*ragged*) fall of the atmospheric pressure, which reached its minimum about $4^h$ $45^m$ P.M. There was then a very abrupt and nearly perpendicular rise of about five hundredths of an inch of pressure, or rather less, after which the rise still went on, but only more gradually.

Through the kindness of the Rev. R. Main, of the Radcliffe Observatory, I have been favoured with a copy of the trace afforded by the Oxford barograph during this squall, in which there appears a very sudden rise of nearly the same extent as that at Kew, but which took place about four o'clock, and therefore, as on the previous occasion, somewhat sooner than at Kew. This change of pressure at Oxford was accompanied by a very rapid fall of temperature of about 8° Fahr.

The minimum atmospheric pressure at Kew was 29·52 inches, while at Oxford it was 29·28 inches.

It will be seen from the Plate that at Kew the electricity of the air fell rapidly from positive to negative about $4^h$ $30^m$ P.M., and afterwards fluctuated a good deal, remaining, however, generally negative until $5^h$ $22^m$ P.M., when it rose rapidly to positive.

We see also from the Plate that there was an increase in the average velocity of the wind at Kew during the continuance of this squall. To conclude, it would appear that in these two squalls there was in both cases an exceedingly rapid rise of the barometer from its minimum both at Oxford and at Kew, this taking place somewhat sooner at the former place than at the latter; and that in both cases the air at Kew remained negatively electrified during the continuance of the squall, while the average velocity of the wind was also somewhat increased.

The Society then adjourned over the Christmas recess to Thursday January 7, 1864.

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## "On the Equations of Rotation of a Solid Body about a Fixed Point." By WILLIAM SPOTTISWOODE, M.A., F.R.S., &c. Received March 21, 1863.*

In treating the equations of rotation of a solid body about a fixed point, it is usual to employ the principal axes of the body as the moving system of coordinates. Cases, however, occur in which it is advisable to employ other systems; and the object of the present paper is to develope the fundamental formulæ of transformation and integration for any system. Adopting the usual notation in all respects, excepting a change of sign in the quantities F, G, H, which will facilitate transformations hereafter to be made, let

$$A=\Sigma m(y^2+z^2),\qquad B=\Sigma m(z^2+x^2),\qquad C=\Sigma m(x^2+y^2),$$
$$-F=\Sigma myz,\qquad -G=\Sigma mzx,\qquad -H=\Sigma mxy;$$

* Read April 16, 1863: see abstract, vol. xii. p. 523.

and if $p, q, r$ represent the components of the angular velocity resolved about the axes fixed in the body, then, as is well known, the equations of motion take the form

$$\left.\begin{aligned} A\frac{dp}{dt}+H\frac{dq}{dt}+G\frac{dr}{dt}&=-F(q^2-r^2)+(B-C)qr+Hrp-Gpq, \\ +H\frac{dp}{dt}+B\frac{dq}{dt}+F\frac{dr}{dt}&=-G(r^2-p^2)-Hqr+(C-A)rp+Fpq, \\ +G\frac{dp}{dt}+F\frac{dq}{dt}+C\frac{dr}{dt}&=-H(p^2-q^2)+Gqr-Frp+(A-B)pq. \end{aligned}\right\} \quad (1)$$

To obtain the two general integrals of this system: multiplying the equations (1) by $p, q, r$, respectively adding and integrating, we have for the first integral

$$Ap^2+Bq^2+Cr^2+2(Fqr+Grp+Hpq)=h, \quad . \quad . \quad . \quad . \quad (2)$$

where $h$ is an arbitrary constant. Again, multiplying (1) by

$$\begin{aligned} &Ap+Hq+Gr, \\ &Hp+Bq+Fr, \\ &Gp+Fq+Cr, \end{aligned}$$

respectively adding and integrating, we have for the second integral

$$(Ap+Hq+Gr)^2+(Hp+Bq+Fr)^2+(Gp+Fq+Cr)^2=k^2, \quad . \quad (3)$$

where $k^2$ is another arbitrary constant. This equation may, however, be transformed into a more convenient form as follows: writing, as usual,

$$\begin{aligned} &\mathfrak{A}=BC-F^2, \quad \mathfrak{B}=CA-G^2, \quad \mathfrak{C}=AB-H^2, \\ &\mathfrak{F}=GH-AF, \quad \mathfrak{G}=HF-BG, \quad \mathfrak{H}=FG-CH, \\ &\qquad A+B+C=S, \end{aligned} \qquad \nabla=\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}. \quad (4)$$

and bearing in mind the inverse system, viz

$$\left.\begin{aligned} &\nabla A=\mathfrak{B}\mathfrak{C}-\mathfrak{F}^2, \quad \nabla B=\mathfrak{C}\mathfrak{A}-\mathfrak{G}^2, \quad \nabla C=\mathfrak{A}\mathfrak{B}-\mathfrak{H}^2, \\ &\nabla F=\mathfrak{G}\mathfrak{H}-\mathfrak{A}\mathfrak{F}, \quad \nabla G=\mathfrak{H}\mathfrak{F}-\mathfrak{B}\mathfrak{G}, \quad \nabla H=\mathfrak{F}\mathfrak{G}-\mathfrak{C}\mathfrak{H}, \\ &\qquad\qquad \mathfrak{A}+\mathfrak{B}+\mathfrak{C}=\mathfrak{S}, \end{aligned}\right\} \quad . \quad ($$

we may transform (3) into the following form:—

$$\left.\begin{aligned} &(AS-\mathfrak{B}-\mathfrak{C})p^2+2(FS+\mathfrak{F})qr \\ &+(BS-\mathfrak{C}-\mathfrak{A})q^2+2(GS+\mathfrak{G})rp \\ &+(CS-\mathfrak{A}-\mathfrak{B})r^2+2(HS+\mathfrak{H})pq=k^2, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

which in virtue of (2) becomes

$$(\mathfrak{A}-\mathfrak{S})p^2+(\mathfrak{B}-\mathfrak{S})q^2+(\mathfrak{C}-\mathfrak{S})r^2+2(\mathfrak{F}qr+\mathfrak{G}rp+\mathfrak{H}pq)=k^2-Sh. \quad (7)$$

This form of the integral is very closely allied with the inverse or reciprocal form of the first integral (2), and is the one used below.

