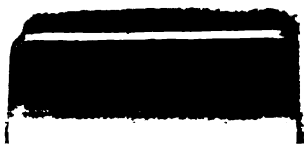


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ELEMENTS

OF

QUATERNIONS.

BY THE LATE

SIR WILLIAM ROWAN HAMILTON, LL. D., M. R. I. A.,

D. C. L. CANTAB. :

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EDITED BY HIS SON,

WILLIAM EDWIN HAMILTON, A. B. T. C. D., C. E.



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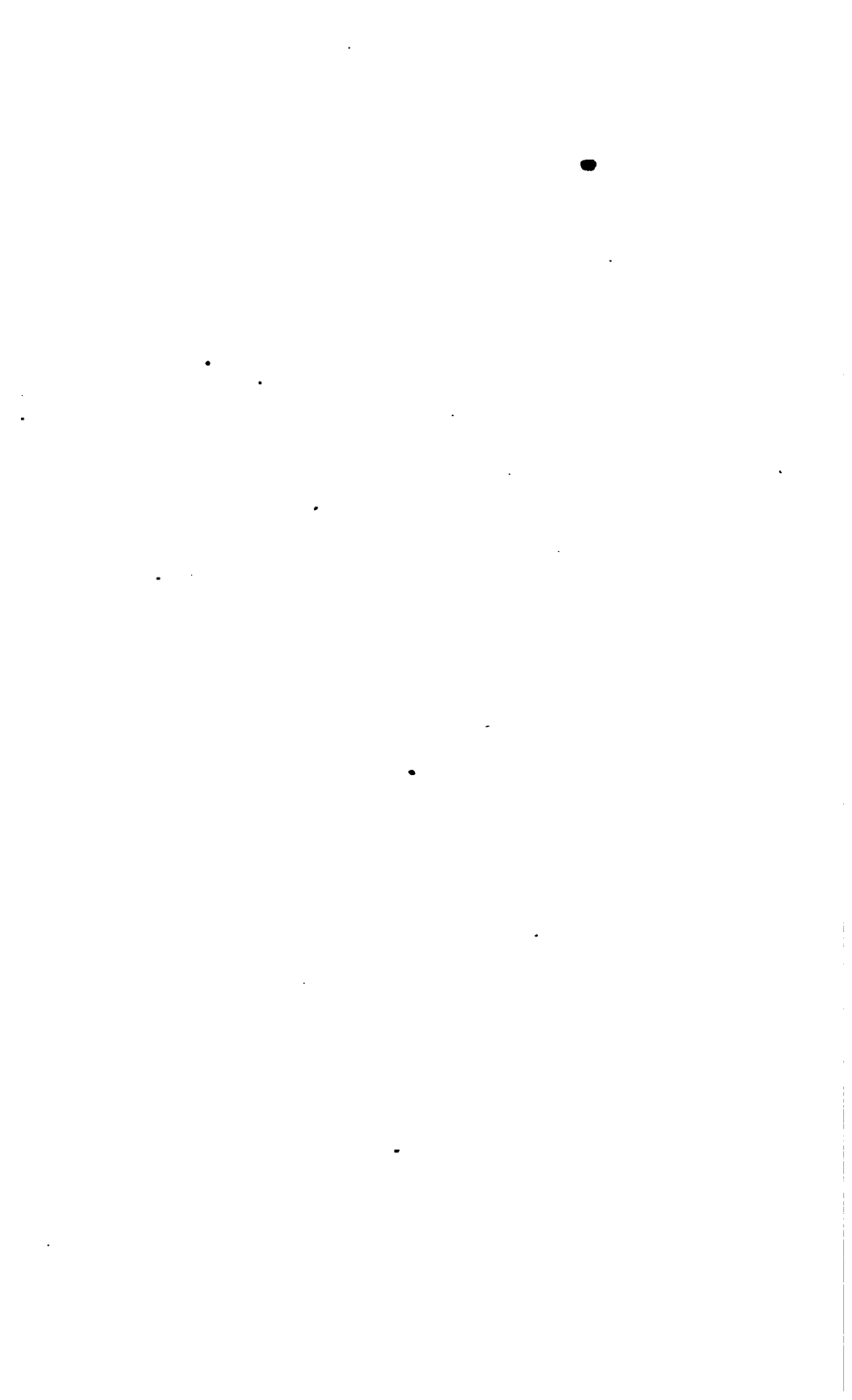
This Volume

IS, BY PERMISSION, DEDICATED,

BY

THE EDITOR.

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IN my late father's Will no instructions were left as to the publication of his Writings, nor specially as to that of the "ELEMENTS OF QUATERNIONS," which, but for his late fatal illness, would have been before now, in all their completeness, in the hands of the Public.

My brother, the Rev. A. H. Hamilton, who was named Executor, being too much engaged in his clerical duties to undertake the publication, deputed this task to me.

It was then for me to consider how I could best fulfil my triple duty in this matter—First, and chiefly, to the dead; secondly, to the present public; and, thirdly, to succeeding generations. I came to the conclusion that my duty was to publish the work as I found it, adding merely proof sheets, partially corrected by my late father and from which I removed a few typographical errors, and editing only in the literal sense of giving forth.

Shortly before my father's death, I had several conversations with him on the subject of the "ELEMENTS." In these he spoke of anticipated applications of Quaternions to Electricity, and to all questions in which the idea of Polarity is involved—applications which he never in his own lifetime expected to be able fully to develop, bows to be reserved for the hands of another Ulysses. He also discussed a good deal the nature of his own forthcoming Preface; and I may intimate, that after dealing with its more important topics, he intended to advert to the great labour which

the writing of the "ELEMENTS" had cost him—labour both mental and mechanical; as, besides a mass of subsidiary and unprinted calculations, he wrote out all the manuscript, and corrected the proof sheets, without assistance.

And here I must gratefully acknowledge the generous act of the Board of Trinity College, Dublin, in relieving us of the remaining pecuniary liability, and thus incurring the main expense, of the publication of this volume. The announcement of their intention to do so, gratifying as it was, surprised me the less, when I remembered that they had, after the publication of my father's former book, "Lectures on Quaternions," defrayed its entire cost; an extension of their liberality beyond what was recorded by him at the end of his Preface to the "Lectures," which doubtless he would have acknowledged, had he lived to complete the Preface of the "ELEMENTS."

He intended also, I know, to express his sense of the care bestowed upon the typographical correctness of this volume by Mr. M. H. Gill of the University Press, and upon the delineation of the figures by the Engraver, Mr. Oldham.

I annex the commencement of a Preface, left in manuscript by my father, and which he might possibly have modified or rewritten. Believing that I have thus best fulfilled my part as trustee of the unpublished "ELEMENTS," I now place them in the hands of the scientific public.

WILLIAM EDWIN HAMILTON.

January 1st, 1866.

P R E F A C E . *

[1.] THE volume now submitted to the public is founded on the same principles as the "LECTURES,"⁽¹⁾ which were published on the same subject about ten years ago : but the plan adopted is entirely new, and the present work can in no sense be considered as a second edition of that former one. The *Table of Contents*, by collecting into one view the headings of the various Chapters and Sections, may suffice to give, to readers already acquainted with the subject, a notion of the course pursued : but it seems proper to offer here a few introductory remarks, especially as regards the method of exposition, which it has been thought convenient on this occasion to adopt.

[2.] The present treatise is divided into Three Books, each designed to develop one guiding conception or view, and to illustrate it by a sufficient but not excessive number of examples or applications. The First Book relates to the *Conception of a Vector*, considered as a *directed right line*, in space of three dimensions. The Second Book introduces a *First Conception of a Quaternion*, considered as the *Quotient of two such Vectors*. And the Third Book treats of *Products and Powers of Vectors*, regarded as constituting a *Second Principal Form of the Conception of Quaternions in Geometry*.

* * * * *

* This fragment, by the Author, was found in one of his manuscript books by the Editor.



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This short First Chapter should be read with care by a beginner; any misconception of the meaning of the word "Vector" being fatal to progress in the Quaternions. The Chapter contains explanations also of the connected, but not all equally important, words or phrases, "revector," "provector," "transvector," "actual and null vectors," "opposite and successive vectors," "origin and term of a vector," "equal and unequal vectors," "addition and subtraction of vectors," "multiples and fractions of vectors," &c.; with the notation $\mathbf{B} - \mathbf{A}$, for the Vector (or directed right line) \mathbf{AB} : and a deduction of the result, essential but *not peculiar*‡ to quaternions, that (what is here called) the *vector-sum*, of two co-initial sides of a parallelogram, is the intermediate and co-initial *diagonal*. The term "Scalar" is also introduced, in connexion with *coefficients of vectors*.

* This Chapter may be referred to, as I. i.; the next as I. ii.; the first Chapter of the Second Book, as II. i.; and similarly for the rest.

† This Section may be referred to, as I. i. 1; the next, as I. i. 2; the sixth Section of the second Chapter of the Third Book, as III. ii. 6; and so on.

‡ Compare the second Note to page 203.

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<p>After reading these two first Sections of the second Chapter, and perhaps the three first Articles (31-33, pages 20-23) of the following Section, a student to whom the subject is new may find it convenient to pass at once, in his first perusal, to the third Chapter of the present Book; and to read only the two first Articles (62, 63, pages 49-51) of the first Section of that Chapter, respecting <i>Vectors in Space</i>, before proceeding to the Second Book (pages 103, &c.), which treats of <i>Quaternions as Quotients of Vectors</i>.</p>	
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<p>Among other results of this Chapter, a theorem is given in page 43, which seems to offer a <i>new geometrical generation</i> of (plane or spherical) <i>curves of the third order</i>. The <i>anharmonic co-ordinates</i> and equations employed, for the plane and for space, were suggested to the writer by some of his own <i>vector forms</i>; but their geometrical <i>interpretations</i> are assigned. The <i>geometrical nets</i> were first discussed by Professor Möbius, in his <i>Barycentric Calculus</i> (Note B), but they are treated in the present work by an entirely new analysis: and, at least for <i>space</i>, their theory has been thereby much extended in the Chapter to which we next proceed.</p>	
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<p>It has already been recommended to the student to read the first two Articles of this Section, even in his first perusal of the Volume; and then to pass to the Second Book.</p>	
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An application of *finite differences*, to a question connected with *barycentres*, occurs in p. 87. The anharmonic generation of a ruled *hyperboloid* (or *paraboloid*) is employed to illustrate anharmonic equations; and (among other examples) certain *cones*, of the second and third orders, have their vector equations assigned. In the last Section, a *definition of differentials* (of *vectors* and *scalars*) is proposed, which is afterwards extended to *differentials of quaternions*, and which is independent of developments and of infinitesimals, but involves the conception of *limits*. *Vectors of Velocity* and *Acceleration* are mentioned; and a hint of *Hodographs* is given.

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Very little, if any, of this Chapter II. I., should be omitted, even in a first perusal; since it contains the most essential conceptions and notations of the Calculus of Quaternions, at least so far as *quotients* of vectors are concerned, with numerous geometrical illustrations. Still there are a few investigations respecting circumscribed cones, imaginary intersections, and ellipsoids, in the thirteenth Section, which a student *may* pass over, and which will be indicated in the proper place in this Table.

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It is shown, by consideration of an *angle on a desk*, or inclined plane, that the *complex relation* of one vector to another, in *length* and

in *direction*, involves generally a system of *four numerical elements*. Many other motives, leading to the adoption of the name, "Quaternion," for the subject of the present Calculus, from its fundamental connexion with the number "Four," are found to present themselves in the course of the work.

SECTION 4.—On Equality of Quaternions; and on the Plane of a Quaternion,	112—117
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In the five foregoing Sections it is shown, among other things, that the *plane* of a quaternion is generally an *essential element* of its constitution, so that *dipplanar quaternions* are *unequal*; but that the *square of every right radial* (or *right versor*) is equal to *negative unity*, whatever its *plane* may be. The Symbol $\sqrt{-1}$ admits then of a *real interpretation*, in this as in several other systems; but when thus treated as *real*, it is in the present Calculus *too vague* to be useful: on which account it is found convenient to *retain the old signification* of that symbol, as denoting the (uninterpreted) *Imaginary of Algebra*, or what may here be called the *scalar imaginary*, in investigations respecting *non-real intersections*, or *non-real contacts*, in geometry.

SECTION 9.—On Vector-Arcs, and Vector-Angles, considered as Representatives of Versors of Quaternions; and on the Multiplication and Division of any one such Versor by another,	142—157
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This Section is important, on account of its *constructions* of multiplication and division; which show that the *product of two dipplanar versors*, and therefore of two such *quaternions*, is *not independent of the order of the factors*.

SECTION 10.—On a System of Three Right Versors, in Three Rectangular Planes; and on the Laws of the Symbols, ijk ,	157—162
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The student ought to make himself *familiar* with these laws, which are all included in the Fundamental Formula,

$$i^2 = j^2 = k^2 = ijk = -1. \quad (A)$$

In fact, a QUATERNION may be *symbolically defined* to be a *Quadrinomial Expression* of the form,

$$q = w + ix + jy + kz, \quad (B)$$

in which w, x, y, z are *four scalars*, or ordinary algebraic quantities, while i, j, k are *three new symbols*, obeying the *laws* contained in the formula (A), and therefore *not subject* to all the usual rules of *algebra*: since we have, for instance,

$$ij = +k, \text{ but } ji = -k; \text{ and } i^2j^2k^2 = -(ijk)^2.$$

SECTION 11.—On the Tensor of a Vector, or of a Quaternion; and on the Product or Quotient of any two Quaternions, 162-174

SECTION 12.—On the Sum or Difference of any two Quaternions; and on the Scalar (or Scalar Part) of a Quaternion, 175-190

SECTION 13.—On the Right Part (or Vector Part) of a Quaternion; and on the Distributive Property of the Multiplication of Quaternions, 190-233

SECTION 14.—On the Reduction of the General Quaternion to a Standard Quadrinomial Form; with a First Proof of the Associative Principle of Multiplication of Quaternions, 233-239

Articles 213-220 (with their sub-articles), in pp. 214-233, may be omitted at first reading.

CHAPTER II.

ON COMPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN ONE PLANE; AND ON POWERS, ROOTS, AND LOGARITHMS OF QUATERNIONS, 240-285

The first six Sections of this Chapter (II. ii.) may be passed over in a first perusal.

SECTION 1.—On Complanar Proportion of Vectors; Fourth Proportional to Three, Third Proportional to Two, Mean Proportional, Square Root; General Reduction of a Quaternion in a given Plane, to a Standard Binomial Form, 240-246

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In this last Section (II. ii. 7) the short first Article 258, and the following Art. 259, as far as the formula VIII. in p. 280, should be read, as a preparation for the Third Book, to which the Student may next proceed.

CHAPTER III.

ON DIPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN SPACE: AND ESPECIALLY ON THE ASSOCIATIVE PRINCIPLE OF MULTIPLICATION OF SUCH QUATERNIONS,	286-300
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This Chapter may be omitted, in a first perusal.

SECTION 1.—On some Enunciations of the Associative Property, or Principle, of Multiplication of Diplanar Quaternions,	286-293
SECTION 2.—On some Geometrical Proofs of the Associative Property of Multiplication of Quaternions, which are independent of the Distributive Principle,	293-297
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CHAPTER I.

ON THE INTERPRETATION OF A PRODUCT OF VECTORS, OR POWER OF A VECTOR, AS A QUATERNION,	301-390
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The first six Sections of this Chapter ought to be read, even in a first perusal of the work.

SECTION 1.—On a First Method of Interpreting a Product of Two Vectors as a Quaternion,	301-303
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SECTION 2.—On some Consequences of the foregoing Interpretation,	303-308
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This *first interpretation* treats the *product* $\beta . a$, as equal to the *quotient* $\beta : a^{-1}$; where a^{-1} (or Ra) is the previously defined *Reciprocal* (II. ii. 7) of the vector a , namely a *second vector*, which has an *inverse length*, and an *opposite direction*. *Multiplication of Vectors* is thus proved to be (like that of Quaternions) a *Distributive*, but not generally a *Commutative Operation*. The *Square of a Vector* is shown to be always a *Negative Scalar*, namely the *negative of the square of the tensor* of that vector, or of the *number* which expresses its *length*; and some geometrical applications of this fertile principle, to *spheres*, &c., are given. The *Index of the Right Part of a Product of Two Co-initial Vectors*, OA, OB , is proved to be a right line, *perpendicular to the Plane of the Triangle* OAB , and representing by its *length* the *Double Area* of that triangle; while the *Rotation round this Index, from the Multiplier to the Multiplicand, is positive*. This *right part, or vector part*, $Va\beta$, of the product *vanishes*, when the *factors are parallel* (to one common line); and the *scalar part*, $Sc\beta$, when they are *rectangular*.

SECTION 3.—On a Second Method of arriving at the same Interpretation, of a Binary Product of Vectors,	308-310
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SECTION 4.—On the Symbolical Identification of a Right Quaternion with its own Index: and on the Construction of a Product of Two Rectangular Lines, by a Third Line, rectangular to both,	310-313
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SECTION 5.—On some Simplifications of Notation, or of Expression, resulting from this Identification; and on the Conception of an Unit-Line as a Right Versor,	313-316
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In this *second interpretation*, which is found to agree in all its results with the first, but is better adapted to an extension of the theory, as in the following Sections, to *ternary products* of vectors, a *product of two vectors* is treated as the product of the two *right quaternions*, of which those vectors are the *indices* (II. i. 5). It is shown that, on the same plan, the *Sum of a Scalar and a Vector is a Quaternion*.

SECTION 6.—On the Interpretation of a Product of Three or more Vectors as a Quaternion, 316-330

This interpretation is effected by the substitution, as in recent Sections, of *Right Quaternions* for *Vectors*, without change of *order* of the *factors*. *Multiplication of Vectors*, like that of *Quaternions*, is thus proved to be an *Associative Operation*. A vector, generally, is reduced to the *Standard Trinomial Form*,

$$\rho = ix + jy + kz; \quad (C)$$

in which i, j, k are the peculiar symbols already considered (II. i. 10), but are regarded *now* as denoting *Three Rectangular Vector-Units*, while the *three scalars* x, y, z are simply *rectangular co-ordinates*; from the known theory of which last, illustrations of results are derived. The *Scalar of the Product of Three coinitial Vectors*, oa, ob, oc , is found to represent, with a *sign* depending on the *direction of a rotation*, the *Volume of the Parallelepiped* under those three lines; so that it *vanishes* when they are *coplanar*. *Constructions* are given also for *products of successive sides of triangles*, and other *closed polygons*, inscribed in *circles*, or in *spheres*; for example, a *characteristic property of the circle* is contained in the theorem, that the product of the *four successive sides* of an *inscribed quadrilateral* is a *scalar*: and an equally *characteristic* (but less obvious) *property of the sphere* is included in this other theorem, that the product of the *five successive sides* of an *inscribed gauche pentagon* is equal to a *tangential vector*, drawn from the point at which the pentagon *begins* (or *ends*). Some general *Formulas of Transformation of Vector Expressions* are given, with which a student ought to render himself *very familiar*, as they are of continual occurrence in the *practice* of this Calculus; especially the four formulæ (pp. 316, 317):

$$V.\gamma V\beta a = aS\beta\gamma - \beta S\gamma a; \quad (D)$$

$$V\gamma\beta a = aS\beta\gamma - \beta S\gamma a + \gamma S a\beta; \quad (E)$$

$$\rho S a\beta\gamma = aS\beta\gamma\rho + \beta S\gamma a\rho + \gamma S a\beta\rho; \quad (F)$$

$$\rho S a\beta\gamma = V\beta\gamma S a\rho + V\gamma a S\beta\rho + V a\beta S\gamma\rho; \quad (G)$$

in which $\alpha, \beta, \gamma, \rho$ are *any four vectors*, while S and V are signs of the operations of taking separately the *scalar* and *vector parts* of a quaternion. On the whole, this Section (III. i. 6) must be considered to be (as regards the present exposition) an important one; and if it have been read with care, after a perusal of the portions previously indicated, no difficulty will be experienced in passing to any subsequent applications of *Quaternions*, in the present or any other work.

SECTION 7.—On the Fourth Proportional to Three Diplanar Vectors,	331–349
SECTION 8.—On an Equivalent Interpretation of the Fourth Proportional to Three Diplanar Vectors, deduced from the Principles of the Second Book,	349–361
SECTION 9.—On a Third Method of interpreting a Product or Function of Vectors as a Quaternion; and on the Consistency of the Results of the Interpretation so obtained, with those which have been deduced from the two preceding Methods of the present Book,	361–364

These three Sections may be passed over, in a first reading. They contain, however, theorems respecting *composition of successive rotations* (pp. 334, 335, see also p. 340); expressions for the *semi-area of a spherical polygon*, or for *half the opening of an arbitrary pyramid*, as the *angle of a quaternion product*, with an extension, by limits, to the *semi-area of a spherical figure bounded by a closed curve*, or to *half the opening of an arbitrary cone* (pp. 340, 341); a construction (pp. 358–360), for a *series of spherical parallelograms*, so called from a *partial analogy to parallelograms in a plane*; a theorem (p. 361), connecting a certain system of *such* (spherical) parallelograms with the *foci of a spherical conic*, inscribed in a certain quadrilateral; and the *conception* (pp. 353, 361) of a *Fourth Unit in Space* (u , or $+1$), which is of a *scalar* rather than a *vector* character, as admitting merely of *change of sign*, through reversal of an *order of rotation*, although it presents itself in this theory as the *Fourth Proportional* ($i^{-1}k$) to *Three Rectangular Vector Units*.

SECTION 10.—On the Interpretation of a Power of a Vector as a Quaternion,	364–384
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It may be well to read this Section (III. i. 10), especially for the *Exponential Connexions* which it establishes, between *Quaternions* and *Spherical Trigonometry*, or rather *Polygonometry*, by a species of *extension of Moivre's theorem*, from the *plane to space*, or to the *sphere*. For example, there is given (in p. 381) an *equation of six terms*, which holds good for *every spherical pentagon*, and is deduced in this way from an *extended exponential formula*. The calculations in the sub-articles to Art. 312 (pp. 375–379) may however be passed over; and perhaps Art. 315, with its sub-articles (pp. 383, 384). But Art. 314, and its sub-articles, pp. 381–383, should be read, on account of the *exponential forms* which they contain, of equations of the *circle, ellipse, logarithmic spirals* (circular and elliptic), *helix*, and *screw surface*.

SECTION 11.—On Powers and Logarithms of Diplanar Quaternions; with some Additional Formulæ,	384–390
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It may suffice to read Art. 816, and its first eleven sub-articles, pp. 384-386. In this Section, the adopted *Logarithm*, lq , of a Quaternion q , is the *simplest root*, q' , of the transcendental equation,

$$1 + q' + \frac{q'^2}{2} + \frac{q'^3}{2 \cdot 3} + \&c. = q;$$

and its expression is found to be,

$$lq = lTq + Lq \cdot UVq, \tag{H}$$

in which T and U are the signs of *tensor* and *versor*, while Lq is the *angle* of q , supposed usually to be between 0 and π . Such *logarithms* are found to be often *useful* in this Calculus, although they do not *generally* possess the elementary property, that the *sum* of the logarithms of two quaternions is equal to the logarithm of their *product*: this apparent paradox, or at least *deviation from ordinary algebraic rules*, arising necessarily from the corresponding property of *quaternion multiplication*, which has been already seen to be *not generally a commutative operation* ($q'q''$ not $= q''q'$, unless q' and q'' be *complanar*). And *here*, perhaps, a student might consider his *first perusal* of this work as *closed*.*

CHAPTER II.

ON DIFFERENTIALS AND DEVELOPMENTS OF FUNCTIONS OF QUATERNIONS; AND ON SOME APPLICATIONS OF QUATERNIONS TO GEOMETRICAL AND PHYSICAL QUESTIONS, 391-495

It has been already said, that this Chapter may be omitted in a first perusal of the work.

SECTION 1.—On the Definition of Simultaneous Differentials, 391-393

* If he should choose to proceed to the *Differential Calculus of Quaternions* in the next Chapter (III. ii.), and to the *Geometrical* and other *Applications* in the third Chapter (III. iii.) of the present Book, it might be useful to read at this stage the last Section (I. iii. 7) of the First Book, which treats of *Differentials of Vectors* (pp. 98-102); and perhaps the omitted parts of the Section II. i. 13, namely Articles 213-220, with their subarticles (pp. 214-233), which relate, among other things, to a *Construction of the Ellipsoid*, suggested by the present Calculus. But the writer will now abstain from making any further suggestions of this kind, after having indicated as above what appeared to him a *minimum* course of study, amounting to rather less than 200 pages (or parts of pages) of this Volume, which will be recapitulated for the convenience of the student at the end of the present Table.

SECTION 2.—Elementary Illustrations of the Definition,
from Algebra and Geometry, 394–398

In the view here adopted (comp. I. iii. 7), *differentials are not necessarily, nor even generally, small.* But it is shown at a later stage (Art. 401, pp. 626–630), that the principles of this Calculus *allow us, whenever any advantage may be thereby gained, to treat differentials as infinitesimals*; and so to *abridge calculation*, at least in many applications.

SECTION 3.—On some general Consequences of the Definition, 398–409

Partial differentials and derivatives are introduced; and differentials of *functions of functions.*

SECTION 4.—Examples of Quaternion Differentiation, . . 409–419

One of the most important *rules* is, to differentiate the *factors* of a quaternion *product, in situ*; thus (by p. 405),

$$d. qq' = dq \cdot q' + q \cdot dq'. \quad (I)$$

The formula (p. 399), $d. q^{-1} = -q^{-1}dq \cdot q^{-1}, \quad (J)$

for the differential of the *reciprocal* of a quaternion (or vector), is also very often useful; and so are the equations (p. 418),

$$\frac{dTq}{Tq} = S \frac{dq}{q}; \quad \frac{dUq}{Uq} = V \frac{dq}{q}; \quad (K)$$

and (p. 411), $d. a^t = \frac{\pi}{2} a^{t+1} dt; \quad (L)$

q being any quaternion, and a any constant vector-unit, while t is a variable scalar. It is important to *remember* (comp. III. i. 11), that we have *not* in *quaternions* the *usual* equation,

$$dlq = \frac{dq}{q},$$

unless q and dq be *coplanar*; and therefore that we have *not generally*,

$$dl\rho = \frac{d\rho}{\rho},$$

if ρ be a *variable vector*; although we *have*, in this Calculus, the scarcely less simple equation, which is useful in questions respecting *orbital motion*,

$$dl \frac{\rho}{a} = \frac{d\rho}{\rho}, \quad (M)$$

if a be any constant vector, and if the *plane* of a and ρ be *given* (or constant).

SECTION 5.—On Successive Differentials and Developments,
of Functions of Quaternions, 420–435

In this Section principles are established (pp. 423–426), respecting quaternion functions which *vanish together*; and a form of development (pp. 427, 428) is assigned, *analogous* to Taylor's Series*, and like it capable of being concisely expressed by the *symbolical equation*, $1 + \Delta = \epsilon^d$ (p. 432). As an example of partial and successive differentiation, the expression (pp. 432, 433),

$$\rho = r k^i j^k f^e k^e,$$

which may represent *any vector*, is operated on; and an application is made, by means of *definite integration* (pp. 434, 435), to deduce the known area and volume of a sphere, or of portions thereof; together with the theorem, that the *vector sum of the directed elements of a spheric segment is zero*: each *element of surface* being represented by an *inward normal*, proportional to the elementary area, and corresponding in hydrostatics to the *pressure of a fluid* on that element.

SECTION 6.—On the Differentiation of Implicit Functions of Quaternions; and on the General Inversion of a Linear Function, of a Vector or a Quaternion: with some connected Investigations, 435–495

In this Section it is shown, among other things, that a *Linear and Vector Symbol*, ϕ , of *Operation on a Vector*, ρ , satisfies (p. 443) a *Symbolic and Cubic Equation*, of the form,

$$0 = m - m'\phi + m''\phi^2 - \phi^3; \tag{N}$$

whence $m\phi^{-1} = m' - m''\phi + \phi^2 = \psi$, (N')

= *another symbol of linear operation*, which it is shown how to deduce otherwise from ϕ , as well as the three scalar constants, m, m', m'' . The connected *algebraical cubic* (pp. 460, 461),

$$M = m + m'e + m''e^2 + e^3 = 0, \tag{O}$$

is found to have important applications; and it is proved† (pp. 460, 462) that if $S\lambda\phi\rho = S\rho\phi\lambda$, independently of λ and ρ , in which case the function ϕ is said to be *self-conjugate*, then this last cubic has *three real roots*, e_1, e_2, e_3 ; while, in the same case, the *vector equation*,

$$V\rho\phi\rho = 0, \tag{P}$$

is satisfied by a system of *Three Real and Rectangular Directions*: namely (compare pp. 468, 469, and the Section III. iii. 7), those of the *axes* of a (*biconcyclic*) *system of surfaces of the second order*, represented by the *scalar equation*,

* At a later stage (Art. 375, pp. 509, 510), a *new Enunciation of Taylor's Theorem* is given, with a *new proof*, but still in a *form* adapted to quaternions.

† A simplified proof, of some of the chief results for this important *case of self-conjugation*, is given at a later stage, in the few first subarticles to Art. 415 (pp. 698, 699).

$$S\rho\phi\rho = C\rho^2 + C', \text{ in which } C \text{ and } C' \text{ are constants.} \quad (Q)$$

Cases are discussed; and *general forms* (called cyclic, rectangular, focal, bifocal, &c., from their chief geometrical uses) are assigned, for the vector and scalar functions $\phi\rho$ and $S\rho\phi\rho$: one useful pair of such (*cyclic*) forms being, with real and constant values of g, λ, μ ,

$$\phi\rho = g\rho + \nabla\lambda\rho\mu, \quad S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu. \quad (R)$$

And finally it is shown (pp. 491, 492) that if fg be a *linear and quaternion function of a quaternion, g* , then the *Symbol of Operation, f* , satisfies a certain *Symbolic and Biquadratic Equation*, analogous to the cubic equation in ϕ , and capable of similar applications.

CHAPTER III.

ON SOME ADDITIONAL APPLICATIONS OF QUATERNIONS, WITH
SOME CONCLUDING REMARKS, . . . 495 to the end.

This Chapter, like the one preceding it, may be omitted in a first perusal of the Volume, as has indeed been already remarked.

SECTION 1.—Remarks Introductory to this Concluding Chapter,	495–496
SECTION 2.—On Tangents and Normal Planes to Curves in Space,	496–501
SECTION 3.—On Normals and Tangent Planes to Surfaces,	501–510
SECTION 4.—On Osculating Planes, and Absolute Normals, to Curves of Double Curvature,	511–515
SECTION 5.—On Geodetic Lines, and Families of Surfaces,	515–531

In these Sections, $d\rho$ usually denotes a *tangent to a curve*, and ν a *normal to a surface*. Some of the theorems or constructions may perhaps be new; for instance, those connected with the *cone of parallels* (pp. 498, 513, &c.) to the *tangents to a curve of double curvature*; and possibly the theorem (p. 525), respecting *reciprocal curves in space*: at least, the deductions here given of these results may serve as exemplifications of the Calculus employed. In treating of *Families of Surfaces* by quaternions, a sort of *analogue* (pp. 529, 530) to the formation and integration of *Partial Differential Equations* presents itself; as indeed it had done, on a similar occasion, in the *Lectures* (p. 574).

SECTION 6.—On Osculating Circles and Spheres, to Curves in Space; with some connected Constructions, . . . 531–630

The analysis, however condensed, of this long Section (III. iii. 6), cannot conveniently be performed otherwise than under the heads of the respective *Articles* (389–401) which compose it: each Article

being followed by several subarticles, which form with it a sort of *Series*.*

ARTICLE 389.—*Osculating Circle defined*, as the *limit* of a circle, which *touches a given curve* (plane or of double curvature) at a *given point P*, and *cuts the curve* at a *near point a* (see Fig. 77, p. 511). Deduction and interpretation of general expressions for the vector κ of the *centre* κ of the circle so defined. The *reciprocal of the radius* κP being called the *vector of curvature*, we have generally,

$$\text{Vector of Curvature} = (\rho - \kappa)^{-1} = \frac{dUd\rho}{Td\rho} = \frac{1}{d\rho} \vee \frac{d^2\rho}{d\rho} = \&c. ; \quad (S)$$

and if the *arc* (s) of the curve be made the independent variable, then

$$\text{Vector of Curvature} = \rho'' = D_s^2\rho. \quad (S')$$

Examples : curvatures of helix, ellipse, hyperbola, logarithmic spiral ; locus of centres of curvature of helix, plane evolute of plane ellipse, 581-535

ARTICLE 390.—Abridged general calculations ; return from (S') to (S), 585, 536

ARTICLE 391.—Centre determined by three scalar equations ; *Polar Axis, Polar Developable*, 587

ARTICLE 392.—*Vector Equation* of osculating circle, 538, 539

ARTICLE 393.—*Intersection* (or intersections) of a circle with a *plane curve* to which it osculates ; example, hyperbola, 589-541

ARTICLE 394.—*Intersection* (or intersections) of a *spherical curve* with a *small circle* osculating thereto ; example, *spherical conic* ; *constructions* for the *spherical centre* (or *pole*) of the circle osculating to such a curve, and for the point of *intersection* above mentioned, 541-549

ARTICLE 395.—*Osculating Sphere*, to a curve of double curvature, defined as the *limit* of a sphere, which *contains the osculating circle* to the curve at a *given point P*, and *cuts the same curve* at a *near point a* (comp. Art. 389). The *centre* s , of the sphere so found, is (as usual) the point in which the *polar axis* (Art. 391) *touches the cusp-edge* of the *polar developable*. Other general construction for the same centre (p. 551, comp. p. 573). General expressions for the *vector*, $\sigma = os$, and for the *radius*, $R = \overline{sp}$; R^{-1} is the *spherical curvature* (comp. Art. 397). *Condition of Sphericity* ($S = 1$), and *Coefficient of Non-sphericity* ($S - 1$), for a curve in space. When this last coefficient is *positive* (as it is for the helix), the *curve* lies *outside the sphere*, at least in the neighbourhood of the point of osculation, 549-553

ARTICLE 396.—Notations τ, τ', \dots for $D_s\rho, D_s^2\rho, \&c.$; properties of a curve depending on the *square* (s^2) of its *arc*, measured from a given point P ; $\tau = \text{unit-tangent}$, $\tau' = \text{vector of curvature}$, $\tau^{-1} = \text{Tr}' = \text{curvature}$ (or *first curvature*, comp. Art. 397), $\nu = \tau\tau' = \text{binormal}$; the

* A *Table of initial Pages* of all the *Articles* will be elsewhere given, which will much facilitate *reference*.

three planes, respectively perpendicular to r , r' , ν , are the *normal plane*, the *rectifying plane*, and the *osculating plane*; general theory of *osculating lines and planes*, *vector of rotation*, *axis of displacement*, *osculating screw surface*; *condition of developability of surface of emanants*,

554-559

ARTICLE 397.—Properties depending on the *cube* (s^3) of the *arc*; *Radius* r (denoted here, for distinction, by a roman letter), and *Vector* r^1r , of *Second Curvature*; this *radius* r may be either positive or negative (whereas the *radius* r of *first curvature* is always treated as positive), and its reciprocal r^{-1} may be thus expressed (pp. 568, 559),

$$\text{Second Curvature}^* = r^{-1} = S \frac{d^3\rho}{Vd\rho d^2\rho}, \quad (T), \quad \text{or, } r^{-1} = S \frac{r''}{rr^2} \quad (T')$$

the independent variable being the *arc* in (T'), while it is *arbitrary* in (T): but quaternions supply a vast variety of *other expressions* for this important scalar (see, for instance, the *Table* in pp. 574, 575). We have also (by p. 560, comp. Arts. 389, 395, 396),

Vector of Spherical Curvature = $sr^{-1} = (\rho - \sigma)^{-1} = \&c.$, (U)
= *projection of vector* (r') of (*simple or first curvature*), on *radius* (R) of *osculating sphere*: and if p and P denote the *linear and angular elevations*, of the centre (s) of this *sphere* above the *osculating plane*, then (by same page 560),

$$p = r \tan P = R \sin P = r'r = rD_s. \quad (U')$$

Again (pp. 560, 561), if we write (comp. Art. 396),

$$\lambda = V \frac{r''}{r} = r^{-1}r + rr' = \text{Vector of Second Curvature plus Binormal}, \quad (V)$$

this line λ may be called the *Rectifying Vector*; and if H denote the *inclination* (considered first by Lancret), of this *rectifying line* (λ) to the *tangent* (r) to the curve, then

$$\tan H = r'^{-1} \tan P = r^{-1}r. \quad (V')$$

Known right cone with *rectifying line* for its *axis*, and with H for its *semiangle*, which *osculates* at r to the *developable locus of tangents* to the curve (or by p. 568 to the *cone of parallels* already mentioned); *new right cone*, with a *new semiangle*, C , connected with H by the relation (p. 562),

$$\tan C = \frac{3}{4} \tan H, \quad (V'')$$

which *osculates* to the *cone of chords*, drawn from the given point r

* In this Article, or Series, 397, and indeed also in 396 and 398, several references are given to a very interesting Memoir by M. de Saint-Venant, "*Sur les lignes courbes non planes*:" in which, however, that able writer objects to such known phrases as *second curvature*, *torsion*, &c., and proposes in their stead a new name "*combrure*," which it has not been thought necessary here to adopt. (*Journal de l'École Polytechnique*, Cahier xxx)

to other points q of the given curve. *Other osculating cones, cylinders, helix, and parabola*; this last being (pp. 562, 566) the parabola which *osculates to the projection of the curve, on its own osculating plane*. *Deviation of curve*, at any near point q , from the osculating circle at P , decomposed (p. 566) into *two rectangular deviations*, from osculating *helix and parabola*. Additional formulæ (p. 576), for the general theory of *emanants* (Art. 396); case of *normally emanant lines*, or of *tangentially emanant planes*. General *auxiliary spherical curve* (pp. 576-578, comp. p. 515); new proof of the second expression (V) for $\tan H$, and of the theorem that if this *ratio of curvatures* be constant, the proposed curve is a *geodetic* on a *cylinder*: new proof that if each curvature (r^{-1} , r'^{-1}) be *constant*, the cylinder is *right*, and therefore the curve a *helix*, 559-578

ARTICLE 398.—Properties of a curve in space, depending on the *fourth and fifth powers* (s^4 , s^5) of its *arc* (s), 578-612

This *Series* 398 is so much longer than any other in the Volume, and is supposed to contain so much original matter, that it seems necessary here to subdivide the analysis under several separate heads, lettered as (a), (b), (c), &c.

(a). Neglecting s^5 , we may write (p. 578, comp. Art. 396),

$$OP_s = \rho_s = \rho + sr + \frac{1}{2}s^2r' + \frac{1}{6}s^3r'' + \frac{1}{24}s^4r''' ; \quad (W)$$

or (comp. p. 587), $\rho_s = \rho + x_s r + y_s r' + z_s r''$, (W')

with expressions (p. 588) for the *coefficients* (or *co-ordinates*) x_s , y_s , z_s , in terms of r' , r'' , r , r' , and s . If s^5 be taken into account, it becomes necessary to add to the expression (W) the term, $\frac{1}{120}s^5r^{(4)}$; with corresponding additions to the scalar coefficients in (W'), introducing r''' and r'' : the laws for forming which additional terms, and for extending them to higher powers of the arc, are assigned in a subsequent Series (399, pp. 612, 617).

(b). Analogous expressions for r'' , v'' , κ'' , λ' , σ' , and p' , R' , P' , H' , to serve in questions in which s^5 is neglected, are assigned (in p. 579); r'' , v'' , κ'' , λ , σ , and p , R , P , H , having been previously expressed (in Series 397); while r''' , v''' , κ''' , λ'' , σ'' , &c. enter into investigations which take account of s^5 : the arc s being treated as the independent variable in all *these* derivations.

(c). One of the chief results of the present Series (398), is the introduction (p. 581, &c.) of a *new auxiliary angle*, J , *analogous* in several respects to the *known angle* H (397), but belonging to a *higher order* of theorems, respecting *curves in space*: because the *new angle* J depends on the *fourth* (and lower) powers of the arc s , while *Lancret's angle* H depends only on s^2 (including s^1 and s^0). In fact, while $\tan H$ is represented by the expressions (V'), whereof one is $r'^{-1} \tan P$, $\tan J$ admits (with many transformations) of the following analogous expression (p. 581),

$$\tan J = R'^{-1} \tan P ; \quad (X)$$

where R depends* by (6) on s^2 , while r' and P depend (397) on no higher power than s^2 .

(d). To give a more distinct *geometrical meaning* to this new angle J , than can be easily gathered from such a formula as (X), respecting which it may be observed, in passing, that J is in general more simply defined by expressions for its *cotangent* (pp. 581, 588), than for its *tangent*, we are to conceive that, at each point r of any proposed curve of double curvature, there is drawn a *tangent plane* to the sphere, which *osculates* (395) to the curve at that point; and that then the *envelope* of all these *planes* is determined, which envelope (for reasons afterwards more fully explained) is called here (p. 581) the "*Circumscribed Developable*:" being a surface *analogous* to the "*Rectifying Developable*" of Lancret, but belonging (e) to a *higher order* of questions. And then, as the *known* angle H denotes (397) the *inclination*, suitably measured, of the *rectifying line* (λ), which is a *generatrix* of the *rectifying developable*, to the *tangent* (r) to the *curve*; so the *new* angle J represents the *inclination* of a *generating line* (ϕ), of what has just been called the *circumscribed developable*, to the *same* tangent (r), measured likewise in a defined direction (p. 581), but in the *tangent plane* to the sphere. It may be noted as another analogy (p. 582), that while H is a *right angle* for a *plane curve*, so J is *right* when the curve is *spherical*. For the *helix* (p. 585), the angles H and J are *equal*; and the *rectifying* and *circumscribed developables* coincide, with each other and with the *right cylinder*, on which the *helix* is a *geodetic line*.

(e). If the recent line ϕ be measured from the given point r , in a suitable direction (as contrasted with the opposite), and with a suitable length, it becomes what may be called (comp. 396) the *Vector of Rotation of the Tangent Plane* (d) to the *Osculating Sphere*; and then it satisfies, among others, the equations (pp. 579, 581, comp. (V)),

$$\phi = V \frac{v''}{v'}, \quad T\phi = R^{-1} \operatorname{cosec} J; \quad (\text{X}')$$

this last being an expression for the *velocity of rotation* of the *plane* just mentioned, or of its *normal*, namely the *spherical radius* R , if the *given curve* be conceived to be described by a point moving with a con-

* In other words, the calculation of r' and P introduces no differentials higher than the *third order*; but that of R requires the *fourth order* of differentials. In the language of modern geometry, the *former* can be determined by the consideration of *four consecutive points* of the curve, or by that of *two consecutive osculating circles*; but the *latter* requires the consideration of *two consecutive osculating spheres*, and therefore of *six consecutive points* of the curve (supposed to be one of double curvature). Other investigations, in the present and immediately following Series (398, 399), especially those connected with what we shall shortly call the *Osculating Twisted Cubic*, will be found to involve the consideration of *six consecutive points* of a curve.

stant velocity, assumed = 1. And if we denote by v the point in which the given radius R or rs is nearest to a consecutive radius of the same kind, or to the radius of a consecutive osculating sphere, then this point v divides the line rs internally, into segments which may (ultimately) be thus expressed (pp. 580, 581),

$$\overline{rv} = R \sin^2 J, \quad \overline{vs} = R \cos^2 J. \quad (X'')$$

But these and other connected results, depending on s^4 , have their known analogues (with H for J , and r for R), in that earlier theory (e) which introduces only s^2 (besides s^1 and s^3): and they are all included in the general theory of emanant lines and planes (396, 397), of which some new geometrical illustrations (pp. 582–584) are here given.

(f). New auxiliary scalar $n (= p^{-1}RR' = \cot J \sec P = \&c.)$, = velocity of centre s of osculating sphere, if the velocity of the point r of the given curve be taken as unity (e); n vanishes with R' , $\cot J$, and (comp. 395) the coefficient $S - 1 (= nr r^{-1})$ of non-sphericity, for the case of a spherical curve (p. 584). Arcs, first and second curvatures, and rectifying planes and lines, of the cusp-edges of the polar and rectifying* developables; these can all be expressed without going beyond s^4 , and some without using any higher power than s^4 , or differentials of the orders corresponding; $r_1 = nr$, and $r_2 = nr$, are the scalar radii of first and second curvature of the former cusp-edge, r_1 being positive when that curve turns its concavity at s towards the given curve at r : determination of the point κ , in which the latter cusp-edge is touched by the rectifying line λ to the original curve (pp. 584–587).

(g). Equation with one arbitrary constant (p. 587), of a cone of the second order, which has its vertex at the given point r , and has contact of the third order (or four-side contact) with the cone of chords (397) from that point; equation (p. 590) of a cylinder of the second order, which has an arbitrary line rs from r as one side, and has contact of the fourth order (or five-point contact) with the curve at r ; the constant above mentioned can be so determined, that the right line rs shall be a side of the cone also, and therefore a part of the intersection of cone and cylinder; and then the remaining or curvilinear part, of the complete intersection of those two surfaces of the second

* The rectifying plane, of the cusp-edge of the rectifying developable, is the plane of λ and r' , of which the formula LIV. in p. 587 is the equation; and the rectifying line rk , of the same cusp-edge, intersects the absolute normal rk to the given curve, or the radius (r) of first curvature, in the point κ in which that radius is nearest (e) to a consecutive radius of the same kind. But this last theorem, which is here deduced by quaternions, had been previously arrived at by M. de Saint-Venant (comp. the Note to p. xv.), through an entirely different analysis, confirmed by geometrical considerations.

and, is (by known principles) a *gauche curve of the third order*, or what is briefly called* a *Twisted Cubic*: and this last curve, in virtue of its construction above described, and whatever the assumed direction of the auxiliary line Px may be, has *contact of the fourth order* (or *six-point contact*) with the given curve of double curvature at P (pp. 587–590, comp. pp. 568, 572).

(A). Determination (p. 590) of the *constant* in the equation of the *cone* (g), so that this cone may have *contact of the fourth order* (or *six-side contact*) with the *cone of chords* from P ; the cone thus found may be called the *Osculating Oblique Cone* (comp. 397), of the second order, to that cone of chords; and the coefficients of its equation involve only r, r', r'', r''', r'''' , but not r''''' , although this last derivative is of no higher order than r'''' , since each depends only on s^2 (and lower powers), or introduces only *fifth differentials*. Again, the *cylinder* (g) will have *contact of the fifth order* (or *six-point contact*) with the given curve at P , if the line Px , which is by construction a *side* of that cylinder, and has hitherto had an *arbitrary direction*, be now obliged to be a *side* of a certain *cubic cone*, of which the equation (p. 590) involves as constants not only $rr'r'r''r''''$, like that of the *osculating cone* just determined, but also r''''' . The two cones last mentioned have the *tangent* (r) to the given curve for a *common side*,† but they have also *three other common sides*, whereof *one* at least is *real*, since they are assigned by a *cubic equation* (same p. 590); and by taking this side for the line Px in (g), there results a *new cylinder* of the second order, which cuts the *osculating oblique cone*, partly in that *right line* Px itself, and partly in a *gauche curve of the third order*, which it is proposed to call an *Osculating Twisted Cubic* (comp. again (g)), because it has *contact of the fifth order* (or *six-point contact*) with the given curve at P (pp. 590, 591).

(V). In general, and independently of any question of *osculation*, a *Twisted Cubic* (g), if passing through the origin o , may be represented by any one of the *vector equations* (pp. 592, 593),

* By Dr. Salmon, in his excellent Treatise on *Analytic Geometry of Three Dimensions* (Dublin, 1862), which is several times cited in the Notes to this final Chapter (III. iii.) of these *Elements*. The *gauche curves*, above mentioned, have been studied with much success, of late years, by M. Chasles, Sig. Cremona, and other geometers: but their existence, and some of their leading properties, appear to have been first perceived and published by Prof. Möbius (see his *Barycentric Calculus*, Leipzig, 1827, pp. 114–122, especially p. 117).

† This side, however, counts as three (p. 614), in the system of the *six lines of intersection* (real or imaginary) of these two cones, which have a common vertex P , and are respectively of the *second and third orders* (or degrees). Additional light will be thrown on this whole subject, in the following Series (899); in which also it will be shown that there is only *one* osculating twisted cubic, at a given point, to a given curve of double curvature; and that this cubic curve can be determined, without resolving any cubic or other equation.

$\nabla \rho + \nabla \rho \phi \rho = 0$, (Y); or $(\phi + e)\rho = a$, (Y')
 or $\rho = (\phi + e)^{-1}a$, (Y''); or $\nabla \rho + \rho \nabla \gamma \rho + \nabla \rho \nabla \lambda \rho \mu = 0$, (Y''')

in which a, γ, λ, μ are *real and constant vectors*, but e is a *variable scalar*; while $\phi \rho$ denotes (comp. the Section III. ii. 8, or pp. xii., xiii.) a *linear and vector function*, which is *here generally not self-conjugate*, of the *variable vector* ρ of the *cubic curve*. The *number of the scalar constants*, in the form (Y'''), or in any other form of the equation, is found to be *ten* (p. 598), with the foregoing supposition that the curve passes *through the origin*, a restriction which it is easy to remove. The curve (Y) is *cut*, as it ought to be, in *three points* (real or imaginary), by an *arbitrary secant plane*; and its *three asymptotes* (real or imaginary) have the *directions* of the *three vector roots* β (see again the last cited Section) of the equation (same p. 593),

$$\nabla \beta \phi \beta = 0: \quad (Z)$$

so that by (P), p. xii., *these three asymptotes compose a real and rectangular system*, for the case of *self-conjugation* of the function ϕ in (Y).

(j). *Deviation* of a near point P , of the given curve, from the *sphere* (395) which *osculates* at the given point P ; this deviation (by p. 593, comp. pp. 558, 584) is

$$\overline{BP}_s - \overline{BP} = \frac{r_1 s^4}{24r\tau^3 R} = \frac{R' s^4}{24r\tau p} = \frac{ns^4}{24r\tau R} = \&c.; \quad (A_1)$$

it is ultimately equal (p. 595) to the *quarter* of the deviation (397) of the same near point P , from the *osculating circle* at P , multiplied by the *sine* of the *small angle* sr_s , which the *small arc* rs , of the *locus* of the *spheric centre* s (or of the *cusp-edge* of the polar developable) *subtends* at the same point P ; and it has an *outward* or an *inward* direction, according as this *last arc* is *concave* or *convex* (f) at s , towards the *given curve* at P (pp. 585, 595). It is also ultimately equal (p. 596) to the deviation $\overline{rs}_s - \overline{P_s s}_s$ of the *given point* P from the *near sphere*, which *osculates* at the *near point* P_s ; and likewise (p. 597) to the *component*, in the direction of sr , of the deviation of that near point from the *osculating circle* at P , measured in a direction parallel to the *normal plane* at that point, if this *last deviation* be *now* expressed to the accuracy of the *fourth order*: whereas it has *hitherto* been considered sufficient to develop this *deviation from the osculating circle* (397) as far as the *third order* (or third dimension of s); and therefore to treat it as having a direction, *tangential to the osculating sphere* (comp. pp. 566, 594).

(k). The deviation (A₁) is also equal to the *third part* (p. 598) of the *deviation* of the *near point* P , from the *given circle* (which *osculates* at P), if measured in the *near normal plane* (at P_s), and *decomposed* in the direction of the *radius* R_s of the *near sphere*; or to the *third part* (with direction preserved) of the deviation of the *new near point* in which the given circle is *cut* by the *near plane*, from the *near sphere*: or finally to the *third part* (as before, and still with an unchanged direc-

tion) of the deviation from the *given sphere*, of that *other near point* c , in which the *near circle* (osculating at r_s) is cut by the *given normal plane* (at r), and which is found to satisfy the equation,

$$\overline{BC} = 3\overline{SP}_s - 2\overline{SP}. \quad (B_1)$$

Geometrical connexions (p. 599) between these various results (j) (k), illustrated by a diagram (Fig. 83).

(l). The *Surface*, which is the *Locus of the Osculating Circle* to a given curve in space, may be represented rigorously by the *vector expression* (p. 600),

$$\omega_{s,u} = \rho_s + r_s r_s' \sin u + r_s^2 r_s' \text{vers } u; \quad (C_1)$$

in which s and u are two independent scalar variables, whereof s is (as before) the *arc* PF_s of the *given curve*, but is *not now treated as small*: and u is the (small or large) *angle subtended at the centre* x_s of the circle, by the *arc of that circle*, measured from its *point of osculation* r_s . But the *same superficial locus* (comp. 892) may be represented also by the *vector equation* (p. 611), involving *apparently only one scalar variable* (σ),

$$V \frac{2r_s}{\omega - \rho_s} + \nu_s = 0, \quad (D_1)$$

in which $\nu_s = r_s r_s'$, and $\omega = \omega_{s,u}$ = the vector of an arbitrary point of the surface. The general method (p. 601), of the Section III. iii. 3, shows that the *normal* to this *surface* (C_1), at any proposed point thereof, has the direction of $\omega_{s,u} - \sigma_s$; that is (p. 600), the direction of the *radius of the sphere*, which contains the *circle* through that point, and has the *same point of osculation* r_s to the given curve. The *locus of the osculating circle* is therefore found, by this little calculation with quaternions, to be at the same time the *Envelope of the Osculating Spheres*, as was to be expected from geometrical considerations (comp. the Note to p. 600).

(m). The *curvilinear locus of the point* c in (k) is *one branch of the section of the surface* (l), made by the normal plane to the given curve at r ; and if D be the projection of c on the tangent at r to this new curve, which tangent PD has a direction *perpendicular* to the radius rs or R of the osculating sphere at r (see again Fig. 88, in p. 599), while the ordinate DC is *parallel* to that radius, then (attending only to principal terms, pp. 598, 599) we have the expressions,

$$PD = \frac{R\sigma^3}{6r^2R} U_T(\sigma - \rho), \quad DC = \frac{-n\sigma^4}{8rR} U(\sigma - \rho), \quad (E_1)$$

and therefore ultimately (p. 600),

$$\frac{DC^2}{PD^4} = \frac{81}{82} \cdot \frac{n^2 r^2 R (\sigma - \rho)}{R^2} = \text{const.}; \quad (F_1)$$

from which it follows that r is a *singular point* of the *section* here considered, but *not a cusp* of that section, although the *curvature* at r is *infinite*: the *ordinate* DC varying ultimately as the *power* with exponent $\frac{1}{2}$ of the *abscissa* PD . Contrast (pp. 600, 601), of this

section, with that of the developable *Locus of Tangents*, made by the same normal plane at r to the given curve; the vectors analogous to PD and DC are in *this* case nearly equal to $-\frac{1}{2}s^2r'$ and $-\frac{1}{2}s^2r''$; so that the latter varies ultimately as the power $\frac{2}{3}$ of the former, and the point r is (as it is known to be) a *cuspidal* of this last section.

(m). A given *Curve* of double curvature is therefore generally a *Singular Line* (p. 601), although not a *cuspidal*, upon that *Surface* (\mathcal{S}), which is at once the *Locus* of its osculating *Circle*, and the *Envelope* of its osculating *Spheres*: and the new developable surface (\mathcal{d}), as being circumscribed to this *superficial locus* (or *envelope*), so as to touch it along this *singular line* (p. 612), may naturally be called, as above, the *Circumscribed Developable* (p. 581).

(o). Additional light may be thrown on this whole theory of the *singular line* (n), by considering (pp. 601–611) a problem which was discussed by Monge, in two distinct Sections (xxii. xxvi.) of his well-known *Analyse* (comp. the Notes to pp. 602, 608, 609, 610 of these *Elements*); namely, to determine the *envelope* of a *sphere* with varying radius R , whereof the centre s traverses a *given curve* in space; or briefly, to find the *Envelope of a Sphere with One varying Parameter* (comp. p. 624): especially for the *Case of Coincidence* (p. 608, &c.), of what are usually two distinct branches (p. 602) of a certain *Characteristic Curve* (or *arête de rebroussement*), namely the *curvilinear envelope* (real or imaginary) of all the *circles*, along which the *superficial envelope* of the *spheres* is touched by those spheres themselves.

(p). Quaternion forms (pp. 603, 604) of the *condition of coincidence* (ϕ); one of these can be at once translated into Monge's *equation of condition* (p. 608), or into an equation slightly more general, as leaving the independent variable arbitrary; but a simpler and more easily interpretable form is the following (p. 604),

$$r_1 dr = \pm R dR, \quad (G_1)$$

in which r is the radius of the *circle of contact*, of a sphere with its envelope (ϕ), while r_1 is the radius of (first) *curvature* of the curve (s), which is the *locus* of the centre s of the sphere.

(q). The *singular line* into which the two branches of the curvilinear envelope are fused, when this *condition* is satisfied, is in general an *orthogonal trajectory* (p. 607) to the *osculating planes* of the curve (s); that curve, which is now the *given* one, is therefore (comp. 891, 895) the *cuspidal* (p. 607) of the *polar developable*, corresponding to the *singular line* just mentioned, or to what may be called the curve (\mathcal{P}), which was formerly the *given* curve. In this way there arise many verifications of formulæ (pp. 607, 608); for example, the equation (G_1) is easily shown to be consistent with the results of (f).

(r). With the geometrical hints thus gained from interpretation of quaternion results, there is now no difficulty in assigning the *Complete and General Integral* of the *Equation of Condition* (ϕ), which was presented by Monge under the form (comp. p. 603) of a *non-linear differential equation of the second order*, involving three variables

(ρ, ϕ, τ) considered as *functions of a fourth* (α), namely the *co-ordinate of the centre of the sphere*, regarded as *varying with the radius*, but which does not appear to have been either *integrated or interpreted* by that illustrious analyst. The general integral here found presents itself at first in a *quaternion form* (p. 609), but is easily translated (p. 610) into the usual language of analysis. A *less general integral* is also assigned, and its geometrical signification exhibited, as answering to a case for which the *singular line* lately considered reduces itself to a *singular point* (pp. 610, 611).

(g). Among the *verifications* (g) of this whole theory, it is shown (pp. 608, 609) that although, when the *two branches* (o) of the general *curvilinear envelope* of the circles of the system are *real and distinct*, each branch is a *cuspidal edge* (or *arête de rebroussement*, as Monge perceived it to be), upon the *superficial envelope* of the spheres, yet in the *case of fusion* (p) this *cuspidal character* is *lost* (as was likewise seen by Monge*): and that then a *section of the surface*, made by a *normal plane* to the *singular line*, has precisely the form (m), expressed by the equation (F_1). In short, the result is in many ways confirmed, by calculation and by geometry, that when the *condition of coincidence* (p) is satisfied, the *Surface* is, as in (n), at once the *Envelope of the osculating Sphere* and the *Locus of the osculating Circle*, to that *Singular Line* on itself, into which by (g) the *two branches* (o) of its *general cuspidal edge* are fused.

(t). Other applications of preceding formulæ might be given; for instance, the formula for κ'' enables us to assign general expressions (p. 611) for the *centre and radius of the circle*, which *osculates at x* to the *locus of the centre of the osculating circle*, to a given curve in space: with an elementary verification, for the case of the *plane evolutes of the plane evolute of a plane curve*. But it is time to conclude this long analysis, which however could scarcely have been much abridged, of the results of Series 398, and to pass to a more brief account of the investigations in the following Series.

ARTICLE 399.—Additional general investigations, respecting that *gauche curve* of the *third order* (or degree), which has been above called an *Osculating Twisted Cubic* (398, (h)), to any proposed curve of double curvature; with applications to the case, where the given curve is a *helix*,

612-621

(a). In general (p. 614), the *tangent PR* to the given curve is a *nodal side* of the *cubic cone* 398, (h); one *tangent plane* to that cone (C_3), along that side, being the *osculating plane* (P) to the curve, and therefore touching also, along the same side, the *osculating oblique cone* (C_2) of the *second order*, to the *cone of chords* (397) from P ; while the other *tangent plane* to the cubic cone (C_3) crosses that *first plane* (P), or the *quadric cone* (C_2), at an angle of which the trigonometric *co-tan-*

* Compare the first Note to p. 609 of these *Elements*.

gent ($\frac{1}{2}r'$) is equal to half the differential of the radius (r) of second curvature, divided by the differential of the arc (s). And the three common sides, PE , PE' , PE'' , of these two cones, which remain when the tangent PT is excluded, and of which one at least must be real, are the parallels through the given point P to the three asymptotes (398, (i)) to the gauche curve sought; being also sides of three quadric cylinders, say (L_2) , (L'_2) , (L''_2) , which contain those asymptotes as other sides (or generating lines): and of which each contains the twisted cubic sought, and is cut in it by the quadric cone (C_2) .

(b). On applying this *First Method* to the case of a given helix, it is found (p. 614) that the general cubic cone (C_2) breaks up into the system of a new quadric cone, (C'_2) , and a new plane (P) ; which latter is the rectifying plane (396) of the helix, or the tangent plane at P to the right cylinder, whereon that given curve is traced. The two quadric cones, (C_2) and (C'_2) , touch each other and the plane (P) along the tangent PT , and have no other real common side: whence two of the sought asymptotes, and two of the corresponding cylinders (a), are in this case imaginary, although they can still be used in calculation (pp. 614, 615, 617). But the plane (P) cuts the cone (C_2) , not only in the tangent PT , but also in a second real side PE , to which the real asymptote is parallel (a); and which is at the same time a side of a real quadric cylinder (L_2) , which has that asymptote for another side (p. 617), and contains the twisted cubic: this gauche curve being thus the curvilinear part (p. 615) of the intersection of the real cone (C_2) , with the real cylinder (L_2) .

(c). Transformations and verifications of this result; fractional expressions (p. 616), for the co-ordinates of the twisted cubic; expression (p. 615) for the deviation of the helix from that osculating curve, which deviation is directed inwards, and is of the sixth order: the least distance, between the tangent PT and the real asymptote, is a right line PE , which is cut internally (p. 617) by the axis of the right cylinder (b), in a point A such that PA is to AE as three to seven.

(d). The *First Method* (a), which had been established in the preceding Series (398), succeeds then for the case of the helix, with a facility which arises chiefly from the circumstance (b), that for this case the general cubic cone (C_2) breaks up into two separate loci, whereof one is a plane (P) . But usually the foregoing method requires, as in 398, (k), the solution of a cubic equation: an inconvenience which is completely avoided, by the employment of a *Second General Method*, as follows.

(e). This *Second Method* consists in taking, for a second locus of the gauche osculatrix sought, a certain Cubic Surface (S_2) , of which every point is the vertex* of a quadric cone, having six-point con-

* It is known that the locus of the vertex of a quadric cone, which passes through six given points of space, A, B, C, D, E, F , whereof no four are in one

cut with the given curve at r : so that this *new surface* is cut by the plane at infinity, in the *same cubic curve* as the cubic cone (C_3). It is found (p. 620) to be a *Ruled Surface*, with the tangent rr for a *Singular Line*; and when this *right line* is set aside, the remaining (that is, the *curvilinear*) part of the intersection of the two loci, (C_3) and (S_3), is the *Osculating Twisted Cubic* sought: which *gauche osculatrix* is thus *completely and generally determined*, without any such difficulty or *apparent variety*, as might be supposed to attend the solution of a cubic equation (d), and with new verifications for the case of the *helix* (p. 621).

ARTICLE 400.—On Involutes and Evolutes in Space, 621—626

(a). The usual points of Monge's theory are deduced from the two fundamental quaternion equations (p. 621),

$$S(\sigma - \rho)\rho' = 0, \quad V(\sigma - \rho)\sigma' = 0, \quad (H_1)$$

in which ρ and σ are corresponding vectors of involute and evolute; together with a theorem of Prof. De Morgan (p. 622), respecting the case when the involute is a spherical curve.

(b). An *involute in space* is generally the *only real part* (p. 624) of the *envelope* of a certain variable *sphere* (comp. 398), which has its *centre* on the *evolute*, while its *radius R* is the *variable intercept* between the two curves: but because we have *here* the relation (p. 622, comp. p. 602),

$$R^2 + \sigma'^2 = 0, \quad (H_1')$$

the *circles of contact* (398, (c)) reduce themselves each to a *point* (or rather to a *pair of imaginary right lines*, intersecting in a real point), and the preceding theory (398), of envelopes of spheres with one varying parameter, undergoes important modifications in its results, the conditions of the application being different. In particular, the *involute* is indeed, as the equations (H_1) express, an *orthogonal trajectory* to the *tangents* of the *evolutes*; but *not* to the *osculating planes*

plane, is generally a *Surface*, say (S_4), of the *Fourth Degree*: in fact, it is *cut* by the plane of the triangle ABC in a system of *four right lines*, whereof three are the *sides* of that triangle, and the fourth is the intersection of the two planes, ABC and DEF . If then we investigate the intersection of this surface (S_4) with the *quadric cone*, ($A.BCDEF$), or say (C_2), which has A for vertex, and passes through the five other given points, we might expect to find (in some sense) a curve of the *eighth degree*. But when we set aside the *five right lines*, AB, AC, AD, AE, AF , which are *common* to the two surfaces here considered, we find that the (remaining or) *curvilinear part* of the *complete intersection* is reduced to a *curve of the third degree*, which is precisely the *twisted cubic through the six given points*. In applying this general (and perhaps new) method, to the problem of the *osculating twisted cubic* to a *curve*, the *osculating planes* to that curve may be *excluded*, as foreign to the question: and then the *quartic surface* (S_4) is reduced to the *cubic surface* (S_3), above described.

of that curve, as the *singular line* (898, (*q*)) of the former envelope was, to those of the curve which was the *locus* of the *centres* of the *spheres* before considered, when a certain *condition of coincidence* (or of *fusion*, 898, (*p*)) was satisfied.

(*e*). *Curvature of hodograph of evolute* (p. 625); if P, P_1, P_2, \dots and S, S_1, S_2, \dots be corresponding points of involute and evolute, and if we draw right lines SP_1, SP_2, \dots in the directions of S_1P_1, S_2P_2, \dots and with a common length = \overline{SP} , the *spherical curve* $PT_1T_2 \dots$ will have contact of the second order at P , with the involute $PP_1P_2 \dots$ (pp. 625, 626).

ARTICLE 401.—Calculations abridged, by the treatment of *quaternion differentials* (which have hitherto been *finite*, comp. p. xi.) as *infinitesimals*.* new deductions of *osculating plane, circle, and sphere*, with the *vector equation* (892) of the circle; and of the first and second curvatures of a curve in space, 626-630

SECTION 7.—On Surfaces of the Second Order; and on Curvatures of Surfaces, 630-706

ARTICLE 402.—References to some equations of *Surfaces*, in earlier parts of the Volume, 630, 631

ARTICLE 403.—Quaternion equations of the *Sphere* ($\rho^2 = -1$, &c.), 631-633

In some of these equations, the notation N for *norm* is employed (comp. the Section II. i. 6).

ARTICLE 404.—Quaternion equations of the *Ellipsoid*, 633-635
One of the simplest of these forms is (pp. 307, 635) the equation,

$$T(\rho + \rho\kappa) = \kappa^2 - \epsilon^2, \quad (I.)$$

* Although, for the sake of brevity, and even of clearness, some phrases have been used in the foregoing analysis of the Series 898 and 899, such as *four-side* or *five-side contact* between cones, and *five-point* or *six-point contact* between curves, or between a curve and a surface, which are borrowed from the doctrine of consecutive points and lines, and therefore from that of infinitesimals; with a few other expressions of modern geometry, such as the *plane at infinity*, &c.; yet the reasonings in the text of these Elements have all been rigorously reduced, so far, or are all obviously reducible, to the fundamental conception of Limits: compare the definitions of the *osculating circle and sphere*, assigned in Articles 889, 895. The object of Art. 401 is to make it visible how, without abandoning such ultimate reference to limits, it is possible to abridge calculation, in several cases, by treating (at this stage) the differential symbols, $d\rho, d^2\rho, \&c.$, as if they represented infinitely small differences, $\Delta\rho, \Delta^2\rho, \&c.$; without taking the trouble to write these latter symbols first, as denoting finite differences, in the rigorous statement of a problem, of which statement it is not always easy to assign the proper form, for the case of points, &c., at finite distances: and then having the additional trouble of reducing the complex expressions so found to simpler forms, in which differentials shall finally appear. In short, it is shown that in Quaternions, as in other parts of Analysis, the rigour of limits can be combined with the facility of infinitesimals.

in which ι and κ are real and constant vectors, in the directions of the cyclic normals. This form (I₁) is intimately connected with, and indeed served to suggest, that *Construction of the Ellipsoid* (II. i. 18), by means of a *Diacentric Sphere* and a *Point* (p. 227, comp. Fig. 58, p. 226), which was among the earliest geometrical results of the Quaternions. The three semiaxes, a, b, c , are expressed (comp. p. 280) in terms of ι, κ as follows :

$$a = T\iota + T\kappa; \quad b = \frac{\kappa^2 - \iota^2}{T(\iota - \kappa)}; \quad c = T\iota - T\kappa; \quad (I_1')$$

whence $ab^{-1}c = T(\iota - \kappa).$ (I₁'')

ARTICLE 405.—*General Central Surface of the Second Order* (or central quadric), $S\phi\phi\rho = f\rho = 1,$ 636-638

ARTICLE 406.—*General Cons of the Second Order* (or quadric cone), $S\phi\phi\rho = f\rho = 0,$ 638-643

ARTICLE 407.—*Bifocal Form* of the equation of a central but non-conical surface of the second order: with some quaternion formulae, relating to *Obfocal Surfaces,* 643-653

(a) The bifocal form here adopted (comp. the Section III. ii. 6) is the equation,

$$Cf\rho = (Sap)^2 - 2eSapSa'\rho + (Sa'\rho)^2 + (1 - e^2)\rho^2 = C, \quad (J_1)$$

in which, $C = (e^2 - 1)(e + Sa'a')^2.$ (J₁')

α, α' are two (real) focal unit-lines, common to the whole system of confocals; the (real and positive) scalar l is also constant for that system: but the scalar e varies, in passing from surface to surface, and may be regarded as a parameter, of which the value serves to distinguish one confocal, say (e), from another (pp. 643, 644).

(b) The squares (p. 644) of the three scalar semiaxes (real or imaginary), arranged in algebraically descending order, are,

$$a^2 = (e + 1)^2, \quad b^2 = (e + Sa'a')^2, \quad c^2 = (e - 1)^2; \quad (K_1)$$

whence $l^2 = \frac{a^2 - c^2}{2}, \quad e = \frac{a^2 + c^2}{a^2 - c^2};$ (L₁)

and the three vector semiaxes corresponding are,

$$aU(\alpha + \alpha'), \quad bUV\alpha\alpha', \quad cU(\alpha - \alpha'). \quad (M_1)$$

(c) Rectangular, unifocal, and cyclic forms (pp. 644, 648, 650), of the scalar function $f\rho$, to each of which corresponds a form of the vector function $\phi\rho$; deduction, by a new analysis, of several known theorems* (pp. 644, 645, 648, 652, 653) respecting confocal surfaces,

* For example, it is proved by quaternions (pp. 652, 653), that the focal lines of the focal cone, which has any proposed point x for vertex, and rests on the focal hyperbola, are generating lines of the single-sheeted hyperboloid (of the given confocal system), which passes through that point: and an extension of this result, to the focal lines of any cone circumscribed to a confocal, is deduced by a similar analysis, in a subsequent Series (408, p. 656). But such known theorems respecting confocals can only be alluded to, in these Contents.

and their focal conics; the lines a, a' are asymptotes to the focal hyperbola (p. 647), whatever the species of the surface may be: references (in Notes to pp. 648, 649) to the Lectures,* for the focal ellipsoid, and for several different generations of this last surface.

(d). General Exponential Transformation (p. 651), of the equation of any central quadric;

$$\rho = xa + y\sqrt{a'}\beta, (N_1), \text{ with } x^2fa + y^2fUVaa' = 1, (N_1')$$

$$\text{and } \beta = \frac{(a' - ea)UVaa'}{e + Saa'}; (N_1'')$$

this auxiliary vector β is constant, for any one confocal (e); the exponent, t , in (N_1) , is an arbitrary or variable scalar; and the coefficients, x and y , are two other scalar variables, which are however connected with each other by the relation (N_1') .

(e). If any fixed value be assigned to t , the equation (N_1) then represents the section made by a plane through a (p. 651), which section is an ellipse if the surface be an ellipsoid, but an hyperbola for either hyperboloid; and the cutting plane makes with the focal plane of a, a' , or with the plane of the focal hyperbola, an angle = $\frac{1}{2}t\pi$.

(f). If, on the other hand, we allow t to vary, but assign to x and y any constant values consistent with (N_1') , the equation (N_1) then represents an ellipse (p. 651), whatever the species of the surface may be; x represents the distance of its centre from the centre o of the surface, measured along the focal line a ; y is the radius of a right cylinder, with a for its axis, of which the ellipse is a section, or the radius of a circle in a plane perpendicular to a , into which that ellipse can be orthogonally projected: and the angle $\frac{1}{2}t\pi$ is now the excentric anomaly. Such elliptic sections of a central quadric may be otherwise obtained from the unifocal form (c) of the equation of the surface; they are, in some points of view, almost as interesting as the known circular sections: and it is proposed (p. 649) to call them Centro-Focal Ellipses.

(g). And it is obvious that, by interchanging the two focal lines a, a' in (d), a Second Exponential Transformation is obtained, with a Second System of centro-focal ellipses, whereof the proposed surface is the locus, as well as of the first system (f), but which have their centres on the line a' , and are projected into circles, on a plane perpendicular to this latter line (p. 649).

(h). Equation of Confocals (p. 652),

$$Vv,\phi v, = Vv\phi, v. (O_1)$$

ARTICLE 408.—On Circumscribed Quadric Cones; and on the Umbilics of a central quadric, 658-663

* Lectures on Quaternions (by the present author), Dublin, Hodges and Smith, 1853.

(b). Equations (p. 653) of *Conjugate Points*, and of *Conjugate Directrices*, with respect to the surface $fp = 1$,

$$f(\rho, \rho') = 1, \quad (P_1), \text{ and } f(\rho, \rho') = 0; \quad (P_1')$$

Condition of *Contact*, of the same surface with the *right line* rr' ,

$$(f(\rho, \rho') - 1)^2 = (f\rho - 1)(f\rho' - 1); \quad (Q_1)$$

this latter is also a form of the equation of the *Cone*, with vertex at r' , which is *circumscribed* to the same quadric ($f\rho = 1$).

(b). The condition (Q_1) may also be thus transformed (p. 654),

$$F\nabla\rho\rho' = a^2b^2c^2f(\rho - \rho'), \quad (Q_1')$$

F being a scalar function, connected with f by certain relations of *reciprocity* (comp. p. 483); and a simple *geometrical interpretation* may be assigned, for this last equation.

(c). The *Reciprocal Cone*, or *Cone of Normals* σ at r' , to the circumscribed cone (Q_1) or (Q_1'), may be represented (p. 655) by the very simple equation,

$$F(\sigma : Sp'\sigma) = 1; \quad (Q_1'')$$

which likewise admits of an extremely simple interpretation.

(d). A given *right line* (p. 656) is *touched* by two *confocals*, and other known results are easy consequences of the present analysis; for example (pp. 658, 659), the cone circumscribed to any surface of the system, from any point of *either* of the two *real focal curves*, is a *cone of revolution* (real or imaginary): but a similar conclusion holds good, when the *vertex* is on the *third* (or *imaginary*) *focal*, and even more generally (p. 668), when that vertex is *any point* of the (known and imaginary) *developable envelope* of the *confocal system*.

(e). A central quadric has in general *Twelve Umbilics* (p. 659), whereof only *four* (at most) can be *real*, and which are its *intersections* with the *three focal curves*: and these *twelve points* are ranged, *three by three*, on *eight imaginary right lines* (p. 662), which *intersect the circle at infinity*, and which it is proposed to call the *Eight Umbilical Generatrices* of the surface.

(f). These (imaginary) *umbilical generatrices* of a quadric are found to possess several interesting properties, especially in relation to the *lines of curvature*: and their *locus*, for a *confocal system*, is a *developable surface* (p. 663), namely the known *envelope* (d) of that system.

ARTICLE 409.—Geodetic Lines on Central Surfaces of the Second Order,

664—667

(a). One form of the *general differential equation of geodetics* on an *arbitrary surface* being, by III. iii. 5 (p. 515),

$$\nabla\rho d^2\rho = 0, \quad (R_1), \text{ if } Td\rho = \text{const.}, \quad (R_1')$$

this is shown (p. 664) to conduct, for central quadrics, to the first integral,

$$P^{-2}D^{-2} = T\nu^2fUd\rho = h = \text{const.}; \quad (S_1)$$

where P is the *perpendicular from the centre* o on the *tangent plane*,

and D is the (real or imaginary) *semidiameter* of the surface, which is *parallel to the tangent* ($d\rho$) to the curve. The known equation of Joachimstal, $P.D = \text{const.}$, is therefore proved anew; this last constant, however, being by no means necessarily *real*, if the surface be not an *ellipsoid*.

(*b*). Deduction (p. 665) of a theorem of M. Chasles, that the *tangents to a geodetic*, on any one central quadric (e), *touch also a common confocal* (e_1); and of an integral (p. 666) of the form,

$$e_1 \sin^2 \vartheta_1 + e_2 \cos^2 \vartheta_1 = e, = \text{const.}, \quad (S_1')$$

which agrees with one of M. Liouville.

(*c*). Without the restriction (R_1'), the differential of the scalar λ in (S_1) may be thus decomposed into factors (p. 666),

$$d\lambda = d.P^{-2}D^{-2} = 2S\nu d\nu d\rho^{-1}.S\nu d\rho^{-1}d^2\rho; \quad (S_1'')$$

but, by the lately cited Section (III. iii. 5, p. 515), the differential equation of the *second order*,

$$S\nu d\rho d^2\rho = 0, \quad (R_1'')$$

with an arbitrary scalar variable, represents the *geodetic lines on any surface*: the theorem (*a*) is therefore in this way reproduced.

(*d*). But we see, at the same time, by (S_1''), that the quantity λ , or $P.D = \lambda^{-1}$, is *constant, not only for the geodetics* on a central quadric, but *also for a certain other set* of curves, determined by the differential equation of the *first order*, $S\nu d\nu d\rho = 0$, which will be seen, in the next Series, to represent the *lines of curvature*.

ARTICLE 410.—On Lines of Curvature generally; and in particular on such lines, for the case of a Central Quadric, 667-674

(*a*). The differential equation (comp. 409, (*d*)),

$$S\nu d\nu d\rho = 0, \quad (T_1)$$

represents (p. 667) the *Lines of Curvature*, upon an *arbitrary surface*; because it is a *limiting form* of this other equation,

$$S\nu \Delta\nu \Delta\rho = 0, \quad (T_1')$$

which is the *condition of intersection* (or of parallelism), of the *normals* drawn at the extremities of the two vectors ρ and $\rho + \Delta\rho$.

(*b*). The normal vector ν , in the equation (T_1), may be multiplied (pp. 673, 700) by any constant or variable scalar n , without any real change in that equation; but in this whole theory, of the treatment of *Curvatures of Surfaces* by Quaternions, it is advantageous to consider the expression $S\nu d\rho$ as denoting the *exact differential* of *some scalar function* of ρ ; for then (by pp. 486, 487) we shall have an equation of the form,

$$d\nu = \phi d\rho = \text{a self-conjugate function of } d\rho, \quad (U_1)$$

which usually involves ρ also. For instance, we may write generally (p. 669, comp. (R), p. xiii),

$$d\nu = g d\rho + V\lambda d\rho\mu; \quad (U_1')$$

the scalar g , and the vectors λ, μ being *real*, and being *generally** functions of ρ , but not involving $d\rho$.

(e). This being understood, the two† *directions* of the *tangent* $d\rho$, which satisfy at once the general equation (T₁) of the lines of curvature, and the differential equation $Svd\rho = 0$ of the surface, are easily found to be represented by the two vector expressions (p. 669),

$$UVv\lambda \pm UVv\mu; \quad (T_1'')$$

they are therefore generally *rectangular* to each other, as they have long been known to be.

(d). The *surface itself* remaining still quite *arbitrary*, it is found useful to introduce the conception of an *Auxiliary Surface of the Second Order* (p. 670), of which the variable vector is $\rho + \rho'$, and the equation is,

$$S\rho'\phi\rho' = g\rho'^2 + S\lambda\rho'\mu\rho' = 1, \quad (U_1'')$$

or more generally = const.; and it is proposed to call *this* surface, of which the *centre* is at the given point \mathbf{r} , the *Index Surface*, partly because its *diametral section*, made by the *tangent plane* to the given surface at \mathbf{r} , is a certain *Index Curve* (p. 668), which may be considered to coincide with the known "*indicatrice*" of Dupin.

(e). The expressions (T₁'') show (p. 670), that whatever the given surface may be, the *tangents* to the lines of curvature *bisect the angles* formed by the *traces* of the two *cyclic planes* of the *Index Surface* (d), on the *tangent plane* to the given surface; these two tangents have also (as was seen by Dupin) the directions of the *axes* of the *Index Curve* (p. 668); and they are distinguished (as he likewise saw) from all *other* tangents to the given surface, at the given point \mathbf{r} , by the condition that each is *perpendicular to its own conjugate*, with respect to that indicating curve: the *equation* of such *conjugation*, of two tangents \mathbf{r} and \mathbf{r}' , being in the present notation (see again p. 668),

$$Sr\phi r' = 0, \quad \text{or} \quad Sr'r = 0. \quad (U_1''')$$

(f). New proof (p. 669) of another theorem of Dupin, namely that *if a developable be circumscribed to any surface, along any curve thereon, its generating lines are everywhere conjugate, as tangents to the surface, to the corresponding tangents to the curve.*

(g). Case of a central quadric; new proof (p. 671) of still another theorem of Dupin, namely that the *curve of orthogonal intersection* (p. 645), of two *confocal surfaces*, is a *line of curvature* on each.

(h). The system of the *eight umbilicar generatrices* (408, (e)), of a central quadric, is the *imaginary envelope* of the *lines of curvature* on that surface (p. 671); and each such *generatrix* is itself an *imaginary*

* For the case of a central quadric, g, λ, μ are constants.

† Generally two; but in some cases more. It will soon be seen, that three lines of curvature pass through an umbilic of a quadric.

lines of curvature thereon: so that through each of the *twelve umbilics* (see again 408, (e)) there pass *three lines of curvature* (comp. p. 677), whereof however only *one*, at most, can be real: namely *two generatrices*, and a *principal section* of the surface. These last results, which are perhaps new, will be illustrated, and otherwise proved, in the following Series (411).

ARTICLE 411.—Additional illustrations and confirmations of the foregoing theory, for the case of a *Central** *Quadric*; and especially of the theorem respecting the *Three Lines of Curvature through an Umbilic*, whereof *two* are always *imaginary* and *rectilinear*, 674–679

(a). The general equation of condition (T_1) , or $S\nabla\Delta\nabla\Delta\rho = 0$, for the intersection of two finitely distant normals, may be easily transformed for the case of a quadric, so as to express (p. 675), that *when the normals at \mathbf{r} and \mathbf{r}' intersect* (or are parallel), the chord $\mathbf{r}\mathbf{r}'$ is *perpendicular to its own polar*.

(b). Under the same conditions, if the point \mathbf{r} be given, the locus of the chord $\mathbf{r}\mathbf{r}'$ is usually (p. 676) a *quadric cone*, say (C) ; and therefore the locus of the point \mathbf{r}' is usually a *quartic curve*, with \mathbf{r} for a *double point*, whereat *two branches* of the curve cut each other at *right angles*, and touch the *two lines* of curvature.

(c). If the point \mathbf{r} be one of a *principal section* of the given surface, but *not an umbilic*, the cone (C) breaks up into a *pair of planes*, whereof one, say (P) , is the plane of the section, and the other, (P') , is perpendicular thereto, and is *not tangential* to the surface; and thus the *quartic* (b) breaks up into a *pair of conics* through \mathbf{r} , whereof one is the *principal section* itself, and the other is perpendicular to it.

(d). But if the given point \mathbf{r} be an *umbilic*, the *second plane* (P') becomes a *tangent plane* to the surface; and the *second conic* (c) breaks up, at the same time, into a *pair of imaginary† right lines*, namely the *two umbilicar generatrices* through \mathbf{r} (pp. 676, 678, 679).

(e). It follows that the *normal* $\mathbf{r}\mathbf{n}$ at a *real umbilic* \mathbf{r} (of an ellipsoid, or a double-sheeted hyperboloid) is *not intersected by any other real normal*, except those which are in the *same principal section*; but that this real normal $\mathbf{r}\mathbf{n}$ is intersected, in an *imaginary sense*, by *all the normals* $\mathbf{r}'\mathbf{n}'$, which are drawn at points \mathbf{r}' of either of the *two imaginary generatrices* through the real umbilic \mathbf{r} ; so that *each* of these

* Many, indeed most, of the results apply, without modification, to the case of the *Paraboloids*; and the rest can easily be adapted to this latter case, by the consideration of infinitely distant points. We shall therefore often, for conciseness, omit the term *central*, and simply speak of *quadrics*, or surfaces of the *second order*.

† It is well known that the *single-sheeted hyperboloid*, which (alone of central quadrics) has *real generating lines*, has at the same time *no real umbilics* (comp. pp. 661, 662).

anyway right lines is seen anew to be a *line* of curvature*, on the surface (comp. 410, (A)), because all the normals $r'n'$, at points of this line are situated in one *common (imaginary) normal plane* (p. 676): and as before, there are thus *three lines of curvature through an umbilic*.

(f). These geometrical results are in various ways deducible from calculation with quaternions; for example, a form of the equation of the lines of curvature on a quadric is seen (p. 677) to become an *identity at an umbilic* ($\nu \parallel \lambda$): while the *differential* of that equation breaks up into two *factors*, whereof *one* represents the tangent to the *principal section*, while the *other* ($S\lambda d^2\rho = 0$) assigns the directions of the *two generatrices*.

(g). The equation of the *cone* (O), which has already presented itself as a certain *locus of chords* (δ), admits of many quaternion transformations; for instance (see p. 675), it may be written thus,

$$\frac{S\alpha\rho\Delta\rho}{S\alpha\Delta\rho} + \frac{S\alpha'\rho\Delta\rho}{S\alpha'\Delta\rho} = 0, \quad (V_1)$$

ρ being the vector of the vertex P , and $\rho + \Delta\rho$ that of any other point P' of the cone; while α, α' are still, as in 407, (a), two real *focal lines*, of which the *lengths* are here arbitrary, but of which the *directions* are constant, as before, for a whole *confocal system*.

(h). This cone (O), or (V_1), is also the *locus* (p. 678) of a system

* It might be natural to suppose, from the known general theory (410, (c)) of the two rectangular directions, that *each* such generatrix PP' is *crossed perpendicularly*, at every one of its *non-umbilic points* P' , by a *second (and distinct, although imaginary) line of curvature*. But it is an almost equally well known and *received result* of modern geometry, paradoxical as it must at first appear, that *when a right line is directed to the circle at infinity*, as (by 408, (e)) the generatrices in question are, then *this imaginary line is everywhere perpendicular to itself*. Compare the Notes to pages 459, 672. Quaternions are not at all responsible for the *introduction* of this principle into geometry, but they *recognise and employ* it, under the following very simple form: that *if a non-evanescent vector be directed to the circle at infinity*, it is an *imaginary value of the symbol* $O\delta$ (comp. pp. 300, 459, 662, 671, 672); and conversely, that *when this last symbol represents a vector which is not null*, the vector thus denoted is an *imaginary line, which cuts that circle*. It may be noted here, that such is the case with the *reciprocal polar* of every chord of a quadric, connecting any two umbilics which are not in one principal plane; and that thus the *quadratic equation* (XXI., in p. 669) from which the two directions (410, (c)) can usually be derived, becomes an *identity for every umbilic, real or imaginary*: as it ought to do, for consistency with the foregoing theory of the *three lines* through that umbilic. And as an additional illustration of the *coincidence* of directions of the lines of curvature at any *non-umbilic point* P' of an umbilic generatrix, it may be added that the *cone of chords* (O), in 411, (b), is found to *touch the quadric along that generatrix*, when its vertex is at any such point P' .

of *three rectangular lines*; and if it be cut by any plane perpendicular to a side, and not passing through the vertex, the *section* is an *equilateral hyperbola*.

(i). The same cone (O) has, for *three* of its sides PP' , the *normals* (p. 677) to the *three confocals* (p. 644) of a given system which pass through its vertex P ; and therefore also, by 410, (g), the *tangents* to the *three lines of curvature* through that point, which are the *intersections* of those three confocals.

(j). And because its equation (V_1) does not involve the constant l , of 407, (a), (b), we arrive at the following theorem (p. 678):—*If indefinitely many quadrics, with a common centre o , have their asymptotic cones biconfocal, and pass through a common point P , their normals at that point have a quadric cone (O) for their locus.*

ARTICLE 412.—On Centres of Curvature of Surfaces, . . . 679–689

(a). If σ be the vector of the *centre* s of *curvature* of a normal section of an *arbitrary surface*, which touches one of the two lines of curvature thereon, at any given point P , we have the two fundamental equations (p. 679),

$$\sigma = \rho + R U \nu, \quad (W_1), \quad \text{and} \quad R^{-1} d\rho + dU \nu = 0; \quad (W_1')$$

whence

$$V \rho dU \nu = 0, \quad (W_1''), \quad \text{and} \quad \frac{T \nu}{R} + S \frac{d\nu}{d\rho} = 0; \quad (W_1''')$$

the equation (W_1''') being a *new form* of the *general differential equation* of the *lines of curvature*.

(b). Deduction (pp. 680, 681, &c.) of some known theorems from these equations; and of some which introduce the new and general conception of the *Index Surface* (410, (d)), as well as that of the known *Index Curve*.

(c). Introducing the auxiliary scalar (p. 682),

$$r = \frac{T \nu}{R} = -S \frac{d\nu}{d\rho} = -S r^{-1} \phi r, \quad (X_1)$$

in which r ($\parallel d\rho$) is a *tangent* to a line of curvature, while $d\nu = \phi d\rho$, as in (U_1), the two values of r , which answer to the two rectangular directions (T_1) in 410, (c), are given (p. 680) by the expression,

$$r = -g - T \lambda \mu \cdot \cos \left(\angle \frac{\nu}{\lambda} \mp \angle \frac{\nu}{\mu} \right), \quad (X_1')$$

in which g , λ , μ are, for any given point P , the constants in the equation (U_1') of the *index surface*; the *difference* of the two curvatures R^{-1} therefore *vanishes at an umbilic* of the *given surface*, whatever the *form* of that surface may be: that is, at a point, where $\nu \parallel \lambda$ or $\parallel \mu$, and where consequently the *index curve* is a *circle*.

(d). At any *other* point P of the given surface, which is as yet *entirely arbitrary*, the values of r may be thus expressed (p. 681),

$$r_1 = a_1^{-2}, \quad r_2 = a_2^{-2}, \quad (X_1'')$$

a_1 , a_2 being the *scalar semiaxes* (real or imaginary) of the *index curve* (defined, comp. 410, (d), by the equations $S \rho' \phi \rho' = 1$, $S \nu \rho' = 0$).

(e). The quadratic equation, of which r_1 and r_2 , or the *inverses* *sines* of the two last semi-axes, are the roots, may be written (p. 683) under the *symbolical form*,

$$S\nu^{-1}(\phi + r)^{-1}\nu = 0; \quad (Y_1)$$

which may be developed (same page) into this other form,

$$r^2 + rS\nu^{-1}\chi\nu + S\nu^{-1}\psi\nu = 0, \quad (Y_1)$$

the linear and vector functions, ψ and χ , being derived from the function ϕ , on the plan of the Section III. ii. 6 (pp. 440, 443).

(f). Hence, generally, the *product of the two curvatures of a surface* is expressed (same p. 683) by the formula,

$$R_1^{-1}R_2^{-1} = r_1 r_2 T\nu^{-2} = -S \frac{1}{\nu} \psi \frac{1}{\nu}; \quad (Z_1)$$

which will be found useful in the following series (418), in connexion with the theory of the *Measure of Curvature*.

(g). The given surface being still quite general, if we write (p. 686),

$$r = U d\rho, \quad r' = U(\nu d\rho), \quad (A_2), \quad \text{and therefore } rr' = U\nu, \quad (A_2)$$

so that r and r' are *unit tangents* to the lines of curvature, it is easily proved that

$$dr' = r Sr' dr, \quad (B_2), \quad \text{or that } Vr dr' = 0; \quad (B_2)$$

this *general parallelism* of dr' to r being geometrically explained, by observing that a line of curvature on any surface is, at the same time, a line of curvature on the *developable normal surface*, which rests upon that line, and to which r' or νr is normal, if r be tangential to the line.

(h). If the *vector of curvature* (389) of a line of curvature be projected on the normal ν to the given surface, the projection (p. 686) is the vector of curvature of the *normal section* of that surface, which has the same tangent r ; but this result, and an analogous one (same page) for the *developable normal surface* (g), are virtually included in Meusnier's theorem, which will be proved by quaternions in Series 414.

(i). The vector σ of a centre s of curvature of the given surface, answering to a given point r thereon, may (by (W₁) and (X₁)) be expressed by the equation,

$$\sigma = \rho + r^{-1}\nu; \quad (C_2)$$

which may be regarded also as a *general form of the Vector Equation of the Surface of Centres*, or of the *locus* of the centre s : the variable vector ρ of the point r of the given surface being supposed (p. 501) to be expressed as a vector function of two independent and scalar variables, whereof therefore ν , r , and σ become also functions, although the two last involve an ambiguous sign, on account of the *Two Sheets* of the surface of centres.

(j). The normal at s , to what may be called the *First Sheet*, has the direction of the tangent r to what may (on the same plan) be called the *First Line of Curvature* at r ; and the vector ν of the point

corresponding to s , on the corresponding sheet of the *Reciprocal* (comp. pp. 507, 508) of the *Surface of Centres*, has (by p. 684) the expression,

$$v = r(Spr)^{-1}; \quad (D_2)$$

which may also be considered (comp. (f)) to be a form of the *Vector Equation* of that *Reciprocal Surface*.

(k). The vector v satisfies generally (by same page) the *equations of reciprocity*,

$$Sv\sigma = S\sigma v = 1, \quad Sv\delta\sigma = 0, \quad S\sigma\delta v = 0, \quad (D_3)$$

$\delta\sigma$, δv denoting any infinitesimal variations of the vectors σ and v , consistent with the equations of the surface of centres and its reciprocal, or any *linear* and *vector elements* of those two surfaces, at two corresponding points; we have also the relations (pp. 684, 685),

$$S\rho v = 1, \quad Sv v = 0, \quad Svv\rho v = 0. \quad (D_3'')$$

(l). The equation $Sv(\omega - \rho) = 0$, or more simply,

$$Sv\omega = 1, \quad (E_2)$$

in which ω is a variable vector, represents (p. 684) the *normal plane* to the *first line* (j) of curvature at r ; or the *tangent plane* at s to the *first sheet* of the surface of centres: or finally, the tangent plane to that *developable normal surface* (g), which rests upon the *second line* of curvature, and touches the *first sheet* along a certain curve, whereof we shall shortly meet with an example. And if v be regarded, comp. (f), as a vector function of two scalar variables, the *envelope of the variable plane* (E_2) is a *sheet of the surface of centres*; or rather, on account of the ambiguous sign (i), it is that surface of centres itself: while, in like manner, the *reciprocal surface* (j) is the *envelope* of this *other plane*,

$$S\sigma\omega = 1. \quad (E_2')$$

(m). The equations (W_1), (W_1') give (comp. the Note to p. 684),

$$d\sigma = dR.Uv; \quad (F_2)$$

combining which with (C_2), we see that the equations (H_1) of p. xxv. are satisfied, when the derived vectors ρ' and σ' are changed to the corresponding differentials, $d\rho$ and $d\sigma$. The known theorem (of Monge), that each *Line of Curvature* is generally an *involute*, with the corresponding *Curve of Centres* for one of its *evolutes* (400), is therefore in this way reproduced: and the connected theorem (also of Monge), that *this evolute* is a *geodesic on its own sheet* of the surface of centres, follows easily from what precedes.

(n). In the foregoing paragraphs of this analysis, the *given surface* has throughout been *arbitrary*, or *general*, as stated in (d) and (g). But if we now consider specially the case of a *central quadric*, several less general but interesting results arise, whereof many, but perhaps not all, are known; and of which some may be mentioned here.

(o). Supposing, then, that not only $dv = \phi d\rho$, but also $v = \phi\rho$, and $S\rho v = \rho = 1$, the *Index Surface* (410, (d)) becomes simply (p. 670) the *given surface*, with its *centre transported* from o to P ; whence many simplifications follow.

(p). For example, the *semiaxes* a_1, a_2 of the *index curve* are now equal (p. 681) to the semiaxes of the *diametral section* of the given surface, made by a plane parallel to the tangent plane; and Tv is, as in 409, the *reciprocal P^{-1}* of the *perpendicular*, from the centre on this latter plane; whence (by (X_1) and (X_1')) these known expressions for the two* curvatures result:

$$R_1^{-1} = Pa_1^{-2}; \quad R_2^{-1} = Pa_2^{-2}. \quad (G_2)$$

(q). Hence, by (d), if a *new surface* be derived from a *given central quadric* (of any species), as the *locus of the extremities of normals*, erected at the centre, to the *planes of diametral sections* of the given surface, each such normal (when real) having the length of one of the *semiaxes* of that section, the *equation of this new surface*† admits (p. 688) of being written thus:

$$S\rho(\phi - \rho^{-2})^{-1}\rho = 0. \quad (H_2)$$

(r). Under the conditions (o), the expression (C_2) for σ gives (p. 684) the two converse forms,

$$\sigma = r^{-1}(\phi + r)\rho, \quad (I_2), \quad \rho = r(\phi + r)^{-1}\sigma; \quad (I_2')$$

whence (pp. 684, 689),

$$v = r(\phi + r)^{-1}\phi\sigma, \quad (J_2), \quad \sigma = (\phi^{-1} + r^{-1})v; \quad (J_2')$$

and therefore (p. 689), by (d), (p), and by the theory (407) of *con-focal surfaces*,

$$\sigma_1 = \phi_2^{-1}v = \phi_2^{-1}\phi\rho, \quad (K_2)$$

if ϕ_2 be formed from ϕ by changing the *semiaxes abc* to $a_2b_2c_2$; it being understood that the given quadric (abc) is cut by the two *confocals* ($a_1b_1c_1$) and ($a_2b_2c_2$), in the *first* and *second lines* of curvature through the given point P : and that σ_1 is here the vector of that *first centre* s of curvature, which answers to the *first line* (comp. (j)). Of course, on the same plan, we have the analogous expression,

* Throughout the present Series 412, we attend only (comp. (a)) to the curvatures of the *two normal sections* of a surface, which have the directions of the *two lines* of curvature: these being in fact what are always regarded as the *two principal curvatures* (or simply as the two curvatures) of the *surface*. But, in a shortly subsequent Series (414), the more general case will be considered, of the curvature of *any section*, normal or oblique.

† When the *given surface* is an *ellipsoid*, the *derived surface* is the celebrated *Wave Surface* of Fresnel: which thus has (H_2) for a *symbolical form* of its equation. When the given surface is an *hyperboloid*, and a *semiaxis* of a section is *imaginary*, the (scalar and now positive) *square*, of the (imaginary) *normal* erected, is still to be made equal to the *square* of that *semiaxis*.

$$\sigma_2 = \phi_1^{-1}\nu = \phi_1^{-1}\phi\rho, \quad (K_2)$$

for the vector of the *second centre*.

(e). These expressions for σ_1, σ_2 include (p. 689) a theorem of Dr. Salmon, namely that the *centres of curvature* of a given quadric at a given point are the *poles of the tangent plane*, with respect to the two confocals through that point; and *either* of them may be regarded, by admission of an ambiguous sign (comp. (i)), as a *new Vector Form** of the *Equation of the Surface of Centres*, for the case (e) of a given central quadric.

(f). In connexion with the same expressions for σ_1, σ_2 , it may be observed that if r_1, r_2 be the corresponding values of the auxiliary scalar r in (e), and if r, r' still denote the unit tangents (g) to the first and second lines of curvature, while $abc, a_1b_1c_1$, and $a_2b_2c_2$ retain their recent significations (r), then (comp. pp. 686, 687, see also p. 652),

$$r_1 = fr = fUd\rho = (a^2 - a_2^2)^{-1} = \&c., \quad (L_2)$$

$$\text{and} \quad r_2 = fr' = fU\nu d\rho = (a^2 - a_1^2)^{-1} = \&c.; \quad (L_2')$$

this association of r_1 and σ_1 with a_2 , &c., and of r_2 and σ_2 with a_1 , &c., arising from the circumstance that the *tangents* r and r' have respectively the directions of the *normals* ν_2 and ν_1 , to the two confocal surfaces, ($a_2b_2c_2$) and ($a_1b_1c_1$).

(u). By the properties of such surfaces, the scalar here called r_2 is therefore *constant*, in the whole extent of a *first line* of curvature; and the same *constancy* of r_2 , or the equation,

$$dfU\nu d\rho = 0, \quad (M_2)$$

may in various ways be proved by quaternions (p. 687).

(v). Writing simply r and r' for r_1 and r_2 , so that r' is constant, but r variable, for a *first line* of curvature, while conversely r is constant and r' variable for a *second line*, it is found (pp. 684, 685, 586), that the *scalar equation* of the *surface of centres* (i) may be regarded as the result of the elimination of r^{-1} between the *two equations*,

$$1 = S.\sigma(1+r^{-1}\phi)^2\phi\sigma, \quad (N_2), \quad \text{and} \quad 0 = S.\sigma(1+r^{-1}\phi)^2\phi^2\sigma; \quad (N_2')$$

whereof the latter is the *derivative* of the former, with respect to the scalar r^{-1} . It follows (comp. p. 688), that the *First Sheet* of the *Surface of Centres* is touched by an *Auxiliary Quadric* (N_2), along a *Quartic Curve* (N_2) (N_2'), which curve is the *Locus of the Centres of First Curvature*, for all the points of a *Line of Second Curvature*; the *same sheet* being also touched (see again p. 688), along the *same curve*, by the *developable normal surface* (l), which rests on the *same second line*: with permission to interchange the words, *first* and *second*, throughout the whole of this enunciation.

(w). The given surface being still a central quadric (o), the vectors ρ, σ, ν can be expressed as functions of ν (comp. (j) (k) (l)),

* Dr. Salmon's result, that this surface of centres is of the *twelfth degree*, may be easily deduced from this form.

and conversely the latter can be expressed as a function of any one of the former; we have, for example, the reciprocal equations (p. 685),

$$v = (1 + r^{-1}\phi)^2 \phi^{-1}v, \quad (O_2), \quad \text{and} \quad w = (1 + r^{-1}\phi)^{-2} \phi w; \quad (O_2')$$

from which last the formula (N₂) may be obtained anew, by observing (k) that $Sv = 1$. Hence also, by (r), we can infer the expressions,*

$$\rho = (\phi^{-1} + r^{-1})v = \phi_2^{-1}v, \quad (P_2), \quad \text{and} \quad v = \phi_2\rho = v_2; \quad (P_2')$$

and in fact it is easy to see otherwise (comp. p. 645), that $v_2 \parallel r \parallel v$, and $S\rho v_2 = 1 = S\rho v$, whence $v_2 = v$ as before.

(x). More fully, the two sheets of the reciprocal (j) of the surface of centres may have their separate vector equations written thus,

$$v_1 = \phi_1\rho = v_1, \quad v_2 = \phi_2\rho = v_2; \quad (P_2'')$$

and the scalar equation† of this reciprocal surface itself, considered as including both sheets, may (by page 685) be thus written, the functions f and F being related as in 408, (b),

$$v^2 = (Fv - 1)fv, \quad (Q_2)$$

with several equivalent forms; one way of obtaining this equation being the elimination of r between the two following (same p. 685):

$$Fv + r^{-1}v^2 = 1, \quad (Q_2'); \quad fv + rv^2 = 0. \quad (Q_2'')$$

(y). The two last equations may also be written thus, for the first sheet of the reciprocal surface,

$$F_2v_1 = 1, \quad (R_2), \quad \text{and} \quad fUv_1 = r, \quad (R_2')$$

in which (comp. pp. 685, 689),

$$F_2v = S v \phi_2^{-1}v = Sv (\phi^{-1} + r^{-1})v; \quad (R_2'')$$

and accordingly (comp. pp. 483, 645), we have $F_2v_2 = Fv = 1$, and $fUv_2 = fv = r$.

(z). For a line of second curvatures on the given surface, the scalar r is constant, as before; and then the two equations (Q₂'), (Q₂''), or (R₂), (R₂'), represent jointly (comp. the slightly different enunciation in p. 688) a certain quartic curve, in which the quadric reciprocal (R₂), of the second confocal (a₁ b₁ c₁), intersects the first sheet (y) of the Reciprocal Surface (Q₂); this quartic curve, being at the same time the intersection of the quadric surface (Q₂'') or (R₂), with the quadric cone (Q₂'') or (R₂'), which is bicyclic with the given quadric, $f\rho = 1$.

* The equation $v = v_2$, = the normal to the confocal (a₂ b₂ c₂) at P , is not actually given in the text of Series 412; but it is easily deduced, as above, from the formulæ and methods of that Series.

† The equation (Q₂) is one of the fourth degree; and, when expanded by coordinates, it agrees perfectly with that which was first assigned by Dr. Booth (see a Note to p. 685), for the Tangential Equation of the Surface of Centres of a quadric, or for the Cartesian equation of the Reciprocal Surface.

ARTICLE 413.—On the Measure of Curvature of a Surface, . . .

The object of this short Series 413 is the deduction by quaternions, somewhat more briefly and perhaps more clearly than in the *Lectures*, of the principal results of Gauss (comp. Note to p. 690), respecting the *Measure of Curvature of a Surface*, and questions therewith connected.

(a). Let P, P_1, P_2 be any three near points on a given but arbitrary surface, and R, R_1, R_2 the three corresponding points (near to each other) on the unit sphere, which are determined by the parallelism of the radii OR, OR_1, OR_2 to the normals PN, P_1N_1, P_2N_2 ; then the areas of the two small triangles thus formed will bear to each other the ultimate ratio p. 690),

$$\lim. \frac{\Delta RR_1R_2}{\Delta PP_1P_2} = \frac{V. dU \nu \delta U \nu}{V d\rho \delta \rho} = -S \frac{1}{\nu} \psi \frac{1}{\nu}; \quad (S_2)$$

whence, with Gauss's definition of the measure of curvature, as the ultimate ratio of corresponding areas on surface and sphere, we have, by the formula (Z_1) in 412, (f), his fundamental theorem,

$$\text{Measure of Curvature} = R_1^{-1} R_2^{-1}, \quad (S_3)$$

= Product of the two Principal Curvatures of Sections.

(b). If the vector ρ of the surface be considered as a function of two scalar variables, t and u , and if derivations with respect to these be denoted by upper and lower accents, this general transformation results (p. 691),

$$\text{Measure of Curvature} = S \frac{\rho''}{\nu} S \frac{\rho''}{\nu} - \left(S \frac{\rho'}{\nu} \right)^2, \quad (T_2)$$

in which $\nu = V\rho' \rho$; (T₂)

with a verification for the notation *pqrst* of Monge.

(c). The square of a linear element ds , of the given but arbitrary surface, may be expressed (p. 691) as follows:

$$ds^2 = (Td\rho^2) = edt^2 + 2fdtdu + gdu^2; \quad (U_2)$$

and with the recent use (b) of accents, the measure (T_2) is proved (same page) to be an explicit function of the ten scalars,

$$e, f, g; \quad e', f', g'; \quad e'', f'', g''; \quad \text{and} \quad e, e', -2f' + g''; \quad (U_2')$$

the form of this function (p. 692) agreeing, in all its details, with the corresponding expression assigned by Gauss.*

(d). Hence follow at once (p. 692) two of the most important results of that great mathematician on this subject; namely, that every Deformation of a Surface, consistent with the conception of it as an infinitely thin and flexible but inextensible solid, leaves unaltered,

* References are given, in Notes to pp. 690, &c. of the present Series 413, to the pages of Gauss's beautiful Memoir, "*Disquisitiones generales circa Superficies Curvas*," as reprinted in the Additions to Liouville's Monge.

to the Measure of Curvature at any Point, and find, the Total Curvature of any Area : this last being the area of the corresponding portion (a) of the unit-sphere.

(f). By a suitable choice of t and u , as certain geodetic co-ordinates, the expression (U_2) may be reduced (p. 692) to the following,

$$ds^2 = dt^2 + n^2 du^2; \quad (U_2'')$$

where t is the length of a geodetic arc AP , from a fixed point A to a variable point P of the surface, and u is the angle BAP which this variable arc makes with a fixed geodetic AB : so that in the immediate neighbourhood of A , we have $n = t$, and $n' = D_t n = 1$.

(f). The general expression (c) for the measure of curvature takes thus the very simple form (p. 692),

$$R_1^{-1} R_2^{-1} = -n^{-1} n'' = -n^{-1} D_t^2 n; \quad (V_2)$$

and we have (comp. (d)) the equation (p. 693),

$$\text{Total Curvature of Area } APQ = \Delta u - \int n' du; \quad (V_2')$$

this area being bounded by two geodesics, AP and AQ , which make with each other an angle $= \Delta u$, and by an arc PQ of an arbitrary curve on the given surface, for which t , and therefore n' , may be conceived to be a given function of u .

(g). If this arc PQ be itself a geodesic, and if we denote by v the variable angle which it makes at P with AP prolonged, so that $\tan v = n du : dt$, it is found that $dv = -n' du$; and thus the equation (V_2') conducts (p. 693) to another very remarkable and general theorem of Gauss, for an arbitrary surface, which may be thus expressed,

$$\text{Total Curvature of a Geodetic Triangle } ABC = A + B + C - \pi, \quad (V_3'')$$

= what may be called the Spheroidal Excess of that triangle, the total area (4π) of the unit-sphere being represented by eight right angles : with extensions to Geodesic Polygons, and modifications for the case of what may on the same plan be called the Spheroidal Defect, when the two curvatures of the surface are oppositely directed.

ARTICLE 414.—On Curvatures of Sections (Normal and Oblique) of Surfaces; and on Geodetic Curvatures,

694—698

(a). The curvatures considered in the two preceding Series having been those of the principal normal sections of a surface, the present Series 414 treats briefly the more general case, where the section is made by an arbitrary plane, such as the osculating plane at p to an arbitrary curve upon the surface.

(b). The vector of curvatures (389) of any such curve or section being $(\rho - \kappa)^{-1} = D_s^2 \rho$, its normal and tangential components are found to be (p. 694),

$$(\rho - \sigma)^{-1} = \nu^{-1} S \frac{d\nu}{d\rho} = (\rho - \sigma_1)^{-1} \cos^2 v + (\rho - \sigma_2)^{-1} \sin^2 v, \quad (W_2)$$

$$\text{and} \quad (\rho - \xi)^{-1} = \nu^{-1} d\rho^{-1} S \nu d\rho^{-1} d^2 \rho; \quad (W_2')$$

the former component being the Vector of Normal Curvature of the

Surface, for the direction of the *tangent* to the curve : and the latter being the *Vector of Geodetic Curvature* of the same *Curve* (or section).

(c). In the foregoing expressions, σ and ξ are the vectors of the points s and x , in which the *axis of the osculating circle* to the curve intersects respectively the *normal* and the *tangent plane* to the *surface* (p. 694); s is also the *centre of the sphere*, which *osculates to the surface* in the direction $d\rho$ of the tangent; σ_1, σ_2 are the vectors of the *two centres* s_1, s_2 , of *curvature of the surface*, considered in Series 412, which are at the same time the centres of the *two osculating spheres*, of which the curvatures are (algebraically) the *greatest and least* : and ν is the *angle* at which the curve here considered crosses the *first line of curvature*.

(d). The equation (W_2) contains a theorem of Euler, under the form (p. 695),

$$R^{-1} = R_1^{-1} \cos^2 \nu + R_2^{-1} \sin^2 \nu ; \quad (W_2')$$

it contains also Meusnier's theorem (same page), under the form (comp. 412, (h)) that the *vector of normal curvature* (b) of a *surface*, for any given direction, is the *projection on the normal* ν , of the *vector of oblique curvature*, whatever the *inclination* of the plane of the section to the tangent plane may be.

(e). The expression (W_2') , for the *vector of geodetic curvature*, admits (p. 697) of various transformations, with corresponding expressions for the *radius* $T(\rho - \xi)$ of geodetic curvature, which is also the *radius of plane curvature* of the *developed curve*, when the developable circumscribed to the given surface along the given curve is *unfolded* into a plane : and when this radius is *constant*, so that the developed curve is a *circle*, or part of one, it is proposed (p. 698) to call the given curve a *Didonia* (as in the *Lectures*), from its possession of a certain *isoperimetrical property*, which was first considered by M. Delaunay, and is represented in quaternions by the formula (p. 697),

$$\delta \int S(U\nu \cdot d\rho d\rho) + e\delta \int Td\rho = 0 ; \quad (X_2)$$

or
$$\sigma^{-1}d\rho = V(U\nu \cdot dUd\rho), \quad (X_2')$$

by the rules of what may be called the *Calculus of Variations in Quaternions* : e being a constant, which represents generally (p. 698) the *radius* of the developed circle, and becomes *infinite* for *geodetic lines*, which are thus included as a *case of Didonias*.

ARTICLE 415.—Supplementary Remarks, 698-706

(a). Simplified proof (referred to in a Note to p. xii), of the general existence of a system of *three real and rectangular directions*, which satisfy the vector equation $V\rho\phi\rho = 0$, (P), when ϕ is a linear, vector, and *self-conjugate* function ; and of a system of *three real roots* of the cubic equation $M=0$ (p. xii), under the same condition (pp. 698-700).

(b). It may happen (p. 701) that the *differential equation*,

$$S\nu d\rho = 0, \quad (Y_2)$$

is integrable, or represents a system of surfaces, without the expression $S\rho d\rho$ being an exact differential, as it was in 410, (b). In this case, there exists some scalar factor, n , such that $Snv d\rho$ is the exact differential of a scalar function of ρ , without the assumption that this vector ρ is itself a function of a scalar variable, t ; and then if we write (pp. 701, 702, comp. p. xxx),

$$d\nu = \phi d\rho, \quad d.n\nu = \phi d\rho, \quad (Y_1')$$

this new vector function ϕ will be self-conjugate, although the function ϕ is not such now, as it was in the equation (U₁).

(c). In this manner it is found (p. 702), that the Condition* of Integrability of the equation (Y₂) is expressed by the very simple formula,

$$S\gamma\nu = 0; \quad (Y_2')$$

in which γ is a vector function of ρ , not generally linear, and deduced from ϕ on the plan of the Section III.ii. 6 (p. 442), by the relation,

$$\phi d\rho - \phi' d\rho = 2V\gamma d\rho; \quad (Y_2'')$$

ϕ' being the conjugate of ϕ , but not here equal to it.

(d). Connexions (pp. 702, 703) of the Mixed Transformations in the last cited Section, with the known Modular and Umbilicar Generations of a surface of the second order.

(e). The equation (p. 704),

$$T(\rho - V.\beta V\gamma a) = T(a - V.\gamma V\beta\rho), \quad (Z_2)$$

in which a , β , γ are any three vector constants, represents a central quadric, and appears to offer a new mode of generation† of such a surface, on which there is not room to enter, at this late stage of the work.

(f). The vector of the centre of the quadric, represented by the equation $S\rho - 2S\varepsilon\rho = \text{const.}$, with $S\rho = S\rho\phi\rho$, is generally $\kappa = \phi^{-1}\varepsilon = \kappa^{-1}\phi\varepsilon$ (p. 704); case of paraboloids, and of cylinders.

(g). The equation (p. 705),

$$Sqpq'\rho^2 + S\rho\phi\rho + S\gamma\rho + C = 0, \quad (Z_2')$$

represents the general surface of the third degree, or briefly the General Cubic Surface; C being a constant scalar, γ a constant vector, and q , q' , q'' three constant quaternions, while $\phi\rho$ is here again a linear, vector, and self-conjugate function of ρ .

(A). The General Cubic Cone, with its vertex at the origin, is thus represented in quaternions by the monomial equation (same page),

* It is shown, in a Note to p. 702, that this monomial equation (Y₂) becomes, when expanded, the known equation of six terms, which expresses the condition of integrability of the differential equation $pdx + qdy + rdx = 0$.

† In a Note to p. 649 (already mentioned in p. xxviii), the reader will find references to the Lectures, for several different generations of the ellipsoid, derived from quaternion forms of its equation.

$$Sq\rho q'\rho = 0. \quad (Z_2'')$$

(i). *Screw Surface, Screw Sections* (p. 705); *Skew Centre of Skew Arch*, with illustration by a diagram (Fig. 85, p. 706).

SECTION 8.—On a few Specimens of Physical Applications of Quaternions, with some Concluding Remarks, 707 to the end.

ARTICLE 416.—On the Statics of a Rigid Body, 707–709
 (a). *Equation of Equilibrium*,

$$\nabla\gamma\Sigma\beta = \Sigma\nabla\alpha\beta; \quad (A_3)$$

each α is a *vector of application*; β the corresponding *vector of applied force*; γ an *arbitrary vector*: and this *one* quaternion formula (A_3) is equivalent to the system of the *six* usual scalar equations ($X=0, Y=0, Z=0, L=0, M=0, N=0$).

(b). When $S(\Sigma\beta. \Sigma\nabla\alpha\beta) = 0$, (B_3), but *not* $\Sigma\beta = 0$, (C_3) the applied forces have an *unique resultant* $= \Sigma\beta$, which acts along the line whereof (A_3) is then the equation, with γ for its variable vector.

(c). When the condition (C_3) is satisfied, the forces compound themselves generally into *one couple*, of which the *axis* $= \Sigma\nabla\alpha\beta$, whatever may be the position of the assumed origin *o* of vectors.

(d). When $\Sigma\nabla\alpha\beta = 0$, (D_3), with or without (C_3), the forces have no tendency to turn the body round *that* point *o*; and when the equation (A_3) holds good, as in (a), for an *arbitrary vector* γ , the forces do not tend to produce a rotation* round *any* point *c*, so that they completely *balance* each other, as before, and *both* the conditions (C_3) and (D_3) are satisfied.

(e). In the general case, when *neither* (C_3) nor (D_3) is satisfied, if q be an *auxiliary quaternion*, such that

$$q\Sigma\beta = \Sigma\nabla\alpha\beta, \quad (E_3)$$

then Vq is the *vector perpendicular* from the origin, on the *central axis* of the system; and if $e = Sq$, then $e\Sigma\beta$ represents, both in quantity and in direction, the *axis of the central couple*.

(f). If Q be *another* auxiliary quaternion, such that

$$Q\Sigma\beta = \Sigma\alpha\beta, \quad (F_3)$$

with $T\Sigma\beta > 0$, then $SQ = e =$ *central moment* divided by *total force*;

* It is easy to prove that the *moment* of the force β , acting at the end of the vector α from *o*, and estimated with respect to any unit-line ι from the same origin, or the *energy* with which the force so acting tends to cause the body to turn round that line ι , regarded as a *fixed axis*, is represented by the scalar, $-S\iota\alpha\beta$, or $S\iota^{-1}\alpha\beta$; so that when the condition (D_3) is satisfied, the applied forces have no tendency to produce rotation round *any axis through the origin*: which origin becomes an *arbitrary point c*, when the *equation of equilibrium* (A_3) holds good.

and VQ is the vector γ of a point c upon the central axis which does not vary with the origin o , and which there are reasons for considering as the *Central Point* of the system, or as the *general centre of applied forces*: in fact, for the case of *parallelism*, this point c coincides with what is usually called the *centre of parallel forces*.

(*g*). Conceptions of the *Total Moment* $\Sigma\alpha\beta$, regarded as being generally a *quaternion*; and of the *Total Tension*, $-\Sigma\alpha\beta$, considered as a *scalar* to which that quaternion with its sign changed reduces itself for the case of *equilibrium* (*a*), and of which the value is in that case *independent of the origin of vectors*.

(*h*). *Principle of Virtual Velocities*,

$$\Sigma S\beta\delta\alpha = 0, \quad (G_3)$$

ARTICLE 417.—On the Dynamics of a Rigid Body, 709-713

(*a*). *General Equation of Dynamics*,

$$\Sigma mS(D_t^2\alpha - \xi)\delta\alpha = 0; \quad (H_3)$$

the vector ξ representing the accelerating force, or $m\xi$ the moving force, acting on a particle m of which the vector at the time t is α ; and $\delta\alpha$ being any infinitesimal variation of this last vector, geometrically compatible with the connexions between the parts of the system, which need not here be a rigid one.

(*b*). For the case of a *free system*, we may change each $\delta\alpha$ to $\epsilon + V_t\alpha$, ϵ and ϵ being any two infinitesimal vectors, which do not change in passing from one particle m to another; and thus the general equation (H_3) furnishes two general vector equations, namely,

$$\Sigma m(D_t^2\alpha - \xi) = 0, \quad (I_3), \quad \text{and} \quad \Sigma mV_t\alpha(D_t^2\alpha - \xi) = 0; \quad (J_3)$$

which contain respectively the law of the motion of the centre of gravity, and the law of description of areas.

(*c*). If a body be supposed to be rigid, and to have a fixed point o , then only the equation (J_3) need be retained; and we may write,

$$D_t\alpha = V_t\alpha, \quad (K_3)$$

ϵ being here a finite vector, namely the *Vector Axis of Instantaneous Rotation*: its versor U_t denoting the direction of that axis, and its tensor T_t representing the angular velocity of the body about it, at the time t .

(*d*). When the forces vanish, or balance each other, or compound themselves into a single force acting at the fixed point, as for the case of a heavy body turning freely about its centre of gravity, then

$$\Sigma mV_t\alpha\xi = 0, \quad (L_3); \quad \text{and if we write, } \phi_t = \Sigma m\alpha V_t\alpha, \quad (M_3)$$

so that ϕ again denotes a linear, vector, and self-conjugate function, we shall have the equations,

$$\phi D_t\alpha + V_t\phi\alpha = 0, \quad (N_3); \quad \phi_t + \gamma = 0, \quad (O_3); \quad S_t\phi_t = h^2; \quad (P_3)$$

$$\text{whence } S_t\gamma + h^2 = 0, \quad (Q_3), \quad \text{and } \phi D_t\alpha = V_t\gamma; \quad (R_3)$$

the vector γ being what we may call the *Constant of Areas*, and the scalar h^2 being the *Constant of Living Force*.

(e). One of Poinso't's representations of the *motion of a body*, under the circumstances last supposed, is thus reproduced under the form, that the *Ellipsoid of Living Force* (P_3), with its *centre* at the *fixed point* o , rolls *without gliding* on the *fixed plane* (Q_3), which is *parallel to the Plane of Areas* ($Sr\gamma = 0$); the *variable semidiameter of contact*, ι , being the *vector-axis* (ϵ) of *instantaneous rotation* of the body.

(f). The *Moment of Inertia*, with respect to *any axis* ι through o , is equal to the *living force* (h^2) divided by the *square* ($T\iota^2$) of the *semidiameter of the ellipsoid* (P_3), which has the direction of that axis; and hence may be derived, with the help of the first *general construction* of an ellipsoid, suggested by quaternions, a simple geometrical representation (p. 711) of the *square-root* of the moment of inertia of a body, with respect to *any axis* ΔD passing through a given point Δ , as a certain *right line* \overline{BD} , if $\overline{CD} = \overline{CA}$, with the help of two other points B and C , which are likewise fixed in the body, but may be chosen in more ways than one.

(g). A *cone* of the second degree,

$$S\iota\nu = 0, \quad (S_2), \quad \text{with } \nu = \gamma^2\phi\iota - h^2\phi^2\iota, \quad (T_3)$$

is *fixed* in the *body*, but *rolls* in *space* on that *other cone*, which is the *locus* of the *instantaneous axis* ι ; and thus a *second representation*, proposed by Poinso't, is found for the *motion of the body*, as the *rolling of one cone on another*.

(h). Some of Mac Cullagh's results, respecting the motion here considered, are obtained with equal ease by the same quaternion analysis; for example, the line γ , although *fixed* in *space*, describes in the *body* an easily assigned *cone of the second degree* (p. 712), which cuts the *reciprocal ellipsoid*,

$$S\gamma\phi^{-1}\gamma = h^2, \quad (U_3)$$

in a certain *sphero-conic*: and the *cone of normals* to the last mentioned cone (or the locus of the line $\iota + h^2\gamma^{-1}$) *rolls on the plane of areas* ($Sr\gamma = 0$).

(i). The *Three (Principal) Axes of Inertia* of the body, for the given point o , have the *directions* (p. 712) of the *three rectangular and vector roots* (comp. (P), p. xii., and the paragraph 415, (a), p. xlii.) of the equation

$$V\iota\phi\iota = 0, \quad (V_3), \quad \text{because, for each, } D\iota = 0; \quad (V_3')$$

and if A, B, C denote the three *Principal Moments* of inertia corresponding, then the *Symbolical Cubic* in ϕ (comp. the formula (N) in page xii.) may be thus written,

$$(\phi + A)(\phi + B)(\phi + C) = 0. \quad (W_3)$$

(j). Passage (p. 713), from moments referred to axes passing through a given point o , to those which correspond to respectively parallel axes, through any other point Ω of the body.

ARTICLE 418.—On the motions of a System of Bodies, considered as free particles m, m', \dots which attract each other according to the law of the Inverse Square 718-717

(a). Equation of motion of the system,

$$\Sigma m S D^2 a \delta a + \delta P = 0, \quad (X_2), \quad \text{if } P = \Sigma m m' T (a - a')^{-1}; \quad (Y_2)$$

a is the vector, at the time t , of the mass or particle m ; P is the potential (or force-function); and the infinitesimal variations δa are arbitrary.

(b). Extension of the notation of derivatives,

$$\delta P = \Sigma S (D_a P \cdot \delta a). \quad (Z_2)$$

(c). The differential equations of motion of the separate masses m, \dots become thus,

$$m D_t^2 a + D_a P = 0, \dots; \quad (A_4)$$

and the laws of the centre of gravity, of areas, and of living force, are obtained under the forms,

$$\Sigma m D_t a = \beta, \quad (B_4); \quad \Sigma m V a D_t a = \gamma; \quad (C_4)$$

$$\text{and} \quad T = -\frac{1}{2} \Sigma m (D_t a)^2 = P + H; \quad (D_4)$$

β, γ being two vector constants, and H a scalar constant.

(d). Writing,

$$F = \int_0^t (P + T) dt, \quad (E_4), \quad \text{and} \quad V = \int_0^t 2 T dt = F + tH, \quad (F_4)$$

F may be called the *Principal* Function*, and V the *Characteristic Function*, of the motion of the system; each depending on the final vectors of position, a, a', \dots and on the initial vectors, a_0, a'_0, \dots ; but F depending also (explicitly) on the time, t , while V (= the Action) depends instead on the constant H of living force, in addition to those final and initial vectors: the masses m, m', \dots being supposed to be known, or constant.

(e). We are led thus to equations of the forms,

$$m D_t a + D_a F = 0, \dots (G_4); \quad -m D_0 a + D_{a_0} F = 0, \dots (H_4);$$

$$(D_t F) = -H, \quad (I_4)$$

whereof the system (G_4) contains what may be called the *Intermediate Integrals*, while the system (H_4) contains the *Final Integrals*, of the differential *Equations of Motion* (A_4) .

(f). In like manner we find equations of the forms,

$$D_a F = -m D_t a, \dots (J_4); \quad D_{a_0} F = m D_0 a, \dots (K_4); \quad D_H V = t; \quad (L_4)$$

the *intermediate integrals* (e) being here the result of the elimination

* References are given to two Essays by the present writer, "On a General Method in Dynamics," in the *Philosophical Transactions* for 1884 and 1885, in which the Action (V), and a certain other function (S), which is here denoted by F , were called, as above, the *Characteristic* and *Principal Functions*. But the analysis here used, as being founded on the *Calculus of Quaternions*, is altogether unlike the analysis which was employed in those former Essays.

of H , between the system (J_4) and the equation (L_4); and the *final integrals*, of the same system of differential equations (A_4), being now (theoretically) obtained, by eliminating the same constant H between (K_4) and (L_4).

(g). The functions F and V are obliged to satisfy certain *Partial Differential Equations in Quaternions*, of which those relative to the final vectors a, a', \dots are the following,

$$(D_t F) - \frac{1}{2} \Sigma m^{-1} (D_a F)^2 = P, (M_4); \quad \frac{1}{2} \Sigma m^{-1} (D_a V)^2 + P + H = 0; (N_4)$$

and they are subject to certain geometrical conditions, from which can be deduced, in a new way, and as new verifications, the law of motion of the centre of gravity, and the law of description of areas.

(h). General approximate expressions (p. 717) for the functions F and V , and for their derivatives H and t , for the case of a *short motion* of the system.

ARTICLE 419.—On the Relative Motion of a Binary System; and on the Law of the Circular Hodograph,

717-733

(a). The vector of one body from the other being a , and the distance being $r (= T a)$, while the sum of the masses is M , the differential equation of the relative motion is, with the law of the inverse square,

$$D^2 a = M a^{-1}, -1; \quad (O_4)$$

D being here used as a characteristic of derivation, with respect to the time t .

(b). As a first integral, which holds good also for any *other law of central force*, we have

$$V a D a = \beta = \text{a constant vector}; \quad (P_4)$$

which includes the two usual laws, of the *constant planes* ($\perp \beta$), and of the *constant areal velocity* $\left(\frac{c}{2} = \frac{1}{2} T \beta\right)$.

(c). Writing $r = D a = \text{vector of relative velocity}$, and conceiving this new vector r to be drawn from that one of the two bodies which is here selected for the origin o , the locus of the extremities of the vector r is (by earlier definitions) the *Hodograph of the Relative Motion*; and this hodograph is proved to be, for the *Law of the Inverse Square*, a *Circle*.

(d). In fact, it is shown (p. 720), that for any *law of central force*, the *radius of curvature* of the hodograph is equal to the force, multiplied into the square of the distance, and divided by the doubled areal velocity; or by the *constant parallelogram* c , under the vectors (a and r) of *position* and *velocity*, or of the *orbit* and the *hodograph*.

(e). It follows then, conversely, that the law of the inverse square is the *only law* which renders the hodograph generally a *circle*; so that the law of nature may be characterized, as the *Law of the Circular Hodograph*: from which latter law, however, it is easy to deduce the *form of the Orbit*, as a *conic section* with a *focus* at o .

(f). If the *semiparameter* of this orbit be denoted, as usual, by p , and if h be the *radius* of the *hodograph*, then (p. 719),

$$h = Mc^{-1} = cp^{-1} = (Mp^{-1})h. \quad (Q_4)$$

(g). The orbital *eccentricity* e is also the *hodographic eccentricity*, in the sense that $e\bar{h}$ is the distance of the centre π of the *hodograph*, from the point o which is here treated as the centre of force.

(A). The orbit is an *ellipse*, when the point o is *interior* to the *hodographic circle* ($e < 1$); it is a *parabola*, when o is *on the circumference* of that circle ($e = 1$); and it is an *hyperbola*, when o is an *exterior point* ($e > 1$). And in all these cases, if we write

$$a = p(1 - e^2)^{-1} = e\bar{h}^{-1}(1 - e^2)^{-1}, \quad (R_4)$$

the constant a will have its usual signification, relatively to the orbit.

(i). The quantity Mr^{-1} being here called the *Potential*, and denoted by P , *geometrical constructions* for this quantity P are assigned, with the help of the *hodograph* (p. 723); and for the *harmonic mean*, $2M(r + r')^{-1}$, between the *two potentials*, P and P' , which answer to the *extremities* τ , τ' of any proposed *chord* of that circle: all which *constructions* are illustrated by a new diagram (Fig. 86).

(j). If v be the *pole* of the chord $\tau\tau'$; m , m' the points in which the line ov cuts the circle; l the middle point, and n the pole, of the *secant chord* mm' , one secant from which last pole is thus the line $nm\tau\tau'$; v' the intersection of *this secant* with the chord mm' , or the *harmonic conjugate* of the point v , with respect to the same chord; and $n\tau, n\tau'$ any *near secant* from n , while u , (on the line ov) is the pole of the *near chord* τ, τ' : then the *two small arcs*, τ, τ' and $\tau\tau'$, of the *hodograph*, intercepted *between these two secants*, are proved to be ultimately *proportional* to the *two potentials*, P and P' ; or to the *two ordinates* τv , $\tau' v'$, namely the perpendiculars let fall from τ and τ' , on what may here be called the *hodographic axis* LN . Also, the *harmonic mean* between these two ordinates is obviously (by the construction) the line $u'L$; while $u\tau$, $u\tau'$, and u, τ, u, τ' are *four tangents* to the *hodograph*, so that *this circle is cut orthogonally*, in the *two pairs of points*, τ, τ' and τ, τ' , by *two other circles*, which have the two near points u, u' for their *centres* (pp. 724, 725).

(k). In general, for *any motion of a point* (absolute or relative, in one plane or in space, for example, in the motion of the centre of the *moon* about that of the *earth*, under the perturbations produced by the attractions of the sun and planets), with a for the *variable vector* (418) of *position* of the point, the *time* dt which corresponds to any *vector-element* $dD\alpha$ of the *hodograph*, or what may be called the *time of hodographically describing* that element, is the *quotient* obtained by *dividing* the same element of the *hodograph*, by the *vector of acceleration* D^2a in the orbit; because we may write generally (p. 724),

$$dt = \frac{dD\alpha}{D^2a}, \quad \text{or} \quad dt = \frac{TdD\alpha}{TD^2a}, \quad \text{if } dt > 0. \quad (S_4)$$

(l). For the law of the *inverse square* (comp. (a) and (f)), the measure of the *forces* is,

$$TD^2\alpha = Mr^{-2} = M^{-1}P^2; \quad (T_4)$$

the *times* dt , dt' , of hodographically describing the small circular arcs $\tau\tau$ and $\tau'\tau'$, of the hodograph, being found by multiplying the lengths (j) of those two arcs by the mass, and dividing each product by the square of the potential corresponding, are therefore *inversely* as those *two potentials*, P , P' , or *directly* as the *distances*, r , r' , in the orbit: so that we have the proportion,

$$dt : dt' : dt + dt' = r : r' : r + r'. \quad (U_4)$$

(m). If we suppose that the *mass*, M , and the *five points* o , l , m , v , v' , upon the chord mm' are *given*, or *constant*, but that the *radius*, h , of the *hodograph*, or the position of the *centre* κ on the *hodographic axis* LN , is *altered*, it is found in this way (p. 725) that although the *two elements of time*, dt , dt' , *separately vary*, yet their *sum* remains *unchanged*: from which it follows, that even if the *two circular arcs*, $\tau\tau$, $\tau'\tau'$, be *not small*, but still intercepted (j) between *two secants* from the *pole* κ of the *fixed chord* mm' , the *sum* (say, $\Delta t + \Delta t'$) of the *two times* is *independent of the radius*, h .

(n). And hence may be deduced (p. 726), by supposing *one secant* to become a *tangent*, this *Theorem of Hodographic Isochronism*, which was communicated without demonstration, several years ago, to the Royal Irish Academy,* and has since been treated as a subject of investigation by several able writers:

If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.

(o). This *common time* can easily be expressed (p. 726), under the form of the definite integral,

$$\text{Time of } \tau\tau\tau' = \frac{2M}{g^2} \int_0^w \frac{dw}{(1 - e' \cos w)^2}; \quad (V_4)$$

$2g$ being the length of the *fixed chord* mm' ; e' the quotient $LO : LM$, which reduces itself to -1 when o is at m' , that is for the case of a *parabolic orbit*; e' lying between ± 1 for an *ellipse*, and *outside* those limits for an *hyperbola*, but being, in all these cases, *constant*; while w is a certain *auxiliary angle*, of which the *sine* $= \overline{OT} : \overline{OL}$ (p. 727), or $= s(r + r')^{-1}$, if s denote the length $\overline{PP'}$ of the *chord of the orbit*, corresponding to the *chord* $\tau\tau'$ of the *hodograph*; and w varies from 0 to π , when the whole *periodic time* $2\pi n^{-1}$ for a *closed orbit* is to be computed: with the verification, that the integral (V₄) gives, in this last case,

$$M = a^2 n^2, \text{ as usual.} \quad (W_4)$$

* See the *Proceedings* of the 16th of March, 1847. It is understood that the *common centre* o of *force* is occupied by a *common mass*, M .

(p). By examining the general composition of the definite integral (V_1), or by more purely geometrical considerations, which are illustrated by Fig. 87, it is found that, with the law of the inverse square, the time t of describing an *arc* FF' of the orbit (closed or un-
closed) is a function (p. 729) of the three ratios,

$$\frac{a^2}{M}, \quad \frac{r+r'}{a}, \quad \frac{s}{r+r'}; \quad (X_4)$$

and therefore simply a function of the chord (s , or $\overline{FF'}$) of the orbit, and of the sum of the distances ($r+r'$, or $\overline{OF} + \overline{OF'}$) when M and a are given: which is a form of the *Theorem of Lambert*.

(q). The same important theorem may be otherwise deduced, through a quite different analysis, by an employment of *partial derivatives*, and of *partial differential equations in quaternions*, which is analogous to that used in a recent investigation (418), respecting the motions of an *attracting system of any number of bodies*, $m, m', \&c.$

(r). Writing now (comp. p. xlvi) the following expression for the *relative living force*, or for the mass ($M = m + m'$), multiplied into the square of the *relative velocity* ($TD\alpha$),

$$2T = -MD\alpha^2 = 2(P+H) = M(2r^{-1} - a^{-1}); \quad (Y_4)$$

introducing the two new integrals (p. 729),

$$F = \int_0^t (P+T) dt, \quad (Z_4), \quad \text{and} \quad V = \int_0^t 2T dt = F + tH, \quad (A_4)$$

which have thus (comp. (E₄) and (F₄)) the same forms as before, but with *different* (although *analogous*) *significations*, and may still be called the *Principal* and *Characteristic Functions* of the motion; and denoting by a, a' (instead of a_0, a) the *initial* and *final vectors* of position, or of the orbit, while r, r' are the two distances, and τ, τ' the two corresponding vectors of velocity, or of the *hodograph*: it is found that when M is given, F may be treated as a function of a, a', t , or of r, r', s, t , and V as a function of a, a', a , or of r, r', s , and H ; and that their *partial derivatives*, in the first view of these two functions, are (p. 729),

$$D_a F = D_a V = \tau, \quad (B_5); \quad D_{a'} F = D_{a'} V = -\tau'; \quad (C_5)$$

$$(D_1) F = -H, \quad (D_5); \quad \text{and} \quad D_H V = \frac{2a^2}{M} D_a V = t; \quad (E_5)$$

while, in the second view of the same functions, they satisfy the two *partial differential equations* (p. 730),

$$D_r F = D_{r'} F, \quad (F_5), \quad \text{and} \quad D_r V = D_{r'} V; \quad (G_5)$$

along with two other equations of the same kind, but of the second degree, for each of the functions here considered, which are analogous to those mentioned in p. xlvi.

(s). The equations (F₅) (G₅) express, that the two distances, r and r' , enter into each of the two functions only by their *sum*; so that, if M be still treated as given, F may be regarded as a function of the

three quantities, $r+r'$, s , and t ; while V , and therefore also t by (E_5) , is found in like manner to be a function of the three scalars, $r+r'$, s , and a : which last result respecting the time agrees with (p) , and furnishes a new proof of Lambert's Theorem.

(t). The three partial differential equations (r) in V conduct, by merely algebraical combinations, to expressions for the three partial derivatives, $D_r V$, $D_{r'} V (= D_{r'} V)$, and $D_s V$; and thus, with the help of (E_5) , to two new definite integrals* (p. 731), which express respectively the Action and the Time, in the relative motion of a binary system here considered, namely, the two following :

$$V = \int_{-s}^s \left(\frac{M}{r+r'+s} - \frac{M}{4a} \right)^{\frac{1}{2}} ds; \tag{H_5}$$

$$t = \frac{1}{2} \int_{-s}^s \left(\frac{4M}{r+r'+s} - \frac{M}{a} \right)^{-\frac{1}{2}} ds; \tag{I_5}$$

whereof the latter is not to be extended, without modification, beyond the limits within which the radical is finite.

ARTICLE 420.—On the determination of the Distance of a Comet, or new Planet, from the Earth, 733, 734

(a). The masses of earth and comet being neglected, and the mass of the sun being denoted by M , let r and w denote the distances of earth and comet from sun, and s their distance from each other, while a is the heliocentric vector of the earth ($Ta=r$), known by the theory of the sun, and ρ is the unit-vector, determined by observation, which is directed from the earth to the comet. Then it is easily proved by quaternions, that we have the equation (p. 734),

$$\frac{8\rho D\rho D^2\rho}{8\rho D\rho Ua} = \frac{r}{z} \left(\frac{M}{r^3} - \frac{M}{w^3} \right), \tag{J_5}$$

with $w^2 = r^2 + s^2 - 2s\rho a\rho; \tag{K_5}$

eliminating w between these two formulæ, clearing of fractions, and dividing by z , we are therefore conducted in this way to an algebraical equation of the seventh degree, whereof one root is the sought distance, s .

(b). The final equation, thus obtained, differs only by its notation, and by the facility of its deduction, from that assigned for the same purpose in the *Mécanique Céleste*; and the rule of Laplace there given, for determining, by inspection of a celestial globe, which of the two

* References are given to the *First Essay*, &c., by the present writer (comp. the Note to p. xlvii.), in which were assigned integrals, substantially equivalent to (H_5) and (I_5) , but deduced by a quite different analysis. It has recently been remarked to him, by his friend Professor Tait of Edinburgh, that while the area described, with Newton's Law, about the full focus of an orbit, has long been known to be proportional to the time corresponding, so the area about the empty focus represents (or is proportional to) the action.

direction (earth and comet) is the *measurer to the sun*, results at sight from the formula (J₅).

734-736

ARTICLE 421.—On the Development of the Disturbing Force of the Sun on the Moon; or of one Planet on another, which is nearer than itself to the Sun,

(a) Let α , σ be the geocentric vectors of moon and sun; $r(=T\alpha)$, and $s(=T\sigma)$, their geocentric distances; M the sum of the masses of earth and moon; S the mass of the sun; and D (as in recent Series) the mark of derivation with respect to the time: then the differential equation of the disturbed motion of the moon about the earth is,

$$D^2\alpha = M\phi\alpha + \eta, \quad (L_5), \quad \text{if } \phi\alpha = \phi(\alpha) = \alpha^{-1}T\alpha^{-1}, \quad (M_5)$$

$$\text{and } \eta = \text{Vector of Disturbing Force} = S(\phi\sigma - \phi(\sigma - \alpha)); \quad (N_5)$$

ϕ denoting here a vector function, but not a linear one.

(b) If we neglect η , the equation (L₅) reduces itself to the form $D^2\alpha = M\phi\alpha$; which contains (comp. (O₄)) the laws of undisturbed elliptic motion.

(c) If we develop the disturbing vector η , according to ascending powers of the quotient $r : s$, of the distances of moon and sun from the earth, we obtain an infinite series of terms, each representing a *series group* of partial disturbing forces, which may be thus denoted,

$$\eta = \eta_1 + \eta_2 + \eta_3 + \&c.; \quad (O_5)$$

$$\eta_1 = \eta_{1,1} + \eta_{1,2}, \quad \eta_2 = \eta_{2,1} + \eta_{2,2} + \eta_{2,3}, \quad \&c.; \quad (P_5)$$

these partial forces increasing in number, but diminishing in intensity, in the passage from any one group to the following; and being connected with each other, within any such group, by simple numerical ratios and angular relations.

(d) For example, the two forces $\eta_{1,1}$, $\eta_{1,2}$ of the first group are, rigorously, proportional to the numbers 1 and 3; the three forces $\eta_{2,1}$, $\eta_{2,2}$, $\eta_{2,3}$ of the second group are as the numbers 1, 2, 5; and the four forces of the third group are proportional to 5, 9, 15, 35: while the separate intensities of the first forces, in these three first groups, have the expressions,

$$T\eta_{1,1} = \frac{Sr}{2s^3}; \quad T\eta_{2,1} = \frac{3Sr^2}{8s^4}; \quad T\eta_{3,1} = \frac{5Sr^3}{16s^5}. \quad (Q_5)$$

(e) All these partial forces are conceived to act at the moon; but their directions may be represented by the respectively parallel unit-lines $U\eta_{1,1}$, &c., drawn from the earth, and terminating on a great circle of the celestial sphere (supposed here to have its radius equal to unity), which passes through the geocentric (or apparent) places, \odot and \textcircled{D} , of the sun and moon in the heavens.

(f) Denoting then the geocentric elongation $\odot\textcircled{D}$ of moon from sun (in the plane of the three bodies) by $+\theta$; and by \odot_1 , \odot_2 , and \textcircled{D}_1 , \textcircled{D}_2 , \textcircled{D}_3 what may be called two fictitious suns, and three fictitious moons, of which the corresponding elongations from \odot , in the same great

circle, are $+2\theta$, -2θ , and $-\theta$, $+3\theta$, -3θ , as illustrated by Fig. 88 (p. 785); it is found that the *directions* of the *two forces* of the *first group* are represented by the *two radii* of this *unit-circle*, which terminate in \mathcal{D} and \mathcal{D}_1 ; those of the *three forces* of the *second group*, by the three radii to \odot_1 , \odot , and \odot_2 ; and those of the *four forces* of the *third group*, by the radii to \mathcal{D}_2 , \mathcal{D} , \mathcal{D}_1 , and \mathcal{D}_3 ; with facilities for *extending* all these results (with the requisite modifications), to the *fourth* and *subsequent groups*, by the *same quaternion analysis*.

(g). And it is important to observe, that *no supposition* is here made respecting any *smallness* of *excentricities* or *inclinations* (p. 786); so that *all the formulæ apply*, with the necessary changes of *geocentric* to *heliocentric* vectors, &c., to the *perturbations* of the *motion* of a *comet about the sun*, produced by the *attraction* of a *planet*, which is (at the time) *more distant* than the comet from the sun.

ARTICLE 422.—On Fresnel's Wave, 786-756

(a). If ρ and μ be two corresponding vectors, of *ray-velocity* and *wave-slowness*, or briefly *Ray* and *Index*, in a *biaxial crystal*, the velocity of light in a vacuum being unity; and if $\delta\rho$ and $\delta\mu$ be any infinitesimal *variations* of these two vectors, consistent with the equations (supposed to be as yet unknown), of the *Wave* (or *wave-surface*), and its *reciprocal*, the *Index-Surface* (or *surface of wave-slowness*): we have then first the *fundamental Equations of Reciprocity* (comp. p. 417),

$$S\mu\rho = -1, (R_s); \quad S\mu\delta\rho = 0, (S_s); \quad S\rho\delta\mu = 0, (T_s)$$

which are independent of any hypothesis respecting the *vibrations* of the *ether*.

(b). If $\delta\rho$ be next regarded as a *displacement* (or *vibration*), *tangential* to the *wave*, and if $\delta\varepsilon$ denote the *elastic force* resulting, there exists then, on Fresnel's principles, a *relation* between these two small vectors; which relation may (with our notations) be expressed by either of the two following equations,

$$\delta\varepsilon = \phi^{-1}\delta\rho, (U_s), \quad \text{or} \quad \delta\rho = \phi\delta\varepsilon; \quad (V_s)$$

the function ϕ being of that *linear*, *vector*, and *self-conjugate* kind, which has been frequently employed in these *Elements*.

(c). The *fundamental connexion*, between the *functional symbol* ϕ , and the *optical constants* *abc* of the *crystal*, is expressed (p. 741, comp. the formula (W_s) in p. xlvi) by the *symbolic and cubic equation*,

$$(\phi + a^{-2})(\phi + b^{-2})(\phi + c^{-2}) = 0; \quad (W_s)$$

of which an *extensive use* is made in the present *Series*.

(d). The *normal component*, $\mu^{-1}S\mu\delta\varepsilon$, of the *elastic force* $\delta\varepsilon$, is *ineffective* in Fresnel's theory, on account of the supposed *incompressibility* of the *ether*; and the *tangential component*, $\phi^{-1}\delta\rho - \mu^{-1}S\mu\delta\varepsilon$, is (in the same theory, and with present notations) to be equated to

ρ^2 , for the propagation of a *rectilinear vibration* (p. 737); we obtain then thus, for such a vibration or *tangential displacement*, $\delta\rho$, the expression,

$$\delta\rho = (\phi^{-1} - \mu^{-2})^{-1} \mu^{-1} S\mu \delta\varepsilon; \quad (X_4)$$

and therefore by (S₂) the equation,

$$0 = S\mu^{-1} (\phi^{-1} - \mu^{-2})^{-1} \mu^{-1}, \quad (Y_4)$$

which is a *Symbolical Form* of the scalar Equation of the Index-Surface, and may be thus transformed,

$$1 = S\mu (\mu^2 - \phi)^{-1} \mu. \quad (Z^2)$$

(r). The *Wave-Surface*, as being the reciprocal (a) of the index-surface (s), is easily found (p. 738) to be represented by this other *Symbolical Equation*,

$$0 = S\rho^{-1} (\phi - \rho^{-2})^{-1} \rho^{-1}; \quad (A_4)$$

$$1 = S\rho (\rho^2 - \phi^{-1})^{-1} \rho. \quad (B_4)$$

(f). In such transitions, from one of these reciprocal surfaces to the other, it is found convenient to introduce *two auxiliary vectors*, v and $\omega (= \phi v)$, namely the lines ov and $o\omega$ of Fig. 89; both drawn from the *common centre* o of the two surfaces; but v terminating (p. 735) on the *tangent plane* to the *wave*, and being *parallel* to the direction of the *elastic force* $\delta\varepsilon$; whereas ω terminates (p. 739) on the *tangent plane* to the *index-surface*, and is parallel to the *displacement* $\delta\rho$.

(g). Besides the relation,

$$\omega = \phi v, \text{ or } v = \phi^{-1} \omega, \quad (C_4)$$

connecting the two new vectors (f) with each other, they are connected with ρ and μ by the equations (pp. 738, 739),

$$S\mu v = -1, \quad (D_4); \quad S\rho v = 0; \quad (E_4)$$

$$S\rho \omega = -1, \quad (F_4); \quad S\mu \omega = 0; \quad (G_4)$$

and generally (p. 739), the following *Rule of the Interchanges* holds good: In any formula involving ρ , μ , v , ω , and ϕ , or some of them, it is permitted to exchange ρ with μ , v with ω , and ϕ with ϕ^{-1} ; provided that we at the same time interchange $\delta\rho$ with $\delta\varepsilon$, but not generally $\delta\mu$ with $\delta\rho$, when these variations, or any of them occur.

(A). We have also the relations (pp. 739, 740),

$$-\rho^{-1} = v^{-1} \nabla v \mu = \mu + v^{-1}; \quad (H_4)$$

$$-\mu^{-1} = \omega^{-1} \nabla \omega \rho = \rho + \omega^{-1}; \quad (I_4)$$

* This apparent exception arises (pp. 739, 740) from the circumstance, that $\delta\rho$ and $\delta\varepsilon$ have their directions generally fixed, in this whole investigation (although subject to a common reversal by \pm), when ρ and μ are given; whereas $\delta\mu$ continues to be used, as in (a), to denote any infinitesimal vector, tangential to the index-surface at the end of μ .

with others easily deduced, which may all be illustrated by the above-cited Fig. 89.

(*i*). Among such deductions, the following equations (p. 740) may be mentioned,

$$(\nabla v \phi v)^2 + S v \phi v = 0, \quad (J_6); \quad (\nabla \omega \phi^{-1} \omega)^2 + S \omega \phi^{-1} \omega = 0; \quad (K_6)$$

which show that the *Locus* of each of the two *Auxiliary Points*, v and ω , wherein the two vectors v and ω terminate (*f*), is a *Surface of the Fourth Degree*, or briefly, a *Quartic Surface*; of which *two loci* the constructions may be connected (as stated in p. 741) with those of the two *reciprocal ellipsoids*,

$$S \rho \phi \rho = 1, \quad (L_6), \quad \text{and} \quad S \rho \phi^{-1} \rho = 1; \quad (M_6)$$

ρ denoting, for each, an *arbitrary semidiameter*.

(*j*). It is, however, a much more interesting use of these two *ellipsoids*, of which (by (W_6) , &c.) the *scalar semiaxes* are a, b, c for the *first*, and a^{-1}, b^{-1}, c^{-1} for the *second*, to observe that they may be employed (pp. 738, 739) for the *Constructions* of the *Wave* and the *Index-Surface*, respectively, by a very simple rule, which (at least for the *first* of these two reciprocal surfaces (*a*)) was assigned by Fresnel himself.

(*k*). In fact, on comparing the *symbolical form* (A_6) of the equation of the *Wave*, with the form (H_6) in p. xxxvii, or with the equation 412, XLI., in p. 683, we derive at once *Fresnel's Construction*: namely, that if the *ellipsoid* (abc) be cut, by an arbitrary *plane* through its centre, and if *perpendiculars* to that plane be erected at that central point, which shall have the *lengths* of the *semiaxes* of the *section*, then the *locus* of the *extremities*, of the *perpendiculars* so erected, will be the *sought Wave-Surface*.

(*l*). A precisely similar construction applies, to the derivation of the *Index-Surface* from the ellipsoid $(a^{-1}b^{-1}c^{-1})$: and thus the two *auxiliary surfaces*, (L_6) and (M_6) , may be briefly called the *Generating Ellipsoid*, and the *Reciprocal Ellipsoid*.

(*m*). The cubic (W_6) in ϕ enables us easily to express (p. 741) the *inverse function* $(\phi + e)^{-1}$, where e is any scalar; and thus, by changing e to $-\rho^{-2}$, &c., *new forms* of the equation (A_6) of the *wave* are obtained, whereof one is,

$$0 = (\phi^{-1} \rho)^2 + (\rho^2 + a^2 + b^2 + c^2) S \rho \phi^{-1} \rho - a^2 b^2 c^2; \quad (N_6)$$

with an analogous equation in μ (comp. the *rule* in (*g*)), to represent the *index-surface*: so that each of these two surfaces is of the *fourth degree*, as indeed is otherwise known.

(*n*). If either $S \rho \phi^{-1} \rho$ or ρ^2 be treated as *constant* in (N_6) , the *degree* of that equation is depressed from the *fourth* to the *second*; and therefore the *Wave* is cut, by each of the two *concentric quadrics*,

$$S \rho \phi^{-1} \rho = h^2, \quad (O_6), \quad \rho^2 + r^2 = 0, \quad (P_6)$$

in a (real or imaginary) *curve* of the *fourth degree*: of which two *quar-*

curves, answering to all scalar values of the constants h and r , the wave is the common locus.

(c). The new ellipsoid (O_e) is similar to the ellipsoid (M_e), and similarly placed, while the sphere (P_e) has r for radius; and every part of the second system (n) is a sphero-conic, because it is, by the equation (A_e) of the wave, the intersection of that sphere (P_e) with the concentric and quadric cone,

$$0 = S\rho (\phi + r^2)^{-1}\rho; \quad (Q_e)$$

or, by (B_e), with this other concentric quadric,*

$$-1 = S\rho (\phi^{-1} + r^2)^{-1}\rho, \quad (R_e)$$

whereof the conjugate (obtained by changing -1 to $+1$ in the last equation) has

$$a^2 - r^2, b^2 - r^2, c^2 - r^2, \quad (S_e)$$

for the squares of its scalar semiaxes, and is therefore confocal with the generating ellipsoid (L_e).

(p). For any point p of the wave, or at the end of any ray ρ , the tangents to the two curves (n) have the directions of ω and $\mu\omega$; so that these two quartics cross each other at right angles, and each is a common orthogonal in all the curves of the other system.

(q). But the vibration $\delta\rho$ is easily proved to be parallel to ω ; hence the curves of the first system (n) are Lines of Vibration of the Wave: and the curves of the second system are the Orthogonal Trajectories† to those Lines.

(r). In general, the vibration $\delta\rho$ has (on Fresnel's principles) the direction of the projection of the ray ρ on the tangent plane to the wave; and the elastic force $\delta\varepsilon$ has in like manner the direction of the projection of the index-vector μ on the tangent plane to the index-surface: so that the ray is thus perpendicular to the elastic force

ARTICLE 423.—Mac Cullagh's Theorem of the Polar Plane, . . . 757-762

* * * * *
* * * * *

* For real curves of the second system (n), this new quadric (R_e) is an hyperboloid, with one sheet or with two, according as the constant r lies between a and b , or between b and c ; and, of course, the conjugate hyperboloid (ρ) has two sheets or one, in the same two cases respectively.

† In a different theory of light (comp. the next Series, 423), these sphero-conics on the wave are themselves the lines of vibration.

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* This Table was mentioned in the Note to p. xiv. of the Contents, as one likely to facilitate reference. In fact, the references in the text of the Elements are almost entirely to Articles (with their sub-articles), and not to pages.

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NOTE.—It appears by these Tables that the Author intended to have completed the work by the addition of Seven Articles, and Two Figures.—Ed.



ELEMENTS OF QUATERNIONS.

BOOK I.

ON VECTORS, CONSIDERED WITHOUT REFERENCE TO ANGLES,
OR TO ROTATIONS.

CHAPTER I.

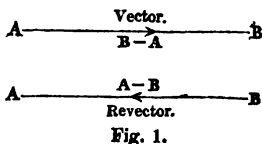
FUNDAMENTAL PRINCIPLES RESPECTING VECTORS.

SECTION 1.— *On the Conception of a Vector; and on Equality of Vectors.*

ART. 1.—A right line AB , considered as having not only *length*, but also *direction*, is said to be a **VECTOR**. Its initial point A is said to be its *origin*; and its final point B is said to be its *term*. A vector AB is conceived to be (or to construct) the *difference* of its two extreme points; or, more fully, to be the result of the *subtraction* of its own origin from its own term; and, in conformity with this *conception*, it is also denoted by the *symbol* $B - A$: a notation which will be found to be extensively useful, on account of the analogies which it serves to express between geometrical and algebraical operations. When the extreme points A and B are *distinct*, the vector AB or $B - A$ is said to be an *actual* (or an *effective*) vector; but when (as a limit) those two points are conceived to *coincide*, the vector AA or $A - A$, which then results, is said to be *null*.

Opposite vectors, such as AB and BA , or $B - A$ and $A - B$, are sometimes called *vector* and *revector*. *Successive* vectors, such as AB and BC , or $B - A$ and $C - B$, are occasionally said

to be *vector* and *provector*: the line AC , or $C - A$, which is



drawn from the origin A of the first to the term C of the second, being then said to be the *trans-vector*. At a later stage, we shall have to consider *vector-arcs* and *vector-angles*; but at present, our only vectors are (as above) *right lines*.

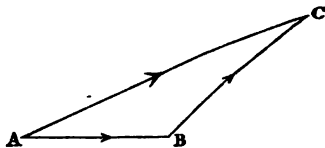


Fig. 2.

2. Two vectors are said to be **EQUAL** to each other, or the equation $AB = CD$, or $B - A = D - C$, is said to hold good, when (and only when) the origin and term of the one can be brought to *coincide* respectively with the corresponding points of the other, by *transports* (or by *translations*) *without rotation*. It follows that *all null vectors are equal*, and may therefore be denoted by a *common symbol*, such as that used for *zero*; so that we may write,

$$A - A = B - B = \&c. = 0;$$

but that two *actual* vectors, AB and CD , are *not* (in the present *full sense*) *equal* to each other, unless they have not merely *equal lengths*, but also *similar directions*. If then they do not happen to be *parts* of one *common line*, they must be *opposite sides* of a *parallelogram*, $ABDC$; the two lines AD , BC becoming thus the two *diagonals* of such a figure, and consequently *bisecting* each other, in some point E . Conversely, if the two equations,

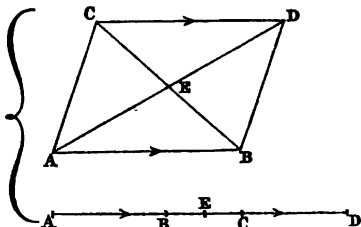


Fig. 8.

$$D - E = E - A, \quad \text{and} \quad C - E = E - B,$$

are satisfied, so that the two lines AD and BC are *commedial*, or have a *common middle point* E , then *even* if they be parts of *one* right line, the equation $D - C = B - A$ is satisfied. *Two radii*, AB , AC , of any *one circle* (or *sphere*), can never be *equal vectors*; because their *directions differ*.

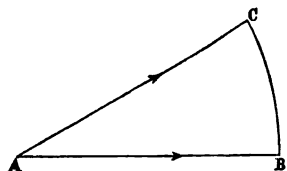


Fig. 4.

3. An equation between vectors, considered as an equidifference of points, admits of inversion and alternation ; or in symbols, if

$$D - C = B - A,$$

then

$$C - D = A - B,$$

and

$$D - B = C - A.$$

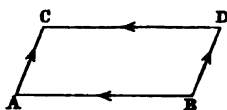


Fig. 5.

Two vectors, CD and EF, which are equal to the same third vector, AB, are also equal to each other ; and these three equal vectors are, in general, the three parallel edges of a prism.

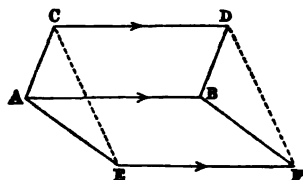


Fig. 6.

SECTION 2.—On Differences and Sums of Vectors taken two by two.

4. In order to be able to write, as in algebra,

$$(C' - A') - (B - A) = C - B, \text{ if } C' - A' = C - A,$$

we next define, that when a first vector AB is subtracted from a second vector AC which is co-initial with it, or from a third vector A'C' which is equal to that second vector, the remainder is that fourth vector BC, which is drawn from the term B of the first to the term C of the second vector : so that if a vector be subtracted from a transvector (Art. 1), the remainder is the provector corresponding. It is evident that this geometrical subtraction of vectors answers to a decomposition of vections (or of motions) ; and that, by such a decomposition of a null vection into two opposite vections, we have the formula,

$$0 - (B - A) = (A - A) - (B - A) = A - B ;$$

so that, if an actual vector AB be subtracted from a null vector AA, the remainder is the revector BA. If then we agree to abridge, generally, an expression of the form $0 - a$ to the shorter form, $-a$, we may write briefly, $-AB = BA$; a and $-a$ being thus symbols of opposite vectors, while a and $-(-a)$ are,

for the same reason, symbols of one *common* vector: so that we may write, as in algebra, the *identity*,

$$-(-a) = a.$$

5. Aiming still at agreement with algebra, and adopting on that account the *formula of relation* between the *two signs*, + and -,

$$(b - a) + a = b,$$

in which we shall say as usual that $b - a$ is *added* to a , and that their *sum* is b , while relatively to it they may be jointly called *summands*, we shall have the two following consequences:

I. If a *vector*, AB or $B - A$, be *added* to its own *origin*, A , the *sum* is its *term* B (Art. 1); and

II. If a *provector* BC be added to a vector AB , the sum is the *transvector* AC ; or in symbols,

$$\text{I. } (B - A) + A = B; \text{ and II. } (C - B) + (B - A) = C - A.$$

In fact, the first equation is an *immediate* consequence of the general formula which, as above, *connects* the signs + and -, when combined with the *conception* (Art. 1) of a *vector* as a *difference* of *two points*; and the second is a result of the same formula, combined with the *definition* of the *geometrical subtraction* of *one* such vector from *another*, which was assigned in Art. 4, and according to which we have (as in algebra) for *any three points*, A , B , C , the *identity*,

$$(C - A) - (B - A) = C - B.$$

It is clear that this *geometrical addition* of *successive vectors* corresponds (comp. Art. 4) to a *composition* of *successive vections*, or *motions*; and that the *sum* of two *opposite vectors* (or of vector and *revector*) is a *null line*; so that

$$BA + AB = 0, \text{ or } (A - B) + (B - A) = 0.$$

It follows also that the *sums* of *equal pairs* of *successive vectors* are *equal*; or more fully that

$$\text{if } B' - A' = B - A, \text{ and } C' - B' = C - B, \text{ then } C' - A' = C - A;$$

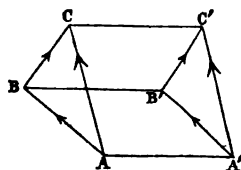


Fig. 7.

the two triangles, ABC and $A'B'C'$, being in general the two opposite faces of a prism (comp. Art. 3).

6. Again, in order to have, as in algebra,

$$(c' - b') + (b - a) = c - a, \text{ if } c' - b' = c - b,$$

we shall define that if there be two successive vectors, AB, BC , and if a third vector $B'C'$ be equal to the second, but not successive to the first, the sum obtained by adding the third to the first is that fourth vector, AC , which is drawn from the origin A of the first to the term C of the second. It follows that the sum of any two co-initial sides, AB, AC , of any parallelogram $ABDC$, is the intermediate and co-initial diagonal AD ; or, in symbols,

$$(c - a) + (b - a) = d - a, \text{ if } d - c = b - a;$$

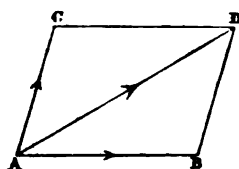


Fig. 8.

because we have then (by 3) $c - a = d - b$.

7. The sum of any two given vectors has thus a value which is independent of their order; or, in symbols, $a + \beta = \beta + a$. If equal vectors be added to equal vectors, the sums are equal vectors, even if the summands be not given as successive (comp. 5); and if a null vector be added to an actual vector, the sum is that actual vector; or, in symbols, $0 + a = a$. If then we agree to abridge generally (comp. 4) the expression $0 + a$ to $+a$, and if a still denote a vector, then $+a$, and $+(+a)$, &c., are other symbols for the same vector; and we have, as in algebra, the identities,

$$-(-a) = +a, \quad +(-a) = -(+a) = -a, \quad (+a) + (-a) = 0, \text{ \&c.}$$

SECTION 3.—On Sums of three or more Vectors.

8. The sum of three given vectors, a, β, γ , is next defined to be that fourth vector,

$$\delta = \gamma + (\beta + a), \text{ or briefly, } \delta = \gamma + \beta + a,$$

which is obtained by adding the third to the sum of the first and second; and in like manner the sum of any number of vectors is formed by adding the last to the sum of all that

precede it: also, for any four vectors, a, β, γ, δ , the sum $\delta + (\gamma + \beta + a)$ is denoted simply by $\delta + \gamma + \beta + a$, without parentheses, and so on for any number of summands.

9. The sum of any number of *successive* vectors, AB, BC, CD , is thus the line AD , which is drawn from the origin A of the first, to the term D of the last; and because, when there are *three* such vectors, we can draw (as in Fig. 9) the two *diagonals* AC, BD of the (plane or gauche) quadrilateral $ABCD$, and may then at pleasure regard AD , either as the sum of AB, BD , or as the sum of AC, CD ; we are allowed to establish the following *general formula of association*, for the case of *any three summand lines*, a, β, γ :

$$(\gamma + \beta) + a = \gamma + (\beta + a) = \gamma + \beta + a;$$

by combining which with the *formula of commutation* (Art. 7), namely, with the equation,

$$a + \beta = \beta + a,$$

which had been previously established for the case of any *two* such summands, it is easy to conclude that the *Addition of Vectors* is always both an *Associative* and a *Commutative Operation*. In other words, the *sum of any number of given vectors* has a *value* which is independent of their *order*, and of the mode of *grouping* them; so that if the *lengths and directions* of the summands be *preserved*, the length and direction of the *sum* will *also* remain unchanged: except that this last *direction* may be regarded as *indeterminate*, when the *length* of the *sum-line* happens to *vanish*, as in the case which we are about to consider.

10. When any n summand-lines, AB, BC, CA , or AB, BC, CD, DA , &c., arranged in any one order, are the n *successive sides* of a triangle ABC , or of a quadrilateral $ABCD$, or of any other *closed polygon*, their *sum* is a *null line*, AA ; and conversely,

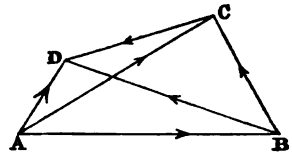


Fig. 9.

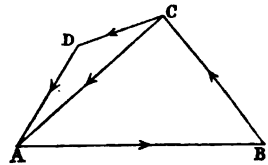


Fig. 10.

when the sum of any given system of n vectors is thus equal to zero, they may be made (in any order, by transports without rotation) the n successive sides of a closed polygon (plane or gauche). Hence, if there be given any such polygon (P), suppose a pentagon $ABCDE$, it is possible to construct another closed polygon (P'), such as $A'B'C'D'E'$, with an arbitrary initial point A' , but with the same number of sides, $A'B', \dots E'A'$, which *new* sides shall be equal (as vectors) to the old sides $AB, \dots EA$, taken in any arbitrary order. For example, if we draw four successive vectors, as follows,

$$A'B' = CD, \quad B'C' = AB, \quad C'D' = EA, \quad D'E' = BC,$$

and then complete the new pentagon by drawing the line $E'A'$, this closing side of the second figure (P') will be equal to the remaining side DE of the first figure (P).

11. Since a closed figure $ABC \dots$ is still a closed one, when all its points are projected on any assumed plane, by any system of parallel ordinates (although the area of the projected figure $A'B'C' \dots$ may happen to vanish), it follows that if the sum of any number of given vectors $\alpha, \beta, \gamma, \dots$ be zero, and if we project them all on any one plane by parallel lines drawn from their extremities, the sum of the projected vectors $\alpha', \beta', \gamma', \dots$ will likewise be null; so that these latter vectors, like the former, can be so placed as to become the successive sides of a closed polygon, even if they be not already such. (In Fig. 11, $A''B''C''$ is considered as such a polygon, namely, as a triangle with evanescent area; and we have the equation,

$$A''B'' + B''C'' + C''A'' = 0,$$

as well as

$$A'B' + B'C' + C'A' = 0, \text{ and } AB + BC + CA = 0.)$$

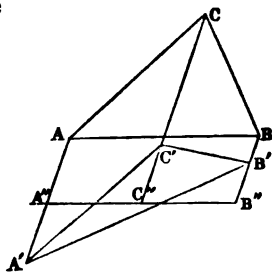


Fig. 11.

Handwritten note:
 The sides
 may be
 made in
 any order

SECTION 4.—On Coefficients of Vectors.

12. The *simple* or *single* vector, a , is also denoted by $1a$, or by $1 \cdot a$, or by $(+1)a$; and in like manner, the *double* vector, $a + a$, is denoted by $2a$, or $2 \cdot a$, or $(+2)a$, &c.; the rule being, that for any algebraical integer, m , regarded as a *coefficient* by which the vector a is *multiplied*, we have always,

$$1a + ma = (1 + m)a;$$

the symbol $1 + m$ being here interpreted as in algebra. Thus, $0a = 0$, the zero on the one side denoting a *null coefficient*, and the zero on the other side denoting a *null vector*; because by the rule,

$$1a + 0a = (1 + 0)a = 1a = a, \text{ and } \therefore 0a = a - a = 0.$$

Again, because $(1)a + (-1)a = (1 - 1)a = 0a = 0$, we have $(-1)a = 0 - a = -a = -(1a)$; in like manner, since $(1)a + (-2)a = (1 - 2)a = (-1)a = -a$, we infer that $(-2)a = -a - a = -(2a)$; and generally, $(-m)a = -(ma)$, whatever whole number m may be: so that we may, without danger of confusion, *omit the parentheses* in these last symbols, and write simply, $-1a$, $-2a$, $-ma$.

13. It follows that *whatever two whole numbers* (positive or negative, or null) may be represented by m and n , and *what-*

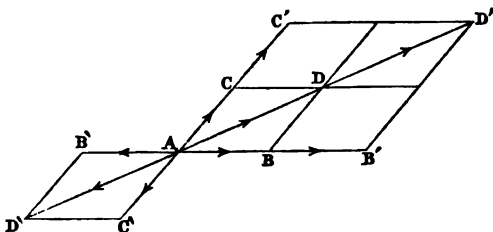


Fig. 12.

ever two vectors may be denoted by a and β , we have always, as in algebra, the formulæ,

$$na \pm ma = (n \pm m)a, \quad n(ma) = (nm)a = nma,$$

and (compare Fig. 12),

$$m(\beta \pm a) = m\beta \pm ma;$$

so that the *multiplication of vectors by coefficients* is a *doubly distributive operation*, at least if the *multipliers* be *whole numbers*; a restriction which, however, will soon be removed.

14. If $ma = \beta$, the coefficient m being still *whole*, the vector β is said to be a *multiple* of a ; and conversely (at least if the integer m be different from *zero*), the vector a is said to be a *sub-multiple* of β . A *multiple of a sub-multiple* of a vector is said to be a *fraction* of that vector; thus, if $\beta = ma$, and $\gamma = na$, then γ is a fraction of β , which is denoted as follows, $\gamma = \frac{n}{m}\beta$; also β is said to be *multiplied* by the fractional coefficient $\frac{n}{m}$, and γ is said to be the *product* of this multiplication. It follows that if x and y be *any two fractions* (positive or negative or null, whole numbers being included), and if a and β be *any two vectors*, then

$$y\alpha \pm x\alpha = (y \pm x)\alpha, \quad y(x\alpha) = (yx)\alpha = yx\alpha, \quad x(\beta \pm \alpha) = x\beta \pm x\alpha;$$

results which include those of Art. 13, and may be extended to the case where x and y are *incommensurable coefficients*, considered as *limits of fractional ones*.

15. For any actual vector a , and for any coefficient x , of any of the foregoing kinds, the *product* xa , interpreted as above, represents always a *vector* β , which has the *same direction* as the *multiplicand-line* a , if $x > 0$, but has the *opposite direction* if $x < 0$, becoming *null* if $x = 0$. Conversely, if a and β be *any two actual vectors*, with directions *either similar or opposite*, in *each* of which two cases we shall say that they are *parallel vectors*, and shall write $\beta \parallel a$ (because *both* are then *parallel*, in the *usual sense* of the word, to *one common line*), we can always find, or conceive as found, a *coefficient* $x \geq 0$, which shall satisfy the equation $\beta = xa$; or, as we shall also write it, $\beta = \alpha x$; and the positive or negative *number* x , so found, will bear to ± 1 the same *ratio*, as that which the *length* of the line β bears to the length of a .

16. Hence it is natural to say that this *coefficient* x is the *quotient* which results, from the *division of the vector* β , *by the parallel vector* a ; and to write, accordingly,

$$x = \beta \div a, \quad \text{or } x = \beta : a, \quad \text{or } x = \frac{\beta}{a};$$

so that we shall have, identically, as in algebra, at least if the *divisor-line* a be an *actual vector*, and if the *dividend-line* β be *parallel* thereto, the equations,

$$(\beta : a) \cdot a = \frac{\beta}{a} a = \beta, \quad \text{and} \quad xa : a = \frac{xa}{a} = x;$$

which will afterwards be *extended*, by *definition*, to the case of *non-parallel* vectors. We may write also, under the same conditions, $a = \frac{\beta}{x}$, and may say that the *vector* a is the *quotient* of the *division* of the other vector β *by the number* x ; so that we shall have these other identities,

$$\frac{\beta}{x} \cdot x = (ax =) \beta, \quad \text{and} \quad \frac{ax}{x} = a.$$

17. The positive or negative *quotient*, $x = \frac{\beta}{a}$, which is thus obtained by the *division* of one of two *parallel vectors* by another, including *zero* as a *limit*, may also be called a *SCALAR*; because it can always be found, and in a certain sense *constructed*, by the *comparison of positions upon one common scale* (or *axis*); or can be put under the form,

$$x = \frac{C - A}{B - A} = \frac{AC}{AB},$$

where the *three points*, A , B , C , are *collinear* (as in the figure annexed). Such *scalars* are, therefore, simply the *REALS* (or *real quantities*) of *Algebra*; but, in combination with the *not less real* *VECTORS* above considered, they form *one of the main elements* of the *System*, or *Calculus*, to

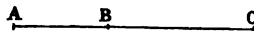


Fig. 18.

which the present work relates. In fact it will be shown, at a later stage, that there is an important sense in which we can conceive a scalar to be *added* to a vector; and that the *sum* so obtained, or the combination,

“*Scalar plus Vector,*”

is a QUATERNION.

CHAPTER II.

APPLICATIONS TO POINTS AND LINES IN A GIVEN PLANE.

SECTION 1.—*On Linear Equations connecting two Co-initial Vectors.*

18. WHEN several vectors, OA, OB, \dots are all drawn from one common point o , that point is said to be the *Origin of the System*; and each particular vector, such as OA , is said to be the *vector of its own term*, A . In the present and future sections we shall always suppose, if the contrary be not expressed, that all the vectors a, β, \dots which we may have occasion to consider, are thus drawn from one *common origin*. But if it be desired to *change* that origin o , without changing the *term-points* A, \dots we shall only have to *subtract*, from *each* of their *old* vectors a, \dots one *common vector* ω , namely, the *old vector* oo' of the *new origin* o' ; since the *remainders*, $a - \omega, \beta - \omega, \dots$ will be the *new vectors* a', β', \dots of the *old points* A, B, \dots . For example, we shall have

$$a' = o'A = A - o' = (A - o) - (o' - o) = OA - oo' = a - \omega.$$

19. If *two* vectors a, β , or OA, OB , be thus drawn from a given origin o , and if their *directions* be either similar or opposite, so that the *three* points, o, A, B , are situated on *one right line* (as in the figure



Fig. 14.

annexed), then (by 16, 17) their *quotient* $\frac{\beta}{a}$ is some positive or negative *scalar*, such as x ; and conversely, the equation $\beta = xa$, interpreted with this reference to an *origin*, expresses the *condition of collinearity*, of the points O, A, B ; the particular values, $x = 0, x = 1$, corresponding to the particular *positions*, O and A , of the *variable point* B , whereof the *indefinite right line* OA is the *locus*.

20. The *linear equation*, connecting the *two* vectors a and β , acquires a more symmetric *form*, when we write it thus :

$$aa + b\beta = 0;$$

where a and b are *two* scalars, of which however only the *ratio* is important. The *condition of coincidence*, of the two points A and B , answering above to $x = 1$, is now $\frac{-a}{b} = 1$; or, more symmetrically,

$$a + b = 0.$$

Accordingly, when $a = -b$, the *linear equation* becomes

$$b(\beta - a) = 0, \quad \text{or} \quad \beta - a = 0,$$

since we do not suppose that *both* the coefficients vanish; and the equation $\beta = a$, or $OB = OA$, requires that the *point* B should *coincide* with the *point* A : a case which may also be conveniently expressed by the formula,

$$B = A;$$

coincident points being thus treated (in *notation* at least) as *equal*. In general, the *linear equation* gives,

$$a \cdot OA + b \cdot OB = 0, \quad \text{and therefore} \quad a : b = BO : OA.$$

SECTION 2.—On Linear Equations between three co-initial Vectors.

21. If two (actual and co-initial) vectors, a, β , be not connected by *any* equation of the form $aa + b\beta = 0$, with *any two* scalar coefficients a and b whatever, their *directions* can neither be similar nor opposite to each other; they therefore *determine*

a plane $\triangle OB$, in which the (now actual) vector, represented by the sum $aa + b\beta$, is situated. For if, for the sake of symmetry, we denote this sum by the symbol $-c\gamma$, where c is some *third scalar*, and $\gamma = OC$ is some *third vector*, so that the three co-initial vectors, a , β , γ , are connected by the *linear equation*,

$$aa + b\beta + c\gamma = 0;$$

and if we make

$$OA' = \frac{-aa}{c}, \quad OB' = \frac{-b\beta}{c};$$

then the two auxiliary points, A' and B' , will be situated (by 19) on the two indefinite right lines, OA , OB , respectively: and we shall have the equation,

$$OC = OA' + OB',$$

so that the figure $A'OB'C$ is (by 6) a parallelogram, and consequently plane.

22. Conversely, if c be *any point in the plane* $\triangle OB$, we can draw from it the *ordinates*, CA' and CB' , to the lines OA and OB , and can determine the ratios of the three scalars, a , b , c , so as to satisfy the two equations,

$$\frac{a}{c} = -\frac{OA'}{OA}, \quad \frac{b}{c} = -\frac{OB'}{OB};$$

after which we shall have the recent expressions for OA' , OB' , with the relation $OC = OA' + OB'$ as before; and shall thus be brought back to the linear equation $aa + b\beta + c\gamma = 0$, which equation may therefore be said to express the *condition of coplanarity* of the four points, O , A , B , C . And if we write it under the form,

$$xa + y\beta + z\gamma = 0,$$

and consider the vectors a and β as *given*, but γ as a *variable vector*, while x , y , z are *variable scalars*, the *locus* of the *variable point* C will then be the *given plane*, OAB .

