos^2\alpha + x_z \sin^2\alpha \\ y_1 &= y_0 + (x_1 - x_0)Y/X \\ z_1 &= z_0 + (x_1 - x_0)Z/X \end{aligned}\right\} . \tag{13}$$

This approximation is usually so good that only one iteration is needed to obtain six significant figures in the coordinates of P.

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[8] The method of obtaining P_1 will be discussed below.

[9] W. Weinstein, Proc. Phys. Soc. (London) [B] **65**, 731 (1952).

[10] (x_0, y_0, z_0) are taken in the coordinate system referring to the current surface, not that on which P_0 lies.

4.2. Refraction Process

The refraction calculation for the chief ray can now be carried out by well-known formulas. Let I be the angle of incidence and n the index of refraction of the medium in which the ray is incident; let primes denote corresponding refracted quantities.   Then

$$\cos I = \boldsymbol{Q} \cdot \boldsymbol{U}, \tag{14}$$

$$\cos I' = + \left(1 - \frac{n^2}{n'^2} (1 - \cos^2 I) \right)^{1/2}, \tag{15}$$

$$\boldsymbol{Q}' = \frac{n}{n'} \boldsymbol{Q} + \left(\cos I' - \frac{n}{n'} \cos I \right) \boldsymbol{U}. \tag{16}$$

In eq (14) $\boldsymbol{Q} \cdot \boldsymbol{U}$ denotes the scalar product of $\boldsymbol{Q}$ and $\boldsymbol{U}$.

This completes the refraction process and the coordinates and direction cosines of the refracted ray are then used, with a suitable change of origin, for the incident ray at the next surface.

5. Astigmatism Calculation

In order to determine equations giving $\boldsymbol{R}'$, $\boldsymbol{S}'$, c_r', and c_s', in terms of known quantities, a general theorem is required.   Referring to figure 4, let P be the point of incidence of the chief ray, and let B be the point of incidence of another ray of the pencil.   Suppose the incident and refracted wavefronts through P to be constructed, and let the ray through B meet these in C and C', respectively.[11]   Then, since the optical path length between two wavefronts is the same along any ray, it follows that $n\overline{CB} = n'\overline{C'B'}$ or, if $\overline{CB}$ is denoted by q,

$$\Delta(nq) = 0.$$

Here the symbol, Δ, means that the refracted quantity minus the incident quantity is to be taken.   This relationship is true for any relative positions of P and B and for any shape of refracting surface.

In order to use eq (17) for the present purpose, q and q' will be expressed in terms of the coordinates of the point B on the refracting surface and the parameters of the wavefronts. The resulting equation will give the normal curvature of the refracted wavefront in a section containing the ray through B and by differentiating with respect to the azimuth of B the maximum and minimum normal curvatures, c_r' and c_s', can be found.   The same calculation gives the directions $\boldsymbol{R}'$ and $\boldsymbol{S}'$ of the principal sections.
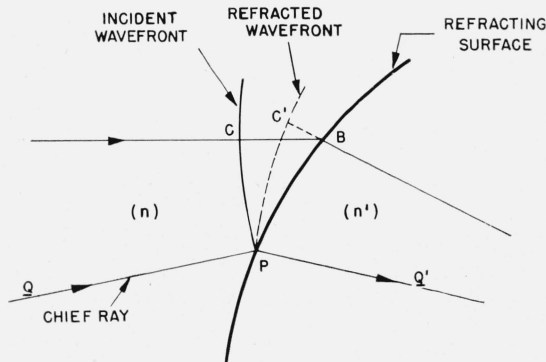

FIGURE 4.

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[11] The part of the refracted wavefront shown in the diagram is virtual.

5.1. Normal Curvature of the Refracted Wavefront

Figure 5 shows the refracting surface near the point of incidence. Let P and B have the same significance as in figure 4 and let V and W be the unit vectors along the principal directions. Let B_1 be the projection of B on the VW plane; let α be the angle between PB_1 and V, and let PB be denoted by p. Then the position of B relative to P will be given by p and α as coordinates.
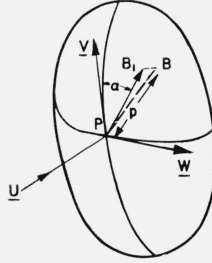

FIGURE 5.