In order to find the third integral, we must find two of the variables in terms of the third by means of (2) and (7), and substitute in the corresponding equation of motion. The most elegant method of effecting this is to transform (2) and (7) simultaneously into their canonical forms. If

$$\begin{matrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

be the coefficients of transformation, and if $\square$ be the determinant formed by them, the terms involving the products of the variables will be destroyed by the conditions

$$\left.\begin{array}{l} (A \quad \ldots \quad F \ldots )(\beta\ \beta_1\beta_2)(\gamma\ \gamma_1\gamma_2)=0, \\ (A \quad \ldots \quad F \ldots )(\gamma\ \gamma_1\gamma_2)(\alpha\ \alpha_1\alpha_2)=0, \\ (A \quad \ldots \quad F \ldots )(\alpha\ \alpha_1\alpha_2)(\beta\ \beta_1\beta_2)=0, \\ (\mathfrak{A}-\mathfrak{S} \ldots \mathfrak{F} \ldots )(\beta\beta_1\ \beta_2)(\gamma\ \gamma_1\gamma_2)=0, \\ (\mathfrak{A}-\mathfrak{S} \ldots \mathfrak{F} \ldots )(\gamma\ \gamma_1\gamma_2)(\alpha\ \alpha_1\alpha_2)=0, \\ (\mathfrak{A}-\mathfrak{S} \ldots \mathfrak{F} \ldots )(\alpha\ \alpha_1\ \alpha_2)(\beta\ \beta_1\beta_2)=0, \end{array}\right\} \quad . \quad . \quad . \quad (8)$$

from the last two of which we have

$$\left.\begin{array}{l} \beta_1\gamma_2-\beta_2\gamma : \beta_2\gamma-\beta\gamma_2 : \beta\gamma_1+\beta_1\gamma \\ =A\alpha+H\alpha_1+G\alpha_2=(\mathfrak{A}-\mathfrak{S})\alpha+\mathfrak{H}\alpha_1+\mathfrak{G}\alpha_2 \\ : H\alpha+B\alpha_1+F\alpha_2 : \mathfrak{H}\alpha+(\mathfrak{B}-\mathfrak{S})\alpha_1+\mathfrak{F}\alpha_2 \\ : G\alpha+F\alpha_1+C\alpha_2 : \mathfrak{G}\alpha+\mathfrak{F}\alpha_1+(\mathfrak{C}-\mathfrak{S})\alpha_2 ; \end{array}\right\} \quad . \quad . \quad (9)$$

whence, $\theta$ being a quantity to be determined,

$$\left.\begin{vmatrix} \mathfrak{A}-\mathfrak{S}-A\theta, & \mathfrak{H}\quad -H\theta, & \mathfrak{G}\quad -G\theta \\ \mathfrak{H}\quad -H\theta, & \mathfrak{B}-\mathfrak{S}-B\theta, & \mathfrak{F}\quad -F\theta \\ \mathfrak{G}\quad -G\theta, & \mathfrak{F}\quad -F\theta, & \mathfrak{C}-\mathfrak{S}-C\theta \end{vmatrix}=0.\right\} \quad . \quad (10)$$

Proceeding to develope this expression, we have the term independent of $\theta$

$$\begin{aligned} &=\nabla^2-(\mathfrak{B}\mathfrak{C}+\mathfrak{C}\mathfrak{A}+\mathfrak{A}\mathfrak{B})\mathfrak{S}+\mathfrak{S}^3-\mathfrak{S}^3 \\ &\qquad -(\mathfrak{F}^2+\mathfrak{G}^2+\mathfrak{H}^2)\mathfrak{S} \\ &=\nabla^2-S\mathfrak{S}\nabla. \end{aligned}$$

The coefficient of $-\theta$

$$\begin{aligned} &=A\{\nabla A-(\mathfrak{B}+\mathfrak{C})\mathfrak{S}+\mathfrak{S}^2\}+H(\nabla H+\mathfrak{H}\mathfrak{S})+G(\nabla G+\mathfrak{G}\mathfrak{S}) \\ &\quad + \ldots . \\ &=\nabla(A^2+H^2+G^2)+\nabla\mathfrak{S} \\ &+\nabla(H^2+B^2+C^2)+\nabla\mathfrak{S} \\ &+\nabla(G^2+F^2+C^2)+\nabla\mathfrak{S} \\ &=\nabla\{A^2+B^2+C^2+3(BC+CA+AB)-F^2-G^2-H^2\} \\ &=\nabla(S^2+\mathfrak{S}). \end{aligned}$$

The coefficient of $-\theta^3$

$$=\nabla.$$

Hence (dividing throughout by $\nabla$) (10) becomes

$$\theta^3+2.\mathrm{S}\theta^2+(\mathrm{S}^2+\mathfrak{S})\theta+\mathrm{S}\mathfrak{S}-\nabla=0;$$

or, what is the same thing,

$$(\theta+\mathrm{S})^3-\mathrm{S}(\theta+\mathrm{S})^2+\mathfrak{S}(\theta+\mathrm{S})-\nabla=0; \quad . \quad . \quad (11)$$

or, as it may also be written,

$$\begin{vmatrix} \mathrm{A}-(\theta+\mathrm{S}), & \mathrm{H}, & \mathrm{G} \\ \mathrm{H}, & \mathrm{B}-(\theta+\mathrm{S}), & \mathrm{F} \\ \mathrm{G}, & \mathrm{F}, & \mathrm{C}-(\theta+\mathrm{S}) \end{vmatrix}=0.$$

It will be seen by reference to (9) that the values of $\theta$ determined by this equation are equal to the ratios of the coefficients of the squares of the new variables respectively in the equivalents of (2) and (7). The coefficients of transformation are nine in number ; if therefore to the six equations of condition (8) we add three more, the system will be determinate.

Let three new conditions be

$$\left.\begin{aligned} &(\mathrm{A}\ldots\mathrm{F}\ldots\between\alpha\,\alpha_1\alpha_2)^2=1, \\ &(\mathrm{A}\ldots\mathrm{F}\ldots\between\beta\,\beta_1\beta_2)^2=1, \\ &(\mathrm{A}\ldots\mathrm{F}\ldots\between\gamma\,\gamma_1\gamma_2)^2=1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

then the variable terms of (2) will take the form of the sum of three squares, and the roots of (11) will be the coefficients of the transformed expression for (7). Or, if $\theta$, $\theta_1$, $\theta_2$ be the roots of (11), (2) and (7) take the forms

$$\left.\begin{aligned} &p_1^2+q_1^2+r_1^2=h, \\ &\theta p_1^2+\theta_1 q_1^2+\theta_2 r_1^2=k^2-\mathrm{S}h. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In order to determine the values of the coefficients of transformation $\alpha$, $\alpha_1$, $\alpha_2$, we have from (9),

$$\left.\begin{aligned} &(\mathfrak{A}-\mathfrak{S}-\mathrm{A}\theta)\alpha+(\mathfrak{H}-\mathrm{H}\theta)\alpha+(\mathfrak{G}-\mathrm{G}\theta)\alpha_2=0, \\ &(\mathfrak{H}-\mathrm{H}\theta)\alpha+(\mathfrak{B}-\mathfrak{S}-\mathrm{B}\theta)\alpha+(\mathfrak{F}-\mathrm{F}\theta)\alpha_2=0, \\ &(\mathfrak{G}-\mathrm{G}\theta)\alpha+(\mathfrak{F}-\mathrm{F}\theta)\alpha+(\mathfrak{C}-\mathfrak{S}-\mathrm{C}\theta)\alpha_2=0; \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