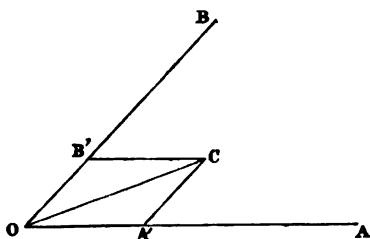


Fig. 15.

23. It may happen that the point c is situated *on the right line* AB , which is here considered as a *given* one. In that case (comp. Art. 17, Fig. 13), the quotient $\frac{AC}{AB}$ must be equal to some scalar, suppose t ; so that we shall have an equation of the form,

$$\frac{\gamma - a}{\beta - a} = t, \quad \text{or } \gamma = a + t(\beta - a), \quad \text{or } (1 - t)a + t\beta - \gamma = 0;$$

by comparing which last form with the linear equation of Art. 21, we see that the *condition of collinearity* of the three points A, B, c , in the given plane OAB , is expressed by the formula,

$$a + b + c = 0.$$

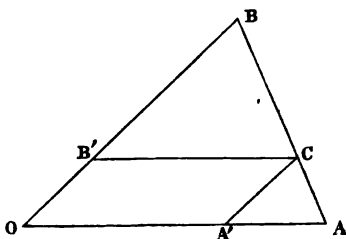


Fig. 16.

This condition may also be thus written,

$$1 = \frac{-a}{c} + \frac{-b}{c}, \quad \text{or } \frac{OA'}{OA} + \frac{OB'}{OB} = 1;$$

and under this last form it expresses a *geometrical relation*, which is otherwise known to exist.

24. When we have thus the *two* equations,

$$aa + b\beta + c\gamma = 0, \quad \text{and } a + b + c = 0,$$

so that the three co-initial vectors a, β, γ terminate on one right line, and may on that account be said to be *termino-collinear*, if we eliminate, successively and separately, each of the three scalars a, b, c , we are conducted to these three other equations, expressing certain *ratios of segments* :

$$\begin{aligned} b(\beta - a) + c(\gamma - a) &= 0, & c(\gamma - \beta) + a(a - \beta) &= 0, \\ a(a - \gamma) + b(\beta - \gamma) &= 0; \end{aligned}$$

or

$$0 = b \cdot AB + c \cdot AC = c \cdot BC + a \cdot BA = a \cdot CA + b \cdot CB.$$

Hence follows this *proportion*, between *coefficients* and *segments*,

$$a : b : c = BC : CA : AB.$$

We might also have observed that the proposed equations give,

$$a = \frac{b\beta + c\gamma}{b + c}, \quad \beta = \frac{c\gamma + aa}{c + a}, \quad \gamma = \frac{aa + b\beta}{a + b};$$

whence

$$\frac{AC}{AB} = \frac{\gamma - a}{\beta - a} = \frac{b}{a + b} = -\frac{b}{c}, \text{ \&c.}$$

25. If we still treat a and β as *given*, but regard γ and $\frac{y}{x}$ as *variable*, the equation

$$\gamma = \frac{xa + y\beta}{x + y}$$

will express that the *variable point* c is situated *somewhere* on the *indefinite right line* AB , or that it has this *line* for its *locus*: while it *divides* the *finite line* AB into *segments*, of which the *variable quotient* is,

$$\frac{AC}{CB} = \frac{y}{x}.$$

Let c' be *another point* on the *same line*, and let its vector be,

$$\gamma' = \frac{x'a + y'\beta}{x' + y'};$$

then, in like manner, we shall have this *other ratio of segments*,

$$\frac{AC'}{C'B} = \frac{y'}{x'}.$$

If, then, we agree to employ, generally, for any group of four collinear points, the notation,

$$(ABCD) = \frac{AB}{BC} \cdot \frac{CD}{DA} = \frac{AB}{BC} : \frac{AD}{DC};$$

so that this *symbol*,

$$(ABCD),$$

may be said to denote the *anharmonic function*, or *anharmonic quotient*, or simply the *anharmonic of the group*, A, B, C, D : we shall have, in the present case, the equation,

$$(ACBC') = \frac{AC}{CB} : \frac{AC'}{C'B} = \frac{yx'}{xy'}.$$

26. When the *anharmionic quotient* becomes equal to *negative unity*, the *group* becomes (as is well known) *harmonic*. If then we have the two equations,

$$\gamma = \frac{xa + y\beta}{x + y}, \quad \gamma' = \frac{xa - y\beta}{x - y},$$

the two points c and c' are *harmonically conjugate* to each other, with respect to the *two given points*, A and B ; and when they *vary together*, in consequence of the variation of the value of $\frac{y}{x}$, they form (in a well-known sense), on the indefinite right line AB , *divisions in involution*; the *double points* (or *foci*) of this involution, namely, the points of which each is *its own* conjugate, being the points A and B themselves. As a verification, if we denote by μ the vector of the middle point M of the given interval AB , so that

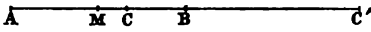
$$\beta - \mu = \mu - \alpha, \text{ or } \mu = \frac{1}{2}(\alpha + \beta),$$


Fig. 17.

we easily find that

$$\frac{\gamma - \mu}{\beta - \mu} = \frac{y - x}{y + x} = \frac{\beta - \mu}{\gamma' - \mu}, \quad \text{or} \quad \frac{MC}{MB} = \frac{MB}{MC'};$$

so that the *rectangle* under the *distances* MC , MC' , of the two *variable but conjugate points*, c , c' , from the *centre* M of the involution, is equal to the *constant square* of *half the interval* between the two *double points*, A , B . More generally, if we write

$$\gamma = \frac{xa + y\beta}{x + y}, \quad \gamma' = \frac{lx + my\beta}{lx + my},$$

where the *anharmionic quotient* $\frac{l}{m} = \frac{yx'}{xy}$ is *any constant scalar*,

then in another known and modern* phraseology, the points c and c' will form, on the indefinite line AB , *two homographic divisions*, of which A and B are still the *double points*. More generally still, if we establish the two equations,

* See the *Géométrie Supérieure* of M. Chasles, p. 107. (Paris, 1852.)

$$\gamma = \frac{x\alpha + y\beta}{x + y}, \quad \text{and} \quad \gamma' = \frac{lx\alpha' + my\beta'}{lx + my},$$

$\frac{1}{m}$ being still constant, but $\frac{y}{x}$ variable, while $\alpha' = OA'$, $\beta' = OB'$, and $\gamma' = OC'$, the two given lines, AB and $A'B'$, are then homographically divided, by the two variable points, c and c' , not now supposed to move along one common line.

27. When the linear equation $a\alpha + b\beta + c\gamma = 0$ subsists, without the relation $a + b + c = 0$ between its coefficients, then the three co-initial vectors α, β, γ are still coplanar, but they no longer terminate on one right line; their term-points A, B, C being now the corners of a triangle.

In this more general case, we may propose to find the vectors α', β', γ' of the three points,

$$\begin{aligned} \alpha' &= OA \cdot BC, & \beta' &= OB \cdot CA, \\ & & \gamma' &= OC \cdot AB; \end{aligned}$$

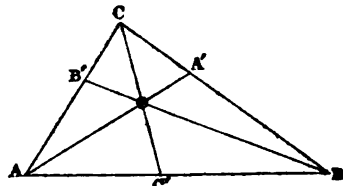


Fig. 18.

that is to say, of the points in which the lines drawn from the origin O to the three corners of the triangle intersect the three

respectively opposite sides. The three collineations OAA' , &c., give (by 19) three expressions of the forms,

$$\alpha' = x\alpha, \quad \beta' = y\beta, \quad \gamma' = z\gamma,$$

where x, y, z are three scalars, which it is required to determine by means of the three other collineations, $A'BC$, &c., with the help of relations derived from the principle of Art. 23. Substituting therefore for α its value $x^{-1}\alpha'$, in the given linear equation, and equating to zero the sum of the coefficients of the new linear equation which results, namely,

$$x^{-1}a\alpha' + b\beta + c\gamma;$$

and eliminating similarly β, γ , each in its turn, from the original equation; we find the values,

$$x = \frac{-a}{b+c}, \quad y = \frac{-b}{c+a}, \quad z = \frac{-c}{a+b};$$

whence the sought vectors are expressed in either of the **tv** following ways :

$$\text{I. . . } a' = \frac{-aa}{b+c}, \quad \beta' = \frac{-b\beta}{c+a}, \quad \gamma' = \frac{-c\gamma}{a+b};$$

or

$$\text{II. . . } a' = \frac{b\beta + c\gamma}{b+c}, \quad \beta' = \frac{c\gamma + aa}{c+a}, \quad \gamma' = \frac{aa + b\beta}{a+b}.$$

In fact we see, by *one* of these expressions for a' , that A' is on the line oA ; and by the *other* expression for the same vector a' , that the same point A' is on the line BC . As another verification, we may observe that the last expressions for a' , β' , γ' , coincide with those which were found in Art. 24, for a , β , γ themselves, on the particular supposition that the three points A , B , C were collinear.

28. We may next propose to determine the *ratios* of the *segments* of the *sides* of the triangle ABC , made by the points A' , B' , C' . For this purpose, we may write the last equations for a' , β' , γ' under the form,

$$0 = b(a' - \beta) - c(\gamma - a') = c(\beta' - \gamma) - a(a - \beta') = a(\gamma' - a) - b(\beta - \gamma');$$

and we see that they then give the required ratios, as follows :

$$\frac{BA'}{A'C} = \frac{c}{b}, \quad \frac{CB'}{B'A} = \frac{a}{c}, \quad \frac{AC'}{C'B} = \frac{b}{a};$$

whence we obtain at once the known *equation of six segments*,

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1,$$

as the *condition of concurrence* of the three right lines AA' , BB' , CC' , in a *common point*, such as o . It is easy also to infer, from the same ratios of segments, the following *proportion of coefficients and areas*,

$$a : b : c = OBC : OCA : OAB,$$

in which we must, in general, attend to algebraic *signs*; a *triangle* being conceived to *pass (through zero)* from *positive* to *negative*, or *vice versa*, as compared with any *given triangle* in

its own plane, when (in the course of any continuous change) its *vertex crosses its base*. It may be observed that *with this convention* (which is, in fact, a *necessary* one, for the establishment of *general formulæ*) we have, for *any three points*, the equation

$$ABC + BAC = 0,$$

exactly as we had (in Art. 5) for any *two points*, the equation

$$AB + BA = 0.$$

More fully, we have, on this plan, the formulæ,

$$ABC = -BAC = BCA = -CBA = CAB = -ACB;$$

and *any two coplanar triangles*, $ABC, A'B'C'$, bear to each other a *positive* or a *negative ratio*, according as the two *rotations*, which may be conceived to be denoted by the same *symbols* $ABC, A'B'C'$, are *similarly* or *oppositely directed*.

29. If A' and B' bisect respectively the sides BC and CA , then

$$a = b = c,$$

and c' bisects AB ; whence the known theorem follows, that *the three bisectors of the sides of a triangle concur*, in a point which is often called the *centre of gravity*, but which we prefer to call the *mean point* of the triangle, and which is here the *origin o*. At the same time, the first expressions in Art. 27 for a', β', γ' become,

$$a' = -\frac{a}{2}, \quad \beta' = -\frac{\beta}{2}, \quad \gamma' = -\frac{\gamma}{2};$$

whence this other known theorem results, that *the three bisectors trisect each other*.

30. The linear equation between a, β, γ reduces itself, in the case last considered, to the form,

$$a + \beta + \gamma = 0, \quad \text{or} \quad OA + OB + OC = 0;$$

the three vectors a, β, γ , or OA, OB, OC , are therefore, in this case, adapted (by Art. 10) to become the *successive sides* of a

triangle, by transports without rotation; and accordingly, if we complete (as in Fig. 19) the parallelogram ΔOBD , the triangle ΔOAD will have the property in question. It follows (by 11) that if we project the four points O, A, B, C , by any system of parallel ordinates, into four other points, O', A', B', C' , on any assumed plane, the sum of the three projected vectors, α, β, γ , or $O'A, \&c.$, will be null; so that we shall have the new linear equation,

$$\alpha + \beta + \gamma = 0,$$

or,

$$O'A + O'B + O'C = 0;$$

and in fact it is evident (see Fig. 20) that the projected mean point O' will be the mean point of the projected triangle, A', B', C' . We shall have also the equation,

$$(\alpha - \alpha') + (\beta - \beta') + (\gamma - \gamma') = 0;$$

where

$$\alpha - \alpha' = O'A - O'A' = (O'A + AA') - (OO' + O'A) = AA' - OO';$$

hence

$$OO' = \frac{1}{3}(AA' + BB' + CC'),$$

or the ordinate of the mean point of a triangle is the mean of the ordinates of the three corners.

SECTION 3.—On Plane Geometrical Nets.

31. Resuming the more general case of Art. 27, in which the coefficients a, b, c are supposed to be unequal, we may next inquire, in what points A'', B'', C'' do the lines $B'C', C'A', A'B'$ meet respectively the sides BC, CA, AB , of the triangle; or may seek to assign the vectors $\alpha'', \beta'', \gamma''$ of the points of intersection (comp. 27),

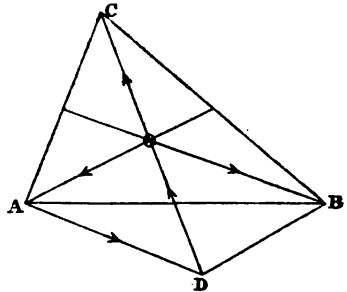


Fig. 19.

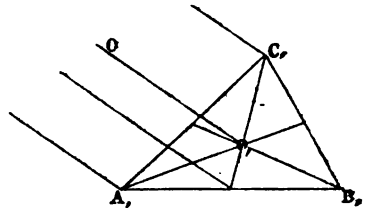


Fig. 20.

$$A'' = B'C' \cdot BC, \quad B'' = C'A' \cdot CA, \quad C'' = A'B' \cdot AB.$$

The first expressions in Art. 27 for β' , γ' , give the equations,

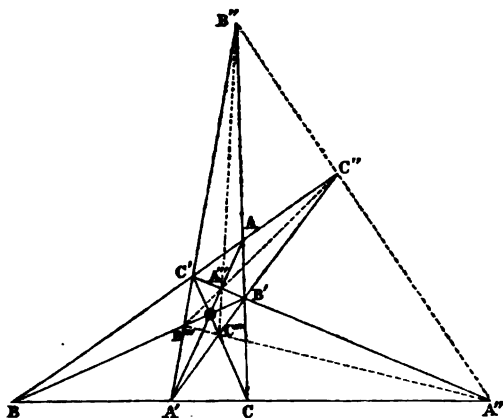


Fig. 21.

$$(c+a)\beta' + b\beta = 0, \quad (a+b)\gamma' + c\gamma = 0;$$

whence

$$\frac{b\beta - c\gamma}{b - c} = \frac{(a+b)\gamma' - (c+a)\beta'}{(a+b) - (c+a)};$$

but (by 25) one member is the vector of a point on BC , and the other of a point on $B'C'$; each therefore is a value for the vector a'' of A'' , and similarly for β'' and γ'' . We may therefore write,

$$a'' = \frac{b\beta - c\gamma}{b - c}, \quad \beta'' = \frac{c\gamma - aa}{c - a}, \quad \gamma'' = \frac{aa - b\beta}{a - b};$$

and by comparing these expressions with the second set of values of a' , β' , γ' in Art. 27, we see (by 26) that the points A'' , B'' , C'' are, respectively, the *harmonic conjugates* (as they are indeed known to be) of the points A' , B' , C' , with respect to the three pairs of points, B, C ; C, A ; A, B ; so that, in the notation of Art. 25, we have the equations,

$$(BA'CA'') = (CB'AB'') = (AC'BC'') = -1.$$

And because the expressions for a'' , β'' , γ'' conduct to the following linear equation between those three vectors,

$$(b-c)\alpha'' + (c-a)\beta'' + (a-b)\gamma'' = 0,$$

with the relation

$$(b-c) + (c-a) + (a-b) = 0$$

between its coefficients, we arrive (by 23) at this other known theorem, that *the three points A'', B'', C'' are collinear*, as indicated by one of the *dotted lines* in the recent Fig. 21.

32. The line A''B''C'' may represent *any rectilinear transversal*, cutting the sides of a triangle ABC; and because we have

$$\frac{BA''}{A''C} = \frac{a'' - \beta}{\gamma - a''} = -\frac{c}{b},$$

while $\frac{CB'}{B'A} = \frac{a}{c}$, and $\frac{AC'}{C'B} = \frac{b}{a}$, as before, we arrive at this *other equation of six segments*, for any triangle cut by a right line (comp. 28),

$$\frac{BA''}{A''C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1;$$

which again agrees with known results.

33. Eliminating β and γ between either set of expressions (27) for β' and γ' , with the help of the given linear equation, we arrive at this other equation, connecting the three vectors a, β', γ' :

$$0 = -aa + (c+a)\beta' + (a+b)\gamma'.$$

Treating this on the same plan as the given equation between a, β, γ , we find that if (as in Fig. 21) we make,

$$A''' = OA \cdot B'C', \quad B''' = OB \cdot C'A', \quad C''' = OC \cdot A'B',$$

the vectors of these three new points of intersection may be expressed in either of the two following ways, whereof the first is shorter, but the second is, for some purposes (comp. 34, 36) more convenient:

$$\text{I.} \dots a''' = \frac{aa}{2a+b+c}, \quad \beta''' = \frac{b\beta}{2b+c+a}, \quad \gamma''' = \frac{c\gamma}{2c+a+b};$$

or

$$\text{II.} \dots a''' = \frac{2aa + b\beta + c\gamma}{2a+b+c}, \quad \beta''' = \frac{2b\beta + c\gamma + aa}{2b+c+a},$$

$$\gamma''' = \frac{2c\gamma + aa + b\beta}{2c+a+b}.$$

And the three equations, of which the following is one,

$$(b - c)\alpha'' - (2b + c + a)\beta'' + (2c + a + b)\gamma'' = 0,$$

with the relations between their coefficients which are evident on inspection, show (by 23) that we have the three additional *collineations*, $A''B''C''$, $B''C''A''$, $C''A''B''$, as indicated by three of the dotted lines in the figure. Also, because we have the two expressions,

$$\alpha'' = \frac{(a + b)\gamma' + (c + a)\beta'}{(a + b) + (c + a)}, \quad \alpha'' = \frac{(a + b)\gamma' - (c + a)\beta'}{(a + b) - (c + a)},$$

we see (by 26) that the two points A'' , A''' are *harmonically conjugate* with respect to B' and c' ; and similarly for the two other pairs of points, B'' , B''' , and C'' , C''' , compared with c' , A' , and with A' , B' : so that, in a notation already employed (25, 31), we may write,

$$(B'A''C'A''') = (C'B''A'B''') = (A'C''B'C''') = -1.$$

34. If we *begin*, as above, with any *four coplanar points*, O , A , B , C , of which no three are collinear, we can (as in Fig. 18), by what may be called a *First Construction*, derive from them six lines, connecting them two by two, and intersecting each other in three new points, A' , B' , C' ; and then by a *Second Construction* (represented in Fig. 21), we may connect these by three new lines, which will give, by their intersections with the former lines, six new points, A'' , . . . C''' . We might proceed to connect these with each other, and with the given points, by sixteen new lines, or lines of a *Third Construction*, namely, the four dotted lines of Fig. 21, and twelve other lines, whereof three should be drawn from each of the four given points: and these would be found to determine eighty-four new points of intersection, of which some may be seen, although they are not marked, in the figure.

But *however far* these processes of *linear construction* may be continued, so as to form what has been called* a *plane*

* By Prof. A. F. MÖBIUS, in page 274 of his *Barycentric Calculus* (der barycentrische Calcul, Leipzig, 1827).

geometrical net, the *vectors* of the points thus determined have all one *common property*: namely, that each can be represented by an expression of the form,

$$\rho = \frac{x\alpha + y\beta + z\gamma}{x + y + z};$$

where the *coefficients* x, y, z are some *whole numbers*. In fact we see (by 27, 31, 33) that such expressions can be assigned for the *nine* derived vectors, $\alpha', \dots \gamma''$, which alone have been hitherto considered; and it is not difficult to perceive, from the nature of the calculations employed, that a similar result must hold good, for every vector subsequently deduced. But this and other connected results will become more completely evident, and their *geometrical signification* will be better understood, after a somewhat closer consideration of *anharmonic quotients*, and the introduction of a certain system of *anharmonic co-ordinates*, for points and lines in one plane, to which we shall next proceed: reserving, for a subsequent Chapter, any applications of the same theory to *space*.

SECTION 4.—On Anharmonic Co-ordinates and Equations of Points and Lines in one Plane.

35. If we compare the last equations of Art. 33 with the corresponding equations of Art. 31, we see that the *harmonic group* $BA'CA''$, on the side BC of the triangle ABC in Fig. 21, has been simply *reflected* into another such group, $B'A'''C'A''$, on the line $B'C'$, by a *harmonic pencil* of four rays, all passing through the point o ; and similarly for the other groups. More generally, let OA, OB, OC, OD , or briefly $o.ABCD$, be any pencil, with the point o for *vertex*; and let the *new ray* OD be cut, as in Fig. 22, by the three sides of the triangle ABC , in the three points A_1, B_1, C_1 ; let also

$$oA_1 = a_1 = \frac{y\beta + z\gamma}{y + z},$$

so that (by 25) we shall have the anharmonic quotients,

$$(BA'CA_1) = \frac{y}{z}, \quad (CA'B_1A_1) = \frac{z}{y};$$

and let us seek to express the two other vectors of intersection, β_1 and γ_1 , with a view to determining the anharmonic ratios of the groups on the two other sides. The given equation (27),

$$aa + b\beta + c\gamma = 0,$$

shows us at once that these two vectors are,

$$OB_1 = \beta_1 = \frac{(y-z)c\gamma + yaa}{(y-z)c + ya};$$

$$OC_1 = \gamma_1 = \frac{(z-y)b\beta + zaa}{(z-y)b + za};$$

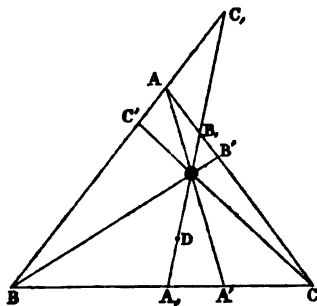


Fig. 22.

whence we derive (by 25) these two other anharmonics,

$$(CB'AB_1) = \frac{y-z}{y}; \quad (BC'AC_1) = \frac{z-y}{z};$$

so that we have the relations,

$$(CB'AB_1) + (CA'BA_1) = (BC'AC_1) + (BA'CA_1) = 1.$$

But in general, for any four collinear points A, B, C, D, it is not difficult to prove that

$$\frac{AB}{BC} \cdot CD + \frac{AC}{CB} \cdot BD = DA;$$

whence by the definition (25) of the signification of the symbol (ABCD), the following identity is derived,

$$(ABCD) + (ACBD) = 1.$$

Comparing this, then, with the recently found relations, we have, for Fig. 22, the following anharmonic equations:

$$(CAB'B_1) = (CA'BA_1) = \frac{z}{y};$$

$$(BAC'C_1) = (BA'CA_1) = \frac{y}{z};$$

and we see that (as was to be expected from known princi-

ples) the anharmonic of the *group* does not change, when we pass from one side of the triangle, considered as a *transversal* of the pencil, to another such side, or transversal. We may therefore speak (as usual) of such an anharmonic of a *group*, as being at the same time the *Anharmonic of a Pencil*; and, with attention to the *order of the rays*, and to the definition (25), may denote the two last anharmonics by the two following reciprocal expressions:

$$(O \cdot CABD) = \frac{z}{y}; \quad (O \cdot BACD) = \frac{y}{z};$$

with other resulting values, when the *order* of the rays is changed; it being understood that

$$(O \cdot CABD) = (C'A'B'D'),$$

if the rays OC, OA, OB, OD be cut, in the points c', a', b', d' , by any one right line.

36. The expression (34),

$$\rho = \frac{xaa + yb\beta + zc\gamma}{xa + yb + zc},$$

may represent the vector of any point P in the given plane, by a suitable choice of the coefficients x, y, z , or simply of their ratios. For since (by 22) the three coplanar vectors PA, PB, PC must be connected by some linear equation, of the form

$$a' \cdot PA + b' \cdot PB + c' \cdot PC = 0,$$

or

$$a'(a - \rho) + b'(\beta - \rho) + c'(\gamma - \rho) = 0,$$

which gives

$$\rho = \frac{a'a + b'\beta + c'\gamma}{a' + b' + c'},$$

we have only to write

$$\frac{a'}{a} = x, \quad \frac{b'}{b} = y, \quad \frac{c'}{c} = z,$$

and the proposed expression for ρ will be obtained. Hence it is easy to infer, on principles already explained, that if we write (compare the annexed Fig. 23),

$$P_1 = PA \cdot BC, \quad P_2 = PB \cdot CA, \quad P_3 = PC \cdot AB,$$

we shall have, with the same coefficients xyz , the following expressions for the vectors OP_1, OP_2, OP_3 or ρ_1, ρ_2, ρ_3 , of these three points of intersection, P_1, P_2, P_3 :

$$\rho_1 = \frac{yb\beta + zc\gamma}{yb + zc}, \quad \rho_2 = \frac{zc\gamma + xaa}{zc + xa},$$

$$\rho_3 = \frac{xaa + yb\beta}{xa + yb};$$

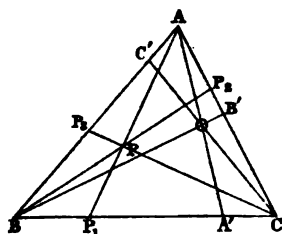


Fig. 28.

which give at once the following anharmonics of pencils, or of groups,

$$(A \cdot BOCF) = (BA'CP_1) = \frac{y}{z};$$

$$(B \cdot COAF) = (CB'AP_2) = \frac{z}{x};$$

$$(C \cdot AOBP) = (AC'BP_3) = \frac{x}{y};$$

whereof we see that the product is unity. *Any two of these three pencils suffice to determine the position of the point P, when the triangle ABC, and the origin O are given; and therefore it appears that the three coefficients x, y, z, or any scalars proportional to them, of which the quotients thus represent the anharmonics of those pencils, may be conveniently called the ANHARMONIC CO-ORDINATES of that point, P, with respect to the given triangle and origin: while the point P itself may be denoted by the Symbol,*

$$P = (x, y, z).$$

With this notation, the thirteen points of Fig. 21 come to be thus symbolized:

$$A = (1, 0, 0), \quad B = (0, 1, 0), \quad C = (0, 0, 1), \quad O = (1, 1, 1);$$

$$A' = (0, 1, 1), \quad B' = (1, 0, 1), \quad C' = (1, 1, 0);$$

$$A'' = (0, 1, -1), \quad B'' = (-1, 0, 1), \quad C'' = (1, -1, 0);$$

$$A''' = (2, 1, 1), \quad B''' = (1, 2, 1), \quad C''' = (1, 1, 2).$$

37. If P_1 and P_2 be any two points in the given plane,

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2),$$

and if t and u be any two scalar coefficients, then the following third point,

$$P = (tx_1 + ux_2, ty_1 + uy_2, tz_1 + uz_2),$$

is *collinear* with the two former points, or (in other words) is situated *on the right line* P_1P_2 . For, if we make

$$x = tx_1 + ux_2, \quad y = ty_1 + uy_2, \quad z = tz_1 + uz_2,$$

and

$$\rho_1 = \frac{x_1aa + \dots}{x_1a + \dots}, \quad \rho_2 = \frac{x_2aa + \dots}{x_2a + \dots}, \quad \rho = \frac{xaa + \dots}{xa + \dots},$$

these *vectors* of the three points P_1P_2P are connected by the *linear equation*,

$$t(x_1a + \dots)\rho_1 + u(x_2a + \dots)\rho_2 - (xa + \dots)\rho = 0;$$

in which (comp. 23), the *sum* of the *coefficients* is zero. Conversely, the point P cannot be collinear with P_1, P_2 , unless its co-ordinates admit of being thus expressed in terms of theirs. It follows that if a *variable point* P be obliged to move along a *given right line* P_1P_2 , or if it have such a *line* (in the given plane) for its *locus*, its co-ordinates xyz must satisfy a *homogeneous equation of the first degree, with constant coefficients*; which, in the known notation of determinants, may be thus written,

$$0 = \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix};$$

or, more fully,

$$0 = x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(x_1y_2 - y_1x_2);$$

or briefly,

$$0 = lx + my + nz,$$

where l, m, n are *three constant scalars*, whereof the *quotients* determine the *position* of the right line Δ , which is thus the *locus* of the point P . It is natural to call the *equation*, which

thus connects the *co-ordinates* of the point P, the *Anharmonic Equation of the Line* Λ ; and we shall find it convenient also to speak of the *coefficients* l, m, n , in that equation, as being the *Anharmonic Co-ordinates of that Line*: which line may also be denoted by the *Symbol*,

$$\Lambda = [l, m, n].$$

38. For example, the three sides BC, CA, AB of the given triangle have thus for their *equations*,

$$x = 0, \quad y = 0, \quad z = 0,$$

and for their *symbols*,

$$[1, 0, 0], \quad [0, 1, 0], \quad [0, 0, 1].$$

The three additional lines OA, OB, OC, of Fig. 18, have, in like manner, for their equations and symbols,

$$\begin{aligned} y - z = 0, & \quad z - x = 0, & \quad x - y = 0, \\ [0, 1, -1], & \quad [-1, 0, 1], & \quad [1, -1, 0]. \end{aligned}$$

The lines B'C'A", C'A'B", A'B'C", of Fig. 21, are

$$\begin{aligned} y + z - x = 0, & \quad z + x - y = 0, & \quad x + y - z = 0, \\ \text{or} & & \\ [-1, 1, 1], & \quad [1, -1, 1], & \quad [1, 1, -1]; \end{aligned}$$

the lines A"B"C"', B"C"A"', C"A"B"', of the same figure, are in like manner represented by the equations and symbols,

$$\begin{aligned} y + z - 3x = 0, & \quad z + x - 3y = 0, & \quad x + y - 3z = 0, \\ [-3, 1, 1], & \quad [1, -3, 1], & \quad [1, 1, -3]; \end{aligned}$$

and the line A"B"C" is

$$x + y + z = 0, \quad \text{or} \quad [1, 1, 1].$$

Finally, we may remark that on the same plan, the equation and the symbol of what is often called the *line at infinity*, or of the *locus of all the infinitely distant points in the given plane*, are respectively,

$$ax + by + cz = 0, \quad \text{and} \quad [a, b, c];$$

because the *linear function*, $ax + by + cz$, of the *co-ordinates* x, y, z of a point P in the plane, is the *denominator* of the expression (34, 36) for the *vector* ρ of that point: so that the *point* P is at an infinite distance from the origin o , when, and only when, this linear function *vanishes*.

39. These *anharmenic co-ordinates of a line*, although above interpreted (37) with reference to the *equation* of that line, considered as connecting the co-ordinates of a variable *point* thereof, are capable of receiving an independent geometrical interpretation. For the three points L, M, N , in which the line Λ , or $[l, m, n]$, or $lx + my + nz = 0$, intersects the three sides BC, CA, AB of the given triangle ABC , or the three given lines $x = 0, y = 0, z = 0$ (38), may evidently (on the plan of 36) be thus denoted:

$$L = (0, n, -m); \quad M = (-n, 0, l); \quad N = (m, -l, 0).$$

But we had also (by 36),

$$A'' = (0, 1, -1); \quad B'' = (-1, 0, 1); \quad C'' = (1, -1, 0);$$

whence it is easy to infer, on the principles of recent articles, that

$$\frac{n}{m} = (BA''CL); \quad \frac{l}{n} = (CB''AM); \quad \frac{m}{l} = (AC''BN);$$

with the resulting relation,

$$(BA''CL) \cdot (CB''AM) \cdot (AC''BN) = 1.$$

40. Conversely, this last equation is easily proved, with the help of the known and general relation between *segments* (32), applied to *any two transversals*, $A''B''C''$ and LMN , of any triangle ABC . In fact, we have thus the two equations,

$$\frac{BA''}{A''C} \cdot \frac{CB''}{B''A} \cdot \frac{AC''}{C''B} = -1, \quad \frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1;$$

on dividing the former of which by the latter, the last formula of the last article results. We might therefore in this way have been led, *without any consideration of a variable point* P ,

to introduce *three auxiliary scalars*, l, m, n , defined as having their quotients $\frac{n}{m}, \frac{l}{n}, \frac{m}{l}$ equal respectively, as in 39, to the three anharmonics of groups,

$$(BA''CL), \quad (CB''AM), \quad (\Delta C''BN);$$

and then it would have been evident that these three scalars, l, m, n (or any others proportional thereto), are sufficient to determine the position of the right line Δ , or LMN , considered as a transversal of the given triangle ABC : so that they might naturally have been called, on this account, as above, the *anharmonic co-ordinates* of that line. But although the anharmonic co-ordinates of a point and of a line may thus be *independently defined*, yet the *geometrical utility* of such definitions will be found to depend mainly on their combination: or on the formula $lx + my + nz = 0$ of 37, which may at pleasure be considered as expressing, either that the *variable point* (x, y, z) is situated *somewhere* upon the given right line $[l, m, n]$; or else that the *variable line* $[l, m, n]$ passes, in some direction, through the given point (x, y, z) .

41. If Δ_1 and Δ_2 be any two right lines in the given plane,

$$\Delta_1 = [l_1, m_1, n_1], \quad \Delta_2 = [l_2, m_2, n_2],$$

then any *third* right line Δ in the same plane, which passes through the intersection $\Delta_1 \cdot \Delta_2$, or (in other words) which *con- curs* with them (at a finite or infinite distance), may be represented (comp. 37) by a symbol of the form,

$$\Delta = [tl_1 + ul_2, tm_1 + um_2, tn_1 + un_2],$$

where t and u are scalar coefficients. Or, what comes to the same thing, if l, m, n be the anharmonic co-ordinates of the line Δ , then (comp. again 37), the equation

$$0 = l(m_1n_2 - n_1m_2) + \&c. = \begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix},$$

must be satisfied; because, if (X, Y, Z) be the supposed point common to the three lines, the three equations

$$lX + mY + nZ = 0, \quad l_1X + m_1Y + n_1Z = 0, \quad l_2X + m_2Y + n_2Z = 0,$$

must co-exist. Conversely, this coexistence will be possible, and the three lines will have a common point (which may be infinitely distant), if the recent *condition of concurrence* be satisfied. For example, because $[a, b, c]$ has been seen (in 38) to be the symbol of the line at infinity (at least if we still retain the same significations of the scalars a, b, c as in articles 27, &c.), it follows that

$$\Lambda = [l, m, n], \quad \text{and} \quad \Lambda' = [l + ua, m + ub, n + uc],$$

are symbols of two *parallel lines*; because they *concur at infinity*. In general, all problems respecting intersections of right lines, collineations of points, &c., in the given plane, when treated by this *anharmonic method*, conduct to easy *eliminations* between *linear equations* (of the *scalar kind*), on which we need not here delay: the *mechanism* of such *calculations* being for the most part the *same* as in the known method of *trilinear co-ordinates*: although (as we have seen) the *geometrical interpretations* are altogether *different*.

SECTION 5.—On Plane Geometrical Nets, resumed.

42. If we now *resume*, for a moment, the consideration of those plane geometrical *nets*, which were mentioned in Art. 34; and agree to call those points and lines, in the given plane, *rational points* and *rational lines*, respectively, which have their *anharmonic co-ordinates equal* (or *proportional*) to *whole numbers*; because then the *anharmonic quotients*, which were discussed in the last Section, are *rational*; but to say that a point or line is *irrational*, or that it is *irrationally related* to the given system of *four initial points* O, A, B, C , when its anharmonic co-ordinates are *not thus all equal* (or *proportional*) to *integers*; it is clear that *whatever four points* we may assume as *initial*, and *however far* the construction of the net may be carried, the *net-points* and *net-lines* which result will *all be rational*, in the sense just now defined. In fact, we *begin* with such; and the subsequent *eliminations* (41) can never after-

lines conduct to any, that are of the contrary kind: the right line which connects two rational points being always a rational line; and the point of intersection of two rational lines being necessarily a rational point. The assertion made in Art. 34 is therefore fully justified.

43. Conversely, every rational point of the given plane, with respect to the four assumed initial points $OABC$, is a point of the net which those four points determine. To prove this, it is evidently sufficient to show that every rational point $A_1 = (0, y, z)$, on any one side BC of the given triangle ABC , can be so constructed. Making, as in Fig. 22,

$$B_1 = OA_1 \cdot CA, \quad \text{and} \quad C_1 = OA_1 \cdot AB,$$

we have (by 35, 36) the expressions,

$$B_1 = (y, 0, y - z), \quad C_1 = (z, z - y, 0);$$

from which it is easy to infer (by 36, 37), that

$$C_1 B_1 \cdot BC = (0, y, z - y), \quad B_1 C_1 \cdot BC = (0, y - z, z);$$

and thus we can reduce the linear construction of the rational point $(0, y, z)$, in which the two whole numbers y and z may be supposed to be prime to each other, to depend on that of the point $(0, 1, 1)$, which has already been constructed as A' . It follows that although no irrational point Q of the plane can be a net-point, yet every such point can be indefinitely approached to, by continuing the linear construction; so that it can be included within a quadrilateral interstice $P_1 P_2 P_3 P_4$, or even within a triangular interstice $P_1 P_2 P_3$, which interstice of the net can be made as small as we may desire. Analogous remarks apply to irrational lines in the plane, which can never coincide with net-lines, but may always be indefinitely approximated to by such.

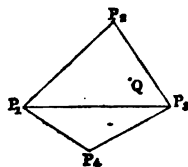


Fig. 24.

44. If P, P_1, P_2 be any three collinear points of the net, so that the formulæ of 37 apply, and if P' be any fourth net-point (x', y', z') upon the same line, then writing

$$x_1 a + y_1 b + z_1 c = v_1, \quad x_2 a + y_2 b + z_2 c = v_2,$$

we shall have two expressions of the forms,

$$\rho = \frac{tv_1\rho_1 + uv_2\rho_2}{tv_1 + uv_2}, \quad \rho' = \frac{t'v_1\rho_1 + u'v_2\rho_2}{t'v_1 + u'v_2},$$

in which the coefficients $tu'u'$ are *rational*, because the co-ordinates xyz , &c., are such, whatever the constants abc may be. We have therefore (by 25) the following rational expression for the anharmonic of this *net-group*:

$$(P_1PP_2P') = \frac{ut'}{tu'} = \frac{(yx_1 - xy_1)(y'x_2 - x'y_2)}{(xy_1 - yx_2)(x'y_1 - y'x_2)};$$

and similarly for every other group of the same kind. Hence *every group* of four collinear net-points, and consequently also *every pencil* of four concurrent net-lines, has a *rational value* for its *anharmonic function*; which value depends *only* on the *processes* of linear construction employed, in arriving at that group or pencil, and is quite *independent* of the *configuration* or *arrangement* of the *four initial points*: because the *three initial constants*, a, b, c , disappear from the expression which results. It was thus that, in Fig. 21, the *nine pencils*, which had the nine derived points $A' \dots C'''$ for their vertices, were all *harmonic pencils*, in whatever manner the four points O, A, B, C might be arranged. In general, it may be said that plane geometrical nets are all *homographic figures**; and conversely, in any two such plane *figures*, *corresponding points* may be considered as either *coinciding*, or at least (by 43) as indefinitely approaching to coincidence, with *similarly constructed points* of two plane *nets*: that is, with points of which (in their respective systems) the *anharmonic co-ordinates* (36) are *equal integers*.

45. Without entering here† on any *general* theory of *transformation* of anharmonic co-ordinates, we may already see that if we select *any four net-points* O_1, A_1, B_1, C_1 , of which no three are collinear, *every other point* P of the same net is *rationally related* (42) to these; because (by 44) the three new anhar-

* Compare the *Géométrie Supérieure* of M. Chales, p. 362.

† See Note A, on *Anharmonic Co-ordinates*.

mechanics of pencils, $(A_1 \cdot B_1 O_1 C_1 P) = \frac{y_1}{z_1}$, &c., are rational: and therefore (comp. 36) the *new co-ordinates* x_1, y_1, z_1 of the point P , as well its *old co-ordinates* xyz , are equal or proportional to *whole numbers*. It follows (by 43) that *every point* P of the net can be *linearly constructed*, if *any four* such points be given (no three being collinear, as above); or, in other words, that the *whole net* can be *reconstructed*,* if *any one* of its *quadrilaterals* (such as the *interstice* in Fig. 24) be *known*. As an example, we may suppose that the four points $OA'B'C'$ in Fig. 21 are given, and that it is required to *recover* from them the three points ABC , which had previously been among the *data* of the construction. For this purpose, it is only necessary to determine first the three auxiliary points A''', B''', C''' , as the intersections $OA' \cdot B'C'$, &c.; and next the three other auxiliary points A'', B'', C'' , as $B'C' \cdot B'''C'''$, &c.: after which the formulae, $A = B'B'' \cdot C'C''$, &c., will enable us to return, as required, to the points A, B, C , as intersections of known right lines.

SECTION 6.—*On Anharmonic Equations, and Vector Expressions, for Curves in a given Plane.*

46. When, in the expressions 34 or 36 for a variable vector $\rho = OP$, the three variable scalars (or anharmonic co-ordinates) x, y, z are connected by any given algebraic equation, such as

$$f_p(x, y, z) = 0,$$

supposed to be rational and integral, and homogeneous of the p^{th} degree, then the *locus* of the term P (Art. 1) of that vector is a *plane curve* of the p^{th} order; because (comp. 37) it is cut

* This theorem (45) of the possible reconstruction of a plane net, from any one of its quadrilaterals, and the theorem (43) respecting the possibility of indefinitely approaching by net-lines to the points above called irrational (42), without ever reaching such points by any processes of linear construction of the kind here considered, have been taken, as regards their substance (although investigated by a totally different analysis), from that highly original treatise of MÖBIUS, which was referred to in a former note (p. 23). Compare Note B, upon the *Barycentric Calculus*; and the remarks in the following Chapter, upon *nets in space*.

in p points (distinct or coincident, and real or imaginary), to any given right line, $lx + my + nz = 0$, in the given plane.

For example, if we write

$$\rho = \frac{t^2 a a + u^2 b \beta + v^2 c \gamma}{t^2 a + u^2 b + v^2 c},$$

where t, u, v are three new variable scalars, of which we shall suppose that the sum is zero, then, by eliminating these between the four equations,

$$x = t^2, \quad y = u^2, \quad z = v^2, \quad t + u + v = 0,$$

we are conducted to the following equation of the *second* degree,

$$0 = f_p = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy;$$

so that here $p = 2$, and the locus of P is a *conic section*. In fact, it is the conic which *touches* the *sides* of the given triangle ABC , at the points above called A', B', C' ; for if we seek its *intersections* with the side BC , by making $x = 0$ (38), we obtain a *quadratic* with *equal* roots, namely, $(y - z)^2 = 0$; which shows that there is *contact* with this side at the point $(0, 1, 1)$, or A' (36): and similarly for the two other sides.

47. If the point o , in which the three right lines AA', BB', CC' *concur*, be (as in Fig. 18, &c.) *interior* to the triangle ABC , the sides of that triangle are then all cut *internally*, by the points A', B', C' of contact with the conic; so that in this case (by 28) the ratios of the constants a, b, c are all *positive*, and the *denominator* of the recent expression (46) for ρ cannot *vanish*, for any *real* values of the variable scalars t, u, v ; and consequently no *such* values can render *infinite* that *vector* ρ . The *conic* is therefore generally in this case, as in Fig. 25, an *inscribed ellipse*; which becomes however the *inscribed circle*, when

$$a^{-1} : b^{-1} : c^{-1} = s - a : s - b : s - c;$$

a, b, c denoting here the lengths of the sides of the triangle, and s being their semi-sum.

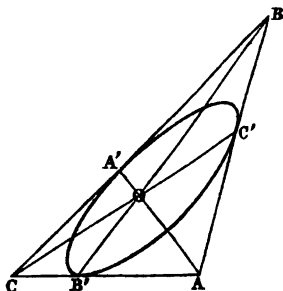


Fig. 25.

48. But if the point of concurrence o be exterior to the triangle of tangents ABC , so that two of its sides are cut externally, then two of the three ratios of segments (28) are negative; and therefore one of the three constants a, b, c may be treated as < 0 , but each of the two others as > 0 . Thus if we suppose that

$$b > 0, \quad c > 0, \quad a < 0, \quad a + b > 0, \quad a + c > 0,$$

A' will be a point on the side B itself, but the points B', c', o will be on the lines AC, AB, AA' prolonged, as in Fig. 26; and then the conic $A'B'C'$ will be an ellipse (including the case of a circle), or a parabola, or an hyperbola, according as the roots of the quadratic,

$$(a + c)t^2 + 2ctu + (b + c)u^2 = 0,$$

obtained by equating the denominator (46) of the vector ρ to zero, are either, I st, imaginary; or II nd, real and equal; or III rd, real and unequal: that is, according as we have

$$bc + ca + ab > 0, \quad \text{or} = 0, \quad \text{or} < 0;$$

or (because the product abc is here negative), according as

$$a^{-1} + b^{-1} + c^{-1} < 0, \quad \text{or} = 0, \quad \text{or} > 0.$$

For example, if the conic be what is often called the *exscribed circle*, the known ratios of segments give the proportion,

$$a^{-1} : b^{-1} : c^{-1} = -s : s - c : s - b;$$

and

$$-s + s - c + s - b < 0.$$

49. More generally, if c , be (as in Fig. 26) a point upon the side AB , or on that side prolonged, such that cc' , is parallel to the chord $B'C'$, then

$$C'C' : AC' = CB' : AB' = -a : c, \quad \text{and} \quad AB : AC' = a + b : b;$$

writing then the condition (48) of ellipticity (or circularity)

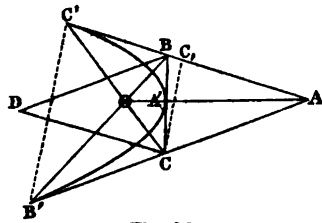


Fig. 26.

under the form, $\frac{-a}{c} < \frac{a+b}{b}$, we see that the conic is an ellipse, parabola, or hyperbola, according as $c, c' < \text{or} = \text{or} > AB$; the arrangement being *still*, in other respects, that which is represented in Fig. 26. Or, to express the same thing more symmetrically, if we complete the parallelogram $CABD$, then according as the point D falls, Ist, *beyond the chord* $B'C'$, with respect to the point A ; or IInd, *on that chord*; or IIIrd, *within the triangle* $AB'C'$, the general arrangement of the same Figure being retained, the curve is *elliptic*, or *parabolic*, or *hyperbolic*. In that *other* arrangement or configuration, which answers to the system of inequalities, $b > 0, c > 0, a + b + c < 0$, the point A' is still upon the side BC *itself*, but o is on the line $A'A$ prolonged *through* A ; and *then* the inequality,

$$a(b+c) + bc < -(b^2 + bc + c^2) < 0,$$

shows that the conic is *necessarily* an hyperbola; whereof it is easily seen that *one branch* is touched by the side BC at A' , while the *other branch* is touched in B' and c' , by the sides CA and BA prolonged through A . The curve is also hyperbolic, if either $a + b$ or $a + c$ be negative, while b and c are positive as before.

50. When the quadratic (48) has its roots real and unequal, so that the conic is an *hyperbola*, then the *directions* of the *asymptotes* may be found, by substituting those roots, or the values of t, u, v which correspond to them (or any scalars *proportional* thereto), in the *numerator* of the expression (46) for ρ ; and similarly we can find the direction of the *axis* of the *parabola*, for the case when the roots are real but equal: for we shall thus obtain the directions, or direction, in which a right line OP must be drawn from o , so as to *meet the conic at infinity*. And the same *conditions* as before, for distinguishing the *species* of the conic, may be otherwise obtained by combining the *anharmonic equation*, $f = 0$ (46), of that conic, with the corresponding equation $ax + by + cz = 0$ (38) of the *line at infinity*; so as to inquire (on known principles of modern geometry) whether that *line* meets that *curve* in *two*

imaginary points, or *touches* it, or *cuts* it, in points which (although *infinitely distant*) are here to be considered as *real*.

51. In general, if $f(x, y, z) = 0$ be the anharmonic equation (46) of *any plane curve*, considered as the *locus* of a variable point P ; and if the *differential** of this equation be thus denoted,

$$0 = df(x, y, z) = Xdx + Ydy + Zdz;$$

then because, by the supposed *homogeneity* (46) of the function f , we have the relation

$$Xx + Yy + Zz = 0,$$

we shall have also this other but analogous relation,

$$Xx' + Yy' + Zz' = 0,$$

if

$$x' - x : y' - y : z' - z = dx : dy : dz;$$

that is (by the principles of Art. 37), if $P' = (x', y', z')$ be *any point upon the tangent to the curve*, drawn at the point $P = (x, y, z)$, and regarded as the *limit of a secant*. The *symbol* (37) of this *tangent* at P may therefore be thus written,

$$[X, Y, Z], \text{ or } [D_x f, D_y f, D_z f];$$

where D_x, D_y, D_z are known *characteristics of partial derivation*.

52. For example, when f has the form assigned in 46, as answering to the conic lately considered, we have $D_x f = 2(x - y - z)$, &c.; whence the tangent at any point (x, y, z) of this curve may be denoted by the symbol,

$$[x - y - z, \quad y - z - x, \quad z - x - y];$$

in which, as usual, the co-ordinates of the line may be replaced by any others proportional to them. Thus at the point A , or (by 36) at $(0, 1, 1)$, which is evidently (by the form of f) a point upon the curve, the tangent is the line $[-2, 0, 0]$, or $[1, 0, 0]$; that is (by 38), the side BC of the given triangle, as

* In the theory of *quaternions*, as distinguished from (although including) that of *vectors*, it will be found necessary to introduce a *new definition of differentials*, on account of the *non-commutative* property of *quaternion-multiplication*: but, for the present, the *usual* significations of the signs d and D are sufficient.

was otherwise found before (46). And in general it is easy to see that the recent symbol denotes the right line, which is (in a well known sense) the *polar* of the point (x, y, z) , with respect to the same given conic; or that the line $[X', Y', Z']$ is the polar of the point (x', y', z') : because the equation

$$Xx' + Yy' + Zz' = 0,$$

which for a *conic* may be written as $X'x + Y'y + Z'z = 0$, expresses (by 51) the condition requisite, in order that a point (x, y, z) of the curve* should belong to a tangent which passes through the point (x', y', z') . Conversely, the point (x, y, z) is (in the same well-known sense) the *pole* of the line $[X, Y, Z]$; so that the *centre* of the conic, which is (by known principles) the *pole of the line at infinity* (38), is the point which satisfies the conditions $a^{-1}X = b^{-1}Y = c^{-1}Z$; it is therefore, for the present conic, the point $\kappa = (b + c, c + a, a + b)$, of which the vector $o\kappa$ is easily reduced, by the help of the linear equation, $aa + b\beta + c\gamma = 0$ (27), to the form,

$$\kappa = -\frac{a^2\alpha + b^2\beta + c^2\gamma}{2(bc + ca + ab)};$$

with the verification that the *denominator vanishes*, by 48, when the conic is a *parabola*. In the more general case, when this denominator is different from zero, it can be shown that *every chord* of the curve, which is drawn *through the extremity* κ of the vector κ , is *bisected* at that point κ : which point would therefore in this way be seen again to be the *centre*.

53. Instead of the *inscribed conic* (46), which has been the subject of recent articles, we may, as another example, consider that *exscribed* (or *circumscribed*) conic, which passes through the three corners A, B, C of the given triangle, and touches there the lines $\Lambda\Lambda''$, BB'' , CC'' of Fig. 21. The anharmonic equation of this new conic is easily seen to be,

$$yz + zx + xy = 0;$$

* If the curve $f=0$ were of a degree *higher* than the *second*, then the two equations above written would represent what are called the *first polar*, and the *last* or the *line-polar*, of the point (x', y', z') , with respect to the given curve.

the vector of a variable point P of the curve may therefore be expressed as follows,

$$\rho = \frac{t^1 a a + u^1 b \beta + v^1 c \gamma}{t^1 a + u^1 b + v^1 c},$$

with the condition $t + u + v = 0$, as before. The vector of its centre κ' is found to be,

$$\kappa' = \frac{2(a^2 a + b^2 \beta + c^2 \gamma)}{a^2 + b^2 + c^2 - 2bc - 2ca - 2ab};$$

and it is an ellipse, a parabola, or an hyperbola, according as the denominator of this last expression is negative, or null, or positive. And because these two recent vectors, κ, κ' , bear a scalar ratio to each other, it follows (by 19) that the three points o, κ, κ' are collinear; or in other words, that the line of centres $\kappa \kappa'$, of the two conics here considered, passes through the point of concurrence o of the three lines AA', BB', CC' . More generally, if L be the pole of any given right line $\Lambda = [l, m, n]$ (37), with respect to the inscribed conic (46), and if L' be the pole of the same line Λ with respect to the exscribed conic of the present article, it can be shown that the vectors oL, oL' , or λ, λ' , of these two poles are of the forms,

$$\lambda = k(la a + mb \beta + nc \gamma), \quad \lambda' = k'(la a + mb \beta + nc \gamma),$$

where k and k' are scalars; the three points o, L, L' are therefore ranged on one right line.

54. As an example of a vector-expression for a curve of an order higher than the second, the following may be taken:

$$oP = \rho = \frac{t^3 a a + u^3 b \beta + v^3 c \gamma}{t^3 a + u^3 b + v^3 c};$$

with $t + u + v = 0$, as before. Making $x = t^3, y = u^3, z = v^3$, we find here by elimination of t, u, v the anharmonic equation,

$$(x + y + z)^3 - 27xyz = 0;$$

the locus of the point P is therefore, in this example, a curve of the third order, or briefly a cubic curve. The mechanism (41)

of calculations with *anharmionic co-ordinates* is so much the same as that of the known *trilinear method*, that it may suffice to remark briefly here that the *sides* of the given triangle ABC are the *three (real) tangents of inflexion*; the *points of inflexion* being those which are marked as A'' , B'' , C'' in Fig. 21; and the *origin* of vectors o being a *conjugate point*.* If $a=b=c$, in which case (by 29) this origin o becomes (as in Fig. 19) the *mean point* of the triangle, the *chord of inflexion* $A''B''C''$ is then the *line at infinity*, and the curve takes the form represented in Fig. 27; hav-

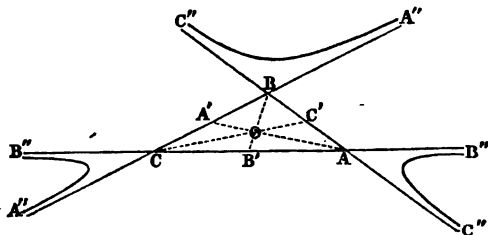


Fig. 27.

ing *three infinite branches*, inscribed within the angles vertically opposite to those of the given triangle ABC , of which the sides are the *three asymptotes*.

55. It would be improper to enter here into any details of discussion of such cubic curves, for which the reader will naturally turn to other works.† But it may be remarked, in passing, that because the *general cubic* may be represented, on the present plan, by combining the general expression of Art. 34 or 36 for the vector ρ , with the scalar equation

$$s^3 = 27kxyz, \quad \text{where } s = x + y + z;$$

k denoting an arbitrary constant, which becomes equal to unity, when the origin is (as in 54) a conjugate point; it follows that if $P = (x, y, z)$ and $P' = (x', y', z')$ be *any two points of the curve*, and if we make $s' = x' + y' + z'$, we shall have the relation,

$$xyzs^3 = x'y'z's^3, \quad \text{or} \quad \frac{xs'}{sx'} \cdot \frac{ys'}{sy'} \cdot \frac{zs'}{sz'} = 1:$$

* Answering to the values $t = 1$, $u = \theta$, $v = \theta^2$, where θ is one of the imaginary cube-roots of unity; which values of t, u, v give $x = y = z$, and $\rho = 0$.

† Especially the excellent Treatise on *Higher Plane Curves*, by the Rev. George Salmon, F. T. C. D., &c. Dublin, 1852.

in which it is not difficult to prove that

$$\frac{zs'}{xz} = (A'' \cdot PBP'B''); \quad \frac{ys'}{sy} = (B'' \cdot PCP'C''); \quad \frac{zs'}{sz} = (C'' \cdot PAP'A'');$$

the notation (35) of *anharmolics of pencils* being retained. We obtain therefore thus the following *Theorem*:—"If the sides of any given plane* triangle ABC be cut (as in Fig. 21) by any given rectilinear transversal A''B''C'', and if any two points P and P' in its plane be such as to satisfy the anharmonic relation

$$(A'' \cdot PBP'B'') \cdot (B'' \cdot PCP'C'') \cdot (C'' \cdot PAP'A'') = 1,$$

then these two points P, P' are on one common cubic curve, which has the three collinear points A'', B'', C'' for its three real points of inflexion, and has the sides BC, CA, AB of the triangle for its three tangents at those points;" a result which seems to offer a new geometrical generation for curves of the third order.

56. Whatever the order of a plane curve may be, or whatever may be the degree p of the function f in 46, we saw in 51 that the tangent to the curve at any point $P = (x, y, z)$ is the right line

$$\Delta = [l, m, n], \quad \text{if } l = D_x f, \quad m = D_y f, \quad n = D_z f;$$

expressions which, by the supposed *homogeneity* of f , give the relation, $lx + my + nz = 0$, and therefore enable us to establish the system of the two following differential equations,

$$l dx + m dy + n dz = 0, \quad x dl + y dm + z dn = 0.$$

If then, by elimination of the ratios of x, y, z , we arrive at a new *homogeneous equation* of the form,

$$0 = F(D_x f, D_y f, D_z f),$$

as one that is true for all values of x, y, z which render the function $f = 0$ (although it may require to be cleared of *factors*, introduced by this *elimination*), we shall have the equation

$$F(l, m, n) = 0,$$

* This Theorem may be extended, with scarcely any modification, from *plane* to *spherical curves*, of the third order.

as a *condition* that must be satisfied by the *tangent* Λ to the curve, in all the positions which can be assumed by that right line. And, by comparing the two differential equations,

$$dF(l, m, n) = 0, \quad xdl + ydm + zdn = 0,$$

we see that we may write the *proportion*,

$$x : y : z = D_l F : D_m F : D_n F, \quad \text{and the symbol } P = (D_l F, D_m F, D_n F),$$

if (x, y, z) be, as above, the *point of contact* P of the *variable line* $[l, m, n]$, in any one of its positions, with the curve which is its *envelope*. Hence we can pass (or return) from the *tangential equation* $F = 0$, of a curve considered as the *envelope of a right line* Λ , to the *local equation* $f = 0$, of the same curve considered (as in 46) as the *locus of a point* P : since, if we obtain, by elimination of the ratios of l, m, n , an equation of the form

$$0 = f(D_l F, D_m F, D_n F),$$

(cleared, if it be necessary, of foreign factors) as a consequence of the homogeneous equation $F = 0$, we have only to substitute for these *partial derivatives*, $D_l F$, &c., the *anharmonic co-ordinates* x, y, z , to which they are proportional. And when the functions f and F are not only *homogeneous* (as we shall always suppose them to be), but also *rational and integral* (which it is sometimes convenient *not* to assume them as being), then, while the *degree* of the function f , or of the *local equation*, marks (as before) the *order* of the curve, the degree of the *other homogeneous function* F , or of the *tangential equation* $F = 0$, is easily seen to denote, in this *anharmonic method* (as, from the analogy of *other* and older methods, it might have been expected to do), the *class* of the curve to which that equation belongs: or the *number of tangents* (distinct or coincident, and real or imaginary), which can be drawn to that curve, from an *arbitrary point* in its plane.

57. As an example (comp. 52), if we eliminate x, y, z between the equations,

$$l = x - y - z, \quad m = y - z - x, \quad n = z - x - y, \quad lx + my + nz = 0,$$

where l, m, n are the co-ordinates of the tangent to the *inscribed*

conic of Art. 46, we are conducted to the following tangential equation of that conic, or *curve of the second class*,

$$F(l, m, n) = mn + nl + lm = 0;$$

with the verification that the sides $[1, 0, 0]$, &c. (38), of the triangle $\triangle ABC$ are among the lines which satisfy this equation. Conversely, if this *tangential equation* were given, we might (by 56) *derive* from it expressions for the *co-ordinates of contact* x, y, z , as follows:

$$x = D_l F = m + n, \quad y = n + l, \quad z = l + m;$$

with the verification that the side $[1, 0, 0]$ touches the conic, considered now as an *envelope*, in the point $(0, 1, 1)$, or A' , as before: and then, by eliminating l, m, n , we should be brought back to the *local equation*, $f = 0$, of 46. In like manner, from the local equation $f = yz + zx + xy = 0$ of the *exscribed conic* (53), we can derive by differentiation the *tangential co-ordinates*,*

$$l = D_x f = y + z, \quad m = z + x, \quad n = x + y,$$

and so obtain by elimination the tangential equation, namely,

$$F(l, m, n) = l^2 + m^2 + n^2 - 2mn - 2nl - 2lm = 0;$$

from which we could in turn *deduce* the *local equation*. And (comp. 40), the very simple formula

$$lx + my + nz = 0,$$

which we have so often had occasion to employ, as *connecting two sets* of anharmonic co-ordinates, may not only be considered (as in 37) as the *local equation of a given right line* Λ , along which a point P moves, but also as the *tangential equation of a given point*, round which a right line turns: according as we suppose the set l, m, n , or the set x, y, z , to be given. Thus, while the right line $A''B''C''$, or $[1, 1, 1]$, of Fig. 21, was

* This name of "*tangential co-ordinates*" appears to have been first introduced by Dr. Booth in a Tract published in 1840, to which the author of the present Elements cannot now more particularly refer: but the *system* of Dr. Booth was entirely different from his own. See the reference in Salmon's *Higher Plane Curves*, note to page 16.

represented in 38 by the equation $x + y + z = 0$, the point o of the same figure, or the point $(1, 1, 1)$, may be represented by the analogous equation,

$$l + m + n = 0;$$

because the co-ordinates l, m, n of every line, which passes through this point o , must satisfy this equation of the first degree, as may be seen exemplified, in the same Art. 38, by the lines oA, oB, oC .

58. To give an instance or two of the use of forms, which, although homogeneous, are yet not rational and integral (56), we may write the local equation of the inscribed conic (46) as follows:

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0;$$

and then (suppressing the common numerical factor $\frac{1}{2}$), the partial derivatives are

$$l = x^{\frac{1}{2}}, \quad m = y^{\frac{1}{2}}, \quad n = z^{\frac{1}{2}};$$

so that a form of the tangential equation for this conic is,

$$l^{-1} + m^{-1} + n^{-1} = 0;$$

which evidently, when cleared of fractions, agrees with the first form of the last Article: with the verification (48), that $a^{-1} + b^{-1} + c^{-1} = 0$ when the curve is a parabola; that is, when it is touched (50) by the line at infinity (38). For the exscribed conic (53), we may write the local equation thus,

$$x^{-1} + y^{-1} + z^{-1} = 0;$$

whence it is allowed to write also,

$$l = x^{-2}, \quad m = y^{-2}, \quad n = z^{-2},$$

and

$$l^{\frac{1}{2}} + m^{\frac{1}{2}} + n^{\frac{1}{2}} = 0;$$

a form of the tangential equation which, when cleared of radicals, agrees again with 57. And it is evident that we could return, with equal ease, from these tangential to these local equations.

59. For the cubic curve with a conjugate point (54), the local equation may be thus written,*

* Compare Salmon's *Higher Plane Curves*, page 172.

$$x^{\dagger} + y^{\dagger} + z^{\dagger} = 0;$$

we may therefore assume for its tangential co-ordinates the expressions,

$$l = x^{\dagger}, \quad m = y^{\dagger}, \quad n = z^{\dagger};$$

and a form of its tangential equation is thus found to be,

$$l^{\dagger} + m^{\dagger} + n^{\dagger} = 0.$$

Conversely, if this tangential form were *given*, we might return to the local equation, by making

$$x = l^{\dagger}, \quad y = m^{\dagger}, \quad z = n^{\dagger},$$

which would give $x^{\dagger} + y^{\dagger} + z^{\dagger} = 0$, as before. The tangential equation just now found becomes, when it is cleared of *radicals*,

$$0 = l^2 + m^2 + n^2 - 2m^{-1}n^{-1} - 2n^{-1}l^{-1} - 2l^{-1}m^{-1};$$

or, when it is also cleared of *fractions*,

$$0 = \mathfrak{F} = m^2n^2 + n^2l^2 + l^2m^2 - 2nl^2m - 2lm^2n - 2mn^2l;$$

of which the *biquadratic* form shows (by 56) that *this cubic* is a *curve of the fourth class*, as indeed it is known to be. The *inflectional* character (54) of the points Λ'' , B'' , C'' upon this curve is here recognised by the circumstance, that when we make $m - n = 0$, in order to find the four tangents from $\Lambda'' = (0, 1, -1)$ (36), the resulting biquadratic, $0 = m^4 - 4lm^3$, has *three equal roots*; so that the line $[1, 0, 0]$, or the side BC , *counts as three*, and is therefore a *tangent of inflexion*: the *fourth tangent* from Λ'' being the line $[1, 4, 4]$, which touches the cubic at the point $(-8, 1, 1)$.

60. In general, the two equations (56),

$$nD_x f - lD_z f = 0, \quad nD_y f - mD_z f = 0,$$

may be considered as expressing that the homogeneous equation,

$$f(nx, ny, -lx - my) = 0,$$

which is obtained by eliminating z with the help of the relation $lx + my + nz = 0$, from $f(x, y, z) = 0$, and which we may

denote by $\phi(x, y) = 0$, has *two equal roots* $x : y$, if l, m, n be still the co-ordinates of a *tangent* to the curve f ; an *equality* which obviously corresponds to the *coincidence of two intersections* of that line with that curve. Conversely, if we seek by the usual methods the *condition of equality* of two roots $x : y$ of the homogeneous equation of the p^{th} degree,

$$0 = \phi(x, y) = f(nx, ny, -lx - my),$$

by eliminating the ratio $x : y$ between the two *derived* homogeneous equations, $0 = D_x\phi$, $0 = D_y\phi$, we shall in general be conducted to a result of the *dimension* $2p(p-1)$ in l, m, n , and of the *form*,

$$0 = n^{p(p-1)} F(l, m, n);$$

and so, by the rejection of the *foreign factor* $n^{p(p-1)}$, introduced by this *elimination*,* we shall obtain the *tangential equation* $F=0$, which will be in general of the *degree* $p(p-1)$; such being generally the known *class* (56) of the curve of which the *order* (46) is denoted by p : with (of course) a similar mode of passing, reciprocally, from a tangential to a local equation.

61. As an example, when the function f has the *cubic form* assigned in 54, we are thus led to investigate the condition for the existence of two equal roots in the *cubic equation*,

$$0 = \phi(x, y) = \{(n-l)x + (m-l)y\}^3 + 27n^2xy(lx + my),$$

by eliminating $x : y$ between *two derived* and *quadratic equations*; and the result presents itself, in the first instance, as of the *twelfth dimension* in the tangential co-ordinates l, m, n ; but it is found to be *divisible by* n^6 , and when this division is effected, it is reduced to the *sixth degree*, thus appearing to imply that the curve is of the *sixth class*, as in fact the *general cubic* is well known to be. A *further reduction* is however possible in the present case, on account of the *conjugate point* o (54), which introduces (comp. 57) the *quadratic factor*,

* Compare the method employed in Salmon's *Higher Plane Curves*, page 98, to find the equation of the *reciprocal* of a given curve, with respect to the imaginary conic, $x^2 + y^2 + z^2 = 0$. In general, if the function F be deduced from f as above, then $F(xyz) = 0$, and $f(xyz) = 0$ are equations of *two reciprocal curves*.

$$(l + m + n)^2 = 0;$$

and when *this factor also is set aside*, the tangential equation is found to be *reduced to the biquadratic form** already assigned in 59; the *algebraic division*, last performed, corresponding to the known *geometric depression of a cubic curve with a double point*, from the *sixth to the fourth class*. But it is time to close this Section on *Plane Curves*; and to proceed, as in the next Chapter we propose to do, to the consideration and comparison of *vectors of points in space*.

CHAPTER III.

APPLICATIONS OF VECTORS TO SPACE.

SECTION 1.—On Linear Equations between Vectors not Coplanar.

62. When three given and actual vectors OA, OB, OC , or a, β, γ , are *not* contained in any common plane, and when the three scalars a, b, c do not *all* vanish, then (by 21, 22) the expression $aa + b\beta + c\gamma$ cannot become equal to *zero*; it must therefore represent *some actual vector* (1), which we may, for the sake of symmetry, denote by the symbol $-d\delta$: where the *new (actual) vector* δ , or OD , is not contained in any one

* If we multiply that form $v = 0$ (59) by x^2 , and then change ax to $-lx - my$, we obtain a biquadratic equation in $l : m$, namely,

$$0 = \psi(l, m) = (l - m)^2 (lx + my)^2 + 2lm(l + m)(lx + my)z + l^2 m^2 z^2;$$

and if we then eliminate $l : m$ between the two derived cubics, $0 = D_l \psi$, $0 = D_m \psi$, we are conducted to the following equation of the twelfth degree, $0 = x^2 y^2 z^2 f(x, y, z)$, where f has the same cubic form as in 54. We are therefore thus brought *back* (comp. 59) from the *tangential* to the *local* equation of the cubic curve (54); complicated, however, as we see, with the *factor* $x^2 y^2 z^2$, which corresponds to the system of the three real tangents of inflexion to that curve, each tangent being taken three times. The reason why we have not here been obliged to reject *also* the foreign factor, $x^2 z$, as by the general theory (60) we might have expected to be, is that we multiplied the biquadratic function v *only* by x^2 , and *not* by x^1 .

of the three given and distinct planes, BOC, COA, AOB, unless some one, at least, of the three given coefficients a, b, c , vanishes; and where the *new scalar*, d , is either greater or less than zero. We shall thus have a *linear equation between four vectors*,

$$a\alpha + b\beta + c\gamma + d\delta = 0;$$

which will give

$$\delta = \frac{-a\alpha}{d} + \frac{-b\beta}{d} + \frac{-c\gamma}{d}, \quad \text{OR} \quad OD = OA' + OB' + OC';$$

where OA', OB', OC' , or $\frac{-a\alpha}{d}, \frac{-b\beta}{d}, \frac{-c\gamma}{d}$, are the

vectors of the three points A', B', C' , into which the point D is projected, on the three given lines OA, OB, OC , by planes drawn parallel to the three given planes, BOC, &c.; so that they are the three *co-initial edges* of a

parallelepiped, whereof the *sum*, OD or δ , is the *internal and co-initial diagonal* (comp. 6). Or we may project D on the three planes, by lines DA'', DB'', DC'' parallel to the three given lines, and then shall have $OA'' = OB' + OC' = \frac{b\beta + c\gamma}{-d}$, &c., and

$$\delta = OD = OA' + OA'' = OB' + OB'' = OC' + OC''.$$

And it is evident that this construction will apply to *any fifth point* D of space, if the four points $OABC$ be still supposed to be given, and *not coplanar*: but that some at least of the *three ratios* of the four scalars a, b, c, d (which last letter is not here used as a mark of differentiation) will vary with the position of the point D , or with the *value* of its vector δ . For example, we shall have $a = 0$, if D be situated in the plane BOC; and similarly for the two other given planes through O .

63. We may inquire (comp. 23), *what relation* between these scalar coefficients must exist, in order that the point D

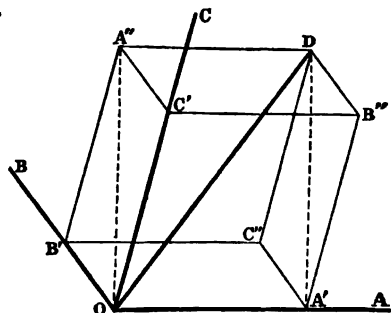


Fig. 28.

may be situated in the fourth given plane ABC ; or what is the condition of *complanarity* of the four points, A, B, C, D . Since the three vectors DA, DB, DC are now supposed to be *complanar*, they must (by 22) be connected by a *linear equation*, of the form

$$a(a - \delta) + b(\beta - \delta) + c(\gamma - \delta) = 0;$$

comparing which with the recent and more general form (62), we see that the required condition is,

$$a + b + c + d = 0.$$

This equation may be written (comp. again 23) as

$$\frac{-a}{d} + \frac{-b}{d} + \frac{-c}{d} = 1, \quad \text{or} \quad \frac{OA'}{OA} + \frac{OB'}{OB} + \frac{OC'}{OC} = 1;$$

and, under this last form, it expresses a known *geometrical property* of a plane $ABCD$, referred to three *co-ordinate axes* OA, OB, OC , which are drawn from any common *origin* O , and *terminate* upon the plane. We have also, in this case of *complanarity* (comp. 28), the following *proportion of coefficients and areas* :

$$a : b : c : -d = DBC : DCA : DAB : ABC;$$

or, more symmetrically, with attention to *signs of areas*,

$$a : b : c : d = BCD : -CDA : DAB : -ABC;$$

where Fig. 18 may serve for illustration, if we conceive O in that Figure to be replaced by D .

64. When we have thus at once the *two equations*,

$$aa + b\beta + c\gamma + d\delta = 0, \quad \text{and} \quad a + b + c + d = 0,$$

so that the four *co-initial vectors* a, β, γ, δ *terminate* (as above) on one common plane, and may therefore be said (comp. 24) to be *termino-complanar*, it is evident that the *two right lines*, DA and BC , which connect *two pairs* of the *four complanar points*, must *intersect* each other in some point A' of the plane, at a finite or infinite distance. And there is no difficulty in perceiving, on the plan of 31, that the *vectors* of the *three*

points A' , B' , C' of intersection, which thus result, are the following:

$$\left\{ \begin{array}{l} \text{for } A' = BC \cdot DA, \quad a' = \frac{b\beta + c\gamma}{b+c} = \frac{aa + d\delta}{a+d}; \\ \text{for } B' = CA \cdot DB, \quad \beta' = \frac{c\gamma + aa}{c+a} = \frac{b\beta + d\delta}{b+d}; \\ \text{for } C' = AB \cdot DC, \quad \gamma' = \frac{aa + b\beta}{a+b} = \frac{c\gamma + d\delta}{c+d}; \end{array} \right.$$

expressions which are *independent* of the *position* of the arbitrary origin o , and which accordingly coincide with the corresponding expressions in 27, when we place that origin in the point D , or make $\delta = 0$. Indeed, these last results hold good (comp. 31), even when the *four vectors* a , β , γ , δ , or the *five points* o , A , B , C , D , are *all* complanar. For, although there then exist *two* linear equations between those four vectors, which may in general be written thus,

$$a'a + b'\beta + c'\gamma + d'\delta = 0, \quad a''a + b''\beta + c''\gamma + d''\delta = 0,$$

without the relations, $a' + \&c. = 0$, $a'' + \&c. = 0$, between the coefficients, yet if we form from these *another* linear equation, of the form,

$$(a'' + ta')a + (b'' + tb')\beta + (c'' + tc')\gamma + (d'' + td')\delta = 0,$$

and determine t by the condition,

$$t = -\frac{a'' + b'' + c'' + d''}{a' + b' + c' + d'},$$

we shall only have to make $a = a'' + ta'$, &c., and the two equations written at the commencement of the present article will then both be satisfied; and will conduct to the expressions assigned above, for the three vectors of intersection: which *vectors* may thus be found, without its being *necessary* to employ those processes of *scalar elimination*, which were treated of in the foregoing Chapter.

As an *Example*, let the two given equations be (comp. 27, 38),

$$aa + b\beta + c\gamma = 0, \quad (2a + b + c)a''' - aa = 0;$$

and let it be required to determine the vectors of the intersections of the three pairs of lines $BC, \Delta\Delta''$; CA, BA'' ; and AB, CA'' . Forming the combination,

$$(2\alpha + b + c)\alpha'' - a\alpha + t(\alpha\alpha + b\beta + c\gamma) = 0,$$

and determining t by the condition,

$$(2\alpha + b + c) - \alpha + t(\alpha + b + c) = 0,$$

which gives $t = -1$, we have for the three sought vectors the expressions,

$$\frac{b\beta + c\gamma}{b + c}, \quad \frac{c\gamma + 2a\alpha}{c + 2a}, \quad \frac{2a\alpha + b\beta}{2a + b};$$

whereof the first = α' , by 27. Accordingly, in Fig. 21, the line $\Delta\Delta''$ intersects BC in the point Δ' ; and although the two *other* points of intersection here considered, which belong to what has been called (in 84) a *Third Construction*, are not marked in that Figure, yet their *anharmonic symbols* (86), namely, $(2, 0, 1)$ and $(2, 1, 0)$, might have been otherwise found by combining the equations $y = 0$ and $x = 2z$ for the two lines CA, BA'' ; and by combining $z = 0, x = 2y$ for the remaining pair of lines.

65. In the more general case, when the *four given points* A, B, C, D , are *not in any common plane*, let ε be any *fifth given point* of space, not situated on any one of the *four faces* of the *given pyramid* $ABCD$, nor on any such face prolonged; and let its vector $OE = \varepsilon$. Then the *four co-initial vectors*, EA, EB, EC, ED , whereof (by supposition) no three are coplanar, and which do not terminate upon one plane, must be (by 62) connected by some equation of the form,

$$a \cdot EA + b \cdot EB + c \cdot EC + d \cdot ED = 0;$$

where the *four scalars*, a, b, c, d , and their *sum*, which we shall denote by $-e$, are *all different from zero*. Hence, because $EA = \alpha - \varepsilon$, &c., we may establish the following *linear equation between five co-initial vectors*, $\alpha, \beta, \gamma, \delta, \varepsilon$, whereof *no four are termino-coplanar* (64),

$$a\alpha + b\beta + c\gamma + d\delta + e\varepsilon = 0;$$

with the *relation*, $a + b + c + d + e = 0$, between the *five scalars* a, b, c, d, e , whereof no one now separately vanishes. Hence also, $\varepsilon = (a\alpha + b\beta + c\gamma + d\delta) : (a + b + c + d)$, &c.

66. Under these conditions, if we write

$$D_1 = DE \cdot ABC, \quad \text{and} \quad OD_1 = \delta_1,$$

that is, if we denote by δ_1 the vector of the point D_1 in which the right line DE intersects the plane ABC , we shall have

$$\delta_1 = \frac{aa + b\beta + c\gamma}{a + b + c} = \frac{d\delta + e\epsilon}{d + e}.$$

In fact, these two expressions are *equivalent*, or represent one *common* vector, in virtue of the given equations; but the first shows (by 63) that this vector δ_1 terminates on the *plane* ABC , and the second shows (by 25) that it terminates on the *line* DE ; its extremity D_1 must therefore be, as required, the *intersection* of this line with that plane. We have therefore the two equations,

$$\text{I. . . } a(a - \delta_1) + b(\beta - \delta_1) + c(\gamma - \delta_1) = 0;$$

$$\text{II. . . } d(\delta - \delta_1) + e(\epsilon - \delta_1) = 0;$$

whence (by 28 and 24) follow the two proportions,

$$\text{I'. . . } a : b : c = D_1BC : D_1CA : D_1AB;$$

$$\text{II'. . . } d : e = ED_1 : D_1D;$$

the arrangement of the points, in the annexed Fig. 29, answering to the case where all the four coefficients a, b, c, d are *positive* (or have one *common sign*), and when therefore the remaining coefficient e is *negative* (or has the *opposite sign*).

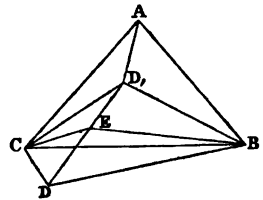


Fig. 29.

67. For the three *complanar triangles*, in the first proportion, we may substitute any three *pyramidal volumes*, which rest upon those triangles as their *bases*, and which have one *common vertex*, such as D or E ; and because the collineation DED_1 gives $DD_1BC - ED_1BC = DEBC$, &c., we may write this other proportion,

$$\text{I''. . . } a : b : c = DEBC : DECA : DEAB.$$

Again, the same collineation gives

$$ED_1 : DD_1 = EABC : DABC;$$

we have therefore, by II' , the proportion,

$$\text{II''. . . } d : -e = EABC : DABC.$$

But

$$DEBC + DECA + DEAB + EABC = DABC,$$

and

$$a + b + c + d = -e;$$

we may therefore establish the following *fuller* formula of proportion, between *coefficients* and *volumes* :

$$\text{III. . . } a : b : c : d : -e = \text{DEBC} : \text{DECA} : \text{DEAB} : \text{EABC} : \text{DABC};$$

the *ratios* of all these five *pyramids* to each other being considered as *positive*, for the particular *arrangement* of the *points* which is represented in the recent figure.

68. The *formula* III. may however be regarded as perfectly *general*, if we agree to say that a *pyramidal volume* changes *sign*, or rather that it *changes its algebraical character*, as *positive* or *negative*, in *comparison* with a *given pyramid*, and with a *given arrangement* of points, in *passing through zero* (comp. 28); namely *when*, in the course of any continuous change, *any one of its vertices crosses the corresponding base*. With this *convention** we shall have, generally,

$$\text{DABC} = -\text{ADEBC} = \text{ABDC} = -\text{ABCD}, \quad \text{DEBC} = \text{BCDE}, \quad \text{DECA} = \text{CDEA};$$

the proportion III. may therefore be expressed in the following more *symmetric*, but equally *general form* :

$$\text{III'. . . } a : b : c : d : e = \text{BCDE} : \text{CDEA} : \text{DEAB} : \text{EABC} : \text{ABCD};$$

the *sum* of these five *pyramids* being *always* equal to *zero*, when *signs* (as above) are attended to.

69. We saw (in 24) that the two equations,

$$aa + b\beta + c\gamma = 0, \quad a + b + c = 0,$$

gave the proportion of *segments*,

$$a : b : c = \text{BC} : \text{CA} : \text{AB},$$

whatever might be the position of the *origin* o. In like manner we saw (in 63) that the two other equations,

* Among the consequences of this convention respecting *signs* of *volumes*, which has already been adopted by some modern geometers, and which indeed is *necessary* (comp. 28) for the establishment of *general formula*, one is that *any two pyramids*, ABCD, A'B'C'D', bear to each other a *positive* or a *negative ratio*, according as the two *relations*, BCD and B'C'D', supposed to be *seen* respectively from the points A and A', have *similar* or *opposite directions*, as right-handed or left-handed.

$$a\alpha + b\beta + c\gamma + d\delta = 0, \quad a + b + c + d = 0,$$

gave the proportion of *areas*,

$$a : b : c : d = \text{BCD} : -\text{CDA} : \text{DAB} : -\text{ABC};$$

where again the origin is arbitrary. And we have just deduced (in 68) a corresponding proportion of *volumes*, from the two analogous equations (65),

$$a\alpha + b\beta + c\gamma + d\delta + e\epsilon = 0, \quad a + b + c + d + e = 0,$$

with an equally arbitrary origin. If then we conceive these *segments*, *areas*, and *volumes* to be replaced by the *scalars* to which they are thus *proportional*, we may establish the *three general formulæ* :

$$\text{I. } o\alpha \cdot \text{BC} + o\beta \cdot \text{CA} + o\gamma \cdot \text{AB} = 0;$$

$$\text{II. } o\alpha \cdot \text{BCD} - o\beta \cdot \text{CDA} + o\gamma \cdot \text{DAB} - o\delta \cdot \text{ABC} = 0;$$

$$\text{III. } o\alpha \cdot \text{BCDE} + o\beta \cdot \text{CDEA} + o\gamma \cdot \text{DEAB} + o\delta \cdot \text{EABC} + o\epsilon \cdot \text{ABCD} = 0;$$

where in I., A, B, C are *any three collinear points* ;

in II., A, B, C, D are *any four complanar points* ;

and in III., A, B, C, D, E are *any five points of space* ;

while o is, in *each* of the three formulæ, an entirely *arbitrary point*. It must, however, be remembered, that the *additions* and *subtractions* are supposed to be performed according to the *rules of vectors*, as stated in the First Chapter of the present Book ; the segments, or areas, or volumes, which the equations indicate, being treated as *coefficients* of those vectors. We might still further *abridge* the *notations*, while retaining the *meaning* of these formulæ, by *omitting the symbol* of the *arbitrary origin* o ; and by thus writing,*

$$\text{I'. } \quad \quad \quad A \cdot \text{BC} + B \cdot \text{CA} + C \cdot \text{AB} = 0,$$

for any three collinear points ; with corresponding formulæ II'. and III', for any four complanar points, and for any five points of space.

* We should thus have some of the *notations* of the *Barycentric Calculus* (see Note B), but employed here with different *interpretations*.

SECTION 2.— *On Quinary Symbols for Points and Planes in Space.*

70. The equations of Art. 65 being still supposed to hold good, the vector ρ of any point P of space may, in indefinitely many ways, be expressed (comp. 36) under the form:

$$I. \dots \rho = \frac{x\alpha a + y\beta b + z\gamma c + w\delta d + ve e}{x\alpha + y\beta + z\gamma + w\delta + ve}$$

in which the ratios of the differences of the five coefficients, $x\alpha z\gamma w\delta$, determine the position of the point. In fact, because the four points $ABCD$ are not in any common plane, there necessarily exists (comp. 65) a determined linear relation between the four vectors drawn to them from the point P , which may be written thus,

$$x'a \cdot PA + y'b \cdot PB + z'e \cdot PC + w'd \cdot PD = 0,$$

giving the expression,

$$II. \dots \rho = \frac{x'aa + y'\beta\beta + z'\gamma\gamma + w'\delta\delta}{x'a + y'b + z'e + w'd},$$

in which the ratios of the four scalars $x'y'z'w'$, depend upon, and conversely determine, the position of P ; writing, then,

$$x = tx' + v, \quad y = ty' + v, \quad z = tz' + v, \quad w = tw' + v,$$

where t and v are two new and arbitrary scalars, and remembering that $aa + \dots + ee = 0$, and $a + \dots + e = 0$ (65), we are conducted to the form for ρ , assigned above.

71. When the vector ρ is thus expressed, the point P may be denoted by the Quinary Symbol (x, y, z, w, v) ; and we may write the equation,

$$P = (x, y, z, w, v).$$

But we see that the same point P may also be denoted by this other symbol, of the same kind, (x', y', z', w', v') , provided that the following proportion between differences of coefficients (70) holds good:

$$x' - v' : y' - v' : z' - v' : w' - v' = x - v : y - v : z - v : w - v.$$

Under this condition, we shall therefore write the following formula of congruence,

$$(x', y', z', w', v') \equiv (x, y, z, w, v),$$

to express that these two quinary symbols, although not identical in composition, have yet the same geometrical signification, or denote one common point. And we shall reserve the symbolic equation,

$$(x', y', z', w', v') = (x, y, z, w, v),$$

to express that the *five coefficients*, $x' \dots v'$, of the one symbol, are *separately equal* to the *corresponding coefficients* of the other, $x' = x, \dots v' = v$.

72. Writing also, generally,

$$(tx, ty, tz, tw, tv) = t(x, y, z, w, v), \\ (x' + x, \dots v' + v) = (x', \dots v') + (x, \dots v), \text{ \&c.,}$$

and abridging the particular symbol* $(1, 1, 1, 1, 1)$ to (U) , while (Q) , (Q') , \dots may briefly denote the quinary symbols $(x, \dots v)$, $(x', \dots v')$, \dots we may thus establish the *congruence* (71),

$$(Q') \equiv (Q), \text{ if } (Q) = t(Q') + u(U);$$

in which t and u are arbitrary coefficients. For example,

$$(0, 0, 0, 0, 1) \equiv (1, 1, 1, 1, 0), \text{ and } (0, 0, 0, 1, 1) \equiv (1, 1, 1, 0, 0);$$

each symbol of the first pair denoting (65) the given point \mathbf{x} ; and each symbol of the second pair denoting (66) the derived point \mathbf{D}_1 . When the coefficients are *so simple* as in these last expressions, we may occasionally *omit the commas*, and thus write, still more briefly,

$$(00001) \equiv (11110); \quad (00011) \equiv (11100).$$

73. If *three vectors*, ρ, ρ', ρ'' , expressed each under the *first form* (70), be *termino-collinear* (24) and if we denote their denominators, $xa + \dots, x'a + \dots, x''a + \dots$, by m, m', m'' , they must then (23) be connected by a *linear equation*, with a *null sum* of coefficients, which may be written thus:

$$tm\rho + t'm'\rho' + t''m''\rho'' = 0; \quad tm + t'm' + t''m'' = 0.$$

We have, therefore, the two *equations of condition*,

$$t(xaa + \dots + v\epsilon\epsilon) + t'(x'aa + \dots + v'\epsilon\epsilon) + t''(x''aa + \dots + v''\epsilon\epsilon) = 0; \\ t(xa + \dots + v\epsilon) + t'(x'a + \dots + v'\epsilon) + t''(x''a + \dots + v''\epsilon) = 0;$$

where t, t', t'' are three new scalars, while the five vectors $a \dots \epsilon$, and the five scalars $a \dots \epsilon$, are subject only to the two equations (65): but these equations of condition are satisfied by supposing that

$$tx + t'x' + t''x'' = \dots = tv + t'v' + t''v'' = -u,$$

where u is some new scalar, and they cannot be satisfied otherwise. Hence the *condition of collinearity* of the *three points* $\mathbf{x}, \mathbf{x}', \mathbf{x}''$, in which the three vectors ρ, ρ', ρ'' terminate, and of which the quinary symbols are $(Q), (Q'), (Q'')$, may briefly be expressed by the equation,

* This quinary symbol (U) denotes *no determined point*, since it corresponds (by 70, 71) to the *indeterminate vector* $\rho = \frac{0}{0}$; but it admits of useful combinations with other quinary symbols, as above.

$$t(Q) + t'(Q') + t''(Q'') = -u(U);$$

so that if any four scalars, t, t', t'', u , can be found, which satisfy this last symbolic equation, then, but not in any other case, those three points $PP'P''$ are ranged on one right line. For example, the three points D, E, D_1 , which are denoted (72) by the quinary symbols, (00010), (00001), (11100), are collinear; because the sum of these three symbols is (U) . And if we have the equation,

$$(Q'') = t(Q) + t'(Q') + u(U),$$

where t, t', u are any three scalars, then (Q'') is a symbol for a point P'' , on the right line PP' . For example, the symbol $(0, 0, 0, t, t')$ may denote any point on the line DE .

74. By reasonings precisely similar it may be proved, that if $(Q)(Q')(Q'')(Q''')$ be quinary symbols for any four points $PP'P''P'''$ in any common plane, so that the four vectors $pp'p''p'''$ are termino-complanar (64), then an equation, of the form

$$t(Q) + t'(Q') + t''(Q'') + t'''(Q''') = -u(U),$$

must hold good; and conversely, that if the fourth symbol can be expressed as follows,

$$(Q''') = t(Q) + t'(Q') + t''(Q'') + u(U),$$

with any scalar values of t, t', t'', u , then the fourth point P''' is situated in the plane $PP'P''$ of the other three. For example, the four points,

$$(10000), \quad (01000), \quad (00100), \quad (11100),$$

or A, B, C, D_1 (66), are complanar; and the symbol $(t, t', t'', 0, 0)$ may represent any point in the plane ABC .

75. When a point P is thus complanar with three given points, P_0, P_1, P_2 , we have therefore expressions of the following forms, for the five coefficients x, \dots, v of its quinary symbol, in terms of the fifteen given coefficients of their symbols, and of four new and arbitrary scalars:

$$x = t_0x_0 + t_1x_1 + t_2x_2 + u; \dots \quad v = t_0v_0 + t_1v_1 + t_2v_2 + u.$$

And hence, by elimination of these four scalars, $t_0 \dots u$, we are conducted to a linear equation of the form

$$l(x-v) + m(y-v) + n(z-v) + r(w-v) = 0,$$

which may be called the Quinary Equation of the Plane $P_0P_1P_2$, or of the supposed locus of the point P : because it expresses a common property of all the points of that locus; and because the three ratios of the four new coefficients l, m, n, r , determine the position of the plane

in space. It is, however, more *symmetrical*, to write the quinary equation of a plane Π as follows,

$$lx + my + ns + rw + sv = 0,$$

where the *fifth coefficient*, s , is connected with the others by the relation,

$$l + m + n + r + s = 0;$$

and then we may say that $[l, m, n, r, s]$ is (comp. 37) the *Quinary Symbol of the Plane* Π , and may write the equation,

$$\Pi = [l, m, n, r, s].$$

For example, the coefficients of the symbol for a point P in the plane ABC may be thus expressed (comp. 74):

$$x = t_0 + u, \quad y = t_1 + u, \quad z = t_2 + u, \quad w = u, \quad v = u;$$

between which the only relation, *independent of the four arbitrary scalars* $t_0 \dots u$, is $w - v = 0$; this therefore is the equation of the plane ABC , and the *symbol* of that plane is $[0, 0, 0, 1, -1]$; which may (comp. 72) be sometimes written more briefly, without commas, as $[0001\bar{1}]$. It is evident that, in any such symbol, the *coefficients* may all be multiplied by any *common factor*.

76. The symbol of the *plane* $P_0P_1P_2$ having been thus determined, we may next propose to find a symbol for the *point*, P , in which that plane is *intersected* by a given *line* P_3P_4 : or to *determine the coefficients* $x \dots v$, or at least the *ratios of their differences* (70), in the quinary symbol of that point,

$$(x, y, z, w, v) = P = P_0P_1P_2 \cdot P_3P_4.$$

Combining, for this purpose, the expressions,

$$x = t_2x_2 + t_4x_4 + u', \dots \quad v = t_3v_3 + t_4v_4 + u',$$

(which are included in the symbolical equation (73),

$$(Q) = t_2(Q_2) + t_4(Q_4) + u'(U),$$

and express the *collinearity* PP_3P_4) with the equations (75),

$$lx + \dots + sv = 0, \quad l + \dots + s = 0,$$

(which express the *complanarity* $PP_0P_1P_2$) we are conducted to the formula,

$$t_2(lx_2 + \dots + sv_2) + t_4(lx_4 + \dots + sv_4) = 0;$$

which determines the *ratio* $t_2 : t_4$, and contains the solution of the problem. For example, if P be a point on the *line* DE , then (comp. 73),

$$x = y = z = u', \quad w = t_3 + u', \quad v = t_4 + u';$$

but if it be also a point in the plane ABC , then $w - v = 0$ (75), and therefore $t_3 - t_4 = 0$; hence

$$(Q) = t_3(00011) + w(11111), \text{ or } (Q) \equiv (00011);$$

which last symbol had accordingly been found (72) to represent the intersection (66), $D_1 = ABC \cdot DE$.

77. When the five coefficients, $xyzvw$, of any given quinary symbol (Q) for a point P , or those of any congruent symbol (71), are any whole numbers (positive or negative, or zero), we shall say (comp. 42) that the point P is rationally related to the five given points, $A \dots E$; or briefly, that it is a Rational Point of the System, which those five points determine. And in like manner, when the five coefficients, $lmnrs$, of the quinary symbol (75) of a plane Π are either equal or proportional to integers, we shall say that the plane is a Rational Plane of the same System; or that it is rationally related to the same five points. On the contrary, when the quinary symbol of a point, or of a plane, has not thus already whole coefficients, and cannot be transformed (comp. 72) so as to have them, we shall say that the point or plane is irrationally related to the given points; or briefly, that it is irrational. A right line which connects two rational points, or is the intersection of two rational planes, may be called, on the same plan, a Rational Line; and lines which cannot in either of these two ways be constructed, may be said by contrast to be Irrational Lines. It is evident from the nature of the eliminations employed (comp. again 42), that a plane, which is determined as containing three rational points, is necessarily a rational plane; and in like manner, that a point, which is determined as the common intersection of three rational planes, is always a rational point: as is also every point which is obtained by the intersection of a rational line with a rational plane; or of two rational lines with each other (when they happen to be complanar).

78. Finally, when two points, or two planes, differ only by the arrangement (or order) of the coefficients in their quinary symbols, those points or planes may be said to have one common type; or briefly to be syntypical. For example, the five given points, A, \dots, E , are thus syntypical, as being represented by the quinary symbols $(10000), \dots, (00001)$; and the ten planes, obtained by taking all the ternary combinations of those five points, have in like manner one common type. Thus, the quinary symbol of the plane ABC has been seen (75) to be $[0001\bar{1}]$; and the analogous symbol $[1\bar{1}000]$ represents the plane CDE , &c. Other examples will present themselves, in a

shortly subsequent Section, on the subject of *Nets in Space*. But it seems proper to say here a few words, respecting those *Anharmonic Co-ordinates, Equations, Symbols, and Types, for Space*, which are obtained from the theory and expressions of the present Section, by *reducing* (as we are allowed to do) the *number of the coefficients*, in each symbol or equation, from *five to four*.

SECTION 3.—On Anharmonic Co-ordinates in Space.

79. When we adopt the *second form* (70) for ρ , or suppose (as we may) that the *fifth coefficient* in the *first form vanishes*, we get this *other general expression* (comp. 34, 36), for the *vector of a point in space*:

$$OP = \rho = \frac{x\alpha\alpha + y\beta\beta + z\gamma\gamma + w\delta\delta}{x\alpha + y\beta + z\gamma + w\delta};$$

and may then write the symbolic equation (comp. 36, 71),

$$P = (x, y, z, w),$$

and call this last the *Quaternary Symbol of the Point P*: although we shall soon see cause for calling it also the *Anharmonic Symbol* of that point. Meanwhile we may remark, that the *only congruent symbols* (71), of this *last form*, are those which differ merely by the introduction of a *common factor*: the *three ratios* of the *four coefficients*, $x \dots w$, being *all* required, in order to *determine the position of the point*; whereof those four coefficients may accordingly be said (comp. 36) to be the *Anharmonic Coordinates in Space*.

80. When we thus suppose that $v = 0$, in the *quinary symbol* of the *point P*, we may *suppress the fifth term sv*, in the *quinary equation* of a *plane Π* , $lx + \dots + sv = 0$ (75); and therefore may *suppress also* (as here unnecessary) the *fifth coefficient, s*, in the *quinary symbol* of that plane, which is thus reduced to the *quaternary form*,

$$\Pi = [l, m, n, r].$$

This last may also be said (37, 79), to be the *Anharmonic Symbol of the Plane*, of which the *Anharmonic Equation* is

$$lx + my + nz + rw = 0;$$

the *four coefficients, lmnr*, which we shall call also (comp. again 37) the *Anharmonic Co-ordinates of that Plane Π* , being not connected among themselves by any *general relation* (such as $l + \dots + s = 0$): since their *three ratios* (comp. 79) are *all* in general necessary, in order to determine the *position of the plane in space*.

81. If we suppose that the *fourth coefficient, w*, also *vanishes*, in

the recent symbol of a point, that point P is in the plane ABC ; and may then be sufficiently represented (as in 36) by the *Ternary Symbol* (x, y, z) . And if we attend only to the points in which an *arbitrary plane* Π intersects the *given plane* ABC , we may suppress its *fourth coefficient*, r , as being for such points unnecessary. In this manner, then, we are reconducted to the equation, $lx + my + nz = 0$, and to the symbol, $\Lambda = [l, m, n]$, for a *right line* (37) in the plane ABC , considered here as the *trace*, on that plane, of an *arbitrary plane* Π in space. If this plane Π be given by its *quinary symbol* (75), we thus obtain the *ternary symbol* for its *trace* Λ , by simply suppressing the two last coefficients, r and s .

82. In the more general case, when the point P is not confined to the plane ABC , if we denote (comp. 72) its *quaternary symbol* by (Q) , the lately established formulæ of *collineation* and *complanarity* (73, 74) will still hold good: provided that we now suppress the symbol (U) , or suppose its coefficient to be zero. Thus, the formula,

$$(Q) = t'(Q') + t''(Q'') + t'''(Q'''),$$

expresses that the point P is in the plane $P'P''P'''$; and if the coefficient t''' vanish, the equation which then remains, namely,

$$(Q) = t'(Q') + t''(Q''),$$

signifies that P is thus *complanar* with the two given points P' , P'' , and with an *arbitrary third point*; or, in other words, that it is on the *right line* $P'P''$; whence (comp. 76) problems of *intersections of lines with planes* can easily be resolved. In like manner, if we denote briefly by $[R]$ the *quaternary symbol* $[l, m, n, r]$ for a *plane* Π , the formula

$$[R] = t'[R'] + t''[R''] + t'''[R''']$$

expresses that the plane Π passes through the *intersection* of the three planes, Π' , Π'' , Π''' ; and if we suppose $t''' = 0$, so that

$$[R] = t'[R'] + t''[R''],$$

the formula thus found denotes that the plane Π passes through the point of intersection of the two planes, Π' , Π'' , with any third plane; or (comp. 41), that this plane Π contains the *line of intersection* of Π' , Π'' ; in which case the three planes, Π , Π' , Π'' , may be said to be *collinear*. Hence it appears that either of the two expressions,

$$\text{I. . . } t'(Q') + t''(Q''), \quad \text{II. . . } t'[R'] + t''[R''],$$

may be used as a *Symbol of a Right Line in Space*: according as we consider that line Λ either, 1st, as connecting two given points, or

Ind, as being the *intersection of two given planes*. The remarks (77) on *rational and irrational points, planes, and lines* require no modification here; and those on *types* (78) adapt themselves as easily to *quaternary* as to *quinary* symbols.

83. From the foregoing general formulæ of collineation and coplanarity, it follows that the point P' , in which the line AB intersects the plane CDE through CD and any proposed point $P = (xyzw)$ of space, may be denoted thus:

$$P' = AB \cdot CDE = (xy00);$$

for example, $E = (1111)$, and $C' = AB \cdot CDE = (1100)$. In general, if $ABCDEF$ be any six points of space, the four collinear planes (82), ABC , ABD , ABE , ABF , are said to form a *pencil* through AB ; and if this be cut by any rectilinear transversal, in four points, C' , D' , E' , F' , then (comp. 35) the *anharmonic function* of this group of points (25) is called also the *Anharmonic of the Pencil of Planes*: which may be thus denoted,

$$(AB \cdot CDEF) = (C'D'E'F').$$

Hence (comp. again 25, 35), by what has just been shown respecting C' and P' , we may establish the important formula:

$$(CD \cdot AEBP) = (AC'BF') = \frac{x}{y};$$

so that this *ratio of coefficients*, in the symbol $(xyzw)$ for a *variable point* P (79), represents the *anharmonic of a pencil of planes*, of which the *variable plane* CDE is *one*; the *three other planes* of this pencil being *given*. In like manner,

$$(\Delta D \cdot BECP) = \frac{y}{z}, \quad \text{and} \quad (BD \cdot CEAP) = \frac{z}{x};$$

so that (comp. 36) the *product* of these three last anharmonics is *unity*. On the same plan we have also,

$$(BC \cdot AEDP) = \frac{x}{w}, \quad (CA \cdot BEDP) = \frac{y}{w}, \quad (\Delta B \cdot CEDP) = \frac{z}{w};$$

so that the *three ratios*, of the three first coefficients xyz to the fourth coefficient w , suffice to *determine the three planes*, BCP , CAE , ABF , whereof the *point* P is the common *intersection*, by means of the *anharmonics of three pencils of planes*, to which the three planes respectively belong. And thus we see a *motive* (besides that of *analogy* to expressions already used for *points in a given plane*), for calling the *four coefficients*, $xyzw$, in the *quaternary symbol* (79) for a *point in space*, the *Anharmonic Co-ordinates of that Point*.

84. In general, if there be any four collinear points, P_0, \dots, P_3 , so

that (comp. 82) their *symbols* are connected by two linear equations, such as the following,

$$(Q_1) = t(Q_0) + u(Q_2), \quad (Q_3) = t'(Q_0) + u'(Q_2),$$

then the anharmonic of their *group* may be expressed (comp. 25, 44) as follows:

$$(P_0P_1P_2P_3) = \frac{uu'}{tu'};$$

as appears by considering the *pencil* (CD. $P_0P_1P_2P_3$), and the *transversal* AB (83). And in like manner, if we have (comp. again 82) the two other symbolic equations, connecting four collinear planes $\Pi_0 \dots \Pi_3$,

$$[R_1] = t[R_0] + u[R_2], \quad [R_3] = t'[R_0] + u'[R_2],$$

the anharmonic of their *pencil* (83) is expressed by the precisely similar formula,

$$(\Pi_0\Pi_1\Pi_2\Pi_3) = \frac{uu'}{tu'};$$

as may be proved by supposing the pencil to be cut by the same transversal line AB.

85. It follows that if $f(xyzw)$ and $f_1(xyzw)$ be any two homogeneous and linear functions of x, y, z, w ; and if we determine four collinear planes $\Pi_0 \dots \Pi_3$ (82), by the four equations,

$$f = 0, \quad f_1 = f, \quad f_1 = 0, \quad f_1 = kf,$$

where k is any scalar; we shall have the following value of the anharmonic function, of the pencil of planes thus determined:

$$(\Pi_0\Pi_1\Pi_2\Pi_3) = k = \frac{f_1}{f}.$$

Hence we derive this *Theorem*, which is important in the application of the present system of co-ordinates to space:—

“The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates (79) of a variable Point P in space, may be expressed as the Anharmonic $(\Pi_0\Pi_1\Pi_2\Pi_3)$ of a Pencil of Planes; whereof three are given, while the fourth passes through the variable point P , and through a given right line Λ which is common to the three former planes.”

86. And in like manner may be proved this other but analogous Theorem:—

“The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates (80) of a variable Plane Π , may be expressed as the Anharmonic $(P_0P_1P_2P_3)$ of a Group of Points; whereof three are given and collinear, and the fourth is the intersection, $\Lambda \cdot \Pi$, of their common and given right line Λ , with the variable plane Π .”

More fully, if the two given functions of $lmnr$ be F and F_1 , and if we determine three points $P_0P_1P_2$ by the equations (comp. 57) $F=0$, $F_1=F$, $F_1=0$, and denote by P_3 the intersection of their common line Λ with Π , we shall have the quotient,

$$\frac{F_1}{F} = (P_0P_1P_2P_3).$$

For example, if we suppose that

$$A_2 = (1001), \quad B_2 = (0101), \quad C_2 = (0011),$$

$$A'_2 = (10\bar{0}1), \quad B'_2 = (01\bar{0}1), \quad C'_2 = (00\bar{1}1),$$

so that

$$A_2 = DA \cdot BCE, \text{ \&c., and } (DA_2AA'_2) = -1, \text{ \&c.,}$$

we find that the three ratios of l , m , n to r , in the symbol $\Pi = [lmnr]$, may be expressed (comp. 39) under the form of anharmonics of groups, as follows:

$$\frac{l}{r} = (DA'_2AQ); \quad \frac{m}{r} = (DB'_2BR); \quad \frac{n}{r} = (DC'_2CS);$$

where Q , R , S denote the intersections of the plane Π with the three given right lines, DA , DB , DC . And thus we have a *motive* (comp. 83) besides that of *analogy to lines* in a given plane (37), for calling (as above) the *four coefficients* l , m , n , r , in the *quaternary symbol* (80) for a *plane* Π , the *Anharmonic Co-ordinates of that Plane in Space* .

87. It may be added, that if we denote by L , M , N the points in which the same plane Π is cut by the three given lines BC , CA , AB , and retain the notations A'' , B'' , C'' for those other points on the same three lines which were so marked before (in 31, &c.), so that we may now write (comp. 36)

$$A'' = (01\bar{1}0), \quad B'' = (\bar{1}010), \quad C'' = (1\bar{1}00),$$

we shall have (comp. 39, 83) these three other anharmonics of groups, with their product equal to unity:

$$\frac{m}{n} = (CA''BL); \quad \frac{n}{l} = (AB''CM); \quad \frac{l}{m} = (BC''AN);$$

and the *six given points* , A'' , B'' , C'' , A'_2 , B'_2 , C'_2 , are all *in one given plane* $[E]$, of which the *equation and symbol* are:

$$x + y + z + w = 0; \quad [E] = [11111].$$

The *six groups* of points, of which the anharmonic functions thus represent the *six ratios* of the four anharmonic co-ordinates, $lmnr$, of a *variable plane* Π , are therefore situated *on the six edges* of the *given pyramid* , $ABCD$; *two points* in each group being *corners* of that

pyramid, and the two others being the intersections of the edges with the two planes, [E] and II. Finally, the plane [E] is (in a known modern sense) the plane of homology,* and the point E is the centre of homology, of the given pyramid ABCD, and of an inscribed pyramid $A_1B_1C_1D_1$, where $A_1 = EA \cdot BCD$, &c.; so that D_1 retains its recent signification (66, 76); and we may write the anharmonic symbols,

$$A_1 = (0111), \quad B_1 = (1011), \quad C_1 = (1101), \quad D_1 = (1110).$$

And if we denote by A'_1, B'_1, C'_1, D'_1 the harmonic conjugates to these last points, with respect to the lines EA, EB, EC, ED, so that

$$(EA_1AA'_1) = \dots = (ED_1DD'_1) = -1,$$

we have the corresponding symbols,

$$A'_1 = (2111), \quad B'_1 = (1211), \quad C'_1 = (1121), \quad D'_1 = (1112).$$

Many other relations of position exist, between these various points, lines, and planes, of which some will come naturally to be noticed, in that theory of nets in space to which in the following Section we shall proceed.

SECTION 4.—On Geometrical Nets in Space.

88. When we have (as in 65) five given points $A \dots E$, whereof no four are complanar, we can connect any two of them by a right line, and the three others by a plane, and determine the point in which these last intersect one another: deriving thus a system of ten lines Λ_1 , ten planes Π_1 , and ten points P_1 , from the given system of five points P_0 , by what may be called (comp. 34) a First Construction. We may next propose to determine all the new and distinct lines, Λ_2 , and planes, Π_2 , which connect the ten derived points P_1 with the five given points P_0 , and with each other; and may then inquire what new and distinct points P_2 arise (at this stage) as intersections of lines with planes, or of lines in one plane with each other: all such new lines, planes, and points being said (comp. again 34) to belong to a Second Construction. And then we might proceed to a Third Construction of the same kind, and so on for ever: building up thus what has been called† a Geometrical Net in Space. To express this geometrical process by quinary symbols (71, 75, 82) of points, planes, and lines, and by quinary types (78), so far at least as to the end of the second construction, will be found to be an useful exercise in the

* See Poncelet's *Traité des Propriétés Projectives* (Paris, 1822).

† By Möbius, in p. 291 of his already cited *Barycentric Calculus*.

application of principles lately established: and therefore ultimately in that METHOD OF VECTORS, which is the subject of the present Book. And the *quinary form* will here be more convenient than the *quaternary*, because it will exhibit more clearly the geometrical dependence of the *derived* points and planes on the *five* given points, and will thereby enable us, through a principle of *symmetry*, to reduce the number of distinct types.

89. Of the *five given points*, P_0 , the *quinary type* has been seen (78) to be (10000); while of the *ten derived points* P_1 , of first construction, the corresponding type may be taken as (00011); in fact, considered as *symbols*, these two represent the points A and D_1 . The nine other points P_1 are $A'B'C'A_1B_1C_1A_2B_2C_2$; and we have now (comp. 83, 87, 86) the symbols,

$$\begin{aligned} A' &= BC \cdot ADE = (01100), & A_1 &= EA \cdot BCD = (10001), \\ A_2 &= DA \cdot BCE = (10010); \end{aligned}$$

also, in any symbol or equation of the present form, it is permitted to change A, B, C to B, C, A , provided that we at the same time write the third, first, and second co-efficients, in the places of the first, second, and third: thus, $B' = CA \cdot BDE = (10100)$, &c. The symbol ($xy000$) represents an *arbitrary point on the line* AB ; and the symbol $[00nrs]$, with $n + r + s = 0$, represents an *arbitrary plane through that line*: each therefore may be regarded (comp. 82) as a *symbol* also of the *line* AB itself, and at the same time as a *type* of the *ten lines* A_1 ; while the symbol $[0001\bar{1}]$, of the plane ABC (75), may be taken (78) as a type of the *ten planes* Π_1 . Finally, the *five pyramids*,

$$BCDE, \quad CADE, \quad ABDE, \quad ABCE, \quad ABCD,$$

and the *ten triangles*, such as ABC , whereof each is a *common face* of two such pyramids, may be called *pyramids* R_1 , and *triangles* T_1 , of the *First Construction*.

90. Proceeding to a *Second Construction* (88), we soon find that the *lines* A_2 may be arranged in *two distinct groups*; one group consisting of *fifteen lines* $A_{2,1}$, such as the line* $AA'D_1$, whereof each connects two points P_1 , and passes also through one point P_0 , being the intersection of two planes Π_1 through that point, as here of ABC, ADE ; while the *other group* consists of *thirty lines* $A_{2,2}$, such as $B'C'$, each connecting two points P_1 , but not passing through any point P_0 , and being one of the *thirty edges* of *five new pyramids* R_2 , namely,

$$C'B'A_2A_1, \quad A'C'B_2B_1, \quad B'A'C_2C_1, \quad A_2B_2C_2D_1, \quad A_1B_1C_1D_1:$$

* $AB_1C_2, AB_2C_1, DA'A_1, EA'A_2$, are other lines of this group.

which pyramids R_2 may be said (comp. 87) to be *inscribed homologues* of the five former pyramids R_1 , the *centres of homology* for these *five pairs of pyramids* being the five given points $A \dots E$; and the *planes of homology* being five planes $[A] \dots [E]$, whereof the last has been already mentioned (87), but which belong properly to a *third construction* (88). The *planes Π_2* , of *second construction*, form in like manner *two groups*; one consisting of *fifteen planes $\Pi_{2,1}$* , such as the plane of the *five points*, $AB_1B_2C_1C_2$, whereof each passes through *one point P_0* , and through *four points P_1* , and contains *two lines $\Lambda_{2,1}$* , as here the lines AB_1C_2 , AC_1B_2 , besides containing *four lines $\Lambda_{2,2}$* , as here B_1A_2 , &c.; while the *other group* is composed of *twenty planes $\Pi_{2,2}$* , such as $A_1B_1C_1$, namely, the *twenty faces* of the five recent pyramids R_2 , whereof each contains *three points P_1* , and *three lines $\Lambda_{2,2}$* , but does not pass through any point P_0 . It is now required to *express these geometrical conceptions** of the *forty-five lines Λ_2* ; the *thirty-five planes Π_2* ; and the *five planes of homology* of pyramids, $[A] \dots [E]$, by *quinary symbols and types*, before proceeding to determine the *points P_2* of *second construction*.

91. An arbitrary point on the right line $AA'D_1$ (90) may be represented by the symbol $(tuu00)$; and an arbitrary plane through that line by this other symbol, $[0mm\bar{r}\bar{r}]$, where \bar{m} and \bar{r} are written (to save commas) instead of $-m$ and $-r$; hence these two symbols may also (comp. 82) denote the line $AA'D_1$ itself, and may be used as types (78) to represent the *group* of lines $\Lambda_{2,1}$. The particular symbol $[01\bar{1}\bar{1}\bar{1}]$, of the last form, represents that particular plane through the last-mentioned line, which contains also the line AB_1C_2 of the same group; and may serve as a type for the group of planes $\Pi_{2,1}$. The line $B'C'$, and the group $\Lambda_{2,2}$, may be represented by $(stu00)$ and $[\bar{t}\bar{t}\bar{t}u\bar{s}]$, if we agree† to write $s = t + u$, and $s = -s$; while the plane $B'C'A_2$, and the group $\Pi_{2,2}$, may be denoted by $[\bar{1}\bar{1}\bar{1}\bar{1}\bar{2}]$. Finally, the plane $[E]$ has for its symbol $[1111\bar{4}]$; and the four other planes $[A]$, &c., of homology of pyramids (90), have this last for their common type.

92. The *points P_2* , of *second construction* (88), are more nume-

* Möbius (in his *Barycentric Calculus*, p. 284, &c.) has very clearly pointed out the existence and chief properties of the foregoing *lines and planes*; but besides that his *analysis* is altogether different from ours, he does not appear to have aimed at *enumerating*, or even at *classifying*, all the *points* of what has been above called (88) the *second construction*, as we propose shortly to do.

† With this convention, the line AB , and the group Λ_1 , may be denoted by the *plane-symbol* $[00tus]$ their *point-symbol* being $(tu000)$.

rous than the *lines* Λ_2 and *planes* Π_2 , of that construction: yet with the help of *types*, as above, it is not difficult to classify and to enumerate them. It will be sufficient here to write down these *types*, which are found to be *eight*, and to offer some remarks respecting them; in doing which we shall avail ourselves of the eight following *typical points*, whereof the two first have already occurred, and which are all situated in the plane of ABC :

$$\begin{aligned} \Lambda'' &= (01\bar{1}00); & \Lambda''' &= (21100); & \Lambda^v &= (\bar{2}1100); & \Lambda^v &= (02100); \\ \Lambda^v &= (02\bar{1}00); & \Lambda^{vi} &= (12\bar{1}00); & \Lambda^{vii} &= (32100); & \Lambda^{viii} &= (23\bar{1}00); \end{aligned}$$

the second and third of these having $(\bar{1}0011)$ and (30011) for *conjuguent symbols* (71). It is easy to see that these *eight types* represent, respectively, ten, thirty, thirty, twenty, twenty, sixty, sixty, and sixty distinct points, belonging to *eight groups*, which we shall mark as $F_{2,1}, \dots, F_{2,8}$; so that the total number of the points F_2 is 290. If then we consent (88) to *close* the present inquiry, at the end of what we have above *defined* to be the *Second Construction*, the *total number of the net points*, F_1, F_2 , which are thus *derived by lines and planes* from the *five given points* F_0 , is found to be exactly *three hundred*: while the *joint number of the net-lines*, Λ_1, Λ_2 , and of the *net-planes*, Π_1, Π_2 , has been seen to be *one hundred*, so far.

(1.) To the type $F_{2,1}$ belong the *ten points*,

$$\Lambda''B''C'', \quad \Lambda'_2B'_2C'_2, \quad \Lambda'_1B'_1C'_1D'_1,$$

with the quinary symbols,

$$\Lambda'' = (01\bar{1}00), \dots \Lambda'_2 = (100\bar{1}0), \dots \Lambda'_1 = (1000\bar{1}), \dots D'_1 = (0001\bar{1}),$$

which are the *harmonic conjugates* of the ten points F_1 , namely, of

$$\Lambda'B'C', \quad \Lambda_2B_2C_2, \quad \Lambda_1B_1C_1D_1,$$

with respect to the ten lines Λ_1 , on which those points are situated; so that we have ten harmonic equations, $(BA'CA'') = -1$, &c., as already seen (81, 86, 87). Each point $F_{2,1}$ is the *common intersection* of a line Λ_1 with *three lines* $\Lambda_{2,2}$; thus we may establish the four following *formulae of concurrence* (equivalent, by 89, to *ten* such formulæ):

$$\begin{aligned} \Lambda'' &= BC \cdot B'C' \cdot B_1C_1 \cdot B_2C_2; & \Lambda'_2 &= DA \cdot D_1A_1 \cdot B'C_2 \cdot C'B_2; \\ \Lambda'_1 &= EA \cdot D_1A_2 \cdot B'C_1 \cdot C'B_1; & D'_1 &= DE \cdot A_1A_2 \cdot B_1B_2 \cdot C_1C_2. \end{aligned}$$

Each point $F_{2,1}$ is also situated in *three planes* Π_1 ; in *three* other planes, of the group $\Pi_{2,1}$; and in *six* planes $\Pi_{2,2}$; for example, Λ'' is a point common to the *twelve* planes,

$$\begin{aligned} ABC, BCD, BCE; & \quad AB_1C_2C_1B_2, & \quad DB'_1C'_1C_1, & \quad EB'_2C'_1C_2; \\ B'C'A_1, & \quad B_1C_1A_1, & \quad B_2C_2A_2, & \quad B'C'A_2, & \quad B_1C_1D_1, & \quad B_2C_2D_1. \end{aligned}$$

Each line, Λ_1 or $\Lambda_{2,2}$, contains *one* point $F_{2,1}$; but no line $\Lambda_{2,1}$ contains any. Each plane, Π_1 or $\Pi_{2,2}$, contains *three* such points; and each plane $\Pi_{2,1}$ contains *two*,

which are the *intersections of opposite sides of a quadrilateral* Q_2 in that plane, whereof the *diagonals* intersect in a point P_0 : for example, the diagonals B_1C_2, B_2C_1 of the quadrilateral $B_1B_2C_2C_1$, which is (by 90) in one of the planes $\Pi_{2,1}$, intersect each other in the point A ; while the opposite sides C_1B_1, B_2C_2 intersect in A'' ; and the two other opposite sides, B_1B_2, C_2C_1 have the point D_1 for their intersection. The ten points $P_{2,1}$ are also ranged, *three by three, on ten lines of third construction* A_3 , namely, on the *axes of homology*,

$$A''B_1C_1', \dots A''B_2C_2', \dots A_1A_2D_1', \dots A''B''C'',$$

of ten pairs of triangles T_1, T_2 , which are situated in the ten planes Π_1 , and of which the *centres of homology* are the ten points P_1 : for example, the dotted line $A''B''C''$, in Fig. 21, is the axis of homology of the two triangles, $ABC, A''B''C''$, whereof the latter is *inscribed* in the former, with the point o in that figure (replaced by D_1 in Fig. 29), to represent their centre of homology. The same ten points $P_{2,1}$ are also ranged *six by six*, and the ten last lines A_3 are ranged *four by four*, in five planes Π_2 , namely in the *planes of homology of five pairs of pyramids*, R_1, R_2 , already mentioned (90): for example, the plane $[E]$ contains (87) the six points $A''B''C''A_2B_2C_2'$, and the four right lines,

$$A''B_2C_2', \quad B''C_2A_2', \quad C''A_2B_2', \quad A''B''C'';$$

which latter are the intersections of the *four faces*,

$$DCB, \quad DAC, \quad DBA, \quad ABC,$$

of the pyramid $ABCD$, with the corresponding faces,

$$D_1C_1B_1, \quad D_1A_1C_1, \quad D_1B_1A_1, \quad A_1B_1C_1,$$

of its *inscribed homologue* $A_1B_1C_1D_1$; and are contained, besides, in the four other planes,

$$A_2B_2C_2', \quad B_2C_2A_2', \quad C_2A_2B_2', \quad A_2B_2C_2:$$

the three triangles, $ABC, A_1B_1C_1, A_2B_2C_2'$, for instance, being *all homologous*, although in *different planes*, and having the line $A''B''C''$ for their *common axis* of homology. We may also say, that this line $A''B''C''$ is the *common trace* (81) of two planes $\Pi_{2,2}$, namely of $A_1B_1C_1$ and $A_2B_2C_2'$, on the plane ABC ; and in like manner, that the point A'' is the *common trace*, on that plane Π_1 , of two lines $A_{2,2}$, namely of B_1C_1 and B_2C_2 : being also the common trace of the two lines B_1C_1' and B_2C_2' , which belong to the *third construction*.

(2.) On the whole, *these ten points, of second construction, $A'' \dots$* , may be considered to be already well known to geometers, in connexion with the theory of *transversal† lines and planes* in space: but it is important here to observe, with what simplicity and clearness their geometrical relations are *expressed* (88), by the *quinary symbols* and *quinary types* employed. For example, the *collinearity* (82) of the *four planes*, $ABC, A_1B_1C_1, A_2B_2C_2'$, and $[E]$, becomes evident from mere *inspection of their four symbols*,

* Compare the Note to page 68.

† The collinear, coplanar, and harmonic relations between the ten points, which we have above marked as $P_{2,1}$, and which have been considered by Möbius also, in connexion with his theory of *nets in space*, appear to have been first noticed by Carnot, in a Memoir upon *transversals*.

$$[0001\bar{1}], \quad [1112\bar{1}], \quad [1111\bar{2}], \quad [1111\bar{4}],$$

which represent (75) the four quinary equations,

$$w - v = 0, \quad x + y + z - 2w - v = 0, \quad x + y + z - w - 2v = 0, \quad x + y + z + w - 4v = 0;$$

with this additional consequence, that the ternary symbol (81) of the common trace, of the three latter on the former, is $[111]$: so that this trace is (by 88) the line $A''B''C''$ of Fig. 21, as above. And if we briefly denote the quinary symbols of the four planes, taken in the same form and order as above, by $[R_0] [R_1] [R_2] [R_3]$, we see that they are connected by the two relations,

$$[R_1] = -[R_0] + [R_2]; \quad [R_3] = 2[R_0] + [R_2];$$

whence if we denote the planes themselves by $\Pi_1, \Pi_2, \Pi_3, \Pi_4$, we have (comp. 84) the following value for the anharmonic of their pencil,

$$(\Pi_1 \Pi_2 \Pi_3 \Pi_4) = -2;$$

a result which can be very simply verified, for the case when $ABCD$ is a regular pyramid, and π (comp. 29) is its mean point: the plane Π_3 , or $[\mathbb{E}]$, becoming in this case (comp. 88) the plane at infinity, while the three other planes, $ABC, A_1B_1C_1, A_2B_2C_2$, are parallel; the second being intermediate between the other two, but twice as near to the third as to the first.

(3.) We must be a little more concise in our remarks on the seven other types of points F_2 , which indeed, if not so well known,* are perhaps also, on the whole, not quite so interesting: although it seems that some circumstances of their arrangement in space may deserve to be noted here, especially as affording an additional exercise (88), in the present system of symbols and types. The type $F_{2,2}$ represents, then, a group of thirty points, of which A'' , in Fig. 21, is an example; each being the intersection of a line $A_{2,1}$ with a line $A_{2,2}$, as A'' is the point in which AA' intersects $B'C'$: but each belonging to no other line, among those which have been hitherto considered. But without aiming to describe here all the lines, planes, and points, of what we have called the third construction, we may already see that they must be expected to be numerous: and that the planes Π_3 , and the lines A_3 , of that construction, as well as the pyramids R_2 , and the triangles T_2 , of the second construction, above noticed, can only be regarded as specimens, which in a closer study of the subject, it becomes necessary to mark more fully, on the present plan, as $\Pi_{3,1}, \dots, T_{2,1}$. Accordingly it is found that not only is each point $F_{2,2}$ one of the corners of a triangle $T_{3,1}$ of third construction (as A'' is of $A''B''C''$ in Fig. 21), the sides of which new triangle are lines $A_{3,2}$, passing each through one point $F_{2,1}$ and through two points $F_{2,2}$ (like the dotted line $A''B''C''$ of Fig. 21); but also each such point $F_{2,2}$ is the intersection of two new lines of third construction, $A_{3,3}$, whereof each connects a point F_0 with a

* It does not appear that any of these other types, or groups, of points F_2 , have hitherto been noticed, in connexion with the net in space, except the one which we have ranked as the *first*, $F_{2,5}$, and which represents two points on each line A_1 , as the type $F_{2,1}$ has been seen to represent one point on each of those ten lines of first construction: but that *first* group, which may be exemplified by the intersections of the line DE with the two planes $A_1B_1C_1$ and $A_2B_2C_2$, has been indicated by Möbius (in page 290 of his already cited work), although with a different notation, and as the result of a different analysis.

point $P_{2,1}$. For example, the point A''' is the common trace (on the plane ABC) of the two new lines, DA'_1, EA'_2 : because, if we adopt for this point A''' the second of its two congruent symbols, we have (comp. 73, 82) the expressions,

$$A''' = (\bar{1}0011) = (D) - (A'_1) = (E) - (A'_2).$$

We may therefore establish the *formula of concurrence* (comp. the first sub-article):

$$A''' = AA' \cdot B'O' \cdot DA'_1 \cdot EA'_2;$$

which represents a system of *thirty* such formulæ.

(4.) It has been remarked that the point A''' may be represented, not only by the quinary symbol (21100), but also by the congruent symbol, ($\bar{1}0011$); if then we write,

$$A_0 = (\bar{1}1100), \quad B_0 = (1\bar{1}100), \quad C_0 = (11\bar{1}00),$$

these three new points $A_0B_0C_0$, in the plane of ABC , must be considered to be *syntypical*, in the quinary sense (78), with the three points $A'''B'''C'''$, or to belong to the *same* group $P_{2,2}$, although they have (comp. 88) a different *ternary type*. It is easy to see that, while the triangle $A'''B'''C'''$ is (comp. again Fig. 21) an *inscribed homologue* $T_{3,1}$ of the triangle $A'B'C'$, which is *itself* (comp. sub-article 1) an inscribed homologue $T_{2,1}$ of a triangle T_1 , namely of ABC , with $A''B''C''$ for their *common axis* of homology, the new triangle $A_0B_0C_0$ is on the contrary an *exscribed homologue* $T_{2,2}$, with the *same axis* $A_{3,1}$, of the same given triangle T_1 . But from the *syntypical relation*, existing as above for *space* between the points A''' and A_0 , we may expect to find that these two points $P_{2,2}$ admit of being *similarly constructed*, when the *five* points P_0 are treated as entering *symmetrically* (or similarly), as *geometrical elements*, into the constructions. The point A_0 must therefore be situated, not only on a line $A_{2,1}$, namely, on AA' , but also on a line $A_{2,2}$, which is easily found to be A_1A_2 , and on two lines $A_{2,3}$, each connecting a point P_0 with a point $P_{2,1}$; which latter lines are soon seen to be BB'' and CC'' . We may therefore establish the formula of *concurrence* (comp. the last sub-article):

$$A_0 = AA' \cdot A_1A_2 \cdot BB'' \cdot CC'';$$

and may consider the three points A_0, B_0, C_0 as the *traces* of the three lines A_1A_2, B_1B_2, C_1C_2 : while the three new lines AA'', BB'', CC'' , which coincide in position with the sides of the exscribed triangle $A_0B_0C_0$, are the traces $A_{3,3}$ of three *planes* $\Pi_{2,1}$, such as $AB_1C_2B_2C_1$, which pass through the three given points A, B, C , but do not contain the lines $A_{2,1}$ whereon the six points $P_{2,2}$ in their plane Π_1 are situated. Every *other* plane Π_1 contains, in like manner, *six* points P_2 of the present group; every plane $\Pi_{2,1}$ contains *eight* of them; and every plane $\Pi_{2,2}$ contains *three*; each line $A_{2,1}$ passing through *two* such points, but each line $A_{2,2}$ only through *one*. But besides being (as above) the intersection of *two* lines A_2 , each point of this group $P_{2,2}$ is common to *two* planes Π_1 , *four* planes $\Pi_{2,1}$, and *two* planes $\Pi_{2,2}$; while each of these thirty points is also a *common corner* of *two* different triangles of *third* construction, of the lately mentioned kinds $T_{3,1}$ and $T_{3,2}$, situated respectively in the two planes of *first* construction which contain the point itself. It may be added that each of the two points $P_{2,2}$, on a line $A_{2,1}$, is the *harmonic conjugate* of one of the two points P_1 , with respect to the point P_0 , and to the *other* point P_1 on that line; thus we have here the two harmonic equations,

$$(AA'D_1A''') = (AD_1A'A_0) = -1,$$

by which the positions of the two points A''' and A_0 might be determined.

(5.) A *third group*, $P_{2,3}$, of *second construction*, consists (like the preceding group) of *thirty points*, ranged *two by two* on the fifteen lines $A_{3,1}$, and *six by six* on the ten planes Π_1 , but so that each is common to *two* such planes; each is also situated in *two planes* $\Pi_{2,1}$, in *two planes* $\Pi_{2,2}$, and on *one line* $A_{3,1}$ in which (by sub-art. 1) these two last planes intersect each other, and two of the five planes $\Pi_{3,1}$; each plane $\Pi_{2,1}$ contains *four* such points, and each plane $\Pi_{2,2}$ contains *three* of them; but no point of this group is on any line A_1 , or $A_{2,2}$. The *six points* $P_{2,3}$, which are in *the plane* ABC , are represented (like the corresponding points of the last group) by *two ternary types*, namely by (211) and (311) ; and may be exemplified by the two following points, of which these last are the ternary symbols:

$$\begin{aligned} A_1^{IV} &= AA' \cdot A''B''C'' = AA' \cdot A_1B_1C_1 \cdot A_2B_2C_2; \\ A_1^{IV} &= AA' \cdot D'_1A'_2A_1 = AA' \cdot B'_1C'_2 \cdot C'_1B_1B_2. \end{aligned}$$

The three points of the first sub-group $A_1^{IV} \dots$ are collinear; but the three points $A_1^{IV} \dots$ of the second sub-group are the corners of a *new triangle*, $T_{3,3}$, which is homologous to the triangle ABC , and to all the other triangles in its plane which have been hitherto considered, as well as to the two triangles $A_1B_1C_1$ and $A_2B_2C_2$; the line of the three former points being their *common axis* of homology; and the *sides* of the new triangle, $A_1^{IV}B_1^{IV}C_1^{IV}$, being the *traces* of the *three planes* (comp. 90) of homology of pyramids, $[A]$, $[B]$, $[C]$; as (comp. sub-art. 2) the line $A_1^{IV}B_1^{IV}C_1^{IV}$ or $A''B''C''$ is the *common trace* of the *two other planes* of the same group $\Pi_{3,1}$, namely of $[D]$ and $[E]$. We may also say that the *point* A_1^{IV} is the *trace* of the *line* $A'_1A'_2$; and because the *lines* B'_1C_0 , C'_1B_0 are the traces of the *two planes* $\Pi_{2,2}$ in which that point is contained, we may write the formula of concurrence,

$$A_1^{IV} = AA' \cdot A'_1A'_2 \cdot B'_1C'_0 \cdot C'_1B_0.$$

(6.) It may be also remarked, that each of the two points $P_{2,3}$, on any line $A_{3,1}$, is the harmonic conjugate of a point $P_{2,2}$, with respect to the point P_0 , and to *one* of the two points P_1 on that line; being *also* the harmonic conjugate of this last point, with respect to the same point P_0 , and the *other* point $P_{2,2}$: thus, on the line $AA'D_1$, we have the *four harmonic equations*, which are not however all *independent*, since two of them can be deduced from the two others, with the help of the two analogous equations of the fourth sub-article:

$$(AA''A'A^{IV}) = (AA'A_0A^{IV}) = (AA_0D_1A_1^{IV}) = (AD_1A''A_1^{IV}) = -1.$$

And the *three pairs of derived points* $P_1, P_{2,2}, P_{2,3}$, on any such line $A_{3,1}$, will be found (comp. 26) to compose an *involution*, with the *given point* P_0 on the line for *one* of its two *double points* (or *foci*): the *other* double point of this involution being a point P_3 of *third construction*; namely, the point in which the line $A_{3,1}$ meets *that one* of the *five planes* of homology $\Pi_{3,1}$, which *corresponds* (comp. 90) to the particular point P_0 as *centre*. Thus, in the present example, if we denote by A^* the point in which the line AA' meets the plane $[A]$, of which (by 81, 91) the trace on ABC is the line $[411]$, and therefore is (as has been stated) the side $B_1^{IV}C_1^{IV}$ of the lately mentioned triangle $T_{3,3}$, so that

$$A^* = (122) = AA' \cdot BC''' \cdot CB''' \cdot B_1^{IV}C_1^{IV},$$

we shall have the *three harmonic equations*,

$$(AA'A^*D_1) = (AA'''A^*A_0) = (AA^{IV}A^*A_1^{IV}) = -1;$$

which express that this new point A^* is the *common harmonic conjugate* of the given

point A , with respect to the three pairs of points, $A'D_1$, $A''A_0$, $A^vA_1^v$; and therefore that these three pairs form (as has been said) an involution, with A and A^x for its two double points.

(7.) It will be found that we have now exhausted all the types of points of second construction, which are situated upon lines $A_{2,1}$; there being only four such points on each such line. But there are still to be considered two new groups of points P_2 on lines A_1 , and three others on lines $A_{2,2}$. Attending first to the former set of lines, we may observe that each of the two new types, $P_{2,4}$, $P_{2,5}$, represents twenty points, situated two by two on the ten lines of first construction, but not on any line A_2 ; and therefore six by six in the ten planes Π_1 , each point however being common to three such planes: also each point $P_{2,4}$ is common to three planes $\Pi_{2,2}$, and each point $P_{2,5}$ is situated in one such plane; while each of these last planes contains three points $P_{2,4}$, but only one point $P_{2,5}$. If we attend only to points in the plane ABC , we can represent these two new groups by the two ternary types, (021) and $(0\bar{2}1)$, which as symbols denote the two typical points,

$$A^v = BC \cdot C' A_1 A_2 \cdot D_1 A_1 B_1 \cdot D_1 A_2 B_2; \quad A^{v'} = BC \cdot C' B_1 B_2 = BC \cdot C' B_0;$$

we have also the concurrence,

$$A^v = BC \cdot C' A_0 \cdot D_1 C'' \cdot AB''.$$

It may be noted that A^v is the harmonic conjugate of C' , with respect to A_0 and B_1'' , which last point is on the same trace $C' A_0$, of the plane $C' A_1 A_2$; and that $A^{v'}$ is harmonically conjugate to B_1^v , with respect to C' and B_0 , on the trace of the plane $C' B_1 B_2$, where B_1^v denotes (by an analogy which will soon become more evident) the intersection of that trace with the line CA : so that we have the two equations,

$$(A_0 C' B_1^v A^v) = (B_0 B_1^v C' A^{v'}) = -1.$$

(8.) Each line A_1 , contains thus two points P_2 , of each of the two last new groups, besides the point $P_{2,1}$, the point P_1 , and the two points P_0 , which had been previously considered: it contains therefore eight points in all, if we still abstain (88) from proceeding beyond the Second Construction. And it is easy to prove that these eight points can, in two distinct modes, be so arranged as to form (comp. sub-art. 6) an involution, with two of them for the two double points thereof. Thus, if we attend only to points on the line BC , and represent them by ternary symbols, we may write,

$$\begin{aligned} B &= (010), & C &= (001), & A' &= (011), & A'' &= (0\bar{1}\bar{1}); \\ A^v &= (021), & A^{v'} &= (0\bar{2}\bar{1}), & A_1^v &= (012), & A_1^{v'} &= (0\bar{1}\bar{2}); \end{aligned}$$

and the resulting harmonic equations

$$I. \dots (BA'CA'') = (BA^vCA^{v'}) = (BA_1^vCA_1^{v'}) = -1,$$

$$II. \dots (A'BA''C) = (A^vA^vA''A_1^v) = (A^vA^{v'}A''A_1^{v'}) = -1,$$

will then suffice to show: Ist., that the two points P_0 , on any line A_1 , are the double points of an involution, in which the points P_1 , $P_{2,1}$ form one pair of conjugates, while the two other pairs are of the common form, $P_{2,4}$, $P_{2,5}$; and IInd., that the two points P_1 and $P_{2,1}$, on any such line A_1 , are the double points of a second involution, obtained by pairing the two points of each of the three other groups. Also each of the two points P_0 , on a line A_1 , is the harmonic conjugate of one of the two points $P_{2,5}$ on that line, with respect to the other point of the same group, and to the point P_1 on the same line; thus,

$$(BA'A_1''A_1'') = (GA'A''A_1'') = -1.$$

(9.) It remains to consider briefly *three other groups* of points F_3 , each group containing *sixty points*, which are situated, two by two, on the thirty lines $A_{2,3}$, and six by six in the ten planes Π_1 . Confining our attention to those which are in the plane ABC , and denoting them by their ternary symbols, we have thus, on the line $B'C'$, the three new typical points, of the three remaining groups, $F_{2,6}$, $F_{2,7}$, $F_{2,8}$:

$$\Delta^{VI} = (1\bar{2}\bar{1}); \quad \Delta^{VII} = (321); \quad \Delta^{IX} = (2\bar{3}\bar{1});$$

with which may be combined these three others, of the same three types, and on the same line $B'C'$:

$$\Delta_1^{VI} = (1\bar{1}\bar{2}); \quad \Delta_1^{VII} = (312); \quad \Delta_1^{IX} = (2\bar{1}\bar{3}).$$

Considered as intersections of a line $A_{2,3}$ with lines A_3 in the same plane Π_1 , or with planes Π_2 (in which *latter* character alone they belong to the *second* construction), the three points Δ^{VI} , &c., may be thus denoted:

$$\begin{aligned} \Delta^{VI} &= B'C' \cdot BB'' \cdot CB''' \cdot \Delta A^{VI} = B'C' \cdot BC_1A_2A_1C_2; \\ \Delta^{VII} &= B'C' \cdot D_1B'' \cdot \Delta B^{VII} \cdot A' = B'C' \cdot D_1C_1A_1 \cdot D_1C_2A_2; \\ \Delta^{IX} &= B'C' \cdot A'C_0B_1''C_1'' \cdot B^{IX} \cdot BA^{IX}B_1''B_1'' = B'C' \cdot A'C_1C_2; \end{aligned}$$

with the harmonic equation,

$$(C_0A'C_1'A^{IX}) = -1,$$

and with analogous expressions for the three other points, Δ_1^{VI} , &c. The line $B'C'$ thus intersects *one* plane $\Pi_{2,1}$ (or its trace BB'' on the plane ABC), in the point Δ^{VI} ; it intersects *two* planes $\Pi_{2,2}$ (or their common trace D_1B'') in Δ^{VII} ; and *one* other plane $\Pi_{2,2}$ (or its trace $A'C_0$) in Δ^{IX} ; and similarly for the other points, Δ_1^{VI} , &c., of the same three groups. Each plane $\Pi_{2,1}$ contains *twelve points* $F_{2,6}$, *eight points* $F_{2,7}$, and *eight points* $F_{2,8}$; while every plane $\Pi_{2,2}$ contains *six points* $F_{2,6}$, *twelve points* $F_{2,7}$, and *nine points* $F_{2,8}$. Each point $F_{2,6}$ is contained in *one plane* Π_1 ; in *three planes* $\Pi_{2,1}$; and in *two planes* $\Pi_{2,2}$. Each point $F_{2,7}$ is in *one plane* Π_1 , in *two planes* $\Pi_{2,1}$, and in *four planes* $\Pi_{2,2}$. And each point $F_{2,8}$ is situated in *one plane* Π_1 , in *two planes* $\Pi_{2,1}$, and in *three planes* $\Pi_{2,2}$.

(10.) The points of the three last groups are situated *only* on lines $A_{2,3}$; but, on each such line, *two points* of each of those three groups are situated; which, along with *one point* of each of the *two* former groups, $F_{2,1}$ and $F_{2,2}$, and with the *two points* F_1 , whereby the line itself is determined, make up a system of *ten points* upon that line. For example, the line $B'C'$ contains, besides the *six points* mentioned in the last sub-article, the *four others*:

$$b' = (101); \quad c' = (110); \quad \Delta'' = (01\bar{1}); \quad \Delta''' = (211).$$

Of these *ten points*, the *two last* mentioned, namely the points $F_{2,1}$ and $F_{2,2}$ upon the line $A_{2,3}$, are the *double points* (comp. sub-art. 8) of a *new involution*, in which the *two points* of each of the *four other groups* compose a conjugate pair, as is expressed by the harmonic equations,

$$(\Delta''B'A'''C') = (\Delta''A^{VI}\Delta'''A_1^{VI}) = (\Delta''A^{VII}\Delta'''A_1^{VII}) = (\Delta''A^{IX}\Delta'''A_1^{IX}) = -1.$$

And the analogous equations,

$$(B'A''C'\Delta''') = (B'A^{VI}C'A^{VII}) = (B'A^{VII}C'A^{VIII}) = -1,$$

show that the *two points* F_1 on any line $A_{2,3}$ are the *double points* of *of another involution* (comp. again sub-art. 8), whereof the *two points* $F_{2,1}$, $F_{2,2}$ on that line form

one conjugate pair, while each of the two points $F_{2,5}$ is paired with one of the points $F_{2,7}$ as its conjugate. In fact, the *eight-rayed pencil* $(A, C'B'A''A'''A^{IV}A^{VII}A_1^{VII}A_1^{VII})$ coincides in position with the pencil $(A, BCA''A'A^{VII}A_1^{VII})$, and may be said to be a *pencil in double involution*; the third and fourth, the fifth and sixth, and the seventh and eighth rays forming *one* involution, whereof the first and second are the two *double^o rays*; while the first and second, the fifth and seventh, and the sixth and eighth rays compose *another* involution, whereof the double rays are the third and fourth of the pencil.

(11.) If we proceeded to connect systematically the points F_2 among themselves, and with the points F_1 and F_0 , we should find many remarkable lines and planes of *third* construction (88), besides those which have been incidentally noticed above; for example, we should have a group $\Pi_{2,2}$ of *twenty new planes*, exemplified by the two following,

$$[E_0] = [1110\bar{8}], \quad [D_0] = [11\bar{1}80],$$

which have the same common trace $A_{2,1}$, namely the line $A''B''C''$, on the plane ABC , as the two planes $A_1B_1C_1, A_2B_2C_2$, and the two planes $[D], [E]$, of the groups $\Pi_{2,2}$ and $\Pi_{2,1}$, which have been considered in former sub-articles; and each of these new planes $\Pi_{2,2}$ would be found to contain *one* point F_0 , *three* points $F_{2,1}$, *six* points $F_{2,2}$, and *three* points $F_{2,3}$. It might be proved also that these twenty new planes are the *twenty faces of five new pyramids* E_0 , which are the *escribed homologues* of the five old pyramids E_1 (89), with the five given points F_0 for the corresponding *centres of homology*. But it would lead us beyond the proposed limits, to pursue this discussion further: although a few additional remarks may be useful, as serving to establish the *completeness* of the *enumeration* above given, of the lines, planes, and points of *second* construction.

93. In general, if there be any n given points, whereof no four are situated in any common plane, the number N of the *derived* points, which are immediately obtained from them, as *intersections $A \cdot \Pi$ of line with plane* (each *line* being drawn through *two* of the given points, and each *plane* through *three others*), or the number of points of the *form* $AB \cdot CDE$, is easily seen to be,

$$N = f(n) = \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 2 \cdot 3};$$

so that $N = 10$, as before, when $n = 5$. But if we were to apply this formula to the case $n = 15$, we should find, for that case, the value,

$$N = f(15) = 15 \cdot 14 \cdot 13 \cdot 11 = 30030;$$

and thus *fifteen given and independent points of space* would conduct, by what might (relatively to *them*) be called a *First Construction* (comp. 88), to a system of *more than thirty thousand* points. Yet it has been lately stated (92), that from the fifteen points above called F_0, F_1 , there can be derived, in this way, *only* two hundred and ninety

* Compare page 172 of the *Geom. Exposition* of M. Chasles.

points P_2 , as intersections of the form* $\Lambda \cdot \Pi$; and therefore *fewer* than *three hundred*. That this *reduction* of the *number of derived points*, at the end of what has been called (88) the *Second Construction* for the *net in space*, arising from the *dependence* of the *ten points* P_1 on the *five points* P_0 , would be found to be *so considerable*, might not perhaps have been anticipated; and although the foregoing examination proves that *all the eight types* (92) do really represent points P_2 , it may appear *possible*, at this stage, that some *other type* of such points has been *omitted*. A study of the manner in which the *types of points* result, from those of the *lines* and *planes* of which they are the intersections, would indeed decide this question; and it was, in fact, in that way that the eight types, or groups, $P_{2,1}, \dots, P_{2,8}$, of points of second construction for space, were investigated, and found to be sufficient: yet it may be useful (compare the last sub-art.) to *verify*, as below, the *completeness* of the foregoing *enumeration*.

(1.) The *fifteen points*, P_0, P_1 , admit of 105 *binary*, and of 455 *ternary combinations*; but these are far from determining so many distinct *lines* and *planes*. In fact, those 15 points are connected by 25 *collineations*, represented by the 25 lines $\Lambda_1, \Lambda_{2,1}$; which *lines* therefore count as 75, among the 105 *binary* combinations of points: and there *remain* only 30 combinations of this sort, which are constructed by the 30 *other* lines, $\Lambda_{2,2}$. Again, there are 25 *ternary* combinations of points, which are represented (as above) by *lines*, and therefore do not *determine* any *plane*. Also, in *each* of the *ten* planes Π_1 , there are 29 ($= 35 - 6$) *triangles* T_1, T_2 , because each of those planes contains 7 *points* P_0, P_1 , connected by 6 *relations* of collinearity. In like manner, each of the *fifteen* planes $\Pi_{2,1}$ contains 8 ($= 10 - 2$) *other triangles* T_2 , because it contains 5 *points* P_0, P_1 , connected by *two* collineations. There remain therefore only 20 ($= 455 - 25 - 290 - 120$) *ternary* combinations of points to be accounted for; and these are represented by the 20 planes $\Pi_{2,2}$. The *completeness* of the enumeration of the *lines* and *planes* of the *second* construction is therefore *verified*; and it only remains to verify that the 805 *points*, P_0, P_1, P_2 , above considered, *represent all* the intersections $\Lambda \cdot \Pi$, of the 55 lines Λ_1, Λ_2 , with the 45 planes Π_1, Π_2 .

(2.) Each plane Π_1 contains three lines of each of the three groups, $\Lambda_1, \Lambda_{2,1}, \Lambda_{2,2}$; each plane $\Pi_{2,1}$ contains two lines $\Lambda_{2,1}$, and four lines $\Lambda_{2,2}$; and each plane $\Pi_{2,2}$ contains three lines $\Lambda_{2,2}$. Hence (or because each line Λ_1 is contained in three planes Π_1 ; each line $\Lambda_{2,1}$ in two planes Π_1 , and in two planes $\Pi_{2,1}$; and each line $\Lambda_{2,2}$ in one plane Π_1 , in two planes $\Pi_{2,1}$, and in two planes $\Pi_{2,2}$), it follows that, without going beyond the *second* construction, there are 240 ($= 30 + 30 + 30 + 30$

* The definition (88) of the points P_2 admits, indeed, intersections $\Lambda \cdot \Lambda$ of coplanar lines, when they are not already points P_0 or P_1 ; but all such intersections are also points of the form $\Lambda \cdot \Pi$; so that no generality is lost, by confining ourselves to this last form, as in the present discussion we propose to do.

+60 ÷ 60) cases of coincidence of line and plane; so that the number of cases of intersection is reduced, hereby, from 55 · 45 = 2475, to 2235 (= 2475 - 240).

(3.) Each point P_0 represents twelve intersections of the form $\Delta_1 \cdot \Pi_1$; because it is common to four lines A_1 , and to six planes Π_1 , each plane containing two of those four lines, but being intersected by the two others in that point P_0 ; as the plane ABC , for example, is intersected in A by the two lines, AD and AE . Again, each point P_0 is common to three planes $\Pi_{2,1}$, no one of which contains any of the four lines A_1 through that point; it represents therefore a system of twelve other intersections, of the form $A_1 \cdot \Pi_{2,1}$. Again, each point P_0 is common to three lines $A_{2,1}$, each of which is contained in two of the six planes Π_1 , but intersects the four others in that point P_0 ; which therefore counts as twelve intersections, of the form $A_{2,1} \cdot \Pi_1$. Finally, each of the points P_0 represents three intersections, $A_{2,1} \cdot \Pi_{2,1}$; and it represents no other intersection, of the form $\Delta \cdot \Pi$, within the limits of the present inquiry. Thus, each of the five given points is to be considered as representing, or constructing, thirty-nine (= 12 + 12 + 12 + 3) intersections of line with plane; and there remain only 2040 (= 2235 - 195) other cases of such intersection $\Delta \cdot \Pi$, to be accounted for (in the present verification) by the 800 derived points, F_1, F_2 .

(4.) For this purpose, the nine columns, headed as I. to IX. in the following Table, contain the numbers of such intersections which belong respectively to the nine forms,

$$\begin{aligned} & \Delta_1 \cdot \Pi_1, \quad \Delta_1 \cdot \Pi_{2,1}, \quad \Delta_1 \cdot \Pi_{2,2}; \quad \Delta_{2,1} \cdot \Pi_1, \quad \Delta_{2,1} \cdot \Pi_{2,1}, \quad \Delta_{2,1} \cdot \Pi_{2,2}; \\ & \Delta_{2,2} \cdot \Pi_1, \quad \Delta_{2,2} \cdot \Pi_{2,1}, \quad \Delta_{2,2} \cdot \Pi_{2,2}, \end{aligned}$$

for each of the nine typical derived points, $\Delta' \dots \Delta^{IX}$, of the nine groups $F_1, F_2, \dots, F_{2,9}$. Column X. contains, for each point, the sum of the nine numbers, thus tabulated in the preceding columns; and expresses therefore the entire number of intersections, which any one such point represents. Column XI. states the number of the points for each type; and column XII. contains the product of the two last numbers, or the number of intersections $\Delta \cdot \Pi$ which are represented (or constructed) by the group. Finally, the sum of the numbers in each of the two last columns is written at its foot; and because the 800 derived points, of first and second constructions, are thus found to represent the 2040 intersections which were to be accounted for, the verification is seen to be complete: and no new type, of points F_3 , remains to be discovered.

(5.) TABLE of Intersections $\Delta \cdot \Pi$.

Type.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
Δ'	1	6	6	6	12	18	18	24	24	115	10	1150
Δ''	0	3	6	0	0	0	6	3	12	30	10	300
Δ'''	0	0	0	0	2	2	1	2	0	7	30	210
Δ^{IV}	0	0	0	0	0	2	0	0	0	2	30	60
Δ^V	0	0	3	0	0	0	0	0	0	3	20	60
Δ^{VI}	0	0	1	0	0	0	0	0	0	1	20	20
Δ^{VII}	0	0	0	0	0	0	0	1	0	1	60	60
Δ^{VIII}	0	0	0	0	0	0	0	0	2	2	60	120
Δ^{IX}	0	0	0	0	0	0	0	0	1	1	60	60
											800	2040

(6.) It is to be remembered that we have *not admitted*, by our *definition* (86), any *points* which can *only* be determined by *intersections of three planes* Π_1, Π_2 , as belonging to the *second construction*: nor have we counted, as *lines* Λ_2 of that construction, any lines which can *only* be found as intersections of *two* such planes. For example, we do not regard the *traces* AA' , &c., of certain *planes* $\Lambda_{2,1}$ considered in recent sub-articles, as among the lines of *second construction*, although they would present themselves early in an enumeration of the lines Λ_2 of the *third*. And any point in the plane ABC , which can *only* be determined (at the present stage) as the intersection of *two* such *traces*, is not regarded as a point F_2 . A student might find it however to be not useless, as an exercise, to investigate the expressions for *such* intersections; and for that reason it may be noted here, that the *ternary types* (comp. 81) of the *forty-four traces* of *planes* Π_1, Π_2 , on the plane ABC , which are found to compose a system of only *twenty-two distinct lines* in that plane, whereof *nine* are lines Λ_1, Λ_2 , are the seven following (comp. 88):

$$[100], [01\bar{1}], [\bar{1}11], [111], [011], [\bar{2}11], [\bar{2}\bar{1}\bar{1}];$$

which, as *ternary symbols*, represent the *seven lines*,

$$BC, AA', B'C', A'B'C', AA'', D_1A'', A'C_0.$$

(7.) Again, on the same principle, and with reference to the same definition, that new point, say F , which may be denoted by *either* of the two *congruent quinary symbols* (71),

$$F = (43210) \equiv (01234),$$

and which, as a *quinary type* (78), represents a *new group of sixty points of space* (and of *no more*, on account of this last *congruence*, whereas a quinary type, with *all its five coefficients unequal*, represents *generally* a group of 120 distinct points), is *not* regarded by us as a point F_2 ; although this new point F is easily seen to be the *intersection of three planes of second construction*, namely, of the three following, which all belong to the group $\Pi_{2,1}$:

$$[01\bar{1}\bar{1}1], [1\bar{1}0\bar{1}\bar{1}], [1\bar{1}\bar{1}10],$$

or $AA'D_1C_1B_2, OC'D_1B_1A_2, EB'B_2C'C_2$. It may, however, be remarked in passing, that *each plane* $\Pi_{2,1}$ contains *twelve points* F_2 of this new group: every such point being common (as is evident from what has been shown) to *three* such planes.

94. From the foregoing discussion it appears that the *five given points* F_0 , and the *three hundred derived points* F_1, F_2 , are *arranged in space*, upon the *fifty-five lines* Λ_1, Λ_2 , and in the *forty-five planes* Π_1, Π_2 , as follows. Each line Λ_1 contains *eight* of the 305 points, forming on it what may be called (see the sub-article (8.) to 92) a *double involution*. Each line $\Lambda_{2,1}$ contains *seven* points, whereof *one*, namely the *given point*, F_0 , has been seen (in the earlier sub-art. (6.)) to be a *double point* of *another* involution, to which the *three derived pairs* of points, F_1, F_2 , on the same line belong. And each line $\Lambda_{2,2}$ contains *ten* points, forming on it a *new involution*; while *eight* of these ten points, with a different *order* of succession, compose still *another*

involution* (92, (10.)). Again, each plane Π_1 contains *fifty-two* points, namely three *given* points, four points of *first*, and 45 points of *second* construction. Each plane $\Pi_{2,1}$ contains *forty-seven* points, whereof *one* is a given point, four are points P_1 , and 42 are points

* These theorems respecting the *relations of involution*, of given and derived points on lines of *first* and *second* constructions, for a *net in space*, are perhaps new; although some of the *harmonic relations*, above mentioned, have been noticed under other forms by Möbius: to whom, indeed, as has been stated, the *conception* of such a *net* is due. Thus, if we consider (compare the Note to page 72) the two intersec-

$$E_1 = DE \cdot A_1B_1C_1, \quad E_2 = DE \cdot A_2B_2C_2,$$

we easily find that they may be denoted by the quinary symbols,

$$E_1 = (000\bar{1}2), \quad E_2 = (0002\bar{1});$$

they are, therefore, by Art. 92, the two points $F_{2,6}$ on the line DE : and consequently, by the theorem stated at the end of sub-art. 8, the harmonic conjugate of *each*, taken with respect to the *other* and to the point D_1 , must be one of the two points D, E on that line. Accordingly, we soon derive, by comparison of the symbols of these *five* points, $DED_1E_1E_2$, the two following harmonic equations, which belong to the *same* type as the *two last* of that sub-art. 8:

$$(D_1DE_2E_1) = -1; \quad (D_1EE_1E_2) = -1;$$

but *these* two equations have been assigned (with notations slightly different) in the formerly cited page 290 of the Barycentric Calculus. (Comp. again the recent Note to page 72.) The *geometrical meaning* of the last equation may be illustrated, by conceiving that $ABCD$ is a *regular pyramid*, and that E is its *mean point*; for then (comp. 92, sub-art. (2.)), D_1 is the mean point of the *base* ABC ; D_1D is the *altitude* of the pyramid; and the *three segments* D_1E, D_1E_1, D_1E_2 are, respectively, the *quarter*, the *third part*, and the *half* of that altitude: they compose therefore (as the formula expresses) a *harmonic progression*; or D_1 and E_1 are *conjugate points*, with respect to E and E_2 . But in order to exemplify the *double involution* of the same sub-art. (8.), it would be necessary to consider *three other points* F_2 , on the *same line* DE ; whereof *one*, above called D'_1 , belongs to a *known group* $F_{2,1}$ (92, (2.)); but the *two others* are of the group $F_{2,4}$, and do not seem to have been previously noticed. As an example of an involution on a line of *third* construction, it may be remarked that on each line of the group $\Delta_{3,3}$, or on each of the sides of any one of the ten triangles $T_{3,2}$, in addition to one given point F_0 , and one derived point $F_{2,1}$, there are two points $F_{2,2}$, and two points $F_{2,6}$; and that the two first points are the double points of an involution, to which the two last pairs belong: thus, on the side $A_0B_0C_0$ of the exscribed triangle $A_0B_0C_0$, or on the trace of the plane $BC_1A_2A_1C_2$, we have the two harmonic equations,

$$(BA_0B''C_0) = (BA''B''C_1''') = -1.$$

Again, on the trace $A'C_0$ of the plane $A'C_1C_2$, (which latter trace is a line not passing through any one of the given points,) C_0 and B_1''' are the double points of an involution, wherein A' is conjugate to C_1'' and A'' to B'' . But it would be tedious to multiply such instances.

P_2 : of which last, 38 are situated *on the six lines* Λ_2 in the plane, but four are *intersections* of that plane $\Pi_{2,1}$ with *four other lines* of second construction. Finally, each plane $\Pi_{2,2}$ passes through *no given point*, but contains *forty-three derived points*, whereof 40 are points of second construction. And because the planes of *first construction alone contain specimens of all the ten groups of points*, $P_0, P_1, P_{2,1}, \dots P_{2,9}$, given or derived, and of *all the three groups of lines*, $\Lambda_1, \Lambda_{2,1}, \Lambda_{2,2}$, at the close of that *second construction* (since the types $P_{2,4}, P_{2,5}, \Lambda_1$ are not represented by any points or lines in any plane $\Pi_{2,1}$, nor are the types $P_0, \Lambda_1, \Lambda_{2,1}$ represented in a plane $\Pi_{2,2}$), it has been thought convenient to prepare the annexed *diagram* (Fig. 30), which may serve to illustrate, by some selected instances, the *arrangement of the fifty-two points* P_0, P_1, P_2 in a plane Π_1 , namely, in the plane ABC ; as well as the *arrangement of the nine lines* Λ_1, Λ_2 in that plane, and the *traces* Λ_2 of other planes upon it.

View of the Arrangement of the Principal Points and Lines in a Plane of First Construction.

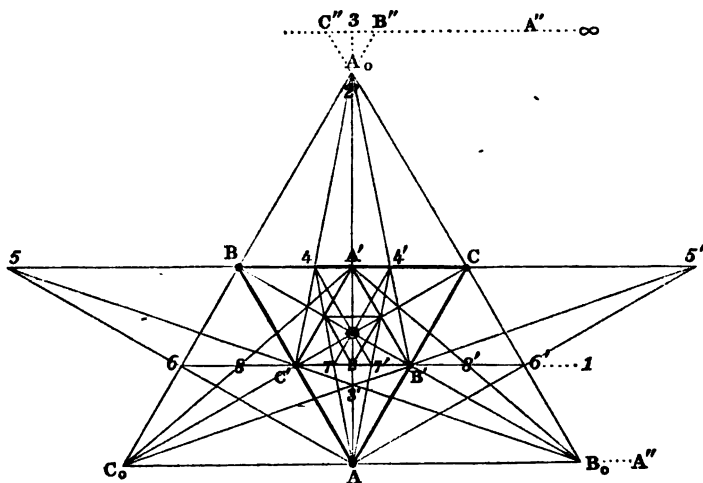


Fig. 30.

In this Figure, the triangle ABC is supposed, for simplicity, to be the *equilateral base of a regular pyramid* $ABCD$ (comp. sub-art. (2,) to 92); and D_1 , again replaced by O , is supposed to be its *mean point* (29). The *first inscribed triangle*, $A'B'C'$, therefore, *bisects the three sides*; and the *axis of homology* $A''B''C''$ is the *line at infinity* (88): the number 1, on the line $c'b'$ prolonged, being designed to suggest that

the point Δ'' , to which that *line tends*, is of the type $F_{2,1}$, or belongs to the *first group* of points of *second construction*. A *second inscribed triangle*, $\Delta''B''C''$, for which Fig. 21 may be consulted, is only *indicated* by the number 2 placed at the middle of the side $B''C''$, to suggest that *this bisecting point Δ''* belongs to the *second group* of points F_2 . The *same number 2*, but with an *accent, 2'*, is placed near the corner Δ_0 of the *escribed triangle* $\Delta_0B_0C_0$, to remind us that this corner *also* belongs (by a *syntypical relation in space*) to the group $F_{2,2}$. The point Δ'' , which is now infinitely distant, is indicated by the number 8, on the dotted line at the top; while the same number with an *accent, lower down*, marks the position of the point Δ_1'' . Finally, the ten other numbers, unaccented or accented, 4, 4', 5, 5', 6, 6', 7, 7', 8, 8', denote the places of the ten points, Δ' , Δ_1' , Δ'' , Δ_1'' , Δ''' , Δ_1''' , Δ^{IV} , Δ_1^{IV} , Δ^V , Δ_1^V . And the principal *harmonic relations*, and relations of *involution*, above mentioned, may be verified by inspection of this Diagram.

95. However far the series of construction of the net in space may be continued, we may now regard it as evident, at least on comparison with the analogous property (42) of the *plane net*, that *every point, line, or plane*, to which such constructions can conduct, must necessarily be *rational* (77); or that it must be *rationally related* to the system of the *five given points*: because the *anharmonic co-ordinates* (79, 80) of every *net-point*, and of every *net-plane*, are equal or proportional to *whole numbers*. Conversely (comp. 43) *every point, line, or plane*, in space, which is thus *rationally related* to the system of points $ABCDE$, is a point, line, or plane of the *net*, which those five points determine. Hence (comp. again 43), every *irrational point, line, or plane* (77), is indeed incapable of being *rigorously constructed*, by any processes of the kind above described; but it admits of being *indefinitely approximated to*, by points, lines, or planes of the net. *Every anharmonic ratio*, whether of a *group of net-points*, or of a *pencil of net-lines*, or of *net-planes*, has a *rational value* (comp. 44), which depends *only* on the *processes of linear construction* employed, in the generation of that group or pencil, and is entirely *independent* of the *arrangement, or configuration*, of the five given points in space. Also, all *relations of collineation*, and of *complanarity*, are *preserved*, in the passage from *one net to another*, by a change of the given system of points: so that it may be briefly said (comp. again 44) that *all geometrical nets in space are homographic figures*. Finally, *any five points** of such a net, of which no four are in one plane, are *sufficient* (comp.

* These *general properties* (95) of the *space-net* are in substance taken from Möbius, although (as has been remarked before) the *analysis* here employed appears to be new: as do also most of the *theorems* above given, respecting the *points of second construction* (92), at least after we pass beyond the *first group* $F_{2,1}$ of ten such points, which (as already stated) have been known comparatively long.

45) for the determination of the *whole net*: or for the *linear construction* of all its points, including the five *given* ones.

(1.) As an Example, let the five points $A_1B_1C_1D_1$ and E be now supposed to be *given*; and let it be required to *derive* the four points $ABCD$, by linear constructions, from these new data. In other words, we are now required to *exscribe* a *pyramid* $ABCD$ to a given pyramid $A_1B_1C_1D_1$, so that it may be *homologous* thereto, with the point E for their *given centre of homology*. An obvious process is (comp. 45) to *in-scribe* another homologous pyramid, $A_2B_2C_2D_2$, so as to have $A_2 = EA_1 \cdot B_1C_1D_1$, &c.; and then to determine the *intersections of corresponding faces*, such as $A_1B_1C_1$ and $A_2B_2C_2$; for these *four lines* of intersection will be in the *common plane* [E], of homology of the *three pyramids*, and will be the *traces* on that plane of the *four sought planes*, ABC , &c., drawn through the four given points D_1 , &c. If it were only required to construct *one corner* A of the exscribed pyramid, we might find the point above called A^{IV} as the common intersection of *three planes*, as follows,

$$A^{IV} = A_1B_1C_1 \cdot A_1D_1E \cdot A_2B_2C_2;$$

and then should have this other formula of intersection,

$$A = EA_1 \cdot D_1A^{IV}.$$

Or the point A might be determined by the anharmonic equation,

$$(EAA_1A_2) = 8,$$

which for a regular pyramid is easily verified.

(2.) As regards the general *passage* from *one net* in space to *another*, let the symbols $F_1 = (x_1 \dots v_1), \dots F_5 = (x_5 \dots v_5)$ denote *any five given points*, whereof no four are coplanar; and let $a'b'c'd'e'$ and w' be six coefficients, of which the five ratios are such as to satisfy the symbolical equation (comp. 71, 72),

$$a'(F_1) + b'(F_2) + c'(F_3) + d'(F_4) + e'(F_5) = -w'(U);$$

or the five ordinary equations which it includes, namely,

$$a'x_1 + \dots + e'x_5 = \dots = a'v_1 + \dots + e'v_5 = -w'.$$

Let F' be any sixth point of space, of which the quinary symbol satisfies the equation,

$$(F') = xa'(F_1) + yb'(F_2) + zc'(F_3) + wd'(F_4) + ve'(F_5) + w(U);$$

then it will be found that this last point F' can be *derived* from the five points $F_1 \dots F_5$ by precisely the *same constructions*, as those by which the point $F = (xyzwv)$ is derived from the five points $ABCDE$. As an example, if $v' = a + y + z + w - 8v$, then the point $(xyzwv')$ is derived from $A_1B_1C_1D_1E$, by the same constructions as $(xyzwv)$ from $ABCDE$; thus A itself may be constructed from $A_1 \dots E$, as the point $F = (30001)$ is from $A \dots E$; which would conduct anew to the anharmonic equation of the last sub-article.

(3.) It may be briefly added here, that instead of *anharmonic ratios*, as connected with a net in space, or indeed generally in relation to *spatial problems*, we are permitted (comp. 68) to substitute products (or quotients) of *quotients of volumes* of pyramids; as a *specimen* of which substitution, it may be remarked, that the anharmonic relation, just referred to, admits of being replaced by the following equation, involving *one* such quotient of pyramids, but introducing *no auxiliary point*:

$$EA : A_1A = 3EB_1C_1D_1 : A_1B_1C_1D_1.$$

In general, if $xyzw$ be (as in 79, 83) the *anharmonic co-ordinates* of a point P in space, we may write,

$$\frac{x}{y} = \frac{FB_1CD}{PCDA} : \frac{EB_1CD}{ECDA};$$

with other equations of the same type, on which we cannot here delay.

SECTION 5.—*On Barycentres of Systems of Points ; and on Simple and Complex Means of Vectors.*

96. In general, when the sum Σa of any number of co-initial vectors,

$$a_1 = OA_1, \dots a_m = OA_m,$$

is divided (16) by their number, m , the resulting vector,

$$\mu = OM = \frac{1}{m} \Sigma a = \frac{1}{m} \Sigma OA,$$

is said to be the *Simple Mean* of those m vectors; and the point M , in which this *mean vector* terminates, and of which the position (comp. 18) is easily seen to be *independent* of the position of the common *origin* O , is said to be the *Mean Point* (comp. 29), of the system of the m points, $A_1, \dots A_m$. It is evident that we have the equation,

$$O = (a_1 - \mu) + \dots + (a_m - \mu) = \Sigma (a - \mu) = \Sigma MA;$$

or that the *sum* of the m vectors, drawn from the *mean point* M , to the points A of the system, is equal to *zero*. And hence (comp. 10, 11, 30), it follows, Ist., that these m vectors are equal to the m *successive sides* of a *closed polygon*; IInd., that if the system and its mean point be *projected*, by any *parallel ordinates*, on any assumed *plane* (or *line*), the *projection* M' , of the *mean point* M , is the *mean point* of the *projected system*; and IIIrd., that the *ordinate* MM' , of the *mean point*, is the *mean* of all the *other ordinates*, $A_1A'_1, \dots A_mA'_m$. It follows, also, that if N be the *mean point* of *another system*, $B_1, \dots B_n$; and if s be the *mean point* of the *total system*, $A_1 \dots B_n$, of the $m+n$ points obtained by *combining* the *two former*, considered as *partial systems*; while ν and σ may denote the *vectors*, ON and OS , of these *two last mean points*: then we shall have the equations,

$$m\mu = \Sigma a, \quad n\nu = \Sigma \beta, \quad s\sigma = \Sigma a + \Sigma \beta = m\mu + n\nu, \\ m(\sigma - \mu) = n(\nu - \sigma), \quad m.MS = n.BN;$$

so that the *general mean point*, s , is situated on the *right line* MN , which connects the *two partial mean points*, M and N ; and divides

that line (internally), into two segments MS and SN , which are inversely proportional to the two whole numbers, m and n .

(1.) As an Example, let $ABCD$ be a *gauche quadrilateral*, and let E be its mean point; or more fully, let

$$\circ E = \frac{1}{4}(OA + OB + OC + OD),$$

or

$$\epsilon = \frac{1}{4}(\alpha + \beta + \gamma + \delta);$$

that is to say, let $a = b = c = d$, in the equations of Art. 65. Then, with notations lately used, for certain *derived points* D_1 , &c., if we write the *vector-formulae*,

$$\begin{aligned} OA_1 = a_1 &= \frac{1}{2}(\beta + \gamma + \delta), & \delta_1 &= \frac{1}{2}(\alpha + \beta + \gamma), \\ OA_2 = a_2 &= \frac{1}{2}(\alpha + \delta), & \gamma_2 &= \frac{1}{2}(\gamma + \delta), \\ OA' = a' &= \frac{1}{2}(\beta + \gamma), & \gamma' &= \frac{1}{2}(\alpha + \beta), \end{aligned}$$

we shall have seven different expressions for the mean vector, ϵ ; namely, the following:

$$\begin{aligned} \epsilon &= \frac{1}{4}(\alpha + 3a_1) = \dots = \frac{1}{4}(\delta + 3\delta_1) \\ &= \frac{1}{4}(\alpha' + a_2) = \dots = \frac{1}{4}(\gamma' + \gamma_2). \end{aligned}$$

And these conduct to the seven equations between segments,

$$\begin{aligned} AE &= 3EA_1, & DE &= 3ED_1; \\ A'E &= EA_2, & C'E &= EC_2; \end{aligned}$$

which prove (what is otherwise known) that the four right lines, here denoted by AA_1, \dots, DD_1 , whereof each connects a corner of the pyramid $ABCD$ with the mean point of the opposite face, intersect and quadrisect each other, in one common point, E ; and that the three common bisectors $A'A_2, B'B_2, C'C_2$, of pairs of opposite edges, such as BC and DA , intersect and bisect each other, in the same mean point: so that the four middle points, C', A', C_2, A_2 , of the four successive sides AB , &c., of the gauche quadrilateral $ABCD$, are situated in one common plane, which bisects also the common bisector, $B'B_2$, of the two diagonals, AC and BD .

(2.) In this example, the number s of the points $A \dots D$ being four, the number of the derived lines, which thus cross each other in their general mean point E is seen to be seven; and the number of the derived planes through that point is nine: namely, in the notation lately used for the *net* in space, four lines Λ_1 , three lines $\Lambda_{2,1}$, six planes Π_1 , and three planes $\Pi_{2,1}$. Of these nine planes, the six former may (in the present connexion) be called *triple planes*, because each contains three lines (as the plane ABE , for instance, contains the lines $AA_1, BB_1, C'C_2$), all passing through the mean point E ; and the three latter may be said, by contrast, to be *non-triple planes*, because each contains only two lines through that point, determined on the foregoing principles.

(3.) In general, let $\phi(s)$ denote the number of the lines, through the general mean point s of a total system of s given points, which is thus, in all possible ways, decomposed into partial systems; let $f(s)$ denote the number of the triple planes, obtained by grouping the given points into three such partial systems; let $\psi(s)$ denote the number of non-triple planes, each determined by grouping these s points in two different ways into two partial systems; and let $\mathfrak{r}(s) = f(s) + \psi(s)$ represent the entire number of distinct planes through the point s : so that

$$\phi(4) = 7, \quad f(4) = 6, \quad \psi(4) = 3, \quad \mathfrak{r}(4) = 9.$$

Then it is easy to perceive that if we introduce a *new point* C , each *old line* MN furnishes *two new lines*, according as we group the new point with one or other of the two old partial systems, (M) and (N) ; and that there is, besides, *one other new line*, namely CS : we have, therefore, the *equation in finite differences*,

$$\phi(s+1) = 2\phi(s) + 1;$$

which, with the *particular value* above assigned for $\phi(4)$, or even with the simpler and more obvious value, $\phi(2) = 1$, conducts to the *general expression*,

$$\phi(s) = 2^{s-1} - 1.$$

(4.) Again, if $(M)(N)(P)$ be any *three partial systems*, which jointly make up the old or given *total system* (S) ; and if, by grouping a *new point* C with each of these in turn, we form *three new partial systems*, $(M')(N')(P')$; then each *old triple plane* such as MNP , will furnish *three new triple planes*,

$$M'NP, \quad MN'P, \quad MNP';$$

while each *old line*, KL , will give *one new triple plane*, CKL : nor can any new triple plane be obtained in any other way. We have, therefore, this *new equation in differences*:

$$f(s+1) = 3f(s) + \phi(s).$$

But we have seen that

$$\phi(s+1) = 2\phi(s) + 1;$$

if then we write, for a moment,

$$f(s) + \phi(s) = \chi(s),$$

we have this other equation in finite differences,

$$\chi(s+1) = 3\chi(s) + 1.$$

Also,

$$f(3) = 1, \quad \phi(3) = 3, \quad \chi(3) = 4:$$

therefore,

$$2\chi(s) = 3^{s-1} - 1,$$

and

$$2f(s) = 3^{s-1} - 2^s + 1.$$

(5.) Finally, it is clear that we have the relation,

$$3f(s) + \psi(s) = \frac{1}{2}\phi(s) \cdot (\phi(s) - 1) = (2^{s-1} - 1)(2^{s-2} - 1);$$

because the *triple planes*, each treated as *three*, and the *non-triple planes*, each treated as *one*, must jointly represent all the *binary combinations* of the *lines*, drawn through the mean point s of the whole system. Hence,

$$2\psi(s) = 2^{2s-2} + 3 \cdot 2^{s-1} - 3^s - 1;$$

and

$$\mathcal{F}(s) = 2^{2s-3} + 2^{s-2} - 3^{s-1};$$

so that

$$\mathcal{F}(s+1) - 4\mathcal{F}(s) = 3^{s-1} - 2^{s-1},$$

and

$$\psi(s+1) - 4\psi(s) = 3f(s);$$

which last equation in finite differences admits of an independent geometrical interpretation.

(6.) For instance, these general expressions give,

$$\phi(5) = 15; \quad f(5) = 25; \quad \psi(5) = 30; \quad \mathcal{F}(5) = 55;$$

so that if we assume a *gauche pentagon*, or a system of *five points in space*, A . . . E,

and determine the *mean point* F of this system, there will in general be a set of *fifteen lines*, of the kind above considered, all passing through this sixth point F : and these will be arranged generally in *fifty-five distinct planes*, whereof *twenty-five* will be what we have called *triple*, the *thirty others* being of the *non-triple* kind.

97. More generally, if $a_1 \dots a_m$ be, as before, a system of m given and co-initial vectors, and if a_1, \dots, a_m be any system of m given scalars (17), then that *new co-initial vector* β , or OB , which is deduced from these by the formula,

$$\beta = \frac{a_1 a + \dots + a_m a_m}{a_1 + \dots + a_m} = \frac{\Sigma a a}{\Sigma a}, \text{ or } OB = \frac{\Sigma a O A}{\Sigma a},$$

or by the equation

$$\Sigma a(a - \beta) = 0, \text{ or } \Sigma a B A = 0,$$

may be said to be the *Complex Mean* of those m given vectors a , or OA , considered as *affected* (or combined) with that system of given scalars, a , as *coefficients*, or as *multipliers* (12, 14). It may also be said that the *derived point* B , of which (comp. 96) the *position* is independent of that of the *origin* O , is the *Barycentre* (or *centre of gravity*) of the given system of points $A_1 \dots$, considered as *loaded* with the given *weights* $a_1 \dots$; and theorems of *intersections* of lines and planes arise, from the comparison of these *complex means*, or *barycentres*, of *partial* and *total systems*, which are entirely analogous to those lately considered (96), for *simple means* of vectors and of points.

(1.) As an Example, in the case of Art. 24, the point c is the barycentre of the system of the *two* points, A and B , with the weights a and b ; while, under the conditions of 27, the origin O is the barycentre of the *three* points A, B, C , with the three weights a, b, c ; and if we use the formula for ρ , assigned in 34 or 36, the same three given points A, B, C , when loaded with xa, yb, zc as weights, have the point F in their plane for their barycentre. Again, with the equations of 65, π is the barycentre of the system of the *four* given points, A, B, C, D , with the weights a, b, c, d ; and if the expression of 79 for the vector or be adopted, then xa, yb, zc, wd are equal (or proportional) to the weights with which the same four points $A \dots D$ must be loaded, in order that the point F of space may be their barycentre. In all these cases, the *weights* are thus *proportional* (by 69) to certain *segments*, or *areas*, or *volumes*, of kinds which have been already considered; and what we have called the *anharmenic co-ordinates* of a *variable point* F , in a *plane* (86), or in *space* (79), may be said, on the same plan, to be *quotients of quotients of weights*.

(2.) The circumstance that the *position* of a *barycentre* (97), like that of a *simple mean point* (96), is *independent* of the position of the assumed *origin* of vectors, might induce us (comp. 69) to *suppress* the *symbol* O of that *arbitrary* and *foreign point*; and therefore to write* simply, under the lately supposed conditions,

* We should thus have some of the principal *notations* of the *Barycentric Calculus*: but used mainly with a reference to *vectors*. Compare the Note to page 56.

$$\mathbf{B} = \frac{\sum aA}{\sum a} \quad \text{or} \quad bB = \sum aA, \quad \text{if } b = a.$$

It is easy to prove (comp. 96), by principles already established, that the *ordinate of the barycentre* of any given system of points is the *complex mean* (in the sense above defined, and with the same system of weights), of the *ordinates of the points* of that system, with reference to *any given plane*: and that the *projection of the barycentre*, on any such plane, is the *barycentre of the projected system*.