In order to express q in terms of p and α, use is made of the fact that CB in figure 4 is a normal to the wavefront. It is found (see section 5.1, a) that

$$q = p(\boldsymbol{Q} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{Q} \cdot \boldsymbol{W} \sin \alpha) + \tfrac{1}{2} p^2 \{ \boldsymbol{Q} \cdot \boldsymbol{U}(c_v \cos^2 \alpha + c_w \sin^2 \alpha) - c_r(\boldsymbol{R} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{R} \cdot \boldsymbol{W} \sin \alpha)^2$$
$$- c_s(\boldsymbol{S} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{S} \cdot \boldsymbol{W} \sin \alpha)^2 \} + \text{terms in } p^3, \tag{18}$$

and there is a similar equation for q'.

Substituting for q and q' in eq (17),

$$p \Delta n(\boldsymbol{Q} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{Q} \cdot \boldsymbol{W} \sin \alpha) + \tfrac{1}{2} p^2 \Delta n \{ \boldsymbol{Q} \cdot \boldsymbol{U}(c_v \cos^2 \alpha + c_w \sin^2 \alpha) - c_r(\boldsymbol{R} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{R} \cdot \boldsymbol{W} \sin \alpha)^2$$
$$- c_s(\boldsymbol{S} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{S} \cdot \boldsymbol{W} \sin \alpha)^2 \} + \text{terms in } p^3 = 0. \tag{19}$$

Only terms up to and including the square of p are taken since, as explained in section 2, this degree of approximation takes account of astigmatism.

Equation 19 holds good for any values of p and α, so that the coefficients of different powers of p may be separately equated to zero. If this is done, the linear term yields an alternative form of Snell's law and on putting $\alpha = 0$ and $\pi/2$, the equations

$$\Delta[n \boldsymbol{Q} \cdot \boldsymbol{V}] = \Delta[n \boldsymbol{Q} \cdot \boldsymbol{W}] = 0 \tag{20}$$

are obtained. These could serve as a check on the refraction calculation.

Equating the coefficient of p^2 in eq (19) to zero and dividing through by $1 - (\boldsymbol{Q}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{Q}' \cdot \boldsymbol{W} \sin \alpha)^2$ gives

$$\frac{c_r'(\boldsymbol{R}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{R}' \cdot \boldsymbol{W} \sin \alpha)^2 + c_s' (\boldsymbol{S}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{S}' \cdot \boldsymbol{W} \sin \alpha)^2}{1 - (\boldsymbol{Q}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{Q}' \cdot \boldsymbol{W} \sin \alpha)^2}$$
$$= \frac{nc_r(\boldsymbol{R} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{R} \cdot \boldsymbol{W} \sin \alpha)^2 + nc_s(\boldsymbol{S} \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{S} \cdot \boldsymbol{W} \sin \alpha)^2 + (c_v \cos^2 \alpha + c_w \sin \alpha) \Delta n \boldsymbol{Q} \cdot \boldsymbol{U}}{n' [1 - (\boldsymbol{Q}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{Q}' \cdot \boldsymbol{W} \sin \alpha)^2]}. \tag{21}$$

Now the equation of the refracted wavefront can be shown to be

$$x = \frac{[c_r'(\boldsymbol{R}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{R}' \cdot \boldsymbol{W} \sin \alpha)^2 + c_s'(\boldsymbol{S}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{S}' \cdot \boldsymbol{W} \sin \alpha)^2] (x^2 + y^2)}{(\boldsymbol{R}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{R}' \cdot \boldsymbol{W} \sin \alpha)^2 + (\boldsymbol{S}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{S}' \cdot \boldsymbol{W} \sin \alpha)^2},$$

taking the x, y, and z axes along $\boldsymbol{Q}'$, $\boldsymbol{R}'$, and $\boldsymbol{S}'$. Thus the left hand side of eq (21) is $c'(\alpha)$, the

normal curvature of the refracted wavefront in a section containing the point B with angular coordinate α on the toric.

a. Derivation of Equation (18)

Two auxiliary coordinate systems are required, each with origin at P, the point of incidence. One, the x, y, z system, has its axes respectively parallel to $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{S}$; the other, x', y', and z' has its axes parallel to $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$. The two are related by the equations

$$x=(\boldsymbol{Q}\cdot\boldsymbol{U})x'+(\boldsymbol{Q}\cdot\boldsymbol{V})y'+(\boldsymbol{Q}\cdot\boldsymbol{W})z', \quad y=(\boldsymbol{R}\cdot\boldsymbol{U})x'+(\boldsymbol{R}\cdot\boldsymbol{V})y'+(\boldsymbol{R}\cdot\boldsymbol{W})z',$$
$$z=(\boldsymbol{S}\cdot\boldsymbol{U})x'+(\boldsymbol{S}\cdot\boldsymbol{V})y'+(\boldsymbol{S}\cdot\boldsymbol{W})z'.$$