from the last two of which

$$\begin{aligned} \alpha:\ &\mathfrak{B}\mathfrak{C}-(\mathfrak{B}+\mathfrak{G})\mathfrak{S}+\mathfrak{S}^2-(\mathfrak{B}\mathrm{C}+\mathfrak{C}\mathrm{B}-\overline{\mathrm{B}+\mathrm{C}}\mathfrak{S})+\mathrm{BC}\theta^2 \\ &-\mathfrak{F}^2 \qquad\qquad\qquad\qquad -2\mathfrak{F}\mathrm{F}\theta \qquad\qquad\qquad -\mathrm{F}^2\theta^2 \\ =\alpha:\ &\nabla\mathrm{A}+\mathfrak{A}\mathfrak{S}+(\overline{\mathrm{B}+\mathrm{C}}\mathfrak{A}+\mathrm{B}\mathfrak{B}+\mathrm{C}\mathfrak{C}+2\mathrm{F}\mathfrak{F})\theta+\mathfrak{A}\theta^2 \\ =\alpha:\ &\nabla\mathrm{A}+\mathfrak{A}\mathfrak{S}+(2\nabla-\mathrm{H}\mathfrak{H}-\mathrm{G}\mathfrak{G}-\mathrm{A}\mathfrak{A}+\mathrm{S}\mathfrak{A})\theta+\mathfrak{A}\theta^2 \\ =\alpha:\ &\nabla\mathrm{A}+\mathfrak{A}\mathfrak{S}+(\nabla+\mathrm{S}\mathfrak{A})+\mathfrak{A}\theta^2 \\ =\alpha:\ &\nabla(\mathrm{A}+\theta)+\mathfrak{A}(\mathfrak{S}+\mathrm{S}\theta+\theta^2); \end{aligned}$$

or, writing for brevity

$$\mathfrak{S}+\mathrm{S}\theta+\theta^2=\mathrm{T},$$

the expression becomes

$$
\begin{aligned}
&\alpha : \nabla(\mathrm{A}+\theta)+\mathrm{T}\mathfrak{A} \\
&=\alpha : \mathfrak{F}\mathfrak{G}-(\mathrm{F}\mathfrak{G}+\mathrm{G}\mathfrak{F})\theta+\mathrm{FG}\theta \\
&\quad -\mathfrak{C}\mathfrak{H}\mathfrak{F}+\mathfrak{H}\mathfrak{S}+(\mathrm{C}\mathfrak{H}+\mathrm{H}\mathfrak{S})\theta-\mathrm{CH}\theta^2 \\
&=\alpha : \nabla\mathrm{H}+\mathfrak{H}\mathfrak{S}+\mathrm{S}\mathfrak{H}\theta+\mathfrak{H}\theta^2 \\
&=\alpha_1 : \nabla\mathrm{H}+\mathrm{T}\mathfrak{H} \\
&=\alpha_2 : \nabla\mathrm{G}+\mathrm{T}\mathfrak{G},
\end{aligned}
$$

whence the system

$$
\left.\begin{aligned}
&\alpha : \alpha_1 : \alpha_2 \\
&=\nabla(\mathrm{A}+\theta)+\mathrm{T}\mathfrak{A} = \nabla\mathrm{H}+\mathrm{T}\mathfrak{H} = \nabla\mathrm{G}+\mathrm{T}\mathfrak{G} \\
&: \nabla\mathrm{H}+\mathrm{T}\mathfrak{H} : \nabla(\mathrm{B}+\theta)+\mathrm{T}\mathfrak{B} : \nabla\mathrm{F}+\mathrm{T}\mathfrak{F} \\
&: \nabla\mathrm{G}+\mathrm{T}\mathfrak{G} : \nabla\mathrm{F}+\mathrm{T}\mathfrak{F} : \nabla(\mathrm{C}+\theta)+\mathrm{T}\mathfrak{C},
\end{aligned}\right\} \quad . \quad (15)
$$

with similar expressions for $\beta$, $\beta_1$, $\beta_2$; $\gamma$, $\gamma_1$, $\gamma_2$, obtained by writing $\theta_1$, $\mathrm{T}_1$; $\theta_2$, $\mathrm{T}_2$ respectively for $\theta$, T.

Returning to the equations of motion (1), and transforming by the formulæ

$$
\left.\begin{aligned}
p&=\alpha\, p_1+\beta\, q_1+\gamma\, r_1, \\
q&=\alpha_1 p_1+\beta_1 q_1+\gamma_1 r_1, \\
r&=\alpha_2 p_1+\beta_2 q_1+\gamma_2 r_1,
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)
$$

we have

$$
\left.\begin{aligned}
&(\mathrm{A}\alpha+\mathrm{H}\alpha_1+\mathrm{G}\alpha_2)p'_1=[-\mathrm{F}(\alpha_1^2-\alpha_2^2)+(\mathrm{B}-\mathrm{C})\alpha_1\alpha_2+\mathrm{H}\alpha_2\alpha-\mathrm{G}\alpha\alpha_1]p_1^2 \\
&+(\mathrm{A}\beta+\mathrm{H}\beta_1+\mathrm{G}\beta_2)q'_1+[-\mathrm{F}(\beta_1^2-\beta_2^2)+(\mathrm{B}-\mathrm{C})\beta_1\beta_2+\mathrm{H}\beta_2\beta-\mathrm{G}\beta\beta_1]q_1^2 \\
&+(\mathrm{A}\gamma+\mathrm{A}\gamma_1+\mathrm{G}\gamma_2)r'_1+[-\mathrm{F}(\gamma_1^2-\gamma_2^2)+(\mathrm{B}-\mathrm{C})\gamma_1\gamma_2+\mathrm{H}\gamma_2\gamma-\mathrm{G}\gamma\gamma_1]r_1^2 \\
&\qquad +[-2\mathrm{F}(\beta_1\gamma_1-\beta_2\gamma_2)+(\mathrm{B}-\mathrm{C})(\beta_1\gamma_2+\beta_2\gamma_1) \\
&\qquad\quad +\mathrm{H}(\beta_2\gamma+\beta\gamma_2)-\mathrm{G}(\beta\gamma_1+\beta_1\gamma)]q_1r_1 \\
&\qquad +[-2\mathrm{F}(\gamma_1\alpha_1-\gamma_2\alpha_2)+(\mathrm{B}-\mathrm{C})(\gamma_1\alpha_2+\gamma_2\alpha_1) \\
&\qquad\quad +\mathrm{H}(\gamma_2\alpha+\gamma\alpha_2)-\mathrm{G}(\gamma\alpha_1+\gamma_1\alpha)]r_1p_1 \\
&\qquad +[-2\mathrm{F}(\alpha_1\beta_1-\alpha_2\beta_2)+(\mathrm{B}-\mathrm{C})(\alpha_1\beta_2+\alpha_2\beta_1) \\
&\qquad\quad +\mathrm{H}(\alpha_2\beta+\alpha\beta_2)-\mathrm{G}(\alpha\beta_1+\alpha_1\beta)]p_1q_1 \\
&\qquad =[\alpha_2(\mathrm{H}\alpha+\mathrm{B}\alpha_1+\mathrm{F}\alpha_2)-\alpha_1(\mathrm{G}\alpha+\mathrm{F}\alpha_1+\mathrm{C}\alpha_2)]p_1^2 \\
&\qquad +[\beta_2(\mathrm{H}\beta+\mathrm{B}\beta_1+\mathrm{F}\beta_2)-\beta_1(\mathrm{G}\beta+\mathrm{F}\beta_1+\mathrm{C}\beta_2)]q_1^2 \\
&\qquad +[\gamma_2(\mathrm{H}\gamma+\mathrm{B}\gamma+\mathrm{F}\gamma_2)-\gamma_1(\mathrm{G}\gamma+\mathrm{F}\gamma_1+\mathrm{C}\gamma_2)]r_1^2 \\
&\qquad +[\beta_2(\mathrm{H}\gamma+\mathrm{B}\gamma_1+\mathrm{F}\gamma_2)-\beta_1(\mathrm{G}\gamma+\mathrm{F}\gamma+\mathrm{C}\gamma_2) \\
&\qquad\quad +\gamma_2(\mathrm{H}\beta+\mathrm{B}\beta_1+\mathrm{F}\beta_2)-\gamma_1(\mathrm{G}\beta+\mathrm{F}\beta+\mathrm{C}\beta_2)]q_1r_1 \\
&\qquad +[\gamma_2(\mathrm{H}\alpha+\mathrm{B}\alpha_1+\mathrm{F}\alpha_2)-\gamma_1(\mathrm{G}\alpha+\mathrm{F}\alpha_1+\mathrm{C}\alpha_2) \\
&\qquad\quad +\alpha_2(\mathrm{H}\gamma+\mathrm{B}\gamma_1+\mathrm{F}\gamma_2)-\alpha_1(\mathrm{G}\gamma+\mathrm{F}\gamma_1+\mathrm{C}\gamma_2)]r_1p_1 \\
&\qquad +[\alpha_2(\mathrm{H}\beta+\mathrm{B}\beta_1+\mathrm{F}\beta_2)-\alpha_1(\mathrm{G}\beta+\mathrm{F}\beta_1+\mathrm{C}\beta_2) \\
&\qquad\quad +\beta_2(\mathrm{H}\alpha+\mathrm{B}\alpha_1+\mathrm{F}\alpha_2)-\beta_1(\mathrm{G}\alpha+\mathrm{F}\alpha_1+\mathrm{C}\alpha_2)]p_1q_1,
\end{aligned}\right\} (17)
$$