(3.) Without any reference to *ordinates*, or to any foreign origin, the *barycentric notation* $\mathbf{B} = \frac{\sum aA}{\sum a}$ may be interpreted, by means of our *fundamental convention*

(Art. 1) respecting the geometrical signification of the *symbol* $\mathbf{B} - \mathbf{A}$, considered as denoting the *vector* from \mathbf{A} to \mathbf{B} : together with the rules for *multiplying* such vectors by *scalars* (14, 17), and for taking the *sums* (6, 7, 8, 9) of those (generally new) vectors, which are (15) the *products* of such multiplications. For we have only to write the formula as follows,

$$\sum a(\mathbf{A} - \mathbf{B}) = 0,$$

in order to perceive that it may be considered as signifying, that the system of the *vectors from the barycentre* \mathbf{B} , to the system of the *given points* $\mathbf{A}_1, \mathbf{A}_2, \dots$ when multiplied respectively by the *scalars* (or coefficients) of the given system a_1, a_2, \dots becomes (generally) a new system of vectors with a *null sum*: in such a manner that these last vectors, $a_1 \cdot \mathbf{BA}_1, a_2 \cdot \mathbf{BA}_2, \dots$ can be made (10) the *successive sides of a closed polygon*, by transports without rotation.

(4.) Thus if we meet the formula,

$$\mathbf{B} = \frac{1}{2}(\mathbf{A}_1 + \mathbf{A}_2),$$

we may indeed interpret it as an *abridged form* of the equation,

$$\mathbf{OB} = \frac{1}{2}(\mathbf{OA}_1 + \mathbf{OA}_2);$$

which implies that if \mathbf{O} be any *arbitrary point*, and if \mathbf{O}' be the point which *completes* (comp. 6) the *parallelogram* $\mathbf{A}_1\mathbf{OA}_2\mathbf{O}'$, then \mathbf{B} is the point which *bisects the diagonal* \mathbf{OO}' , and therefore also the *given line* $\mathbf{A}_1\mathbf{A}_2$, which is here the *other diagonal*. But we may also regard the formula as a mere *symbolical transformation* of the equation,

$$(\mathbf{A}_2 - \mathbf{B}) + (\mathbf{A}_1 - \mathbf{B}) = 0;$$

which (by the earliest principles of the present Book) expresses that the *two vectors*, from \mathbf{B} to the two given points \mathbf{A}_1 and \mathbf{A}_2 , have a *null sum*; or that they are *equal in length*, but *opposite in direction*: which can only be, by \mathbf{B} bisecting $\mathbf{A}_1\mathbf{A}_2$, as before.

(5.) Again, the formula, $\mathbf{B}_1 = \frac{1}{3}(\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3)$, may be interpreted as an *abridgment* of the equation,

$$\mathbf{OB}_1 = \frac{1}{3}(\mathbf{OA}_1 + \mathbf{OA}_2 + \mathbf{OA}_3),$$

which expresses that the point \mathbf{B} *trisects the diagonal* \mathbf{OO}' of the *parallelepiped* (comp. 62), which has $\mathbf{OA}_1, \mathbf{OA}_2, \mathbf{OA}_3$ for *three co-initial edges*. But the same formula may also be considered to express, in full consistency with the foregoing interpretation, that the *sum of the three vectors*, from \mathbf{B} to the three points $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$, *vanishes*: which is the characteristic property (30) of the *mean point of the triangle* $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$. And similarly in more complex cases: the *legitimacy* of such *transformations* being here regarded as a consequence of the original *interpretation* (1) of the *symbol* $\mathbf{B} - \mathbf{A}$, and of the rules for *operations on vectors*, so far as as they have been hitherto established.

SECTION 6.—*On Anharmonic Equations, and Vector-Expressions, of Surfaces and Curves in Space.*

98. When, in the expression 79 for the vector ρ of a variable point P of space, the four variable scalars, or anharmonic co-ordinates, $xyzw$, are connected (comp. 46) by a given algebraic equation,

$$f_p(x, y, z, w) = 0, \text{ or briefly } f = 0,$$

supposed to be rational and integral, and homogeneous of the p^{th} dimension, then the point P has for its locus a surface of the p^{th} order, whereof $f = 0$ may be said (comp. 56) to be the local equation. For if we substitute instead of the co-ordinates $x \dots w$, expressions of the forms,

$$x = tx_0 + ux_1, \dots \quad w = tw_0 + uw_1,$$

to indicate (82) that P is collinear with two given points, P_0, P_1 , the resulting algebraic equation in $t : u$ is of the p^{th} degree; so that (according to a received modern mode of speaking), the surface may be said to be cut in p points (distinct or coincident, and real or imaginary*), by any arbitrary right line, P_0P_1 . And in like manner, when the four anharmonic co-ordinates $lmnr$ of a variable plane Π (80) are connected by an algebraical equation, of the form,

$$F_q(l, m, n, r) = 0, \text{ or briefly } F = 0,$$

where F denotes a rational and integral function, supposed to be homogeneous of the q^{th} dimension, then this plane Π has for its envelope (comp. 56) a surface of the q^{th} class, with $F = 0$ for its tangential equation: because if we make

$$l = tl_0 + ul_1, \dots \quad r = tr_0 + ur_1,$$

to express (comp. 82) that the variable plane Π passes through a given right line $\Pi_0 \cdot \Pi_1$, we are conducted to an algebraical equation of the q^{th} degree, which gives q (real or imaginary) values for the ratio $t : u$, and thereby assigns q (real or imaginary)† tangent planes to the sur-

* It is to be observed, that no interpretation is here proposed, for imaginary intersections of this kind, such as those of a sphere with a right line, which is wholly external thereto. The language of modern geometry requires that such imaginary intersections should be spoken of, and even that they should be enumerated: exactly as the language of algebra requires that we should count what are called the imaginary roots of an equation. But it would be an error to confound geometrical imaginaries, of this sort, with those square roots of negatives, for which it will soon be seen that the Calculus of Quaternions supplies, from the outset, a definite and real interpretation.

† As regards the uninterpreted character of such imaginary contacts in geometry, the preceding Note to the present Article, respecting imaginary intersections, may be consulted.

face, drawn through any such given but arbitrary right line. We may add (comp. 51, 56), that if the functions f and F be only homogeneous (without necessarily being rational and integral), then

$$[D_x f, D_y f, D_z f, D_w f]$$

is the anharmonic symbol (80) of the tangent plane to the surface $f=0$, at the point $(xyzw)$; and that

$$(D_l F, D_m F, D_n F, D_r F)$$

is in like manner, a symbol for the point of contact of the plane $[lmnr]$, with its enveloped surface, $F=0$; $D_x, \dots D_l, \dots$ being characteristics of partial derivation.

(1.) As an Example, the surface of the second order, which passes through the nine points called lately

$$A, C', B, A', C, C_2, D, A_2, E,$$

has for its local equation,

$$0 = f = xz - yw;$$

which gives, by differentiation,

$$l = D_x f = z; \quad m = D_y f = -w;$$

$$n = D_z f = x; \quad r = D_w f = -y;$$

so that

$$[z, -w, x, -y]$$

is a symbol for the tangent plane, at the point (x, y, z, w) .

(2.) In fact, the surface here considered is the ruled (or hyperbolic) hyperboloid, on which the gauche quadrilateral ABCD is superscribed, and which passes also through the point E. And if we write

$$P = (xyzw), \quad Q = (xy00), \quad R = (0yz0), \quad s = (00zw), \quad T = (x00w),$$

then QS and RT (see the annexed Figure 81), namely, the lines drawn through P to intersect the two pairs, AB, CD, and BC, DA, of opposite sides of that quadrilateral ABCD, are the two generating lines, or generatrices, through that point; so that their plane, QRST, is the tangent plane to the surface, at the point P. If, then, we denote that tangent plane by the symbol $[lmnr]$, we have the equations of condition,

$$0 = lx + my + nz + rw = rz + rw + lx;$$

whence follows the proportion,

$$l : m : n : r = x^{-1} : -y^{-1} : z^{-1} : -w^{-1};$$

or, because $xz = yw$,

$$l : m : n : r = z : -w : x : -y,$$

as before.

(3.) At the same time we see that

$$(\Delta C'BQ) = \frac{x}{y} = \frac{w}{z} = (\Delta C_2CS);$$

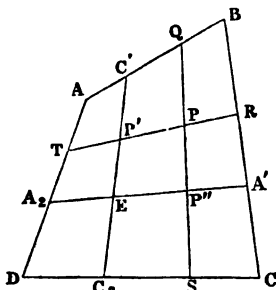


Fig. 81.

so that the *variable generatrix* qs divides (as is known) the two *fixed generatrices* AB and DC *homographically**; AD , BC , and $c'O_2$ being three of its positions. Conversely, if it were proposed to find the locus of the right line qs , which thus divides homographically (comp. 26) two given right lines in space, we might take AB and DC for those two given lines, and AD , BC , $c'O_2$ (with the recent meanings of the letters) for three given positions of the variable line; and then should have, for the two variable but corresponding (or homologous) points q , s themselves, and for any arbitrary point P collinear with them, anharmonic symbols of the forms,

$$Q = (s, u, 0, 0), \quad S = (0, 0, u, s), \quad P = (st, tu, uv, vs);$$

because, by 82, we should have, between these three symbols, a relation of the form,

$$(P) = t(Q) + v(S);$$

if then we write $P = (x, y, z, w)$, we have the anharmonic equation $xz = yw$, as before; so that the locus, whether of the line qs , or of the point P , is (as is known) a ruled surface of the second order.

(4.) As regards the known *double generation* of that surface, it may suffice to observe that if we write, in like manner,

$$R = (0t0), \quad T = (t00), \quad (P) = u(R) + s(T),$$

we shall have again the expression,

$$P = (st, tu, uv, vs), \quad \text{giving } xz = yw,$$

as before: so that the same hyperboloid is also the locus of that other line RT , which divides the other pair of opposite sides BC , AD of the same gauche quadrilateral $ABCD$ homographically; BA , CD , and $A'A_2$ being three of its positions; and the lines $A'A_2$, $c'O_2$ being still supposed to intersect each other in the given point E .

(5.) The symbol of an arbitrary point on the variable line RT is (by sub-art. 2) of the form, $t(0, y, z, 0) + u(x, 0, 0, w)$, or (ux, ty, tz, uw) ; while the symbol of an arbitrary point on the given line $c'O_2$ is (t', t', u', u') . And these two symbols represent one common point (comp. Fig. 81),

$$P' = RT \cdot c'O_2 = (y, y, z, z),$$

when we suppose

$$t' = y, \quad u' = z, \quad t = 1, \quad u = \frac{y}{w} = \frac{z}{w}.$$

Hence the known theorem results, that a *variable generatrix*, RT , of one system, intersects three fixed lines, BC , AD , $c'O_2$, which are generatrices of the other system. Conversely, by the same comparison of symbols, for points on the two lines RT and $c'O_2$, we should be conducted to the equation $xz = yw$, as the condition for their intersection; and thus should obtain this other known theorem, that the locus of a right line, which intersects three given right lines in space, is generally an hyperboloid with those three lines for generatrices. A similar analysis shows that qs intersects $A'A_2$, in a point (comp. again Fig. 81) which may be thus denoted:

$$P'' = qs \cdot A'A_2 = (xyyx).$$

(6.) As another example of the treatment of surfaces by their anharmonic and local equations, we may remark that the recent symbols for P' and P'' , combined with

* Compare p. 298 of the *Géométrie Supérieure*.

those of sub-art. 2 for F, Q, R, S, T ; with the symbols of 86, 86 for C', A', C_2, A_2, E ; and with the equation $xz = yw$, give the expressions :

$$(F) = (Q) + (S) = (R) + (T); \quad (F') = y(C') + z(C_2) = (R) + \frac{y}{x}(T);$$

$$(E) = (C') + (C_2) = (A') + (A_2); \quad (E') = y(A') + x(A_2) = (Q) + \frac{y}{z}(S);$$

whence it follows (84) that the two points F', E' , and the sides of the quadrilateral ABCD, divide the four generating lines through F and E in the following anharmonic ratios :

$$(C'BC_2F') = (QF'EF) = \frac{y}{z} = (BA'OR) = (AA_2DT);$$

$$(A'EA_2E') = (EF'TE) = \frac{y}{x} = (BC'AQ) = (OC_2DS);$$

so that (as again is known) the *variable* generatrices, as well as the *fixed* ones, of the hyperboloid, are all divided *homographically*.

(7.) The *tangential equation* of the present surface is easily found, by the expressions in sub-art. 1 for the co-ordinates $lmnr$ of the tangent plane, to be the following :

$$0 = F = ln - mr;$$

which may be interpreted as expressing, that this hyperboloid is the *surface of the second class*, which *touches the nine planes*,

$$[1000], [0100], [0010], [0001], [1100], [0110], [0011], [1001], [1111];$$

or with the literal symbols lately employed (comp. 86, 87),

$$BCD, CDA, DAB, ABC, CDC'', DAA'', ABC', BCA', \text{ and } [E].*$$

Or we may interpret the same tangential equation $F = 0$ as expressing (comp. again 86, 87, where q, l, n are now replaced by T, R, Q), that the surface is the *envelope of a plane QRST*, which satisfies *either of the two connected conditions of homography* :

$$(BC'AQ) = -\frac{l}{m} = -\frac{r}{n} = (OC_2DS);$$

$$(CA'ER) = -\frac{m}{n} = -\frac{l}{r} = (DA_2AT);$$

a *double generation* of the hyperboloid thus showing itself in a new way. And as regards the *passage* (or *return*), from the *tangential* to the *local equation* (comp. 56), we have in the present example the formulæ :

$$x = D_l F = n; \quad y = D_m F = -r; \quad z = D_n F = l; \quad w = D_r F = -m;$$

whence

$$xz - yw = 0,$$

as before.

(8.) More generally, when the surface is of the *second order*, and therefore also of the *second class*, so that the two functions f and F , when presented under rational and integral forms, are both homogeneous of the *second dimension*, then whether we derive $l \dots r$ from $x \dots w$ by the formulæ,

* In the anharmonic symbol of Art. 87, for the plane of homology $[E]$, the coefficient 1 occurred, through inadvertence, *five times*.

$$l = D_x f, \quad m = D_y f, \quad n = D_z f, \quad r = D_w f,$$

or $x \dots w$ from $l \dots r$ by the *converse* formulae,

$$x = D_l F, \quad y = D_m F, \quad z = D_n F, \quad w = D_r F,$$

the point $F = (xyzw)$ is, relatively to that surface, what is usually called (comp. 52) the *pole* of the plane $\Pi = [lmnr]$; and conversely, the plane Π is the *polar* of the point F ; wherever in space the point F and plane Π , thus related to each other, may be situated. And because the *centre* of a surface of the second order is known to be (comp. again 52) the *pole* of (what is called) the *plane at infinity*; while (comp. 88) the *equation* and the *symbol* of this last plane are, respectively,

$$ax + by + cz + dw = 0, \quad \text{and} \quad [a, b, c, d],$$

if the four constants $abcd$ have still the same significations as in 65, 70, 79, &c., with reference to the system of the five given points $ABCDE$: it follows that we may denote this *centre* by the symbol,

$$\kappa = (D_a F_0, D_b F_0, D_c F_0, D_d F_0);$$

where F_0 denotes, for abridgment, the function $F(abcd)$, and d is still a scalar constant.

(9.) In the recent example, we have $F_0 = ac - bd$; and the anharmonic symbol for the centre of the hyperboloid becomes thus,

$$\kappa = (c, -d, a, -b).$$

Accordingly if we assume (comp. sub-arts. 8, 4),

$$F = (st, tu, uv, vs), \quad F' = (s't', -t'u', u'v', = r's'),$$

where s, t, u, v are any four scalars, and F' is a new point, while

$$s' = bt + cv, \quad t' = cu + ds, \quad u' = dv + at, \quad v' = as + bw;$$

if also we write, for abridgment,

$$e' = ac - bd, \quad w' = ast + btu + cvv + dws;$$

we shall then have the symbolic relations,

$$e'(F) + (F) = w'(\kappa), \quad e'(F) - (F) = (F''),$$

if $F'' = (x''y''z''w'')$ be that new point, of which the co-ordinates are,

$$x'' = 2e'st - cw', \quad y'' = 2e'tu + dw', \quad z'' = 2e'uv - aw', \quad w'' = 2e'vs + bw',$$

and therefore,

$$ax'' + by'' + cz'' + dw'' = 0.$$

That is to say, if FF' be any chord of the hyperboloid, which passes through the fixed point κ , and if F'' be the harmonic conjugate of that fixed point, with respect to that variable chord, so that $(\kappa F F' F'') = -1$, then this conjugate point F'' is on the infinitely distant plane $[abcd]$: or in other words, the fixed point κ bisects all the chords FF' which pass through it, and is therefore (as above asserted) the centre of the surface.

(10.) With the same meanings (65, 79) of the constants a, b, c, d , the mean point (96) of the quadrilateral $ABCD$, or of the system of its corners, may be denoted by the symbol,

$$\mu = (a^{-1}, b^{-1}, c^{-1}, d^{-1});$$

if then this mean point be on the surface, so that

$$ac - bd = 0,$$

the centre κ is on the plane $[a, b, c, d]$; or in other words, it is infinitely distant: so

that the surface becomes, in this case, a *ruled (or hyperbolic) paraboloid*. In general (comp. sub-art. 8), if $r_0 = 0$, the surface of the second order is a *paraboloid* of some kind, because its centre is then at *infinity*, in virtue of the equation

$$(aD_x + bD_y + cD_z + dD_4)r_0 = 0;$$

or because (comp. 50, 58) the *plane [abcd]* at *infinity* is then one of its *tangent planes*, as satisfying its *tangential equation*, $r = 0$.

(11.) It is evident that a *curve in space* may be represented by a system of *two anharmonic and local equations*; because it may be regarded as the *intersection of two surfaces*. And then its *order*, or the *number of points* (real or imaginary*), in which it is cut by an *arbitrary plane*, is obviously the *product* of the *orders* of those two surfaces; or the *product of the degrees* of their two local equations, supposed to be rational and integral.

(12.) A *curve of double curvature* may also be considered as the *edge of regression* (or *arête de rebroussement*) of a *developable surface*, namely of the *locus of the tangents to the curve*; and this surface may be supposed to be *circumscribed* at once to *two given surfaces*, which are *envelopes of variable planes* (98), and are represented, as such, by their *tangential equations*. In this view, a *curve of double curvature* may itself be represented by a system of *two anharmonic and tangential equations*; and if the *class* of such a curve be defined to be the *number of its osculating planes*, which pass through an *arbitrary point of space*, then this class is the *product of the classes* of the *two curved surfaces* just now mentioned: or (what comes to the same thing) it is the *product of the dimensions* of the two *tangential equations*, by which the curve is (on this plan) symbolized. But we cannot enter further into these details; the *mechanism* of calculation respecting which would indeed be found to be the same, as that employed in the known method (comp. 41) of *quadriplanar co-ordinates*.

99. Instead of anharmonic co-ordinates, we may consider *any other system* of *n variable scalars*, x_1, \dots, x_n , which enter into the expression of a *variable vector*, ρ ; for example, into an expression of the form (comp. 96, 97),

$$\rho = x_1 a_1 + x_2 a_2 + \dots = \sum x a.$$

And then, if those *n scalars* x be all *functions of one independent and variable scalar*, t , we may regard this *vector* ρ as being *itself a function* of that *single scalar*; and may write,

$$\text{I. } \rho = \phi(t).$$

But if the *n scalars* $x \dots$ be *functions of two independent and scalar variables*, t and u , then ρ becomes a *function* of those *two scalars*, and we may write accordingly,

$$\text{II. } \rho = \phi(t, u).$$

In the 1st case, the *term* r (comp. 1) of the *variable vector* ρ has

* Compare the Notes to page 90.

generally for its locus a *curve in space*, which may be plane or of double curvature, or may even become a *right line*, according to the form of the vector-function ρ ; and ρ may be said to be the *vector of this line, or curve*. In the IInd case, ρ is the *vector of a surface*, plane or curved, according to the form of $\rho(t, u)$; or to the manner in which this vector ρ depends on the *two independent scalars* that enter into its expression.

(1.) As Examples (comp. 25, 63), the expressions,

$$\text{I. } \dots \rho = \frac{\alpha + t\beta}{1+t}; \quad \text{II. } \dots \rho = \frac{\alpha + t\beta + u\gamma}{1+t+u},$$

signify, Ist, that ρ is the vector of a variable point P on the right line AB; or that it is the vector of that line itself, considered as the locus of a point; and IInd, that ρ is the vector of the plane ABC, considered in like manner as the locus of an arbitrary point P thereon.

(2.) The equations,

$$\text{I. } \dots \rho = x\alpha + y\beta, \quad \text{II. } \dots \rho = x\alpha + y\beta + z\gamma,$$

with

$$x^2 + y^2 = 1 \text{ for the Ist, and } x^2 + y^2 + z^2 = 1 \text{ for the IInd,}$$

signify Ist, that ρ is the vector of an ellipse, and IInd, that it is the vector of an ellipsoid, with the origia o for their common centre, and with OA, OB, or OA, OB, OC, for conjugate semi-diameters.

(3.) The equation (comp. 46),

$$\rho = t^2\alpha + u^2\beta + (t+u)^2\gamma,$$

expresses that ρ is the vector of a cone of the second order, with o for its vertex (or centre), which is touched by the three planes OBC, OCA, OAB; the section of this cone, made by the plane ABC, being an ellipse (comp. Fig. 25), which is inscribed in the triangle ABC; and the middle points A', B', C', of the sides of that triangle, being the points of contact of those sides with that conic.

(4.) The equation (comp. 53),

$$\rho = t^{-1}\alpha + u^{-1}\beta + v^{-1}\gamma, \text{ with } t+u+v=0,$$

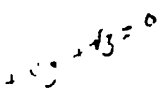
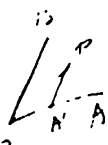
expresses that ρ is the vector of another cone of the second order, with o still for vertex, but with OA, OB, OC for three of its sides (or rays). The section by the plane ABC is a new ellipse, circumscribed to the triangle ABC, and having its tangents at the corners of that triangle respectively parallel to the opposite sides thereof.

(5.) The equation (comp. 54),

$$\rho = t^3\alpha + u^3\beta + v^3\gamma, \text{ with } t+u+v=0,$$

signifies that ρ is the vector of a cone of the third order, of which the vertex is still the origin; its section (comp. Fig. 27) by the plane ABC being a cubic curve, whereof the sides of the triangle ABC are at once the asymptotes, and the three (real) tangents of inflexion; while the mean point (say o') of that triangle is a conjugate point of the curve; and therefore the right line oo', from the vertex o to that mean point, may be said to be a conjugate ray of the cone.

(6.) The equation (comp. 93, sub-art. (8.)),



$$\rho = \frac{sta + t\beta + uv\gamma + vs\delta}{st + tu + uv + vs},$$

in which $\frac{s}{u}$ and $\frac{t}{v}$ are two variable scalars, while a, b, c, d are still four constant scalars, and $\alpha, \beta, \gamma, \delta$ are four constant vectors, but ρ is still a variable vector, expresses that ρ is the vector of a ruled (or single-sheeted) hyperboloid, on which the gauche quadrilateral ABCD is superscribed, and which passes through the given point κ , whereof the vector ϵ is assigned in 65.

(7.) If we make (comp. 98, sub-art (9)),

$$\rho' = \frac{s't'a\alpha - t'u'b\beta + u'v'c\gamma - v's'd\delta}{s't'a - t'u'b + u'v'c - v's'd},$$

where

$$s' = bt + cv, \quad t' = cu + ds, \quad u' = dv + at, \quad v' = as + bu,$$

then $\rho' = \text{OP}'$ is the vector of another point P' on the same hyperboloid; and because it is found that the sum of these two last vectors is constant,

$$\rho + \rho' = 2\kappa, \text{ if } \kappa = \frac{ac(\alpha + \gamma) - bd(\beta + \delta)}{2(ac - bd)},$$

it follows that κ is the vector of a fixed point κ , which bisects every chord PP' that passes through it: or in other words (comp. 52), that this point κ is the centre of the surface.

(8.) The three vectors,

$$\kappa, \quad \frac{\alpha + \gamma}{2}, \quad \frac{\beta + \delta}{2},$$

are *termino-collinear* (24); if then a gauche quadrilateral ABCD be superscribed on a ruled hyperboloid, the common bisector of the two diagonals, AC, BD, passes through the centre κ .

(9.) When $ac = bd$, or when we have the equation,

$$\rho = \frac{sta + t\beta + uv\gamma + vs\delta}{st + tu + uv + vs},$$

or simply,

$$\rho = sta + t\beta + uv\gamma + vs\delta, \text{ with } s + u = t + v = 1,$$

ρ is then the vector of a ruled paraboloid, of which the centre (comp. 52, and 98, sub-art. (10.)), is infinitely distant, but upon which the quadrilateral ABCD is still superscribed. And this surface passes through the mean point μ of that quadrilateral, or of the system of the four given points A...D; because, when $s = t = u = v = \frac{1}{4}$, the variable vector ρ takes the value (comp. 96, sub-art. (1.)),

$$\mu = \frac{1}{4}(\alpha + \beta + \gamma + \delta).$$

(10.) In general, it is easy to prove, from the last vector-expression for ρ , that this paraboloid is the locus of a right line, which divides similarly the two opposite sides AB and DC of the same gauche quadrilateral ABCD; or the other pair of opposite sides, BC and AD.

SECTION 7.—On Differentials of Vectors.

100. The equation (99, I.),

$$\rho = \phi(t),$$

in which $\rho = \text{OP}$ is generally the vector of a point P of a curve in space, $\text{PQ} \dots$, gives evidently, for the vector OQ of another point Q of the same curve, an expression of the form

$$\rho + \Delta\rho = \phi(t + \Delta t);$$

so that the chord PQ , regarded as being itself a vector, comes thus to be represented (4) by the finite difference,

$$\text{PQ} = \Delta\rho = \Delta\phi(t) = \phi(t + \Delta t) - \phi(t).$$

Suppose now that the other finite difference, Δt , is the n^{th} part of a new scalar, u ; and that the chord $\Delta\rho$, or PQ , is in like manner (comp. Fig. 32), the n^{th} part of a new vector, σ_n , or PR ; so that we may write,

$$n\Delta t = u, \text{ and } n\Delta\rho = n \cdot \text{PQ} = \sigma_n = \text{PR}.$$

Then, if we treat the two scalars, t and u , as constant, but the number n as variable (the form of the vector-function ϕ , and the origin o , being given), the vector ρ and the point P will be fixed: but the two points Q and R, the two differences Δt and $\Delta\rho$, and the multiple vector $n\Delta\rho$, or σ_n , will (in general) vary together. And if this number n be indefinitely increased, or made to tend to infinity, then each of the two differences Δt , $\Delta\rho$ will in general tend to zero; such being the common limit, of $n^{-1}u$, and of $\phi(t + n^{-1}u) - \phi(t)$: so that the variable point Q of the curve will tend to coincide with the fixed point P. But although the chord PQ will thus be indefinitely shortened, its n^{th} multiple, PR or σ_n , will tend (generally) to a finite limit,* depending on the supposed continuity of the function $\phi(t)$; namely, to a certain definite vector, PT , or σ_∞ , or (say) τ , which vector PT will evidently be (in general) tangential to the curve: or, in other words, the variable point R will tend to a fixed position T, on the tangent to that curve at P. We shall thus have a limiting equation, of the form

$$\tau = \text{PT} = \lim. \text{PR} = \sigma_\infty = \lim_{n \rightarrow \infty} n\Delta\phi(t), \text{ if } n\Delta t = u;$$

t and u being, as above, two given and (generally) finite scalars. And

* Compare Newton's *Principia*.

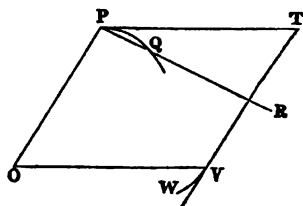


Fig. 32.

if we then agree to call the *second* of these two given scalars the *differential* of the first, and to denote it by the symbol dt , we shall define that the *vector-limit*, τ or σ_∞ , is the (corresponding) *differential of the vector* ρ , and shall denote it by the *corresponding symbol*, $d\rho$; so as to have, under the supposed conditions,

$$u = dt, \text{ and } \tau = d\rho.$$

Or, eliminating the two symbols u and τ , and *not necessarily supposing* that P is a *point of a curve*, we may express our *Definition** of the *Differential of a Vector* ρ , considered as a *Function* ϕ of a *Scalar* t , by the following *General Formula* :

$$d\rho = d\phi(t) = \lim_{n \rightarrow \infty} n \left\{ \phi\left(t + \frac{dt}{n}\right) - \phi(t) \right\},$$

in which t and dt are *two arbitrary and independent scalars*, both *generally finite*; and $d\rho$ is, *in general*, a *new and finite vector*, depending on those two scalars, according to a *law expressed by the formula*, and *derived from that given law*, whereby the *old or former vector*, ρ or $\phi(t)$, depends upon the *single scalar*, t .

(1.) As an example, let the given *vector-function* have the form,

$$\rho = \phi(t) = \frac{1}{2}t^2\alpha, \text{ where } \alpha \text{ is a given vector.}$$

Then, making $\Delta t = \frac{u}{n}$, where u is *any given scalar*, and n is a *variable whole number*, we have

$$\Delta\rho = \Delta\phi(t) = \frac{\alpha}{2} \left\{ \left(t + \frac{u}{n} \right)^2 - t^2 \right\} = \frac{\alpha u}{n} \left(t + \frac{u}{2n} \right);$$

$$\sigma_n = n\Delta\rho = \alpha u \left(t + \frac{u}{2n} \right); \quad \sigma_\infty = \alpha t u;$$

and finally, writing dt and $d\rho$ for u and σ_∞ ,

$$d\rho = d\phi(t) = d\left(\frac{t^2\alpha}{2}\right) = \alpha t dt.$$

(2.) In general, let $\phi(t) = \alpha f(t)$, where α is still a given or *constant vector*, and $f(t)$ denotes a *scalar function of the scalar variable*, t . Then because α is a *common factor* within the brackets $\{ \}$ of the recent general formula (100) for $d\rho$, we may write,

$$d\rho = d\phi(t) = d.\alpha f(t) = \alpha df(t);$$

provided that we now define that the *differential of a scalar function*, $f(t)$, is a *new scalar function of two independent scalars*, t and dt , determined by the precisely similar formula :

$$df(t) = \lim_{n \rightarrow \infty} n \left\{ f\left(t + \frac{dt}{n}\right) - f(t) \right\};$$

* Compare the Note to page 89.

which can easily be proved to agree, in all its consequences, with the usual rules for differentiating functions of one variable.

(3.) For example, if we write $dt = n\lambda$, where λ is a new variable scalar, namely, the n^{th} part of the given and (generally) finite differential, dt , we shall thus have the equation,

$$\frac{df(t)}{dt} = \lim_{\lambda \rightarrow 0} \frac{f(t + \lambda) - f(t)}{\lambda};$$

in which the first member is here considered as the actual quotient of two finite scalars, $df(t):dt$, and not merely as a differential coefficient. We may, however, as usual, consider this quotient, from the expression of which the differential dt disappears, as a derived function of the former variable, t ; and may denote it, as such, by either of the two usual symbols,

$$f'(t) \text{ and } D_t f(t).$$

(4.) In like manner we may write, for the derivative of a vector-function,* $\phi(t)$, the formula :

$$\rho' = \phi'(t) = D_t \rho = D_t \phi(t) = \frac{d\rho}{dt} = \frac{d\phi(t)}{dt};$$

these two last forms denoting that actual and finite vector, ρ' or $\phi'(t)$, which is obtained, or derived, by dividing (comp. 16) the not less actual (or finite) vector, $d\rho$ or $d\phi(t)$, by the finite scalar, dt . And if again we denote the n^{th} part of this last scalar by λ , we shall thus have the equally general formula :

$$D_t \rho = D_t \phi(t) = \lim_{\lambda \rightarrow 0} \frac{\phi(t + \lambda) - \phi(t)}{\lambda};$$

with the equations,

$$d\rho = D_t \rho \cdot dt = \rho' dt; \quad d\phi(t) = D_t \phi(t) \cdot dt = \phi'(t) \cdot dt,$$

exactly as if the vector-function, ρ or ϕ , were a scalar function, f .

(5.) The particular value, $dt = 1$, gives thus $d\rho = \rho'$; so that the derived vector ρ' is (with our definitions) a particular but important case of the differential of a vector. In applications to mechanics, if t denote the time, and if the term ρ of the variable vector ρ be considered as a moving point, this derived vector ρ' may be called the Vector of Velocity: because its length represents the amount, and its direction is the direction of the velocity. And if, by setting off vectors $OV = \rho'$ (comp. again Fig. 82) from one origin, to represent thus the velocities of a point moving in space according to any supposed law, expressed by the equation $\rho = \phi(t)$, we construct a new curve $vw\dots$ of which the corresponding equation may be written as $\rho' = \phi'(t)$, then this new curve has been defined to be the HODOGRAPH,† as the old curve $PQ\dots$ may be called the orbit of the motion, or of the moving point.

* In the theory of *Differentials of Functions of Quaternions*, a definition of the differential $d\phi(q)$ will be proposed, which is expressed by an equation of precisely the same form as those above assigned, for $df(t)$, and for $d\phi(t)$; but it will be found that, for quaternions, the quotient $d\phi(q):dq$ is not generally independent of dq ; and consequently that it cannot properly be called a derived function, such as $\phi'(q)$, of the quaternion q alone. (Compare again the Note to page 89.)

† The subject of the *Hodograph* will be resumed, at a subsequent stage of this work. In fact, it almost requires the assistance of *Quaternions*, to connect it, in what appears to be the best mode, with Newton's Law of Gravitation.

(6.) We may differentiate a vector-function twice (or oftener), and so obtain its successive differentials. For example, if we differentiate the derived vector ρ' , we obtain a result of the form,

$$d\rho' = \rho'' dt, \text{ where } \rho'' = D\rho' = D^2\rho,$$

by an obvious extension of notation; and if we suppose that the second differential, ddt or d^2t , of the scalar t is zero, then the second differential of the vector ρ is,

$$d^2\rho = dd\rho = d.\rho' dt = d\rho' . dt = \rho'' . dt^2;$$

where dt^2 , as usual, denotes $(dt)^2$; and where it is important to observe that, with the definitions adopted, $d^2\rho$ is as finite a vector as $d\rho$, or as ρ itself. In applications to motion, if t denote the time, ρ'' may be said to be the Vector of Acceleration.

(7.) We may also say that, in mechanics, the finite differential $d\rho$, of the Vector of Position ρ , represents, in length and in direction, the right line (suppose PR in Fig. 32) which would have been described, by a freely moving point P , in the finite interval of time dt , immediately following the time t , if at the end of this time t all foreign forces had ceased to act.*

(8.) In geometry, if $\rho = \phi(t)$ be the equation of a curve of double curvature, regarded as the edge of regression (comp. 98, (12.)) of a developable surface, then the equation of that surface itself, considered as the locus of the tangents to the curve, may be thus written (comp. 99, II.):

$$\rho = \phi(t) + u\phi'(t); \text{ or simply, } \rho = \phi(t) + d\phi(t),$$

if it be remembered that u , or dt , may be any arbitrary scalar.

(9.) If any other curved surface (comp. again 99, II.) be represented by an equation of the form, $\rho = \phi(x, y)$, where ϕ now denotes a vector-function of two independent and scalar variables, x and y , we may then differentiate this equation, or this expression for ρ , with respect to either variable separately, and so obtain what may be called two partial (but finite) differentials, $d_x\rho$, $d_y\rho$, and two partial derivatives, $D_x\rho$, $D_y\rho$, whereof the former are connected with the latter, and with the two arbitrary (but finite) scalars, dx , dy , by the relations,

$$d_x\rho = D_x\rho . dx; \quad d_y\rho = D_y\rho . dy.$$

And these two differentials (or derivatives) of the vector ρ of the surface denote two tangential vectors, or at least two vectors parallel to two tangents to that surface at the point P : so that their plane is (or is parallel to) the tangent plane at that point.

(10.) The mechanism of all such differentiations of vector-functions is, at the present stage, precisely the same as in the usual processes of the Differential Calculus; because the most general form of such a vector-function, which has been considered in the present Book, is that of a sum of products (comp. 99) of the form xa , where a is a constant vector, and x is a variable scalar: so that we have only to operate on these scalar coefficients x . . . by the usual rules of the calculus, the vectors a . . . being treated as constant factors (comp. sub-art. 2). But when we shall come to consider quotients or products of vectors, or generally those new functions of vectors which can only be expressed (in our system) by Quaternions, then some few new rules of differentiation become necessary, although deduced from the same (or nearly the same) definitions, as those which have been established in the present Section.

* As is well illustrated by Atwood's machine.

(11.) As an example of *partial differentiation* (comp. sub-art. 9), of a *vector function* (the word "vector" being *here* used as an *adjective*) of *two scalar variables*, let us take the equation,

$$\rho = \phi(x, y) = \frac{1}{2}\{x^2\alpha + y^2\beta + (x+y)^2\gamma\};$$

in which ρ (comp. 99, (8.)) is the vector of a certain *cone of the second order*; or more precisely, the vector of *one sheet* of such a cone, if x and y be supposed to be *real scalars*. Here, the two partial derivatives of ρ are the following:

$$D_x\rho = x\alpha + (x+y)\gamma; \quad D_y\rho = y\beta + (x+y)\gamma;$$

and therefore,

$$2\rho = xD_x\rho + yD_y\rho;$$

so that the *three vectors*, ρ , $D_x\rho$, $D_y\rho$, if drawn (18) from one *common origin*, are contained (22) in one *common plane*; which implies that the *tangent plane* to the surface, at any point P , passes *through the origin* O : and thereby verifies the *conical character* of the *locus* of that point P , in which the *variable vector* ρ , or OP , *terminates*.

(12.) If, in the same example, we make $x = 1$, $y = -1$, we have the values,

$$\rho = \frac{1}{2}(\alpha + \beta), \quad D_x\rho = \alpha, \quad D_y\rho = -\beta;$$

whence it follows that the middle point, say C' , of the right line AB , is one of the points of the conical locus; and that (comp. again the sub-art. 8 to Art. 99, and the recent sub-art. 9) the right lines OA and OB are parallel to two of the tangents to the surface at that point; so that the *cone* in question is *touched by the plane* AOB , *along the side* (or ray) OC' . And in like manner it may be proved, that the *same cone* is touched by the *two other planes*, BOC and COA , at the middle points A' and B' of the *two other lines* BC and CA ; and therefore along the *two other sides* (or rays), OA' and OB' : which again agrees with former results.

(13.) It will be found that a *vector function* of the *sum of two scalar variables*, t and dt , may generally be *developed*, by an extension of *Taylor's Series*, under the form,

$$\begin{aligned} \phi(t + dt) &= \phi(t) + d\phi(t) + \frac{1}{2}d^2\phi(t) + \frac{1}{2.3}d^3\phi(t) + \dots \\ &= \left(1 + d + \frac{d^2}{2} + \frac{d^3}{2.3} + \dots\right)\phi(t) = e^d\phi(t); \end{aligned}$$

it being supposed that $d^2t = 0$, $d^3t = 0$, &c. (comp. sub-art. 6). Thus, if $\phi t = \frac{1}{2}at^2$, (as in sub-art. 1), where a is a *constant vector*, we have $d\phi t = atdt$, $d^2\phi t = adt^2$, $d^3\phi t = 0$, &c.; and

$$\phi(t + dt) = \frac{1}{2}a(t + dt)^2 = \frac{1}{2}at^2 + atdt + \frac{1}{2}adt^2,$$

rigorously, without any supposition that dt is *small*.

(14.) When we thus suppose $\Delta t = dt$, and *develop* the *finite difference*, $\Delta\phi(t) = \phi(t + dt) - \phi(t)$, the *first term* of the development so obtained, or the term of *first dimension* relatively to dt , is hence (by a *theorem*, which holds good for *vector-functions*, as well as for *scalar functions*) the *first differential* $d\phi t$ of the function; but we do not choose to *define* that this *Differential* is (or means) that *first term*: because the Formula (100), which we prefer, does not postulate the *possibility*, nor even suppose the *conception*, of any such *development*. Many recent remarks will perhaps appear more clear, when we shall come to *connect* them, at a later stage, with that theory of *Quaternions*, to which we next proceed.

BOOK II.

ON QUATERNIONS, CONSIDERED AS QUOTIENTS OF VECTORS,
AND AS INVOLVING ANGULAR RELATIONS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING QUOTIENTS OF VECTORS.

SECTION I.—*Introductory Remarks; First Principles adopted from Algebra.*

ART. 101. The only *angular relations*, considered in the foregoing Book, have been those of *parallelism* between *vectors* (Art. 2, &c.); and the only *quotients*, hitherto employed, have been of the three following kinds:

I. *Scalar quotients of scalars*, such as the *arithmetical fraction* $\frac{n}{m}$ in Art. 14;

II. *Vector quotients*, of *vectors divided by scalars*, as $\frac{\beta}{x} = a$ in Art. 16;

III. *Scalar quotients of vectors*, with *directions* either *similar* or *opposite*, as $\frac{\beta}{a} = x$ in the last cited Article. But we now propose to treat of *other geometric QUOTIENTS* (or *geometric Fractions*, as we shall also call them), such as

$$\frac{OB}{OA} = \frac{\beta}{a} = q, \text{ with } \beta \text{ not } \parallel a \text{ (comp. 15);}$$

for each of which the *Divisor* (or *denominator*), a or OA , and the *Dividend* (or *numerator*), β or OB , shall not only *both* be

Vectors, but shall also be *inclined* to each other at an **ANGLE**, *distinct* (in general) from *zero*, and from *two** *right angles*.

102. In introducing this *new conception*, of a *General Quotient of Vectors*, with *Angular Relations* in a given plane, or in space, it will obviously be necessary to employ some properties of *circles* and *spheres*, which were not wanted for the purpose of the former Book. But, on the other hand, it will be possible and useful to suppose a much less degree of acquaintance with many important theories† of *modern geometry*, than that of which the possession was assumed, in several of the foregoing Sections. Indeed it is hoped that a very moderate amount of geometrical, algebraical, and trigonometrical preparation will be found sufficient to render the present Book, as well as the early parts of the preceding one, fully and easily intelligible to any attentive reader.

103. It may be proper to premise a few general principles respecting quotients of vectors, which are indeed *suggested* by *algebra*, but are here *adopted* by *definition*. And 1st, it is evident that the *supposed operation* of *division* (whatever its *full geometrical import* may afterwards be found to be), by which we here conceive ourselves to pass from a given *divisor-line* a , and from a given *dividend-line* β , to what we have called (provisionally) their *geometric quotient*, q , may (or rather must) be *conceived* to correspond to *some converse act* (as yet not *fully* known) of *geometrical multiplication*: in which new act the former *quotient*, q , becomes a **FACTOR**, and *operates on the line* a , so as to *produce* (or *generate*) the *line* β . We shall therefore *write*, as in algebra,

$$\beta = q \cdot a, \text{ or simply, } \beta = qa, \text{ when } \beta : a = q ;$$

* More generally speaking, from *every even multiple* of a *right angle*.

† Such as *homology*, *homography*, *involution*, and generally whatever depends on *anharmonic ratio*: although all that is needful to be known respecting such ratio, for the applications subsequently made, may be learned, without reference to any other treatise, from the *definitions* incidentally given, in Art. 25, &c. It was, perhaps, not strictly *necessary* to introduce any of these modern geometrical theories, in any part of the present work; but it was thought that it might interest one class, at least, of students, to see how they could be *combined* with that fundamental *conception* of the **VECTOR**, which the First Book was designed to develop.

even if the two lines α and β , or OA and OB , be supposed to be inclined to each other, as in Fig. 33. And this very simple and natural notation (comp. 16) will then allow us to treat as identities the two following formulæ :

$$\left(\frac{\beta}{\alpha} \cdot a\right) \frac{\beta}{\alpha} a = \beta; \quad \frac{qa}{a} = q;$$

although we shall, for the present, abstain from writing also such formulæ* as the following:

$$\frac{\beta a}{a} = \beta, \quad \frac{q}{a} a = q,$$

where α , β still denote two vectors, and q denotes their geometrical quotient: because we have not yet even begun to consider the multiplication of one vector by another, or the division of a quotient by a line.

104. As a IInd general principle, suggested by algebra, we shall next lay it down, that if

$$\frac{\beta'}{a'} = \frac{\beta}{a}, \quad \text{and} \quad a' = a, \quad \text{then} \quad \beta' = \beta;$$

or in words, and under a slightly varied form, that *unequal vectors, divided by equal vectors, give unequal quotients*. The importance of this very natural and obvious assumption will soon be seen in its applications.

105. As a IIIrd principle, which indeed may be considered to pervade the whole of *mathematical language*, and without adopting which we could not usefully speak, in any case, of EQUALITY as existing between any two geometrical quotients, we shall next assume that two such quotients can never be equal to the same third† quotient, without being at the same time equal to each other: or in symbols, that

$$\text{if } q' = q, \quad \text{and} \quad q'' = q, \quad \text{then} \quad q'' = q'.$$

* It will be seen, however, at a later stage, that these two formulæ are permitted, and even required, in the development of the Quaternion System.

† It is scarcely necessary to add, what is indeed included in this IIIrd principle, in virtue of the identity $q = q$, that if $q' = q$, then $q = q'$; or in words, that we shall never admit that any two geometrical quotients, q and q' , are equal to each other in one order, without at the same time admitting that they are equal, in the opposite order also.

106. In the IVth place, as a preparation for *operations on geometrical quotients*, we shall say that any two such **QUOTIENTS**, or *fractions* (101), which have a *common divisor-line*, or (in more familiar words) a *common denominator*, are *added*, *subtracted*, or *divided*, among themselves, by adding, subtracting, or dividing their *numerators*: the common denominator being *retained*, in each of the two former of these three cases. In symbols, we thus define (comp. 14) that, for any three (actual) *vectors*, a , β , γ ,

$$\frac{\gamma}{a} + \frac{\beta}{a} = \frac{\gamma + \beta}{a}; \quad \frac{\gamma}{a} - \frac{\beta}{a} = \frac{\gamma - \beta}{a};$$

and

$$\frac{\gamma}{a} : \frac{\beta}{a} = \frac{\gamma}{\beta};$$

aiming still at agreement with algebra.

107. Finally, as a Vth principle, designed (like the foregoing) to assimilate, so far as can be done, the present Calculus to Algebra, in its *operations* on geometrical quotients, we shall define that the following formula holds good :

$$\left(\frac{\gamma}{\beta} \cdot \frac{\beta}{a} \right) = \frac{\gamma \beta}{\beta a} = \frac{\gamma}{a};$$

or that if two *geometrical fractions*, q and q' , be so related, that the *denominator*, β , of the *multiplier* q' (here written towards the left-hand) is equal to the *numerator* of the *multiplicand* q , then the *product*, $q' \cdot q$ or $q'q$, is that *third fraction*, whereof the numerator is the *numerator* γ of the *multiplier*, and the denominator is the *denominator* a of the *multiplicand*: all such denominators, or *divisor-lines*, being still supposed (16) to be *actual* (and not null) *vectors*.

SECTION 2.—*First Motive for naming the Quotient of two Vectors a Quaternion.*

108. Already we may see grounds for the application of the name, QUATERNION, to such a *Quotient of two Vectors* as has been spoken of in recent articles. In the first place, such a quotient cannot *generally* be what we have called (17) a SCA-

LAR: or in other words, it cannot generally be equal to any of the (so-called) *reals of algebra*, whether of the *positive* or of the *negative* kind. For let x denote any such (actual*) *scalar*, and let a denote any (actual) *vector*; then we have seen (15) that the *product* xa denotes *another* (actual) *vector*, say β' , which is either *similar* or *opposite* in direction to a , according as the scalar coefficient, or *factor*, x , is positive or negative; in *neither* case, then, can it represent any vector, such as β , which is *inclined* to a , at any actual *angle*, whether acute, or right, or obtuse: or in other words (comp. 2), the *equation* $\beta' = \beta$, or $xa = \beta$, is impossible, under the conditions here supposed. But we have agreed (16, 103) to write, as in algebra,

$\frac{xa}{a} = x$; we must, therefore (by the IIInd principle' of the foregoing Section, stated in Art. 104), *abstain* from writing also $\frac{\beta}{a} = x$, under the same conditions: x still denoting a *scalar*.

Whatever *else* a *quotient of two inclined vectors* may be found to be, it is thus, at least, a NON-SCALAR.

109. Now, in forming the conception of the *scalar itself*, as the *quotient of two parallel† vectors* (17), we took into account not only *relative length*, or *ratio* of the usual kind, but also *relative direction*, under the form of *similarity* or *opposition*. In passing from a to xa , we *altered* generally (15) the *length* of the line a , in the ratio of $\pm x$ to 1; and we *preserved* or *reversed* the *direction* of that line, according as the *scalar coefficient* x was *positive* or *negative*. And in like manner, in proceeding to form, more definitely than we have yet done, the conception of the *non-scalar quotient* (108), $q = \beta : a = OB : OA$, of *two inclined vectors*, which for simplicity may be supposed (18) to be co-

* By an *actual scalar*, as by an *actual vector* (comp. 1), we mean here one that is *different from zero*. An *actual vector*, multiplied by a *null scalar*, has for product (15) a *null vector*; it is therefore unnecessary to prove that the *quotient of two actual vectors* cannot be a *null scalar*, or zero.

† It is to be remembered that we have proposed (15) to extend the use of this term *parallel*, to the case of two vectors which are (in the *usual* sense of the word) parallel to one *common line*, even when they happen to be *parts* of one and the *same* right line.

initial, we have *still* to take account both of the *relative length*, and of the *relative direction*, of the two lines compared. But while the *former element* of the *complex relation* here considered, between these two lines or vectors, is *still* represented by a simple RATIO (of the kind commonly considered in geometry), or by a *number** expressing that ratio; the *latter element* of the same complex relation is *now* represented by an ANGLE, $\angle O B$: and not simply (as it was before) by an *algebraical sign*, + or -.

110. Again in estimating this *angle*, for the purpose of *distinguishing* one quotient of vectors from another, we must consider not only its *magnitude* (or *quantity*), but also its PLANE: since otherwise, in violation of the principle stated in Art. 104, we should have $OB' : OA = OB : OA$, if OB and OB' were *two distinct rays* or sides of a *cone* of revolution, with OA for its *axis*; in which case (by 2) they would necessarily be *unequal vectors*. For a similar reason, we must attend also to the *contrast* between two *opposite angles*, of equal magnitudes, and in one *common plane*. In short, for the purpose of knowing *fully* the *relative direction* of two co-initial lines OA , OB in *space*, we ought to know not only *how many degrees*, or other *parts* of some *angular unit*, the angle $\angle O B$ contains; but also (comp. Fig. 33) the *direction of the rotation* from OA to OB : including a knowledge of the *plane*, in which the rotation is performed; and of the *hand* (as *right* or *left*, when *viewed* from a *known side* of the plane), *towards which* the rotation is *directed*.

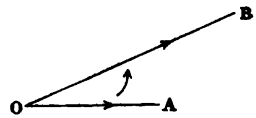


Fig. 33.

111. Or, if we agree to *select* some *one fixed hand* (suppose the *right* † hand), and to call all *rotations positive* when they

* This number, which we shall presently call the *tensor* of the quotient, may be *whole* or *fractional*, or even *incommensurable* with unity; but it may always be *equated*, in calculation, to a *positive scalar*: although it might perhaps more *properly* be said to be a *signless number*, as being derived solely from comparison of *lengths*, without any reference to *directions*.

† If *right-handed rotation* be thus considered as *positive*, then the *positive axis* of the rotation $\angle O B$, in Fig. 33, must be conceived to be directed *downward*, or *below* the plane of the paper.

are directed towards *this* selected hand, but all rotations *negative* when they are directed towards the *other hand*, then, for *any given angle* $\angle O B$, supposed for simplicity to be less than two right angles, and considered as representing a *rotation in a given plane* from OA to OB , we may speak of *one perpendicular* OC to that plane AOB as being the *positive axis* of that rotation; and of the *opposite perpendicular* OC' to the same plane as being the *negative axis* thereof: the rotation round the positive axis being *itself* positive, and *vice versa*. And then the *rotation* $\angle O B$ may be considered to be entirely *known*, if we know, Ist, its *quantity*, or the *ratio* which it bears to a *right rotation*; and IInd, the *direction* of its *positive axis*, OC : but not without a knowledge of these *two things*, or of some data equivalent to them. But whether we consider the *direction of an AXIS*, or the *aspect of a PLANE*, we find (as indeed is well known) that the *determination* of such a *direction*, or of such an *aspect*, depends on two *polar co-ordinates**, or other *angular elements*.

112. It appears, then, from the foregoing discussion, that for the *complete determination*, of what we have called the *geometrical QUOTIENT of two co-initial Vectors*, a *System of Four Elements*, admitting each separately of numerical expression, *is generally required*. Of these four elements, *one* serves (109) to determine the *relative length* of the two lines compared; and the other *three* are in general necessary, in order to determine *fully* their *relative direction*. Again, of these three latter elements, *one* represents the mutual *inclination*, or *elongation*, of the two lines; or the *magnitude* (or quantity) of the *angle* between them; while the *two others* serve to determine the *direction* of the *axis*, perpendicular to their common *plane*, round which a *rotation* through that angle is to be performed, in a *sense* previously selected as the *positive one* (or towards a fixed and previously selected *hand*), for the purpose of *passing* (in the simplest way, and therefore in the plane of the two lines) *from the direction* of the *divisor-line*, to the direction of

* The actual (or at least the frequent) use of such co-ordinates is foreign to the spirit of the present System: but the mention of them here seems likely to assist a student, by suggesting an appeal to results, with which his previous reading can scarcely fail to have rendered him familiar.

the *dividend-line*. And no more than four numerical elements are necessary, for our present purpose: because the *relative length* of two lines is not changed, when their two lengths are altered *proportionally*, nor is their *relative direction* changed, when the *angle* which they form is merely *turned about*, in its own plane. On account, then, of this *essential connexion* of that *complex relation* (109) between two lines, which is compounded of a *relation of lengths*, and of a *relation of directions*, and to which we have given (by an *extension* from the theory of *scalars*) the name of a *geometrical quotient*, with a *System of Four numerical Elements*, we have already a *motive** for saying, that “the *Quotient of two Vectors is generally a Quaternion.*”

SECTION 3.—Additional Illustrations.

113. Some additional light may be thrown, on this first *conception* of a *Quaternion*, by the annexed Figure 34. In that Figure, the letters CDEFG are designed to indicate corners of a *prismatic desk*, resting upon a *horizontal table*. The angle HCD (supposed to be one of *thirty degrees*) represents a (left-handed) *rotation*, whereby the horizontal *ledge* CD of the desk is conceived to be *elongated* (or

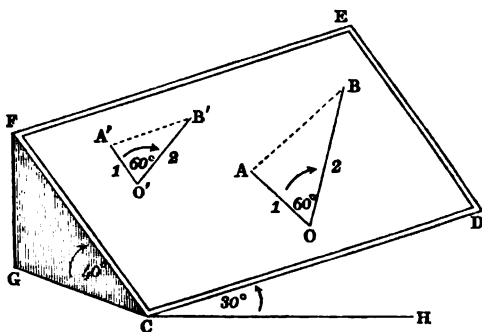


Fig. 84.

removed) from a *given horizontal line* CH, which may be imagined to be an *edge* of the table. The angle GCF (supposed here to contain *forty degrees*) represents the *slope*† of the desk, or the amount of its *inclination* to the table. On the *face* CDEF of the desk are drawn two *similar* and *similarly turned triangles*, AOB and A'O'B', which are supposed to be halves of two *equilateral triangles*; in such a manner that *each*

* Several other reasons for thus speaking will offer themselves, in the course of the present work.

† These two angles, HCD and GCF, may thus be considered to correspond to *longitude of node*, and *inclination of orbit*, of a planet or comet in astronomy.

rotation, $\angle AOB$ or $\angle A'O'B'$ is one of *sixty* degrees, and is directed towards *one common hand* (namely the *right* hand in the Figure): while if *lengths* alone be attended to, the *side* OB is to the side OA , in *one triangle*, as the side $O'B'$ is to the side $O'A'$, in the *other*; or as the *number two to one*.

114. Under these conditions of construction, we consider the *two quotients*, or the *two geometric fractions*,

$$OB:OA \text{ and } OB':OA', \text{ or } \frac{OB}{OA} \text{ and } \frac{O'B'}{O'A'}$$

as being *equal to each other*; because we regard the *two lines*, OA and OB , as having the *same relative length*, and the *same relative direction*, as the *two other lines*, $O'A'$ and $O'B'$. And we consider and speak of *each Quotient*, or *Fraction*, as a *Quaternion*: because its *complete construction* (or *determination*) depends, for all that is *essential* to its *conception*, and requisite to *distinguish* it from others, on a system of *four numerical elements* (comp. 112); which are, in this Example, the *four numbers*,

$$2, 60, 30, \text{ and } 40.$$

115. Of these *four elements* (to recapitulate what has been above supposed), the *Ist*, namely the number 2, expresses that the *length* of the *dividend-line*, OB or $O'B'$, is *double* of the *length* of the *divisor-line*, OA or $O'A'$. The *IInd* numerical element, namely 60, expresses here that the *angle* $\angle AOB$ or $\angle A'O'B'$, is one of *sixty degrees*; while the corresponding *rotation*, from OA to OB , or from $O'A'$ to $O'B'$, is *towards a known hand* (in this case the *right* hand, as seen by a person looking at the *face CDEF* of the desk), which *hand* is the *same* for both of these *two equal angles*. The *IIIrd* element, namely 30, expresses that the *horizontal ledge* CD of the desk makes an angle of *thirty degrees* with a known horizontal line CH , being removed from it, by that angular quantity, in a known direction (which in this case happens to be towards the *left* hand, as seen from above). Finally, the *IVth* element, namely 40, expresses here that the desk has an *elevation* of *forty degrees* as before.

116. Now an *alteration* in any one of these *Four Elements*, such as an alteration of the *slope* or *aspect* of the desk, would make (in the view here taken) an *essential change* in the *Quaternion*, which is (in the same view) the *Quotient of the two lines* compared: although (as the Figure is in part designed to suggest) *no such change* is conceived to take place, when the triangle $\triangle AOB$ is merely turned about, in its own plane, without being turned over (comp. Fig. 86); or when the *sides* of that triangle are *lengthened* or *shortened proportionally*, so as to *preserve the ratio* (in the old sense of that word), of any one to any other of those sides. We may then briefly say, in this mode of illustrating the notion of a QUATERNION* in geometry, by refe-

* As to the mere word, *Quaternion*, it signifies primarily (as is well known), like its Latin original, "Quaternio," or the Greek noun τετρακτύς, a *Set of Four*: but it is obviously used *hæc*, and elsewhere in the present work, in a *technical sense*.

rence to an *angle on a desk*, that the *Four Elements* which it involves are the following:

Ratio, Angle, Ledge, and Slope;

although the *two latter elements* are in fact *themselves angles also*, but are not immediately obtained as such, from the simple comparison of the *two lines*, of which the *Quaternion* is the *Quotient*.

SECTION 4.—*On Equality of Quaternions; and on the Plane of a Quaternion.*

117. It is an immediate consequence of the foregoing *conception* of a Quaternion, that *two quaternions*, or *two quotients of vectors*, supposed for simplicity to be all *co-initial* (18), are regarded as being **EQUAL** to each other, or that the **EQUATION**,

$$\frac{\delta}{\gamma} = \frac{\beta}{\alpha}, \quad \text{or} \quad \frac{OD}{OC} = \frac{OB}{OA},$$

is by us considered and *defined* to hold good, when the *two triangles*, $\triangle OAB$ and $\triangle OCD$, are *similar and similarly turned*, and in *one common plane*, as represented in the annexed Fig. 35: the **RELATIVE LENGTH** (109), and the **RELATIVE DIRECTION** (110), of the *two lines*, OA , OB , being then in all respects the *same* as the relative length and the relative direction of the *two other lines*, OC , OD .

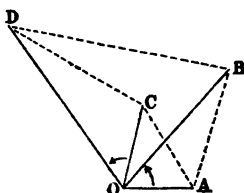


Fig. 35.

118. Under the same conditions, we shall write the following *formula of direct similitude*,

$$\triangle OAB \propto \triangle OCD;$$

reserving this *other formula*,

$$\triangle OAB \propto' \triangle OAB', \quad \text{or} \quad \triangle A'OB \propto' \triangle A'OB',$$

which we shall call a *formula of inverse similitude*, to denote that the two triangles, $\triangle OAB$ and $\triangle OAB'$, or $\triangle A'OB$ and $\triangle A'OB'$, although otherwise *similar* (and even, in this case, *equal**, on account of their having a *common side*, OA or OA'), are

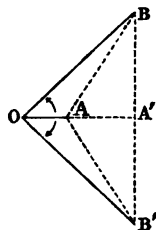


Fig. 36.

* That is to say, *equal* in *absolute amount of area*, but with *opposite algebraic signs* (28). The *two quotients* $OB : OA$, and $OB' : OA$, although *not equal* (110), will soon be defined to be *conjugate quaternions*. Under the same conditions, we shall write also the formula,

$$\triangle OAB' \propto' \triangle OAB.$$

oppositely turned (comp. Fig. 36), as if one were the *reflexion* of the other in a mirror; or as if the one triangle were *derived* (or generated) from the other, by a *rotation of its plane* through *two right angles*. We may therefore write,

$$\frac{OB}{OA} = \frac{OD}{OC}, \text{ if } \Delta AOB \propto COD.$$

119. When the vectors are thus all drawn from one common origin *o*, the *plane* *AOB* of any two of them may be called the *Plane of the Quaternion* (or of the Quotient), *OB : OA*; and of course also the plane of the *inverse* (or *reciprocal*) *quaternion* (or of the inverse quotient), *OA : OB*. And any two *quaternions*, which have a *common plane* (through *o*), may be said to be *Complanar** *Quaternions*, or *complanar quotients*, or *fractions*; but any two *quaternions* (or *quotients*), which have *different planes* (*intersecting* therefore in a right line through the origin), may be said, by contrast, to be *Diplanar*.

120. Any two *quaternions*, considered as *geometric fractions* (101), can be *reduced to a common denominator* without change of the *value*† of either of them, as follows. Let $\frac{OB}{OA}$ and $\frac{OD}{OC}$ be the two given fractions, or *quaternions*; and if they be *complanar* (119), let *OE* be any line in their *common plane*; but if they be *diplanar* (see again 119), then let *OE* be any assumed part of the *line of intersection* of the two planes: so that, in each case, the line *OE* is situated at once in the plane *AOB*, and also in the plane *COD*. We can then always conceive two other lines, *OF*, *OG*, to be determined so as to satisfy the two conditions of direct similitude (118),

$$\Delta EOF \propto AOB, \quad \Delta EOG \propto COD;$$

* It is, however, convenient to extend the use of this word, *complanar*, so as to include the case of *quaternions* represented by *angles in parallel planes*. Indeed, as all *vectors* which have *equal lengths*, and *similar directions*, are equal (2), so the *quaternion*, which is a *quotient of two such vectors*, ought not to be considered as undergoing any change, when either vector is merely changed in *position*, by a *transport without rotation*.

† That is to say, the *new* or *transformed* *quaternions* will be respectively *equal* to the old or *given* ones.

and therefore also the *two equations* between quotients (117, 118),

$$\frac{OF}{OE} = \frac{OB}{OA}, \quad \frac{OG}{OE} = \frac{OD}{OC};$$

and thus the required *reduction* is effected, OE being the *common denominator* sought, while OF, OG are the new or *reduced numerators*. It may be added that if H be a new point in the plane AOB, such that $\Delta HOE \propto \Delta AOB$, we shall have also,

$$\frac{OE}{OH} = \frac{OB}{OA} = \frac{OF}{OE};$$

and therefore, by 106, 107,

$$\frac{OD}{OC} \pm \frac{OB}{OA} = \frac{OG \pm OF}{OE}; \quad \frac{OD}{OC} : \frac{OB}{OA} = \frac{OG}{OF}; \quad \frac{OD}{OC} \cdot \frac{OB}{OA} = \frac{OG}{OH};$$

whatever *two geometric quotients* (complanar or diplanar) may be represented by $OB : OA$ and $OD : OC$.

121. If now the two triangles AOB, COD are not only *complanar* but *directly similar* (118), so that $\Delta AOB \propto \Delta COD$, we shall evidently have $\Delta EOF \propto \Delta EOG$; so that we may write $OF = OG$ (or $F = G$, by 20), the *two new lines* OF, OG (or the *two new points* F, G) in this case *coinciding*. The general construction (120), for the reduction to a common denominator, gives therefore here only *one new triangle*, EOF, and *one new quotient*, $OF : OE$, to which in this case *each* (comp. 105) of the *two given equal and complanar quotients*, $OB : OA$ and $OD : OC$, is equal.

122. But if these two latter symbols (or the *fractional forms* corresponding) denote *two diplanar** quotients, then the *two new numerator-lines*, OF and OG, have *different directions*, as being situated in *two different planes*, drawn through the *new denominator-line* OE, without having either the direction of that line *itself*, or the direction *opposite* thereto; they are therefore (by 2) *unequal vectors*, even if they should happen to be *equally long*; whence it follows (by 104) that the *two new quotients*, and therefore also (by 105) that the *two old or given quotients*, are *unequal*, as a consequence of their *diplanarity*.

* And therefore *non scalar* (108); for a *scalar*, considered as a *quotient* (17), has no *determined plane*, but must be considered as *complanar with every geometric quotient*; since it may be represented (or constructed) by the quotient of two similarly or oppositely directed lines, in *any proposed plane* whatever.

It results, then, from this analysis, that *diplanar quotients of vectors*, and therefore that *Diplanar Quaternions* (119), are *always unequal*; a new and comparatively *technical* process thus *confirming* the conclusion, to which we had arrived by general considerations, and in (what might be called) a *popular* way before, and which we had sought to *illustrate* (comp. Fig. 34) by the consideration of *angles on a desk*: namely, that a *Quaternion*, considered as the quotient of *two mutually inclined lines in space*, involves generally a *Plane*, as an *essential part* (comp. 110) of its constitution, and as necessary to the *completeness* of its conception.

123. We propose to use the mark

$$\parallel\parallel$$

as a *Sign of Complanarity*, whether of *lines* or of *quotients*; thus we shall write the formula,

$$\gamma \parallel\parallel a, \beta,$$

to express that the *three vectors*, a, β, γ , supposed to be (or to be made) *co-initial* (18), are situated *in one plane*; and the analogous formula,

$$q' \parallel\parallel q, \quad \text{or} \quad \frac{\delta}{\gamma} \parallel\parallel \frac{\beta}{a},$$

to express that the *two quaternions*, denoted here by q and q' , and therefore that the *four vectors*, a, β, γ, δ , are *complanar* (119). And because we have just found (122) that *diplanar quotients are unequal*, we see that *one equation of quaternions includes two complanarities of vectors*; in such a manner that we may write,

$$\gamma \parallel\parallel a, \beta, \quad \text{and} \quad \delta \parallel\parallel a, \beta, \quad \text{if} \quad \frac{\delta}{\gamma} = \frac{\beta}{a};$$

the *equation of quotients*, $\frac{OD}{OC} = \frac{OB}{OA}$, being impossible, *unless all the four lines from o be in one common plane*. We shall also employ the notation

$$\gamma \parallel\parallel q,$$

to express that the *vector* γ is *in* (or *parallel to*) the *plane of the quaternion* q .

124. With the same notation for complanarity, we may write generally,

$$xa \parallel\parallel a, \beta;$$

a and β being any two vectors, and x being any scalar; because, if $a = OA$ and $\beta = OB$ as before, then (by 15, 17) $xa = OA'$, where A' is some point on the indefinite right line through the points O and A : so that the plane AOB contains the line OA' . For a similar reason, we have generally the following formula of complanarity of quotients,

$$\frac{y\beta}{xa} \parallel\parallel \frac{\beta}{a},$$

whatever two scalars x and y may be; a and β still denoting any two vectors.

125. It is evident (comp. Fig. 35) that

if $\triangle AOB \propto \triangle COD$, then $\triangle BOA \propto \triangle DOC$, and $\triangle AOC \propto \triangle BOD$; whence it is easy to infer that for quaternions, as well as for ordinary or algebraic quotients,

$$\text{if } \frac{\beta}{a} = \frac{\delta}{\gamma}, \text{ then, inversely, } \frac{a}{\beta} = \frac{\gamma}{\delta}, \text{ and alternately, } \frac{\gamma}{a} = \frac{\delta}{\beta};$$

it being permitted now to establish the converse of the last formula of 118, or to say that

$$\text{if } \frac{OB}{OA} = \frac{OD}{OC}, \text{ then } \triangle AOB \propto \triangle COD.$$

Under the same condition, by combining inversion with alternation, we have also this other equation, $\frac{a}{\gamma} = \frac{\beta}{\delta}$.

126. If the sides, OA , OB , of a triangle AOB , or those sides either way prolonged, be cut (as in Fig. 37) by any parallel, $A'B'$ or $A''B''$, to the base AB , we have evidently the relations of direct similarity (118),

$$\triangle A'OB' \propto \triangle AOB, \quad \triangle A''OB'' \propto \triangle AOB;$$

whence (comp. Art. 13 and Fig. 12) it follows that we may write, for quaternions as in algebra, the general equation, or identity,

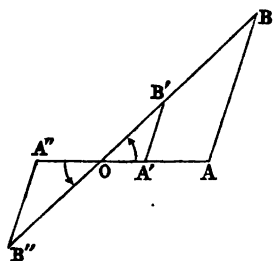


Fig. 37.

$$\frac{x\beta}{x\alpha} = \frac{\beta}{\alpha};$$

where x is again *any scalar*, and α, β are *any two vectors*. It is easy also to see, that for any quaternion q , and any scalar x , we have the *product* (comp. 107),

$$xq = \frac{x\beta}{\beta} \cdot \frac{\beta}{\alpha} = \frac{x\beta}{\alpha} = \frac{\beta}{x^{-1}\alpha} = \frac{\beta}{\alpha} \cdot \frac{\alpha}{x^{-1}\alpha} = qx;$$

so that, in the *multiplication of a quaternion by a scalar* (as in the multiplication of a *vector by a scalar*, 15), the *order of the factors is indifferent*.

SECTION 5.—*On the Axis and Angle of a Quaternion; and on the Index of a Right Quotient, or Quaternion.*

127. From what has been already said (111, 112), we are naturally led to define that the **AXIS**, or more fully that the *positive axis, of any quaternion* (or *geometric quotient*) $OB : OA$, is a *right line perpendicular to the plane $\triangle OAB$* of that quaternion; and is such that the *rotation round this axis, from the divisor-line OA , to the dividend-line OB , is positive*: or (as we shall henceforth assume) *directed towards the right-hand,** like the motion of the hands of a watch.

128. To render still more *definite* this conception of the *axis of a quaternion*, we may add, Ist, that the *rotation*, here spoken of, is supposed (112) to be the *simplest possible*, and therefore to be *in the plane of the two lines* (or of the quaternion), being also *generally less than a semi-revolution* in that plane; IInd, that the axis shall be usually supposed to be a line ox drawn *from the assumed origin o* ; and IIIrd, that the *length* of this line shall be supposed to be *given, or fixed*, and to be equal to some assumed *unit* of length: so that the *term x* , of this *axis ox* , is situated (by its construction) *on a given spheric surface* described about the *origin o* as *centre*, which surface we may call the **SURFACE OF THE UNIT-SPHERE**.

129. In this manner, for every given *non-scalar quotient*

* This is, of course, merely conventional, and the reader may (if he pleases) substitute the *left-hand* throughout.

(108), or for every given *quaternion* q which does not reduce itself (or degenerate) to a mere *positive or negative number*, the *axis* will be an entirely *definite vector*, which may be called an **UNIT-VECTOR**, on account of its assumed *length*, and which we shall *denote**, for the present, by the *symbol* $\text{Ax} \cdot q$. Employing then the usual *sign of perpendicularity*, \perp , we may now write, for any two vectors a, β , the formula :

$$\text{Ax} \cdot \frac{\beta}{a} \perp a; \quad \text{Ax} \cdot \frac{\beta}{a} \perp \beta; \quad \text{or briefly,} \quad \text{Ax} \cdot \frac{\beta}{a} \perp \left\{ \frac{\beta}{a} \right\}.$$

130. The **ANGLE of a quaternion**, such as $\text{OB} : \text{OA}$, shall simply be, with us, the *angle* $\angle \text{AOB}$ *between the two lines*, of which the quaternion is the quotient; *this angle* being supposed here to be one of the *usual kind* (such as are considered by Euclid): and therefore being *acute*, or *right*, or *obtuse* (but not of any class *distinct* from these), when the quaternion is a *non-scalar* (108). We shall *denote* this *angle of a quaternion* q , by the *symbol*, $\angle q$; and thus shall have, *generally*, the two *inequalities*† following :

$$\angle q > 0; \quad \angle q < \pi;$$

where π is used as a symbol for *two right angles*.

131. When the *general quaternion*, q , *degenerates* into a *scalar*, x , then the *axis* (like the *plane*‡) becomes entirely *indeterminate* in its *direction*; and the *angle* takes, at the same time, either *zero* or *two right angles* for its value, according as the scalar is *positive* or *negative*. Denoting then, as above, any such *scalar* by x , we have :

* At a later stage, reasons will be assigned for denoting this *axis*, $\text{Ax} \cdot q$, of a quaternion q , by the *less arbitrary* (or more systematic) *symbol*, $\text{UV}q$; but for the present, the notation in the text may suffice.

† In some investigations respecting *complanar quaternions*, and *powers or roots* of quaternions, it is convenient to consider *negative angles*, and angles *greater than two right angles*: but these may then be called **AMPLITUDES**; and the word "Angle," like the word "Ratio," may thus be restricted, at least for the present, to its *ordinary geometrical sense*.

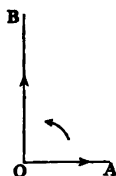
‡ Compare the Note to page 114. The *angle*, as well as the *axis*, becomes *indeterminate*, when the quaternion reduces itself to *zero*; unless we happen to know a *law*, according to which the *dividend-line tends* to become *null*, in the transition

from $\frac{\beta}{a}$ to $\frac{o}{a}$.

$Ax \cdot x$ = an indeterminate unit-vector ;
 $\angle x = 0$, if $x > 0$; $\angle x = \pi$, if $x < 0$.

132. Of *non-scalar quaternions*, the most important are those of which the *angle* is *right*, as in the annexed Figure 38 ; and when we have thus,

$$q = \frac{OB}{OA}, \text{ and } OB \perp OA, \text{ or } \angle q = \frac{\pi}{2},$$



the quaternion q may then be said to be a **RIGHT QUOTIENT** ;* or sometimes, a *Right Quaternion*.

(1.) If then $a = OA$ and $\rho = OP$, where O and A are two given (or fixed) points, but P is a variable point, the equation

$$\angle \frac{\rho}{a} = \frac{\pi}{2}$$

expresses that the *locus* of this point P is the *plane* through O , *perpendicular* to the *line* OA ; for it is equivalent to the *formula of perpendicularity* $\rho \perp a$ (129).

(2.) More generally, if $\beta = OB$, B being any third given point, the equation,

$$\angle \frac{\rho}{a} = \angle \frac{\beta}{a}$$

expresses that the *locus* of P is *one sheet of a cone of revolution*, with O for *vertex*, and OA for *axis*, and passing *through the point* B ; because it implies that the *angles* AOB and AOP are *equal in amount*, but *not necessarily* in one common *plane*.

(3.) The equation (comp. 128, 129),

$$Ax \cdot \frac{\rho}{a} = Ax \cdot \frac{\beta}{a},$$

expresses that the *locus* of the variable point P is the *given plane* AOB ; or rather the *indefinite half-plane*, which contains all the points P that are at once *complanar* with the *three given points* O, A, B , and are also *at the same side of the indefinite right line* OA , as the point B .

(4.) The system of the two equations,

$$\angle \frac{\rho}{a} = \angle \frac{\beta}{a}, \quad Ax \cdot \frac{\rho}{a} = Ax \cdot \frac{\beta}{a},$$

expresses that the point P is situated, *either on the finite right line* \overline{OB} , or on that line *prolonged through* B , but *not through* O ; so that the *locus* of P may in this case be said to be the *indefinite half-line*, or *ray*, which sets out from O in the *direction* of the *vector* OB or β ; and we may write $\rho = x\beta$, $x > 0$ (x being understood to be a *scalar*), instead of the equations assigned above.

* Reasons will afterwards be assigned, for *equating such a quotient*, or quaternion, to a *Vector* ; namely to the *line* which will presently (138) be called the *Index of the Right Quotient*.

(5.) This *other* system of two equations,

$$\angle \frac{\rho}{\alpha} = \pi - \angle \frac{\beta}{\alpha}, \quad \text{Ax} \cdot \frac{\rho}{\alpha} = -\text{Ax} \cdot \frac{\beta}{\alpha},$$

expresses that the *locus* of ρ is the *opposite ray* from o ; or that ρ is situated on the *prolongation of the vector* BO (1); or that $\rho = \alpha\beta$, $\alpha < 0$; or that

$$\rho = \alpha\beta', \quad \alpha > 0, \quad \text{if } \beta' = o\beta' = -\beta.$$

(Comp. Fig. 8B, *bis*.)

(6.) *Other notations*, for representing these and other geometrical *loci*, will be found to be supplied, in great abundance, by the Calculus of Quaternions; but it seemed proper to point out these, at the present stage, as serving already to show that even the two symbols of the present Section, Ax and \angle , when considered as *Characteristics of Operation* on *quotients of vectors*, enable us to *express*, very simply and concisely, several useful geometrical *conceptions*.

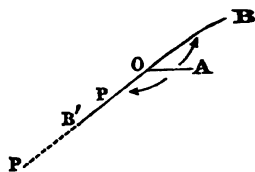


Fig. 33, *bis*.

133. If a *third line*, oi , be drawn in the *direction* of the *axis* ox of such a right quotient (and therefore *perpendicular*, by 127, 129, to *each* of the *two given rectangular lines*, OA , OB); and if the *length* of this new line oi bear to the *length* of that axis ox (and therefore also, by 128, to the assumed *unit* of length) the *same ratio*, which the *length of the dividend-line*, OB , bears to the *length of the divisor-line*, OA ; then the line oi , thus determined, is said to be the **INDEX of the Right Quotient**. And it is evident, from this definition of such an *Index*, combined with our general definition (117, 118) of *Equality between Quaternions*, that *two right quotients are equal or unequal to each other, according as their two index-lines (or indices) are equal or unequal vectors*.

SECTION 6.—*On the Reciprocal, Conjugate, Opposite, and Norm of a Quaternion; and on Null Quaternions.*

134. The **RECIPROCAL** (or the *Inverse*, comp. 119) of a quaternion, such as $q = \frac{\beta}{\alpha}$, is that *other* quaternion,

$$q' = \frac{\alpha}{\beta},$$

which is formed by *interchanging the divisor-line and the dividend-line*; and in thus passing from any *non-scalar* quaternion to its *reciprocal*, it is evident that the *angle* (as lately

defined in 130) remains *unchanged*, but that the *axis* (127, 128) is *reversed* in direction: so that we may write generally,

$$\angle \frac{a}{\beta} = \angle \frac{\beta}{a}; \quad \text{Ax.} \frac{a}{\beta} = - \text{Ax.} \frac{\beta}{a}.$$

135. The *product* of two reciprocal quaternions is always equal to *positive unity*; and each is equal to the *quotient* of *unity* divided by the other; because we have, by 106, 107,

$$1 : \frac{\beta}{a} = \frac{a}{a} : \frac{\beta}{a} = \frac{a}{\beta}, \quad \text{and} \quad \frac{a}{\beta} : \frac{\beta}{a} = \frac{a}{a} = 1.$$

It is therefore unnecessary to introduce any *new* or peculiar notation, to express the mutual relation existing between a quaternion and its *reciprocal*; since, if *one* be denoted by the symbol q , the *other* may (in the present System, as in Algebra) be denoted by the connected symbol,* $1 : q$, or $\frac{1}{q}$. We have thus the two general formulæ (comp. 134):

$$\angle \frac{1}{q} = \angle q; \quad \text{Ax.} \frac{1}{q} = - \text{Ax.} q.$$

136. Without yet entering on the *general* theory of multiplication and division of quaternions, beyond what has been done in Art. 120, it may be here remarked that if any two quaternions q and q' be (as in 134) *reciprocal* to each other, so that $q' \cdot q = 1$ (by 135), and if q'' be *any third* quaternion, then (as in algebra), we have the general formula,

$$q'' : q = q'' \cdot q' = q'' : \frac{1}{q};$$

because if (by 120) we *reduce* q and q' to a *common denominator* a , and denote the *new numerators* by β and γ , we shall have (by the definitions in 106, 107),

$$q'' : q = \frac{\gamma}{a} : \frac{\beta}{a} = \frac{\gamma}{\beta} = \frac{\gamma}{a} \cdot \frac{a}{\beta} = q'' \cdot q'.$$

137. When two *complanar triangles* AOB , AOB' , with a *com-*

* The symbol q^{-1} , for the *reciprocal* of a quaternion q , is also permitted in the present Calculus; but we defer the use of it, until its legitimacy shall have been established, in connexion with a general theory of powers of Quaternions.

mon side OA , are (as in Fig. 36) *inversely similar* (118), so that the formula $\Delta AOB' \alpha' AOB$ holds good, then the *two unequal* ^c *quotients*,* $\frac{OB}{OA}$ and $\frac{OB'}{OA}$, are said to be **CONJUGATE QUATERNIONS**; and if the *first* of them be still denoted by q , then the *second*, which is thus *the conjugate* of that *first*, or of any *other* quaternion which is *equal* thereto, is denoted by the *new symbol*, Kq : in which the letter K may be said to be the *Characteristic of Conjugation*. Thus, with the construction above supposed (comp. again Fig. 36), we may write,

$$\frac{OB}{OA} = q; \quad \frac{OB'}{OA} = Kq = K \frac{OB}{OA}.$$

138. From this definition of conjugate quaternions, it follows,

Ist, that if the equation $\frac{OB'}{OA} = K \frac{OB}{OA}$ hold good, then the *line* OB' may be called (118) the *reflexion of the line* OB (and conversely, the *latter line* the reflexion of the *former*), *with respect to the line* OA ; IInd, that, under the same condition, the *line* OA (prolonged if necessary) *bisects perpendicularly the line* BB' , in some point A' (as represented in Fig. 36); and IIIrd, that *any two conjugate quaternions* (like any two *reciprocal* quaternions, comp. 134, 135) have *equal angles*, but *opposite axes*: so that we may write, generally,

$$\angle Kq = \angle q; \quad Ax.Kq = -Ax.q;$$

and therefore† (by 135),

$$\angle Kq = \angle \frac{1}{q}; \quad Ax.Kq = Ax.\frac{1}{q}.$$

139. The *reciprocal* of a *scalar*, x , is simply *another scalar*, $\frac{1}{x}$, or x^{-1} , having the *same algebraic sign*, and in all other respects related to x as in algebra. But the *conjugate* Kx , of a *scalar* x , considered as a *limit* of a *quaternion*, is equal to that *scalar* x *itself*; as may be seen by supposing the *two equal* but *opposite angles*, $\angle AOB$ and $\angle AOB'$, in Fig. 36, to tend together to

* Compare the Note to page 112.

† It will soon be seen that these two last equations (138) express, that the *conjugate* and the *reciprocal*, of any proposed quaternion q , have always *equal versors*, although they have in general *unequal tensors*.



zero, or to two right angles. We may therefore write, generally,

$Kx = x$, if x be any scalar;
and conversely*,

$$q = \text{a scalar, if } Kq = q;$$

because then (by 104) we must have $OB = OB'$, $BB' = 0$; and therefore each of the two (now coincident) points, B, B' , must be situated somewhere on the indefinite right line OA .

140. In general, by the construction represented in the same Figure, the *sum* (comp. 6) of the *two numerators* (or *dividend-lines*, OB and OB'), of the *two conjugate fractions* (or *quotients*, or *quaternions*), q and Kq (137), is equal to the *double* of the line OA' ; whence (by 106), the *sum of those two conjugate quaternions* themselves is,

$$Kq + q = q + Kq = \frac{2OA'}{OA};$$

this sum is therefore *always scalar*, being *positive* if the angle $\angle q$ be *acute*, but *negative* if that angle be *obtuse*.

141. In the *intermediate case*, when the angle $\angle OAB$ is *right*, the *interval* OA' between the origin o and the line BB' *vanishes*; and the two lately mentioned *numerators*, OB, OB' , become two *opposite vectors*, of which the *sum* is *null* (5). Now, in general, it is natural, and will be found useful, or rather *necessary* (for consistency with *former* definitions), to admit that a *null vector*, *divided by an actual vector*, gives always a NULL QUATERNION as the *quotient*; and to *denote* this *null quotient* by the usual symbol for *Zero*. In fact, we have (by 106) the equation,

$$\frac{0}{a} = \frac{a - a}{a} = \frac{a}{a} - \frac{a}{a} = 1 - 1 = 0;$$

the zero in the numerator of the *left-hand* fraction representing here a *null line* (or a *null vector*, 1, 2); but the zero on the *right-hand* side of the equation denoting a *null quotient* (or quaternion). And thus we are entitled to infer that the *sum*,

* Somewhat later it will be seen that the equation $Kq = q$ may also be written as $Vq = 0$; and that this last is another mode of expressing that the *quaternion*, q , *degenerates* (131) *into a scalar*.

$Kq + q$, or $q + Kq$, of a *right-angled quaternion*, or *right quotient* (132), and of its *conjugate*, is always equal to zero.

142. We have, therefore, the three following formulæ, whereof the *second* exhibits a *continuity* in the transition from the *first* to the *third* :

$$\text{I. . . } q + Kq > 0, \quad \text{if } \angle q < \frac{\pi}{2};$$

$$\text{II. . . } q + Kq = 0, \quad \text{if } \angle q = \frac{\pi}{2};$$

$$\text{III. . . } q + Kq < 0, \quad \text{if } \angle q > \frac{\pi}{2}.$$

And because a quaternion, or *geometric quotient*, with an *actual* and *finite divisor-line* (as here oA), *cannot become equal to zero unless its dividend-line vanishes*, because (by 104) the equation

$$\frac{\beta}{\alpha} = 0 = \frac{0}{\alpha} \text{ requires the equation } \beta = 0,$$

if α be any actual and finite vector, we may infer, conversely, that *the sum* $q + Kq$ *cannot vanish*, without the line oA' *also vanishing*; that is, without the lines oB , oB' becoming *opposite vectors*, and therefore the quaternion q becoming a *right quotient* (132). We are therefore entitled to establish the three following *converse* formulæ (which indeed result from the three former) :

$$\text{I'. . . if } q + Kq > 0, \quad \text{then } \angle q < \frac{\pi}{2};$$

$$\text{II'. . . if } q + Kq = 0, \quad \text{then } \angle q = \frac{\pi}{2};$$

$$\text{III'. . . if } q + Kq < 0, \quad \text{then } \angle q > \frac{\pi}{2}.$$

143. When *two opposite vectors* (1), as β and $-\beta$, are both divided by *one common* (and actual) vector, α , we shall say that the *two quotients*, thus obtained are **OPPOSITE QUATERNIONS**; so that the *opposite* of any quaternion q , or of any quotient $\beta : \alpha$, may be denoted as follows (comp. 4) :

$$-\frac{\beta}{\alpha} = \frac{0 - \beta}{\alpha} = \frac{0}{\alpha} - \frac{\beta}{\alpha} = 0 - q = -q;$$

while the quaternion q itself may, on the same plan, be denoted (comp. 7) by the symbol $0 + q$, or $+q$. The sum of any two opposite quaternions is zero, and their quotient is negative unity; so that we may write, as in algebra (comp. again 7),

$(-q) + q = (+q) + (-q) = 0$; $(-q) : q = -1$; $-q = (-1)q$;
because, by 106 and 141,

$$\frac{-\beta}{\alpha} + \frac{\beta}{\alpha} = \frac{\beta - \beta}{\alpha} = \frac{0}{\alpha} = 0, \quad \frac{-\beta}{\alpha} : \frac{\beta}{\alpha} = \frac{-\beta}{\beta} = -1, \text{ \&c.}$$

The reciprocals of opposite quaternions are themselves opposite; or in symbols (comp. 126),

$$\frac{1}{-q} = -\frac{1}{q}, \text{ because } \frac{\alpha}{-\beta} = \frac{-\alpha}{\beta} = -\frac{\alpha}{\beta}.$$

Opposite quaternions have opposite axes, and supplementary angles (comp. Fig. 33, bis); so that we may establish (comp. 132, (5.)) the two following general formulæ,

$$\angle(-q) = \pi - \angle q; \quad \text{Ax.}(-q) = -\text{Ax.}q.$$

144. We may also now write, in full consistency with the recent formulæ II. and II'. of 142, the equation,

$$\text{II}'' \dots \text{K}q = -q, \text{ if } \angle q = \frac{\pi}{2};$$

and conversely* (comp. 138),

$$\text{II}''' \dots \text{if } \text{K}q = -q, \text{ then } \angle \text{K}q = \angle q = \frac{\pi}{2}.$$

In words, the conjugate of a right quotient, or of a right-angled (or right) quaternion (132), is the right quotient opposite thereto; and conversely, if an actual quaternion (that is, one which is not null) be opposite to its own conjugate, it must be a right quotient.

(1.) If then we meet the equation,

$$\text{K} \frac{\rho}{\alpha} = -\frac{\rho}{\alpha}, \text{ or } \frac{\rho}{\alpha} + \text{K} \frac{\rho}{\alpha} = 0,$$

we shall know that $\rho \perp \alpha$; and therefore (if $\alpha = \text{OA}$, and $\rho = \text{OP}$, as before), that the

* It will be seen at a later stage, that the equation $\text{K}q = -q$, or $q + \text{K}q = 0$, may be transformed to this other equation, $\text{S}q = 0$; and that, under this last form, it expresses that the scalar part of the quaternion q vanishes: or that this quaternion is a right quotient (132).

locus of the point P is the plane through O , perpendicular to the line OA (as in 132, (1.)).

(2.) On the other hand, the equation,

$$K \frac{\rho}{a} = + \frac{\rho}{a}, \quad \text{or} \quad \frac{\rho}{a} - K \frac{\rho}{a} = 0,$$

expresses (by 189) that the quotient $\rho : a$ is a scalar; and therefore (by 131) that its angle $\angle(\rho : a)$ is either 0 or π ; so that in this case, the locus of P is the indefinite right line through the two points O and A .

145. As the *opposite of the opposite*, or the *reciprocal of the reciprocal*, so also the *conjugate of the conjugate*, of any quaternion, is that quaternion *itself*; or in symbols,

$$-(-q) = +q; \quad 1:(1:q) = q; \quad KKq = q = 1q;$$

so that, by *abstracting from the subject of the operation*, we may write briefly,

$$K^2 = KK = 1.$$

It is easy also to prove, that the *conjugates of opposite quaternions* are themselves *opposite* quaternions; and that the *conjugates of reciprocals* are *reciprocal*: or in symbols, that

$$\text{I.} \dots K(-q) = -Kq, \quad \text{or} \quad Kq + K(-q) = 0;$$

and

$$\text{II.} \dots K \frac{1}{q} = 1 : Kq, \quad \text{or} \quad Kq \cdot K \frac{1}{q} = 1.$$

(1.) The equation $K(-q) = -Kq$ is included (comp. 143) in this more general formula, $K(xq) = xKq$, where x is any scalar; and this last equation (comp. 126) may be proved, by simply conceiving that the two lines OB , OB' , in Fig. 86, are multiplied by any common scalar; or that they are both cut by any parallel to the line BB' .

(2.) To prove that conjugates of reciprocals are reciprocal, or that $Kq \cdot K \frac{1}{q} = 1$, we may conceive that, as in the annexed Figure 86, *bis*, while we have still the relation of *inverse* similitude,

$$\Delta AOB' \propto \Delta OAB \quad (118, 137),$$

as in the former Figure 86, a new point C is determined, either on the line OA itself, or on that line prolonged through A , so as to satisfy either of the two following connected conditions of *direct* similitude:

$$\Delta BOC \propto \Delta OAB'; \quad \Delta B'OC \propto \Delta OAB;$$

or simply, as a relation between the four points O, A, B, C , the formula,

$$\Delta BOC \propto \Delta OAB.$$

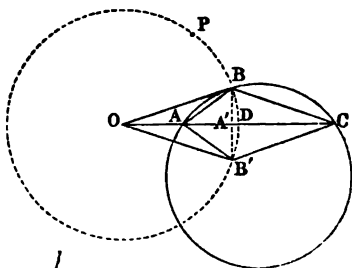


Fig. 86, bis.

For then we shall have the transformations,

$$K \frac{1}{q} = K \frac{OA}{OB} = K \frac{OB'}{OC} = \frac{OB}{OC} = \frac{OA}{OB'} = \frac{1}{Kq}.$$

(3.) The two quotients, $OB : OA$, and $OB : OC$, that is to say, the *quaternion* q itself, and the *conjugate of its reciprocal*, or^a the *reciprocal of its conjugate*, have the same angle, and the same axis; we may therefore write, generally,

$$\angle K \frac{1}{q} = \angle q; \quad Ax \cdot K \frac{1}{q} = Ax \cdot q.$$

(4.) Since $OA : OB$ and $OA : OB'$ have thus been proved (by sub-art. 2) to be a pair of *conjugate quotients*, we can now infer this theorem, that any two *geometric fractions*, $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta'}$, which have a common numerator α , are *conjugate quaternions*, if the denominator β' of the second be the reflexion of the denominator β of the first, with respect to that common numerator (comp. 138, I.); whereas it had only been previously assumed, as a definition (137), that such conjugation exists, under the same geometrical condition, between the two other (or inverse) fractions, $\frac{\beta}{\alpha}$ and $\frac{\beta'}{\alpha}$; the three vectors α, β, β' being supposed to be all co-initial (18).

(5.) Conversely, if we meet, in any investigation, the formula

$$OA : OB' = K (OA : OB),$$

we shall know that the point B' is the reflexion of the point B , with respect to the line OA ; or that this line, OA , prolonged if necessary in either of two opposite directions, bisects at right angles the line BB' , in some point A' , as in either of the two Figures 36 (comp. 138, II.).

(6.) Under the recent conditions of construction, it follows from the most elementary principles of geometry, that the circle, which passes through the three points A, B, C , is touched at B , by the right line OB ; and that this line is, in length, a mean proportional between the lines OA, OC . Let then OD be such a geometric mean, and let it be set off from O in the common direction of the two last mentioned lines, so that the point D falls between A and C ; also let the vectors OC, OD be denoted by the symbols, γ, δ ; we shall then have expressions of the forms,

$$\delta = a\alpha, \quad \gamma = a^2\alpha,$$

where a is some positive scalar, $a > 0$; and the vector β of B will be connected (comp. sub-art. 2) with this scalar a , and with the vector α , by the formula,

$$\frac{OB}{OC} = K \frac{OA}{OB}, \quad \text{or} \quad \frac{OC}{OB} = K \frac{OB}{OA}, \quad \text{or} \quad \frac{a^2\alpha}{\beta} = K \frac{\beta}{\alpha}.$$

(7.) Conversely, if we still suppose that $\gamma = a^2\alpha$, this last formula expresses the inverse similitude of triangles, $\Delta BOC \propto \Delta OAB$; and it expresses nothing more: or in other

^a It will be seen afterwards, that the common value of these two equal quaternions, $K \frac{1}{q}$ and $\frac{1}{Kq}$, may be represented by either of the two new symbols, $Uq : Tq$, or $q : Nq$; or in words, that it is equal to the versor divided by the tensor; and also to the quaternion itself divided by the norm.

words, it is satisfied by the vector β of every point B, which gives that inverse similitude. But for this purpose it is only requisite that the length of OB should be (as above) a geometric mean between the lengths of OA, OC; or that the two lines, OB, OD (sub-art. 6), should be equally long: or finally, that B should be situated somewhere on the surface of a sphere, which is described so as to pass through the point D (in Fig. 36, bis), and to have the origin o for its centre.

(8.) If then we meet an equation of the form,

$$\frac{a^2\alpha}{\rho} = K\frac{\rho}{\alpha}, \quad \text{or} \quad \frac{\rho}{\alpha} K \frac{\rho}{\alpha} = a^2,$$

in which $\alpha = OA$, $\rho = OP$, and a is a scalar, as before, we shall know that the locus of the point P is a spheric surface, with its centre at the point o, and with the vector oa for a radius; and also that if we determine a point o by the equation $oc = a^2\alpha$, this spheric locus of P is a common orthogonal to all the circles APC, which can be described, so as to pass through the two fixed points, A and C: because every radius OP of the sphere is a tangent, at the variable point P, to the circle APC, exactly as OB is to ABC in the recent Figure.

(9.) In the same Fig. 36, bis, the similar triangles show (by elementary principles) that the length of BC is to that of AB in the sub-duplicate ratio of OC to OA; or in the simple ratio of OD to OA; or as the scalar a to 1. If then we meet, in any research, the recent equation in ρ (sub-art. 8), we shall know that

$$\text{length of } (\rho - a^2\alpha) = a \times \text{length of } (\rho - \alpha);$$

while the recent interpretation of the same equation gives this other relation of the same kind:

$$\text{length of } \rho = a \times \text{length of } \alpha.$$

(10.) At a subsequent stage, it will be shown that the *Calculus of Quaternions* supplies *Rules of Transformation*, by which we can pass from any one to any other of these last equations respecting ρ , without (at the time) constructing any Figure, or (immediately) appealing to *Geometry*: but it was thought useful to point out, already, how much geometrical meaning* is contained in so simple a formula, as that of the last sub-art. 8.

(11.) The product of two conjugate quaternions is said to be their common NORM, † and is denoted thus:

$$qKq = Nq.$$

* A student of ancient geometry may recognise, in the two equations of sub-art. 9, a sort of translation, into the language of vectors, of a celebrated local theorem of APOLLONIUS of Perga, which has been preserved through a citation made by his early commentator, Eutocius, and may be thus enunciated: Given any two points (as here A and C) in a plane, and any ratio of inequality (as here that of 1 to a), it is possible to construct a circle in the plane (as here the circle BDB'), such that the (lengths of the) two right lines (as here AB and CB, or AP and CP), which are inflected from the two given points to any common point (as B or P) of the circumference, shall be to each other in the given ratio. (*Δύο δοθέντων σημείων, κ. τ. λ.* Page 11 of Halley's Edition of Apollonius, Oxford, MDCCX.)

† This name, NORM, and the corresponding characteristic, N, are here adopted, as suggestions from the *Theory of Numbers*; but, in the present work, they will not

It follows that $NKq = Nq$; and that the norm of a quaternion is generally a positive scalar: namely, the square of the quotient of the lengths of the two lines, of which (as vectors) the quaternion itself is the quotient (112). In fact we have, by sub-art. 6, and by the definition of a norm, the transformations:

$$N \frac{OB}{OA} = N \frac{OB'}{OA} = \frac{OC}{OB'} \cdot \frac{OB'}{OA} = \frac{OC}{OB} \cdot \frac{OB}{OA} = \frac{OC}{OA} = \left(\frac{OD}{OA} \right)^2;$$

$$Nq = N \frac{\beta}{a} = \frac{\beta}{a} K \frac{\beta}{a} = \left(\frac{\text{length of } \beta}{\text{length of } a} \right)^2.$$

As a limit, we may say that the norm of a null quaternion is zero; or in symbols, $N0 = 0$.

(12.) With this notation, the equation of the spheric locus (sub-art. 8), which has the point o for its centre, and the vector aa for one of its radii, assumes the shorter form:

$$N \frac{\rho}{a} = a^2; \text{ or } N \frac{\rho}{aa} = 1.$$

SECTION 7.—On Radial Quotients; and on the Square of a Quaternion.

146. It was early seen (comp. Art. 2, and Fig. 4) that any two radii, AB, AC , of any one circle, or sphere, are necessarily unequal vectors; because their directions differ. On the other hand, when we are attending only to relative direction (110), we may suppose that all the vectors compared are not merely co-initial (18), but are also equally long; so that if their common length be taken for the unit, they are all radii, OA, OB, \dots of what we have called the Unit-Sphere (128), described round the origin as centre; and may all be said to be Unit-Vectors (129). And then the quaternion, which is the quotient of any one such vector divided by any other, or generally the quotient of any two equally long vectors, may be called a Radial Quotient; or sometimes simply a RADIAL. (Compare the annexed Figure 39.)

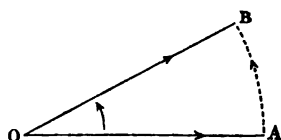


Fig. 39.

be often wanted, although it may occasionally be convenient to employ them. For we shall soon introduce the conception, and the characteristic, of the Tensor, Tq , of a quaternion, which is of greater geometrical utility than the Norm, but of which it will be proved that this norm is simply the square,

$$qKq = Nq = (Tq)^2.$$

Compare the Note to sub-art. 3.

147. The two *Unit-Scalars*, namely, *Positive and Negative Unity*, may be considered as *limiting cases* of *radial quotients*, corresponding to the *two extreme values*, 0 and π , of the *angle AOB*, or $\angle q$ (131). In the *intermediate*

case, when AOB is a *right angle*, or $\angle q = \frac{\pi}{2}$,

as in Fig. 40, the resulting quotient, or quaternion, may be called (comp. 132) a *Right Radial Quotient*; or simply, a *RIGHT RADIAL*. The consideration of such *right radials*

will be found to be of great importance, in the whole theory and practice of Quaternions.

148. The most important *general property* of the quotients last mentioned is the following: that *the Square of every Right Radial is equal to Negative Unity*; it being understood that we write generally, as in algebra,

$$q \cdot q = qq = q^2,$$

and call this *product of two equal quaternions* the *SQUARE* of each of them. For if, as in Fig. 41, we describe a *semicircle ABA'*, with O for *centre*, and with OB for the *bisecting radius*, then the two right quotients, OB : OA, and OA' : OB, are *equal* (comp. 117); and therefore their *common square* is (comp. 107) the *product*,

$$\left(\frac{OB}{OA}\right)^2 = \frac{OA'}{OB} \cdot \frac{OB}{OA} = \frac{OA'}{OA} = -1;$$

where OA and OB may represent any two equally long, but mutually rectangular lines. More generally, the *Square of every Right Quotient*

(132) is equal to a *Negative Scalar*; namely, to the *negative of the square of the number*, which represents the *ratio of the lengths** of the two rectangular lines compared; or to zero

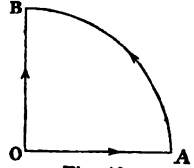


Fig. 40.

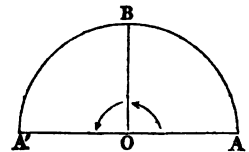


Fig. 41.

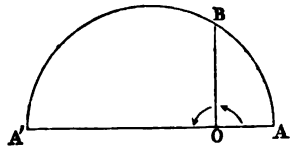


Fig. 41, bis.

* Hence, by 145, (11.), $q^2 = -Ng$, if $\angle q = \frac{\pi}{2}$.

minus the square of the number which denotes (comp. 133) the length of the Index of that Right Quotient: as appears from Fig. 41, bis, in which OB is only an ordinate, and not (as before) a radius, of the semicircle ABA'; for we have thus,

$$\left(\frac{OB}{OA}\right)^2 = \frac{OA'}{OA} = -\left(\frac{\text{length of } OB}{\text{length of } OA}\right)^2, \text{ if } OB \perp OA.$$

149. Thus every Right Radial is, in the present System, one of the Square Roots of Negative Unity; and may therefore be said to be one of the Values of the Symbol $\sqrt{-1}$; which celebrated symbol has thus a certain degree of vagueness, or at least of indetermination, of meaning in this theory, on account of which we shall not often employ it. For although it thus admits of a perfectly clear and geometrically real Interpretation, as denoting what has been above called a Right Radial Quotient, yet the Plane of that Quotient is arbitrary; and therefore the symbol itself must be considered to have (in the present system) indefinitely many values; or in other words the Equation,

$$q^2 = -1,$$

has (in the Calculus of Quaternions) indefinitely many Roots,* which are all Geometrical Reals: besides any other roots, of a purely symbolical character, which the same equation may be conceived to possess, and which may be called Geometrical Imaginaries.† Conversely, if q be any real quaternion, which

* It will be subsequently shown, that if x, y, z be any three scalars, of which the sum of the squares is unity, so that

$$x^2 + y^2 + z^2 = 1;$$

and if i, j, k be any three right radials, in three mutually rectangular planes; then the expression,

$$q = ix + jy + kz,$$

denotes another right radial, which satisfies (as such, and by symbolical laws to be assigned) the equation $q^2 = -1$; and is therefore one of the geometrically real values of the symbol $\sqrt{-1}$.

† Such imaginaries will be found to offer themselves, in the treatment by Quaternions (or rather by what will be called Biquaternions), of ideal intersections, and of ideal contacts, in geometry; but we confine our attention, for the present, to geometrical reals alone. Compare the Notes to page 90.

satisfies the equation $q^2 = -1$, it must be a right radial; for if, as in Fig. 42, we suppose that $\Delta AOB \propto BOC$, we shall have

$$q^2 = \left(\frac{OB}{OA}\right)^2 = \frac{OC}{OB} \cdot \frac{OB}{OA} = \frac{OC}{OA};$$

and this square of q cannot become equal to negative unity, except by OC being $= -OA$, or $= OA'$ in Fig. 41; that is, by the line OB being at right angles to the line OA , and being at the same time equally long, as in Fig. 40.

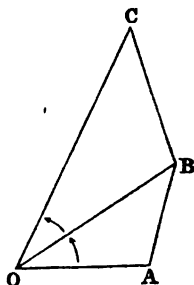


Fig. 42.

(1.) If then we meet the equation,

$$\left(\frac{\rho}{a}\right)^2 = -1,$$

where $a = OA$, and $\rho = OP$, as before, we shall know that the locus of the point P is the circumference of a circle, with O for its centre, and with a radius which has the same length as the line OA ; while the plane of the circle is perpendicular to that given line. In other words, the locus of P is a great circle, on a sphere of which the centre is the origin; and the given point A , on the same spheric surface, is one of the poles of that circle.

(2.) In general, the equation $q^2 = -a^2$, where a is any (real) scalar, requires that the quaternion q (if real) should be some right quotient (132); the number a denoting the length of the index (138), of that right quotient or quaternion (comp. Art. 148, and Fig. 41, bis). But the plane of q is still entirely arbitrary; and therefore the equation

$$q^2 = -a^2,$$

like the equation $q^2 = -1$, which it includes, must be considered to have (in the present system) indefinitely many geometrically real roots.

(8.) Hence the equation,

$$\left(\frac{\rho}{a}\right)^2 = -a^2,$$

in which we may suppose that $a > 0$, expresses that the locus of the point P is a (new) circular circumference, with the line OA for its axis,* and with a radius of which the length $= a \times$ the length of OA .

150. It may be added that the index (133), and the axis (128), of a right radial (147), are the same; and that its reciprocal (184), its conjugate (137), and its opposite (143), are all equal to each other. Conversely, if the reciprocal of a given quaternion q be equal to the opposite

* It being understood, that the axis of a circle is a right line perpendicular to the plane of that circle, and passing through its centre.

of that quaternion, then q is a *right radial*; because its square, q^2 , is then equal (comp. 136) to the quaternion itself, *divided by its opposite*; and therefore (by 143) to *negative unity*. But the *conjugate of every radial quotient* is equal to the *reciprocal of that quotient*; because if, in Fig. 36, we conceive that the *three lines* OA , OB , OB' are *equally long*, or if, in Fig. 39, we *prolong the arc* BA , by an *equal arc* AB' , we have the equation,

$$Kq = \frac{OB'}{OA} = \frac{OA}{OB} = \frac{1}{q}.$$

And conversely,*

$$\text{if } Kq = \frac{1}{q}, \quad \text{or if } qKq = 1,$$

then the quaternion q is a radial quotient.

SECTION 8.—*On the Versor of a Quaternion, or of a Vector ; and on some General Formulæ of Transformation.*

151. When a quaternion $q = \beta : a$ is thus a *radial quotient* (146), or when the *lengths* of the two lines a and β are *equal*, the *effect* of this quaternion q , considered as a FACTOR (103), in the equation $qa = \beta$, is simply the *turning* of the *multiplicand-line* a , in the *plane* of q (119), and *towards the hand* determined by the direction of the *positive axis* $Ax \cdot q$ (129), *through the angle* denoted by $\angle q$ (130); so as to bring that line a (or a revolving line which *had* coincided therewith) *into a new direction*: namely, into that of the *product-line* β . And with reference to this conceived *operation of turning*, we shall now say that *every Radial Quotient is a VERSOR*.

152. A *Versor* has thus, in general, a *plane*, an *axis*, and an *angle*; namely, those of the *Radial* (146) to which it *corresponds*, or is *equal*: the *only difference* between them being a difference in the *points of view*† from which they are respectively regarded; namely, the *radial* as the *quotient*, q , in the

* Hence, in the *notation of norms* (145, (11.)), if $Nq = 1$, then q is a *radial*; and conversely, the *norm of a radial quotient* is always equal to *positive unity*.

† In a slightly *metaphysical* mode of expression it may be said, that the *radial quotient* is the *result of an analysis*, wherein *two radii* of one sphere (or circle) are *compared*, as regards their *relative direction*; and that the *equal versor* is the *instrument of a corresponding synthesis*, wherein *one radius* is conceived to be *generated*, by a certain *rotation*, from the *other*.

formula, $q = \beta : a$; and the *versor* as the (equal) *factor*, q , in the *converse formula*, $\beta = q . a$; where it is still supposed that the two vectors, a and β , are *equally long*.

153. A *versor*, like a *radial* (147), cannot *degenerate* into a *scalar*, except by its *angle* acquiring one or other of the two *limit-values*, 0 and π . In the first case, it becomes *positive unity*; and in the second case, it becomes *negative unity*: each of these two *unit-scalars* (147) being here regarded as a *factor* (or *coefficient*, comp. 12), which *operates on a line*, to *preserve* or to *reverse* its *direction*. In this view, we may say that -1 is an *Inversor*; and that every *Right Versor* (or *versor* with an angle $= \frac{\pi}{2}$) is a *Semi-inversor*:* because it *half-inverts* the *line* on which it *operates*, or *turns it through half of two right angles* (comp. Fig. 41). For the same reason, we are led to consider every *right versor* (like every *right radial*, 149, from which indeed we have just seen, in 152, that it differs only as *factor* differs from *quotient*), as being *one of the square-roots of negative unity*: or as *one of the values of the symbol* $\sqrt{-1}$.

154. In fact we may observe that the *effect* of a *right versor*, considered as *operating on a line* (in its own plane), is to *turn that line, towards a given hand, through a right angle*. If then q be such a *versor*, and if $qa = \beta$, we shall have also (comp. Fig. 41), $q\beta = -a$; so that, if a be any line in the plane of a right versor q , we have the equation,

$$q . qa = -a;$$

whence it is natural to write, under the same condition,

$$q^2 = -1,$$

as in 149. On the other hand, *no versor, which is not right-angled, can be a value of* $\sqrt{-1}$; or can satisfy the equation $q^2 a = -a$, as Fig. 42 may serve to illustrate. For it is included in the meaning of this last equation, as applied to the theory of *versors*, that a *rotation through* $2 \angle q$, or through the *double of the angle* of q itself, is equi-

* This word, "semi-inversor," will not be often used; but the introduction of it here, in passing, seems adapted to throw light on the *view* taken, in the present work, of the symbol $\sqrt{-1}$, when regarded as denoting a certain important class (149) of *Reals in Geometry*. There are uses of that symbol, to denote *Geometrical Imaginaries* (comp. again Art. 149, and the Notes to page 90), considered as connected with *ideal intersections*, and with *ideal contacts*; but with such uses of $\sqrt{-1}$ we have, at present, nothing to do.

valent to an *inversion of direction*; and therefore to a rotation through *two right angles*.

155. In general, if a be *any vector*, and if a be used as a temporary* symbol for the *number* expressing its *length*; so that a is here a *positive scalar*, which bears to *positive unity*, or to the scalar $+1$, the *same ratio* as that which the length of the line a bears to the assumed *unit of length* (comp. 128); then the *quotient* $a : a$ denotes generally (comp. 16) a *new vector*, which has the *same direction* as the *proposed vector* a , but has its *length* equal to that assumed *unit*: so that it is (comp. 146) *the Unit-Vector in the direction of* a . We shall denote this *unit-vector* by the *symbol*, Ua ; and so shall write, generally,

$$Ua = \frac{a}{a}, \text{ if } a = \text{length of } a;$$

that is, more fully, if a be, as above supposed, the *number* (commensurable or incommensurable, but *positive*) which *represents* that length, with reference to some selected standard.

156. Suppose now that $q = \beta : a$ is (as at first) a *general quaternion*, or the *quotient of any two vectors*, a and β , whether equal or unequal in length. Such a *Quaternion* will not (generally) be a *Versor* (or at least not *simply such*), according to the definition lately given; because its *effect*, when operating as a *factor* (103) on a , will not in general be *simply to turn* that line (151): but will (generally) *alter the length*,† as well as the *direction*. But if we *reduce* the two proposed vectors, a and β , to the two *unit-vectors* Ua and $U\beta$ (155), and form the *quotient of these*, we shall then have taken account of *relative direction alone*: and the *result* will therefore be a *versor*, in the sense lately defined (151). We propose to call the quotient, or the versor, thus obtained, the *versor-element*, or briefly, the *VERSOR*, of the *Quaternion* q ; and shall find it convenient to em-

* We shall soon propose a general notation for representing the *lengths of vectors*, according to which the symbol Ta will denote what has been above called a ; but are unwilling to introduce more than *one new characteristic of operation*, such as K , or T , or U , &c., at one time.

† By what we shall soon call an *act of tension*, which will lead us to the consideration of the *tensor* of a quaternion.

ploy the same* *Characteristic*, U , to denote the *operation of taking the versor of a quaternion*, as that employed above to denote the operation (155) of *reducing a vector to the unit of length*, without any change of its *direction*. On this plan, the symbol Uq will denote the *versor of q* ; and the foregoing *definitions* will enable us to establish the *General Formula* :

$$Uq = U \frac{\beta}{a} = \frac{U\beta}{Ua};$$

in which the *two unit-vectors*, Ua and $U\beta$, may be called, by analogy, and for other reasons which will afterwards appear, the *versors*† of the *vectors*, a and β .

157. In thus passing from a given *quaternion*, q , to its *versor*, Uq , we have only changed (in general) the *lengths* of the two lines compared, namely, by *reducing* each to the assumed *unit of length* (155, 156), without making any change in their *directions*. Hence the *plane* (119), the *axis* (127, 128), and the *angle* (130), of the quaternion, remain *unaltered* in this passage; so that we may establish the two following general formulæ :

$$\angle Uq = \angle q; \quad Ax \cdot Uq = Ax \cdot q.$$

More generally we may write,

* For the moment, this double use of the characteristic U , to assist in denoting both the *unit-vector* Ua derived from a given *line* a , and also the *versor* Uq derived from a *quaternion* q , may be regarded as established here by arbitrary *definition*; but as *permitted*, because the *difference* of the *symbols*, as here a and q , which serve for the present to denote *vectors* and *quaternions*, considered as the *subjects* of these *two operations* U , will prevent such *double use* of that *characteristic* from giving rise to any *confusion*. But we shall further find that several important *analogies* are by anticipation *expressed*, or at least *suggested*, when the proposed *notation* is employed. Thus it will be found (comp. the Note to page 119), that *every vector* a may usefully be *equated to that right quotient*, of which it is (138) the *index*; and that then the *unit-vector* Ua may be, on the same plan, *equated to that right radial* (147), which is (in the sense lately defined) the *versor of that right quotient*. We shall also find ourselves led to regard *every unit-vector* as the *axis of a quadrantal* (or *right*) *rotation*, in a plane perpendicular to that axis; which will supply another inducement, to *speak* of every *such* vector as a *versor*. On the whole, it appears that there will be no inconvenience, but rather a prospective advantage, in our already *reading* the symbol Ua as "*versor of a* ;" just as we may *read* the analogous symbol Uq , as "*versor of q* ."

† Compare the Note immediately preceding.

$$\angle q' = \angle q, \text{ and } \text{Ax. } q' = \text{Ax. } q, \text{ if } Uq' = Uq;$$

the *versor* of a quaternion depending solely on, but conversely being sufficient to determine, the *relative direction* (156) of the two lines, of which (as *vectors*) the quaternion itself is the *quotient* (112); or the axis and angle of the *rotation*, in the plane of those two lines, from the divisor to the dividend (128): so that any two quaternions, which have *equal versors*, must also have *equal angles*, and *equal* (or coincident) *axes*, as is expressed by the last written formula. Conversely, from this dependence of the *versor* Uq on *relative direction** alone, it follows that any two quaternions, of which the angles and the axes are equal, have also *equal versors*; or in symbols, that

$$Uq' = Uq, \text{ if } \angle q' = \angle q, \text{ and } \text{Ax. } q' = \text{Ax. } q.$$

For example, we saw (in 138) that the *conjugate* and the *reciprocal* of any quaternion have thus their *angles* and their *axes* the same; it follows, therefore, that the *versor of the conjugate* is always *equal* to the *versor of the reciprocal*; so that we are permitted to establish the following general formula, †

$$UKq = U \frac{1}{q}.$$

158. Again, because

$$U \left(1 : \frac{\beta}{\alpha} \right) = U \frac{\alpha}{\beta} = \frac{U\alpha}{U\beta} = 1 : \frac{U\beta}{U\alpha} = 1 : U \frac{\beta}{\alpha},$$

it follows that the *versor of the reciprocal* of any quaternion is, at the same time, the *reciprocal of the versor*; so that we may write,

* The *unit-vector* $U\alpha$, which we have recently proposed (156) to call the *versor of the vector* α , depends in like manner on the *direction of that vector alone*; which *exclusive reference*, in each of these two cases, to *DIRECTION*, may serve as an additional *motive* for employing, as we have lately done, one *common name*, *VERSOR*, and one *common characteristic*, U , to assist in describing or denoting both the *Unit-Vector* $U\alpha$ itself, and the *Quotient of two such Unit-Vectors*, $Uq = U\beta : U\alpha$; all danger of *confusion* being sufficiently guarded against (comp. the Note to Art. 156), by the *difference of the two symbols*, α and q , employed to denote the *vector* and the *quaternion*, which are respectively the *subjects of the two operations* U ; while those two operations agree in this *essential point*, that each serves to *eliminate the quantitative element*, of absolute or relative length.

† Compare the Note to Art. 138.

$$U \frac{1}{q} = \frac{1}{Uq}; \text{ or } Uq \cdot U \frac{1}{q} = 1.$$

Hence, by the recent result (157), we have also, generally,

$$UKq = \frac{1}{Uq}; \text{ or, } Uq \cdot UKq = 1.$$

Also, because the *versor* Uq is always a *radial* quotient (151, 152), it is (by 150) the *conjugate of its own reciprocal*; and therefore at the same time (comp. 145), the *reciprocal of its own conjugate*; so that the *product of two conjugate versors*, or what we have called (145, (11.)) their *common NORM*, is always equal to *positive unity*; or in symbols (comp. 150),

$$NUq = Uq \cdot KUq = 1.$$

For the same reason, the *conjugate of the versor* of any quaternion is equal to the *reciprocal of that versor*, or (by what has just been seen) to the *versor of the reciprocal* of that quaternion; and therefore also (by 157), to the *versor of the conjugate*; so that we may write generally, as a summary of recent results, the formula:

$$KUq = \frac{1}{Uq} = U \frac{1}{q} = UKq;$$

each of these four symbols denoting a *new versor*, which has the *same plane*, and the *same angle*, as the *old or given versor* Uq , but has an *opposite axis*, or an *opposite direction of rotation*: so that, with respect to that given *Versor*, it may naturally be called a **REVERSOR**.

159. As regards the *versor itself*, whether of a vector or of a quaternion, the definition (155) of Ua gives,

$$Uxa = + Ua, \text{ or } - Ua, \text{ according as } x > \text{ or } < 0;$$

because (by 15) the *scalar coefficient* x *preserves*, in the first case, but *reverses*, in the second case, the *direction* of the vector a ; whence also, by the definition (156) of Uq , we have generally (comp. 126, 143),

$$Uxq = + Uq, \text{ or } - Uq, \text{ according as } x > \text{ or } < 0.$$

The *versor of a scalar*, regarded as the *limit of a quaternion* (131, 139), is equal to *positive or negative unity* (comp. 147,

153), according as the scalar itself is positive or negative ; or in symbols,

$$Ux = +1, \text{ or } -1, \text{ according as } x > \text{ or } < 0;$$

the *plane* and *axis* of each of these two *unit scalars* (147), considered as *versors* (153), being (as we have already seen) *indeterminate*. The *versor of a null quaternion* (141) must be regarded as *wholly arbitrary*, unless we happen to know a *law*,* according to which the quaternion *tends to zero*, before actually *reaching* that limit; in which latter case, the *plane*, the *axis*, and the *angle* of the *versor* † $U0$ may *all* become *determined*, as *limits* deduced from that law. The *versor of a right quotient* (132), or of a *right-angled quaternion* (141), is always a *right radial* (147), or a *right versor* (153); and therefore is, as such, *one of the square roots of negative unity* (149), or *one of the values of the symbol* $\sqrt{-1}$; while (by 150) the *axis* and the *index* of such a versor *coincide*; and in like manner its *reciprocal*, its *conjugate*, and its *opposite* are all *equal* to each other.

160. It is evident that if a proposed *quaternion* q be *already* a *versor* (151), in the sense of being a *radial* (146), the operation of *taking its versor* (156) produces *no change*; and in like manner that, if a given *vector* a be *already* an *unit-vector*, it remains the *same* vector, when it is *divided* (155) by its *own length*; that is, in this case, by the *number one*. For example, we have assumed (128, 129), that the *axis* of *every* quaternion is an *unit-vector*; we may therefore write, generally, in the notation of 155, the equation,

$$U(Ax . q) = Ax . q.$$

A *second operation* U leaves thus the *result* of the *first operation* U *unchanged*, whether the *subject* of such successive operations be a *line*, or a *quaternion*; we have therefore the two

* Compare the Note to Art. 131.

† When the *zero* in this symbol, $U0$, is considered as denoting a *null vector* (2), the symbol itself denotes *generally*, by the foregoing principles, an *indeterminate unit-vector*; although the *direction* of this unit-vector may, in certain questions, become determined, as a *limit* resulting from a *law*.

following general formulæ, differing only in the symbols of that subject:

$$UUa = Ua; \quad UUq = Uq;$$

whence, by *abstracting* (comp. 145) from the subject of the operation, we may write, briefly and symbolically,

$$U^2 = UU = U.$$

161. Hence, with the help of 145, 158, 159, we easily deduce the following (among other) *transformations of the versor of a quaternion*:

$$\begin{aligned} Uq &= \frac{1}{U \frac{1}{q}} = \frac{1}{KUq} = \frac{1}{UKq} = KU \frac{1}{q} = K \frac{1}{Uq} = KUKq \\ &= U \frac{1}{Kq} = UK \frac{1}{q} = U^2q = UKU \frac{1}{q} = UK \frac{1}{Uq} = (UK)^2q; \\ Uq &= Uxq, \text{ if } x > 0; \quad = -Uxq, \text{ if } x < 0. \end{aligned}$$

We may also write, generally,

$$\frac{q}{Kq} = \frac{Uq}{KUq} = (Uq)^2 = U(q^2) = Uq^2;$$

the *parentheses* being here unnecessary, because (as will soon be more fully seen) the *symbol* Uq^2 denotes *one common versor*, whether we interpret it as denoting the *square of the versor*, or as the *versor of the square*, of q . The present Calculus will be found to abound in *General Transformations* of this sort; which all (or nearly all), like the foregoing, depend ultimately on very *simple geometrical conceptions*; but which, notwithstanding (or rather, perhaps, on account of) this extreme *simplicity of their origin*, are often *useful, as elements of a new kind of Symbolical Language in Geometry*: and generally, as *instruments of expression*, in all those mathematical or physical researches to which the *Calculus of Quaternions* can be applied. It is, however, by no means necessary that a student of the subject, at the present stage, should make himself *familiar* with *all* the recent transformations of Uq ; although it may be well that he should satisfy himself of their correctness, in doing which the following remarks will perhaps be found to assist.

(1.) To give a *geometrical illustration*, which may also serve as a *proof*, of the recent equation,

$$q : Kq = (Uq)^2,$$

we may employ Fig. 36, *bis*; in which, by 145, (2.), we have

$$q \cdot \frac{1}{Kq} = \frac{OB}{OA} \cdot \frac{OA}{OB'} = \frac{OB}{OB'} = \left(\frac{OB}{OD} \right)^2 = \left(U \frac{OB}{OA} \right)^2 = (Uq)^2.$$

(2.) As regards the equation, $U(q^2) = (Uq)^2$, we have only to conceive that the three lines OA, OB, OC , of Fig. 42, are cut (as in Fig. 42, *bis*) in three new points, A', B', C' , by an *unit-circle* (or by a circle with a radius equal to the unit of length), which is described about their common origin o as centre, and in their common plane; for then if these three lines be called α, β, γ , the three new lines OA', OB', OC' are (by 155) the three unit-vectors denoted by the symbols, $U\alpha, U\beta, U\gamma$; and we have the transformations (comp. 148, 149),

$$U(q^2) = U \cdot \left(\frac{\beta}{\alpha} \right)^2 = U \frac{\gamma}{\alpha} = \frac{U\gamma}{U\alpha} = \frac{OC'}{OA'} = \left(\frac{OB'}{OA'} \right)^2 = (Uq)^2.$$

(3.) As regards *other* recent transformations (161), although we have seen (185) that it is *not necessary* to invent any *new* or peculiar *symbol*, to represent the *reciprocal* of a quaternion, yet if, for the sake of present convenience, and as a merely *temporary notation*, we write

$$Rq = \frac{1}{q},$$

employing thus, for a moment, the letter R as a *characteristic of reciprocation*, or of the operation of *taking the reciprocal*, we shall then have the *symbolical equations* (comp. 145, 158):

$$R^2 = K^2 = 1; \quad RK = KR; \quad RU = UR = KU = UK;$$

but we have also (by 160), $U^2 = U$; whence it easily follows that

$$\begin{aligned} U &= RUR = RKU = RUK = KUR = KRU = KUK \\ &= URK = UKR = UKUR = UKRU = (UK)^2 = \&c. \end{aligned}$$

(4.) The equation

$$U \frac{\rho}{\alpha} = U \frac{\beta}{\alpha}, \quad \text{or simply,} \quad U\rho = U\beta,$$

expresses that the *locus* of the point P is the *indefinite right line*, or *ray* (comp. 132, (4.)), which is drawn *from* o in the *direction of* OB ,* but *not* in the *opposite direction*; because it is equivalent to

$$U \frac{\rho}{\beta} = 1; \quad \text{or} \quad \angle \frac{\rho}{\beta} = 0; \quad \text{or} \quad \rho = x\beta, \quad x > 0.$$

(5.) On the other hand the equation,

$$U \frac{\rho}{\alpha} = -U \frac{\beta}{\alpha}, \quad \text{or} \quad U\rho = -U\beta,$$

expresses (comp. 182, (5.)) that the *locus* of P is the *opposite ray* from o ; or that it is the *indefinite prolongation of the revector* BO ; because it may be transformed to

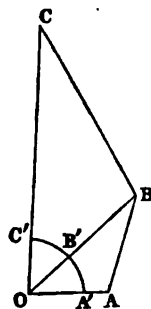


Fig. 42, *bis*.

* In 132, (4.), p. 119, OA and A ought to have been OB and B .

$$U \frac{\rho}{\beta} = -1; \quad \text{or} \quad \angle \frac{\rho}{\beta} = \pi; \quad \text{or} \quad \rho = x\beta, \quad x < 0.$$

(6.) If α, β, γ denote (as in sub-art. 2) the three lines OA, OB, OC of Fig. 42 (or of Fig. 42, bis), so that (by 149) we have the equation $\frac{\gamma}{\alpha} = \left(\frac{\beta}{\alpha}\right)^2$, then this other equation,

$$\left(U \frac{\rho}{\alpha}\right)^2 = U \frac{\gamma}{\alpha},$$

expresses *generally* that the locus of ρ is the *system* of the two last loci; or that it is the *whole indefinite right line*, both ways prolonged, through the two points o and π (comp. 144, (2.)).

(7.) But if it happen that the line γ , or OC , like OA' in Fig. 41 (or in Fig. 41, bis), has the direction *opposite* to that of α , or of OA , so that the last equation takes the *particular form*,

$$\left(U \frac{\rho}{\alpha}\right)^2 = -1,$$

then $U \frac{\rho}{\alpha}$ must be (by 154) a *right versor*; and reciprocally, *every right versor*, with a plane containing α , will be (by 158) a value satisfying the equation. In *this case*, therefore, the locus of the point ρ is (as in 182, (L), or in 144, (1.)) the *plane through o , perpendicular to the line OA* ; and the recent equation itself, if supposed to be satisfied by a *real** vector ρ , may be put under either of these two earlier but equivalent forms:

$$\angle \frac{\rho}{\alpha} = \frac{\pi}{2}; \quad \rho \perp \alpha.$$

SECTION 9.—*On Vector-Arcs, and Vector-Angles, considered as Representatives of Versors of Quaternions; and on the Multiplication and Division of any one such Versor by another.*

162. Since every *unit-vector* OA (129), drawn from the origin o , terminates in some point A on the surface of what we have called the *unit-sphere* (128), that *term* A (1) may be considered as a *Representative Point*, of which the *position* on that *surface* determines, and may be said to *represent*, the *direction of the line OA in space*; or of that line *multiplied* (12, 17) by *any positive scalar*. And then the *Quaternion* which is the *quotient* (112) of any *two* such unit-vectors, and which is in one view a *Radial* (146), and in another view a *Versor* (151), may be said to have the *arc of a great circle, AB , upon the unit sphere, which connects the terms of the two*

* Compare 149, (2.); also the second Note to the same Article; and the Notes to page 90.

vectors, for its *Representative Arc*. We may also call this arc a VECTOR ARC, on account of its having a *definite direction* (comp. Art. 1), such as is indicated (for example) by a *curved arrow* in Fig. 39, and as being thus *contrasted* with its own *opposite*, or with what may be called by analogy the *Revector Arc* BA (comp. again 1): this *latter* arc representing, on the present plan, at once the *reciprocal* (134), and the *conjugate* (137), of the former *versor*; because it represents the corresponding *Reversor* (158).

163. This mode of *representation*, of versors of quaternions by vector arcs, would obviously be very imperfect, unless *equals* were to be represented by *equals*. We shall therefore define, as it is otherwise natural to do, that a vector arc, AB, upon the unit sphere, is *equal* to every *other* vector arc CD which can be derived from it, by simply causing (or conceiving) it to *slide** in its own great circle, *without any change of length*, or *reversal of direction*. In fact, the two isosceles and plane triangles AOB, COD, which have the origin o for their common vector, and rest upon the chords of these two arcs as bases, are thus coplanar, similar, and similarly turned; so that (by 117, 118) we may here write,

$$\Delta AOB \propto COD, \quad \frac{OB}{OA} = \frac{OD}{OC};$$

the condition of the *equality of the quotients* (that is, here, of the *versors*), *represented by the two arcs*, being thus satisfied. We shall sometimes *denote* this sort of *equality of two vector arcs*, AB and CD, by the formula,

$$\sphericalangle AB = \sphericalangle CD;$$

and then it is clear (comp. 125, and the earlier Art. 3) that we shall also have, by what may be called *inversion* and *alternation*, these two *other* formulæ of *arcual equality*,

$$\sphericalangle BA = \sphericalangle DC; \quad \sphericalangle AC = \sphericalangle BD.$$

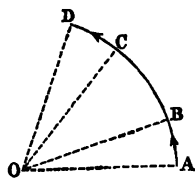


Fig. 85, bis.

(Compare the annexed Figure 35, bis.)

* Some aid to the conception may here be derived from the inspection of Fig 34; in which *two equal angles* are supposed to be traced on the surface of *one com-*

164. Conversely, *unequal versors* ought to be represented (on the present plan) by *unequal vector arcs*; and accordingly, we purpose to regard any *two such arcs*, as being, for the present purpose, *unequal* (comp. 2), even when they agree in quantity, or contain the *same number of degrees*, provided that they *differ in direction*: which may happen in either of two principal ways, as follows. For, 1st, they may be *opposite arcs* of *one great circle*; as, for example, a *vector arc* AB, and the corresponding *revector arc* BA; and so may represent (162) a *versor*, OB : OA, and the corresponding *reversor*, OA : OB, respectively. Or, 2nd, the *two arcs* may belong to *different great circles*, like AB and BC in Fig. 43; in which latter case, they represent two *radial quotients* (146) in *different planes*; or (comp. 119) *two diplanar versors*, OB : OA, and OC : OB; but it has been shown generally (122), that *diplanar quaternions* are *always unequal*: we consider therefore, here again the arcs, AB and BC, *themselves*, to be (as has been said) *unequal vectors*.

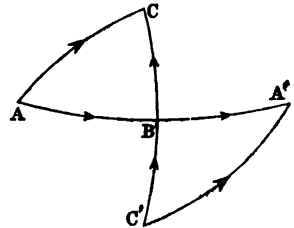


Fig. 43.

165. In this manner, then, we may be led (comp. 122) to regard the *conception of a plane*, or of the *position of a great circle* on the unit sphere, as entering, *essentially*, in general,* into the *conception of a vector-arc*, considered as the *representative of a versor* (162). But even without expressly referring to *versors*, we may see that if, in Fig. 43, we suppose that B is the middle point of an arc AA' of a great circle, so that in a recent notation (163) we may establish the *arcual equation*,

$$\frown AB = \frown BA',$$

we ought then (comp. 105) not to write *also*,

$$\frown AB = \frown BC;$$

mon desk. Or the four lines OA, OB, OC, OD, of Fig. 35, may now be conceived to be *equally long*; or to be cut by a circle with o for centre, as in the modification of that Figure, which is given in Article 163, a little lower down.

* We say, *in general*; for it will soon be seen that there is a sense in which *all great semicircles*, considered as *vector arcs*, may be said to be *equal* to each other.

because the *two co-initial arcs*, BA' and BC , which *terminate differently*, must be considered (comp. 2) to be, as *vector-arcs*, *unequal*. On the other hand, if we should refuse to admit (as in 163) that any *two coplanar arcs*, if *equally long*, and *similarly* (not *oppositely*) *directed*, like AB and CD in the recent Fig. 35, *bis*, are *equal vectors*, we could not usefully speak of *equality* between *vector-arcs* as existing under any circumstances. We are then thus led again to *include*, generally, the conception of a *plane*, or of one *great circle* as *distinguished* from another, as an *element* in the *conception* of a *Vector-Arc*. And hence an *equation* between *two* such arcs must in general be conceived to include *two relations of co-arcuality*. For example, the equation $\sphericalangle AB = \sphericalangle CD$, of Art. 163, includes generally, as a *part* of its signification, the assertion (comp. 123) that the *four points* A, B, C, D belong to one *common great circle* of the unit-sphere; or that *each* of the *two points*, C and D , is *co-arcual* with the *two other points*, A and B .

166. There is, however, a remarkable case of *exception*, in which two vector arcs may be said to be *equal*, although situated in *different planes*: namely, when they are both *great semicircles*. In fact, upon the present plan, *every* great semicircle, AA' , considered as a vector arc, *represents* an *inversor* (153); or it represents *negative unity* ($OA' : OA = -a : a = -1$), considered as one *limit* of a *versor*; but we have seen (159) that *such* a versor has in general an *indeterminate plane*. Accordingly, whereas the *initial and final points*, or (comp. 1) the *origin* A and the *term* B , of a vector arc AB , are in *general* sufficient to determine the plane of that arc, considered as the *shortest* or the *most direct path* (comp. 112, 128) from the one point to the other on the sphere; in the particular *case* when one of the two given points is diametrically *opposite* to the other, as A' to A , the *direction* of this *path* becomes, on the contrary, *indeterminate*. If then we only attend to the *effect produced*, in the way of *change of position of a point*, by a conceived *vection* (or *motion*) upon the *sphere*, we are permitted to say that *all great semicircles are equal vector arcs*; each serving simply, in the present view, to *transport a point* from *one position* to the *opposite*; and thereby to *reverse* (like the factor -1 , of which it is here the *representative*) the *direction* of the *radius* which is drawn to that point of the unit sphere.

(1.) The equation,

$$\circ \Delta \Delta' = \circ \text{BB}',$$

in which it is here supposed that Δ' is opposite to Δ , and B' to B , satisfies evidently the general conditions of co-arcuality (165); because the four points $\Delta \Delta' \text{B} \text{B}'$ are all on one great circle. It is evident that the same arcual equation admits (as in 163) of inversion and alternation; so that

$$\circ \Delta' \Delta = \circ \text{B}' \text{B}, \quad \text{and} \quad \circ \Delta \text{B} = \circ \Delta' \text{B}'.$$

(2.) We may also say (comp. 2) that all null arcs are equal, as producing no effect on the position of a point upon the sphere; and thus may write generally,

$$\circ \Delta \Delta = \circ \text{BB} = 0,$$

with the alternate equation, or identity, $\circ \Delta \text{B} = \circ \Delta \text{B}$.

(3.) Every such null vector arc $\Delta \Delta$ is a representative, on the present plan, of the other unit scalar, namely positive unity, considered as another limit of a versor (158); and its plane is again indeterminate (159), unless some law be given, according to which the arcual vection may be conceived to begin, from a given point Δ , to an indefinitely near point B upon the sphere.

167. The principal use of *Vector Arcs*, in the present theory, is to assist in representing, and (so to speak) in constructing, by means of a *Spherical Triangle*, the *Multiplication* and *Division* of any two *Diplanar Versors* (comp. 119, 164). In fact, any two such versors of quaternions (156), considered as radial quotients (152), can easily be reduced (by the general process of Art. 120) to the forms,

$$q = \beta : \alpha = \text{OB} : \text{OA}, \quad q' = \gamma : \beta = \text{OC} : \text{OB},$$

where Δ , B , C are corners of such a triangle on the unit sphere; and then (by 107), the former quotient multiplied by the latter will give for product:

$$q' \cdot q = \gamma : \alpha = \text{OC} : \text{OA}.$$

If then (on the plan of Art. 1) any two successive arcs, as ΔB and BC in Fig. 43, be called (in relation to each other) *vector* and *provector*; while that third arc ΔC , which is drawn from the initial point of the first to the final point of the second, shall be called (on the same plan) the *transvector*: we may now say that in the multiplication of any one versor (of a quaternion) by any other, if the *multiplicand** q be represented (162) by a *vector-arc* ΔB , and if the *multiplier* q' be in like manner

* Here, as in 107, and elsewhere, we write the symbol of the multiplier towards the left-hand, and that of the multiplicand towards the right.

represented by a *provector-arc* BC, which mode of representation is always possible, by what has been already shown, then the *product* $q' \cdot q$, or $q'q$, is represented, at the same time, by the *transvector-arc* AC corresponding.

168. One of the most remarkable consequences of this construction of the multiplication of versors is the following: that the value of the product of two diplanar versors (164) depends upon the order of the factors; or that $q'q$ and qq' are unequal, unless q' be *complanar* (119) with q . For let aa' and cc' be any two arcs of great circles, in different planes, bisecting each other in the point B, as Fig. 43 is designed to suggest; so that we have the two arcual equations (163),

$$\overset{\curvearrowright}{\cap} AB = \overset{\curvearrowright}{\cap} BA', \quad \text{and} \quad \overset{\curvearrowright}{\cap} BC = \overset{\curvearrowright}{\cap} C'B;$$

then one or other of the two following alternatives will hold good. Either, I_{st}, the two mutually bisecting arcs will both be *semicircles*, in which case the two new arcs, AC and $C'A'$, will indeed both belong to one great circle, namely to that of which B is a *pole*, but will have opposite directions therein; because, in this case, A' and C' will be diametrically opposite to A and C, and therefore (by 166, (1.)) the equation

$$\overset{\curvearrowright}{\cap} AC = \overset{\curvearrowright}{\cap} A'C',$$

but not the equation

$$\overset{\curvearrowright}{\cap} AC = \overset{\curvearrowright}{\cap} C'A',$$

will be satisfied. Or, II_{nd}, the arcs aa' and cc' , which are supposed to bisect each other in B, will not both be semicircles, even if one of them happen to be such; and in this case, the arcs AC, $C'A'$ will belong to two distinct great circles, so that they will be *diplanar*, and therefore *unequal*, when considered as *vectors*. (Compare the I_{st} and II_{nd} cases of Art. 164.) In each case, therefore, AC and $C'A'$ are unequal vector arcs; but the former has been seen (167) to represent the product $q'q$; and the latter represents, in like manner, the other product, qq' , of the same two versors taken in the opposite order, because it is the new transvector arc, when $C'B (= BC)$ is treated as the new vector arc, and $BA' (= AB)$ as the new provector arc, as is indicated by the curved arrows in Fig. 43. The two products,

$q'q$ and qq' , are therefore *themselves unequal*, as above asserted, under the supposed condition of *diplanarity*.

169. On the other hand, when the two factors, q and q' , are *complanar versors*, it is easy to prove, in several different ways, that their products, $q'q$ and qq' , are *equal*, as in algebra. Thus we may conceive that the arc cc' , in Fig. 43, is made to *turn* round its middle point B , until the spherical angle cBA' *vanishes*; and then the two *new transvector-arcs*, ac and $c'A'$, will evidently become not only *complanar* but *equal*, in the sense of Art. 163, as being *still equally long*, and being *now similarly directed*. Or, in Fig. 35, *bis*, of the last cited Article, we may conceive a point E , bisecting the arc BC , and therefore also the arc AD , which is *commedial* therewith (comp. Art. 2, and the second Figure 3 of that Article); and then, if we represent the one versor q by either of the two equal arcs, AE , ED , we may at the same time represent the other versor q' by either of the two other equal arcs, EC , BE ; so that the one product, $q'q$, will be represented by the arc AC , and the other product, qq' , by the equal arc BD . Or, without reference to *vector arcs*, we may suppose that the two *factors* are,

$$q = \beta : a = OB : OA, \quad q' = \gamma : a = OC : OA,$$

OA , OB , OC being any three *complanar* and *equally long* right lines (see again Fig. 35, *bis*); for thus we have only to determine a fourth line, δ or OD , of the same length, and in the same plane, which shall satisfy the equation $\delta : \gamma = \beta : a$ (117), and therefore also (by 125) the alternate equation, $\delta : \beta = \gamma : a$; and it will then immediately follow* (by 107), that

$$q' \cdot q = \frac{\delta}{\beta} \cdot \frac{\beta}{a} = \frac{\delta}{a} = \frac{\delta}{\gamma} \cdot \frac{\gamma}{a} = q \cdot q'.$$

We may therefore infer, for *any two versors* of quaternions, q and q' , the two following reciprocal relations :

* It is evident that, in this last process of reasoning, we make *no use* of the supposed *equality of lengths* of the four lines compared; so that we might prove, in exactly the same way, that $q'q = qq'$ if $q' ||| q$ (128), *without* assuming that these two *complanar factors*, or *quaternions*, q and q' , are *versors*.

- I. . . $q'q = qq'$, if $q' \parallel\parallel q$ (123);
 II. . . if $q'q = qq'$, then $q' \parallel\parallel q$ (168);

convertibility of factors (as regards their *places* in the *product*) being thus at once a *consequence* and a *proof* of *complanarity*.

170. In the 1st case of Art. 168, the *factors* q and q' are both *right versors* (153); and because we have seen that then their *two products*, $q'q$ and qq' , are versors *represented* by equally long but *oppositely directed arcs* of one great circle, as in the 1st case of 164, it follows (comp. 162), that these two products are at once *reciprocal* (134), and *conjugate* (137), to each other; or that they are related as *versor* and *reversor* (158). We may therefore write, generally,

$$\text{I. . . } qq' = Kq'q, \text{ and II. . . } qq' = \frac{1}{q'q},$$

if q and q' be *any two right versors*; because the multiplication of any two *such* versors, in two opposite orders, may always be represented or constructed by a Figure such as that lately numbered 43, in which the bisecting arcs $\Delta\Delta'$ and cc' are *semicircles*. The IInd formula may also be thus written (comp. 135, 154):

$$\text{III. . . if } q^2 = -1, \text{ and } q'^2 = -1, \text{ then } q'q \cdot qq' = +1;$$

and under this form it evidently agrees with ordinary algebra, because it expresses that, *under the supposed conditions*,

$$q'q \cdot qq' = q^2 \cdot q'^2;$$

but it will be found that this last equation is *not an identity*, in the general theory of *quaternions*.

171. If the two bisecting semicircles cross each other *at right angles*, the conjugate products are represented by *two quadrants, oppositely turned*, of one great circle. It follows that *if two right versors, in two mutually rectangular planes, be multiplied together in two opposite orders, the two resulting products will be two opposite right versors, in a third plane, rectangular to the two former*; or in symbols, that

$$\text{if } q^2 = -1, q'^2 = -1, \text{ and Ax. } q' \perp \text{Ax. } q,$$

then

$$(q'q)^2 = (qq')^2 = -1, \quad q'q = -qq';$$

and

$$\text{Ax. } q'q \perp \text{Ax. } q, \quad \text{Ax. } q'q \perp \text{Ax. } q'.$$

In *this* case, therefore, we have what would be in algebra a *paradox*, namely the equation,

$$(q'q)^2 = -q^2 \cdot q'^2,$$

if q and q' be any two right versors, in two rectangular planes; but we see that this result is *not more* paradoxical, in appearance, than the equation

$$q'q = -qq'$$

which exists, *under the same conditions*. And when we come to examine what, in the last analysis, may be said to be the *meaning* of this last equation, we find it to be simply this: that any two quadrantal or right rotations, in planes perpendicular to each other, compound themselves into a third right rotation, as their resultant, in a plane perpendicular to each of them: and that this third or resultant rotation has one or other of two opposite directions, according to the order in which the two component rotations are taken, so that one shall be successive to the other.

172. We propose to return, in the next Section, to the consideration of such a *System of Right Versors*, as that which we have here briefly touched upon: but desire at present to remark (comp. 167) that a *spherical triangle* ABC may serve to construct, by means of *representative arcs* (162), not only the *multiplication*, but also the *division*, of any one of two diplanar versors (or radial quotients) by the other. In fact, we have only to conceive (comp. Fig. 43) that the *vector arc* AB represents a given *divisor*, say q , or $\beta : a$, and that the *transvector arc* AC (167) represents a given *dividend*, suppose q'' , or $\gamma : a$; for then the *provector arc* BC (comp. again 167) will represent, on the same plan, the *quotient of these two versors*, namely $q'' : q$, or $\gamma : \beta$ (106), or the versor lately called q' ; since we have generally, by 106, 107, 120, for quaternions, as in algebra, the two identities:

$$(q'' : q) \cdot q = q''; \quad q'q : q = q'.$$

173. It is however to be observed that, for reasons already assigned, we must *not* employ, for *diplanar versors*, such an equation as $q \cdot (q'' : q) = q''$; because we have found (168) that, for *such* versors, the ordinary *algebraic identity*, $qq' = q'q$, *ceases to be true*. In fact by 169, we may now establish the two converse formulæ:

$$\text{I.} \quad \dots \quad q(q'' : q) = q'', \quad \text{if} \quad q'' \parallel\parallel q \quad (123);$$

$$\text{II.} \quad \dots \quad \text{if} \quad q(q'' : q) = q'', \quad \text{then} \quad q'' \parallel\parallel q.$$

Accordingly, in Fig. 43, if q , q' , q'' be still represented by the arcs AB , BC , AC , the product $q(q'' : q)$, or qq' , is *not* represented by

AC, but by the *different arc* $C'A'$ (168), which as a *vector arc* has been seen to be *unequal* thereto: although it is true that these two last arcs, AC and $C'A'$, are always *equally long*, and therefore subtend *equal angles* at the centre O of the unit sphere; so that we may write, generally, for *any two versors* (or indeed for *any two quaternions*),* q and q'' , the formula,

$$\angle q (q'' : q) = \angle q''.$$

174. *Another mode of Representation of Versors*, or rather two such new modes, although intimately connected with each other, may be briefly noticed here.

Ist. We may consider the *angle* $\angle AOB$, at the centre O of the unit-sphere, when conceived to have not only a definite *quantity*, but also a determined *plane* (110), and a given *direction* therein (as indicated by one of the curved arrows in Fig. 39, or by the arrow in Fig. 33), as being what may be called by analogy a *Vector-Angle*; and may say that it *represents*, or that it is the *Representative Angle* of, the *Versor* $OB : OA$, where OA, OB are radii of the unit-sphere.

Ind. Or we may replace this *rectilinear angle* $\angle AOB$ at the centre, by the equal *Spherical Angle* $\angle AC'B$, at what may be called the *Positive Pole* of the *representative arc* AB; so that $C'A$ and $C'B$ are *quadrants*; and the *rotation*, at this *pole* C' , from the first of these two quadrants to the second (as seen from a point outside the sphere), has the *direction* which has been selected (111, 127) for the *positive one*, as indicated in the annexed Figure 44: and then we may consider this *spherical angle* as a *new Angular Representative* of the same *versor* q , or $OB : OA$, as before.

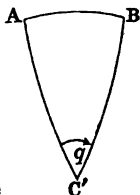


Fig. 44.

175. Conceive now that after employing a *first spherical triangle* ABC, to *construct* (as in 167) the *multiplication* of any one given *versor* q , by any other given *versor* q' , we form a *second* or *polar triangle*, of which the corners A' , B' , C' shall be respectively (in the sense just stated) the *positive poles* of the three *successive sides*, BC, CA, AB, of the former triangle; and that then we pass to a *third triangle* $A'B''C'$, as part of the *same lune* $B'B''$ with the second, by taking for B'' the point diametrically *opposite* to B' ; so that B'' shall be

* It will soon be seen that several of the formulæ of the present Section, respecting the *multiplication* and *division* of *versors*, considered as *radial quotients* (151), require little or no modification, in the passage to the corresponding *operations on quaternions*, considered as *general quotients of vectors* (112).

the *negative* pole of the arc CA , or the *positive* pole of what was lately called (167) the *transvector-arc* AC : also let c'' be, in like manner, the point opposite to c' on the unit sphere. Then we may not only write (comp. 129),

$Ax. q = oc'$, $Ax. q' = oA'$, $Ax. q'q = oB''$,
but shall also have the equations,

$\angle q = B''C'A'$, $\angle q' = C'A'B''$, $\angle q'q = C'B''A'$;

these three spherical angles, namely the two *base-angles* at c' and A' , and the *external vertical angle* at B'' , of the new or third triangle $A'B''C'$, will therefore represent, respectively, on the plan of 174, II., the *multiplicand*, q , the *multiplier*, q' , and the *product*, $q'q$. (Compare the annexed Figure 45.)

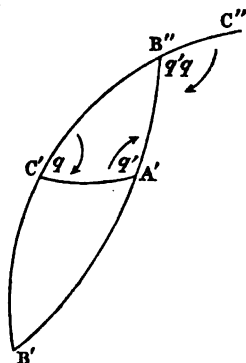


Fig. 45.

176. Without expressly referring to the former triangle ABC , we can connect this last construction of multiplication of versors (175) with the general formula (107), as follows.

Let a and β be now conceived to be two *unit-tangents** to the sphere at c' , perpendicular respectively to the two arcs $c'B''$ and $c'A'$, and drawn towards the same sides of those arcs as the points A' and B' respectively; and let two other unit-tangents, equal to these, and denoted by the same letters, be drawn (as in the annexed Figure 45, *bis*) at the points B'' and A' , so as to be normal there to the same arcs $c'B''$ and $c'A'$, and to fall towards the same sides of them as before. Let also two other unit-tangents, equal to each B' other, and each denoted by γ , be drawn at the two last points B'' and A' , so as to be both perpendicular to the arc $A'B''$, and to fall towards the same side of it as the point c' . Then (comp. 174, II.) the two *quotients*, $\beta : a$ and $\gamma : \beta$, will be equal to the two *versors*, q and q' , which were lately represented (in Fig. 45) by the

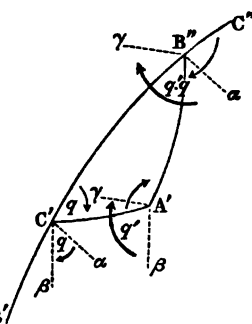


Fig. 45, bis.

* By an *unit tangent* is here meant simply an *unit line* (or *unit vector*, 129) so drawn as to be *tangential to the unit-sphere*, and to have its *origin*, or its *initial point* (1), on the *surface* of that sphere, and not (as we have usually supposed) at the *centre* thereof.

two base angles, at C' and A' , of the spherical triangle $A'B'C'$; the product, $q'q$, of these two versors, is therefore (by 107) equal to the third quotient, $\gamma : a$; and consequently it is represented, as before, by the external vertical angle $C''B'A'$ of the same triangle, which is evidently equal in quantity to the angle of this third quotient, and has the same axis OB'' , and the same direction of rotation, as the arrows in Fig. 45, *bis*, may assist to show.

177. In each of the two last Figures, the internal vertical angle at B'' is thus equal to the Supplement, $\pi - \angle q'q$, of the angle of the product; and it is important to observe that the corresponding rotation at the vertex B'' , from the side $B''A'$ to the side $B''C'$, or (as we may briefly express it) from the point A' to the point C' , is positive; a result which is easily seen to be a general one, by the reasoning of the foregoing Article.* We may then infer, generally, that when the multiplication of any two versors is constructed by a spherical triangle, of which the two base angles represent (as in the two last Articles) the factors, while the external vertical angle represents the product, then the rotation round the axis (OB'') of that product $q'q$, from the axis (OA') of the multiplier q' , to the axis (OC') of the multiplicand q , is positive: whence it follows that the rotation round the axis Ax . q' of the multiplier, from the axis Ax . q of the multiplicand, to the axis Ax . $q'q$ of the product, is also positive. Or, to express the same thing more fully, since the only rotations hitherto considered have been plane ones (as in 128, &c.), we may say that if the two latter axes be projected on a plane perpendicular to the former, so as still to have a common origin o , then the rotation round Ax . q' , from the projection of Ax . q to the projection of Ax . $q'q$, will be directed (with our conventions) towards the right hand.

178. We have therefore thus a new mode of geometrically exhibiting the inequality of the two products, $q'q$ and qq' , of two diplanar versors (168), when taken as factors in two different orders. For this purpose, let

$$Ax.q = OP, Ax.q' = OQ, Ax.q'q = OR;$$

and prolong to some point s the arc PR of a great circle on the unit sphere. Then, for the spherical triangle PQR , by prin-

* If a person be supposed to stand on the sphere at B'' , and to look towards the arc $A'C'$, it would appear to him to have a right-handed direction, which is the one here adopted as positive (127).

principles lately established, we shall have (comp. 175) the following values of the two internal base angles at P and Q, and of the external vertical angle at R :

$$RPQ = \angle q; \quad PQR = \angle q'; \quad SRQ = \angle q'q;$$

and the rotation at Q, from the side QP to the side QR will be right-handed. Let fall an arcual perpendicular, RT, from the vertex R on the base PQ, and prolong this perpendicular to r', in such a manner as to have

$$\sphericalangle RT = \sphericalangle TR';$$

also prolong PR' to some point s'. We shall then have a new triangle PQR', which will be a sort of *reflexion* (comp. 138) of the old one with respect to their common base PQ; and this *new triangle* will serve to *construct* the *new product*, qq' . For the rotation at P from PQ to PR' will be right-handed, as it ought to be; and we shall have the equations,

$$QPR' = \angle q; \quad R'QP = \angle q'; \quad QR'S' = \angle qq'; \quad OR' = Ax.qq';$$

so that the *new external and spherical angle*, QR'S', will represent the *new versor*, qq' , as the *old angle* SRQ represented the *old versor*, $q'q$, obtained from a different order of the factors. And although, no doubt, these *two angles*, at R and R', are always equal in quantity, so that we may establish (comp. 173) the *general formula*,

$$\angle q'q = \angle qq',$$

yet as *vector angles* (174), and therefore as *representatives of versors*, they must be considered to be *unequal*: because they have *different planes*, namely, the *tangent planes* to the sphere at the *two vertices* R and R'; or the two planes respectively *parallel* to these, which are drawn *through the centre* o.

179. *Division of Versors* (comp. 172) can be constructed by means of *Representative Angles* (174), as well as by *representative arcs* (162). Thus to *divide* q'' by q , or rather to *represent* such division *geometrically*, on a plan entirely similar to that last employed for

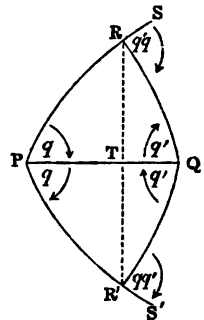


Fig. 46.

multiplication, we have only to determine the two points P and R , in Fig. 46, by the two conditions,

$$OP = Ax. q, \quad OR = Ax. q',$$

and then to find a third point Q by the two angular equations,

$$RPQ = \angle q, \quad QRP = \varphi - \angle q',$$

the rotation round P from PR towards PQ being positive; after which we shall have,

$$Ax. (q'' : q) = OQ; \quad \angle (q'' : q) = PQR.$$

(1.) Instead of conceiving, in Fig. 46, that the dotted line PRR' , which connects the vertices of the two triangles, with PQ for their common base (178), is an arc of a great circle, perpendicularly bisected by that base, we may imagine it to be an arc of a small circle, described with the point P for its positive pole (comp. 174, II.). And then we may say that the passage (comp. 173) from the versor q'' , or $q'q$, to the unequal versor $q (q'' : q)$, or qq' , is geometrically performed by a Conical Rotation of the Axis $Ax. q''$, round the axis $Ax. q$, through an angle $= 2 \angle q$, without any (quantitative) change of the angle $\angle q''$; so that we have, as before, the general formula (comp. again 173),

$$\angle q (q'' : q) = \angle q''.$$

(2.) Or if we prefer to employ the construction of multiplication and division by representative arcs, which Fig. 43 was designed to illustrate, and conceive that a new point c'' is determined in that Figure by the condition $\wedge A'c'' = \wedge c'A'$, we may then say that in the passage from the versor q'' , which is represented by ΔC , to the versor $q (q'' : q)$, represented by $c'A'$ or by $A'c''$, the representative arc of q'' is made to move, without change of length, so as to preserve a constant inclination* to the representative arc ΔB of q , while its initial point describes the double of that arc ΔB , in passing from A to A' .

(3.) It may be seen, by these few Examples, that if, even independently of some new characteristics of operation, such as K and U , new combinations of old symbols, such as $q (q'' : q)$, occur in the present Calculus, which are not wanted in Algebra, they admit for the most part of geometrical interpretations, of an easy and interesting kind; and in fact represent conceptions, which cannot well be dispensed with, and which it is useful to be able to express, with so much simplicity and conciseness. (Compare the remarks in Art. 161; and the sub-articles to 132, 145.)

180. In connexion with the construction indicated by the two Figures 45, it may be here remarked, that if ΔBC be any spherical triangle, and if A', B', c' be (as in 175) the positive poles of its three successive sides, $BC, CA, \Delta B$, then the rotation (comp. 177, 179) round A' from B' to c' , or that round B' from

* In a manner analogous to the motion of the equator on the ecliptic, by lunar precession, in astronomy.

c' to A' , &c., is *positive*. The easiest way, perhaps, of *seeing* the truth of this assertion, is to conceive that if the rotation round A from B to C be not *already positive*, we *make* it such, by passing to the diametrically *opposite triangle* on the sphere, which will not change the *poles* A' , B' , C' . Assuming then that these poles are thus the *near* ones to the corresponding corners of the given triangle, we arrive without any difficulty at the conclusion stated above: which has been virtually employed in our *construction* of *multiplication* (and *division*) of *versors*, by means of *Representative Angles* (175, 176); and which may be otherwise justified (as before), by the consideration of the *unit-tangents* of Fig. 45, *bis*.

(1.) Let then α , β , γ be *any three given unit vectors*, such that the rotation round the first, from the second to the third, is *positive* (in the sense of Art. 177); and let α' , β' , γ' be three other unit vectors, derived from these by the equations,

$$\alpha' = \text{Ax.}(\gamma : \beta), \quad \beta' = \text{Ax.}(\alpha : \gamma), \quad \gamma' = \text{Ax.}(\beta : \alpha);$$

then the rotation round α' , from β' to γ' , will be *positive also*; and we shall have the converse formulæ,

$$\alpha = \text{Ax.}(\gamma' : \beta'), \quad \beta = \text{Ax.}(\alpha' : \gamma'), \quad \gamma = \text{Ax.}(\beta' : \alpha').$$

(2.) If the rotation round α from β to γ were given to be *negative*, α' , β' , γ' being still deduced from those three vectors by the same three equations as before, then the *signs* of α , β , γ would all require to be *changed*, in the three last (or *reciprocal*) formulæ; but the rotation round α' , from β' to γ' , would still be *positive*.

(3.) Before closing this Section, it may be briefly noticed, that it is sometimes convenient, from motives of analogy (comp. Art. 5), to speak of the *Transvector-Arc* (167), which has been seen to *represent a product* of two versors, as being the *ARCUAL SUM* of the two *successive vector-arcs*, which represent (on the same plan) the *factors*; *Provector* being still said to be *added to Vector*: but the *Order* of such *Addition of Diplanar Arcs* being *not now indifferent* (168), as the corresponding *order* had been early found (in 7) to be, when the *vectors* to be added were *right lines*.

(4.) We may also speak occasionally, by an extension of the same analogy, of the *External Vertical Angle* of a spherical triangle, as being the *SPHERICAL SUM* of the two *Base Angles* of that triangle, taken in a suitable *order of summation* (comp. Fig. 46); the *Angle* which represents (174) the *Multiplier* being then said to be *added* (as a sort of *Angular Provector*) to that *other Vector-Angle* which represents the *Multiplicand*; whilst what is here called the *sum* of these two angles (and is, with respect to them, a species of *Transvector-Angle*) represents, as has been proved, the *Product*.

(5.) This conception of *angular transvection* becomes perhaps a little more clear, when (on the plan of 174, 1.) we assume the *centre* O as the *common vertex* of three angles AOB , BOC , AOC , situated generally in *three different planes*. For then we may

conceive a revolving radius to be either carried by two successive angular motions, from OA to OB, and thence to OC; or to be transported immediately, by one such motion, from the first to the third position.

(6.) Finally, as regards the construction indicated by Fig. 45, bis, in which tangents instead of radii were employed, it may be well to remark distinctly here, that $\Delta''C'$, in that Figure, may be any given spherical triangle, for which the rotation round B'' from A' to C' is positive (177); and that then, if the two factors, q and q' , be defined to be the two versors, of which the internal angles at C' and A' are (in the sense of 174, II.) the representatives, the reasonings of Art. 176 will prove, without necessarily referring, even in thought, to any other triangle (such as ΔBC), that the external angle at B'' is (in the same sense) the representative of the product, $q'q$, as before.

SECTION 10.—On a System of Three Right Versors, in Three Rectangular Planes; and on the Laws of the Symbols, i, j, k .

181. Suppose that OI, OJ, OK are any three given and co-initial but rectangular unit-lines, the rotation round the first from the second to the third being positive; and let OI', OJ', OK' be the three unit-vectors respectively opposite to these, so that

$$OI' = -OI, \quad OJ' = -OJ, \quad OK' = -OK.$$

Let the three new symbols i, j, k denote a system (comp. 172) of three right versors, in three mutually rectangular planes, with the three given lines for their respective axes; so that

$$Ax.i = OI, \quad Ax.j = OJ, \quad Ax.k = OK,$$

and

$$i = OK : OJ, \quad j = OI : OK, \quad k = OJ : OI,$$

as Figure 47 may serve to illustrate.

We shall then have these other expressions for the same three versors:

$$i = OJ' : OK = OK' : OJ' = OJ : OK';$$

$$j = OK' : OI = OI' : OK' = OK : OI';$$

$$k = OI' : OJ = OJ' : OI' = OI : OJ';$$

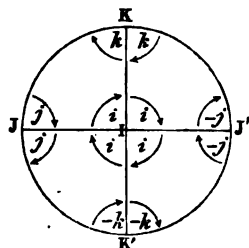


Fig. 47.

while the three respectively opposite versors may be thus expressed:

$$-i = OJ : OK = OK' : OJ = OJ' : OK' = OK : OJ';$$

$$-j = OK : OI = OI' : OK = OK' : OI' = OI : OK';$$

$$-k = OI : OJ = OJ' : OI = OI' : OJ' = OJ : OI'.$$

And from the comparison of these different expressions several important symbolical consequences follow, which it will be worth while to enunciate separately here, although some of them are virtually included in the results of former Sections.

182. In the *first* place, since

$$i^2 = (OJ' : OK) \cdot (OK : OJ) = OJ' : OJ, \text{ \&c.,}$$

we deduce (comp. 148) the following equal values for the *squares* of the new symbols:

$$\text{I. . . } i^2 = -1; \quad j^2 = -1; \quad k^2 = -1;$$

as might indeed have been at once inferred (154), from the circumstance that the three radial quotients (146), denoted here by i, j, k , are all *right versors* (181).

In the *second* place, since

$$ij = (OJ : OK') \cdot (OK' : OI) = OJ : OI, \text{ \&c.,}$$

we have the following values for the *products* of the same three symbols, or versors, when taken *two by two*, and in a certain *order of succession* (comp. 168, 171):

$$\text{II. . . } ij = k; \quad jk = i; \quad ki = j.$$

But in the *third* place (comp. again 171), since

$$j \cdot i = (OI : OK) \cdot (OK : OJ) = OI : OJ, \text{ \&c.,}$$

we have these other and *contrasted* formulæ, for the *binary products* of the *same* three right versors, when taken as factors with an *opposite order*:

$$\text{III. . . } ji = -k; \quad kj = -i; \quad ik = -j.$$

Hence, while the *square* of each of the *three right versors*, denoted by these *three new symbols*, ijk , is equal (154) to *negative unity*, the *product* of any *two* of them is equal either to the *third itself*, or to the *opposite* (171) of that third versor, according as the *multiplier precedes* or *follows* the *multiplier*, in the *cyclical succession*,

$$i, j, k, \quad i, j, \dots$$

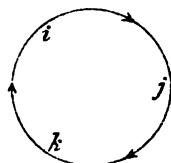


Fig. 47, bis.

which the annexed Figure 47, *bis*, may give some help towards remembering.

(1.) To connect such multiplications of i, j, k with the theory of representative arcs (162), and of representative angles (174), we may regard any one of the four *quadrantal arcs*, $JK, KJ', J'K', K'J$, in Fig. 47, or any one of the four *spherical right angles*, $JIK, KIJ', J'IK', K'IJ$, which those arcs subtend at their common pole I , as representing the versor i ; and similarly for j and k , with the introduction of the point i' opposite to i , which is to be conceived as being at the back of the Figure.

(2.) The *squaring* of i , or the equation $i^2 = -1$, comes thus to be geometrically constructed by the *doubling* (comp. Arts. 148, 154, and Figs. 41, 42) of an arc, or of an angle. Thus, we may conceive the *quadrant* KJ' to be added to the equal arc JK , their sum being the *great semicircle* JJ' , which (by 166) represents an *inversor* (158), or *negative unity* considered as a *factor*. Or we may add the *right angle* KIJ' to the equal angle JIK , and so obtain a *rotation* through two right angles at the pole I , or at the centre O ; which rotation is equivalent (comp. 154, 174) to an *inversion of direction*, or to a passage from the radius OJ , to the opposite radius OJ' .

(3.) The *multiplication* of j by i , or the equation $ij = k$, may in like manner be *arcually constructed*, by the addition of $K'J$, as a *provector-arc* (167), to IK' as a *vector-arc* (162), giving IJ , which is a *representative* of k , as the *transvector-arc*, or *arcual-sum* (180, (3.)). Or the same multiplication may be *angularly constructed*, with the help of the *spherical triangle* IJK ; in which the *base-angles* at I and J represent respectively the *multiplier*, i , and the *multiplicand*, j , the rotation round I from J to K being *positive*: while their *spherical sum* (180, (4.)), or the *external vertical angle* at K (comp. 175, 176), represents the same *product*, k , as before.

(4.) The *contrasted multiplication* of i by j , or of j into* i , may in like manner be *constructed*, or geometrically represented, either by the addition of the arc KI , as a *new provector*, to the arc JK as a new vector, which new process gives JI (instead of IJ) as the *new transvector*; or with the aid of the *new triangle* IJK' (comp. Figs. 46, 47), in which the rotation round I from J to the *new vertex* K' is *negative*, so that the angle at I represents now the *multiplicand*, and the resulting angle at the *new pole* K' represents the *new and opposite product*, $ji = -k$.

183. Since we have thus $ji = -ij$ (as we had $q'q = -qq'$ in 171), we see that the *laws of combination of the new symbols*, i, j, k , are *not in all respects the same* as the corresponding laws in *algebra*; since the *Commutative Property of Multiplication*, or the *convertibility* (169) of the places of the *factors* without change of value of the *product*, does *not here hold good*: which arises (168) from the circumstance, that the factors to be combined are here *diplanar versors* (181). It is therefore important to observe, that there *is* a respect in which

* A multiplicand is said to be multiplied *by* the multiplier; while, on the other hand, a multiplier is said to be multiplied *into* the multiplicand: a *distinction* of this sort between the *two factors* being necessary, as we have seen, for *quaternions*, although it is not needed for algebra.

the laws of i, j, k agree with usual and algebraic laws: namely, in the *Associative Property of Multiplication*; or in the property that the new symbols always obey the *associative formula* (comp. 9),

$$\iota \cdot \kappa \lambda = \iota \kappa \cdot \lambda,$$

whichever of them may be substituted for ι , for κ , and for λ ; in virtue of which equality of values we may omit the point, in any such symbol of a *ternary product* (whether of equal or of unequal factors), and write it simply as $\iota \kappa \lambda$. In particular we have thus,

$$i \cdot jk = i \cdot i = i^2 = -1; \quad ij \cdot k = k \cdot k = k^2 = -1;$$

or briefly,

$$ijk = -1.$$

We may, therefore, by 182, establish the following important *Formula*:

$$i^2 = j^2 = k^2 = ijk = -1; \quad (A)$$

to which we shall occasionally refer, as to "Formula A," and which we shall find to contain (virtually) *all the laws of the symbols ijk* , and therefore to be a *sufficient symbolical basis* for the whole *Calculus of Quaternions*:* because it will be shown that *every quaternion can be reduced to the Quadrinomial Form*,

$$q = w + ix + jy + kz,$$

where w, x, y, z compose a *system of four scalars*, while i, j, k are the same *three right versors* as above.

(1.) A direct proof of the equation, $ijk = -1$, may be derived from the definitions of the symbols in Art. 181. In fact, we have only to remember that those definitions were seen to give,

* This formula (A) was accordingly made the *basis* of that Calculus in the first communication on the subject, by the present writer, to the Royal Irish Academy in 1843; and the letters, i, j, k , continued to be, for some time, the *only peculiar symbols* of the calculus in question. But it was gradually found to be useful to incorporate with these a few other *notations* (such as K and U , &c.), for representing *Operations on Quaternions*. It was also thought to be instructive to establish the *principles* of that Calculus, on a more *geometrical* (or less exclusively *symbolical*) *foundation* than at first; which was accordingly afterwards done, in the volume entitled: *Lectures on Quaternions* (Dublin, 1858); and is again attempted in the present work, although with many differences in the adopted *plan* of exposition, and in the *applications* brought forward, or suppressed.

$$i = o\j' : o\kappa, \quad j = o\kappa : o\j', \quad k = o\j' : o\j;$$

and to observe that, by the general formula of multiplication (107), whatever four lines may be denoted by $\alpha, \beta, \gamma, \delta$, we have always,

$$\frac{\delta}{\gamma} \cdot \frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\delta}{\gamma} \cdot \frac{\gamma}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\delta}{\gamma} \cdot \frac{\beta}{\alpha};$$

or briefly, as in algebra,

$$\frac{\delta}{\gamma} \cdot \frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\delta}{\alpha},$$

the point being thus omitted without danger of confusion : so that

$$ijk = o\j' : o\j = -1, \text{ as before.}$$

Similarly, we have these two other ternary products :

$$jki = (o\kappa' : o\j) (o\j : o\j') (o\j' : o\kappa) = o\kappa' : o\kappa = -1 ;$$

$$kij = (o\j' : o\j) (o\j : o\kappa') (o\kappa' : o\j) = o\j' : o\j = -1.$$

(2.) On the other hand,

$$kji = (o\j : o\j') (o\j' : o\kappa) (o\kappa : o\j) = o\j : o\j = +1 ;$$

and in like manner,

$$ikh = +1, \text{ and } jik = +1.$$

(3.) The equations in 182 give also these other ternary products, in which the law of association of factors is still obeyed :

$$i . ij = ik = -j = i^2j = ii . j, \quad iij = -j ;$$

$$i . ji = i . -k = -ik = j = ki = ij . i, \quad iji = +j ;$$

$$i . jj = i . -1 = -i = hj = ij . j, \quad ijj = -i ;$$

with others deducible from these, by mere cyclical permutation of the letters, on the plan illustrated by Fig. 47, bis.

(4.) In general, if the Associative Law of Combination exist for any three symbols whatever of a given class, and for a given mode of combination, as for addition of lines in Art. 9, or for multiplication of ijk in the present Article, the same law exists for any four (or more) symbols of the same class, and combinations of the same kind. For example, if each of the four letters $\iota, \kappa, \lambda, \mu$ denote some one of the three symbols i, j, k (but not necessarily the same one), we have the formula,

$$\iota . \kappa\lambda\mu = \iota . \kappa . \lambda\mu = \iota\kappa . \lambda\mu = \iota\kappa . \lambda . \mu = \iota\kappa\lambda . \mu = \iota\kappa\lambda\mu.$$

(5.) Hence, any multiple (or complex) product of the symbols ijk , in any manner repeated, but taken in one given order, may be interpreted, with one definite result, by any mode of association, or of reduction to partial factors, which can be performed without commutation, or change of place of the given factors. For example, the symbol $ijkkji$ may be interpreted in either of the two following (among other) ways :

$$ij . kk . ji = ij . -ji = i . -j^2 . i = ii = -1 ; \quad ijk . kji = -1 . 1 = -1.$$

184. The formula (Δ) of 183 includes obviously the three equations (I.) of 182. To show that it includes also the six other equations, (II.), (III.), of the last cited Article, we may observe that it gives, with the help of the associative principle of multiplication (which may be suggested to the memory by the absence of the point in the symbol ijk),

$$\begin{aligned} ij &= -ji, \quad kk = -ijk, \quad k = +k; & jk &= -i, \quad ijk = +i; \\ ji &= j, \quad jk = j^2k = -k; & ik &= i, \quad ij = i^2j = -j; \\ kj &= ij, \quad j = ij^2 = -i; & ki &= -k^2j = -ji^2 = +j. \end{aligned}$$

And then it is easy to prove, *without any reference to geometry*, if the foregoing laws of the symbols be admitted, that we have also,

$$jki = kij = -1, \quad kji = jik = ikj = +1,$$

as otherwise and *geometrically* shown in recent sub-articles. It may be added that the mere inspection of the formula (A) is sufficient to show that the *three** square roots of negative unity, denoted in it by i, j, k , cannot be subject to all the ordinary rules of algebra: because that formula gives, at sight,

$$i^2 j^2 k^2 = (-1)^2 = -1 = -(ijk)^2;$$

the *non-commutative character* (183), of the multiplication of such roots among themselves, being thus put in evidence.

SECTION 11.—*On the Tensor of a Vector, or of a Quaternion; and on the Product or Quotient of any two Quaternions.*

185. Having now sufficiently availed ourselves, in the two last Sections, of the conceptions (alluded to, so early as in the First Article of these Elements) of a *vector-arc* (162), and of a *vector-angle* (174), in illustration† of the laws of multiplication and division of versors of quaternions; we propose to return to that use of the word, VECTOR, with which alone the First Book, and the first eight Sections of this First Chapter of the Second Book, have been concerned: and shall therefore henceforth mean again, *exclusively*, by that word “vector,” a *Directed Right Line* (as in 1). And because we have already considered and expressed the *Direction* of any such line, by

* It is evident that $-i, -j, -k$ are also, on the same principles, values of the symbol $\sqrt{-1}$; because they also are *right versors* (153); or because $(-q)^2 = q^2$. More generally (comp. a Note to page 181), if x, y, z be any three scalars which satisfy the condition $x^2 + y^2 + z^2 = 1$, it will be proved, at a later stage, that

$$(ix + jy + kz)^2 = -1.$$

† One of the chief uses of such vectors, in connexion with those laws, has been to illustrate the *non-commutative property* (168) of multiplication of versors, by exhibiting a corresponding property of what has been called, by analogy to the earlier operation of the same kind on linear vectors (5), the *addition of arcs and angles* on a sphere. Compare 180, (3.), (4.).

introducing the conception and notation (155) of the *Unit-Vector*, U_a , which has the *same direction* with the line a , and which we have proposed (156) to call *the Versor of that Vector*, a ; we now propose to consider and express the *Length* of the same line a , by introducing the *new name* TENSOR, and the *new symbol*,* T_a ; which latter symbol we shall read, as *the Tensor of the Vector* a : and shall define it to be, or to denote, the *Number* (comp. again 155) which represents the *Length* of that line a , by expressing the *Ratio* which that length bears to some assumed standard, or *Unit* (128).

186. To connect more closely these two conceptions, of the *versor* and the *tensor* of a *vector*, we may remember that when we employed (in 155) the letter a as a temporary symbol for the number which thus expresses the length of the line a , we had the equation, $U_a = a : a$, as one form of the *definition* of the *unit-vector* denoted by U_a . We might therefore have written also these two other forms of equation (comp. 15, 16),

$$a = a \cdot U_a, \quad a = a : U_a,$$

to express the dependence of the *vector*, a , and of the *scalar*, a , on each other, and on what has been called (156) the *versor*, U_a . For example, with the construction of Fig. 42, *bis* (comp. 161, (2.)), we may write the three equations,

$$a = OA : OA', \quad b = OB : OB', \quad c = OC : OC',$$

if a, b, c be thus the *three positive scalars*, which denote the *lengths* of the three lines, OA, OB, OC ; and these three scalars may then be considered as *factors*, or as *coefficients* (12), by which the *three unit-vectors* U_a, U_β, U_γ , or OA', OB', OC' (in the cited Figure), are to be respectively *multiplied* (15), in order to change them into the three other vectors a, β, γ , or OA, OB, OC , by *altering their lengths*, without any change in their *directions*. But such an exclusive *Operation*, on the *Length* (or on the *extension*) of a line, may be said to be an *Act of Tension*;† as an operation on *direction alone* may be called (comp. 151) an *act of version*. We have then thus a *motive*

* Compare the Note to Art. 155.

† Compare the Note to Art. 156, in page 135.

for the introduction of the *name, Tensor*, as applied to the *positive number* which (as above) represents the *length* of a line. And when the *notation* Ta (instead of a) is employed for such a *tensor*, we see that we may write generally, for any *vector* a , the equations (compare again 15, 16) :

$$Ua = a : Ta; \quad Ta = a : Ua; \quad a = Ta \cdot Ua = Ua \cdot Ta.$$

For example, if a be an unit-vector, so that $Ua = a$ (160), then $Ta = 1$; and therefore, generally, *whatever vector* may be denoted by a , we have always,

$$TUa = 1.$$

For the same reason, *whatever quaternion* may be denoted by q , we have always (comp. again 160) the equation,

$$T(Ax \cdot q) = 1.$$

(1.) Hence the equation

$$T\rho = 1,$$

where $\rho = OP$, expresses that the *locus* of the variable point P is the surface of the *unit sphere* (128).

(2.) The equation $T\rho = Ta$ expresses that the *locus* of P is the spheric surface with o for centre, which passes through the point A .

(8.) On the other hand, for the sphere through o , which has its *centre* at A , we have the equation,

$$T(\rho - a) = Ta; \quad AP = OP - OA$$

which expresses that the lengths of the two lines, AP , AO , are equal. $\therefore T(AP) = T(\rho - a)$

(4.) More generally, the equation,

$$T(\rho - a) = T(\beta - a), \quad T(AP) = T(\beta - a)$$

expresses that the *locus* of P is the spheric surface through B , which has its centre at A .