If the point B on the refracting surface (fig. 4) has polar coordinates (p, α), its coordinates in the x', y', z' system are [12]

$$x'=\tfrac{1}{2}(c_v\cos^2\alpha+c_w\sin^2\alpha)p^2+O(p^4), \quad y'=p\cos\alpha+O(p^3), \quad z'=p\sin\alpha+O(p^3).$$

Let B have coordinates (u, v, w) in the (x, y, z) system; then

$$u=\tfrac{1}{2}(\boldsymbol{Q}\cdot\boldsymbol{U})(c_v\cos^2\alpha+c_w\sin^2\alpha)p^2+(\boldsymbol{Q}\cdot\boldsymbol{V})p\cos\alpha+(\boldsymbol{Q}\cdot\boldsymbol{W})p\sin\alpha+O(p^3),$$

$$v=(\boldsymbol{R}\cdot\boldsymbol{V})p\cos\alpha+(\boldsymbol{R}\cdot\boldsymbol{W})p\sin\alpha+O(p^2),$$

$$w=(\boldsymbol{S}\cdot\boldsymbol{V})p\cos\alpha+(\boldsymbol{S}\cdot\boldsymbol{W})p\sin\alpha+O(p^2).$$

Let (x, y, z) be the coordinates of C in the x, y, z system, so that

$$x=\tfrac{1}{2}(c_ry^2+c_sz^2)+O(y^3,z^3).$$

The direction cosines of the normal to the wavefront at C are, to the approximation $O(y^2,z^2)$, $(1,-c_ry,-c_sz)$, and since CB is a ray, or wavefront normal, it follows that

$$u=x+q, \quad v=y-c_ryq, \quad w=z-c_szq.$$

There are now enough relations to eliminate x, y, z, u, v, and w. The resulting equation involving p, q, and α can be solved for q, remembering that terms which are $O(p^3,q^3)$ can be neglected, since only astigmatism is in question. The solution for q is found to be eq (18).

5.2. The Principal Curvatures of the Refracted Wavefront

The principal curvatures c_s' and c_r' are the maximum and minimum values of $c'(\alpha)$. Since all quantities on the right of eq (21) are known with the exception of α, c_r' and c_s' can be obtained in the usual way by differentiating with respect to α and equating to zero. Let

$$a=\tfrac{1}{2}[nc_r((\boldsymbol{R}\cdot\boldsymbol{V})^2+(\boldsymbol{R}\cdot\boldsymbol{W})^2)+nc_s((\boldsymbol{S}\cdot\boldsymbol{V})^2+(\boldsymbol{S}\cdot\boldsymbol{W})^2)+(c_v+c_w)\Delta n\,\boldsymbol{Q}\cdot\boldsymbol{U}], \tag{22}$$

$$b=\tfrac{1}{2}[nc_r((\boldsymbol{R}\cdot\boldsymbol{V})^2-(\boldsymbol{R}\cdot\boldsymbol{W})^2)+nc_s((\boldsymbol{S}\cdot\boldsymbol{V})^2-(\boldsymbol{S}\cdot\boldsymbol{W})^2)+(c_v+c_w)\Delta n\,\boldsymbol{Q}\cdot\boldsymbol{U}], \tag{23}$$

$$c=nc_r(\boldsymbol{R}\cdot\boldsymbol{V})(\boldsymbol{R}\cdot\boldsymbol{W})+nc_s(\boldsymbol{S}\cdot\boldsymbol{V})(\boldsymbol{S}\cdot\boldsymbol{W}), \tag{24}$$

$$d=\tfrac{1}{2}((2-(\boldsymbol{Q}'\cdot\boldsymbol{V})^2-(\boldsymbol{Q}'\cdot\boldsymbol{W})^2), \tag{25}$$

$$e=\tfrac{1}{2}((\boldsymbol{Q}'\cdot\boldsymbol{V})^2-(\boldsymbol{Q}'\cdot\boldsymbol{W})^2), \tag{26}$$

$$f=-(\boldsymbol{Q}'\cdot\boldsymbol{V})(\boldsymbol{Q}'\cdot\boldsymbol{W}). \tag{27}$$