with similar expressions for the two other equations. Multiplying the system so formed by $\gamma$, $\gamma_1$, $\gamma_2$ respectively and adding, the coefficients of $p'_1$, $q'_1$ will vanish, and that of $r'_1$ will $=1$ in virtue of (12); and as regards the right-hand side of the equation, the coefficient of $p_1^2$

$$=\begin{vmatrix} A\alpha+H\alpha_1+G\alpha_2, & \alpha, & \gamma \\ H\alpha+B\alpha_1+F\alpha_2, & \alpha_1, & \gamma_1 \\ G\alpha+F\alpha_2+C\alpha_2, & \alpha_2, & \gamma_2, \end{vmatrix}$$

which, omitting common factors,

$$\begin{vmatrix} (S+\theta)A+\mathfrak{A}+(S+\theta)\theta, & \nabla A+T\mathfrak{A}+\nabla\theta, & \nabla A+T_2\mathfrak{A}+\nabla\theta_2 \\ (S+\theta)H+\mathfrak{H}, & \nabla H+T\mathfrak{H}, & \nabla H+T_2\mathfrak{H} \\ (S+\theta)G+\mathfrak{G}, & \nabla G+T\mathfrak{G} & \nabla G+T_2\mathfrak{G} \end{vmatrix}$$

$$=\{(S+\theta)\theta\begin{vmatrix} \nabla H+T\mathfrak{H} & \nabla H+T_2\mathfrak{H} \\ \nabla G+T\mathfrak{G} & \nabla G+T_2\mathfrak{G} \end{vmatrix}+\nabla\theta\begin{vmatrix} \nabla H+T_2\mathfrak{H} & (S+\theta)H+\mathfrak{H} \\ \nabla G+T_2\mathfrak{G} & (S+\theta)G+\mathfrak{G} \end{vmatrix}$$

$$+\nabla\theta_2\begin{vmatrix} (S+\theta)H+\mathfrak{H} & \nabla H+T\mathfrak{H} \\ (S+\theta)G+\mathfrak{G} & \nabla G+T\mathfrak{G} \end{vmatrix}$$

$$=\{(S+\theta)\theta\nabla(T_2-T)+\nabla\theta\big(\nabla-T_2(S+\theta)\big)+\nabla\theta_2\big(T(S+\theta)-\nabla\big)\}(H\mathfrak{G}-\mathfrak{F}G)$$

$$=\nabla(\theta_2-\theta)\{T(S+)\theta-\nabla\}(H\mathfrak{G}-\mathfrak{H}G).$$

But

$$T(S+\theta)-\nabla=(S+\theta)(\theta^2+S\theta+\mathfrak{S})-\nabla=(S+\theta)\{(S+\theta)^2-(S+\theta)+\mathfrak{S}\}-\nabla$$
$$=(S+\theta)^3-S(S+\theta)^2+\mathfrak{S}(S+\theta)-\nabla$$
$$=0.$$

Hence, finally, the coefficient of $p_1^2$ vanishes.

So likewise the coefficient of $q_1^2$

$$=\begin{vmatrix} A\beta+H\beta_1+G\beta_2 & \beta & \gamma \\ H\beta+B\beta_1+F\beta_2 & \beta_1 & \gamma_1 \\ G\beta+F\beta_1+C\beta_2 & \beta_2 & \gamma \end{vmatrix}=0.$$

And that of $r_1^2$,

$$\begin{vmatrix} A\gamma+H\gamma_1+G\gamma_2 & \gamma & \gamma \\ H\gamma+B\gamma_1+F\gamma_2 & \gamma_1 & \gamma_1 \\ G\gamma+F\gamma_1+C\gamma_2 & \gamma_2 & \gamma_2 \end{vmatrix}=0.$$

Similarly the coefficients of $q_1 r_1$, and $r_1 p_1$ will be found to vanish; and lastly, the coefficient of $p_1 q_1$

$$=\alpha\{A(\beta_1\gamma_2-\beta_2\gamma_1)+H(\beta_2\gamma-\beta\gamma_2)+G(\beta\gamma_1-\beta_1\gamma)\}$$
$$+\alpha_1\{H(\beta_1\gamma_2-\beta_2\gamma_1)+B(\beta_2\gamma-\beta\gamma_2)+F(\beta\gamma_1-\beta_1\gamma)\}$$
$$+\alpha_2\{G(\beta_1\gamma_2-\beta_2\gamma_1)+F(\beta_2\gamma-\beta\gamma_2)+C(\beta\gamma_1-\beta_1\gamma)\}$$
$$-\beta\{A(\gamma_1\alpha_2-\gamma_2\alpha_1)+H(\gamma_2\alpha-\gamma\alpha_2)+G(\gamma\alpha_1-\gamma_1\alpha)\}$$
$$-\beta_1\{H(\gamma_1\alpha_2-\gamma_2\alpha_1)+B(\gamma_2\alpha-\gamma\alpha_2)+F(\gamma\alpha_1-\gamma_1\alpha)\}$$
$$-\beta_2\{G(\gamma_1\alpha_2-\gamma_2\alpha_1)+F(\gamma_2\alpha-\gamma\alpha_2)+C(\gamma\alpha_1-\gamma_1\alpha)\},$$

which, by reference to (9), may be transformed into

$$\square\ \{(A\alpha+H\alpha_1+G\alpha_2)^2+(H\alpha+B\alpha_1+F\alpha_2)^2+(G\alpha+F\alpha_1+C\alpha_2)^2$$
$$-(A\beta+H\beta_1+G\beta_2)^2+(H\beta+B\beta_1+F\beta_2)^2+(G\beta+F\beta_1+C\beta_2)^2\}$$