(5.) The equation of the Apollonian* Locus, 145, (8.), (9.), may be written under either of the two following forms :

$$T(\rho - a^2a) = aT(\rho - a); \quad T\rho = aTa; \quad OA = a \quad OC = a^{-1}$$

from each of which we shall find ourselves able to pass to the other, at a later stage, by general *Rules of Transformation*, without appealing to *geometry* (comp. 145, (10.)).

(6.) The equation,

$$T(\rho + a) = T(\rho - a),$$

expresses that the *locus* of P is the plane through o , perpendicular to the line oa ; because it expresses that if $OA' = -OA$, then the point P is equally distant from the two points A and A' . It represents therefore the *same locus* as the equation,

* Compare the first Note to page 128.

$$\angle \frac{\rho}{\alpha} = \frac{\pi}{2}, \text{ of 132, (1.);}$$

or as the equation,

$$\frac{\rho}{\alpha} + K \frac{\rho}{\alpha} = 0, \text{ of 144, (1.);}$$

or as

$$\left(U \frac{\rho}{\alpha} \right)^2 = -1, \text{ of 161, (7.);}$$

α as the simple geometrical formula, $\rho \perp \alpha$ (129). And in fact it will be found possible, by *General Rules* of this Calculus, to transform any one of these five forms into any other of them; or into this sixth form,

$$S \frac{\rho}{\alpha} = 0,$$

which expresses that the scalar part* of the quaternion $\frac{\rho}{\alpha}$ is zero, and therefore that this quaternion is a *right quotient* (132).

(7.) In like manner, the equation

$$T(\rho - \beta) = T(\rho - \alpha)$$

expresses that the locus of ρ is the plane which perpendicularly bisects the line AB ; because it expresses that ρ is equally distant from the two points A and B .

(8.) The tensor, $T\alpha$, being generally a *positive scalar*, but vanishing (as a limit) with α , we have,

$$T x \alpha = \pm x T \alpha, \text{ according as } x > \text{ or } < 0;$$

thus, in particular,

$$T(-\alpha) = T\alpha; \text{ and } T0\alpha = T0 = 0.$$

(9.) That

$$T(\beta + \alpha) = T\beta + T\alpha, \text{ if } U\beta = U\alpha,$$

but not otherwise (α and β being any two actual vectors), will be seen, at a later stage, to be a symbolical consequence from the *rules* of the present Calculus; but in the mean time it may be *geometrically* proved, by conceiving that while $\alpha = OA$, as usual, we make $\beta + \alpha = OC$, and therefore $\beta = OC - OA = AC$ (4); for thus we shall see that while, in *general*, the three points O, A, C are corners of a triangle, and therefore the length of the side OC is less than the sum of the lengths of the two other sides OA and AC , the former length becomes, on the contrary, equal to the latter sum, in the particular case when the triangle vanishes, by the point A falling on the finite line OC ; in which case, OA and AC , or α and β , have one common direction, as the equation $U\alpha = U\beta$ implies.

(10.) If α and β be any actual vectors, and if their versors be unequal ($U\alpha \neq U\beta$), then

$$T(\beta + \alpha) < T\beta + T\alpha;$$

an inequality which results at once from the consideration of the recent triangle OAC ; but which (as it will be found) may also be *symbolically* proved, by rules of the calculus of quaternions.

* Compare the Note to page 125; and the following Section of the present Chapter.

(11.) If $U\beta = -U\alpha$, then $T(\beta + \alpha) = \pm(T\beta - T\alpha)$, according as $T\beta >$ or $< T\alpha$; but

$$T(\beta + \alpha) > \pm(T\beta - T\alpha), \text{ if } U\beta \text{ not } = -U\alpha.$$

187. The *quotient*, $U\beta : U\alpha$, of the *versors* of the two vectors, α and β , has been called (in 156) the *Versor of the Quotient*, or quaternion, $q = \beta : \alpha$; and has been denoted, as such, by the symbol, Uq . On the same plan, we propose now to call the quotient, $T\beta : T\alpha$, of the *tensors* of the same two vectors, the *Tensor* of the Quaternion* q , or $\beta : \alpha$, and to denote it by the corresponding *symbol*, Tq . And then, as we have called the letter U (in 156) the characteristic of the operation of *taking the versor*, so we may now speak of T as the *Characteristic of the (corresponding) Operation of taking the Tensor*, whether of a *Vector*, α , or of a *Quaternion*, q . We shall thus have, generally,

$$T(\beta : \alpha) = T\beta : T\alpha, \text{ as we had } U(\beta : \alpha) = U\beta : U\alpha \text{ (156);}$$

and may say that as the *versor* Uq depended solely on, but conversely was sufficient to determine, the *relative direction* (157), so the *tensor* Tq depends on and determines the *relative length*† (109), of the two vectors, α and β , of which the *quaternion* q is the *quotient* (112).

(1.) Hence the equation $T\frac{\rho}{\alpha} = 1$, like $T\rho = T\alpha$, to which it is equivalent, expresses that the locus of ρ is the sphere with o for centre, which passes through the point α .

* Compare the Note to Art. 109, in page 108; and that to Art. 156, in page 155.

† It has been shown, in Art. 112, and in the *Additional Illustrations* of the third Section of the present Chapter (118-116), that *Relative Length*, as well as *relative direction*, enters as an *essential element* into the very *Conception* of a *Quaternion*. Accordingly, in Art. 117, an *agreement* of *relative lengths* (as well as an *agreement* of *relative directions*) was made one of the *conditions of equality*, between any two quaternions, considered as *quotients of vectors*: so that we may now say, that the *tensors* (as well as the *versors*) of *equal quaternions are equal*. Compare the first Note to page 187, as regards what was there called the *quantitative element*, of absolute or *relative length*, which was *eliminated* from α , or from q , by means of the *characteristic* U ; whereas the *new characteristic*, T , of the present Section, serves on the contrary to *retain that element alone*, and to eliminate what may be called by contrast the *qualitative element*, of absolute or *relative direction*.

(2.) The equation comp. 186, (6.),

$$T \frac{\rho + a}{\rho - a} = 1,$$

expresses that the locus of r is the plane through o , perpendicular to the line oa .

(3.) Other examples of the same sort may easily be derived from the sub-articles to 186, by introducing the notation (187) for the *tensor of a quotient*, or *quaternion*, as additional to that for the *tensor of a vector* (185).

(4.) $T(\beta : a) >, =, \text{ or } < 1$, according as $T\beta >, =, \text{ or } < Ta$.

(5.) The *tensor of a right quotient* (182) is always equal to the tensor of its *index* (138).

(6.) The *tensor of a radial* (146) is always *positive unity*; thus we have, generally, by 156,

$$TUq = 1;$$

and in particular, by 181,

$$Ti = Tj = Tk = 1.$$

(7.) $Txq = \pm xTq$, according as $x > \text{ or } < 0$;

thus, in particular, $T(-q) = Tq$, or the tensors of *opposite* quaternions are *equal*.

(8.) $Tx = \pm x$, according as $x > \text{ or } < 0$;

thus, the tensor of a *scalar* is that scalar *taken positively*.

(9.) Hence,

$$TTa = Ta, \quad TTq = Tq;$$

so that, by abstracting from the *subject* of the operation T (comp. 145, 160), we may establish the symbolical equation,

$$T^2 = TT = T.$$

(10.) Because the tensor of a quaternion is generally a positive scalar, such a tensor is *its own conjugate* (189); its *angle* is *zero* (181); and its *versor* (159) is *positive unity*: or in symbols,

$$KTq = Tq; \quad \angle Tq = 0; \quad UTq = 1.$$

(11.) $T(1 : q) = T(a : \beta) = Ta : T\beta = 1 : Tq$;

or in words, the *tensor of the reciprocal* of a quaternion is equal to the *reciprocal of the tensor*.

(12.) Again, since the two lines, ob and ob' , in Fig. 86, are *equally long*, the definition (187) of a conjugate gives

$$TKq = Tq;$$

or in words, the tensors of *conjugate* quaternions are *equal*.

(13.) It is scarcely necessary to remark, that any two quaternions which have *equal tensors*, and *equal versors*, are themselves *equal*: or in symbols, that

$$q' = q, \text{ if } Tq' = Tq, \text{ and } Uq' = Uq.$$

188. Since we have, generally,

$$\frac{\beta}{a} = \frac{T\beta \cdot U\beta}{T\alpha \cdot U\alpha} = \frac{T\beta}{T\alpha} \cdot \frac{U\beta}{U\alpha} = \frac{U\beta}{U\alpha} \cdot \frac{T\beta}{T\alpha} \text{ (comp. 126, 186),}$$

we may establish the two following general formulæ of *decom-*

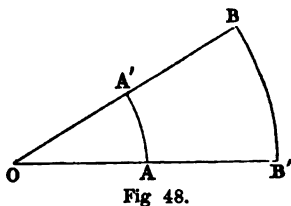
position of a quaternion into two factors, of the tensor and versor kinds :

$$\text{I.} \dots q = Tq \cdot Uq; \quad \text{II.} \dots q = Uq \cdot Tq;$$

which are exactly analogous to the formulæ (186) for the corresponding decomposition of a vector, into factors of the same two kinds : namely,

$$\text{I.} \dots a = Ta \cdot Ua; \quad \text{II.} \dots a = Ua \cdot Ta.$$

To illustrate this last decomposition of a quaternion, q , or $OB : OA$, into factors, we may conceive that AA' and BB' are two concentric and circular, but oppositely directed arcs, which terminate respectively on the two lines OB and OA , or rather on the longer of those two lines itself, and on the shorter of them prolonged, as in the annexed Figure 48 ; so that OA' has the length of OA , but the direction of OB , while OB' , on the contrary, has the length of OB , but the direction of OA ; and that therefore we may write, by what has been defined respecting versors and tensors of vectors (155, 156, 185, 186),



$$OA' = Ta \cdot U\beta; \quad OB' = T\beta \cdot Ua.$$

Then, by the definitions in 156, 187, of the versor and tensor of a quaternion,

$$Uq = U(OB : OA) = OA' : OA = OB : OB';$$

$$Tq = T(OB : OA) = OB' : OA = OB : OA';$$

whence, by the general formula of multiplication of quotients (107),

$$\text{I.} \dots q = OB : OA = (OB : OA') \cdot (OA' : OA) = Tq \cdot Uq;$$

and

$$\text{II.} \dots q = OB : OA = (OB : OB') \cdot (OB' : OA) = Uq \cdot Tq,$$

as above.

189. In words, if we wish to pass from the vector a to the vector β , or from the line OA to the line OB , we are at liberty either, 1st, to begin by turning, from OA to OA' , and then to end by stretching,

from OA' to OB , as Fig. 48 may serve to illustrate; or, *I*nd, to *begin* by stretching, from OA to OB' , and *end* by turning, from OB' to OB . The *act of multiplication* of a line a by a quaternion q , considered as a *factor* (103), which affects *both* length and direction (109), may thus be *decomposed* into *two* distinct and *partial acts*, of the kinds which we have called *Version* and *Tension*; and these two acts may be performed, at pleasure, in *either* of two *orders* of succession. And although, if we attended *merely to lengths*, we might be led to say that the *tensor* of a quaternion was a *signless number*,* expressive of a geometrical *ratio* of magnitudes, yet when the recent *construction* (Fig. 48) is adopted, we see, by either of the two resulting expressions (188) for Tq , that there is a *propriety* in treating this tensor as a *positive scalar*, as we have lately done, and propose systematically to do.

190. Since $TKq = Tq$, by 187, (12.), and $UKq = 1 : Uq$, by 158, we may write, generally, for any quaternion and its conjugate, the two connected expressions:

$$I. \dots q = Tq \cdot Uq; \quad II. \dots Kq = Tq : Uq;$$

whence, by multiplication and division,

$$III. \dots q \cdot Kq = (Tq)^2; \quad IV. \dots q : Kq = (Uq)^2.$$

This last formula had occurred before; and we saw (161) that in it the *parentheses* might be omitted, because $(Uq)^2 = U(q^2)$. In like manner (comp. 161, (2.)), we have also

$$(Tq)^2 = T(q^2) = Tq^2,$$

parentheses being again omitted; or in words, the *tensor of the square* of a quaternion is always equal to the *square of the tensor*: as appears (among other ways) from inspection of Fig. 42, *bis*, in which the lengths of OA , OB , OC form a *geometrical progression*; whence

$$T \cdot \left(\frac{OB}{OA} \right)^2 = T \frac{OC}{OA} = \frac{T \cdot OC}{T \cdot OA} = \left(\frac{T \cdot OB}{T \cdot OA} \right)^2 = \left(T \frac{OB}{OA} \right)^2.$$

At the same time, we see again that the *product* qKq of two *conjugate quaternions*, which has been called (145, (11.)) their common *Norm*, and denoted by the symbol Nq , represents geometrically the *square of the quotient of the lengths* of the two lines, of which (when considered as *vectors*) the quaternion q is itself the quotient (112).

We may therefore write generally,†

$$V. \dots qKq = Tq^2 = Nq; \quad VI. \dots Tq = \sqrt{Nq} = \sqrt{(qKq)}.$$

* Compare the Note in page 108, to Art. 109.

† Compare the Note in page 129.

(1.) We have also, by II., the following other general transformations for the tensor of a quaternion :

$$\text{VII.} \dots Tq = Kq \cdot Uq; \quad \text{VIII.} \dots Tq = Uq \cdot Kq;$$

of which the geometrical significations might easily be exhibited by a diagram, but of which the validity is sufficiently proved by what precedes.

(2.) Also (comp. 158),

$$\frac{1}{Uq} = \frac{Kq}{Tq} = K \frac{q}{Tq} = KUq; \quad K \frac{1}{Uq} = \frac{q}{Tq} = Uq.$$

(3.) The reciprocal of a quaternion, and the conjugate* of that reciprocal, may now be thus expressed :

$$\frac{1}{q} = \frac{Kq}{Tq^2} = \frac{Kq}{Nq} = \frac{KUq}{Tq} = \frac{1}{Uq} \cdot \frac{1}{Tq} = \frac{1}{Tq} \cdot \frac{1}{Uq};$$

$$K \frac{1}{q} = \frac{q}{Nq} = \frac{q}{Tq^2} = \frac{Uq}{Tq} = \frac{1}{Kq}.$$

(4.) We may also write, generally,

$$\text{IX.} \dots Kq = Tq \cdot KUq = Nq : q.$$

191. In general, let *any two* quaternions, q and q' , be considered as multiplicand and multiplier, and let them be reduced (by 120) to the forms $\beta : a$ and $\gamma : \beta$; then the tensor and versor of that *third* quaternion, $\gamma : a$, which is (by 107) their *product* $q'q$, may be thus expressed :

$$\text{I.} \dots Tq'q = T(\gamma : a) = T\gamma : Ta = (T\gamma : T\beta) \cdot (T\beta : Ta) = Tq' \cdot Tq;$$

$$\text{II.} \dots Uq'q = U(\gamma : a) = U\gamma : Ua = (U\gamma : U\beta) \cdot (U\beta : Ua) = Uq' \cdot Uq;$$

where $Tq'q$ and $Uq'q$ are written, for simplicity, instead of $T(q' \cdot q)$ and $U(q' \cdot q)$. Hence, in any such multiplication, the *tensor of the product* is the *product of the tensors*²; and the *versor of the product* is the *product of the versors*²; the order of the factors being generally *retained* for the latter (comp. 168, &c.), although it may be *varied* for the former, on account of the *scalar* character of a *tensor*. In like manner, for the *division* of any one quaternion q' , by any other q , we have the analogous formulæ :

$$\text{III.} \dots T(q' : q) = Tq' : Tq; \quad \text{IV.} \dots U(q' : q) = Uq' : Uq;$$

or in words, the *tensor of the quotient* of any two quaternions is equal to the *quotient of the tensors*; and similarly, the *versor of the quotient* is equal to the *quotient of the versors*. And because multiplication and division of *tensors* are performed according to the rules of *algebra*, or rather of *arithme-*

* Compare Art. 145, and the Note to page 127.

tic (a tensor being always, by what precedes, a *positive number*), we see that the difficulty (whatever it may be) of the general *multiplication and division of quaternions* is thus reduced to that of the corresponding *operations on versors*: for which *latter operations geometrical constructions* have been assigned, in the ninth Section of the present Chapter.

(1.) The two products, $q'q$ and qq' , of any two quaternions taken as factors in two different orders, are *equal or unequal*, according as those two factors are *complanar or diplanar*; because such equality (169), or inequality (168), has been already proved to exist, for the case* when each tensor is unity: but we have always (comp. 178),

$$Tq'q = Tqq', \text{ and } \angle q'q = \angle qq'.$$

(2.) If $\angle q = \angle q' = \frac{\pi}{2}$, then $qq' = Kq'q$ (170); so that the products of two *right* quotients, or *right quaternions* (182), taken in *opposite orders*, are always *conjugate quaternions*.

(3.) If $\angle q = \angle q' = \frac{\pi}{2}$, and $\text{Ax} \cdot q' \perp \text{Ax} \cdot q$, then $qq' = -q'q$,

$$\angle qq' = \angle q'q = \frac{\pi}{2}, \quad \text{Ax} \cdot q'q \perp \text{Ax} \cdot q, \quad \text{Ax} \cdot q'q \perp \text{Ax} \cdot q' \text{ (171);}$$

so that the *product of two right quaternions, in two rectangular planes, is a third right quaternion, in a plane rectangular to both*; and is *changed to its own opposite*, when the *order of the factors is reversed*: as we had $ij = k = -ji$ (182).

(4.) In general, if q and q' be any two *diplanar* quaternions, the *rotation round* $\text{Ax} \cdot q'$, from $\text{Ax} \cdot q$ to $\text{Ax} \cdot q'q$, is *positive* (177).

(5.) Under the same condition, $q'(q' : q)$ is a quaternion with the *same tensor*, and *same angle*, as q' , but with a *different axis*; and this new axis, $\text{Ax} \cdot q(q' : q)$, may be derived (179, (1.)) from the old axis, $\text{Ax} \cdot q'$, by a *conical rotation* (in the positive direction) round $\text{Ax} \cdot q$, through an angle $= 2 \angle q$.

(6.) The product or quotient of two *complanar* quaternions is, in general, a *third quaternion* *complanar* with both; but if they be both *scalar*, or both *right*, then this product or quotient *degenerates* (181) into a *scalar*.

(7.) Whether q and q' be *complanar* or *diplanar*, we have always as in algebra (comp. 106, 107, 186) the two identical equations:

$$\text{V} \dots (q' : q) \cdot q = q'; \quad \text{VI} \dots (q' \cdot q) : q = q'.$$

(8.) Also, by 190, V., and 191, I., we have this other general formula:

$$\text{VII} \dots Nq'q = Nq' \cdot Nq;$$

or in words, the *norm of the product* is equal to the *product of the norms*.

192. Let $q = \beta : a$, and $q' = \gamma : \beta$, as before; then

$$1 : q'q = 1 : (\gamma : a) = a : \gamma = (a : \beta) \cdot (\beta : \gamma) = (1 : q) \cdot (1 : q');$$

so that the *reciprocal of the product* of any two quaternions is

* Compare the Notes to pages 148, 151.

equal to the *product of the reciprocals*, taken in an *inverted order*: or briefly,

$$I. \dots Rq'q = Rq \cdot Rq',$$

if R be again used (as in 161, (3.)) as a (temporary) *characteristic of reciprocation*. And because we have then (by the same sub-article) the symbolical equation, $KU = UR$, or in words, the *conjugate of the versor* of any quaternion q is equal (158) to the *versor of the reciprocal* of that quaternion; while the *versor of a product* is equal (191) to the product of the versors: we see that

$$KUq'q = URq'q = URq \cdot URq' = KUq \cdot KUq'.$$

But

$Kq = Tq \cdot KUq$, by 190, IX.; and $Tq'q = Tq' \cdot Tq = Tq \cdot Tq'$, by 191; we arrive then thus at the following other important and general formula:

$$II. \dots Kq'q = Kq \cdot Kq';$$

or in words, the *conjugate of the product* of any two quaternions is equal to the *product of the conjugates*, taken (still) in an *inverted order*.

(1.) These two results, I., II., may be illustrated, for *versors* ($Tq = Tq' = 1$), by the consideration of a *spherical triangle* ABC (comp. Fig. 48); in which the sides AB and BC (comp. 167) may represent q and q' , the arc AC then representing $q'q$. For then the new multiplier $Rq = Kq$ (158) is represented (162) by BA , and the new multiplicand $Rq' = Kq'$ by CB ; whence the new product, $Rq \cdot Rq' = Kq \cdot Kq'$, is represented by the *inverse arc* CA , and is therefore at once the *reciprocal* $Rq'q$, and the *conjugate* $Kq'q$, of the old product $q'q$.

(2.) If q and q' be *right quaternions*, then $Kq = -q$, $Kq' = -q'$ (by 144); and the recent formula II. becomes, $Kq'q = qq'$, as in 170.

(3.) In general, that formula II. (of 192) may be thus written:

$$III. \dots K \frac{\gamma}{\alpha} = K \frac{\beta}{\alpha} \cdot K \frac{\gamma}{\beta};$$

where α, β, γ may denote any three vectors.

(4.) Suppose then that, as in the annexed Fig. 49, we have the two following relations of *inverse similitude* of triangles (118),

$$\Delta AOB \propto BOC, \quad \Delta BOE \propto DOB;$$

and therefore (by 187) the two equations,

$$\frac{\gamma}{\beta} = K \frac{\beta}{\alpha}, \quad \frac{\beta}{\delta} = K \frac{\epsilon}{\beta};$$

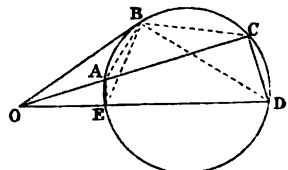


Fig. 49.

we shall have, by III.,

$$\frac{\gamma}{\delta} = K \frac{a}{a}, \text{ or } \Delta \text{DOC} \propto \Delta \text{AOE};$$

so that this *third* formula of inverse similitude is a *consequence* from the other two.

(5.) If then (comp. 145, (6.)) any two circles, whether in one plane or in space, touch one another at a point B; and if from any point O, on the common tangent BO, two secants OAC, OED be drawn, to these two circles; the four points of section, A, C, D, E, will be on one common circle: for such concircularity is an easy consequence (through equal angles, &c.), from the last inverse similitude.

(6.) The same conclusion (respecting concircularity, &c.) may be otherwise and geometrically drawn, from the equality of the two rectangles, ΔOC and ΔOE , each being equal to the square of the tangent OB; which may serve as an instructive verification of the recent formula III., and as an example of the consistency of the results, to which calculations with quaternions conduct.

(7.) It may be noticed that the construction would in general give three circles, although only one is drawn in the Figure; but that if the two triangles ABC and DBE be situated in different planes, then these three circles, and of course the five points ABCDE, are situated on one common sphere.

193. An important application of the foregoing general theory of Multiplication and Division, is to the case of *Right Quaternions* (132), taken in connexion with their *Index-Vectors*, or *Indices* (133).

Considering *division* first, and employing the general formula of 106, let β and γ be each $\perp a$; and let β' and γ' be the respective indices of the two right quotients, $q = \beta : a$, and $q' = \gamma : a$. We shall thus have the two complanarities, $\beta' \parallel \beta, \gamma$, and $\gamma' \parallel \beta, \gamma$ (comp. 123), because the four lines $\beta, \gamma, \beta', \gamma'$ are all perpendicular to a ; and within their common plane it is easy to see, from definitions already given, that these four lines form a *proportion of vectors*, in the same sense in which a, β, γ, δ did so, in the fourth Section of the present Chapter: so that we may write the *equation of quotients*,

$$\gamma' : \beta' = \gamma : \beta.$$

In fact, we have (by 133, 185, 187) the following relations of length,

$$T\beta' = T\beta : Ta, \quad T\gamma' = T\gamma : Ta, \quad \text{and } \therefore T(\gamma' : \beta') = T(\gamma : \beta);$$

while the relation of *directions*, expressed by the formula,

$$U(\gamma' : \beta') = U(\gamma : \beta), \quad \text{or } U\gamma' : U\beta' = U\gamma : U\beta,$$

is easily established by means of the equations,

$$\angle(\gamma' : \gamma) = \angle(\beta' : \beta) = \frac{\pi}{2}; \quad \text{Ax} \cdot (\gamma' : \gamma) = \text{Ax} \cdot (\beta' : \beta) = U\alpha.$$

We arrive, then, at this general Theorem (comp. again 133): that "*the Quotient of any two Right Quaternions is equal to the Quotient of their Indices.*"*

(1.) For example (comp. 150, 159, 181), the indices of the right versors i, j, k are the axes of those three versors, namely, the lines oi, oj, ok ; and we have the equal quotients,

$$j : i = oi : oj' = k = oj : oi, \&c.$$

(2.) In like manner, the indices of $-i, -j, -k$ are oi', oj', ok' ; and

$$i : -j = oj' : oi' = k = oi : oj', \&c.$$

(3.) In general the *quotient of any two right versors* is equal to the *quotient of their axes*; as the theory of *representative arcs*, and of their *poles*, may easily serve to illustrate.

194. As regards the *multiplication* of two right quaternions, in connexion with their indices, it may here suffice to observe that, by 106 and 107, the *product* $\gamma : \alpha = (\gamma : \beta) \cdot (\beta : \alpha)$ is equal (comp. 136) to the *quotient*, $(\gamma : \beta) : (\alpha : \beta)$; whence it is easy to infer that "*the Product, $q'q$, of any two Right Quaternions, is equal to the Quotient of the Index of the Multiplier, q' , divided by the Index of the Reciprocal of the Multiplicand, q .*"

It follows that the *plane*, whether of the product or of the quotient of two right quaternions, coincides with the *plane of their indices*; and therefore also with the plane of their *axes*; because we have, generally, by principles already established, the transformation,

$$\text{if } \angle q = \frac{\pi}{2}, \text{ then Index of } q = Tq \cdot \text{Ax} \cdot q.$$

* We have thus a new point of *agreement*, or of *connexion*, between *right quaternions*, and their *index-vectors*, tending to justify the ultimate assumption (not yet made), of *equality* between the former and the latter. In fact, we shall soon *prove* that the *index of the sum* (or difference), of any two right quotients (182), is equal to the *sum* (or difference) of their *indices*; and shall find it convenient subsequently to *interpret* the *product* $\beta\alpha$ of any two vectors, as being the *quaternion-product* (194) of the two right quaternions, of which those two lines are the *indices* (188): after which, the above-mentioned assumption of *equality* will appear natural, and be found to be useful. (Compare the Notes to pages 119, 136.)

SECTION 12.—*On the Sum or Difference of any two Quaternions ; and on the Scalar (or Scalar Part) of a Quaternion.*

195. The *Addition* of any given quaternion q' , considered as a geometrical quotient or *fraction* (101), to any other given quaternion q , considered also as a fraction, can always be accomplished by the first general formula of Art. 106, when these two fractions have a *common denominator* ; and if they be not already *given* as having such, they can always be *reduced* so as to have one, by the process of Art. 120. And because the *addition of any two lines* was early seen to be a *commutative operation* (7, 9), so that we have always $\gamma + \beta = \beta + \gamma$, it follows (by 106) that the addition of any two quaternions is *likewise* a commutative operation, or in symbols, that

$$\text{I. . . } q + q' = q' + q ;$$

so that the *SUM of any two* Quaternions* has a *Value*, which is independent of their *Order* : and which (by what precedes) must be considered to be *given*, or at least *known*, or *definite*, when the two *summand* quaternions are given. It is easy also to see that the *conjugate* of any such *sum* is equal to the *sum of the conjugates*, or in symbols, that

$$\text{II. . . } K(q + q') = Kq' + Kq.$$

(1.) The important formula last written becomes geometrically evident, when it is presented under the following form. Let $OBDC$ be any parallelogram, and let OA be any right line, drawn from one corner of it, but not generally in its plane. Let the three other corners, B, C, D , be *reflected* (in the sense of 145, (5.)) with respect to that line OA , into three new points, B', C', D' ; or let the three lines OB, OC, OD be *reflected* (in the sense of 186) with respect to the same line OA ; which thus bisects at right angles the three joining lines, BB', CC', DD' , as it does BB' in Fig. 36. Then *each of the lines* OB, OC, OD , and therefore also the whole *plane figure* $OBDC$, may be considered to have simply *revolved* round the line OA as an *axis*, by a *conical rotation* through *two right angles* ; and consequently the *new figure* $OB'D'C'$, like that *old one* $OBDC$, must be a *parallelogram*. Thus (comp. 106, 137), we have

$$OD' = OC' + OB', \quad \delta' = \gamma' + \beta', \quad \delta' : a = (\gamma' : a) + (\beta' : a) ;$$

and the recent formula II. is justified.

* It will be found that this result admits of being extended to the case of *three* (or *more*) quaternions ; but, for the moment, we content ourselves with *two*.

(2.) Simple as this last reasoning is, and unnecessary as it appears to be to draw any new Diagram to illustrate it, the reader's attention may be once more invited to the great *simplicity of expression*, with which many important *geometrical conceptions*, respecting *space of three dimensions*, are stated in the present Calculus: and are thereby kept *ready* for future application, and for easy combination with *other* results of the same kind. Compare the remarks already made in 132, (6.); 145, (10.); 161; 179, (8.); 192, (6.); and some of the shortly following sub-articles to 196, respecting properties of an *oblique cone* with circular base.

196. One of the most important *cases of addition*, is that of *two conjugate summands*, q and Kq ; of which it has been seen (in 140) that the *sum* is always a *scalar*. We propose now to denote the *half* of this sum by the *symbol*,

$$Sq;$$

thus writing generally,

$$I. \dots q + Kq = Kq + q = 2Sq;$$

or defining the new symbol Sq by the formula,

$$II. \dots Sq = \frac{1}{2}(q + Kq); \text{ or briefly, } II'. \dots S = \frac{1}{2}(1 + K).$$

For reasons which will soon more fully appear, we shall also call this new quantity, Sq , the *scalar part*, or simply *the SCALAR, of the Quaternion, q* ; and shall therefore call the letter S , thus used, the *Characteristic of the Operation of taking the Scalar* of a quaternion. (Comp. 132, (6.); 137; 156; 187.) It follows that not only *equal quaternions*, but also *conjugate quaternions*, have *equal scalars*; or in symbols,

$$III. \dots Sq' = Sq, \text{ if } q' = q; \text{ and } IV. \dots SKq = Sq;$$

or briefly,

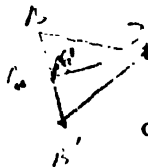
$$IV'. \dots SK = S.$$

And because we have seen that $Kq = +q$, if q be a *scalar* (139), but that $Kq = -q$, if q be a *right quotient* (144), we find that the *scalar of a scalar* (considered as a *degenerate quaternion*, 131) is equal to that *scalar itself*, but that the *scalar of a right quaternion is zero*. We may therefore now write (comp. 160):

$$V. \dots Sx = x, \text{ if } x \text{ be a scalar; } \quad VI. \dots SSq = Sq, \quad S^2 = SS = S;$$

$$\text{and } VII. \dots Sq = 0, \text{ if } \angle q = \frac{\pi}{2}.$$

Again, because oa' in Fig. 36 is multiplied by x , when ob is multiplied thereby, we may write, generally,



VIII. . . $Sxq = xSq$, if x be any scalar;

and therefore in particular (by 188),

$$IX. \dots Sq = S(Tq \cdot Uq) = Tq \cdot SUq.$$

Also because $SKq = Sq$, by IV., while $KUq = U \frac{1}{q}$, by 158, we have the general equation,

$$X. \dots SUq = SU \frac{1}{q}; \quad \text{or} \quad X'. \dots SU \frac{\beta}{\alpha} = SU \frac{\alpha}{\beta};$$

whence, by IX.,

$$XI. \dots Sq = Tq \cdot SU \frac{1}{q}; \quad \text{or} \quad XI'. \dots S \frac{\beta}{\alpha} = T \frac{\beta}{\alpha} \cdot SU \frac{\alpha}{\beta};$$

and therefore also, by 190, (V.), since $Tq \cdot T \frac{1}{q} = 1$,

$$XII. \dots Sq = Tq^2 \cdot S \frac{1}{q} = Nq \cdot S \frac{1}{q}; \quad XII'. \dots S \frac{\beta}{\alpha} = N \frac{\beta}{\alpha} \cdot S \frac{\alpha}{\beta}.$$

The results of 142, combined with the recent definition I. or II., enable us to extend the recent formula VII., by writing,

$$XIII. \dots Sq >, =, \text{ or } < 0, \text{ according as } \angle q <, =, \text{ or } > \frac{\pi}{2};$$

and conversely,

$$XIV. \dots \angle q <, =, \text{ or } > \frac{\pi}{2}, \text{ according as } Sq >, =, \text{ or } < 0.$$

In fact, if we compare that *definition* I. with the formula of 140, and with Fig. 36, we see at once that because, in that Figure,

$$S(OB : OA) = OA' : OA,$$

we may write, generally,

$$XV. \dots Sq = Tq \cdot \cos \angle q; \quad \text{or} \quad XVI. \dots SUq = \cos \angle q;$$

equations which will be found of great importance, as serving to *connect quaternions with trigonometry*; and which show that

$$XVII. \dots \angle q' = \angle q, \text{ if } SUq' = SUq,$$

the angle $\angle q$ being still taken (as in 130), so as not to fall outside the limits 0 and π ; whence also,

XVIII. . . $\angle q' = \angle q$, if $Sq' = Sq$, and $Tq' = Tq$, the *angle of a quaternion* being thus given, when the *scalar* and the *tensor* of that quaternion are given, or known. Finally because, in the same Figure 36 (comp. 15, 103), the *line*,

$$OA' = (OA' : OA) \cdot OA = OA \cdot S(OB : OA),$$

may be said to be the *projection* of OB on OA , since A' is the *foot of the perpendicular* let fall from the *point* B upon this latter line OA , we may establish this other general formula :

$$\text{XIX. . . } aS\frac{\beta}{a} = S\frac{\beta}{a} \cdot a = \text{projection of } \beta \text{ on } a;$$

a result which will be found to be of great utility, in investigations respecting *geometrical loci*, and which may be also written thus :

$$\text{XX. . . Projection of } \beta \text{ on } a = U_a \cdot T\beta \cdot SU\frac{\beta}{a};$$

with other transformations deducible from principles stated above. It is scarcely necessary to remark that, on account of the *scalar* character of Sq , we have, generally, by 159, and 187, (8.), the expressions,

$$\text{XXI. . . } USq = \pm 1; \quad \text{XXII. . . } TSq = \pm Sq;$$

while, for the same reason, we have always, by 139, the equation (comp. IV.),

$$\text{XXIII. . . } KSq = Sq; \quad \text{or} \quad \text{XXIII'. . . } KS = S;$$

and, by 131,

$$\text{XXIV. . . } \angle Sq = 0, \text{ or } = \pi, \text{ unless } \angle q = \frac{\pi}{2};$$

in which last case $Sq = 0$, by VII., and therefore $\angle Sq$ is indeterminate :* USq becoming at the same time indeterminate, by 159, but TSq vanishing, by 186, 187.

(1.) The equation,

$$S\frac{\rho}{a} = 0,$$

is now seen to be equivalent to the formula, $\rho \perp a$; and therefore to denote the

* Compare the Note in page 118, to Art. 181.

6A
A

same plane locus for P , as that which is represented by any one of the four other equations of 186, (6.); or by the equation,

$$T \frac{\rho + \alpha}{\rho - \alpha} = 1, \text{ of 187, (2.)}$$

(2.) The equation,

$$S \frac{\rho - \beta}{\alpha} = 0, \text{ or } S \frac{\rho}{\alpha} = S \frac{\beta}{\alpha},$$

expresses that $BP \perp OA$; or that the points B and P have the same projection on OA ; or that the locus of P is the plane through B , perpendicular to the line OA .

(3.) The equation,

$$SU \frac{\rho}{\alpha} = SU \frac{\beta}{\alpha},$$

expresses (comp. 182, (2.)) that P is on one sheet of a cone of revolution, with O for vertex, and OA for axis, and passing through the point B .

(4.) The other sheet of the same cone is represented by this other equation,

$$SU \frac{\rho}{\alpha} = -SU \frac{\beta}{\alpha};$$

and both sheets jointly by the equation,

$$\left(SU \frac{\rho}{\alpha} \right)^2 = \left(SU \frac{\beta}{\alpha} \right)^2.$$

(5.) The equation,

$$S \frac{\rho}{\alpha} = 1, \text{ or } SU \frac{\rho}{\alpha} = T \frac{\alpha}{\rho},$$

expresses that the locus of P is the plane through A , perpendicular to the line OA ; because it expresses (comp. XIX.) that the projection of OP on OA is the line OA itself; or that the angle OAP is right; or that $S \frac{\rho - \alpha}{\alpha} = 0$.

(6.) On the other hand the equation,

$$S \frac{\beta}{\rho} = 1, \text{ or } SU \frac{\beta}{\rho} = T \frac{\rho}{\beta},$$

expresses that the projection of OB on OP is OP itself; or that the angle OPB is right; or that the locus of P is that spheric surface, which has the line OB for a diameter.

(7.) Hence the system of the two equations,

$$S \frac{\rho}{\alpha} = 1, \quad S \frac{\beta}{\rho} = 1,$$

represents the circle, in which the sphere (6.), with OB for a diameter, is cut by the plane (5.), with OA for the perpendicular let fall on it from O .

(8.) And therefore this new equation,

$$S \frac{\rho}{\alpha} \cdot S \frac{\beta}{\rho} = 1,$$

obtained by multiplying the two last, represents the *Cyclic* Cone* (or cone of the

* Historically speaking, the oblique cone with circular base may deserve to be named the *Apollonian Cone*, from Apollonius of Perga, in whose great work on Co-

second order, but not generally of revolution), which rests on this last circle (7.) as its base, and has the point o for its vertex. In fact, the equation (8.) is evidently satisfied, when the two equations (7.) are so; and therefore every point of the circular circumference, denoted by those two equations, must be a point of the locus, represented by the equation (8.). But the latter equation remains unchanged, at least essentially, when ρ is changed to $x\rho$, x being any scalar; the locus (8.) is, therefore, some conical surface, with its vertex at the origin, o ; and consequently it can be none other than that particular cone (both ways prolonged), which rests (as above) on the given circular base (7.).

(9.) The system of the two equations,

$$S \frac{\rho}{\alpha} S \frac{\beta}{\rho} = 1, \quad S \frac{\rho}{\gamma} = 1,$$

(in writing the first of which the point may be omitted,) represents a conic section; namely that section, in which the cone (8.) is cut by the new plane, which has oc for the perpendicular let fall upon it, from the origin of vectors o .

(10.) Conversely, every plane ellipse (or other conic section) in space, of which the plane does not pass through the origin, may be represented by a system of two equations, of this last form (9.); because the cone which rests on any such conic as its base, and has its vertex at any given point o , is known to be a cyclic cone.

(11.) The curve (or rather the pair of curves), in which an oblique but cyclic cone (8.) is cut by a concentric sphere (that is to say, a cone resting on a circular base by a sphere which has its centre at the vertex of that cone), has come, in modern times, to be called a Spherical Conic. And any such conic may, on the foregoing plan, be represented by the system of the two equations,

$$S \frac{\rho}{\alpha} S \frac{\beta}{\rho} = 1, \quad T\rho = 1;$$

the length of the radius of the sphere being here, for simplicity, supposed to be the unit of length. But, by writing $T\rho = a$, where a may denote any constant and positive scalar, we can at once remove this last restriction, if it be thought useful or convenient to do so.

(12.) The equation (8.) may be written, by XII. or XII', under the form (comp. 191, VII.):

$$S \frac{\alpha}{\rho} S \frac{\rho}{\beta} = N \frac{\alpha}{\beta} = \left(T \frac{\alpha}{\beta} \right)^2;$$

or briefly,

$$S \frac{\beta'}{\rho} S \frac{\rho}{\alpha'} = 1,$$

nics ($\kappa\omega\nu\iota\kappa\tilde{\omega}\nu$), already referred to in a Note to page 128, the properties of such a cone appear to have been first treated systematically; although the cone of revolution had been studied by Euclid. But the designation "cyclic cone" is shorter; and it seems more natural, in geometry, to speak of the above-mentioned oblique cone thus, for the purpose of marking its connexion with the circle, than to call it, as is now usually done, a cone of the second order, or of the second degree: although these phrases also have their advantages.

$$\text{if } \alpha' = \beta T \frac{\alpha}{\beta} = T\alpha \cdot U\beta, \quad \text{and } \beta' = \alpha T \frac{\beta}{\alpha} = T\beta \cdot U\alpha;$$

so that α' and β' are here the lines oA' and oB' , of Art. 188, and Fig. 48.

(13.) Hence the cone (8.) is cut, not only by the plane (5.) in the circle (7.), which is on the sphere (6.), but also by the (generally) *new plane*, $S \frac{\rho}{\alpha} = 1$, in the (generally) *new circle*, in which this new plane cuts the (generally) *new sphere*, $S \frac{\beta'}{\rho} = 1$; or in the circle which is represented by the system of the two equations,

$$S \frac{\rho}{\alpha} = 1, \quad S \frac{\beta'}{\rho} = 1.$$

(14.) In the *particular case* when $\beta \parallel \alpha$ (15), so that the quotient $\beta : \alpha$ is a *scalar*, which must be positive and greater than unity, in order that the plane (5.) may (*really*) *cut* the sphere (6.), and therefore that the circle (7.) and the cone (8.) may be *real*, we may write

$$\beta = a^2\alpha, \quad a > 1, \quad T(\beta : \alpha) = a^2, \quad \alpha' = \alpha, \quad \beta' = \beta;$$

and the circle (13.) *coincides* with the circle (7.).

(15.) In the same *case*, the cone is one of *revolution*; every point P of its circular *base* (that is, of the *circumference* thereof) being at one *constant distance* from the *vertex* o , namely at a distance $= aT\alpha$. For, in the case supposed, the equations (7.) give, by XII,

$$N \frac{\rho}{\alpha} = S \frac{\rho}{\alpha} : S \frac{\alpha}{\rho} = 1 : S \frac{\alpha}{\rho} = a^2 : S \frac{\beta}{\rho} = a^2; \quad \text{or } T\rho = aT\alpha.$$

(Compare 145, (12.), and 186, (5).)

(16.) Conversely, if the cone *be* one of *revolution*, the equations (7.) must con-
duct to a result of the form,

$$\alpha^2 = N \frac{\rho}{\alpha} = S \frac{\rho}{\alpha} : S \frac{\alpha}{\rho} = S \frac{\beta}{\rho} : S \frac{\alpha}{\rho}, \text{ or (comp. (2.)), } S \frac{\beta - \alpha^2\alpha}{\rho} = 0;$$

which can only be by the line $\beta - \alpha^2\alpha$ vanishing; or by our having $\beta = \alpha^2\alpha$, as in (14.); since otherwise we should have, by XIV., $\rho \perp \beta - \alpha^2\alpha$, and *all the points of the base* would be situated in *one plane* passing *through the vertex* o , which (for any actual cone) would be absurd.

(17.) Supposing, then, that we have *not* $\beta \parallel \alpha$, and therefore *not* $\alpha' = \alpha$, $\beta' = \beta$, as in (14.), nor even $\alpha' \parallel \alpha$, $\beta' \parallel \beta$, we see that the cone (8.) is *not* a cone of *revolution* (or what is often called a *right cone*); but that it is, on the contrary, an *oblique* (or *scalene*) cone, although still a *cyclic* one. And we see that *such* a cone is cut in *two distinct series** of *circular sections*, by planes parallel to the two distinct (and mutually non-parallel) planes, (5.) and (18.); or to *two new planes*, drawn *through the vertex* o , which have been called† the *two Cyclic Planes* of the cone, namely, the two following:

* These *two series* of *sub-contrary* (or *antiparallel*) but *circular sections* of a *cyclic cone*, appear to have been first discovered by Apollonius: see the Fifth Proposition of his First Book, in which he says, *καλίσθω δὲ ἡ τοιαύτη τομὴ ὑπεραντία* (page 22 of Halley's Edition).

† By M. Chasles.

$$S \frac{\rho}{\alpha} = 0; \quad S \frac{\rho}{\beta} = 0;$$

while the *two lines* from the vertex, OA and OB, which are *perpendicular to these two planes* respectively, may be said to be the *two Cyclic Normals*.

(18.) Of these two *lines*, α and β , the *second* has been seen to be a *diameter* of the sphere (6.), which may be said to be *circumscribed* to the cone (8.), when that cone is considered as having the circle (7.) for its *base*; the *second cyclic plane* (17.) is therefore the *tangent plane at the vertex* of the cone, to that *first circumscribed sphere* (6.).

(19.) The sphere (18.) may in like manner be said to be circumscribed to the cone, if the latter be considered as resting on the new circle (18.), or as terminated by that circle as its *new base*; and the *diameter* of this *new sphere* is the line OB', or β' , which has by (12.) the *direction* of the line α , or of the *first cyclic normal* (17.); so that (comp. (18.)) the *first cyclic plane* is the *tangent plane at the vertex*, to the *second circumscribed sphere* (18.).

(20.) *Any other sphere* through the vertex, which *touches the first cyclic plane*, and which therefore has its *diameter from the vertex* = $b'\beta'$, where b' is some scalar co-efficient, is represented by the equation,

$$S \frac{b'\beta'}{\rho} = 1, \quad \text{or} \quad S \frac{\beta'}{\rho} = \frac{1}{b'};$$

it therefore *cuts the cone in a circle*, of which (by (12.)) the *equation of the plane* is

$$S \frac{\rho}{\alpha'} = b', \quad \text{or} \quad S \frac{\rho}{b'\alpha'} = 1,$$

so that the *perpendicular from the vertex* is $b'\alpha' \parallel \beta$ (comp. (5.)); and consequently this *plane of section* of sphere and cone is *parallel to the second cyclic plane* (17.).

(21.) In like manner *any sphere*, such as

$$S \frac{b\beta}{\rho} = 1, \quad \text{where } b \text{ is any scalar,}$$

which *touches the second cyclic plane at the vertex*, *intersects the cone* (8.) *in a circle*, of which the plane has for equation,

$$S \frac{\rho}{b\alpha} = 1,$$

and is therefore *parallel to the first cyclic plane*.

(22.) The equation of the cone (by IX., X., XVI.) may also be thus written :

$$S U \frac{\rho}{\alpha} \cdot S U \frac{\beta}{\rho} = T \frac{\alpha}{\beta}; \quad \text{or,} \quad \cos \angle \frac{\rho}{\alpha} \cdot \cos \angle \frac{\rho}{\beta} = T \frac{\alpha}{\beta};$$

it expresses, therefore, that the *product of the cosines of the inclinations, of any variable side* (ρ) *of an oblique cyclic cone, to two fixed lines* (α and β), namely to the *two cyclic normals* (17.), is *constant*; or that the *product of the sines of the inclinations, of the same variable side* (or ray, ρ) *of the cone, to two fixed planes*, namely to the *two cyclic planes*, is thus a constant quantity.

(23.) The *two great circles*, in which the *concentric sphere* $T\rho = 1$ is cut by the *two cyclic planes*, have been called the *two Cyclic Arcs** of the *Spherical Conic* (11.), in

* By M. Chales.

which that sphere is cut by the cone. It follows (by (22.)) that the *product of the sines of the (arcual) perpendiculars, let fall from any point P of a given spherical conic, on its two cyclic arcs, is constant.*

(24.) These properties of *cyclic cones*, and of *spherical conics*, are not put forward as *new*; but they are of importance enough, and have been here deduced with sufficient facility, to show that we are already in possession of a *Calculus*, with its own *Rules* of Transformation*, whereby *one enunciation* of a geometrical theorem, or problem, or construction, can be *translated* into several others, of which some may be clearer, or simpler, or more elegant, than the one first proposed.

197. Let α, β, γ be any three co-initial vectors, $oA, \&c.$, and let $oD = \delta = \gamma + \beta$, so that $OBDC$ is a parallelogram (6); then, if we write

$$\beta : \alpha = q, \quad \gamma : \alpha = q', \quad \text{and} \quad \delta : \alpha = q'' = q' + q \quad (106),$$

and suppose that B', C', D' are the feet of perpendiculars let fall from the points B, C, D on the line oA , we shall have, by 196, XIX., the expressions,

$$(oB' =) \beta' = \alpha Sq, \quad \gamma' = \alpha Sq', \quad \delta' = \alpha Sq'' = \alpha S(q' + q).$$

But also $OB = OD$, and therefore $OB' = C'D'$, the *similar projections of equal lines being equal*; hence (comp. 11) the *sum of the projections of the lines β, γ must be equal to the projection of the sum*, or in symbols,

$$oD' = oC' + oB', \quad \delta' = \gamma' + \beta', \quad \delta' : \alpha = (\gamma' : \alpha) + (\beta' : \alpha).$$

Hence, generally, for *any two quaternions, q and q'* , we have the formula:

$$I. \dots S(q' + q) = Sq' + Sq;$$

or in words, the *scalar of the sum* is equal to the *sum of the scalars*. It is easy to extend this result to the case of any *three* (or more) quaternions, with their respective scalars; thus, if q'' be a third *arbitrary* quaternion, we may write

$$S(q'' + (q' + q)) = Sq'' + S(q' + q) = Sq'' + (Sq' + Sq);$$

where, on account of the *scalar* character of the summands, the last parentheses may be omitted. We may therefore write, generally,

$$II. \dots S\Sigma q = \Sigma Sq, \quad \text{or briefly,} \quad S\Sigma = \Sigma S;$$

where Σ is used as a *sign of Summation*: and may say that

* Comp. 145, (10.), &c.

a formula which holds even when $\angle q$ is 0, or $\frac{\pi}{2}$, or π , and which gives,

$$\text{IV.} \dots \text{SU}(q^2) = 2(\text{SU}q)^2 - 1.$$

Hence, generally, the scalar of q^2 may be put under either of the two following forms :

$$\text{V.} \dots \text{S}(q^2) = \text{T}q^2 \cdot \cos 2 \angle q; \quad \text{VI.} \dots \text{S}(q^2) = 2(\text{S}q)^2 - \text{T}q^2;$$

- where we see that it would not be safe to omit the parentheses, without some convention previously made, and to write simply $\text{S}q^2$, without first deciding whether this last symbol shall be understood to signify the *scalar of the square*, or the *square of the scalar* of q : these two things being generally unequal. The latter of them, however, occurring rather oftener than the former, it appears convenient to fix on it as that which is to be understood by $\text{S}q^2$, while the other may occasionally be written with a point thus, $\text{S}.q^2$; and then, with these conventions respecting notation,* we may write :

$$\text{VII.} \dots \text{S}q^2 = (\text{S}q)^2; \quad \text{VIII.} \dots \text{S}.q^2 = \text{S}(q^2).$$

But the *square of the conjugate* of any quaternion is easily seen to be the *conjugate of the square*; so that we have generally (comp. 190, II.) the formula :

$$\text{IX.} \dots \text{K}q^2 = \text{K}(q^2) = (\text{K}q)^2 = \text{T}q^2 : \text{U}q^2.$$

(1.) A quaternion, like a positive scalar, may be said to have in general two opposite square roots; because the squares of opposite quaternions are always equal (comp. (8.)). But of these two roots the principal (or simpler) one, and that which we shall denote by the symbol \sqrt{q} , or \sqrt{q} , and shall call by eminence the *Square Root* of q , is that which has its angle acute, and not obtuse. We shall therefore write, generally,

$$\text{X.} \dots \angle \sqrt{q} = \frac{1}{2} \angle q; \quad \text{Ax.} \sqrt{q} = \text{Ax}.q;$$

* As, in the *Differential Calculus*, it is usual to write dx^2 instead of $(dx)^2$; while $d(x^2)$ is sometimes written as $d.x^2$. But as d^2x denotes a *second differential*, so it seems safest not to denote the square of $\text{S}q$ by the symbol S^2q , which properly signifies $\text{SS}q$, or $\text{S}q$, as in 196, VI.; the *second scalar* (like the *second tensor*, 187, (9.), or the *second versor*, 160) being equal to the *first*. Still every calculator will of course use his own discretion; and the employment of the notation S^2q for $(\text{S}q)^2$, as $\cos^2 x$ is often written for $(\cos x)^2$, may sometimes cause a *saving of space*.

with the reservation that, when $\angle q = 0$, or $= \pi$, this common axis of q and \sqrt{q} becomes (by 181, 149) an *indeterminate* unit-line.

(2.) Hence,

$$XI. \dots S\sqrt{q} > 0, \text{ if } \angle q < \pi;$$

while this *scalar of the square root* of a quaternion may, by VI., be thus transformed:

$$XII. \dots S\sqrt{q} = \sqrt{\frac{1}{2}(Tq + Sq)};$$

a formula which holds good, even at the limit $\angle q = \pi$.

(3.) The principle* (1.), that in quaternions, as in algebra, the equation,

$$XIII. \dots (-q)^2 = q^2,$$

is an *identity*, may be illustrated by conceiving that, in Fig. 42, a point α' is determined by the equation $o\alpha' = \alpha o$; for then we shall have (comp. Fig. 38, bis),

$$(-q)^2 = \left(\frac{o\alpha'}{o\alpha}\right)^2 = \frac{o\alpha}{o\alpha} = q^2, \text{ because } \Delta \alpha o \alpha' \alpha' o \alpha.$$

200. Another useful connexion between *scalars* and *tensors* (or *norms*) of quaternions may be derived as follows. In any plane triangle $\alpha o \beta$, we have† the relation,

$$(T. \alpha\beta)^2 = (T. o\alpha)^2 - 2(T. o\alpha) \cdot (T. o\beta) \cdot \cos \alpha o \beta + (T. o\beta)^2;$$

in which the symbols $T. o\alpha$, &c., denote (by 185, 186) the *lengths* of the sides $o\alpha$, &c.; but if we still write $q = o\beta : o\alpha$, we have $q - 1 = \alpha\beta : o\alpha$; dividing therefore by $(T. o\alpha)^2$, the formula becomes (by 196, &c.),

$$I. \dots T(q - 1)^2 = 1 - 2Tq \cdot S\sqrt{q} + Tq^2 = Tq^2 - 2Sq + 1;$$

or

$$II. \dots N(q - 1) = Nq - 2Sq + 1.$$

But q is here a perfectly *general* quaternion; we may therefore change its *sign*, and write,

$$III. \dots T(1 + q)^2 = 1 + 2Sq + Tq^2; \quad IV. \dots N(1 + q) = 1 + 2Sq + Nq.$$

And since it is easy to prove (by 106, 107) that

$$V. \dots \left(\frac{q'}{q} + 1\right) q = q' + q,$$

whatever two quaternions q and q' may be, while

$$VI. \dots S\frac{q'}{q} \cdot Nq = S \cdot q'Kq = S \cdot qKq',$$

we easily infer this other general formula,

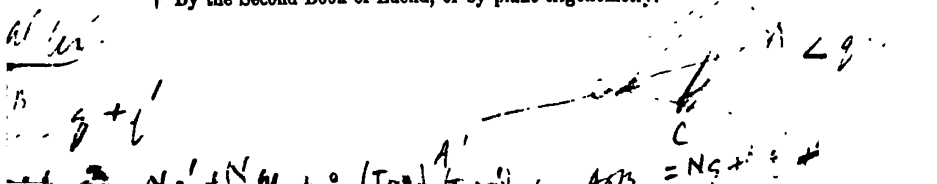
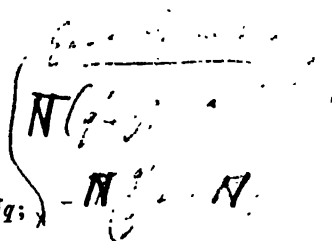
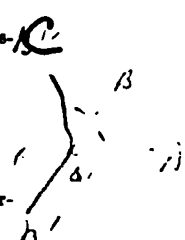
$$VII. \dots N(q' + q) = Nq' + 2S \cdot qKq' + Nq;$$

which gives, if x be any scalar,

$$VIII. \dots N(q + x) = Nq + 2xSq + x^2.$$

* Compare the first Note to page 162.

† By the Second Book of Euclid, or by plane trigonometry.



(1.) We are now prepared to effect, by *rules* of transformation*, some other *passages* from one mode of *expression* to another, of the kind which has been alluded to, and partly exemplified, in former sub-articles. Take, for example, the formula,

$$T \frac{\rho + \alpha}{\rho - \alpha} = 1, \text{ of 187, (2.);}$$

or the equivalent formula,

$$T(\rho + \alpha) = T(\rho - \alpha), \text{ of 186, (6.);}$$

which has been seen, on *geometrical* grounds, to represent a certain *locus*, namely the plane through o , perpendicular to the line oA ; and therefor the *same locus* as that which is represented by the equation,

$$S \frac{\rho}{\alpha} = 0, \text{ of 196, (1.).}$$

To *pass* now from the former equations to the latter, by *calculation*, we have only to denote the quotient $\rho : \alpha$ by q , and to observe that the first or second form, as just now cited, becomes then,

$$T(q + 1) = T(q - 1); \text{ or } N(q + 1) = N(q - 1);$$

or finally, by II. and IV.,

$$Sq = 0,$$

which gives the third form of equation, as required.

(2.) Conversely, from $S \frac{\rho}{\alpha} = 0$, we can *return*, by the same general formulæ II.

and IV., to the equation $N\left(\frac{\rho}{\alpha} - 1\right) = \overline{N}\left(\frac{\rho}{\alpha} + 1\right)$, or by I. and III. to $T\left(\frac{\rho}{\alpha} - 1\right) = T\left(\frac{\rho}{\alpha} + 1\right)$ or to $T(\rho - \alpha) = T(\rho + \alpha)$, or to $T \frac{\rho + \alpha}{\rho - \alpha} = 1$, as above; and generally,

$$Sq = 0 \text{ gives } T(q - 1) = T(q + 1), \text{ or } T \frac{q + 1}{q - 1} = 1;$$

while the latter equations, in turn, involve, as has been seen, the former.

(3.) Again, if we take the Apollonian Locus, 145, (8.), (9.), and employ the *first* of the two forms 186, (5.) of its equation, namely,

$$T(\rho - a^2\alpha) = aT(\rho - \alpha),$$

where a is a given positive scalar different from unity, we may write it as

$$T(q - a^2) = aT(q - 1), \text{ or as } N(q - a^2) = a^2N(q - 1);$$

or by VIII.,

$$Ng - 2a^2Sq + a^4 = a^2(Ng - 2Sq + 1);$$

or, after suppressing $-2a^2Sq$, transposing, and dividing by $a^2 - 1$,

$$Ng = a^2; \text{ or, } N\rho = a^2N\alpha; \text{ or, } T\rho = aTa;$$

which last is the *second form* 186, (5.), and is thus *deduced from the first*, by *calculation alone*, without any immediate appeal to *geometry*, or the construction of any *diagram*.

* Compare 145, (10.); and several subsequent sub-articles.

(4.) Conversely if we take the equation,

$$N \frac{\rho}{a} = a^2, \text{ of 145, (12.),}$$

$$N \sqrt{q-1} = N \cdot \frac{1}{2} - \dots$$

which was there seen to represent the same locus, considered as a spheric surface, with o for centre, and aa for one of its radii, and write it as $Nq = a^2$, we can then by calculation return to the form

$$N(q - a^2) = a^2 N(q - 1), \text{ or } T(q - a^2) = aT(q - 1),$$

or finally,

$$T(\rho - a^2 a) = aT(\rho - a), \text{ as in 186, (5.);}$$

this first form of that sub-article being thus deduced from the second, namely from

$$T\rho = aT\alpha, \text{ or } T \frac{\rho}{a} = \alpha.$$

(5.) It is far from being the intention of the foregoing remarks, to discourage attention to the geometrical interpretation of the various forms of expression, and general rules of transformation, which thus offer themselves in working with quaternions; on the contrary, one main object of the present Chapter has been to establish a firm geometrical basis, for all such forms and rules. But when such a foundation has once been laid, it is, as we see, not necessary that we should continually recur to the examination of it, in building up the superstructure. That each of the two forms, in 186, (5.), involves the other, may be proved, as above, by calculation; but it is interesting to inquire what is the meaning of this result: and in seeking to interpret it, we should be led anew to the theorem of the Apollonian Locus.

(6.) The result (4.) of calculation, that

$$N(q - a^2) = a^2 N(q - 1), \text{ if } Nq = a^2,$$

may be expressed under the form of an identity, as follows:

$$\text{IX. . . } N(q - Nq) = Nq \cdot N(q - 1);$$

in which q may be any quaternion.

(7.) Or, by 191, VII., because it will soon be seen that

$$q(q - 1) = q^2 - q, \text{ as in algebra,}$$

we may write it as this other identity:

$$\text{X. . . } N(q - Nq) = N(q^2 - q).$$

(8.) If $T(q - 1) = 1$, then $S \frac{1}{q} = \frac{1}{2}$; and conversely, the former equation follows from the latter; because each may be put under the form (comp. 196, XII.),

$$Nq = 2Sq.$$

(9.) Hence, if $T(\rho - a) = T\alpha$, then $S \frac{2a}{\rho} = 1$, and reciprocally. In fact (comp. 196, (6.)), each of these two equations expresses that the locus of ρ is the sphere which passes through o , and has its centre at Δ ; or which has $OB = 2a$ for a diameter.

(10.) By changing q to $q + 1$ in (8), we find that

$$\text{if } Tq = 1, \text{ then } S \frac{q - 1}{q + 1} = 0, \text{ and reciprocally.}$$

(11.) Hence if $T\rho = T\alpha$, then $S\frac{\rho - \alpha}{\rho + \alpha} = 0$, and reciprocally; because (by 106)

$$\frac{\rho - \alpha}{\rho + \alpha} = \frac{\rho - \alpha}{\alpha} : \frac{\rho + \alpha}{\alpha} = \left(\frac{\rho}{\alpha} - 1\right) : \left(\frac{\rho}{\alpha} + 1\right).$$

(12.) Each of these two equations (11.) expresses that the locus of ρ is the sphere through A , which has its centre at o ; and their proved agreement is a recognition, by quaternions, of the elementary geometrical theorem, that the angle in a semicircle is a right angle.

SECTION 13.—On the Right Part (or Vector Part) of a Quaternion; and on the Distributive Property of the Multiplication of Quaternions.

201. A given vector OB can always be decomposed, in one but in only one way, into two component vectors, of which it is the *sum* (6); and of which *one*, as OB' in Fig. 50, is *parallel* (15) to another given vector OA , while the *other*, as OB'' in the same Figure, is *perpendicular* to that given line OA ; namely, by letting fall the perpendicular BB' on OA , and drawing $OB'' = B'B$, so that $OB'BB''$ shall be a rectangle. In other words, if a and β be any two given, actual, and co-initial vectors, it is always possible to deduce from them, in one definite way, two other co-initial vectors, β' and β'' , which need not however *both* be actual (1); and which shall satisfy (comp. 6, 15, 129) the conditions,

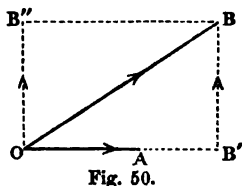
$$\beta = \beta' + \beta'' = \beta'' + \beta', \quad \beta' \parallel a, \quad \beta'' \perp a;$$

β' vanishing, when $\beta \perp a$; and β'' being null, when $\beta \parallel a$; but both being (what we may call) *determinate vector-functions* of a and β . And of these two functions, it is evident that β' is the orthographic *projection* of β on the line a ; and that β'' is the corresponding *projection* of β on the plane through o , which is *perpendicular* to a .

202. Hence it is easy to infer, that there is always one, but only one way, of *decomposing a given quaternion*,

$$q = OB : OA = \beta : a,$$

into two parts or summands (195), of which *one* shall be, as in



196, a *scalar*, while the *other* shall be a *right quotient* (132). Of these two parts, the *former* has been already called (196) *the scalar part*, or simply *the Scalar* of the Quaternion, and has been denoted by the symbol Sq ; so that, with reference to the recent Figure 50, we have

$$I. . . Sq = S(OB : OA) = OB' : OA ; \text{ or, } S(\beta : a) = \beta' : a.$$

And we now propose to call the *latter* part the **RIGHT PART*** of the same quaternion, and to denote it by the new symbol

$$Vq ;$$

writing thus, in connexion with the same Figure,

$$II. . . Vq = V(OB : OA) = OB'' : OA ; \text{ or, } V(\beta : a) = \beta'' : a.$$

The *System of Notations*, peculiar to the present Calculus, will thus have been completed; and we shall have the following general *Formula of Decomposition of a Quaternion into two Summands* (comp. 188), of the *Scalar* and *Right* kinds :

$$III. . . q = Sq + Vq = Vq + Sq,$$

or, briefly and symbolically,

$$IV. . . 1 = S + V = V + S.$$

(1.) In connexion with the same Fig. 50, we may write also,

$$V(OB : OA) = B'B : OA,$$

because, by construction, $B'B = OB''$.

(2.) In like manner, for Fig. 86, we have the equation,

$$V(OB : OA) = A'B : OA.$$

(3.) Under the recent conditions,

$$V(\beta' : a) = 0, \text{ and } S(\beta'' : a) = 0.$$

(4.) In general, it is evident that

$$V. . . q = 0, \text{ if } Sq = 0, \text{ and } Vq = 0; \text{ and reciprocally.}$$

(5.) More generally,

$$VI. . . q' = q, \text{ if } Sq' = Sq, \text{ and } Vq' = Vq; \text{ with the converse.}$$

(6.) Also VII. . . $Vq = 0$, if $\angle q = 0$, or $= \pi$;

or VIII. . . $V(\beta : a) = 0$, if $\beta \parallel a$;

the *right part* of a *scalar* being zero.

* This *Right Part*, Vq , will come to be also called *the Vector Part*, or simply *the VECTOR*, of the Quaternion; because it will be found possible and useful to identify such part with its own Index-Vector (183). Compare the Notes to pages 119, 136, 174.

(7.) On the other hand,

$$\text{IX.} \dots Vq = q, \text{ if } \angle q = \frac{\pi}{2};$$

a right quaternion being its own right part.

203. We had (196, XIX.) a formula which may now be written thus,

$$\text{I.} \dots OB' = S(OB : OA) \cdot OA, \text{ or } \beta' = S \frac{\beta}{a} \cdot a,$$

to express the *projection of OB on OA*, or of the vector β on a ; and we have evidently, by the definition of the new symbol Vq , the analogous formula,

$$\text{II.} \dots OB'' = V(OB : OA) \cdot OA, \text{ or } \beta'' = V \frac{\beta}{a} \cdot a,$$

to express the *projection of β on the plane* (through o), which is drawn so as to be *perpendicular to a* ; and which has been considered in several former sub-articles (comp. 186, (6.), and 196, (1.)). It follows (by 186, &c.) that

$$\text{III.} \dots T\beta'' = TV \frac{\beta}{a} \cdot T\alpha = \text{perpendicular distance of B from } OA;$$

this perpendicular being *here* considered with reference to its *length* alone, as the characteristic T of the *tensor* implies. It is to be observed that because the *factor*, $V \frac{\beta}{a}$, in the recent formula II. for the projection β'' , is *not a scalar*, we must write that factor as a *multiplier*, and *not* as a *multiplicand*; although we were at liberty, in consequence of a general convention (15), respecting the *multiplication of vectors and scalars*, to denote the *other* projection β' under the form,

$$\text{I'.} \dots \beta' = aS \frac{\beta}{a} \text{ (196, XIX.)}$$

(1.) The equation,

$$V \frac{\rho}{a} = 0,$$

expresses that the locus of ρ is the *indefinite right line* OA .

(2.) The equation,

$$V \frac{\rho - \beta}{a} = 0, \text{ or } V \frac{\rho}{a} = V \frac{\beta}{a},$$

expresses that the locus of P is the indefinite right line an'' , in Fig. 50, which is drawn through the point n , parallel to the line OA .

(3.) The equation

$$S \frac{\rho - \beta}{\alpha} = 0, \text{ or } S \frac{\rho}{\alpha} = S \frac{\beta}{\alpha}, \text{ of 196, (2.),}$$

has been seen to express that the locus of P is the plane through n , perpendicular to the line OA ; if then we combine it with the recent equation (2.), we shall express that the point P is situated at the intersection of the two last mentioned loci; or that it coincides with the point n .

(4.) Accordingly, whether we take the two first or the two last of these recent forms (2.), (3.), namely,

$$V \frac{\rho - \beta}{\alpha} = 0, \quad S \frac{\rho - \beta}{\alpha} = 0, \quad \text{or} \quad V \frac{\rho}{\alpha} = V \frac{\beta}{\alpha}, \quad S \frac{\rho}{\alpha} = S \frac{\beta}{\alpha},$$

we can infer this position of the point P : in the first case by inferring, through 202, V ., that $\frac{\rho - \beta}{\alpha} = 0$, whence $\rho - \beta = 0$, by 142; and in the second case by inferring, through 202, VI ., that $\frac{\rho}{\alpha} = \frac{\beta}{\alpha}$; so that we have in each case (comp. 104), or as a consequence from each system, the equality $\rho = \beta$, or $OP = OB$; or finally (comp. 20) the coincidence, $P = n$.

(5.) The equation,

$$TV \frac{\rho}{\alpha} = TV \frac{\beta}{\alpha},$$

expresses that the locus of the point P is the cylindrical surface of revolution, which passes through the point n , and has the line OA for its axis; for it expresses, by III., that the perpendicular distances of P and n , from this latter line, are equal.

(6.) The system of the two equations,

$$TV \frac{\rho}{\alpha} = TV \frac{\beta}{\alpha}, \quad S \frac{\rho}{\gamma} = 0,$$

expresses that the locus of P is the (generally) elliptic section of the cylinder (5.), made by the plane through o , which is perpendicular to the line OC .

(7.) If we employ an analogous decomposition of ρ , by supposing that

$$\rho = \rho' + \rho'', \quad \rho' \parallel \alpha, \quad \rho'' \perp \alpha,$$

the three rectilinear or plane loci, (1.), (2.), (3.), may have their equations thus briefly written:

$$\rho'' = 0; \quad \rho'' = \beta''; \quad \rho' = \beta':$$

while the combination of the two last of these gives $\rho = \beta$, as in (4.).

(8.) The equation of the cylindrical locus, (5.), takes at the same time the form,

$$Tp'' = T\beta'';$$

which last equation expresses that the projection P'' of the point P , on the plane through o perpendicular to OA , falls somewhere on the circumference of a circle, with o for centre, and OB'' for radius: and this circle may accordingly be considered as the base of the right cylinder, in the sub-article last cited.

204. From the mere circumstance that Vq is always a right quotient (132), whence UVq is a right versor (153), of

which the *plane* (119), and the *axis* (127), coincide with those of q , several general consequences easily follow. Thus we have generally, by principles already established, the relations :

$$\text{I.} \dots \angle Vq = \frac{\pi}{2}; \quad \text{II.} \dots \text{Ax. } Vq = \text{Ax. } UVq = \text{Ax. } q;$$

$$\text{III.} \dots KVq = -Vq, \text{ or } KV = -V \text{ (144);}$$

$$\text{IV.} \dots SVq = 0, \text{ or } SV = 0 \text{ (196, VII.);}$$

$$\text{V.} \dots (UVq)^2 = -1 \text{ (153, 159);}$$

and therefore,

$$\text{VI.} \dots (Vq)^2 = -(TVq)^2 = -NVq,*$$

because, by the general decomposition (188) of a quaternion into *factors*, we have

$$\text{VII.} \dots Vq = TVq \cdot UVq.$$

We have also (comp. 196, VI.),

$$\text{VIII.} \dots VSq = 0, \text{ or } VS = 0 \text{ (202, VII.);}$$

$$\text{IX.} \dots VVq = Vq, \text{ or } V^2 = VV = V \text{ (202, IX.);}$$

and

$$\text{X.} \dots VKq = -Vq, \text{ or } VK = -V,$$

because *conjugate quaternions* have *opposite right parts*, by the definitions in 137, 202, and by the construction of Fig. 36. For the same reason, we have this other general formula,

$$\text{XI.} \dots Kq = Sq - Vq, \text{ or } K = S - V;$$

but we had

$$q = Sq + Vq, \text{ or } 1 = S + V, \text{ by 202, III., IV.};$$

hence not only, by addition,

$$q + Kq = 2Sq, \text{ or } 1 + K = 2S, \text{ as in 196, I.,}$$

but also, by subtraction,

$$\text{XII.} \dots q - Kq = 2Vq, \text{ or } 1 - K = 2V;$$

whence the *Characteristic*, V , of the *Operation of taking the Right Part of a Quaternion* (comp. 132, (6.); 137; 156; 187; 196), may be *defined* by either of the two following symbolical equations :

$$\text{XIII.} \dots V = 1 - S \text{ (202, IV.);} \quad \text{XIV.} \dots V = \frac{1}{2}(1 - K);$$

whereof the former connects it with the characteristic S , and

* Compare the Note to page 180.

the latter with the characteristic K ; while the dependence of K on S and V is expressed by the recent formula XI.; and that of S on K by 196, II'. Again, if the line OB , in Fig. 50, be multiplied (15) by any scalar coefficient, the perpendicular BB' is evidently multiplied by the same; hence, generally,

$$\text{XV.} \dots Vxq = xVq, \text{ if } x \text{ be any scalar;}$$

and therefore, by 188, 191,

$$\text{XVI.} \dots Vq = Tq \cdot VUq, \text{ and XVII.} \dots TVq = Tq \cdot TVUq.$$

But the consideration of the right-angled triangle, $OB'B$, in the same Figure, shows that

$$\text{XVIII.} \dots TVq = Tq \cdot \sin \angle q,$$

because, by 202, II., we have

$$TVq = T(OB'' : OA) = T.OB'' : T.OA,$$

and

$$T.OB'' = T.OB \cdot \sin \angle AOB;$$

we arrive then thus at the following general and useful formula, connecting *quaternions* with *trigonometry* anew:

$$\text{XIX.} \dots TVUq = \sin \angle q;$$

by combining which with the formula,

$$SUq = \cos \angle q \text{ (196, XVI.)},$$

we arrive at the general relation:

$$\text{XX.} \dots (SUq)^2 + (TVUq)^2 = 1;$$

which may also (by XVII., and by 196, IX.) be written thus:

$$\text{XXI.} \dots (Sq)^2 + (TVq)^2 = (Tq)^2;$$

and might have been immediately deduced, *without sines and cosines*, from the right-angled triangle, by the property of the square of the hypotenuse, under the form,

$$(T.OB')^2 + (T.B'B)^2 = (T.OB)^2.$$

The same important relation may be expressed in various other ways; for example, we may write,

$$\text{XXII.} \dots Nq = Tq^2 = Sq^2 - Vq^2,$$

where it is assumed, as an abridgment of *notation* (comp. 199, VII., VIII.), that

$$\text{XXIII.} \dots Vq^2 = (Vq)^2, \text{ but that XXIV.} \dots V.q^2 = V(q^2),$$

the import of this last symbol remaining to be examined. And because, by the definition of a *norm*, and by the properties of Sq and Vq ,

$$\text{XXV.} \dots NSq = Sq^2, \text{ but } \text{XXVI.} \dots NVq = -Vq^2,$$

we may write also,

$$\text{XXVII.} \dots Nq = N(Sq + Vq) = NSq + NVq;$$

a result which is indeed included in the formula 200, VIII., since that equation gives, generally,

$$\text{XXVIII.} \dots N(q + x) = Nq + Nx, \text{ if } \angle q = \frac{\pi}{2};$$

x being, as usual, any scalar. It may be added that because (by 106, 143) we have, as in algebra, the identity,

$$\text{XXIX.} \dots -(q' + q) = -q' - q,$$

the *opposite of the sum* of any two quaternions being thus equal to the *sum of the opposites*, we may (by XI.) establish this other general formula :

$$\text{XXX.} \dots -Kq = Vq - Sq;$$

the *opposite of the conjugate* of any quaternion q having thus the *same right part* as that quaternion, but an *opposite scalar part*.

(1.) From the last formula it may be inferred, that

$$\text{if } q' = -Kq, \text{ then } Vq' = +Vq, \text{ but } Sq' = -Sq;$$

and therefore that

$$Tq' = Tq, \text{ and } Ax.q' = Ax.q, \text{ but } \angle q' = \pi - \angle q;$$

which two last relations might have been deduced from 138 and 143, without the introduction of the characteristics S and V .

(2.) The equation,

$$\left(V \frac{\rho}{\alpha} \right)^2 = \left(V \frac{\beta}{\alpha} \right)^2, \text{ or (by XXVI.), } NV \frac{\rho}{\alpha} = NV \frac{\beta}{\alpha},$$

like the equation of 203, (5.), expresses that the locus of P is the *right cylinder*, or *cylinder of revolution*, with OA for its axis, which passes through the point α .

(3.) The system of the two equations,

$$\left(V \frac{\rho}{\alpha} \right)^2 = \left(V \frac{\beta}{\alpha} \right)^2, \quad S \frac{\rho}{\gamma} = 0,$$

like the corresponding system in 203, (6.), represents generally an *elliptic section* of the same right cylinder; but if it happen that $\gamma \parallel \alpha$, the section then becomes *circular*.

(4.) The system of the two equations,

$$S \frac{\rho}{\alpha} = x, \quad \left(\sqrt{\frac{\rho}{\alpha}} \right)^2 = x^2 - 1, \quad \text{with } x > -1, \quad x < 1,$$

represents the *circle*,* in which the cylinder of revolution, with oA for axis, and with $(1-x^2)^{1/2}T\alpha$ for radius, is perpendicularly cut by a plane at a distance $= \pm xT\alpha$ from o ; the vector of the centre of this circular section being αx .

(5.) While the scalar x increases (algebraically) from -1 to 0 , and thence to $+1$, the connected scalar $\sqrt{(1-x^2)}$ at first increases from 0 to 1 , and then decreases from 1 to 0 ; the *radius* of the circle (4.) at the same time enlarging from zero to a maximum $= T\alpha$, and then again diminishing to zero; while the position of the *centre* of the circle varies continuously, in one constant direction, from a *first limit-point* A' , if $oA' = -\alpha$, to the point A , as a *second limit*.

(6.) The *locus* of all such *circles* is the *sphere*, with AA' for a diameter, and therefore with o for centre; namely, the sphere which has already been represented by the equation $T\rho = T\alpha$ of 186, (2.); or by $T \frac{\rho}{\alpha} = 1$, of 187, (1.); or by

$$S \frac{\rho - \alpha}{\rho + \alpha} = 0, \quad \text{of 200, (11.);}$$

but which now presents itself under the new form,

$$\left(S \frac{\rho}{\alpha} \right)^2 - \left(\sqrt{\frac{\rho}{\alpha}} \right)^2 = 1,$$

obtained by *eliminating* x between the two recent equations (4.).

(7.) It is easy, however, to *return* from the last form to the second, and thence to the first, or to the third, by rules of calculation already established, or by the general relations between the symbols used. In fact, the last equation (6.) may be written, by XXII., under the form,

$$N \frac{\rho}{\alpha} = 1;$$

whence

$$T \frac{\rho}{\alpha} = 1, \quad \text{by 190, VI.};$$

and therefore also $T\rho = T\alpha$, by 187, and $S \frac{\rho - \alpha}{\rho + \alpha} = 0$, by 200, (11.).

(8.) Conversely, the sphere through A , with o for centre, might already have been seen, by the first definition and property of a *norm*, stated in 145, (11.), to admit (comp. 145, (12.)) of being represented by the equation $N \frac{\rho}{\alpha} = 1$; and therefore, by XXII., under the recent form (6.); in which if we write x to denote the variable scalar $S \frac{\rho}{\alpha}$, as in the first of the two equations (4.), we recover the second of those equations: and thus might be led to consider, as in (6.), the *sphere* in question

* By the word "circle," in these pages, is usually meant a *circumference*, and not an *area*; and in like manner, the words "sphere," "cylinder," "cone," &c., are usually here employed to denote *surfaces*, and not *volumes*.

as the *locus of a variable circle*, which is (as above) the *intersection of a variable cylinder*, with a *variable plane* perpendicular to its axis.

(9.) The same sphere may also, by XXVII., have its equation written thus,

$$N\left(S\frac{\rho}{a} + V\frac{\rho}{a}\right) = 1; \quad \text{or} \quad T\left(S\frac{\rho}{a} + V\frac{\rho}{a}\right) = 1.$$

(10.) If, in each variable plane represented by the first equation (4.), we conceive the radius of the circle, or that of the variable cylinder, to be multiplied by any constant and positive scalar a , the centre of the circle and the axis of the cylinder remaining unchanged, we shall pass thus to a *new system of circles*, represented by this new system of equations,

$$S\frac{\rho}{a} = x, \quad \left(V\frac{\rho}{aa}\right)^2 = x^2 - 1.$$

(11.) The *locus of these new circles* will evidently be a *Spheroid of Revolution*; the centre of this new surface being the centre O , and the *axis* of the same surface being the *diameter* $\Delta\Delta'$, of the *sphere* lately considered: which sphere is therefore either *inscribed* or *circumscribed* to the spheroid, according as the constant $a >$ or $<$ 1; because the *radii* of the new circles are in the first case *greater*, but in the second case *less*, than the radii of the old circles; or because the *radius of the equator* of the spheroid = aTa , while the radius of the sphere = Ta .

(12.) The equations of the *two co-axial cylinders* of revolution, which *envelope* respectively the sphere and spheroid (or are *circumscribed* thereto) are:

$$\left(V\frac{\rho}{a}\right)^2 = -1; \quad \left(V\frac{\rho}{aa}\right)^2 = -1;$$

or

$$NV\frac{\rho}{a} = 1, \quad NV\frac{\rho}{a} = a^2;$$

or

$$TV\frac{\rho}{a} = 1, \quad TV\frac{\rho}{a} = a.$$

(13.) The system of the two equations,

$$S\frac{\rho}{a} = x, \quad \left(V\frac{\rho}{\beta}\right)^2 = x^2 - 1, \quad \text{with } \beta \text{ not } \parallel a,$$

represents (comp. (8.)) a *variable ellipse*, if the scalar x be still treated as a variable.

(14.) The result of the elimination of x between the two last equations, namely this new equation,

$$\left(S\frac{\rho}{a}\right)^2 - \left(V\frac{\rho}{\beta}\right)^2 = 1;$$

or

$$NS\frac{\rho}{a} + NV\frac{\rho}{\beta} = 1, \quad \text{by XXV., XXVI.};$$

or

$$N\left(S\frac{\rho}{a} + V\frac{\rho}{\beta}\right) = 1, \quad \text{by XXVII.};$$

or finally,

$$T\left(S\frac{\rho}{a} + V\frac{\rho}{\beta}\right) = 1, \quad \text{by 190, VI.},$$

represents the *locus of all such ellipses* (18.), and will be found to be an adequate representation, through quaternions, of the *general ELLIPSOID* (with *three unequal axes*): that celebrated surface being here referred to its *centre*, as the *origin* o of vectors to its points; and the *six scalar* (or algebraic) *constants*, which enter into the usual *algebraic equation* (by co-ordinates) of such a *central ellipsoid*, being here virtually included in the *two independent vectors*, a and β , which may be called its *two Vector-Constants*.*

||| . . .
12!

(15.) The equation (comp. (12.)),

$$\left(\nabla \frac{\rho}{\beta} \right)^2 = -1, \text{ or } NV \frac{\rho}{\beta} = 1, \text{ or } TV \frac{\rho}{\beta} = 1,$$

represents a *cylinder of revolution*, circumscribed to the *ellipsoid*, and touching it along the *ellipse* which answers to the value $x = 0$, in (18.); so that the *plane* of this *ellipse of contact* is represented by the equation,

$$S \frac{\rho}{a} = 0;$$

the *normal* to this *plane* being thus (comp. 196, (17.)) the vector a , or oa ; while the *axis* of the lately mentioned enveloping *cylinder* is β , or ob .

(16.) Postponing any further discussion of the recent *quaternion equation of the ellipsoid* (14.), it may be noted here that we have generally, by XXII., the two following useful transformations for the *squares*, of the *scalar Sq*, and of the *right part Vq*, of any quaternion q :

$$\text{XXXI. . . } Sq^2 = Tq^2 + Vq^2; \quad \text{XXXII. . . } Vq^2 = Sq^2 - Tq^2.$$

(17.) In referring briefly to these, and to the connected formula XXII., upon occasion, it may be somewhat safer to write:

$$(S)^2 = (T)^2 + (V)^2, \quad (V)^2 = (S)^2 - (T)^2, \quad (T)^2 = (S)^2 - (V)^2,$$

than $S^2 = T^2 + V^2$, &c.; because these last forms of notation, S^2 , &c., have been otherwise interpreted already, in analogy to the known *Functional Notation*, or *Notation of the Calculus of Functions*, or of *Operations* (comp. 187, (9.); 196, VI.; and 204, IX.).

(18.) In pursuance of the same analogy, any *scalar* may be denoted by the *general symbol*,

$$V^{-1}0;$$

because *scalars* are the *only* quaternions of which the *right parts vanish*.

(19.) In like manner, a *right quaternion*, generally, may be denoted by the *symbol*,

$$S^{-1}0;$$

and since this includes (comp. 204, I.) the *right part of any quaternion*, we may establish this *general symbolic transformation of a Quaternion*:

$$q = V^{-1}0 + S^{-1}0.$$

(20.) With this form of notation, we should have generally, at least for *real†* quaternions, the inequalities,

* It will be found, however, that *other pairs of vector-constants*, for the *central ellipsoid*, may occasionally be used with advantage.

† Compare Art. 149; and the Notes to pages 90, 134.

$$(\sqrt{-1}0)^2 > 0; \quad (8^{-1}0)^2 < 0;$$

so that a (geometrically real) Quaternion is generally of the form:

Square-root of a Positive, plus Square-root of a Negative.

(21.) The equations 196, XVI. and 204, XIX. give, as a new link between quaternions and trigonometry, the formulæ:

$$\text{XXXIII.} \dots \tan \angle q = \text{TVU}q : \text{SU}q = \text{TV}q : \text{S}q.$$

(22.) It may not be entirely in accordance with the *theory* of that *Functional* (or *Operational*) *Notation*, to which allusion has lately been made, but it will be found to be convenient in *practice*, to write this last result under one or other of the *abridged forms* :*

$$\text{XXXIV.} \dots \tan \angle q = \frac{\text{TV}}{\text{S}} \cdot q; \quad \text{or} \quad \text{XXXIV'.} \dots \tan \angle q = (\text{TV} : \text{S}) q;$$

which have the advantage of *saving the repetition of the symbol of the quaternion*, when that symbol happens to be a *complex expression*, and not, as here, a single letter, *g*.

(23.) The transformation 194, for the *index* of a right quotient, gives generally, by II., for any quaternion *q*, the formulæ:

$$\text{XXXV.} \dots \text{IV}q = \text{TV}q \cdot \text{Ax} \cdot q; \quad \text{XXXVI.} \dots \text{IUV}q = \text{Ax} \cdot q;$$

so that we may establish generally the symbolical† equation,

$$\text{XXXVI'.} \dots \text{IUV} = \text{Ax}.$$

(24.) And because $\text{Ax} \cdot (1 : \sqrt{q}) = -\text{Ax} \cdot \sqrt{q}$, by 185, and therefore $= -\text{Ax} \cdot q$, by II., we may write also, by XXXV.,

$$\text{XXXV'.} \dots \text{I}(1 : \sqrt{q}) = -\text{Ax} \cdot q : \text{TV}q.$$

205. If any parallelogram *oBDC* (comp. 197) be projected on the plane through *o*, which is perpendicular to *oA*, the projected figure *oB''d''c''* (comp. 11) is still a parallelogram; so that

$$oD'' = oC'' + oB'' \quad (6), \quad \text{or} \quad \delta'' = \gamma'' + \beta'';$$

and therefore, by 106,

$$\delta'' : a = (\gamma'' : a) + (\beta'' : a).$$

Hence, by 120, 202, for *any two quaternions*, *q* and *q'*, we have the general formula,

$$\text{I.} \dots \text{V}(q' + q) = \text{V}q' + \text{V}q;$$

* Compare the Note to Art. 199.

† At a later stage it will be found possible (comp. the Note to page 174, &c.), to write, generally,

$$\text{IV}q = \text{V}q, \quad \text{IUV}q = \text{UV}q;$$

and then (comp. the Note in page 118 to Art. 129) the recent equations, XXXVI., XXXVI', will take these shorter forms:

$$\text{Ax} \cdot q = \text{UV}q; \quad \text{Ax} = \text{UV}.$$

with which it is easy to connect this other,

$$\text{II.} \dots V(q' - q) = Vq' - Vq.$$

Hence also, for any *three* quaternions, q, q', q'' ,

$$V\{q'' + (q' + q)\} = Vq'' + V(q' + q) = Vq'' + (Vq' + Vq);$$

and similarly for any greater number of summands: so that we may write generally (comp. 197, II.),

$$\text{III.} \dots V\Sigma q = \Sigma Vq, \text{ or briefly III'.} \dots V\Sigma = \Sigma V;$$

while the formula II. (comp. 197, IV.) may, in like manner, be thus written,

$$\text{IV.} \dots V\Delta q = \Delta Vq, \text{ or IV'.} \dots V\Delta = \Delta V;$$

the *order* of the terms added, and the mode of *grouping* them, in III., being *as yet* supposed to remain unaltered, although both those restrictions will soon be removed. We conclude then, that the *characteristic* V, of the operation of *taking the right part* (202, 204) of a quaternion, like the characteristic S of *taking the scalar* (196, 197), and the characteristic K of *taking the conjugate* (137, 195*), is a *Distributive Symbol*, or represents a *distributive operation*: whereas the characteristics, Ax., \angle , N, U, T, of the operations of taking respectively the *axis* (128, 129), the *angle* (130), the *norm* (145, (11.)), the *versor* (156), and the *tensor* (187), are *not* thus distributive symbols (comp. 186, (10.), and 200, VII.); or do *not* operate upon a *whole* (or *sum*), by operating on its *parts* (or *summands*).

(1.) We may now recover the symbolical equation $K^2 = 1$ (145), under the form (comp. 196, VI.; 202, IV.; and 204, IV. VIII. IX. XI.):

$$V \dots K^2 = (S - V)^2 = S^2 - SV - VS + V^2 = S + V = 1.$$

(2.) In like manner we can recover each of the expressions for S^2, V^2 from the other, under the forms (comp. again 202, IV.):

$$\text{VI.} \dots S^2 = (1 - V)^2 = 1 - 2V + V^2 = 1 - V = S, \text{ as in 196, VI.};$$

$$\text{VII.} \dots V^2 = (1 - S)^2 = 1 - 2S + S^2 = 1 - S = V, \text{ as in 204, IX.};$$

or thus (comp. 196, II', and 204, XIV.), from the expressions for S and V in terms of K:

* Indeed, it has only been proved as yet (comp. 195, (1.)), that $K\Sigma q = \Sigma Kq$, for the case of *two* summands; but this result will soon be extended.

$$\text{VIII.} \dots S^2 = \frac{1}{4}(1+K)^2 = \frac{1}{4}(1+2K+K^2) = \frac{1}{4}(1+K) = S;$$

$$\text{IX.} \dots V^2 = \frac{1}{4}(1-K)^2 = \frac{1}{4}(1-2K+K^2) = \frac{1}{4}(1-K) = V.$$

(8.) Similarly,

$$\text{X.} \dots SV = \frac{1}{4}(1+K)(1-K) = \frac{1}{4}(1-K^2) = 0, \text{ as in 204, IV.};$$

$$\text{and XI.} \dots VS = \frac{1}{4}(1-K)(1+K) = \frac{1}{4}(1-K^2) = 0, \text{ as in 204, VIII.}$$

206. As regards the *addition* (or subtraction) of such *right parts*, Vq , Vq' , or generally of any two right quaternions (132), we may connect it with the addition (or subtraction) of their *indices* (133), as follows. Let $obdc$ be again any parallelogram (197, 205), but let oA be now an unit-vector (129) perpendicular to its plane; so that

$$Ta = 1, \quad \angle(\beta : a) = \angle(\gamma : a) = \angle(\delta : a) = \frac{\pi}{2}, \quad \delta = \gamma + \beta.$$

Let $ob'd'c'$ be another parallelogram in the same plane, obtained by a positive rotation of the former, through a right angle, round oA as an axis; so that

$$\angle(\beta' : \beta) = \angle(\gamma' : \gamma) = \angle(\delta' : \delta) = \frac{\pi}{2};$$

$$Ax.(\beta' : \beta) = Ax.(\gamma' : \gamma) = Ax.(\delta' : \delta) = a.$$

Then the three right quotients, $\beta : a$, $\gamma : a$, and $\delta : a$, may represent *any two right quaternions*, q , q' , and their *sum*, $q' + q$, which is always (by 197, (2.)) *itself* a right quaternion; and the *indices* of these three right quotients are (comp. 133, 193) the three lines β , γ , δ , so that we may write, under the foregoing conditions of construction,

$$\beta' = I(\beta : a), \quad \gamma' = I(\gamma : a), \quad \delta' = I(\delta : a).$$

But this third index is (by the second parallelogram) the *sum* of the two former indices, or in symbols, $\delta' = \gamma' + \beta'$; we may therefore write,

$$\text{I.} \dots I(q' + q) = Iq' + Iq, \quad \text{if } \angle q = \angle q' = \frac{\pi}{2};$$

or in words *the Index of the Sum* of any two Right Quaternions is equal to the Sum of their Indices*. Hence, generally, for any two quaternions, q and q' , we have the formula,

$$\text{II.} \dots IV(q' + q) = IVq' + IVq,$$

* Compare the Note to page 174.

because Vq, Vq' are *always* right quotients (202, 204), and $V(q' + q)$ is always their *sum* (205, I.); so that the *index of the right part of the sum of any two quaternions* is the *sum of the indices of the right parts*. In like manner, there is no difficulty in proving that

$$\text{III.} \dots I(q' - q) = Iq' - Iq, \text{ if } \angle q' = \angle q = \frac{\pi}{2};$$

and generally, that

$$\text{IV.} \dots IV(q' - q) = IVq' - IVq;$$

the *Index of the Difference* of any two right quotients, or of the right parts of any two quaternions, being thus equal to the *Difference of the Indices*.* We may then *reduce* the *addition* or *subtraction* of any two such quotients, or parts, to the *addition* or *subtraction* of their *indices*; a right quaternion being always (by 133) determined, when its index is given, or known.

207. We see, then, that as the *MULTIPLICATION* of any two Quaternions was (in 191) reduced to (Ist) the *arithmetical operation* of multiplying their tensors, and (IIInd) the *geometrical operation* of multiplying their versors, which latter was constructed by a certain *composition of rotations*, and was represented (in either of two distinct but connected ways, 167, 175) by sides or angles of a *spherical triangle*: so the *ADDITION* of any two Quaternions may be reduced (by 197, I., and 206, II.) to, Ist, the *algebraical addition* of their scalar parts, considered as two positive or negative numbers (16); and, IIInd, the *geometrical addition* of the indices of their right parts, considered as certain *vectors* (1): this latter *Addition of Lines* being performed according to the *Rule of the Parallelogram* (6).† In

* Compare again the Note to page 174.

† It does not fall within the plan of these Notes to allude often to the history of the subject; but it ought to be distinctly stated that this celebrated *Rule*, for what may be called *Geometrical Addition of right lines*, considered as *analogous* to *composition of motions* (or of *forces*), had occurred to several writers, *before* the invention of the quaternions: although the method adopted, in the present and in a former work, of deducing that rule, by algebraical analogies, from the *symbol* $B - A$ (1) for the line AB , may possibly not have been anticipated. The reader may compare the Notes to the Preface to the author's Volume of Lectures on Quaternions (Dublin, 1853).

like manner, as the general *Division of Quaternions* was seen (in 191) to admit of being reduced to an *arithmetical division of tensors*, and a *geometrical division of versors*, so we may now (by 197, III., and 206, IV.) reduce, generally, the *Subtraction of Quaternions* to (Ist) an *algebraical subtraction of scalars*, and (IInd) a *geometrical subtraction of vectors*: this last operation being again constructed by a parallelogram, or even by a *plane triangle* (comp. Art. 4, and Fig. 2). And because the *sum* of any given set of *vectors* was early seen to have a *value* (9), which is independent of their *order*, and of the mode of *grouping* them, we may now infer that the *Sum of any number of given Quaternions* has, in like manner, a *Value* (comp. 197, (1.)), which is *independent of the Order, and of the Grouping of the Summands*: or in other words, that the *general Addition of Quaternions is a Commutative* and an Associative Operation*.

(1.) The formula,

$$V\Sigma q = \Sigma Vq, \text{ of 205, III.,}$$

is now seen to hold good, for *any number* of quaternions, independently of the *arrangement* of the terms in each of the two sums, and of the manner in which they may be *associated*.

(2.) We can infer anew that

$$K(q' + q) = Kq' + Kq, \text{ as in 195, II.,}$$

under the form of the equation or identity,

$$S(q' + q) - V(q' + q) = (Sq' - Vq') + (Sq - Vq).$$

(3.) More generally, it may be proved, in the same way, that

$$K\Sigma q = \Sigma Kq, \text{ or briefly, } K\Sigma = \Sigma K,$$

whatever the number of the summands may be.

208. As regards the *quotient or product of the right parts*, Vq and Vq' , of any two quaternions, let t and t' denote the *tensors* of those two parts, and let x denote the *angle of their indices*, or of their *axes*, or the mutual *inclination* of the axes, or of the *planes*, † of the two quaternions q and q' themselves, so that (by 204, XVIII.),

* Compare the Note to page 175.

† Two planes, of course, make with each other, in general, two unequal and supplementary angles; but we here suppose that these are mutually distinguished, by taking account of the *aspect* of each plane, as distinguished from the *opposite aspect*: which is most easily done (111.), by considering the *axes* as above.

$$t = TVq = Tq \cdot \sin \angle q, \quad t' = TVq' = Tq' \cdot \sin \angle q',$$

and

$$x = \angle (IVq' : IVq) = \angle (Ax \cdot q' : Ax \cdot q).$$

Then, by 193, 194, and by 204, XXXV., XXXV',

$$I. \dots Vq' : Vq = IVq' : IVq = + (TVq' : TVq) \cdot (Ax \cdot q' : Ax \cdot q);$$

$$II. \dots Vq' \cdot Vq = IVq' : I \frac{1}{Vq} = - (TVq' \cdot TVq) \cdot (Ax \cdot q' : Ax \cdot q);$$

and therefore (comp. 198), with the temporary abridgments proposed above,

$$III. \dots S(Vq' : Vq) = t't^{-1} \cos x; \quad IV. \dots SU(Vq' : Vq) = + \cos x;$$

$$V. \dots S(Vq' \cdot Vq) = -t't \cos x; \quad VI. \dots SU(Vq' \cdot Vq) = - \cos x;$$

$$VII. \dots \angle(Vq' : Vq) = x; \quad VIII. \dots \angle(Vq' \cdot Vq) = \pi - x.$$

We have also generally (comp. 204, XVIII., XIX.),

$$IX. \dots TV(Vq' : Vq) = t't^{-1} \sin x; \quad X. \dots TVU(Vq' : Vq) = \sin x;$$

$$XI. \dots TV(Vq' \cdot Vq) = t't \sin x; \quad XII. \dots TVU(Vq' \cdot Vq) = \sin x;$$

and in particular,

$$XIII. \dots V(Vq' : Vq) = 0, \quad \text{and} \quad XIV. \dots V(Vq' \cdot Vq) = 0, \\ \text{if } q' \parallel q \text{ (123);}$$

because (comp. 191, (6.), and 204, VI.) the quotient or product of the right parts of two *complanar* quaternions (supposed here to be both *non-scalar* (108), so that t and t' are each > 0) degenerates (131) into a *scalar*, which may be thus expressed :

$$XV. \dots Vq' : Vq = +t't^{-1}, \quad \text{and} \quad XVI. \dots Vq' \cdot Vq = -t't, \quad \text{if } x = 0;$$

but

$$XVII. \dots Vq' : Vq = -t't^{-1}, \quad \text{and} \quad XVIII. \dots Vq' \cdot Vq = +t't, \quad \text{if } x = \pi;$$

the first case being that of *coincident*, and the second case that of *opposite axes*. In the more general case of *dipplanarity* (119), if we denote by δ the unit-line which is *perpendicular to both their axes*, and therefore *common to their two planes*, or in which those planes *intersect*, and which is so directed that the *rotation* round it from $Ax \cdot q$ to $Ax \cdot q'$ is *positive* (comp. 127, 128), the recent formulæ I., II. give easily,

$$XIX. \dots Ax \cdot (Vq' : Vq) = +\delta; \quad XX. \dots Ax \cdot (Vq' \cdot Vq) = -\delta;$$

and therefore (by IX., XI., and by 204, XXXV.), the *indices of the right parts*, of the quotient and product of the right parts of any two dipplanar quaternions, may be expressed as follows:

$$XXI. \dots IV(Vq' : Vq) = +\delta \cdot t't^{-1} \sin x;$$

$$XXII. \dots IV(Vq' \cdot Vq) = -\delta \cdot t't \sin x.$$

(1.) Let ABC be any triangle upon the unit-sphere (128), of which the spherical angles and the corners may be denoted by the same letters A, B, C , while the sides shall as usual be denoted by a, b, c ; and let it be supposed that the rotation (comp. 177) round A from C to B , and therefore that round B from A to C , &c., is positive, as in Fig. 48. Then writing, as we have often done,

$$q = \beta : a, \quad \text{and} \quad q' = \gamma : \beta, \quad \text{where} \quad a = OA, \quad \&c.,$$

we easily obtain the following expressions for the three scalars t, t', x , and for the vector δ :

$$t = \sin c; \quad t' = \sin a; \quad x = \pi - B; \quad \delta = -\beta.$$

(2.) In fact we have here,

$$Tq = Tq' = 1, \quad \angle q = c, \quad \angle q' = a;$$

whence t and t' are as just stated. Also if A', B', C' be (as in 175) the positive poles of the three successive sides BC, CA, AB , of the given triangle, and therefore the points A, B, C the negative poles (comp. 180, (2.)) of the new arcs $B'C', C'A', A'B'$, then

$$Ax. q = OC', \quad Ax. q' = OA';$$

but x and δ are the angle and the axis of the quotient of these two axes, or of the quaternion which is represented (162) by the arc $C'A'$; therefore x is, as above stated, the supplement of the angle B , and δ is directed to the point upon the sphere, which is diametrically opposite to the point B .

(3.) Hence, by III. V. VII. VIII. IX. XI., for any triangle ABC on the unit-sphere, with $a = OA$, &c., we have the formulæ:

$$\text{XXIII.} \dots S \left(\sqrt{\frac{\gamma}{\beta}} : \sqrt{\frac{\beta}{a}} \right) = -\sin a \operatorname{cosec} c \cos B;$$

$$\text{XXIV.} \dots S \left(\sqrt{\frac{\gamma}{\beta}} \cdot \sqrt{\frac{\beta}{a}} \right) = +\sin a \sin c \cos B;$$

$$\text{XXV.} \dots \angle \left(\sqrt{\frac{\gamma}{\beta}} : \sqrt{\frac{\beta}{a}} \right) = \pi - B; \quad \text{XXVI.} \dots \angle \left(\sqrt{\frac{\gamma}{\beta}} \cdot \sqrt{\frac{\beta}{a}} \right) = B;$$

$$\text{XXVII.} \dots TV \left(\sqrt{\frac{\gamma}{\beta}} : \sqrt{\frac{\beta}{a}} \right) = +\sin a \operatorname{cosec} c \sin B;$$

$$\text{XXVIII.} \dots TV \left(\sqrt{\frac{\gamma}{\beta}} \cdot \sqrt{\frac{\beta}{a}} \right) = +\sin a \sin c \sin B.$$

(4.) Also, by XIX. XX. XXI. XXII., if the rotation round B from A to C be still positive,

$$\text{XXIX.} \dots Ax. \left(\sqrt{\frac{\gamma}{\beta}} : \sqrt{\frac{\beta}{a}} \right) = -\beta; \quad \text{XXX.} \dots Ax. \left(\sqrt{\frac{\gamma}{\beta}} \cdot \sqrt{\frac{\beta}{a}} \right) = +\beta;$$

$$\text{XXXI.} \dots IV \left(\sqrt{\frac{\gamma}{\beta}} : \sqrt{\frac{\beta}{a}} \right) = -\beta \sin a \operatorname{cosec} c \sin B;$$

$$\text{XXXII.} \dots IV \left(\sqrt{\frac{\gamma}{\beta}} \cdot \sqrt{\frac{\beta}{a}} \right) = +\beta \sin a \sin c \sin B.$$

(5.) If, on the other hand, the rotation round B from A to C were negative, then writing for a moment $\alpha_1 = -\alpha, \beta_1 = -\beta, \gamma_1 = -\gamma$, we should have a new and opposite triangle, $A_1B_1C_1$, in which the rotation round B_1 from A_1 to C_1 would be positive, but the angle at B_1 equal in magnitude to that at B ; so that by treating (as usual) all the angles of a spherical triangle as positive, we should have $B_1 = B$, as well as $c_1 = c$, and $a_1 = a$; and therefore, for example, by XXXI.

$$I \vee \left(\vee \frac{\gamma_1}{\beta_1} : \vee \frac{\beta_1}{\alpha_1} \right) = -\beta_1 \sin \alpha_1 \operatorname{cosec} c_1 \sin B_1,$$

$$\text{or } IV \left(\vee \frac{\gamma}{\beta} : \vee \frac{\beta}{\alpha} \right) = +\beta \sin \alpha \operatorname{cosec} c \sin B ;$$

the four formulæ of (4.) would therefore still subsist, provided that, for this new direction of rotation in the given triangle, we were to *change the sign of β , in the second member of each.*

(6.) Abridging, generally $IVq : Sq$ to $(IV : S)q$, as $TVq : Sq$ was abridged, in 204, XXXIV., to $(TV : S)q$, we have by (5.), and by XXIV., XXXII., this other general formula, for any three unit-vectors α, β, γ , considered still as terminating at the corners of a spherical triangle ABC :

$$\text{XXXIII.} \dots (IV : S) \left(\vee \frac{\gamma}{\beta} \cdot \vee \frac{\beta}{\alpha} \right) = \pm \beta \tan B ;$$

the upper or the lower sign being taken, according as the rotation round B from A to C , or that round β from α to γ , which might perhaps be denoted by the symbol $\hat{\alpha}\beta\gamma$, and which in quantity is equal to the spherical angle B , is positive or negative.

209. When the *planes* of any three quaternions q, q', q'' , considered as all passing through the origin o (119), contain any *common line*, those three may then be said to be *Collinear* Quaternions* ; and because the *axis* of each is then perpendicular to that line, it follows that *the Axes of Collinear Quaternions are Complanar* : while conversely, the *complanarity of the axes* insures the *collinearity of the quaternions*, because the *perpendicular to the plane* of the axes is a line common to the planes of the quaternions.

(1.) Complanar quaternions are always collinear ; but the converse proposition does not hold good, collinear quaternions being not necessarily complanar.

(2.) Collinear quaternions, considered as *fractions* (101), can always be reduced to a *common denominator* (120) ; and conversely, if three or more quaternions can be so reduced, as to appear under the form of fractions with a common denominator ϵ , those quaternions must be *collinear* : because the line ϵ is then common to all their planes.

(3.) *Any two quaternions are collinear with any scalar* ; the *plane of a scalar* being *indeterminate*† (131).

(4.) Hence the *scalar and right parts*, $Sq, Sq', \vee q, \vee q'$, of any two quaternions, are always *collinear* with each other.

(5.) The *conjugates* of collinear quaternions are themselves collinear.

* Quaternions of which the planes are *parallel* to any common line may also be said to be *collinear*. Compare the first Note to page 113.

† Compare the Note to page 114.

210. Let q, q', q'' be any three collinear quaternions; and let α denote a line common to their planes. Then we may determine (comp. 120) three other lines β, γ, δ , such that

$$q = \frac{\beta}{\alpha}, \quad q' = \frac{\gamma}{\alpha}, \quad q'' = \frac{\alpha}{\delta};$$

and thus may conclude that (as in algebra),

$$\text{I.} \dots (q' + q) q'' = q' q'' + q q'',$$

because, by 106, 107,

$$\left(\frac{\gamma}{\alpha} + \frac{\beta}{\alpha}\right) \frac{\alpha}{\delta} = \frac{\gamma + \beta}{\alpha} \cdot \frac{\alpha}{\delta} = \frac{\gamma + \beta}{\delta} = \frac{\gamma}{\delta} + \frac{\beta}{\delta} = \frac{\gamma \alpha}{\alpha \delta} + \frac{\beta \alpha}{\alpha \delta}.$$

In like manner, at least under the same condition of collinearity,* it may be proved that

$$\text{II.} \dots (q' - q) q'' = q' q'' - q q''.$$

Operating by the characteristic \mathbf{K} upon these two equations, and attending to 192, II., and 195, II., we find that

$$\text{III.} \dots \mathbf{K} q'' \cdot (\mathbf{K} q' + \mathbf{K} q) = \mathbf{K} q'' \cdot \mathbf{K} q' + \mathbf{K} q'' \cdot \mathbf{K} q;$$

$$\text{IV.} \dots \mathbf{K} q'' \cdot (\mathbf{K} q' - \mathbf{K} q) = \mathbf{K} q'' \cdot \mathbf{K} q' - \mathbf{K} q'' \cdot \mathbf{K} q;$$

where (by 209, (5.)) the three *conjugates* of arbitrary collinears, $\mathbf{K} q, \mathbf{K} q', \mathbf{K} q''$, may represent *any three* collinear quaternions. We have, therefore, with the same degree of generality as before,

$$\text{V.} \dots q'' (q' + q) = q'' q' + q'' q; \quad \text{VI.} \dots q'' (q' - q) = q'' q' - q'' q.$$

If, then, q, q', q'', q''' be *any four collinear quaternions*, we may establish the formula (again agreeing with algebra):

$$\text{VII.} \dots (q''' + q'') (q' + q) = q''' q' + q'' q' + q''' q + q'' q;$$

and similarly for any greater number, so that we may write briefly,

$$\text{VIII.} \dots \Sigma q' \cdot \Sigma q = \Sigma q' q,$$

where

$$\Sigma q' = q_1 + q_2 + \dots + q_m, \quad \Sigma q = q'_1 + q'_2 + \dots + q'_n,$$

and

$$\Sigma q' q = q'_1 q_1 + \dots + q'_1 q_m + q'_2 q_1 + \dots + q'_n q_m,$$

m and n being any positive whole numbers. In words (comp. 13), *the Multiplication of Collinear† Quaternions is a Doubly Distributive Operation.*

* It will soon be seen, however, that this condition is unnecessary.

† This *distributive property of multiplication* will soon be found (compare the last Note) to extend to the more general case, in which the quaternions are *not collinear*.

(1.) Hence, by 209, (4.), and 202, III., we have this general transformation, for the product of any two quaternions :

$$\text{IX.} \dots q'q = Sq'.Sq + Vq'.Sq + Sq'.Vq + Vq'.Vq.$$

(2.) Hence also, for the square of any quaternion, we have the transformation (comp. 126; 199, VII.; and 204, XXIII.):

$$\text{X.} \dots q^2 = Sq^2 + 2Sq.Vq + Vq^2.$$

(3.) Separating the scalar and right parts of this last expression, we find these other general formulæ :

$$\text{XI.} \dots S.q^2 = Sq^2 + Vq^2; \quad \text{XII.} \dots V.q^2 = 2Sq.Vq;$$

whence also, dividing by Tq^2 , we have

$$\text{XIII.} \dots SU(q^2) = (SUq)^2 + (VUq)^2; \quad \text{XIV.} \dots VU(q^2) = 2SUq.VUq.$$

(4.) By supposing $q' = Kq$, in IX., and therefore $Sq' = Sq$, $Vq' = -Vq$, and transposing the two conjugate and therefore complanar factors (comp. 191, (1.)), we obtain this general transformation for a norm, or for the square of a tensor (comp. 190, V.; 202, III.; and 204, XI.):

$$\text{XV.} \dots Tq^2 = Nq = qKq = (Sq + Vq)(Sq - Vq) = Sq^2 - Vq^2;$$

which had indeed presented itself before (in 204, XXII.) but is now obtained in a new way, and without any employment of sines, or cosines, or even of the well-known theorem respecting the square of the hypotenuse.

(5.) Eliminating Vq^2 , by XV., from XI., and dividing by Tq^2 , we find that

$$\text{XVI.} \dots S.q^2 = 2Sq^2 - Tq^2; \quad \text{XVII.} \dots SU(q^2) = 2(SUq)^2 - 1;$$

agreeing with 199, VI. and IV., but obtained here without any use of the known formula for the cosine of the double of an angle.

(6.) Taking the scalar and right parts of the expression IX., we obtain these other general expressions :

$$\text{XVIII.} \dots Sq'q = Sq'.Sq + S(Vq'.Vq);$$

$$\text{XIX.} \dots Vq'q = Vq'.Sq + Vq.Sq' + V(Vq'.Vq);$$

in the latter of which we may (by 126) transpose the two factors, Vq' , Sq , or Vq , Sq' . We may also (by 206, 207) write, instead of XIX., this other formula :

$$\text{XIX'.} \dots IVq'q = IVq'.Sq + IVq.Sq' + IV(Vq'.Vq).$$

(7.) If we suppose, in VII., that $q'' = Kq$, $q''' = Kq'$, and transpose (comp. (4.)) the two complanar (because conjugate) factors, $q' + q$ and $K(q' + q)$, we obtain the following general expression for the norm of a sum :

$$(q' + q) K(q' + q) = q'Kq' + qKq' + q'Kq + qKq;$$

or briefly,

$$\text{XX.} \dots N(q' + q) = Nq' + 2S.qKq' + Nq, \text{ as in 200, VII.};$$

because

$$q'Kq = K.qKq', \text{ by 192, II., and } (1 + K).qKq' = 2S.qKq', \text{ by 196, II'.$$

(8.) By changing q' to x in XX., or by forming the product of $q + x$ and $Kq + x$, where x is any scalar, we find that

$$\text{XXI.} \dots N(q + x) = Nq + 2xSq + x^2, \text{ as in 200, VIII.};$$

whence, in particular,

$$\text{XXI'.} \dots N(q - 1) = Nq - 2Sq + 1, \text{ as in 200, II.}$$

(9.) Changing q to β : a , and multiplying by the square of Ta , we get, for any two vectors, a and β , the formula,

$$\text{XXII.} \dots T(\beta - a)^2 = T\beta^2 - 2T\beta \cdot Ta \cdot \text{SU} \frac{\beta}{a} + Ta^2,$$

in which Ta^2 denotes* $(Ta)^2$; because (by 190, and by 196, IX.),

$$N\left(\frac{\beta}{a} - 1\right) = N\frac{\beta - a}{a} = \left(\frac{T(\beta - a)}{Ta}\right)^2, \text{ and } S\frac{\beta}{a} = \frac{T\beta}{Ta} \text{SU} \frac{\beta}{a}.$$

(10.) In any plane triangle, ABC , with sides of which the *lengths* are as usual denoted by a, b, c , let the vertex C be taken as the origin o of vectors; then

$$a = CA, \quad \beta = CB, \quad \beta - a = AB, \quad Ta = b, \quad T\beta = a, \quad T(\beta - a) = c, \quad \text{SU} \frac{\beta}{a} = \cos c;$$

we recover therefore, from XXII., the *fundamental formula of plane trigonometry*, under the form,

$$\text{XXIII.} \dots c^2 = a^2 - 2ab \cos c + b^2.$$

(11.) It is important to observe that we have not here been arguing in a circle; because although, in Art. 200, we assumed, for the convenience of the student, a previous knowledge of the last written formula, in order to arrive more rapidly at certain applications, yet in these recent deductions from the *distributive property VIII. of multiplication* of (at least) collinear quaternions, we have founded nothing on the results of that former Article; and have made no use of any properties of *oblique-angled triangles*, or even of *right-angled* ones, since the theorem of the square of the hypotenuse has been virtually proved anew in (4.): nor is it necessary to the argument, that any *properties* of trigonometric *functions* should be known, beyond the mere *definition* of a *cosine*, as a certain *projecting factor*, from which the formula 196, XVI. was derived, and which justifies us in writing $\cos c$ in the last equation (10.). The geometrical Examples, in the sub-articles to 200, may therefore be read again, and their validity be seen anew, *without any appeal to even plane trigonometry* being now supposed.

(12.) The formula XV. gives $Sq^2 = Tq^2 + Vq^2$, as in 204, XXXI.; and we know that Vq^2 , as being generally the *square of a right quaternion*, is equal to a *negative scalar* (comp. 204, VI.), so that

$$\text{XXIV.} \dots Vq^2 < 0, \text{ unless } \angle q = 0, \text{ or } = \pi,$$

in each of which two cases $Vq = 0$, by 202, (6.), and therefore its square vanishes; hence,

$$\text{XXV.} \dots Sq^2 < Tq^2, \quad (\text{SU}q)^2 < 1,$$

in every other case.

* We are not yet at liberty to interpret the symbol Ta^2 as denoting also $T(a^2)$; because we have not yet assigned any meaning to the *square of a vector*, or generally to the *product of two vectors*. In the Third Book of these Elements it will be shown, that such a square or product can be interpreted as being a *quaternion*: and then it will be found (comp. 190), that

$$T(a^2) = (Ta)^2 = Ta^2,$$

whatever vector a may be.

(13.) It might therefore have been thus proved, without any use of the transformation $SUq = \cos \angle q$ (196, XVI.), that (for any *real* quaternion q) we have the inequalities,

$$\text{XXVI.} \dots SUq < +1, \quad SUq > -1, \quad \text{and} \quad Sq < +Tq, \quad Sq > -Tq,$$

unless it happen that $\angle q = 0$, or π ; SUq being $= +1$, and $Sq = +Tq$, in the first case; whereas $SUq = -1$, and $Sq = -Tq$, in the second case.

(14.) Since $Tq^2 = Nq$, and $Tq \cdot Tq' = T \cdot qKq' = T \cdot q'Kq = Nq \cdot T(q' : q)$, while $S \cdot qKq' = S \cdot q'Kq = Nq \cdot S(q' : q)$, the formula XX. gives, by XXVI.,

$$\text{XXVII.} \dots (Tq' + Tq)^2 - T(q' + q)^2 = 2(T - S)qKq' = 2Nq \cdot (T - S)(q' : q) > 0,$$

if we adopt the abridged notation,

$$\text{XXVIII.} \dots Tq - Sq = (T - S)q,$$

and suppose that the quotient $q' : q$ is *not* a positive scalar; hence,

$$\text{XXIX.} \dots Tq' + Tq > T(q' + q), \quad \text{unless} \quad q' = xq, \quad \text{and} \quad x > 0;$$

in which excepted case, each member of this last inequality becomes $= (1 + x)Tq$.

(15.) Writing $q = \beta : a$, $q' = \gamma : a$, and multiplying by Ta , the formula XXIX. becomes,

$$\text{XXX.} \dots T\gamma + T\beta > T(\gamma + \beta), \quad \text{unless} \quad \gamma = x\beta, \quad x > 0;$$

in which latter case, but not in any other, we have $U\gamma = U\beta$ (156). We therefore arrive anew at the results of 186, (9.), (10.), but without its having been necessary to consider any *triangle*, as was done in those former sub-articles.

(16.) On the other hand, with a corresponding abridgment of notation, we have, by XXVI.,

$$\text{XXXI.} \dots Tq + Sq = (T + S)q > 0, \quad \text{unless} \quad \angle q = \pi;$$

also, by XX., &c.,

$$\text{XXXII.} \dots T(q' + q)^2 - (Tq' - Tq)^2 = 2(T + S)qKq' = 2Nq \cdot (T + S)(q' : q);$$

hence,

$$\text{XXXIII.} \dots T(q' + q) > \pm(Tq' - Tq), \quad \text{unless} \quad q' = -xq, \quad x > 0;$$

where either sign may be taken.

(17.) And hence, on the plan of (15.), for any two vectors β, γ ,

$$\text{XXXIV.} \dots T(\gamma + \beta) > \pm(T\gamma - T\beta), \quad \text{unless} \quad U\gamma = -U\beta,$$

whichever sign be adopted; but, on the contrary,

$$\text{XXXV.} \dots T(\gamma + \beta) = \pm(T\gamma - T\beta), \quad \text{if} \quad U\gamma = -U\beta,$$

the upper or the lower sign being taken, according as $T\gamma >$ or $< T\beta$: all which agrees with what was inferred, in 186, (11.), from *geometrical* considerations alone, combined with the definition of Ta . In fact, if we make $\beta = OB$, $\gamma = OC$, and $-\gamma = OC'$, then OBC' will be in general a *plane triangle*, in which the length of the side BC' exceeds the difference of the lengths of the two other sides; but if it happen that the directions of the two lines OB, OC' coincide, or in other words that the lines OB, OC have opposite directions, then the difference of lengths of these two lines becomes equal to the length of the line BC' .

(18.) With the representations of q and q' , assigned in 208, (1.), by two sides of a *spherical triangle* ABC , we have the values,

$$Sq = \cos c, \quad Sq' = \cos a, \quad Sq'q = S(\gamma : a) = \cos b;$$

the equation XVIII. gives therefore, by 208, XXIV., the *fundamental formula of spherical trigonometry* (comp. (10.)), as follows :

$$\text{XXXVI.} \dots \cos b = \cos a \cos c + \sin a \sin c \cos \mathfrak{B}.$$

(19.) To interpret, with reference to the same spherical triangle, the connected equation XIX., or XIX', let it be now supposed, as in 208, (5.), that the rotation round \mathfrak{B} from c to A is positive, so that \mathfrak{B} and \mathfrak{B}' are situated at the same side of the arc cA , if \mathfrak{B}' be still, as in 208, (2.), the positive pole of that arc. Then writing $a' = oA'$, &c., we have

$$\text{IV } q = \gamma' \sin c; \quad \text{IV } q' = a' \sin a; \quad \text{IV } q'q = -\beta' \sin b;$$

and $\text{IV } (Vq' \cdot Vq) = -\beta \sin a \sin c \sin \mathfrak{B}$ (comp. 208, (5.)),

with the recent values (18.), for Sq and Sq' ; thus the formula XIX'. becomes, by transposition of the two terms last written :

$$\text{XXXVII.} \dots \beta \sin a \sin c \sin \mathfrak{B} = a' \sin a \cos c + \beta' \sin b + \gamma' \sin c \cos a.$$

(20.) Let $\rho = oP$ be any unit-vector; then, dividing each term of the last equation by ρ , and taking the scalar of each of the four quotients, we have, by 196, XVI., this new equation :

$$\text{XXXVIII.} \dots \sin a \sin c \sin \mathfrak{B} \cos P\mathfrak{B} = \sin a \cos c \cos P\mathfrak{A}' + \sin b \cos P\mathfrak{B}' \\ + \sin c \cos a \cos P\mathfrak{C}';$$

where a, b, c are as usual the sides of the spherical triangle ΔBC , and A', B', C' are still, as in 208, (2.), the positive poles of those sides; but P is an arbitrary point, upon the surface of the sphere. Also $\cos P\mathfrak{A}'$, $\cos P\mathfrak{B}'$, $\cos P\mathfrak{C}'$, are evidently the sines of the arcual perpendiculars, let fall from that point upon those sides; being positive when P is, relatively to them, in the same hemispheres as the opposite corners of the triangle, but negative in the contrary case; so that $\cos \Delta A'$, &c., are positive, and are the sines of the three altitudes of the triangle.

(21.) If we place P at \mathfrak{B} , two of these perpendiculars vanish, and the last formula becomes, by 208, XXVIII.,

$$\text{XXXIX.} \dots \sin b \cos \mathfrak{B}\mathfrak{B}' = \sin a \sin c \sin \mathfrak{B} = \text{TV} \left(\sqrt{\frac{\gamma}{\beta}} \cdot \sqrt{\frac{\beta}{\alpha}} \right);$$

such then is the quaternion expression for the product of the sine of the side cA , multiplied by the sine of the perpendicular let fall upon that side, from the opposite vertex \mathfrak{B} .

(22.) Placing P at A , dividing by $\sin a \cos c$, and then interchanging \mathfrak{B} and c , we get this other fundamental formula of spherical trigonometry,

$$\text{XL.} \dots \cos \Delta A' = \sin c \sin \mathfrak{B} = \sin b \sin c;$$

and we see that this is included in the interpretation of the quaternion equation XIX., or XIX', as the formula XXXVI. was seen in (18.) to be the interpretation of the connected equation XVIII.

(23.) By assigning other positions to P , other formulæ of spherical trigonometry may be deduced, from the recent equation XXXVIII. Thus if we suppose P to coincide with \mathfrak{B}' , and observe that (by the supplementary* triangle),

* No previous knowledge of spherical trigonometry, properly so called, is here supposed; the supplementary relations of two polar triangles to each other forming rather a part, and a very elementary one, of spherical geometry.

while $B'O' = \pi - A, \quad C'A' = \pi - B, \quad A'B' = \pi - C,$
 $\cos BB' = \sin a \sin c = \sin c \sin A,$ by XL,

we easily deduce the formula,

XLl. . . $\sin a \sin c \sin A \sin B \sin C = \sin B - \cos c \cos C \sin A - \cos a \cos A \sin C;$

which obviously agrees, at the plane limit, with the elementary relation,

$$A + B + C = \pi.$$

(24.) Again, by placing P at A', the general equation becomes,

XLlIl. . . $\sin a \cos c = \sin b \cos C + \sin c \cos a \cos B;$

with the verification that, at the plane limit,

$$a = b \cos C + c \cos B.$$

But we cannot here delay on such deductions, or verifications : although it appeared to be worth while to point out, that the whole of spherical trigonometry may thus be developed, from the fundamental equation of multiplication of quaternions (107), when that equation is operated on by the two characteristics S and V, and the results interpreted as above.

211. It may next be proved, as follows, that the distributive formula I. of the last Article holds good, when the three quaternions, q, q', q'' , which enter into it, without being now necessarily *collinear*, are *right*; in which case their *reciprocals* (135), and their *sums* (197, (2.)), will be right also. Let then

$$\angle q = \angle q' = \angle q'' = \frac{\pi}{2}, \quad qq' = 1;$$

and therefore,

$$\angle q, = \angle (q'' + q') = \frac{\pi}{2}.$$

We shall then have, by 106, 194, 206,

$$(q'' + q') q = I (q'' + q') : I q,$$

$$= (Iq'' : Iq,) + (Iq' : Iq,) = q''q + q'q;$$

and the distributive property in question is proved.

(1.) By taking conjugates, as in 210, it is easy hence to infer, that the *other* distributive formula, 210, V., holds good for any three right quaternions; or that

$$q(q'' + q') = qq'' + qq', \quad \text{if } \angle q = \angle q' = \angle q'' = \frac{\pi}{2}.$$

(2.) For any three quaternions, we have therefore the two equations :

$$(Vq'' + Vq') \cdot Vq = Vq'' \cdot Vq + Vq' \cdot Vq;$$

$$Vq \cdot (Vq'' + Vq') = Vq \cdot Vq'' + Vq \cdot Vq'.$$

(3.) The quaternions q, q', q'' being still arbitrary, we have thus, by 210, IX.,

$$\begin{aligned} (q'' + q)q &= (Sq'' + Sq') \cdot Sq + (Vq'' + Vq') \cdot Sq + Vq \cdot (Sq'' + Sq') + (Vq'' + Vq') \cdot Vq \\ &= (Sq'' \cdot Sq + Vq'' \cdot Sq + Vq \cdot Sq'' + Vq' \cdot Vq) + (Sq' \cdot Sq + Vq' \cdot Sq + Vq \cdot Sq' + Vq' \cdot Vq) \\ &= q''q + q'q; \end{aligned}$$

so that the formula 210, I., and therefore also (by conjugates) the formula 210, V., is valid *generally*.

212. The *General* Multiplication of Quaternions* is therefore (comp. 13, 210) a *Doubly Distributive Operation*; so that we may *extend*, to quaternions *generally*, the formula (comp. 210, VIII.),

$$I. \dots \Sigma q' \cdot \Sigma q = \Sigma q'q :$$

however many the summands of each set may be, and whether they be, or be not, *collinear* (209), or *right* (211).

(1.) Hence, as an extension of 210, XX., we have now,

$$II. \dots N\Sigma q = \Sigma Nq + 2\Sigma SqKq' ;$$

where the second sign of summation refers to all possible binary combinations of the quaternions q, q', \dots

(2.) And, as an extension of 210, XXIX., we have the inequality,

$$III. \dots \Sigma Tq > T\Sigma q,$$

unless *all* the quaternions q, q', \dots bear *scalar* and *positive* ratios to each other, in which case the two members of this inequality become equal: so that the *sum of the tensors*, of any set of quaternions, is *greater than the tensor of the sum*, in every other case.

(3.) In general, as an extension of 210, XXVII.,

$$IV. \dots (\Sigma Tq)^2 - (T\Sigma q)^2 = 2\Sigma (T - S)qKq'.$$

(4.) The formulae, 210, XVIII., XIX., admit easily of analogous extensions.

(5.) We have also (comp. 168) the general equation,

$$V. \dots (\Sigma q)^2 - \Sigma (q^2) = \Sigma (qq' + q'q) ;$$

in which, by 210, IX.,

$$VI. \dots qq' + q'q = 2(Sq \cdot Sq' + Vq \cdot Sq' + Vq' \cdot Sq + S(Vq' \cdot Vq)) ;$$

because, by 208, we have generally

$$VII. \dots V(Vq' \cdot Vq) = -V(Vq \cdot Vq') ;$$

or

$$VIII. \dots Vq'q = -Vqq', \quad \text{if } \angle q = \angle q' = \frac{\pi}{2}.$$

(Comp. 191, (2.), and 204, X.)

213. Besides the advantage which the Calculus of Quaternions gains, from the general establishment (212) of the *Distributive Principle*, or *Distributive Property of Multiplication*, by being, so far,

* Compare the Notes to page 208.

assimilated to Algebra, in processes which are of continual occurrence, this principle or property will be found to be of great importance, in applications of that calculus to Geometry; and especially in questions respecting the (real or ideal*) intersections of right lines with spheres, or other surfaces of the second order, including contacts (real or ideal), as limits of such intersections. The following Examples may serve to give some notion, how the general distributive principle admits of being applied to such questions: in some of which however the less general principle (210), respecting the multiplication of collinear quaternions (209), would be sufficient. And first we shall take the case of chords of a sphere, drawn from a given point upon its surface.

(1.) From a point A, of a sphere with O for centre, let it be required to draw a chord AP, which shall be parallel to a given line OB; or more fully, to assign the vector, $\rho = OP$, of the extremity of the chord so drawn, as a function of the two given vectors, $\alpha = OA$, and $\beta = OB$; or rather of α and $U\beta$, since it is evident that the length of the line β cannot affect the result of the construction, which Fig. 51 may serve to illustrate.

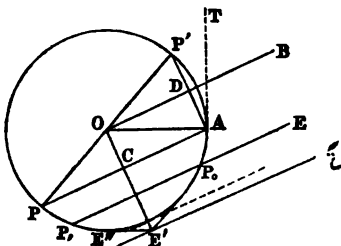


Fig. 51.

(2.) Since $AP \parallel OB$, or $\rho - \alpha \parallel \beta$, we may begin by writing the expression,

$$\rho = \alpha + x\beta \quad (15),$$

which may be considered (comp. 28, 99) as a form of the equation of the right line AP; and in which it remains to determine the scalar coefficient x , so as to satisfy the equation of the sphere,

$$T\rho = T\alpha \quad (186, (2.)).$$

In short, we are to seek to satisfy the equation,

$$T(\alpha + x\beta) = T\alpha,$$

$$\therefore N\left(\frac{\alpha}{\beta} + x\right) = N\alpha$$

by some scalar x which shall be (in general) different from zero; and then to substitute this scalar in the expression $\rho = \alpha + x\beta$, in order to determine the required vector ρ .

$$N\alpha + 2x\frac{\alpha}{\beta} + x^2 = N\alpha$$

(3.) For this purpose, an obvious process is, after dividing both sides by $T\beta$, to square, and to employ the formula 210, XXI., which had indeed occurred before, as 200, VIII., but not then as a consequence of the distributive property of multiplication. In this manner we are conducted to a quadratic equation, which admits of division by x , and gives then,

$$x = -2S\frac{\alpha}{\beta}; \quad \rho = \alpha - 2\beta S\frac{\alpha}{\beta};$$

* Compare the Notes to page 90, &c.

the problem (1.) being thus resolved, with the verification that β may be replaced by $U\beta$, in the resulting expression for ρ .

(4.) As a mere exercise of calculation, we may vary the last process (3.), by dividing the last equation (2.) by $T\alpha$, instead of $T\beta$, and then going on as before. This last procedure gives,

$$1 = N \left(1 + x \frac{\beta}{\alpha} \right) = 1 + 2xS \frac{\beta}{\alpha} + x^2 N \frac{\beta}{\alpha};$$

and therefore,

$$x = -2S \frac{\beta}{\alpha} : N \frac{\beta}{\alpha} = -2S \frac{\alpha}{\beta} \text{ (by 196, XII'), as before.}$$

(5.) In general, by 196, II',

$$1 - 2S = -K;$$

hence, by (3.),

$$\frac{\rho}{\beta} = -K \frac{\alpha}{\beta};$$

and finally,

$$\rho = -K \frac{\alpha}{\beta} \cdot \beta;$$

a new expression for ρ , in which it is not permitted generally, as it was in (3.), to treat the vector β as the multiplier,* instead of the multiplicand.

(6.) It is now easy to see that the second equation of (2.) is satisfied; for the expression (5.) for ρ gives (by 186, 187, &c.),

$$T\rho = T \frac{\alpha}{\beta} \cdot T\beta = T\alpha,$$

as was required.

(7.) To interpret the solution (3.), let O in Fig. 51 be the middle point of the chord AP , and let D be the foot of the perpendicular let fall from A on OB ; then the expression (3.) for ρ gives, by 196, XIX.,

$$OA = \frac{1}{2}(\alpha - \rho) = \beta S \frac{\alpha}{\beta} = OD;$$

and accordingly, $OCAD$ is a parallelogram.

(8.) To interpret the expression (5.), which gives

$$\frac{-\rho}{\beta} = K \frac{\alpha}{\beta}, \text{ or } \frac{OP'}{OB} = K \frac{OA}{OB}, \text{ if } OP' = PO,$$

we have only to observe (comp. 188) that the angle $\angle AOP'$ is bisected internally, or the supplementary angle $\angle AOP$ externally, by the indefinite right line OB (see again Fig. 51).

(9.) Conversely, the *geometrical considerations* which have thus served in (7.) and (8.) to *interpret* or to *verify* the two forms of solution (3.), (5.), might have been employed to *deduce* those two forms, if we had not seen how to obtain them, by *rules of calculation*, from the proposed conditions of the question. (Comp. 145, (10.), &c.)

(10.) It is evident, from the nature of that question, that α ought to be deduci-

* Compare the Note to page 159.

ble from β and ρ , by exactly the same processes as those which have served us to deduce ρ from β and α . Accordingly, the form (3.) of ρ gives,

$$S \frac{\rho}{\beta} = -S \frac{\alpha}{\beta} \quad \alpha = \rho + 2\beta S \frac{\alpha}{\beta} = \rho - 2\beta S \frac{\rho}{\beta};$$

and the form (5.) gives,

$$K \frac{\rho}{\beta} = -\frac{\alpha}{\beta} \quad \alpha = -K \frac{\rho}{\beta} \cdot \beta.$$

And since the first form can be recovered from the second, we see that each leads us back to the parallelism, $\rho - \alpha \parallel \beta$ (2.).

(11.) The solution (3.) for x shows that

$$x = 0, \quad \rho = \alpha, \quad r = \alpha, \quad \text{if } S(\alpha : \beta) = 0, \quad \text{or if } \beta \perp \alpha.$$

And the geometrical meaning of this result is obvious; namely, that a right line drawn at the extremity of a radius OA of a sphere, so as to be perpendicular to that radius, does not (in strictness) *intersect* the sphere, but *touches* it: its *second* point of meeting the surface *coinciding*, in this case, as a *limit*, with the *first*.

(12.) Hence we may infer that the plane represented by the equation,

$$S \frac{\rho - \alpha}{\alpha} = 0, \quad \text{or } S \frac{\rho}{\alpha} = 1,$$

is the *tangent plane* (comp. 196, (5.)) to the sphere here considered, at the point A .

(13.) Since β may be replaced by any vector parallel thereto, we may substitute for it $\gamma - \alpha$, if $\gamma = OC$ be the vector of *any given point* C upon the chord AP , whether (as in Fig. 51) the middle point, or not; we may therefore write, by (3.) and (5.),

$$\rho = \alpha - 2(\gamma - \alpha) S \frac{\alpha}{\gamma - \alpha} = -K \frac{\alpha}{\gamma - \alpha} \cdot (\gamma - \alpha).$$

214. In the Examples of the foregoing Article, there was no room for the occurrence of *imaginary roots* of an equation, or for *ideal intersections* of line and surface. To give now a case in which such imaginary intersections may occur, we shall proceed to consider the question of drawing a *secant* to a sphere, in a given direction, from a given *external point*; the recent Figure 51 still serving us for illustration.

(1.) Suppose then that ϵ is the vector of any given point ϵ , through which it is required to draw a chord or secant $\epsilon P_0 \epsilon_1$, parallel to the same given line β as before. We have now, if $\rho_0 = OP_0$,

$$\rho_0 = \epsilon + x_0 \beta, \quad T \alpha = T \rho_0 = T(\epsilon + x_0 \beta),$$

$$x_0^2 + 2x_0 S \frac{\epsilon}{\beta} + N \frac{\epsilon}{\beta} - N \frac{\alpha}{\beta} = 0,$$

$$x_0 = -S \frac{\epsilon}{\beta} \mp \sqrt{\left\{ \left(T \frac{\alpha}{\beta} \right)^2 + \left(V \frac{\epsilon}{\beta} \right)^2 \right\}},$$

x_0 being a new scalar; and similarly, if $\rho_1 = OP_1$,

$$\rho_1 = \epsilon + x_1 \beta, \quad x_1 = -S \frac{\epsilon}{\beta} \pm \sqrt{\left\{ \left(T \frac{\alpha}{\beta} \right)^2 + \left(V \frac{\epsilon}{\beta} \right)^2 \right\}},$$

by transformations* which will easily occur to any one who has read recent articles with attention. And the points F_0, F_1 will be together *real*, or together *imaginary*, according as the quantity under the radical sign is positive or negative; that is, according as we have one or other of the two following inequalities,

$$T\frac{\alpha}{\beta} > \text{ or } < TV\frac{\epsilon}{\beta}.$$

(2.) The equation (comp. 208, (5.)),

$$TV\frac{\rho}{\beta} = T\frac{\alpha}{\beta}, \text{ or } \left(T\frac{\alpha}{\beta}\right)^2 + \left(V\frac{\rho}{\beta}\right)^2 = 0,$$

represents a cylinder of revolution, with OB for its axis, and with $T\alpha$ for the radius of its base. If κ be a point of this cylindrical surface, the quantity under the radical sign in (1.) vanishes; and the two roots x_0, x_1 of the quadratic become *equal*. In this case, then, the line through κ , which is parallel to OB , *touches* the given sphere; as is otherwise evident geometrically, since the cylinder *envelopes* the sphere (comp. 204, (12.)), and the line is one of its generatrices. If κ be *internal* to the cylinder, the intersections F_0, F_1 are *real*; but if κ be *external* to the same surface, those intersections are *ideal*, or *imaginary*.

(3.) In this last case, if we make, for abridgment,

$$s = -S\frac{\epsilon}{\beta}, \text{ and } t = \sqrt{\left\{\left(TV\frac{\epsilon}{\beta}\right)^2 - \left(T\frac{\alpha}{\beta}\right)^2\right\}},$$

s and t being thus two given and *real scalars*, we may write,

$$x_0 = s - t\sqrt{-1}; \quad x_1 = s + t\sqrt{-1};$$

where $\sqrt{-1}$ is the *old and ordinary imaginary symbol of Algebra*, and is *not invested here* with any sort of *Geometrical Interpretation*.† We merely express thus the *fact of calculation*, that (with these meanings of the symbols $\alpha, \beta, \epsilon, s$ and t) the formula $T\alpha = T(\epsilon + x\beta)$, (1.), *when treated by the rules of quaternions, conducts to the quadratic equation,*

$$(x - s)^2 + t^2 = 0,$$

which has *no real root*; the reason being that *the right line* through κ is, in the present case, *wholly external to the sphere*, and therefore *does not really intersect it at all*; although, for the sake of *generalization of language*, we may agree to say, as usual, that the line intersects the sphere in *two imaginary points*.

(4.) We must however agree, then, for *consistency of symbolical expression*, to consider these two ideal points as having *determinate but imaginary vectors*, namely, the two following:

$$\rho_0 = s + s\beta - t\beta\sqrt{-1}; \quad \rho_1 = s + s\beta + t\beta\sqrt{-1};$$

in which it is easy to prove, 1st, that the *real part* $s + s\beta$ is the *vector s' of the foot κ' of the perpendicular* let fall from the centre O on the line through κ which is drawn (as above) parallel to OB ; and 2nd, that the *real tensor* $t\beta$ of the *coefficient* of

* It does not seem to be necessary, at the present stage, to supply so many references to former Articles, or Sub-articles, as it has hitherto been thought useful to give; but such may still, from time to time, be given.

† Compare again the Notes to page 90, and Art. 149.

$\sqrt{-1}$ in the *imaginary part* of each expression, represents the *length of a tangent* $\kappa\kappa'$ to the sphere, drawn from that external point, or foot, κ' .

(5.) In fact, if we write $o\kappa' = \epsilon' = \epsilon + s\beta$, we shall have

$$\kappa\kappa' = \epsilon - \epsilon' = -s\beta = \beta S \frac{\epsilon}{\beta} = \text{projection of } o\epsilon \text{ on } o\beta;$$

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which proves the 1st assertion (4.), whether the points F_0, F_1 be real or imaginary. And because

$$\left(T \frac{\epsilon'}{\beta}\right)^2 = N \frac{\epsilon'}{\beta} = N \left(\frac{\epsilon}{\beta} + s\right) = N \frac{\epsilon}{\beta} + 2sS \frac{\epsilon}{\beta} + s^2$$

$$= \left(T \frac{\epsilon}{\beta}\right)^2 - \left(S \frac{\epsilon}{\beta}\right)^2 = \left(TV \frac{\epsilon}{\beta}\right)^2 = \epsilon^2 + \left(T \frac{\alpha}{\beta}\right)^2, \quad \text{by (3.)}$$

we have, for the case of imaginary intersections,

$$\epsilon T\beta = \sqrt{(T\epsilon^2 - T\alpha^2)} = T.\kappa\kappa',$$

and the 2nd assertion (4.) is justified.

(6.) An expression of the form (4.), or of the following,

$$\rho' = \beta + \sqrt{-1}\gamma,$$

in which β and γ are two real vectors, while $\sqrt{-1}$ is the (scalar) *imaginary of algebra*, and not a symbol for a *geometrically real right versor* (149, 153), may be said to be a **BIVECTOR**.

(7.) In like manner, an expression of the form (8.), or $x' = s + t\sqrt{-1}$, where s and t are two real scalars, but $\sqrt{-1}$ is still the ordinary imaginary of algebra, may be said by analogy to be a **BISCALAR**. *Imaginary roots of algebraic equations* are thus, in general, *biscalars*.

(8.) And if a bivector (6.) be divided by a (real) vector, the quotient, such as

$$q' = \frac{\rho'}{\alpha} = \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}\sqrt{-1} = q_0 + q_1\sqrt{-1},$$

in which q_0 and q_1 are two real quaternions, but $\sqrt{-1}$ is, as before, *imaginary*, may be said to be a **QUATERNION**.*

new
see!

215. The same distributive principle (212) may be employed in investigations respecting *circumscribed cones*, and the *tangents* (real or ideal), which can be drawn to a given sphere from a given point.

(1.) Instead of conceiving that o, A, B are three given points, and that *limits of position* of the point κ are sought, as in 214, (2.), which shall allow the points of intersection F_0, F_1 to be real, we may suppose that o, A, κ (which may be assumed to be collinear, without loss of generality, since α enters only by its tensor) are now the data of the question; and that *limits of direction* of the line $o\kappa$ are to be assigned, which shall permit the same reality: $\kappa F_0 F_1$ being still drawn parallel to oB , as in 214, (1.).

(2.) Dividing the equation $T\alpha = T(\epsilon + s\beta)$ by $T\epsilon$, and squaring, we have

* Compare the second Note to page 181.

$$N \frac{\alpha}{\epsilon} = \left(N \left(1 + x \frac{\beta}{\epsilon} \right) \right) = 1 + 2xS \frac{\beta}{\epsilon} + x^2 N \frac{\beta}{\epsilon};$$

the quadratic in x may therefore be thus written,

$$\left(xT \frac{\beta}{\epsilon} + SU \frac{\beta}{\epsilon} \right)^2 = \left(T \frac{\alpha}{\epsilon} \right)^2 + \left(\nabla U \frac{\beta}{\epsilon} \right)^2;$$

and its roots are real and unequal, or real and equal, or imaginary, according as

$$TVU \frac{\beta}{\epsilon} < \text{or} = \text{or} > T \frac{\alpha}{\epsilon};$$

that is, according as

$$\sin \text{EOB} < \text{or} = \text{or} > T.OA : T.OE.$$

(3.) If κ be interior to the sphere, then $T\epsilon < T\alpha$, $T(\alpha : \epsilon) > 1$; but $TVUq$ can never exceed unity (by 204, XIX., or by 210, XV., &c.); we have, therefore, in this case, the first of the three recent alternatives, and the two roots of the quadratic are necessarily real and unequal, whatever the direction of β may be. Accordingly it is evident, geometrically, that every indefinite right line, drawn through an internal point, must cut the spheric surface in two distinct and real points.

(4.) If the point κ be superficial, so that $T\epsilon = T\alpha$, $T(\alpha : \epsilon) = 1$, then the first alternative (2.) still exists, except at the limit for which $\beta \perp \epsilon$, and therefore $TVU(\beta : \epsilon) = 1$, in which case we have the second alternative. One root of the quadratic in x is now $= 0$, for every direction of β ; and the other root, namely $x = -2S(\epsilon : \beta)$, is likewise always real, but vanishes for the case when the angle EOB is right. In short, we have here the same system of chords and of tangents, from a point upon the surface, as in 218; the only difference being, that we now write κ for A , or ϵ for α .

(5.) But finally, if κ be an external point, so that $T\epsilon > T\alpha$, and $T(\alpha : \epsilon) < 1$, then $TVU(\beta : \epsilon)$ may either fall short of this last tensor, or equal, or exceed it; so that any one of the three alternatives (2.) may come to exist, according to the varying direction of β .

(6.) To illustrate geometrically the law of passage from one such alternative to another, we may observe that the equation,

$$TVU \frac{\rho}{\epsilon} = T \frac{\alpha}{\epsilon},$$

or

$$\sin \text{EOP} = T.OA : T.OE,$$

represents (when κ is thus external) a real cone of revolution, with its vertex at the centre O of the sphere; and according as the line OB lies inside this cone, or on it, or outside it, the first or the second or the third of the three alternatives (2.) is to be adopted; or in other words, the line through κ , drawn parallel (as before) to OB , either cuts the sphere, or touches it, or does not (really) meet it at all. (Compare the annexed Fig. 52.)

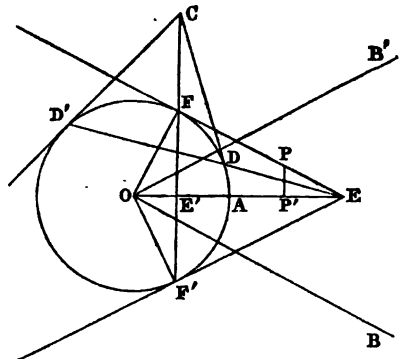


Fig. 52.

(7.) If ε be still an external point, the cone of tangents which can be drawn from it to the sphere is real; and the equation of this enveloping or circumscribed cone, with its vertex at ε , may be obtained from that of the recent cone (6.), by simply changing ρ to $\rho - \varepsilon$; it is, therefore, or at least one form of it is,

$$TVU \frac{\rho - \varepsilon}{\varepsilon} = T \frac{\alpha}{\varepsilon}; \quad \text{or} \quad \sin OKP = T.OA : T.O\varepsilon.$$

(8.) In general, if q be any quaternion, and x any scalar,

$$VU(q+x) = Vq : T(q+x);$$

the recent equation (7.) may therefore be thus written :

$$T \frac{V(\rho : \varepsilon) \cdot \varepsilon}{\rho - \varepsilon} = T \frac{\alpha}{\varepsilon};$$

or

$$T.F'P : T.EP = T.OA : T.O\varepsilon,$$

if F' be the foot of the perpendicular let fall from P on $O\varepsilon$; and in fact the first quotient is evidently = $\sin OKP$.

(9.) We may also write,

$$TV \frac{\rho}{\varepsilon} = T \frac{\alpha}{\varepsilon} \cdot T \left(\frac{\rho}{\varepsilon} - 1 \right); \quad \text{or} \quad 0 = \left(S \frac{\rho}{\varepsilon} \right)^2 - N \frac{\rho}{\varepsilon} + N \frac{\alpha}{\varepsilon} \left(N \frac{\rho}{\varepsilon} - 2S \frac{\rho}{\varepsilon} + 1 \right);$$

or

$$\left(S \frac{\rho}{\varepsilon} - N \frac{\alpha}{\varepsilon} \right)^2 = \left(1 - N \frac{\alpha}{\varepsilon} \right) \left(N \frac{\rho}{\varepsilon} - N \frac{\alpha}{\varepsilon} \right),$$

as another form of the equation of the circumscribed cone.

(10.) If then we make also

$$N \frac{\rho}{\alpha} = 1, \quad \text{or} \quad N \frac{\rho}{\varepsilon} = N \frac{\alpha}{\varepsilon},$$

to express that the point P is on the enveloped sphere, as well as on the enveloping cone, we find the following equation of the plane of contact, or of what is called the polar plane of the point ε , with respect to the given sphere :

$$\left(S \frac{\rho}{\varepsilon} - N \frac{\alpha}{\varepsilon} \right)^2 = 0; \quad \text{or} \quad S \frac{\rho}{\varepsilon} - N \frac{\alpha}{\varepsilon} = 0;$$

while the fact that it is a plane of contact* is exhibited by the occurrence of the exponent 2, or by its equation entering through its square.

(11.) The vector,

$$\varepsilon' = \varepsilon S \frac{\rho}{\varepsilon} = \varepsilon N \frac{\alpha}{\varepsilon} = O\varepsilon',$$

is that of the point ε' in which the polar plane (10.) of ε cuts perpendicularly the right line $O\varepsilon$; and we see that

$$T\varepsilon \cdot T\varepsilon' = T\alpha^2, \quad \text{or} \quad T.O\varepsilon \cdot T.O\varepsilon' = (T.OA)^2,$$

as was to be expected from elementary theorems, of spherical or even of plane geometry.

* In fact a modern geometer would say, that we have here a case of two coincident planes of intersection, merged into a single plane of contact.

(12.) The equation (10.), of the polar plane of κ , may easily be thus transformed :

$$S \frac{\varepsilon}{\rho} = \left(S \frac{\rho}{\varepsilon} \cdot N \frac{\varepsilon}{\rho} \right) N \frac{\alpha}{\rho}, \quad \text{or} \quad S \frac{\varepsilon}{\rho} - N \frac{\alpha}{\rho} = 0;$$

it continues therefore to hold good, when ε and ρ are *interchanged*. If then we take, as the vertex of a *new enveloping cone*, any point c external to the sphere, and situated on the polar plane $\mathcal{P}\mathcal{P}' \dots$ of the former external point κ , the *new plane of contact*, or the polar plane $\mathcal{D}\mathcal{D}' \dots$ of the new point c , will pass through the former vertex κ : a geometrical relation of *reciprocity*, or of *conjugation*, between the two points c and κ , which is indeed well-known, but which it appeared useful for our purpose to prove by quaternions* anew.

(13.) In general, each of the two connected equations,

$$S \frac{\rho'}{\rho} = N \frac{\alpha}{\rho}, \quad S \frac{\rho}{\rho'} = N \frac{\alpha}{\rho'},$$

which may also be thus written,

$$1 = \left(S \frac{\rho'}{\alpha} \cdot N \frac{\rho}{\alpha} \right) S \frac{\rho}{\alpha} K \frac{\rho'}{\alpha}, \quad 1 = S \frac{\rho}{\alpha} K \frac{\rho'}{\alpha},$$

may be said to be a form of the *Equation of Conjugation* between any two points P and P' (not those so marked in Fig. 52), of which the vectors satisfy it: because it expresses that those two points are, in a well-known sense, *conjugate* to each other, with respect to the given sphere, $T\rho = T\alpha$.

(14.) If one of the two points, as P' , be given by its vector ρ' , while the other point P and vector ρ are *variable*, the equation then represents a *plane locus*; namely, what is still called the *polar plane* of the given point, whether that point be external or internal, or on the surface of the sphere.

(15.) Let P, P' be thus two conjugate points; and let it be proposed to find the points s, s' , in which the right line PP' intersects the sphere. Assuming (comp. 25) that

$$Os = \sigma = x\rho + y\rho', \quad x + y = 1, \quad T\sigma = T\alpha,$$

and attending to the equation of conjugation (13.), we have, by 210, XX., or by 200, VII., the following quadratic equation in $y : x$,

$$(x + y)^2 = N \left(x \frac{\rho}{\alpha} + y \frac{\rho'}{\alpha} \right) = x^2 N \frac{\rho}{\alpha} + 2xy + y^2 N \frac{\rho'}{\alpha};$$

which gives,

$$x^2 \left(N \frac{\rho}{\alpha} - 1 \right) = y^2 \left(1 - N \frac{\rho'}{\alpha} \right).$$

(16.) Hence it is evident that, if the points of intersection s, s' are to be *real*, one of the two points P, P' must be interior, and the other must be exterior to the sphere; because, of the *two norms* here occurring, one must be greater and the other less than unity. And because the *two roots* of the quadratic, or the two values of $y : x$, differ

* In fact, it will easily be seen that the investigations in recent sub-articles are put forward, almost entirely, as exercises in the Language and Calculus of Quaternions, and not as offering any geometrical novelty of result.

only by their signs, it follows (by 26) that the right line pp' is *harmonically divided* (as indeed it is well known to be), at the two points a, a' at which it meets the sphere : or that in a notation already several times employed (25, 81, &c.), we have the *harmonic formula*,

$$(pap's) = -1.$$

(17.) From a real but *internal point* P , we can still *speak of a cone of tangents*, as being drawn to the sphere : but if so, we must say that those tangents are *ideal*, or *imaginary* ;* and must consider them as terminating on an *imaginary circle of contact* ; of which the *real but wholly external plane* is, by quaternions, as by modern geometry, recognised as being (comp. (14.)) the polar plane of the supposed internal point.

216. Some readers may find it useful, or at least interesting, to see here a few examples of the application of the General Distributive Principle (212) of multiplication to the *Ellipsoid*, of which some forms of the Quaternion Equation were lately assigned (in 204, (14.)) ; especially as those forms have been found to conduct† to a Geometrical Construction, previously unknown, for that celebrated and important Surface : or rather to several such constructions. In what follows, it will be supposed that any such reader has made himself already sufficiently familiar with the chief formulæ of the preceding Articles ; and therefore comparatively few references‡ will be given, at least upon the present subject.

(1.) To prove, first, that the locus of the variable ellipse,

$$I. \dots S \frac{\rho}{a} = x, \quad \left(\sqrt{\frac{\rho}{\beta}} \right)^2 = x^2 - 1, \quad 204, (18.)$$

which locus is represented by the equation,

$$II. \dots \left(S \frac{\rho}{a} \right)^2 - \left(\sqrt{\frac{\rho}{\beta}} \right)^2 = 1, \quad 204, (14.)$$

the two constant vectors a, β being supposed to be real, and to be inclined to each other at some acute or obtuse (but not right§) angle, is a *surface of the second order*,

* Compare again the second Note to page 90, and others formerly referred to.

† See the Proceedings of the Royal Irish Academy, for the year 1846.

‡ Compare the Note to page 218.

§ If $\beta \perp a$, the system I. represents (not an ellipse but) a *pair of right lines*, real or ideal, in which the *cylinder of revolution*, denoted by the second equation of that system, is cut by a *plane parallel to its axis*, and represented by the first equation.

in the sense that it is cut by an arbitrary rectilinear transversal in two (real or imaginary) points, and in no more than two, let us assume two points L, M , or their vectors $\lambda = OL, \mu = OM$, as given; and let us seek to determine the points P (real or imaginary), in which the indefinite right line LM intersects the locus II.; or rather the number of such intersections, which will be sufficient for the present purpose.

(2.) Making then $\rho = \frac{y\lambda + z\mu}{y+z}$ (25), we have, for $y : z$, the following quadratic equation,

$$\text{III.} \dots \left(yS \frac{\lambda}{a} + zS \frac{\mu}{a} \right)^2 - \left(yV \frac{\lambda}{a} + zV \frac{\mu}{a} \right)^2 = (y+z)^2;$$

without proceeding to *resolve* which, we see already, by its mere *degrees*, that the number sought is *two*; and therefore that the locus II. is, as above stated, a surface of the *second order*.

(3.) The equation II. remains unchanged, when $-\rho$ is substituted for ρ ; the surface has therefore a *centre*, and this centre is at the *origin* O of vectors.

(4.) It has been seen that the equation of the surface may also be thus written :

$$\text{IV.} \dots T \left(S \frac{\rho}{a} + V \frac{\rho}{\beta} \right) = 1; \quad 204, (14.)$$

it gives therefore, for the reciprocal of the radius vector from the centre, the expression,

$$\text{V.} \dots \frac{1}{T\rho} = T \left(S \frac{U\rho}{a} + V \frac{U\rho}{\beta} \right);$$

and this expression has a real value, which never vanishes,* whatever real value may be assigned to the versor $U\rho$, that is, whatever direction may be assigned to ρ : the surface is therefore *closed*, and *finite*.

(5.) Introducing two new constant and auxiliary vectors, determined by the two expressions,

$$\text{VI.} \dots \gamma = \frac{2\beta}{\beta+a} \cdot a, \quad \delta = \frac{2\beta}{\beta-a} \cdot a,$$

which give (by 125) these other expressions,

$$\text{VI.} \dots \gamma = \frac{2a}{\beta+a} \cdot \beta, \quad \delta = \frac{2a}{\beta-a} \cdot \beta,$$

we have

$$\text{VII.} \dots \frac{\gamma}{a} + \frac{\gamma}{\beta} = 2, \quad \frac{\delta}{a} - \frac{\delta}{\beta} = 2;$$

$$\text{VII.} \dots \frac{a}{\gamma} + \frac{a}{\delta} = 1, \quad \frac{\beta}{\gamma} - \frac{\beta}{\delta} = 1;$$

and under these conditions, γ is said to be the *harmonic mean* between the two former vectors, a and β ; and in like manner, δ is the harmonic mean between a and $-\beta$; while $2a$ is the corresponding mean between γ, δ ; and 2β is so, between γ and $-\delta$.

* It is to be remembered that we have excluded in (1.) the case where $\beta \perp a$; in which case it can be shown that the equation II. represents an *elliptic cylinder*.

all the vectors a, β, γ, δ must be coplanar from above conditions.

(6.) Under the same conditions, for any arbitrary vector ρ , we have the transformations,

$$\text{VIII.} \dots \frac{\rho}{\gamma} = \frac{1}{2} \left(\frac{\rho}{\alpha} + \frac{\rho}{\beta} \right); \quad \frac{\rho}{\delta} = \frac{1}{2} \left(\frac{\rho}{\alpha} - \frac{\rho}{\beta} \right);$$

$$\text{IX.} \dots \frac{\rho}{\gamma} + \kappa \frac{\rho}{\delta} = \frac{S \rho}{\alpha} + \nu \frac{\rho}{\beta};$$

the equation IV. of the surface may therefore be thus written :

$$\text{X.} \dots T \left(\frac{\rho}{\gamma} + \kappa \frac{\rho}{\delta} \right) = 1; \quad \text{or thus,} \quad \text{X'.} \dots T \left(\frac{\rho}{\delta} + \kappa \frac{\rho}{\gamma} \right) = 1;$$

the geometrical meaning of which new forms will soon be seen.

(7.) The system of the two planes through the origin, which are respectively perpendicular to the new vectors γ and δ , is represented by the equation,

$$\text{XI.} \dots S \frac{\rho}{\gamma} S \frac{\rho}{\delta} = 0, \quad \text{or} \quad \text{XII.} \dots \left(S \frac{\rho}{\alpha} \right)^2 = \left(S \frac{\rho}{\beta} \right)^2;$$

combining which with the equation II. we get

$$\text{XIII.} \dots 1 - \left(S \frac{\rho}{\beta} \right)^2 - \left(\nu \frac{\rho}{\beta} \right)^2 = N \frac{\rho}{\beta}; \quad \text{or,} \quad \text{XIV.} \dots T \rho = T \beta.$$

These two diametral planes therefore cut the surface in two circular sections, with $T\beta$ for their common radius; and the normals γ and δ , to the same two planes, may be called (comp. 196, (17.)) the *cyclic normals* of the surface; while the planes themselves may be called its *cyclic planes*.

(8.) Conversely, if we seek the intersection of the surface with the concentric sphere XIV., of which the radius is $T\beta$, we are conducted to the equation XII. of the system of the two cyclic planes, and therefore to the two circular sections (7.); so that every radius vector of the surface, which is not drawn in one or other of these two planes, has a length either greater or less than the radius $T\beta$ of the sphere.

(9.) By all these marks, it is clear that the locus II., or 204, (14.), is (as above asserted) an *Ellipsoid*; its centre being at the origin (3.), and its mean semiaxis being = $T\beta$; while $U\beta$ has, by 204, (15.), the direction of the axis of a circumscribed cylinder of revolution, of which cylinder the radius is $T\beta$; and α is, by the last cited sub-article, perpendicular to the plane of the ellipse of contact.

(10.) Those who are familiar with modern geometry, and who have caught the notations of quaternions, will easily see that this ellipsoid II., or IV., is a deformation of what may be called the *mean sphere* XIV., and is homologous thereto; the infinitely distant point in the direction of β being a centre of homology, and either of the two planes XI. or XII. being a plane of homology corresponding.

217. The recent form, X. or X', of the quaternion equation of the ellipsoid, admits of being interpreted, in such a way as to conduct (comp. 216) to a simple construction of that surface; which we shall first investigate by calculation, and then illustrate by geometry.

(1.) Carrying on the Roman numerals from the sub-articles to 216, and observing that (by 190, &c.),

$$\frac{\rho}{\gamma} = K \frac{\gamma}{\rho} \cdot N \frac{\rho}{\gamma}, \text{ and } K \frac{\rho}{\delta} = \frac{\delta}{\rho} \cdot N \frac{\rho}{\delta},$$

the equation X. takes the form,

$$\text{XV.} \dots 1 = T \left\{ \left(\frac{\delta}{T\delta^2} + K \frac{\gamma}{\rho} \cdot \frac{\rho}{T\gamma^2} \right); \frac{\rho}{T\rho^2} \right\};$$

or

$$\text{XVI.} \dots \frac{\rho^2}{T\rho} = T \left(\epsilon + K \frac{\kappa}{\rho} \cdot \rho \right),$$

if we make

$$\text{XVII.} \dots \frac{\delta}{T\delta^2} = \frac{\epsilon}{\rho^2} \text{ and } \frac{\gamma}{T\gamma^2} = \frac{\kappa}{\rho^2},$$

when ϵ and κ are two new constant vectors, and ϵ is a new constant scalar, which we shall suppose to be positive, but of which the value may be chosen at pleasure.

(2.) The comparison of the forms X. and X' shows that γ and δ may be interchanged, or that they enter symmetrically into the equation of the ellipsoid, although they may not at first seem to do so; it is therefore allowed to assume that

$$\text{XVIII.} \dots T\gamma > T\delta, \text{ and therefore that } \text{XVIII'.} \dots T\epsilon > T\kappa;$$

for the supposition $T\gamma = T\delta$ would give, by VI.,

$$T(\beta + \alpha) = T(\beta - \alpha), \text{ and } \therefore \text{(by 186, (6.) \&c.) } \beta \perp \alpha,$$

which latter case was excluded in 216, (1.).

(3.) We have thus,

$$\text{XIX.} \dots U\epsilon = U\delta; \quad U\kappa = U\gamma;$$

$$\text{XX.} \dots T\epsilon = \frac{\rho^2}{T\delta}; \quad T\kappa = \frac{\rho^2}{T\gamma};$$

$$\text{XXI.} \dots \frac{T\epsilon^2 - T\kappa^2}{\epsilon^2} = \left(\frac{\epsilon}{T\delta} \right)^2 - \left(\frac{\epsilon}{T\gamma} \right)^2.$$

(4.) Let ABC be a plane triangle, such that

$$\text{XXII.} \dots CB = \epsilon, \quad CA = \kappa;$$

let also

$$AE = \rho.$$

Then if a sphere, which we shall call the *diacentric sphere*, be described round the point C as centre, with a radius = $T\kappa$, and therefore so as to pass through the centre A (here written instead of o) of the ellipsoid, and if D be the point in which the line AE meets this sphere again, we shall have, by 218, (6.), (18.),

$$\text{XXIII.} \dots CD = -K \frac{\kappa}{\rho} \cdot \rho,$$

and therefore

$$\text{XXIII'.} \dots DB = \epsilon + K \frac{\kappa}{\rho} \cdot \rho;$$

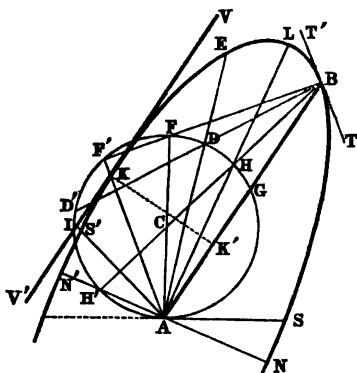


Fig. 53.

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$CD = -K \frac{\kappa}{\rho} \cdot \rho$
 $DB = \epsilon + K \frac{\kappa}{\rho} \cdot \rho$
 $DA = \epsilon + K \frac{\kappa}{\rho} \cdot \rho$
 $CP = \rho$
 $AC = \kappa$

so that the equation XVI. becomes,

$$\text{XXIV.} \dots t^2 = T. \overline{AE}. T. DB.$$

(5.) The point B is *external* to the diacentric sphere (4.), by the assumption (2.); a real tangent (or rather cone of tangents) to this sphere can therefore be drawn from that point; and if we select the length of such a tangent as the value (1.) of the scalar t , that is to say, if we make each member of the formula XXI. equal to unity, and denote by D' the second intersection of the right line BD with the sphere, as in Fig. 53, we shall have (by Euclid III.) the elementary relation,

$$\text{XXV.} \dots t^2 = T. DB. T. BD';$$

whence follows this *Geometrical Equation of the Ellipsoid*,

$$\text{XXVI.} \dots T. \overline{AE} = T. \overline{BD'};$$

or in a somewhat more familiar notation,

$$\text{XXVII.} \dots \overline{AE} = \overline{BD'};$$

where \overline{AE} denotes the *length* of the line AE, and similarly for $\overline{BD'}$.

(6.) The following very simple *Rule of Construction* (comp. the recent Fig. 53) results therefore from our quaternion analysis:—

From a fixed point A, on the surface of a given sphere, draw any chord AD; let D' be the second point of intersection of the same spheric surface with the secant BD, drawn from a fixed external point B; and take a radius vector AE, equal in length to the line BD', and in direction either coincident with, or opposite to, the chord AD: the locus of the point E will be an ellipsoid, with A for its centre, and with B for a point of its surface.*

(7.) Or thus:—

If, of a plane but variable quadrilateral ABED', of which one side AB is given in length and in position, the two diagonals AE, BD' be equal to each other in length, and if their intersection D be always situated upon the surface of a given sphere, whereof the side AD' of the quadrilateral is a chord, then the opposite side BE is a chord of a given ellipsoid.

218. From either of the two foregoing statements, of the *Rule of Construction for the Ellipsoid* to which quaternions have conducted, many *geometrical consequences* can easily be inferred, a few of which may be mentioned here, with their proofs by *calculation* annexed: the present Calculus being, of course, still employed.

(1.) That the corner B, of what may be called the *Generating Triangle* ABC, is in fact a point of the generated surface, with the construction 217, (6.), may be

* It is merely to fix the conceptions, that the point B is here supposed to be *external* (5.); the calculations and the construction would be almost the same, if we assumed B to be an *internal* point, or $T_t < T_k$, $T_\gamma < T\delta$.

proved, by conceiving the variable chord AD of the given diacentric sphere to take the position AG; where G is the second intersection of the line AB with that spheric surface.

(2.) If D be conceived to approach to A (instead of G), and therefore D' to G (instead of A), the direction of AE (or of AD) then tends to become tangential to the sphere at A, while the length of AE (or of AD) tends, by the construction, to become equal to the length of BG; the surface has therefore a *diametral and circular section*, in a plane which touches the diacentric sphere at A, and with a radius = BG.

(3.) Conceive a circular section of the sphere through A, made by a plane perpendicular to BC; if D move along this circle, D' will move along a parallel circle through G, and the length of BD', or that of AE, will again be equal to BG; such then is the radius of a *second diametral and circular section* of the ellipsoid, made by the lately mentioned plane.

(4.) The construction gives us thus two *cyelic planes* through A; the perpendiculars to which planes, or the two *cyelic normals* (216, (7.)) of the ellipsoid, are seen to have the directions of the two sides, CA, CB, of the generating triangle ABC (1.).

(5.) Again, since the rectangle

$$\overline{BA} \cdot \overline{BQ} = \overline{BD} \cdot \overline{BD'} = \overline{BD} \cdot \overline{AB} = \text{double area of triangle } \triangle ABE : \sin BDE,$$

we have the equation,

$$\text{XXVIII.} \dots \text{perpendicular distance of } \kappa \text{ from } \overline{AB} = \overline{BQ} \cdot \sin BDE;$$

the third side, AB, of the generating triangle (1.), is therefore the *axis of revolution* of a cylinder, which envelopes the ellipsoid, and of which the radius has the same length, BQ, as the radius of each of the two diametral and circular sections.

(6.) For the points of contact of ellipsoid and cylinder, we have the geometrical relation,

$$\text{XXIX.} \dots BDE = \text{a right angle; or } \text{XXIX'}. \dots ADB = \text{a right angle;}$$

the point D is therefore situated on a *second spheric surface*, which has the line AB for a diameter, and intersects the diacentric sphere in a circle, whereof the plane passes through A, and cuts the enveloping cylinder in an *ellipse of contact* (comp. 204, (15.), and 216, (9.)), of that cylinder with the ellipsoid.

(7.) Let AC meet the diacentric sphere again in F, and let BF meet it again in F' (as in Fig. 58); the common plane of the last-mentioned circle and ellipse (6.) can then be easily proved to cut perpendicularly the plane of the generating triangle ABC in the line AF'; so that the line F'B is normal to this plane of contact; and therefore also (by conjugate diameters, &c.) to the ellipsoid, at B.

(8.) These geometrical consequences of the construction (217), to which many others might be added, can all be shown to be consistent with, and confirmed by, the quaternion analysis from which that construction itself was derived. Thus, the two circular sections (2.) (3.) had presented themselves in 216, (7.); and their two cyelic normals (4.), or the sides CA, CB of the triangle, being (by 217, (4.)) the two vectors κ , ι , have (by 217, (1.) or (3.)) the directions of the two former vectors γ , δ ; which again agrees with 216, (7.).

(9.) Again, it will be found that the assumed relations between the three pairs of constant vectors, α , β ; γ , δ ; and ι , κ , any one of which pairs is sufficient to deter-

mine the ellipsoid, conduct to the following expressions (of which the investigation is left to the student, as an exercise):

$$\text{XXX.} \dots \alpha = \frac{\delta}{\delta + \gamma} \quad \gamma = \frac{\gamma}{\delta + \gamma} \quad \delta = \frac{+\epsilon^2}{T(\epsilon + \kappa)} \quad U(\epsilon + \kappa) = \mathcal{V}'\mathcal{B};$$

$$\text{XXXI.} \dots \beta = \frac{\delta}{\delta - \gamma} \quad \gamma = \frac{\gamma}{\delta - \gamma} \quad \delta = \frac{-\epsilon^2}{T(\epsilon - \kappa)} \quad U(\epsilon - \kappa) = \mathcal{B}\mathcal{G};$$

the letters \mathcal{B} , \mathcal{V}' , \mathcal{G} referring here to Fig. 58, while $\alpha\beta\gamma\delta$ retain their former meanings (216), and are not interpreted as vectors of the points $\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}$ in that Figure. Hence the recent geometrical inferences, that $\mathcal{A}\mathcal{B}$ (or $\mathcal{B}\mathcal{G}$) is the axis of revolution of an enveloping cylinder (5.), and that $\mathcal{V}'\mathcal{B}$ is normal to the plane of the ellipse of contact (7.), agree with the former conclusions (216, (9.), or 204, (15.)), that β is such an axis, and that α is such a normal.

(10.) It is easy to prove, generally, that

$$S \frac{q-1}{q+1} = S \frac{(q-1)(Kq+1)}{(q+1)(Kq+1)} = \frac{Nq-1}{N(q+1)}, \quad S \frac{q+1}{q-1} = \frac{Nq-1}{N(q-1)};$$

whence

$$\text{XXXII.} \dots S \frac{\epsilon - \kappa}{\epsilon + \kappa} = \frac{T\epsilon^2 - T\kappa^2}{T(\epsilon + \kappa)^2}, \quad S \frac{\epsilon + \kappa}{\epsilon - \kappa} = \frac{T\epsilon^2 - T\kappa^2}{T(\epsilon - \kappa)^2},$$

whatever two vectors ϵ and κ may be. But we have here,

$$\text{XXXIII.} \dots \epsilon^2 = T\epsilon^2 - T\kappa^2, \text{ by 217, (5.)};$$

the recent expressions (9.) for α and β become, therefore,

$$\text{XXXIV.} \dots \alpha = +(\epsilon + \kappa) S \frac{\epsilon - \kappa}{\epsilon + \kappa}; \quad \beta = -(\epsilon - \kappa) S \frac{\epsilon + \kappa}{\epsilon - \kappa}.$$

The last form 204, (14.), of the equation of the ellipsoid, may therefore be now thus written:

$$\text{XXXV.} \dots T \left(S \frac{\rho}{\epsilon + \kappa} : S \frac{\epsilon - \kappa}{\epsilon + \kappa} - \mathcal{V} \frac{\rho}{\epsilon - \kappa} : S \frac{\epsilon + \kappa}{\epsilon - \kappa} \right) = 1;$$

in which the sign of the right part may be changed. And thus we verify by calculation the recent result (1.) of the construction, namely that \mathcal{B} is a point of the surface; for we see that the last equation is satisfied, when we suppose

$$\text{XXXVI.} \dots \rho = \mathcal{A}\mathcal{B} = \epsilon - \kappa = \beta : S \frac{\beta}{\alpha};$$

a value of ρ which evidently satisfies also the form 216, IV.

(11.) From the form 216, II., combined with the value XXXIV. of α , it is easy to infer that the plane,

$$\text{XXXVII.} \dots S \frac{\rho}{\alpha} = 1, \text{ or XXXVII.} \dots S \frac{\rho}{\epsilon + \kappa} = S \frac{\epsilon - \kappa}{\epsilon + \kappa},$$

which corresponds to the value $x = 1$ in 216, I., touches the ellipsoid at the point \mathcal{B} , of which the vector ρ has been thus determined (10); the normal to the surface, at that point, has therefore the direction of $\epsilon + \kappa$, or of α , that is, of $\mathcal{V}'\mathcal{B}$, or of $\mathcal{V}'\mathcal{B}$: so that the last geometrical inference (7.) is thus confirmed, by calculation with quaternions.

219. A few other consequences of the construction (217) may be here noted; especially as regards the geometrical determination

of the *three principal semiaxes* of the ellipsoid, and the major and minor semiaxes of any elliptic and *diametral section*; together with the assigning of a certain *system of spherical conics*, of which the *surface* may be considered to be the *locus*.

(1.) Let a, b, c denote the lengths of the greatest, the mean, and the least semi-axes of the ellipsoid, respectively; then if the side BC of the generating triangle cut the diacentric sphere in the points H and H' , the former lying (as in Fig. 53) between the points B and C , we have the values,

$$\text{XXXVIII.} \dots a = \overline{BH'}; \quad b = \overline{BG}; \quad c = \overline{BH};$$

so that the lengths of the sides of the triangle ABC may be thus expressed, in terms of these semiaxes,

$$\text{XXXIX.} \dots \overline{BC} = T_t = \frac{a+c}{2}; \quad \overline{CA} = T_\kappa = \frac{a-c}{2}; \quad \overline{AB} = T(t-\kappa) = \frac{ac}{b};$$

and we may write,

$$\text{XL.} \dots a = T_t + T_\kappa; \quad b = \frac{T_t^2 - T_\kappa^2}{T(t-\kappa)}; \quad c = T_t - T_\kappa.$$

(2.) If, in the respective directions of the two supplementary chords $\Delta H, \Delta H'$ of the sphere, or in the opposite directions, we set off lines $\Delta L, \Delta \kappa$, with the lengths of BH', BH , the points L, κ , thus obtained, will be respectively a *major* and a *minor summit* of the surface. And if we erect, at the centre Δ of that surface, a perpendicular ΔM to the plane of the triangle, with a length $= \overline{BG}$, the point M (which will be common to the two circular sections, and will be situated on the enveloping cylinder) will be a *mean summit* thereof.

(3.) Conceive that the sphere and ellipsoid are both cut by a plane through Δ , on which the points B' and C' shall be supposed to be the projections of B and C ; then O' will be the centre of the circular section of the sphere; and if the line $B'C'$ cut this new circle in the points D_1, D_2 , of which D_1 may be supposed to be the nearer to B' , the two supplementary chords $\Delta D_1, \Delta D_2$ of the circle have the *directions of the major and minor semiaxes* of the *elliptic section* of the ellipsoid; while the *lengths* of those semiaxes are, respectively, $\overline{BA} \cdot \overline{BG} : \overline{BD_1}$, and $\overline{BA} \cdot \overline{BG} : \overline{BD_2}$; or $\overline{BD_1}'$ and $\overline{BD_2}'$, if the secants BD_1 and BD_2 meet the sphere again in D_1' and D_2' .

(4.) If these two semiaxes of the section be called a , and c , and if we still denote by t the tangent from B to the sphere, we have thus,

$$\text{XLI.} \dots \overline{BD_1} = t^2 : a, = aca^{-1}; \quad \overline{BD_2} = t^2 : c, = acc^{-1};$$

but if we denote by p_1 and p_2 the inclinations of the plane of the section to the two cyclic planes of the ellipsoid, whereto CA and CB are perpendicular, so that the projections of these two sides of the triangle are

$$\text{XLII.} \dots \begin{cases} \overline{C'A} = \overline{CA} \cdot \sin p_1 = \frac{1}{2}(a-c) \sin p_1, \\ \overline{C'B} = \overline{CB} \cdot \sin p_2 = \frac{1}{2}(a+c) \sin p_2, \end{cases}$$

we have

$$\text{XLIII.} \dots \overline{BD_2}^2 - \overline{BD_1}^2 = \overline{B'D_2}^2 - \overline{B'D_1}^2 = 4\overline{B'C'} \cdot \overline{C'A} = (a^2 - c^2) \sin p_1 \sin p_2;$$

whence follows the important formula,

$$\text{XLIV.} \dots c^{-2} - a^{-2} = (c^2 - a^2) \sin p_1 \sin p_2;$$

or in words, the known and useful theorem, that "the difference of the inverse squares of the semi-axes, of a plane and diametral section of an ellipsoid, varies as the product of the sines of the inclinations of the cutting plane, to the two planes of circular section.

(5.) As verifications, if the plane be that of the generating triangle ABC, we have

$$p_1 = p_2 = \frac{\pi}{2}, \text{ and } a_1 = a, \quad c_1 = c;$$

but if the plane be perpendicular to either of the two sides, CA, CB, then either p_1 or $p_2 = 0$, and $c_1 = a_1$.