The equation for α is

$$(ea-bd)\sin^2\alpha+(dc-af)\cos^2\alpha=(bf-ec),$$

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[12] The expression $O(p^n)$ used in this section denotes terms of order of magnitude p^n.

and the two solutions are

$$\alpha_1 = \frac{\gamma - \beta}{2}, \quad \alpha_2 = \frac{\pi}{2} - \frac{\gamma + \beta}{2}, \tag{28}$$

where [13]

$$\beta = \arcsin \frac{dc - af}{+[(ea - bd)^2 + (dc - af)^2]^{1/2}} \tag{29}$$

and

$$\gamma = \arcsin \frac{bf - ec}{[(ea - bd)^2 + (dc - af)^2]^{1/2}}. \tag{30}$$

The values of c'_s and c'_r are then obtained by putting $\alpha = \alpha_1$ and α_2 in the right hand side of eq (21).

5.3. Principal Directions on the Refracted Wavefront

In figure 6, let P be the point of incidence, PB the refracting surface, and PC' the refracted wavefront. Let $\boldsymbol{Q}'$ be the unit vector along the chief ray and let $C'B$ be another ray of the astigmatic pencil. Let the projections of C' and B on the tangent planes to the wavefront and refracting surface be C'_1 and B_1 respectively. Then $\boldsymbol{V} \cos \alpha + \boldsymbol{W} \sin \alpha$ is a vector in the direction PB_1, so that a unit vector in the direction PC_1' is given by

$$\frac{\boldsymbol{V} \cos \alpha + \boldsymbol{W} \sin \alpha - \boldsymbol{Q}'(\boldsymbol{Q}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{Q}' \cdot \boldsymbol{W} \sin \alpha)}{[1 - (\boldsymbol{Q}' \cdot \boldsymbol{V} \cos \alpha + \boldsymbol{Q}' \cdot \boldsymbol{W} \sin \alpha)^2]^{1/2}}. \tag{31}$$

Thus, the vectors $\boldsymbol{R}'$ and $\boldsymbol{S}'$ are obtained by putting $\alpha = \alpha_1$ and α_2 in the expression (31). It should be noticed that in this derivation it was assumed that the points C' and B lie on a plane through $\boldsymbol{Q}'$, i. e., that the ray through C' intersects the chief ray. This is not strictly true in general, but the error depends on terms in the equation of the wavefront of the third degree in the aperture, so that astigmatism calculations are not affected.
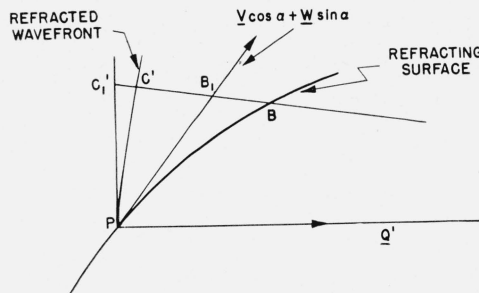

FIGURE 6.

This completes the refraction computation for astigmatism. The $\boldsymbol{R}'$ and $\boldsymbol{S}'$ vectors are transferred without change to be the new $\boldsymbol{R}$ and $\boldsymbol{S}$ vectors at the next surface. In order to transfer the curvatures the distance L between surfaces along the chief ray is required; this is easily found to be $(x - x_0)/X$, in the notation of section 4.1. Then the new curvatures are given by

$$\left. \begin{array}{l} (1/c_r)_{+1} = (1/c_r)' - L \\ (1/c_s)_{+1} = (1/c_s)' - L \end{array} \right\}. \tag{32}$$

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[13] Of the possible solutions for β, one must be chosen which satisfies

$$\beta = \arccos \frac{ea - bd}{+[(ea - bd)^2 + (dc - af)^2]^{1/2}} + \text{a multiple of } 2\pi.$$

6. Input Data

If the wavefront incident on the first surface is not already astigmatic, some arbitrary directions for R and S must be assigned. Similarly if a refracting surface is spherical, directions for V and W must be assigned.