$$=\square\ \{(A\alpha^2+B\alpha_1^2+C\alpha_2^2+2F\alpha_1\alpha_2+2G\alpha_2\alpha+2H\alpha\alpha_1)S$$
$$-(A\beta^2+B\beta_1^2+C\beta_2^2+2F\beta_1\beta_2+2G\beta_2\beta+2H\beta\beta_1)S$$
$$+(\mathfrak{A}-\mathfrak{S})(\alpha^2-\beta^2)+(\mathfrak{B}-\mathfrak{S})(\alpha_1^2-\beta_1^2)+(\mathfrak{C}-\mathfrak{S})(\alpha_2^2-\beta_2^2)$$
$$+2\mathfrak{F}(\alpha_1\alpha_2-\beta_1\beta_2)+2\mathfrak{G}(\alpha_2\alpha-\beta_2\beta)+2\mathfrak{H}(\alpha\alpha_1-\beta\beta_1)\};$$

in which the coefficient of S vanishes in virtue of (12); so that the coefficient of $p_1$, $q_1$

$$=\square\{(\mathfrak{A}-\mathfrak{S},\ \mathfrak{B}-\mathfrak{S},\ \mathfrak{C}-\mathfrak{S},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\alpha,\ \alpha_1,\ \alpha_2)^2$$
$$-(\mathfrak{A}-\mathfrak{S},\ \mathfrak{B}-\mathfrak{S},\ \mathfrak{C}-\mathfrak{S},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\beta,\ \beta_1,\ \beta_2)^2\};$$

but, by (12),

$$(\mathfrak{A}-\mathfrak{S},\ \mathfrak{B}-\mathfrak{S},\ \mathfrak{C}-\mathfrak{S},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\alpha\alpha_1\alpha_2)^2=\theta,$$
$$(\mathfrak{A}-\mathfrak{S},\ \mathfrak{B}-\mathfrak{S},\ \mathfrak{C}-\mathfrak{S},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H}\between\beta\beta_1\beta_2)^2=\theta_1.$$

Hence the coefficient in question

$$=\square\,(\theta-\theta_1), \quad \ldots\ldots\ldots \quad (18)$$

and the equations of motion become

$$\left.\begin{aligned} p_1'&=\square\,(\theta_1-\theta_2)q_1r_1,\\ q_1'&=\square\,(\theta_2-\theta)r_1p_1,\\ r_1'&=\square\,(\theta-\theta_1)p_1q_1.\end{aligned}\right\} \quad \ldots\ldots \quad (19)$$

To find the value of $\square$ in terms of A, B, C, F, G, H, we have from (12)

$$\begin{aligned} A\alpha+H\alpha_1+G\alpha_2&=\square^{-1}(\beta_1\gamma_2-\beta_2\gamma_1),\\ A\beta+H\beta_1+G\beta_2&=\square^{-1}(\gamma_1\alpha_2-\gamma_2\alpha_1),\\ A\gamma+H\gamma_1+G\gamma_2&=\square^{-1}(\alpha_1\beta_2-\alpha_2\beta_1),\\ G\alpha+B\,\alpha_1+F\alpha_2&=\square^{-1}(\beta_2\gamma-\beta\,\gamma_2),\\ H\beta+B\beta_1+F\beta_2&=\square^{-1}(\gamma_2\alpha-\gamma\,\alpha_2),\\ H\gamma+B\gamma_1+F\gamma_2&=\square^{-1}(\alpha_2\beta-\alpha\beta_2),\\ H\alpha+F\alpha_1+C\alpha_2&=\square^{-1}(\beta\,\gamma_1-\beta_1\gamma),\\ G\beta+F\beta_1+C\beta_2&=\square^{-1}(\gamma\alpha_1-\gamma_1\alpha),\\ G\gamma+F\gamma_1+C\gamma_2&=\square^{-1}(\alpha\beta_1-\alpha_1\beta).\end{aligned}$$

And forming the determinant of each side of this system, there results

$$\nabla\square=\square^{-3}\square^2,$$

or

$$\nabla\ =\square^{-2}; \quad \ldots\ldots\ldots \quad (20)$$

whence the equations of motion (19) become

$$\left.\begin{aligned} p'_1 &= \nabla^{-\frac{1}{2}}(\theta_1-\theta_2)q_1\,r_1, \\ q'_1 &= \nabla^{-\frac{1}{2}}(\theta_2-\theta)\,r_1 p_1, \\ r'_1 &= \nabla^{-\frac{1}{2}}(\theta-\theta_1)\,p_1\,q_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

In order to compare these results with the ordinary known form, we must make

$$\mathrm{F}=0, \qquad \mathrm{G}=0, \qquad \mathrm{H}=0,$$

$$p_1=\mathrm{A}^{\frac{1}{2}}p, \quad q_1=\mathrm{B}^{\frac{1}{2}}q, \quad r_1=\mathrm{C}^{\frac{1}{2}}r;$$

which values reduce (13) to the following:

$$(\mathrm{A}^{\frac{1}{2}}p)^2+(\mathrm{B}^{\frac{1}{2}}q)^2+(\mathrm{C}^{\frac{1}{2}}r)^2=h,$$

$$-(\mathrm{B}+\mathrm{C})\mathrm{A}p^2-(\mathrm{C}+\mathrm{A})\mathrm{B}q^2-(\mathrm{A}+\mathrm{B})\mathrm{C}r^2=k^2-\mathrm{S}h;$$

which last is equivalent to

$$(\mathrm{A}-\mathrm{S})(\mathrm{A}^{\frac{1}{2}}p)+(\mathrm{B}-\mathrm{S}(\mathrm{B}^{\frac{1}{2}}q)^2+(\mathrm{C}-\mathrm{S})(\mathrm{C}^{\frac{1}{2}}r)^2=\underline{k-\mathrm{S}h},$$

or

$$\mathrm{A}(\mathrm{A}^{\frac{1}{2}}p)^2+\mathrm{B}(\mathrm{B}^{\frac{1}{2}}q)^2+\mathrm{C}(\mathrm{C}^{\frac{1}{2}}r)^2=k^2.$$

Also, on the same supposition,

$$\nabla=\mathrm{ABC}, \quad \theta=-(\mathrm{B}+\mathrm{C}), \quad \theta_1=-(\mathrm{C}+\mathrm{A}), \quad \theta_2=-(\mathrm{A}+\mathrm{B}),$$

which, when substituted in the above, give

$$\mathrm{A}p^{\frac{1}{2}\prime}=(\mathrm{ABC})^{-\frac{1}{2}}(\mathrm{B}-\mathrm{C})\mathrm{B}^{\frac{1}{2}}\mathrm{C}^{\frac{1}{2}}qr, \quad \mathrm{B}^{\frac{1}{2}}q'=\ldots, \quad \mathrm{C}^{\frac{1}{2}}r'=\ldots,$$

or

$$\mathrm{A}p'=(\mathrm{B}-\mathrm{C})qr, \quad \mathrm{B}q'=(\mathrm{C}-\mathrm{A})rp, \quad \mathrm{C}r'=(\mathrm{A}-\mathrm{B})pq,$$

as usual.