(6.) If the ellipsoid be cut by any concentric sphere, distinct from the mean sphere XIV., so that

$$\text{XLV.} \dots \overline{AE} = T\rho = r < b, \text{ where } r \text{ is a given positive scalar;}$$

then

$$\text{XLVI.} \dots \overline{BD} = r^2 r^{-1} < acb^{-1}, \text{ that is, } < \overline{BA};$$

so that the locus of what may be called the *guide-point* D, through which, by the construction, the variable semidiameter AE of the ellipsoid (or one of its prolongations) passes, and which is still at a constant distance from the given external point B, is now again a circle of the diacentric sphere, but one of which the plane does not pass (as it did in 218, (3.)) through the centre A of the ellipsoid. The point E has therefore here, for one locus, the cyclic cone which has A for vertex, and rests on the last-mentioned circle as its base; and since it is also on the concentric sphere XLV., it must be on one or other of the two spherical conics, in which (comp. 196, (11.)) the cone and sphere last mentioned intersect.

(7.) The intersection of an ellipsoid with a concentric sphere is therefore, generally, a system of two such conics, varying with the value of the radius r, and becoming, as a limit, the system of the two circular sections, for the particular value $r = b$; and the ellipsoid itself may be considered as the locus of all such spherical conics, including those two circles.

(8.) And we see, by (6.), that the two cyclic planes (comp. 196, (17.), &c.) of any one of the concentric cones, which rest on any such conic, coincide with the two cyclic planes of the ellipsoid: all this resulting, with the greatest ease, from the construction (217) to which quaternions had conducted.

(9.) With respect to the Figure 58, which was designed to illustrate that construction, the signification of the letters ABCDD'EFF'GHH'LN has been already explained. But as regards the other letters we may here add, Ist, that N' is a second minor summit of the surface, so that AN' = NA; IInd, that K is a point in which the chord AN', of what we may here call the diacentric circle ACF, intersects what may be called the principal ellipse,* or the section NBLEN' of the ellipsoid, made by the plane of the greatest and least axes, that is by the plane of the generating triangle ABC, so that the lengths of AK and BF are equal; IIIrd, that the tangent, VKV', to this ellipse at this point, is parallel to the side AB of the triangle, or to the axis of

* In the plane of what is called, by many modern geometers, the focal hyperbola of the ellipsoid.

revolution of the enveloping cylinder 218, (5.), being in fact one side (or generatrix) of that cylinder; IVth, that ΔK , ΔB are thus two conjugate semidiameters of the ellipse, and therefore the tangent $\tau \sigma \tau'$, at the point B of that ellipse, is parallel to the line $\Delta K F'$, or perpendicular to the line $B F F'$; Vth, that this latter line is thus the normal (comp. 218, (7.), (11.)) to the same elliptic section, and therefore also to the ellipsoid, at B ; VIth, that the least distance $K K'$ between the parallels ΔB , $K V$, being = the radius b of the cylinder, is equal in length to the line $B G$, and also to each of the two semidiameters, ΔS , $\Delta S'$, of the ellipse, which are radii of the two circular sections of the ellipsoid, in planes perpendicular to the plane of the Figure; VIIth, that ΔS touches the circle at A ; and VIIIth, that the point s' is on the chord ΔI of that circle, which is drawn at right angles to the side $B C$ of the triangle.

220. The reader will easily conceive that the quaternion equation of the ellipsoid admits of being put under several other forms; among which, however, it may here suffice to mention one, and to assign its geometrical interpretation.

(1.) For any three vectors, ι , κ , ρ , we have the transformations,

$$\begin{aligned} \text{XLVII.} \dots N\left(\frac{\iota}{\rho} + K \frac{\kappa}{\rho}\right) &= N \frac{\iota}{\rho} + N \frac{\kappa}{\rho} + 2S \frac{\iota \kappa}{\rho \rho} \\ &= N \frac{\iota}{\kappa} N \frac{\kappa}{\rho} + N \frac{\kappa}{\iota} N \frac{\iota}{\rho} + 2S \frac{\iota \kappa}{\rho \rho} T \frac{\kappa}{\iota} T \frac{\iota}{\kappa} \\ &= N\left(\frac{\iota}{\rho} T \frac{\kappa}{\iota} + K \frac{\kappa}{\rho} T \frac{\iota}{\kappa}\right) = N\left(\frac{\kappa}{\rho} T \frac{\iota}{\kappa} + K \frac{\iota}{\rho} T \frac{\kappa}{\iota}\right) \\ &= N\left(\frac{U \iota \cdot T \kappa}{\rho} + K \frac{U \kappa \cdot T \iota}{\rho}\right) = N\left(\frac{U \kappa \cdot T \iota}{\rho} + K \frac{U \iota \cdot T \kappa}{\rho}\right); \end{aligned}$$

whence follows this other general transformation :

$$\text{XLVIII.} \dots T\left(\iota + K \frac{\kappa}{\rho} \cdot \rho\right) = T\left(U \kappa \cdot T \iota + K \frac{U \iota \cdot T \kappa}{\rho} \cdot \rho\right)$$

(2.) If then we introduce two new auxiliary and constant vectors, ι' and κ' , defined by the equations,

$$\text{XLIX.} \dots \iota' = -U \kappa \cdot T \iota, \quad \kappa' = -U \iota \cdot T \kappa,$$

which give,

$$\text{L.} \dots T \iota' = T \iota, \quad T \kappa' = T \kappa, \quad T(\iota' - \kappa') = T(\iota - \kappa), \quad T \iota'^2 - T \kappa'^2 = \iota^2,$$

we may write the equation XVI. (in 217) of the ellipsoid under the following precisely similar form :

$$\text{LI.} \dots \frac{\rho^2}{T \rho} = T\left(\iota' + K \frac{\kappa'}{\rho} \cdot \rho\right);$$

in which ι' and κ' have simply taken the places of ι and κ .

(3.) Retaining then the centre A of the ellipsoid, construct a new diocentric sphere, with a new centre O' , and a new generating triangle $\Delta B' C'$, where B' is a new fixed external point, but the lengths of the sides are the same, by the conditions,

$$\text{LII.} \dots \Delta O' = -\kappa', \quad O' B' = +\iota', \quad \text{and therefore } \Delta B' = \iota' - \kappa';$$

draw any secant $B' d'' d'''$ (instead of $B D D'$), and set off a line ΔE in the direction of

AD'' , or in the opposite direction, with a length equal to that of BD''' ; the locus of the point E will be the same ellipsoid as before.

(4.) The only inference which we shall here* draw from this new construction is, that there exists (as is known) a second enveloping cylinder of revolution, and that its axis is the side AB' of the new triangle $AB'C'$; but that the radius of this second cylinder is equal to that of the first, namely to the mean semiaxis, b , of the ellipsoid; and that the major semiaxis, a , or the line AL in Fig. 58, bisects the angle BAB' , between the two axes of revolution of these two circumscribed cylinders: the plane of the new ellipses of contact being geometrically determined by a process exactly similar to that employed in 218, (7.); and being perpendicular to the new vector, $\epsilon' + \epsilon'$, as the old plane of contact was (by 218, (11.)) to $\epsilon + \epsilon$.

SECTION 14.—On the Reduction of the General Quaternion to a Standard Quadrinomial Form; with a First Proof of the Associative Principle of Multiplication of Quaternions.

221. Retaining the significations (181) of the three rectangular unit-lines OI, OJ, OK , as the axes, and therefore also the indices (159), of three given right versors i, j, k , in three mutually rectangular planes, we can express the index oq of any other right quaternion, such as Vq , under the trinomial form (comp. 62),

$$I. \dots IVq = oq = x \cdot OI + y \cdot OJ + z \cdot OK;$$

where xyz are some three scalar coefficients, namely, the three rectangular co-ordinates of the extremity q of the index, with respect to the three axes OI, OJ, OK . Hence we may write also generally, by 206 and 126,

$$II. \dots Vq = xi + yj + zk = ix + jy + kz;$$

and this last form, $ix + jy + kz$, may be said to be a *Standard Trinomial Form*, to which every right quaternion, or the right part Vq of any proposed quaternion q , can be (as above) reduced. If then we denote by w the scalar part, Sq , of the same general quaternion q , we shall have, by 202, the following *General Reduction of a Quaternion to a STANDARD QUADRINOMIAL FORM* (183):

* If room shall allow, a few additional remarks may be made, on the relations of the constant vectors ϵ, ϵ' , &c., to the ellipsoid, and on some other constructions of that surface, when, in the following Book, its equation shall come to be put under the new form,

$$T(\epsilon\rho + \rho\epsilon) = \epsilon^2 - \rho^2.$$

$$\text{III. . . } q = (Sq + Vq)w + ix + jy + kz;$$

in which the *four scalars*, $wxyz$, may be said to be the *Four Constituents of the Quaternion*. And it is evident (comp. 202, (5.), and 133), that if we write in like manner,

$$\text{IV. . . } q' = w' + ix' + jy' + kz',$$

where ijk denote the same three given right versors (181) as before, then *the equation*

$$\text{V. . . } q' = q,$$

between these two quaternions, q and q' , includes the four following scalar equations between the constituents :

$$\text{VI. . . } w' = w, \quad x' = x, \quad y' = y, \quad z' = z;$$

which is a new justification (comp. 112, 116) of the *propriety of naming*, as we have done throughout the present Chapter, the *General Quotient of two Vectors* (101) a QUATERNION.

222. When the *Standard Quadrinomial Form* (221) is adopted, we have then not only

$$\text{I. . . } Sq = w, \quad \text{and} \quad Vq = ix + jy + kz,$$

as before, but also, by 204, XI.,

$$\text{II. . . } Kq = (Sq - Vq)w - ix - jy - kz.$$

And because the *distributive property of multiplication* of quaternions (212), combined with the *laws of the symbols ijk* (182), or with the *General and Fundamental Formula of this whole Calculus* (183), namely with the formula,

$$i^2 = j^2 = k^2 = ijk = -1, \quad (\text{A})$$

gives the transformation,

$$\text{III. . . } (ix + jy + kz)^2 = -(x^2 + y^2 + z^2),$$

we have, by 204, &c., the following new expressions :

$$\text{IV. . . } NVq = (TVq)^2 = -Vq^2 = x^2 + y^2 + z^2;$$

$$\text{V. . . } TVq = \sqrt{(x^2 + y^2 + z^2)};$$

$$\text{VI. . . } UVq = (ix + jy + kz) : \sqrt{(x^2 + y^2 + z^2)};$$

$$\text{VII. . . } Nq = Tq^2 = Sq^2 - Vq^2 = w^2 + x^2 + y^2 + z^2;$$

$$\text{VIII. . . } Tq = \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\text{IX. . . } Uq = (w + ix + jy + kz) : \sqrt{(w^2 + x^2 + y^2 + z^2)};$$

$$\begin{aligned} \text{X.} \dots \text{SU}q &= w : \sqrt{(w^2 + x^2 + y^2 + z^2)}; \\ \text{XI.} \dots \text{VU}q &= (ix + jy + kz) : \sqrt{(w^2 + x^2 + y^2 + z^2)}; \\ \text{XII.} \dots \text{TVU}q &= \sqrt{\frac{x^2 + y^2 + z^2}{w^2 + x^2 + y^2 + z^2}}. \end{aligned}$$

(1.) To prove the recent formula III., we may arrange as follows the steps of the multiplication (comp. again 182):

$$\begin{aligned} \text{V}q &= ix + jy + kz, \\ \text{V}q &= ix + jy + kz; \\ ix \cdot \text{V}q &= -x^2 + kxy - jxz; \\ jy \cdot \text{V}q &= -y^2 - kyx + iyz, \\ kz \cdot \text{V}q &= -z^2 + jzx - izy; \\ \text{V}q^2 &= \text{V}q \cdot \text{V}q = -x^2 - y^2 - z^2. \end{aligned}$$

(2.) We have, therefore,

$$\text{XIII.} \dots (ix + jy + kz)^2 = -1, \text{ if } x^2 + y^2 + z^2 = 1,$$

a result to which we have already alluded,* in connexion with the partial *indeterminateness* of signification, in the present calculus, of the symbol $\sqrt{-1}$, when considered as denoting a *right radial* (149), or a *right versor* (158), of which the *plane* or the *axis* is *arbitrary*.

(3.) If $q'' = q'q$, then $Nq'' = Nq' \cdot Nq$, by 191, (8.); but if $q = w + \&c.$, $q' = w' + \&c.$, $q'' = w'' + \&c.$, then

$$\text{XIV.} \dots \begin{cases} w'' = w'w - (x'x + y'y + z'z), \\ x'' = (w'x + x'w) + (y'z - z'y), \\ y'' = (w'y + y'w) + (x'z - x'z), \\ z'' = (w'z + z'w) + (x'y - y'x); \end{cases}$$

and conversely these four scalar equations are jointly equivalent to, and may be summed up in, the quaternion formula,

$$\text{XV.} \dots w'' + ix'' + jy'' + kz'' = (w' + ix' + jy' + kz')(w + ix + jy + kz);$$

we ought therefore, under these conditions XIV., to have the equation,

$$\text{XVI.} \dots w''^2 + x''^2 + y''^2 + z''^2 = (w'^2 + x'^2 + y'^2 + z'^2)(w^2 + x^2 + y^2 + z^2);$$

which can in fact be verified by so easy an algebraical calculation, that its truth may be said to be obvious upon mere inspection, at least when the terms in the four quadrinomial expressions $w'' \dots z''$ are arranged† as above.

* Compare the first Note to page 131; and that to page 162.

† From having somewhat otherwise *arranged* those terms, the author had some little trouble at first, in verifying that the twenty-four *double products*, in the expansion of $w''^2 + \&c.$, destroy each other, leaving only the sixteen *products of squares*, or that XVI. follows from XIV., when he was led to anticipate that result through quaternions, in the year 1843. He believes, however, that the *algebraic theorem* XVI., as distinguished from the *quaternion formula* XV., with which it is here connected, had been discovered by the celebrated EULER.

223. The principal use which we shall here make of the standard quadrinomial form (221), is to prove by it the general *associative property of multiplication* of quaternions; which can now with great ease be done, the *distributive* property* (212) of such multiplication having been already proved. In fact, if we write, as in 222, (3.),

$$\text{I. . . } \begin{cases} q = w + ix + jy + kz, \\ q' = w' + ix' + jy' + kz', \\ q'' = w'' + ix'' + jy'' + kz'', \end{cases}$$

without now assuming that the relation $q'' = q'q$, or any other relation, exists between the three quaternions q, q', q'' , and inquire whether it be true that the *associative formula*,

$$\text{II. . . } q''q' \cdot q = q'' \cdot q'q,$$

holds good, we see, by the distributive principle, that we have only to try whether this last formula is valid when the three quaternion factors q, q', q'' are replaced, in any one common order on both sides of the equation, and with or without repetition, by the three given right versors ijk ; but this has already been proved, in Art. 183. We arrive then, thus, at the important conclusion, that *the General Multiplication of Quaternions is an Associative Operation*, as it had been previously seen (212) to be a *Distributive* one: although we had also found (168, 183, 191) that *such Multiplication is not* (in general) *Commutative*: or that *the two products, $q'q$ and qq' , are generally unequal*. We may therefore omit the point (as in 183), and may denote each member of the equation II. by the symbol $q''q'q$.

(1.) Let $v = \nabla q, v' = \nabla q', v'' = \nabla q''$; so that v, v', v'' are any three right quaternions, and therefore, by 191, (2.), and 196, 204,

$$\text{--- } Kv'v = vv', \quad Sv'v = \frac{1}{2}(v'v + vv'), \quad Vv'v = \frac{1}{2}(v'v - vv').$$

Let this last right quaternion be called v_s , and let $Sv'v = s_s$, so that $v'v = s_s + v_s$; we shall then have the equations,

* At a later stage, a sketch will be given of at least one proof of this *Associative Principle of Multiplication*, which will not presuppose the *Distributive Principle*.

$$2Vv''v = v''v - vv''; \quad 0 = v''s - sv'';$$

whence, by addition,

$$\begin{aligned} 2Vv''v &= v''v - vv'' \\ &= (v''v + vv'')v - v'(v''v + vv'') \\ &= 2vSv''v - 2v'Sv''v; \end{aligned}$$

and therefore generally, if v, v', v'' be still right, as above,

$$\text{III.} \dots V.v''Vv''v = vSv''v - v'Sv''v;$$

+ imp'

a formula with which the student ought to make himself completely familiar, on account of its extensive utility.

(2.) With the recent notations,

$$V.v''Sv''v = Vv''s = v''s = v'Sv''v;$$

we have therefore this other very useful formula,

$$\text{IV.} \dots V.v''v''v = vSv''v'' - v'Sv''v'' + v''Sv''v'';$$

+ ...

where the point in the first member may often for simplicity be dispensed with; and in which it is still supposed that

$$\angle v = \angle v' = \angle v'' = \frac{\pi}{2}.$$

(3.) The formula III. gives (by 206),

$$\text{V.} \dots \text{IV.} v''Vv''v = Iv.Sv''s - Iv'.Sv''s;$$

hence this last vector, which is evidently *complanar* with the two indices Iv and Iv' , is at the same time (by 208) *perpendicular* to the third index Iv'' , and therefore (by (1.)) *complanar* with the third quaternion q'' .

(4.) With the recent notations, the vector,

$$\text{VI.} \dots Iv = IVv''v = IV(Vq'.Vq),$$

is (by 208, XXII.) a line perpendicular to both Iv and Iv' ; or *common* to the planes of q and q' ; being also such that the rotation round it from Iv' to Iv is *positive*: while its length,

$$T_{Iv}, \text{ or } T_{v}, \text{ or } TV.v''v, \text{ or } TV(Vq'.Vq),$$

bears to the unit of length the same ratio, as that which the parallelogram under the indices, Iv and Iv' , bears to the unit of area.

(5.) To interpret (comp. IV.) the scalar expression,

$$\text{VII.} \dots Sv''v''v = Sv''v, = S.v''Vv''v,$$

(because $Sv''s = 0$), we may employ the formula 208, V.; which gives the the transformation,

$$\text{VIII.} \dots Sv''v''v = Tv''.Tv. \cos(\pi - x);$$

where Tv'' denotes the length of the line Iv'' , and Tv , represents by (4.) the area (positively taken) of the parallelogram under Iv' and Iv ; while x is (by 208), the angle between the two indices Iv'' , Iv . This angle will be obtuse, and therefore the cosine of its supplement will be positive, and equal to the sine of the inclination of the line Iv'' to the plane of Iv and Iv' , if the rotation round Iv'' from Iv' to Iv be negative, that is, if the rotation round Iv from Iv' to Iv'' be positive; but that cosine will be equal the negative of this sine, if the direction of this rotation be reversed. We have therefore the important interpretation:

$$\text{IX.} \dots Sv''v''v = \pm \text{volume of parallelepiped under } Iv, Iv', Iv'';$$

the upper or the lower sign being taken, according as the rotation round Ii , from Iv' to Iv'' , is *positively* or *negatively directed*.

(6.) For example, we saw that the ternary products ijk and kji have scalar values, namely,

$$ijk = -1, \quad kji = +1, \text{ by 188, (1.), (2.)};$$

and accordingly the *parallelepiped of indices* becomes, in this case, an *unit-cube*; while the rotation round the index of i , from that of j to that of k , is *positive* (181).

(7.) In general, for any three *right* quaternions vv'' , we have the formula,

$$X. \dots Sv''v'v = -Sv''v'v;$$

and when the three indices are *coplanar*, so that the *volume* mentioned in IX. *vanishes*, then each of these two last scalars becomes *zero*; so that we may write, as a new *Formula of Coplanarity*;

$$XI. \dots Sv''v'v = 0, \text{ if } Iv'' \parallel Iv', Iv \text{ (123):}$$

while, on the other hand, this scalar cannot vanish in any *other* case, if the quaternions (or their indices) be still supposed to be *actual* (1, 144); because it then represents an actual volume.

(8.) Hence also we may establish the following *Formula of Collinearity*, for any three quaternions:

$$XII. \dots S(Vq'' \cdot Vq' \cdot Vq) = 0, \text{ if } IVq'' \parallel IVq', IVq;$$

that is, by 209, if the *planes* of q, q', q'' have any *common line*.

(9.) In general, if we employ the *standard trinomial form* 221, II., namely,

$$v = Vq = ix + jy + kz, \quad v' = ix' + \&c., \quad v'' = ix'' + \&c.,$$

the laws (182, 183) of the symbols i, j, k give the transformation,

$$XIII. \dots Sv''v'v = x''(x'y - y'x) + y''(x'z - z'x) + z''(y'x - x'y);$$

and accordingly this is the known expression for the volume (with a suitable sign) of the parallelepiped, which has the three lines OP, OP', OP'' for three co-initial edges, if the rectangular co-ordinates* of the four corners, o, P, P', P'' be $000, xyz, x'y'z', x''y''z''$.

(10.) Again, as another important consequence of the general associative property of multiplication, it may be here observed, that although products of *more than two* quaternions have *not* generally *equal scalars*, for *all* possible permutations of the factors, since we have just seen a case X. in which such a change of arrangement produces a change of *sign* in the result, yet *cyclical permutation is permitted*, under the sign S ; or in symbols, that for any three quaternions (and the result is easily extended to any greater number of such factors) the following formula holds good:

$$XIV. \dots Sq''q'q = Sq'q''q'.$$

In fact, to prove this equality, we have only to write it thus,

$$XIV'. \dots S(q''q'.q) = S(q.q''q'),$$

and to remember that the scalar of the product of any *two* quaternions remains unaltered (198, I.), when the order of those two factors is changed.

* This result may serve as an example of the manner in which *quaternions*, although *not based* on any usual doctrine of *co-ordinates*, may yet be employed to *deduce*, or to *recover*, and often with great ease, important co-ordinate expressions.

(11.) In like manner, by 192, II., it may be inferred that

$$\text{XV.} \dots Kq'q = K(q'.q) = Kq'.Kq'' = Kq.Kq'.Kq'',$$

with a corresponding result for any greater number of factors; whence by 192, I., if Πq and $\Pi'q$ denote the *products* of any one set of quaternions taken in two *opposite orders*, we may write,

$$\text{XVI.} \dots K\Pi q = \Pi'Kq; \quad \text{XVII.} \dots R\Pi q = \Pi'Rq.$$

(12.) But if v be *right*, as above, then $Kv = -v$, by 144; hence,

$$\text{XVIII.} \dots K\Pi v = \pm \Pi'v; \quad \text{XIX.} \dots S\Pi v = \pm S\Pi'v; \quad \text{XX.} \dots V\Pi v = \mp V\Pi'v;$$

upper or lower signs being taken, according as the *number* of the *right factors* is *even or odd*; and under the same conditions,

$$\text{XXI.} \dots S\Pi v = \frac{1}{2}(\Pi v \pm \Pi'v); \quad \text{XXII.} \dots V\Pi v = \frac{1}{2}(\Pi v \mp \Pi'v);$$

as was lately exemplified (1.), for the case where the number is *two*.

(13.) For the case where that number is *three*, the four last formulæ give,

$$\text{XXIII.} \dots Sv'v = -Sv'v' = \frac{1}{2}(v'v'v - vv'v);$$

$$\text{XXIV.} \dots Vv'v = +Vv'v' = \frac{1}{2}(v'v'v + vv'v);$$

results which obviously agree with X. and IV.

224. For the case of *Complanar Quaternions* (119), the power of reducing each (120) to the form of a fraction (101) which shall have, at pleasure, for its denominator or for its numerator, any arbitrary line in the given plane, furnishes some peculiar facilities for proving the *commutative* and *associative* properties of *Addition* (207), and the *distributive* and *associative* properties of *Multiplication* (212, 223); while, for this *case* of multiplication of quaternions, we have already seen (191, (1.)) that the *commutative* property *also* holds good, as it does in algebraic multiplication. It may therefore be not irrelevant nor useless to insert here a short Second Chapter on the subject of such *complanars*: in treating briefly of which, while assuming as proved the existence of all the foregoing properties, we shall have an opportunity to say something of Powers and Roots and Logarithms; and of the connexion of Quaternions with Plane Trigonometry, and with Algebraical Equations. After which, in the Third and last Chapter of this Second Book, we propose to resume, for a short time, the consideration of *Diplanar Quaternions*; and especially to show how the *Associative Principle of Multiplication* can be established, for them, *without* employing the Distributive Principle*.

* Compare the Note to page 236.

CHAPTER II.

ON COMPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN ONE PLANE; AND ON POWERS, ROOTS, AND LOGARITHMS OF QUATERNIONS.

SECTION 1.—*On Complanar Proportion of Vectors; Fourth Proportional to Three, Third Proportional to Two, Mean Proportional, Square Root; General Reduction of a Quaternion in a given Plane, to a Standard Binomial Form.*

225. The Quaternions of the present Chapter shall all be supposed to be *complanar* (119); their *common plane* being assumed to coincide with that of the given right versor i (181). And *all the lines*, or vectors, such as α, β, γ , &c., or $\alpha_0, \alpha_1, \alpha_2$, &c., to be here employed, shall be conceived to be *in that given plane* of i ; so that we may write (by 123), for the purposes of this Chapter, the *formulae of complanarity*:

$$q \parallel q' \parallel q'' \dots \parallel i; \quad \alpha \parallel i, \quad \beta \parallel i, \quad \alpha_0 \parallel i, \quad \&c.$$

226. Under these conditions, we can always (by 103, 117) interpret any symbol of the form $(\beta : \alpha) \cdot \gamma$, as denoting a *line* δ in the given plane; which line may also be denoted (125) by the symbol $(\gamma : \alpha) \cdot \beta$, but *not** (comp. 103) by either of the two apparently equivalent symbols, $(\beta \cdot \gamma) : \alpha$, $(\gamma \cdot \beta) : \alpha$; so that we may write,

$$I. \dots \delta = \frac{\beta}{\alpha} \gamma = \frac{\gamma}{\alpha} \beta,$$

and may say that this line δ is the *Fourth Proportional* to the

* In fact the symbols $\beta \cdot \gamma$, $\gamma \cdot \beta$, or $\beta\gamma$, $\gamma\beta$, have not as yet received with us any interpretation; and even when they shall come to be interpreted as representing certain quaternions, it will be found (comp. 168) that the two combinations, $\frac{\beta}{\alpha} \gamma$ and $\frac{\beta\gamma}{\alpha}$, have generally different significations.

three lines α, β, γ ; or to the three lines α, γ, β ; the *two Means*, β and γ , of any such *Complanar Proportion of Four Vectors*, admitting thus of being *interchanged*, as in algebra. Under the same conditions we may write also (by 125),

$$\text{II.} \dots \alpha = \frac{\beta}{\delta} \gamma = \frac{\gamma}{\delta} \beta; \quad \beta = \frac{\alpha}{\gamma} \delta = \frac{\delta}{\gamma} \alpha; \quad \gamma = \frac{\delta}{\beta} \alpha = \frac{\alpha}{\beta} \delta;$$

so that (still as in algebra) the *two Extremes*, α and δ , of any such proportion of four lines $\alpha, \beta, \gamma, \delta$, may likewise change places among themselves: while we may also make the *means* become the *extremes*, if we at the same time change the *extremes* to *means*. More generally, if $\alpha, \beta, \gamma, \delta, \epsilon \dots$ be any *odd number* of vectors in the given plane, we can always find *another* vector ρ in that plane, which shall satisfy the equation,

$$\text{III.} \dots \dots \frac{\epsilon \gamma}{\delta \beta} \alpha = \rho, \quad \text{or} \quad \text{III'.} \dots \dots \frac{\epsilon \gamma \alpha}{\delta \beta \rho} = 1;$$

and when such a formula holds good, for any *one* arrangement of the *numerator-lines* $\alpha, \gamma, \epsilon, \dots$ and of the *denominator-lines* $\rho, \beta, \delta \dots$ it can easily be proved to hold good also for any *other* arrangement of the numerators, and any other arrangement of the denominators. For example, whatever four (complanar) vectors may be denoted by $\beta\gamma\delta\epsilon$, we have the transformations,

$$\text{IV.} \dots \frac{\epsilon \gamma}{\delta \beta} = \frac{\epsilon}{\delta} \gamma : \beta = \frac{\gamma}{\delta} \epsilon : \beta = \frac{\gamma \epsilon}{\delta \beta},$$

the two numerators being thus interchanged. Again,

$$\text{IV'.} \dots \frac{\epsilon \gamma}{\delta \beta} = \frac{\gamma \epsilon}{\beta \delta} = \frac{\epsilon \gamma}{\beta \delta}, \text{ by IV.};$$

so that the two denominators also may change places.

227. An interesting *case* of such *proportion* (226) is that in which the *means coincide*; so that only *three distinct lines*, such as α, β, γ , are involved: and that we have (comp. Art. 149, and Fig. 42) an equation of the form,

$$\text{I.} \dots \gamma = \frac{\beta}{\alpha} \beta, \quad \text{or} \quad \alpha = \frac{\beta}{\gamma} \beta,$$

but not* $\gamma = \beta\beta : a$, nor $a = \beta\beta : \gamma$. In this case, it is said that the *three lines* $a\beta\gamma$ form a *Continued Proportion*; of which a and γ are now the *Extremes*, and β is the *Mean*: this line β being also said to be a† *Mean Proportional* between the two others, a and γ ; while γ is the *Third Proportional* to the two lines a and β ; and a is, at the same time, the third proportional to γ and β . Under the same conditions, we have

$$\text{II. . . } \beta = \frac{a}{\beta} \gamma = \frac{\gamma}{\beta} a;$$

so that this *mean*, β , between a and γ , is also the *fourth proportional* (226) to itself, as first, and to those two other lines. We have also (comp. again 149),

$$\text{III. . . } \left(\frac{\beta}{a}\right)^2 = \frac{\gamma}{a}, \quad \left(\frac{\beta}{\gamma}\right)^2 = \frac{a}{\gamma};$$

whence it is natural to write,

$$\text{IV. . . } \frac{\beta}{a} = \left(\frac{\gamma}{a}\right)^{\frac{1}{2}}, \quad \frac{\beta}{\gamma} = \left(\frac{a}{\gamma}\right)^{\frac{1}{2}},$$

and therefore (by 103),

$$\text{V. . . } \beta = \left(\frac{\gamma}{a}\right)^{\frac{1}{2}} a, \quad \beta = \left(\frac{a}{\gamma}\right)^{\frac{1}{2}} \gamma;$$

although we are *not* here to write $\beta = (\gamma a)^{\frac{1}{2}}$, nor $\beta = (a\gamma)^{\frac{1}{2}}$. But because we have always, as in algebra (comp. 199, (3.)), the equation or identity, $(-q)^2 = q^2$, we are equally well entitled to write,

$$\text{VI. . . } \left(\frac{-\beta}{a}\right)^2 = \frac{\gamma}{a}, \quad \left(\frac{-\beta}{\gamma}\right)^2 = \frac{a}{\gamma}, \quad -\beta = \left(\frac{\gamma}{a}\right)^{\frac{1}{2}} a = \left(\frac{a}{\gamma}\right)^{\frac{1}{2}} \gamma;$$

the symbol $q^{\frac{1}{2}}$ denoting thus, in general, *either of two opposite quaternions*, whereof however one, namely that one of which the *angle is acute*, has been already selected in 199, (1.), as that which shall be called by us *the Square Root* of the quaternion

* Compare the Note to the foregoing Article.

† We say, a *mean proportional*; because we shall shortly see that the *opposite line*, $-\beta$, is in the same sense *another mean*; although a *rule* will presently be given, for distinguishing between them, and for *selecting one*, as that which may be called, by eminence, *the mean proportional*.

q , and denoted by \sqrt{q} . We may therefore establish the formula,

$$\text{VII.} \dots \beta = \pm \sqrt{\left(\frac{\gamma}{a}\right)} \cdot a = \pm \sqrt{\left(\frac{a}{\gamma}\right)} \cdot \gamma,$$

* if a, β, γ form, as above, a continued proportion; the *upper signs* being taken when (as in Fig. 42) the *angle* $\angle OC$, between the extreme lines a, γ , is *bisected* by the line OB , or β , *itself*; but the *lower signs*, when that angle is bisected by the *opposite line*, $-\beta$, or when β bisects the *vertically opposite angle* (comp. again 199, (3.)): but the *proportion of tensors*,

$$\text{VIII.} \dots T\gamma : T\beta = T\beta : Ta,$$

and the resulting formulæ,

$$\text{IX.} \dots T\beta^2 = Ta \cdot T\gamma, \quad T\beta = \sqrt{(Ta \cdot T\gamma)},$$

in *each* case holding good. And when we shall speak simply of the *Mean Proportional between two vectors*, a and γ , which make any acute, or right, or obtuse angle with each other, we shall always henceforth understand the *former* of these two bisectors; namely, the *bisector* OB of that *angle* $\angle OC$ *itself*, and *not* that of the *opposite* angle: thus taking upper signs, in the recent formula VII.

(1.) At the limit when the angle $\angle OC$ *vanishes*, so that $U\gamma = Ua$, then $U\beta =$ each of these two unit-lines; and the mean proportional β has the *same common direction* as each of the two given extremes. This comes to our agreeing to write,

$$\text{X.} \dots \sqrt{1} = +1, \quad \text{and generally,} \quad \text{XI.} \dots \sqrt{a^2} = +a,$$

if a be any positive scalar.

(2.) At the *other* limit, when $\angle OC = \pi$, or $U\gamma = -Ua$, the *length* of the mean proportional β is still determined by IX., as the *geometric mean* (in the usual sense) between the lengths of the two given extremes (comp. the two Figures 41); but, even with the supposed restriction (225) on the *planes* in which all the lines are situated, an *ambiguity* arises in this case, from the doubt *which* of the two *opposite perpendiculars* at O , to the line $\angle OC$, is to be taken as the *direction of the mean vector*. To remove this ambiguity, we shall suppose that the *rotation* round the axis of i (to which *axis* all the lines considered in this Chapter are, by 225, perpendicular), from the first line OA to the second line OB , is in this case *positive*; which supposition is equivalent to writing, for present purposes,

$$\text{XI.} \dots \sqrt{-1} = +i; \quad \text{and} \quad \text{XI.} \dots \sqrt{-a^2} = ia, \quad \text{if } a > 0.$$

* It is to be carefully observed that *this square root of negative unity* is *not*, in any sense, *imaginary*, nor even *ambiguous*, in its geometrical interpretation, but denotes a *real and given right versor* (181).

And thus the *mean proportional* between two vectors (in the given plane) becomes, in *all* cases, *determined*: at least if their *order* (as first and third) be given.

(8.) If the *restriction* (225) on the *common plane* of the lines, were *removed*, we might *then*, on the recent plan (227), fix *definitely* the *direction*, as well as the *length*, of the *mean* OB , in *every* case *but one*: this excepted case being that in which, as in (2.), the *two given extremes*, OA, OC , have exactly opposite directions; so that the *angle* ($\angle AOC = \pi$) between them has *no one definite bisector*. In this case, the sought point B would have *no one determined position*, but only a *locus*: namely the *circumference of a circle*, with O for centre, and with a radius equal to the geometric mean between $\overline{OA}, \overline{OC}$, while its plane would be perpendicular to the given right line AOC . (Comp. again the Figures 41; and the remarks in 148, 149, 153, 154, on the *square of a right radial*, or *versor*, and on the partially indeterminate character of the *square root of a negative scalar*, when interpreted, on the plan of this Calculus, as a *real* in geometry.)

228. The quotient of any two coplanar and right quaternions has been seen (191, (6.)) to be a scalar; since then we here suppose (225) that $q \parallel i$, we are at liberty to write,

$$I. \dots Sq = x; \quad Vq : i = y; \quad Vq = yi = iy;$$

and consequently may establish the following *Reduction of a Quaternion in the given Plane* (of i) to a *Standard Binomial Form** (comp. 221):

$$II. \dots q = x + iy, \quad \text{if } q \parallel i;$$

x and y being some *two scalars*, which may be called the *two constituents* (comp. again 221) of this binomial. And then an *equation* between *two* quaternions, considered as binomials of this form, such as the equation,

$$III. \dots q' = q, \quad \text{or} \quad III'. \dots x' + iy' = x + iy,$$

breaks up (by 202, (5.)) into *two scalar equations* between their respective *constituents*, namely,

$$IV. \dots x' = x, \quad y' = y,$$

notwithstanding the *geometrical reality* of the right versor, i .

(1.) On comparing the recent equations II., III., IV., with those marked as III., V., VI., in 221, we see that, in thus passing from *general* to *coplanar* quaternions, we have merely *suppressed the coefficients of j and k* , as being for our present purpose, *null*; and have then written x and y , instead of w and x .

* It is *permitted*, by 227, XI., to write this expression as $x + y\sqrt{-1}$; but the form $x + iy$ is shorter, and perhaps less liable to any ambiguity of interpretation.

(2.) As the word "binomial" has other meanings in algebra, it may be convenient to call the form II. a **COUPLE**; and the two constituent scalars x and y , of which the values serve to distinguish one such couple from another, may not unnaturally be said to be the *Co-ordinates* of that *Couple*, for a reason which it may be useful to state.

(3.) Conceive, then, that the plane of Fig. 50 coincides with that of i , and that positive rotation round $Ax.i$ is, in that Figure, directed towards the *left-hand*; which may be reconciled with our general convention (127), by imagining that this axis of i is directed from o towards the *back* of the Figure; or *below** it, if horizontal. This being assumed, and perpendiculars BB' , BB'' being let fall (as in the Figure) on the indefinite line oA itself, and on a normal to that line at o , which normal we may call oA' , and may suppose it to have a length equal to that of oA , with a left-handed rotation AOA' , so that

$$V. \dots oA' = i.oA, \text{ or briefly, } V' \dots a' = ia,$$

while $\beta' = oB'$, and $\beta'' = oB''$, as in 201, and $q = \beta : a$, as in 202;

then, on whichever side of the indefinite right line oA the point B may be situated, a comparison of the quaternion q with the binomial form II. will give the two equations,

$$VI. \dots x (= Sq) = \beta' : a; \quad y (= \sqrt{q} : i = \beta'' : ia) = \beta'' : a';$$

so that *these two scalars, x and y , are precisely the two rectangular co-ordinates of the point B , referred to the two lines oA and oA' , as two rectangular unit-axes, of the ordinary (or Cartesian) kind.* And since every other quaternion, $q' = x' + iy'$, in the given plane, can be reduced to the form $\gamma : a$, or $oC : oA$, where o is a point in that plane, which can be projected into o' and o'' in the same way (comp. 197, 205), we see that the two *new scalars*, or constituents, x' and y' , are simply (for the same reason) the *co-ordinates of the new point C* , referred to the same pair of axes.

(4.) It is evident (from the principles of the foregoing Chapter), that if we thus express as *couples* (2.) any two coplanar quaternions, q and q' , we shall have the following general transformations for their *sum, difference, and product*:

$$VII. \dots q' \pm q = (x' \pm x) + i(y' \pm y);$$

$$VIII. \dots q' . q = (x'x - y'y) + i(x'y + y'x).$$

(5.) Again, for any *one* such couple, q , we have (comp. 222) not only $Sq = x$, and $\sqrt{q} = iy$, as above, but also,

$$IX. \dots Kq = x - iy; \quad X. \dots Nq = x^2 + y^2; \quad XI. \dots Tq = \sqrt{(x^2 + y^2)};$$

$$XII. \dots Uq = \frac{x + iy}{\sqrt{(x^2 + y^2)}}; \quad XIII. \dots \frac{1}{q} = \frac{x - iy}{x^2 + y^2}; \text{ \&c.}$$

(6.) Hence, for the *quotient* of any two such couples, we have,

$$XIV. \dots \begin{cases} \frac{q'}{q} = \frac{x' + iy'}{x + iy} = \frac{x'' + iy''}{x^2 + y^2}, & x'' + iy'' = q'Kg, \\ x'' = x'x + y'y, & y'' = y'x - x'y. \end{cases}$$

* Compare the second Note to page 108.

(7.) The *law of the norms* (191, (8.)), or the formula, $Nq'q = Nq' \cdot Nq$, is expressed here (comp. 222, (8.)) by the well-known algebraic equation, or identity,

$$\text{XV.} \dots (x'^2 + y'^2)(x^2 + y^2) = (x'x - y'y)^2 + (x'y + y'x)^2;$$

in which $xyz'y'$ may be any four scalars.

SECTION 2.—On Continued Proportion of Four or more Vectors; Whole Powers and Roots of Quaternions; and Roots of Unity.

229. The conception of *continued proportion* (227) may easily be extended from the case of *three* to that of *four* or more (complanar) vectors; and thus a theory may be formed of *cubes* and *higher whole powers of quaternions*, with a correspondingly extended theory of *roots* of quaternions, including roots of *scalars*, and in particular of *unity*. Thus if we suppose that the *four* vectors $a\beta\gamma\delta$ form a continued proportion, expressed by the formulæ,

$$\text{I.} \dots \frac{\delta}{\gamma} = \frac{\gamma}{\beta} = \frac{\beta}{a}, \quad \text{whence} \quad \text{II.} \dots \frac{\delta}{a} = \frac{\delta}{\gamma} \frac{\gamma}{\beta} \frac{\beta}{a} = \left(\frac{\beta}{a}\right)^3,$$

(by an obvious extension of usual algebraic notation,) we may say that the quaternion $\delta : a$ is *the cube*, or the *third power*, of $\beta : a$; and that the latter quaternion is, conversely, a *cube-root* (or *third root*) of the former; which last relation may naturally be denoted by writing,

$$\text{III.} \dots \frac{\beta}{a} = \left(\frac{\delta}{a}\right)^{\frac{1}{3}}, \quad \text{or} \quad \text{III'}. \dots \beta = \left(\frac{\delta}{a}\right)^{\frac{1}{3}} a \quad (\text{comp. 227, IV., V.}).$$

230. But it is important to observe that as the equation $q^3 = Q$, in which q is a sought and Q is a given quaternion, was found to be satisfied by *two* opposite quaternions q , of the form $\pm \sqrt[3]{Q}$ (comp. 227, VII.), so the slightly less simple equation $q^3 = Q$ is satisfied by *three* distinct and real quaternions, if Q be actual and real; whereof *each*, divided by *either* of the other two, gives for *quotient* a *real* quaternion, which is equal to *one of the cube-roots of positive unity*. In fact, if we conceive (comp. the annexed Fig. 54) that β' and β'' are two other but equally long vectors in the given plane, ob-

tained from β by two successive and positive rotations, each through the third part of a circumference, so that

$$\text{IV.} \dots \frac{\beta}{\beta''} = \frac{\beta'}{\beta'} = \frac{\beta'}{\beta'}$$

or

$$\text{IV}' \dots \frac{\beta}{\beta'} = \frac{\beta'}{\beta''} = \frac{\beta''}{\beta'}$$

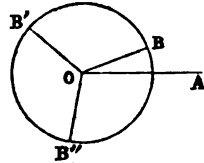


Fig. 54.

and therefore

$$\text{V.} \dots \left(\frac{\beta'}{\beta}\right)^3 = \left(\frac{\beta''}{\beta}\right)^3 = 1, \text{ while V}' \dots \frac{\beta''}{\beta} = \left(\frac{\beta'}{\beta}\right)^2, \frac{\beta'}{\beta} = \left(\frac{\beta''}{\beta}\right)^2,$$

we shall have

$$\text{VI.} \dots \left(\frac{\beta'}{a}\right)^3 = \left(\frac{\beta''}{\beta}\right)^3 \left(\frac{\beta}{a}\right)^3 = \frac{\delta}{a}, \text{ and VI}' \dots \left(\frac{\beta''}{a}\right)^3 = \frac{\delta}{a};$$

so that we are equally entitled, at this stage, to write, instead of III. or III', these other equations:

$$\text{VII.} \dots \frac{\beta'}{a} = \left(\frac{\delta}{a}\right)^{\frac{1}{3}}, \beta' = \left(\frac{\delta}{a}\right)^{\frac{1}{3}} a;$$

or

$$\text{VII}' \dots \frac{\beta''}{a} = \left(\frac{\delta}{a}\right)^{\frac{1}{3}}, \beta'' = \left(\frac{\delta}{a}\right)^{\frac{1}{3}} a.$$

231. A (real and actual) quaternion Q may thus be said to have *three* (real, actual, and) *distinct cube-roots*; of which however only *one* can have an angle less than sixty degrees; while *none* can have an angle equal to sixty degrees, unless the proposed quaternion Q degenerates into a *negative scalar*. In every other case, *one of the three* cube-roots of Q , or one of the three values of the symbol $Q^{\frac{1}{3}}$, may be considered as *simpler* than either of the other two, because it has a *smaller angle* (comp. 199, (1.)); and if we, for the present, denote this one, which we shall call the *Principal Cube-Root* of the quaternion Q , by the symbol $\surd Q$, we shall thus be enabled to establish the formula of inequality,

$$\text{VIII.} \dots \angle \surd Q < \frac{\pi}{3}, \text{ if } \angle Q < \pi.$$

232. At the limit, when Q degenerates, as above, into a negative scalar, *one of its cube-roots is itself* a negative scalar, and has there-

fore its angle = π ; while *each* of the two other roots has its angle = $\frac{\pi}{3}$. In *this case*, among these two roots of which the angles are equal to each other, and are less than that of the third, we shall consider as *simpler*, and therefore as *principal*, the one which answers (comp. 227, (2.)) to a *positive rotation* through sixty degrees; and so shall be led to write,

$$\text{IX.} \dots \sqrt[3]{-1} = \frac{1+i\sqrt{3}}{2}; \quad \text{and} \quad \text{X.} \dots \angle \sqrt[3]{-1} = \frac{\pi}{3};$$

using thus the *positive sign* for the radical $\sqrt{3}$, by which i is multiplied in the expression IX. for $2\sqrt[3]{-1}$; with the connected formula,

$$\text{IX}'. \dots \sqrt[3]{(-a^3)} = \frac{a}{2}(1+i\sqrt{3}), \quad \text{if } a > 0;$$

although it might at first have seemed more natural to adopt as *principal* the *scalar value*, and to write thus,

$$\sqrt[3]{-1} = -1;$$

which latter is in fact *one value* of the symbol, $(-1)^{\frac{1}{3}}$.

(1.) We have, however, on the present plan, as in arithmetic,

$$\text{XI.} \dots \sqrt[3]{1} = 1; \quad \text{and} \quad \text{XI}'. \dots \sqrt[3]{(a^3)} = a, \quad \text{if } a > 0.$$

(2.) The equations,

$$\text{XII.} \dots \left(\frac{1+i\sqrt{3}}{2}\right)^3 = -1, \quad \text{and} \quad \text{XIII.} \dots \left(\frac{-1+i\sqrt{3}}{2}\right)^3 = +1,$$

can be verified in *calculation*, by actual *cubing*, exactly as in algebra; the only difference being, as regards the *conception* of the subject, that although i satisfies the equation $i^2 = -1$, it is regarded *here* as altogether *real*; namely, as a real *right vector** (181).

233. There is no difficulty in conceiving how the same general principles may be extended (comp. 229) to a *continued proportion* of $n + 1$ coplanar vectors,

$$\text{I.} \dots a, a_1, a_2, \dots a_n,$$

* *This conception differs fundamentally* from one which had occurred to several able writers, *before* the invention of the quaternions; and according to which the symbols 1 and $\sqrt{-1}$ were interpreted as representing a *pair of equally long and mutually rectangular right lines, in a given plane*. In *Quaternions*, no line is represented by the number, ONE, except as regards its *length*; the *reason* being, mainly, that we require, in the present Calculus, to be able to deal with *all possible planes*; and that *no one right line is common to all such*.

when n is a whole number greater than three; nor in interpreting, in connexion therewith, the equations,

$$\text{II.} \dots \frac{a_n}{a} = \left(\frac{a_1}{a}\right)^n; \quad \text{III.} \dots \frac{a_1}{a} = \left(\frac{a_n}{a}\right)^{\frac{1}{n}}; \quad \text{IV.} \dots a_1 = \left(\frac{a_n}{a}\right)^{\frac{1}{n}} a.$$

Denoting, for the moment, what we shall call the *principal* n^{th} root of a quaternion Q by the symbol $\sqrt[n]{Q}$, we have, on this plan (comp. 231, VIII.),

$$\text{V.} \dots \angle \sqrt[n]{Q} < \frac{\pi}{n}, \quad \text{if } \angle Q < \pi;$$

$$\text{VI.} \dots \angle (\sqrt[n]{-1}) = \frac{\pi}{n}; \quad \text{VII.} \dots \angle (\sqrt[n]{-1}) : i > 0;$$

this last condition, namely that there shall be a *positive* (scalar) coefficient y of i , in the *binomial* (or *couple*) form $x + iy$ (228), for the quaternion $\sqrt[n]{-1}$, thus serving to complete the determination of that *principal* n^{th} root of *negative unity*; or of any *other* negative scalar, since -1 may be changed to $-a$, if $a > 0$, in each of the two last formulæ. And as to the *general* n^{th} root of a quaternion, we may write, on the same principles,

$$\text{VIII.} \dots Q^{\frac{1}{n}} = 1^{\frac{1}{n}} \cdot \sqrt[n]{Q};$$

where the factor $1^{\frac{1}{n}}$, representing the *general* n^{th} root of *positive unity*, has n different values, depending on the division of the circumference of a circle into n equal parts, in the way lately illustrated, for the case $n=3$, by Figure 54; and only differing from ordinary algebra by the *reality* here attributed to i . In fact, each of these n^{th} roots of unity is with us a *real versor*; namely the *quotient of two radii of a circle*, which make with each other an *angle*, equal to the n^{th} part of some whole number of circumferences.

(1.) We propose, however, to interpret the particular symbol $i^{\frac{1}{n}}$, as always denoting the *principal value* of the n^{th} root of i ; thus writing,

$$\text{IX.} \dots i^{\frac{1}{n}} = \sqrt[n]{i};$$

whence it will follow that when this root is expressed under the form of a *couple* (228), the two constituents x and y shall both be positive, and the quotient $y : x$ shall have a smaller value than for any other couple $x + iy$ (with constituents thus positive), of which the n^{th} power equals i .

(2.) For example, although the equation

$$q^n = (x + iy)^n = i,$$

is satisfied by the two values, $\pm(1 + i) : \sqrt{2}$, we shall write *definitely*,

$$\text{X.} \dots i^2 = +\sqrt{i} = \frac{1+i}{\sqrt{2}}.$$

(3.) And although the equation,

$$q^2 = (x + iy)^2 = i,$$

is satisfied by the *three* distinct and real couples, $(i \pm \sqrt{3})$: 2, and $-i$, we shall adopt only the *one* value,

$$\text{XI.} \dots i^{\frac{1}{2}} = \sqrt[3]{i} = \frac{i + \sqrt{3}}{2}.$$

(4.) In general, we shall thus have the expression,

$$\text{XII.} \dots i^{\frac{1}{n}} = \cos \frac{\pi}{2n} + i \sin \frac{\pi}{2n};$$

which we shall occasionally *abridge* to the following:

$$\text{XII'.} \dots i^{\frac{1}{n}} = \text{cis } \frac{\pi}{2n};$$

and this root, $i^{\frac{1}{n}}$, thus interpreted, denotes a *versor*, which *turns* any line on which it operates, through an angle equal to the n^{th} part of a right angle, in the positive direction of rotation, round the given axis of i .

234. If m and n be any two positive whole numbers, and q any quaternion, the definition contained in the formula 233, II., of the *whole power*, q^n , enables us to write, as in algebra, the two equations:

$$\text{I.} \dots q^m q^n = q^{m+n}; \quad \text{II.} \dots (q^n)^m = q^{mn};$$

and we propose to extend the former to the case of *null* and *negative* whole exponents, writing therefore,

$$\text{III.} \dots q^0 = 1; \quad \text{IV.} \dots q^{m-n} = q^m : q^n;$$

and in particular,

$$\text{V.} \dots q^{-1} = 1 : q = \frac{1}{q} = \text{reciprocal}^* \text{ (134) of } q.$$

We shall also extend the formula II., by writing

$$\text{VI.} \dots (q^{\frac{1}{n}})^m = q^{\frac{m}{n}},$$

whether m be positive or negative; so that this last symbol, if m and n be still *whole* numbers, whereof n may be supposed to be *positive*, has as many distinct values as there are *units* in the denominator of its fractional exponent, when reduced to its

* Compare the Note to page 121.

least terms; among which values of $q^{\frac{m}{n}}$, we shall naturally consider as the *principal* one, that which is the m^{th} power of the principal n^{th} root (233) of q .

(1.) For example, the symbol $q^{\frac{1}{2}}$ denotes, on this plan, the *square of any cube-root of q* ; it has therefore *three* distinct values, namely, the three values of the *cube-root of the square* of the same quaternion q ; but among these we regard as principal, the *square of the principal cube-root* (231) of that proposed quaternion.

(2.) Again, the symbol $q^{\frac{1}{4}}$ is interpreted, on the same plan, as denoting the *square of any fourth root of q* ; but because $(1^{\frac{1}{2}})^2 = 1^{\frac{1}{2}} = \pm 1$, this *square has only two* distinct values, namely those of the *square root $q^{\frac{1}{2}}$* , the fractional exponent $\frac{1}{4}$ being thus reduced to its least terms; and among these the *principal* value is the *square of the principal fourth root*, which square is, at the same time, the *principal square root* (199, (1.), or 227) of the quaternion q .

(3.) The symbol q^{-1} denotes, as in algebra, the reciprocal of a square-root of q ; while q^{-2} denotes the reciprocal of the square, &c.

(4.) If the exponent t , in a symbol of the form q^t , be still a *scalar*, but a *surd* (or incommensurable), we may consider this *surd exponent, t* , as a *limit*, towards which a *variable fraction tends*: and the symbol itself may then be interpreted as the corresponding limit of a *fractional power* of a quaternion, which has however (in this case) *indefinitely many values*, and can therefore be of little or no use, until a *selection* shall have been made, of *one* value of this *surd power* as *principal*, according to a law which will be best understood by the introduction of the conception of the *amplitude* of a quaternion, to which in the next Section we shall proceed.

(5.) Meanwhile (comp. 233, (4.)), we may already *definitely interpret* the symbol i^t as denoting a *versor*, which *turns* any line in the given plane, *through t right angles*, round Ax. i , in the positive or negative direction, according as this *scalar exponent, t* , whether rational or irrational, is itself positive or negative; and thus may establish the formula,

$$\text{VII. . . } i^t = \cos \frac{t\pi}{2} + i \sin \frac{t\pi}{2};$$

or briefly (comp. 233, XII'),

$$\text{VIII. . . } i^t = \text{cis } \frac{t\pi}{2}.$$

SECTION 3.—*On the Amplitudes of Quaternions in a given Plane; and on Trigonometric Expressions for such Quaternions, and for their Powers.*

235. Using the binomial or *couple form* (228) for a quaternion in the plane of i (225), if we introduce two new and real scalars, r and z , whereof the former shall be supposed to be positive, and which are connected with the two former scalars x and y by the equations,

$$\text{I. . . } x = r \cos z, \quad y = r \sin z, \quad r > 0,$$

we shall then evidently have the formulæ (comp. 228, (5.)) :

$$\text{II. . . } Tq = T(x + iy) = r;$$

$$\text{III. . . } Uq = U(x + iy) = \cos z + i \sin z;$$

which last expression may be conveniently abridged (comp. 233, XII', and 234, VIII.) to the following :

$$\text{IV. . . } Uq = \text{cis } z; \quad \text{so that} \quad \text{V. . . } q = r \text{cis } z.$$

And the arcual or angular quantity, z , may be called the *Amplitude** of the quaternion q ; this name being here preferred by us to "*Angle*," because we have already appropriated the latter name, and the corresponding symbol $\angle q$, to denote (130) an *angle of the Euclidean kind*, or at least one not exceeding, in either direction, the *limits* 0 and π ; whereas the *amplitude*, z , considered as obliged only to satisfy the equations I., may have *any real and scalar value*. We shall denote this amplitude, at least for the present, by the *symbol*, † $\text{am. } q$, or simply, $\text{am } q$; and thus shall have the following formula, of *connexion between amplitude and angle*,

$$\text{VI. . . } (z =) \text{am. } q = 2\pi n \pm \angle q;$$

* Compare the Note to Art. 180.

† The symbol V was spoken of, in 202, as completing the *system of notations* peculiar to the present Calculus; and in fact, besides the *three letters*, i, j, k , of which the laws are expressed by the *fundamental formula* (Δ) of Art. 188, and which were originally (namely in the year 1843, and in the two following years) the *only peculiar symbols of quaternions* (see Note to page 160), that Calculus does not *habitually* employ, with peculiar significations, any more than the *five characteristics of operation*, K, S, T, U, V, for *conjugate*, *scalar*, *tensor*, *versor*, and *vector* (or *right part*): although perhaps the mark N for *norm*, which in the present work has been adopted from the *Theory of Numbers*, will gradually come more into use than it has yet done, in connexion with quaternions also. As to the marks, \angle , Ax., I, R, and now am. (or am.), for *angle*, *axis*, *index*, *reciprocal*, and *amplitude*, they are to be considered as chiefly available for the present *exposition* of the system, and as not often wanted, nor employed, in the subsequent *practice* thereof; and the same remark applies to the recent *abridgment* cis , for $\cos + i \sin$; to some notations in the present Section for *powers* and *roots*, serving to express the conception of one n^{th} root, &c., as *distinguished* from another; and to the characteristic P, of what we shall call in the next section the *potential* of a quaternion, though not requiring that notation afterwards. No apology need be made for employing the purely *geometrical signs*, \perp , \parallel , \llcorner , for *perpendicularity*, *parallelism*, and *complanarity*: although the *last* of them was perhaps first introduced by the present writer, who has found it frequently useful.

the upper or the lower *sign* being taken, according as $Ax. q = \pm Ax. i$; and n being *any whole number*, positive or negative or null. We may then write also (for any quaternion $q ||| i$) the general transformations following :

$$\text{VII.} \dots Uq = \text{cis am } q; \quad \text{VIII.} \dots q = Tq \cdot \text{cis am } q.$$

(1.) Writing $q = \beta : a$, the amplitude $\text{am } q$, or $\text{am } (\beta : a)$, is thus a scalar quantity, expressing (*with its proper sign*) the amount of rotation, round $Ax. i$, from the line a to the line β ; and admitting, in general, of being increased or diminished by *any whole number of circumferences*, or of entire revolutions, when only the directions of the two lines, a and β , in the given plane of i , are given.

(2.) But the particular quaternion, or right versor, i itself, shall be considered as having *definitely*, for its amplitude, *one right angle*; so that we shall establish the particular formula,

$$\text{IX.} \dots \text{am } i = \angle i = \frac{\pi}{2}.$$

(3.) When, for any other given quaternion q , the generally arbitrary integer n in VI. receives any one determined value, the corresponding value of the amplitude may be denoted by either of the two following temporary symbols,* which we here treat as equivalent to each other,

$$\text{am}_n \cdot q, \quad \text{or} \quad \angle_n q;$$

so that (with the same rule of signs as before) we may write, as a more definite formula than VI., the equation :

$$\text{X.} \dots \text{am}_n \cdot q = \angle_n q = 2n\pi \pm \angle q;$$

and may say that this last quantity is the n^{th} value of the amplitude of q ; while the zero-value, $\text{am}_0 q$, may be called the *principal amplitude* (or the *principal value* of the amplitude).

(4.) With these notations, and with the convention, $\text{am}_0(-1) = +\pi$, we may write,

$$\text{XI.} \dots \text{am}_0 q = \angle_0 q = \pm \angle q;$$

$$\text{XII.} \dots \text{am}_n a = \text{am}_n 1 = \angle_n 1 = 2n\pi, \quad \text{if } a > 0;$$

and

$$\text{XIII.} \dots \text{am}_n(-a) = \text{am}_n(-1) = \angle_n(-1) = (2n+1)\pi,$$

if a be still a positive scalar.

236. From the foregoing definition of amplitude, and from the formerly established connexion of *multiplication of versors* with *composition of rotations* (207), it is obvious that (within the given plane, and with abstraction made of tensors) *multiplication and division of quaternions* answer respectively to

* Compare the recent Note, respecting the notations employed.

(algebraical) *addition and subtraction of amplitudes*: so that, if the symbol $\text{am}.q$ be interpreted in the *general* (or indefinite) sense of the equation 235, VI., we may write:

I. . . $\text{am}(q'.q) = \text{am} q' + \text{am} q$; II. . . $\text{am}(q':q) = \text{am} q' - \text{am} q$; implying hereby that, in each formula, *one* of the values of the first member is *among* the values of the second member; but not here specifying *which* value. With the same generality of signification, it follows evidently that, for a *product* of any number of (complanar) quaternions, and for a *whole power* of any one quaternion, we have the analogous formulæ:

$$\text{III. . . } \text{am } \Pi q = \Sigma \text{ am } q; \quad \text{IV. . . } \text{am } q^p = p . \text{am } q;$$

where the exponent p may be any positive or negative integer, or zero.

(1.) It was proved, in 191, II., that for any two quaternions, the formula $Uq'q = Uq'.Uq$ holds good; a result which, by the associative principle of multiplication (223), is easily extended to any number of quaternion factors (complanar or diplanar), with an analogous result for tensors: so that we may write, generally,

$$\text{V. . . } U\Pi q = \Pi Uq; \quad \text{VI. . . } T\Pi q = \Pi Tq.$$

(2.) Confining ourselves to the first of these two equations, and combining it with III., and with 235, VII., we arrive at the important formula:

$$\text{VII. . . } \Pi \text{ cis am } q (= \Pi Uq = U\Pi q = \text{cis am } \Pi q) = \text{cis } \Sigma \text{ am } q;$$

whence in particular (comp. IV.),

$$\text{VIII. . . } (\text{cis am } q)^p = \text{cis}(p . \text{am } q),$$

at least if the exponent p be still any whole number.

(3.) In these last formulæ, the amplitudes $\text{am}.q$, $\text{am}.q'$, &c., may represent any angular quantities, z , z' , &c.; we may therefore write them thus,

$$\text{IX. . . } \Pi \text{ cis } z = \text{cis } \Sigma z; \quad \text{X. . . } (\text{cis } z)^p = \text{cis } pz;$$

including thus, under *abridged forms*, some known and useful theorems, respecting *cosines and sines of sums and multiples of arcs*.

(4.) For example, if the number of factors of the form $\text{cis } z$ be two, we have thus,

$$\text{IX'. . . } \text{cis } z' . \text{cis } z = \text{cis}(z' + z); \quad \text{X'. . . } (\text{cis } z)^2 = \text{cis } 2z;$$

whence

$$\begin{aligned} \cos(z' + z) &= S(\text{cis } z' . \text{cis } z) = \cos z' \cos z - \sin z' \sin z; \\ \sin(z' + z) &= i^{-1}V(\text{cis } z' . \text{cis } z) = \cos z' \sin z + \sin z' \cos z; \\ \cos 2z &= (\cos z)^2 - (\sin z)^2; \quad \sin 2z = 2 \cos z \sin z; \end{aligned}$$

with similar results for more factors than two.

(5.) Without expressly introducing the conception, or at least the notation of *amplitude*, we may derive the recent formulæ IX. and X., from the consideration of the power i^t (234), as follows. That power of i , with a scalar exponent, t , has been

interpreted in 234, (5.), as a symbol satisfying an equation which may be written thus:

$$\text{XI.} \dots i^z = \text{cis } z, \text{ if } z = \frac{1}{2}t\pi;$$

or geometrically as a *versor*, which *turns a line through t right angles*, where *t* may be any scalar. We see then at once, from this interpretation, that if *t'* be either the same or any other scalar, the formula,

$$\text{XII.} \dots i^{t'} = i^{t' t}, \text{ or } \text{XIII.} \dots \Pi. i^z = i^{z^2},$$

must hold good, as in algebra. And because the number of the factors *i* is easily seen to be arbitrary in this last formula, we may write also,

$$\text{XIV.} \dots (i^t)^p = i^{t^p};$$

if *p* be any whole* number. But the two last formulæ may be changed by XI., to the equations IX. and X., which are therefore thus again obtained; although the later forms, namely XIII. and XIV., are perhaps somewhat simpler: having indeed the appearance of being mere algebraical identities, although we see that their geometrical interpretations, as given above, are important.

(6.) In connexion with the same interpretation XI. of the same useful symbol *i*, it may be noticed here that

$$\text{XV.} \dots \text{K. } i^z = i^{-z};$$

and that therefore,

$$\text{XVI.} \dots \cos \frac{t\pi}{2} = \text{S. } i^z = \frac{1}{2}(i^z + i^{-z});$$

$$\text{XVII.} \dots \sin \frac{t\pi}{2} = i^{-1} \text{V. } i^z = \frac{1}{2}i^{-1}(i^z - i^{-z}).$$

(7.) Hence, by raising the double of each member of XVI. to any positive whole power *p*, halving, and substituting *z* for $\frac{1}{2}t\pi$, we get the equation,

$$\begin{aligned} \text{XVIII.} \dots 2^{p-1}(\cos z)^p &= \frac{1}{2}(i^z + i^{-z})^p = \frac{1}{2}(i^{pz} + i^{-pz}) + \frac{1}{2}p(i^{(p-2)z} + i^{-(p-2)z}) + \&c. \\ &= \cos pz + p \cos(p-2)z + \frac{p(p-1)}{2} \cos(p-4)z + \&c., \end{aligned}$$

with the usual rule for halving the coefficient of $\cos 0z$, if *p* be an even integer; and with analogous processes for obtaining the known expansions of $2^{p-1}(\sin z)^p$, for any positive whole value, even or odd, of *p*; and many other known results of the same kind.

237. If *p* be still a whole number, we have thus the transformation,

$$\text{I.} \dots q^p = (r \text{cis } z)^p = r^p \text{cis } pz = (\text{T}q)^p \text{cis } (p. \text{am } q);$$

in which (comp. 190, 161) the two factors, of the tensor and versor kinds, may be thus written:

$$\text{II.} \dots \text{T}(q)^p = (\text{T}q)^p = \text{T}q^p; \quad \text{III.} \dots \text{U}(q)^p = (\text{U}q)^p = \text{U}q^p;$$

and any value (235) of the amplitude *am. q* may be taken, since *all*

* It will soon be seen that there is a sense, although one not quite so definite, in which this formula holds good, even when the exponent *p* is fractional, or surd; namely, that the second member is then one of the values of the first.

will conduct to one *common* value of this *whole power* q^p . And if, for I., we substitute this slightly different formula (comp. 235, (3.)),

$$\text{IV.} \dots (q^p)_n = Tq^p \cdot \text{cis}(p \cdot \text{am}_n q), \text{ with } p = \frac{m'}{n'}, n' > 0,$$

m', n', n being whole numbers whereof the first is supposed to be *prime* to the second, so that the *exponent* p is here a *fraction in its least terms*, with a *positive denominator* n' , while the factor Tq^p is interpreted as a *positive scalar* (of which the positive or negative logarithm, in any given system, is equal to $p \times$ the logarithm of Tq), then the expression in the second member admits of n' *distinct values*, answering to different values of n ; which are precisely the n' values (comp. 234) of the *fractional power* q^p , on principles already established: the *principal value* of that power corresponding to the value $n = 0$.

(1.) For any value of the integer n , we may say that the symbol $(q^p)_n$, defined by the formula IV., represents the n^{th} value of the power q^p ; such values, however, recurring periodically, when p is, as above, a *fraction*.

(2.) Abridging $(1^p)_n$ to 1^p_n , we have thus, generally, by 235, XII.,

$$\text{V.} \dots 1^p_n = \text{cis } 2pn\pi, \text{ if } p \text{ be any fraction,}$$

a restriction which however we shall soon remove; and in particular,

$$\text{VI.} \dots \text{Principal value of } 1^p = 1^p_0 = 1.$$

(3.) Thus, making successively $p = \frac{1}{2}, p = \frac{1}{3}$, we have

$$\text{VII.} \dots 1^{\frac{1}{2}}_n = \text{cis } n\pi, \quad 1^{\frac{1}{2}}_0 = +1, \quad 1^{\frac{1}{2}}_1 = -1, \quad 1^{\frac{1}{2}}_2 = +1, \text{ \&c.};$$

$$\text{VIII.} \dots 1^{\frac{1}{3}}_n = \text{cis } \frac{2n\pi}{3}, \quad 1^{\frac{1}{3}}_0 = 1, \quad 1^{\frac{1}{3}}_1 = \frac{-1 + i\sqrt{3}}{2}, \quad 1^{\frac{1}{3}}_2 = \frac{-1 - i\sqrt{3}}{2}, \quad 1^{\frac{1}{3}}_3 = 1, \text{ \&c.}$$

(4.) Denoting in like manner the n^{th} value of $(-1)^p$ by the abridged symbol $(-1)^p_n$, we have, on the same plan (comp. 235, XIII.), for any fractional* value of p ,

$$\text{IX.} \dots (-1)^p_n = \text{cis } p(2n + 1)\pi; \text{ whence (comp. 232),}$$

$$\text{X.} \dots (-1)^{\frac{1}{2}}_0 = \text{cis } \frac{\pi}{2} = +i, \quad (-1)^{\frac{1}{2}}_1 = \text{cis } \frac{3\pi}{2} = -i, \quad (-1)^{\frac{1}{2}}_2 = +i, \text{ \&c.};$$

and

$$\text{XI.} \dots (-1)^{\frac{1}{3}}_0 = \frac{1 + i\sqrt{3}}{2}, \quad (-1)^{\frac{1}{3}}_1 = -1, \quad (-1)^{\frac{1}{3}}_2 = \frac{1 - i\sqrt{3}}{2}, \text{ \&c.},$$

these three values of $(-1)^{\frac{1}{3}}$ recurring periodically.

(5.) The formula IV. gives, generally, by V., the transformation,

$$\text{XII.} \dots (q^p)_n = (q^p)_0 \text{cis } 2pn\pi = 1^p_n (q^p)_0;$$

so that the n^{th} value of q^p is equal to the *principal value* of that power of q , multi-

* As before, this restriction is only a temporary one.

plied by the *corresponding value of the same power of positive unity*; and it may be remarked, that if the *base* a be any *positive scalar*, the *principal* p^{th} power, $(a^p)_0$, is simply, by our definitions, the *arithmetical value of* a^p .

(6.) The n^{th} value of the p^{th} power of any *negative scalar*, $-a$, is in like manner equal to the *arithmetical* p^{th} power of the positive opposite, $+a$, multiplied by the corresponding value of the same power of *negative unity*; or in symbols,

$$\text{XIII.} \dots (-a)^p_n = (-1)^p_n (a^p)_0 = (a^p)_0 \text{cis } p(2n+1)\pi.$$

(7.) The formula IV., with its consequences V. VI. IX. XII. XIII., may be extended so as to include, as a *limit*, the case when the *exponent* p being still *scalar*, becomes *incommensurable*, or surd; and although the *number of values* of the power q^p becomes thus *unlimited* (comp. 234, (4.)), yet we can still consider *one* of them as the *principal value* of this (now) *surd power*: namely the value,

$$\text{XIV.} \dots (q^p)_0 = Tq^p \cdot \text{cis}(p \text{ am } q),$$

which answers to the *principal amplitude* (235, (3.)) of the proposed quaternion q .

238. We may therefore consider the *symbol*,

$$q^p,$$

in which the *base*, q , is any *quaternion*, while the *exponent*, p , is any *scalar*, as being now fully *interpreted*; but no interpretation has been as yet assigned to this *other* symbol of the same kind,

$$q^q,$$

in which *both* the base q , and the exponent q' , are supposed to be (generally) *quaternions*, although for the purposes of this Chapter *complanar* (225). To do this, in a way which shall be completely *consistent* with the foregoing conventions and conclusions, or rather which shall *include* and *reproduce* them, for the case where the new *quaternion exponent*, q' , *degenerates* (131) *into a scalar*, will be one main object of the following Section: which however will also contain a theory of *logarithms of quaternions*, and of the *connexion* of both *logarithms* and *powers* with the properties of a certain function, which we shall call the *potential* of a quaternion, and to consider which we next proceed.

SECTION 4.—On the Potential and Logarithm of a Quaternion; and on Powers of Quaternions, with Quaternions for their Exponents.

239. If we consider the polynomial function,

$$\text{I.} \dots P(q, m) = 1 + q_1 + q_2 + \dots q_m,$$

2 L

in which q is any quaternion, and m is any positive whole number, while it is supposed (for conciseness) that

$$\text{II.} \dots q_m = \frac{q^m}{1.2.3\dots m} \left(= \frac{q^m}{\Gamma(m+1)} \right),$$

then it is not difficult to prove that *however great*, but *finite* and *given*, the tensor Tq may be, a *finite number* m can be assigned, for which the inequality

$$\text{III.} \dots T(P(q, m+n) - P(q, m)) < \alpha, \quad \text{if } \alpha > 0,$$

shall be satisfied, *however large* the (positive whole) number n may be, and *however small* the (positive) scalar α , provided that this last is *given*. In other words, if we write (comp. 228),

$$\text{IV.} \dots q = x + iy, \quad P(q, m) = X_m + iY_m,$$

a finite value of the number m can always be assigned, such that the following inequality,

$$\text{V.} \dots (X_{m+n} - X_m)^2 + (Y_{m+n} - Y_m)^2 < \alpha^2,$$

shall hold good, *however large* the number n , and *however small* (but *given* and > 0) the scalar α may be. It follows evidently that *each of the two scalar series*, or succession of scalar functions,

$$\text{VI.} \dots X_0 = 1, \quad X_1 = 1 + x, \quad X_2 = 1 + x + \frac{x^2 - y^2}{2}, \dots X_m, \dots$$

$$\text{VII.} \dots Y_0 = 0, \quad Y_1 = y, \quad Y_2 = y + xy, \dots Y_m, \dots$$

converges ultimately to a fixed and finite limit, whereof the one may be called X_∞ , or simply X , and the latter Y_∞ , or Y , and of which each is a certain *function of the two scalars*, x and y . Writing then

$$\text{VIII.} \dots Q = X_\infty + iY_\infty = X + iY,$$

we must consider this *quaternion* Q (namely the *limit* to which the following *series of quaternions*,

$$\text{IX.} \dots P(q, 0) = 1, \quad P(q, 1) = 1 + q, \quad P(q, 2) = 1 + q + \frac{q^2}{2}, \dots P(q, m), \dots$$

converges ultimately) as being in like manner a certain *function*, which we shall call the *potential function*, or simply *the Potential of* q , in consequence of its possessing certain *exponential properties*; and which may be denoted by any one of the three symbols,

$$P(q, \infty), \quad \text{or } P(q), \quad \text{or simply } Pq.$$

We have therefore the equation,

$$\text{X.} \dots \text{Potential of } q = Q = Pq = 1 + q_1 + q_2 + \dots + q_\infty,$$

with the signification II. of the term q_m .

(1.) In connexion with the convergence of this ponential series, or with the inequality (II.), it may be remarked that if we write (comp. 235) $r = Tq$, and $r_m = Tq_m$, we shall have, by 212, (2.),

$$XI. \dots T(P(q, m + n) - P(q, m)) \leq P(r, m + n) - P(r, m);$$

it is sufficient then to prove that this last difference, or the sum of the n positive terms, r_{m+1}, \dots, r_{m+n} , can be made $< a$. Now if we take a number $p > 2r - 1$, we shall have $r_{p+1} < \frac{1}{2}r_p$, $r_{p+2} < \frac{1}{4}r_{p+1}$, &c., so that a finite number $m > p > 2r - 1$ can be assigned, such that $r_m < a$; and then,

$$XII. \dots P(r, m + n) - P(r, m) < a(1 + 2^{-p} + \dots + 2^{-n}) < a;$$

the asserted inequality is therefore proved to exist.

(2.) In general, if an ascending series with positive coefficients, such as

$$XIII. \dots A_0 + A_1q + A_2q^2 + \&c., \text{ where } A_0 > 0, A_1 > 0, \&c.,$$

be convergent when q is changed to a positive scalar, it will *à fortiori* converge, when q is a quaternion.

240. Let q and q' be any two complanar quaternions, and let q'' be their sum, so that

$$I. \dots q'' = q' + q, \quad q'' \parallel q' \parallel q;$$

then, as in algebra, with the signification 239, II. of q_m , and with corresponding significations of q'_m and q''_m , we have

$$II. \dots q''_m = \frac{(q' + q)^m}{1.2.3\dots m} = q'_m q_0 + q'_{m-1} q_1 + q'_{m-2} q_2 + \dots + q'_0 q_m,$$

where $q_0 = q'_0 = 1$. Hence, writing again $r = Tq$, $r_m = Tq_m$, and in like manner $r' = Tq'$, $r'' = Tq''$, &c., the two differences,

$$III. \dots P(r', m).P(r, m) - P(r'', m),$$

and

$$IV. \dots P(r'', 2m) - P(r', m).P(r, m),$$

can be expanded as sums of positive terms of the form $r'_p r_p$ (one sum containing $\frac{1}{2}m(m+1)$, and the other containing $m(m+1)$ such terms); but, by 239, III., the sum of these two positive differences can be made less than any given small positive scalar a , since

$$V. \dots P(r'', 2m) - P(r', m) < a, \text{ if } a > 0,$$

provided that the number m is taken large enough; each difference, therefore, separately tends to 0, as m tends to ∞ ; a tendency which must exist *à fortiori*, when the tensors, r, r', r'' , are replaced by the quaternions, q, q', q'' . The function Pq is therefore subject to the Exponential Law,

$$VI. \dots P(q' + q) = Pq'.Pq = Pq.Pq', \text{ if } q' \parallel q.$$

(1.) If we write (comp. 237, (5.)),

$$\text{VII.} \dots P1 = \epsilon, \text{ then VIII.} \dots Px = (\epsilon^x)_0 = \text{arithmetical value of } \epsilon^x;$$

where ϵ is the known base of the natural system of logarithms, and x is any scalar. We shall henceforth write simply ϵ^x to denote this *principal* (or arithmetical) value of the x^{th} power of ϵ , and so shall have the simplified equation,

$$\text{VIII}'. \dots Px = \epsilon^x.$$

(2.) Already we have thus a motive for writing, *generally*,

$$\text{IX.} \dots Pq = \epsilon^q;$$

but this formula is *here* to be considered merely as a *definition* of the sense in which we interpret this *exponential symbol*, ϵ^q ; namely as what we have lately called the *ponential function*, Pq , considered as the sum of the infinite but converging *series*, 289, X. It will however be soon seen to be *included* in a *more general definition* (comp. 238) of the symbol q^e .

(3.) For any scalar x , we have by VIII. the transformation :

$$\text{X.} \dots x = 1Px = \text{natural logarithm of ponential of } x.$$

241. The exponential law (240) gives the following general *decomposition of a ponential into factors*,

$$\text{I.} \dots Pq = P(x + iy) = Px.Piy;$$

in which we have just seen that the factor Px is a positive scalar. The other factor, Piy , is easily proved to be a versor, and therefore to be *the versor of* Pq , while Px is the *tensor* of the same ponential; because we have in general,

II. $\dots Pq.P(-q) = P0 = 1$, and III. $\dots PKq = KPq$,
since IV. $\dots (Kq)^m = K(q^m) = (\text{say}) Kq^m$ (comp. 199, IX.);
and therefore, in particular (comp. 150, 158), $\dots Nq = qKq$

$$\text{V.} \dots 1 : Piy = P(-iy) = KPiy, \text{ or VI.} \dots NPiy = 1. \text{ - Hence?}$$

We may therefore write (comp. 240, IX., X.),

$$\text{VII.} \dots TPq = PSq = Px = \epsilon^x; \quad \text{VIII.} \dots x = Sq = 1TPq;$$

$$\text{IX.} \dots UPq = PVq = Piy = \epsilon^y = \text{cis } y \text{ (comp. 235, IV.);}$$

this last transformation being obtained from the two series,

$$\text{X.} \dots SPiy = 1 - \frac{y^2}{2} + \&c. = \cos y;$$

$$\text{XI.} \dots i^{-1}VPiy = y - \frac{y^3}{2.3} + \&c. = \sin y.$$

Hence the ponential Pq may be thus transformed:

$$\text{XII.} \dots Pq = P(x + iy) = \epsilon^x \text{cis } y.$$

(1) If we had not chosen to assume as known the series for cosine and sine, nor to select (at first) any one unit of angle, such as that known one on which their validity depends, we might then have proceeded as follows. Writing

$$\text{XIII.} \dots Piy = fy + i\phi y, \quad f(-y) = +fy, \quad \phi(-y) = -\phi y,$$

we should have, by the exponential law (240),

$$\text{XIV.} \dots f(y+y') = S(Piy \cdot Piy') = fy \cdot fy' - \phi y \cdot \phi y';$$

$$\text{XV.} \dots f(y-y') = fy \cdot fy' + \phi y \cdot \phi y';$$

and then the functional equation, which results, namely,

$$\text{XVI.} \dots f(y+y') + f(y-y') = 2fy \cdot fy',$$

would show that

$$\text{XVII.} \dots fy = \cos \left(\frac{y}{c} \times \text{a right angle} \right),$$

whatever unit of angle may be adopted, provided that we determine the constant c by the condition,

$$\text{XVIII.} \dots c = \text{least positive root of the equation } fy (= SPiy) = 0;$$

or nearly,

$$\text{XVIII}'. \dots c = 1.5708, \text{ as the study of the series* would show.}$$

(2.) A motive would thus arise for representing a right angle by this numerical constant, c ; or for so selecting the angular unit, as to have the equation (π still denoting two right angles),

$$\text{XIX.} \dots \pi = 2c = \text{least positive root of the equation } fy = -1;$$

giving nearly,

$$\text{XIX}'. \dots \pi = 3.14159, \text{ as usual;}$$

for thus we should reduce XVII. to the simpler form,

$$\text{XX.} \dots fy = \cos y.$$

(3.) As to the function ϕy , since

$$\text{XXI.} \dots (fy)^2 + (\phi y)^2 = Piy \cdot P(-iy) = 1,$$

it is evident that $\phi y = \pm \sin y$; and it is easy to prove that the upper sign is to be taken. In fact, it can be shown (without supposing any previous knowledge of cosines or sines) that ϕc is positive, and therefore that

$$\text{XXII.} \dots \phi c = +1, \text{ or } \text{XXIII.} \dots Pic = i;$$

whence

$$\text{XXIV.} \dots \phi y = S \cdot i^{-1} Piy = SPi(y-c) = f(y-c),$$

and

$$\text{XXV.} \dots Piy = fy + if(y-c).$$

If then we replace c by $\frac{\pi}{2}$, we have

* In fact, the value of the constant c may be obtained to this degree of accuracy, by simple interpolation between the two approximate values of the function f ,

$$f(1.5) = +0.070787, \quad f(1.6) = -0.029200;$$

and of course there are artifices, not necessary to be mentioned here, by which a far more accurate value can be found.

XXVI. . . $\phi y = \cos \left(y - \frac{\pi}{2} \right) = \sin y$; and XXVII. . . $Piy = \text{cis } y$, as in IX.

! (4.) The series X. XI. for cosine and sine might thus be *deduced*, instead of being *assumed* as known: and since we have the limiting value,

XXIX. . . $\lim_{y \rightarrow 0} y^{-1} \sin y = \lim_{y \rightarrow 0} y^{-1} i^{-1} \nabla Piy = 1$,

it follows that the *unit of angle*, which thus gives $Piy = \text{cis } y$, is (as usual) the angle subtended at the centre by the *arc equal to radius*; or that the *number π* (or $2c$) is to 1, as the *circumference* is to the *diameter* of a circle.

(5.) If any *other angular unit* had been, for any reason, chosen, then a *right angle* would of course be represented by a *different number*, and not by 1.5708 nearly; but we should *still* have the *transformation*,

XXX. . . $Piy = \text{cis} \left(\frac{y}{c} \times \text{a right angle} \right)$,

though *not* the same *series* as before, for $\cos y$ and $\sin y$.

242. The usual unit being retained, we see, by 241, XII., that

I. . . $P. 2in\pi = 1$, and II. . . $P(q + 2in\pi) = Pq$,

if n be any whole number; it follows, then, that the *inverse potential function*, $P^{-1}q$, or what we may call the *Imponential*, of a given quaternion q , has *indefinitely many values*, which may all be represented by the formula,

III. . . $P_n^{-1}q = lTq + i \text{am}_n q$; $\therefore \dots$

and of which *each* satisfies the equation,

IV. . . $PP_n^{-1}q = q$;

while the one which corresponds to $n=0$ may be called the *Principal Imponential*. It will be found that when the *exponent p* is any *scalar*, the definition already given (237, IV., XII.) for the n^{th} value of the p^{th} power of q enables us to establish the formula,

V. . . $(q^p)_n = P(pP_n^{-1}q)$;

and we now propose to *extend* this last formula, by a *new definition*, to the *more general case* (238), when the *exponent* is a *quaternion q'* : thus writing generally, for any two *complanar quaternions*, q and q' , the *General Exponential Formula*,

VI. . . $(q^q)_n = P(q'P_n^{-1}q)$;

the *principal value* of q^q being still conceived to correspond to $n=0$, or to the *principal amplitude* of q (comp. 235, (3)).

(1.) For example,

$$\text{VII.} \dots (\epsilon^q)_0 = P(qP_0^{-1}\epsilon) = Pq, \text{ because } P_0^{-1}\epsilon = \epsilon = 1;$$

the *ponential* Pq , which we agreed, in 240, (2.), to denote simply by ϵ^q , is therefore now seen to be in fact, by our general definition, the *principal value of that power*, or exponential.

(2.) With the same notations,

$$\text{VIII.} \dots \epsilon^{iy} = \text{cis } y, \quad \cos y = \frac{1}{2}(\epsilon^{iy} + \epsilon^{-iy}), \quad \sin y = \frac{1}{2i}(\epsilon^{iy} - \epsilon^{-iy});$$

these two last only differing from the usual imaginary expressions for cosine and sine, by the geometrical *reality** of the versor i .

(3.) The *cosine and sine of a quaternion* (in the given plane) may now be defined by the equations:

$$\text{IX.} \dots \cos q = \frac{1}{2}(\epsilon^{iq} + \epsilon^{-iq}); \quad \text{X.} \dots \sin q = \frac{1}{2i}(\epsilon^{iq} - \epsilon^{-iq});$$

and we may write (comp. 241, IX.),

$$\text{XI.} \dots \text{cis } q = \epsilon^{iq} = Piq.$$

(4.) With this interpretation of $\text{cis } q$, the exponential properties, 236, IX., X., continue to hold good; and we may write,

$$\text{XII.} \dots (q^p)_n = P(q^p T_p q) \cdot P(iq^p \text{am}_n q) = (T_p q)_0^n \text{cis}(q^p \text{am}_n q);$$

a formula which evidently includes the corresponding one, 237, IV., for the n^{th} value of the p^{th} power of q , when p is scalar.

(5.) The definitions III. and VI., combined with 235, XII., give generally,

$$\text{XIII.} \dots 1_n q^p = (1^q)_n = P.2i\pi q^p; \quad \text{XIV.} \dots (q^p)_n = 1_n q^p \cdot (q^p)_0;$$

this last equation including the formula 237, XII.

(6.) The same definitions give,

$$\text{XV.} \dots P_0^{-i} = \frac{i\pi}{2}; \quad \text{XVI.} \dots (i^p)_0 = \epsilon^{-\frac{\pi}{2}};$$

which last equation agrees with a known interpretation of the symbol,

$$\sqrt{-1}^{p-1},$$

considered as denoting in algebra a real quantity.

(7.) The formula VI. may even be extended to the case where the *exponent* q' is a *quaternion*, which is *not in the given plane of i* , and therefore *not complanar with the base q* ; thus we may write,

$$\text{XVII.} \dots (i^q)_0 = P(jP_0^{-1}i) = P\left(-\frac{k\pi}{2}\right) = -k;$$

but it would be foreign (225) to the plan of this Chapter to enter into any further details, on the subject of the interpretation of the exponential symbol q^p , for this case of *diplanar quaternions*, though we see that there would be no difficulty in treating it, after what has been shown respecting *complanars*.

* Compare 232, (2.), and the Notes to pages 248, 248.

243. As regards the *general logarithm* q' of a quaternion q (in the given plane), we may regard it as any quaternion which satisfies the equation,

$$I. \dots e^{q'} = Pq' = q;$$

and in this view it is simply the *Imponential* $P^{-1}q$, of which the n^{th} value is expressed by the formula 242, III. But the *principal imponential*, which answers (as above) to $n=0$, may be said to be the *principal logarithm*, or simply the *Logarithm*, of the quaternion q , and may be denoted by the symbol,

$$lq;$$

so that we may write,

$$I. \dots lq = P_0^{-1}q = lTq + i \text{ am}_0 q;$$

or still more simply,

$$II. \dots lq = l(Tq \cdot Uq) = lTq + lUq,$$

because $lTUq = l1 = 0$, and therefore,

$$III. \dots lUq = i \text{ am}_0 q.$$

We have thus the two general equations,

$$IV. \dots Slq = lTq; \quad V. \dots Vlq = lUq;$$

in which lTq is still the scalar and natural logarithm of the positive scalar Tq .

(1.) As examples (comp. 235, (2.) and (4.)),

$$VI. \dots li = \frac{1}{2}i\pi; \quad VII. \dots l(-1) = i\pi.$$

(2.) The *general logarithm* of q may be denoted by any one of the symbols,

$$\log \cdot q, \text{ or } \log q, \text{ or } (\log q)_n,$$

this last denoting the n^{th} value; and then we shall have,

$$VIII. \dots (\log q)_n = lq + 2in\pi.$$

(3.) The formula,

$$IX. \dots \log \cdot q'q = \log q' + \log q, \text{ if } q' ||| q,$$

holds good, in the sense that every value of the first member is one of the values of the second (comp. 236).

(4.) *Principal value* of $q^q = e^{q'q}$; and one value of $\log \cdot q^q = q'lq$.

(5.) The *quotient* of two general logarithms,

$$X. \dots (\log q')_n : (\log q)_n = \frac{lq' + 2in'\pi}{lq + 2in\pi},$$

may be said to be the *general logarithm of the quaternion*, q' , to the *complanar quaternion base*, q ; and we see that its expression involves* two arbitrary and independent integers, while its *principal value* may be defined to be $lq' : lq$.

* As the corresponding expression in algebra, according to Graves and Ohm.

SECTION 5.—*On Finite* (or Polynomial) Equations of Algebraic Form, involving Complanar Quaternions; and on the Existence of n Real Quaternion Roots, of any such Equation of the nth Degree.*

244. We have seen (233) that an equation of the form,

$$I. \dots q^n - Q = 0,$$

where n is any given positive integer, and Q is any† given, real, and actual quaternion (144), has always n real, actual, and unequal quaternion roots, q , complanar with Q ; namely, the n distinct and real values of the symbol $Q^{\frac{1}{n}}$ (233, VIII.), determined on a plan lately laid down. This result is, however, included in a much more general *Theorem*, respecting *Quaternion Equations of Algebraic Form*; namely, that if q_1, q_2, \dots, q_n be any n given, real, and complanar quaternions, then the equation,

$$II. \dots q^n + q_1 q^{n-1} + q_2 q^{n-2} + \dots + q_n = 0,$$

has always n real quaternion roots, $q', q'', \dots, q^{(n)}$, and no more in the given plane; of which roots it is possible however that some, or all may become equal, in consequence of certain relations existing between the n given coefficients.

245. As another statement of the same *Theorem*, if we write,

$$I. \dots F_n q = q^n + q_1 q^{n-1} + \dots + q_n,$$

the coefficients $q_1 \dots q_n$ being as before, we may say that every such polynomial function, $F_n q$, is equal to a product of n real, complanar, and linear (or binomial) factors, of the form $q - q'$; or that an equation of the form,

$$II. \dots F_n q = (q - q')(q - q'') \dots (q - q^{(n)}),$$

can be proved in all cases to exist: although we may not be

* By saying finite equations, we merely intend to exclude here equations with infinitely many terms, such as $Pq = 1$, which has been seen (242) to have infinitely many roots, represented by the expression $q = 2i\pi$, where n may be any whole number.

† It is true that we have supposed $Q ||| i$ (225); but nothing hinders us, in any other case, from substituting for i the versor UVQ , and then proceeding as before.

able, with our present methods, to assign expressions for the roots, $q', \dots q^{(n)}$, in terms of the coefficients $q_1, \dots q_n$.

246. Or we may say that there is always a certain system of n real quaternions, $q', \&c.$, $\| \| i$, which satisfies the system of equations, of known algebraic form,

$$\text{III} \dots \begin{cases} q' + q'' + \dots + q^{(n)} = -q_1; \\ q'q'' + q'q''' + q''q''' + \dots = +q_2; \\ q'q''q''' + \dots = -q_3; \&c. \end{cases}$$

247. Or because the difference $F_n q - F_n q'$ is divisible by $q - q'$, as in algebra, under the supposed conditions of *complanarity* (224), it is sufficient to say that *at least one real quaternion q' always exists* (whether we can assign it or not), which satisfies the equation,

$$\text{IV} \dots F_n q' = 0,$$

with the foregoing form (245, I.) of the polynomial function F .

248. Or finally, because the theorem is evidently true for the case $n = 1$, while the case 244, I., has been considered, and the case $q_n = 0$ is satisfied by the supposition $q = 0$, we may, without essential loss of generality, reduce the enunciation to the following:

*Every equation of the form,**

$$\text{I} \dots q(q - q')(q - q'') \dots (q - q^{(n-1)}) = Q,$$

in which q', q'', \dots and Q are any n real and given quaternions in the given plane, whereof at least Q and q' may be supposed actual (144), is satisfied by at least one real, actual, and *complanar* quaternion, q .

* The corresponding form, of the algebraical equation of the n^{th} degree, was proposed by Mourey, in his very ingenious and original little work, entitled *La vraie théorie des Quantités Négatives, et des Quantités prétendues Imaginaires* (Paris, 1828). Suggestions also, towards the geometrical proof of the theorem in the text have been taken from the same work; in which, however, the curve here called (in 251) an *oval* is not perhaps defined with sufficient precision: the *inequality*, here numbered as 251, XII., being not employed. It is to be observed that Mourey's book contains no hint of the present calculus, being confined, like the *Double Algebra* of Prof. De Morgan (London, 1849), and like the earlier work of Mr. Warren (Cambridge, 1828), to questions within the plane: whereas the very conception of the Quaternion involves, as we have seen, a reference to *Tridimensional SPACE*.

249. Supposing that the $m-1$ last of the $n-1$ given quaternions $q' \dots q^{(n-1)}$ vanish, but that the $n-m$ first of them are actual, where m may be any whole number from 1 to $n-1$, and introducing a new real, known, complanar, and actual quaternion q_0 , which satisfies the condition,

$$\text{II.} \dots q_0^m = \frac{Q}{q'q'' \dots q^{(n-m)}},$$

we may write thus the recent equation I.,

$$\text{III.} \dots fq = \left(\frac{q}{q_0}\right)^m \left(\frac{q}{q'} - 1\right) \left(\frac{q}{q''} - 1\right) \dots \left(\frac{q}{q^{(n-m)}} - 1\right) = 1;$$

and may (by 187, 159, 235) decompose it into the two following:

$$\text{IV.} \dots Tf q = 1; \text{ and } \text{V.} \dots Uf q = 1, \text{ or } \text{VI.} \dots am f q = 2p\pi;$$

in which p is some whole number (negatives and zero included).

250. To give a more *geometrical form* to the equation, let λ be any given or assumed line $||| i$, and let it be supposed that α, β, \dots and ρ, σ , or OA, OB, \dots and OP, OS , are $n-m+2$ other lines in the same planes, and that $\phi\rho$ is a known scalar function of ρ , such that

$$\text{VII.} \dots \alpha = q'\lambda, \quad \beta = q''\lambda, \dots \quad \rho = q\lambda, \quad \sigma = q_0\lambda,$$

and

$$\text{VIII.} \dots \phi\rho = fq = \left(\frac{\rho}{\sigma}\right)^m \cdot \frac{\rho-\alpha}{\alpha} \cdot \frac{\rho-\beta}{\beta} \dots = \left(\frac{OP}{OS}\right)^m \cdot \frac{AP}{OA} \cdot \frac{BP}{OB} \dots;$$

the theorem to be proved may then be said to be, that *whatever system of real points, o, A, B, \dots and s , in a given plane, and whatever positive whole number m , may be assumed, or given, there is always at least one real point P , in the same plane, which satisfies the two conditions:*

$$\text{IX.} \dots T\phi\rho = 1; \quad \text{X.} \dots am \phi\rho = 2p\pi.$$

251. Whatever value $\epsilon ||| i$ we may assume for the *versor* (or unit-vector) $U\rho$, there always exists *at least one* value of the *tensor* $T\rho$, which satisfies the condition IX.; because the function $T\phi\rho$ vanishes with $T\rho$, and becomes infinite when $T\rho = \infty$, having varied continuously (although perhaps with fluctuations) in the interval. Attending then only to the *least value* (if there be more than one) of $T\rho$, which thus renders $T\phi\rho$ equal to unity, we can conceive a real, unambiguous, and scalar function $\psi\epsilon$, which shall have the two following properties:

$$\text{XI.} \dots T\phi(\epsilon\psi\epsilon) = 1; \quad \text{XII.} \dots T\phi(x\epsilon\psi\epsilon) < 1, \text{ if } x > 0, < 1.$$

And in this way the equation, or system of equations,

XIII. . . $\rho = \epsilon\psi\iota$, or XIV. . . $U\rho = \iota$, $T\rho = \psi\iota$,

may be conceived to *determine a real, finite, and plane closed curve*, which we shall call generally an *Oval*, and which shall have the two following properties: Ist, *every right line, or ray, drawn from the origin o, in any arbitrary direction within the plane, meets the curve once, but once only*; and IInd, *no one of the $n - m$ other given points A, B, . . . is on the oval, because $\phi\alpha = \phi\beta = \dots = 0$.*

252. This being laid down, let us conceive a point P to perform *one circuit* of the oval, moving in the *positive* direction relatively to the given interior point o; so that, whatever the given direction of the line os may be, the *amplitude* $\text{am}(\rho : \sigma)$, if supposed to vary *continuously*,* will have *increased by four right angles, or by 2π* , in the course of this *one positive circuit*; and consequently, the amplitude of the left-hand factor $(\rho : \sigma)^m$, of $\phi\rho$, will have increased, at the same time, by $2m\pi$. Then, if the point A be also *interior* to the oval, so that the line OA must be prolonged to meet that curve, the ray AP will have likewise made *one positive revolution*, and the amplitude of the factor $(\rho - a) : a$ will have increased by 2π . But if A be an *exterior* point, so that the *finite line* OA *intersects* the curve in a point M, and therefore *never meets it again* if prolonged, although the prolongation of the *opposite line* AO must meet it *once* in some point N, then while the *point P* performs first what we may call the *positive half-circuit* from M to N, and afterwards the *other positive half-circuit* from N to M *again*, the ray AP has only *oscillated* about its initial and final direction, namely that of the line AO, without ever attaining the *opposite direction*; in *this case*, therefore, the amplitude $\text{am}(AP : OA)$, if still supposed to vary *continuously*, has only *fluctuated* in its value, and has (upon the whole) undergone *no change* at all. And since precisely similar remarks apply to the other given points, B, &c., it follows that the amplitude, $\text{am} \phi\rho$, of the *product* (VIII.) of all these factors, has (by 236) received a *total increment* $= 2(m + t)\pi$, if *t* be the *number* (perhaps zero) of *given internal points*, A, B, . . .; while the number *m* is (by 249) *at least* = 1. Thus, while P performs (as above) *one positive circuit*, the *amplitude* $\text{am} \phi\rho$ has *passed at least m times*, and therefore *at least once*, through a *value of the form $2p\pi$* ; and consequently the *condition X.* has been *at least once satisfied*. But the *other condition, IX.*, is satisfied *throughout*, by the

* That is, so as not to receive any *sudden* increment, or decrement, of one or more whole circumferences (comp. 235, (1.)).

supposed *construction* of the oval: there is therefore *at least one real position* P, upon that curve, for which $\phi\rho$ or $f\bar{q} = 1$; so that, for *this position* of that point, the equation 249, III., and therefore also the equation 248, I., is satisfied. The *theorem* of Art. 248, and consequently also, by 247, the theorem of 244, with its transformations 245 and 246, is therefore in this manner *proved*.

253. This conclusion is so important, that it may be useful to illustrate the general reasoning, by applying it to the case of a *quadratic equation*, of the form,

$$I. \dots f\bar{q} = \frac{q}{q_0} \left(\frac{q}{q} - 1 \right) = 1; \text{ or } II. \dots \phi\rho = \frac{\rho}{\sigma} \left(\frac{\rho}{a} - 1 \right) = \frac{OP}{OS} \cdot \frac{AP}{OA} = 1.$$

We have now to prove (comp. 250, VIII.) that a (real) point P exists, which renders the fourth proportional (226) to the three lines OA, OP, AP equal to a given line os, or AB, if this latter be drawn = OS; or which satisfies the following condition of similarity of triangles (118),

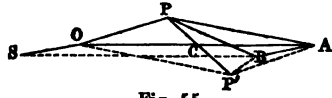


Fig. 55.

$$III. \dots \Delta AOP \propto PAB;$$

which *includes* the equation of rectangles,

$$IV. \dots \overline{OP} \cdot \overline{AP} = \overline{OA} \cdot \overline{AB}.$$

(Compare the annexed Figures, 55, and 55, bis.) Conceive, then, that a continuous curve* is described as a *locus* (or as *part* of the locus) of P, by means of this equality IV., with the additional condition when necessary, that o shall be *within* it; in such a manner that when (as in Fig. 56) a right line from o meets the general or total locus in *several* points, M,

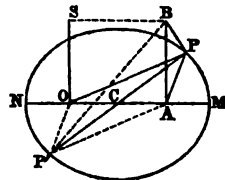


Fig. 55, bis.

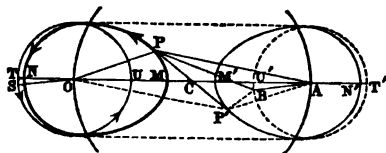


Fig. 56.

* This curve of the fourth degree is the well-known *Cassian*; but when it breaks up, as in Fig. 56, into *two separate ovals*, we here retain, as the oval of the proof, *only* the one round o, rejecting for the present that round A.

m' , n' , we reject all but the point m which is nearest to o , as not belonging (comp. 251, XII.) to the oval here considered. Then while p moves upon that oval, in the positive direction relatively to o , from m to n , and from n to m again, so that the ray op performs one positive revolution, and the amplitude of the factor $op : os$ increases continuously by 2π , the ray ap performs in like manner one positive revolution, or (on the whole) does not revolve at all, and the amplitude of the factor $ap : oa$ increases by 2π or by 0 , according as the point a is interior or exterior to the oval. In the one case, therefore, the amplitude $am \phi\rho$ of the product increases by 4π (as in Fig. 55, bis); and in the other case, it increases by 2π (as in Fig. 56); so that in each case, it passes at least once through a value of the form $2p\pi$, whatever its initial value may have been. Hence, for at least one real position, p , upon the oval, we have

$$\text{V. . . } am \phi\rho = 1, \text{ and therefore VI. . . } U\phi\rho = 1;$$

but

$$\text{VII. . . } T\phi\rho = 1,$$

throughout, by the construction, or by the equation of the locus IV.; the geometrical condition $\phi\rho = 1$ (II.) is therefore satisfied by at least one real vector ρ ; and consequently the quadratic equation $f\rho = 1$ (I.) is satisfied by at least one real quaternion root, $q = \rho : \lambda$ (250, VII.). But the recent form I. has the same generality as the earlier form,

$$\text{VIII. . . } F_2q = q^2 + q_1q + q_2 = 0 \text{ (comp. 245),}$$

where q_1 and q_2 are any two given, real, actual, and complanar quaternions; thus there is always a real quaternion q' in the given plane, which satisfies the equation,

$$\text{VIII'. . . } F_2q' = q'^2 + q_1q' + q_2 = 0 \text{ (comp. 247);}$$

subtracting, therefore, and dividing by $q - q'$, as in algebra (comp. 224), we obtain the following depressed or linear equation q ,

$$\text{IX. . . } q + q' + q_1 = 0, \text{ or IX'. . . } q = q'' = -q' - q_1 \text{ (comp. 246).}$$

The quadratic VIII. has therefore a second real quaternion root, q'' , related in this manner to the first; and because the quadratic function F_2q (comp. again 245) is thus decomposable into two linear factors, or can be put under the form,

$$X. \dots F_2q = (q - q') (q - q''),$$

it cannot vanish for any third real quaternion, q ; so that (comp. 244) the quadratic equation has no more than two such real roots.

(1.) The cubic equation may therefore be put under the form (comp. 248),

$$X. \dots F_3q = q^3 + q_1q^2 + q_2q + q_3 = q(q - q')(q - q'') + q_3 = 0;$$

it has therefore one real root, say q' , by the general proof (252), which has been above illustrated by the case of the quadratic equation; subtracting therefore (compare 247) the equation $F_3q' = 0$, and dividing by $q - q'$, we can depress the cubic to a quadratic, which will have two new real roots, q'' and q''' ; and thus the cubic function may be put under the form,

$$XI. \dots F_3q = (q - q')(q - q'')(q - q'''),$$

which cannot vanish for any fourth real value of q ; the cubic equation X. has therefore no more than three real quaternion roots (comp. 244): and similarly for equations of higher degrees.

(2.) The existence of two real roots q of the quadratic I., or of two real vectors, ρ and ρ' , which satisfy the equation II., might have been geometrically anticipated, from the recently proved increase $= 4\pi$ of amplitude $\phi\rho$, in the course of one circuit, for the case of Fig. 55, bis, in consequence of which there must be two real positions, ρ and ρ' , on the one oval of that Figure, of which each satisfies the condition of similarity III.; and for the case of Fig. 56, from the consideration that the second (or lighter) oval, which in this case exists, although not employed above, is related to Δ exactly as the first (or dark) oval of the Figure is related to o ; so that, to the real position ρ on the first, there must correspond another real position ρ' , upon the second.

(3.) As regards the law of this correspondence, if the equation II. be put under the form,

$$XII. \dots \left(\frac{\rho}{\alpha}\right)^2 - \left(\frac{\rho'}{\alpha}\right)^2 - \frac{\sigma}{\alpha} = 0,$$

and if we now write

$$XIII. \dots \rho = qa, \quad \text{we may write} \quad XIV. \dots q_1 = -1, \quad q_2 = -\sigma : \alpha,$$

for comparison with the form VIII.; and then the recent relation IX'. (or 246) between the two roots will take the form of the following relation between vectors,

$$XV. \dots \rho + \rho' = \alpha; \quad \text{or} \quad XV'. \dots o\rho' = \rho' = \alpha - \rho = \rho A;$$

so that the point ρ' completes (as in the cited Figures) the parallelogram $o\rho A\rho'$, and the line $\rho\rho'$ is bisected by the middle point c of oA . Accordingly, with this position of ρ' , we have (comp. III.) the similarity, and (comp. II. and 226) the equation,

$$XVI. \dots \Delta o\rho\rho' \sim \rho' A B; \quad XVII. \dots \phi\rho' = \phi(\alpha - \rho) = \phi\rho = 1.$$

(4.) The other relation between the two roots of the quadratic VIII., namely (comp. 246),

$$XVIII. \dots q'q'' = q_2, \quad \text{gives} \quad XIX. \dots \frac{\rho}{\alpha} \rho' = -\sigma;$$

and accordingly, the line σ , or os , is a fourth proportional to the three lines OA , OP , and ΔP , or a , ρ , and $-\rho'$.

(5.) The *actual solution*, by calculation, of the quadratic equation VIII. in *complex quaternions*, is performed *exactly as in algebra*; the formula being,

$$\text{XX.} \dots q = -\frac{1}{2}q_1 \pm \sqrt{\left(\frac{1}{4}q_1^2 - q_2\right)},$$

in which, however, the *square root* is to be interpreted as a *real quaternion*, on principles already laid down.

(6.) *Cubic* and *biquadratic* equations, with quaternion coefficients of the kind considered in 244, are in like manner *resolved* by the known *formula* of algebra; but we have now (as has been proved) *three real* (quaternion) *roots* for the former, and *four* such real roots for the latter.

254. The following is another mode of presenting the geometrical reasonings of the foregoing Article, without expressly introducing the notation or conception of *amplitude*. The equation $\phi\rho = 1$ of 253 being written as follows,

$$\text{I.} \dots \sigma = \chi\rho = \frac{\rho}{a}(\rho - a), \text{ or II.} \dots T\sigma = T\chi\rho, \text{ and III.} \dots U\sigma = U\chi\rho,$$

we may thus regard the *vector* σ as a *known function* of the *vector* ρ , or the *point* s as a *function* of the *point* P ; in the sense that, while o and A are *fixed*, P and s *vary together*: although it may (and does) happen, that s may *return* to a former position without P having similarly returned. Now the essential property of the *oval* (253) may be said to be this: that it is *the locus* of the *points* P *nearest* to o , for which the *tensor* $T\chi\rho$ has a given value, say b ; namely the given value of $T\sigma$, or of os , when the *point* s , like o and A , is *given*. If then we conceive the *point* P to *move*, as before, *along the oval*, and the *point* s *also* to *move*, according to the *law* expressed by the recent formula I., this latter *point* must *move* (by II.) *on the circumference* of a given circle (comp. again Fig. 56), with the given origin o for *centre*; and the *theorem* is, that *in so moving*, s will *pass*, at least *once*, through *every position* on that circle, while P performs *one circuit* of the oval. And *this* may be proved by observing that (by III.) the *angular motion* of the *radius* os is equal to the *sum* of the *angular motions* of the *two rays*, OP and ΔP ; but this latter *sum* amounts to *eight right angles* for the case of Fig. 55, *bis.*, and to *four right angles* for the case of Fig. 56; the *radius* os , and the *point* s , must therefore have *revolved twice* in the first case, and *once* in the second case, which proves the *theorem* in question.

(1.) In the first of these two cases, namely when A is an interior point, each of the three angular velocities is *positive* throughout, and the *mean angular velocity* of

the radius os is double of that of each of the two rays OP , AP . But in the second case, when Δ is exterior, the mean angular velocity of the ray AP is zero; and we might for a moment doubt, whether the sometimes negative velocity of that ray might not, for parts of the circuit, exceed the always positive velocity of the ray OP , and so cause the radius os to move backwards, for a while. This cannot be, however; for if we conceive P to describe, like P' , a circuit of the other (or lighter) oval, in Fig. 56, the point s (if still dependent on it by the law I.) would again traverse the whole of the same circumference as before; if then it could ever fluctuate in its motion, it would pass more than twice through some given series of real positions on that circle, during the successive description of the two ovals by P ; and thus, within certain limiting values of the coefficients, the quadratic equation would have more than two real roots: a result which has been proved to be impossible.

(2.) While s thus describes a circle round o , we may conceive the connected point B to describe an equal circle round Δ ; and in the case at least of Fig. 56, it is easy to prove geometrically, from the constant equality (258, IV.) of the rectangles $\overline{OP} \cdot \overline{AP}$ and $\overline{OA} \cdot \overline{AB}$, that these two circles (with $r'u$ and $r'u'$ as diameters), and the two ovals (with MX and $M'X'$ as axes), have two common tangents, parallel to the line $o\Delta$; which connects what we may call the two given foci (or focal points), o and Δ : the new or third circle, which is described on this focal interval $o\Delta$ as diameter, passing through the four points of contact on the ovals, as the Figure may serve to exhibit.

(3.) To prove the same things by quaternions, we shall find it convenient to change the origin (18), for the sake of symmetry, to the central point o ; and thus to denote now OP by ρ , and $o\Delta$ by α , writing also $\overline{OA} = T\alpha = a$, and representing still the radius of each of the two equal circles by b . We shall then have, as the joint equation of the system of the two ovals, the following:

$$IV. \dots T(\rho + \alpha) \cdot T(\rho - \alpha) = 2ab;$$

or

$$V. \dots T(q^2 - 1) = 2c, \quad \text{if } q = \frac{\rho}{\alpha} \quad \text{and} \quad c = \frac{b}{\alpha}.$$

But because we have generally (by 199, 204, &c.) the transformations,

$$VI. \dots S \cdot q^2 = 2Sq^2 - Tq^2 = Tq^2 + 2Vq^2 = 2NSq - Nq = Nq - 2NVq,$$

the square of the equation V. may (by 210, (8.)) be written under either of the two following forms:

$$VII. \dots (Nq - 1)^2 + 4NVq = 4c^2; \quad VIII. \dots (Nq + 1)^2 - 4NSq = 4c^2;$$

whereof the first shows that the maximum value of TVq is c , at least if $2c < 1$, as happens for this case of Fig. 56; and that this maximum corresponds to the value $Tq = 1$, or $T\rho = \alpha$: results which, when interpreted, reproduce those of the preceding sub-article.

(4.) When $2c > 1$, it is permitted to suppose $Sq = 0$, $NVq = Nq = 2c - 1$; and then we have only one continuous oval, as in the case of Fig. 56, bis; but if $c < 1$, though $> \frac{1}{2}$, there exists a certain undulation in the form of the curve (not represented in that Figure), TVq being a minimum for $Sq = 0$, or for $\rho = \alpha$, but becoming (as before) a maximum when $Tq = 1$, and vanishing when $Sq^2 = 2c + 1$, namely at the two summits x , x' , where the oval meets the axis.

(5.) In the intermediate case, when $2c = 1$, the Cassinian curve IV. becomes (as is known) a lemniscata; of which the quaternion equation may, by V., be written (comp. 200, (8.)) under any one of the following forms:

IX. . . $T(q^2 - 1) = 1$; or X. . . $Nq^2 = 2S.q^2$; or XI. . . $Tq^2 = 2SU.q^2$; or finally,

$$\text{XII. . . } T\rho^2 = 2Ta^2 \cos 2 \angle \frac{\rho}{a};$$

which last, when written as

$$\text{XII'. . . } \overline{CP}^2 = 2\overline{OA}^2 \cdot \cos 2\text{ACP},$$

agrees evidently with known results.

(6.) This corresponds to the case when

$$\text{XIII. . . } \sigma = \frac{-a}{4}, \text{ and XIV. . . } \rho = \rho' = +\frac{a}{2}, \text{ in 258, XII.,}$$

that *quadratic equation* having thus its roots *equal*; and in general, for all degrees, cases of *equal roots* answer to some interesting *peculiarities of form* of the *ovals*, on which we cannot here delay.

(7.) It may, however, be remarked, in passing, that if we *remove the restriction* that the vector ρ , or CP , shall be in a *given plane* (225), drawn through the line which connects the *two foci*, O and A , the recent equation V. will then represent the *surface* (or *surfaces*) generated by the *revolution* of the *oval* (or *ovals*), or *lemniscata*, about that line OA as an *axis*.

255. If we look back, for a moment, on the formula of *similarity*, 253, III., we shall see that it involves not merely an *equality of rectangles*, 253, IV., but also an *equality of angles*, $\angle OP$ and $\angle PAB$; so that the angle $\angle OAB$ represents (in the Figures 55) a *given difference of the base angles* $\angle OAP$, $\angle PAO$ of the triangle OAP : but to *construct a triangle*, by means of *such a given difference*, combined with a *given base*, and a *given rectangle of sides*, is a known problem of elementary geometry. To solve it briefly, as an exercise, *by quaternions*, let the given base be the line AA' , with O for its middle point, as in the annexed Figure 57; let BAA' represent the given difference of base angles, $\angle PAA' - \angle AA'P$; and let $\overline{OA} \cdot \overline{AB}$ be equal to the given rectangle of sides, $\overline{AP} \cdot \overline{A'P}$. We shall then have the similarity and equation,

$$\text{I. . . } \triangle OAP \propto \triangle PAB; \quad \text{II. . . } \frac{\rho + a}{a} = \frac{\beta - a}{\rho - a};$$

whence it follows by the simplest calculations, that

$$\text{III. . . } \left(\frac{\rho}{a}\right)^2 = \left(\frac{\rho}{a} + 1\right) \left(\frac{\rho}{a} - 1\right) + 1 = \frac{\beta - a}{a} + 1 = \frac{\beta}{a};$$

or that ρ is a mean proportional (227) between a and β . Draw, therefore, a line OP , which shall be in length a *geometric mean* between the two given lines, OA , OB , and shall also *bisect their angle*

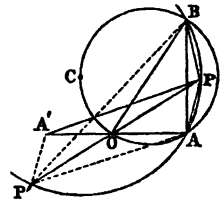


Fig. 57.

AOB; its extremity will be the required *vertex*, P, of the sought triangle AA'P: a result of the *quaternion analysis*, which *geometrical synthesis** easily confirms.

(1.) The equation III. is however satisfied also (comp. 227) by the *opposite* vector, OP' = PO, or $\rho' = -\rho$; and because $\beta = (\rho : a) \cdot \rho$, we have

$$\text{IV.} \dots \frac{\rho + \beta}{\rho + a} = \frac{\rho}{a} = \frac{\beta}{\rho} = \frac{\rho'}{a'} \quad \text{or} \quad \text{IV'.} \dots \frac{P'B}{P'A} = \frac{OP}{OA} = \frac{OB}{OP} = \frac{OP'}{OA'};$$

so that the *four* following triangles are *similar* (the *two first* of them indeed being *equal*):

$$\text{V.} \dots \Delta A'OP' \propto \Delta OPB \propto \Delta P'AB;$$

as geometry again would confirm.

(2.) The angles AP'B, BPA, are therefore *supplementary*, their sum being equal to the sum of the angles in the triangle OAP; whence it follows that *the four points A, P, B, P' are concircular*:† or in other words, *the quadrilateral APBP' is inscriptible in a circle*, which (we may add) passes *through the centre C of the circle OAB* (see again Fig. 57), because the angle AOB is *double* of the angle AP'B, by what has been already proved.

(3.) Quadratic equations in quaternions may also be employed in the solution of many other geometrical problems; for example, to decompose a given vector into two others, which shall have a given geometrical mean, &c.

SECTION 6.—On the $n^2 - n$ Imaginary (or Symbolical) Roots of a Quaternion Equation of the n^{th} Degree, with Coefficients of the kind considered in the foregoing Section.

256. The polynomial function $F_n q$ (245), like the quaternions q, q_1, \dots, q_n on which it depends, may always be reduced to the form of a couple (228); and thus we may establish the transformation (comp. 239),

$$\text{I.} \dots F_n q = F_n(x + iy) = X_n + iY_n = G_n(x, y) + iH_n(x, y),$$

X_n and Y_n , or G_n and H_n , being *two* known, real, finite, and scalar *functions* of the *two sought scalars*, x and y ; which functions, rela-

* In fact, the two triangles I. are similar, as required, because their angles at O and P are equal, and the sides about them are proportional.

† Geometrically, the construction gives at once the similarity,

$$\Delta AOP \propto \Delta POB, \quad \text{whence} \quad \angle BPA = \angle OPA + \angle PAO = \angle OPA';$$

and if we complete the parallelogram APA'P', the new similarity,

$$\Delta OA'P' \propto \Delta P'OB, \quad \text{gives} \quad \angle AP'B = \angle OA'P' + \angle A'PO = \angle AOP;$$

thus the opposite angles BPA, AP'B are *supplementary*, and the quadrilateral APBP' is *inscriptible*. It will be shown, in a shortly subsequent Section, that these four points, A, P, B, P', form a *harmonic group* upon their common circle.

tively to *them*, are each of the n^{th} dimension, but which involve also, though only in the *first* dimension, the $2n$ given and real scalars, $x_1, y_1, \dots, x_n, y_n$. And since the *one quaternion* (or *couple*) equation, $F_n q = 0$, is equivalent (by 228, IV.) to the *system* of the *two scalar equations*;

II. . . $X_n = 0, Y_n = 0$, or III. . . $G_n(x, y) = 0, H_n(x, y) = 0$, we see (by what has been stated in 244, and proved in 252) that *such a system*, of *two equations* of the n^{th} dimension, can *always be satisfied by n systems* (or *pairs*) of *real scalars*, and by *not more than n* , such as

$$\text{IV. . . } x', y'; \quad x'', y''; \dots \quad x^{(n)}, y^{(n)};$$

although it may happen that *two or more* of these systems shall *coincide* with (or *become equal to*) each other.

(1.) If x and y be treated as *co-ordinates* (comp. 228, (8.)), the two equations II. or III. represent a *system of two curves*, in the given plane; and then the *theorem* is, that these two curves *intersect each other* (*generally**) in *n real points*, and in *no more*: although two or more of these n points may happen to *coincide* with each other.

(2.) Let h denote, as a temporary abridgment, the *old or ordinary imaginary*, $\sqrt{-1}$, of *algebra*, considered as an *uninterpreted symbol*, and as *not equal to any real versor*, such as i (comp. 181, and 214, (8.)), but as following the *rules of scalars*, especially as regards the *commutative property* of multiplication (126); so that

$$\text{V. . . } h^2 + 1 = 0, \text{ and VI. . . } h i = i h, \text{ but VII. . . } h \text{ not } = \pm i.$$

(3.) Let q denote still a *real quaternion*, or *real couple*, $x + iy$; and with the meaning just now proposed of h , let $[q]$ denote the *connected but imaginary algebraic quantity*, or *bi-scalar* (214, (7.)), $x + hy$; so that

$$\text{VIII. } q = x + iy, \text{ but IX. . . } [q] = x + hy;$$

and let any *biquaternion* (214), (8.), or (as we may *here* call it) *BI-COUPLE*, of the form $[q'] + i[q'']$, be said to be *complanar* with i ; with the *old notation* (123) of *complanarity*.

(4.) Then, for the *polynomial equation in real and complanar quaternions*, $F_n q = 0$ (244, 245), we may be led to *substitute* the following *connected algebraical equation*, of the *same degree, n* , and involving *real scalars similarly*:

$$\text{X. . . } [F_n q] = [q]^n + [q_1] [q]^{n-1} + \dots + [q_n] = 0;$$

* Cases of *equal roots* may cause points of intersection, which are *generally imaginary*, to *become real*, but *coincident* with each other, and with *former real roots*: for instance the *hyperbola*, $x^2 - y^2 = a$, is intersected in *two real and distinct points*, by the *pair of right lines* $xy = 0$, if the scalar $a > 0$ or < 0 ; but for the case $a = 0$, the *two pairs of lines*, $x^2 - y^2 = 0$ and $xy = 0$, may be considered to have *four coincident intersections* at the origin.

which, after the reductions depending on the substitution V. of -1 for k^2 , receives the form,

$$\text{XI.} \dots [F_n q] = X_n + k Y_n = 0;$$

where X_n and Y_n are the same real and scalar functions as in I.

(5.) But we have seen in II., that these two real functions can be made to vanish together, by selecting any one of n real pairs IV. of scalar values, x and y ; the General Algebraical Equation X., of the n^{th} Degree, has therefore n Real or Imaginary Roots,* of the Form $x + y \sqrt{-1}$; and it has no more than n such roots.

(6.) Elimination of y , between the two equations II. or III., conducts generally to an algebraic equation in x , of the degree n^2 ; which equation has therefore n^2 algebraic roots (6.), real or imaginary; namely, by what has been lately proved, n real and scalar roots, $x', \dots x^{(n)}$, with real and scalar values $y', \dots y^{(n)}$ (comp. IV.) of y to correspond; and $n(n-1)$ other roots, with the same number of corresponding values of y , which may be thus denoted,

$$\text{XII.} \dots [x^{(n+1)}, \dots [x^{(n^2)}]; \quad \text{XIII.} \dots [y^{(n+1)}], \dots [y^{(n^2)}];$$

and which are either themselves imaginary (or bi-scalar, 214, (7.)), or at least correspond, by the supposed elimination, to imaginary or bi-scalar values of y ; since if $x^{(n+1)}$ and $y^{(n+1)}$, for example, could both be real, the quaternion equation $F_n q = 0$ would then have an $(n+1)$ st real root, of the form, $q^{(n+1)} = x^{(n+1)} + iy^{(n+1)}$, contrary to what has been proved (252).

257. On the whole, then, it results that the equation $F_n q = 0$ in complanar quaternions, of the n^{th} degree, with real coefficients, while it admits of only n real quaternion roots,

$$\text{I.} \dots q', q'', \dots q^{(n)} \text{ (244, \&c.),}$$

is symbolically satisfied also (comp. 214, (3.)) by $n(n-1)$ imaginary quaternion roots, or by $n^2 - n$ bi-quaternions (214, (8.)), or bi-couples (256, (3.)), which may be thus denoted,

$$\text{II.} \dots [q^{(n+1)}], \dots [q^{(n^2)}];$$

and of which the first, for example, has the form,

$$\text{III.} \dots [q^{(n+1)}] = [x^{(n+1)}] + i[y^{(n+1)}] = x_i^{(n+1)} + h x_{ii}^{(n+1)} + i(y_i^{(n+1)} + h y_{ii}^{(n+1)});$$

where $x_i^{(n+1)}$, $x_{ii}^{(n+1)}$, $y_i^{(n+1)}$, and $y_{ii}^{(n+1)}$ are four real scalars, but h is the imaginary of algebra (256, (2.)).

(1.) There must, for instance, be $n(n-1)$ imaginary n^{th} roots of unity, in the given plane of i (comp. 256, (3.)), besides the n real roots already determined (233,

* This celebrated Theorem of Algebra has long been known, and has been proved in other ways; but it seemed necessary, or at least useful, for the purpose of the present work, to prove it anew, in connexion with Quaternions: or rather to establish the theorem (244, 252), to which in the present Calculus it corresponds. Compare the Note to page 266.

237); and accordingly in the case $n=2$, we have the *four* following *square-roots* of 1 $||| i$, two real and two imaginary :

$$\text{IV.} \dots +1, -1; \quad +hi, -hi;$$

for, by 256, (2.), we have

$$\text{V.} \dots (\pm hi)^2 = h^2 i^2 = (-1)(-1) = +1.$$

And the *two imaginary roots* of the *quadratic* equation $F_2 q = 0$, which *generally* exist, at least as *symbols* (214, (8.)), may be obtained by *multiplying the square-root* in the formula 258, XX. by hi ; so that in the *particular case*, when that *radical vanishes*, the *four roots* of the equation become *real and equal*: zero having thus *only itself for a square-root*.

(2.) Again, if we write (comp. 287, (3.)),

$$\text{VI.} \dots q = 1\frac{1}{2} = \frac{-1 + i\sqrt{3}}{2}, \quad q^2 = 1\frac{2}{3} = \frac{-1 - i\sqrt{3}}{2},$$

so that 1, q , q^2 are the *three real cube-roots of positive unity*, in the given plane; and if we write also,

$$\text{VII.} \dots \theta = [q] = \frac{-1 + h\sqrt{3}}{2}, \quad \theta^2 = [q^2] = \frac{-1 - h\sqrt{3}}{2},$$

so that θ and θ^2 are (as usual) the *two ordinary* (or *algebraical*) *imaginary cube-roots of unity*; then the *nine cube-roots of 1* ($||| i$) are the following :

$$\text{VIII.} \dots 1; \quad q, q^2; \quad \theta, \theta^2; \quad \theta q, \theta^2 q; \quad \theta^2 q, \theta q^2;$$

whereof the first is a *real scalar*; the two next are *real couples*, or *quaternions* $||| i$; the two following are *imaginary scalars*, or *biscalars*; and the four that remain are *imaginary couples*, or *bi-couples*, or *biquaternions*.

(3.) The *sixteen fourth roots of unity* ($||| i$) are:

$$\text{IX.} \dots \pm 1; \quad \pm i; \quad \pm h; \quad \pm hi; \quad \pm \frac{1}{2}(1 \pm h)(1 \pm i);$$

the three ambiguous signs in the last expression being all independent of each other.

(4.) *Imaginary roots*, of this sort, are sometimes *useful*, or rather *necessary*, in calculations respecting *ideal intersections*,* and *ideal contacts*, in *geometry*: although in what remains of the present Volume, we shall have little or no occasion to employ them.

(5.) We may, however, here observe, that when the *restriction* (225) on the *plane* of the quaternion q is *removed*, the *General Quaternion Equation of the n^{th} Degree* admits, by the foregoing principles, no fewer than n^4 *Roots*, *real or imaginary*: because, when that general equation is reduced, by 221, to the *Standard Quadrinomial Form*,

$$\text{X.} \dots F_n q = W_n + iX_n + jY_n + hZ_n = 0,$$

it breaks up (comp. 221, VI.) into a *System of Four Scalar Equations*, each (generally) of the n^{th} *dimension*, in w, x, y, z ; namely,

$$\text{XI.} \dots W_n = 0, \quad X_n = 0, \quad Y_n = 0, \quad Z_n = 0;$$

and if x, y, z be eliminated between these four, the result is (generally) a *scalar* (or *algebraical*) *equation of the degree n^4* , relatively to the remaining *constituent*, w ;

* Comp. Art. 214, and the Notes there referred to.

which therefore has n^4 (algebraical) values, real or imaginary: and similarly for the three other constituents, x, y, z , of the sought quaternion q .

(6.) It may even happen, when no plane is given, that the number of roots (or solutions) of a finite* equation in quaternions shall become infinite; as has been seen to be the case for the equation $q^2 = -1$ (149, 154), even when we confine ourselves to what we have considered as real roots. If imaginary roots be admitted, we may write, still more generally, besides the two biscalar values, $\pm h$, the expression,

$$\text{XII.} \dots (-1)^{\frac{1}{2}} = v + hv', \quad Sv = Sv' = Svv' = 0, \quad Nv - Nv' = 1;$$

v and v' being thus any two real and right quaternions, in rectangular planes, provided that the norm of the first exceeds that of the second by unity.

(7.) And in like manner, besides the two real and scalar values, ± 1 , we have this general symbolical expression for a square root of positive unity, with merely the difference of the norms reversed:

$$\text{XIII.} \dots 1^{\frac{1}{2}} = v + hv', \quad Sv = Sv' = Svv' = 0, \quad Nv' - Nv = 1.$$

SECTION 7.—On the Reciprocal of a Vector, and on Harmonic Means of Vectors; with Remarks on the Anharmonic Quaternion of a Group of Four Points, and on Conditions of Concircularity.

258. When two vectors, a and a' , are so related that

$$\text{I.} \dots a' = -Ua : Ta, \quad \text{and therefore} \quad \text{II.} \dots a = -Ua' : Ta,$$

or that

$$\text{III.} \dots Ta \cdot Ta' = 1, \quad \text{and} \quad \text{IV.} \dots Ua + Ua' = 0,$$

we shall say that each of these two vectors is the *Reciprocal*† of the other; and shall (at least for the present) denote this relation between them, by writing

$$\text{V.} \dots a' = Ra, \quad \text{or} \quad \text{VI.} \dots a = Ra';$$

so that for every vector a , and every right quotient v ,

$$\text{VII.} \dots Ra = -Ua : Ta; \quad \text{VIII.} \dots R^2a = RRa = a;$$

and

$$\text{IX.} \dots RIV = IRv \text{ (comp. 161, (3.), and 204, XXXV').}$$

259. One of the most important properties of such reciprocals is contained in the following theorem:

* Compare the Notes to page 265.

† Accordingly, under these conditions, we shall afterwards denote this reciprocal of a vector a by the symbol a^{-1} ; but we postpone the use of this notation, until we shall be prepared to connect it with a general theory of products and powers of vectors. Compare 284, V., and the Note to page 121. And as regards the temporary use of the characteristic R , compare the second Note to page 252.

If any two vectors OA, OB , have OA', OB' for their reciprocals, then (comp. Fig. 58) the right line $A'B'$ is parallel to the tangent OD , at the origin O , to the circle OAB ; and the two triangles, $OAB, OB'A'$, are inversely similar (118). Or in symbols,

I. . . if $OA' = R.OA$, and $OB' = R.OB$,
then

$$\Delta OAB \propto \Delta OB'A'.$$

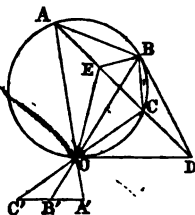


Fig. 58.

(1.) Of course, under the same conditions, the tangent at O to the circle $OA'B'$ is parallel to the line AB .

(2.) The angles BAO and $OB'A'$ or BOD being equal, the fourth proportional (226) to AB, AO , and OB , or to BA, OA , and OB , has the direction of OD , or the direction op ; *posite* to that of $A'B'$; and its length is easily proved to be the reciprocal (or inverse) of the length of the same line $A'B'$, because the similar triangles give,

$$II. \dots (\overline{OA} : \overline{BA}) \cdot \overline{OB} = (\overline{OB'} : \overline{A'B'}) \cdot \overline{OB} = 1 : \overline{A'B'},$$

it being remembered that

$$III. \dots \overline{OA} \cdot \overline{OA'} = \overline{OB} \cdot \overline{OB'} = 1;$$

we may therefore write,

$$IV. \dots (\overline{OA} : \overline{BA}) \cdot \overline{OB} = R.A'B', \text{ or } V. \dots \frac{\alpha}{\alpha - \beta} \beta = R(R\beta - R\alpha),$$

whatever two vectors α and β may be.

(3.) Changing α and β to their reciprocals, the last formula becomes,

$$VI. \dots R(\beta - \alpha) = \frac{R\alpha}{R\alpha - R\beta} \cdot R\beta; \text{ or } VII. \dots (\overline{OA'} : \overline{B'A'}) \cdot \overline{OB'} = R.AB.$$

(4.) The inverse similarity I. gives also, generally, the relation,

$$VIII. \dots K \frac{\beta}{\alpha} = \frac{R\alpha}{R\beta}.$$

(5.) Since, then, by 195, II., or 207, (2.),

$$IX. \dots K \frac{\beta}{\alpha} \pm 1 = K \frac{\beta \pm \alpha}{\alpha}, \text{ we have } X. \dots \frac{R\alpha \pm R\beta}{R\beta} = \frac{R\alpha}{R(\beta \pm \alpha)};$$

the lower signs agreeing with VI.

(6.) In general, the reciprocals of opposite vectors are themselves opposite; or in symbols,

$$XI. \dots R(-\alpha) = -R\alpha.$$

(7.) More generally,

$$XII. \dots Rxa = x^{-1}R\alpha,$$

if x be any scalar.

(8.) Taking lower signs in X., changing α to γ , dividing, and taking conjugates, we find for any three vectors α, β, γ (coplanar or diplanar) the formula:

$$XIII. \dots K \frac{R\gamma - R\beta}{R\alpha - R\beta} = K \left(\frac{R\gamma}{R(\beta - \gamma)} \cdot \frac{R(\beta - \alpha)}{R\alpha} \right) = \frac{\alpha}{\beta - \alpha} \cdot \frac{\gamma - \beta}{-\gamma} = \frac{OA}{AB} \cdot \frac{BC}{OO'}$$

if $\alpha = OA, \beta = OB$, and $\gamma = OC$, as usual.

(9.) If then we extend, to any four points of space, the notation (25),

$$\text{XIV.} \dots (\text{ABCD}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DA}},$$

interpreting each of these two factor-quotients as a quaternion, and defining that their product (in this order) is the anharmonic quaternion function, or simply the Anharmonic, of the Group of four points A, B, C, D, or of the (plane or gauche) Quadrilateral ABCD, we shall have the following general and useful formula of transformation:

$$\text{XV.} \dots (\text{OA'BO}) = \text{K} \frac{\text{R}\gamma - \text{R}\beta}{\text{R}\alpha - \text{R}\beta} = \text{K} \frac{\text{B}'\text{C}'}{\text{B}'\text{A}'},$$

where OA', OB', OC' are supposed to be reciprocals of OA, OB, OC.

(10.) With this notation XIV., we have generally, and not merely for collinear groups (85), the relations:

$$\text{XVI.} \dots (\text{ABCD}) + (\text{ACBD}) = 1; \quad \text{XVII.} \dots (\text{ABCD}) \cdot (\text{ADCB}) = 1.$$

(11.) Let O, A, B, C, D be any five points, and OA', .. OD' the reciprocals of OA, .. OD; we shall then have, by XV.,

$$\text{XVIII.} \dots \frac{\text{B}'\text{A}'}{\text{B}'\text{C}'} = \text{K} (\text{OCBA}), \quad \frac{\text{D}'\text{C}'}{\text{D}'\text{A}'} = \text{K} (\text{OADC});$$

and therefore,

$$\text{XIX.} \dots \text{K} (\text{A}'\text{B}'\text{C}'\text{D}') = (\text{OADC}) (\text{OCBA}) = -(\text{OADCBA}),$$

if we agree to write generally, for any six points, the formula,*

$$\text{XX.} \dots (\text{ABCDEF}) = \frac{\text{AB}}{\text{BC}} \cdot \frac{\text{CD}}{\text{DE}} \cdot \frac{\text{EF}}{\text{FA}}.$$

(12.) If then the five points O .. D be coplanar (225), we have, by 226, and by XIV.,

$$\text{XXI.} \dots \text{K} (\text{A}'\text{B}'\text{C}'\text{D}') = (\text{ABCD}), \quad \text{or} \quad \text{XXI}'. \dots (\text{A}'\text{B}'\text{C}'\text{D}') = \text{K} (\text{ABCD});$$

the anharmonic quaternion (ABCD) being thus changed to its conjugate, when the four rays OA, .. OD are changed to their reciprocals.

260. Another very important consequence from the definition (258) of reciprocals of vectors, or from the recent theorem (259), may be expressed as follows:

If any three coincital vectors, OA, OB, OC, be chords of one common circle, then (see again Fig. 58) their three coincital re-

* There is a convenience in calling, generally, this product of three quotients, (ABCDEF), the evolutionary quaternion, or simply the Evolutionary, of the Group of Six Points, A .. F, or (if they be not collinear) of the plane or gauche Hexagon ABCDEF: because the equation,

$$(\text{ABCA}'\text{B}'\text{C}') = -1,$$

expresses either I st, that the three pairs of points, AA', BB', CC', form a collinear involution (26) of a well-known kind; or II nd, that those three pairs, or the three corresponding diagonals of the hexagon, compose a coplanar or a homospheric Involution, of a new kind suggested by quaternions (comp. 261, (11.)).

reciprocals, OA' , OB' , OC' , are *termino-collinear* (24): or, in other words, if the *four points* O , A , B , C be *conconcircular*, then the *three points* A' , B' , C' are situated on *one right line*.

And conversely, if *three coinitial vectors*, OA' , OB' , OC' , thus terminate on *one right line*, then their three coinitial *reciprocals*, OA , OB , OC , are *chords* of one circle; the *tangent* to which circle, at the origin, is *parallel* to the *right line*; while the *anharmonic function* (259, (9.)), of the *inscribed quadrilateral* $OABC$, reduces itself to a *scalar quotient of segments* of that line (which therefore is its own conjugate, by 139): namely,

$$I. \dots (OABC) = B'C' : B'A' = (\infty A'B'C') = (O.OABC),$$

if the symbol ∞ be used *here* to denote the *point at infinity* on the right line $A'B'C'$; and if, in thus employing the notation (35) for the *anharmonic of a plane pencil*, we consider the *null chord*, oo , as having the *direction** of the *tangent*, OD .

(1.) If $\rho = OP$ be the *variable vector* of a point P upon the circle OAB , the *quaternion equation* of that circle may be thus written:

$$II. \dots R\rho = R\beta + x(R\alpha - R\beta), \quad \text{where} \quad III. \dots x = (OABP);$$

the coefficient x being thus a *variable scalar* (comp. 99, I.), which depends on the *variable position* of the point P on the circumference.

(2.) Or we may write,

$$IV. \dots R\rho = \frac{tR\alpha + uR\beta}{t + u},$$

as another form of the equation of the same circle OAB ; with which may usefully be contrasted the earlier form (comp. 25), of the equation of the *line* AB ,

$$V. \dots \rho = \frac{t\alpha + u\beta}{t + u}.$$

(3.) Or, dividing the second member of IV. by the first, and taking conjugates, we have for the circle,

$$VI. \dots \frac{t\rho}{\alpha} + \frac{u\rho}{\beta} = t + u; \quad \text{while} \quad VII. \dots \frac{t\alpha}{\rho} + \frac{u\beta}{\rho} = t + u,$$

for the right line.

(4.) Or we may write, by II.,

$$VIII. \dots V \frac{R\rho - R\beta}{R\alpha - R\beta} = 0; \quad \text{or} \quad VIII'. \dots \frac{R\rho - R\beta}{R\alpha - R\beta} = V^{-1}0;$$

this latter symbol, by 204, (18.), denoting *any scalar*.

* Compare the remarks in the second Note to page 189, respecting the possible indeterminateness of signification of the symbol $U0$, when the zero denotes a *line*, which *vanishes* according to a *law*.

(5.) Or still more briefly,

$$\text{IX.} \dots V(OABP) = 0; \text{ or } \text{IX}' \dots (OABP) = V^{-1}0.$$

(6.) If the four points O, A, B, C be still *concircular*, and if P be any *fifth point* in their plane, while PO_1, \dots, PC_1 are the reciprocals of PO, \dots, PC , then by 259, XXI., we have the relation,

$$\text{X.} \dots (O_1A_1B_1C_1) = K(OABC) = (OABC) = V^{-1}0;$$

the four new points $O_1 \dots C_1$ are therefore *generally concircular*.

(7.) If, however, the point P be again placed on the circle $OABC$, those four new points are (by the present Article) *collinear*; being the intersections of the pencil $P.OABC$ with a parallel to the tangent at P . In this case, therefore, we have the equation,

$$\text{XI.} \dots (P.OABC) = (O_1A_1B_1C_1) = (OABC);$$

so that the constant anharmonic of the pencil (85) is thus seen to be equal to what we have defined (259, (9.)) to be the anharmonic of the group.

(8.) And because the anharmonic of a circular group is a scalar, it is equal (by 187, (8.)) to its own tensor, either positively or negatively taken: we may therefore write, for any inscribed quadrilateral $OABC$, the formula,

$$\text{XII.} \dots (OABC) = \mp T(OABC) = \mp (\overline{OA} \cdot \overline{BC}) : (\overline{AB} \cdot \overline{CO}),$$

= \mp a quotient of rectangles of opposite sides; the upper or the lower sign being taken, according as the point B' falls, or does not fall, between the points A' and C' : that is, according as the quadrilateral $OABC$ is an *uncrossed* or a *crossed* one.

§(9.) Hence it is easy to infer that for any circular group O, A, B, C , we have the equation,

$$\text{XIII.} \dots U \frac{OA}{AB} = \pm U \frac{CO}{CB};$$

the upper sign being taken when the succession $OABC$ is a *direct* one, that is, when the quadrilateral $OABC$ is *uncrossed*; and the lower sign, in the contrary case, namely, when the succession is (what may be called) *indirect*, or when the quadrilateral is *crossed*: while conversely this equation XIII. is sufficient to prove, whenever it occurs, that the anharmonic $(OABC)$ is a negative or a positive scalar, and therefore by (5.) that the group is *circular* (if not *linear*), as above.

(10.) If A, B, C, D, E be any five *homospheric points* (or points upon the surface of one sphere), and if o be any *sixth point of space*, while oA', \dots, oE' are the reciprocals of oA, \dots, oE , then the five new points $A' \dots E'$ are *generally homospheric* (with each other); but if o happens to be on the sphere $ABCDE$, then $A' \dots E'$ are *complanar*, their common plane being parallel to the tangent plane to the given sphere at o : with resulting anharmonic relations, on which we cannot here delay.

261. An interesting case of the foregoing theory is that when the generally scalar anharmonic of a circular group becomes equal to *negative unity*: in which case (comp. 26), the group is said to be *harmonic*. A few remarks upon such *circular and harmonic groups* may here be briefly made: the stu-

dent being left to fill up hints for himself, as what must be now to him an easy exercise of calculation.

(1.) For such a group (comp. again Fig. 58), we have thus the equation,

$$\text{I.} \dots (\text{OABC}) = -1; \text{ and therefore II.} \dots \text{A'B' = B'C'}$$

or

$$\text{III.} \dots \text{R}\beta = \frac{1}{2}(\text{Ra} + \text{R}\gamma);$$

and under this condition, we shall say (comp. 216, (5.)) that the *Vector* β is the *Harmonic Mean* between the two vectors, a and γ .

(2.) Dividing, and taking conjugates (comp. 260, (3.), and 216, (5.)), we thus obtain the equation,

$$\text{IV.} \dots \frac{\beta}{a} + \frac{\beta}{\gamma} = 2; \text{ or V.} \dots \beta = \frac{2a}{\epsilon\gamma + a} \gamma = \frac{2\gamma}{\gamma + a} a;$$

or

$$\text{VI.} \dots \beta = \frac{a}{\epsilon} \gamma = \frac{\gamma}{\epsilon} a, \text{ if VII.} \dots \epsilon = \frac{1}{2}(\gamma + a);$$

ϵ thus denoting here the vector OE (Fig. 58) of the middle point of the chord AC . We may then say that the *harmonic mean* between any two lines is (as in algebra) the *fourth proportional to their semisum, and to themselves*.

(8.) Geometrically, we have thus the similar triangles,

$$\text{VIII.} \dots \triangle \text{AOB} \propto \triangle \text{EOC}; \quad \text{VIII'.} \dots \triangle \text{AOE} \propto \triangle \text{BOC};$$

whence, either because the angles OBA and OCA , or because the angles OAC and OBC are equal, we may infer (comp. 260, (5.)) that, when the equation I. is satisfied, the four points O, A, B, C , if not *collinear*, are *concurrent*.

(4.) We have also the similarities,

$$\text{IX.} \dots \triangle \text{OEC} \propto \triangle \text{CEB}, \text{ and IX'.} \dots \triangle \text{OEA} \propto \triangle \text{AEB};$$

or the equations,

$$\text{X.} \dots \frac{\beta - \epsilon}{\gamma - \epsilon} = \frac{\gamma - \epsilon}{-\epsilon}, \text{ and X'.} \dots \frac{\beta - \epsilon}{a - \epsilon} = \frac{a - \epsilon}{-\epsilon};$$

in fact we have, by VI. and VII.,

$$\text{XI.} \dots \frac{a}{\epsilon} + \frac{\gamma}{\epsilon} = 2; \quad \text{XII.} \dots \frac{\beta - \epsilon}{-\epsilon} \left(= 1 - \frac{\beta}{a} \frac{a}{\epsilon} = 1 - \frac{\gamma}{\epsilon} \frac{a}{\epsilon} \right) = \left(1 - \frac{a}{\epsilon} \right) \frac{\epsilon}{\epsilon}.$$

(5.) Hence the line EC , in Fig. 58, is the *mean proportional* (227) between the lines EO and EB ; or in words, the *semisum* (OE), the *semidifference* (EC), and the *excess* (BE) of the *semisum over the harmonic mean* (OB), form (as in algebra) a *continued proportion* (227).

(6.) Conversely, if any three coinitial vectors, EO, EC, EB , form thus a continued proportion, and if we take $\text{EA} = \text{CB}$, then the four points OABC will compose a circular and harmonic group; for example, the points APB'P' of Fig. 57 are arranged so as to form such a group.*

(7.) It is easy to prove that, for the *inscribed quadrilateral* OABC of Fig. 58, the *rectangles under opposite sides* are each equal to *half* of the rectangle under the

* Compare the Note to 255, (2.). In that sub-article, the text should have run thus: of which (we may add) the centre C is on the circle OAB , &c. In Fig. 58, the centre of the circle OABC is concurrent with the three points O, E, B .

diagonals; which geometrical relation answers to either of the two anharmonic equations (comp. 259, (10.)) :

$$\text{XIII.} \dots (\text{OBAC}) = + 2; \quad \text{XIII'.} \dots (\text{OCAB}) = + \frac{1}{2}.$$

(8.) Hence, or in other ways, it may be inferred that these diagonals, OB, AC, are *conjugate chords* of the circle to which they belong : in the sense that *each* passes through the *pole* of the *other*, and that thus the line DB is the *second tangent* from the point D, in which the chord AC prolonged intersects the tangent at O.

(9.) Under the same conditions, it is easy to prove, either by quaternions or by geometry, that we have the harmonic equations :

$$\text{XIV.} \dots (\text{ABCO}) = (\text{BOCA}) = (\text{COAB}) = - 1;$$

so that AC is the harmonic mean between AB and AO; BO is such a mean between BC and BA; and CA between CO and CB.

(10.) In any such group, any two *opposite points* (or opposite corners of the quadrilateral), as for example O and B, may be said to be *harmonically conjugate* to each other, with respect to the two other points, A and C; and we see that when these two points A and C are given, then to every third point O (whether in a given plane, or in space) there always corresponds a fourth point B, which is in this sense *conjugate* to that third point : this fourth point being always *coplanar* with the three points A, C, O, and being even *concircular* with them, unless they happen to be *collinear* with each other; in which extreme (or limiting) case, the fourth point B is still *determined*, but is now *collinear* with the others (as in 26, &c.).

(11.) When, after thus selecting two* points, A and C, or treating them as given or fixed, we determine (10.) the harmonic conjugates B, B', B'', with respect to them, of any three assumed points, O, O', O'', then the three pairs of points, O, B; O', B'; O'', B'', may be said to form an *Involution*,† either on the right line AC, (in which case it will only be one of an already well-known kind), or in a plane through that line, or even generally in space : and the two points A, C may in all these cases be said to be the two *Double Points* (or *Foci*) of this Involution. But the field thus opened, for geometrical investigation by Quaternions, is far too extensive to be more than mentioned here.

(12.) We shall therefore only at present add, that the conception of the *harmonic mean* between two vectors may easily be extended to any number of such, and need not be limited to the plane : since we may define that η is the harmonic mean of the n arbitrary vectors $a_1, \dots a_n$, when it satisfies the equation,

$$\text{XV.} \dots R\eta = \frac{1}{n} (Ra_1 + \dots + Ra_n); \quad \text{or} \quad \text{XVI.} \dots nR\eta = \Sigma Ra.$$

(13.) Finally, as regards the notation Ra , and the definition (258) of the reciprocal of a vector, it may be observed that if we had chosen to define reciprocal vectors as having *similar* (instead of *opposite*) directions, we should indeed have had the positive sign in the equation 258, VII. ; but should have been obliged to write, instead of 258, IX., the much less simple formula,

$$RI\sigma = - IR\sigma.$$

* There is a sense in which the geometrical process here spoken of can be applied, even when the two fixed points, or foci, are imaginary. Compare the *Géométrie Supérieure* of M. Chasles, page 136.

† Compare the Note to 259, (11.).

CHAPTER III.

ON DIPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN SPACE: AND ESPECIALLY ON THE ASSOCIATIVE PRINCIPLE OF MULTIPLICATION OF SUCH QUATERNIONS.

SECTION I.—*On some Enunciations of the Associative Property, or Principle, of Multiplication of Diplanar Quaternions.*

262. In the preceding Chapter we have confined ourselves almost entirely, as had been proposed (224, 225), to the consideration of quaternions *in a given plane* (that of i); alluding only, in some instances, to possible extensions* of results so obtained. But we must now return to consider, as in the First Chapter of this Second Book, the subject of *General Quotients of Vectors*: and especially their *Associative Multiplication* (223), which has hitherto been only proved in connexion with the *Distributive Principle* (212), and with the *Laws of the Symbols i, j, k* (183). And first we shall give a few *geometrical enunciations* of that associative principle, which shall be independent of the distributive one, and in which it will be sufficient to consider (comp. 191) the *multiplication of versors*; because the multiplication of *tensors* is *evidently* an associative operation, as corresponding simply to *arithmetical multiplication*, or to the *composition of ratios* in geometry.† We shall therefore suppose, throughout the present Chapter, that q, r, s are some *three given but arbitrary versors*, in *three given and distinct planes*;‡ and our object will be to throw

* As in 227, (8.); 242, (7.); 254, (7.); 257, (6.) and (7.); 259, (8.), (9.), (10.), (11.); 260, (10.); and 261, (11.) and (12.).

† Or, more generally, for any three pairs of magnitudes, each pair separately being homogeneous.

‡ If the factors q, r, s were *complanar*, we could always (by 120) put them

some additional light, by new enunciations in this Section, and by new demonstrations in the next, on the very important, although very simple, *Associative Formula* (223, II.), which may be written thus :

$$\text{I. . . } sr \cdot q = s \cdot rq;$$

or thus, more fully,

$$\text{II. . . } q'q = t, \text{ if } q' = sr, \ s' = rq, \text{ and } t = ss';$$

q' , s' , and t being here *three new and derived versors*, in *three new and derived planes*.

263. Already we may see that this *Associative Theorem of Multiplication*, in all its forms, has an essential reference to a *System of Six Planes*, namely the planes of these *six versors*,

$$\text{IV. . . } q, r, s, rq, sr, srq, \text{ or } \text{IV}' \text{ . . } q, r, s, s', q', t;$$

on the judicious selection and arrangement of which, the clearness and elegance of every geometrical statement or proof of the theorem must very much depend : while the *versor character* of the factors (in the only part of the theorem for which *proof* is required) suggests a reference to a *Sphere*, namely to what we have called the *unit-sphere* (128). And the *three following arrangements* of the six planes appear to be the most natural and simple that can be considered : namely, I_{st}, the arrangement in which the planes all pass *through the centre* of the sphere ; II_{nd}, that in which they all *touch* its surface ; and III_{rd}, that in which they are the *six faces of an inscribed solid*. We proceed to consider successively these three arrangements.

264. When the *first arrangement* (263) is adopted, it is natural to employ *arcs of great circles*, as *representatives* of the *versors*, on the

under the forms,

$$q = \frac{\beta}{\alpha}, \quad r = \frac{\gamma}{\beta}, \quad s = \frac{\delta}{\gamma};$$

and then should have (comp. 183, (1.)) the *two equal ternary products*,

$$sr \cdot q = \frac{\delta}{\beta} \frac{\beta}{\alpha} = \frac{\delta}{\alpha} = \frac{\delta}{\gamma} \frac{\gamma}{\alpha} = s \cdot rq;$$

so that in *this case* (comp. 224) the associative property would be proved without any difficulty.

plan of Art. 162. Representing thus the factor q by the arc AB , and r by the successive arc BC , we represent (167) their *product* rq , or s' , by AC ; or by any *equal arc* (165), such as DE , in Fig. 59, may be supposed to be. Again, representing s by EF , we shall have DF as the representative of the *ternary product* $s.rq$, or ss' , or t , taken in *one order of association*. To represent the *other ternary product*, $sr.q$, or $q'q$, we may first determine three new points, G, H, I , by arcual equations (165), between GH, BC , and between HI, EF , so that BC, EF

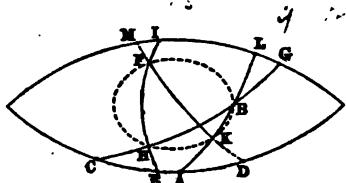


Fig. 59.

intersect in H , as the arcs representing s' and s had intersected in E ; and then, after thus finding an arc GI which represents sr , or q' , may determine three other points, K, L, M , by equations between KL, AB , and between LM, GI , so that these two new arcs, KL, LM , represent q and q' , and that AB, GI intersect in L ; for in this way we shall have an arc, namely KM , which represents $q'q$ as required. And the *theorem* then is, that this last arc KM is *equal* to the former arc DF , in the *full sense* of Art. 165; or that *when* (as under the foregoing conditions of construction) *the five arcual equations*,

I. . . $\sphericalangle AB = \sphericalangle KL, \sphericalangle BC = \sphericalangle GH, \sphericalangle EF = \sphericalangle HI, \sphericalangle AC = \sphericalangle DE, \sphericalangle GI = \sphericalangle LM$,
exist, then this sixth equation of the same kind is satisfied also,

$$\text{II. . . } \sphericalangle DF = \sphericalangle KM:$$

the *two points*, K and M , being both *on the same great circle* as the two previously determined points, D and F ; or D and M being on the great circle through F and K : and the *two arcs*, DF and KM , of that great circle, or the two *dotted arcs*, DK, FM in the Figure, being *equally long*, and *similarly directed* (165).

(1.) Or, after determining the *nine points* $A . . . I$ so as to satisfy the three middle equations I., we might determine the *three other points*, K, L, M , without any other arcual equations, as *intersections* of the three pairs of arcs AB, DF ; AB, GI ; DF, GI ; and then the theorem would be, that (if these three last points be suitably distinguished from their own opposites upon the sphere) the two extreme equations I., and the equation II., are satisfied.

(2.) The same geometrical theorem may also be thus enunciated: *If the first, third, and fifth sides* (KL, GH, ED) *of a spherical hexagon* $KLGHED$ *be respectively and arcually equal* (165) *to the first, second, and third sides* (AB, BC, CA) *of a spherical triangle* ABC , *then the second, fourth, and sixth sides* (LG, HE, DK) *of the same hexagon are equal to the three successive sides* (MI, IF, FM) *of another spherical triangle*, MIF .

(3.) It may also be said, that if *five successive sides* ($KL, \dots ED$) of one spherical hexagon be respectively and arcually equal to the *five successive diagonals* (AB, MI, BC, IF, CA) of another such hexagon ($AMBCIF$), then the *sixth side* (DK) of the first is equal to the *sixth diagonal* (FM) of the second.

(4.) Or, if we adopt the conception mentioned in 180, (3.), of an *arcual sum*, and denote such a sum by inserting + between the symbols of the two summands, that of the *added arc* being written to the *left-hand*, we may state the theorem, in connexion with the recent Fig. 59, by the formula :

$$\text{III.} \dots \cap DF + \cap BA = \cap EF + \cap BC, \text{ if } \cap DA = \cap EC;$$

where B and F may denote any two points upon the sphere.

(5.) We may also express* the same principle, although somewhat less simply, as follows (see again Fig. 59, and compare sub-art. (2.)) :

$$\text{IV.} \dots \text{if } \cap ED + \cap GH + \cap HL = 0, \text{ then } \cap DE + \cap HE + \cap LG = 0.$$

(6.) If, for a moment, we agree to write (comp. Art. 1),

$$\text{V.} \dots \cap AB = \widehat{B-A},$$

we may then express the recent statement IV. a little more lucidly thus :

$$\text{VI.} \dots \text{if } \widehat{D-E} + \widehat{H-G} + \widehat{L-K} = 0, \text{ then } \widehat{K-D} + \widehat{E-H} + \widehat{G-L} = 0.$$

(7.) Or still more simply, if \cap, \cap', \cap'' be supposed to denote any three *dip-sar arcs*, which are to be *added* according to the *rule* (180, (3.)) above referred to, the *theorem* may be said to be, that

$$\text{VII.} \dots (\cap'' + \cap') + \cap = \cap'' + (\cap' + \cap);$$

or in words, that *Addition of Arcs on a Sphere is an Associative Operation.*

(8.) Conversely, if any independent demonstration be given, of the truth of any one of the foregoing statements, considered as expressing a *theorem of spherical geometry*, † a *new proof* will thereby be furnished, of the associative property of *multiplication of quaternions.*

265. In the *second arrangement* (263) of the *six planes*, instead of representing the three given versors, and their partial or total products, by *arcs*, it is natural to represent them (174, II.) by *angles* on the sphere. Conceive then that the two versors, q and r , are represented, in Fig. 60, by the two spherical angles, $\mathbb{E}AB$ and $\mathbb{A}BE$; and therefore (175) that their product, rq or s' , is represented by the external vertical angle at \mathbb{E} , of the triangle $\mathbb{A}BE$. Let the

* Some of these formulæ and figures, in connexion with the associative principle, are taken, though for the most part with modifications, from the author's Sixth Lecture on Quaternions, in which that whole subject is very fully treated. Comp. the Note to page 160.

† Such a demonstration, namely a deduction of the equation II. from the five equations I., by known properties of *spherical conics*, will be briefly given in the ensuing Section.

second versor r be also represented by the angle FBC , and the third versor s by BCF ; then the other binary product, sr or q' , will be represented by the external angle at F , of the new triangle BCF . Again, to represent the *first* ternary product, $t = ss' = s.r.q$, we have only to take the external angle at D of the triangle ECD , if D be a point determined

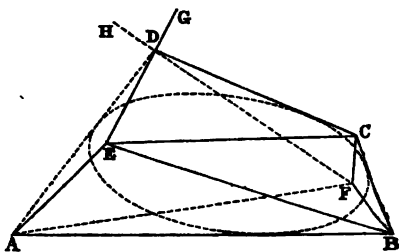


Fig. 60.

by the two conditions, that the angle ECD shall be *equal* to BCF , and DEC *supplementary* to BEA . On the other hand, if we conceive a point D' determined by the conditions that $D'AF$ shall be equal to EAB , and AFD' supplementary to CFB , then the external angle at D' , of the triangle AFD' , will represent the *second* ternary product, $q'q = sr.q$, which (by the associative principle) must be *equal* to the *first*. Conceiving then that ED is prolonged to G , and FD' to H , the *two spherical angles*, GDC and $AD'H$, must be *equal in all respects*; their *vertices* D and D' *coinciding*, and the *rotations* (174, 177) which they represent being not only *equal in amount*, but also *similarly directed*. Or, to express the same thing otherwise, we may *enunciate* (262) the *Associative Principle* by saying, that *when the three angular equations,*

$$\text{I. . . } ABE = FBC, \quad BOF = ECD, \quad DEC = \pi - BEA,$$

are satisfied, then these three other equations,

$$\text{II. . . } DAF = EAB, \quad FDA = CDE, \quad AFD = \pi - CFB,$$

are satisfied also. For not only is this *theorem of spherical geometry* a *consequence* of the associative principle of *multiplication of quaternions*, but conversely any independent demonstration* of the theorem is, at the same time, a *proof* of the principle.

266. The *third arrangement* (263) of the six planes may be illustrated by conceiving a *gauche hexagon*, $AB'CA'BC'$, to be *inscribed in a sphere*, in such a manner that the intersection D of the three planes, $c'AB'$, $B'CA'$, $A'BC'$, is on the surface; and therefore that the *three small circles*, denoted by these three last trilateral symbols, *concur*

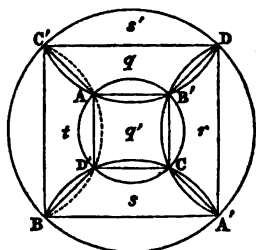


Fig. 61.

* Such as we shall sketch, in the following Section, with the help of the known properties of the *spherical conics*. Compare the Note to the foregoing Article.

in one point D; while the second intersection of the two other small circles, AB'C, CA'B, may be denoted by the letter D', as in the annexed Fig. 61. Let it be also for simplicity at first supposed, that (as in the Figure) the *five circular successions*,

$$I \dots C'AB'D, AB'CD', B'CA'D, CA'BD', A'BC'D,$$

are all *direct*; or that the *five inscribed quadrilaterals*, denoted by these symbols I., are all *uncrossed* ones. Then (by 260, (9.)) it is allowed to introduce *three versors*, q, r, s , each having *two expressions*, as follows:

$$II \dots q = U \frac{B'D}{DC'} = + U \frac{AB'}{AC'}; \quad r = U \frac{DA'}{B'D} = + U \frac{CA'}{CB'};$$

$$s = U \frac{CD'}{CA'} = + U \frac{BD'}{A'B'};$$

although (by the cited sub-article) the last members of these three formulæ should receive the *negative* sign, if the first, third, and fourth of the successions I. were to become *indirect*, or if the corresponding quadrilaterals were *crossed* ones. We have thus (by 191) the derived expressions,

$$III \dots s' = rq = U \frac{DA'}{DC'} = U \frac{A'B'}{BC'}; \quad q' = sr = U \frac{CD'}{CB'} = U \frac{D'A}{AB'};$$

whereof, however, the two versors in the first formula would differ in their signs, if the fifth succession I. were indirect; and those in the second formula, if the second succession were such. Hence,

$$IV \dots t = ss' = s.rq = U \frac{BD'}{BC'}; \quad q'q = sr.q = U \frac{D'A}{AC'};$$

and since, by the associative principle, these two last versors are to be *equal*, it follows that, under the supposed conditions of construction, the *four points*, B, C', A, D', compose a *circular and direct succession*; or that the *quadrilateral*, BC'AD', is *plane, inscriptible*,* and *uncrossed*.

267. It is easy, by suitable changes of sign, to adapt the recent reasoning to the case where some or all of the successions I. are indirect; and thus to *infer*, from the associative principle, this *theorem of spherical geometry*: *If* AB'CA'BC'

* Of course, since the four points BC'AD' are known to be *homospheric* (comp. 260. (10.)), the *inscriptibility* of the quadrilateral in a *circle* would *follow* from its being *plane*, if the latter were otherwise proved: but it is *here* deduced from the *equality* of the two versors IV., on the plan of 260, (9.).

be a spherical hexagon, such that the three small circles $C'AB'$, $B'CA'$, $A'BC'$ concur in one point D , then, Ist, the three other small circles, $AB'C$, $CA'B$, $BC'A$, concur in another point, D' ; and IInd, of the six circular successions, 266, I., and $BC'AD'$, the number of those which are indirect is always even (including zero). And conversely, any independent demonstration* of this geometrical theorem will be a *new proof* of the associative principle.

268. The same fertile principle of *associative multiplication* may be enunciated in other ways, without limiting the factors to be *versors*, and without introducing the conception of a *sphere*. Thus we may say (comp. 264, (2.)), that if $O.ABCDEF$ (comp. 35) be any pencil of six rays in space, and $O.A'B'C'$ any pencil of three rays, and if the three angles AOB , COB , EOB of the first pencil be respectively equal to the angles $B'OC'$, $C'OA'$, $A'OB'$ of the second, then another pencil of three rays, $O.A''B''C''$, can be assigned, such that the three other angles BOC , DOE , FOA of the first pencil shall be equal to the angles $B''OC''$, $C''OA''$, $A''OB''$ of the third: equality of angles (with one vertex) being here understood (comp. 165) to include *coplanarity*, and *similarity of direction of rotations*.

(1.) Again (comp. 264, (4.)), we may establish the following formula, in which the four vectors $a\beta\gamma\delta$ form a coplanar proportion (226), but ϵ and ζ are any two lines in space:

$$\text{I.} \dots \frac{\zeta\delta}{\gamma\epsilon} = \frac{\zeta\beta}{\alpha\epsilon}, \text{ if } \frac{\delta}{\gamma} = \frac{\beta}{\alpha};$$

for, under this last condition, we have (comp. 125),

$$\text{II.} \dots \frac{\zeta\delta}{\gamma\epsilon} = \frac{\zeta\alpha}{\alpha\gamma} \cdot \frac{\delta}{\epsilon} = \frac{\zeta}{\alpha} \cdot \frac{\beta\delta}{\delta\epsilon}.$$

(2.) Another enunciation of the associative principle is the following:

$$\text{III.} \dots \text{if } \frac{\delta\beta}{\gamma\alpha} = \frac{\zeta}{\epsilon}, \text{ then } \frac{\epsilon\beta}{\alpha\gamma} = \frac{\zeta}{\delta};$$

for if we determine (120) six new vectors, $\eta\theta$, and $\kappa\lambda\mu$, so that

$$\text{IV.} \dots \left\{ \begin{array}{l} \frac{\theta}{\eta} = \frac{\delta}{\gamma}, \quad \frac{\eta}{\epsilon} = \frac{\beta}{\alpha}, \quad \text{whence } \frac{\theta}{\epsilon} = \frac{\zeta}{\epsilon}, \\ \text{and} \\ \frac{\lambda}{\kappa} = \frac{\epsilon}{\alpha}, \quad \frac{\kappa}{\mu} = \frac{\beta}{\gamma}, \end{array} \right.$$

* An elementary proof, by *stereographic projection*, will be proposed in the following Section.

we shall have the transformations,

$$V. \dots \frac{\lambda}{\zeta} = \frac{\lambda \epsilon}{\epsilon \theta} = \frac{\lambda}{\epsilon} \cdot \frac{\epsilon \eta}{\eta \theta} = \frac{\lambda \epsilon}{\epsilon \eta} \cdot \frac{\eta}{\theta} = \frac{\kappa \gamma}{\beta \delta} = \frac{\mu}{\delta}, \quad \text{or VI.} \dots \frac{\lambda}{\mu} = \frac{\zeta}{\delta}.$$

(3.) Conversely, the assertion that this last equation or proportion VI. is true, whenever the twelve vectors $\alpha \dots \mu$ are connected by the five proportions IV., is a form of enunciation of the associative principle; for it conducts (comp. IV. and V.) to the equation,

$$VII. \dots \frac{\lambda}{\epsilon} \cdot \frac{\epsilon \eta}{\eta \theta} = \frac{\lambda \epsilon}{\epsilon \eta} \cdot \frac{\eta}{\theta}, \quad \text{at least if } \epsilon \parallel \iota, \theta;$$

but, even with this last restriction, the three factor-quotients in VII. may represent any three quaternions.

SECTION 2.—On some Geometrical Proofs of the Associative Property of Multiplication of Quaternions, which are independent of the Distributive* Principle.

269. We propose, in this Section, to furnish *three* geometrical Demonstrations of the Associative Principle, in connexion with the three Figures (59–61) which were employed in the last Section for its Enunciation; and with the *three arrangements of six planes*, which were described in Art. 263. The two first of these proofs will suppose the knowledge of a few properties of *spherical conics* (196, (11.)); but the third will only employ the doctrine of *stereographic projection*, and will therefore be of a more strictly *elementary character*. The Principle itself is, however, of such great importance in this Calculus, that its nature and its evidence can scarcely be put in too many different points of view.

270. The only properties of a spherical conic, which we shall in this Article assume as known,† are the three following: Ist, that *through any three given points* on a given sphere, which are not on a great circle, a *conic* can be described (consisting generally of *two opposite ovals*), which shall have a *given great circle* for one of its *two cyclic arcs*; IIInd, that if a *transversal arc* cut *both* these arcs, and the conic, the *intercepts* (suitably measured) on this transversal are *equal*; and IIIrd, that if the *vertex* of a spherical angle *move along the conic*, while its *legs* pass always *through two fixed points* thereof, those legs

* Compare 224 and 262; and the Note to page 286.

† The reader may consult the Translation (Dublin, 1841, pp. 46, 50, 55) by the present Dean Graves, of two Memoirs by M. Chasles, on *Cones of the Second Degree, and Spherical Conics*.

intercept a *constant interval*, upon *each* cyclic arc, separately taken. Admitting these three properties, we see that if, in Fig. 59, we conceive a spherical conic to be described, so as to pass through the three points B, F, H, and to have the great circle DAEC for *one* cyclic arc, the second and third equations I. of 264 will prove that the arc GLIM is the *other* cyclic arc for this conic; the first equation I. proves next that the conic passes through K; and if the arcual chord FK be drawn and prolonged, the two remaining equations prove that it meets the cyclic arcs in D and M; after which, the equation II. of the same Art. 264 immediately results, at least with the arrangement* adopted in the Figure.

(1.) The 1st property is easily seen to correspond to the possibility of circumscribing a circle about a given plane triangle, namely that of which the corners are the intersections of a plane parallel to the plane of the given cyclic arc, with the three radii drawn to the three given points upon the sphere: but it may be worth while, as an exercise, to prove here the 2nd property *by quaternions*.

(2.) Take then the equation of a *cyclic cone*, 196, (8.), which may (by 196, XII.) be written thus:

$$\text{I. . . } S \frac{\rho}{\alpha} S \frac{\rho}{\beta} = N \frac{\rho}{\beta}; \quad \text{and let} \quad \text{II. . . } S \frac{\rho'}{\alpha} S \frac{\rho'}{\beta} = N \frac{\rho'}{\beta},$$

ρ and ρ' being thus *two rays* (or *sides*) of the cone, which may also be considered to be the vectors of two points P and P' of a *spherical conic*, by supposing that their lengths are each unity. Let r and r' be the vectors of the two points r and r' on the two cyclic arcs, in which the arcual chord PP' of the conic cuts them; so that

$$\text{III. . . } S \frac{r}{\alpha} = 0, \quad S \frac{r'}{\beta} = 0, \quad \text{and} \quad \text{IV. . . } Tr = Tr' = 1.$$

The theorem may then be stated thus: that

$$\text{V. . . if } \rho = xr + x'r', \quad \text{then} \quad \text{VI. . . } \rho' = x'r + xr';$$

or that this expression VI. satisfies II., if the equations I. III. IV. V. be satisfied. Now, by III. V. VI., we have

$$\text{VII. . . } S \frac{\rho}{\alpha} = x'S \frac{r}{\alpha} = \frac{x}{x'} S \frac{\rho'}{\alpha}, \quad S \frac{\rho}{\beta} = xS \frac{r}{\beta} = \frac{x}{x'} S \frac{\rho'}{\beta};$$

whence it follows that the first members of I. and II. are equal, and it only remains to prove that their second members are equal also, or that $T\rho' = T\rho$, if $Tr' = Tr$. Accordingly we have, by V. and VI.,

$$\text{VIII. . . } \frac{\rho' - \rho}{\rho' + \rho} = \frac{x' - x}{x' + x} \cdot \frac{r - r'}{r + r'} = S^{-1}0, \quad \text{by 200, (11.), and 204, (19.);}$$

and the property in question is proved.

* Modifications of that arrangement may be conceived, to which however it would be easy to adapt the reasoning.

271. To prove the associative principle, with the help of Fig. 60, three other properties of a spherical conic shall be supposed known: * Ist, that for every such curve *two focal points* exist, possessing several important relations to it, one of which is, that if these *two foci* and *one tangent arc* be given, the conic can be constructed; IInd, that if, from any point upon the sphere, *two tangents* be drawn to the conic, and also *two arcs to the foci*, then *one focal arc* makes with *one tangent* the *same angle* as the *other focal arc* with the *other tangent*; and IIIrd, that if a spherical quadrilateral be circumscribed to such a conic (supposed here for simplicity to be a spherical *ellipse*, or the *opposite ellipse* being neglected), *opposite sides subtend supplementary angles*, at *either* of the two (interior) foci. Admitting these known properties, and supposing the arrangement to be as in Fig. 60, we may conceive a conic described, which shall have \mathfrak{E} and \mathfrak{F} for its two focal points, and shall touch the arc BC ; and then the two first of the equations I., in 265, will prove that it touches also the arcs AB and CD , while the third of those equations proves that it touches AD , so that $ABCD$ is a circumscribed† quadrilateral: after which the three equations II., of the same article, are consequences of the same properties of the curve.

272. Finally, to prove the same important Principle in a more completely elementary way, by means of the arrangement represented in Fig. 61, or to prove the theorem of spherical geometry enunciated in Art. 267, we may assume the point D as the *pole* of a *stereographic projection*, in which the three small circles through that point shall be represented by *right lines*, but the three others by *circles*,

all being in one *common plane*. And then (interchanging accents) the theorem comes to be thus stated:

If A', B', C' be any three points (comp. Fig. 62) on the sides BC, CA, AB of any plane triangle, or on those sides prolonged, then, Ist, the three circles,

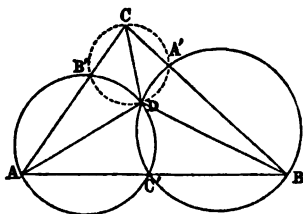


Fig. 62.

* The reader may again consult pages 46 and 50 of the Translation lately cited. In strictness, there are of course *four foci*, opposite two by two.

† The writer has elsewhere proposed the notation, $\mathfrak{EF}(\dots)ABCD$, to denote the relation of the focal points $\mathfrak{E}, \mathfrak{F}$ to this circumscribed quadrilateral.

I. . . C'AB', A'BC', B'CA',

will meet in one point D; and IInd, an even number (if any) of the six (linear or circular) successions,

II. . . AB'C, BC'A, CA'B, and II'. . . C'AB'D, A'BC'D, B'CA'D, will be *direct*; an *even* number therefore *also* (if any) being *indirect*. But, under this form,* the theorem can be proved by very elementary considerations, and still without any employment of the *distributive principle* (224, 262).

(1.) The *first part* of the theorem, as thus stated, is evident from the Third Book of Euclid; but to prove *both parts* together, it may be useful to proceed as follows, admitting the conception (235) of *amplitudes*, or of angles as representing *rotations*, which may have any values, positive or negative, and are to be *added* with attention to their *signs*.

(2.) We may thus write the three equations,

$$\text{III. . . } AB'C = n\pi, \quad BC'A = n'\pi, \quad CA'B = n''\pi,$$

to express the three *collineations*, AB'C, &c. of Fig. 62; the integer, n , being *odd* or *even*, according as the point B' is on the finite line AC, or on a prolongation of that line; or in other words, according as the first *succession* II. is *direct* or *indirect*: and similarly for the two other coefficients, n' and n'' .

(3.) Again, if OPQR be any four points in one plane, we may establish the formula,

$$\text{IV. . . } POQ + QOR = POR + 2m\pi,$$

with the same conception of addition of amplitudes; if then D be any point in the plane of the triangle ABC, we may write,

$$\text{V. . . } AB'D + DB'C = n\pi, \quad BC'D + DC'A = n'\pi, \quad CA'D + DA'B = n''\pi;$$

and therefore,

$$\text{VI. . . } (AB'D + DC'A) + (BC'D + DA'B) + (CA'D + DB'C) = (n + n' + n'')\pi.$$

(4.) Again, if any four points OPQR be not merely *coplanar* but *concurrent*, we have the general formula,

$$\text{VII. . . } OPQ + QRO = p\pi,$$

the integer p being *odd* or *even*, according as the succession OPQR is *direct* or *indirect*.

* The Associative Principle of Multiplication was stated nearly under this form, and was illustrated by the same simple diagram, in paragraph XXII. of a communication by the present author, which was entitled *Letters on Quaternions*, and has been printed in the First and Second Editions of the late Dr. Nichol's *Cyclopædia of the Physical Sciences* (London and Glasgow, 1857 and 1860). The same communication contained other illustrations and consequences of the same principle, which it has not been thought necessary here to reproduce (compare however Note C); and others may be found in the Sixth of the author's already cited *Lectures on Quaternions* (Dublin, 1853), from which (as already observed) some of the formulæ and figures of this Chapter have been taken.

rect; if then we denote by D the *second intersection* of the first and second circles I., whereof C' is a *first intersection*, we shall have

$$\text{VIII.} \dots AB'D + DC'A = p\pi, \quad BC'D + DA'B = p'\pi,$$

p and p' being *odd*, when the two first successions II'. are *direct*, but *even* in the *contrary case*.

(5.) Hence, by VI., we have,

$$\text{IX.} \dots CA'D + DB'C = p''\pi, \quad \text{where X.} \dots p + p' + p'' = \pi + \pi' + \pi'';$$

the *third succession* II'. is therefore *always circular*, or the *third circle* I. passes *through the intersection* D of the *two first*; and it is *direct* or *indirect*, that is to say, p'' is *odd* or *even*, according as the *number of even coefficients*, among the *five* previously considered, is itself *even* or *odd*; or in other words, according as the *number of indirect successions*, among the *five* previously considered, is *even* (including zero), or *odd*.

(6.) In every case, therefore, the *total number of successions of each kind* is *even*, and *both parts* of the theorem are proved: the importance of the *second part* of it (respecting the *even partition*, if any, of the *six successions* II. II'.) arising from the necessity of proving that we have *always*, as in algebra,

$$\text{XI.} \dots sr.q = +s.rq, \quad \text{and never XII.} \dots sr.q = -s.rq,$$

if q, r, s be any three actual quaternions.

(7.) The *associative principle* of multiplication may also be proved, without the *distributive principle*, by certain considerations of *rotations of a system*, on which we cannot enter here.

SECTION 3.—On some Additional Formulæ.

273. Before concluding the Second Book, a few additional remarks may be made, as regards some of the notations and transformations which have already occurred, or others analogous to them. And first as to *notation*, although we have reserved for the Third Book the *interpretation* of such expressions as βa , or a^2 , yet we have agreed, in 210, (9.), to *abridge* the frequently occurring *symbol* $(Ta)^2$ to Ta^2 ; and we *now* propose to *abridge* it still further to Na , and to *call* this *square of the tensor* (or of the *length*) of a vector, a , the *Norm of that Vector*: as we had (in 190, &c.), the equation $Tq^2 = Nq$, and called Nq the *norm* of the *quaternion* q (in 145, (11.)). We shall therefore now write generally, for any vector a , the formula,

$$\text{I.} \dots (Ta)^2 = Ta^2 = Na.$$

(1.) The equations (comp. 186, (1.) (2.) (3.) (4.)),

$$\text{II.} \dots N\rho = 1; \quad \text{III.} \dots N\rho = Na; \quad \text{IV.} \dots N(\rho - a) = Na;$$

$$\text{V.} \dots N(\rho - a) = N(\beta - a),$$

represent, respectively, the *unit-sphere*; the sphere through A , with o for centre; the sphere through o , with A for centre; and the sphere through B , with the same centre A .

(2.) The equations (comp. 186, (6.) (7.)),

$$\text{VI.} \dots N(\rho + \alpha) = N(\rho - \alpha); \quad \text{VII.} \dots N(\rho - \beta) = N(\rho + \alpha),$$

represent, respectively, the *plane* through o, perpendicular to the line oA; and the *plane* which *perpendicularly bisects* the line AB.

274. As regards *transformations*, the few following may here be added, which relate partly to the *quaternion forms* (204, 216, &c.) of the *Equation* of the Ellipsoid*.