These pairs of vectors must form trirectangular sets with Q and U, respectively, but subject to this condition they can be chosen to suit computational convenience. Choosing R to be coplanar with Q and the y axis, it is found that

$$\left. \begin{aligned} R &= \frac{(-XY, 1-Y^2, -YZ)}{(1-Y^2)^{1/2}} \\ S &= \frac{(Z, 0, -X)}{(1-Y^2)^{1/2}} \end{aligned} \right\} . \tag{33}$$

Since Y^2 is rarely greater than 0.1, the vectors so defined are well determined numerically. In a similar way V and W can be taken as

$$\left. \begin{aligned} V &= \frac{(-\lambda\mu, 1-\mu^2, -\mu\nu)}{(1-\mu^2)^{1/2}} \\ W &= \frac{(\nu, 0, -\lambda)}{(1-\mu^2)^{1/2}} \end{aligned} \right\} . \tag{34}$$

It is also necessary to choose between ring and barrel shapes for the toric surfaces. In the equations of section 3.1 the rotation axis of the toric is taken parallel to the y axis and the toric will be ring or barrel shape according as c_y is numerically greater or less than c_z. Thus, if it is required to have a barrel shape with $|c_y| > |c_z|$ or a ring shape with $|c_y| < |c_z|$, the equations must be changed so that the rotation axis is parallel to the z axis. This can be done by interchanging y and z in the symbols and subscripts of eq (4) to (8), inclusive, as mentioned in section 3.1.

7. Presentation of Results

7.1. A Numerical Example

In the application to spectacle lenses the results are presented in the form of a diagram giving the astigmatism of a number of pencils and the directions of the principal sections on the wave front. An example is given in figure 7.
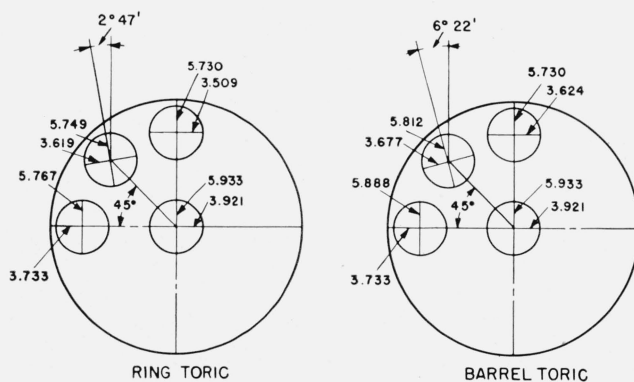

FIGURE 7.

The wavefront curvatures are marked on the appropriate sections, in diopters with respect to the vertex sphere. The apparent rotation of the principal sections of the wavefront can be found from the consideration that the vertical plane containing the chief ray contains the direction which is apparently vertical to the eye. This plane is the plane through Q and parallel to the y axis. From this it can be shown that the angle, A, through which the R and S vectors appear to be turned is given by

$$\tan A = \frac{r_2}{s_2}, \tag{35}$$

where r_2 and s_2 are the y components of R and S.

The nominal powers of the lens in this example were 5.5 diopters in the vertical section and 3.62 diopters in the horizontal section. The curvatures of the front (toric) surface were $+0.2199$ cm^{-1} and $+0.1839$ cm^{-1}, that of the back (spherical) surface was -0.1147 cm^{-1}, the center thickness was 0.48 cm and the refractive index was taken as 1.523. The chief ray was made to intersect the axis 27 mm behind the lens at an angle of 30°. The results for both ring and barrel shapes with the same nominal curvatures are given.

Computations have also been done on the prescriptions of a number of lenses and measurements have been made on these with a vertex power measuring instrument. The agreement is satisfactory within the limits of experimental error.

7.2. Discussion

In the numerical example given above the rotation of the focal lines in intermediate azimuths is negligible; this may be expected to be the case for all but very heavily cylindrical spectacle lenses of best-form type. It can also be seen that the variation of astigmatism over the field is less for the barrel than for the ring shape (0.14 D and 0.21 D, respectively). This, of course, is not a general conclusion, but it suggests that further computation would be desirable. It is possible that for heavy cylinders a considerable improvement might be obtained by using barrel shapes, but this might be outweighed by the increased manufacturing costs. This would depend also on visual tolerances for astigmatism; it seems that more experimental work is required on this subject.

A different question is that of the accuracy with which torics are produced. The principal curvatures check within the limits of accuracy of a lens gage, and the sagittal curvatures change or remain constant along the meridan or equatorial sections, respectively, to the same accuracy. However, this may not be enough to prove that the lens surface is a true toric to an accuracy which justifies the calculations for chief rays in intermediate azimuths. The mode of manufacture is certainly such as to produce a very close approximation to a toric if the surfacing machine really acts in the intended manner, but it is quite possible that distortions and non-uniform wear in the tools and machine parts may cause departures from the toric shape. No measurements other than checks on the principal curvatures seem to have been made. A detailed check on the x coordinates of a network of points would be needed.

WASHINGTON, June 29, 1954.