It remains only to determine the absolute values of the coefficients of transformation, the ratios of which are given in (15). For this purpose let

$$\left.\begin{aligned} \nabla(\mathrm{A}+\theta_0)+\mathrm{T}_0\mathfrak{A}&=\mathfrak{A}_0, & \nabla\mathrm{F}+\mathrm{T}_0\mathfrak{F}&=\mathfrak{F}_0, \\ \nabla(\mathrm{B}+\theta_0)+\mathrm{T}_0\mathfrak{B}&=\mathfrak{B}_0, & \nabla\mathrm{G}+\mathrm{T}_0\mathfrak{G}&=\mathfrak{G}_0, \\ \nabla(\mathrm{C}+\theta_0)+\mathrm{T}_0\mathfrak{C}&=\mathfrak{C}_0, & \nabla\mathrm{H}+\mathrm{T}_0\mathfrak{H}&=\mathfrak{H}_0. \end{aligned}\right\} \quad . \quad . \quad . \quad (22)$$

Then, from (15),

$$\alpha=\frac{\mathfrak{A}_0}{(\mathrm{A}\ldots\mathrm{H}\ldots\between\mathfrak{A}_0\mathfrak{H}_0\mathfrak{G}_0)^2}=\frac{\mathfrak{H}_0}{(\mathrm{A}\ldots\between\mathfrak{H}_0\mathfrak{B}_0\mathfrak{F}_0)^2}=\frac{\mathfrak{G}_0}{(\mathrm{A}\ldots\between\mathfrak{G}_0\mathfrak{F}_0\mathfrak{C}_0)^2},$$

$$\alpha_1=\frac{\mathfrak{H}_0}{(\mathrm{A}\ldots\mathrm{H}\ldots\between\mathfrak{A}_0\mathfrak{H}_0\mathfrak{G}_0)^2}=\frac{\mathfrak{B}_0}{(\mathrm{A}\ldots\between\mathfrak{H}_0\mathfrak{B}_0\mathfrak{F}_0)^2}=\frac{\mathfrak{F}_0}{\mathrm{A}\ldots\between\mathfrak{G}_0\mathfrak{F}_0\mathfrak{C}_0)^2},$$

$$\alpha_2=\frac{\mathfrak{G}_0}{(\mathrm{A}\ldots\mathrm{H}\ldots\between\mathfrak{A}_0\mathfrak{H}_0\mathfrak{G}_0)^2}=\frac{\mathfrak{F}_0}{(\mathrm{A}\ldots\between\mathfrak{H}_0\mathfrak{B}_0\mathrm{F}_0)^2}=\frac{\mathfrak{C}_0}{(\mathrm{A}\ldots\between\mathfrak{G}_0\mathfrak{H}_0\mathfrak{C}_0)^2}.$$

From these relations it follows that

$$\left.\begin{array}{ll}\mathfrak{B}_0\mathfrak{C}_0-\mathfrak{F}_0^2=0, & \mathfrak{G}_0\mathfrak{H}_0-\mathfrak{A}_0\mathfrak{F}_0=0,\\ \mathfrak{C}_0\mathfrak{A}_0-\mathfrak{G}_0^2=0, & \mathfrak{H}_0\mathfrak{F}_0-\mathfrak{B}_0\mathfrak{G}_0=0,\\ \mathfrak{A}_0\mathfrak{B}_0-\mathfrak{H}_0^2=0, & \mathfrak{F}_0\mathfrak{G}_0-\mathfrak{C}_0\mathfrak{H}_0=0,\end{array}\right\} \quad . \quad . \quad . \quad . \quad (23)$$

which relations may be also verified as follows:—

$$\mathfrak{G}_0\mathfrak{H}_0-\mathfrak{A}_0\mathfrak{F}_0=(\nabla G+T_0\mathfrak{G})(\nabla H+T_0\mathfrak{H})-(\nabla A+T_0\mathfrak{A}+\nabla\theta_0)(\nabla F+T_0\mathfrak{F})$$

$$=\nabla^2\mathfrak{F}+\nabla T_0(G\mathfrak{H}+HG-A\mathfrak{F}-F\mathfrak{A})+T_0^2\nabla F-\nabla\theta_0(\nabla F+T_0\mathfrak{F})$$

$$\nabla\{\nabla\mathfrak{F}-T_0(S\mathfrak{F}+\mathfrak{S}F)+T_0^2F-\nabla\theta_0F-\nabla\mathfrak{F}+ST_0\mathfrak{F}\mathfrak{G}\};$$

Since

$$G\mathfrak{H}+F\mathfrak{B}+C\mathfrak{F}=0,$$

$$H\mathfrak{G}+B\mathfrak{F}+F\mathfrak{C}=0,$$

and

$$(\theta+S)T-\nabla=0,$$

or

$$\theta T=\nabla-ST.$$

Hence

$$\begin{aligned}\mathfrak{G}_0\mathfrak{H}_0-\mathfrak{A}_0\mathfrak{F}_0&=\nabla F\{T_0^2-T_0\mathfrak{S}-\nabla\theta_0\}\\&=\nabla F\{T\theta_0(S+\theta_0)-\nabla\theta_0\}\\&=0.\end{aligned}$$

From these relations it follows that the first denominator, viz.

$$\begin{aligned}&(A,\ B,\ C,\ F,\ G,\ H,\between\mathfrak{A}_0\mathfrak{H}_0\mathfrak{G}_0)^2\\&=A\mathfrak{A}_0^2+B\mathfrak{H}_0^2+C\mathfrak{G}_0^2+r(F\mathfrak{H}_0\mathfrak{G}_0+G\mathfrak{G}_0\mathfrak{A}_0+H\mathfrak{A}_0\mathfrak{H}_0)\\&=\mathfrak{A}_0\{A\mathfrak{A}_0+B\mathfrak{B}_0+C\mathfrak{C}_0+2(F\mathfrak{F}_0+G\mathfrak{G}_0+H\mathfrak{H}_0\}\\&=\mathfrak{A}_0\nabla\{A^2+B^2+C^2+2(F^2+G^2+H^2)+3T_0+S\theta_0\}\\&=\mathfrak{A}_0\nabla\{S^2-2\mathfrak{S}+3T_0+S\theta_0\}\\&=\mathfrak{A}_0\nabla\ \ 3\theta_0^2+4S\theta_0+\mathfrak{S}+S^2\}\\&=\mathfrak{A}_0\nabla\{(S+\theta_0)(S+3\theta_0)+\mathfrak{S}\}.\end{aligned}$$

Hence, writing $(S+\theta_0)(S+3\theta_0)+S=\mathfrak{T}_0$, we have, finally,

$$\alpha=\frac{1}{\mathfrak{T}_0},\ \alpha_1=\frac{\mathfrak{H}_0}{\mathfrak{A}_0\mathfrak{T}_0},\ \alpha_2=\frac{\mathfrak{G}_0}{\mathfrak{A}_0\mathfrak{T}_0}.$$