(1.) Changing $K(\kappa : \rho)$ to $R\rho : R\kappa$, by 259, VIII., in the equation 217, XVI. of the ellipsoid, and observing that the three vectors ρ , $R\rho$, and $R\kappa$ are coplanar, while $1 : T\rho = TR\rho$ by 258, that equation becomes, when divided by $TR\rho$, and when the value 217, (5.) for ι^2 is taken, and the notation 273 is employed :

$$\text{I.} \dots T \left(\frac{\iota}{R\rho} + \frac{\rho}{R\kappa} \right) = N\iota - N\kappa;$$

of which the first member will soon be seen to admit of being written† as $T(\iota\rho + \rho\kappa)$, and the second member as $\kappa^2 - \iota^2$.

(2.) If, in connexion with the earlier forms (204, 216) of the equation of the same surface, we introduce a *new auxiliary vector*, σ or os , such that (comp. 216, VIII.)

$$\text{II.} \dots \sigma = \left(S \frac{\rho}{\alpha} + V \frac{\rho}{\beta} \right) \beta = \rho + 2\beta S \frac{\rho}{\delta},$$

the equation may, by 204, (14.), be reduced to the following extremely simple form :

$$\text{III.} \dots T\sigma = T\beta;$$

which expresses that the *locus* of the *new auxiliary point* s is what we have called the *mean sphere*, 216, XIV.; while the *line* rs , or $\sigma - \rho$, which *connects* any two *corresponding points*, r and s, on the ellipsoid and sphere, is seen to be *parallel to* the *fixed line* β ; which is *one element of the homology*, mentioned in 216, (10.).

(3.) It is easy to prove that

$$\text{IV.} \dots S \frac{\sigma}{\delta} = S \frac{\beta}{\alpha} S \frac{\rho}{\delta}, \quad \text{and therefore} \quad \text{V.} \dots S \frac{\sigma'}{\delta} : S \frac{\sigma}{\delta} = S \frac{\rho'}{\delta} : S \frac{\rho}{\delta}$$

if ρ' and σ' be the vectors of two new but corresponding points, r' and s' , on the ellipsoid and sphere; whence it is easy to infer this *other element of the homology*, that any two *corresponding chords*, rr' and ss' , of the two surfaces, *intersect each other on the cyclic plane* which has δ for its *cyclic normal* (comp. 216, (7.)) : in fact, they intersect in the point τ of which the vector is,

$$\text{VI.} \dots \tau = \frac{x\rho + x'\rho'}{x + x'} = \frac{x\sigma + x'\sigma'}{x + x'}, \quad \text{if} \quad x = S \frac{\rho'}{\delta}, \quad \text{and} \quad x' = -S \frac{\rho}{\delta};$$

* In the verification 216, (2.) of the equation 216, (1.), considered as representing a *surface of the second order*, $V \frac{\lambda}{\beta}$ and $V \frac{\mu}{\beta}$ ought to have been printed, instead of $V \frac{\lambda}{\alpha}$ and $V \frac{\mu}{\alpha}$; but this does not affect the reasoning.

† Compare the Note to page 283.

and this point is on the plane just mentioned (comp. 216, XI.), because

$$\text{VII.} \dots S \frac{r}{\delta} = 0.$$

(4.) Quite similar results would have followed, if we had assumed

$$\text{VIII.} \dots \sigma = \left(-S \frac{\rho}{\alpha} + V \frac{\rho}{\beta} \right) \beta = \rho - 2\beta S \frac{\rho}{\gamma},$$

which would have given again, as in III.,

$$\text{IX.} \dots T\sigma = T\beta, \quad \text{but with} \quad \text{X.} \dots S \frac{\sigma}{\gamma} = -S \frac{\beta}{\alpha} S \frac{\rho}{\gamma};$$

the other cyclic plane, with γ instead of δ for its normal, might therefore have been taken (as asserted in 216, (10.)), as another plane of homology of ellipsoid and sphere, with the same centre of homology as before: namely, the point at infinity on the line β , or on the axis (204, (15.)) of one of the two circumscribed cylinders of revolution (comp. 220, (4.)).

(5.) The same ellipsoid is, in two other ways, homologous to the same mean sphere, with the same two cyclic planes as planes of homology, but with a new centre of homology, which is the infinitely distant point on the axis of the second circumscribed cylinder (or on the line AB' of the sub-article last cited).

(6.) Although not specially connected with the ellipsoid, the following general transformations may be noted here (comp. 199, XII., and 204, XXXIV.):

$$\text{XI.} \dots TV \vee q = \vee \left\{ \frac{1}{2}(Tq - Sq) \right\}; \quad \text{XII.} \dots \tan \frac{1}{2} \angle q = (TV : S) \vee q = \sqrt{\frac{Tq - Sq}{Tq + Sq}}.$$

(7.) The equations 204, XVI. and XXXV., give easily,

$$\text{XIII.} \dots UVq = UVUq; \quad \text{XIV.} \dots UIVq = Ax. q; \quad \text{XV.} \dots TIVq = TVq;$$

or the more symbolical forms,

$$\text{XIII'.} \dots UVU = UV; \quad \text{XIV'.} \dots UIV = Ax.; \quad \text{XV'.} \dots TIV = TV;$$

and the identity 200, IX. becomes more evident, when we observe that

$$\text{XVI.} \dots q - Nq = q(1 - Kq).$$

(8.) We have also generally (comp. 200, (10.) and 218, (10.)),

$$\text{XVII.} \dots \frac{q-1}{q+1} = \frac{(q-1)(Kq+1)}{(q+1)(Kq+1)} = \frac{Nq-1+2Vq}{Nq+1+2Sq}.$$

(9.) The formula,*

$$\text{XVIII.} \dots U(rq + Kqr) = U(Sr. Sq + Vr. Vq) = r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1},$$

in which q and r may be any two quaternions, is not perhaps of any great importance in itself, but will be found to furnish a student with several useful exercises in transformation.

(10.) When it was said, in 257, (1.), that zero had only itself for a square-root, the meaning was (comp. 225), that no binomial expression of the form $x + iy$ (228) could satisfy the equation,

$$\text{XIX.} \dots 0 = q^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy,$$

* This formula was given, but in like manner without proof, in page 587 of the author's Lectures on Quaternions.

for any real or imaginary values of the two scalar coefficients x and y , different from zero;* for if *biquaternions* (214, (8.)) be admitted, and if \mathbf{h} again denote, as in 256, (2.), the *imaginary of algebra*, then (comp. 257, (6.) and (7.)) we may write, generally, besides the real value $0\mathbf{h} = 0$, the *imaginary expression*,

$$\text{XX.} \dots 0\mathbf{h} = v + hv', \text{ if } Sv = Sv' = Svv' = Nv' - Nv = 0;$$

v and v' being thus any two real right quaternions, with equal norms (or with equal tensors), in planes perpendicular to each other.

(11.) For example, by 256, (2.) and by the laws (188) of ijk , we have the transformations,

$$\text{XXI.} \dots (i + hj)^2 = i^2 - j^2 + h(ij + ji) = 0 + h0 = 0;$$

so that the bi-quaternion $i + hj$ is one of the imaginary values of the symbol $0\mathbf{h}$.

(12.) In general, when *bi-quaternions* are admitted into calculation, not only the square of one, but the product of two such factors may vanish, without either of them separately vanishing: a circumstance which may throw some light on the existence of those *imaginary (or symbolical) roots of equations*, which were treated of in 257.

(13.) For example, although the equation

$$\text{XXII.} \dots q^2 - 1 = (q - 1)(q + 1) = 0$$

has no real roots except ± 1 , and therefore cannot be verified by the substitution of any other real scalar, or real quaternion, for q , yet if we substitute for q the *bi-quaternion* $\dagger v + hv'$, with the conditions 257, XIII., this equation XXII. is verified.

(14.) It will be found, however, that when two *imaginary* but *non-evanescent* factors give thus a null product, the norm of each is zero; provided that we agree to extend to *bi-quaternions* the formula $Nq = Sq^2 - Vq^2$ (204, XXI.); or to define that the Norm of a Biquaternion (like that of an ordinary or real quaternion) is equal to the Square of the Scalar Part, minus the Square of the Right Part: each of these two parts being generally *imaginary*, and the former being what we have called a *Bi-scalar*.

(15.) With this definition, if q and q' be any two real quaternions, and if \mathbf{h} be, as above, the ordinary imaginary of algebra, we may establish the formula:

$$\text{XXIII.} \dots N(q + hq') = (Sq + hSq')^2 - (Vq + hVq')^2;$$

or (comp. 200, VII., and 210, XX.),

$$\text{XXIV.} \dots N(q + hq') = Nq - Nq' + 2\mathbf{h}S.qKq'.$$

(16.) As regards the norm of the sum of any two real quaternions, or real vectors (278), the following transformations are occasionally useful (comp. 220, (2.)):

$$\text{XXV.} \dots N(q' + q) = N(Tq'.Uq + Tq.Uq');$$

$$\text{XXVI.} \dots N(\beta + \alpha) = N(T\beta.U\alpha + T\alpha.U\beta);$$

in each of which it is permitted to change the norms to the tensors of which they are the squares, or to write T for N .

* Compare the Note to page 276.

† This includes the expression $\pm hi$, of 257, (1.), for a *symbolical square-root of positive unity*. Other such roots are $\pm hj$, and $\pm hk$.

BOOK III.

ON QUATERNIONS, CONSIDERED AS PRODUCTS OR POWERS OF VECTORS; AND ON SOME APPLICATIONS OF QUATERNIONS.

CHAPTER I.

ON THE INTERPRETATION OF A PRODUCT OF VECTORS, OR POWER OF A VECTOR, AS A QUATERNION.

SECTION 1.—*On a First Method of interpreting a Product of Two Vectors as a Quaternion.*

ART. 275. In the First Book of these *Elements* we interpreted, Ist, the *difference* of any two directed right lines in space (4); IInd, the *sum* of two or more such lines (5-9); IIIrd, the *product* of one such line, multiplied by or into a positive or negative *number* (15); IVth, the *quotient* of such a line, divided by such a number (16), or by what we have called generally a SCALAR (17); and Vth, the sum of a *system* of such lines, each affected (97) with a scalar *coefficient* (99), as being in each case *itself* (generally) a *Directed Line** in Space, or what we have called a VECTOR (1).

276. In the Second Book, the fundamental principle or pervading conception has been, that the *Quotient of two such Vectors* is, generally, a QUATERNION (112, 116). It is however to be remembered, that we have *included* under this general conception, which *usually* relates to what may be called an *Oblique Quotient*, or the quotient of two lines in space making either an *acute* or an *obtuse angle* with each other

* The *Fourth Proportional* to any three coplanar lines has also been since interpreted (226), as being *another line* in the same plane.

(130), the *three* following particular cases: Ist, the *limiting case*, when the angle becomes *null*, or when the two lines are *similarly directed*, in which case the quotient *degenerates* (131) into a *positive scalar*; IInd, the *other limiting case*, when the angle is equal to *two right angles*, or when the lines are *oppositely directed*, and when in consequence the quotient *again degenerates*, but now into a *negative scalar*; and IIIrd, the *intermediate case*, when the angle is *right*, or when the two lines are *perpendicular* (132), instead of being *parallel* (15), and when therefore their quotient becomes what we have called (132) a *Right Quotient*, or a **RIGHT QUATERNION**: which has been seen to be a case not less important than the two former ones.

277. But *no Interpretation* has been assigned, in either of the two foregoing Books, for a **PRODUCT** of *two or more Vectors*; or for the **SQUARE**, or other **POWER** of a *Vector*: so that the *Symbols*,

$$\text{I.} \dots \beta a, \gamma \beta a, \dots \quad \text{II.} \dots a^2, a^3, \dots a^{-1}, \dots a'$$

in which $a, \beta, \gamma \dots$ denote *vectors*, but t denotes a *scalar*, remain as yet entirely *uninterpreted*; and we are therefore *free* to assign, at this stage, *any meanings* to these *new symbols*, or *new combinations* of symbols, which shall *not contradict each other*, and shall appear to be consistent with *convenience and analogy*. And to do so will be the chief object of this First Chapter of the Third (and last) Book of these *Elements*: which is designed to be a much shorter one than either of the foregoing.

278. As a commencement of such *Interpretation* we shall here *define*, that a *vector* a is multiplied by another vector β , or that the *latter* vector is multiplied *into** the *former*, or that the *product* βa is obtained, when the *multiplier-line* β is divided by the *reciprocal* Ra (258) of the *multiplicand-line* a ; as we had *proved* (136) that one *quaternion* is multiplied *into another*, when it is divided by the *reciprocal* thereof. In symbols, we shall therefore write, as a *first definition*, the formula:

* Compare the Notes to pages 146, 159.

I. . . $\beta\alpha = \beta : R\alpha$; where II. . . $R\alpha = -U\alpha : T\alpha$ (258, VII.).
 And we proceed to consider, in the following Section, some of the general consequences of this definition, or interpretation, of a *Product of two Vectors*, as being equal to a certain *Quotient*, or *Quaternion*.

SECTION 2.—On some Consequences of the foregoing Interpretation.

279. The definition (278) gives the formula :

$$\text{I. . . } \beta\alpha = \frac{\beta}{R\alpha}; \text{ and similarly, } \text{I'. . . } \alpha\beta = \frac{\alpha}{R\beta};$$

it gives therefore, by 259, VIII., the general relation,

$$\text{II. . . } \beta\alpha = K\alpha\beta; \text{ or } \text{II'. . . } \alpha\beta = K\beta\alpha.$$

The *Products of two Vectors*, taken in two opposite orders, are therefore *Conjugate Quaternions*; and the *Multiplication of Vectors*, like that of *Quaternions* (168), is (generally) a *Non-Commutative Operation*.

(1.) It follows from II. (by 196, comp. 223, (1.)), that

$$\text{III. . . } S\beta\alpha = +S\alpha\beta = \frac{1}{2}(\beta\alpha + \alpha\beta).$$

(2.) It follows also (by 204, comp. again 223, (1.)), that

$$\text{IV. . . } V\beta\alpha = -V\alpha\beta = \frac{1}{2}(\beta\alpha - \alpha\beta).$$

280. Again, by the same general formula 259, VIII., we have the transformations,

$$\text{I. . . } \frac{\beta}{R(\alpha + \alpha')} = K \frac{\alpha + \alpha'}{R\beta} = K \frac{\alpha}{R\beta} + K \frac{\alpha'}{R\beta} = \frac{\beta}{R\alpha} + \frac{\beta}{R\alpha'};$$

it follows, then, from the definition (278), that

$$\text{II. . . } \beta(\alpha + \alpha') = \beta\alpha + \beta\alpha';$$

whence also, by taking conjugates (279), we have this other general equation,

$$\text{III. . . } (\alpha + \alpha')\beta = \alpha\beta + \alpha'\beta.$$

Multiplication of Vectors is, therefore, like that of *Quaternions* (212), a *Doubly Distributive Operation*.

281. As we have not yet assigned any signification for a *ternary product of vectors*, such as $\gamma\beta\alpha$, we are not yet pre-

pared to pronounce, whether the *Associative Principle* (223) of *Multiplication of Quaternions* does or does not extend to *Vector-Multiplication*. But we can already derive several other consequences from the definition (278) of a *binary product*, βa ; among which, attention may be called to the *Scalar character* of a *Product of two Parallel Vectors*; and to the *Right character* of a *Product of two Perpendicular Vectors*, or of two lines at right angles with each other.

(1.) The definition (278) may be thus written,

$$\text{I. . . } \beta a = -T\beta . Ta . U(\beta : a);$$

it gives, therefore,

$$\text{II. . . } T\beta a = T\beta . Ta; \quad \text{III. . . } U\beta a = -U(\beta : a) = U\beta . Ua;$$

the *tensor* and *versor* of the *product* of two *vectors* being thus *equal* (as for quaternions, 191) to the *product of the tensors*, and to the *product of the versors*, respectively.

(2.) Writing for abridgment (comp. 208),

$$\text{IV. . . } a = Ta, \quad b = T\beta, \quad \gamma = Ax . (\beta : a), \quad x = \angle(\beta : a),$$

we have thus,

$$\text{V. . . } T\beta a = ba; \quad \text{VI. . . } S\beta a = Sa\beta = -ba \cos x;$$

$$\text{VII. . . } SU\beta a = SUa\beta = -\cos x; \quad \text{VIII. . . } \angle \beta a = \pi - x;$$

so that (comp. 198) the *angle of the product* of any two vectors is the *supplement of the angle of the quotient*.

(3.) We have next the transformations (comp. again 208),

$$\text{IX. . . } TV\beta a = TVa\beta = ba \sin x; \quad \text{X. . . } TVU\beta a = TVUa\beta = \sin x;$$

$$\text{XI. . . } IV\beta a = -\gamma ba \sin x; \quad \text{XI'. . . } IVa\beta = +\gamma ab \sin x;$$

$$\text{XII. . . } IUV\beta a = Ax . \beta a = -\gamma; \quad \text{XII'. . . } IUVa\beta = Ax . a\beta = +\gamma;$$

so that the *rotation round the axis of a product of two vectors, from the multiplier to the multiplicand, is positive*.

(4.) It follows also, by IX., that the *tensor of the right part* of such a *product*, βa , is equal to the *parallelogram under the factors*; or to the *double of the area of the triangle OAB*, whereof those two factors a, β , or OA, OB , are two *coinitial sides*: so that if we denote here this last-mentioned *area* by the symbol

$$\Delta OAB,$$

we may write the equation,

$$\text{XIII. . . } TV\beta a = \text{parallelogram under } a, \beta = 2\Delta OAB;$$

and the *index*, $IV\beta a$, is a *right line perpendicular to the plane of this parallelogram*, of which line the *length represents its area*, in the sense that they bear *equal ratios to their respective units* (of length and of area).

(5.) Hence, by 279, IV.,

$$\text{XIV. . . } T(\beta a - a\beta) = 2 \times \text{parallelogram} = 4 \Delta OAB.$$

(6.) For any two vectors, a, β ,

XV. . . $S\beta a = -Na.S(\beta : a)$; XVI. . . $V\beta a = -Na.V(\beta : a)$;

or briefly,*

XVII. . . $\beta a = -Na.(\beta : a)$,

with the signification (278) of Na , as denoting $(Ta)^2$.

(7.) If the two factor-lines be *perpendicular* to each other, so that x is a *right angle*, then the *parallelogram* (4.) becomes a *rectangle*, and the *product* βa becomes a *right quaternion* (182); so that we may write,

XVIII. . . $S\beta a = S\alpha\beta = 0$, if $\beta \perp a$, and reciprocally.

(8.) Under the same condition of perpendicularity,

XIX. . . $\angle \beta a = \angle \alpha\beta = \frac{\pi}{2}$; XX. . . $I\beta a = -\gamma b a$; XXI. . . $I\alpha\beta = +\gamma ab$.

(9.) On the other hand, if the two factor-lines be *parallel*, the *right part* of their product vanishes, or that product reduces itself to a *scalar*, which is *negative* or *positive* according as the two vectors multiplied have *similar* or *opposite directions*; for we may establish the formula,

XXII. . . if $\beta \parallel a$, then $V\beta a = 0$, $V\alpha\beta = 0$;

and, under the same condition of parallelism,

XXIII. . . $\beta a = \alpha\beta = S\beta a = S\alpha\beta = \mp ba$,

the upper or the lower sign being taken, according as $x = 0$, or π .

(10.) We may also write (by 279, (1.) and (2.)) the following formula of perpendicularity, and formula of parallelism :

XXIV. . . if $\beta \perp a$, then $\beta a = -\alpha\beta$, and reciprocally;

XXV. . . if $\beta \parallel a$, then $\beta a = +\alpha\beta$, with the converse.

(11.) If α, β, γ be any three unit-lines, considered as vectors of the corners A, B, C of a spherical triangle, with sides equal to three new positive scalars, a, b, c , then because, by XVII., $\beta a = -\beta : a$, and $\gamma\beta = -\gamma : \beta$, the sub-articles to 208 allow us to write,

XXVI. . . $S(V\gamma\beta.V\beta a) = \sin a \sin c \cos b$;

XXVII. . . $IV(V\gamma\beta.V\beta a) = \pm \beta \sin a \sin c \sin b$;

XXVIII. . . $(IV : S)(V\gamma\beta.V\beta a) = \pm \beta \tan b$;

upper or lower signs being taken, in the two last formulae, according as the rotation round β from a to γ , or that round b from A to C , is positive or negative.

(12.) The equation 274, I., of the *Ellipsoid*, may now be written thus :

XXIX. . . $T(\iota\rho + \rho\kappa) = T\iota^2 - T\kappa^2$; or XXX. . . $T(\iota\rho + \rho\kappa) = N\iota - N\kappa$.

282. Under the general head of a *product of two parallel vectors*, two interesting cases occur, which furnish two first examples of *Powers of Vectors* : namely, 1st, the case when

* All the consequences of the interpretation (278), of the product βa of two vectors, might be deduced from this formula XVII.; which, however, it would not have been so natural to have assumed for a definition of that symbol, as it was to assume the formula 278, I.

the two factors are *equal*, which gives this remarkable result, that *the Square of a Vector is always equal to a Negative Scalar*; and IInd, the case when the factors are (in the sense already defined, 258) *reciprocal* to each other, in which case it follows from the definition (278) that their *product* is equal to *Positive Unity*: so that *each* may, in this case, be considered as equal to *unity divided by the other*, or to the *Power* of that other which has *Negative Unity* for its *Exponent*.

(1.) When $\beta = \alpha$, the product $\beta\alpha$ reduces itself to what we may call the *square* of α , and may denote by α^2 ; and thus we may write, as a particular but important case of 281, XXIII., the formula (comp. 278),

$$\text{I. . . } \alpha^2 = -\alpha^2 = -(\text{Ta})^2 = -\text{Na};$$

so that the *square of any vector* α is equal to the *negative of the norm* (273) of that vector; or to the *negative of the square of the number* Ta , which expresses (185) the *length* of the same vector.

(2.) More immediately, the definition (278) gives,

$$\text{II. . . } \alpha^2 = \alpha\alpha = \alpha : \text{Ra} = -(\text{Ta})^2 = -\text{Na}, \text{ as before.}$$

(3.) Hence (compare the notations 161, 190, 199, 204),

$$\text{III. . . } \text{S. } \alpha^2 = -\text{Na}; \quad \text{IV. . . } \text{V. } \alpha^2 = 0;$$

and

$$\text{V. . . } \text{T. } \alpha^2 = \text{T}(\alpha^2) = +\text{Na} = (\text{Ta})^2 = \text{Ta}^2;$$

the omission of the *parentheses*, or of the *point*, in this last symbol of a tensor,* for the square of a *vector*, as well as for the square of a *quaternion* (190), being thus justified: and in like manner we may write,

$$\text{VI. . . } \text{U. } \alpha^2 = \text{U}(\alpha^2) = -1 = (\text{U}\alpha)^2 = \text{U}\alpha^2;$$

the *square of an unit-vector* (129) being always equal to *negative unity*, and *parentheses* (or *points*) being *again* omitted.

(4.) The equation

$$\text{VII. . . } \rho^2 = \alpha^2, \text{ gives VII'. . . } \text{N}\rho = \text{Na}, \text{ or VII''. . . } \text{T}\rho = \text{Ta};$$

it represents therefore, by 186, (2.), the *sphere* with o for centre, which passes through the point A .

(5.) The more general equation,

$$\text{VIII. . . } (\rho - \alpha)^2 = (\beta - \alpha)^2, \quad (\text{comp. } \dagger \text{ 186, (4.),})$$

represents the sphere with A for centre, which passes through the point B .

(6.) For example, the equation,

$$\text{IX. . . } (\rho - \alpha)^2 = \alpha^2, \quad (\text{comp. } 186, (3.),)$$

represents the sphere with A for centre, which passes through the origin o .

* Compare the Note to page 210.

† Compare also the sub-articles to 278.

(7.) The equations (comp. 186, (6.), (7.)),

$$\text{X.} \dots (\rho + \alpha)^2 = (\rho - \alpha)^2; \quad \text{XI.} \dots (\rho - \beta)^2 = (\rho - \alpha)^2,$$

represent, respectively, the *plane* through *o*, perpendicular to the line *oA*; and the plane which perpendicularly bisects the line *AB*.

(8.) The *distributive principle of vector-multiplication* (280), and the formula 279, III., enable us to establish generally (comp. 210, (9.)) the formula,

$$\text{XII.} \dots (\beta \pm \alpha)^2 = \beta^2 \pm 2S\beta\alpha + \alpha^2;$$

the recent equations IX. and X. may therefore be thus transformed :

$$\text{IX'.} \dots \rho^2 = 2S\alpha\rho; \quad \text{and} \quad \text{X'.} \dots S\alpha\rho = 0.$$

(9.) The equations,

$$\text{XIII.} \dots \rho^2 + \alpha^2 = 0; \quad \text{XIV.} \dots \rho^2 + 1 = 0,$$

represent the spheres with *o* for centre, which have *a* and 1 for their respective radii ; so that this very simple formula, $\rho^2 + 1 = 0$, is (comp. 186, (1.)) a *form of the Equation of the Unit-Sphere* (128), and is, as such, of great importance in the present Calculus.

(10.) The equation,

$$\text{XV.} \dots \rho^2 - 2S\alpha\rho + c = 0,$$

may be transformed to the following,

$$\text{XVI.} \dots N(\rho - \alpha) = -(\rho - \alpha)^2 = c - \alpha^2 = e + N\alpha;$$

or

$$\text{XVI'.} \dots T(\rho - \alpha) = \sqrt{c - \alpha^2} = \sqrt{e + N\alpha};$$

it represents therefore a (real or imaginary) sphere, with *A* for *centre*, and with this last radical (if real) for *radius*.

(11.) This *sphere* is therefore necessarily *real*, if *c* be a *positive* scalar ; or if this scalar constant, *c*, though *negative*, be (algebraically) *greater than* α^2 , or than $-N\alpha$; but it becomes *imaginary*, if $c + N\alpha < 0$.

(12.) The *radical plane of the two spheres*,

$$\text{XVII.} \dots \rho^2 - 2S\alpha\rho + c = 0, \quad \rho^2 - 2S\alpha'\rho + c' = 0,$$

has for equation,

$$\text{XVIII.} \dots 2S(\alpha' - \alpha)\rho = c' - c;$$

it is therefore *always real*, if the given *vectors* *a*, *a'* and the given *scalars* *e*, *e'* be such, even if one or both of the *spheres themselves* be *imaginary*.

(13.) The equation 281, XXIX., or XXX., of the *Central Ellipsoid* (or of the ellipsoid with its *centre* taken for the *origin* of vectors), may now be still further simplified,* as follows :

$$\text{XIX.} \dots T(\rho + \rho\kappa) = \kappa^2 - \iota^2.$$

(14.) The definition (278) gives also,

$$\text{XX.} \dots aRa = a : a = 1; \quad \text{or} \quad \text{XX'.} \dots Ra.a = Ra : Ra = 1;$$

whence it is natural to write,†

* Compare the Notes to page 288.

† Compare the second Note to page 279.

$$\text{XXI.} \dots R\alpha = 1 : \frac{1}{\alpha} = \alpha^{-1},$$

if we so far anticipate here the general theory of *powers of vectors*, above alluded to (277), as to use this last symbol to denote the *quotient*, of *unity divided by the vector* α ; so as to have *identically*, or for *every* vector, the equation,

$$\text{XXII.} \dots \alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1.$$

(15.) It follows, by 258, VII., that

$$\text{XXIII.} \dots \alpha^{-1} = -U\alpha : T\alpha; \text{ and } \text{XXIV.} \dots \beta\alpha = \beta : \alpha^{-1}.$$

(16.) If we had adopted the equation XXIII. as a *definition** of the symbol α^{-1} , then the formula XXIV. might have been used, as a *formula of interpretation* for the symbol $\beta\alpha$. But we proceed to consider an entirely *different method*, of arriving at the same (or an *equivalent*) *Interpretation* of this latter symbol: or of a *Binary Product of Vectors*, considered as equal to a *Quaternion*.

SECTION 3.—On a Second Method of arriving at the same Interpretation, of a Binary Product of Vectors.

283. It cannot fail to have been observed by any attentive reader of the Second Book, how close and intimate a *connexion* † has been found to exist, between a *Right Quaternion* (132), and its *Index*, or *Index-Vector* (133). Thus, if v and v' denote (as in 223, (1.), &c.) any two *right* quaternions, and if Iv , Iv' denote, as usual, their *indices*, we have already seen that

$$\text{I.} \dots Iv' = Iv, \text{ if } v' = v, \text{ and conversely (133);}$$

$$\text{II.} \dots I(v' \pm v) = Iv' \pm Iv \text{ (206);}$$

$$\text{III.} \dots Iv' : Iv = v' : v \text{ (193);}$$

to which may be added the more recent formula,

$$\text{IV.} \dots RIv = IRv \text{ (258, IX.).}$$

284. It could not therefore have appeared strange, if we had proposed to establish this new formula of the same kind,

$$\text{I.} \dots Iv' \cdot Iv = v' \cdot v = v'v,$$

as a *definition* (supposing that the recent definition 278 had not occurred to us), whereby to *interpret the product of any two indices of right quaternions*, as being equal to the *product of those two quaternions themselves*. And then, to *interpret the product* $\beta\alpha$, of any two given vectors, taken in a given order,

* Compare the Note to page 305.

† Compare the Note to page 174.

we should only have had to conceive (as we always may), that the two proposed *factors*, α and β , are the *indices* of two right quaternions, v and v' , and to multiply these latter, in the same order. For thus we should have been led to establish the formula,

$$\text{II.} \dots \beta\alpha = v'v, \text{ if } \alpha = Iv, \text{ and } \beta = Iv';$$

or we should have this slightly more *symbolical* equation,

$$\text{III.} \dots \beta\alpha = \beta \cdot \alpha = I^{-1}\beta \cdot I^{-1}\alpha;$$

in which the symbols,

$$I^{-1}\alpha \text{ and } I^{-1}\beta,$$

are understood to denote the two right quaternions, whereof the two lines α and β are the indices.

(1.) To establish now the substantial *identity* of these two interpretations, 278 and 284, of a binary product of vectors $\beta\alpha$, notwithstanding the difference of form of the *definitional equations* by which they have been expressed, we have only to observe that it has been found, as a *theorem* (194), that

$$\text{IV.} \dots v'v = Iv' : I(1 : v) = Iv' : IRv;$$

but the definition (258) of $R\alpha$ gave us the lately cited equation, $RIv = IRv$; we have therefore, by the recent formula II., the equation,

$$\text{V.} \dots Iv' \cdot Iv = Iv' : RIv; \text{ or VI.} \dots \beta \cdot \alpha = \beta : Ra,$$

as in 278, I.; α and β still denoting any two vectors. The two interpretations therefore coincide, at least in their results, although they have been obtained by different processes, or suggestions, and are expressed by two different formulæ.

(2.) The result 279, II., respecting conjugate products of vectors, corresponds thus to the result 191, (2.), or to the first formula of 223, (1.).

(3.) The two formulæ of 279, (1.) and (2.), respecting the scalar and right parts of the product $\beta\alpha$, answer to the two other formulæ of the same sub-article, 223, (1.), respecting the corresponding parts of $v'v$.

(4.) The doubly distributive property (280), of vector-multiplication, is on this plan seen to be included in the corresponding but more general property (212), of multiplication of quaternions.

(5.) By changing IVq, IVq', t , t' , and δ , to α , β , a , b , and γ , in those formulæ of Art. 208 which are previous to its sub-articles, we should obtain, with the recent definition (or interpretation) II. of $\beta\alpha$, several of the consequences lately given (in sub-arts. to 281), as resulting from the former definition, 278, I. Thus, the equations,

$$\text{VI., VII., VIII., IX., X., XI., XII., XXII., and XXIII.,}$$

of 281, correspond to, and may (with our last definition) be deduced from, the formulæ,

$$\text{V., VI., VIII., XI., XII., XXII., XX., XIV., and XVI., XVIII.,}$$

of 208. (Some of the consequences from the sub-articles to 208 have been already considered, in 281, (11.))

(6.) The *geometrical properties of the line* $IV\beta a$, deduced from the *first definition* (278) of βa in 281, (3.) and (4.), (namely, the *positive rotation* round that line, from β to a ; its *perpendicularity to their plane*; and the *representation* by the same line of the *parallelogram under those two factors*, regard being had to *units of length* and of *area*.) might also have been deduced from 228, (4.), by means of the *second definition* (284), of the same product, βa .

SECTION 4.—*On the Symbolical Identification of a Right Quaternion with its own Index: and on the Construction of a Product of Two Rectangular Lines, by a Third Line, rectangular to both.*

285. It has been seen, then, that the recent formula 284, II. or III., may replace the formula 278, I., as a *second definition* of a *product of two vectors*, which conducts to the same *consequences*, and therefore ultimately to the same *interpretation* of such a product, as the *first*. Now, in the *second* formula, we have *interpreted that product*, βa , by *changing the two factor-lines*, a and β , to the *two right quaternions*, v and v' , or $I^{-1}a$ and $I^{-1}\beta$, of which they are the *indices*; and by then *defining* that the sought product βa is *equal* to the product $v'v$, of those two right quaternions. It becomes, therefore, important to inquire, at this stage, *how far such substitution*, of $I^{-1}a$ for a , or of v for Iv , together with the *converse* substitution, is permitted in this Calculus, consistently with principles already established. For it is evident that if such substitutions can be shown to be generally legitimate, or allowable, we shall thereby be enabled to *enlarge* greatly the existing field of *interpretation*: and to treat, in *all cases*, *Functions of Vectors*, as being, at the same time, *Functions of Right Quaternions*.

286. We have first, by 133 (comp. 283, I.), the *equality*,

$$I \dots I^{-1}\beta = I^{-1}a, \text{ if } \beta = a.$$

In the next place, by 206 (comp. 283, II.), we have the formula of *addition or subtraction*,

$$II \dots I^{-1}(\beta \pm a) = I^{-1}\beta \pm I^{-1}a;$$

with these more general results of the same kind (comp. 207 and 99),

$$III \dots I^{-1}\Sigma a = \Sigma I^{-1}a; \quad IV \dots I^{-1}\Sigma xa = \Sigma xI^{-1}a.$$

In the third place, by 193 (comp. 283, III.), we have, for *division*, the formula,

$$V. \dots I^{-1}\beta : I^{-1}a = \beta : a;$$

while the *second definition* (284) of *multiplication of vectors*, which has been proved to be *consistent* with the *first definition* (278), has given us the analogous equation,

$$VI. \dots I^{-1}\beta \cdot I^{-1}a = \beta \cdot a = \beta a.$$

It would seem, then, that we might at once proceed to *define*, for the purpose of *interpreting any proposed Function of Vectors as a Quaternion*, that the following general *Equation* exists:

$$VII. \dots I^{-1}a = a; \text{ or } VIII. \dots I v = v, \text{ if } v = \frac{\pi}{2};$$

or still more briefly and *symbolically*, if it be understood that the *subject* of the *operation I* is always a *right quaternion*,

$$IX. \dots I = 1.$$

But, before finally adopting this conclusion, there is a *case* (or rather a *class* of cases), which it is necessary to examine, in order to be certain that *no contradiction to former results* can ever be thereby caused.

287. The *most general form of a vector-function*, or of a vector regarded as a function of other vectors and of scalars, which was considered in the First Book, was the form (99, comp. 275),

$$I. \dots \rho = \Sigma x a;$$

and we have seen that *if we change, in this form, each vector a to the corresponding right quaternion I⁻¹a*, and then take the *index* of the new right quaternion which *results*, we shall thus be conducted to precisely the *same vector ρ*, as that which had been otherwise obtained before; or in symbols, that

$$II. \dots \Sigma x a = I \Sigma x I^{-1} a \text{ (comp. 286, IV.)}$$

But *another form of a vector-function* has been considered in the Second Book; namely, the form,

$$III. \dots \rho = \dots \frac{\epsilon \gamma}{\delta \beta} a \text{ (226, III.)};$$

in which *a, β, γ, δ, ε . . .* are *any odd number of complanar vectors*. And before we accept, as *general*, the equation VII. or VIII. or IX. of 286, we must inquire whether we are at liberty to write, under the *same conditions of complanarity*, and with the *same signification* of the *vector ρ*, the equation,

$$\text{IV.} \dots \rho = I \left(\dots \frac{I^{-1}\epsilon}{I^{-1}\delta} \cdot \frac{I^{-1}\gamma}{I^{-1}\beta} \cdot I^{-1}a \right).$$

288. To examine this, let there be at first only *three given* coplanar vectors, $\gamma \parallel a, \beta$; in which case there will always be (by 226) a *fourth vector* ρ , in the same plane, which will represent or construct the function $(\gamma : \beta) \cdot a$; namely, the *fourth proportional* to β, γ, a . Taking then what we may call the *Inverse Index-Functions*, or *operating* on these four vectors a, β, γ, ρ by the *characteristic* I^{-1} , we obtain *four collinear and right quaternions* (209), which may be denoted by v, v', v'', v''' ; and we shall have the equation,

$$\text{V.} \dots v''' : v = (\rho : a = \gamma : \beta =) v'' : v';$$

or

$$\text{VI.} \dots v''' = (v'' : v') \cdot v;$$

which proves what was required. Or, more symbolically,

$$\text{VII.} \dots \frac{I^{-1}\rho}{I^{-1}a} = \frac{\rho}{a} = \frac{\gamma}{\beta} = \frac{I^{-1}\gamma}{I^{-1}\beta};$$

$$\text{VIII.} \dots \frac{\gamma}{\beta} \cdot a = \rho = I(I^{-1}\rho) = I \left(\frac{I^{-1}\gamma}{I^{-1}\beta} \cdot I^{-1}a \right).$$

And it is so easy to extend this reasoning to the case of any greater odd number of given vectors in one plane, that we may now consider the recent formula IV. as proved.

289. We shall therefore *adopt*, as *general*, the *symbolical equations* VII. VIII. IX. of 286; and shall thus be enabled, in a shortly subsequent Section, to *interpret ternary* (and other) *products of vectors*, as well as *powers* and other *Functions of Vectors*, as being *generally Quaternions*; although they may, in particular cases, *degenerate* (131) into *scalars*, or may become *right quaternions* (132): in which latter event they may, in virtue of the same principle, be *represented by*, and *equated* to, *their own indices* (133), and so be treated as *vectors*. In symbols, we shall write *generally*, for *any set of vectors* a, β, γ, \dots and *any function* f , the equation,

$$\text{I.} \dots f(a, \beta, \gamma, \dots) = f(I^{-1}a, I^{-1}\beta, I^{-1}\gamma, \dots) = q,$$

q being some *quaternion*; while in the particular case when this quaternion is *right*, or when

$$q = v = S^{-1}0 = I^{-1}\rho,$$

we shall write *also*, and usually *by preference* (for *that case*), the formula,

$$\text{II. } \dots f(a, \beta, \gamma, \dots) = I f(I^{-1}a, I^{-1}\beta, I^{-1}\gamma, \dots) = \rho,$$

ρ being a *vector*.

290. For example, instead of saying (as in 281) that the *Product of any two Rectangular Vectors* is a *Right Quaternion*, with certain properties of its *Index*, already pointed out (284, (6.)), we may now say that *such a product* is equal to that *index*. And hence will follow the important consequence, that the *Product of any Two Rectangular Lines in Space* is equal to (or may be constructed by) a *Third Line*, rectangular to both; the *Rotation round this Product-Line*, from the *Multiplier-Line* to the *Multiplicand-Line*, being *Positive*: and the *Length of the Product* being equal to the *Product of the Lengths of the Factors*, or representing (with a suitable reference to *units*) the *Area of the Rectangle* under them. And generally we may now, for all purposes of calculation and *expression*, identify* a *Right Quaternion* with its own *Index*.

SECTION 5.—On some Simplifications of Notation, or of Expression, resulting from this Identification; and on the Conception of an Unit-Line as a Right Versor.

291. An immediate consequence of the symbolical equation 286, IX., is that we may now *suppress the Characteristic I*, of the *Index of a Right Quaternion*, in all the formulæ into which it has entered; and so may *simplify the Notation*. Thus, instead of writing,

$$\begin{aligned} \text{Ax. } q &= IU V q, & \text{or } \text{Ax.} &= IU V, & \text{as in } 204, (23.), \\ \text{or } \text{Ax. } q &= UI V q, & \text{Ax.} &= UI V, & \text{as in } 274, (7.), \end{aligned}$$

we may now write simply†,

$$\text{I. } \dots \text{Ax. } q = UV q; \quad \text{or} \quad \text{II. } \dots \text{Ax.} = UV.$$

The *Characteristic Ax.*, of the *Operation of taking the Axis of a Quaternion* (132, (6.)), may therefore henceforth be replaced

* Compare the Notes to pages 119, 186, 174, 191, 200.

† Compare the first Note to page 118, and the second Note to page 200.

whenever we may think fit to *dispense with it*, by this *combination of two other characteristics*, U and V, which are of greater and more *general utility*, and indeed *cannot** be dispensed with, in the *practice* of the present Calculus.

292. We are now enabled also to *diminish*, to some extent, the *number of technical terms*, which have been employed in the foregoing Book. Thus, whereas we defined, in 202, that the right quaternion Vq was the *Right Part* of the Quaternion q , or of the *sum* $Sq + Vq$, we may now, by 290, *identify* that *part* with its own *index-vector* IVq , and so may be led to call it *the vector part*, or simply *the VECTOR*, † of that Quaternion q , without henceforth *speaking of the right part*: although the *plan of exposition*, adopted in the Second Book, required that we should do so for some time. And thus an *enunciation*, which was put forward at an early stage of the present work, namely, at the end of the First Chapter of the First Book, or the assertion (17) that

“*Scalar plus Vector equals Quaternion,*”

becomes entirely intelligible, and acquires a perfectly *definite signification*. For we are in this manner led to conceive a *Number* (positive or negative) as being *added to a Line*, ‡ *when it is added* (according to rules already established) *to that right quotient* (132), of which the *line* is the *Index*. In symbols, we are thus led to establish the formula,

$$\text{I. . . } q = a + a, \quad \text{when} \quad \text{II. . . } q = a + I^{-1}a;$$

* Of course, any one who chooses may *invent new symbols*, to denote the same operations on quaternions, as those which are denoted in these *Elements*, and in the elsewhere cited *Lectures*, by the letters U and V; but, under *some form*, such symbols *must be used*: and it appears to have been hitherto thought expedient, by other writers, not hastily to innovate on notations which have been already employed in several published researches, and have been found to answer their purpose. As to the *type* used for these, and for the analogous characteristics K, S, T, that must evidently be a mere affair of taste and convenience: and in fact they have all been printed as small italic capitals, in some examination-papers by the author.

† Compare the Note to page 191.

‡ On account of this possibility of conceiving a quaternion to be the *sum* of a number and a line, it was at one time suggested by the present author, that a *Quaternion* might also be called a *Grammarithm*, by a combination of the two Greek words, γραμμή and ἀριθμός, which signify respectively a *Line* and a *Number*.

whatever scalar, and whatever vector, may be denoted by v and a . And because either of these two parts, or summands, may vanish separately, we are entitled to say, that both Scalars and Vectors, or Numbers and Lines, are included in the Conception of a Quaternion, as now enlarged or modified.

293. Again, the same *symbolical identification* of Iv with v (286, VIII.) leads to the forming of a *new conception of an Unit-Line, or Unit-Vector* (129), as being also a *Right Versor* (153); or an *Operator*, of which the effect is to turn a line, in a plane perpendicular to itself, through a *positive quadrant of rotation*: and thereby to oblige the *Operand-Line* to take a *new direction, at right angles* to its old direction, but *without any change of length*. And then the remarks (154) on the equation $q^2 = -1$, where q was a *right versor* in the former sense (which is still a permitted one) of its being a *right radial quotient* (147), or the *quotient of two equally long but mutually rectangular lines*, become immediately applicable to the interpretation of the equation,

$$\rho^2 = -1, \quad \text{or} \quad \rho^2 + 1 = 0 \quad (282, \text{XIV.});$$

where ρ is still an *unit-vector*.

(1.) Thus (comp. Fig. 41), if α be any line perpendicular to such a vector ρ , we have the equations,

$$\text{I.} \dots \rho\alpha = \beta; \quad \text{II.} \dots \rho^2\alpha = \rho\beta = \alpha' = -\alpha;$$

β being another line perpendicular to ρ , which is, at the same time, at right angles to α , and of the same length with it; and from which a *third line* α' , or $-\alpha$, opposite to the line α , but still *equally long*, is formed by a *repetition of the operation*, denoted by (what we may here call) the *characteristic* ρ ; or having that *unit-vector* ρ for the *operator, or instrument employed*, as a sort of *handle, or axis* of rotation*.

(2.) More generally (comp. 290), if α, β, γ be any three lines at right angles to each other, and if the *length* of γ be numerically equal to the *product of the lengths* of α and β , then (by what precedes) the line γ represents, or constructs, or is equal to, the *product of the two other lines*, at least if a certain order of the factors (comp. 279) be observed: so that we may write the equation (comp. 281, XXI.),

$$\text{III.} \dots \alpha\beta = \gamma, \quad \text{if} \quad \text{IV.} \dots \beta \perp \alpha, \quad \gamma \perp \alpha, \quad \gamma \perp \beta, \quad \text{and} \quad \text{V.} \dots T\alpha. T\beta = T\gamma,$$

* Compare the first Note to page 186.

provided that the rotation round a , from β to γ , or that round γ from a to β , &c., has the direction taken as the positive one.

(3.) In this more general case, we may still conceive that the multiplier-line a has operated on the multiplicand-line β , so as to produce (or generate) the product-line γ ; but not now by an operation of version alone, since the tensor of β is (generally) multiplied by that of a , in order to form, by V., the tensor of the product, γ .

(4.) And if (comp. Fig. 41, bis, in which a was first changed to β , and then to a') we repeat this compound operation, of tension and version combined (comp. 189), or if we multiply again by a , we obtain a fourth line β' , in the plane of β , γ , but with a direction opposite to that of β , and with a length generally different: namely the line,

$$\text{VI.} \dots a\gamma = aa\beta = a^2\beta = \beta' = -a^2\beta, \text{ if } a = T\alpha.$$

(5.) The operator a^2 , or aa , is therefore equivalent, in its effect on β , to the negative scalar, $-a^2$, or $-(T\alpha)^2$, or $-Na$, considered as a coefficient, or as a (scalar) multiplier (15): whence the equation,

$$\text{VII.} \dots a^2 = -Na \text{ (282, I.),}$$

may be again deduced, but now with a new interpretation, which is, however, as we see, completely consistent, in all its consequences, with the one first proposed (282).

SECTION 6.—On the Interpretation of a Product of Three or more Vectors, as a Quaternion.

294. There is now no difficulty in interpreting a ternary product of vectors (comp. 277, I.), or a product of more vectors than three, taken always in some given order; namely, as the result (289, I.) of the substitution of the corresponding right quaternions in that product: which result is generally what we have lately called (276) an *Oblique Quotient*, or a Quaternion with either an acute or an obtuse angle (130); but may degenerate (131) into a scalar, or may become itself a right quaternion (132), and so be constructed (289, II.) by a new vector. It follows (comp. 281), that *Multiplication of Vectors*, like that of *Quaternions* (223), in which indeed we now see that it is included, is an *Associative Operation*: or that we may write generally (comp. 223, II.), for any three vectors, a , β , γ , the Formula,

$$\text{I.} \dots \gamma\beta \cdot a = \gamma \cdot \beta a.$$

(1.) The formulæ 228, III. and IV., are now replaced by the following:

$$\text{II.} \dots V. \gamma V\beta a = aS\beta\gamma - \beta S\gamma a;$$

$$\text{III.} \dots V\gamma\beta a = aS\beta\gamma - \beta S\gamma a + \gamma S a\beta;$$

in which $V\gamma\beta\alpha$ is written, for simplicity, instead of $V(\gamma\beta\alpha)$, or $V.\gamma\beta\alpha$; and with which, as with the earlier equations referred to, a student of this Calculus will find it useful to render himself *very familiar*.

(2.) Another useful form of the equation II. is the following :

$$\text{IV.} \dots V(Va\beta.\gamma) = aS\beta\gamma - \beta S\gamma a.$$

(3.) The equations IX. X. XIV. of 223 enable us now to write, for any three vectors, the formula :

$$\begin{aligned} \text{V.} \dots S\gamma\beta\alpha &= -Sa\beta\gamma = S\alpha\gamma\beta = -S\beta\gamma\alpha = S\beta\alpha\gamma = -S\gamma\alpha\beta \\ &= \pm \text{volume of parallelepiped under } a, \beta, \gamma, \\ &= \pm 6 \times \text{volume of pyramid } OABC; \end{aligned}$$

upper or lower signs being taken, according as the rotation round a from β to γ is positive or negative: or in other words, the scalar $S\gamma\beta\alpha$, of the ternary product of vectors $\gamma\beta\alpha$, being positive in the first case, but negative in the second.

(4.) The condition of complanarity of three vectors, a, β, γ , is therefore expressed by the equation (comp. 223, XI.):

$$\text{VI.} \dots S\gamma\beta\alpha = 0; \text{ or } \text{VI.} \dots Sa\beta\gamma = 0; \text{ \&c.}$$

(5.) If a, β, γ be any three vectors, complanar or diplanar, the expression,

$$\text{VII.} \dots \delta = aS\beta\gamma - \beta S\gamma a,$$

gives $\text{VIII.} \dots S\gamma\delta = 0$, and $\text{IX.} \dots Sa\beta\delta = 0$;

it represents therefore (comp. II. and IV.) a fourth vector δ , which is perpendicular to γ , but complanar with a and β : or in symbols,

$$\text{X.} \dots \delta \perp \gamma, \text{ and } \text{XI.} \dots \delta \parallel a, \beta.$$

(Compare the notations 123, 129.)

(6.) For any four vectors, we have by II. and IV. the transformations,

$$\text{XII.} \dots V(Va\beta.V\gamma\delta) = \delta Sa\beta\gamma - \gamma Sa\beta\delta;$$

$$\text{XIII.} \dots V(Va\beta.V\gamma\delta) = aS\beta\gamma\delta - \beta Sa\gamma\delta;$$

and each of these three equivalent expressions represents a fifth vector ϵ , which is at once complanar with a, β , and with γ, δ ; or a line $O\epsilon$, which is in the intersection of the two planes, OAB and OCD .

(7.) Comparing them, we see that any arbitrary vector ρ may be expressed as a linear function of any three given diplanar vectors, a, β, γ , by the formula :

$$\text{XIV.} \dots \rho Sa\beta\gamma = aS\beta\gamma\rho + \beta S\gamma a\rho + \gamma Sa\beta\rho;$$

which is found to be one of extensive utility.

(8.) Another very useful formula, of the same kind, is the following :

$$\text{XV.} \dots \rho Sa\beta\gamma = V\beta\gamma.Sa\rho + V\gamma a.S\beta\rho + Va\beta.S\gamma\rho;$$

in the second member of which, the points may be omitted.

(9.) One mode of proving the correctness of this last formula XV., is to operate on both members of it, by the three symbols, or characteristics of operation,

$$\text{XVI.} \dots S.a, S.\beta, S.\gamma;$$

the common results on both sides being respectively the three scalar products,

$$\text{XVII.} \dots Sa\rho.Sa\beta\gamma, S\beta\rho.Sa\beta\gamma, S\gamma\rho.Sa\beta\gamma;$$

where again the points may be omitted.

(10.) We here employ the principle, that if the three vectors α, β, γ be actual and diplanar, then no actual vector λ can satisfy at once the three scalar equations,

$$\text{XVIII.} \dots S\alpha\lambda = 0, \quad S\beta\lambda = 0, \quad S\gamma\lambda = 0;$$

because it cannot be perpendicular at once to those three diplanar vectors.

(11.) If, then, in any investigation with quaternions, we meet a system of this form XVIII., we can at once infer that

$$\text{XIX.} \dots \lambda = 0, \quad \text{if} \quad \text{XX.} \dots S\alpha\beta\gamma > 0;$$

while, conversely, if λ be an actual vector, then α, β, γ must be coplanar vectors, or $S\alpha\beta\gamma = 0$, as in VI'.

(12.) Hence also, under the same condition XX., the three scalar equations,

$$\text{XXI.} \dots S\alpha\lambda = S\alpha\mu, \quad S\beta\lambda = S\beta\mu, \quad S\gamma\lambda = S\gamma\mu,$$

give

$$\text{XXII.} \dots \lambda = \mu.$$

(13.) Operating (comp. (9.)) on the equation XV. by the symbol, or characteristic, $S.\delta$, in which δ is any new vector, we find a result which may be written thus (with or without the points):

$$\text{XXIII.} \dots 0 = S\alpha\rho.S\beta\gamma\delta - S\beta\rho.S\gamma\delta\alpha + S\gamma\rho.S\delta\alpha\beta - S\delta\rho.S\alpha\beta\gamma;$$

where $\alpha, \beta, \gamma, \delta, \rho$ may denote any five vectors.

(14.) In drawing this last inference, we assume that the equation XV. holds good, even when the three vectors α, β, γ are coplanar: which in fact must be true, as a limit, since the equation has been proved, by (9.) and (12.), to be valid, if γ be ever so little out of the plane of α and β .

(15.) We have therefore this new formula:

$$\text{XXIV.} \dots V\beta\gamma S\alpha\rho + V\gamma\alpha S\beta\rho + V\alpha\beta S\gamma\rho = 0, \quad \text{if} \quad S\alpha\beta\gamma = 0;$$

in which ρ may denote any fourth vector, whether in, or out of, the common plane of α, β, γ .

(16.) If ρ be perpendicular to that plane, the last formula is evidently true, each term of the first member vanishing separately, by 281, (7.); and if we change ρ to a vector δ in the plane of α, β, γ , we are conducted to the following equation, as an interpretation of the same formula XXIV., which expresses a known theorem of plane trigonometry, including several others under it:

$$\text{XXV.} \dots \sin \text{BOC} \cdot \cos \text{AOD} + \sin \text{COA} \cdot \cos \text{BOD} + \sin \text{AOB} \cdot \cos \text{COD} = 0,$$

for any four coplanar and co-initial lines, OA, OB, OC, OD.

(17.) By passing from OD to a line perpendicular thereto, but in their common plane, we have this other known* equation:

$$\text{XXVI.} \dots \sin \text{BOC} \sin \text{AOD} + \sin \text{COA} \sin \text{BOD} + \sin \text{AOB} \sin \text{COD} = 0;$$

which, like the former, admits of many transformations, but is only mentioned here as offering itself naturally to our notice, when we seek to interpret the formula XXIV. obtained as above by quaternions.

(18.) Operating on that formula by $S.\delta$, and changing ρ to ϵ , we have this new equation:

* Compare page 20 of the *Géométrie Supérieure* of M. Chasles.

$$\text{XXVII.} \dots 0 = S\alpha\epsilon S\beta\gamma\delta + S\beta\epsilon S\gamma\alpha\delta + S\gamma\epsilon S\alpha\beta\delta, \text{ if } S\alpha\beta\gamma = 0;$$

which might indeed have been at once deduced from XXIII.

(19.) The equation XIV., as well as XV., must hold good at the limit, when α, β, γ are *coplanar*; hence

$$\text{XXVIII.} \dots \alpha S\beta\gamma\rho + \beta S\gamma\alpha\rho + \gamma S\alpha\beta\rho = 0, \text{ if } S\alpha\beta\gamma = 0.$$

(20.) This last formula is evidently true, by (4.), if ρ be in the common plane of the three other vectors; and if we suppose it to be *perpendicular* to that plane, so that

$$\text{XXIX.} \dots \rho \parallel V\beta\gamma \parallel V\gamma\alpha \parallel V\alpha\beta,$$

and therefore, by 281, (9.), since $S(S\beta\gamma.\rho) = 0$,

$$\text{XXX.} \dots S\beta\gamma\rho = S(V\beta\gamma.\rho) = V\beta\gamma.\rho, \text{ \&c.},$$

we may *divide each term by* ρ , and so obtain this other formula,

$$\text{XXXI.} \dots \alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta = 0, \text{ if } S\alpha\beta\gamma = 0.$$

(21.) In general, the *vector* (292) of this last expression *vanishes* by II.; the expression is therefore equal to its own *scalar*, and we may write,

$$\text{XXXII.} \dots \alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta = 3S\alpha\beta\gamma,$$

whatever three vectors may be denoted by α, β, γ .

(22.) For the *case of coplanarity*, if we suppose that the three vectors are *equally long*, we have the proportion,

$$\text{XXXIII.} \dots V\beta\gamma : V\gamma\alpha : V\alpha\beta = \sin \text{BOC} : \sin \text{COA} : \sin \text{AOB};$$

and the formula XXXI. becomes thus,

$$\text{XXXIV.} \dots \text{OA} \cdot \sin \text{BOC} + \text{OB} \cdot \sin \text{COA} + \text{OC} \cdot \sin \text{AOB} = 0;$$

where OA, OB, OC are *any three radii of one circle*, and the equation is interpreted as in Articles 10, 11, &c.

(23.) The equation XXIII. might have been deduced from XIV., instead of XV., by first operating with $S.\delta$, and then interchanging δ and ρ .

(24.) A *vector* ρ may in general be considered (221) as *depending on three scalars* (the *co-ordinates of its term*); it cannot then be *determined by fewer than three scalar equations*; nor can it be *eliminated between fewer than four*.

(25.) As an example of such *determination of a vector*, let α, β, γ be again *any three given and coplanar vectors*; and let the three given equations be,

$$\text{XXXV.} \dots S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c;$$

in which a, b, c are supposed to denote *three given scalars*. Then the *sought vector* ρ has for its expression, by XV.,

$$\text{XXXVI.} \dots \rho = e^{-1}(\alpha V\beta\gamma + \beta V\gamma\alpha + \gamma V\alpha\beta), \text{ if } \text{XXXVII.} \dots e = S\alpha\beta\gamma.$$

(26.) As another example, let the three equations be,

$$\text{XXXVIII.} \dots S\beta\gamma\rho = a', \quad S\gamma\alpha\rho = b', \quad S\alpha\beta\rho = c';$$

then, with the same signification of the scalar e , we have, by XIV.,

$$\text{XXXIX.} \dots \rho = e^{-1}(a'\alpha + b'\beta + c'\gamma).$$

(27.) As an example of *elimination of a vector*, let there be the *four scalar equations*,

$$\text{XL.} \dots S\alpha\rho = a, \quad S\beta\rho = b, \quad S\gamma\rho = c, \quad S\delta\rho = d;$$

then, by XXIII., we have this *resulting equation*, into which ρ does not enter, but only the *four vectors*, $a \dots \delta$, and the *four scalars*, $a \dots d$:

$$\text{XLL} \dots a \cdot S\beta\gamma\delta - b \cdot S\gamma\delta a + c \cdot S\delta a\beta - d \cdot Sa\beta\gamma = 0.$$

(28.) This last equation may therefore be considered as the *condition of concurrence of the four planes*, represented by the *four scalar equations* XL., in one *common point*; for, although it has not been expressly stated before, it follows evidently from the *definition* 278 of a *binary product of vectors*, combined with 196, (5.), that *every scalar equation of the linear form* (comp. 282, XVIII.),

$$\text{XLII} \dots Sa\rho = a, \text{ or } S\rho a = a,$$

in which $a = OA$, and $\rho = OP$, as usual, *represents a plane locus of the point P*; the *vector of the foot s*, of the *perpendicular on that plane from the origin*, being

$$\text{XLIII} \dots os = \sigma = aRa = aa^{-1} \text{ (282, XXI.)}$$

(29.) If we conceive a *pyramidal volume* (68) as having an *algebraical* (or *scalar*) *character*, so as to be capable of bearing either a *positive* or a *negative ratio* to the volume of a *given pyramid*, with a *given order of its points*, we may then omit the *ambiguous sign*, in the last expression (3.) for the *scalar of a ternary product of vectors*: and so may write, generally, $OABC$ denoting such a *volume*, the formula,

$$\text{XLIV} \dots Sa\beta\gamma = 6 \cdot OABC,$$

= a *positive* or a *negative scalar*, according as the *rotation* round OA from OB to OC is *negative* or *positive*.

(30.) More generally, changing o to D , and OA or a to $a - \delta$, &c., we have thus the formula:

$$\text{XLV} \dots 6 \cdot DABC = S(a - \delta)(\beta - \delta)(\gamma - \delta) = Sa\beta\gamma - S\beta\gamma\delta + S\gamma\delta a - S\delta a\beta;$$

in which it may be observed, that the expression is *changed to its own opposite*, or *negative*, or is *multiplied by -1*, when *any two of the four vectors*, a, β, γ, δ , or when *any two of the four points*, A, B, C, D , *change places with each other*; and therefore is *restored to its former value*, by a *second such binary interchange*.

(31.) Denoting then the *new origin* of a, β, γ, δ by Σ , we have first, by XLIV., XLV., the equation,

$$\text{XLVI} \dots DABC = EABC - EBCD + ECDA - EDAB;$$

and may then write the result (comp. 68) under the more *symmetric form* (because $-EBCD = BECD = \&c.$):

$$\text{XLVII} \dots BCDE + CDEA + DEAB + EABC + ABCD = 0;$$

in which A, B, C, D, E may denote *any five points of space*.

(32.) And an analogous formula (69, III.) of the *First Book*, for *any six points* $OABCDE$, namely the equation (comp. 65, 70),

$$\text{XLVIII} \dots OA \cdot BCDE + OB \cdot CDEA + OC \cdot DEAB + OD \cdot EABC + OE \cdot ABCD = 0,$$

in which the *additions* are performed according to the rules of *vectors*, the *volumes* being treated as *scalar coefficients*, is easily recovered from the foregoing principles and results. In fact, by XLVII., this last formula may be written as

$$\text{XLIX} \dots ED \cdot EABC = EA \cdot EBOD + EB \cdot ECAD + EC \cdot EABD;$$

or, substituting a, β, γ, δ for EA, EB, EC, ED , as

$$L. \dots \delta Sa\beta\gamma = aS\beta\gamma\delta + \beta S\gamma a\delta + \gamma Sa\beta\delta;$$

which is only another form of XIV., and ought to be familiar to the student.

(83.) The formula 69, II. may be deduced from XXXI., by observing that, when the three vectors a, β, γ are coplanar, we have the proportion,

$$LI. \dots V\beta\gamma : V\gamma a : Va\beta : V(\beta\gamma + \gamma a + a\beta) = OBC : OCA : OAB : ABC,$$

if signs (or algebraic or scalar ratios) of areas be attended to (28, 63); and the formula 69, I., for the case of three collinear points A, B, C, may now be written as follows:

$$LII. \dots a(\beta - \gamma) + \beta(\gamma - a) + \gamma(a - \beta) = 2V(\beta\gamma + \gamma a + a\beta) \\ = 2V(\beta - a)(\gamma - a) = 0,$$

if the three coincital vectors a, β, γ be termino-collinear (24).

(84.) The case when four coincital vectors a, β, γ, δ are termino-coplanar (64), or when they terminate in four coplanar points A, B, C, D, is expressed by equating to zero the second or the third member of the formula XLV.

(85.) Finally, for ternary products of vectors in general, we have the formula:

$$LIII. \dots a^3\beta^3\gamma^3 + (Sa\beta\gamma)^3 = (Va\beta\gamma)^3 = (aS\beta\gamma - \beta S\gamma a + \gamma Sa\beta)^3 \\ = a^2(S\beta\gamma)^3 + \beta^2(S\gamma a)^3 + \gamma^2(Sa\beta)^3 - 2S\beta\gamma S\gamma a S a\beta.$$

295. The identity (290) of a right quaternion with its index, and the conception (293) of an unit-line as a right versor, allow us now to treat the three important versors, i, j, k , as constructed by, and even as (in our present view) identical with, their own axes; or with the three lines OI, OJ, OK of 181, considered as being each a certain instrument, or operator, or agent in a right rotation (293, (1.)), which causes any line, in a plane perpendicular to itself, to turn in that plane, through a positive quadrant, without any change of its length. With this conception, or construction, the Laws of the Symbols ijk are still included in the Fundamental Formula of 183, namely,

$$i^2 = j^2 = k^2 = ijk = -1; \quad (A)$$

and if we now, in conformity with the same conception, transfer the Standard Trinomial Form (221) from Right Quaternions to Vectors, so as to write generally an expression of the form,

$$I. \dots \rho = ix + jy + kz, \quad \text{or} \quad I'. \dots a = ia + jb + kc, \text{ \&c.},$$

where xyz and abc are scalars (namely, rectangular co-ordinates), we can recover many of the foregoing results with ease: and can, if we think fit, connect them with co-ordinates.

(1.) As to the laws (182), included in the Fundamental Formula A, the law $i^2 = -1$, &c., may be interpreted on the plan of 293, (1.), as representing the reversal which results from two successive quadrantal rotations.

(2.) The two contrasted laws, or formulae,

$$ij = +k, \quad ji = -k, \quad (182, \text{II. and III.})$$

may now be interpreted as expressing, that although a *positive rotation through a right angle, round the line i as an axis, brings a revolving line from the position j to the position k , or $+k$, yet, on the contrary, a positive quadrantal rotation round the line j , as a new axis, brings a new revolving line from a new initial position, i , to a new final position, denoted by $-k$, or opposite* to the old final position, $+k$.*

(3.) Finally, the law $ijk = -1$ (183) may be interpreted by conceiving, that we operate on a line α , which has at first the direction of $+j$, by the three lines, k, j, i , in succession; which gives three new but equally long lines, β, γ, δ , in the directions of $-i, +k, -j$, and so conducts at last to a line $-\alpha$, which has a direction opposite to the initial one.

(4.) The foregoing laws of ijk , which are all (as has been said) included (184) in the Formula A, when combined with the recent expression I. for ρ , give (comp. 222, (1.)) for the square of that vector the value:

$$\text{II.} \dots \rho^2 = (ix + jy + kz)^2 = -(x^2 + y^2 + z^2);$$

this square of the line ρ is therefore equal to the negative of the square of its length $T\rho$ (185), or to the negative of its norm $N\rho$ (278), which agrees with the former result† 282, (1.) or (2.).

(5.) The condition of perpendicularity of the two lines ρ and α , when they are represented by the two trinomials I. and I', may be expressed (281, XVIII.) by the formula,

$$\text{III.} \dots 0 = S\alpha\rho = -(ax + by + cz);$$

which agrees with a well-known theorem of rectangular co-ordinates.

(6.) The condition of complanarity of three lines, ρ, ρ', ρ'' , represented by the trinomial forms,

$$\text{IV.} \dots \rho = ix + jy + kz, \quad \rho' = ix' + \&c., \quad \rho'' = ix'' + \&c.,$$

is (by 294, VI.) expressed by the formula (comp. 223, XIII.),

$$\text{V.} \dots 0 = Sp''\rho'\rho = x''(z'y - y'z) + y''(x'z - z'x) + z''(y'x - x'y);$$

agreeing again with known results.

(7.) When the three lines ρ, ρ', ρ'' , or OP, OP', OP'' , are not in one plane, the recent expression for $Sp''\rho'\rho$ gives, by 294, (8.), the volume of the parallelepiped

* In the Lectures, the three rectangular unit-lines, i, j, k , were supposed (in order to fix the conceptions, and with a reference to northern latitudes) to be directed, respectively, towards the south, the west, and the zenith; and then the contrast of the two formulae, $ij = +k, ji = -k$, came to be illustrated by conceiving, that we at one time turn a moveable line, which is at first directed westward, round an axis (or handle) directed towards the south, with a right-handed (or screwing) motion, through a right angle, which causes the line to take an upward position, as its final one; and that at another time we operate, in a precisely similar manner, on a line directed at first southward, with an axis directed to the west, which obliges this new line to take finally a downward (instead of, as before, an upward) direction.

† Compare also 222, IV.

(comp. 223, (9.)) of which they are *edges*; and this volume, thus expressed, is a *positive* or a *negative scalar*, according as the rotation round ρ from ρ' to ρ'' is itself *positive* or *negative*: that is, according as it has the *same direction* as that round $+x$ from $+y$ to $+z$ (or round i from j to k), or the direction *opposite* thereto.

(8.) It may be noticed here (comp. 223, (13.)), that if α, β, γ be any three vectors, then (by 294, III. and V.) we have :

$$\text{VI.} \dots S\alpha\beta\gamma = -S\gamma\beta\alpha = \frac{1}{2}(a\beta\gamma - \gamma\beta a);$$

$$\text{VII.} \dots V\alpha\beta\gamma = +V\gamma\beta\alpha = \frac{1}{2}(a\beta\gamma + \gamma\beta a).$$

(9.) More generally (comp. 223, (12.)), since a vector, considered as representing a *right quaternion* (290), is always (by 144) the *opposite of its own conjugate*, so that we have the important formula,*

$$\text{VIII.} \dots K\alpha = -\alpha, \text{ and therefore IX.} \dots K\Pi\alpha = \pm\Pi'\alpha,$$

we may write for any number of vectors, the transformations,

$$\text{X.} \dots S\Pi\alpha = \pm S\Pi'\alpha = \frac{1}{2}(\Pi\alpha \pm \Pi'\alpha),$$

$$\text{XI.} \dots V\Pi\alpha = \mp V\Pi'\alpha = \frac{1}{2}(\Pi\alpha \mp \Pi'\alpha),$$

upper or lower signs being taken, according as that number is *even* or *odd*: it being understood that

$$\text{XII.} \dots \Pi'\alpha = \dots \gamma\beta\alpha, \text{ if } \Pi\alpha = a\beta\gamma \dots$$

(10.) The relations of *rectangularity*,

$$\text{XIII.} \dots A_x i \perp A_x j; A_x j \perp A_x k; A_x k \perp A_x i,$$

which result at once from the definitions (181), may now be written more briefly, as follows :

$$\text{XIV.} \dots i \perp j; j \perp k, k \perp i;$$

and similarly in other cases, where the *axes*, or the *planes*, of any two right quaternions are at *right angles* to each other.

(11.) But, with the notations of the Second Book, we might also have written, by 123, 181, such formulæ of *complanarity* as the following, $A_x j \parallel\parallel i$, to express (comp. 225) that the *axis* of j was a *line* in the *plane* of i ; and it might cause some confusion, if we were now to *abridge* that formula to $j \parallel\parallel i$. In general, it seems convenient that we should not henceforth employ the *sign* $\parallel\parallel$, except as connecting either *symbols of three lines*, considered still as *complanar*; or else *symbols of three right quaternions*, considered as being *collinear* (209), because their *indices* (or *axes*) are *complanar*: or finally, any two *complanar quaternions* (123).

(12.) On the other hand, no inconvenience will result, if we now insert the *sign of parallelism*, between the symbols of two right quaternions which are, in the former sense (123), *complanar*: for example, we may write, on our present plan,

$$\text{XV.} \dots xi \parallel i, yj \parallel j, zk \parallel k,$$

if xyz be any three scalars.

* If, in like manner, we interpret, on our present plan, the symbols $U\alpha, T\alpha, N\alpha$ as equivalent to $UI^{-1}\alpha, TI^{-1}\alpha, NI^{-1}\alpha$, we are reconducted (compare the Notes to page 186) to the same significations of those symbols as before (155, 185, 278); and it is evident that on the same plan we have now,

$$S\alpha = 0, \quad V\alpha = \alpha.$$

296. There are a few particular but remarkable *cases*, of *ternary* and other *products of vectors*, which it may be well to mention here, and of which some may be worth a student's while to remember: especially as regards the *products of successive sides of closed polygons, inscribed in circles, or in spheres.*

(1.) If A, B, C, D be any four *concircular points*, we know, by the sub-articles to 260, that their *anharmomic function* $(ABCD)$, as defined in 259, (9.), is *scalar*; being also *positive or negative*, according to a law of *arrangement* of those four points, which has been already stated.

(2.) But, by that definition, and by the *scalar* (though *negative*) character of the *square of a vector* (282), we have generally, for *any plane or gauche quadrilateral* $ABCD$, the formula:

$$I. \dots e^2(ABCD) = AB \cdot BC \cdot CD \cdot DA = \text{the continued product of the four sides};$$

in which the coefficient e^2 is a *positive scalar*, namely the product of two negative or of two positive squares, as follows:

$$II. \dots e^2 = BC^2 \cdot DA^2 = \overline{BC}^2 \cdot \overline{DA}^2 > 0.$$

(3.) If then $ABCD$ be a *plane* and *inscribed quadrilateral*, we have, by 260, (8.), the formula,

$$III. \dots AB \cdot BC \cdot CD \cdot DA = \text{a positive or negative scalar},$$

according as this *quadrilateral in a circle* is a *crossed* or an *uncrossed* one.

(4.) The product $a\beta\gamma$ of any three *complanar vectors* is a *vector*, because its *scalar part* $Sa\beta\gamma$ *vanishes*, by 294, (3.) and (4.); and if the factors be three *successive sides* AB, BC, CD of a quadrilateral thus *inscribed in a circle*, their product has either the *direction of the fourth successive side*, DA , or else the *opposite direction*, or in symbols,

$$IV. \dots AB \cdot BC \cdot CD : DA > \text{ or } < 0,$$

according as the *quadrilateral* $ABCD$ is an *uncrossed* or a *crossed* one.

(5.) By conceiving the *fourth point* D to *approach*, continuously and indefinitely, to the *first point* A , we find that the *product of the three successive sides of any plane triangle*, ABC , is given by an equation of the form:

$$V. \dots AB \cdot BC \cdot CA = AT;$$

AT being a line (comp. Fig. 63) which *touches the circumscribed circle*, or (more fully) which *touches the segment* ABC of that circle, at the point A ; or represents the *initial direction of motion, along the circumference, from* A *through* B *to* C : while the length of this *tangential product-line*, AT , is equal to, or represents, with the usual reference to an *unit of length*, the *product of the lengths of the three sides*, of the same inscribed triangle.

(6.) Conversely, if this theorem respecting the product of the sides of an *inscribed triangle* be supposed to have been *otherwise proved*, and if it be *remembered*, then since it will give in like manner the equation,

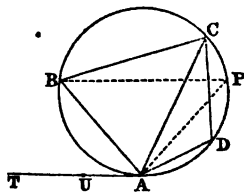


Fig. 63.

$$VI. \dots AC \cdot CD \cdot DA = AU,$$

if D be any fourth point, concircular with A, B, C, while AU is, as in the annexed Figures 63, a tangent to the new segment ACD, we can recover easily the theorem (3.), respecting the product of the sides of an inscribed quadrilateral; and thence can return to the corresponding theorem (260, (8.)), respecting the anharmonic function of any such figure ABCD: for we shall thus have, by V. and VI., the equation,

$$VII. \dots AB \cdot BC \cdot CD \cdot DA = (AT \cdot AU) : (CA \cdot AC),$$

in which the divisor CA.AC or N.AC, or \overline{AC}^2 , is always positive (282, (1.)), but the dividend AT.AU is negative (281, (9.)) for the case of an uncrossed quadrilateral (Fig. 63), being on the contrary positive for the other case of a crossed one (Fig. 63, bis).

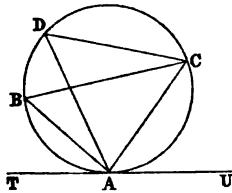


Fig. 63, bis.

(7.) If P be any point on the circle through a given point A, which touches at a given origin O a given line OT = r, as represented in Fig. 64, we shall then have by (5.) an equation of the form,

$$VIII. \dots OA \cdot AP \cdot PO = x \cdot OT,$$

in which x is some scalar coefficient, which varies with the position of P. Making then OA = a, and OP = ρ, as usual, we shall have

$$IX. \dots a(\rho - a)\rho = -xr,$$

or

$$IX'. \dots \rho^{-1} - a^{-1} = xr : a^2\rho^2,$$

or

$$IX''. \dots \sqrt{\tau}\rho^{-1} = \sqrt{\tau}a^{-1};$$

and any one of these may be considered as a

form of the equation of the circle, determined by the given conditions.

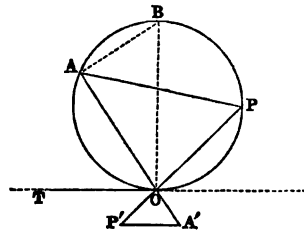


Fig. 64.

(8.) Geometrically, the last formula IX." expresses, that the line $\rho^{-1} - a^{-1}$, or $B\rho - Ra$, or $A'P'$ (see again Fig. 64), if $OA' = a^{-1} = Ra = R \cdot OA$, and $OP' = \rho^{-1} = R \cdot OP$, is parallel to the given tangent r at O; which agrees with Fig. 58, and with Art. 260.

(9.) If B be the point opposite to O upon the circle, then the diameter OB, or β, as being $\perp r$, so that $\tau\beta^{-1}$ is a vector, is given by the formula,

$$X. \dots \tau\beta^{-1} = \sqrt{\tau}a^{-1}; \text{ or } X'. \dots \beta = -\tau : \sqrt{\tau}a^{-1};$$

in which the tangent r admits, as it ought to do, of being multiplied by any scalar, without the value of β being changed.

(10.) As another verification, the last formula gives,

$$XL. \dots \overline{OB} = T\beta = T\alpha : TVU\tau a^{-1} = \overline{OA} : \sin \Delta OT.$$

(11.) If a quadrilateral OABC be not inscriptible in a circle, then, whether it be plane or gauche, we can always circumscribe (as in Fig. 65) two circles, OAB and OBC, about the two triangles, formed by drawing the diagonal OB; and then, on the plan of (6.), we can draw two tangents OT, OU, to the two segments OAB, OBC, so as to represent the two ternary products,

$$OA \cdot AB \cdot BO, \text{ and } OB \cdot BC \cdot CO;$$

after which we shall have the *quaternary product*,

$$\text{XII.} \dots OA \cdot AB \cdot BC \cdot CO = OT \cdot OU : OB^2;$$

where the *divisor*, OB^2 , or $BO \cdot OB$, or $N \cdot OB$, is a *positive scalar*, but the *dividend* $OT \cdot OU$, and therefore also the *quotient* in the *second member*, or the *product* in the *first member*, is a *quaternion*.

(12.) The *axis* of this quaternion is *perpendicular to the plane* TOU of the two *tangents*; and therefore to the *plane itself* of the quadrilateral $OABC$, if that be a *plane figure*; but if it be *gauche*, then the *axis* is *normal to the circumscribed sphere at the point* O : being also in all cases such, that the *rotation* round it, from OT to OU , is *positive*.

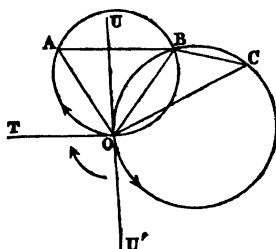


Fig. 65.

(13.) The *angle* of the same quaternion is the *supplement* of the angle TOU between the two tangents above mentioned; it is therefore *equal* to the angle $U'OT$, if OU' touch the *new segment* OCB , or proceed in a *new and opposite direction* from O (see again Fig. 65); it may therefore be said to be the *angle between the two arcs*, OAB and OCB , *along which a point should move*, in order to go from O , on the two circumferences, to the *opposite corner* B of the quadrilateral $OABC$, *through the two other corners*, A and C , respectively: or the angle between the arcs OCB , OAB .

(14.) These results, respecting the *axis* and *angle* of the *product of the four successive sides*, of any quadrilateral $OABC$, or $ABCD$, apply without any modification to the *anharmonic quaternion* (259, (9.)) of the same quadrilateral; and although, for the case of a quadrilateral in a circle, the *axis* becomes indeterminate, because the quaternary product and the anharmonic function degenerate together into *scalars*, or because the figure may then be conceived to be inscribed in *indefinitely many spheres*, yet the *angle* may still be determined by the *same rule* as in the *general case*: this angle being $=\pi$, for the inscribed and *uncrossed* quadrilateral (Fig. 68); but $=0$, for the inscribed and *crossed* one (Fig. 68, bis).

(15.) For the *gauche* quadrilateral $OABC$, which may always be conceived to be inscribed in a *determined sphere*, we may say, by (13.), that the *angle of the quaternion product*, $\angle(OA \cdot AB \cdot BC \cdot CO)$, is equal to the *angle of the lune*, bounded (generally) by the two arcs of *small circles* OAB , OCB ; with the *same construction* for the equal *angle of the anharmonic*,

$$\angle(OABC), \text{ or } \angle(OA : AB \cdot BC : CO).$$

(16.) It is evident that the general principle 223, (10.), of the permissibility of *cyclical permutation* of quaternion factors under the sign S , must hold good for the case when those quaternions *degenerate* (294) into *vectors*; and it is still more obvious, that *every permutation* of factors is allowed, under the sign T : whence *cyclical permutation* is *again allowed*, under this other sign SU ; and consequently also (comp. 196, XVI.) under the sign \angle .

(17.) Hence generally, for any four vectors, we have the three equations,

$$\text{XIII.} \dots S\alpha\beta\gamma\delta = S\beta\gamma\delta\alpha; \quad \text{XIV.} \dots SU\alpha\beta\gamma\delta = SU\beta\gamma\delta\alpha;$$

$$\text{XV.} \dots \angle\alpha\beta\gamma\delta = \angle\beta\gamma\delta\alpha;$$

and in particular, for the successive sides of any plane or gauche quadrilateral ABCD, we have the four equal angles,

$$\text{XVI.} \dots \angle (AB \cdot BC \cdot CD \cdot DA) = \angle (BC \cdot CD \cdot DA \cdot AB) = \&c. ;$$

with the corresponding equality of the angles of the four anharmonics,

$$\text{XVII.} \dots \angle (ABCD) = \angle (BCDA) = \angle (CDAB) = \angle (DABC) ;$$

or of those of the four reciprocal anharmonics (259, XVII.),

$$\text{XVII.} \dots \angle (ADCB) = \angle (BADC) = \angle (CBAD) = \angle (DCBA).$$

¶ (18.) Interpreting now, by (13.) and (15.), these last equations, we derive from them the following theorem, for the plane, or for space:—

Let ABCD be any four points, connected by four circles, each passing through three of the points: then, not only is the angle at A, between the arcs ABC, ADC, equal to the angle at C, between CDA and CBA, but also it is equal (comp. Fig. 66) to the angle at B, between the two other arcs BCD and BAD, and to the angle at D, between the arcs DAB, DCB.

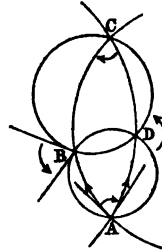


Fig. 66.

(19.) Again, let ABCDE be any pentagon, inscribed in a sphere; and conceive that the two diagonals AC, AD are drawn. We shall then have three equations, of the forms,

$$\begin{aligned} \text{XVIII.} \dots AB \cdot BC \cdot CA &= AT; \quad AC \cdot CD \cdot DA = AU; \\ AD \cdot DE \cdot EA &= AV; \end{aligned}$$

where AT, AU, AV are three tangents to the sphere at A, so that their product is a fourth tangent at that point. But the equations XVIII. give

$$\begin{aligned} \text{XIX.} \dots AB \cdot BC \cdot CD \cdot DE \cdot EA &= (AT \cdot AU \cdot AV) : (\overline{AC}^2 \cdot \overline{AD}^2) \\ &= AW = a \text{ new vector, which touches the sphere at A.} \end{aligned}$$

We have therefore this Theorem, which includes several others' under it:—

“The product of the five successive sides, of any (generally gauche) pentagon inscribed in a sphere, is equal to a tangential vector, drawn from the point at which the pentagon begins and ends.”

(20.) Let then P be a point on the sphere which passes through o, and through three given points A, B, C; we shall have the equation,

$$\begin{aligned} \text{XX.} \dots 0 &= S(OA \cdot AB \cdot BC \cdot CP \cdot PO) = S\alpha(\beta - a)(\gamma - \beta)(\rho - \gamma)(-\rho) \\ &= a^2S\beta\gamma\rho + \beta^2S\gamma a\rho + \gamma^2S\alpha\beta\rho - \rho^2S\alpha\beta\gamma. \end{aligned}$$

(21.) Comparing with 294, XIV., we see that the condition for the four co-initial vectors $\alpha, \beta, \gamma, \rho$ thus terminating on one spheric surface, which passes through their common origin o, may be thus expressed:

$$\text{XXI.} \dots \text{if } \rho = x\alpha + y\beta + z\gamma, \text{ then } \rho^2 = x\alpha^2 + y\beta^2 + z\gamma^2.$$

(22.) If then we project (comp. 62) the variable point P into points A', B', C' on the three given chords OA, OB, OC, by three planes through that point P, respectively parallel to the planes BOC, COA, AOB, we shall have the equation:

$$\text{XXII.} \dots OP^2 = OA \cdot OA' + OB \cdot OB' + OC \cdot OC'.$$

(23.) That the equation XX. does in fact represent a spheric locus for the point P, is evident from its mere form (comp. 282, (10.)); and that this sphere passes

through the four given points, O, A, B, C , may be proved by observing that the equation is satisfied, when we change ρ to any one of the four vectors, $0, a, \beta, \gamma$.

(24.) Introducing an auxiliary vector, OD or δ , determined by the equation,

$$\text{XXIII.} \dots \delta Sa\beta\gamma = \alpha^2 V\beta\gamma + \beta^2 V\gamma\alpha + \gamma^2 Va\beta,$$

or by the system of the three scalar equations (comp. 294, (25.)),

$$\text{XXIV.} \dots \alpha^2 = S\delta\alpha, \quad \beta^2 = S\delta\beta, \quad \gamma^2 = S\delta\gamma,$$

or

$$\text{XXIV'.} \dots S\delta\alpha^{-1} = S\delta\beta^{-1} = S\delta\gamma^{-1} = 1,$$

the equation XX. of the sphere becomes simply,

$$\text{XXV.} \dots \rho^2 = S\delta\rho, \quad \text{or} \quad \text{XXV'.} \dots S\delta\rho^{-1} = 1;$$

so that D is the point of the sphere opposite to O , and δ is a diameter (comp. 282, IX'.; and 196, (6.)).

(25.) The formula XXIII., which determines this diameter, may be written in this other way :

$$\text{XXVI.} \dots \delta Sa\beta\gamma = Va(\beta - a)(\gamma - \beta)\gamma;$$

or

$$\text{XXVI'.} \dots \delta \cdot OABC \cdot OD = -V(OA \cdot AB \cdot BC \cdot CO);$$

where the symbol $OABC$, considered as a coefficient, is interpreted as in 294, XLIV.; namely, as denoting the volume of the pyramid $OABC$, which is here an inscribed one.

(26.) This result of calculation, so far as it regards the direction of the axis of the quaternion $OA \cdot AB \cdot BC \cdot CO$, agrees with, and may be used to confirm, the theorem (12.), respecting the product of the successive sides of a gauche quadrilateral, $OABC$; including the rule of rotation, which distinguishes that axis from its opposite.

(27.) The formula XXIII. for the diameter δ may also be thus written :

$$\begin{aligned} \text{XXVII.} \dots \delta \cdot Sa^{-1}\beta^{-1}\gamma^{-1} &= V(\beta^{-1}\gamma^{-1} + \gamma^{-1}\alpha^{-1} + \alpha^{-1}\beta^{-1}) \\ &= V(\beta^{-1} - \alpha^{-1})(\gamma^{-1} - \alpha^{-1}); \end{aligned}$$

and the equation XX. of the sphere may be transformed to the following :

$$\text{XXVIII.} \dots 0 = S(\beta^{-1} - \alpha^{-1})(\gamma^{-1} - \alpha^{-1})(\rho^{-1} - \alpha^{-1});$$

which expresses (by 294, (34.), comp. 260, (10.)), that the four reciprocal vectors,

$$\text{XXIX.} \dots oA' = \alpha' = \alpha^{-1}, \quad oB' = \beta' = \beta^{-1}, \quad oC' = \gamma' = \gamma^{-1}, \quad o\rho' = \rho' = \rho^{-1},$$

are termino-complanar (64); the plane $A'B'C'\rho'$, in which they all terminate, being parallel to the tangent plane to the sphere at O : because the perpendicular let fall on this plane from O is

$$\text{XXX.} \dots \delta' = \delta^{-1},$$

as appears from the three scalar equations,

$$\text{XXXI.} \dots Sa'\delta = S\beta'\delta = S\gamma'\delta = 1.$$

(28.) In general, if D be the foot of the perpendicular from O , on the plane ABC , then

$$\text{XXXII.} \dots \delta = Sa\beta\gamma : V(\beta\gamma + \gamma\alpha + \alpha\beta);$$

because this expression satisfies, and may be deduced from, the three equations,

$$\text{XXXIII.} \dots Sa\delta^{-1} = S\beta\delta^{-1} = S\gamma\delta^{-1} = 1.$$

As a verification, the formula shows that the length $T\delta$, of this perpendicular, or altitude, OD , is equal to the sextuple volume of the pyramid $OABC$, divided by the double area of the triangular base ABC . (Compare 281, (4.), and 294, (8.), (33.).)

(29.) The equation XX., of the sphere OABC, might have been obtained by the elimination of the vector δ , between the four scalar equations XXIV. and XXV., on the plan of 294, (27.).

(30.) And another form of equation of the same sphere, answering to the development of XXVIII., may be obtained by the analogous elimination of the same vector δ , between the four other equations, XXIV'. and XXV'.

(31.) The product of any even number of complanar vectors is generally a quaternion with an axis perpendicular to their plane; but the product of the successive sides of a hexagon ABCDEF, or any other even-sided figure, inscribed in a circle, is a scalar: because by drawing diagonals AC, AD, AE from the first (or last) point A of the polygon, we find as in (6.) that it differs only by a scalar coefficient, or divisor, from the product of an even number of tangents, at the first point.

(32.) On the other hand, the product of any odd number of complanar vectors is always a line, in the same plane; and in particular (comp. (19.)), the product of the successive sides of a pentagon, or heptagon, &c., inscribed in a circle, is equal to a tangential vector, drawn from the first point of that inscribed and odd-sided polygon: because it differs only by a scalar coefficient from the product of an odd number of such tangents.

(33.) The product of any number of lines in space is generally a quaternion (289); and if they be the successive sides of a hexagon, or other even-sided polygon, inscribed in a sphere, the axis of this quaternion (comp. (12.)) is normal to that sphere, at the initial (or final) point of the polygon.

(34.) But the product of the successive sides of a heptagon, or other odd-sided polygon in a sphere, is equal (comp. (19.)) to a vector, which touches the sphere at the initial or final point; because it bears a scalar ratio to the product of an odd number of vectors, in the tangent plane at that point.

(35.) The equation XX., or its transformation XXVIII., may be called the condition or equation of homosphericity (comp. 260, (10.)) of the five points O, A, B, C, P; and the analogous equation for the five points ABCDE, with vectors $\alpha\beta\gamma\delta\epsilon$ from any arbitrary origin o, may be written thus:

$$\text{XXXIV.} \dots 0 = S(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \epsilon)(\epsilon - \alpha);$$

or thus, $\text{XXXV.} \dots 0 = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\epsilon^2,$

six times the second member of this last formula being found to be equal to the second member of the one preceding it, if

$$\text{XXXVI.} \dots a = BCDE, \quad b = ODEA, \quad c = DEAB, \quad d = EABC, \quad e = ABCD,$$

or more fully,

$$\text{XXXVII.} \dots 6a = S(\gamma - \beta)(\delta - \beta)(\epsilon - \beta) = S(\gamma\delta\epsilon - \delta\epsilon\beta + \epsilon\beta\gamma - \beta\gamma\delta), \text{ \&c. ;}$$

so that, by 294, XLVIII. and XLVII., we have also (comp. 65, 70) the equation,

$$\text{XXXVIII.} \dots 0 = a\alpha + b\beta + c\gamma + d\delta + e\epsilon,$$

with the relation between the coefficients,

$$\text{XXXIX.} \dots 0 = a + b + c + d + e,$$

which allows (as above) the origin of vectors to be arbitrary.

(36.) The equation or condition XXXV. may be obtained as the result of an elimination (294, (27.)), of a vector κ , and of a scalar g , between five scalar equations of the form 282, (10.), namely the five following,

$$\text{XL.} \dots \alpha^2 - 2S\kappa\alpha + g = 0, \quad \beta^2 - 2S\kappa\beta + g = 0, \dots \epsilon^2 - 2S\kappa\epsilon + g = 0;$$

κ being the vector of the centre κ of the sphere ABCD, of which the equation may be written as

$$\text{XLI.} \dots \rho^2 - 2S\kappa\rho + g = 0,$$

g being some scalar constant; and on which, by the condition referred to, the fifth point ϵ is situated.

(37.) By treating this fifth point, or its vector ϵ , as arbitrary, we recover the condition or equation of concircularity (8.), of the four points A, B, C, D; or the formula,

$$\text{XLII.} \dots 0 = V(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \alpha).$$

(38.) The equation of the circle ABC, and the equation of the sphere ABCD, may in general be written thus:

$$\text{XLIII.} \dots 0 = V(\alpha - \beta)(\beta - \gamma)(\gamma - \rho)(\rho - \alpha);$$

$$\text{XLIV.} \dots 0 = S(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \rho)(\rho - \alpha);$$

ρ being as usual the vector of a variable point P, on the one or the other locus.

(39.) The equations of the tangent to the circle ABC, and of the tangent plane to the sphere ABCD, at the point A, are respectively,

$$\text{XLV.} \dots 0 = V(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\rho - \alpha),$$

and

$$\text{XLVI.} \dots 0 = S(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \alpha)(\rho - \alpha).$$

(40.) Accordingly, whether we combine the two equations XLIII. and XLV., or XLIV. and XLVI., we find in each case the equation,

$$\text{XLVII.} \dots (\rho - \alpha)^2 = 0, \quad \text{giving } \rho = \alpha, \quad \text{or } P = A \quad (20);$$

it being supposed that the three points A, B, C are not collinear, and that the four points A, B, C, D are not coplanar.

(41.) If the centre of the sphere ABCD be taken for the origin O, so that

$$\text{XLVIII.} \dots \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = -r^2, \quad \text{or } \text{XLIX.} \dots T\alpha = T\beta = T\gamma = T\delta = r,$$

the positive scalar r denoting the radius, then after some reductions we obtain the transformation,

$$\text{L.} \dots V(\alpha - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \alpha) = 2\alpha S(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha).$$

(42.) Hence, generally, if κ be, as in (86.), the centre of the sphere, we have the equation (comp. XXVI.),

$$\text{LI.} \dots V(AB \cdot BC \cdot CD \cdot DA) = 12\kappa A \cdot ABCD.$$

(43.) We may therefore enunciate this theorem:—

“The vector part of the product of four successive sides, of a gauche quadrilateral inscribed in a sphere, is equal to the diameter drawn to the initial point of the polygon, multiplied by the sextuple volume of the pyramid, which its four points determine.”

(44.) In effecting the reductions (41.), the following general formulae of transformation have been employed, which may be useful on other occasions:

$$\text{LII.} \dots aq + qa = 2(aSq + Sqa); \quad \text{LII'.} \dots aq\alpha = a^2Kq + 2aSqa;$$

where a may be any vector, and q may be any quaternion.

SECTION 7.—*On the Fourth Proportional to Three Diplanar Vectors.*

297. In general, when any four quaternions, q, q', q'', q''' , satisfy the equation of quotients,

$$I. . . q''' : q'' = q' : q,$$

or the equivalent formula,

$$II. . . q''' = (q' : q) \cdot q'' = q' q^{-1} q'',$$

we shall say that they form a *Proportion*; and that the fourth, namely q''' , is the *Fourth Proportional* to the first, second, and third quaternions, namely to q, q' , and q'' , taken in this given order. This definition will include (by 288) the one which was assigned in 226, for the fourth proportional to three coplanar vectors, a, β, γ , namely that fourth vector in the same plane, $\delta = \beta a^{-1} \gamma$, which has been already considered; and it will enable us to interpret (comp. 289) the symbol

$$III. . . \beta a^{-1} \gamma, \text{ when } \gamma \text{ not } ||| a, \beta,$$

as denoting not indeed a *Vector*, in this new case, but at least a *Quaternion*, which may be called (on the present general plan) the *Fourth Proportional to these Three Diplanar Vectors, a, \beta, \gamma*. Such fourth proportionals possess some interesting properties, especially with reference to their vector parts, which it will be useful briefly to consider, and to illustrate by showing their connexion with spherical trigonometry, and generally with spherical geometry.

(1.) Let a, β, γ be (as in 208, (1.), &c.) the vectors of the corners of a triangle $\triangle ABC$ on the unit-sphere, whereof the sides are a, b, c ; and let us write,

$$IV. . . \begin{cases} l = \cos a = S\gamma\beta^{-1} = -S\beta\gamma, \\ m = \cos b = S\alpha\gamma^{-1} = -S\gamma\alpha, \\ n = \cos c = S\beta\alpha^{-1} = -S\alpha\beta; \end{cases}$$

where it is understood that

$$V. . . a^2 = \beta^2 = \gamma^2 = -1, \text{ or VI. . . } T\alpha = T\beta = T\gamma = 1;$$

it being also at first supposed, for the sake of fixing the conceptions, that each of these three cosines, l, m, n , is greater than zero, or that each side of the triangle $\triangle ABC$ is less than a quadrant.

(2.) Then, introducing three new vectors, δ, ϵ, ζ , defined by the equations,

$$VII. . . \begin{cases} \delta = V\beta a^{-1} \gamma = V\gamma a^{-1} \beta = m\beta + n\gamma - la, \\ \epsilon = V\gamma \beta^{-1} a = V\alpha \beta^{-1} \gamma = n\gamma + la - m\beta, \\ \zeta = V\alpha \gamma^{-1} \beta = V\beta \gamma^{-1} a = la + m\beta - n\gamma, \end{cases}$$

we find that these *three derived vectors* have all one common length, say r , because they have one common norm; namely,

$$\text{VIII.} \dots N\delta = N\varepsilon = N\zeta = l^2 + m^2 + n^2 - 2lmn = r^2;$$

so that $\text{IX.} \dots T\delta = T\varepsilon = T\zeta = r = \sqrt{l^2 + m^2 + n^2 - 2lmn}$.

(8.) This common length, r , is less than unity; for if we write,

$$\text{X.} \dots S\alpha\beta\gamma = S\beta\alpha^{-1}\gamma = \varepsilon,$$

we shall have the relation,

$$\text{XI.} \dots \varepsilon^2 + r^2 = N\beta\alpha^{-1}\gamma = 1;$$

and the scalar ε is different from zero, because the vectors α, β, γ are diplanar.

(4.) Dividing the three lines $\delta, \varepsilon, \zeta$ by their length, r , we change them to their *versors* (155, 156); and so obtain a new triangle, DEF, on the unit-sphere, of which the corners are determined by the three new unit-vectors,

$$\text{XII.} \dots OD = U\delta = r^{-1}\delta; \quad OE = U\varepsilon = r^{-1}\varepsilon;$$

$$OF = U\zeta = r^{-1}\zeta.$$

(5.) The sides opposite to D, E, F, in this new or derived triangle, are bisected, as in Fig. 67, by the corners A, B, C of the old or given triangle; because we have the three equations,

$$\text{XIII.} \dots \varepsilon + \zeta = 2la; \quad \zeta + \delta = 2m\beta; \quad \delta + \varepsilon = 2n\gamma.$$

(6.) Denoting the halves of the new sides by a', b', c' (so that the arc EF = $2a'$, &c.), the equations XIII. show also, by IV. and IX., that

$$\text{XIV.} \dots \cos a = r \cos a', \quad \cos b = r \cos b', \quad \cos c = r \cos c';$$

the cosines of the half-sides of the new (or bisected) triangle, DEF, are therefore proportional to the cosines of the sides of the old (or bisecting) triangle ABC.

(7.) The equations IV. give, by 279, (1.),

$$\text{XV.} \dots 2l = -(\beta\gamma + \gamma\beta), \quad 2m = -(\gamma\alpha + \alpha\gamma), \quad 2n = -(\alpha\beta + \beta\alpha);$$

we have therefore, by VII., the three following equations between quaternions,

$$\text{XVI.} \dots \alpha\varepsilon = \zeta\alpha, \quad \beta\zeta = \delta\beta, \quad \gamma\delta = \varepsilon\gamma;$$

which may also be thus written,

$$\text{XVI.} \dots \varepsilon\alpha = \alpha\zeta, \quad \zeta\beta = \beta\delta, \quad \delta\gamma = \gamma\varepsilon,$$

and express in a new way the relations of bisection (5.).

(8.) We have therefore the equations between vectors,

$$\text{XVII.} \dots \varepsilon = \alpha\zeta\alpha^{-1}, \quad \zeta = \beta\delta\beta^{-1}, \quad \delta = \gamma\varepsilon\gamma^{-1};$$

or

$$\text{XVII.} \dots \zeta = \alpha\varepsilon\alpha^{-1}, \quad \delta = \beta\zeta\beta^{-1}, \quad \varepsilon = \gamma\delta\gamma^{-1}.$$

(9.) Hence also, by V., or because α, β, γ are unit-vectors,

$$\text{XVIII.} \dots \varepsilon = -\alpha\zeta\alpha, \quad \zeta = -\beta\delta\beta, \quad \delta = -\gamma\varepsilon\gamma;$$

or

$$\text{XVIII.} \dots \zeta = -\alpha\varepsilon\alpha, \quad \delta = -\beta\zeta\beta, \quad \varepsilon = -\gamma\delta\gamma.$$

(10.) In general, whatever the length of the vector α may be, the first equation XVII. expresses that the line ε is (comp. 188) the reflexion of the line ζ , with respect to that vector α ; because it may be put (comp. 279) under the form,

$$\text{XIX.} \dots \zeta\alpha^{-1} = \alpha^{-1}\varepsilon = K\varepsilon\alpha^{-1}, \quad \text{or} \quad \text{XIX.} \dots \varepsilon\alpha^{-1} = K\zeta\alpha^{-1}.$$

(11.) Another mode of arriving at the same interpretation of the equation

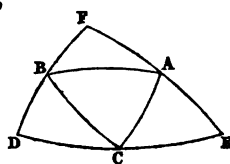


Fig. 67.

$\epsilon = \alpha\zeta\alpha^{-1}$, is to conceive ζ decomposed into two summand vectors, ζ' and ζ'' , one parallel and the other perpendicular to α , in such a manner that

$$\text{XX.} \dots \zeta = \zeta' + \zeta'', \quad \zeta' \parallel \alpha, \quad \zeta'' \perp \alpha;$$

for then we shall have, by 281, (10.), the transformations,

$$\text{XXI.} \dots \epsilon = \alpha\zeta'\alpha^{-1} + \alpha\zeta''\alpha^{-1} = \zeta'\alpha\alpha^{-1} - \zeta''\alpha\alpha^{-1} = \zeta' - \zeta'';$$

the *parallel part* of ζ being thus *preserved*, but the *perpendicular part* being *reversed*, by the operation $\alpha(\)\alpha^{-1}$.

(12.) Or we may *return* from $\epsilon = \alpha\zeta\alpha^{-1}$ to the form $\epsilon\alpha = \alpha\zeta$, that is, to the first equation XVI'.; and then this equation between quaternions will show, as suggested in (7.), that whatever may be the *length* of α , we must have,

$$\text{XXII.} \dots T\epsilon = T\zeta, \quad \text{Ax.}^* \epsilon\alpha = \text{Ax.} \alpha\zeta, \quad \angle \epsilon\alpha = \angle \alpha\zeta;$$

so that the *two lines* ϵ , ζ are *equally long*, and the *rotation* from ϵ to α is *equal* to that from α to ζ ; these two rotations being *similarly directed*, and in one *common plane*.

(13.) We may also write the equations XVII. XVII'. under the forms,

$$\text{XXIII.} \dots \epsilon = \alpha^{-1}\zeta\alpha, \text{ \&c.}; \quad \text{XXIII}'. \dots \zeta = \alpha^{-1}\epsilon\alpha, \text{ \&c.}$$

(14.) Substituting this last expression for ζ in the second equation XVII'., we derive this new equation,

$$\text{XXIV.} \dots \delta = \beta\alpha^{-1}\epsilon\alpha\beta^{-1}; \quad \text{or} \quad \text{XXIV}'. \dots \epsilon = \alpha\beta^{-1}\delta\beta\alpha^{-1};$$

that is, more briefly,

$$\text{XXV.} \dots \delta = q\epsilon q^{-1}, \quad \text{and} \quad \text{XXV}'. \dots \epsilon = q^{-1}\delta q, \quad \text{if} \quad \text{XXVI.} \dots q = \beta\alpha^{-1}.$$

(15.) An expression of this *form*, namely one with such a symbol as

$$\text{XXVII.} \dots q(\)q^{-1}$$

for an *operator*, occurred before, in 179, (1.), and in 191, (6.); and was seen to indicate a *conical rotation of the axis of the operand quaternion* (of which the *symbol* is to be conceived as being written *within the parentheses*), *round the axis of q , through an angle = $2\angle q$* , without any change of the *angle*, or of the *tensor*, of that *operand*; so that a *vector* must remain a *vector*, after any *operation* of this sort, as being *still a right-angled quaternion* (290); or (comp. 223, (10.)) because

$$\text{XXVIII.} \dots Sq\rho q^{-1} = Sq^{-1}q\rho = S\rho = 0.$$

(16.) If then we conceive two *opposite points*, r' and r , to be determined on the unit-sphere, by the conditions of being respectively the *positive poles* of the two *opposite arcs*, ΔB and BA , so that

$$\text{XXIX.} \dots \text{Op}' = \text{Ax.} \beta\alpha^{-1} = \text{Ax.} q, \quad \text{and} \quad \text{Op} = \text{P}'\text{O} = \text{Ax.} \alpha\beta^{-1} = \text{Ax.} q^{-1},$$

we can infer from XXIV. that the line OD may be derived from the line OE , by a *conical rotation round the line OP' as an axis, through an angle equal to the double of the angle ΔOB* (if O be still the *centre of the sphere*).

(17.) And in like manner we can infer from XXIV'., that the line OE admits

* It was remarked in 291, that this *characteristic Ax.* can be *dispensed with*, because it admits of being *replaced* by UV ; but there may still be a convenience in employing it occasionally.

of being derived from OD, by an *equal but opposite conical rotation*, round the line or as a *new positive axis*, through an angle equal to twice the angle BOA.

(18.) To illustrate these and other connected results, the annexed Figure 68 is drawn; in which P represents, as above, the positive pole of the arc BA, and arcs are drawn from it to D, E, F, meeting the great circle through A and B in the points R, S, T. (The other letters in the Figure are not, for the moment, required, but their significations will soon be explained.)

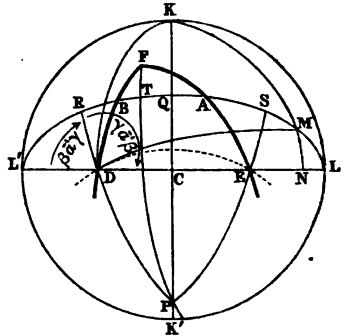


Fig. 68.

(19.) This being understood, we see, first, that because the arcs EF and FD are bisected (5.) at A and B, the *three arcual perpendiculars*, ES, FT, DR, let fall from E, F, D, on the great circle through A and B, are *equally long*: and that therefore the point P is the *interior pole of the small circle DEF'*, if F' be the point diametrically opposite to F: so that a *conical rotation round this pole P*, or round the *axis OP*, would in fact bring the point D, or the line OD, to the position E, or OE, which is *one part of the theorem* (17.).

(20.) Again, the *quantity of this conical rotation*, is evidently measured by the arc ES of the great circle with P for pole; but the *bisections* above mentioned give (comp. 165) the two *arcual equations*,

$$\text{XXX.} \dots \circ RB = \circ BT, \quad \circ TA = \circ AS; \text{ whence } \text{XXXI.} \dots \circ RS = 2 \circ BA,$$

and the *other part* of the same theorem (17.) is proved.

(21.) The point P may be said to be the *reflexion, on the sphere, of the point D, with respect to the point B*, which bisects the interval between them; and thus we may say that *two successive reflexions of an arbitrary point upon a sphere* (as here from D to F, and then from F to E), *with respect to two given points* (B and A) *of a given great circle*, are jointly equivalent to *one conical rotation, round the pole (P) of that great circle*; or to the description of an arc of a small circle, round that pole, or parallel to that great circle: and that the *angular quantity* (DFE) of this rotation is double of that represented by the arc (BA) connecting the two given points; or is the double of the angle (BPA), which that given arc subtends, at the same pole (P).

(22.) There is, as we see, no difficulty in *geometrically proving this theorem of rotation*: but it is remarkable *how simply quaternions express it*: namely by the formula,

$$\text{XXXII.} \dots a \cdot \beta^{-1} \rho \beta \cdot a^{-1} = a \beta^{-1} \cdot \rho \cdot \beta a^{-1},$$

in which α, β, ρ may denote any three vectors; and which, as we see by the points, involves essentially the *associative principle of multiplication*.

(23.) Instead of conceiving that the point D, or the line OD, has been *reflected* into the position F, or OF, with respect to the point B, or to the line OB, with a similar *successive reflexion* from F to E, we may conceive that a point has moved along a small semicircle, with B for pole, from D to F, as indicated in Fig. 69, and then along

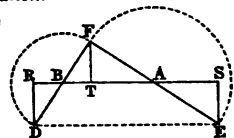


Fig. 69.

another small semicircle, with A for pole, from F to E ; and we see that the result, or effect, of these two successive and semicircular motions is equivalent to a motion along an arc DE of a third small circle, which is parallel (as before) to the great circle through B and A , and has a projection ES thereon, which (still as before) is double of the given arc BA .

(24.) And instead of thus conceiving two successive arcual motions of a point D upon a sphere, or two successive conical rotations of a radius OD , considered as compounding themselves into one resultant motion of that point, or rotation of that radius, we may conceive an analogous composition of two successive rotations of a solid body (or rigid system), round axes passing through a point O , which is fixed in space (and in the body): and so obtain a theorem respecting such rotation, which easily suggests itself from what precedes, and on which we may perhaps return.

(25.) But to draw some additional consequences from the equations VII., &c., and from the recent Fig. 68, especially as regards the Construction of the Fourth Proportional to three diplanar vectors, let us first remark, generally, that when we have (as in 62) a linear equation, of the form

$$aa + b\beta + c\gamma + d\delta = 0,$$

connecting four co-initial vectors $a \dots \delta$, whereof no three are coplanar, then this fifth vector,

$$\epsilon = aa + b\beta = -c\gamma - d\delta,$$

is evidently coplanar (22) with a, β , and also with γ, δ (comp. 294, (6.)); it is therefore part of the indefinite line of intersection of the plane AOB , COD , of these two pairs of vectors.

(26.) And if we divide this fifth vector ϵ by the two (generally unequal) scalars,

$$a + b, \text{ and } -c - d,$$

the two (generally unequal) vectors,

$$(aa + b\beta) : (a + b), \text{ and } (c\gamma + d\delta) : (c + d),$$

which are obtained as the quotients of these two divisions, are (comp. 25, 64) the vectors of two (generally distinct) points of intersection, of lines with planes, namely the two following:

$$AB \cdot OCD, \text{ and } CD \cdot OAB.$$

(27.) When the two lines, AB and CD , happen to intersect each other, the two last-mentioned points coincide; and thus we recover, in a new way, the condition (68), for the coplanarity of the four points O, A, B, C , or for the termino-coplanarity of the four vectors a, β, γ, δ ; namely the equation

$$a + b + c + d = 0,$$

which may be compared with 294, XLV. and L.

(28.) Resuming now the recent equations VII., and introducing the new vector,

$$\text{XXXIII.} \dots \lambda = la - m\beta = \frac{1}{2}(\epsilon - \delta),$$

which gives,

$$\text{XXXIV.} \dots S\gamma\lambda = 0, \text{ and } \text{XXXV.} \dots T\lambda = \sqrt{(r^2 - \pi^2)} = r \sin c',$$

we see that the two arcs BA, DE , prolonged, meet in a point L (comp. Fig. 68), for which $OL = O\lambda$, and which is distant by a quadrant from O : a result which may be confirmed by elementary considerations, because (by a well-known theorem respect-

ing transversal arcs) the common bisector BA of the two sides, DE and EF, must meet the third side in a point L, for which

$$\sin DL = \sin EL.$$

(29.) To prove by quaternions this last equality of sines, and to assign their common value, we have only to observe that by XXXIII.,

$$\text{XXXVI.} \dots V\delta\lambda = V\epsilon\lambda = \frac{1}{2}V\delta\epsilon;$$

in which,

$$T\delta\lambda = T\epsilon\lambda = r^2 \sin c', \quad \text{and} \quad TV\delta\epsilon = r^2 \sin 2c';$$

the sines in question are therefore (by 204, XIX.),

$$\text{XXXVI.} \dots TVU\delta\lambda = TVU\epsilon\lambda = \frac{1}{2}r^2 \sin 2c' : r^2 \sin c' = \cos c'.$$

(30.) On similar principles, we may interpret the two vector-equations,

$$\text{XXXVII.} \dots V\beta\lambda = lV\beta\alpha, \quad V\alpha\lambda = mV\beta\alpha,$$

in which

$$\text{XXXVIII.} \dots T\lambda : TV\beta\alpha = r \sin c' : \sin c = \tan c' : \tan c,$$

an equivalent to the trigonometric equations,

$$\text{XXXIX.} \dots \frac{\tan CD}{\tan AB} = \frac{\cos BC}{\sin BL} = \frac{\cos AC}{\sin AL}.$$

(31.) Accordingly, if we let fall the perpendicular CQ on AB (see again Fig. 68), so that Q bisects BA, and if we determine two new points M, N by the arcual equations,

$$\text{XL.} \dots \odot LM = \odot AB = \odot QR, \quad \odot LN = \odot OD,$$

the arcs MR, MD will be quadrants; and because the angle at R is right by construction (18.), M is the pole of DR, and DM is a quadrant; whence D is the pole of MN and the angle LNM is right: conceiving then that the arcs CA and CB are drawn, we have three triangles, right-angled at Q and N, which show, by elementary principles, that the three trigonometric quotients in XXXIX. have in fact a common value, namely $\cos CQ$, or $\cos L$.

(32.) To prove this last result by quaternions, and without employing the auxiliary points M, N, Q, R, we have the transformations,

$$\text{XLI.} \dots \cos L = SU \frac{V\beta\alpha}{V\delta\epsilon} = SU \frac{V\beta\alpha}{\gamma\lambda} = T \frac{\lambda}{V\beta\alpha} \cdot S \frac{\beta\alpha}{\gamma\lambda} = T \frac{\lambda}{V\beta\alpha};$$

because

$$\text{XLII.} \dots \delta = n\gamma - \lambda, \quad \epsilon = n\gamma + \lambda, \quad V\delta\epsilon = 2n\gamma\lambda, \quad UV\delta\epsilon = U\gamma\lambda,$$

and

$$\text{XLIII.} \dots S \frac{\beta\alpha}{\gamma\lambda} = \frac{S\beta\alpha\gamma\lambda}{(\gamma\lambda)^2} = -S\beta\alpha^{-1}\gamma\lambda^{-1} = -S\delta\lambda^{-1} = 1;$$

it being remembered that $\lambda \perp \gamma$, whence .

$$V\gamma\lambda = \gamma\lambda = -\lambda\gamma, \quad (\gamma\lambda)^2 = -\gamma^2\lambda^2 = \lambda^2, \quad S\gamma\lambda^{-1} = \phi.$$

(33.) At the same time we see that if P be (as before) the positive pole of BA, and if K, K' be the negative and positive poles of DE, while L' is the negative (as L is the positive) pole of CQ, whereby all the letters in Fig. 68 have their significations determined, we may write,

$$\text{XLIV.} \dots \odot P = UV\beta\alpha; \quad \odot K' = \gamma U\lambda; \quad \odot K = -\gamma U\lambda; \quad \odot L' = -U\lambda;$$

while

$$\odot L = +U\lambda, \text{ as before.}$$

(34.) Writing also,

$$\text{XLV.} \dots \kappa = -\gamma\lambda, \text{ or } \lambda = \gamma\kappa, \text{ and } \mu = \beta\alpha^{-1}\lambda,$$

so that $\text{XLV.} \dots \text{OK} = \text{U}\kappa, \text{ and } \text{OM} = \text{U}\mu,$

we have $\text{XLVI.} \dots \beta\alpha^{-1}\gamma = \mu\lambda^{-1}\lambda\kappa^{-1} = \mu\kappa^{-1};$

this fourth proportional, to the three equally long but diplanar vectors, $\alpha, \beta, \gamma,$ is therefore a versor, of which the representative arc (162) is $\text{KM},$ and the representative angle (174) is $\text{KDM},$ or $\text{L'DR},$ or $\text{EDP};$ and we may write for this versor, or quaternion, the expression :

$$\text{XLVII.} \dots \beta\alpha^{-1}\gamma = \cos \text{L'DR} + \text{OD. sin L'DR.}$$

(35.) The double of this representative angle is the sum of the two base-angles of the isosceles triangle $\text{DEF};$ and because the two other triangles, $\text{EFF'}, \text{F'PD},$ are also isosceles (19.), the line FF' shows that this sum is what remains, when we subtract the vertical angle $\text{F},$ of the triangle $\text{DEF},$ from the sum of the supplements of the two base-angles D and E of that triangle; or when we subtract the sum of the three angles of the same triangle from four right angles. We have therefore this very simple expression for the Angle of the Fourth Proportional :

$$\text{XLVIII.} \dots \angle \beta\alpha^{-1}\gamma = \text{L'DR} = \pi - \frac{1}{2}(\text{D} + \text{E} + \text{F}).$$

(36.) Or, if we introduce the area, or the spherical excess, say $\Sigma,$ of the triangle $\text{DEF},$ writing thus

$$\text{XLIX.} \dots \Sigma = \text{D} + \text{E} + \text{F} - \pi,$$

we have these other expressions :

$$\text{L.} \dots \angle \beta\alpha^{-1}\gamma = \frac{1}{2}\pi - \frac{1}{2}\Sigma; \quad \text{LI.} \dots \beta\alpha^{-1}\gamma = \sin \frac{1}{2}\Sigma + r^{-1}\delta \cos \frac{1}{2}\Sigma;$$

because

$$\text{OD} = \text{U}\delta = r^{-1}\delta, \text{ by XII.}$$

(37.) Having thus expressed $\beta\alpha^{-1}\gamma,$ we require no new appeal to the Figure, in order to express this other fourth proportional, $\gamma\alpha^{-1}\beta,$ which is the negative of its conjugate, or has an opposite scalar, but an equal vector part (comp. 204, (1.), and 295, (9.)) : the geometrical difference being merely this, that because the rotation round α from β to γ has been supposed to be negative, the rotation round α from γ to β must be, on the contrary, positive.

(38.) We may thus write, at once,

$$\text{LII.} \dots \gamma\alpha^{-1}\beta = -\text{K}\beta\alpha^{-1}\gamma = -\sin \frac{1}{2}\Sigma + r^{-1}\delta \cos \frac{1}{2}\Sigma;$$

and we have, for the angle of this new fourth proportional, to the same three vectors $\alpha, \beta, \gamma,$ of which the second and third have merely changed places with each other, the formula :

$$\text{LIII.} \dots \angle \gamma\alpha^{-1}\beta = \text{EDL} = \frac{1}{2}(\text{D} + \text{E} + \text{F}) = \frac{1}{2}\pi + \frac{1}{2}\Sigma.$$

(39.) But the common vector part of these two fourth proportionals is $\delta,$ by VII ; we have therefore, by XI,

$$\text{LIV.} \dots r = \cos \frac{1}{2}\Sigma; \quad \epsilon = \pm \sin \frac{1}{2}\Sigma;$$

the upper sign being taken, when the rotation round α from β to γ is negative, as above supposed.

(40.) It follows by (6.) that when the sides $2a', 2b', 2c',$ of a spherical triangle

DEF, of which the area is Σ , are bisected by the corners A, B, C of another spherical triangle, of which the sides* are a, b, c , then.

$$\text{LV.} \dots \cos a : \cos a' = \cos b : \cos b' = \cos c : \cos c' = \cos \frac{1}{2}\Sigma.$$

(41.) It follows also, from what has been recently shown, that the angle RDK, or MDN, or the arc MN in Fig. 68, represents the semi-area of the bisecting triangle DEF; whence, by the right-angled triangle LMN, we can infer that the sine of this semi-area is equal to the sine of a side of the bisecting triangle ABC, multiplied into the sine of the perpendicular, let fall upon that side from the opposite corner of the latter triangle; because we have

$$\text{LVI.} \dots \sin \frac{1}{2}\Sigma = \sin MN = \sin LM \cdot \sin L = \sin AB \cdot \sin OQ.$$

(42.) The same conclusion can be drawn immediately, by quaternions, from the expression,

$$\text{LVII.} \dots \sin \frac{1}{2}\Sigma = \epsilon = S\alpha\beta\gamma = S(\sqrt{\beta\alpha} \cdot \gamma^{-1}) = TV\beta\alpha \cdot SU(\sqrt{\beta\alpha} : \gamma);$$

in which one factor is the sine of AB, and the other factor is the cosine of CP, or the sine of OQ.

(43.) Under the same conditions, since

$$\text{LVIII.} \dots a = U(\epsilon + \zeta) = \frac{1}{2}I^{-1}(\epsilon + \zeta), \text{ \&c.,}$$

we may write also,

$$\text{LIX.} \dots \sin \frac{1}{2}\Sigma = SU(\epsilon + \zeta) (\zeta + \delta) (\delta + \epsilon) = S\delta\epsilon\zeta : 4lmn;$$

in which, by IV. and XIII.,

$$\text{LX.} \dots 4lmn = -S(\delta + \epsilon) (\epsilon + \zeta) = r^2 - S(\epsilon\zeta + \zeta\delta + \delta\epsilon).$$

(44.) Hence also, by LIV.,

$$\text{LXI.} \dots \cos \frac{1}{2}\Sigma = r = (r^2 - rS(\epsilon\zeta + \zeta\delta + \delta\epsilon)) : 4lmn;$$

$$\text{LXII.} \dots \tan \frac{1}{2}\Sigma = \frac{\epsilon}{r} = \frac{S\delta\epsilon\zeta}{r^2 - rS(\epsilon\zeta + \zeta\delta + \delta\epsilon)} = \frac{SU\delta\epsilon\zeta}{1 - SU\epsilon\zeta - SU\zeta\delta - SU\delta\epsilon};$$

and under this last form, we have a general expression for the tangent of half the spherical opening at O, of any triangular pyramid ODEF, whatever the lengths T δ , T ϵ , T ζ of the edges at O may be.

(45.) As a verification, we have

$$\begin{aligned} \text{LXIII.} \dots (4lmn)^2 &= -\frac{1}{2}(\epsilon + \zeta)^2 (\zeta + \delta)^2 (\delta + \epsilon)^2 \\ &= 2(r^2 - S\epsilon\zeta) (r^2 - S\zeta\delta) (r^2 - S\delta\epsilon); \end{aligned}$$

but the elimination of $\frac{1}{2}\Sigma$ between LIX. LXI. gives,

$$\text{LXIV.} \dots (4lmn)^2 = (S\delta\epsilon\zeta)^2 + (r^2 - r(S\epsilon\zeta + S\zeta\delta + S\delta\epsilon))^2;$$

we ought then to find that

$$\text{LXV.} \dots (S\delta\epsilon\zeta)^2 = r^6 - r^2 \{(S\epsilon\zeta)^2 + (S\zeta\delta)^2 + (S\delta\epsilon)^2\} - 2S\epsilon\zeta S\zeta\delta S\delta\epsilon,$$

if $\delta^2 = \epsilon^2 = \zeta^2 = -r^2$; and in fact this equality results immediately from the general formula 294, LIII.

(46.) Under the same condition, respecting the equal lengths of δ, ϵ, ζ , we have also the formula,

* These sides abc , of the bisecting triangle ABC, have been hitherto supposed for simplicity (1.) to be each less than a quadrant, but it will be found that the formula LV. holds good, without any such restriction.

LXVI. . . $V(\delta + \epsilon) (\epsilon + \zeta) (\zeta + \delta) = 2\delta(r^2 - S\epsilon\zeta - S\zeta\delta - S\delta\epsilon) = 8lmnd$;

whence other verifications may be derived.

(47.) If σ denote the *area** of the bisecting triangle ABC, the general principle LXII. enables us to infer that

$$\begin{aligned} \text{LXVII.} \dots \tan \frac{\sigma}{2} &= \frac{Sa\beta\gamma}{1 - S\beta\gamma - S\gamma\alpha - S\alpha\beta} = \frac{e}{1 + l + m + n} \\ &= \frac{\sin c \sin p}{1 + \cos a + \cos b + \cos c}, \end{aligned}$$

if p denote the perpendicular CQ from C on AB, so that

$e = \sin c \sin p = \sin b \sin c \sin A = \&c.$ (comp. 210, (21.).)

(48.) But, by (IX.) and (XI.),

$$\begin{aligned} \text{LXVIII.} \dots e^2 + (1 + l + m + n)^2 &= 2(1 + l)(1 + m)(1 + n) \\ &= \left(4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}\right)^2; \end{aligned}$$

hence the cosine and sine of the *new* semi-area are,

$$\text{LXIX.} \dots \cos \frac{\sigma}{2} = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}};$$

$$\text{LXX.} \dots \sin \frac{\sigma}{2} = \frac{\sin \frac{a}{2} \sin \frac{b}{2} \sin c}{\cos \frac{c}{2}} = \&c.$$

(49.) Returning to the bisected triangle, DEF, the last formula gives,

$$\text{LXXI.} \dots \sin \frac{1}{2}\Sigma = \frac{\sin a' \sin b' \sin F}{\cos c'} = \sin p' \sin c \sec c',$$

if p' denote the perpendicular from F on the bisecting arc AB, or FT in Fig. 68; but $\cos \frac{1}{2}\Sigma = \cos c \sec c'$, by LV.; hence

$$\text{LXXII.} \dots \tan \frac{1}{2}\Sigma = \sin p' \tan c = \sin FT \cdot \tan AB.$$

Accordingly, in Fig. 68, we have, by spherical trigonometry,

$$\sin FT = \sin MS = \sin LE \sin L = \cos LN \sin MN \operatorname{cosec} LM = \tan MN \cot AB.$$

(50.) The arc MN, which thus represents in quantity the semiarea of DEF, has its pole at the point D, and may be considered as the *representative arc* (162) of a certain *new quaternion*, Q, or of its *versor*, of which the axis is the radius OD, or U δ ; and this new quaternion may be thus expressed:

$$\text{LXXIII.} \dots Q = \delta\gamma\alpha\beta = -\delta^2 + \delta Sa\beta\gamma = r^2 + e\delta;$$

its tensor and versor being, respectively,

$$\text{LXXIV.} \dots TQ = r = \cos \frac{1}{2}\Sigma; \quad \text{LXXV.} \dots UQ = \cos \frac{1}{2}\Sigma + OD \cdot \sin \frac{1}{2}\Sigma.$$

(51.) An important transformation of this last *versor* may be obtained as follows:

* The reader will observe that the more usual symbol Σ , for this area of ABC, is here employed (36.) to denote the area of the *exscribed* triangle DEF.

LXXVI. . . $UQ = U(\delta\gamma^{-1} \cdot a\zeta^{-1} \cdot \zeta\beta^{-1}) = (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}}$;
so that

$$\text{LXXVII. . . } \frac{1}{2}\Sigma = \angle Q = \angle \delta\gamma\alpha\beta = \angle (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}};$$

these powers of quaternions, with exponents each = $\frac{1}{2}$, being interpreted as square roots (199, (1.)), or as equivalent to the symbols $\sqrt{(\delta\epsilon^{-1})}$, &c.

(52.) The conjugate (or reciprocal) versor, UQ^{-1} , which has \mathbb{M} for its representative arc, may be deduced from UQ by simply interchanging β and γ , or ϵ and ζ ; the corresponding quaternion is,

$$\text{LXXVIII. . . } Q' = KQ = \delta\beta\alpha\gamma = r^* - e\delta;$$

and we have

$$\text{LXXIX. . . } UQ' = \cos \frac{1}{2}\Sigma - oD \cdot \sin \frac{1}{2}\Sigma = (\delta\zeta^{-1})^{\frac{1}{2}} (\zeta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\delta^{-1})^{\frac{1}{2}};$$

the rotation round D , from \mathbb{E} to \mathbb{F} , being still supposed to be negative.

(53.) Let H be any other point upon the sphere, and let $OH = \eta$; also let Σ' be the area of the new spherical triangle, $D\mathbb{F}H$; then the same reasoning shows that

$$\text{LXXX. . . } \cos \frac{1}{2}\Sigma' + oD \cdot \sin \frac{1}{2}\Sigma' = (\delta\zeta^{-1})^{\frac{1}{2}} (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\delta^{-1})^{\frac{1}{2}},$$

if the rotation round D from \mathbb{F} to H be negative; and therefore, by multiplication of the two co-axial versors, LXXVI. and LXXX., we have by LXXV. the analogous formula:

$$\text{LXXXI. . . } \cos \frac{1}{2}(\Sigma + \Sigma') + oD \cdot \sin \frac{1}{2}(\Sigma + \Sigma') = (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\eta^{-1})^{\frac{1}{2}} (\eta\delta^{-1})^{\frac{1}{2}};$$

where $\Sigma + \Sigma'$ denotes the area of the spherical quadrilateral, $D\mathbb{E}\mathbb{F}H$.

(54.) It is easy to extend this result to the area of any spherical polygon, or to the spherical opening (44.) of any pyramid; and we may even conceive an extension of it, as a limit, to the area of any closed curve upon the sphere, considered as decomposed into an indefinite number of indefinitely small triangles, with some common vertex, such as the point D , on the spheric surface, and with indefinitely small arcs $\mathbb{E}\mathbb{F}$, $\mathbb{F}H$, . . of the curve, for their respective bases: or to the spherical opening of any cone, expressed thus as the Angle of a Quaternion, which is the limit* of the product of indefinitely many factors, each equal to the square-root of a quaternion, which differs indefinitely little from unity.

(55.) To assist the recollection of this result, it may be stated as follows (comp. 180, (3.) for the definition of an arcual sum):—

“The Arcual Sum of the Halves of the successive Sides, of any Spherical Polygon, is equal to an arc of a Great Circle, which has the Initial (or Final) Point of

* This Limit is closely analogous to a definite integral, of the ordinary kind; or rather, we may say that it is a Definite Integral, but one of a new kind, which could not easily have been introduced without Quaternions. In fact, if we did not employ the non-commutative property (168) of quaternion multiplication, the Products here considered would evidently become each equal to unity: so that they would furnish no expressions for spherical or other areas, and in short, it would be useless to speak of them. On the contrary, when that property or principle of multiplication is introduced, these expressions of product-form are found, as above, to have extremely useful significations in spherical geometry; and it will be seen that they suggest and embody a remarkable theorem, respecting the resultant of rotations of a system, round any number of successive axes, all passing through one fixed point, but in other respects succeeding each other with any gradual or sudden changes.

the Polygon for its Pole, and represents the Semi-area of the Figure;" it being understood that this resultant arc is reversed in direction, when the half-sides are (arcually) added in an opposite order.

(56.) As regards the order thus referred to, it may be observed that in the arcual addition, which corresponds to the quaternion multiplication in LXXVI, we conceive a point to move, first, from B to F, through half the arc DF; which half-side of the triangle DEF answers to the right-hand factor, or square-root, $(\zeta\delta^{-1})^{\frac{1}{2}}$. We then conceive the same point to move next from F to A, through half the arc FA, which answers to the factor placed immediately to the left of the former; having thus moved, on the whole, so far, through the resultant arc BA (as a transvector, 180, (8.)), or through any equal arc (163), such as ML in Fig. 68. And finally, we conceive a motion through half the arc ED, or through any arc equal to that half, such as the arc LW in the same Figure, to correspond to the extreme left-hand factor in the formula; the final resultant (or total transvector arc), which answers to the product of the three square-roots, as arranged in the formula, being thus represented by the final arc MN, which has the point D for its positive pole, and the half-area, $\frac{1}{2}\Sigma$, for the angle (51.) of the quaternion (or versor) product which it represents.

(57.) Now the direction of positive rotation on the sphere has been supposed to be that round D, from F to E; and therefore along the perimeter, in the order DFE, as seen* from any point of the surface within the triangle: that is, in the order in which the successive sides DF, FE, ED have been taken, before adding (or compounding) their halves. And accordingly, in the conjugate (or reciprocal) formula LXXIX, we took the opposite order, DEF, in proceeding as usual from right-hand to left-hand factors, whereof the former are supposed to be multiplied by† the latter; while the result was, as we saw in (52.), a new versor, in the expression for which, the area Σ of the triangle was simply changed to its own negative.

(58.) To give an example of the reduction of the area to zero, we have only to conceive that the three points D, E, F are co-arcual (165), or situated on one great circle; or that the three lines δ , ϵ , ζ are complanar. For this case, by the laws‡ of complanar quaternions, we have the formula,

$$\text{LXXXII.} \dots (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}} = 1, \quad \text{if } S\delta\epsilon\zeta = 0;$$

thus $\cos \frac{1}{2}\Sigma = 1$, and $\Sigma = 0$.

* In this and other cases of the sort, the spectator is imagined to stand on the point of the sphere, round which the rotation on the surface is conceived to be performed; his body being outside the sphere. And similarly when we say, for example, that the rotation round the line, or radius, OA, from the line OB to the line OC, is negative (or left-handed), as in the recent Figures, we mean that such would appear to be the direction of that rotation, to a person standing thus with his feet on A, and with his body in the direction of OA prolonged: or else standing on the centre (or origin) O, with his head at the point A. Compare 174, II.; 177; and the Note to page 158.

† Compare the Notes to pages 146, 159.

‡ Compare the Second Chapter of the Second Book.

(59.) Again, in (53.) let the point H be co-arcual with D and F , or let $S\delta\zeta\eta = 0$; then, because

$$\text{LXXXII}' \dots (\zeta\eta^{-1})^\dagger (\eta\delta^{-1})^\ddagger = (\zeta\delta^{-1})^\ddagger, \text{ if } S\delta\zeta\eta = 0,$$

the product of *four factors* LXXXI. reduces itself to the product of *three factors* LXXVI.; the *geometrical reason* being evidently that in this case the *added area* Σ' vanishes; so that the *quadrilateral* $DEFH$ has only the *same area* as the *triangle* DEF .

(60.) But this *added area* (58.) may even have a *negative** effect, as for example when the new point H falls on the old side DE . Accordingly, if we write

$$\text{LXXXIII} \dots Q_1 = (\epsilon\zeta^{-1})^\dagger (\zeta\eta^{-1})^\ddagger (\eta\epsilon^{-1})^\ddagger,$$

and denote the product LXXXI. of four square-roots by Q_2 , we shall have the transformation,

$$\text{LXXXIV} \dots Q_2 = (\delta\epsilon^{-1})^\dagger Q_1 (\epsilon\delta^{-1})^\ddagger, \text{ if } S\delta\epsilon\eta = 0;$$

which shows (comp. (15.)) that in this case the *angle* of the *quaternary product* Q_2 is that of the *ternary product* Q_1 , or the half-area of the *triangle* EFH ($= DEF - DHF$), although the *axis* of Q_2 is *transferred* from the position of the axis of Q_1 , by a *rotation* round the pole of the arc ED , which brings it from OE to OD .

(61.) From this example, it may be considered to be sufficiently evident, how the formula LXXXI. may be applied and extended, so as to represent (comp. (54.)) the *area of any closed figure on the sphere*, with *any assumed point* D on the surface as a sort of *spherical origin*; even when this *auxiliary point* is not situated on the *perimeter*, but is either *external* or *internal* thereto.

(62.) A *new quaternion* Q_0 , with the *same axis* OD as the quaternion Q of (50.), but with a *double angle*, and with a *tensor* equal to *unity*, may be formed by simply *squaring the versor* UQ ; and although this *squaring* cannot be effected by *removing the fractional exponents*,† in the formula LXXVI., yet it can easily be accomplished in other ways. For example we have, by LXXIII. LXXIV., and by VII. IX. X., the transformations:‡

$$\begin{aligned} \text{LXXXV} \dots Q_0 = UQ^2 = r^{-2}(\delta\gamma\alpha\beta)^2 &= -\delta^{-2} \cdot \gamma\alpha\beta\delta \cdot \delta\gamma\alpha\beta \\ &= -(\gamma\alpha\beta)^2 = -(e-\delta)^2 = r^2 - e^2 + 2e\delta; \end{aligned}$$

and in fact, because $\delta = r \cdot OD$, by XII., the trigonometric values LIV. for r and e enable us to write this last result under the form,

$$\text{LXXXVI} \dots Q_0 = -(\gamma\alpha\beta)^2 = \cos \Sigma + OD \cdot \sin \Sigma.$$

(63.) To show its geometrical signification, let us conceive that ABC and LMN

* In some investigations respecting *areas on a sphere*, it may be convenient to *distinguish* (comp. 28, 68) between the *two symbols* DEF and DFE , and to consider them as denoting *two opposite triangles*, of which the *sum* is zero. But for the present, we are content to express this *distinction*, by means of the *two conjugate quaternion products*, (51.) and (52.).

† Compare the Note to (54.).

‡ The equation $\delta\gamma\alpha\beta = \gamma\alpha\beta\delta$ is *not valid generally*; but we have here $\delta = -\sqrt{\gamma\alpha\beta}$; and in general, $q\rho = \rho q$, if $\rho \parallel \sqrt{q}$.

have the same meanings in the new Fig. 70, as in Fig. 68; and that $\triangle B_1M_1$ are three new points, determined by the three arcual equations (168),

$$\begin{aligned} \text{LXXXVII. } \sphericalangle AC = \sphericalangle CA_1, \quad \sphericalangle BC = \sphericalangle CB_1, \\ \sphericalangle MN = \sphericalangle NM_1; \end{aligned}$$

which easily conduct to this fourth equation of the same kind,

$$\text{LXXXVII.} \dots \sphericalangle LM_1 = \sphericalangle B_1A_1.$$

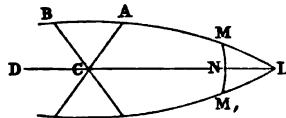


Fig. 70.

This *new arc* LM_1 represents thus (comp. 167, and Fig. 43) the product $a_1\gamma^{-1}.\gamma\beta^{-1} = \gamma\alpha^{-1}.\beta\gamma^{-1}$; while the old arc ML , or its equal BA (31.), represents $\alpha\beta^{-1}$; whence the arc MM_1 , which has its *pole* at D , and is numerically equal to the *whole area* Σ of DEF (because MN was seen to be equal (50.) to *half* that area), represents the product $\gamma\alpha^{-1}\beta\gamma^{-1}.\alpha\beta^{-1}$, or $-(\gamma\alpha\beta)^2$, or Q_0 . The formula LXXXVI. has therefore been *interpreted*, and may be said to have been *proved anew*, by these simple geometrical considerations.

(64.) We see, at the same time, how to *interpret the symbol*,

$$\text{LXXXVIII.} \dots Q_0 = \frac{\gamma}{a} \frac{\beta}{\gamma} \frac{\alpha}{\beta};$$

namely as denoting a *versor*, of which the *axis* is directed *to*, or *from*, the *corner* D of a certain *auxiliary spherical triangle* DEF , whereof the *sides*, respectively *opposite* to D, E, F , are *bisected* (5.) by the *given points* A, B, C , according as the *rotation* round a from β to γ is *negative* or *positive*; and of which the *angle* represents, or is numerically equal to, the *area* Σ of that auxiliary triangle: at least if we *still* suppose, as we have hitherto for simplicity done (1.), that the *sides* of the *given triangle* ABC are *each less than a quadrant*.

298. The *case* when the *sides* of the *given triangle* are *all greater*, instead of being *all less*, than *quadrants*, may deserve next to be (although more briefly) considered; the *case* when they are *all equal to quadrants*, being reserved for a short subsequent Article: and *other cases* being easily referred to *these*, by *limits*, or by passing from a *given line* to its *opposite*.

(1.) Supposing now that

$$\text{I.} \dots l < 0, \quad m < 0, \quad n < 0,$$

or that

$$\text{II.} \dots a > \frac{\pi}{2}, \quad b > \frac{\pi}{2}, \quad c > \frac{\pi}{2},$$

we may still retain the recent equations IV. to XI.; XIII.; and XV. to XXVI., of 297; but we must *change the sign of the radical*, r , in the equations XII. and XIV., and also the *signs of the versors* $U\delta, U\epsilon, U\zeta$ in XII., if we desire that the *sides* of the *auxiliary triangle*, DEF , may still be *bisected* (as in Figures 67, 68) by the *corners* of the *given triangle* ABC , of which the *sides* a, b, c are now each *greater than a quadrant*. Thus, r being still the *common tensor* of δ, ϵ, ζ , and therefore being still supposed to be *itself* > 0 , we must write now, under these *new conditions* I. or II., the new equations,

$$\text{III.} \dots \text{OD} = -\text{U}\delta = -r^{-1}\delta; \quad \text{OE} = -\text{U}\epsilon = -r^{-1}\epsilon; \quad \text{OF} = -\text{U}\zeta = -r^{-1}\zeta;$$

$$\text{IV.} \dots \cos a = -r \cos a', \quad \cos b = -r \cos b', \quad \cos c = -r \cos c'.$$

(2.) The equations IV. and VIII. of 297 still holding good, we may now write,

$$\text{V.} \dots \pm 2r \cos a' \cos b' \cos c' = \cos a^2 + \cos b^2 + \cos c^2 - 1,$$

according as we adopt positive values (297), or negative values (298), for the cosines l, m, n of the sides of the bisecting triangle; the value of r being still supposed to be positive.

(8.) It is not difficult to prove (comp. 297, LIV., LXIX.), that

$$\text{VI.} \dots r = \pm \cos \frac{1}{2}\Sigma, \text{ according as } l > 0, \text{ \&c.}, \text{ or } l < 0, \text{ \&c.};$$

the recent formula V. may therefore be written *unambiguously* as follows:

$$\text{VII.} \dots 2 \cos a' \cos b' \cos c' \cos \frac{1}{2}\Sigma = \cos a^2 + \cos b^2 + \cos c^2 - 1;$$

and the formula 297, LV. continues to hold good.

(4.) In like manner, we may write, without an ambiguous sign (comp. 297, LI.), the following *expression for the fourth proportional $\beta a^{-1}\gamma$ to three unit-vectors a, β, γ , the rotation round the first from the second to the third being negative:*

$$\text{VIII.} \dots \beta a^{-1}\gamma = \sin \frac{1}{2}\Sigma + \text{OD.} \cos \frac{1}{2}\Sigma;$$

where the scalar part changes sign, when the rotation is reversed.

(5.) It is, however, to be observed, that although this *formula VIII.* holds good, not only in the cases of the last article and of the present, but also in that which has been reserved for the next, namely when $l = 0$, &c.; yet because, in the *present case* (298) we have the area $\Sigma > \pi$, the radius OD is no longer the (positive) axis U δ of the fourth proportional $\beta a^{-1}\gamma$; nor is $\frac{1}{2}\pi - \frac{1}{2}\Sigma$ any longer, as in 297, L., the (positive) *angle* of that versor. On the contrary we have *now*, for this axis and angle, the expressions:

$$\text{IX.} \dots \text{Ax. } \beta a^{-1}\gamma = \text{DO} = -\text{OD}; \quad \text{X.} \dots \angle \beta a^{-1}\gamma = \frac{1}{2}(\Sigma - \pi).$$

(6.) To illustrate these results by a construction, we may remark that if, in Fig. 67, the bisecting arcs BC, CA, AB be supposed each greater than a quadrant, and if we proceed to form from it a new Figure, analogous to 68, the perpendicular CQ will also exceed a quadrant, and the poles P and K will fall *between* the points C and Q; also M and R will fall on the arcs LQ and QL' *prolonged*: and although the arc KM, or the angle KDM, or L'DR, or EDF, may *still* be considered, as in 297, (84.), to represent the versor $\beta a^{-1}\gamma$, yet the corresponding rotation round the point D is now of a *negative* character.

(7.) And as regards the *quantity* of this rotation, or the magnitude of the angle at D, it is again, as in Fig. 68, a base-angle of one of three isosceles triangles, with P for their common vertex; but we have now, as in Fig. 71, a *new arrangement*, in virtue of which this angle is to be found by halving what remains, when the sum of the supplements of the angles at D and E, in the triangle DER, is subtracted from the angle at P, instead of our subtracting (as in 297, (85.)) the latter angle from the former sum; it is therefore *now*, in agreement with the recent expression X.,

$$\text{XI.} \dots \angle \beta a^{-1}\gamma = \frac{1}{2}(D + E + P) - \pi.$$

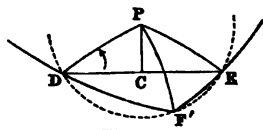


Fig. 71.

(8.) The negative of the conjugate of the formula VIII. gives,

$$\text{XII.} \dots \gamma\alpha^{-1}\beta = -\sin \frac{1}{2}\Sigma + \text{OD.} \cos \frac{1}{2}\Sigma;$$

and by taking the negative of the square of this equation, we are conducted to the following:

$$\text{XIII.} \dots \frac{\gamma}{\alpha} \frac{\beta}{\gamma} \frac{\alpha}{\beta} = -(\gamma\alpha^{-1}\beta)^2 = \cos \Sigma + \text{OD.} \sin \Sigma;$$

a result which had only been proved before (comp. 297, (62.), (64.)) for the case $\Sigma < \pi$; and in which it is still supposed that the rotation round α from β to γ is negative.

(9.) With the same direction of rotation, we have also the *conjugate* or *reciprocal* formula,

$$\text{XIV.} \dots \frac{\beta}{\alpha} \frac{\gamma}{\beta} \frac{\alpha}{\gamma} = -(\beta\alpha^{-1}\gamma)^2 = \cos \Sigma - \text{OD.} \sin \Sigma.$$

(10.) If it happened that only *one* side, as AB , of the *given* triangle ABC , was greater, while each of the two others was less than a quadrant, or that we had $l > 0$, $m > 0$, but $n < 0$; and if we wished to represent the fourth proportional to α, β, γ by means of the foregoing constructions: we should only have to introduce the point c' opposite to c , or to change γ to $\gamma' = -\gamma$; for thus the *new* triangle ABC' would have each side greater than a quadrant, and so would fall under the case of the present Article; after employing the construction for which, we should only have to change the resulting versor to its negative.

(11.) And in like manner, if we had l and m negative, but n positive, we might again substitute for c its opposite point c' , and so fall back on the construction of Art. 297: and similarly in other cases.

(12.) In general, if we begin with the equations 297, XII., attributing any *arbitrary* (but positive) value to the *common tensor*, r , of the three co-initial vectors ξ, ϵ, ζ , of which the *versors*, or the *unit-vectors* $U\xi, U\epsilon, U\zeta$, terminate at the corners of a *given* or *assumed* triangle DEF , with sides $= 2a', 2b', 2c'$, we may then suppose (comp. Fig. 67) that *another* triangle ABC , with sides denoted by a, b, c , and with their cosines denoted by l, m, n , is *derived* from this one, by the condition of *bisecting its sides*; and therefore by the equations (comp. 297, LVIII.),

$$\text{XV.} \dots \text{OA} = \alpha = U(\epsilon + \zeta), \quad \text{OB} = \beta = U(\zeta + \delta), \quad \text{OC} = \gamma = U(\delta + \epsilon),$$

with the relations 297, IV. V. VI., as before; or by these other equations (comp. 297, XIII. XIV.),

$$\text{XVI.} \dots \epsilon + \zeta = 2r\alpha \cos a', \quad \zeta + \delta = 2r\beta \cos b', \quad \delta + \epsilon = 2r\gamma \cos c'.$$

(13.) When *this* simple construction is adopted, we have at once (comp. 297, LX.), by merely taking *scalars of products of vectors*, and *without any reference to areas* (compare however 297, LXIX., and 298, VII.), the equations,

$$\text{XVII.} \dots 4 \cos a \cos b' \cos c' = 4 \cos b \cos c' \cos a' = 4 \cos c \cos a' \cos b' \\ = -r^2 S(\zeta + \delta)(\delta + \epsilon) = \&c. = 1 + \cos 2a' + \cos 2b' + \cos 2c';$$

or

$$\text{XVIII.} \dots \frac{\cos a}{\cos a'} = \frac{\cos b}{\cos b'} = \frac{\cos c}{\cos c'} = \frac{\cos a'^2 + \cos b'^2 + \cos c'^2 - 1}{2 \cos a' \cos b' \cos c'};$$

which can indeed be otherwise deduced, by the known formulæ of spherical trigonometry.

(14.) We see, then, that according as the sum of the squares of the cosines of the half-sides, of a given or assumed spherical triangle, DEF, is greater than unity, or equal to unity, or less than unity, the sides of the inscribed and bisecting triangle, ABC, are together less than quadrants, or together equal to quadrants, or together greater than quadrants.

(15.) Conversely, if the sides of a given spherical triangle ABC be thus all less, or all greater than quadrants, a triangle DEF, but only one* such triangle, can be exscribed to it, so as to have its sides bisected, as above: the simplest process being to let fall a perpendicular, such as CQ in Fig. 68, from C on AB, &c.; and then to draw new arcs, through C, &c., perpendicular to these perpendiculars, and therefore coinciding in position with the sought sides DE, &c., of DEF.

(16.) The *trigonometrical results* of recent sub-articles, especially as regards the area† of a spherical triangle, are probably all well known, as certainly some of them are; but they are here brought forward only in connexion with *quaternion formulae*; and as one of that class, which is not irrelevant to the present subject, and includes the formula 294, LIII., the following may be mentioned, wherein α, β, γ denote any three vectors, but the order of the factors is important:

$$\text{XIX.} \dots (\alpha\beta\gamma)^2 = 2\alpha^2\beta^2\gamma^2 + \alpha^2(\beta\gamma)^2 + \beta^2(\alpha\gamma)^2 + \gamma^2(\alpha\beta)^2 - 4\alpha\gamma S\alpha\beta S\beta\gamma.$$

(17.) And if, as in 297, (1.), &c., we suppose that α, β, γ are three unit-vectors, OA, OB, OC, and denote, as in 297, (47.), by σ the area of the triangle ABC, the principle expressed by the recent formula XIII. may be stated under this apparently different, but essentially equivalent form:

$$\text{XX.} \dots \frac{\alpha + \beta}{\beta + \gamma} \cdot \frac{\gamma + \alpha}{\alpha + \beta} \cdot \frac{\beta + \gamma}{\gamma + \alpha} = \cos \sigma + \alpha \sin \sigma;$$

which admits of several verifications.

(18.) We may, for instance, transform it as follows (comp. 297, LXVII.):

$$\begin{aligned} \text{XXI.} \dots \frac{-(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)}{K(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)} &= \frac{-2e + 2a(1 + l + m + n)}{+2e + 2a(1 + l + m + n)} \\ &= \frac{1 + l + m + n + ea}{1 + l + m + n - ea} = \frac{1 + \alpha \tan \frac{\sigma}{2}}{1 - \alpha \tan \frac{\sigma}{2}} = \frac{\cos \frac{\sigma}{2} + \alpha \sin \frac{\sigma}{2}}{\cos \frac{\sigma}{2} - \alpha \sin \frac{\sigma}{2}} \\ &= \left(\cos \frac{\sigma}{2} + \alpha \sin \frac{\sigma}{2} \right)^2 = \cos \sigma + \alpha \sin \sigma, \text{ as above.} \end{aligned}$$

* In the next Article, we shall consider a case of *indeterminateness*, or of the existence of indefinitely many exscribed triangles DEF: namely, when the sides of ABC are all equal to quadrants.

† This opportunity may be taken of referring to an interesting Note, to pages 96, 97 of *Luby's Trigonometry* (Dublin, 1852); in which an elegant construction, connected with the area of a spherical triangle, is acknowledged as having been mentioned to Dr. Luby, by a since deceased and lamented friend, the Rev. William Digby Sadleir, F.T.C.D. A construction nearly the same, described in the sub-articles to 297, was suggested to the present writer by quaternions, several years ago.

(19.) This seems to be a natural place for observing (comp. (16.)), that if $\alpha, \beta, \gamma, \delta$ be any four vectors, the lately cited equation 294, LIII., and the square of the equation 294, XV., with δ written in it instead of ρ , conduct easily to the following very general and symmetric formula :

$$\begin{aligned} \text{XXII.} \dots & \alpha^2 \beta^2 \gamma^2 \delta^2 + (\text{S}\beta\gamma\text{S}\alpha\delta)^2 + (\text{S}\gamma\alpha\text{S}\beta\delta)^2 + (\text{S}\alpha\beta\text{S}\gamma\delta)^2 \\ & + 2\alpha^2\text{S}\beta\gamma\text{S}\beta\delta\text{S}\gamma\delta + 2\beta^2\text{S}\gamma\alpha\text{S}\gamma\delta\text{S}\alpha\delta + 2\gamma^2\text{S}\alpha\beta\text{S}\alpha\delta\text{S}\beta\delta + 2\delta^2\text{S}\alpha\beta\text{S}\beta\gamma\text{S}\gamma\alpha \\ & = 2\text{S}\gamma\alpha\text{S}\alpha\beta\text{S}\beta\delta\text{S}\gamma\delta + 2\text{S}\alpha\beta\text{S}\beta\gamma\text{S}\gamma\delta\text{S}\alpha\delta + 2\text{S}\beta\gamma\text{S}\gamma\alpha\text{S}\alpha\delta\text{S}\beta\delta \\ & \quad + \beta^2\gamma^2(\text{S}\alpha\delta)^2 + \gamma^2\alpha^2(\text{S}\beta\delta)^2 + \alpha^2\beta^2(\text{S}\gamma\delta)^2 \\ & \quad + \alpha^2\delta^2(\text{S}\beta\gamma)^2 + \beta^2\delta^2(\text{S}\gamma\alpha)^2 + \gamma^2\delta^2(\text{S}\alpha\beta)^2. \end{aligned}$$

(20.) If then we take any spherical quadrilateral ABCD, and write

$$\text{XXIII.} \dots l' = \cos AD = -\text{S}U\alpha\delta, \quad m' = \cos BD = -\text{S}U\beta\delta, \quad n' = \cos CD = \&c.,$$

treating α, β, γ as the unit-vectors of the points A, B, C, and l, m, n as the cosines of the arcs BC, CA, AB, as in 297, (1.), we have the equation,

$$\begin{aligned} \text{XXIV.} \dots & 1 + l'^2 + m'^2 + n'^2 + 2lm'n' + 2mn'l' + 2n'l'm' + 2lmn \\ & = 2mnm'n' + 2nlm'l' + 2lm'l'm' \\ & \quad + l^2 + m^2 + n^2 + l'^2 + m'^2 + n'^2; \end{aligned}$$

which can be confirmed by elementary considerations,* but is here given merely as an interpretation of the quaternion formula XXII.

(21.) In squaring the lately cited equation 294, XV., we have used the two following formulæ of transformation (comp. 204, XXII., and 210, XVIII.), in which α, β, γ may be any three vectors, and which are often found to be useful :

$$\text{XXV.} \dots (\text{V}\alpha\beta)^2 = (\text{S}\alpha\beta)^2 - \alpha^2\beta^2; \quad \text{XXVI.} \dots \text{S}(\text{V}\beta\gamma \cdot \text{V}\gamma\alpha) = \gamma^2\text{S}\alpha\beta - \text{S}\beta\gamma\text{S}\gamma\alpha.$$

299. The two cases, for which the three sides a, b, c , of the given triangle ABC, are all less, or all greater, than quadrants, having been considered in the two foregoing Articles, with a reduction, in 298, (10.) and (11.), of certain other cases to these, it only remains to consider that third principal case, for which the sides of that given triangle are all equal to quadrants: or to inquire what is, on our general principles, the Fourth Proportional to Three Rectangular Vectors. And we shall find, not only that this fourth proportional is not itself a Vector, but that it does not even contain any vector part (292) different from zero: although, as being found to be equal to a Scalar, it is still included (131, 276) in the general conception of a Quaternion.

(1.) In fact, if we suppose, in 297, (1.), that

$$\text{I.} \dots l = 0, \quad m = 0, \quad n = 0, \quad \text{or that} \quad \text{II.} \dots a = b = c = \frac{\pi}{2},$$

* A formula equivalent to this last equation of seventeen terms, connecting the six cosines of the arcs which join, two by two, the corners of a spherical quadrilateral ABCD, is given at page 407 of Carnot's *Géométrie de Position* (Paris, 1808).

or III. . . $S\beta\gamma = S\gamma\alpha = S\alpha\beta = 0$, while IV. . . $T\alpha = T\beta = T\gamma = 1$,
the formulæ 297, VII. give,

$$V. . . \delta = 0, \quad \epsilon = 0, \quad \zeta = 0;$$

but these are the *vector parts* of the *three pairs of fourth proportionals* to the *three rectangular unit-lines*, α, β, γ , taken in all possible orders; and the same *evanescence of vector parts* must evidently take place, if the three given lines be only at *right angles* to each other, without being *equally long*.

(2.) Continuing, however, for simplicity, to suppose that they are unit lines, and that the rotation round α from β to γ is negative, as before, we see that we have now $r = 0$, and $e = 1$, in 297, (3.); and that thus *the six fourth proportionals reduce themselves to their scalar parts*, namely (here) to *positive or negative unity*. In this manner we find, under the supposed conditions, the values:

$$VI. . . \beta\alpha^{-1}\gamma = \gamma\beta^{-1}\alpha = \alpha\gamma^{-1}\beta = +1; \quad VII. . . \gamma\alpha^{-1}\beta = \alpha\beta^{-1}\gamma = \beta\gamma^{-1}\alpha = -1.$$

(3.) For example (comp. 295) we have, by the laws (182) of i, j, k , the values,

$$VII. . . ij^{-1}k = jk^{-1}i = ki^{-1}j = +1; \quad VII'. . . kj^{-1}i = ik^{-1}j = ji^{-1}k = -1.$$

In fact, the *two fourth proportionals*, $ij^{-1}k$ and $kj^{-1}i$, are respectively equal to the *two ternary products*, $-ijk$ and $-kji$, and therefore to $+1$ and -1 , by the laws included in the *Fundamental Formula A* (188).

(4.) To connect this important result with the *constructions* of the two last Articles, we may observe that when we seek, on the general plan of 298, (15.), to *exscribe a spherical triangle, DEF, to a given tri-quadrantal (or tri-rectangular) triangle, ABC*, as for instance to the triangle IJK (or JIK) of 181, in such a manner that the *sides of the new triangle shall be bisected by the corners of the old*, the problem is found to admit of *indefinitely many solutions*. Any point P may be assumed, in the *interior* of the given triangle ABC ; and then, if its *reflexions D, E, F* be taken, with respect to the three *sides a, b, c*, so that (comp. Fig. 72) the *arcs PD, PE, PF* are *perpendicularly bisected* by those three sides, the three *other arcs EF, FD, DE* will be bisected by the *points A, B, C*, as required: because the arcs AE, AF have each the same length as AP , and the angles subtended at A by PE and PF are together equal to two right angles, &c.

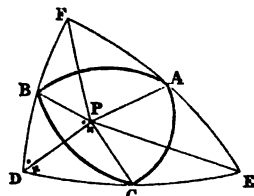


Fig. 72.

(5.) The *positions of the auxiliary points, D, E, F*, are therefore, in the present case, *indeterminate, or variable*; but the *sum of the angles at those three points is constant*, and equal to *four right angles*; because, by the *six isosceles triangles* on PD, PE, PF as bases, that sum of the three angles D, E, F is equal to the sum of the angles subtended by the sides of the given triangle ABC , at the assumed interior point P . The *spherical excess* of the triangle DEF is therefore equal to *two right angles*, and its area $\Sigma = \pi$; as may be otherwise seen from the same Figure 72, and might have been inferred from the formula 297, LV., or LVI.

(6.) The *radius OD*, in the formula 297, XLVII., for the fourth proportional $\beta\alpha^{-1}\gamma$, becomes therefore, in the present case, *indeterminate*; but because the *angle L'DB*, or $\frac{1}{2}(\pi - \Sigma)$, in the same equation, *vanishes*, the formula becomes simply

$\beta\alpha^{-1}\gamma = 1$, as in the recent equations VI. ; and similarly in other examples, of the class here considered.

(7.) The conclusion, that the *Fourth Proportional to Three Rectangular Lines is a Scalar*, may in several other ways be deduced, from the principles of the present Book. For example, with the recent suppositions, we may write,

$$\begin{aligned} \text{VIII.} \dots \beta\alpha^{-1} &= -\gamma, & \gamma\beta^{-1} &= -\alpha, & \alpha\gamma^{-1} &= -\beta; \\ \text{VIII}'. \dots \gamma\alpha^{-1} &= +\beta, & \alpha\beta^{-1} &= +\gamma, & \beta\gamma^{-1} &= +\alpha; \end{aligned}$$

the three fourth proportionals VI. are therefore equal, respectively, to $-\gamma^2$, $-\alpha^2$, $-\beta^2$, and consequently to $+1$; while the corresponding expressions VI'. are equal to $+\beta^2$, $+\gamma^2$, $+\alpha^2$, and therefore to -1 .

(8.) Or (comp. (8.)) we may write *generally* the transformation (comp. 282, XXI.*),

$$\text{IX.} \dots \beta\alpha^{-1}\gamma = a^{-2}.\beta\alpha\gamma, \text{ if } a^{-2} = 1 : a^2,$$

in which the factor a^{-2} is *always* a scalar, whatever vector a may be; while the *vector part* of the ternary product $\beta\alpha\gamma$ vanishes, by 294, III., when the recent conditions of rectangularity III. are satisfied.

(9.) Conversely, this ternary product $\beta\alpha\gamma$, and this fourth proportional $\beta\alpha^{-1}\gamma$, can never reduce themselves to scalars, unless the three vectors α , β , γ (supposed to be all actual (Art. 1)) are perpendicular each to each.

SECTION 8.—*On an equivalent Interpretation of the Fourth Proportional to Three Diplanar Vectors, deduced from the Principles of the Second Book.*

300. In the foregoing Section, we naturally employed the results of preceding Sections of the present Book, to assist ourselves in attaching a definite signification to the Fourth Proportional (297) to Three Diplanar Vectors; and thus, in order to interpret the symbol $\beta\alpha^{-1}\gamma$, we availed ourselves of the interpretations previously obtained, in this Third Book, of α^{-1} as a *line*, and of $a\beta$, $a\beta\gamma$ as *quaternions*. But it may be interesting, and not uninteresting, to inquire how the equivalent symbol,

$$\text{I.} \dots (\beta : \alpha) . \gamma, \text{ or } \frac{\beta}{\alpha} \gamma, \text{ with } \gamma \text{ not } ||| \alpha, \beta,$$

might have been interpreted, on the principles of the Second Book, without at first assuming as known, or even seeking to discover, any interpretation of the three lately mentioned symbols,

$$\text{II.} \dots \alpha^{-1}, \alpha\beta, \alpha\beta\gamma.$$

It will be found that the inquiry conducts to an expression of the form,

* The formula here referred to should have been printed as $R\alpha = 1 : \alpha = \alpha^{-1}$.

$$\text{III. . . } (\beta : a) . \gamma = \delta + \epsilon u ;$$

where δ is the *same vector*, and ϵ is the *same scalar*, as in the recent sub-articles to 297; while u is employed as a temporary symbol, to denote a certain *Fourth Proportional to Three Rectangular Unit Lines*, namely, to the three lines oq , ol' , and op in Fig. 68; so that, with reference to the construction represented by that Figure, we should be led, by the principles of the Second Book, to write the equation:

$$\text{IV. . . } (ob : oa) . oc = od . \cos \frac{1}{2} \Sigma + (ol' : oq) . op . \sin \frac{1}{2} \Sigma .$$

And when we proceed to consider *what signification* should be attached, on the principles of the same Second Book, to *that particular fourth proportional*, which is here the coefficient of $\sin \frac{1}{2} \Sigma$, and has been provisionally denoted by u , we find that although it may be regarded as being *in one sense* a *Line*, or at least *homogeneous with a line*, yet it *must not be equated to any Vector*: being rather *analogous, in Geometry, to the Scalar Unit of Algebra*, so that it may be naturally and conveniently denoted by the *usual symbol* 1, or + 1, or be equated to *Positive Unity*. But when we thus write $u = 1$, the last term of the formula III. or IV., of the present Article, becomes simply ϵ , or $\sin \frac{1}{2} \Sigma$; and while this *term* (or *part*) of the result comes to be considered as a species of *Geometrical Scalar*, the *complete Expression for the General Fourth Proportional to Three Diplanar Vectors* takes the *Form of a Geometrical Quaternion*: and thus the *formula* 297, XLVII., or 298, VIII., is *reproduced*, at least if we substitute in it, for the present, $(\beta : a) . \gamma$ for $\beta a^{-1} \gamma$, to avoid the necessity of interpreting *here* the recent symbols II.

(1.) The construction of Fig. 68 being retained, but no principles peculiar to the Third Book being employed, we may write, with the same significations of c , p , &c., as before,

$$\text{V. . . } ob : oa = or : oq = \cos c + (ol' : oq) \sin c ;$$

$$\text{VI. . . } oc = oq . \cos p + op . \sin p .$$

(2.) Admitting then, as is natural, for the purposes of the sought *interpretation*, that *distributive property* which has been *proved* (212) to hold good for the *multiplication of quaternions* (as it does for multiplication in algebra); and writing for abridgment,

$$\text{VII. . . } \kappa = (ol' : oq) . op ;$$

we have the *quadrinomial expression* :

$$\text{VIII. . . } (ob : oa) . oc = ol' . \sin c \cos p + oq . \cos c \cos p \\ + op . \cos c \sin p + \kappa . \sin c \sin p ;$$

in which it may be observed that *the sum of the squares of the four coefficients of the*

three rectangular unit-vectors, OQ , OL' , OP , and of their fourth proportional, u , is equal to unity.

(3.) But the coefficient of this fourth proportional, which may be regarded as a species of fourth unit, is

$$\text{IX.} \dots \sin c \sin p = \sin MN = \sin \frac{1}{2}\Sigma = e;$$

we must therefore expect to find that the three other coefficients in VIII., when divided by $\cos \frac{1}{2}\Sigma$, or by r , give quotients which are the cosines of the arcual distances of some point x upon the unit-sphere, from the three points L' , Q , P ; or that a point x can be assigned, for which

$$\text{X.} \dots \sin c \cos p = r \cos L'x; \quad \cos c \cos p = r \cos Qx; \quad \cos c \sin p = r \cos Px.$$

(4.) Accordingly it is found that these three last equations are satisfied, when we substitute D for x ; and therefore that we have the transformation,

$$\text{XI.} \dots OL' \cdot \sin c \cos p + OQ \cdot \cos c \cos p + OP \cdot \cos c \sin p = OD \cdot \cos \frac{1}{2}\Sigma = \delta,$$

whence follow the equations IV. and III.; and it only remains to study and interpret the fourth unit, u , which enters as a factor into the remaining part of the quadrinomial expression VIII., without employing any principles except those of the *Second Book*: and therefore without using the Interpretations 278, 284, of βa , &c.

301. In general, when two sets of three vectors, α, β, γ , and α', β', γ' , are connected by the relation,

$$\text{I.} \dots \frac{\beta}{\alpha} \frac{\gamma}{\gamma'} \frac{\alpha'}{\beta'} = 1, \quad \text{or} \quad \text{II.} \dots \frac{\beta}{\alpha} \frac{\gamma}{\gamma'} = \frac{\beta'}{\alpha'}$$

it is natural to write this other equation,

$$\text{III.} \dots \frac{\beta}{\alpha} \gamma = \frac{\beta'}{\alpha'} \gamma';$$

and to say that these two fourth proportionals (297), to α, β, γ , and to α', β', γ' , are equal to each other: whatever the full signification of each of these two last symbols III., supposed for the moment to be not yet fully known, may be afterwards found to be. In short, we may propose to make it a condition of the sought Interpretation, on the principles of the Second Book, of the phrase,

“Fourth Proportional to Three Vectors,”

and of either of the two equivalent Symbols 300, I., that the recent Equation III. shall follow from I. or II.; just as, at the commencement of that Second Book, and before concluding (112) that the general Geometric Quotient $\beta : \alpha$ of any two lines in space is a Quaternion, we made it a condition (103) of the interpretation of such a quotient, that the equation $(\beta : \alpha) \cdot \alpha = \beta$ should be satisfied.

302. There are however two tests (comp. 287), to which the recent equation III. must be submitted, before its final adoption; in

order that we may be sure of its *consistency*, Ist, with the *previous interpretation* (226) of a Fourth Proportional to *Three Coplanar Vectors*, as a *Line* in their common plane; and IInd, with the *general principle* of all *mathematical language* (105), that things equal to the same thing, are to be considered as equal to each other. And it is found, on trial, that *both* these tests are borne: so that they form no objection to our *adopting the equation* 301, III., as true by *definition*, whenever the preceding equation II., or I., is satisfied.

(1.) It may happen that the first member of that equation III. is equal to a *line* δ , as in 226; namely, when α, β, γ are *coplanar*. In this case, we have by II. the equation,

$$\text{IV.} \dots \frac{\delta}{\gamma'} = \frac{\delta \gamma}{\gamma \gamma'} = \frac{\beta'}{\alpha''} \quad \text{or} \quad \text{IV'.} \dots \frac{\beta'}{\alpha''} \gamma' = \delta = \frac{\beta}{\alpha} \gamma;$$

so that α', β', γ' are also coplanar (among themselves), and the line δ is their fourth proportional likewise: and the equation III. is satisfied, both members being symbols for one common line, δ , which is in general situated in the intersection of the two planes, $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$; although those planes may happen to coincide, without disturbing the truth of the equation.

(2.) Again, for the more general case of *dipolarity* of α, β, γ , we may conceive that the equation* II. co-exists with this other of the same form,

$$\text{V.} \dots \frac{\beta}{\alpha} \frac{\gamma}{\gamma''} = \frac{\beta''}{\alpha''}; \quad \text{which gives} \quad \text{VI.} \dots \frac{\beta}{\alpha} \gamma = \frac{\beta''}{\alpha''} \gamma'';$$

if the definition 801 be adopted. If then that definition be consistent with general principles of equality, we ought to find, by III. and VI., that this third equation between two fourth proportionals holds good:

$$\text{VII.} \dots \frac{\beta'}{\alpha'} \gamma' = \frac{\beta''}{\alpha''} \gamma''; \quad \text{or that} \quad \text{VIII.} \dots \frac{\beta'}{\alpha'} \frac{\gamma'}{\gamma''} = \frac{\beta''}{\alpha''};$$

when the equations II. and V. are satisfied. And accordingly, those two equations give, by the general principles of the Second Book, respecting quaternions considered as *quotients* of vectors, the transformation,

$$\frac{\beta'}{\alpha'} \frac{\gamma'}{\gamma''} = \frac{\beta}{\alpha} \frac{\gamma}{\gamma'} \cdot \frac{\gamma'}{\gamma''} = \frac{\beta}{\alpha} \frac{\gamma}{\gamma''} = \frac{\beta''}{\alpha''}, \quad \text{as required.}$$

303. It is then permitted to *interpret the equation* 301, III., on the principles of the Second Book, as being simply a *transformation* (as it is in algebra) of the immediately preceding equation II., or I.; and therefore to write, generally,

$$\text{I.} \dots q\gamma = q'\gamma', \quad \text{if} \quad \text{II.} \dots q(\gamma:\gamma') = q';$$

* In this and other cases of reference, the numeral cited is always supposed to be the one which (with the same number) has last occurred before, although perhaps it may have been in connexion with a shortly preceding Article. Compare 217, (1.).

where γ, γ' are any two vectors, and q, q' are any two quaternions, which satisfy this last condition. Now, if v and v' be any two right quaternions, we have (by 193, comp. 283) the equation,

$$\text{III.} \dots Iv : Iv' = v : v' = vv'^{-1};$$

or

$$\text{IV.} \dots v^{-1}(Iv : Iv') = v'^{-1}; \quad \text{whence} \quad \text{V.} \dots v^{-1} \cdot Iv = v'^{-1} \cdot Iv',$$

by the principle which has just been enunciated. It follows, then, that "if a right Line (Iv) be multiplied by the Reciprocal (v^{-1}) of the Right Quaternion (v), of which it is the Index, the Product ($v^{-1}Iv$) is independent of the Length, and of the Direction, of the Line thus operated on;" or, in other words, that this Product has one common Value, for all possible Lines (a) in Space: which common or constant value may be regarded as a kind of new Geometrical Unit, and is equal to what we have lately denoted, in 300, III., and VII., by the temporary symbol u ; because, in the last cited formula, the line or is the index of the right quotient $OQ : OL'$. Retaining, then, for the moment, this symbol, u , we have, for every line a in space, considered as the index of a right quaternion, v , the four equations:

$$\text{VI.} \dots v^{-1}a = u; \quad \text{VII.} \dots a = vu; \quad \text{VIII.} \dots v = a : u;$$

$$\text{IX.} \dots v^{-1} = u : a;$$

in which it is understood that $a = Iv$, and the three last are here regarded as being merely transformations of the first, which is deduced and interpreted as above. And hence it is easy to infer, that for any given system of three rectangular lines a, β, γ , we have the general expression:

$$\text{X.} \dots (\beta : a) \cdot \gamma = xu, \quad \text{if} \quad a \perp \beta, \beta \perp \gamma, \gamma \perp a;$$

where the scalar co-efficient, x , of the new unit, u , is determined by the equation,

$$\text{XI.} \dots x = \pm (T\beta : T\alpha) \cdot T\gamma, \quad \text{according as} \quad \text{XII.} \dots U\gamma = \pm Ax \cdot (a : \beta).$$

This coefficient x is therefore always equal, in magnitude (or absolute quantity), to the fourth proportional to the lengths of the three given lines $a\beta\gamma$; but it is positively or negatively taken, according as the rotation round the third line γ , from the second line β , to the first line a , is itself positive or negative: or in other words, according as the rotation round the first line, from the second to the third, is on the contrary negative or positive (compare 294, (3.)).

(1.) In illustration of the constancy of that fourth proportional which has been, for the present, denoted by u , while the system of the three rectangular unit-lines

from which it is conceived to be derived is in any manner *turned about*, we may observe that the *three* equations, or proportions,

$$\text{XIII.} \dots \alpha : \gamma = \beta : a; \quad \gamma : a = \alpha : -\gamma; \quad \beta : -\gamma = \gamma : \beta,$$

conduct immediately to this *fourth* equation of the same kind,

$$\text{XIV.} \dots \alpha : a = \gamma : \beta, \quad \text{or}^* \quad \alpha = (\gamma : \beta) \cdot a;$$

if we admit that this new quantity, or symbol, α , is to be operated on *at all*, or combined with *other* symbols, according to the general rules of vectors and quaternions.

(2.) It is, then, permitted to change the *three letters* a, β, γ , by a *cyclical permutation*, to the three other letters β, γ, a (considered again as representing *unit-lines*), without altering the *value* of the *fourth proportional*, α ; or in other words, it is allowed to make the *system of the three rectangular lines revolve, through the third part of four right angles, round the interior and co-initial diagonal of the unit-cube, of which they are three co-initial edges.*

(3.) And it is still more evident, that no such change of value will take place, if we merely cause the system of the *two first lines* to revolve, *through any angle*, in its own plane, *round the third line* as an axis; since thus we shall merely substitute, for the factor $\beta : a$, another factor *equal* thereto. But by *combining* these two last *modes* of rotation, we can represent *any rotation whatever*, round an origin supposed to be fixed.

(4.) And as regards the *scalar ratio* of any *one* fourth proportional, such as $\beta' : a' \cdot \gamma'$, to any *other*, of the kind here considered, such as $\beta : a \cdot \gamma$, or α , it is sufficient to suggest that, without any real change in the former, we are allowed to suppose it to be so *prepared*, that we shall have

$$\text{XV.} \dots a' = a; \quad \beta' = \beta; \quad \gamma' = \alpha \gamma;$$

α being some scalar coefficient, and representing the ratio required.

304. In the more general case, when the three given lines are *not* rectangular, *nor* unit-lines, we may on similar principles determine their fourth proportional, without referring to Fig. 68, as follows. Without any real loss of generality, we may suppose that the planes of a, β and a, γ are perpendicular to each other; since this comes merely to substituting, if necessary, for the quotient $\beta : a$, another quotient equal thereto. Having thus

I. . . $Ax. (\beta : a) \perp Ax. (\gamma : a)$, let II. . . $\beta = \beta' + \beta'', \quad \gamma = \gamma' + \gamma''$, where β' and γ' are parallel to a , but β'' and γ'' are perpendicular to it, and to each other; so that, by 203, I. and II., we shall have the expressions,

$$\text{III.} \dots \beta' = S \frac{\beta}{a} \cdot a, \quad \gamma' = S \frac{\gamma}{a} \cdot a,$$

* In equations of this form, the parentheses may be omitted, though for greater clearness they are here retained.

and IV. . . $\beta'' = V \frac{\beta}{a} \cdot a, \quad \gamma'' = V \frac{\gamma}{a} \cdot a.$

We may then deduce, by the distributive principle (300, (2.)), the transformations,

$$\begin{aligned} \text{V. . . } \frac{\beta}{a} \cdot \gamma &= \left(\frac{\beta'}{a} + \frac{\beta''}{a} \right) (\gamma' + \gamma'') \\ &= \frac{\beta'}{a} \gamma' + \frac{\beta'}{a} \gamma'' + \frac{\beta''}{a} \gamma' + \frac{\beta''}{a} \gamma'' = \delta + xu; \end{aligned}$$

where

VI. . . $\delta = \beta S \frac{\gamma}{a} + \gamma'' S \frac{\beta}{a} = \gamma S \frac{\beta}{a} + \beta'' S \frac{\gamma}{a},$ and VII. . . $xu = \frac{\beta''}{a} \gamma''.$

The latter part, xu , is what we have called (300) the (geometrically) *scalar part*, of the sought fourth proportional; while the former part δ may (still) be called its *vector part*: and we see that *this part* is represented by a *line*, which is at once in the two planes, of β, γ'' , and of γ, β'' ; or in two planes which may be generally constructed as follows, *without now assuming* that the planes $a\beta$ and $a\gamma$ are *rectangular*, as in I. Let γ' be the projection of the line γ on the plane of a, β , and operate on this projection by the quotient $\beta : a$ as a multiplier; the *plane* which is drawn through the line $\beta : a \cdot \gamma'$ so obtained, at right angles to the plane $a\beta$, is *one locus* for the sought line δ : and the plane through γ , which is perpendicular to the plane $\gamma\gamma'$, is *another locus* for that line. And as regards the *length* of this line, or *vector part* δ , and the *magnitude* (or quantity) of the scalar part xu , it is easy to prove that

VIII. . . $T\delta = t \cos s,$ and IX. . . $x = \pm t \sin s,$

where

X. . . $t = T\beta : Tu \cdot T\gamma,$ and XI. . . $\sin s = \sin c \sin p,$

if c denote the angle between the two given lines a, β , and p the inclination of the third given line γ to their plane: the *sign* of the scalar coefficient, x , being positive or negative, according as the *rotation* round a from β to γ is negative or positive.

(1.) Comparing the recent construction with Fig. 68, we see that when the condition I. is satisfied, the four unit-lines $U\gamma, Ua, U\beta, U\delta$ take the directions of the four radii oc, oq, or, od , which terminate at the four corners of what may be called a *tri-rectangular quadrilateral* $cqrd$ on the sphere.

(2.) It may be remarked that the *area* of this *quadrilateral* is exactly equal to *half* the area Σ of the *triangle* DEF ; which may be inferred, either from the circum-

stance that its *spherical excess* (over four right angles) is constructed by the angle MDN; or from the triangles DBR and KAS being together equal to the triangle ABF, so that the area of DESR is Σ , and therefore that of CQRD is $\frac{1}{2}\Sigma$, as before.

(3.) The two sides CQ, QR of this quadrilateral, which are *remote* from the obtuse angle at D, being still called p and c , and the side CD which is *opposite* to c being still denoted by c' , let the side DR which is opposite to p be now called p' ; also let the *diagonals* CR, QD be denoted by d and d' ; and let s denote the *spherical excess* (CDR - $\frac{1}{2}\pi$), or the *area* of the quadrilateral. We shall then have the relations,

$$\text{XII. . . } \begin{cases} \cos d = \cos p \cos c; & \cos d' = \cos p \cos c'; \\ \tan c' = \cos p \tan c; & \tan p' = \cos c \tan p; \\ \cos s = \cos p \sec p' = \cos c \sec c' = \cos d \sec d'; \end{cases}$$

of which some have virtually occurred before, and all are easily proved by right-angled triangles, arcs being when necessary prolonged.

(4.) If we take now two points, A and B, on the side QR, which satisfy the arcual equation (comp. 297, XL., and Fig. 68),

$$\text{XIII. . . } \cap AB = \cap QR;$$

and if we then join AC, and let fall on this new arc the perpendiculars BB', DD'; it is easy to prove that the *projection* B'D' of the side BD on the arc AC is equal to that arc, and that the angle DBB' is right: so that we have the two new equations,

$$\text{XIV. . . } \cap B'D' = \cap AC; \quad \text{XV. . . } \text{DBB}' = \frac{1}{2}\pi;$$

and the *new quadrilateral* BB'D'D is also *tri-rectangular*.

(5.) Hence the point D may be derived from the three points ABC, by any two of the four following conditions: Ist, the equality XIII. of the arcs AB, QR; IInd, the corresponding equality XIV. of the arcs AC, B'D'; IIIrd, the *tri-rectangular character* of the quadrilateral CQRD; IVth, the corresponding character of BB'D'D.

(6.) In other words, this *derived point* D is the *common intersection* of the four *perpendiculars*, to the four arcs AB, AC, CQ, BB', erected at the four points R, D', C, B; CQ, BB' being still the perpendiculars from C and B, on AB and AC; and R and D' being deduced from Q and B', by equal arcs, as above.

305. These consequences of the construction employed in 297, &c., are here mentioned merely in connexion with that theory of *fourth proportionals to vectors*, which they have thus served to illustrate; but they are perhaps numerous and interesting enough, to justify us in suggesting the name, "*Spherical Parallelogram*,"* for the quadrilateral CABD, or BACD, in Fig. 68 (or 67); and in proposing to say that D is the *Fourth Point*, which *completes* such a *parallelogram*, when the *three points* C, A, B, or B, A, C, are *given* upon the sphere, *us first, second, and third*. It must however be carefully observed, that the *analogy to the plane* is here thus far *imperfect*, that in the

* By the same analogy, the quadrilateral CQRD, in Fig. 68, may be called a *Spherical Rectangle*.

general case, when the three given points are not co-arcual, but on the contrary are corners of a spherical triangle ABC, then if we take C, D, B, or B, D, C, for the three first points of a new spherical parallelogram, of the kind here considered, the new fourth point, say A_1 , will not coincide with the old second point A; although it will very nearly do so, if the sides of the triangle ABC be small: the deviation AA_1 being in fact found to be small of the third order, if those sides of the given triangle be supposed to be small of the first order; and being always directed towards the foot of the perpendicular, let fall from A on BC.

(1.) To investigate the law of this deviation, let β, γ be still any two given unit-vectors, OB, OC, making with each other an angle equal to α , of which the cosine is l ; and let ρ or OP be any third vector. Then, if we write,

$$I. \dots \rho_1 = \phi(\rho) = \frac{1}{2}N\rho \cdot \left(\frac{\beta}{\rho} \gamma + \frac{\gamma}{\rho} \beta \right), \quad OQ = U\rho, \quad OQ_1 = U\rho_1,$$

the new or derived vector, $\phi\rho$ or ρ_1 , or OP_1 , will be the common vector part of the two fourth proportionals, to ρ, β, γ , and to ρ, γ, β , multiplied by the square of the length of ρ ; and $BQOQ_1$ will be what we have lately called a spherical parallelogram. We shall also have the transformation (compare 297, (2.)),

$$II. \dots \rho_1 = \phi\rho = \beta S \frac{\rho}{\gamma} + \gamma S \frac{\rho}{\beta} - \rho S \frac{\gamma}{\beta};$$

and the distributive symbol of operation ϕ will be such that

$$III. \dots \phi\rho ||| \beta, \gamma, \text{ and } \phi^2\rho = \rho, \text{ if } \rho ||| \beta, \gamma;$$

but

$$IV. \dots \phi\rho = -l\rho, \text{ if } \rho \parallel Ax. (\gamma : \beta).$$

(2.) This being understood, let

$$V. \dots \rho = \rho' + \rho''; \quad \phi\rho' = \rho'_1; \quad \rho' ||| \beta, \gamma; \quad \rho'' \parallel Ax. (\gamma : \beta);$$

so that ρ' , or OP' , is the projection of ρ on the plane of $\beta\gamma$; and ρ'' , or OP'' , is the part (or component) of ρ , which is perpendicular to that plane. Then we shall have an indefinite series of derived vectors, $\rho_1, \rho_2, \rho_3, \dots$ or rather two such series, succeeding each other alternately, as follows:

$$VI. \dots \begin{cases} \rho_1 = \phi\rho = \rho'_1 - l\rho''; & \rho_2 = \phi^2\rho = \rho' + l^2\rho''; \\ \rho_3 = \phi^3\rho = \rho'_1 - l^3\rho''; & \rho_4 = \phi^4\rho = \rho' + l^4\rho''; \text{ \&c.}; \end{cases}$$

the two series of derived points, $P_1, P_2, P_3, P_4, \dots$ being thus ranged, alternately, on the two perpendiculars, PF' and $F_1F'_1$, which are let fall from the points P and P_1 , on the given plane BOC ; and the intervals, $FP_2, P_1P_3, P_2P_4, \dots$ forming a geometrical progression, in which each is equal to the one before it, multiplied by the constant factor $-l$, or by the negative of the cosine of the given angle BOC .

(3.) If then this angle be still supposed to be distinct from 0 and π , and also in general from the intermediate value $\frac{1}{2}\pi$, we shall have the two limiting values,

$$VII. \dots \rho_{2n} = \rho', \quad \rho_{2n+1} = \rho'_1, \text{ if } n = \infty;$$

or in words, the derived points P_2, P_4, \dots of even orders, tend to the point P' , and the other derived points, P_1, P_3, \dots of odd orders, tend to the other point P'_1 , as limiting

positions: these two limit points being the feet of the two (rectilinear) perpendiculars, let fall (as above) from P and P' on the plane BOC .

(4.) But even the first deviation PP_2 , is small of the third order, if the length TP of the line OP be considered as neither large nor small, and if the sides of the spherical triangle BQC be small of the first order. For we have by VI. the following expressions for that deviation,

$$\text{VIII.} \dots PP_2 = \rho_2 - \rho = (\beta^2 - 1)\rho'' = -\sin a^2 \cdot \sin p_a \cdot TP \cdot U\rho'';$$

where p_a denotes the inclination of the line ρ to the plane $\beta\gamma$; or the arcual perpendicular from the point Q on the side BC , or a , of the triangle. The statements lately made (305) are therefore proved to have been correct.

(5.) And if we now resume and extend the spherical construction, and conceive that D_1 is deduced from BA_1C , as A_1 was from BDC , or D from BAC ; while A_2 may be supposed to be deduced by the same rule from BD_1C , and D_2 from BA_2C , &c., through an indefinite series of spherical parallelograms, in which the fourth point of any one is treated as the second point of the next, while the first and third points remain constant: we see that the points A_1, A_2, \dots are all situated on the arcual perpendicular let fall from A on BC ; and that in like manner the points D_1, D_2, \dots are all situated on that other arcual perpendicular, which is let fall from D on BC . We see also that the ultimate positions, A_∞ and D_∞ , coincide precisely with the feet of those two perpendiculars: a remarkable theorem, which it would perhaps be difficult to prove, by any other method than that of the Quaternions, at least with calculations so simple as those which have been employed above.

(6.) It may be remarked that the construction of Fig. 68 might have been otherwise suggested (comp. 228, IV.), by the principles of the Second Book, if we had sought to assign the fourth proportional (297) to three right quaternions; for example, to three right versors, v, v', v'' , whereof the unit lines a, β, γ should be supposed to be the axes. For the result would be in general a quaternion $v'v^{-1}v''$, with e for its scalar part, and with δ for the index of its right part: e and δ denoting the same scalar, and the same vector, as in the sub-articles to 297.

306. Quaternions may also be employed to furnish a new construction, which shall complete (comp. 305, (5.)) the graphical determination of the two series of derived points,

$$\text{I.} \dots D, A_1, D_1, A_2, D_2, \&c.,$$

when the three points A, B, C are given upon the unit-sphere; and thus shall render visible (so to speak), with the help of a new Figure, the tendencies of those derived points to approach, alternately and indefinitely, to the feet, say D' and A' , of the two arcual perpendiculars let fall from the two opposite corners, D and A , of the first spherical parallelogram, $BACD$, on its given diagonal BC ; which diagonal (as we have seen) is common to all the successive parallelograms.

(1.) The given triangle ABC being supposed for simplicity to have its sides abc less than quadrants, as in 297, so that their cosines lmn are positive, let A', B', C' be

the feet of the perpendiculars let fall on these three sides from the points A, B, C; also let M and N be two auxiliary points, determined by the equations,

$$\text{II.} \dots \odot BM = \odot MC, \quad \odot AM = \odot MN;$$

so that the arcs AN and BC bisect each other in M. Let fall from N a perpendicular ND' on BC, so that

$$\text{III.} \dots \odot BD' = \odot A'C;$$

and let B'', C'' be two other auxiliary points, on the sides b and c, or on those sides prolonged, which satisfy these two other equations,

$$\text{IV.} \dots \odot B''B'' = \odot AC, \quad \odot C''C'' = \odot AB.$$

(2.) Then the perpendiculars to these last sides, CA and AB, erected at these last points, B'' and C'', will intersect each other in the point D, which completes (305) the spherical parallelogram BACD; and the foot of the perpendicular from this point D, on the third side BC of the given triangle, will coincide (comp. 305, (2.)) with the foot D' of the perpendicular on the same side from N; so that this last perpendicular ND' is one locus of the point D.

(3.) To obtain another locus for that point, adapted to our present purpose, let K denote now* that new point in which the two diagonals, AD and BC, intersect each other; then because (comp. 297, (2.)) we have the expression,

$$\text{V.} \dots OD = v(m\beta + \pi\gamma - la),$$

we may write (comp. 297, (25.), and (30.)),

VI. . . OE = v(m\beta + \pi\gamma), whence VII. . . sin BE : sin EC = \pi : m = cos BA' : cos A'C; the diagonal AD thus dividing the arc BC into segments, of which the sines are proportional to the cosines of the adjacent sides of the given triangle, or to the cosines of their projections BA' and A'C on BC; so that the greater segment is adjacent to the lesser side, and the middle point M of BC (1.) lies between the points A' and K.

(4.) The intersection K is therefore a known point, and the great circle through A and K is a second known locus for D; which point may therefore be found, as the intersection of the arc AK prolonged, with the perpendicular ND' from N (1.). And because K lies (3.) beyond the middle point M of BC, with respect to the foot A' of the perpendicular on BC from A, but (as it is easy to prove) not so far beyond M as the point D', or in other words falls between M and D' (when the arc BC is, as above supposed, less than a quadrant), the prolonged arc AK cuts ND' between N and D'; or in other words, the perpendicular distance of the sought fourth point D, from the diagonal BC of the parallelogram, is less than the distance of the given second point A, from the same given diagonal. (Compare the annexed Fig. 78.)

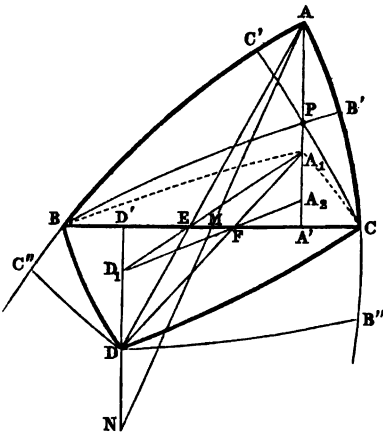


Fig. 78.

* It will be observed that M, N, K have not here the same significations as in

(5.) Proceeding next (305) to derive a new point A_1 from B, D, C , as D has been derived from B, A, C , we see that we have only to determine a new* auxiliary point F , by the equation,

$$\text{VIII.} \dots \circ EM = \circ MF;$$

and then to draw DF , and prolong it till it meets AA' in the required point A_1 , which will thus complete the second parallelogram, $BDCA_1$, with BC (as before) for a given diagonal.

(6.) In like manner, to complete (comp. 305, (5.)), the third parallelogram, BA_1CD_1 , with the same given diagonal BC , we have only to draw the arc A_1E , and prolong it till it cuts ND' in D_1 ; after which we should find the point A_2 of a fourth successive parallelogram BD_1CA_2 , by drawing D_1F , and so on for ever.

(7.) The constant and indefinite tendency, of the derived points D, D_1, \dots to the limit-point D' , and of the other (or alternate) derived points A_1, A_2, \dots to the other limit-point A' , becomes therefore evident from this new construction; the final (or limiting) results of which, we may express by these two equations (comp. again 305, (5.)),

$$\text{IX.} \dots D_\infty = D'; \quad A_\infty = A'.$$

(8.) But the smallness (305) of the first deviation AA_1 , when the sides of the given triangle ABC are small, becomes at the same time evident, by means of the same construction, with the help of the formula VII.; which shows that the interval κM , or the equal interval MF (5.), is small of the third order, when the sides of the given triangle are supposed to be small of the first order: agreeing thus with the equation 305, VIII.

(9.) The theory of such spherical parallelograms admits of some interesting applications, especially in connexion with spherical conics; on which however we cannot enter here, beyond the mere enunciation of a Theorem, † of which (comp. 271) the proof by quaternions is easy:—

Fig. 68; and that the present letters c' and c'' correspond to q and x in that Figure.

* This new point, and the intersection of the perpendiculars of the given triangle, are evidently not the same in the new Figure 73, as the points denoted by the same letters, F and F , in the former Figure 68; although the four points A, B, C, D are conceived to bear to each other the same relations in the two Figures, and indeed in Fig. 67 also; $BACD$ being, in that Figure also, what we have proposed to call a spherical parallelogram. Compare the Note to (3.).

† The formula VII. gives easily the relation,

$$\text{VII.} \dots \tan \kappa M = \tan MA' \left(\tan \frac{\alpha}{2} \right)^2;$$

hence the interval κM is small of the third order, in the case (8.) here supposed; and generally, if $\alpha < \frac{\pi}{2}$, as in (1.), while b and c are unequal, the formula shows that this interval κM is less than MA' , or than $D'M$, so that κ falls between M and D' , as in (4.).

‡ This Theorem was communicated to the Royal Irish Academy in June, 1845, as a consequence of the principles of Quaternions. See the Proceedings of that date (Vol. III., page 109).

"If $KLMN$ be any spherical quadrilateral, and I any point on the sphere; if also we complete the spherical parallelograms,

$$X \dots KILA, LIMB, MINC, NIKD,$$

and determine the poles E and F of the diagonals KM and LN of the quadrilateral: then these two poles are the foci* of a spherical conic, inscribed in the derived quadrilateral $ABCD$, or touching its four sides."

(10.) Hence, in a notation† elsewhere proposed, we shall have, under these conditions of construction, the formula:

$$XI \dots EF(\dots) ABCD; \text{ or } XI' \dots EF(\dots) BCDA; \text{ \&c.}$$

(11.) Before closing this Article and Section, it seems not irrelevant to remark, that the projection γ' of the unit-vector γ , on the plane of α and β , is given by the formula,

$$XII \dots \gamma' = \frac{\alpha \sin \alpha \cos \beta + \beta \sin \beta \cos \alpha}{\sin c};$$

and that therefore the point P , in which (see again Fig. 73) the three arcual perpendiculars of the triangle ABC intersect, is on the vector,

$$XIII \dots \rho = \alpha \tan A + \beta \tan B + \gamma \tan C.$$

(12.) It may be added, as regards the construction in 805, (2.), that the right lines,

$$XIV \dots PF_1, F_1F_2, F_2F_3, F_3F_4, \dots$$

however far their series may be continued, intersect the given planes BOC , alternately, in two points s and t , of which the vectors are,

$$XV \dots os = \frac{\rho' + l\rho'}{1+l}, \text{ or } ot = \frac{\rho' + l\rho'}{1+l};$$

and which thus become two fixed points in the plane, when the position of the point P in space is given, or assumed.

SECTION 9.—On a Third Method of interpreting a Product or Function of Vectors as a Quaternion; and on the Consistency of the Results of the Interpretation so obtained, with those which have been deduced from the two preceding Methods of the present Book.

307. The Conception of the Fourth Proportional to Three Rectangular Unit-Lines, as being itself a species of Fourth Unit in Geometry, is eminently characteristic of the present Calculus; and offers a Third Method of interpreting a Product of two Vectors as a Quaternion: which is however found to be

* In the language of modern geometry, the conic in question may be said to touch eight given arcs; four real, namely the sides AB, BC, CD, DA ; and four imaginary, namely two from each of the focal points, E and F .

† Compare the Second Note to page 295.

consistent, in all its *results*, with the two former methods (278, 284) of the present Book; and admits of being easily extended to products of *three* or more *lines in space*, and generally to *Functions of Vectors* (289). In fact we have only to conceive*

* It was in a somewhat analogous way that *Des Cartes* showed, in his *Geometria* (Schooten's Edition, Amsterdam, 1659), that all *products* and *powers of lines*, considered relatively to their *lengths* alone, and without any reference to their *directions*, could be *interpreted as lines*, by the suitable introduction of a line taken for *unity*, however high the *dimension* of the product or power might be. Thus (at page 3 of the cited work) the following remark occurs:—

“Ubi notandum est, quòd per a^2 vel b^3 , similésve, communiter, non nisi lineas omnino simplices concipiam, licèt illas, ut nominibus in Algebra usitatis utar, Quadrata aut Cubos, &c. appellem.”

But it was much more difficult to accomplish the corresponding *multiplication of directed lines in space*; on account of the *non-existence of any such line*, which is *symmetrically related to all other lines*, or *common to all possible planes* (comp. the Note to page 248). The *Unit of Vector-Multiplication* cannot properly be itself a *Vector*, if the *conception of the Symmetry of Space* is to be retained, and duly combined with the other elements of the question. This difficulty however disappears, at least in theory, when we come to consider that *new Unit*, of a *scalar kind* (300), which has been above denoted by the temporary symbol u , and has been obtained, in the foregoing Section, as a certain *Fourth Proportional to Three Rectangular Unit-Lines*, such as the *three co-initial edges*, AB , AC , AD of what we have called an *Unit-Cube*: for *this fourth proportional*, by the proposed *conception* of it, undergoes *no change*, when the cube $ABCD$ is in any manner *moved*, or *turned*; and therefore may be considered to be *symmetrically related to all directions of lines in space*, or to all possible *vections* (or *translations*) of a *point*, or *body*. In fact, we *conceive its determination*, and the *distinction* of it ($as + u$) from the *opposite unit* of the same kind ($-u$), to depend *only* on the *usual assumption* of an *unit of length*, combined with the *selection of a hand* (as , for example, the *right hand*), *rotation towards which hand* shall be considered to be *positive*, and *contrasted* (as such) with rotation towards the *other hand*, round the *same arbitrary axis*. Now in whatever manner the supposed *cube* may be thrown about in space, the *conceived rotation round the edge* AB , *from* AC *to* AD , will have the *same character*, as *right-handed* or *left-handed*, at the *end* as at the *beginning* of the motion. If then the *fourth proportional* to these *three edges*, taken in *this order*, be denoted by $+u$, or simply by $+1$, at *one stage* of that arbitrary motion, it may (on the plan here considered) be denoted by the *same symbol*, at *every other stage*: while the *opposite character* of the (conceived) *rotation*, round the *same edge* AB , *from* AD *to* AC , leads us to regard the fourth proportional to AB , AD , AC as being on the contrary equal to $-u$, or to -1 . It is true that this *conception of a new unit for space*, *symmetrically related* (as above) *to all linear directions* therein, may appear somewhat abstract and metaphysical; but readers who think it such can of course confine their attention to the *rules of calculation*, which have been above derived from it, and from other connected considerations: and which have (it is hoped) been stated and exemplified, in this and in a former Volume, with sufficient clearness and fullness.

that *each proposed vector, a, is divided by the new or fourth unit, u, above alluded to; and that the quotient so obtained, which is always (by 303, VIII.) the right quaternion $I^{-1}a$, whereof the vector a is the index, is substituted for that vector: the resulting quaternion being finally, if we think it convenient, multiplied into the same fourth unit. For in this way we shall merely reproduce the process of 284, or 289, although now as a consequence of a different train of thought, or of a distinct but Consistent Interpretation: which thus conducts, by a new Method, to the same Rules of Calculation as before.*

(1.) The equation of the unit-sphere, $\rho^2 + 1 = 0$ (282, XIV.), may thus be conceived to be an abridgment of the following fuller equation:

$$I. \dots \left(\frac{\rho}{u}\right)^2 = -1;$$

the quotient $\rho : u$ being considered as equal (by 303) to the right quaternion, $I^{-1}\rho$, which must here be a right versor (154), because its square is negative unity.

(2.) The equation of the ellipsoid,

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2 \text{ (282, XIX.),}$$

may be supposed, in like manner, to be abridged from this other equation:

$$II. \dots T\left(\frac{\iota}{u}\frac{\rho}{u} + \frac{\rho}{u}\frac{\kappa}{u}\right) = \left(\frac{\kappa}{u}\right)^2 - \left(\frac{\iota}{u}\right)^2;$$

and similarly in other cases.

(3.) We might also write these equations, of the sphere and ellipsoid, under these other, but connected forms:

$$III. \dots \frac{\rho}{u}\rho = -u; \quad IV. \dots T\left(\frac{\iota}{u}\rho + \frac{\rho}{u}\kappa\right) = \frac{\kappa}{u}\kappa - \frac{\iota}{u}\iota;$$

with interpretations which easily offer themselves, on the principles of the foregoing Section.

(4.) It is, however, to be distinctly understood, that we do not propose to adopt this Form of Notation, in the practice of the present Calculus: and that we merely suggest it, in passing, as one which may serve to throw some additional light on the Conception, introduced in this Third Book, of a Product of two Vectors as a Quaternion.

(5.) In general, the Notation of Products, which has been employed throughout the greater part of the present Book and Chapter, appears to be much more convenient, for actual use in calculation, than any Notation of Quotients: either such as has been just now suggested for the sake of illustration, or such as was employed in the Second Book, in connexion with that First Conception of a Quaternion (112), to which that Book mainly related, as the Quotient of two Vectors (or of two directed lines in space). The notations of the two Books are, however, intimately connected, and the former was judged to be an useful preparation for the latter, even as

regarded the *quotient-forms* of many of the expressions used : while the *Characteristics of Operation*, such as

$$S, V, T, U, K, N,$$

are employed according to exactly the *same laws* in both. In short, a reader of the Second Book has *nothing to unlearn* in the Third; although he may be supposed to have become prepared for the use of somewhat shorter and more convenient *processes*, than those before employed.

SECTION 10.—*On the Interpretation of a Power of a Vector as a Quaternion.*

308. The only symbols, of the kinds mentioned in 277, which we have not yet interpreted, are the *cube* a^3 , and the *general power* a^t , of an *arbitrary vector base*, a , with an *arbitrary scalar exponent*, t ; for we have already assigned interpretations (282, (1.), (14.), and 299, (8.)) for the *particular symbols* a^2 , a^{-1} , a^{-2} , which are *included* in this last *form*. And we shall preserve those particular interpretations if we now *define*, in full consistency with the principles of the present and preceding Books, that this *Power* a^t is generally a *Quaternion*, which may be decomposed into *two factors*, of the *tensor* and *versor* kinds, as follows :

$$I. \dots a^t = T a^t \cdot U a^t;$$

$T a^t$ denoting the *arithmetical value* of the t^{th} *power* of the *positive number* $T a$, which represents (as usual) the *length* of the *base-line* a ; and $U a^t$ denoting a *versor*, which *causes any line* ρ , *perpendicular to that line* a , to *revolve round it as an axis*, *through t right angles*, or quadrants, and in a *positive or negative direction*, according as the *scalar exponent*, t , is *itself a positive or negative number* (comp. 234, (5.)).

(1.) As regards the omission of parentheses in the formula I., we may observe that the recent *definition*, or *interpretation*, of the symbol a^t , enables us to write (comp. 237, II. III.),

$$II. \dots T(a^t) = (T a)^t = T a^t; \quad III. \dots U(a^t) = (U a)^t = U a^t.$$

(2.) The *axis* and *angle* of the *power* a^t , considered as a quaternion, are generally determined by the two following formulæ :

$$IV. \dots Ax. a^t = \pm U a; \quad V. \dots \angle. a^t = 2n\pi \pm \frac{1}{2}t\pi;$$

the *signs* accompanying each other, and the (*positive or negative or null*) integer, n , being so chosen as to bring the *angle* within the usual limits, 0 and π .

(3.) In general (comp. 285), we may speak of the (positive or negative) product $\frac{1}{2}t\pi$, as being the *amplitude* of the same power, with reference to the line a as an axis of rotation; and may write accordingly,

$$\text{VI. . . am. } a^t = \frac{1}{2}t\pi.$$

(4.) We may write also (comp. 284, VII. VIII.),

$$\text{VII. . . } Ua^t = \cos \frac{t\pi}{2} + Ua \cdot \sin \frac{t\pi}{2}; \quad \text{or briefly, VIII. . . } Ua^t = \text{cas } \frac{t\pi}{2}.$$

(5.) In particular,

$$\text{IX. . . } Ua^{2n} = \text{cas } n\pi = \pm 1; \quad \text{IX'. . . } Ua^{2n+1} = \pm Ua;$$

upper or lower signs being taken, according as the number n (supposed to be whole) is even or odd. For example, we have thus the cubes,

$$\text{X. . . } Ua^3 = -Ua; \quad \text{X'. . . } a^3 = -aNa.$$

(6.) The *conjugate* and *norm* of the power a^t may be thus expressed (it being remembered that to turn a line $\perp a$ through $-\frac{1}{2}t\pi$ round $+a$, is equivalent to turning that line through $+\frac{1}{2}t\pi$ round $-a$):

$$\text{XI. . . } Ka^t = Ta^t \cdot Ua^{-t} = (-a)^t; \quad \text{XII. . . } Na^t = Ta^{2t};$$

parentheses being unnecessary, because (by 295, VIII.) $Ka = -a$.

(7.) The *scalar*, *vector*, and *reciprocal* of the same power are given by the formulae:

$$\text{XIII. . . } S.a^t = Ta^t \cdot \cos \frac{t\pi}{2}; \quad \text{XIV. . . } V.a^t = Ta^t \cdot Ua \cdot \sin \frac{t\pi}{2};$$

$$\text{XV. . . } 1 : a^t = Ta^{-t} \cdot Ua^{-t} = a^{-t} = Ka^t : Na^t \text{ (comp. 190, (3.)).}$$

(8.) If we decompose any vector ρ into parts ρ' and ρ'' , which are respectively parallel and perpendicular to a , we have the general transformation:*

$$\text{XVI. . . } a^t \rho a^{-t} = a^t (\rho' + \rho'') a^{-t} = \rho' + Ua^{2t} \cdot \rho'',$$

= the new vector obtained by causing ρ to revolve conically through an angular quantity expressed by $t\pi$, round the line a as an axis (comp. 297, (15.)).

(9.) More generally (comp. 191, (5.)), if q be any quaternion, and if

$$\text{XVII. . . } a^t q a^{-t} = q',$$

the new quaternion q' is formed from q by such a conical rotation of its own axis Ax . q , through $t\pi$, round a , without any change of its angle $\angle q$, or of its tensor Tq .

(10.) Treating ijk as three rectangular unit-lines (295), the symbol, or expression,

$$\text{XVIII. . . } \rho = rk^t j^s k^t, \quad \text{or} \quad \text{XIX. . . } \rho = rk^t j^{2s} k^{t-1},$$

in which

$$\text{XX. . . } r \geq 0, \quad s \geq 0, \quad s \leq 1, \quad t \geq 0, \quad t \leq 2,$$

may represent any vector; the length or tensor of this line ρ being r ; its inclination† to k being $s\pi$; and the angle through which the variable plane $k\rho$ may be

* Compare the shortly following sub-article (11.).

† If we conceive (compare the first Note to page 822) that the two lines i and j are directed respectively towards the south and west points of the horizon, while the third line k is directed towards the zenith, then $s\pi$ is the zenith-distance of ρ ; and $t\pi$ is the azimuth of the same line, measured from south to west, and thence (if necessary) through north and east, to south again.

conceived to have *revolved*, from the initial position ki , with an initial direction towards the position kj , being $t\pi$.

(11.) In accomplishing the transformation XVI., and in passing from the expression XVIII. to the less symmetric but equivalent expression XIX., we employ the principle that

$$\text{XXI.} \dots kj^{-s} = S^{-1}0 = -K(kj^{-s}) = j^s k;$$

which easily admits of extension, and may be confirmed by such transformations as VII. or VIII.

(12.) It is scarcely necessary to remark, that the definition or interpretation I., of the *power* a^t of any vector a , gives (as in algebra) the *exponential property*,

$$\text{XXII.} \dots a^s a^t = a^{s+t},$$

whatever scalars may be denoted by s and t ; and similarly when there are more than two factors of this form.

(13.) As verifications of the expression XVIII., considered as representing a vector, we may observe that it gives,

$$\text{XXIII.} \dots \rho = -K\rho; \text{ and } \text{XXIV.} \dots \rho^2 = -r^2.$$

(14.) More generally, it will be found that if u^* be any scalar, we have the eminently simple transformation :

$$\text{XXV.} \dots \rho^u = (r k^t j^s k j^{-s} k^{-t})^u = r^u k^t j^s k^u j^{-s} k^{-t}.$$

In fact, the two last expressions denote generally two *equal quaternions*, because they have, Ist, *equal tensors*, each = r^u ; IInd, *equal angles*, each = $\angle(k^u)$; and IIIrd, *equal* (or coincident) *axes*, each formed from $\pm k$ by one *common system* of two *successive rotations*, one through $s\pi$ round j , and the other through $t\pi$ round k .

309. Any quaternion, q , which is not simply a scalar, may be brought to the form a^t , by a suitable choice of the *base*, a , and of the *exponent*, t ; which latter may moreover be supposed to fall between the limits 0 and 2; since for this purpose we have only to write,

$$\text{I.} \dots t = \frac{2 \angle q}{\pi}; \quad \text{II.} \dots T a = T q^{\frac{1}{2}}; \quad \text{III.} \dots U a = A x. q;$$

and thus the *general dependence of a Quaternion, on a Scalar and a Vector Element, presents itself in a new way* (comp. 17, 207, 292). When the proposed quaternion is a *versor*, $Tq = 1$,

* The employment of this letter u , to denote what we called, in the two preceding Sections, a *fourth unit*, &c., was stated to be a merely temporary one. In general, we shall henceforth simply *equate that scalar unit to the number one*; and denote it (when necessary to be denoted at all) by the *usual symbol*, 1, for that number.

we have thus $T_a = 1$; or in other words, the *base* a , of the equivalent *power* a^t , is an *unit-line*. Conversely, every versor may be considered as a *power of an unit-line*, with a *scalar exponent*, t , which may be supposed to be in *general positive*, and *less than two*; so that we may write generally,

$$\text{IV.} \dots Uq = a^t, \quad \text{with} \quad \text{V.} \dots a = \mathbf{Ax}. q = \mathbf{T}^{-1},$$

and $\text{VI.} \dots t > 0, \quad t < 2;$

although if this versor *degenerate* into 1 or -1 , the *exponent* t becomes 0 or 2, and the *base* a has an indeterminate or *arbitrary direction*. And from such *transformations of versors* new methods may be deduced, for treating questions of *spherical trigonometry*, and generally of *spherical geometry*.

(1.) Conceive that r, q, π , in Fig. 46, are replaced by $\mathbf{A}, \mathbf{B}, \mathbf{C}$, with unit-vectors a, β, γ as usual; and let x, y, z be three scalars between 0 and 2, determined by the three equations,

$$\text{VII.} \dots x\pi = 2\mathbf{A}, \quad y\pi = 2\mathbf{B}, \quad z\pi = 2\mathbf{C};$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ denote the angles of the spherical triangle. The three versors, indicated by the three arrows in the upper part of the Figure, come then to be thus denoted:

$$\text{VIII.} \dots q = a^x; \quad q' = \beta^y; \quad q'q = \gamma^{z-x};$$

so that we have the equation,

$$\text{IX.} \dots \beta^y a^x = \gamma^{z-x}; \quad \text{or} \quad \text{X.} \dots \gamma^z \beta^y a^x = -1;$$

from which last, by easy divisions and multiplications, these two others immediately follow:

$$\text{X'.} \dots a^x \gamma^z \beta^y = -1; \quad \text{X''.} \dots \beta^y a^x \gamma^z = -1;$$

the rotation round a from β to γ being again supposed to be negative.

(2.) In X. we may write (by 308, VIII.),

$$\text{XI.} \dots a^x = c a s \mathbf{A}; \quad \beta^y = c \beta s \mathbf{B}; \quad \gamma^z = c \gamma s \mathbf{C};$$

and then the formula becomes, for any spherical triangle, in which the *order of rotation* is as above:

$$\text{XII.} \dots c \gamma s \mathbf{C} . c \beta s \mathbf{B} . c a s \mathbf{A} = -1;$$

or (comp. IX.),

$$\text{XIII.} \dots -\cos \mathbf{C} + \gamma \sin \mathbf{C} = (\cos \mathbf{B} + \beta \sin \mathbf{B}) (\cos \mathbf{A} + a \sin \mathbf{A}).$$

(3.) Taking the scalars on both sides of this last equation, and remembering that $S\beta a = -\cos c$, we thus immediately derive *one form* of the *fundamental equation of spherical trigonometry*; namely, the equation,

$$\text{XIV.} \dots \cos \mathbf{C} + \cos \mathbf{A} \cos \mathbf{B} = \cos c \sin \mathbf{A} \sin \mathbf{B}.$$

(4.) Taking the vectors, we have this other formula:

$$\text{XV.} \dots \gamma \sin \mathbf{C} = a \sin \mathbf{A} \cos \mathbf{B} + \beta \sin \mathbf{B} \cos \mathbf{A} + \mathbf{V}\beta a \sin \mathbf{A} \sin \mathbf{B};$$

which is easily seen to agree with 306, XII., and may also be usefully compared with the equation 210, XXXVII.

(5.) The result XV. may be enunciated in the form of a *Theorem*, as follows:—

“If there be any spherical triangle $\triangle ABC$, and three lines be drawn from the centre O of the sphere, one towards the point A , with a length = $\sin A \cos B$; another towards the point B , with a length = $\sin B \cos A$; and the third perpendicular to the plane $\triangle OAB$, and towards the same side of it as the point C , with a length = $\sin c \sin A \sin B$; and if, with these three lines as edges, we construct a parallelepiped: the intermediate diagonal from O will be directed towards C , and will have a length = $\sin c$.”

(6.) Dividing both members of the same equation XV. by ρ , and taking scalars, we find that if P be any fourth point on the sphere, and Q the foot of the perpendicular let fall from this point on the arc AB , this perpendicular PQ being considered as positive when C and P are situated at one common side of that arc (or in one common hemisphere, of the two into which the great circle through A and B divides the spheric surface), we have then,

$$\text{XVI.} \dots \sin C \cos PO = \sin A \cos B \cos PA + \sin B \cos A \cos PB + \sin A \sin B \sin c \sin PQ;$$

a formula which might have been derived from the equation 210, XXXVIII., by first cyclically changing $abc\triangle ABC$ to $bc a\triangle BCA$, and then passing from the former triangle to its polar, or supplementary: and from which many less general equations may be deduced, by assigning particular positions to P .

(7.) For example, if we conceive the point P to be the centre of the circumscribed small circle $\triangle ABC$, and denote by R the arcual radius of that circle, and by s the semisum of the three angles, so that $2s = A + B + C = \pi + \sigma$, if σ again denote, as in 297, (47.), the area* of the triangle $\triangle ABC$, whence

$$\text{XVII.} \dots PA = PB = PC = R, \text{ and } \sin PQ = \sin R \sin (s - c),$$

the formula XVI. gives easily,

$$\text{XVIII.} \dots 2 \cot R \sin \frac{\sigma}{2} = \sin A \sin B \sin c;$$

a relation between radius and area, which agrees with known results, and from which we may, by 297, LXX., &c., deduce the known equation:

$$\text{XIX.} \dots e \tan R = 4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2};$$

in which we have still, as in 297, (47.), &c.,

$$\text{XX.} \dots e = (S a \beta \gamma) \sin a \sin b \sin c = \&c.$$

(8.) In like manner we might have supposed, in the corresponding general equation 210, XXXVIII., that P was placed at the centre of the inscribed small circle, and that the arcual radius of that circle was r , the semisum of the sides being s ; and thus should have with ease deduced this other known relation, which is a sort of polar reciprocal of XVIII.,

$$\text{XXI.} \dots 2 \tan r \cdot \sin s = e.$$

But these results are mentioned here, only to exemplify the fertility of the formulæ, to which the present calculus conducts, and from which the theorem in (5.) was early seen to be a consequence.

* Compare the Note to the cited sub-article.

(9.) We might *develop* the ternary product in the equation XII., as we developed the binary product XIII.; compare scalar and vector parts; and operate on the latter, by the symbol $S.\rho^{-1}$. *New general theorems*, or at least new general forms, would thus arise, of which it may be sufficient in this place to have merely suggested the investigation.

(10.) As regards the order of rotation (1.) (2.), it is clear, from a mere inspection of the formula XV., that the rotation round γ from β to α , or that round c from B to A , must be positive, when that equation XV. holds good; at least if the angles A, B, C , of the triangle ABC , be (as usual) treated as positive: because the rotation round the line $\sqrt{\beta\alpha}$ from β to α is always positive (by 281, (8.)).

(11.) If, then, for any given spherical triangle, ABC , with angles still supposed to be positive, the rotation round c from B to A should happen to be (on the contrary) negative, we should be obliged to modify the formula XV.; which could be done, for example, so as to restore its correctness, by interchanging α with β , and at the same time A with B .

(12.) There is, however, a sense in which the formula might be considered as still remaining true, without any change in the mode of writing it; namely, if we were to interpret the symbols A, B, C as denoting negative angles, for the case last supposed (11.). Accordingly, if we take the reciprocal of the equation X., we get this other equation,

$$\text{XXII.} \dots \alpha^{-x}\beta^{-y}\gamma^{-z} = -1;$$

where x, y, z are positive, as before, and therefore the new exponents, $-x, -y, -z$, are negative, if the rotation round α from β to γ be itself negative, as in (1.).

(13.) On the whole, then, if α, β, γ be any given system of three co-initial and diplanar unit-lines, OA, OB, OC , we can always assign a system of three scalars, x, y, z , which shall satisfy the exponential equation X., and shall have relations of the form VII. to the spherical angles A, B, C ; but these three scalars, if determined so as to fall between the limits ± 2 , will be all positive, or all negative, according as the rotation round α from β to γ is negative, as in (1.), or positive, as in (11.).

(14.) As regards the limits just mentioned, or the inequalities,

$$\text{XXIII.} \dots x < 2, \quad y < 2, \quad z < 2; \quad x > -2, \quad y > -2, \quad z > -2,$$

they are introduced with a view to render the problem of finding the exponents xyz in the formula X. determinate; for since we have, by 808,

$$\text{XXIV.} \dots \alpha^4 = \beta^4 = \gamma^4 = +1, \quad \text{if } T\alpha = T\beta = T\gamma = 1,$$

we might otherwise add any multiple (positive or negative) of the number four, to the value of the exponent of any unit-line, and the value of the resulting power would not be altered.

(15.) If we admitted exponents $= \pm 2$, we might render the problem of satisfying the equation X. indeterminate in another way; for it would then be sufficient to suppose that any one of the three exponents was thus equal to $+2$, or -2 , and that the two others were each $= 0$; or else that all three were of the form ± 2 .

(16.) When it was lately said (13.), that the exponents, x, y, z , in the formula X., if limited as above, would have one common sign, the case was tacitly excluded, for which those exponents, or some of them, when multiplied each by a quadrant, give angles not equal to those of the spherical triangle ABC , whether positively or

negatively taken; but equal to the *supplements* of those angles, or to the *negatives* of those supplements.

(17.) In fact, it is evident (because $\alpha^2 = \beta^2 = \gamma^2 = -1$), that the equation X., or the reciprocal equation XXII., if it be satisfied by any *one system* of values of xyz , will *still* be satisfied, when we divide or multiply *any two* of the three exponential factors, by the *squares* of the two unit-vectors, of which those factors are supposed to be *powers*: or in other words, if we *subtract* or *add* the number two, in *each of two exponents*.

(18.) We may, for example, derive from XXII. this other equation :

$$\text{XXV.} \dots \alpha^{2-z} \beta^{2-y} \gamma^z = -1; \text{ or } \text{XXVI.} \dots \alpha^{2-z} \beta^{2-y} = \gamma^{z-2};$$

which, when the rotation is as supposed in (1.), so that xyz are *positive*, may be interpreted as follows.

(19.) Conceive a *lune* OC' , with points A and B on its two bounding semicircles, and with a negative rotation round A from B to C ; or, what comes to the same thing, with a positive rotation round A from B to C' . Then, on the plan illustrated by Figures 45 and 46, the *supplements* $\pi - A$, $\pi - B$, of the angles A and B in the triangle ABC , or the angles at the *same points* A and B in the *co-lunar* triangle ABC' , will represent two *versors*, a *multiplier*, and a *multiplicand*, which are precisely those denoted, in XXVI., by the two factors, α^{2-z} and β^{2-y} ; and the *product* of these two factors, taken in *this order*, is that *third versor*, which has its *axis* directed to C' , and is *represented*, on the same general plan (177), by the *external angle of the lune, at that point* C' ; which, in *quantity*, is equal to the external angle of the same lune at C , or to the angle $\pi - C$. This *product* is therefore equal to that *power of the unit-line* OC' , or $-\gamma$, which has its *exponent* $= \frac{2}{\pi} (\pi - C) = 2 - z$; we have therefore, by this construction, the equation,

$$\text{XXVII.} \dots \alpha^{2-z} \beta^{2-y} = (-\gamma)^{2-z};$$

which (by 308, (6.)) agrees with the recent formula XXVI.

310. The equation,

$$\text{I.} \dots \gamma^{\frac{2x}{\pi}} \beta^{\frac{2y}{\pi}} \alpha^{\frac{2z}{\pi}} = -1,$$

which results from 309, (1.), and in which α , β , γ are the unit-vectors OA , OB , OC of any three points on the unit-sphere; while the three scalars x , y , z , in the exponents of the three factors, represent generally the angular quantities of rotation, round those three unit-lines, or radii, α , β , γ , from the plane AOC to the plane AOB , from BOA to BOC , and from COB to COA , and are positive or negative according as these rotations of planes are themselves positive or negative: must be regarded as an important formula, in the applications of the present Calculus. It *includes*, for example, the whole doctrine of *Spherical Triangles*; not merely because it conducts, as we

have seen (309, (3.)), to one form of the fundamental scalar equation of spherical trigonometry, namely to the equation,

$$\text{II.} \dots \cos C + \cos A \cos B = \cos c \sin A \sin B ;$$

but also because it gives a vector equation (309, (4.)), which serves to connect the angles, or the rotations, A, B, C, with the directions* of the radii, α, β, γ , or OA, OB, OC, for any system of three diverging right lines from one origin. It may, therefore, be not improper to make here a few additional remarks, respecting the nature, evidence and extension of the recent formula I.

(1.) Multiplying both members of the equation I., by the inverse exponential $\gamma^{-\frac{2\alpha}{\pi}}$, we have the transformation (comp. 309, (1.)) :

$$\text{III.} \dots \beta^{\frac{2\alpha}{\pi}} \alpha^{\frac{2\alpha}{\pi}} = -\gamma^{-\frac{2\alpha}{\pi}} = \gamma^{\frac{2(\pi-\alpha)}{\pi}} .$$

(2.) Again, multiplying both members of I. into $\dagger a^{-\frac{2\alpha}{\pi}}$, we obtain this other formula :

$$\text{IV.} \dots \gamma^{\frac{2\alpha}{\pi}} \beta^{\frac{2\alpha}{\pi}} = -\alpha^{-\frac{2\alpha}{\pi}} = \alpha^{\frac{2(\pi-\alpha)}{\pi}} .$$

(3.) Multiplying this last equation IV. by $\alpha^{\frac{2\alpha}{\pi}}$, and the equation III. into $\gamma^{\frac{2\alpha}{\pi}}$, we derive these other forms :

* This may be considered to be another instance of that habitual reference to *direction*, as distinguished from mere *quantity* (or magnitude), although combined therewith, which pervades the present Calculus, and is eminently characteristic of it ; whereas *Des Cartes*, on the contrary, had aimed to reduce all problems of geometry to the determination of the *lengths of right lines* : although (as all who use his *co-ordinates* are of course well aware) a certain reference to *direction* is even in his theory inevitable, in connexion with the interpretation of *negative roots* (by him called *inverse* or *false roots*) of equations. Thus in the first sentence of Schooten's recently cited translation (1659) of the *Geometry* of *Des Cartes*, we find it said :

“ Omnia Geometriæ Problemata facillè ad hujusmodi terminos reduci possunt, ut deinde ad illorum constructionem, opus tantum sit rectarum quarundam longitudinem cognoscere.”

The very different *view of geometry*, to which the present writer has been led, makes it the more proper to express here the profound admiration with which he regards the cited Treatise of *Des Cartes* : containing as it does the germs of so large a portion of all that has since been done in mathematical science, even as concerns *imaginary roots* of equations, considered as marks of *geometrical impossibility*.

† For the distinction between multiplying a quaternion *into* and *by* a factor, see the Notes to pages 146, 159.

$$V. \dots \alpha^{\frac{2a}{\pi}} \gamma^{\frac{2b}{\pi}} \beta^{\frac{2c}{\pi}} = -1; \quad VI. \dots \beta^{\frac{2a}{\pi}} \alpha^{\frac{2b}{\pi}} \gamma^{\frac{2c}{\pi}} = -1;$$

so that *cyclical permutation of the letters, $\alpha, \beta, \gamma,$ and $A, B, C,$ is allowed in the equation I.*; as indeed was to be expected, from the nature of the theorem which that equation expresses.

(4.) From either V. or VI. we can deduce the formula:

$$VII. \dots \alpha^{\frac{2a}{\pi}} \gamma^{\frac{2b}{\pi}} = -\beta^{\frac{2c}{\pi}} = \beta^{\frac{2(\pi-c)}{\pi}};$$

by comparing which with III. and IV., we see that *cyclical permutation of letters is permitted, in these equations also.*

(5.) Taking the *reciprocal* (or *conjugate*) of the equation I., we obtain (compare 309, XXII.) this other equation:

$$VIII. \dots \alpha^{-\frac{2a}{\pi}} \beta^{-\frac{2b}{\pi}} \gamma^{-\frac{2c}{\pi}} = -1;$$

or

$$IX. \dots \alpha^{\frac{2(\pi-a)}{\pi}} \beta^{\frac{2(\pi-b)}{\pi}} \gamma^{\frac{2(\pi-c)}{\pi}} = +1;$$

in which cyclical permutation of letters is again allowed, and from which (or from III.) we can at once derive the formula,

$$X. \dots \alpha^{-\frac{2a}{\pi}} \beta^{-\frac{2b}{\pi}} = -\gamma^{\frac{2c}{\pi}}.$$

(6.) The equation X. may also be thus written (comp. 309, XXVII.):

$$XI. \dots \alpha^{\frac{2(\pi-a)}{\pi}} \beta^{\frac{2(\pi-b)}{\pi}} = \gamma^{\frac{2(\pi-c)}{\pi}} = (-\gamma)^{\frac{2(\pi-c)}{\pi}}.$$

(7.) And all the foregoing equations may be *interpreted* (comp. 309, (19.)), and at the same time *proved*, by a reference to that general *construction* (177) for the *multiplication of versors*, which the Figures 45 and 46 were designed to illustrate; if we bear in mind that a *power a^t* , of an *unit-line a* , with a *scalar exponent, t* , is (by 308, 309) a *versor*, which has the *effect of turning a line $\perp a$, through t right angles, round a as an axis of rotation.*

(8.) The principle expressed by the equation I., from which all the subsequent equations have been deduced, may be stated in the following manner, if we adopt the *definition* proposed in an earlier part of this work (180, (4.)), for the *spherical sum* of two angles on a spheric surface:

"For any spherical triangle, the Spherical Sum of the three angles, if taken in a suitable Order, is equal to Two Right Angles."

(9.) In fact, when the rotation round A from B to C is negative, if we *spherically add* the angle B to the angle A , the *spherical sum* so obtained is (by the definition referred to) equal to the *external angle at C* ; if then we *add to this sum*, or *supplement of C* , the angle C itself, we get a *final or total sum*, which is exactly equal to π ; *addition of spherical angles at one vertex*, and therefore in *one plane*, being accomplished in the *usual manner*; but the *spherical summation* of angles with *different vertices* being performed according to those *new rules*, which were deduced in the Ninth Section of Book II., Chapter I.; and were connected (180, (5.)) with the conception of *angular transvection*, or of the *composition of angular motions, in different and successive planes.*

(10.) Without pretending to attach importance to the following *notation*, we may just propose it in passing, as one which may serve to recall and represent the *conception* here referred to. Using a *plus in parentheses*, as a *symbol or characteristic* of such *spherical addition of angles*, the formula I. may be *abridged* as follows:

$$\text{XII.} \dots \circ(+)\text{B}(+)\text{A} = \pi;$$

the *symbol of an added angle* being written to the *left* of the symbol of the *angle to which it is added* (comp. 264, (4.)); because *such addition corresponds* (as above) to a *multiplication of versors*, and we have agreed to write the *symbol of the multiplier* to the *left* of the symbol of the *multiplicand*, in every *multiplication of quaternions*.

311. There is, however, *another view* of the important equation 310, I., according to which it is connected rather with *addition of arcs* (180, (3.)), than with *addition of angles* (180, (4.)); and may be *interpreted*, and *proved anew*, with the help of the *supplementary or polar triangle*, $\text{A}'\text{B}'\text{C}'$, as follows.

(1.) The rotation round A from B to C being still supposed to be negative, let A' , B' , C' be (as in 175) the positive poles of the sides BC , CA , AB ; and let α' , β' , γ' be their unit-vectors. Then, because the rotation round α from γ' to β' is positive (by 180, (2.)), and is in quantity the supplement of the spherical angle A , the *product* $\gamma'\beta'$ will be (by 281, (2.), (3.)) a *versor*, of which α is the *axis*, and A the *angle*; with similar results for the two other products, $\alpha'\gamma'$, $\beta'\alpha'$.

(2.) If then we write (comp. 291),

$$\text{I.} \dots \alpha' = \text{UV}\beta\gamma, \quad \beta' = \text{UV}\gamma\alpha, \quad \gamma' = \text{UV}\alpha\beta,$$

supposing that

$$\text{II.} \dots \text{T}\alpha = \text{T}\beta = \text{T}\gamma = 1, \quad \text{and} \quad \text{III.} \dots \text{S}\alpha\beta\gamma > 0,$$

we shall have (comp. again 180, (2.)),

$$\text{IV.} \dots \alpha = \text{UV}\gamma'\beta', \quad \beta = \text{UV}\alpha'\gamma', \quad \gamma = \text{UV}\beta'\alpha',$$

and

$$\text{V.} \dots \text{A} = \angle \gamma'\beta', \quad \text{B} = \angle \alpha'\gamma', \quad \text{C} = \angle \beta'\alpha';$$

whence (by 308 or 309) we have the following *exponential expressions* for these three last *products of unit-lines*,

$$\text{VI.} \dots \gamma'\beta' = \alpha^{\frac{2\text{A}}{\pi}}; \quad \alpha'\gamma' = \beta^{\frac{2\text{B}}{\pi}}; \quad \beta'\alpha' = \gamma^{\frac{2\text{C}}{\pi}}.$$

(3.) Multiplying these three expressions, in an inverted order, we have, therefore, the new product:

$$\text{VII.} \dots \gamma^{\frac{2\text{C}}{\pi}} \beta^{\frac{2\text{B}}{\pi}} \alpha^{\frac{2\text{A}}{\pi}} = \beta'\alpha' \cdot \alpha'\gamma' \cdot \gamma'\beta' = \gamma'^2\beta'^2\alpha'^2 = -1;$$

and the equation 310, I. is in this way *proved anew*.

(4.) And because, instead of VI., we might have written,

* Compare the Note to page 146.

$$\text{VIII.} \dots a^{\frac{2a}{\beta}} = -\frac{\gamma'}{\beta}; \quad \beta^{\frac{2b}{\gamma}} = -\frac{a'}{\gamma}; \quad \gamma^{\frac{2c}{a}} = -\frac{\beta'}{a},$$

we see that the equation to be proved may be reduced to the form of the *identity*

$$\text{IX.} \dots \frac{\beta'}{a} \frac{a'}{\gamma} \frac{\gamma'}{\beta} = +1;$$

and may be *interpreted* as expressing, what is evident, that if a point be supposed to move first along the side $B'C'$, of the polar triangle $A'B'C'$, from B' to C' ; then along the successive side $C'A'$, from C' to A' ; and finally along the remaining side $A'B'$, from A' to B' , it will thus have *returned* to the position from which it *set out*, or will *on the whole* have *not changed place* at all.

(5.) In *this view*, then, we perform what we have elsewhere called an *addition of arcs* (instead of *angles* as in 810); and in a *notation* already used (264, (4.)), we may express the result by the formula,

$$\text{X.} \dots \circ A'B' + \circ C'A' + \circ B'C' = 0;$$

each of the the two *left-hand symbols* denoting an *arc*, which is conceived to be *added* (as a *successive vector-arc*, 180, (3.)), to the arc whose symbol immediately *follows* it, or is written *next* it, but towards the *right-hand*.

(6.) The expressions VI. or VIII., for the *exponential factors* in 810, I., show in a new way the necessity of attending to the *order* of those factors, in that formula: for if we should *invert that order*, *without altering* (as in 810, VIII.) *the exponents*, we may now see that we should obtain this *new product*:

$$\text{XI.} \dots a^{\frac{2a}{\beta}} \beta^{\frac{2b}{\gamma}} \gamma^{\frac{2c}{a}} = -\frac{\gamma'}{\beta} \frac{a'}{\gamma} \frac{\beta'}{a} = +(\gamma'\beta'a')^2;$$

which, on account of the *dipolarity* of the lines a' , β' , γ' , is *not equal to negative unity*, but to a certain *other versor*; the properties of which may be inferred from what was shown in 297, (64.), and in 298, (8.), but upon which we cannot here delay.

312. In general (comp. 221), *an equation*, such as

$$\text{I.} \dots q' = q,$$

between two quaternions, includes a *system of four* scalar equations*, such as the following:

$$\text{II.} \dots Sq' = Sq; \quad Saq' = Saq; \quad S\beta q' = S\beta q; \quad S\gamma q' = S\gamma q;$$

where a , β , γ may be *any three actual and dipolar vectors*: and conversely, if a , β , γ be *any three such vectors*, then the *four scalar equations* II. reproduce, and are sufficiently re-

* The *propriety*, which such results as this establish, for the use of the *name*, QUATERNIONS, as applied to this whole Calculus, on account of its essential connexion with the *number* FOUR, does not require to be again insisted on.

placed by, the *one* quaternion equation I. But an *equation between two vectors* is equivalent only to a system of *three scalar equations*, such as the *three last* equations II.; for example, in 294, (12.), the *one vector equation* XXII. is equivalent to the *three scalar equations* XXI., under the immediately preceding *condition of diplanarity* XX. In like manner, an *equation between two versors of quaternions*,* such as the equation

$$\text{III.} \dots Uq' = Uq,$$

includes generally a system of *three*, but of *not more than three*, scalar equations; because the *versor* Uq depends generally (comp. 157) on a *system of three scalars*, namely the *two* which determine its *axis* $Ax.q$, and the *one* which determines its *angle* $\angle q$; or because the *versor equation* III. requires to be *combined* with the *tensor equation*,

$$\text{IV.} \dots Tq' = Tq, \quad \text{compare 187 (13.),}$$

in order to reproduce the *quaternion equation* I. Now the recent equation, 310, I., is evidently of this *versor-form* III., if α, β, γ be still supposed to be *unit-lines*. If then we *met that equation*, or if one of its *form* had occurred to us, without any knowledge of its *geometrical signification*, we might propose to *resolve it*, with respect to the *three scalars* A, B, C , treated as *three unknown quantities*. The few following remarks, on the problem thus proposed, may be not out of place, nor uninteresting, here.

(1.) Writing for abridgment,

$$\text{V.} \dots \cot A = t, \quad \cot B = u, \quad \cot C = v,$$

and

$$\text{VI.} \dots s = -\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C,$$

the equation to be resolved becomes (by 808, VII., or 809, XII.),

$$\text{VII.} \dots (v + \gamma) (u + \beta) (t + \alpha) = s;$$

in which the *tensors* on both sides are already equal, because

* An equation, $U\rho' = U\rho$, or $UVq' = UVq$, between two *versors of vectors* (156), or between the *axes* of two quaternions (291), is equivalent only to a system of *two scalar equations*; because the *direction of an axis*, or of a *vector*, depends on a system of *two angular elements* (111).

$$\text{VIII.} \dots s^2 = (v^2 + 1)(u^2 + 1)(t^2 + 1).$$

(2.) Multiplying the equation VII. by $t + \alpha$, and into $t - \alpha$, and dividing the result by $t^2 + 1$, we have this new equation of the same form, but differing by cyclical permutation (comp. 310, (3.)) :

$$\text{IX.} \dots (t + \alpha)(v + \gamma)(u + \beta) = s;$$

and in like manner,

$$\text{X.} \dots (u + \beta)(t + \alpha)(v + \gamma) = s.$$

(3.) Taking the half difference of the two last equations, and observing that (by 279, IV., and 294, II.)

$$\text{XI.} \dots \begin{cases} \frac{1}{2}(\beta\alpha\gamma - \alpha\gamma\beta) = V. \beta V\alpha\gamma = \gamma S\alpha\beta - \alpha S\beta\gamma, \\ \frac{1}{2}(\beta\alpha - \alpha\beta) = V\beta\alpha, \quad \frac{1}{2}(\beta\gamma - \gamma\beta) = V\beta\gamma, \end{cases}$$

we arrive at this new equation, of vector form :

$$\text{XII.} \dots 0 = vV\beta\alpha + tV\beta\gamma + \gamma S\alpha\beta - \alpha S\beta\gamma;$$

which is equivalent only to a system of two scalar equations, because it gives $0 = 0$, when operated on by $S. \beta$ (comp. 294, (9.)).

(4.) It enables us, however, to determine the two scalars, t and v ; for if we operate on it by $S. \alpha$, we get (comp. 298, XXVI.),

$$\text{XIII.} \dots tS\alpha\beta\gamma = \alpha^2 S\beta\gamma - S\beta\alpha S\alpha\gamma = S(V\beta\alpha. V\alpha\gamma);$$

and if we operate on the same equation XII. by $S. \gamma$, we get in like manner,

$$\text{XIV.} \dots vS\alpha\beta\gamma = \gamma^2 S\alpha\beta - S\alpha\gamma S\gamma\beta = S(V\alpha\gamma. V\gamma\beta).$$

(5.) Processes quite similar give the analogous result,

$$\text{XV.} \dots uS\alpha\beta\gamma = \beta^2 S\gamma\alpha - S\gamma\beta S\beta\alpha = S(V\gamma\beta. V\beta\alpha);$$

and thus the problem is resolved, in the sense that expressions have been found for the three sought scalars t, u, v , or for the cotangents $V.$ of the three sought angles α, β, γ : whence the fourth scalar, s , in the quaternion equation VII., can easily be deduced, as follows.

(6.) Since (by 294, (6.), changing δ to α , and afterwards cyclically permuting) we have, for any three vectors α, β, γ , the general transformations,

$$\text{XVI.} \dots \begin{aligned} \alpha S\alpha\beta\gamma &= V(V\beta\alpha. V\alpha\gamma), & \beta S\alpha\beta\gamma &= V(V\gamma\beta. V\beta\alpha), \\ \gamma S\alpha\beta\gamma &= V(\alpha\gamma. V\gamma\beta), \end{aligned}$$

the expressions XIII. XV. XIV. give,

$$\text{XVII.} \dots \begin{cases} (t + \alpha)S\alpha\beta\gamma = V\beta\alpha. V\alpha\gamma; \\ (u + \beta)S\alpha\beta\gamma = V\gamma\beta. V\beta\alpha; \\ (v + \gamma)S\alpha\beta\gamma = V\alpha\gamma. V\gamma\beta; \end{cases}$$

whence, by VII.,

$$\text{XVIII.} \dots s(S\alpha\beta\gamma)^3 = (V\gamma\beta)^2 (V\beta\alpha)^2 (V\alpha\gamma)^2;$$

and thus the remaining scalar, s , is also entirely determined.

(7.) And the equation VIII. may be verified, by observing that the expressions XVII. give,

$$\text{XIX.} \dots \begin{cases} (t^2 + 1)(S\alpha\beta\gamma)^2 = (V\beta\alpha)^2 (V\alpha\gamma)^2; \\ (u^2 + 1)(S\alpha\beta\gamma)^2 = (V\gamma\beta)^2 (V\beta\alpha)^2; \\ (v^2 + 1)(S\alpha\beta\gamma)^2 = (V\alpha\gamma)^2 (V\gamma\beta)^2. \end{cases}$$

(8.) The equations XIII. XIV. XV. XVI. give, by elimination of $S\alpha\beta\gamma$, these new expressions :

$$\text{XX.} \dots \alpha t^{-1} = (V : S) (\sqrt{\beta a} \cdot \sqrt{\alpha \gamma}); \quad \beta w^{-1} = (V : S) (\sqrt{\gamma \beta} \cdot \sqrt{\beta a});$$

$$\gamma v^{-1} = (V : S) (\sqrt{\alpha \gamma} \cdot \sqrt{\gamma \beta});$$

by comparing which with the formula 281, XXVIII., after suppressing (291) the characteristic I, we find that the three scalars, t, u, v , are either Ist, the cotangents of the angles opposite to the sides a, b, c , of the spherical triangle in which the three given unit-lines α, β, γ terminate, or IIInd, the negatives of those cotangents, the angles themselves of that triangle being as usual supposed to be positive (809, (10.)), according as the rotation round a from β to γ is negative or positive: that is (294, (8.)), according as $S\alpha\beta\gamma > \text{or} < 0$; or finally, by XVIII., according as the fourth scalar, s , is negative or positive, because the second member of that equation XVIII. is always negative, as being the product of three squares of vectors (282, 292).

(9.) In the Ist case, which is that of 809, (1.), we see then anew, by V. and VI., that we are permitted to interpret the scalars A, B, C , in the exponential formula 810, I., as equal to the angles of the spherical triangle (8.), which are usually denoted by the same letters. But we see also, that we may add any even multiples of π to those three angles, without disturbing the exponential equation; or any one even, and two odd multiples of π , in any order, so as to preserve a positive product of cosecants, because s is, for this case, negative in VI., by (8.).

(10.) In the IIInd case, which is that of 809, (11.), we may, for similar reasons, interpret the scalars A, B, C , in the formula 810, I., as equal to the negatives of the angles of the triangle; and as thus having, what VI. now requires, because s is now positive (8.), a negative product of cosecants, while their cotangents have the values required. But we may also add, as in (9.), any multiples of π , to the scalars thus found for the formula, provided that the number of the odd multiples, so added, is itself even (0 or 2).

(11.) The conclusions of 809, or 810, respecting the interpretation of the exponential formula, are therefore confirmed, and might have been anticipated, by the present new analysis: in conducting which it is evident that we have been dealing with real scalars, and with real vectors, only.

(12.) If this last restriction were removed, and imaginary values admitted, in the solution of the quaternion equation VII., we might have begun by operating, as in II., on that equation, by the four characteristics,

$$\text{XXI.} \dots S, S.a, S.\beta, \text{ and } S.\gamma;$$

which would have given, with the significations 297, (1.), (3.), of l, m, n , and e , and therefore with the following relation between those four scalar data,

$$\text{XXII.} \dots e^2 = 1 - l^2 - m^2 - n^2 + 2lmn,$$

a system of four scalar equations, involving the four sought scalars, s, t, u, v ; from which it might have been required to deduce the (real or imaginary) values of those four scalars, by the ordinary processes of algebra.

(13.) The four scalar equations, so obtained, are the following:

$$\text{XXIII.} \dots \begin{cases} 0 = e + lt + mu + nv - tuv + s; \\ 0 = et + mtu + ntv + uv - l; \\ 0 = -eu + ltu + tv + nuv + m - 2ln; \\ 0 = ev + tu + ltv + muv - n; \end{cases}$$

eliminating wv and u between the three last of which, we find, with the help of XXII., the determinant,

$$\text{XXIV.} \dots 0 = \begin{vmatrix} 1, & mt, & ntv + et - l \\ m, & t, & ltv + ev - n \\ n, & lt - e, & tv + m - 2ln \end{vmatrix} = e(t^2 + 1)(ev - n + lm);$$

and analogous eliminations give,

$$\text{XXV.} \dots 0 = e(t^2 + 1)(eu - m + nl),$$

and $\text{XXVI.} \dots 0 = (t^2 + 1)\{e^2uv - (m - nl)(n - lm) + (1 - t^2)(et - l + mn)\}$.

(14.) Rejecting then the factor $t^2 + 1$ we find, as the *only real solution* of the problem (12.), the following system of values:

$$\text{XXVII.} \dots et = l - mn; \quad eu = m - nl; \quad ev = n - lm;$$

and $\text{XXVIII.} \dots e^2s = -(1 - t^2)(1 - m^2)(1 - n^2);$

which correspond precisely to those otherwise found before, in (4.) (5.) (6.), and might therefore serve to *reproduce the interpretation* of the *exponential formula* (810).

(15.) But on the purely *algebraic side*, it is found, by a similar analysis, that the *four equations* XXIII. are *satisfied also* by a *system of four imaginary solutions*, represented by the following formulæ:

$$\text{XXIX.} \dots \begin{cases} t^2 + 1 = 0; & v^2 + 1 = 0; \\ s = twv - lt - mu - nv - e = 0; \end{cases}$$

which it may be sufficient to have mentioned in passing, since they do not appear to have any such *geometrical interest*, as to deserve to be dwelt on here: though, as regards the *consistency* of the different processes employed, it may be remembered that in passing (2.) from the equation VII. to IX., after certain preliminary multiplications, we *divided by* $t^2 + 1$, as we were entitled to do, when seeking only for *real solutions*, because t was supposed to be a *scalar*.

(16.) This seems to be a natural occasion for remarking that the following *general transformation* exists, *whatever three vectors* may be denoted by α, β, γ :

$$\text{XXX.} \dots S(V\beta\gamma.V\gamma\alpha.V\alpha\beta) = -(S\alpha\beta\gamma)^2;$$

which proves in a new way (comp. 180), that *the rotation round the line* $V\beta\gamma$, *from* $V\gamma\alpha$ *to* $V\alpha\beta$, *is always positive*; or is directed in the *same sense* (281, (8.)), as the rotation round $V\alpha\beta$ from α to β , &c.

(17.) In like manner we have generally,

$$\text{XXXI.} \dots S(V\alpha\beta.V\gamma\alpha.V\beta\gamma) = +(S\alpha\beta\gamma)^2,$$

and $\text{XXXII.} \dots S(V\gamma\beta.V\alpha\gamma.V\beta\alpha) = +(S\alpha\beta\gamma)^2;$

so that *the rotation round* $V\gamma\beta$ *from* $V\alpha\gamma$ *to* $V\beta\alpha$ *is negative*, whatever arrangement the three diplanar vectors α, β, γ may have among themselves.

(18.) If then A'', B'', C'' be the *negative poles* of the *three successive sides*, BC, CA, AB , of any *spherical triangle*, the rotation round A'' from B'' to C'' is *negative*: which is entirely consistent with the *opposite result* (180), respecting the system of the *three positive poles* A', B', C' .

(19.) A *quantitative interpretation* of the equation XXX. may also be easily assigned: for we may infer from it (by 281, (4.), and 294, (3.)) that if $OABC$ be any pyramid, and if normals OA', OB', OC' to the three faces BOC, COA, AOB have their lengths numerically equal to the areas of those faces (as bearing the same ratios to

units, &c.), then (with a similar reference to units) the volume of the new pyramid, $OA'B'C'$, will be three quarters of the square of the volume of the old pyramid, $OABC$.

313. But an allusion was made, in 310, to an *extension* of the *exponential formula* which has lately been under discussion; and in fact, that formula admits of being easily extended, from *triangles to polygons* upon the sphere: for we may write, generally,

$$I. \dots a_n^{\frac{2A_n}{\pi}} a_{n-1}^{\frac{2A_{n-1}}{\pi}} \dots a_2^{\frac{2A_2}{\pi}} a_1^{\frac{2A_1}{\pi}} = (-1)^n,$$

if $A_1A_2 \dots A_{n-1}A_n$ be any *spherical polygon*, and if the scalars A_1, A_2, \dots in the exponents denote the positive or negative *angles* of that polygon, considered as the *rotations* $A_nA_1A_2, A_1A_2A_3, \dots$ namely those from A_1A_n to A_1A_2 , &c.; while n is any positive whole number* > 2 .

(1.) One mode of proving this extended formula is the following. Let $oc = \gamma$ be the unit-vector of an arbitrary point c on the spheric surface; and conceive that arcs of great circles are drawn from this point c to the n successive corners of the polygon. We shall thus have a system of n spherical triangles, and each angle of the polygon will (generally) be decomposed into two (positive or negative) *partial angles*, which may be thus denoted:

$$II. \dots CA_1A_2 = A_1', \quad CA_2A_3 = A_2', \dots;$$

$$III. \dots A_nA_1c = A_1'', \quad A_1A_2c = A_2'', \dots;$$

so that, with attention to signs of angles in the additions,

$$IV. \dots A_1 = A_1' + A_1'', \quad A_2 = A_2' + A_2'', \text{ \&c.}$$

Also let

$$V. \dots A_2cA_1 = C_1, \quad A_3cA_2 = C_2, \text{ \&c.};$$

and therefore

$$VI. \dots C_1 + C_2 + \dots + C_n = \text{an even multiple of } \pi,$$

which reduces itself to 2π in the simple case of a polygon with no re-entrant angles, and with the point c in its interior.

(2.) Then, for the triangle CA_1A_2 , of which the angles are C_1, A_1', A_2'' , we have, by §10, III., the equation,

$$VII. \dots a_2^{\frac{2A_2''}{\pi}} a_1^{\frac{2A_1'}{\pi}} = -\gamma^{\frac{2C_1}{\pi}};$$

and in like manner, for the triangle CA_2A_3 , we have

* The formula admits of interpretation, even for the case $n = 2$.

$$\text{VIII.} \dots \alpha_3 \frac{\partial \Delta_3}{\pi} \alpha_2 \frac{\partial \Delta_2'}{\pi} = -\gamma \frac{\partial C_2}{\pi}, \&c.$$

But, when we multiply VII. by VIII., we obtain, by IV., the product,

$$\text{IX.} \dots \alpha_3 \frac{\partial \Delta_3''}{\pi} \alpha_2 \frac{\partial \Delta_2}{\pi} \alpha_1 \frac{\partial \Delta_1'}{\pi} = +\gamma \frac{\partial (C_1 + C_2)}{\pi};$$

and so proceeding, we have at last, by VI., a product of the form,

$$\text{X.} \dots \alpha_1 \frac{\partial \Delta_1''}{\pi} \alpha_n \frac{\partial \Delta_n}{\pi} \dots \alpha_2 \frac{\partial \Delta_2}{\pi} \alpha_1 \frac{\partial \Delta_1'}{\pi} = (-1)^n;$$

which reduces itself to I., when it is multiplied by $\alpha \frac{\partial \Delta_1''}{\pi}$, and into $\alpha \frac{\partial \Delta_1''}{\pi}$ (comp. 310, (8.)). The theorem is therefore proved.

(3.) In words (comp. 310, (8.)), "*the spherical sum of the successive angles of any spherical polygon, if taken in a suitable order, is equal to a multiple of two right angles, which is odd or even, according as the number of the sides (or corners) of the polygon is itself odd or even*": the definition formerly given (180, (4.)), of a *Spherical Sum of Angles*, being of course retained. And the reasoning may be briefly stated thus. When an arbitrary point O is taken on the spherical surface, as in (1.), the *spherical sum* of the two partial angles, at the ends of any one side, is the supplement of the angle which that side subtends, at the point O ; but the sum of all such subtended angles is either four right angles, or some whole multiple thereof: therefore the sum of their supplements can differ only by some such multiple from $n\pi$, if n be the number of the sides.

(4.) Whatever that number may be, if we denote by p_n the exponential product in the formula I., we have for every vector ρ , and for every quaternion q , the equations:

$$\text{XI.} \dots p_n \rho p_n^{-1} = \rho; \quad \text{XII.} \dots p_n q p_n^{-1} = q;$$

whereof the former may (by 308, (8.)), be thus interpreted:—

"If any line OP , drawn from the centre O of a sphere, be made to revolve conically round any n radii, OA_1, \dots, OA_n , as n successive axes of rotation, through angles equal respectively to the doubles of the angles of the spherical polygon $\Delta_1 \dots \Delta_n$, the line will be brought back to its initial position, by the composition of these n rotations."

(5.) Another way of proving the extended formula I., for any spherical polygon, is analogous to that which was employed in 311 for the case of a triangle on a sphere, and may be stated as follows. Let $\Delta_1', \Delta_2', \dots, \Delta_n'$ be the positive poles of the arcs $\Delta_1 \Delta_2, \Delta_2 \Delta_3, \dots, \Delta_n \Delta_1$; and let $\alpha_1', \alpha_2', \dots, \alpha_n'$ be the unit-vectors of those n poles. Then the point Δ_1 is the positive pole of the new arc $\Delta_1' \Delta_n'$, and the angle Δ_1 of the polygon at that point is measured by the supplement of that arc; with similar results for other corners of the polygon. Thus we have the system of expressions (comp. 311, VI.):

$$\text{XIII.} \dots \alpha_1 \frac{\partial \Delta_1}{\pi} = \alpha_1' \alpha_n'; \dots \alpha_n \frac{\partial \Delta_n}{\pi} = \alpha_n' \alpha_{n-1}';$$

so that the product of powers in I. is equal to the following product of n squares of unit-lines, and therefore to the n^{th} power of negative unity,

$$\text{XIV.} \dots \alpha'_n \alpha'_{n-1} \cdot \alpha'_{n-1} \alpha'_{n-2} \dots \alpha'_2 \alpha'_1 \cdot \alpha'_1 \alpha'_n = (-1)^n;$$

and thus the extended theorem is proved anew.

(6.) This latter process may be translated into another *theorem of rotation*, on which it is possible that we may briefly return,* in the Second and last Chapter of this Third Book, but upon which we cannot here delay.

(7.) It may be remarked however here (comp. 309, XII.), that the *extended exponential formula I.* may be thus written:

$$\text{XV.} \dots \cos \alpha \beta \Delta_n \cdot \cos \alpha_{n-1} \beta \Delta_{n-1} \dots \cos \alpha_2 \beta \Delta_2 \cdot \cos \alpha_1 \beta \Delta_1 = (-1)^n.$$

(8.) For example, if ABCD be any *spherical quadrilateral*, of which the *angles* (suitably measured) are denoted by Δ, \dots, D , so that Δ represents the positive or negative *rotation* from AD to AB, &c., while $\alpha, \beta, \gamma, \delta$ are the unit vectors of its corners, then

$$\text{XVI.} \dots \cos \delta \beta \Delta \cdot \cos \gamma \beta \Delta \cdot \cos \alpha \beta \Delta \cdot \cos \Delta = +1.$$

(9.) Hence (comp. 309, XIII.), we may write also,

$$\text{XVII.} \dots (\cos \gamma - \gamma \sin \delta) (\cos \delta - \delta \sin \gamma) = (\cos \beta + \beta \sin \alpha) (\cos \Delta + \alpha \sin \Delta);$$

and therefore, by taking scalars on both sides, and changing signs,

$$\text{XVIII.} \dots -\cos \delta \cos \gamma + \sin \delta \sin \gamma = -\cos \beta \cos \Delta + \sin \beta \sin \Delta \cos \alpha;$$

in fact, each member of this last formula is equal (by 309, XIV.) to the cosine of the angle $\alpha \beta \Delta$, or $\gamma \delta \Delta$, if the opposite sides AD, BC of the quadrilateral intersect in Δ .

(10.) Let $\rho = OP$ be the unit vector of any *fifth point*, P, upon the spheric surface; then operating by S. ρ on XVII., we obtain this other general formula,

$$\text{XIX.} \dots \begin{cases} 0 = \sin \Delta \cos \beta \cos \alpha \rho + \sin \beta \cos \alpha \cos \beta \rho + \sin \alpha \sin \beta \sin \alpha \rho \sin \Delta \\ \quad + \sin \gamma \cos \delta \cos \gamma \rho + \sin \delta \cos \gamma \cos \delta \rho + \sin \gamma \sin \delta \sin \gamma \rho \sin \Delta; \end{cases}$$

in which the *sines of the sides* AB, CD are treated as *always positive*; but the *sines of the perpendiculars* PQ and PR, on those two sides, are regarded as *positive or negative*, according as the *rotations* round P, from A to B and from C to D, are *negative or positive*: and hence, by assigning particular positions to P, several other but less general equations of *spherical tetragonometry* can be derived.

(11.) For example, if we place P at the *intersection*, say F, of the *opposite sides* AB, CD, the two last perpendiculars will vanish, and *two of the six terms* will *disappear*, from the general formula XIX.; and a similar *reduction to four terms* will occur, if we make the arbitrary point P the *pole of a side*, or of a *diagonal*.

314. The definition of the *power* a^2 , which was assigned in 308, enables us to form some useful expressions, by quaternions, for *circular, elliptic, and spiral loci*, in a given plane, or in space, a few of which may be mentioned here.

(1.) Let α be any given unit-vector OA, and β any other given line OB, perpendicular to it; then, by the definition (308), if we write,

* Compare 297, (24.).

$$\text{I. . . } OP = \rho = a^t \beta, \quad Ta = 1, \quad Sa\beta = 0,$$

the *locus of the point P* will be the *circumference of a circle*, with *O* for *centre*, and *OB* for *radius*, and in a *plane perpendicular to OA*.

(2.) If we retain the condition $Ta = 1$, but *not* the condition $Sa\beta = 0$, then the product $a^t \beta$ will be in general a *quaternion*, and not merely a *vector*; but if we take its *vector-part* (292), we can form this *new vector-expression*,

$$\text{II. . . } OP = \rho = V. a^t \beta = \beta \cos x + \gamma \sin x,$$

where

$$\text{III. . . } 2x = t\pi, \quad \text{and} \quad \text{IV. . . } \gamma = OC = Va\beta;$$

and now the *locus of P* is a *plane ellipse*, with its *centre* at *O*, and with *OB* and *OC* for its *major* and *minor semi-axes*: while the angular quantity, x , is what is often called the *eccentric anomaly*.

(3.) If we write, under the same conditions (2.),

$$\text{V. . . } OB' = \beta' = V\beta a: a = a^{-1} \gamma, \quad \text{and} \quad \text{VI. . . } OP' = \rho' = V\rho a: a = a V\rho a,$$

so that B' and P' are the *projections* (208) of B and P on a *plane drawn through O*, at *right angles to the unit-line OA*, we have then, by II., the equation,

$$\text{VII. . . } \rho' = \beta' \cos x + \gamma \sin x = a^t \beta';$$

so that the *locus of this projected point P'* is a *circle*, with OB' and OC for two *rectangular radii*.

(4.) Under the same conditions, the *elliptic locus* (2.), of the point P itself, is the *section of the right cylinder* (compare 208, (5.)),

$$\text{VIII. . . } TVap = TVa\beta = T\gamma,$$

made by the *plane*,

$$\text{IX. . . } 0 = S\gamma\beta\rho, \quad \text{or} \quad \text{IX'. . . } \beta^2 Sap = Sa\beta S\beta\rho \quad (\text{comp. 298, XXVI.});$$

as a confirmation of which last form we have, by II. and IV.,

$$\text{X. . . } Sap = Sa\beta \cos x, \quad S\beta\rho = \beta^2 \cos x.$$

(5.) If we retain the condition $Sa\beta = 0$ (1.), but *not now* the condition $Ta = 1$, we may again write the equation I. for ρ ; but the *locus of P* will now be a *logarithmic spiral*, with *O* for its *pole*, in the *plane perpendicular to OA*; because *equal angular motions*, of the *turning line OP*, correspond now to *equal multiplications of the length* of that line ρ .

(6.) For example, when the scalar exponent t is increased by 4, so that the *revolving unit line*,

$$\text{XI. . . } U\rho = Ua^4. U\beta$$

returns (comp. 809, XXIV.) to the *direction* which it had before the increase of t was made, the *length* $T\rho$ of its *turning line* ρ itself, or of the *radius vector of the locus*, is *multiplied by* Ta^4 ; which constant and positive scalar is *not now* equal to *unity*.

(7.) If we reject both the conditions (1.),

$$Ta = 1, \quad \text{and} \quad Sa\beta = 0,$$

so that the *line a*, or the *base of the power* a^t , is now *neither an unit-line, nor perpendicular to* β , namely to the *line on which that power operates*, as a *factor*, we must again take *vector parts*, but we have now this *new expression*:

$$\text{XII. . . } OP = \rho = V. a^t \beta = a^t (\beta \cos x + \gamma \sin x)$$

in which we have written, for abridgment,

$$\text{XIII.} \dots a = Ta, \quad \gamma = V(Ua.\beta).$$

(8.) In this more complex case, the *locus* of r is still a *plane curve*, and may be said to be now an *elliptic* logarithmic spiral*; for if we suppress the *scalar factor*, a^t , we fall back on the *form* II., and have again an *ellipse* as the *locus*: but when we take account of that factor, we find (comp. (2.)) that *equal increments of eccentric anomaly* (x), in the *auxiliary ellipse* so determined, correspond to *equal multiplications of the length* ($T\rho$), of the vector of the *new spiral*.

(9.) We may also project β and r , as in (3.), into points β' and r' , on the plane through o perpendicular to oa , which plane still contains the extremity o of the auxiliary vector γ ; and then, since it is easily proved that $\gamma = Ua.\beta$, the equation of the *projected spiral* becomes (with $Ta >$ or < 1),

$$\text{XIV.} \dots \rho' = a^t(\beta' \cos x + \gamma \sin x) = a^t\beta';$$

so that we are brought back to the case (5.), and the *projected curve* is seen to be a logarithmic spiral, of the known and *ordinary kind*.

(10.) Several *spirals of double curvature* are easily represented, on the same general plan, by merely introducing a *vector-term proportional to t* , combined or not with a *constant vector-term*, in each of the expressions above given, for the *variable vector* ρ . For example, the equation,

$$\text{XV.} \dots \rho = cta + a^t\beta, \quad \text{with } Ta = 1, \quad \text{and } Sa\beta = 0,$$

while c is any *constant scalar* different from zero, represents a *helix*, on the right circular cylinder VIII.

(11.) And if we introduce a new and variable scalar, u , as a *factor* in the right-hand term, and so write,

$$\text{XVI.} \dots \rho = cta + ua^t\beta,$$

we shall have an expression for a *variable vector* ρ , considered as depending on two *variable scalars* (t and u), which thus becomes (99) the expression for a *vector of a surface*: namely of that important *Screw Surface*, which is the *locus of the perpendiculars*, let fall from the various points of a *given helix*, on the *axis* of the cylinder of revolution, on which that helix, or spiral curve, is traced.

315. Without at present pursuing farther the study of these *loci* by quaternions, it may be remarked that the definition (308) of the power a^t , especially for the case when $Ta = 1$, combined with the laws (182) of i, j, k , and with the identification (295) of those three important right versors with their own indices, enables us to establish the following among other *transformations*, which will be found useful on several occasions.

(1.) Let a be any *unit-vector*, and let t be any *scalar*; then,

$$\text{I.} \dots S. a^{-t} = S. a^t; \quad \text{II.} \dots S. a^{t-1} = S. a^{t+1} = -S. a^{t-1};$$

* The usual logarithmic spiral might perhaps be called, by contrast to this one, a *circular logarithmic spiral*. Compare the following sub-article (9.), respecting the *projection* of what is here called an *elliptic logarithmic spiral*.

$$\text{III.} \dots \alpha^t = S. \alpha^t + aS. \alpha^{t-1}; \quad \text{IV.} \dots \alpha^t = S. \alpha^t - aS. \alpha^{t-1}; \\ \text{V.} \dots (S. \alpha^t)^2 + (S. \alpha^{t-1})^2 = \alpha^t \alpha^{-t} = 1.$$

(2.) Let α and ϵ be any two unit-vectors, and let t be still any scalar; then

$$\text{VI.} \dots S. \alpha^t = S. \epsilon^t; \quad \text{VII.} \dots V. \alpha^t = aS. \alpha^{t-1}; \\ \text{VIII.} \dots aV. \alpha^t = a^2 S. \alpha^{t-1} = S. \alpha^{t+1}.$$

(3.) Hence, by the laws of i, j, k ,

$$\text{IX.} \dots iV. \epsilon^t = jV. j^t = kV. k^t = S. \alpha^{t+1}.$$

(4.) We have also, by the same principles and laws,

$$\text{X.} \dots iV. j^t = V. k^t; \quad jV. k^t = V. i^t; \quad kV. i^t = V. j^t; \\ \text{XI.} \dots jV. i^t = -V. k^t; \quad kV. j^t = -V. i^t; \quad iV. k^t = -V. j^t.$$

(5.) The expression 308, (10.), for an arbitrary vector ρ , may be put under the following form:

$$\text{XII.} \dots \rho = rV. k^{2s+1} + r'k^s V. \theta^s.$$

(6.) And it may be expanded as follows:

$$\text{XIII.} \dots \rho = r \{ (i \cos t\pi + j \sin t\pi) \sin s\pi + k \cos s\pi \}.$$

(7.) We shall return, briefly, in the Second Chapter of this Book, on some of these last expressions, in connexion with *differentials* and *derivatives* of powers of vectors; but, for the purposes of the present Section, they may suffice.

SECTION 11.—On Powers and Logarithms of Diplanar Quaternions; with some Additional Formulæ.

316. We shall conclude the present Chapter with a short Supplementary Section, in which the recent definition (308) of a *power of a vector*, with a *scalar exponent*, shall be extended so as to include the *general case*, of a *Power of a Quaternion*, with a *Quaternion Exponent*, even when the two quaternions so combined are *diplanar*: and a connected *definition* shall be given (consistent with the less general one of the same kind, which was assigned in the Second Chapter of the Second Book), for the *Logarithm of a Quaternion in an arbitrary Plane*:* together with a few additional Formulæ, which could not be so conveniently introduced before.

(1.) We propose, then, to write, *generally*,

$$\text{I.} \dots \epsilon^q = 1 + \frac{q}{1} + \frac{q^2}{1.2} + \frac{q^3}{1.2.3} + \&c.;$$

q being any quaternion, and ϵ being the real and known base of the natural (or Napierian) system of logarithms, of real and positive scalars: so that (as usual),

* The quaternions considered, in the Chapter referred to, were all supposed to be in the plane of the right versor i . But see the Second Note to page 265.

$$\text{II.} \dots \varepsilon = \varepsilon^1 = 1 + \frac{1}{1} + \frac{1^2}{1 \cdot 2} + \&c. = 2 \cdot 71828 \dots$$

(Compare 240, (1.) and (2).)

(2.) We shall also write, for any quaternion q , the following expression for what we shall call its *principal logarithm*, or simply its *Logarithm* :

$$\text{III.} \dots \text{I}q = \text{I}Tq + \angle q \cdot \text{UV}q;$$

and thus shall have (comp. 243) the equation,

$$\text{IV.} \dots \varepsilon^q = q.$$

(3.) When q is any *actual* quaternion (144), which does not *degenerate* (131) into a *negative scalar*, the formula III. assigns a *definite value* for the *logarithm*, $\text{I}q$; which is such (comp. again 243) that

$$\begin{aligned} \text{V.} \dots \text{S} \text{I}q &= \text{I}Tq; & \text{VI.} \dots \text{V} \text{I}q &= \angle q \cdot \text{UV}q; \\ \text{VII.} \dots \text{U} \text{V} \text{I}q &= \text{UV}q; & \text{VIII.} \dots \text{T} \text{V} \text{I}q &= \angle q; \end{aligned}$$

the *scalar part of the logarithm* being thus the (natural) *logarithm of the tensor*; and the *vector part* of the same logarithm $\text{I}q$ being constructed by a *line* in the direction of the *axis* $\text{Ax. } q$, of which the *length* bears, to the assumed *unit of length*, the same *ratio* as that which the *angle* $\angle q$ bears, to the usual *unit of angle* (comp. 241, (2.), (4.)).

(4.) If it were merely required to *satisfy the equation*,

$$\text{IX.} \dots \varepsilon^q = q,$$

in which q is supposed to be a *given* and *actual* quaternion, which is not equal to any *negative scalar* (3.), we might do this by writing (compare again 243),

$$\text{X.} \dots q' = (\log q)^n = \text{I}q + 2n\pi \text{UV}q,$$

where n is any *whole number*, positive or negative or null; and in this view, what we have called the *logarithm*, $\text{I}q$, of the quaternion q , is only what may be considered as the *simplest solution* of the *exponential equation* IX., and may, as such, be thus denoted :

$$\text{XI.} \dots \text{I}q = (\log q)_0.$$

(5.) The *excepted case* (3.), where q is a *negative scalar*, becomes on this plan a *case of indetermination*, but *not of impossibility*: since we have, for example, by the definition III., the following expression for the *logarithm of negative unity*,

$$\text{XII.} \dots \text{I}(-1) = \pi \sqrt{-1};$$

which in its *form* agrees with old and well-known results, but is here *interpreted* as signifying any *unit-vector*, of which the *length* bears to the *unit* of length the ratio of π to 1 (comp. 243, VII.).

(6.) We propose also to write, generally, for any *two quaternions*, q and q' , even if *dipplanar*, the following expression (comp. 243, (4.)) for what may be called the *principal value of the power*, or simply the *Power*, in which the former quaternion q is the *base*, while the latter quaternion q' is the *exponent* :

$$\text{XIII.} \dots q^q = \varepsilon^{q'q};$$

and thus this *quaternion power* receives, in *general*, with the help of the definitions I. and III., a perfectly *definite signification*.

(7.) When the *base*, q , becomes a *vector*, ρ , its *angle* becomes a *right angle*; the definition III. gives therefore, for this case,

$$\text{XIV.} \dots l\rho = lT\rho + \frac{\pi}{2} U\rho;$$

and this is the quaternion which is to be multiplied by q' , in the expression,

$$\text{XV.} \dots \rho' = \varepsilon^{q'} l\rho.$$

(8.) When, for the same *vector-base*, the *exponent* q' becomes a *scalar*, t , the last formula becomes :

$$\text{XVI.} \dots \rho^t = \varepsilon^{t\rho} = T\rho^t \cdot \varepsilon^{xU\rho}, \quad \text{if } 2x = t\pi;$$

and because, by I., the relation $(U\rho)^2 = -1$ gives,

$$\text{XVII.} \dots \varepsilon^{xU\rho} = \cos x + U\rho \sin x, \quad \text{or briefly,} \quad \text{XVII.} \dots \varepsilon^{xU\rho} = \text{cpx},$$

we see that the former definition, 308, I., of the power a^t , is in this way *reproduced*, as one which is *included* in the *more general definition* XIII., of the power q^t ; for we may write, by the last mentioned definition,

$$\text{XVIII.} \dots (U\rho)^t = \varepsilon^{tU\rho} = \text{cps } \frac{t\pi}{2} \quad (\text{comp. 234, VIII.}),$$

with the recent values XVI. and XVII., of x and $\varepsilon^{xU\rho}$.

(9.) In the present theory of *diplanar quaternions*, we cannot expect to find that the *sum of the logarithms* of any two proposed *factors*, shall be *generally equal* to the *logarithm of the product*; but for the simpler and earlier case of *complanar* quaternions, that *algebraic property* may be considered to exist, with due modifications for *multiplicity of value*.*

(10.) The definition III. enables us, however, to establish *generally* the very simple formula (comp. 243, II. III.) :

$$\text{XIX.} \dots lq = l(Tq \cdot Uq) = lTq + lUq;$$

in which (comp. (3).),

$$\text{XX.} \dots lUq = \angle q. UVq = \nabla q; \quad \text{XXI.} \dots lTq = \angle q; \quad \text{XXII.} \dots lUq = UVq.$$

(11.) We have also generally, by XIII., for any *scalar exponent*, t , and any *quaternion base*, q , the *power*,

$$\text{XXIII.} \dots q^t = \varepsilon^{tq} = (Tq)^t \cdot (\cos t \angle q + UVq \cdot \sin t \angle q);$$

or briefly,

$$\text{XXIII.} \dots q^t = Tq^t \cdot \text{cvs } t \angle q, \quad \text{if } v = UVq;$$

in which the parentheses about Tq may be omitted, because

$$\text{XXIV.} \dots T(q^t) = (Tq)^t = Tq^t \quad (\text{comp. 237, II.}).$$

(12.) When the base and exponent of a power are *two rectangular vectors*, ρ and ρ' , then, whatever their *lengths* may be, the product $\rho' l\rho$ is, by XIV., a *vector*; but ε^a is always a *versor*,

$$\text{XXV.} \dots \varepsilon^a = \cos Ta + Ua \sin Ta, \quad \text{if } a \text{ be any vector};$$

we have therefore,

* In 243, (3.), it might have been observed, that *every value of each member of the formula IX.*, there given, is *one of the values of the other member*; and a similar remark applies to the formulæ I. and II. of 236.

XXVI. . . $T. \rho \rho' = 1$, if $S. \rho \rho' = 0$;

or in words, the power $\rho \rho'$ is a versor, under this condition of rectangularity.

(13.) For example (comp. 242, (7).)* and the shortly following formula XXVIII.),

XXVII. . . $i^j = \epsilon^{j^i} = -k$; $j^i = \epsilon^{i^j} = +k$;

and generally, if the base be an unit-line, and the exponent a line of any length, but perpendicular to the base, the axis of the power is a line perpendicular to both; unless the direction of that axis becomes indeterminate, by the power reducing itself to a scalar, which in certain cases may happen.

(14.) Thus, whatever scalar c may be, we may write,

XXVIII. . . $i^c = \epsilon^{c^i} = \epsilon^{-k^c} = \cos \frac{c\pi}{2} - k \sin \frac{c}{2}$;

this power, then, is a versor (12.), and its axis is generally the line $\mp k$; but in the case when c is any whole and even number, this versor degenerates into positive or negative unity (153), and the axis becomes indeterminate (131).

(15.) If, for any real quaternion q , we write again,

XXIX. . . $UVq = v$, and therefore XXX. . . $vq = qv$, and XXXI. . . $v^2 = -1$, the process of 239 will hold good, when we change i to v ; the series, denoted in I. by ϵ^i , is therefore always at last convergent, † however great (but finite) the tensor Tq may be; and in like manner the two following other series, derived from it, which represent (comp. 242, (8.)) what we shall call, generally, by analogy to known expressions, the cosine and sine of the quaternion q , are always ultimately convergent:

XXXII. . . $\cos q = \frac{1}{2}(\epsilon^{vq} + \epsilon^{-vq}) = 1 - \frac{q^2}{1.2} + \frac{q^4}{1.2.3.4} - \&c.$;

XXXIII. . . $\sin q = \frac{1}{2v}(\epsilon^{vq} - \epsilon^{-vq}) = \frac{q}{1} - \frac{q^3}{1.2.3} + \frac{q^5}{1.2.3.4.5} - \&c.$

(16.) We shall also define that the secant, cosecant, tangent, and cotangent of a quaternion, supposed still to be real, are the functions:

XXXIV. . . $\sec q = \frac{2}{\epsilon^{vq} + \epsilon^{-vq}}$; $\operatorname{cosec} q = \frac{2v}{\epsilon^{vq} - \epsilon^{-vq}}$;

XXXV. . . $\tan q = \frac{v^{-1}(\epsilon^{vq} - \epsilon^{-vq})}{\epsilon^{vq} + \epsilon^{-vq}}$; $\cot q = \frac{v(\epsilon^{vq} + \epsilon^{-vq})}{\epsilon^{vq} - \epsilon^{-vq}}$;

and thus shall have the usual relations, $\sec q = 1 : \cos q$, &c.

(17.) We shall also have,

XXXVI. . . $\epsilon^{vq} = \cos q + v \sin q$, $\epsilon^{-vq} = \cos q - v \sin q$;

* In the theory of *complanar quaternions*, it was found convenient to admit a certain multiplicity of value for a power, when the exponent was not a whole number; and therefore a notation for the principal value of a power was employed, with which the conventions of the present Section enable us now to dispense.

† In fact, it can be proved that this final convergence exists, even when the quaternion is *imaginary*, or when it is replaced by a *biquaternion* (214, (8.)); but we have no occasion here to consider any but *real* quaternions.

and therefore, as in trigonometry (comp. 315, (1.)),

$$\text{XXXVII.} \dots (\cos q)^2 + (\sin q)^2 = \epsilon^{vq} \cdot \epsilon^{-vq} = \epsilon^0 = 1,$$

whatever quaternion q may be.

(18.) And all the formulae of trigonometry, for cosines and sines of sums of two or more arcs, &c., will thus hold good for quaternions also, provided that the quaternions to be combined are in any common plane; for example,

$$\text{XXXVIII.} \dots \cos(q' + q) = \cos q' \cos q - \sin q' \sin q, \text{ if } q' \parallel q.$$

(19.) This condition of coplanarity is here a necessary one; because (comp. (9.)) it is necessary for the establishment of the exponential relation between sums and powers.

(20.) Thus, we may indeed write,

$$\text{XXXIX.} \dots \epsilon^{q'+q} = \epsilon^{q'} \cdot \epsilon^q, \text{ if } q' \parallel q;$$

but, in general, the developments of these two expressions give the difference,

$$\text{XL.} \dots \epsilon^{q'+q} - \epsilon^{q'} \epsilon^q = \frac{qq' - q'q}{2} + \text{terms of third and higher dimensions};$$

and

$$\text{XLI.} \dots \frac{1}{2}(qq' - q'q) = \nabla(\nabla q \cdot \nabla q'),$$

an expression which does not vanish, when the quaternions q and q' are diplanar.

(21.) A few supplementary formulae, connected with the present Chapter, may be appended here, as was mentioned at the commencement of this Article (316). And first it may be remarked, as connected with the theory of powers of vectors, that if α, β, γ be any three unit-lines, OA, OB, OC, and if σ denote the area of the spherical triangle ABC, then the formula 298, XX. may be thus written:

$$\text{XLII.} \dots \frac{\alpha + \beta}{\beta + \gamma} \cdot \frac{\gamma + \alpha}{\alpha + \beta} \cdot \frac{\beta + \gamma}{\gamma + \alpha} = a^{\frac{2\sigma}{\pi}};$$

the exponent being here a scalar.

(22.) The immediately preceding formula, 298, XIX., gives for any three vectors, the relation:

$$\text{XLIII.} \dots (U\alpha\beta\gamma)^2 + (U\beta\gamma)^2 + (U\alpha\gamma)^2 + (U\alpha\beta)^2 + 4U\alpha\gamma \cdot S U\alpha\beta \cdot S U\beta\gamma = -2;$$

for example, if α, β, γ be made equal to i, j, k , the first member of this equation becomes, $1 - 1 - 1 - 1 + 0 = -2$.

(23.) The following is a much more complex identity, involving as it does not only three arbitrary vectors α, β, γ , but also four arbitrary scalars, a, b, c , and r ; but it has some geometrical applications, and a student would find it a good exercise in transformations, to investigate a proof of it for himself. To abridge notation, the three vectors α, β, γ , and the three scalars a, b, c , are considered as each composing a cycle, with respect to which are formed sums Σ , and products Π , on a plan which may be thus exemplified:

$$\text{XLIV.} \dots \Sigma a \nabla \beta \gamma = a \nabla \beta \gamma + b \nabla \gamma \alpha + c \nabla \alpha \beta; \quad \Pi a^2 = a^2 b^2 c^2.$$

This being understood, the formula to be proved is the following:

$$\begin{aligned} \text{XLV.} \dots & (S\alpha\beta\gamma)^2 + (\Sigma a \nabla \beta \gamma)^2 + r^2 (\Sigma \nabla \beta \gamma)^2 - r^2 (\Sigma a (\beta - \gamma))^2 \\ & + 2\Pi (r^2 + S\beta\gamma + bc) = 2\Pi (r^2 + a^2) + 2\Pi a^2 \\ & + \Sigma (r^2 + a^2 + a^2) \{ (\nabla \beta \gamma)^2 + 2bc(r^2 + S\beta\gamma) - r^2 (\beta - \gamma)^2 \}; \end{aligned}$$

the sign of summation in the last line governing all that follows it.

(24.) For example, by making the *four scalars* a, b, c, r each = 0, this formula gives, for any *three vectors* α, β, γ , the relation,

$$\text{XLVI.} \dots (\text{Sa}\beta\gamma)^2 + 2\Pi\text{S}\beta\gamma = 2\Pi a^2 + \Sigma . a^2(\text{V}\beta\gamma)^2 ;$$

which agrees with the very useful equation 294, LIII., because

$$\text{XLVII.} \dots a^2(\text{V}\beta\gamma)^2 = a^2\{(\text{S}\beta\gamma)^2 - \beta^2\gamma^2\} = (a\text{S}\beta\gamma)^2 - \Pi a^2.$$

(25.) Let α, β, γ be the *vectors of three points* A, B, C, which are exterior to a given sphere, of which the radius is r , and the equation is,

$$\text{XLVIII.} \dots \rho^2 + r^2 = 0 \text{ (comp. 282, XIII.)};$$

and let a, b, c denote the *lengths of the tangents* to that sphere, which are drawn from those three points respectively. We shall then have the relations:

$$\text{XLIX.} \dots a^2 + a^2 = \beta^2 + b^2 = \gamma^2 + c^2 = -r^2 ;$$

thus $r^2 + a^2 = -a^2$, &c., and the second member of the formula XLV. vanishes; the first member of that formula is therefore also equal to zero, for these significations of the letters: and thus a *theorem* is obtained, which is found to be extremely useful, in the investigation by quaternions of the system of the *eight* (real or imaginary) *small circles, which touch a given set of three small circles on a sphere.*

(26.) We cannot enter upon that investigation here; but may remark that because the vector ρ of the foot P, of the perpendicular or let fall the origin o on the right line AB, is given by the expression,

$$\text{L.} \dots \rho = a\text{S} \frac{\beta}{\beta - a} + \beta\text{S} \frac{a}{a - \beta} = \frac{\text{V}\beta a}{a - \beta}$$

as may be proved in various ways, the *condition of contact* of that right line AB with the sphere XLVIII. is expressed by the equation,

$$\text{LI.} \dots \text{TV}\beta a = r\text{T}(a - \beta); \text{ or } \text{LII.} \dots (\text{V}\beta a)^2 = r^2(a - \beta)^2 ;$$

or by another easy transformation, with the help of XLIX.,

$$\text{LIII.} \dots (r^2 + \text{Sa}\beta)^2 = (r^2 + a^2)(r^2 + \beta^2) = a^2b^2.$$

(27.) This last equation evidently admits of decomposition into *two factors*, representing *two alternative conditions*, namely,

$$\text{LIV.} \dots r^2 + \text{Sa}\beta - ab = 0; \quad \text{LV.} \dots r^2 + \text{Sa}\beta + ab = 0;$$

and if we still consider the *tangents* a and b (25.) as *positive*, it is easy to prove, in several different ways, that the *first* or the *second* factor is to be selected, according as the point P, at which the line AB touches the sphere, does or does not fall between the points A and B; or in other words, according as the *length* of that line is equal to the *sum*, or to the *difference*, of those two tangents.

(28.) In fact we have, for the first case,

$$\text{LVI.} \dots \text{T}(\beta - a) = b + a, \text{ or } 0 = (\beta - a)^2 + (b + a)^2 = -2(r^2 + \text{Sa}\beta - ab),$$

in virtue of the relations XLIX.; but, for the second case,

$$\text{LVII.} \dots \text{T}(\beta - a) = \pm(b - a), \text{ or } 0 = (\beta - a)^2 + (b - a)^2 = -2(r^2 + \text{Sa}\beta + ab);$$

and it may be remarked, that we might in this way have been led to find the system of the *two conditions* (27.), and thence the equation LIII., or its transformations, LII. and LI.

(29.) We may conceive a *cone of tangents* from Λ , *circumscribing* the sphere XLVIII., and touching it *along a small circle*, of which the *plane*, or the *polar plane of the point Λ* , is easily found to have for its equation,

$$\text{LVIII.} \dots S\alpha\rho + r^2 = 0 \text{ (comp. 294, (28.), and 215, (10.))};$$

and in like manner the equation,

$$\text{LIX.} \dots S\beta\rho + r^2 = 0,$$

represents the polar plane of the point \mathbf{B} , which plane cuts the sphere in a *second small circle*: and *these two circles touch each other*, when *either* of the two conditions (27.) is satisfied; such *contact* being *external* for the case LIV., but *internal* for the case LV.

(30.) The *condition of contact* (26.), of the *line and sphere*, might have been otherwise found, as the condition of *equality of roots* in the *quadratic equation* (comp. 216, (2.)),

$$\text{LX.} \dots 0 = (x\alpha + y\beta)^2 + (x + y)^2 r^2,$$

$$\text{or LXI.} \dots 0 = x^2(r^2 + \alpha^2) + 2xy(r^2 + S\alpha\beta) + y^2(r^2 + \beta^2);$$

the *contact* being thus considered here as a case of *coincidence of intersections*.

(31.) The *equation of conjugation* (comp. 215, (13.)), which expresses that each of the two points Λ and \mathbf{B} is in the polar plane of the other, is (with the present notations),

$$\text{LXII.} \dots r^2 + S\alpha\beta = 0;$$

the *equal but opposite roots* of LXI., which then exist if the line cuts the sphere, answering here to the well-known *harmonic division* of the *secant line \mathbf{AB}* (comp. 215, (16.)), which thus connects *two conjugate points*.

(32.) In like manner, from the quadratic equation* 216, III., we get this analogous equation,

$$\text{LXIII.} \dots S \frac{\lambda}{\alpha} S \frac{\mu}{\alpha} - S \left(\sqrt{\frac{\lambda}{\beta}} \cdot \sqrt{\frac{\mu}{\beta}} \right) = 1,$$

connecting the vectors λ , μ of any two points \mathbf{L} , \mathbf{M} , which are *conjugate relatively to the ellipsoid* 216, II.; and if we place the point \mathbf{L} on *the surface*, the equation LXIII. will represent the *tangent plane at that point \mathbf{L}* , considered as the *locus of the conjugate point \mathbf{M}* ; whence it is easy to deduce the *normal*, at any point of the ellipsoid. But all researches respecting *normals to surfaces* can be better conducted, in connexion with the *Differential Calculus of Quaternions*, to which we shall next proceed.

(33.) It may however be added here, as regards *Powers of Quaternions with scalar exponents* (11.), that the symbol $q^t q^{-t}$ represents a quaternion formed from r , by a conical rotation of its axis round that of q , through an angle $= 2t \angle q$; and that both members of the equation,

$$\text{LXIV.} \dots (qrq^{-1})^t = qr^t q^{-1},$$

are symbols of one common quaternion.

* Corrected as in the first Note to page 298.

CHAPTER II.

ON DIFFERENTIALS AND DEVELOPMENTS OF FUNCTIONS OF QUATERNIONS; AND ON SOME APPLICATIONS OF QUATERNIONS, TO GEOMETRICAL AND PHYSICAL QUESTIONS.

SECTION I.—*On the Definition of Simultaneous Differentials.*

317. IN the foregoing Chapter of the present Book, and in several parts of the Book preceding it, we have taken occasion to exhibit, as we went along, a considerable variety of *Examples*, of the *Geometrical Application of Quaternions*: but these have been given, chiefly as assisting to impress on the reader the *meanings of new notations*, or of *new combinations of symbols*, when such presented themselves in turn to our notice. In this concluding Chapter, we desire to offer a few *additional examples*, of the same *geometrical* kind, but dealing, more freely than before, with *tangents* and *normals* to *curves* and *surfaces*; and to give at least some *specimens*, of the application of quaternions to *Physical Inquiries*. But it seems necessary that we should first establish here some *Principles*, and some *Notations*, respecting *Differentials of Quaternions*, and of their *Functions*, generally.

318. The *usual definitions*, of *differential coefficients*, and of *derived functions*, are found to be inapplicable generally to the present Calculus, on account of the (generally) *non-commutative* character of quaternion-multiplication (168, 191). It becomes, therefore, necessary to have recourse to a *new Definition of Differentials*, which yet ought to be so framed, as to be *consistent with*, and to *include*, the *usual Rules of Differentiation*: because *scalars* (131), as well as *vectors* (292), have been seen to be *included*, under the general *Conception of Quaternions*.

319. In seeking for such a new definition, it is natural to

go back to the first principles of the whole subject of Differentials: and to consider how the great Inventor of *Fluxions* might be supposed to have dealt with the question, if he had been *deprived* of that powerful resource of *common calculation*, which is supplied by the *commutative property* of *algebraic multiplication*; or by the familiar equation,

$$xy = yx,$$

considered as a *general* one, or as subsisting for *every pair of factors*, x and y ; while *limits* should still be *allowed*, but *infinitesimals* be still *excluded*: and indeed the *fluxions themselves* should be regarded as *generally finite*,* according to what seems to have been the ultimate *view* of NEWTON.

320. The answer to this question, which a study of the Principia appears to suggest, is contained in the following *Definition*, which we believe to be a perfectly general one, as regards the *older Calculus*, and which we propose to *adopt* for Quaternions:—

“*Simultaneous Differentials* (or *Corresponding Fluxions*) are *Limits of Equimultiples* † of *Simultaneous and Decreasing Differences*.”

* Compare the remarks annexed to the Second Lemma of the Second Book of the Principia (Third Edition, London, 1726); and especially the following passage (page 244):

“Neque enim spectatur in hoc Lemmate magnitudo momentorum, sed prima nascentium proportio. Eodem recidit si loco momentorum usurpentur vel velocitates incrementorum ac decrementorum (quas etiam motus, mutationes et fluxiones quantitatum nominare licet) vel finitæ quævis quantitates velocitatibus hisce proportionales.”

† As regards the notion of *multiplying* such *differences*, or generally any quantities which all *diminish together*, in order to render their *ultimate relations* more evident, it may be suggested by various parts of the *Principia* of Sir Isaac Newton; but especially by the First Section of the First Book. See for example the Seventh Lemma (p. 81), under which such expressions as the following occur: “intelligantur semper AB et AD ad puncta longinqua b et d producti,” . . . “ideoque rectæ semper finitæ Ab , Ad , . . .” The direction, “ad puncta longinqua producti,” is repeated in connexion with the Eighth and Ninth Lemmas of the same Book and Section; while under the former of those two Lemmas we meet the expression, “triangula semper finita,” applied to the *magnified representations of three triangles*, which all *diminish indefinitely together*: and under the latter Lemma the words occur, “manente longitudine Ae ,” where Ae is a *finite and constant line*, obtained by a *constantly increasing multiplication* of a *constantly diminishing line* AE (page 83 of the edition cited).

And conversely, whenever any *simultaneous differences*, of any system of variables, all *tend to vanish together*, according to any *law*, or system of laws; then, if any *equimultiples* of those decreasing differences all *tend together* to any system of finite *limits*, those *Limits* are *said to be Simultaneous Differentials* of the related *Variables* of the *System*; and are *denoted*, as such, by prefixing the *letter d*, as a *characteristic of differentiation*, to the *Symbol* of each such *variable*.

321. More fully and symbolically, let

$$\text{I. . . } q, r, s, \dots$$

denote *any system of connected variables* (quaternions or others); and let

$$\text{II. . . } \Delta q, \Delta r, \Delta s, \dots$$

denote, as usual, a system of their *connected* (or *simultaneous*) *differences*; in such a manner that the sums,

$$\text{III. . . } q + \Delta q, \quad r + \Delta r, \quad s + \Delta s, \dots$$

shall be a *new system of variables*, satisfying the *same laws of connexion*, whatever they may be, as those which are satisfied by the *old system I*. Then, in *returning gradually* from the new system to the old one, or in *proceeding gradually* from the old to the new, the *simultaneous differences II*. can all be made (in general) to *approach together to zero*, since it is evident that they *may all vanish together*. But *if*, while the *differences themselves* are thus supposed to *decrease* indefinitely together*, we *multiply them all* by some one *common but increasing number, n*, the *system of their equimultiples*,

$$\text{IV. . . } n\Delta q, \quad n\Delta r, \quad n\Delta s, \dots$$

may tend to become equal to some determined *system of finite limits*. And *when this happens*, as in all ordinary cases it may be *made to do*, by a *suitable adjustment* of the *increase* of *n* to the *decrease* of Δq , &c., the *limits* thus obtained are *said to be simultaneous differentials* of the related variables, *q, r, s*; and are *denoted*, as such, by the symbols,

$$\text{V. . . } dq, \quad dr, \quad ds, \dots$$

* A *quaternion* may be said to *decrease*, when its *tensor* decreases; and to *decrease indefinitely*, when that *tensor tends to zero*.

SECTION 2.—*Elementary Illustrations of the Definition, from Algebra and Geometry.*

322. To leave no possible doubt, or obscurity, on the import of the foregoing *Definition*, we shall here apply it to determine the *differential of a square*, in algebra, and that of a *rectangle*, in geometry; in doing which we shall show, that while for such cases the *old rules* are reproduced, the *differentials* treated of need not be small; and that it would be a *vitiatio*, and not a *correction*, of the results, if any additional terms were introduced into their expressions, for the purpose of rendering *all the differentials equal* to the corresponding differences: though some of them may be assumed to be so, namely, in the first Example, *one*, and in the second Example, *two*.

(1.) In Algebra, then, let us consider the equation,

$$\text{I. . . } y = x^2,$$

which gives,

$$\text{II. . . } y + \Delta y = (x + \Delta x)^2,$$

and therefore, as usual,*

$$\text{III. . . } \Delta y = 2x\Delta x + \Delta x^2;$$

or what comes to the same thing,

$$\text{IV. . . } \pi \Delta y = 2x\pi \Delta x + \pi^{-1}(\pi \Delta x)^2,$$

where π is an *arbitrary multiplier*, which may be supposed, for simplicity, to be a positive whole number.

(2.) Conceive now that while the differences Δx and Δy , remaining always connected with each other and with x by the equation III., decrease, and tend together to zero, the number π increases, in the transformed equation IV., and tends to infinity, in such a manner that the product, or multiple, $\pi \Delta x$, tends to some finite limit a ; which may happen, for example, by our obliging Δx to satisfy always the condition,

$$\text{V. . . } \Delta x = \pi^{-1}a, \text{ or } \pi \Delta x = a,$$

after a previous selection of some given and finite value for a .

* We write here, as is common, Δx^2 to denote $(\Delta x)^2$; while $\Delta . x^2$ would be written, on the same known plan, for $\Delta(x^2)$, or Δy . In like manner we shall write dx^2 , as usual, for $(dx)^2$; and shall denote $d(x^2)$ by $d . x^2$. Compare the notations Sq^2 , $S . q^2$, and Vq^2 , $V . q^2$, in 199 and 204.

(3.) We shall then have, with this last condition V., the following expression by IV., for the equimultiple $n\Delta y$, of the other difference, Δy :

$$\text{VI.} \dots n\Delta y = 2ax + n^{-1}a^2 = b + n^{-1}a^2, \text{ if } b = 2ax.$$

But because a , and therefore a^2 , is given and finite, (2.), while the number n increases indefinitely, the term $n^{-1}a^2$, in this expression VI. for $n\Delta y$, indefinitely tends to zero, and its limit is rigorously null. Hence the two finite quantities, a and b (since x is supposed to be finite), are two simultaneous limits, to which, under the supposed conditions, the two equimultiples, $n\Delta x$ and $n\Delta y$, tend;* they are, therefore, by the definition (320), simultaneous differentials of x and y : and we may write accordingly (321),

$$\text{VII.} \dots dx = a, \quad dy = b = 2ax;$$

or, as usual, after elimination of a ,

$$\text{VIII.} \dots dy = d.x^2 = 2xdx.$$

(4.) And it would not improve, but vitiate, according to the adopted definition (320), this usual expression for the differential of the square of a variable x in algebra, if we were to add to it the term dx^2 , in imitation of the formula III. for the difference $\Delta.x^2$. For this would come to supposing that, for a given and finite value, a , of dx , or of $n\Delta x$, the terms $n^{-1}a^2$, or $n^{-1}dx^2$, in the expression VI. for $n\Delta y$, could fail to tend to zero, while the number, n , by which the square of dx is divided, increases without limit, or tends (as above) to infinity.

(5.) As an arithmetical example, let there be the given values,

$$\text{IX.} \dots x = 2, \quad y = x^2 = 4, \quad dx = 1000;$$

and let it be required to compute, as a consequence of the definition (320), the arithmetical value of the simultaneous differential, dy . We have now the following equimultiples of simultaneous differences,

$$\text{X.} \dots n\Delta x = dx = 1000; \quad n\Delta y = 4000 + 1000000n^{-1};$$

but the limit of the n^{th} part of a million (or of any greater, but given and finite number) is exactly zero, if n increase without limit; the required value of dy is, therefore, rigorously, in this example,

$$\text{XI.} \dots dy = 4000.$$

(6.) And we see that these two simultaneous differentials,

$$\text{XII.} \dots dx = 1000, \quad dy = 4000,$$

are not, in this example, even approximately equal to the two simultaneous differences,

$$\text{XIII.} \dots \Delta x = dx = 1000, \quad \Delta y = 1002^2 - 2^2 = 1004000,$$

which answer to the value $n = 1$; although, no doubt, from the very conception of simultaneous differentials, as embodied in the definition (320), they must admit of having such equimultiples of themselves taken,

$$\text{XIV.} \dots n^{-1}dx \text{ and } n^{-1}dy,$$

* In this case, indeed, the multiple $n\Delta x$ has by V. a constant value, namely a ; but it is found convenient to extend the use of the word, limit, so as to include the case of constants: or to say, generally, that a constant is its own limit.

as to be *nearly equal*, for *large values of the number n*, to some system of simultaneous and *decreasing differences*,

$$\text{XV.} \dots \Delta x \text{ and } \Delta y;$$

and *more and more nearly equal* to such a system, even in the way of ratio, as they all become *smaller and smaller together*, and *tend together to vanish*.

(7.) For example, while the *differentials themselves* retain the *constant values XII.*, their *millionth parts* are, respectively,

$$\text{XVI.} \dots n^{-1}dx = 0.001, \text{ and } n^{-1}dy = 0.004, \text{ if } n = 1000000;$$

and the same value of the number *n* gives, by X., the equally rigorous values of two simultaneous differences, as follows,

$$\text{XVII.} \dots \Delta x = 0.001, \text{ and } \Delta y = 0.004001;$$

so that *these values of the decreasing differences XV.* may already be considered to be *nearly equal* to the two *equisubmultiples, XIV. or XVI.*, of the two simultaneous differentials, XII. And it is evident that *this approximation would be improved*, by taking *higher values of the number, n*, without the rigorous and constant values XII., of *dx* and *dy*, being at all affected thereby.

(8.) It is, however, evident also, that after assuming $y = x^2$, and $x = 2$, as in IX., we might have assumed any other finite value for the differential *dx*, instead of the value 1000; and should then have deduced a different (but still finite) value for the other differential, *dy*, and not the formerly deduced value, 4000: but there would always exist, in this example, or for this form of the function, *y*, and for this value of the variable, *x*, the rigorous relation between the two simultaneous differentials, *dx* and *dy*,

$$\text{XVIII.} \dots dy = 4dx,$$

which is obviously a case of the equation VIII., and can be proved by similar reasonings.

323. Proceeding to the promised *Example from Geometry* (322), we shall again see that differences and differentials are not in general to be *confounded* with each other, and that the latter (like the former) *need not be small*. But we shall also see that the *differentials* (like the differences), which enter into a statement of relation, or into the enunciation of a proposition; respecting quantities which *vary together*, according to any law or laws, *need not even be homogeneous among themselves*: it being sufficient that each separately should be homogeneous with the variable to which it corresponds, and of which it is the differential, as line of line, or area of area. It will also be seen that the definition (320) enables us to construct the differential of a rectangle, as the sum of two other (finite) rectangles, without any reference to units of length, or of area, and without even the thought of employing any numerical calculation whatever.

(1.) Let, then, as in the annexed Figure 74, $ABCD$ be any given rectangle, and let BE and DG be any arbitrary but given and finite increments of its sides, AB and AD . Complete the increased rectangle $GAEF$, or briefly AF , which will thus exceed the given rectangle AC , or CA , by the sum of the three partial rectangles, CE , CF , CG ; or by what we may call the *gnomon*,* $CBEFGDC$. On the diagonal CF take a point I , so that the line CI may be any arbitrarily selected submultiple of that diagonal; and draw through I , as in the Figure, lines HM , KL , parallel to the sides AD , AB ; and therefore intercepting, on the sides AB , AD prolonged, equisubmultiples BH , DK of the two given increments, BE , DG , of those two given sides.

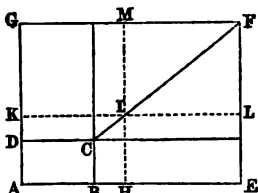


Fig. 74.

(2.) Conceive now that, in this construction, the point I approaches to C , or that we take a series of new points I , on the given diagonal CF , nearer and nearer to the given point C , by taking the line CI successively a smaller and smaller part of that diagonal. Then the two new linear intervals, BH , DK , and the new *gnomon*, $CBHIKDC$, or the sum of the three new partial rectangles, CH , CI , CK , will all indefinitely decrease, and will tend to vanish together: remaining, however, always a system of three simultaneous differences (or increments), of the two given sides, AB , AD , and of the given area, or rectangle, AC .

(3.) But the given increments, BE and DG , of the two given sides, are always (by the construction) equimultiples of the two first of the three new and decreasing differences; they may, therefore, by the definition (820), be arbitrarily taken as two simultaneous differentials of the two sides, AB and AD , provided that we then treat, as the corresponding or simultaneous differential of the rectangle AC , the limit of the equimultiple of the new *gnomon* (2.), or of the decreasing difference between the two rectangles, AC and AI , whereof the first is given.

(4.) We are then, first, to increase this new *gnomon*, or the difference of AC , AI , or the sum (2.) of the three partial rectangles, CH , CI , CK , in the ratio of BE to BH , or of DG to DK ; and secondly, to seek the limit of the area so increased. For this last limit will, by the definition (820), be exactly and rigorously equal to the sought differential of the rectangle AC ; if the given and finite increments, BE and DG , be assumed (as by (3.) they may) to be the differentials of the sides, AB , AD .

(5.) Now when we thus increase the two new partial rectangles, CH and CK , we get precisely the two old partial rectangles, CE and CG ; which, as being given and constant, must be considered to be their own limits.† But when we increase, in the same ratio, the other new partial rectangle CI , we do not recover the old partial rectangle CF , corresponding to it; but obtain the new rectangle CL , or the equal rectangle CM , which is not constant, but diminishes indefinitely as the point I approaches to C ; in such a manner that the limit of the area, of this new rectangle CL or CM , is rigorously null.

* The word, *gnomon*, is here used with a slightly more extended signification, than in the Second Book of Euclid.

† Compare the Note to page 895.

(6.) If, then, the given increments, BE, DG, be still assumed to be the differentials of the given sides AB, AD (an assumption which has been seen to be permitted), the differential of the given area, or rectangle, AC, is proved (not assumed) to be, as a necessary consequence of the definition (320), exactly and rigorously equal to the sum of the two partial rectangles CE and CG; because such is the limit (5.) of the multiple of the new gnomon (2.), in the construction.

(7.) And if any one were to suppose that he could improve this known value for the differential of a rectangle, by adding to it the rectangle CE, as a new term, or part, so as to make it equal to the old or given gnomon (1.), he would (the definition being granted) commit a geometrical error, equivalent to that of supposing that the two similar rectangles CI and CE, bear to each other the simple ratio, instead of bearing (as they do) the duplicate ratio, of their homologous sides.

SECTION 3.—On some general Consequences of the Definition.

324. Let there be any proposed equation of the form,

$$I. \dots Q = F(q, r, \dots);$$

and let dq, dr, \dots be any assumed (but generally finite) and simultaneous differentials of the variables, q, r, \dots whether scalars, or vectors, or quaternions, on which Q is supposed to depend, by the equation I. Then the corresponding (or simultaneous) differential of their function, Q , is equal (by the definition 320, compare 321) to the following limit:

$$II. \dots dQ = \lim_{n \rightarrow \infty} n \{ F(q + n^{-1}dq, r + n^{-1}dr, \dots) - F(q, r, \dots) \};$$

where n is any whole number (or other positive* scalar) which, as the formula expresses, is conceived to become indefinitely greater and greater, and so to tend to infinity. And if, in particular, we consider the function Q as involving only one variable q , so that

$$III. \dots Q = f(q) = fq,$$

then

$$IV. \dots dQ = dfq = \lim_{n \rightarrow \infty} n \{ f(q + n^{-1}dq) - fq \};$$

a formula for the differential of a single explicit function of a single variable, which agrees perfectly with those given, near the end of the First Book, for the differentials of a vector, and of a scalar, considered each as a function (100) of a single sca-

* Except in some rare cases of discontinuity, not at present under our consideration, this scalar n may as well be conceived to tend to negative infinity.

lar variable, t : but which is now extended, as a consequence of the general definition (320), to the case when the connected variables, q , Q , and their differentials, dq , dQ , are quaternions: with an analogous application, of the still more general Formula of Differentiation II., to Functions of several Quaternions.

(1.) As an example of the use of the formula IV., let the function of q be its square, so that

$$V. \dots Q = fq = q^2.$$

Then, by the formula,

$$VI. \dots dQ = dfq = \lim_{n \rightarrow \infty} n \{ (q + n^{-1} dq)^n - q^n \} \\ = \lim_{n \rightarrow \infty} (q \cdot dq + dq \cdot q + n^{-1} dq^2),$$

where dq^2 signifies* the square of dq ; that is,

$$VII. \dots d \cdot q^2 = q \cdot dq + dq \cdot q;$$

or without the points† between q and dq ,

$$VII'. \dots d \cdot q^2 = qdq + dq q;$$

an expression for the differential of the square of a quaternion, which does not in general admit of any further reduction: because q and dq are not generally commutative, as factors in multiplication. When, however, it happens, as in algebra, that $q \cdot dq = dq \cdot q$, by the two quaternions q and dq being coplanar, the expression VII. then evidently reproduces the usual form, §22, VIII., or becomes,

$$VIII. \dots d \cdot q^2 = 2q dq, \text{ if } dq \parallel q \text{ (128).}$$

(2.) As another example, let the function be the reciprocal,

$$IX. \dots Q = fq = q^{-1}.$$

Then, because

$$X. \dots f(q + n^{-1}dq) - fq = (q + n^{-1}dq)^{-1} - q^{-1} \\ = (q + n^{-1}dq)^{-1} \{ q - (q + n^{-1}dq) \} q^{-1} \\ = -n^{-1}(q + n^{-1}dq)^{-1} \cdot dq \cdot q^{-1},$$

of which, when multiplied by n , the limit is $-q^{-1}dq \cdot q^{-1}$, we have the following expression for the differential of the reciprocal of a quaternion,

$$XI. \dots d \cdot q^{-1} = -q^{-1} \cdot dq \cdot q^{-1};$$

* Compare the Note to page 394.

† The point between d and q^2 , in the first member of VII., is indispensable, to distinguish the differential of the square from the square of the differential. But just as this latter square is denoted briefly by dq^2 , so the products, $q \cdot dq$ and $dq \cdot q$, may be written as qdq and $dq q$; the symbol, dq , being thus treated as a whole one, or as if it were a single letter. Yet, for greater clearness of expression, we shall retain the point between q and dq , in several (though not in all) of the subsequent formulae, leaving it to the student to omit it, at his pleasure.

or without the *points** in the second member, dq being treated (as in VII.) as a whole symbol,

$$\text{XI.} \dots d.q^{-1} = -q^{-1}dq q^{-1};$$

an expression which does not generally admit of being any farther reduced, but becomes, as in the ordinary calculus,

$$\text{XII.} \dots d.q^{-1} = -q^{-2}dq, \text{ if } dq \parallel q,$$

that is, for the case of *complanarity*, of the quaternion and its differential.

325. Other *Examples of Quaternion Differentiation* will be given in the following Section; but the two foregoing may serve sufficiently to exhibit the nature of the operation, and to show the *analogy* of its results to those of the older Calculus, while exemplifying also the *distinction* which generally exists between them. And we shall here proceed to explain a *notation*, which (at least in the *statement* of the present theory of differentials) appears to possess some advantages; and will enable us to offer a still more brief *symbolical definition*, of the *differential of a function* fq , than before.

(1.) We have defined (320, 324), that if dq be called the *differential* of a (quaternion or other) *variable*, q , then the *limit of the multiple*,

$$\text{I.} \dots n \{f(q + n^{-1}dq) - fq\},$$

of an indefinitely decreasing difference of the function fq , of that (single) variable q , when taken relatively to an indefinite increase of the multiplying number, n , is the corresponding or simultaneous *differential of that function*, and is denoted, as such, by the symbol dfq .

(2.) But before we thus pass to the *limit*, relatively to n , and while that multiplier, n , is still considered and treated as *finite*, the multiple I. is evidently a *function of that number, n , as well as of the two independent variables, q and dq* . And we propose to denote (at least for the present) *this new function of the three variables*,

$$\text{II.} \dots n, q, \text{ and } dq,$$

of which the form depends, according to the law expressed by the formula I., on the form of the given function, f , by the new symbol,

$$\text{III.} \dots f_n(q, dq);$$

in such a manner as to write, for any two variables, q and q' , and any number, n , the equation,

$$\text{IV.} \dots f_n(q, q') = n \{f(q + n^{-1}q') - fq\};$$

which may obviously be also written thus,

$$\text{V.} \dots f(q + n^{-1}q') = fq + n^{-1}f_n(q, q'),$$

and is here regarded as *rigorously exact*, in virtue of the definitions, and without anything whatever being neglected, as *small*.

* Compare the Note immediately preceding.

(3.) For example, it appears from the little calculation in 324, (1.), that,

$$\text{VI.} \dots f_n(q, q') = qq' + q'q + \pi^{-1}q^2, \text{ if } fq = q^2;$$

and from 324, (2.), that,

$$\text{VII.} \dots f_n(q, q') = -(q + \pi^{-1}q')^{-1}q'q^{-1}, \text{ if } fq = q^{-1}.$$

(4.) And the definition of dfq may now be briefly thus expressed :

$$\text{VIII.} \dots dfq = f_\infty(q, dq);$$

or, if the *sub-index* ∞ be understood, we may write, still more simply,

$$\text{IX.} \dots dfq = f(q, dq);$$

this last expression, $f(q, dq)$, or $f(q, q')$, denoting thus a *function of two independent variables*, q and q' , of which the form is derived* or deduced (comp. (2.)), from the given or proposed form of the function f_q of a single variable, q , according to a law which it is one of the main objects of the *Differential Calculus* (at least as regards Quaternions) to study.

326. One of the most important *general properties*, of the functions of this class $f(q, q')$, is that they are all *distributive* with respect to the *second independent variable*, q' , which is introduced in the foregoing process of what we have called *derivation*,† from some given function f_q , of a single variable, q : a theorem which may be proved as follows, whether the two independent variables be, or be not, quaternions.

(1.) Let q'' be any *third* independent variable, and let π be any number; then the formula 325, V. gives the three following equations, resulting from the law of derivation of $f_n(q, q')$ from f_q :

$$\text{I.} \dots f(q + \pi^{-1}q'') = fq + \pi^{-1}f_n(q, q'');$$

$$\text{II.} \dots f(q + \pi^{-1}q'' + \pi^{-1}q') = f(q + \pi^{-1}q'') + \pi^{-1}f_n(q + \pi^{-1}q'', q');$$

$$\text{III.} \dots f(q + \pi^{-1}q' + \pi^{-1}q'') = fq + \pi^{-1}f_n(q, q' + q'');$$

* It was remarked, or hinted, in 318, that the usual definition of a derived function, namely, that given by Lagrange in the *Calcul des Fonctions*, cannot be taken as a *foundation* for a differential calculus of quaternions: although such derived functions of scalars present themselves occasionally in the applications of that calculus, as in 100, (3.) and (4.), and in some analogous but more general cases, which will be noticed soon. The present Law of Derivation is of an entirely different kind, since it conducts, as we see, from a given function of one variable, to a derived function of two variables, which are in general independent of each other. The function $f_n(q, q')$, of the three variables, π, q, q' , may also be called a derived function, since it is deduced, by the fixed law IV., from the same given function f_q , although it has in general a less simple form than its own limit, $f_\infty(q, q')$, or $f(q, q')$.

† Compare the Note immediately preceding.

by comparing which we see at once that

$$\text{IV.} \dots f_n(q, q' + q'') = f_n(q + \pi^{-1}q'', q') + f_n(q, q'),$$

the form of the original function, f_q , and the values of the four variables, q, q', q'' , and π , remaining altogether arbitrary: except that π is supposed to be a number, or at least a scalar, while q, q', q'' may (or may not) be quaternions.

(2.) For example, if we take the particular function $f_q = q^2$, which gives the form 325, VI. of the derived function $f_n(q, q')$, we have

$$\begin{aligned} \text{V.} \dots f_n(q, q'') &= qq'' + q''q + \pi^{-1}q''^2; \\ \text{VI.} \dots f_n(q, q' + q'') &= q(q' + q'') + (q' + q'')q + \pi^{-1}(q' + q'')^2; \end{aligned}$$

and therefore

$$\begin{aligned} \text{VII.} \dots f_n(q, q' + q'') - f_n(q, q'') &= qq' + q'q + \pi^{-1}(q'^2 + q'q' + q'q) \\ &= (q + \pi^{-1}q'')q' + q'(q + \pi^{-1}q'') + \pi^{-1}q'^2 \\ &= f_n(q + \pi^{-1}q'', q), \end{aligned}$$

as required by the formula IV.

(3.) Admitting then that formula as proved, for all values of the number π , we have only to conceive that number (or scalar) to tend to infinity, in order to deduce this limiting form of the equation:

$$\text{VIII.} \dots f_\infty(q, q' + q'') = f_\infty(q, q') + f_\infty(q, q'');$$

or simply, with the abridged notation of 325, (4.),

$$\text{IX.} \dots f(q, q' + q'') = f(q, q') + f(q, q'');$$

which contains the expression of the functional property, above asserted to exist.

(4.) For example, by what has been already shown (comp. 325, (3.) and (4.)),

$$\begin{aligned} \text{X.} \dots \text{if } f_q = q^2, \text{ then } f(q, q) &= qq' + q'q; \\ \text{and XI.} \dots \text{if } f_q = q^{-1}, \text{ then } f(q, q') &= -q^{-1}q'q^{-1}; \end{aligned}$$

in each of which instances we see that the derived function $f(q, q')$ is distributive relatively to q' , although it is only in the first of them that it happens to be distributive with respect to q also.

(5.) It follows at once from the formula IX. that we have generally*

$$\text{XII.} \dots f(q, 0) = 0;$$

and it is not difficult to prove, as a result including this, that

$$\text{XIII.} \dots f(q, \alpha q') = \alpha f(q, q'), \text{ if } \alpha \text{ be any scalar.}$$

(6.) As a confirmation of this last result, we may observe that the definition of $f(q, q')$ may be expressed by the following formula (comp. 324, IV., and 325, IX.):

$$\text{XIV.} \dots f(q, q') = \lim_{\pi \rightarrow \infty} \pi \{ f(q + \pi^{-1}q') - f_q \};$$

we have therefore, if α be any finite scalar, and $m = \alpha^{-1}\pi$,

$$\text{XV.} \dots f(q, \alpha q') = \alpha \lim_{m \rightarrow \infty} m \{ f(q + m^{-1}q') - f_q \};$$

a transformation which gives the recent property XIII., since it is evident that the letter m may be written instead of π , in the formula of definition XIV.

* We abstract here from some exceptional cases of discontinuity, &c.

327. Resuming then the general expression 325, IX., or writing anew,

$$\text{I. . . } dfq = f(q, dq),$$

we see (by 326, IX.) that *this derived function*, dfq , of q and dq , is always (as in the examples 324, VII. and XI.) *distributive with respect to that differential* dq , considered as an *independent variable*, whatever the *form* of the *given function* fq may be. We see also (by 326, XIII.), that *if the differential* dq *of the variable*, q , *be multiplied by any scalar*, x , *the differential* dfq , *of the function* fq , *comes to be multiplied*, at the same time, *by the same scalar*, or that

$$\text{II. . . } f(q, xdq) = xf(q, dq), \text{ if } x \text{ be any scalar.}$$

And in fact it is evident, from the very *conception and definition* (320) of *simultaneous differentials*, that every *system of such differentials* must admit of being *all changed together* to any system of *equimultiples*, or *equisubmultiples*, of themselves, *without ceasing* to be *simultaneous differentials*: or more generally, that it is *permitted to multiply all the differentials*, of a *system*, *by any common scalar*.

(1.) It follows that the *quotient*,

$$\text{III. . . } dfq : dq = f(q, dq) : dq,$$

of the *two simultaneous differentials*, dfq and dq , *does not change* when the differential dq is thus *multiplied by any scalar*; and consequently that this *quotient* III. is *independent of the tensor* Tdq , although it is *not generally independent of the versor* Udq , if q and dq be *quaternions*: except that it remains in general *unchanged*, when we merely change that *versor* to its own *opposite* (or *negative*), or to $-Udq$, because this comes to multiplying dq by -1 , which is a scalar.

(2.) For example, the *quotient*,

$$\text{IV. . . } d.q^2 : dq = q + dq.q.dq^{-1} = q + Udq.q.Udq^{-1},$$

in which dq^{-1} and Udq^{-1} denote the reciprocals of dq and Udq , is very far from being independent of dq , or at least of Udq ; since it represents, as we see, the *sum* of the *given quaternion* q , and of a certain *other quaternion*, which latter, in its *geometrical interpretation* (comp. 191, (5.)), may be considered as being *derived* from q , by a *conical rotation* of $Ax.q$ round $Ax.dq$, through an *angle* $= 2\angle dq$: so that both the *axis* and the *quantity* of this rotation *depend* on the *versor* Udq , and *vary with that versor*.

(3.) In general we may, if we please, say that the *quotient* III. is a *Differential Quotient*; but we ought not to call it a *Differential Coefficient* (comp. 818), because dfq does *not* generally admit of *decomposition into two factors*, whereof one shall be the *differential* dq , and the other a *function of* q *alone*.

(4.) And for the same reason, we ought not to call that *Quotient* a *Derived Function* (comp. again §18), unless in so speaking we understand a *Function of Two* independent Variables*, namely of q and Udq , as before.

(5.) When, however, a *quaternion*, q , is considered as a *function of a scalar variable*, t , so that we have an equation of the form,

$$V. \dots q = ft, \text{ where } t \text{ denotes a scalar,}$$

it is then permitted (comp. 100, (8.)) and (4.) to write,

$$\begin{aligned} VI. \dots dq : dt = df : dt &= \lim_{n \rightarrow \infty} \frac{n}{n} \left\{ f \left(t + \frac{dt}{n} \right) - ft \right\} \\ &= \lim_{h \rightarrow 0} h^{-1} \{ f(t+h) - ft \} \\ &= f't = D_t ft = D_t q; \end{aligned}$$

and to call this limit, as usual, a *derived function of t* , because it is (in fact) a *function of that scalar variable, t , alone*, and is independent of the scalar differential, dt .

(6.) We may also write, under these circumstances, the *differential equation*,

$$VII. \dots dq = D_t q \cdot dt, \text{ or } VIII. \dots dfq = f't \cdot dt,$$

and may call the *derived quaternion*, $D_t q$, or $f't$, as usual, a *differential coefficient in this formula*, because the *scalar differential, dt* , is (in fact) *multiplied by it*, in the expression thus found for the *quaternion differential, dq or dfq* .

(7.) But as regards the *logic* of the question (comp. again 100, (3.)), it is important to remember that we regard this *derived function*, or *differential coefficient*,

$$IX. \dots f't, \text{ or } D_t ft, \text{ or } D_t q,$$

as being an *actual quotient* VI., obtained by *dividing an actual quaternion*,

$$X. \dots dfq, \text{ or } dq,$$

by an *actual scalar, dt* , of which the *value* is altogether *arbitrary*, and may (if we choose) be supposed to be *large* (comp. 822); while the *dividend quaternion* X. *depends, for its value, on the values of the two independent scalars, t and dt , and on the form of the function ft* , according to the *law* which is expressed by the *general formula* 824, IV., for the *differentiation of explicit functions of any single variable*.

328. It is easy to conceive that similar remarks apply to *quaternion functions of more variables than one*; and that when the *differential of such a function* is expressed (comp. 324, II.) under the form,

$$I. \dots dQ = dF(q, r, s, \dots) = F(q, r, s, \dots) dq, dr, ds, \dots,$$

the *new function F* is always *distributive*, with respect to *each* separately of the *differentials, dq, dr, ds, \dots* ; being also *homogeneous of the first dimension* (comp. 327), with respect to *all* those *differentials*, considered as a *system*; in such a manner

* Compare the Note to 325, (4.).

that, whatever may be the form of the given quaternion function, Q , or F , the derived* function F , or the third member of the formula I., must possess this general functional property (comp. 326, XIII., and 327, II.),

$$\text{II.} \dots F(q, r, s, \dots x dq, x dr, x ds \dots) \\ = x F(q, r, s, \dots dq, dr, ds, \dots),$$

where x may be any scalar: so that products, as well as squares, of the differentials $dq, dr, \&c.$, of $q, r, \&c.$ considered as so many variables on which Q depends, are excluded from the expanded expression of the differential dQ of the function Q .

(1.) For example, if the function to be differentiated be a product of two quaternions,

$$\text{III.} \dots Q = F(q, r) = qr,$$

then it is easily found from the general formula 824, II., that (because the limit of $n^{-1} \cdot dq \cdot dr$ is null, when the number n increases without limit) the differential of the function is,

$$\text{IV.} \dots dQ = d \cdot qr = dF(q, r) = F(q, r, dq, dr) = q \cdot dr + dq \cdot r;$$

with analogous results, for differentials of products of more than two quaternions.

(2.) Again, if we take this other function,

$$\text{V.} \dots Q = F(q, r) = q^{-1}r,$$

then, applying the same general formula 824, II., and observing that we have, for all values of the number (or other scalar), n , and of the four quaternions, q, r, q', r' , the identical transformation (comp. 824, (2.)),

$$\forall \text{I.} \dots n \{ (q + n^{-1}q')^{-1} (r + n^{-1}r') - q^{-1}r \} \\ = q^{-1}r' - (q + n^{-1}q')^{-1} q' q^{-1} (r + n^{-1}r'),$$

we find, as the required limit, when n tends to infinity, the following differential of the function :

$$\text{VII.} \dots dQ = d \cdot q^{-1}r = dF(q, r) = F(q, r, dq, dr) = q^{-1} \cdot dr - q^{-1} \cdot dq \cdot q^{-1}r;$$

which is again, like the expression IV., distributive with respect to each of the differentials dq, dr , of the variables q, r , and does not involve the product of those two differentials: although these two differential expressions, IV. and VII., are both entirely rigorous, and are not in any way dependent on any supposition that the tensors of dq and dr are small (comp. again 822).

329. In thus differentiating a function of more variables than one, we are led to consider what may be called *Partial Differentials of Functions of two or more Quaternions*; which may be thus denoted,

* Compare the Note last referred to.

$$\text{I. . . } d_q Q, d_r Q, d_s Q, \dots$$

if Q be a function, as above, of q, r, s, \dots which is here supposed to be differentiated with respect to *each* variable *separately*, as if the others were constant. And then, if dQ denote, as before, what may be called, by contrast, the *Total Differential* of the function Q , we shall have the *General Formula*,

$$\text{II. . . } dQ = d_q Q + d_r Q + d_s Q + \dots;$$

or, briefly and symbolically,

$$\text{III. . . } d = d_q + d_r + d_s + \dots,$$

if q, r, s, \dots denote the quaternion variables on which the *quaternion function* depends, of which the total differential is to be taken; whether those *variables* be all *independent*, or be *connected* with each other, by any *relation* or relations.

(1.) For example (comp. 828, (1.)),

$$\text{IV. . . if } Q = qr, \text{ then } d_q Q = dq.r, \text{ and } d_r Q = q.dr;$$

and the *sum* of these *two partial differentials* of Q makes up its *total differential* dQ , as otherwise found above.

(2.) Again (comp. 828, (2.)),

$$\text{V. . . if } Q = q^{-1}r, \text{ then } d_q Q = -q^{-1}dq.q^{-1}r; \quad d_r Q = q^{-1}dr;$$

and $d_q Q + d_r Q =$ the same dQ as that which was otherwise found before, for this form of the function Q .

(3.) To exemplify the possibility of a *relation* existing between the *variables* q and r , let those variables be now supposed *equal* to each other in V.; we shall then have $Q = 1, dQ = 0$; and accordingly we have here $d_q Q = -q^{-1}dq = -d_r Q$.

(4.) Again, in IV., let $qr = c =$ any constant quaternion; we shall then again have $0 = dQ = d_q Q + d_r Q$; and may infer that

$$\text{VI. . . } dr = -q^{-1}.dq.r, \quad \text{if } qr = c = \text{const.};$$

a result which evidently agrees with, and includes, the expression 824, XI., for the *differential of a reciprocal*.

(5.) A *quaternion*, q , may happen to be expressed as a *function of two or more scalar variables*, t, u, \dots ; and then it will have, as such, by the present Article, its *partial differentials*, $d_t q, d_u q, \&c.$ But because, by 827, VII., we may in *this* case write,

$$\text{VII. . . } d_t q = D_t q . dt, \quad d_u q = D_u q . du, \dots$$

where the *coefficients* are *independent* of the *differentials* (as in the ordinary calculus), we shall have (by II.) an expression for the *total differential* dq , of the form,

$$\text{VIII. . . } dq = d_t q + d_u q + \dots = D_t q . dt + D_u q . du + \dots;$$

and may at pleasure say, *under the conditions here supposed*, that the *derived quaternions*,

$$\text{IX.} \dots D_t q, D_w q, \dots$$

are either the *Partial Derivatives*, or the *Partial Differential Coefficients*, of the *Quaternion Function*,

$$\text{X.} \dots q = F(t, u, \dots);$$

with analogous remarks for the case, when the *quaternion*, q , *degenerates* (comp. 289) into a *vector*, ρ .

330. In general, it may be considered as evident, from the definition in 320, that the *differential of a constant is zero*; so that if Q be changed to *any constant quaternion*, c , in the equation 324, I., then dQ is to be *replaced by 0*, in the *differentiated equation*, 324, II. And if there be given any *system of equations*, connecting the *quaternion variables*, q, r, s, \dots we may treat the corresponding *system of differentiated equations*, as holding good, for the *system of simultaneous differentials*, dq, dr, ds, \dots ; and may therefore, legitimately in theory, whenever in practice it shall be found to be possible, *eliminate* any one or more of those *differentials*, between the equations of this system.

(1.) As an example, let there be the two equations,

$$\text{I.} \dots qr = c, \quad \text{and} \quad \text{II.} \dots s = r^2,$$

where c denotes a constant quaternion. Then (comp. 328, (1.), and 324, (1.)) we have the two differentiated equations corresponding,

$$\text{III.} \dots q \cdot dr + dq \cdot r = 0; \quad \text{IV.} \dots ds = r \cdot dr + dr \cdot r;$$

in which the *points** might be omitted. The former gives,

$$\text{V.} \dots dr = -q^{-1}dq \cdot r, \text{ as in 329, VI.};$$

and when we substitute this value in the latter, we thereby *eliminate the differential* dr , and obtain this *new differential equation*,

$$\text{VI.} \dots ds = -rq^{-1} \cdot dq \cdot r - q^{-1} \cdot dq \cdot r^2.$$

(2.) The equation I. gives also the expression,

$$\text{VII.} \dots r = q^{-1}c;$$

the equation II. gives therefore this other expression,

$$\text{VIII.} \dots s = (q^{-1}c)^2 = q^{-1}cq^{-1}c,$$

by *elimination before differentiation*. And if, in the formula VI., we substitute the expressions VII. and VIII. for r and s , we get this *other differential equation*,

* Compare the second Note to 324, (1.).

$$\text{IX.} \dots d.(q^{-1}c)^2 = -q^{-1}cq^{-1}.dq.q^{-1}c - q^{-1}.dq.q^{-1}cq^{-1}c;$$

which might have been otherwise obtained (comp. again 324, (1.) and (2.)), under the form,

$$\text{X.} \dots d.(q^{-1}c)^2 = q^{-1}c.d(q^{-1}c) + d(q^{-1}c).q^{-1}c.$$

331. *No special rules* are required, for the *differentiation of functions of functions* of quaternions; but it may be instructive to show, briefly, how the consideration of *such* differentiation conducts (comp. 326) to a *general property of functions of the class* $f(q, q')$; and how that property can be *otherwise* established.

(1.) Let $f, \phi,$ and ψ denote any functional operators, such that

$$\text{I.} \dots \psi q = \phi(fq);$$

then writing

$$\text{II.} \dots r = fq, \text{ and III.} \dots s = \phi r, \text{ we have IV.} \dots s = \psi q;$$

$$\text{whence V.} \dots ds = d\psi q = d\phi r.$$

That is, we may (as usual) *differentiate the compound function, $\phi(fq)$, as if fq were an independent variable, r ; and then, in the expression so found, replace the differential dfq by its value, obtained by differentiating the simple function, fq . For this comes virtually to the elimination of the differential dr , or of the symbol dfq , in a way which we have seen to be permitted (330).*

(2.) But, by the *definitions* of dfq and $f_n(q, q')$, we saw (325, VIII. IX.) that the differential dfq might generally be denoted by $f_n(q, dq)$, or briefly by $f(q, dq)$; whence $d\phi r$ and $d\psi q$ may also, by an extension of the same notation, be represented by the analogous symbols, $\phi_n(r, dr)$ and $\psi_n(q, dq)$, or simply by $\phi(r, dr)$ and $\psi(q, dq)$.

(3.) We ought, therefore, to find that

$$\text{VI.} \dots \psi_n(q, dq) = \phi_n(fq, f_n(q, dq)), \text{ if } \psi q = \phi(fq);$$

or briefly that

$$\text{VII.} \dots \psi(q, q') = \phi(fq, f(q, q')), \text{ if } \psi q = \phi fq,$$

for any two quaternions, q, q' , and any two functions, f, ϕ ; provided that the functions $f_n(q, q')$, $\phi_n(q, q')$, $\psi_n(q, q')$ are deduced (or derived) from the functions $fq, \phi q, \psi q$, according to the law expressed by the formula 325, IV.; and that then the limits to which these *derived functions* $f_n(q, q')$, &c. tend, when the number n tends to infinity, are denoted by these *other functional symbols*, $f(q, q')$, &c.

(4.) To prove this *otherwise*, or to establish this *general property VII.*, of functions of this class $f(q, q')$, without any use of differentials, we may observe that the general and rigorous transformation 325, V., of the formula 325, IV. by which the functions $f_n(q, q')$ are defined, gives for all values of n the equation :

$$\begin{aligned} \text{VIII.} \dots \phi f(q + n^{-1}q') &= \phi(fq + n^{-1}f_n(q, q')) \\ &= \phi fq + n^{-1}\phi_n(fq, f_n(q, q')); \end{aligned}$$

but also, by the same general transformation,

$$\text{IX.} \dots \psi(q + \pi^{-1}q') = \psi q + \pi^{-1}\psi_n(q, q');$$

hence generally, for all values of the number n , as well as for all values of the two independent quaternions, q, q' , and for all forms of the two functions, f, ϕ , we may write,

$$\text{X.} \dots \psi_n(q, q') = \phi_n(fq, f_n(q, q')), \quad \text{if } \psi q = \phi fq;$$

an equation of which the limiting form, for $n = \infty$, is (with the notations used) the equation VII. which was to be proved.

(5.) It is scarcely worth while to verify the general formula X., by any particular example: yet, merely as an exercise, it may be remarked that if we take the forms,

$$\text{XI.} \dots fq = q^2, \quad \phi q = q^2, \quad \psi q = q^4,$$

of which the two first give, by 325, VI., the common derived form,

$$\text{XII.} \dots f_n(q, q') = \phi_n(q, q') = qq' + q'q + \pi^{-1}q^2,$$

the formula X. becomes,

$$\begin{aligned} \text{XIII.} \dots \psi_n(q, q') &= \phi_n(q^2, qq' + q'q + \pi^{-1}q^2) \\ &= q^2(qq' + q'q + \pi^{-1}q^2) + (qq' + q'q + \pi^{-1}q^2)q^2 + \pi^{-1}(qq' + q'q + \pi^{-1}q^2)^2; \end{aligned}$$

which agrees with the value deduced immediately from the function ψq or q^4 , by the definition 325, IV., namely,

$$\begin{aligned} \text{XIV.} \dots \psi_n(q, q') &= \pi \{ (q + \pi^{-1}q')^4 - q^4 \} \\ &= \pi \{ (q^2 + \pi^{-1}(qq' + q'q + \pi^{-1}q^2))^2 - (q^2)^2 \}. \end{aligned}$$

(6.) In general, the theorem, or rule, for differentiating as in (1.) a function of a function, of a quaternion or other variable, may be briefly and symbolically expressed by the formula,

$$\text{XV.} \dots d(\phi f)q = d\phi(fq);$$

and if we did not otherwise know it, a proof of its correctness would be supplied, by the recent proof of the correctness of the equivalent formula VII.

SECTION 4.—Examples of Quaternion Differentiation.

332. It will now be easy and useful to give a short collection of *Examples of Differentiation of Quaternion Functions and Equations*, additional to and inclusive of those which have incidentally occurred already, in treating of the principles of the subject.

(1.) If c be any constant quaternion (as in 380), then

$$\begin{aligned} \text{I.} \dots dc &= 0; & \text{II.} \dots d(fq + c) &= dfq; \\ \text{III.} \dots d.cfq &= cdfq; & \text{IV.} \dots d(fq.c) &= dfq.c. \end{aligned}$$

(2.) In general,

$$\text{V.} \dots d(fq + \phi q + \dots) = dfq + d\phi q + \dots; \quad \text{or briefly, VI.} \dots d\Sigma = \Sigma d,$$

if Σ be used as a mark of summation.

(3.) Also, VII. $\dots d(fq \cdot \phi q) = dfq \cdot \phi q + fq \cdot d\phi q;$

and similarly for a product of more functions than two: the rule being simply, to differentiate each factor separately, in its own place, or without disturbing the order

of the factors (comp. 818, 819); and then to *add together the partial results* (comp. 829).

(4.) In particular, if m be any positive whole number,

$$\text{VIII.} \dots d.q^m = q^{m-1}dq + q^{m-2}dq.q \dots + qdq.q^{m-2} + dq.q^{m-1};$$

and because we have seen (824, (2.)) that

$$\text{IX.} \dots d.q^{-1} = -q^{-1}.dq.q^{-1},$$

we have this analogous expression for the differential of a power of a quaternion, with a *negative but whole exponent*,

$$\text{X.} \dots d.q^{-m} = -q^{-m}(d.q^m).q^{-m} \\ = -q^{-1}dq.q^{-m} - q^{-2}dq.q^{1-m} \dots - q^{1-m}dq.q^{-2} - q^{-m}dq.q^{-1}.$$

(5.) To *differentiate a square root*, we are to *resolve the linear equation*,*

$$\text{XI.} \dots q^{\frac{1}{2}}.d.q^{\frac{1}{2}} + d.q^{\frac{1}{2}}.q^{\frac{1}{2}} = dq; \quad \text{or} \quad \text{XI.} \dots rr' + r'r = q',$$

if we write, for abridgment,

$$\text{XII.} \dots r = q^{\frac{1}{2}}, \quad q' = dq, \quad r' = d.q^{\frac{1}{2}} = dr.$$

(6.) Writing also, for this purpose,

$$\text{XIII.} \dots s = Kr = K.q^{\frac{1}{2}},$$

whence (by 190, 196) it will follow that

$$\text{XIV.} \dots rs = Nr = Tr^2 = Tq, \quad \text{and} \quad \text{XV.} \dots r + s = 2Sr = 2S.q^{\frac{1}{2}},$$

the *product and sum of these two conjugate quaternions*, r and s , being thus *scalars* (140, 145), we have, by XI.,

$$\text{XVI.} \dots r^{-1}q's = r's + sr'; \quad = r's + r^{-1}(rs) \wedge r' \\ = r's + sr'$$

whence, by addition,

$$\text{XVII.} \dots q' + r^{-1}q's = (r+s)r' + r'(r+s) = 2r'(r+s);$$

and finally,

$$\text{XVIII.} \dots r' = \frac{q' + r^{-1}q's}{2(r+s)}, \quad \text{or} \quad \text{XIX.} \dots d.q^{\frac{1}{2}} = \frac{dq + q^{\frac{1}{2}}dq.K.q^{\frac{1}{2}}}{4S.q^{\frac{1}{2}}}; \quad *$$

an expression for the differential of the square-root of a quaternion, which will be found to admit of many transformations, not needful to be considered here.

(7.) In the three last sub-articles, as in the three preceding them, it has been supposed, for the sake of generality, that q and dq are two *dipplanar quaternions*; but if in any application they *happen*, on the contrary, to be *complanar*, the expressions are then *simplified*, and take *usual, or algebraic forms*, as follows:

$$\text{XX.} \dots d.q^m = mq^{m-1}dq; \quad \text{XXI.} \dots d.q^{-m} = -mq^{m-1}dq;$$

and $\text{XXII.} \dots d.q^{\frac{1}{2}} = \frac{1}{2}q^{-\frac{1}{2}}dq$, if $\text{XXIII.} \dots dq \parallel q$ (125);

* Although such *solution of a linear equation*, or equation of the *first degree*, in quaternions, is easily enough accomplished in the present instance, yet in general the problem presents difficulties, without the consideration of which the *theory of differentiation of implicit functions of quaternions* would be entirely incomplete. But a *general method*, for the solution of *all such equations*, will be sketched in a subsequent Section.

when $q \parallel dq$ $d.q^{\frac{1}{2}} = q^{-\frac{1}{2}} \frac{q^{\frac{1}{2}}dq + Kq^{\frac{1}{2}}dq}{2(q^{\frac{1}{2}} + Kq^{\frac{1}{2}})} = \frac{1}{2}q^{-\frac{1}{2}}dq.$

because, when q' is coplanar with q , and therefore with q^i , or with r , in the expression XVIII., the numerator of that expression may be written as $r^{-1}q'(r+s)$.

(8.) More generally, if x be any scalar exponent, we may write, as in the ordinary calculus, but still under the condition of coplanarity XXIII.,

$$\text{XXIV.} \dots d.q^x = xq^{x-1}dq; \quad \text{or} \quad \text{XXV.} \dots qd.q^x = xq^x dx.$$

333. The functions of quaternions, which have been lately differentiated, may be said to be of algebraic form; the following are a few examples of differentials of what may be called, by contrast, transcendental functions of quaternions: the condition of coplanarity ($dq \parallel q$) being however here supposed to be satisfied, in order that the expressions may not become too complex. In fact, with this simplification, they will be found to assume, for the most part, the known and usual forms, of the ordinary differential calculus.

(1.) Admitting the definitions in 816, and supposing throughout that $dq \parallel q$, we have the usual expressions for the differentials of ϵq and lq , namely,

$$\text{I.} \dots d.\epsilon q = \epsilon dq; \quad \text{II.} \dots dlq = q^{-1}dq.$$

(2.) We have also, by the same system of definitions (816),

$$\text{III.} \dots d \sin q = \cos q dq; \quad \text{IV.} \dots d \cos q = -\sin q dq; \quad \&c.$$

(3.) Also, if r and dr be coplanar with q and dq , then, by 816,

$$\text{IV'.} \dots d.q^r = d.\epsilon^{r^i} q = q^r d.r^i q = q^r (lq dr + q^{-1} r dq);$$

or in the notation of partial differentials (829),

$$\text{V.} \dots d_q.q^r = r q^{r-1} dq, \quad \text{and} \quad \text{VI.} \dots d_r.q^r = q^r lq dr.$$

(4.) In particular, if the base q be a given or constant vector, a , and if the exponent r be a variable scalar, t , then (by the value 816, XIV. of lq) the recent formula IV. becomes,

$$\text{VII.} \dots d.a^t = \left(lT a + \frac{\pi}{2} U a \right) a^t dt.$$

(5.) If then the base a be a given unit line, so that $lT a = 0$, and $U a = a$, we may write simply,

$$\text{VIII.} \dots d.a^t = \frac{\pi}{2} a^{t+1} dt, \quad \text{if} \quad da = 0, \quad \text{and} \quad T a = 1.$$

(6.) This useful formula, for the differential of a power of a constant unit line, with a variable scalar exponent, may be obtained more rapidly from the equation 308, VII., which gives,

$$\text{IX.} \dots a^t = \cos \frac{t\pi}{2} + a \sin \frac{t\pi}{2}, \quad \text{if} \quad T a = 1;$$

since it is evident that the differential of this expression is equal to the expression itself multiplied by $\frac{1}{2}\pi a dt$, because $a^2 = -1$.

(7.) The formula VIII. admits also of a simple geometrical interpretation, connected with the rotation through t right angles, in a plane perpendicular to a , of

which rotation, or *version*, the *power* a^t , or the *versor* Ua^t , is considered (308) to be the *instrument*,* or *agent*, or *operator* (comp. 298).

334. Besides *algebraical* and *transcendental forms*, there are *other results of operation* on a quaternion, q , or on a function thereof, which may be regarded as forming a *new class* (or kind) of *functions*, arising out of the *principles* and *rules* of the *Quaternion Calculus itself*: namely those which we have denoted in former Chapters by the *symbols*,

$$\text{I. . . } Kq, Sq, Vq, Nq, Tq, Uq,$$

or by symbols formed through *combinations* of the same *signs of operation*, such as

$$\text{II. . . } SUq, VUq, UVq, \&c.$$

And it is essential that we should know how to *differentiate* expressions of *these forms*, which can be done in the following manner, with the help of the principles of the present and former Chapters, and *without* now assuming the *complanarity*, $dq \parallel q$.

(1.) In general, let f represent, for a moment, *any distributive symbol*, so that for any two quaternions, q and q' , we shall have the equation,

$$\text{III. . . } f(q + q') = fq + fq';$$

and therefore also† (comp. 326, (5.)),

$$\text{IV. . . } f(xq) = xfq, \text{ if } x \text{ be any scalar.}$$

(2.) Then, with the notation 325, IV., we shall have

$$\text{V. . . } f_n(q, q') = n\{f(q + n^{-1}q') - fq\} = fq';$$

and therefore, by 325, VIII., for any *such* function f_q , we shall have the differential expression,

$$\text{VI. . . } dfq = fdq.$$

(3.) But S, V, K have been seen to be *distributive symbols* (197, 207); we can therefore infer at once that

$$\text{VII. . . } dKq = Kdq; \quad \text{VIII. . . } dSq = Sdq; \quad \text{IX. . . } dVq = Vdq;$$

or in words, that the *differentials of the conjugate, the scalar, and the vector of a quaternion are*, respectively, *the conjugate, the scalar, and the vector of the differential of that quaternion*.

(4.) To find the *differential of the norm*, Nq , or to deduce an *expression for* dNq , we have (by VII. and 145) the equation,

* Compare the second Note to page 133.

† In *quaternions* the equation III. is not a necessary consequence of IV., although the latter is so of the former; for example, the equation IV., but *not* the equation III., will be satisfied, if we assume $fq = qcq^{-1}c'q$, where c and c' are any two constant quaternions, which do not degenerate into scalars.

$$X \dots dNq = d \cdot qKq = dq \cdot Kq + q \cdot Kdq;$$

but

$$qKq' = K \cdot q'Kq, \text{ by 145, and 192, II. ;}$$

$$\text{and } (1 + K) \cdot q'Kq = 2S \cdot q'Kq = 2S(Kq \cdot q), \text{ by 196, II., and 198, I. ;}$$

therefore

$$XI \dots dNq = 2S(Kq \cdot dq).$$

(5.) Or we might have deduced this expression XI. for dNq , more immediately, by the *general formula* 324, IV., from the earlier expression 200, VII., or 210, XX., for the *norm of a sum*, under the form,

$$XI \dots dNq = \lim_{\pi \rightarrow 0} \pi \{ N(q + \pi^{-1}dq) - Nq \}$$

$$= \lim_{\pi \rightarrow 0} \{ 2S(Kq \cdot dq) + \pi^{-1}Ndq \}$$

$$= 2S(Kq \cdot dq),$$

as before.

(6.) The *tensor*, Tq , is the *square-root* (190) of the *norm*, Nq ; and because Tq and Nq are scalars, the formula 332, XXII. may be applied; which gives, for the *differential of the tensor of a quaternion*, the expression (comp. 158),

$$XII \dots dTq = \frac{dNq}{2Tq} = S(KUq \cdot dq) = S \frac{dq}{Uq}$$

a result which is more easily remembered, under the form,

$$XIII \dots \frac{dTq}{Tq} = S \frac{dq}{q}.$$

(7.) The *versor* Uq is equal (by 188) to the *quotient*, $q : Tq$, of the quaternion q divided by its tensor Tq ; hence the *differential of the versor* is,

$$XIV \dots dUq = d \frac{q}{Tq} = \left(\frac{dq}{q} - S \frac{dq}{q} \right) \frac{q}{Tq} = V \frac{dq}{q} \cdot Uq;$$

whence follows at once this formula, analogous to XIII., and like it easily remembered,

$$XV \dots \frac{dUq}{Uq} = V \frac{dq}{q}.$$

(8.) We might also have observed that because (by 188), we have generally $q = Tq \cdot Uq$, therefore (by 332, (3.)) we have also,

$$XVI \dots dq = dTq \cdot Uq + Tq \cdot dUq,$$

and

$$XVII \dots \frac{dq}{q} = \frac{dTq}{Tq} + \frac{dUq}{Uq};$$

if then we have in any manner established the equation XIII., we can immediately deduce XV.; and conversely, the former equation would follow at once from the latter.

(9.) It may be considered as remarkable, that we should thus have *generally*, or for any two quaternions, q and dq , the formula :*

* When the connexion of the theory of normals to surfaces, with the differential calculus of quaternions, shall have been (even briefly) explained in a subsequent Section, the student will perhaps be able to perceive, in this formula XVIII., a recognition, though not a very direct one, of the geometrical principle, that the radii of a sphere are its normals.

$$* \quad d \frac{q}{Tq} = \frac{dq}{Tq} - \frac{dTq}{Tq} \frac{q}{Tq} = \frac{dTq}{Tq} \frac{1}{Tq} \cdot q \quad \frac{dq}{Tq} - \frac{dTq}{Tq} \frac{q}{Tq}$$

$$\text{XVIII.} \dots S(dUq; Uq) = 0; \text{ or } \text{XVIII}'. \dots dUq; Uq = S^{-1}0;$$

but this vector character of the quotient $dUq : Uq$ can easily be confirmed, as follows. Taking the conjugate of that quotient, we have, by VII. (comp. 192, II.; 158; and 824, XI.),

$$\text{XIX.} \dots K(dUq; Uq^{-1}) = KUq^{-1}.dKUq = Uq.d(Uq^{-1}) = -dUq.Uq^{-1};$$

whence

$$\text{XX.} \dots (1 + K)(dUq; Uq^{-1}) = 0;$$

which agrees (by 196, II.) with XVIII.

(10.) The scalar character of the tensor, Tq , enables us always to write, as in the ordinary calculus,

$$\text{XXI.} \dots dTq = dTq : Tq;$$

but $ITq = SIq$, by 816, V.; the recent formula XIII. may therefore by VIII. be thus written,

$$\text{XXII.} \dots Sdlq = dSIq = dTq : dq = S(dq : q); \text{ or } \text{XXII}'. \dots dlq - q^{-1}dq = S^{-1}0.$$

(11.) When $dq ||| q$, this last difference vanishes, by 338, II.; and the equation XV. takes the form,

$$\text{XXIII.} \dots dUq = \nabla dlq = dVlq.$$

And in fact we have generally, $lUq = Vlq$, by 816, XX., although the differentials of these two equal expressions do not separately coincide with the members of the recent formula XV., when q and dq are *diplanar*. We may however write generally (comp. XXII.),

$$\text{XXIV.} \dots dUq - dUq : Uq = \nabla(dlq - dq : q) = dlq - dq : q.$$

335. We have now differentiated the *six simple functions* 334, I., which are formed by the operation of the *six characteristics*,

$$K, S, V, N, T, U;$$

and as regards the differentiation of the *compound functions* 334, II., which are formed by combinations of those former operations, it is easy on the same principles to determine them, as may be seen in the few following examples.

(1.) The axis $Ax.q$ of a quaternion has been seen (291) to admit of being represented by the combination UVq ; the differential of this axis may therefore, by 334, IX. and XIV., be thus expressed:

$$\text{I.} \dots d(Ax.q) = dUVq = \nabla(\nabla dq : Vq).UVq;$$

whence

$$\text{II.} \dots \frac{d(Ax.q)}{Ax.q} = \frac{dUVq}{UVq} = \nabla \frac{\nabla dq}{Vq}.$$

The differential of the axis is therefore, generally, a line perpendicular to that axis, or situated in the plane of the quaternion; but it vanishes, when the plane (and therefore the axis) of that quaternion is constant; or when the quaternion and its differential are *complanar*.

(2.) Hence,

$$\text{III.} \dots dUVq = 0, \text{ if } \text{IV.} \dots dq ||| q;$$

and conversely this *complanarity* IV. may be expressed by the equation III.

(3.) It is easy to prove, on similar principles, that

$$V \dots dVUq = VdUq = V \left(V \frac{dq}{q} \cdot Uq \right);$$

and

$$VI \dots dSUq = SdUq = S \left(V \frac{dq}{q} \cdot Uq \right).$$

(4.) But in general, for any two quaternions, q and q' , we have (comp. 228, (5.)) the transformations,

$$VII \dots S(Vq' \cdot q) = S(Vq' \cdot Vq) = S \cdot q'Vq;$$

and when we thus suppress the characteristic V before $dq : q$, and insert it before Uq , under the sign S in the last expression VI , we may replace the new factor VUq by $TVUq \cdot UVUq$ (188), or by $TVUq \cdot UVq$ (274, XIII.), or by $-TVUq : UVq$ (204, V.), where the scalar factor $TVUq$ may be taken outside (by 196, VIII.); also for $q^{-1} : UVq$ we may substitute $1 : (UVq \cdot q)$, or $1 : qUVq$, because $UVq ||| q$; the formula VI may therefore be thus written,

$$VIII \dots dSUq = -S \frac{dq}{qUVq} \cdot TVUq.$$

(5.) Now it may be remembered, that among the earliest *connexions* of quaternions with *trigonometry*, the following formulæ occurred (196, XVI., and 204, XIX.),

$$IX \dots SUq = \cos \angle q, \quad TVUq = \sin \angle q;$$

we had also, in 316, these expressions for the *angle* of a quaternion,

$$X \dots \angle q = TVIq = TIUq;$$

we may therefore establish the following expression for the *differential of the angle* of a quaternion,

$$XI \dots d \angle q = dTVIq = dTIUq = S \frac{dq}{qUVq}.$$

(6.) The following is another way of arriving at the same result, through the differentiation of the *sine* instead of the *cosine* of the angle, or through the calculation of $dTVUq$, instead of $dSUq$. For this purpose, it is only necessary to remark that we have, by 384, XII. XIV., and by some easy transformations of the kind lately employed in (4.), the formula,

$$XII \dots dTVUq = S \frac{VdUq}{UVUq} = S \frac{dUq}{UVq} = S \left(V \frac{dq}{q} \cdot \frac{Uq}{UVq} \right) = S \frac{dq}{qUVq} \cdot SUq;$$

dividing which by SUq , and attending to IX. and X., we arrive again at the expression XI., for the differential of the angle of a quaternion.

(7.) Eliminating $S(dq : qUVq)$ between VIII. and XII., we obtain the *differential equation*,

$$XIII \dots SUq \cdot dSUq + TVUq \cdot dTVUq = 0;$$

of which, on account of the *scalar* character of the differentiated variables, the *integral* is evidently of the *form*,

$$XIV \dots (SUq)^2 + (TVUq)^2 = \text{const.};$$

and accordingly we saw, in 204, XX., that the sum in the first member of this equation is constantly equal to positive unity.

(8.) The formula XI. may also be thus written,

$$\text{XV.} \dots d \angle q = S(V(dq : q) : UVq);$$

with the verification, that when we suppose $dq \parallel q$, as in IV., and therefore $dUVq = 0$ by III., the expression under the sign S becomes the differential of the quotient, $Vq : UVq$, and therefore, by 816, VI., of the angle $\angle q$ itself.

336. An important application of the foregoing principles and rules consists in the *differentiation of scalar functions of vectors*, when those functions are defined and expressed according to the laws and notations of quaternions. It will be found, in fact, that *such* differentiations play a very extensive part, in the applications of quaternions to *geometry*; but, for the moment, we shall treat them *here*, as merely exercises of calculation. The following are a few examples.

(1.) Let ρ denote, in these sub-articles, a *variable vector*; and let the following equation be proposed,

$$\text{I.} \dots r^2 + \rho^2 = 0, \text{ in which } Vr = 0,$$

so that r is a (generally variable) *scalar*. Differentiating, and observing that, by 279, III., $\rho\rho' + \rho'\rho = 2S\rho\rho'$, if ρ' be any *second vector*, such as we suppose $d\rho$ to be, we have, by 822, VIII., and 824, VII., the equation,

$$\text{II.} \dots r dr + S\rho d\rho = 0; \text{ or III.} \dots dr = -r^{-1} S\rho d\rho = r S\rho^{-1} d\rho.$$

In fact, if r be supposed *positive*, it is here, by 282, II., the *tensor* of ρ ; so that this last expression III. for dr is included in the general formula, 834, XIII.

(2.) If this tensor, r , be *constant*, the differential equation II. becomes simply,

$$\text{IV.} \dots S\rho d\rho = 0, \text{ if } -\rho^2 = \text{const.}, \text{ or if } dT\rho = 0.$$

(8.) Again, let the proposed equation be (comp. 282, XIX.),

$$\text{V.} \dots r^2 = T(\iota\rho + \rho\kappa), \text{ with } d\iota = 0, d\kappa = 0,$$

so that ι and κ are here *two constant vectors*. Then, squaring and differentiating, we have (by 834, XI., because $K\rho = \rho\iota$, &c.),

$$\text{VI.} \dots 2r^2 dr = \frac{1}{2} dN(\iota\rho + \rho\kappa) = S(\rho\iota + \kappa\rho)(\iota d\rho + d\rho\kappa)$$

or more briefly,

$$\text{VII.} \dots 2r^{-1} dr = S\nu d\rho,$$

if ν be an *auxiliary vector*, determined by the equation,

$$\text{VIII.} \dots r^4 \nu = (\iota^2 + \kappa^2) \rho + 2V\kappa\rho\iota;$$

which admits of several transformations.

(4.) For example we may write, by 295, VII.,

$$\text{IX.} \dots r^4 \nu = (\iota^2 + \kappa^2) \rho + \kappa\rho\iota + \iota\rho\kappa \\ = \iota(\iota\rho + \rho\kappa) + \kappa(\rho\iota + \kappa\rho);$$

or, by 294, III., and 282, XII.,

$$\text{X.} \dots r^4 \nu = (\iota^2 + \kappa^2) \rho + 2(\kappa S\iota\rho - \rho S\iota\kappa + \iota S\kappa\rho) \\ = (\iota - \kappa)^2 \rho + 2(\iota S\kappa\rho + \kappa S\iota\rho); \text{ \&c.}$$

$$L L \cdot l l' - k l + k' l'$$

(5.) The equation V. gives (comp. 190, V.), when squared without differentiation,

$$\begin{aligned} \text{XI.} \dots r^4 &= N(\iota\rho + \rho\kappa) = (\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho) \\ &= (\iota^2 + \kappa^2)\rho^2 + \iota\rho\kappa\rho + \rho\kappa\rho\iota \\ &= (\iota^2 + \kappa^2)\rho^2 + 2\widehat{S}\rho\kappa\rho \\ &= (\iota - \kappa)^2\rho^2 + 4\widehat{S}\rho\kappa\rho + \&c., \end{aligned}$$

by transformations of the same kind as before; we have therefore, by the recent expressions for r^4 , the following remarkably simple relation between the two variable vectors, ρ and ν ,

$$\text{XII.} \dots S\nu\rho = 1; \text{ or } \text{XII}'. \dots S\rho\nu = 1.$$

(6.) When the scalar, r , is constant, we have, by VII., the differential equation,

$$\text{XIII.} \dots S\nu d\rho = 0; \text{ whence also } \text{XIV.} \dots S\rho d\nu = 0, \text{ by XII.};$$

a relation of reciprocity thus existing, between the two vectors ρ and ν , of which the geometrical signification will soon be seen.

(7.) Meanwhile, supposing r again to vary, we see that the last expression VI. for $2r^2dr$ may be otherwise obtained, by taking half the differential of either of the two last expanded expressions XI. for r^4 ; it being remembered, in all these little calculations, that cyclical permutation of factors, under the sign \widehat{S} , is permitted (228, (10.)), even if those factors be quaternions, and whatever their number may be: and that if they be vectors, and if their number be odd, it is then permitted, under the sign \widehat{V} , to invert their order (295, (9.)), and so to write, for instance, $\widehat{V}\rho\kappa$ instead of $\widehat{V}\kappa\rho$, in the formula VIII.

(8.) As another example of a scalar function of a vector, let p denote the proximity (or nearness) of a variable point P to the origin o ; so that

$$\text{XV.} \dots p = (-\rho^2)^{-\frac{1}{2}} = T\rho^{-1}, \text{ or } \text{XV}'. \dots p^{-2} + \rho^2 = 0.$$

Then,

$$\text{XVI.} \dots dp = S\nu d\rho, \text{ if } \text{XVII.} \dots \nu = p^3\rho = p^3\cup\rho;$$

ν being here a new auxiliary vector, distinct from the one lately considered (VIII.), and having (as we see) the same versor (or the same direction) as the vector ρ itself, but having its tensor equal to the square of the proximity of P to o ; or equal to the inverse square of the distance, of one of those two points from the other.

337. On the other hand, we have often occasion, in the applications, to consider vectors as functions of scalars, as in 99, but now with forms arising out of operations on quaternions, and therefore such as had not been considered in the First Book. And whenever we have thus an expression such as either of the two following,

$$\text{I.} \dots \rho = \phi(t), \text{ or } \text{II.} \dots \rho = \phi(s, t),$$

for the variable vector of a curve, or of a surface (comp. again 99), s and t being two variable scalars, and $\phi(t)$ and $\phi(s, t)$ denoting any functions of vector form, whereof the latter is here supposed to be entirely independent* of the former, we may then employ (comp. 100,

* We are therefore not employing here the temporary notation of some recent Articles, according to which we should have had, $d\phi q = \phi(q, dq)$.

(4.) and (9.), and the more recent sub-articles, 327, (5.), (6.), and 329, (5.)) the *notation of derivatives*, total or partial; and so may write, as the *differentiated equations*, resulting from the forms I. and II. respectively, the following:

$$\text{III.} \dots d\rho = \rho' t. dt = \rho' dt = D_t \rho. dt;$$

$$\text{IV.} \dots d\rho = d_t \rho + d_s \rho = D_t \rho. ds + D_s \rho. dt;$$

of which the geometrical significations have been already partially seen, in the sub-articles to 100, and will soon be more fully developed.

(1.) Thus, for the *circular locus*, 314, (1.), for which

$$\text{V.} \dots \rho = \alpha^t \beta, \quad T\alpha = 1, \quad S\alpha\beta = 0,$$

we have, by 338, VIII., the following *derived vector*,

$$\text{VI.} \dots \rho' = D_t \rho = \frac{\pi}{2} \alpha^{t+1} \beta = \frac{\pi}{2} \alpha \rho.$$

(2.) And for the *elliptic locus*, 314, (3.), for which

$$\text{VII.} \dots \rho = V. \alpha^t \beta, \quad T\alpha = 1, \quad \text{but not } S\alpha\beta = 0,$$

we have, in like manner, this other *derived vector*,

$$\text{VIII.} \dots \rho' = D_t \rho = \frac{\pi}{2} V. \alpha^{t+1} \beta.$$

(8.) As an example of a *vector-function of more scalars than one*, let us resume the expression (308, XVIII.),

$$\text{IX.} \dots \rho = r k^i j^s k^j \rightarrow k^t;$$

in which we shall now suppose that the tensor r is *given*, so that ρ is the *variable vector* of a point upon a *given spheric surface*, of which the *radius* is r , and the *centre* is at the *origin*; while s and t are *two independent scalar variables*, with respect to which the *two partial derivatives* of the vector ρ are to be determined.