From this we may obtain the following system:

$$\left.\begin{array}{lll}\alpha=\dfrac{1}{\mathfrak{T}_0}, & \alpha_1=\dfrac{\mathfrak{H}_0}{\mathfrak{A}_0\mathfrak{T}_0}, & \alpha_2=\dfrac{\mathfrak{G}_0}{\mathfrak{A}_0\mathfrak{T}_0}\\ & \quad=\dfrac{\mathfrak{B}_0}{\mathfrak{H}_0\mathfrak{T}_0} & \quad=\dfrac{\mathfrak{F}_0}{\mathfrak{H}_0\mathfrak{T}_0}\\ & \quad=\dfrac{\mathfrak{F}_0}{\mathfrak{G}_0\mathfrak{T}_0} & \quad=\dfrac{\mathfrak{C}_0}{\mathfrak{G}_0\mathfrak{T}_0},\end{array}\right\} \quad . \quad . \quad . \quad . \quad (24)$$

with similar expressions for $\beta, \beta_1, \beta_2$; $\gamma, \gamma_1, \gamma_2$, obtained by writing the suffixes 1 and 2 respectively for 0. By means of these we may write the equations connecting the variables as follow:—

$$\left.\begin{aligned} p&=\frac{1}{\mathfrak{C}_0}\,p_1+\frac{1}{\mathfrak{C}_1}\,q_1+\frac{1}{\mathfrak{C}_2}r_1, \\ q&=\frac{\mathfrak{H}_0}{\mathfrak{A}_0\mathfrak{C}_0}p_1+\frac{\mathfrak{B}_1}{\mathfrak{H}_1\mathfrak{C}_1}q_1+\frac{\mathfrak{F}_2}{\mathfrak{G}_2\mathfrak{C}_2}r_1, \\ r&=\frac{\mathfrak{G}_0}{\mathfrak{A}_0\mathfrak{C}_0}p_1+\frac{\mathfrak{F}_1}{\mathfrak{H}_0\mathfrak{C}_1}q_1+\frac{\mathfrak{C}_2}{\mathfrak{G}_2\mathfrak{C}_2}r_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

Lastly, to complete the transformations, the values of $p_1$, $q_1$, $r_1$ should be determined in terms of $p$, $q$, $r$. Now

$$\begin{aligned} \mathfrak{A}_0\mathfrak{H}_1+\mathfrak{H}_0\mathfrak{B}_1+\mathfrak{G}_0\mathfrak{F}_1&=(\nabla A+T_0\mathfrak{A}+\nabla\theta_0)(\nabla H+T_1\mathfrak{H}) \\ &\quad+(\nabla H+T_0\mathfrak{H})(\nabla B+T_1B+T\theta_1) \\ &\quad+(\nabla G+T_0\mathfrak{G})(\nabla F+T_1\mathfrak{H}) \\ &=\nabla^2\{(A+B)H+FG\}+T_0T_1\{(\mathfrak{A}+\mathfrak{B})\mathfrak{H}+\mathfrak{F}\mathfrak{G}\}+\nabla^2H(\theta_0+\theta_1) \\ &\quad+\nabla\mathfrak{H}(\theta_0T_1+\theta_1T_0) \\ &=\nabla^2(SH+\mathfrak{H})+T_0T_1(\mathfrak{S}\mathfrak{H}+\nabla H)+\nabla^2H(\theta_0+\theta_1)+\nabla\mathfrak{H}(\theta_0T_1+\theta_1T_0) \\ &=\nabla\{\nabla(S+\theta_0+\theta_1)+T_0T_1\}H+(\nabla\theta_0T_1+\nabla\theta_1T_0+\mathfrak{S}T_0T_1+\nabla^2) \\ &=T_0T_1\{[-(S+\theta_0)(S+\theta_1)(S+\theta_2)+\nabla]H+[\theta_0(S+\theta_0)+\theta_1(S+\theta_1) \\ &\quad+S+(S+\theta_0)(S+\theta_1)]\mathfrak{H}\}, \end{aligned}$$

since

$$\nabla=T_0(S+\theta_0)=T_1(S+\theta_1)=T_2(S+\theta_2).$$

Moreover by (11) we have

$$(S+\theta_0)(S+\theta_1)(S+\theta_2)=\nabla,$$

and consequently the coefficient of H vanishes. And it may be noticed, as a useful formula for verification, that, from the relations last above written, we may at once deduce the following:

$$T_0T_1T_2=\nabla^2.$$

Again, the coefficient of $\mathfrak{H}$ may be thus written:

$$\begin{aligned} &(S+\theta_0+\theta_2)(S+\theta_0)+(S+\theta_0+\theta_1)(S+\theta_1)+S+(S+\theta_0)(S+\theta_1) \\ &-\quad(S+\theta_2)(S+\theta_0)-\quad(S+\theta_0)(S+\theta_1) \\ &=-\quad(S+\theta_1)(S+\theta_0)-(S+\theta_2)(S+\theta_1)-(S+\theta_0)(S+\theta_2)+S \\ &=0, \end{aligned}$$

in virtue of (11). Hence the whole expression vanishes, or

$$\mathfrak{A}_0\mathfrak{H}_1+\mathfrak{H}_0\mathfrak{B}_1+\mathfrak{G}_0\mathfrak{F}_1=0\,; \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and similarly

$$\mathfrak{A}_0\mathfrak{G}_2+\mathfrak{H}_0\mathfrak{F}_2+\mathfrak{G}_0\mathfrak{C}_2=0.$$

Moreover, in virtue of (23), we have

$$\mathfrak{A}_0^{\,2}+\mathfrak{H}_0^{\,2}+\mathfrak{G}_0^{\,2}=\mathfrak{A}_0\mathfrak{S}_0.$$

Hence multiplying (25) first by $\mathfrak{A}_0$, $\mathfrak{H}_0$, $\mathfrak{G}_0$ respectively and adding,

secondly by $\mathfrak{H}_1$, $\mathfrak{B}_1$, $\mathfrak{F}_1$,

thirdly by $\mathfrak{G}_2$, $\mathfrak{F}_2$, $\mathfrak{C}_2$,

we shall obtain the inverse system

$$\left.\begin{aligned}\frac{\mathfrak{S}_0}{\mathfrak{C}_0}p_1&=\mathfrak{A}_0p+\mathfrak{H}_0q+\mathfrak{G}_0r,\\ \frac{\mathfrak{S}_1}{\mathfrak{C}_1}q_1&=\mathfrak{H}_1p+\mathfrak{B}_1q+\mathfrak{F}_1r,\\ \frac{\mathfrak{S}_2}{\mathfrak{C}_2}r_1&=\mathfrak{G}_2p+\mathfrak{F}_2q+\mathfrak{C}_2r.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Returning to the integrals (13), we derive

$$\begin{aligned}(\theta_1-\theta)q_1^{\,2}+(\theta_2-\theta)r_1^{\,2}&=k^2-(\mathrm{S}+\theta\,)h,\\ (\theta_2-\theta_1)r_1^{\,2}+(\theta\,-\theta_1)p_1^{\,2}&=k^2-(\mathrm{S}+\theta_1)h,\\ (\theta\,-\theta_2)p_1^{\,2}+(\theta_1-\theta_2)q_1^{\,2}&=k^2-(\mathrm{S}+\theta_2)h.\end{aligned}$$

Let

$$p_1=\sqrt{\frac{k^2-(\mathrm{S}+\theta_2)h}{\theta-\theta_2}}\cos\chi\,;$$

then

$$q_1=\sqrt{\frac{k^2-(\mathrm{S}+\theta_2)h}{\theta_1-\theta_2}}\sin\chi\,;$$

and

$$\begin{aligned}r_1&=\sqrt{\frac{k^2-(\mathrm{S}+\theta)h}{\theta_2-\theta}}\sqrt{1-\frac{\theta_1-\theta}{k^2-(\mathrm{S}+\theta)h}q_1^{\,2}}\\ &=\sqrt{\frac{k^2-(\mathrm{S}+\theta)h}{\theta_2-\theta}}\sqrt{1-\frac{\theta_1-\theta}{\theta_1-\theta_2}\,\frac{k^2-(\mathrm{S}+\theta_2)h}{k^2-(\mathrm{S}+\theta\,)h}\sin^2\chi}.\end{aligned}$$

Substituting in the equations of motion (21) (*e. g.* the first of them) and dividing throughout by $\sin\chi\sqrt{k^2-(\mathrm{S}+\theta_2)h}$, we have

$$\frac{1}{\sqrt{\theta-\theta_2}}\frac{d\chi}{dt}+\frac{1}{\sqrt{\theta_1-\theta_2}}\sqrt{\frac{k^2-(S+\theta)h}{\theta_2-\theta}}\sqrt{1-\frac{\theta_1-\theta}{\theta_1-\theta_2}\frac{k^2-(S+\theta)h}{k^2-(S+\theta)h}\sin^2\chi},$$

or

$$\frac{d\chi}{dt}+\nabla^{-\frac{1}{2}}(\theta_1-\theta_2)\sqrt{\frac{k^2-(S+\theta)h}{\theta_2-\theta_1}}\sqrt{1-\frac{\theta-\theta_1}{\theta_2-\theta_1}\frac{k^2-(S+\theta_2)h}{k^2-(S+\theta)h}\sin^2\chi},$$

or

$$\frac{d\chi}{\sqrt{1-\frac{\theta_1-\theta}{\theta_1-\theta_2}\frac{k^2-(S+\theta_2)h}{k^2-(S+\theta)h}\sin^2\chi}}=\sqrt{\frac{\theta_2-\theta_1}{\nabla}}\sqrt{k^2-(S+\theta)h}\,dt;$$

then

$$\chi=am\left(\sqrt{\frac{\theta_2-\theta_1}{\nabla}}\sqrt{k^2-(S+\theta)h}\,t+f\right),$$

and

$$p_1=\sqrt{\frac{k^2-(S+\theta_2)h}{\theta-\theta_2}}\cos am\left(\sqrt{\frac{\theta_2-\theta_1}{\nabla}}\sqrt{k^2-(S+\theta)h}\,t+f\right),$$

$$q_1=\sqrt{\frac{k^2-(S+\theta_2)h}{\theta_1-\theta_2}}\sin am\left(\sqrt{\frac{\theta_2-\theta_1}{\nabla}}\sqrt{k^2-(S+\theta)h}\,t+f\right),$$

$$r_1=\sqrt{\frac{k^2-(S+\theta)h}{\theta_2-\theta}}\Delta am\left(\sqrt{\frac{\theta_2-\theta_1}{\nabla}}\sqrt{k^2-(S+\theta)h}\,t+f\right).$$

These, then, are the integrals of the equations of motion when no external forces are acting. The next step is to determine the variations of the arbitrary constants, due to the action of disturbing forces, when, as in the case of nature, those forces are small. With a view to this, it will be con venient to change the arbitrary constants into the following,

$$\sqrt{k^2-(S+\theta_2)h}=m \quad \sqrt{k^2-(S+\theta)h}=n,$$

whence

$$(\theta-\theta_2)h=m^2-n^2,$$

$$(\theta-\theta_2)k^2=(S+\theta)m^2-(S+\theta_2)n^2;$$

also, for brevity, let

$$\sqrt{\frac{\theta_2-\theta_1}{\nabla}}=l, \quad am(lnt+f)=\chi, \quad \frac{\theta_1-\theta}{\theta_1-\theta_2}\frac{m}{n^2}=k_1^2.$$

Then the equations of motion become

$$p_1=\frac{m}{\sqrt{\theta-\theta_2}}\cos am(lnt+f),$$

$$q_1=\frac{m}{\sqrt{\theta_1-\theta_2}}\sin am(lnt+f),$$

$$r_1=\frac{n}{\sqrt{\theta_2-\theta}}\Delta am(lnt+f).$$

Now it is known by the theory of elliptic functions that

$$\frac{d\cos amx}{dx} = -\sin am\,x\Delta\,am\,x,$$

$$\frac{d\sin am\,x}{dx} = \cos am\,x\Delta\,am\,x,$$

$$\frac{d\Delta\,am\,x}{dx} = -k_1^2\sin am\,x\cos am\,x.$$

Whence $P_1$, $Q_1$, $R_1$ being the moments of the disturbing forces about the present axes,

$$P_1 = \frac{1}{\sqrt{\theta-\theta_2}}\left\{\cos\chi\frac{dm}{dt} - m\sin\chi\Delta\chi\left(lt\frac{dn}{dt}+\frac{df}{dt}\right)\right\},$$

$$Q_1 = \frac{1}{\sqrt{\theta_1-\theta_2}}\left\{\sin\chi\frac{dm}{dt} + m\cos\chi\Delta\chi\left(lt\frac{dn}{dt}+\frac{df}{dt}\right)\right\},$$

$$R_1 = \frac{1}{\sqrt{\theta_2-\theta}}\left\{\Delta\chi\frac{dn}{dt} nk_1^2\sin\chi\cos\chi\left(lt\frac{dn}{dt}+\frac{df}{dt}\right)\right\}.$$

From these we derive

$$\frac{dm}{dt} = \sqrt{\theta-\theta_2}P_1\cos\chi + \sqrt{\theta_1-\theta_2}\,Q_1\sin\chi,$$

$$m\Delta\chi\left(lt\frac{dn}{dt}+\frac{df}{dt}\right) = -\sqrt{\theta-\theta_2}P_1\sin\chi + \sqrt{\theta_1-\theta_2}\,Q_1\cos\chi,$$

$$\Delta\chi\frac{dn}{dt} = \sqrt{\theta_2-\theta}R_1 + \frac{n}{m}k_1^2\frac{\sin\chi\cos\chi}{\Delta\chi}\{-\sqrt{\theta-\theta_2}P_1\sin\chi + \sqrt{\theta_1-\theta_2}Q_1\cos\chi\},$$

or

$$\frac{dn}{dt} = \sqrt{\theta_2-\theta}\frac{R_1}{\Delta\chi} + \frac{\theta_1-\theta}{\theta_1-\theta_2}\frac{m}{n}\frac{\sin\chi\cos\chi}{(\Delta\chi)^2}\{-\sqrt{\theta-\theta_2}P_1\sin\chi + \sqrt{\theta_1-\theta_2}Q_1\cos\chi\}$$

$$= \sqrt{\theta_2-\theta}\frac{R_1}{\chi} + \frac{\theta_1-\theta}{\theta_1-\theta_2}\frac{1}{}\frac{\sin\chi\cos\chi}{(\Delta\chi)^2}\{-\sqrt{\theta-\theta_2}P_1\sin\chi + \sqrt{\theta_1-\theta_2}Q_1\cos\chi\}$$

$$\int\{\sqrt{\theta-\theta_2}P_1\cos\chi + \sqrt{\theta_1-\theta_2}Q_1\sin\chi\}dt.$$

And lastly,

$$\frac{df}{dt} = -lt\frac{dn}{dt} - \frac{1}{\Delta\chi}\frac{-\sqrt{\theta-\theta_2}P_1\sin\chi + \sqrt{\theta_1-\theta_2}Q_1\cos\chi}{\int\{\sqrt{\theta-\theta_2}P_1\cos\chi + \sqrt{\theta_1-\theta_2}Q_1\sin\chi\}dt}.$$