(4.) The derivation relatively to t is easy; for, since ijk are *vector-units* (295), and since we have generally, by 338, VIII.,

$$\text{X.} \dots d. \alpha^s = \frac{\pi}{2} \alpha^{s+1} ds, \quad \text{and therefore} \quad \text{XI.} \dots D_t. \alpha^s = \frac{\pi}{2} \alpha^{s+1} D_t \alpha^s,$$

if $T\alpha = 1$, and if z be any scalar function of t , we may write, at once, by 279, IV.,

$$\text{XII.} \dots D_t \rho = \frac{\pi}{2} (k\rho - \rho k) = \pi V k \rho;$$

and we see that

$$\text{XIII.} \dots S \rho D_t \rho = 0,$$

a result which was to be expected, on account of the equation,

$$\text{XIV.} \dots \rho^2 + r^2 = 0,$$

which follows, by 308, XXIV., from the recent expression IX. for ρ .

(5.) To form an expression of about the same degree of simplicity, for the *other partial derivative* of ρ , we may observe that $j^{s+1} k^j \rightarrow$ is equal to its own vector part (its scalar vanishing); hence

XV. . . $D_\rho = \pi k^t j k^{-t} \rho$; or XVI. . . $D_\rho = \pi k^{2t} j \rho = \pi j k^{-2t} \rho$,

by the transformation 308, (11.). And because the scalar of $k^t j k^{-t}$ is zero, we have thus the equation,

XVII. . . $S_\rho D_\rho = 0$,

which is analogous to XIII., and might have been otherwise obtained, by taking the derivative of XIV. with respect to the variable scalar s .

(6.) The partial derivative D_ρ must be a vector; hence, by XV. or XVI., ρ must be perpendicular to the vector $k^t j k^{-t}$, or $k^{2t} j$, or $j k^{-2t}$; a result which, under the last form, is easily confirmed by the expression 315, XII. for ρ . In fact that expression gives, by 315, (8.) and (4.), and by the recent values XII. XVI., these other forms for the two partial derivatives of ρ , which have been above considered:

XVIII. . . $D_\rho = \pi r k^{2t} V . j^{2t}$; XIX. . . $D_\rho = \pi r (k^{2t} V . i^{2t+1} - V . k^{2t})$;

which might have been immediately obtained, by partial derivations, from the expression 315, XII. itself, and of which both are vector-forms.

(7.) And hence, or immediately by deriving the expanded expression 315, XIII., we obtain these new forms for the partial derivatives of ρ :

XX. . . $D_\rho = \pi r (j \cos t\pi - i \sin t\pi) \sin s\pi$;

XXI. . . $D_\rho = \pi r \{ (i \cos t\pi + j \sin t\pi) \cos s\pi - k \sin s\pi \}$.

(8.) We may add that not only is the variable vector ρ perpendicular to each of the two derived vectors, D_ρ and D_ρ , but also they are perpendicular to each other; for we may write, by XII. and XVI.,

XXII. . . $S(D_\rho . D_\rho) = -\pi^2 S . k^{2t} j \rho^{2t} k = \pi^2 r^2 S . k^{2t} i = 0$;

and the same conclusion may be drawn from the expressions XX. and XXI.

(9.) A vector may be considered as a function of three independent scalar variables, such as r, s, t ; or rather it must be so considered, if it is to admit of being the vector of an arbitrary point of space: and then it will have a total differential (329) of the trinomial form,

XXIII. . . $d\rho = d_r \rho + d_s \rho + d_t \rho = D_r \rho . dr + D_s \rho . ds + D_t \rho . dt$;

and will thus have three partial derivatives.

(10.) For example, when ρ has the expression IX., we have this third partial derivative,

XXIV. . . $D_r \rho = r^{-1} \rho = U_\rho$,

which may also be thus more fully written (comp. again 315, XIII.),

XXV. . . $D_r \rho = k^t j^2 k j^{-2} k^{-t} = (i \cos t\pi + j \sin t\pi) \sin s\pi + k \cos s\pi$;

and we see that the three derived vectors,

XXVI. . . $D_r \rho, D_s \rho, D_t \rho$,

compose here a rectangular system.

* That is to say, three of the first order; for we shall soon have occasion to consider successive differentials, of functions of one or more variables, and so shall be conducted to the consideration of orders of differentials and derivatives, higher than the first.

SECTION 5.—*On Successive Differentials, and Developments, of Functions of Quaternions.*

338. There will now be no difficulty in the *successive differentiation*, total or partial, of functions of one or more quaternions; and *such* differentiation will be found to be *useful*, as in the ordinary calculus, in connexion with *developments of functions*: besides that it is *necessary* for many of those *geometrical and physical applications* of differentials of quaternions, on which we have not entered yet. A few *examples* of successive differentiation may serve to show, more easily than any general precepts, the nature and effects of the operation; and we shall begin, for simplicity, with *explicit functions of one quaternion variable*.

(1.) Take then the *square*, q^2 , of a quaternion, as a function $f q$, which is to be *twice* differentiated. We saw, in 824, VII., that a *first* differentiation gave the equation,

$$I. \dots d f q = d. q^2 = q. dq + dq. q;$$

but we are now to differentiate *again*, in order to form the *second differential* $d^2 f q$ of the function q^2 , treating the differential of the variable q as *still* equal to dq , and in *general* writing $d^2 q = d^2 q$, where $d^2 q$ is a *new arbitrary quaternion*, of which the *tensor*, $T d^2 q$, need not be *small* (comp. 822). And thus we get, in *general*, this *twice differentiated expression*, or *differential of the second order*,

$$II. \dots d^2 f q = d^2. q^2 = q. d^2 q + 2dq^2 + d^2 q. q.$$

(2.) The *second differential of the reciprocal* of a quaternion is *generally* (comp. 324, XI.),

$$III. \dots d^2. q^{-1} = 2(q^{-1} dq)^2 q^{-1} - q^{-1} d^2 q. q^{-1}.$$

(3.) If ρ be a *variable vector*, then (comp. 836, (1.)) we have, for the first and second differentials of its square, the expressions:

$$IV. \dots d. \rho^2 = 2S\rho d\rho; \quad V. \dots d^2. \rho^2 = 2S\rho d^2\rho + 2d\rho^2.$$

(4.) If $f\rho$ be any *other scalar function* of a variable vector ρ , and if (comp. again the sub-articles to 836) its *first* differential be put under the *form*,

$$VI. \dots d f\rho = 2S\nu d\rho, \text{ when } \nu \text{ is another variable vector,}$$

then the *second differential* of the same function may be expressed as follows:

$$VII. \dots d^2 f\rho = 2S\nu d^2\rho + 2Sd\nu d\rho;$$

in which we have written, briefly, $Sd\nu d\rho$, instead of $S(d\nu. d\rho)$.

(5.) The following very simple equation will be found useful, in the theory of *motions*, performed under the influence of *central forces*:

$$VIII. \dots dV\rho d\rho = V\rho d^2\rho; \text{ because } V. \rho^2 = 0.$$

(6.) As an example of the second differential of a *quaternion*, considered as a

$$d^2 q^2 = -q^{-1} d q q^{-1} \dots = +2(q^{-1} d q)^2 - q^{-1} d^2 q$$

function of a scalar variable (comp. 333, VIII., and 337, (1.)), the following may be assigned, in which α denotes a given unit line, so that $\alpha^2 = -1$, $d\alpha = 0$, but x is a variable scalar :

$$\text{IX. } \dots d^2 \cdot \alpha^x = d \left(\frac{\pi}{2} \alpha^{x-1} dx \right) = \frac{\pi}{2} \alpha^{x-1} d^2 x - \left(\frac{\pi}{2} \right)^2 \alpha^x dx^2.$$

(7.) The second differential of the product of any two functions of a quaternion q may be expressed as follows (comp. II.):

$$\text{X. } \dots d^2 (fq \cdot \phi q) = d^2 fq \cdot \phi q + 2dfq \cdot d\phi q + fq \cdot d^2 \phi q.$$

339. The second differential, d^2q , of the variable quaternion q , enters generally (as has been seen) into the expression of the second differential d^2fq , of the function fq , as a new and arbitrary quaternion: but, for that very reason, it is permitted, and it is frequently found to be convenient, to assume that this second differential d^2q is equal to zero: or, what comes to the same thing, that the first differential dq is constant. And when we make this new supposition,

$$\text{I. } \dots dq = \text{constant, or } \text{I'. } \dots d^2q = 0,$$

the expressions for d^2fq become of course more simple, as in the following examples.

(1.) With this last supposition, I. or I', we have the following second differentials, of the square and the reciprocal of a quaternion:

$$\text{II. } \dots d^2 \cdot q^2 = 2dq^2; \quad \text{III. } \dots d^2 \cdot q^{-1} = 2(q^{-1}dq)^2 q^{-1} = 2q^{-1}(dq \cdot q^{-1})^2. \quad 2/$$

(2.) Again, if we suppose that e_0, e_1, e_2 are any three constant quaternions, and take the function,

$$\text{IV. } \dots fq = e_0 q e_1 q e_2,$$

we find, under the same condition I. or I', that its first and second differentials are,

$$\text{V. } \dots dfq = e_0 dq \cdot e_1 q e_2 + e_0 q e_1 dq \cdot e_2; \quad \text{VI. } \dots d^2fq = 2e_0 dq \cdot e_1 dq \cdot e_2;$$

in writing which, the points* may be omitted.

(3.) The first differential, dq , remaining still entirely arbitrary (comp. 322, (8.), and 325, (2.)), so that no supposition is made that its tensor Tdq is small, although we now suppose this differential dq to be constant (I.) we have rigorously,

$$\text{VII. } \dots (q + dq)^2 = q^2 + d \cdot q^2 + \frac{1}{2} d^2 \cdot q^2;$$

an equation which may be also written thus,

$$\text{VIII. } \dots (q + dq)^2 = (1 + d + \frac{1}{2} d^2) \cdot q^2.$$

(4.) And in like manner we shall have, more generally, under the same condition of constancy of dq , the equation,

$$\text{IX. } \dots f(q + dq) = (1 + d + \frac{1}{2} d^2) fq,$$

if the function fq be the sum of any number of monomes, each separately of the form

* Compare the second Note to page 399.

IV., and therefore each *rational, integral, and homogeneous of the second dimension*, with respect to the variable quaternion, q ; or of such monomes, combined with others of the *first dimension*, and with *constant terms*: that is, if $a_0, b_0, b_1, b'_0, b'_1, \dots$ and $c_0, c_1, c_2, c'_0, c'_1, c'_2, \dots$ be any constant quaternions, and

$$X. \dots fq = a_0 + \sum b_0 q b_1 + \sum c_0 q c_1 q c_2.$$

340. It is easy to carry on the operation of differentiating, to the *third and higher orders*; remembering only that if, in any former stage, we have denoted the first differentials of q, dq, \dots by dq, d^2q, \dots we then continue so to denote them, in every subsequent stage of the successive differentiation: and that if we find it convenient to treat any one differential as *constant*, we must then treat *all its successive differentials as vanishing*. A few examples may be given, chiefly with a view to the *extension* of the recent formula 339, IX., for the function $f(q + dq)$ of a sum, of any two quaternions, q and dq , to *polynomial forms, of dimensions higher than the second*.

(1.) The third differential of a square is generally (comp. 338, II.),

$$I. \dots d^3 \cdot q^2 = q \cdot d^2q + d^2q \cdot q + 8(dq \cdot d^2q + d^2q \cdot dq).$$

(2.) More generally, the third differential of a product of two quaternion functions (comp. 338, X.) may be thus expressed:

$$II. \dots d^3(fq \cdot \phi q) = d^3fq \cdot \phi q + 3d^2fq \cdot d\phi q + 8dfq \cdot d^2\phi q + fq \cdot d^3\phi q.$$

(3.) More generally still, the n^{th} differential of a product is, as in the ordinary calculus,

$$III. \dots d^n(fq \cdot \phi q) = d^n fq \cdot \phi q + nd^{n-1}fq \cdot d\phi q + n_2 d^{n-2}fq \cdot d^2\phi q + \dots + fq \cdot d^n\phi q,$$

if $n_2 = \frac{n(n-1)}{2}, n_3 = \frac{n(n-1)(n-2)}{2 \cdot 3}, \&c.;$

the only thing peculiar to quaternions being, that we are obliged to retain (generally) the order of the factors, in each term of this expansion III.

(4.) Hence, in particular, denoting briefly the function fq by r , and changing ϕq to q ,

$$IV. \dots d^n \cdot rq = d^n r \cdot q + nd^{n-1} r \cdot dq, \text{ if } d^2q = 0.$$

(5.) Hence also, under this condition that dq is constant, if c be any other constant quaternion, we have the transformation,

$$V. \dots \left(1 + d + \frac{1}{2}d^2 + \frac{1}{2 \cdot 3}d^3 + \dots + \frac{1}{2 \cdot 3 \dots n}d^n \right) \cdot rqc =$$

$$\left(1 + d + \frac{1}{2}d^2 + \frac{1}{2 \cdot 3}d^3 + \dots + \frac{1}{2 \cdot 3 \dots (n-1)}d^{n-1} \right) r \cdot (q + dq)c, \text{ if } d^2r = 0.$$

(6.) Hence, by 339, (4.), it is easy to infer that if we interpret the symbol ϵ^d by the equation (comp. 316, I.),

$$VI. \dots \epsilon^d = 1 + d + \frac{1}{2}d^2 + \frac{1}{2 \cdot 3}d^3 + \&c.,$$

that is, if we interpret this other symbol $\epsilon^d fq$, as concisely denoting the series which

is formed from fq , by *operating* on it with this symbolic development; and if the function fq , thus operated on, be any finite polynome, involving (like the expression 339, X.) no fractional nor negative exponents; we may then write, as an extension of a recent equation (339, IX.), the formula:

$$\text{VII.} \dots \epsilon^d f q = f(q + dq), \text{ if } d^2 q = 0;$$

which is here a perfectly rigorous one, all the terms of this expansion for a function of a sum of two quaternions, q and dq , becoming separately equal to zero, as soon as the symbolic exponent of d becomes greater than the dimension of the polynome.

(7.) We shall soon see that there is a sense, in which this exponential transformation VII. may be extended, to other functional forms which are not composed as above: and that thus an analogue of Taylor's Theorem can be established for Quaternions. Meanwhile it may be observed that by changing dq to Δq , in the finite expansion obtained as above, we may write the formula as follows:

$$\text{VIII.} \dots \epsilon^d f q = f(q + \Delta q) = (1 + \Delta) f q, \text{ or briefly, IX.} \dots \epsilon^d = 1 + \Delta;$$

which last *symbolical equation* may be operated on, or transformed, as in the usual calculus of differences and differentials. For instance, it being understood that we treat $\Delta^2 q$ as well as $d^2 q$ as vanishing, we have thus (for any positive and whole exponent m), the two following transformations of IX.,

$$\text{X.} \dots \Delta^m = (\epsilon^d - 1)^m, \text{ and XI.} \dots d^m = (\log(1 + \Delta))^m;$$

the results of operating, with the symbols thus equated, on any polynomial function $f q$, of the kind above described, being always finite expansions, which are rigorously equal to each other.

341. Let Fx and ϕx be any two functions of a scalar variable, of which both vanish with that variable; so that they satisfy the two conditions,

$$\text{I.} \dots F0 = 0, \quad \phi 0 = 0.$$

Then the three simultaneous values,

$$\text{II.} \dots x, \quad Fx, \quad \phi x,$$

of the variable and the two functions, are at the same time (comp. 320, 321) three simultaneous differences, as compared with this other system of three simultaneous values,

$$\text{III.} \dots 0, \quad F0, \quad \phi 0.$$

If, then, any equimultiples,

$$\text{IV.} \dots nx, \quad nFx, \quad n\phi x,$$

of the three values II., can be made, by any suitable increase of the number, n , combined with a decrease of the variable, x , to tend together to any system of limits, those limits must (by the definition in 320, compare again 321) admit of being considered as a system of simultaneous differentials,

$$V. \dots dx, dFx, d\phi x,$$

answering to the *system of initial values* III.; and must be *proportional* to the *ultimate values* of the connected *system of derivatives*,

$$VI. \dots 1, F'x, \phi'x, \text{ when } x \text{ tends to zero.}$$

We may therefore write, as *expressions* for those *ultimate values* of the two last *derived functions*,

$$VII. \dots F'0 = \lim_{n \rightarrow \infty} nF \frac{1}{n}, \quad \phi'0 = \lim_{n \rightarrow \infty} n\phi \frac{1}{n}, \quad \text{if } F0 = \phi0 = 0.$$

And *even if* these last values *vanish*, or if the *two new conditions*

$$VIII. \dots F'0 = 0, \quad \phi'0 = 0,$$

are satisfied, so that x , $F'x$, and $\phi'x$ are now (comp. II.) a *new system of simultaneous differences*, we may *still* establish the following *equation of limits of quotients*, which is *independent* of these *last conditions* VIII.,

$$IX. \dots \lim_{x \rightarrow 0} (Fx : \phi x) = \lim_{x \rightarrow 0} (F'x : \phi'x), \quad \text{if } F0 = \phi0 = 0;$$

it being understood that, in certain *cases*, these *two quotients* may *both vanish with* x ; or may *tend together to infinity*, when x *tends*, as before, *to zero*.

(1.) This theorem is so important, that it will not be useless to confirm it by a *geometrical illustration*, which may at the same time serve for a *geometrical proof*; at least for the extensive case where *both the functions* fx and ϕx are of *scalar forms*, and consequently may be *represented*, or *constructed*, by the *corresponding ordinates*, XY and XZ (or ordinates answering to one *common abscissa* OX), of *two curves* OyY and OzZ , which are *in one plane*, and set out from (or pass through) one *common origin* O , as in the annexed Figure 75. We shall afterwards see that the result, so obtained, can be *extended to quaternion functions*.

(2.) Suppose then, first, that the ordinates of these two curves are *proportional*, or that they bear to each other one *fixed and constant ratio*; so that the equation,

$$X. \dots XY : XZ = xy : zx,$$

is satisfied for *every pair of abscissæ*, OX and Ox , however *great* or *small* the corresponding *ordinates* may be. Prolonging then (if necessary) the *chord* Yy of the *first curve*, to meet the *axis of abscissæ* in some point t , and so to determine a *subsecant* tX , we see at once (by similar triangles) that the *corresponding chord* Zx of the *second curve* will meet the same *axis* in the *same point*, t ; and therefore that it will determine (*rigorously*) the *same subsecant*, tX .

(3.) Hence, if the point x be conceived to approach to X , so that the *secant* Yzt of the *first* curve tends to coincide with the *tangent* YT to that curve at the point Y , the *secant* Zzt of the *second* curve must tend to coincide with the line ZT , which line therefore must be the *tangent* to that second curve : or in other words, *corresponding subtangents coincide*, and of course are *equal*, under the supposed condition X , of a constant *proportionality of ordinates*.

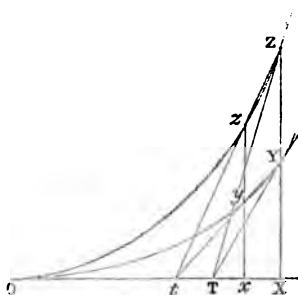


Fig. 75.

(4.) Suppose next that corresponding ordinates only *tend* to bear a *given* or *constant ratio* to each other ; or that their (now) *variable ratio* tends to a *given* or *fixed limit*, when the common abscissa is indefinitely diminished, or when the point X tends to O ; and let T be still the variable point in which the tangent to the *first* curve at Y meets the axis, so that the line TX is still the *first subtangent*. Then the corresponding tangent to the *second* curve at Z will *not* in general pass through the point T , but will meet the axis in some *different* point U . But the *ratio* of the two *corresponding subtangents*, TX and UX , which *had* been a *ratio of equality*, when the condition of *proportionality* X . was satisfied *rigorously*, will now at least *tend to such a ratio* ; so that we shall have, under this *new condition*, of *tendency to proportionality of ordinates*, the *limiting equation*,

$$XI. \dots \lim (TX : UX) = 1 ;$$

whence the equation IX. results, under the *geometrical form*,

$$XII. \dots \lim (\tan XTY : \tan XUZ) = \lim (XY : XZ).$$

(5.) We might also have observed that, when the *proportion* X . is *rigorous*, corresponding *areas** (such as $xXYy$ and $xXZz$) of the two curves are then *exactly* in the *given ratio* of the *ordinates* ; so that this *other equation*, or *proportion*,

$$XIII. \dots OXYyO : OXZzO = XY : XZ,$$

is then *also rigorous*. Hence if we only suppose, as in (4.), that the ordinates *tend* to some *fixed limiting ratio*, the *areas* must *tend to the same* ; so that *if the second member* of the equation IX. have *any definite value*, as a *limit*, the *first member* must have the *same* : whereas the recent proof, by *subtangents*, served rather to show that *if the first* (or left hand) *limit* in IX. *existed*, then the *second limit* in that equation existed *also*, and was *equal* to the first.

(6.) If the *function* Fx be a *quaternion*, we may (by 221) express it as follows,

$$XIV. \dots Fx = W + iX + jY + kZ,$$

where W, X, Y, Z are *four scalar functions* of x , of which *each* separately can be

* Compare the Fourth Lemma of the First Book of the Principia ; and see especially its Corollary, in which the reasoning of the present sub-article is virtually anticipated.

constructed, as the ordinate of a plane curve; and the recent *geometrical** reasoning will thus apply to each of them, and therefore to their *linear combination* Fx : which *quaternion function* reduces itself to a *vector function* of x , when $W = 0$.

(7.) And if ψx were *another* quaternion or vector function, we might first *substitute* it for Fx , and then *eliminate* the *scalar function* ϕx ; so that a limiting equation of the form IX. may thus be proved to hold good, when *both* the functions compared are *vectors*, or *quaternions*, supposed still to *vanish* with x .

(8.) The general considerations, however, on which the equation IX. was lately established, appear to be more simple and direct; and it is evident that they give, in like manner, this other but analogous equation, in which $F''x$ and $\phi''x$ are *second derivatives*, and the conditions VIII. are now supposed to be satisfied:

$$\text{XV.} \dots \lim_{x=0} (F'x : \phi'x) = \lim_{x=0} (F''x : \phi''x), \quad \text{if } F'0 = 0, \phi'0 = 0.$$

And so we might proceed, as long as *successive derivatives*, of *higher orders*, continue to *vanish* together.

(9.) Hence, in particular, if we take this *scalar form*,

$$\text{XVI.} \dots \phi x = \frac{x^m}{2.3\dots m},$$

which evidently gives the values,

$$\text{XVII.} \dots \phi 0 = 0, \quad \phi'0 = 0, \quad \phi''0 = 0, \dots \phi^{(m-1)}0 = 0, \quad \phi^{(m)}0 = 1,$$

and if we suppose that the function Fx is such that

$$\text{XVIII.} \dots F0 = 0, \quad F'0 = 0, \quad F''0 = 0, \dots F^{(m-1)}0 = 0,$$

while $F^{(m)}0$ has any finite value, we may then establish this limiting equation:

$$\text{XIX.} \dots \lim_{x=0} (Fx : \phi x) = F^{(m)}0;$$

in which the *function* Fx , and the *value* $F^{(m)}0$, are *here* supposed to be *generally quaternions*; although they may *happen*, in particular cases, to *reduce themselves* (292) to *vectors*, or to *scalars*.

* Instead of the equation IX., it has become usual, in modern works on the Differential Calculus, to give one of the following form (deduced from principles of Lagrange):

$$\frac{F(x)}{\phi(x)} = \frac{F'(\theta x)}{\phi'(\theta x)}, \quad \text{if } F(0) = \phi(0) = 0;$$

θ denoting *some proper fraction*, or quantity between 0 and 1. And a *geometrical illustration*, which is also a *geometrical proof*, when the *functions* Fx and ϕx can be *constructed* (or conceived to be constructed) as the *ordinates* of *two plane curves*, is sometimes derived from the *axiom* (or *geometrical intuition*), that the *chord* of any finite and *plane arc* must be *parallel to the tangent*, drawn at *some point* of that *finite arc*. But this *parallelism* no longer *exists*, in general, when the *curve* is one of *double curvature*; and accordingly the *equation* in this Note is *not generally true*, when the *functions* are *quaternions*; or even when *one* of them is a *quaternion*, or a *vector*.

342. It will now be easy to *extend* the *Exponential Transformation* 340, VII.; and to show that there is a *sense* in which that very important *Formula*,

$$I. \dots \varepsilon^d f q = f(q + dq), \quad \text{if } d^2 q = 0,$$

which is, in fact, a known* mode of expressing the *Series* or *Theorem* of *Taylor*, holds good for *Quaternion Functions generally*, and not merely for those functions of *finite* and *polynomial form*, with *positive* and *whole exponents*, for which it was lately deduced, in 340, (6.). For let $f q$ and $f(q + dq)$ denote *any two states*, or *values*, of which *neither* is *infinite*, of *any function of a quaternion*; and of the m first differentials,

$$II. \dots dfq, \quad d^2 f q, \dots d^m f q, \quad \text{in which } dq = \text{const.},$$

let it be supposed that *no one* is *infinite*, and that the *last* of them is different from *zero*; while all that precede it, and the functions $f q$ and $f(q + dq)$ themselves, may or may not happen to vanish. Let the first m terms, of the *exponential development* of the symbol $(\varepsilon^d - 1)f q$, be denoted briefly by $q_1, q_2, \dots q_m$; and let r_m denote what may be called the *remainder of the series*, or the *correction* which must be conceived to be added to the *sum* of these m terms, in order to produce the *exact value* of the *difference*,

$$III. \dots \Delta f q = f(q + \Delta q) - f q = f(q + dq) - f q;$$

in such a manner that we shall have *rigorously*, by the *notations* employed, the equation,

$$IV. \dots f(q + dq) = f q + q_1 + q_2 + \dots + q_m + r_m, \quad \text{where } q_m = \frac{d^m f q}{2.3 \dots m};$$

this term q_m being different from *zero*, but *no one* of the terms being *infinite*, by what has been above supposed. Then we shall prove, as a *Theorem*, that

* Lacroix, for instance, in page 168 of the First Volume of his larger Treatise on the Differential and Integral Calculus (Paris, 1810), presents the Theorem of Taylor under the form,

$$x' = x + \frac{dx}{1} + \frac{d^2 x}{1.2} + \frac{d^3 x}{1.2.3} + \frac{d^4 x}{1.2.3.4} + \&c.;$$

where x' denotes the value which the function x receives, when the variable x receives the *arbitrary increment* dx (l'accroissement quelconque dx).

$$V. \dots \lim (Tr_m : Tq_m) = 0, \quad \text{if } \lim. Tdq = 0;$$

or in words, that *the tensor of the remainder may be made to bear as small a ratio as we please, to the tensor of the last term retained, by diminishing the tensor, without changing the versor, of the differential (or difference) dq.* And this very general result, which will soon be seen to extend to functions of several quaternions, is in the present Calculus that *analogue of Taylor's theorem to which we lately alluded (in 340, (7.))*; and it may be called, for the sake of reference, "*Taylor's Theorem adapted to Quaternions.*"

(1.) Writing

$$VI. \dots Fx = f(q + xdq) - fq - xd fq - \frac{x^2}{2} d^2 fq - \dots - \frac{x^{m-1}}{2.8 \dots (m-1)} d^{m-1} fq,$$

we shall have the following successive derivatives with respect to x ,

$$VII. \dots \begin{cases} F'x = df(q + xdq) - dfq - xd^2fq - \dots - \frac{x^{m-2}}{2.8 \dots (m-2)} d^{m-1} fq; \\ F''x = d^2f(q + xdq) - d^2fq - \dots - \frac{x^{m-3}}{2.8 \dots (m-3)} d^{m-1} fq; \dots \\ F^{(m-1)}x = d^{m-1}f(q + xdq) - d^{m-1}fq; \text{ and finally,} \\ F^{(m)}x = d^m f(q + xdq); \end{cases}$$

because, by 327, VI., and 324, IV.,

$$VIII. \dots Df(q + xdq) = \lim_{n \rightarrow \infty} \{ f(q + xdq + n^{-1}dq) - f(q + xdq) \} = df(q + xdq),$$

and in like manner,

$$IX. \dots D^2f(q + xdq) = d^2f(q + xdq), \text{ \&c. ;}$$

the mark of derivation D referring to the scalar variable x , while d operates on q alone, and not here on x , nor on dq .

(2.) We have therefore, by VI. and VII., the values,

$$X. \dots F0 = 0, \quad F'0 = 0, \quad F''0 = 0, \dots F^{(m-1)}0 = 0, \quad F^{(m)}0 = d^m fq;$$

whence, by 341, XIX., we have this limiting equation,

$$XI. \dots \lim_{x \rightarrow 0} \left(Fx : \frac{x^m}{2.8 \dots m} \right) = d^m fq;$$

or

$$XII. \dots \lim_{x \rightarrow 0} (Fx : \psi x) = 1, \quad \text{if } \psi x = \left(\frac{x^m d^m fq}{2.8 \dots m} \right).$$

(3.) But these two functions, Fx and ψx , are formed by IV. from $q_m + r_m$ and q_m , by changing dq to xdq ; and instead of thus *multiplying* dq by a *decreasing scalar*, x , we may *diminish its tensor* Tdq , without changing its *versor* Udq . We may therefore say that, when this is done, the *quotient* $(q_m + r_m) : q_m$ *tends to unity*, or this other quotient $r_m : q_m$ *tends to zero*, as its *limit*; or in other words, the *limiting equation* V. holds good.

(4.) As an *example*, let the function $f q$ be the *reciprocal*, q^{-1} ; then (comp. 389, III.) its m^{th} differential is (for $dq = \text{const.}$),

$$\text{XIII.} \dots d^m f q = d^m . q^{-1} = 2.3 \dots m . q^{-1} (-r)^m, \text{ if } r = dq . q^{-1};$$

and it is easy to prove, *without differentials*, that

XIV. $\dots (q + r q)^{-1} = q^{-1}(1 + r)^{-1} = q^{-1} \{ 1 - r + r^2 - \dots + (-r)^m + (-r)^{m+1} (1 + r)^{-1} \}$;
we have therefore here

$$\text{XV.} \dots q_m = q^{-1} (-r)^m, \quad r_m = -q_m r (1 + r)^{-1}, \quad T(r_m : q_m) = \text{Tr.} T(1 + r)^{-1};$$

and this last tensor indefinitely diminishes with Tdq , the quaternion q being supposed to have some given value different from zero.

(5.) In general, if we establish the following equation,

$$\text{XVI.} \dots f(q + \pi^{-1} dq) = f q + \pi^{-1} d f q + \frac{\pi^{-2}}{2} d^2 f q + \dots + \frac{\pi^{1-m}}{2.3 \dots (m-1)} d^{m-1} f q + \frac{\pi^{-m}}{2.3 \dots m} f_n^{(m)}(q, dq),$$

as a *definitional extension* of the equation 325, V.; and if we suppose that neither the function $f q$ itself, nor any one of its differentials as far as $d^{m-1} f q$ is infinite; the result contained in the limiting equation XI. may then be expressed by the formula,

$$\text{XVII.} \dots f_n^{(m)}(q, dq) = d^m f q;$$

which for the particular value $m = 1$, if we suppress the upper index, coincides with the form 325, VIII. of the definition $d f x$, but for *higher* values of m contains a *theorem*: namely (when $d^m f q$ is supposed *neither to vanish, nor to become infinite*), what we have called *Taylor's Theorem adapted to Quaternions*.

343. That very important theorem may be applied to cases, in which a *quaternion* (as in 327, (5.)), or a *vector* (as in 337), is expressed as a *function of a scalar*; also to *transcendental forms* (333), whenever the differentiations can be effected; and to those *new forms* (334), which result from the *peculiar operations* of the present Calculus itself. A few such applications may here be given.

(1.) Taking first this transcendental and quaternion function of a variable scalar,

$$\text{I.} \dots q = f t = a^t, \text{ with } T a = 1, \quad da = 0, \quad dt = \text{const.},$$

we have, by 383, VIII., the general term,

$$\text{II.} \dots q_m = \frac{d^m . a^t}{2.3 \dots m} = \frac{a^t}{2.3 \dots m} \left(\frac{\pi a dt}{2} \right)^m = \frac{a^t (xa)^m}{2.3 \dots m}, \text{ if } 2x = \pi dt;$$

dividing then $\varepsilon^d . a^t$ by a^t , we obtain an *infinite series*, which is found to be *correct*, and *convergent*; namely (comp. 308, (4.)),

$$\text{III.} \dots a^{d^t} = 1 + xa + \frac{(xa)^2}{2} + \dots + \frac{(xa)^m}{2.3 \dots m} + \dots = \varepsilon^{xa} = \cos \frac{\pi dt}{2} + a \sin \frac{\pi dt}{2}.$$

(2.) Correct and *finite expansions*, for $S(q + dq)$, $V(q + dq)$, $K(q + dq)$, and $N(q + dq)$, are obtained when we operate with ε^d on Sq , Vq , Kq , and Nq ; for example (dq being still constant), the third and higher differentials of Nq vanish by 334, XI., and we have

$$\text{IV.} \dots \varepsilon^d Nq = (1 + d + \frac{1}{2}d^2)Nq = Nq + 2S(Kq \cdot dq) + Ndq = N(q + dq);$$

an expression for the *norm of a sum*, which agrees with 210, XX., and with 200, VII.

(3.) To develop, on like principles, the *tensor and versor of a sum*, let us again write r for $dq : q$, and denote the scalar and vector parts of this quotient by s and v ; so that, by 384, XIII. and XV.,

$$\text{V.} \dots s = Sr = S \frac{dq}{q} = \frac{dTq}{Tq}; \quad \text{VI.} \dots v = Vr = V \frac{dq}{q} = \frac{dUq}{Uq}.$$

(4.) Then writing also, for abridgment, as in a known notation of *factorials*,

$$\text{VII.} \dots [-1]^m = (-1) \cdot (-2) \cdot (-3) \dots (-m),$$

we shall have, by 342, XIII., dq being still treated as constant, the equation,

$$\text{VIII.} \dots d^m(s+v) = d^m r = [-1]^m r^{m+1} = [-1]^m (s+v)^{m+1},$$

of which it is easy to separate the scalar and vector parts; for example,

$$\text{IX.} \dots ds = -S \cdot (s+v)^2 = -(s^2 + v^2); \quad dv = -V \cdot (s+v)^2 = -2sv.$$

(5.) We have also, by V. and VI.,

$$\text{X.} \dots \frac{d^m Tq}{Tq} = (s+d) \frac{d^{m-1} Tq}{Tq} = \dots = (s+d)^m 1; \quad \therefore d^m (Tq) = (s+d)^m Tq$$

$$\text{XI.} \dots \frac{d^m Uq}{Uq} = (v+d) \frac{d^{m-1} Uq}{Uq} = \dots = (v+d)^m 1; \quad \therefore d^m (Uq) = (v+d)^m Uq$$

the notation being such that we have, for instance, by IX.,

$$\text{XII.} \dots (s+d)1 = s; \quad (s+d)^2 1 = (s+d)s = s^2 + ds = -v^2;$$

$$\text{XIII.} \dots (v+d)1 = v; \quad (v+d)^2 1 = (v+d)v = v^2 + dv = v^2 - 2sv.$$

(6.) The *exponential formula* 342, I., gives, therefore,

$$\text{XIV.} \dots T(q+dq) = \varepsilon^d Tq = \varepsilon^{s+d} 1 \cdot Tq;$$

$$\text{XV.} \dots U(q+dq) = \varepsilon^d Uq = \varepsilon^{v+d} 1 \cdot Uq;$$

or, dividing and substituting,

$$\text{XVI.} \dots T(1+s+v) = \varepsilon^{s+d} 1; \quad \text{XVII.} \dots U(1+s+v) = \varepsilon^{v+d} 1; \quad \frac{dq}{q} = s+v$$

s and v being here a *scalar* and a *vector*, which are entirely *independent* of each other; but of which, in the applications, the *tensors* must not be taken *too large*, in order that the *series* may *converge*.

(7.) The *symbolical expressions*, XVI. and XVII., for those two *series*, may be *developed* by (4.) and (5.); thus, if we only write down the terms which do not exceed the *second dimension*, with respect to s and v , we have by XII. and XIII. the *development*,

$$\text{XVIII.} \dots T(1+s+v) = 1 + s - \frac{1}{2}v^2 + \dots,$$

$$\text{XIX.} \dots U(1+s+v) = 1 + v + (\frac{1}{2}v^2 - sv) + \dots;$$

of which accordingly the *product* is $1 + s + v$, to the same order of approximation.

(8.) A *function of a sum* of two quaternions can sometimes be developed, *without differentials*, by processes of a more *algebraical character*; and when this happens, we may *compare* the result with the *form* given by *Taylor's Series*, as adapted to quaternions in 342, and so *deduce* the *values* of the *successive differentials* of the function; for example, we can *infer* the expression 342, XIII. for $d^m \cdot q^{-1}$, from the *series* 342, XIV., for the *reciprocal of a sum*.

(9.) And not only may we *verify* the recent developments, XVIII. and XIX., by comparing them with the more *algebraical forms*,

$$\begin{aligned} \text{XX.} \dots T(1+s+v) &= (1+s+v)^{\frac{1}{2}}(1+s-v)^{\frac{1}{2}}, \\ \text{XXI.} \dots U(1+s+v) &= (1+s+v)^{\frac{1}{2}}(1+s-v)^{-\frac{1}{2}}, \end{aligned}$$

but also, if the first of these for example (when expanded by *ordinary processes*, which are in *this case* applicable) have given us, *without differentials*,

$$\text{XXII.} \dots T(q+q') = (1+s - \frac{1}{2}v^2 \dots)Tq, \text{ where } s = Sq'q^{-1}, \text{ and } v = Vq'q^{-1},$$

we can then *infer* the values of the *first* and *second differentials* of the *tensor* of a quaternion, as follows:

$$\text{XXIII.} \dots dTq = S \frac{dq}{q} \cdot Tq; \quad d^2Tq = - \left(V \frac{dq}{q} \right)^2 Tq;$$

whereof the first agrees with 334, XII or XIII, and the second can be deduced from it, under the form,

$$\text{XXIV.} \dots d^2Tq = d \left(S \frac{dq}{q} \cdot Tq \right) = \left(\left(S \frac{dq}{q} \right)^2 - S \cdot \left(\frac{dq}{q} \right)^2 \right) Tq.$$

(10.) In general, if we can only develop a function $f(q+q')$ as far as the term or terms which are of the *first dimension* relatively to q' , we shall still obtain thus an expression for the *first differential* dfq , by merely writing dq in the place of q' . But we have not chosen (comp. 100, (14.)) to regard *this property of the differential of a function* as the *fundamental one*, or to adopt it as the *definition* of dfq ; because we have not chosen to *postulate* the *general possibility* of such *developments of functions of quaternion sums*, of which in fact it is in many cases *difficult to discover the laws*, or even to *prove the existence*, except in some such way as that above explained.

(11.) This opportunity may be taken to observe, that (with recent notations) we have, by VIII., the symbolical expression,

$$\text{XXV.} \dots e^{s+v+d} 1 = 1 + s + v; \quad \text{or} \quad \text{XXVI.} \dots e^{r+d} 1 = 1 + r.$$

344. *Successive differentials* are also connected with *successive differences*, by laws which it is easy to investigate, and on which only a few words need here be said.

(1.) We can easily prove, from the definition 324, IV. of dfq , that if dq be constant,

$$I. \dots d^2fq = \lim_{n \rightarrow \infty} n^2 \{ f(q + 2n^{-1}dq) - 2f(q + n^{-1}dq) + fq \};$$

with analogous expressions for differentials of higher orders.

(2.) Hence we may say (comp. 340, X.) that the *successive differentials*,

$$II. \dots dfq, \quad d^2fq, \quad d^3fq, \dots \text{ for } d^2q = 0,$$

are *limits* to which the following *multiples of successive differences*,

$$III. \dots n\Delta fq, \quad n^2\Delta^2fq, \quad n^3\Delta^3fq, \dots \text{ for } \Delta^2q = 0,$$

all simultaneously *tend*, when the multiple $n\Delta q$ is either constantly *equal* to dq , or at least *tends* to become equal thereto, while the number n increases indefinitely.

(3.) And hence we might prove, in a new way, that *if the function* $f(q+dq)$

can be developed, in a series proceeding according to ascending and whole dimensions with respect to dq , the parts of this series, which are of those successive dimensions, must follow the law expressed by Taylor's Theorem* adapted to Quaternions (342).

345. It is easy to conceive that the foregoing results may be extended (comp. 338), to the successive differentiations of functions of several quaternions; and that thus there arises, in each such case, a system of successive differentials, total and partial: as also a system of partial derivatives, of orders higher than the first, when a quaternion, or a vector, is regarded (comp. 337) as a function of several scalars.

(1.) The general expression for the second total differential,

$$\text{I. . . } d^2Q = d^2F(q, r, \dots),$$

involves d^3q , d^2r , . . . ; but it is often convenient to suppose that all these second differentials vanish, or that the first differentials dq , dr , . . . are constant; and then d^mQ , or $d^mF(q, r, \dots)$, becomes a rational, integral, and homogeneous function of the m^{th} dimension, of those first differentials dq , dr , . . . , which may (comp. 329, III.) be thus denoted,

$$\text{II. . . } d^mQ = (d_q + d_r + \dots)^m Q; \text{ or briefly, III. . . } d^m = (d_q + d_r + \dots)^m,$$

in developing which symbolical power, the multinomial theorem of algebra may be employed: because we have generally, for quaternions as in the ordinary calculus,

$$\text{IV. . . } d_r d_q = d_q d_r.$$

(2.) For example, if we denote dq and dr by q' and r' , and suppose

$$\text{V. . . } Q = rqr, \text{ then VI. . . } d_q Q = r'q'r; \text{ VII. . . } d_r Q = r'qr' + rqr';$$

and

$$\text{VIII. . . } d_r d_q Q = d_q d_r Q = r'q'r + r'q'r'.$$

And in general, each of the two equated symbols IV. gives, by its operation on $F(q, r)$, the limit of this other function, or product (comp. 344, I.),

$$\text{IX. . . } n\pi' \{ F(q + \pi^{-1}dq, r + \pi^{-1}dr) - F(q, r + \pi^{-1}dr) - F(q + \pi^{-1}dq, r) + F(q, r) \};$$

in which the numbers n and π' are supposed to tend to infinity.

(3.) We may also write, for functions of several quaternions,

$$\text{X. . . } Q + \Delta Q = F(q + dq, r + dr, \dots) = e^{d_q + d_r + \dots} F(q, r);$$

or briefly,

$$\text{XI. . . } 1 + \Delta = e^{d_q + d_r + \dots} = e^d;$$

with interpretations and transformations analogous to those which have occurred already, for functions of a single quaternion.

(4.) Finally, as an example of successive and partial derivation, if we resume the vector expression 308, XVIII. (comp. 315, XII. and XIII.), namely,

$$\text{XII. . . } \rho = rh^i j^k j^{-i} h^{-k},$$

* Some remarks on the adaptation and proof of this important theorem will be found in the Lectures, page 589, &c.

which has been seen to be capable of representing the vector of any point of space, we may observe that it gives, without trigonometry, by the principle mentioned in 308, (11.), and by the sub-articles to 815, not only the form,

$$\text{XIII.} \dots \rho = rk^t j^{2s} k^{-t}, \text{ as in 808, XIX,}$$

but also, if α be any vector unit,

$$\text{XIV.} \dots \rho = rk^{t+1} j^{-2s} k^{-t} = rk^t (kS. a^{2s} + iS. a^{2s-1}). k^{-t};$$

whence $\text{XV.} \dots \rho = rV. k^{2s+1} + rk^{2s} V. i^{2s}$, as in 815, XII.

(5.) We have therefore the following new expressions (compare the sub-articles to 837), for the two partial derivatives of the first order, of this variable vector ρ , taken with respect to s and t :

$$\text{XVI.} \dots D_s \rho = \pi r k^t j^{2s-1} k^{-t} = -\pi \rho k^t j k^{-t},$$

with the verification, that

$$\text{XVII.} \dots \rho D_s \rho = \pi r^2. k^t j^{2s-1} k^{-t}. k^t j^{2s-1} k^{-t} = \pi r^2 k^t j k^{-t};$$

and $\text{XVIII.} \dots D_t \rho = \pi r k^{2s} V. j^{2s} = \pi r k^{2s} j S. a^{2s-1} = r^{-1} \rho D_s \rho. S. a^{2s-1}$,

whence $\text{XIX.} \dots \rho D_t \rho = -r D_s \rho. S. a^{2s-1}$, and $\text{XX.} \dots D_s \rho. D_t \rho = \pi^2 r \rho S. a^{2s-1}$;

while $\text{XXI.} \dots D_r \rho = r^{-1} \rho = k^t j^{2s} k^{-t}$, as in 837, XXV.;

so that we have the following ternary product of these derived vectors of the first order,

$$\text{XXII.} \dots D_r \rho. D_s \rho. D_t \rho = \pi^2 \rho^2 S. a^{2s-1} = \pi r^2 D_s \rho. a^{2s};$$

the scalar character of which product depends (comp. 299, (9.)) on the circumstance, that the vectors thus multiplied compose (837, (10.)) a rectangular system.

(6.) It is easy then to infer, for the six partial derivatives of ρ , of the second order, taken with respect to the same three scalar variables, r, s, t , the expressions:

$$\text{XXIII.} \dots D_r^2 \rho = 0; \quad D_r D_s \rho = D_s D_r \rho = r^{-1} D_s \rho; \quad D_r D_t \rho = D_t D_r \rho = r^{-1} D_t \rho;$$

$\text{XXIV.} \dots D_s^2 \rho = -\pi^2 \rho; \quad D_s D_t \rho = D_t D_s \rho = \pi^2 r k^{2s} V. j^{2s+1}; \quad D_t^2 \rho = -\pi^2 r k^{2s} V. i^{2s}.$

(7.) The three partial differentials of the first order, of the same variable vector ρ , are the following:

$$\text{XXV.} \dots d_r \rho = r^{-1} \rho dr; \quad d_s \rho = D_s \rho. ds; \quad d_t \rho = D_t \rho. dt;$$

with the products,

$$\text{XXVI.} \dots d_s \rho. d_t \rho = -\pi r \rho dS. a^{2s}. dt;$$

$$\text{XXVII.} \dots d_r \rho. d_s \rho. d_t \rho = \pi r^2 dr. dS. a^{2s}. dt.$$

(8.) These differential vectors, $d_r \rho$, $d_s \rho$, $d_t \rho$, are (in the present theory) generally finite; $d_r \rho$, like $D_r \rho$, being a line in the direction of ρ , or of the radius of this sphere round the origin, at least if dr , like r , be positive; while $d_s \rho$, like $D_s \rho$, is (comp. 100, (9.)) a tangent to the meridian of that spheric surface, for which r and t are constant; but $d_t \rho$, like $D_t \rho$, is on the contrary a tangent to the small circle (or parallel), on the same sphere, for which r and s are constant.

(9.) Treating only the radius r as constant, and writing $\rho = OP$, if we pass from the point P , or (s, t) , to another point Q , or $(s + \Delta s, t)$, on the same meridian, the chord PQ is represented by the finite difference, $\Delta_s \rho$; and in like manner, if we pass from P to a point R , or $(s, t + \Delta t)$, on the same parallel, the new chord PR is represented by the other partial and finite difference, $\Delta_t \rho$; while the point $(s + \Delta s, t + \Delta t)$ may be denoted by s .

(10.) If now the two points Q and R be conceived to approach to P , and to come to be very near it, the chords PQ and PR will very nearly coincide with the two cor-

responding arcs of meridian and parallel; or with the tangents to the same two circles at P , so drawn as to have the lengths of those two arcs: or finally with the differential and tangential vectors, d_ρ and d_ρ , if we suppose (as we may, comp. 322) that the two arbitrary and scalar differentials, ds and dt , are so assumed as to be constantly equal to the two differences, Δs and Δt , and consequently to diminish with them.

(11.) Whether the differentials ds and dt be large or small, the product $d_\rho \cdot d_\rho$, like the product $D_\rho \cdot D_\rho$, represents rigorously a normal vector (as in XXVI. and XX.); of which the length bears to the unit of length (comp. 281) the same ratio, as that which the rectangle under the two perpendicular tangents, d_ρ and d_ρ , to the sphere, bears to the unit of area. Hence, with the recent suppositions (10.), we may regard this product $d_\rho \cdot d_\rho$ as representing, with a continually and indefinitely increasing accuracy, even in the way of ratio, what we may call the directed element of spheric surface, $PQRS$, considered as thus represented (or constructed) by a normal at P ; and the tensor of the same product, namely (by XXVI.),

$$\text{XXVIII.} \dots T(d_\rho \cdot d_\rho) = -\pi r^2 dS \cdot a^{2s} \cdot dt,$$

in which the negative sign is retained, because $S \cdot a^{2s}$ decreases from $+1$ to -1 , while s increases from 0 to 1 , is an expression on the same plan for what we may call by contrast the undirected element of spheric area, or that element considered with reference merely to quantity, and not with reference to direction.

(12.) Integrating, then, this last differential expression XXVIII., from $t=0$ to $t=2$, and from $s=s_0$ to $s=s_1$, that is, taking the limit of the sum of all the elements $PQRS$ between these bounding values, we find the following equation:

$$\text{XXIX.} \dots \text{Area of Spheric Zone} = 2\pi r^2 S(a^{2s_0} - a^{2s_1});$$

whence

$$\text{XXX.} \dots \text{Area of Spheric Cap } (s) = 2\pi r^2(1 - S \cdot a^{2s}) = 4\pi r^2(TV \cdot a^s)^2;$$

and finally,

$$\text{XXXI.} \dots \text{Area of Sphere} = 4\pi r^2, \text{ as usual.}$$

(13.) In like manner the expression XXVII., with its sign changed (on account of the decrease of $S \cdot a^{2s}$, as in (11.)), represents the element of volume; and thus, by integrating from $r=r_0$ to $r=r_1$, from $s=0$ to $s=1$, and from $t=0$ to $t=2$, we obtain anew the known values:

$$\text{XXXII.} \dots \text{Volume of Spheric Shell} = \frac{4\pi}{3}(r_1^3 - r_0^3);$$

and

$$\text{XXXIII.} \dots \text{Volume of Sphere } (r) = \frac{4\pi r^3}{3}, \text{ as usual.}$$

(14.) These are however only specimens of what may be called *Scalar Integration*, although connected with *quaternion forms*; and it will be more characteristic of the present Calculus, if we apply it briefly to take the *Vector Integral*, or the limit of the vector-sum of the directed elements (11.), of a portion of a spheric surface: a problem which corresponds, in *hydrostatics*, to calculating the resultant of the pressures on that surface, each pressure having a normal direction, and a quantity proportional to the element of area.

(15.) For this purpose, we may employ the expression XXVI. with its sign changed, in order to denote an inward normal, or a pressure acting from without; and if we then substitute for ρ its value XV., and observe that

XXXIV. . . $\int_0^1 k^{2t} dt = 0$, because $k^2 = -1$,

and remember that $V. k^{2n+1} = kS. a^{2n}$, we easily deduce the expressions :

XXXV. . . *Sum of Directed Elements of Elementary Zone* = $\pi r^2 k d. (S. a^{2n})^2$;
 XXXVI. . . *Sum of Directed Elements of Spheric Cap (s)* = $-\pi r^2 k (1 - (S. a^{2n})^2)$
 = $\pi r^2 k (V. a^{2n})^2 = \pi^{-1} k (Dip)^2 = \pi k (V k \rho)^2$.

(16.) But the radius of the plane and circular base, of the spheric segment corresponding, is $TVk\rho$, so that its area is in quantity = $-\pi (V k \rho)^2$; and the common direction of all its inward normals is that of $+k$; hence if we still represent the directed elements by normals thus drawn inwards, we have this new expression :

XXXVII. . . *Sum of Directed Elements of Circular Base* = $-\pi k (V k \rho)^2$;

comparing which with XXXVI., we arrive at the formula,

XXXVIII. . . *Sum of Directed Elements of Spheric Segment* = Zero ;

a result which may be greatly extended, and which evidently answers to a known case of equilibrium in hydrostatics.

(17.) These few examples may serve to show already, that *Differentials of Quaternions* (or of *Vectors*) may be applied to various geometrical and physical questions ; and that, when so applied, it is permitted to treat them as small, if any convenience be gained thereby, as in cases of integration there always is. But we must now pass to an important investigation of another kind, with which differentials will be found to have only a sort of indirect or suggestive connexion.

SECTION 6.—On the Differentiation of Implicit Functions of Quaternions ; and on the General Inversion of a Linear Function, of a Vector or a Quaternion : with some connected Investigations.

346. We saw, when differentiating the square-root of a quaternion (332, (5.) and (6.)), that it was necessary for that purpose to resolve a linear equation,* or an equation of the first degree ; namely the equation,

$$I \dots r r' + r' r = q',$$

in which r and q' represented two given quaternions, $q^{\frac{1}{2}}$ and dq , while r' represented a sought quaternion, namely dr or $d.q^{\frac{1}{2}}$. And generally, from the linear or distributive form (327), of the quaternion differential

$$II. \dots dQ = dfq = f(q, dq),$$

of any given and explicit function $f q$, when considered as depending on the differential dq of the quaternion variable q , we see that the return from the former differential to the latter,

* Compare the Note to page 410.

that is from dQ to dq , or the *differentiation* of the *inverse* or *implicit* function $f^{-1}Q$, requires for its accomplishment the *Solution of an Equation of the First Degree*: or what may be called the *Inversion of a Linear Function of a Quaternion*. We are therefore led to consider here that general Problem; to which accordingly, and to investigations connected with which, we shall devote the present Section, dismissing however now the *special* consideration of the *Differentials* above mentioned, or treating them only as *Quaternions*, sought or given, of which the *relations* to each other are to be studied.

347. Whatever the *particular form* of the given *linear* or *distributive function*, fq , may be, we can always *decompose* it as follows:

$$\text{I. . . } fq = f(Sq + Vq) = fSq + fVq = Sq \cdot f1 + fVq;$$

taking then separately scalars and vectors, or operating with S and V on the proposed *linear equation*,

$$\text{II. . . } fq = r,$$

where r is a *given* quaternion, and q a *sought* one, we can in general *eliminate* Sq , and so reduce the problem to the solution of a *linear* and *vector equation*, of the form,

$$\text{III. . . } \phi\rho = \sigma;$$

where σ is a *given* vector, but $\rho (= Vq)$ is a *sought* one, and ϕ is used as the characteristic of a *given linear and vector function* of a vector, which function we shall throughout suppose to be a *real* one, or to involve *no imaginary constants* in its composition. But, to every *such* function $\phi\rho$, there always *corresponds* what may be called a *conjugate* linear and vector function $\phi'\rho$, connected with it by the following *Equation of Conjugation*,

$$\text{IV. . . } S\lambda\phi\rho = S\rho\phi'\lambda;$$

where λ and ρ are *any two* vectors. Assuming then, as we may, that μ and ν are *two auxiliary* vectors, so chosen as to satisfy the equation,

$$\text{V. . . } V\mu\nu = \sigma,$$

and therefore also,

$$\text{VI. . . } S\lambda\sigma = S\lambda\mu\nu, \quad S\mu\sigma = 0, \quad S\nu\sigma = 0,$$

where λ is a *third* auxiliary and arbitrary vector, we may (comp. 312) replace the *one* vector equation III. by the *three* scalar equations,

VII. . . $S\rho\phi'\lambda = S\lambda\mu\nu$, $S\rho\phi'\mu = 0$, $S\rho\phi'\nu = 0$.

And these give, by principles with which the reader is supposed to be already familiar,* the expression,

VIII. . . $m\rho = \psi\sigma$, or IX. . . $\rho = \phi^{-1}\sigma = m^{-1}\psi\sigma$,

if m be a *vector-constant*, and ψ an *auxiliary linear and vector function*, of which the *value* and the *form* are determined by the two following equations:

X. . . $mS\lambda\mu\nu = S(\phi'\lambda, \phi'\mu, \phi'\nu)$; $(2) \psi(3) \rho = \frac{1}{V}(\psi' \mu \nu)$
 XI. . . $\psi(V\mu\nu) = V(\phi'\mu, \phi'\nu)$; $\psi' \mu \nu = \dots$

or briefly,

X'. . . $mS\lambda\mu\nu = S.\phi'\lambda\phi'\mu\phi'\nu$,

and

XI'. . . $\psi V\mu\nu = V.\phi'\mu\phi'\nu$.

And thus the proposed *Problem of Inversion*, of the linear and vector function ϕ , may be considered to be, in all its generality, *resolved*; because it is always possible so to *prepare* the second members of the equations X. and XI, that they shall take the *forms* indicated in the first members of those equations.

(1.) For example, if we *assume* any three diplanar vectors a, a', a'' , and *deduce* from them three other vectors $\beta_0, \beta'_0, \beta''_0$, by the equations,

XII. . . $\beta_0 Saa'a'' = Va'a'$, $\beta'_0 Saa'a'' = Va''a$, $\beta''_0 Saa'a'' = Vaa'$;

then any vector ρ may, by 294, XV., be expressed as follows,

XIII. . . $\rho = \beta_0 S\rho\rho + \beta'_0 S\rho\rho + \beta''_0 S\rho\rho$;

if then we write,

XIV. . . $\beta = \phi\beta_0$, $\beta' = \phi\beta'_0$, $\beta'' = \phi\beta''_0$,

we shall have the following *General Expression*, or *Standard Trinomial Form*, for a *Linear and Vector Function of a Vector*,

XV. . . $\phi\rho = \beta S\rho\rho + \beta' S\rho\rho + \beta'' S\rho\rho$;

containing, as we see, *three vector constants*, β, β', β'' , or *nine scalar constants*, such as

XVI. . . $Sa\beta, Sa'\beta, Sa''\beta$; $Sa\beta', Sa'\beta', Sa''\beta'$; $Sa\beta'', Sa'\beta'', Sa''\beta''$;

which may (and generally will) all *vary*, in passing from *one* linear and *vector function* $\phi\rho$ to *another* such function; but which are *all* supposed to be *real*, and *given*, for each *particular form* of that function.

(2.) Passing to what we have called the *conjugate* linear function $\phi'\rho$, the form XV. gives, by IV., the expression,

* A student might find it useful, at this stage, to read again the Sixth Section of the preceding Chapter; or at least the early sub-articles to Art. 294, a *familiar acquaintance* with which is presumed in the present Section.

$$\text{XVII.} \dots \phi' \rho = \alpha S \beta \rho + \alpha' S \beta' \rho + \alpha'' S \beta'' \rho;$$

but

$$\begin{aligned} \text{V.} (\alpha S \beta \mu + \alpha' S \beta' \mu) (\alpha S \beta \nu + \alpha' S \beta' \nu) &= \text{V} \alpha \alpha' S. \beta' (\nu S \beta \mu - \mu S \beta \nu) \\ &= \text{V} \alpha \alpha' S. \beta' \text{V} \beta \nu \mu \nu = \text{V} \alpha \alpha' S. \beta' \beta \nu \mu \nu; \end{aligned}$$

therefore the transformation XI. succeeds, and gives,

$$\text{XVIII.} \dots \psi \rho = \text{V} \alpha' \alpha'' S \beta'' \beta' \rho + \text{V} \alpha'' \alpha S \beta \beta' \rho + \text{V} \alpha \alpha' S \beta' \beta \rho,$$

as an expression for the *auxiliary* function ψ ; of which the *conjugate* may be thus written,

$$\text{XIX.} \dots \psi' \rho = \text{V} \beta' \beta'' S \alpha'' \alpha' \rho + \text{V} \beta'' \beta S \alpha \alpha'' \rho + \text{V} \beta \beta' S \alpha' \alpha \rho;$$

so that ψ is changed to ψ' , when ϕ is changed to ϕ' , by interchanging each of the three *alphas* with the corresponding *betas*.

(8.) If we write, as in this whole investigation we propose to do,

$$\text{XX.} \dots \lambda' = \text{V} \mu \nu, \quad \mu' = \text{V} \nu \lambda, \quad \nu' = \text{V} \lambda \mu,$$

the formulæ XI. and X. become,

$$\text{XXI.} \dots \psi \lambda' = \text{V.} \phi' \mu \phi' \nu, \quad \text{and} \quad \text{XXII.} \dots m S \lambda \lambda' = S. \phi' \lambda \psi \lambda',$$

with the same sort of abridgment of notation as in XI.; and because the coefficient of $S \alpha \alpha' \alpha''$ in this last expression XXII. is by XVII. XVIII.,

$$S \beta \lambda S \beta'' \beta' \lambda' + S \beta' \lambda S \beta \beta'' \lambda' + S \beta'' \lambda S \beta \beta' \lambda' = S \beta'' \beta' S \lambda \lambda',$$

the *division* by $S \lambda \lambda'$, or by $S \lambda \mu \nu$, succeeds, and we find the expression,

$$\text{XXIII.} \dots m = S \alpha \alpha' \alpha'' S \beta'' \beta' \beta;$$

which may also be thus written,

$$\text{XXIII.} \dots m = S \beta \beta'' \beta' S \alpha'' \alpha' \alpha,$$

so that m does not change when we pass from ϕ to ϕ' , on which account we may write also,

$$\text{XXIV.} \dots m S \lambda \lambda' = S. \phi \lambda \psi' \lambda', \quad \text{or} \quad \text{XXIV.} \dots m S \lambda \mu \nu = S. \phi \lambda \phi \mu \phi \nu,$$

because, by (2.), we can deduce from XI. the conjugate expression,

$$\text{XXV.} \dots \psi' \lambda' = \text{V.} \phi \mu \phi \nu.$$

(4.) We ought then to find that the *linear equation*,

$$\text{XXVI.} \dots \sigma = \phi \rho = \beta S \alpha \rho + \beta' S \alpha' \rho + \beta'' S \alpha'' \rho,$$

has its *solution* expressed (comp. VIII.) by the formula,

$$\text{XXVII.} \dots \rho S \alpha \alpha' \alpha'' S \beta'' \beta' \beta = \text{V} \alpha' \alpha'' S \beta'' \beta' \sigma + \text{V} \alpha'' \alpha S \beta \beta' \sigma + \text{V} \alpha \alpha' S \beta' \beta \sigma;$$

and accordingly, if we operate on the expression XXVI. for σ with the three symbols,

$$\text{XXVIII.} \dots S. \beta'' \beta', \quad S. \beta \beta'', \quad S. \beta' \beta,$$

we obtain the three scalar equations,

$$\text{XXIX.} \dots S \beta'' \beta' \sigma = S \beta'' \beta' \beta S \alpha \rho, \quad \&c.,$$

from which the equation XXVII. follows immediately, without any introduction of the auxiliary vectors λ, μ, ν , although these are useful in the theory generally.

(5.) Conversely, if the equation XXVII. were *given*, and the value of σ *sought*, we might operate with the three symbols,

$$\text{XXX.} \dots S. \alpha, \quad S. \beta, \quad S. \gamma,$$

and so obtain the three scalar equations XXIX., from which the expression XXVI. for σ would follow.

(6.) It will be found an useful check on formulæ of this sort, to consider each *beta*, in what we have called the *Standard Form* (1.) of $\phi\rho$, as being of the *first dimension*; for then we may say that ϕ and ϕ' are *also* of the *first dimension*, but ψ and ψ' of the *second*, and m of the *third*; and every formulæ, into which these symbols enter, will thus be *homogeneous*: $\alpha, \alpha', \alpha''$, and λ, μ, ν, ρ , being *not counted*, in this mode of estimating *dimensions*, but σ being treated as of the *first dimension*, when it is taken as representing $\phi\rho$.

(7.) And although the *trinomial form XV.* has been seen to be *sufficiently general*, yet if we choose to take the more expanded form,

$$\text{XXXI.} \dots \phi\rho = \Sigma\beta S\alpha\rho, \text{ which gives } \text{XXXII.} \dots \phi'\rho = \Sigma\alpha S\beta\rho,$$

any number of terms of $\phi\rho$, such as $\beta S\alpha\rho, \beta' S\alpha'\rho$, &c., being now included in the sum Σ , there is no difficulty in proving that the equations VIII. and IX. are satisfied, when we write,

$$\text{XXXIII.} \dots \psi\rho = \Sigma V\alpha\alpha'' S\beta'\beta\rho, \text{ with } \text{XXXIV.} \dots \psi'\rho = \Sigma V\beta\beta' S\alpha'\alpha\rho,$$

and

$$\text{XXXV.} \dots m = \Sigma S\alpha\alpha'\alpha'' S\beta'\beta'\beta = \Sigma S\beta\beta'\beta'' S\alpha''\alpha'\alpha.$$

(8.) The important property (2.), that the auxiliary function ψ is changed to its own *conjugate* ψ , when ϕ is changed to ϕ' , may be proved without any reference to the form $\Sigma\beta S\alpha\rho$ of $\phi\rho$, by means of the definitions IV. and XI., of ϕ' and ψ , as follows. Whatever four vectors μ, ν, μ_1 , and ν_1 may be, if we write

$$\text{XXXVI.} \dots \lambda'_1 = V\mu_1\nu_1, \text{ and } \text{XXXVII.} \dots \psi'V\mu\nu = V.\phi\mu\phi\nu,$$

adopting here this last equation as a *definition* of the function ψ' , we may proceed to prove that it is *conjugate* to ψ , by observing that we have the transformations,

$$\begin{aligned} \text{XXXVIII.} \dots S\lambda'_1\psi'\lambda' &= S(V\mu_1\nu_1.V.\phi\mu\phi\nu) = S.\mu_1(V.\nu_1V.\phi\mu\phi\nu) \\ &= S\mu_1\phi\nu.S\nu_1\phi\mu - S\mu_1\phi\mu.S\nu_1\phi\nu \\ &= S\mu\phi'\nu_1.S\nu\phi'\mu_1 - S\mu\phi'\mu_1.S\nu\phi'\nu_1 \\ &= S.\mu(V.\nu V.\phi'\mu_1\phi'\nu_1) = S(V\mu\nu.V.\phi'\mu_1\phi'\nu_1) = S\lambda'\psi\lambda'_1; \end{aligned}$$

which establish the relation in question, between ψ and ψ' .

(9.) And the not less important property (8.), that m remains *unchanged* when we pass from ϕ to ϕ' , may in like manner be proved, without reference to the form XV. or XXXI. of $\phi\rho$, by observing that we have by XXXVII., &c. the transformations,

$$\text{XXXIX.} \dots S.\phi\lambda\phi\mu\phi\nu = S.\phi\lambda\psi'\lambda' = S\lambda'\psi\phi\lambda = mS\lambda'\lambda = mS\lambda\mu\nu,$$

because the equations III. and VIII. give,

$$\text{XL.} \dots \psi\phi\rho = m\rho, \text{ whatever vector } \rho \text{ may be;}$$

so that the value of this scalar constant m may now be derived from the *original* linear function ϕ , exactly as it was in X. or X'. from the *conjugate* function ϕ' .

348. It is found, then, that the *linear and vector equation*,

$$\text{I.} \dots \phi\rho = \sigma, \text{ gives } \text{II.} \dots m\rho = \psi\sigma,$$

as its *formula of solution*; with the *general method*, above explained, of deducing m and ψ from ϕ . We have therefore the two *identities*,

$$\text{III.} \dots m\sigma = \phi\psi\sigma, \quad m\rho = \psi\phi\rho;$$

or briefly and symbolically,

$$\text{III}'. \dots m = \phi\psi = \psi\phi;$$

with which, by what has been shown, we may connect these *conjugate equations*,

$$\text{III}'' \dots m = \phi'\psi' = \psi'\phi'.$$

Changing then successively μ and ν to $\psi'\mu$ and $\psi'\nu$, in the equation of definition of the auxiliary function ψ , or in the formula,

$$\psi V\mu\nu = V.\phi'\mu\phi'\nu, \quad 347, \text{XI}';$$

we get these two other equations,

$$\text{IV.} \dots -\psi V.\nu\psi'\mu = mV.\mu\phi'\nu; \quad \text{V.} \dots \psi V.\psi'\mu\psi'\nu = m^2V\mu\nu;$$

in the former of which the *points* may be omitted, while in each of them *accented* may be exchanged with *unaccented* symbols of operation: and we see that the *law of homogeneity* (347, (6.)) is preserved. And many other transformations of the same sort may be made, of which the following are a few examples.

(1.) Operating on V. by ψ^{-1} , or by $m^{-1}\phi$, we get this new formula,

$$\text{VI.} \dots V.\psi'\mu\psi'\nu = m\phi V\mu\nu;$$

comparing which with the lately cited *definition* of ψ , we see that we may change ϕ to ψ , if we at the same time change ψ to $m\phi$, and therefore also m to m^2 ; ϕ' being then changed to ψ' , and ψ' to $m\phi'$.

(2.) For example, we may thus pass from IV. and V. to the formulae,

$$\text{VII.} \dots -\phi V\nu\phi'\mu = V\mu\psi'\nu, \quad \text{and} \quad \text{VIII.} \dots \phi V.\phi'\mu\phi'\nu = mV\mu\nu;$$

In which we see that the lately cited *law of homogeneity* is still observed.

(3.) The equation VII. might have been otherwise obtained, by interchanging μ and ν in IV., and operating with $-m^{-1}\phi$, or with $-\psi^{-1}$; and the formula VIII. may be at once deduced from the equation of definition of ψ , by operating on it with ϕ . In fact, our *rule of inversion*, of the *linear function* ϕ , may be said to be contained in the formulae,

$$\text{IX.} \dots \phi^{-1}V\mu\nu = m^{-1}V.\phi'\mu\phi'\nu;$$

where m is a scalar constant, as above.

(4.) By similar operations and substitutions,

$$\text{X.} \dots \phi^2 V.\phi'\mu\phi'\nu = m\phi V\mu\nu = V.\psi'\mu\psi'\nu;$$

$$\text{XI.} \dots m\phi V.\phi'\mu\phi'\nu = m^2 V\mu\nu = \psi V.\psi'\mu\psi'\nu;$$

$$\text{XII.} \dots m^2 V.\phi'\mu\phi'\nu = m^2 \psi V\mu\nu = \psi^2 V.\psi'\mu\psi'\nu;$$

$$\text{XIII.} \dots V.\phi^2 \mu\phi^2 \nu = \psi V.\phi'\mu\phi'\nu = \psi^2 V\mu\nu; \quad \&c.$$

(5.) But we have also,

$$\text{XIV. . . } S. \lambda \phi^2 \rho = S. \phi \rho \phi' \lambda = S. \rho \phi' \epsilon \lambda,$$

so that the *second functions* ϕ^2 and ϕ'^2 are *conjugate* (compare 347, IV.); hence, by XIII., ψ^2 is formed from ϕ^2 , as ψ from ϕ ; and generally it will be found, that if n be any whole number, and if we change ϕ to ϕ^n , we change at the same time ϕ' to ϕ'^n , ψ to ψ^n , ψ' to ψ'^n , and m to m^n .

(6.) It may also be remarked that the changes (1.) conduct to the equation,

$$\text{XV. . . } (S. \phi \lambda \phi \mu \phi \nu)^2 = S \lambda \mu \nu S. \psi \lambda \psi \mu \psi \nu;$$

and to many other analogous formulæ.

349. The expressions,

$$\lambda' \phi \lambda + \mu' \phi \mu + \nu' \phi \nu, \quad \lambda' \psi \lambda + \mu' \psi \mu + \nu' \psi \nu$$

with the significations 347, XX. of λ' , μ' , ν' , and others of the same type, are easily proved to *vanish* when λ , μ , ν are *coplanar*, and therefore to be *divisible* by $S \lambda \mu \nu$, since each such expression involves each of the three auxiliary vectors λ , μ , ν in the *first degree* only; the *quotients* of such divisions being therefore certain *constant quaternions*, independent of λ , μ , ν , and depending only on the *particular form* of ϕ , or on the (scalar or vector, but real) *constants*, which enter into the composition of that given function. Writing, then,

$$\text{I. . . } q_1 = (\lambda' \phi \lambda + \mu' \phi \mu + \nu' \phi \nu) : S \lambda \mu \nu,$$

and

$$\text{II. . . } q_2 = (\lambda' \psi \lambda + \mu' \psi \mu + \nu' \psi \nu) : S \lambda \mu \nu,$$

we shall find it useful to consider separately the scalar and vector *parts* of these two *quaternion constants*, q_1 and q_2 ; which constants are, respectively, of the *first* and *second dimensions*, in a sense lately explained.

(1.) Since $\nabla \lambda' \phi \lambda = \mu S \nu \phi \lambda - \nu S \lambda \phi' \mu$, &c., it follows that the vector parts of q_1 and q_2 change signs, when ϕ is changed to ϕ' , and therefore ψ to ψ' . On the other hand, we may change the arbitrary vectors λ , μ , ν to λ' , μ' , ν' , if we at the same time change λ' to $\nabla \mu' \nu'$, or to $-\lambda S \lambda \mu \nu$, &c., and $S \lambda \mu \nu$, or $S \lambda \lambda'$, to $-(S \lambda \mu \nu)^2$; dividing then by $-S \lambda \mu \nu$, we find these new expressions,

$$\text{III. . . } q_1 = (\lambda \phi \lambda' + \mu \phi \mu' + \nu \phi \nu') : S \lambda \mu \nu,$$

$$\text{IV. . . } q_2 = (\lambda \psi \lambda' + \mu \psi \mu' + \nu \psi \nu') : S \lambda \mu \nu;$$

operating on which by S , we return to the scalars of the expressions I. and II., with ϕ and ψ changed to ϕ' and ψ' .

(2.) Hence the *conjugate quaternion constants*, Kq_1 and Kq_2 , are obtained by passing to the *conjugate linear functions*; and thus we may write,

$$\text{V.} \dots Kq_1 = (\lambda'\phi'\lambda + \mu'\phi'\mu + \nu'\phi'\nu) : S\lambda\mu\nu;$$

$$\text{VI.} \dots Kq_2 = (\lambda'\psi'\lambda + \mu'\psi'\mu + \nu'\psi'\nu) : S\lambda\mu\nu;$$

or, interchanging λ with λ' , &c., in the dividends,

$$\text{VII.} \dots Kq_1 = (\lambda\phi'\lambda' + \mu\phi'\mu' + \nu\phi'\nu') : S\lambda\mu\nu;$$

$$\text{VIII.} \dots Kq_2 = (\lambda\psi'\lambda' + \mu\psi'\mu' + \nu\psi'\nu') : S\lambda\mu\nu;$$

where $\lambda' = \nabla\mu\nu$, &c., as before.

(3.) Operating with $\text{V.}\rho$ on $\text{V}q_1$, and observing that

$$\text{V.}\rho \text{V}\lambda'\phi\lambda = \phi(\lambda S\lambda'\rho) - \lambda' S\lambda\phi'\rho, \text{ \&c.,}$$

while

$$\phi(\lambda S\lambda'\rho + \mu S\mu'\rho + \nu S\nu'\rho) = \phi\rho S\lambda\mu\nu,$$

and

$$\lambda' S\lambda\phi'\rho + \mu' S\mu\phi'\rho + \nu' S\nu\phi'\rho = \phi\rho S\lambda\mu\nu,$$

with similar transformations for $\text{V.}\rho \text{V}q_2$, we find that

$$\text{IX.} \dots \text{V.}\rho \text{V}q_1 = \phi\rho - \phi'\rho;$$

and

$$\text{X.} \dots \text{V.}\rho \text{V}q_2 = \psi\rho - \psi'\rho.$$

(4.) Accordingly, since

$$S\rho(\phi\rho - \phi'\rho) = -S\rho(\phi\rho - \phi'\rho) = 0,$$

the vector $\phi\rho - \phi'\rho$, if it do not vanish, must be a line perpendicular to ρ , and therefore of the form,

$$\text{XI.} \dots \phi\rho - \phi'\rho = 2\text{V}\gamma\rho,$$

in which γ is some constant vector; so that we may write,

$$\text{XII.} \dots \phi\rho = \phi_0\rho + \text{V}\gamma\rho, \quad \phi'\rho = \phi_0\rho - \text{V}\gamma\rho,$$

where the function $\phi_0\rho$ is its own conjugate, or is the common self-conjugate part of $\phi\rho$ and $\phi'\rho$; namely the part,

$$\text{XIII.} \dots \phi_0\rho = \frac{1}{2}(\phi\rho + \phi'\rho).$$

And we see that, with this signification of γ ,

$$\text{XIV.} \dots \text{V}(\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu) = -2\gamma S\lambda\mu\nu, \quad \text{or} \quad \text{XIV'.} \dots \text{V}q_1 = -2\gamma;$$

while we have, in like manner,

$$\text{XV.} \dots \text{V}(\lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu) = -2\delta S\lambda\mu\nu, \quad \text{or} \quad \text{XV'.} \dots \text{V}q_2 = -2\delta,$$

if

$$\text{XVI.} \dots \psi\rho - \psi'\rho = 2\text{V}\delta\rho.$$

As a confirmation, the part ϕ_0 of ϕ has by (1.) no effect on $\text{V}q_1$; and if we change $\phi\lambda$ to $\text{V}\gamma\lambda$, &c., in the first member of XIV., we have thus,

$$(\lambda S\gamma\lambda' + \mu S\gamma\mu' + \nu S\gamma\nu') - \gamma S(\lambda\lambda' + \mu\mu' + \nu\nu') = \gamma S\lambda\mu\nu - \delta\gamma S\lambda\mu\nu.$$

(5.) Since $\text{V}\lambda'\psi\lambda = -\phi\text{V}\lambda\phi'\lambda'$, &c., by 848, VII., while we may write, by (1.), (2.), and (4.),

$$\text{XVII.} \dots \text{V}(\lambda\phi\lambda' + \mu\phi\mu' + \nu\phi\nu') = -2\gamma S\lambda\mu\nu,$$

$$\text{XVIII.} \dots \text{V}(\lambda\psi\lambda' + \mu\psi\mu' + \nu\psi\nu') = -2\delta S\lambda\mu\nu,$$

or

$$\text{XIX.} \dots \text{V}(\lambda\phi'\lambda' + \mu\phi'\mu' + \nu\phi'\nu') = +2\gamma S\lambda\mu\nu,$$

and

$$\text{XX.} \dots \text{V}(\lambda'\psi\lambda + \mu'\psi\mu + \nu'\psi\nu) = +2\delta S\lambda\mu\nu,$$

we have this relation between the two new vector constants,

$$\text{XXI.} \dots \delta = -\phi\gamma = -\phi'\gamma = -\phi_0\gamma;$$

for ϕ , ϕ' , and ϕ_0 have all the same effect, on this particular vector, γ .

(6.) We may add that the vector constant γ is of the first dimension, and that δ is of the second dimension, with respect to the betas of the standard form; in fact, with that form, 847, XV., of $\phi\rho$, we have the expressions,

XXII. . . $\gamma = \frac{1}{2}V(\beta\alpha + \beta'a' + \beta''a'')$,

and XXIII. . . $\delta = \frac{1}{2}V(V\beta\beta'.V\alpha'a'' + V\beta''\beta.V\alpha''a + V\beta\beta'.V\alpha\alpha')$.

(7.) If we denote by ψ_0 and m_0 , what ψ and m become when ϕ is changed to ϕ_0 , we easily find that

XXIV. . . $\psi\rho = \psi_0\rho - \gamma S\gamma\rho + V\delta\rho$; XXV. . . $\psi'\rho = \psi_0\rho - \gamma S\gamma\rho - V\delta\rho$;

so that the *self-conjugate part* of $\psi\rho$ contains a term, $-\gamma S\gamma\rho$, which involves the vector γ , but only in the *second degree*; and in like manner,

XXVI. . . $m = m_0 + S\gamma\delta = m_0 - S\gamma\phi\gamma$;

γ again entering only in an *even degree*, because m remains unchanged, when we pass from ϕ to ϕ' , or from γ to $-\gamma$.

(8.) It is evident that we have the relations,

XXVII. . . $m_0 = \phi_0\psi_0 = \psi_0\phi_0$;

and that, in a sense already explained, ϕ_0 , ψ_0 , and m_0 are of the *first, second, and third dimensions*, respectively.

350. After thus considering the *vector parts* of the two quaternion constants, q_1 and q_2 , we proceed to consider their *scalar parts*; which will introduce two new scalar constants, m'' and m' , and will lead to the employment of two new conjugate auxiliary functions, $\chi\rho$ and $\chi'\rho$; whence also will result the establishment of a certain *Symbolic and Cubic Equation*,

I. . . $0 = m - m'\phi + m''\phi^3 - \phi^3$,

which is *satisfied by the Linear Symbol of Operation*, ϕ , and is of great importance in this whole *Theory of Linear Functions*.

(1.) Writing, then,

II. . . $m'' = Sq_1$, and III. . . $m' = Sq_2$,

we see first that neither of these two new constants changes value, when we pass from ϕ to ϕ' , or from γ to $-\gamma$; because, in such a passage, it has been seen that we only change q_1 and q_2 to Kq_1 and Kq_2 . Accordingly, if we denote by m'_0 and m''_0 what m' and m'' become, when ϕ is changed to ϕ_0 , we easily find the expressions,

IV. . . $m'' = m''_0$; and V. . . $m' = m'_0 - \gamma^2$.

(2.) It may be noted that m'' , or m''_0 , is of the *first dimension*, but that m' and m'_0 are of the *second*, with respect to the standard form of ϕ ; and accordingly, with that form we have,

VI. . . $m'' = S\alpha\beta + S\alpha'\beta' + S\alpha''\beta''$;

and VII. . . $m' = S(V\alpha'a''.V\beta''\beta' + V\alpha'a.V\beta\beta'' + V\alpha\alpha'.V\beta'\beta)$.

(3.) If we introduce two new linear functions, $\chi\rho$ and $\chi'\rho$, such that

VIII. . . $\chi V\mu\nu = V(\mu\phi'\nu - \nu\phi'\mu)$,

and IX. . . $\chi' V\mu\nu = V(\mu\phi\nu - \nu\phi\mu)$,

it is easily proved that these functions are *conjugate* to each other, and that each is of the *first* dimension; in fact, with the standard form of $\phi\rho$, we have the expressions,

$$\text{X.} \dots \chi\rho = \mathbf{V}(\alpha\mathbf{V}\beta\rho + \alpha'\mathbf{V}\beta'\rho + \alpha''\mathbf{V}\beta''\rho),$$

$$\text{XI.} \dots \chi'\rho = \mathbf{V}(\beta\mathbf{V}\alpha\rho + \beta'\mathbf{V}\alpha'\rho + \beta''\mathbf{V}\alpha''\rho);$$

and $\mathbf{S}.\lambda\alpha\mathbf{V}\beta\rho = \mathbf{S}.\rho\beta\mathbf{V}\alpha\lambda$, &c. Also, if χ_0 be formed from ϕ_0 , as χ from ϕ , it will be found that

$$\text{XII.} \dots \chi\rho = \chi_0\rho - \mathbf{V}\gamma\rho, \quad \text{and} \quad \text{XIII.} \dots \chi'\rho = \chi_0\rho + \mathbf{V}\gamma\rho;$$

where χ_0 is of the first dimension.

(4.) Since

$$\mathbf{S}\lambda\chi\lambda' = \mathbf{S}.\lambda(\mu\phi'\nu - \nu\phi'\mu) = \mathbf{S}(\mu'\phi'\mu + \nu'\phi'\nu),$$

the expression II. gives, by 849, V., the equation,

$$\text{XIV.} \dots m''\mathbf{S}\lambda\lambda' = \mathbf{S}.\lambda(\phi + \chi)\lambda',$$

λ and λ' being two *arbitrary* and *independent* vectors; which can only be, by our having the *functional relation*,

$$\text{XV.} \dots \phi\rho + \chi\rho = m''\rho;$$

or briefly and symbolically,

$$\text{XVI.} \dots \chi + \phi = m''.$$

Accordingly it is evident that the relation XV. is verified, by the form X. of $\chi\rho$, combined with the standard form of $\phi\rho$, and with the expression VI. for the constant m'' .

(5.) The formula XVI. gives,

$$\text{XVII.} \dots \chi\phi = m''\phi - \phi^2 = \phi\chi;$$

and accordingly the identity of $\chi\phi$ and $\phi\chi$ may easily be otherwise proved, by changing μ and ν to $\psi'\mu$ and $\psi'\nu$ in the definition VIII. of χ , and remembering that

$$\mathbf{V}.\psi'\mu\psi'\nu = m\phi\mathbf{V}\mu\nu, \quad \phi'\psi' = m, \quad \text{and} \quad \mathbf{V}\mu\psi'\nu = -\phi\mathbf{V}\nu\phi'\mu;$$

for thus we have,

$$\text{XVIII.} \dots \chi\phi\mathbf{V}\mu\nu = \mathbf{V}(\mu\psi'\nu - \nu\psi'\mu) = \phi\mathbf{V}(\mu\phi'\nu - \nu\phi'\mu) = \phi\chi\mathbf{V}\mu\nu,$$

as required.

(6.) Since, then,

$$\mathbf{S}.\lambda\phi\chi\lambda' = \mathbf{S}.\lambda(\mu\psi'\nu - \nu\psi'\mu) = \mathbf{S}(\mu'\psi'\mu + \nu'\psi'\nu),$$

the value III. of m' gives, by 849, VI., the equation,

$$\text{XIX.} \dots m'\mathbf{S}\lambda\lambda' = \mathbf{S}.\lambda(\psi + \phi\chi)\lambda',$$

λ and λ' being independent vectors; hence,

$$\text{XX.} \dots \psi\rho + \phi\chi\rho = m'\rho,$$

or briefly,

$$\text{XXI.} \dots \psi + \phi\chi = m'.$$

And in fact, with the standard form of $\phi\rho$, we have

$$\text{XXII.} \dots \phi\chi\rho = \chi\phi\rho = \mathbf{V}(\mathbf{V}\beta'\beta''.\mathbf{V}\rho\mathbf{V}\alpha'\alpha'' + \mathbf{V}\beta''\beta'.\mathbf{V}\rho\mathbf{V}\alpha''\alpha' + \mathbf{V}\beta\beta'.\mathbf{V}\rho\mathbf{V}\alpha\alpha');$$

which verifies the equation XX., when it is combined with the value VII. of m' , and with the expression 847, XVIII. for $\psi\rho$.

(7.) *Eliminating* the symbol χ , between the two equations XVI. and XXI., and remembering that $\phi\psi = \psi\phi = m$, we find the symbolic expression,

$$\text{XXIII.} \dots m\phi^{-1} = \psi = m' - m''\phi + \phi^2;$$

and thus the *symbolic and cubic equation I.* is proved.

(8.) And because the *coefficients*, m, m', m'' , of that equation, have been seen to remain unaltered, in the passage from ϕ to ϕ' , we may write also this *conjugate equation*,

$$\text{XXIV.} \dots 0 = m - m'\phi' + m''\phi'^2 - \phi'^3.$$

(9.) Multiplying symbolically the equation I. by $-m^{-1}\psi^2$, and reducing by $\psi\phi = m$, we eliminate the symbol ϕ , and obtain this *cubic in* ψ ,

$$\text{XXV.} \dots 0 = m^2 - mm''\psi + m'\psi^2 - \psi^3;$$

in which ψ' may be substituted for ψ .

(10.) In general, it may be remarked, that when we change ϕ to ψ , and therefore ψ to $m\phi$, as before, we change not only m to m^2 , but also m' to mm'' , and m'' to m' ; while χ is at the same time changed to $\phi\chi$, or to $\chi\phi$, and the quaternion q_1 is changed to q_2 . Accordingly, we may thus pass from the relation XVI. to XXI.; and conversely, from the latter to the former.

(11.) And if the two new auxiliary functions, χ and χ' , be considered as *defined* by the equations VIII. and IX., their *conjugate relation* (8.) to each other may be *proved*, without any reference to the *standard form* of $\phi\rho$, by reasonings similar to those which were employed in 847, (8.), to establish the corresponding conjugation of the functions ψ and ψ' .

(12.) It may be added that the relations between ϕ, ϕ', χ, χ' , and m'' give the following additional transformations, which are occasionally useful:

$$\text{XXVI.} \dots \phi'V\mu\nu = V(\mu\chi\nu + \nu\phi\mu) = -V(\nu\chi\mu + \mu\phi\nu);$$

$$\text{XXVII.} \dots \phi V\mu\nu = V(\mu\chi'\nu + \nu\phi'\mu) = -V(\nu\chi'\mu + \mu\phi'\nu);$$

with others on which we cannot here delay.

351. The cubic in ϕ may be thus written :

$$\text{I.} \dots 0 = m\rho - m'\phi\rho + m''\phi^2\rho - \phi^3\rho;$$

where ρ is an arbitrary vector. If then it happen that for some *particular* but *actual* vector, ρ , the linear function $\phi\rho$ vanishes, so that $\phi\rho = 0, \phi^2\rho = 0, \phi^3\rho = 0, \&c.$, the *constant* m must be *zero*; or in symbols,

$$\text{II.} \dots \text{if } \phi\rho = 0, \text{ and } T\rho > 0, \text{ then } m = 0.$$

Hence, by the expression 347, XXIII. for m , when the *standard form* for $\phi\rho$ is adopted, we must have either

$$\text{III.} \dots Saa'' = 0, \text{ or else } \text{IV.} \dots S\beta''\beta'\beta = 0;$$

so that, in *each* case, that *generally trinomial form*, 347, XV., must admit of being *reduced to a binomial*. Conversely, when we have thus a function of the *particular form*,

$$V. \dots \phi\rho = \beta S a\rho + \beta' S a'\rho,$$

we have then,

$$VI. \dots \phi V a a' = 0;$$

so that if a and a' be *actual* and *non-parallel* lines, the *real* and *actual* vector $V a a'$ will be a value of ρ , which will satisfy the equation $\phi\rho = 0$; but *no other real and actual* value of ρ , except $\rho = x V a a'$, will satisfy that equation, if β and β' be *actual*, and *non-parallel*. In this case V., the operation ϕ reduces every other vector to the fixed plane of β , β' , which plane is therefore the *locus* of $\phi\rho$; and since we have also,

$$VII. \dots \phi'\rho = a S \beta\rho + a' S \beta'\rho,$$

we see that the *locus* of the functionally conjugate vector, $\phi'\rho$, is another fixed plane, namely that of a , a' . Also, the normal to the latter plane is the line which is destroyed by the former operation, namely by ϕ ; while the normal to the former plane is in like manner the line, which is annihilated by the latter operation, ϕ' , since we have,

$$VIII. \dots \phi' V \beta \beta' = 0,$$

but not $\phi'\rho = 0$, for any *actual* ρ , in any direction except that of $V \beta \beta'$, or its opposite, which may however, for the present purpose, be regarded as the same*. In this case we have also monomial forms for $\psi\rho$ and $\psi'\rho$, namely

$$IX. \dots \psi\rho = V a a' S \beta' \beta \rho, \quad \text{and} \quad X. \dots \psi'\rho = V \beta \beta' S a' a \rho;$$

so that the operation ψ destroys every line in the first fixed plane (of β , β'), and the conjugate operation ψ' annihilates every line in the second fixed plane (of a , a'). On the other hand, the operation ψ reduces every line, which is out of the first plane, to the fixed direction of the normal to the second plane; and the operation ψ' reduces every line which is out of the second plane, to that other fixed direction, which is normal to the first plane. And thus it comes to pass, that whether we operate first with ψ , and then with ϕ ; or first with ϕ , and then with ψ ; or first with ψ' and then with ϕ' ; or first with ϕ' ,

* Accordingly, in the present investigation, whenever we shall speak of a "fixed direction," or the "direction of a given line," &c., we are always to be understood as meaning, "or the opposite of that direction."

and then with ψ' ; in *all* these cases, we *arrive at last at a null line*, in conformity with the symbolic equations,

$$\text{XI. . . } \phi\psi = \psi\phi = \phi'\psi' = \psi'\phi' = m = 0,$$

which belong to the case here considered.

(1.) Without recurring to the *standard form* of $\phi\rho$, the equation 848, VI., namely $V.\psi'\mu\psi\nu = m\phi V\mu\nu$, and the analogous equation $V.\psi\mu\psi\nu = m\phi'V\mu\nu$, might have enabled us to foresee that $\psi'\rho$ and $\psi\rho$, if they do not both *constantly vanish*, must (if $m = 0$) have each a *fixed direction*; and therefore that each must be expressible by a *monome*, as above: the fixed *direction* of $\psi\rho$ being that of a line which is *annihilated* by the operation ϕ , and similarly for $\psi'\rho$ and ϕ' .

(2.) And because, by 847, XI. and XXV., we have

$$\psi V\mu\nu = V.\phi'\mu\phi'\nu, \text{ and } \psi'V\mu\nu = V.\phi\mu\phi\nu,$$

so that the line $\phi'\mu$, if actual, is perpendicular to $\psi V\mu\nu$, and the line $\phi\mu$ perpendicular to $\psi'V\mu\nu$, we see that *each of the two lines*, $\phi'\rho$ and $\phi\rho$, must have (in the present case) a *plane locus*; whence the *binomial forms* of the two *conjugate vector functions*, $\phi\rho$ and $\phi'\rho$, might have been foreseen: $\psi\rho$ and $\psi'\rho$ being here supposed to be *actual vectors*.

(3.) The *relations of rectangularity*, of the two *fixed lines* (or *directions*), to the two *fixed planes*, might also have been thus deduced, through the two *conjugate binomial forms*, V. and VII., without the *previous establishment* of the more *general trinomial* (or *standard*) *form* of $\phi\rho$.

(4.) The existence of a *plane locus* for $\phi\rho$, and of another for $\phi'\rho$, for the case when $m = 0$, might also have been foreseen from the equations,

$$S.\phi\lambda\phi\mu\phi\nu = S.\phi'\lambda\phi'\mu\phi'\nu = mS\lambda\mu\nu;$$

and the same equations might have enabled us to foresee, that the *scalar constant* m must be *zero*, if for any *one actual vector*, such as λ , either $\phi\lambda$ or $\phi'\lambda$ becomes *null*.

(5.) And the *reducibility* of the *trinomial* to the *binomial form*, when this last condition is satisfied, might have been anticipated, without any reference to the composition of the constant m , from the simple consideration (comp. 294, (10.)), that *no actual vector* ρ can be *perpendicular*, at once, to *three diplanar lines*.

352. It may happen, that besides the recent reduction (351) of the *linear function* $\phi\rho$ to a *binomial form*, when the *relation*

$$\text{I. . . } m = 0$$

exists between the *constants* of that function, in which case the symbolic and *cubic equation* 350, I. reduces itself to the form,

$$\text{II. . . } \phi^3 - m''\phi^2 + m'\phi = 0,$$

thus losing its absolute term, or having *one root* equal to *zero*,

this equation may undergo a further *reduction*, by *two* of its roots becoming *equal* to *each other*; namely either by our having

$$\text{III.} \dots m' = 0, \quad \text{and} \quad \text{IV.} \dots \phi^2(\phi - m'') = 0;$$

or in another way, by the existence of these other equations,

$$\text{V.} \dots m''^2 - 4m' = 0, \quad \text{and} \quad \text{VI.} \dots \phi(\phi - \frac{1}{2}m'')^2 = 0.$$

In *each* of these two cases, we shall find that certain *new geometrical relations* arise, which it may be interesting briefly to investigate; and of which the principal is the mutual *rectangularity* of *two fixed planes*, which are the *loci* (comp. 351) of certain *derived*, and *functionally conjugate vectors*: namely, in the case III. IV., the loci of $\phi\rho$ and $\phi'\rho$; and in the case V. VI., the loci of $\Phi\rho$ and $\Phi'\rho$, if

$$\text{VII.} \dots \Phi = \phi - \frac{1}{2}m'', \quad \text{and} \quad \text{VIII.} \dots \Phi' = \phi' - \frac{1}{2}m'',$$

so that, in this last case, the symbol Φ satisfies this *new cubic*,

$$\text{IX.} \dots 0 = \Phi^2(\Phi + \frac{1}{2}m'');$$

while Φ' satisfies at the same time a cubic equation with the *same coefficients* (comp. 350, (8.)), namely

$$\text{X.} \dots 0 = \Phi'^2(\Phi' + \frac{1}{2}m'').$$

(1.) We saw in 351, (1.), (2.), that when $m = 0$ the line $\psi'\rho$ has *generally a fixed direction*, to which that of the line $\phi\rho$ is *perpendicular*; and that in like manner the line $\psi\rho$ has then *another fixed direction*, to which $\phi'\rho$ is perpendicular. If then the *plane loci* of $\phi\rho$ and $\phi'\rho$ be at *right angles* to each other, we must also have the *fixed lines* $\psi'\lambda$ and $\psi\mu$ *rectangular*, or

$$\text{XI.} \dots 0 = S.\psi'\lambda\psi\mu = S\lambda\psi^2\mu,$$

independently of the directions of λ and μ ; whence

$$\text{XII.} \dots 0 = \psi^2\mu, \quad \text{or} \quad \text{XIII.} \dots \psi^2 = 0,$$

since μ is an arbitrary vector.

(2.) Now *in general*, by the functional relation 350, XXI. combined with $\psi\phi = m$, we have the transformation,

$$\text{XIV.} \dots \psi^2 = \psi(m' - \phi\chi) = m'\psi - m\chi;$$

if then $m = 0$, as in I., the symbol ψ must satisfy the *depressed* or *quadratic equation*,

$$\text{XV.} \dots 0 = m'\psi - \psi^2;$$

which is accordingly a *factor* of the *cubic equation*,

$$\text{XVI.} \dots 0 = m'\psi^2 - \psi^3,$$

whereto the general equation 350, XXV. is *reduced*, by this supposition of m vanishing.

(3.) If then we have *not only* $m = 0$, as in I., but *also* $m' = 0$, as in III., the condition XIII. is satisfied, by XV.; and the *two planes*, above referred to, are generally *rectangular*.

(4.) We might indeed propose to satisfy that condition XIII., by supposing that we had always,

$$\text{XVII.} \dots \psi = 0, \text{ that is, } \text{XVII}'. \dots \psi\rho = 0,$$

for every direction of ρ ; but in *this case*, the *quaternion constant* q_2 would vanish (by 349, II.); and therefore the constant m' , as being its *scalar part* (by 350, III.), would *still* be equal to zero.

(5.) The particular supposition XVII. would however *alter* completely the *geometrical character* of the question; for it would imply (comp. 351, (2.)) that the *directions* of the lines $\phi\rho$ and $\phi'\rho$ (when not *evanescent*) are *fixed*, instead of those lines having only certain *planes* for their *loci*, as before.

(6.) On the side of *calculation*, we should thus have, for the two *conjugate functions*, $\phi\rho$ and $\phi'\rho$, *monomial expressions* of the forms,

$$\text{XVIII.} \dots \phi\rho = \beta\beta a\rho, \quad \phi'\rho = a\beta\beta\rho;$$

whence, by 347, XVIII., and 350, VII., we should recover the equations, $\psi\rho = 0$ and $m' = 0$.

(7.) We should have also, in this particular case,

$$\text{XIX.} \dots \phi\rho = 0, \text{ if } \rho \perp a, \text{ and } \text{XX.} \dots \phi'\rho = 0, \text{ if } \rho \perp \beta;$$

so that $\phi\rho$ now *vanishes*, if ρ be *any line* in the *fixed plane* perpendicular to a ; and in like manner $\phi'\rho$ is a null line, if ρ be in that *other fixed plane*, which is at right angles to the *other given line*, β .

(8.) *These two planes*, or their *normals* a and β , or the *fixed directions* of the two lines $\phi'\rho$ and $\phi\rho$, will be *rectangular* (comp. (1.)), if we have this new equation,

$$\text{XXI.} \dots \phi^2 = 0, \text{ or } \text{XXI}'. \dots \phi^2\rho = 0,$$

for every direction of ρ ; and accordingly the expression XVIII. gives

$$\phi^2\rho = \beta a \beta. \phi\rho = 0, \text{ if } \beta \perp a, \text{ and reciprocally.}$$

(9.) Without expressly introducing a and β , the equation 350, XXIII. shows that when $\psi = 0$, and therefore also $m' = 0$, as in (4.), the symbol ϕ satisfies (comp. (2.)) the *new quadratic or depressed equation*,

$$\text{XXII.} \dots 0 = \phi^2 - m''\phi;$$

which is accordingly a *factor of the cubic IV.*, but to which that cubic is *not reducible*, unless we have thus $\psi = 0$, as well as $m' = 0$.

(10.) The *condition*, then, of the *existence and rectangularity* of the *two planes* (7.), for which we have respectively $\phi\rho = 0$ and $\phi'\rho = 0$, without $\phi\rho$ generally vanishing (a case which it would be useless to consider), is that the four following equations should subsist:

$$\text{XXIII.} \dots m = 0, \quad m' = 0, \quad m'' = 0, \text{ and } \text{XVII.} \dots \psi = 0;$$

or that the *cubic IV.*, and its *quadratic factor XXII.*, should reduce themselves to the very simple forms,

$$\text{XXIV.} \dots \phi^3 = 0, \text{ and } \text{XXV.} \dots \phi^2 = 0;$$

the cubic in ϕ having thus its *three roots equal*, and *null*, and $\psi\rho$ *vanishing*.

(11.) We may also observe that as, when even *one* root of the general cubic 850, I. is zero, that is when $m=0$, the vector equation

$$\text{XXVI.} \dots \phi\rho = 0$$

was seen (in 851) to be satisfied by *one real direction* of ρ , so when we have *also* $m'=0$, or when the cubic in ϕ has *two null roots*, or takes the form IV., then the *two vector equations*,

$$\text{XXVII.} \dots \phi\rho = 0, \quad \psi\rho = 0,$$

are satisfied by one *common direction* of the *real and actual line* ρ ; because we have, by 850, XVII. and XX., the *general relation*,

$$\psi\rho = m'\rho - \chi\phi\rho.$$

(12.) And because, by 850, XV., we have also the relation $\chi\rho = m''\rho - \phi\rho$, it follows that when the *three roots* of the cubic *all vanish*, or when the *three scalar equations* XXIII. are satisfied, then the *three vector equations*,

$$\text{XXVIII.} \dots \phi\rho = 0, \quad \psi\rho = 0, \quad \chi\rho = 0,$$

have a *common (real and actual) vector root*; or are all satisfied by one *common direction* of ρ .

(13.) Since $m'' - \phi = \chi$, the cubic IV. may be written under any one of the following forms,

$$\text{XXIX.} \dots 0 = \phi^2\chi = \phi\chi\phi = \chi\phi^2 = \phi \cdot \phi\chi = \&c.,$$

in which accented may be substituted for unaccented symbols: and its *geometrical signification* may be illustrated by a reference to certain *fixed lines*, and *fixed planes*, as follows.

(14.) Suppose first that m and m' both vanish, but that m'' is different from zero, so that the cubic in ϕ is reducible to the form IV., but *not* to the form XXIV.; and that the operation ψ , which is here equivalent to $-\phi\chi$, or to $-\chi\phi$, does not annihilate *every* vector ρ , so that (comp. (4.) (5.) (6.)) $\phi\rho$ and $\phi'\rho$ have *not* the directions of *two fixed lines*, but have only (comp. (1.) and (8.)) *two fixed and rectangular planes*, Π and Π' , as their *loci*; and let the *normals* to these two planes be denoted by λ and λ' , so that these two rectangular lines, λ and λ' , are situated respectively in the planes Π' and Π .

(15.) Then it is easily shown (comp. 851) that the operation ϕ *destroys* the line λ' *itself*, while it *reduces** every *other* line (that is, every line which is not of the form $x\lambda'$, with $Vx=0$) to the *plane* $\Pi + \lambda$; and that it reduces every line in that plane to a *fixed direction*, μ , in the same plane, which is thus the *common direction* of all the lines $\phi^2\rho$, whatever the direction of ρ may be. And the symbolical equation, $\chi \cdot \phi^2 = 0$, expresses that this fixed direction μ of $\phi^2\rho$ may also be denoted by $\chi^{-1}0$; or that we have the equation,

$$\text{XXX.} \dots 0 = \chi\mu = m''\mu - \phi\mu, \quad \text{if } \mu = \phi^2\rho,$$

which can accordingly be otherwise proved: with similar results for the conjugate symbols, ϕ' and χ' .

* We propose to include the case where an *operation* of this sort *destroys* a line, or reduces it to *zero*, under the case when the same operation *reduces* a line to a *fixed direction*, or to a *fixed plane*.

(16.) For example, we may represent the conditions of the present case by the following system of equations (comp. 851, V. VII. IX. X., and 850, VI. VII. X. XI):

$$\text{XXXI.} \dots \begin{cases} \phi\rho = \beta Sa\rho + \beta' Sa'\rho, & \phi'\rho = aS\beta\rho + a'S\beta'\rho, \\ 0 = m' = S(Va\alpha'.V\beta\beta') = Sa\beta Sa'\beta' - Sa'\beta Sa\beta, \\ m'' = Sa\beta + Sa'\beta'; \end{cases}$$

$$\text{XXXII.} \dots \begin{cases} \chi\rho = V(aV\beta\rho + a'V\beta'\rho) = m''\rho - \phi\rho, \\ \chi'\rho = V(\beta V\alpha\rho + \beta'V\alpha'\rho) = m'''\rho - \phi'\rho, \\ -\psi\rho = \phi\chi\rho = \chi\phi\rho = Va\alpha'S\beta\beta'\rho, \\ -\psi'\rho = \phi'\chi'\rho = \chi'\phi'\rho = V\beta\beta'Sa\alpha'\rho; \end{cases}$$

and may then write (not here supposing $\lambda' = V\mu\nu$, &c.),

$$\text{XXXIII.} \dots \begin{cases} \lambda = V\beta\beta', & \lambda' = Va\alpha', & S\lambda\lambda' = 0, \\ \mu = \phi\beta \parallel \phi\beta', & \mu' = \phi'a' \parallel \phi'a, & S\lambda\mu = S\lambda'\mu' = 0; \end{cases}$$

after which we easily find that

$$\text{XXXIV.} \dots \begin{cases} \phi\lambda' = 0, & \phi^2\rho \parallel \mu, & \phi\mu = m''\mu, & \chi\mu = 0; \\ \phi\lambda = 0, & \phi^2\rho \parallel \mu', & \phi'\mu' = m'''\mu', & \chi'\mu' = 0. \end{cases}$$

(17.) Since we have thus $\chi'\mu' = 0$, where μ' is a line in the fixed direction of $\phi^2\rho$, we have also the equation,

$$\text{XXXV.} \dots 0 = S\rho\chi'\mu' = S\mu'\chi\rho, \text{ or } \chi\rho \perp \mu';$$

the locus of $\chi\rho$ is therefore a plane perpendicular to the line μ' ; and in like manner, μ is the normal to a plane, which is the locus of the line $\chi\rho$. And the symbolical equations, $\phi \cdot \phi\chi = 0$, $\phi^2 \cdot \chi = 0$, may be interpreted as expressing, that the operation ϕ reduces every line in this new plane of $\chi\rho$ to the fixed direction of $\phi^{-1}0$, or of λ' ; and that the operation ϕ^2 destroys every line in this plane $\perp \mu'$; with analogous results, when accented are interchanged with unaccented symbols. Accordingly we see, by XXXII., that $\phi\chi\rho$ has the fixed direction of $Va\alpha'$, or of λ' ; and that $\phi \cdot \phi\chi\rho = 0$, because $\phi\lambda' = 0$.

(18.) We see also, that the operation $\phi\chi$, or $\chi\phi$, destroys every line in the plane Π , to which the operation ϕ reduces every line; and that thus the symbolical equations, $\phi\chi \cdot \phi = 0$, $\chi\phi \cdot \phi = 0$, may be interpreted.

(19.) As a verification, it may be remarked that the fixed direction λ' , of $\phi\chi\rho$ or $\chi\phi\rho$, ought to be that of the line of intersection of the two fixed planes of $\phi\rho$ and $\chi\rho$; and accordingly it is perpendicular by XXXIII. to their two normals, λ and μ' : with similar remarks respecting the fixed direction λ , of $\phi'\chi'\rho$ or $\chi'\phi'\rho$, which is perpendicular to λ' and to μ .

(20.) Let us next suppose, that besides $m = 0$, and $m' = 0$, we have $\psi = 0$, but that m'' is still different from zero. In this case, it has been seen (6.) that the expression for $\phi\rho$ reduces itself to the monomial form, $\beta Sa\rho$; and therefore that the operation ϕ destroys every line in a fixed plane ($\perp a$), while it reduces every other line to a fixed direction ($\parallel \beta$), which is not contained in that plane, because we have not now $Sa\beta = 0$.

(21.) In this case we have by (16.), equating a' or β' to 0, the expressions,

$$\text{XXXVI.} \dots \begin{cases} \phi\rho = \beta Sa\rho, & \phi'\rho = aS\beta\rho, & m'' = Sa\beta > 0, \\ \chi\rho = V.aV\beta\rho = (m'' - \phi)\rho, & \chi'\rho = V.\beta V\alpha\rho = (m'' - \phi')\rho, \end{cases}$$

so that the equations XVIII. are reproduced; and the depressed cubic, or the quadratic XXII. in ϕ , may be written under the very simple form,

XXXVII. . . $0 = \phi\chi = \chi\phi$.

(22.) Accordingly (comp. (6.) and (7.)), the operation ϕ here reduces an arbitrary line to the fixed direction of β , while χ destroys every line in that direction; and conversely, the operation χ reduces an arbitrary line to the fixed plane perpendicular to α , and ϕ destroys every line in that fixed plane. But because we do not *here* suppose that $m'' = 0$, the *fixed direction* of $\phi\rho$ is *not contained* in the *fixed plane* of $\chi\rho$; and (comp. (8.) and (10.)) the directions of $\phi\rho$ and $\phi'\rho$ are *not rectangular* to each other.

(23.) On the other hand, if we suppose that the *three roots* of the cubic in ϕ *vanish*, or that we have $m = 0$, $m' = 0$, and $m'' = 0$, as in XXIII., but that the equation $\psi\rho = 0$ is *not satisfied* for *all* directions of ρ , then the *binomial forms* XXXI. of $\phi\rho$ and $\phi'\rho$ reappear, but with these *two* equations of condition between their *vector constants*, whereof only *one* had occurred before:

$$\text{XXXVIII. . . } 0 = Sa\beta Sa'\beta' - Sa\beta'Sa'\beta', \quad 0 = Sa\beta + Sa'\beta'.$$

(24.) We have also now the expressions,

$$\text{XXXIX. . . } \chi\rho = -\phi\rho, \quad \chi'\rho = -\phi'\rho;$$

and the cubic in ϕ becomes simply $\phi^3 = 0$, as in XXIV.; but it is important to observe that we have *not here* (comp. (9.)) the *depressed* or *quadratic* equation $\phi^2 = 0$, since we have *now* on the contrary the two conjugate expressions,

$$\text{XL. . . } \phi^3\rho = \psi\rho = Vaa'S\beta'\beta\rho, \quad \phi'^3\rho = \psi'\rho = V\beta\beta'Sa'a\rho,$$

which do not generally vanish. And the equation $\phi^3 = 0$ is now *interpreted*, by observing that ϕ^2 here reduces *every* line to the *fixed direction* of $\phi^{-1}0$; while ϕ reduces an arbitrary vector to that *fixed plane*, all lines in which are destroyed by ϕ^2 .

(25.) In this last case (28.), in which *all* the roots of the cubic in ϕ are *equal*, and are *null*, the theorem (12.), of the existence of a *common vector root* of the three equations XXVIII., may be verified by observing that we have now,

$$\text{XLI. . . } \phi Vaa' = 0, \quad \psi Vaa' = 0, \quad \chi Vaa' = 0;$$

the *third* of which would not have here held good, unless we had supposed $m'' = 0$.

(26.) This last condition allows us to write, by (16.),

$$\text{XLII. . . } \phi\mu = 0, \quad \phi'\mu' = 0, \quad V\mu\lambda' = 0, \quad V\mu'\lambda = 0, \quad S\mu\mu' = 0,$$

the lines μ' and μ thus coinciding in direction with the normals λ and λ' , to the planes II and II'; if then we write,

$$\text{XLIII. . . } \nu = V\lambda\lambda' \parallel V\mu\mu', \quad \text{so that } S\mu\nu = 0, \quad S\mu'\nu = 0,$$

this new vector ν will be a line in the *intersection* of those *two rectangular planes*, which were lately seen (14.) to be the *loci* of the lines $\phi\rho$ and $\phi'\rho$, and are *now* (comp. (17.)) the *loci* of $\chi\rho$ and $\chi'\rho$; and the *three lines* μ , μ' , ν (or λ' , λ , ν) will compose a *rectangular system*.

(27.) *In general*, it is easy to prove that the expressions,

$$\text{XLIV. . . } \begin{cases} \beta = a\beta_1 + b\beta'_1, & \beta' = a'\beta_1 + b'\beta'_1, \\ \alpha_1 = a\alpha + a'\alpha', & \alpha'_1 = b\alpha + b'\alpha', \end{cases}$$

in which a , β , a' , β' may be *any four vectors*, and a , b , a' , b' may be *any four scalars*, conduct to the following transformations (in which ρ may be *any vector*):

- XLV. . . $Sa_1\beta_1 + Sa'_1\beta'_1 = Sa\beta + Sa'\beta'$;
- XLVI. . . $\beta_1Sa_1\rho + \beta'_1Sa'_1\rho = \beta Sa\rho + \beta' Sa'\rho$;
- XLVII. . . $\nabla a_1a'_1 \cdot \nabla \beta'_1\beta_1 = \nabla aa' \cdot \nabla \beta\beta'$;

so that the *scalar*, $Sa\beta + Sa'\beta'$; the *vector*, $\beta Sa\rho + \beta' Sa'\rho$; and the *quaternion*,* $\nabla aa' \cdot \nabla \beta\beta'$, remain *unaltered* in value, when we pass from a *given system* of four vectors $a\beta a'\beta'$, to another system of four vectors $a_1\beta_1 a'_1\beta'_1$, by expressions of the forms XLIV.

(28.) With the help of this general principle (27.), and of the remarks in (26.), it may be shown, without difficulty, that in the case (28.) the vector constants of the binomial expression $\beta Sa\rho + \beta' Sa'\rho$ for $\phi\rho$ may, without any real loss of generality, be supposed subject to the four following conditions,

$$\text{XLVIII. . . } 0 = Sa\beta = Sa'\beta' = S\beta\beta' = Sa'\beta';$$

which evidently conduct to these other expressions,

$$\text{XLIX. . . } \phi^2\rho = \beta Sa\beta' Sa'\rho, \quad \phi^3\rho = 0;$$

and thus put in evidence, in a very simple manner, the *general non-depression* of the cubic $\phi^3 = 0$, to the *quadratic*, $\phi^2 = 0$.

(29.) The case, or *sub-case*, when we have not only $m = 0$, $m' = 0$, $m'' = 0$, but also $\psi = 0$, and therefore $\phi^3 = 0$, as a depressed form of $\phi^3 = 0$, by the linear function $\phi\rho$ reducing itself to the monomial $\beta Sa\rho$, with the relation $Sa\beta = 0$ between its constants, has been already considered (in (10.)); and thus the consequences of the supposition III., that there are (at least) *two equal but null roots* of the cubic in ϕ , have been perhaps sufficiently discussed.

(30.) As regards the *other principal case of equal roots*, of the cubic equation in ϕ , namely that in which the vector constants are connected by the relation V., or by the equation of condition,

$$\text{L. . . } 0 = m''^2 - 4m' = (Sa\beta + Sa'\beta')^2 - 4S(\nabla aa' \cdot \nabla \beta\beta') \\ = (Sa\beta - Sa'\beta')^2 + 4Sa\beta'Sa'\beta,$$

it may suffice to remark that it conducts, by VI., or by VII. and IX., to the symmetrical equation,

$$\text{LI. . . } 0 = \phi\Phi^2, \quad \text{if } \Phi = \phi - \frac{1}{2}m'';$$

and that thus its *interpretation* is precisely similar to that of the analogous equation,

$$\chi\phi^2 = 0, \quad \text{where } \chi = m'' - \phi, \quad \text{XXIX.}$$

as given in (14.), and in the following sub-articles.

353. When we have $m = 0$, but *not* $m' = 0$, *nor* $m'' = 4m'$, the *three roots* of the cubic in ϕ are *all unequal*, while *one* of them is still *null*, as before; and the *two roots* of the *quadratic and scalar equation*, with *real coefficients* (347),

$$\text{I. . . } 0 = c^2 + m''c + m',$$

* We have, in these transformations, examples of what may be called *Quaternion Invariants*.

which is formed from the cubic by changing ϕ to $-c$, and then dividing by c , are also necessarily *unequal*, whether they be *real* or *imaginary*. We shall find that when these *two scalar roots*, c_1, c_2 , are *real*, there are then *two real directions*, ρ_1 and ρ_2 , in that *fixed plane* Π which is the *locus* (351, 352) of the line $\phi\rho$, possessing the property that for each of them the *homogeneous and vector equation of the second degree*,

$$\text{II.} \dots V\rho\phi\rho = 0, \text{ or } \phi\rho \parallel \rho,$$

is satisfied, *without* ρ *vanishing*; namely by our having, for the *first* of these two directions, the equation

$$\text{III.} \dots \phi\rho_1 = -c_1\rho_1, \text{ or } \phi_1\rho_1 = 0, \text{ if } \phi_1 = \phi + c_1;$$

and for the *second* of them the analogous equation,

$$\text{IV.} \dots \phi\rho_2 = -c_2\rho_2, \text{ or } \phi_2\rho_2 = 0, \text{ if } \phi_2 = \phi + c_2;$$

but that *no other direction* of the *real* and *actual vector* ρ , satisfies the equation V., except that *third* which has already been considered (351), as satisfying the *linear* and *vector equation*,

$$\text{V.} \dots \phi\rho = 0, \text{ with } T\rho > 0.$$

It will also be shown that these *two directions*, ρ_1, ρ_2 , are not only *real*, but *rectangular*, to each other and to the *third* direction ρ , when the *linear function* $\phi\rho$ is *self-conjugate* (349, (4.)), or when the condition

$$\text{VI.} \dots \phi'\rho = \phi\rho, \text{ or } \text{VI.} \dots S\lambda\phi\rho = S\rho\phi\lambda,$$

is satisfied by the *given form* of ϕ , or by the *constants* which enter into the composition of that *linear symbol*; but that when this *condition of self-conjugation* is *not* satisfied, the roots of the quadratic I. *may* happen to be *imaginary*: and that in *this* case there exists *no real direction* of ρ , for which the vector equation II. of the *second degree* is satisfied, by *actual values* of ρ , except that *one* direction which has been seen before to satisfy the *linear equation* V.

(1.) The most obvious mode of seeking to satisfy II., otherwise than through V., is to assume an expression of the form, $\rho = x\beta + x'\beta'$, and to seek thereby to satisfy the equation, $(\phi + c)\rho = 0$, with $\phi\rho = \beta S\alpha\rho + \beta' S\alpha'\rho$, by satisfying separately the two scalar equations,

$$\text{VII.} \dots 0 = x(c + S\alpha\beta) + x'S\alpha\beta', \quad 0 = x'(c + S\alpha'\beta') + xS\alpha'\beta,$$

which give, by elimination of x' : x , the following quadratic in e ,

$$\text{VIII.} \dots (e + Sa\beta) (e + Sa'\beta') = Sa\beta'Sa'\beta',$$

which is easily seen to be only another form of I. Denoting then, as above, by c_1 and c_2 , the roots of that quadratic I., supposed for the present to be real, we have these two real directions for ρ , in the plane II of β , β' :

$$\text{IX.} \dots \rho_1 = \beta(c_1 + Sa'\beta') - \beta'Sa'\beta = c_1\beta + VaV\beta'\beta;$$

$$\text{X.} \dots \rho_2 = \beta(c_2 + Sa'\beta') - \beta'Sa'\beta = c_2\beta + VaV\beta'\beta;$$

which satisfy the equations III. and IV. In fact, the expression IX. gives

$$\phi\rho_1 = c_1\phi\beta + m'\beta = -c_1\rho_1, \text{ or } \phi_1\rho_1 = 0,$$

because we may write it thus,

$$\text{XI.} \dots \rho_1 = (m'' + c_1)\beta - \phi\beta = -c_2\beta - \phi\beta = -\phi_2\beta = -\phi\beta - m'c_1^{-1}\beta;$$

and in like manner, the expression X. may be thus written,

$$\text{XII.} \dots \rho_2 = (m'' + c_2)\beta - \phi\beta = -c_1\beta - \phi\beta = -\phi_1\beta = -\phi\beta - m'c_2^{-1}\beta,$$

and gives,

$$\phi\rho_2 = c_2\phi\beta + m'\beta = -c_2\rho_2, \text{ or } \phi_2\rho_2 = 0.$$

(2.) We may also write,

$$\text{XIII.} \dots \rho'_1 = \beta'(c_1 + Sa\beta) - \beta Sa\beta' = c_1\beta' + VaV\beta\beta' = -\phi_2\beta' \parallel \rho_1;$$

$$\text{XIV.} \dots \rho'_2 = \beta'(c_2 + Sa\beta) - \beta Sa\beta' = c_2\beta' + VaV\beta\beta' = -\phi_1\beta' \parallel \rho_2;$$

and shall then have the equations,

$$\text{XV.} \dots \phi_1\rho'_1 = 0, \quad \phi_2\rho'_2 = 0;$$

but the directions of ρ'_1 and ρ'_2 will be the same by VIII. as those of ρ_1 and ρ_2 , and so will furnish no new solution of the problem just resolved.

(3.) Since we have thus,

$$\text{XVL.} \dots \phi_2\beta' \parallel \phi_2\beta \parallel \rho_1 \parallel \phi_1^{-1}0, \text{ and } \text{XVII.} \dots \phi_1\beta' \parallel \phi_1\beta \parallel \rho_2 \parallel \phi_2^{-1}0,$$

it follows that the operation ϕ_2 reduces every line in the fixed plane of $\phi\rho$ to the fixed direction of $\phi_1^{-1}0$; and that, in like manner, the operation ϕ_1 reduces every line, in the same fixed plane of $\phi\rho$, to the other fixed direction of $\phi_2^{-1}0$.

(4.) Hence we may write the symbolic equations,

$$\text{XVIII.} \dots \phi_1 \cdot \phi_2\phi = 0, \quad \phi_2 \cdot \phi_1\phi = 0,$$

in which the points may be omitted; and in fact we have the transformations,

$$\text{XIX.} \dots \phi_1\phi_2 = \phi_2\phi_1 = (\phi + c_1)(\phi + c_2) = \phi^2 - m''\phi + m' = \psi,$$

so that $\phi_1\phi_2 \cdot \phi = \phi_2\phi_1 \cdot \phi = \psi\phi = m = 0$.

(5.) If we propose to form ψ_1 from ϕ_1 , by the same general rule (347, XI.) by which ψ is formed from ϕ , we have

$$\text{XX.} \dots \psi_1 V\mu\nu = V \cdot \phi'_1\mu\phi'_1\nu = V \cdot (\phi'_1\mu + c_1\mu)(\phi'_1\nu + c_1\nu),$$

and therefore, by the definition 850, VIII. of χ ,

$$\text{XXI.} \dots \psi_1\rho = \psi\rho + c_1\chi\rho + c_1^2\rho, \text{ or } \text{XXII.} \dots \psi_1 = \psi + c_1\chi + c_1^2;$$

and in like manner,

$$\text{XXIII.} \dots \psi_2 = \psi + c_2\chi + c_2^2,$$

even if m be different from zero, and if c_1, c_2 be arbitrary scalars.

(6.) Accordingly, without assuming that m vanishes, if we operate on $\psi_1\rho$ with

ϕ_1 , or symbolically multiply the expression XXI. for ψ_1 by ϕ_1 , we get the symbolic product,

$$\begin{aligned} \text{XXIII.} \dots \phi_1 \psi_1 &= (\phi + c_1) (\psi + c_1 \chi + c_1^2) \\ &= \phi \psi + c_1 (\phi \chi + \psi) + c_1^2 (\phi + \chi) + c_1^3 \\ &= m + c_1 m' + c_1^2 m'' + c_1^3 = m_1, \end{aligned}$$

where m_1 is what the scalar m becomes, when ϕ is changed to ϕ_1 , or is such that

$$\text{XXIV.} \dots m_1 S \lambda \mu \nu = S. \phi'_1 \lambda \phi'_1 \mu \phi'_1 \nu = S. (\phi' \lambda + c_1 \lambda) (\phi' \mu + c_1 \mu) (\phi' \nu + c_1 \nu);$$

as appears by the definitions of ϕ' , ψ , χ , m , m' , m'' , and by the relations between those symbols which have been established in recent Articles, or in the sub-articles appended to them.

(7.) Supposing now again that $m = 0$, and that c_1, c_2 are the roots of the quadratic I. in c , we have by XXIII.,

$$\text{XXV.} \dots \phi_1 \psi_1 = m_1 = 0; \quad \text{and in like manner} \quad \text{XXVI.} \dots \phi_2 \psi_2 = m_2 = 0,$$

if m_2 be formed from m_1 , by changing c_1 to c_2 .

(8.) Comparing XXV. with XVII., we may be led to suspect the existence of an intimate connexion existing between ψ_1 and $\phi_2 \phi$, since each reduces an arbitrary vector to the fixed direction of $\phi_1^{-1} 0$, or of ρ_1 ; and in fact these two operations are *identical*, because, by XXI., and by the known relations between the symbols, we have the transformations,

$$\begin{aligned} \text{XXVII.} \dots \psi_1 &= \psi + c_1 \chi + c_1^2 = (m' - m'' \phi + \phi^2) + c_1 (m'' - \phi) + c_1^3 \\ &= \phi^2 - (m'' + c_1) \phi = \phi^2 + c_2 \phi = \phi \phi_2; \end{aligned}$$

and similarly,

$$\text{XXVIII.} \dots \psi_2 = \phi^2 + c_1 \phi = \phi \phi_1;$$

while $\psi = \phi_1 \phi_2$, as before.

(9.) We have thus the *new symbolic equation*,

$$\text{XXIX.} \dots \phi \phi_1 \phi_2 = 0,$$

in which the *three symbolic factors* ϕ, ϕ_1, ϕ_2 may be in any manner *grouped and transposed*, so that it *includes* the two equations XVII.; and in which the *subject of operation* is an *arbitrary vector* ρ . Its interpretation has been already partly given; but we may add, that while ϕ *reduces every vector to the fixed plane* Π , ϕ_1 *reduces every line to another fixed plane*, Π_1 , and ϕ_2 *reduces to a third plane*, Π_2 ; thus $\phi_1 \phi_2$, or $\phi_2 \phi_1$, while it *destroys two lines* ρ_1, ρ_2 , and therefore *every line in the plane* Π , *reduces an arbitrary line to the fixed direction of the intersection of the two planes* $\Pi_1 \Pi_2$, which intersection must thus have the direction of $\phi^{-1} 0$; and in like manner, the fixed direction ρ_1 of $\phi_1^{-1} 0$, as being that to which an arbitrary vector is reduced (8.) by the compound operation $\phi_2 \phi$, or $\phi \phi_2$, must be that of the intersection of the planes $\Pi \Pi_2$; and ρ_2 , or $\phi_2^{-1} 0$, has the direction of the intersection of $\Pi \Pi_1$; while on the other hand $\phi \phi_2$ *destroys every line in* Π_1 , and $\phi \phi_1$ *every line in* Π_2 : so that *these three planes, with their three lines of intersection*, are the chief elements in the *geometrical interpretation* of the equation $\phi \phi_1 \phi_2 = 0$.

(10.) The *conjugate equation*,

$$\text{XXX.} \dots \phi' \phi'_1 \phi'_2 = 0,$$

may be interpreted in a similar way, and so conducts to the consideration of a *conjugate system of planes and lines*; namely the planes Π', Π'_1, Π'_2 , which are the *loci of* $\phi' \rho, \phi'_1 \rho, \phi'_2 \rho$, while the operations $\phi'_1 \phi'_2, \phi'_2 \phi'_1$, and $\phi' \phi'_1$ *destroy all lines*

in these three planes respectively, and reduce arbitrary lines to the fixed directions of the intersections, $\Pi'_1\Pi'_2$, $\Pi'_2\Pi'_1$, $\Pi'_1\Pi'_1$, which are also those of $\phi'^{-1}0$, $\phi'_1^{-1}0$, $\phi'_2^{-1}0$.

(11.) It is important to observe that these three last lines are the normals to the three first planes, Π , Π' , Π'' ; and that, in like manner, the three former lines are perpendicular to the three latter planes. To prove this, it is sufficient to observe that

$$\text{XXXI.} \dots S\rho'\phi\rho = S\rho\phi'\rho' = 0, \text{ if } \phi'\rho' = 0, \text{ or that } \phi\rho \perp \phi'^{-1}0;$$

and similarly, $\phi'\rho \perp \phi^{-1}0$, &c.

(12.) Instead of eliminating $x' : x$ between the two equations VII., we might have eliminated c ; which would have given this other quadratic,

$$\text{XXXII.} \dots 0 = x^2 Sa'\beta + xx'(Sa'\beta' - Sa\beta) - x'^2 Sa\beta';$$

also, if $x'_1 : x_1$ and $x'_2 : x_2$ be the two values of $x' : x$, then

$$\text{XXXIII.} \dots \rho_1 \parallel x_1\beta + x'_1\beta', \quad \rho_2 \parallel x_2\beta + x'_2\beta',$$

$$\text{and XXXIV.} \dots x_1x_2 : (x_1x'_2 + x_2x'_1) : x'_1x'_2 = -Sa\beta' : (Sa\beta - Sa'\beta') : Sa'\beta;$$

hence the condition of rectangularity of the two lines ρ_1, ρ_2 , or $\phi_1^{-1}0, \phi_2^{-1}0$, is expressed by the equation,

$$\text{XXXV.} \dots 0 = -\beta^2 Sa\beta' + S\beta\beta'(Sa\beta - Sa'\beta') + \beta'^2 Sa'\beta = S.\beta\beta'V(\beta\alpha + \beta'a');$$

and consequently it is satisfied, if the given function ϕ be self-conjugate (VI.), because we have then the relation,

$$\text{XXXVI.} \dots V\beta\alpha + V\beta'a' = 0;$$

in fact the binomial form of ϕ gives (comp. 849, XXII.),

$$\text{XXXVII.} \dots \phi\rho - \rho\phi = (aS\beta\rho - \beta Sa\rho) + (a'S\beta'\rho - \beta' Sa'\rho) = V.\rho V(\beta\alpha + \beta'a'),$$

which cannot vanish independently of ρ , unless the constants satisfy the condition

$$\text{XXXVI.}$$

(13.) With this condition then, of self-conjugation of ϕ , we have the relation of rectangularity,

$$\text{XXXVIII.} \dots S\rho_1\rho_2 = 0, \text{ or } \phi_1^{-1}0 \perp \phi_2^{-1}0;$$

at least if these directions ρ_1 and ρ_2 be real, which they can easily be proved to be, as follows. The condition XXXVI. gives,

$$\text{XXXIX.} \dots 0 = S.a\alpha V(\beta\alpha + \beta'a') = a^2 Sa'\beta + Sa\alpha'(Sa'\beta' - Sa\beta) - a^2 Sa\beta';$$

hence

$$(a^2 Sa'\beta - a^2 Sa\beta')^2 = (Sa\alpha')^2 (Sa\beta - Sa'\beta')^2,$$

$$a^2\alpha^2(m'^2 - 4m) = a^2\alpha^2\{(Sa\beta - Sa'\beta')^2 + 4Sa\beta'Sa'\beta\}$$

$$= (a^2\alpha^2 - (Sa\alpha')^2)(a\beta - Sa'\beta')^2 + (a^2Sa'\beta + a^2Sa\beta')^2 > 0,$$

and

$$\text{XL.} \dots (Sa\beta - Sa'\beta')^2 + 4Sa\beta'Sa'\beta = m'^2 - 4m > 0;$$

so that each of the two quadratics, I. (or VIII.), and XXXII., has real and unequal roots: a conclusion which may also be otherwise derived, from the expressions $\beta = a\alpha + \beta'a'$, $\beta' = \beta\alpha + a'\alpha'$, which the condition allows us to substitute for β and β' .

(14.) The same condition XXXVI. shows that the four vectors $a\beta a'\beta'$ are coplanar, or that we have the relations,

$$\text{XLI.} \dots Sa\beta\beta' = 0, \quad Sa'\beta\beta' = 0, \quad V(Va\alpha'.V\beta'\beta) = 0;$$

hence $Va\alpha'$, or $\phi^{-1}0$ is now normal to the plane Π ; and therefore by (13.), when the function ϕ is self-conjugate (VI.), the three directions,

$$\text{XLII.} \dots \rho, \rho_1, \rho_2, \text{ or } \phi^{-1}0, \phi_1^{-1}0, \phi_2^{-1}0,$$

compose a real and rectangular system.

(15.) In the present series of subarticles (to 858), we suppose that the three roots of the cubic in ϕ are all unequal, the cases of equal roots (with $m = 0$) having been discussed in a preceding series (852); but it may be remarked in passing, that when a self-conjugate function $\phi\rho$ is reducible to the monomial form $\beta\Sigma a\rho$, we must have the relation $\nabla\beta a = 0$; and that thus the line β , to the fixed direction of which (comp. 852, (5.) and (6.)) the operation ϕ then reduces an arbitrary vector, is perpendicular to the fixed plane (852, (7.)), every line in which is destroyed by that operation ϕ .

(16.) In general, if ϕ be thus self-conjugate, it is evident that the three planes Π' , Π'_1 , Π'_2 , which are (comp. (10.)) the loci of $\phi'\rho$, $\phi'_1\rho$, $\phi'_2\rho$, coincide with the planes Π , Π_1 , Π_2 , which are the loci of $\phi\rho$, $\phi_1\rho$, $\phi_2\rho$.

(17.) When ϕ is not self-conjugate, so that $\phi\rho$ and $\phi'\rho$ are not generally equal, it has been remarked that the scalar quadratic I., and therefore also the symbolical cubic in ϕ , may have imaginary roots; and that, in this case, the vector equation II. of the second degree cannot be satisfied by any real direction of ρ , except that one which satisfies the linear equation V., or causes $\phi\rho$ itself to vanish, while ρ remains real and actual. As an example of such imaginary scalars, as roots of I., and of what may be called imaginary directions, or imaginary vectors (comp. 214, (4.)), which correspond to those scalars, and are themselves imaginary roots of II., we may take the very simple expressions (comp. 849, XII.),

$$\text{XLIII.} \dots \phi\rho = \nabla\gamma\rho, \quad \phi'\rho = -\nabla\gamma\rho;$$

in which γ denotes some real and given vector, and which evidently do not satisfy the condition VI., the function ϕ being here the negative of its own conjugate, so that its self-conjugate part ϕ_0 is zero (comp. 849, XIII.). We have thus,

$$\text{XLIV.} \dots m_0 = 0, \quad m'_0 = 0, \quad m''_0 = 0, \quad \phi_0 = 0, \quad \psi_0 = 0, \quad \chi_0 = 0,$$

and consequently, by the sub-articles to 849 and 850,

$$\text{XLV.} \dots m = 0, \quad m' = -\gamma^2, \quad m'' = 0, \quad \psi\rho = -\gamma\Sigma\gamma\rho, \quad \chi\rho = -\nabla\gamma\rho;$$

the quadratic I., and its roots c_1 , c_2 , become therefore,

$$\text{XLVI.} \dots c^2 - \gamma^2 = 0, \quad c_1 = +\sqrt{-1}.T\gamma, \quad c_2 = -\sqrt{-1}.T\gamma,$$

where $\sqrt{-1}$ is the imaginary of algebra (comp. 214, (8.)); thus by XX. or XXI., and XXII.) we have now

$$\text{XLVII.} \dots \psi_1\sigma = -\gamma\Sigma\gamma\sigma - c_1\nabla\gamma\sigma + c_1^2\sigma = (\gamma - c_1)\nabla\gamma\sigma, \quad \psi_2\sigma = (\gamma - c_2)\nabla\gamma\sigma;$$

hence

$$\Sigma\gamma\psi_1\sigma = 0, \quad \nabla\gamma\psi_1\sigma = \gamma\psi_1\sigma, \text{ \&c.,}$$

and

$$\text{XLVIII.} \dots \phi_1\psi_1\sigma = (\phi + c_1)\psi_1\sigma = (\gamma + c_1)(\gamma - c_1)\nabla\gamma\sigma = (\gamma^2 - c_1^2)\nabla\gamma\sigma = 0;$$

and in like manner XLVIII. $\dots \phi_2\psi_2\sigma = 0$;

If then we take an arbitrary vector σ , and derive (or rather conceive as derived) from it two (imaginary) vectors ρ_1 and ρ_2 by the (imaginary) operations ψ_1 and ψ_2 , we shall have (comp. III. and IV.) the equations,

$$\text{XLIX.} \dots \rho_1 = \psi_1\sigma, \quad \phi_1\rho_1 = 0, \quad \phi\rho_1 = -c_1\rho_1, \quad \nabla\rho_1\phi\rho_1 = 0,$$

and

$$\text{L.} \dots \rho_2 = \psi_2\sigma, \quad \phi_2\rho_2 = 0, \quad \phi\rho_2 = -c_2\rho_2, \quad \nabla\rho_2\phi\rho_2 = 0,$$

as ones which are at least *symbolically true*. We find then that the *two imaginary directions*, ρ_1 and ρ_2 , satisfy (at least in a symbolical sense, or as far as calculation is concerned) the *vector equation* II., or that ρ_1 and ρ_2 are *two imaginary vector roots* of $V\rho\phi\rho = 0$; but that, because the *scalar quadratic* I. has here *imaginary roots*, this vector equation II. has (as above stated) *no real vector root* ρ , except one in the *direction of the given and real vector* γ , which satisfies the *linear equation* V., or gives $\phi\rho = 0$.

(18.) This particular example might have been more simply treated, by a less general method, as follows. We wish to satisfy the equation,

$$\text{LI.} \dots 0 = V.\rho V\gamma\rho = \rho S\gamma\rho - \rho^2\gamma;$$

which gives, when we operate on it by $V.\gamma$ and $V.\rho$, these others,

$$\text{LII.} \dots 0 = V\gamma\rho.S\gamma\rho, \quad 0 = \rho^2 V\gamma\rho;$$

if then we wish to *avoid supposing* $\phi\rho = V\gamma\rho = 0$, we must seek to satisfy the *two scalar equations*,

$$\text{LIII.} \dots S\gamma\rho = 0, \quad \rho^2 = 0;$$

and conversely, if we can satisfy *these* by any (real or imaginary) ρ , we shall have satisfied (really or symbolically) the *vector equation* LI. Now the *first equation* LIII. is satisfied, when we assume the expression,

$$\text{LIV.} \dots \rho = (c + \gamma)V\gamma\sigma = V\gamma\sigma.(c - \gamma),$$

where σ is an *arbitrary vector*, and c is *any scalar*, or *symbol* subject to the *laws of scalars*; and this expression LIV. for ρ , with its transformation just assigned, gives

$$\text{LV.} \dots \rho^2 = (c^2 - \gamma^2)(V\gamma\sigma)^2 = 0, \quad \text{if} \quad c^2 - \gamma^2 = 0;$$

the *quadratic* XLVI. is therefore reproduced, and we have the *same imaginary roots*, and *imaginary directions*, as before.

(19.) *Geometrically*, the *imaginary character* of the recent problem, of satisfying the equation $V.\rho V\gamma\rho = 0$ by any direction of ρ except that of the given line γ , is apparent from the circumstance that $\phi\rho$, or $V\gamma\rho$, is here a *vector perpendicular* to ρ , if *both* be *actual lines*; and that therefore the one cannot be also *parallel* to the other, so long as both are *real*.*

354. In the three preceding Articles, and in the sub-articles annexed, we have supposed throughout that the *absolute term* of the cubic in ϕ is *wanting*, or that the condition $m = 0$ is satisfied; in which case we have seen (351) that it is always possible to satisfy the *linear equation* $\phi\rho = 0$, by at least *one* real and actual value of ρ (with an arbitrary scalar coefficient); or by at least *one* real direction. It will be easy now to show,

* Accordingly the *two imaginary directions*, above found for ρ , are easily seen to be those which in modern geometry are called the *directions of lines drawn in a given plane* (perpendicular here to the given line γ), *to the circular points at infinity*: of which supposed *directions* the *imaginary character* may be said to be precisely this, that *each* is (in the given plane) *its own perpendicular*.

that although conversely (comp. 351, (4.)) the function $\phi\rho$ cannot vanish for any actual vector ρ , unless we have thus $m = 0$, yet there is always at least one real direction for which the vector equation of the second degree,

$$\text{I. . . } \nabla\rho\phi\rho = 0,$$

which has already been considered (353) in combination with the condition $m = 0$, is satisfied; and that if the function ϕ be a self-conjugate one, then this equation I. is always satisfied by at least three real and rectangular directions, but not generally by more directions than three; although, in this case of self-conjugation, namely when

$$\text{II. . . } \phi'\rho = \phi\rho, \quad \text{or} \quad \text{II}'. . . \text{Sl}\phi\rho = \text{S}\rho\phi\lambda,$$

for all values of the vectors ρ and λ , the equation I. may happen to become true, for one real direction of ρ , and for every direction perpendicular thereto: or even for all possible directions, according to the particular system of constants, which enter into the composition of the function $\phi\rho$. We shall show also that the scalar (or algebraic) and cubic equation,

$$\text{III. . . } 0 = m + m'c + m''c^2 + c^3,$$

which is formed from the symbolic and cubic equation 350, I., by changing ϕ to $-c$, enters importantly into this whole theory; and that if it have one real and two imaginary roots, the quadratic and vector equation I. is satisfied by only one real direction of ρ ; but that it may then be said (comp. 353, (17.)) to be satisfied also by two imaginary directions, or to have two imaginary and vector roots: so that this equation I. may be said to represent generally a system of three right lines, whereof one at least must be real. For the case II., the scalar roots of III. will be proved to be always real; so that if m_0 , m'_0 , and m''_0 be formed (as in sub-articles to 349 and 350) from the self-conjugate part $\phi_0\rho$ of any linear and vector function $\phi\rho$, as m , m' , and m'' are formed from that function $\phi\rho$ itself, then the new cubic,

$$\text{IV. . . } 0 = m_0 + m'_0c + m''_0c^2 + c^3,$$

which thus results, can never have imaginary roots.

(1.) If we write,

V. . . $\Phi\rho = \phi\rho + c\rho$, $\Phi'\rho = \phi'\rho + c\rho$, or briefly, V' . . $\Phi = \phi + c$, $\Phi' = \phi' + c$, where c is an arbitrary scalar, and if we denote by Ψ , Ψ' , and M what ψ , ψ' , and m become, by this change of ϕ to $\phi + c$ or Φ , the calculations in 353, (5.), (6.), show that we have the expressions,

$$\text{VI. . . } \Psi = \psi + c\chi + c^2, \quad \Psi' = \psi' + c\chi' + c^2,$$

and

$$\text{VII. . . } M = m + m'c + m''c^2 + c^3,$$

with

$$\text{VIII. . . } M = \Phi\Psi = \Psi\Phi = \Phi'\Psi' = \Psi'\Phi'.$$

(2.) Hence it may be inferred that the functions χ , χ' , and the constants m' , m'' become,

$$\text{IX. . . } X = D_c\Psi = \chi + 2c, \quad X' = D_c\Psi' = \chi' + 2c,$$

$$\text{X. . . } \begin{cases} M' = D_cM = m' + 2m''c + 3c^2, \\ M'' = \frac{1}{2}D_c^2M = m'' + 3c; \end{cases}$$

with the verifications,

$$\text{XI. . . } \Phi + X = \Phi' + X' = M'', \quad \Phi X + \Psi = \Phi'X' + \Psi' = M',$$

as we had, by the sub-articles to 350,

$$\phi + \chi = \phi' + \chi' = m'', \quad \phi\chi + \psi = \phi'\chi' + \psi' = m'.$$

(3.) The *new linear symbol* Φ must satisfy the *new cubic*,

$$\text{XII. . . } 0 = M - M'\Phi + M''\Phi^2 - \Phi^3;$$

which accordingly can be at once derived from the *old cubic* 350, I., under the form,

$$\text{XIII. . . } 0 = m + m'(c - \Phi) + m''(c - \Phi)^2 + (c - \Phi)^3.$$

(4.) Now it is always possible to satisfy the condition,

$$\text{XIV. . . } M = 0,$$

by substituting for c a *real root of the scalar cubic* III.; and thereby to *reduce the new symbolical cubic* XII. to the *form*,

$$\text{XV. . . } 0 = \Phi^3 - M''\Phi^2 + M'\Phi;$$

which is precisely similar to the form,

$$0 = \phi^3 - m''\phi^2 + m'\phi, \quad 352, \text{ II.},$$

and conducts to analogous consequences, which need not here be developed in detail, since they can easily be supplied by any one who will take the trouble to read again the few recent series of sub-articles.

(5.) For example, unless it happen that $\Psi\rho$ *constantly vanishes*, in which case $M' = 0$, and $\Phi\rho$ (if not *identically null*) takes a *monomial form*, which is reduced to zero (comp. 352, (7.)) for *every direction* of ρ in a *given plane*, the operation Ψ *reduces* (comp. 351) an *arbitrary vector* to a *given direction*; and the operation Φ *destroys* every line in *that direction*: so that, *in every case*, there is *at least one real way* of satisfying the *vector equation* $\Phi\rho = 0$, and therefore also (as above asserted) the *equation* I., without causing ρ *itself* to vanish.

(6.) And since that equation I. may be thus written,

$$\text{XVI. . . } V\rho\Phi\rho = 0, \quad \text{or} \quad \Phi\rho \parallel \rho,$$

we see that it can be satisfied *without* $\Phi\rho$ *vanishing*, if this *new scalar and quadratic equation*,

$$\text{XVII. . . } 0 = C^2 + M' C + M', \quad \text{comp. 353, I.},$$

have *real and unequal roots* C_1, C_2 ; for if we then write,

$$\text{XVIII.} \dots \Phi_1 = \Phi + C_1, \quad \Phi_2 = \Phi + C_2,$$

the line $\Phi\rho$ will *generally* have for its *locus* a *given plane*, and there will be *two real and distinct directions* ρ_1 and ρ_2 in that plane, for one of which $\Phi_1\rho_1 = 0$, while $\Phi_2\rho_2 = 0$ for the other, so that *each* satisfies XVI., or I.; and these are precisely the *fixed directions* of $\Psi_1\rho$ and $\Psi_2\rho$, if Ψ_1 and Ψ_2 be formed from Ψ by changing Φ to Φ_1 and Φ_2 respectively.

(7.) Cases of *equal* and of *imaginary roots* need not be dwelt on here; but it may be remarked in passing, that if the function $\phi\rho$ have the *particular form* (g being any scalar constant),

$$\text{XIX.} \dots \phi\rho = g\rho, \quad \text{then} \quad \text{XX.} \dots (g - \phi)^2 = 0, \quad \text{and} \quad \text{XXI.} \dots M = (g + c)^2;$$

the cubic XIV. or III. having thus *all its roots equal*, and the equation I. being satisfied by *every direction* of ρ , in this particular case.

(8.) The *general existence* of a *real and rectangular system* of *three directions* satisfying I., when the *condition* II. is satisfied, may be proved as in 353, (14.); and it is unnecessary to dwell on the case where, by *two roots* of the cubic becoming *equal*, *all lines* in a *given plane*, and *also* the *normal* to that plane, are *vector roots* of I., with the same condition II.

(9.) And because the *quadratic*, $0 = c^2 + m''c + m'$ (353, I.), has been proved to have always *real roots* (353, (18.)) when $\phi'\rho = \phi\rho$, the *analogous quadratic* XVII. must likewise then have *real roots*, C_1, C_2 ; whence it immediately follows (comp. XII. and XIII.), that (under the same *condition of self-conjugation*) the *cubic* III. has *three real roots*, $c, c + C_1, c + C_2$; and therefore that (as above stated) the *other cubic* IV., which is formed from the *self-conjugate part* ϕ_0 of the *general linear and vector function* ϕ , and which may on that account be thus denoted,

$$\text{XXII.} \dots M_0 = 0, \quad \text{has its roots always real.}$$

(10.) If we denote in like manner by Φ_0 the symbol $\phi_0 + c$, the equation $m = m_0 - S\gamma\phi_0\gamma$ (349, XXVI., comp. 349, XXI.) becomes,

$$\text{XXIII.} \dots M = M_0 - S\gamma\Phi_0\gamma;$$

whence, by comparing powers of c , we recover the relations,

$$m' = m'_0 - \gamma^2, \quad \text{and} \quad m'' = m''_0, \quad \text{as in 350, (1.)}$$

(11.) On a similar plan, the equation $m\phi'V\mu\nu = V.\psi\mu\psi\nu$ becomes,

$$\text{XXIV.} \dots M\phi'V\mu\nu = V.\Psi\mu\Psi\nu, \quad \text{comp. 348, (1.)}$$

in which μ and ν are *arbitrary vectors*, and c is an *arbitrary scalar*; or more fully, XXV. $\dots (m + m'c + m''c^2 + c^3)(\phi' + c)V\mu\nu = V.(\psi\mu + c\chi\mu + c^2\mu)(\psi\nu + c\chi\nu + c^2\nu)$; whence follow these new equations,

$$\text{XXVI.} \dots (m + m'\phi')V\mu\nu = V(\psi\mu \cdot \chi\nu - \psi\nu \cdot \chi\mu),$$

$$\text{XXVII.} \dots (m' + m''\phi')V\mu\nu = V(\mu\psi\nu - \nu\psi\mu + \chi\mu \cdot \chi\nu),$$

$$\text{XXVIII.} \dots (m'' + \phi')V\mu\nu = V(\mu\chi\nu - \nu\chi\mu),$$

which can all be otherwise proved, and from the last of which (by changing ϕ to ψ , &c.) we can infer this other of the same kind,

$$\text{XXIX.} \dots (m' + \psi')V\mu\nu = V(\mu\phi\chi\nu - \nu\phi\chi\mu).$$

(12.) As an *example* of the existence of a *real and rectangular system* of *three directions* (8.), represented jointly by an equation of the form I., and of a system of

three real roots of the scalar cubic III., when the condition II. is satisfied, let us take the form,

$$\text{XXX.} \dots \phi\rho = g\rho + \nabla\lambda\rho\mu = \phi'\rho,$$

g being here any real and given scalar, and λ, μ any real and non-parallel given vectors; to which form, indeed, we shall soon find that every self-conjugate function $\phi\rho$ can be brought. We have now (after some reductions),

$$\text{XXXI.} \dots \psi\rho = \nabla\lambda\rho\mu\delta\lambda\mu - \nabla\lambda\mu\delta\lambda\rho\mu - g(\lambda\delta\mu\rho + \mu\delta\lambda\rho) + g^2\rho,$$

$$\text{XXXII.} \dots \chi\rho = -(\lambda\delta\mu\rho + \mu\delta\lambda\rho) + 2g\rho,$$

$$\text{and XXXIII.} \dots m = (g - \delta\lambda\mu)(g^2 - \lambda^2\mu^2), \quad m' = -\lambda^2\mu^2 - 2g\delta\lambda\mu + \delta g^2, \\ m'' = -\delta\lambda\mu + \delta g;$$

where the part of $\psi\rho$ which is independent of g may be put under several other forms, such as the following,

$$\text{XXXIV.} \dots \nabla(\lambda\rho\mu\delta\lambda\mu - \lambda\mu\delta\lambda\rho\mu) = \lambda\rho\mu\delta\lambda\mu - \lambda\mu\delta\lambda\rho\mu$$

$$= \lambda(\rho\delta\lambda\mu + \delta\lambda\mu\rho)\mu = \frac{1}{2}\lambda(\lambda\mu\rho + \rho\lambda\mu)\mu = \lambda(\lambda\delta\mu\rho + \mu\delta\lambda\rho - \lambda\rho\mu)\mu, \text{ \&c.};$$

and $\Phi, \Psi, X, M, M', M''$ may be formed from $\phi, \psi, \chi, m, m', m''$, by simply changing g to $c+g$. The equation $M=0$ has therefore here three real and unequal roots, namely the three following,

$$\text{XXXV.} \dots c = -g + \delta\lambda\mu, \quad c + C_1 = -g + T\lambda\mu, \quad c + C_2 = -g - T\lambda\mu;$$

and the corresponding forms of $\Psi\rho$ are found to be,

$$\text{XXXVI.} \dots \Psi\rho = \nabla\lambda\mu\delta\lambda\mu\rho, \quad \Psi_{1\rho} = -(\lambda T\mu + \mu T\lambda)\delta.\rho(\lambda T\mu + \mu T\lambda), \\ \Psi_{2\rho} = -(\lambda T\mu - \mu T\lambda)\delta.\rho(\lambda T\mu - \mu T\lambda).$$

Thus $\Psi\rho, \Psi_{1\rho}$, and $\Psi_{2\rho}$ have in fact the three fixed and rectangular directions of $\nabla\lambda\mu, \lambda T\mu + \mu T\lambda$, and $\lambda T\mu - \mu T\lambda$, namely of the normal to the given plane of λ, μ , and the bisectors of the angles made by those two given lines; and these are accordingly the only directions which satisfy the vector equation of the second degree,

$$\text{XXXVII.} \dots (\nabla\rho\phi\rho = \nabla.\rho\nabla\lambda\rho\mu) = \nabla\rho\lambda\delta\mu\rho + \nabla\rho\mu\delta\lambda\rho = 0;$$

so that this last equation represents (as was expected) a system of three right lines, in these three respective directions.

(18.) In general, if c_1, c_2, c_3 denote the three roots (real or imaginary) of the cubic equation $M=0$, and if we write,

$$\text{XXXVIII.} \dots \Phi_1 = \phi + c_1, \quad \Phi_2 = \phi + c_2, \quad \Phi_3 = \phi + c_3,$$

the corresponding values of Ψ will be (comp. VI.),

$$\text{XXXIX.} \dots \Psi_1 = \psi + c_1\chi + c_1^2, \quad \Psi_2 = \psi + c_2\chi + c_2^2, \quad \Psi_3 = \psi + c_3\chi + c_3^2;$$

also we have the relations,

$$\text{XL.} \dots \begin{cases} c_1 + c_2 + c_3 = -m'' = -\phi - \chi, \\ c_2c_3 + c_3c_1 + c_1c_2 = +m' = \phi\chi + \psi, \\ c_1c_2c_3 = -m = -\phi\psi; \end{cases}$$

whence it is easy to infer the expressions,

$$\text{XLI.} \dots \Phi_1 = (c_2 - c_3)^{-1}(\Psi_3 - \Psi_2), \quad \Phi_2 = (c_3 - c_1)^{-1}(\Psi_1 - \Psi_3), \\ \Phi_3 = (c_1 - c_2)^{-1}(\Psi_2 - \Psi_1);$$

which enable us to express the functions $\Phi_{1\rho}, \Phi_{2\rho}, \Phi_{3\rho}$ as binomials (comp. 851, &c.), when $\Psi_{1\rho}, \Psi_{2\rho}, \Psi_{3\rho}$ have been expressed as monomes, and to assign the planes (real or imaginary), which are the loci of the lines $\Phi_{1\rho}, \Phi_{2\rho}, \Phi_{3\rho}$.

(14.) Accordingly, the *three operations*, Φ , Φ_1 , Φ_2 , by which lines in the three lately determined *directions* (12.) are *destroyed*, or reduced to zero, and which at first present themselves under the forms,

XLII. . . $\Phi\rho = \lambda S\mu\rho + \mu S\lambda\rho$, $\Phi_1\rho = \nabla\lambda\rho\mu + \rho T\lambda\mu$, $\Phi_2 = \nabla\lambda\rho\mu - \rho T\lambda\mu$,
are found to admit of the transformations,

$$\text{XLIII. . . } \Phi\rho = \frac{\Psi_2\rho - \Psi_1\rho}{2T\lambda\mu}; \quad \Phi_1\rho = \frac{\Psi_2\rho - \Psi\rho}{T\lambda\mu + S\lambda\mu}; \quad \Phi_2\rho = \frac{\Psi\rho - \Psi_1\rho}{T\lambda\mu - S\lambda\mu};$$

where Ψ , Ψ_1 , Ψ_2 have the recent forms XXXVI., and the loci of $\Phi\rho$, $\Phi_1\rho$, $\Phi_2\rho$ compose a system of *three rectangular planes*.

(15.) In general, the relations (18.) give also (comp. 353, (8.)),

$$\text{XLIV. . . } \Psi_1 = \Phi_2\Phi_3, \quad \Psi_2 = \Phi_3\Phi_1, \quad \Psi_3 = \Phi_1\Phi_2,$$

and

$$\text{XLV. . . } \Phi_1\Psi_1 = \Phi_2\Psi_2 = \Phi_3\Psi_3 = \Phi_1\Phi_2\Phi_3 = 0,$$

whence also,

$$\text{XLVI. . . } \Psi_1\Psi_2 = \Psi_2\Psi_3 = \Psi_3\Psi_1 = 0,$$

the symbols (in any one system of this sort) admitting of being *transposed and grouped* at pleasure; if then the roots of $M=0$ be *real and unequal*, there arises a system of *three real and distinct planes*, which are connected with the *interpretation* of the *symbolical equation*, $\Phi_1\Phi_2\Phi_3 = 0$, exactly as the three planes in 353, (9.) were connected with the analogous equation, $\phi\phi_1\phi_2 = 0$.

(16.) And when the cubic has *two imaginary roots*, it may then be said that there is *one real plane* (such as the plane $\perp \gamma$ in 353, (18.), (19.)), containing the *two imaginary directions* which then satisfy the equation I.; and *two imaginary planes*, which respectively contain those two directions, and intersect each other in *one real line* (such as the line γ in the example cited), namely the *one real vector root* of the same equation I.

355. Some additional light may be thrown upon that *vector equation of the second degree*, by considering the system of the *two scalar equations*,

$$\text{I. . . } S\lambda\rho\phi\rho = 0, \quad \text{and} \quad \text{II. . . } S\lambda\rho = 0,$$

and investigating the condition of the *reality* of the *two* directions*, ρ_1 and ρ_2 , by which they are generally satisfied, and for each of which the *plane* of ρ and $\phi\rho$ contains generally the *given line* λ in I., or is *normal* to the *plane locus* II. of ρ . We shall find that these two directions are *always real and rectangular* (except that they may become *indeterminate*), when the linear function ϕ is *its own conjugate*; and that then, if λ be a root ρ_0 of the *vector equation*,

$$\text{III. . . } V\rho\phi\rho = 0,$$

* Geometrically, the equation I. represents a *cone of the second order*, with λ for *one side*, and with the *three lines* ρ which satisfy III. for *three other sides*; and II. represents a *plane* through the vertex, *perpendicular* to the side λ . The *two directions* sought are thus the *two sides*, in which this plane cuts the cone.

which has been already otherwise discussed, the lines ρ_1 and ρ_2 are also roots of that equation; the general existence (354) of a system of three real and rectangular directions, which satisfy this equation III. when $\phi' \rho = \phi \rho$, being thus proved anew: whence also will follow a new proof of the reality of the scalar roots of the cubic $M = 0$, for this case of self-conjugation of ϕ ; and therefore of the necessary reality of the roots of that other cubic, $M_0 = 0$, which is formed (354, IV. or XXII.) from the self-conjugate part ϕ_0 of the general linear and vector function ϕ , as $M = 0$ was formed from ϕ .

(1.) Let λ, μ, ν be a system of three rectangular vector units, following in all respects the laws (182, 188), of the symbols i, j, k . Writing then,

$$\text{IV.} \dots \rho = y\mu + z\nu, \text{ and therefore, } \lambda\rho = y\nu - z\mu, \quad \phi\rho = y\phi\mu + z\phi\nu,$$

the equation II. is satisfied, and I. becomes,

$$\text{V.} \dots 0 = y^2 S\nu\phi\mu + yz (S\nu\phi\nu - S\mu\phi\mu) - z^2 S\mu\phi\nu;$$

the roots of which quadratic will be real and unequal, if

$$\text{VI.} \dots (S\nu\phi\nu - S\mu\phi\mu)^2 + 4S\mu\phi\nu S\nu\phi\mu > 0;$$

and the corresponding directions of ρ will be rectangular, if

$$\text{VII.} \dots 0 = S(y_1\mu + z_1\nu)(y_2\mu + z_2\nu) = -(y_1y_2 + z_1z_2);$$

that is, if

$$\text{VIII.} \dots S\nu\phi\mu = S\mu\phi\nu,$$

at least for this particular pair of vectors, μ and ν .

(2.) Introducing now the expression, $\phi\rho = \phi_0\rho + V\gamma\rho$ (849, XII.), the conditions VI. and VIII. take the forms,

$$\text{IX.} \dots (S\nu\phi_0\nu - S\mu\phi_0\mu)^2 + 4S(\mu\phi_0\nu)^2 > 4(S\gamma\mu\nu)^2, \text{ and X.} \dots S\gamma\mu\nu = 0;$$

which are both satisfied generally when $\gamma = 0$, or $\phi = \phi' = \phi_0$; the only exception being, that the quadratic V. may happen to become an identity, by all its coefficients vanishing: but the opposite inequality (to VI. and IX.) can never hold good, that is to say, the roots of that quadratic can never be imaginary, when ϕ is thus self-conjugate.

(3.) On the other hand, when γ is actual, or $\phi' \rho$ not generally $= \phi \rho$, the condition X. of rectangularity can only accidentally be satisfied, namely by the given or fixed line γ happening to be in the assumed plane of μ, ν ; and when the two directions of ρ are thus not rectangular, or when the scalar $S\gamma\mu\nu$ does not vanish, we have only to suppose that the square of this scalar becomes large enough, in order to render (by IX.) those directions coincident, or imaginary.

(4.) When $\phi' = \phi$, or $\gamma = 0$, we may take μ and ν for the two rectangular directions of ρ , or may reduce the quadratic to the very simple form $yz = 0$; but, for this purpose, we must establish the relations,

$$\text{XI.} \dots S\mu\phi\nu = S\nu\phi\mu = 0.$$

(5.) And if, at the same time, λ satisfies the equation III., so that $\phi\lambda \parallel \lambda$, we shall have these other scalar equations,

$$\text{XII.} \dots 0 = S\mu\phi\lambda = S\nu\phi\lambda = S\lambda\phi\mu = S\lambda\phi\nu;$$

whence

$$\phi\mu \parallel V\nu\lambda \parallel \mu, \text{ and } \phi\nu \parallel V\lambda\mu \parallel \nu,$$

or,

$$\text{XIII.} \dots 0 = V\lambda\phi\lambda = V\mu\phi\mu = V\nu\phi\nu;$$

λ, μ, ν thus forming (as above stated) a *system*, of *three* real and rectangular roots of that *vector* equation III.

(6.) But in general, if III. be satisfied by even *two* real and *distinct* directions of ρ , the *scalar* and *cubic* equation $M=0$ can have *no imaginary* root; for if those two directions give *two unequal* but *real* and *scalar* values, c_1 and c_2 , for the *quotient* $-\phi\rho : \rho$, then c_1 and c_2 are *two real roots* of the cubic, of which therefore the *third* root is *also* real; and if, on the other hand, the two directions ρ_1 and ρ_2 give one *common* real and *scalar* value, such as c_1 , for that quotient, then $\phi\rho = -c_1\rho$, or $\Phi_1\rho = (\phi + c_1)\rho = 0$, for *every line* in the *plane* of ρ_1, ρ_2 ; so that $\phi\rho$ must be of the *form*, $-c_1\rho + \beta S\rho_1\rho_2\rho$, and the *cubic* will have at least *two equal* roots, since it will take the form,

$$\text{XIV.} \dots 0 = (c - c_1)^2 (c - c_1 + S\rho_1\rho_2\beta),$$

as is easily shown from principles and formulæ already established.

(7.) It is then proved anew, that the equation $M=0$ has *all* its roots *real*, if $\phi'\rho = \phi\rho$; and therefore that the equation $M_0=0$ (as above stated) can *never* have an *imaginary* root.

(8.) And we see, at the same time, how the *scalar cubic* $M=0$ might have been deduced from the *symbolical cubic* 850, I., or from the equation 851, I., as the condition for the *vector* equation III. being satisfied by any *actual* ρ ; namely by observing that if $\phi\rho = -c\rho$, then $\phi^2\rho = c^2\rho$, $\phi^3\rho = -c^3\rho$, &c., and therefore $M\rho = 0$, in which ρ , by supposition, is different from zero.

(9.) Finally, as regards the *case** of *indetermination*, above alluded to, when the *quadratic* V. *fails* to assign any *definite* values to $y : z$, or any *definite* directions in the *given* plane to ρ , this case is evidently distinguished by the condition,

$$\text{XV.} \dots S\mu\phi\mu = S\nu\phi\nu,$$

in combination with the equations XI.

356. The existence of the *Symbolic and Cubic Equation* (350), which is satisfied by the *linear and vector symbol* ϕ , suggests a *Theorem†* of *Geometrical Deformation*, which may be thus enunciated:—

* It will be found that this *case* corresponds to the *circular sections* of a *surface* of the *second order*; while the less particular case in which $\phi'\rho = \phi\rho$, but not $S\mu\phi\mu = S\nu\phi\nu$, so that the *two directions* of ρ are *determined, real, and rectangular*, corresponds to the *axes* of a *non-circular section* of such a *surface*.

† This theorem was stated, nearly in the same way, in page 568 of the *Lectures*; and the problem of *inversion* of a *linear and vector* function was treated, in the few preceding pages (559, &c.), though with somewhat less of completeness and perhaps of simplicity than in the present Section, and with a slightly different notation. The *general form* of such a function which was there adopted may now be thus expressed:

$$\phi\rho = \Sigma\beta S a\rho + V r\rho, \text{ } r \text{ being a given quaternion;}$$

the resulting value of m was found to be (page 561),

“If, by any given Mode, or Law, of Linear Derivation, of the kind above denoted by the symbol ϕ , we pass from any assumed Vector ρ to a Series of Successively Derived Vectors, $\rho_1, \rho_2, \rho_3, \dots$ or $\phi^1\rho, \phi^2\rho, \phi^3\rho, \dots$; and if, by constructing a Parallelepiped, we decompose any Line of this Series, such as ρ_3 , into three partial or component lines, $m\rho, -m'\rho_1, m''\rho_2$, in the Directions of the three which precede it, as here of ρ, ρ_1, ρ_2 ; then the Three Scalar Coefficients, $m, -m', m''$, or the Three Ratios which these three Components of the Fourth Line ρ_3 bear to the Three Preceding Lines of the Series, will depend only on the given Mode or Law of Derivation, and will be entirely independent of the assumed Length and Direction of the Initial Vector.”

(1.) As an Example of such successive Derivation, let us take the law,

$$I. \dots \rho_1 = \phi\rho = -V\beta\rho\gamma, \quad \rho_2 = \phi^2\rho = -V\beta\rho_1\gamma, \text{ \&c.,}$$

which answers to the construction in 805, (1.), &c., when we suppose that β and γ are unit-lines. Treating them at first as any two given vectors, our general method conducts to the equation,

$$II. \dots \rho_3 = m\rho - m'\rho_1 + m''\rho_2,$$

with the following values of the coefficients,

$$III. \dots m = -\beta^2\gamma^2S\beta\gamma, \quad m' = -\beta^2\gamma^2, \quad m'' = S\beta\gamma;$$

as may be seen, without any new calculation, by merely changing g, λ , and μ , in 854, XXXIII., to 0, β , and $-\gamma$.

(2.) Supposing next, for comparison with 805, that

$$IV. \dots \beta^2 = \gamma^2 = -1, \quad \text{and } S\beta\gamma = -l,$$

so that β, γ are unit lines, and l is the cosine of their inclination to each other, the values III. become,

$$V. \dots m = l, \quad m' = -1, \quad m'' = -l;$$

and the equation II., connecting four successive lines of the series, takes the form,

$$VI. \dots \rho_3 = l\rho + \rho_1 - l\rho_2, \quad \text{or } VII. \dots \rho_3 - \rho_1 = -l(\rho_2 - \rho);$$

$$m = \Sigma Saa'a''S\beta'\beta'' + \Sigma S(rVaa'.V\beta'\beta) + Sr\Sigma S\alpha\beta r - \Sigma SarS\beta r + SrTr^2;$$

and the auxiliary function which we now denote by ψ was,

$$m\phi^{-1}\sigma = \psi\sigma = \Sigma Vaa'S\beta'\beta\sigma + \Sigma V.aV(V\beta\sigma.r) + (V\sigma rSr - VrS\sigma r);$$

where the sum of the two last terms of $\psi\sigma$ might have been written as $\sigma rSr - rS\sigma r$.

A student might find it an useful exercise, to prove the correctness of these expressions by the principles of the present Section. One way of doing so would be, to treat $\Sigma\beta S\alpha\rho$ and r as respectively equal to $\phi_0\rho + V\gamma\rho$ and $c + \varepsilon$; which would transform m and $\psi\sigma$, as above written, into the following,

$$M_0 - S(\gamma + \varepsilon)(\phi_0 + c)(\gamma + \varepsilon), \quad \text{and } \Psi_0\sigma - (\gamma + \varepsilon)S(\gamma + \varepsilon)\sigma + V\sigma(\phi_0 + c)(\gamma + \varepsilon);$$

that is, into the new values which the M and $\Psi\sigma$ of the Section assume, when $\phi\rho$ takes the new value, $\Phi\rho = (\phi_0 + c)\rho + V(\gamma + \varepsilon)\rho$.

a result which agrees with 805, (3.), since we there found that if $\rho = \text{or}$, &c., the interval P_1P_2 was $= -I \times PP_2$.

(8.) And as regards the *inversion* of a linear and vector function (847), or the *return* from any one line ρ_1 of such a *series* to the line ρ which *precedes* it, our general method gives, for the example I., by 854, (12.),

$$\text{VIII.} \dots \psi\rho_1 = \frac{1}{2}\beta(\beta\gamma\rho_1 + \rho_1\beta\gamma)\gamma,$$

and

$$\text{IX.} \dots \rho = \phi^{-1}\rho_1 = \text{m}^{-1}\psi\rho_1 = -\frac{\beta\rho_1\beta^{-1} + \gamma\rho_1\gamma^{-1}}{\beta\gamma + \gamma\beta};$$

a result which it is easy to verify and to interpret, on principles already explained.

357. We are now prepared to assign some new and general *Forms*, to which the *Linear* and *Vector Function* (with real constants) of a variable vector can be brought, *without* assuming its *self-conjugation*; one of the simplest of which forms is the following,

$$\text{I.} \dots \phi\rho = Vq_0\rho + V\lambda\rho\mu, \quad \text{with} \quad \text{I'.} \dots q_0 = g + \gamma;$$

q_0 being here a *real* and *constant quaternion*, and λ, μ *two real* and *constant vectors*, which can *all* be *definitely assigned*, when the *particular form* of ϕ is given: except that λ and μ may be *interchanged* (by 295, VII.), and that *either* may be *multiplied* by any scalar, if the *other* be *divided* by the same. It will follow that the *scalar, quadratic, and homogeneous function* of a vector, denoted by $S\rho\phi\rho$, can always be thus expressed:

$$\text{II.} \dots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho;$$

or thus,

$$\text{II'.} \dots S\rho\phi\rho = g'\rho^2 + 2S\lambda\rho S\mu\rho, \quad \text{if} \quad g' = g - S\lambda\mu;$$

a *general* and (as above remarked) *definite transformation*, which is found to be one of great utility in the theory of *Surfaces** of the *Second Order*.

(1.) Attending first to the *case* of *self-conjugate* functions $\phi_0\rho$, from which we can pass to the *general* case by merely adding the term $V\gamma\rho$, and supposing (in virtue of what precedes) that $a_1a_2a_3$ are three *real* and *rectangular vector-units*, and $e_1e_2e_3$ three *real scalars* (the roots of the cubic $M_0 = 0$), such that

* In the theory of such surfaces, the two constant and real vectors, λ and μ , have the directions of what are called the *cyclic normals*.

III. . . $\phi_1 a_1 = (\phi_0 + e_1) a_1 = 0$, $\phi_2 a_2 = (\phi_0 + e_2) a_2 = 0$, $\phi_3 a_3 = (\phi_0 + e_3) a_3 = 0$,
we may write

$$\text{IV. . . } \rho = -(a_1 S a_1 \rho + a_2 S a_2 \rho + a_3 S a_3 \rho),$$

and therefore

$$\text{V. . . } \phi_0 \rho = c_1 a_1 S a_1 \rho + c_2 a_2 S a_2 \rho + c_3 a_3 S a_3 \rho;$$

so that

$$\text{VI. . . } \begin{cases} \phi_1 \rho = (c_2 - e_1) a_2 S a_2 \rho + (c_3 - e_1) a_3 S a_3 \rho, \\ \phi_2 \rho = (c_3 - e_2) a_3 S a_3 \rho + (e_1 - c_2) a_1 S a_1 \rho, \\ \phi_3 \rho = (e_1 - e_3) a_1 S a_1 \rho + (e_2 - e_3) a_2 S a_2 \rho, \end{cases}$$

the binomial forms of ϕ_1, ϕ_2, ϕ_3 being thus put in evidence.

(2.) We have thus the general but *scalar* expressions:

$$\text{VII. . . } -\rho^3 = (S a_1 \rho)^2 + (S a_2 \rho)^2 + (S a_3 \rho)^2;$$

$$\begin{aligned} \text{VIII. . . } S \rho \phi \rho &= S \rho \phi_0 \rho = c_1 (S a_1 \rho)^2 + c_2 (S a_2 \rho)^2 + c_3 (S a_3 \rho)^2 \\ &= -c_1 \rho^3 + (c_2 - e_1) (S a_2 \rho)^2 + (c_3 - e_1) (S a_3 \rho)^2 \\ &= -c_2 \rho^3 - (e_2 - e_1) (S a_1 \rho)^2 + (e_3 - e_2) (S a_3 \rho)^2 \\ &= -c_3 \rho^3 - (e_3 - e_1) (S a_1 \rho)^2 - (e_3 - e_2) (S a_2 \rho)^2; \end{aligned}$$

in which it is in general permitted to assume that

$$\text{IX. . . } e_1 < e_2 < e_3, \text{ or that } \text{X. . . } e_3 - e_1 = 2e^2, \quad e_3 - e_2 = 2e'^2,$$

e and e' being *real* scalars, and the numerical *coefficients* being introduced for a motive of convenience which will presently appear.

(3.) Comparing the last but one of the expressions VIII. with II', we see that we may bring $S \rho \phi \rho$ to the proposed form II., by assuming,

$$\text{XI. . . } \lambda = e a_1 + e' a_3, \quad \mu = -e a_1 + e' a_3, \quad g = S \lambda \mu - c_2 = -\frac{1}{2}(c_1 + c_3),$$

because $S \lambda \mu = e^2 - e'^2 = e_2 - \frac{1}{2}(c_1 + c_3)$.

(4.) But in general (comp. 849, (4.)) we cannot have, for *all* values of ρ ,

$$\text{XII. . . } S \rho \phi \rho = S \rho \phi' \rho, \text{ unless } \text{XIII. . . } \phi_0 \rho = \phi' \rho,$$

that is, unless the *self-conjugate parts* of ϕ and ϕ' be *equal*; we can therefore infer from II. that $\phi_0 \rho = g \rho + \nabla \lambda \rho \mu$, because $\nabla \lambda \rho \mu = \nabla \mu \rho \lambda =$ its own conjugate; and thus the transformation I. is proved to be *possible*, and *real*.

(5.) Accordingly, with the values XI. of λ, μ, g , the expression,

$$\text{XIV. . . } \phi_0 \rho = g \rho + \nabla \lambda \rho \mu = \rho (g - S \lambda \mu) + \lambda S \mu \rho + \mu S \lambda \rho,$$

becomes,

$$\begin{aligned} \text{XV. . . } \phi_0 \rho &= -c_2 \rho + (e' a_3 + e a_1) S (e' a_3 - e a_1) \rho + (e' a_3 - e a_1) S (e' a_3 + e a_1) \rho \\ &= -c_2 \rho - 2e^2 a_1 S a_1 \rho + 2e'^2 a_3 S a_3 \rho; \end{aligned}$$

which agrees, by X., with VI.

(6.) Conversely if g, λ , and μ be constants such that $\phi_0 \rho = g \rho + \nabla \lambda \rho \mu$, then $\phi_0 \nabla \lambda \mu = g' \nabla \lambda \mu$, where $g' = g - S \lambda \mu$, as before; hence $-g'$ must be one of the three roots e_1, e_2, e_3 of the cubic $M_0 = 0$, and the normal to the plane of λ, μ must have one of the three directions of a_1, a_2, a_3 ; if then we assume, on trial, that this plane is that of a_1, a_3 , and write accordingly,

$$\text{XVI. . . } \lambda = a a_1 + a' a_3, \quad \mu = b a_1 + b' a_3, \quad \phi_2 \rho = \lambda S \mu \rho + \mu S \lambda \rho,$$

we are, by VI., to seek for scalars $aa'bb'$ which shall satisfy the three conditions,

$$\text{XVII. . . } 2ab = e_1 - e_3, \quad 2a'b' = e_3 - e_2, \quad ab' + b'a' = 0;$$

but these give

$$\text{XVIII. . . } (2ab')^2 = (2ba')^2 = (e_3 - e_2) (e_2 - e_1),$$

so that if the transformation is to be a *real* one, we must suppose that $c_2 - c_1$ and $c_3 - c_2$ are either *both positive*, as in IX., or else *both negative*; or in other words we must so *arrange* the three real roots of the cubic, that c_2 may be (algebraically) *intermediate* in value between the other two. Adopting then the *order IX.*, with the values X., we satisfy the conditions XVII. by supposing that

$$\text{XIX.} \dots a' = b' = e', \quad a = -b = e;$$

and are thus led back from XVI. to the expressions XI., as the *only real ones* for λ , μ , and g which render possible the transformations I. and II.; except that λ and μ may be *interchanged*, &c., as before.

(7.) We see, however, that in an *imaginary sense* there exist *two other solutions* of the problem, to *transform* $\phi\rho$ and $S\rho\phi\rho$ as above; for if we retain the order IX., and equate g' in II' to either $-c_1$ or $-c_3$, we may in each case conceive the corresponding *sum of two squares* in VIII. as being the *product of two imaginary but linear factors*; the *planes* of the two *imaginary pairs* of vectors which result being *real*, and perpendicular respectively to a_1 and a_3 .

(8.) And if the real expression XIV. for $\phi_0\rho$ be *given*, and it be required to pass from it to the expression V., with the order of inequality IX., the investigation in 854, (12.) enables us at once to establish the formulæ:

$$\text{XX.} \dots c_1 = -g - T\lambda\mu, \quad c_2 = -g + S\lambda\mu, \quad c_3 = -g + T\lambda\mu;$$

$$\text{XXI.} \dots a_1 = U(\lambda T\mu - \mu T\lambda), \quad a_2 = UV\lambda\mu, \quad a_3 = U(\lambda T\mu + \mu T\lambda);$$

in which however it is permitted to change the sign of any one of the three vector units. Accordingly the expressions XI. give,

$$T\lambda\mu + S\lambda\mu = 2e^2 = c_2 - c_1, \quad T\lambda\mu - S\lambda\mu = 2e^2 = c_3 - c_2, \quad S\lambda\mu = g + c_2;$$

$$T\lambda = T\mu, \quad \lambda - \mu = 2ea_1, \quad V\lambda\mu = -2ee'a_3a_1 = \mp 2ee'a_2, \quad \lambda + \mu = 2e'a_3.$$

(9.) We have also the two *identical transformations*,

$$\text{XXII.} \dots S\lambda\rho\mu\rho = \rho^2 T\lambda\mu + \{(S\lambda\mu\rho)^2 + (S\lambda\rho T\mu + S\mu\rho T\lambda)^2\} (T\lambda\mu - S\lambda\mu)^{-1},$$

$$\text{XXIII.} \dots S\lambda\rho\mu\rho = -\rho^2 T\lambda\mu - \{(S\lambda\mu\rho)^2 + (S\lambda\rho T\mu - S\mu\rho T\lambda)^2\} (T\lambda\mu + S\lambda\mu)^{-1},$$

which hold good for *any three vectors*, λ , μ , ρ , and may (among other ways) be deduced, through the expressions XX. and XXI., from II. and VIII.

(10.) Finally, as regards the expressions VI. for $\phi_1\rho$, &c., if we denote the corresponding forms of $\psi\rho$ by $\psi_1\rho$, &c., we have (comp. 854, (15.)) these other expressions, which are as usual (comp. 851, &c.) of *monomial form*:

$$\text{XXIV.} \dots \begin{cases} \psi_1\rho = \phi_2\phi_3\rho = (c_2 - c_1)(c_1 - c_3)a_1S a_1\rho; \\ \psi_2\rho = \phi_3\phi_1\rho = (c_3 - c_2)(c_2 - c_1)a_2S a_2\rho; \\ \psi_3\rho = \phi_1\phi_2\rho = (c_1 - c_3)(c_3 - c_2)a_3S a_3\rho; \end{cases}$$

and which verify the relations 854, XLI., and several other parts of the whole foregoing theory.

358. The *general linear and vector function* $\phi\rho$ of a *vector* has been seen (347, (1.)) to contain, at least implicitly, *nine scalar constants*; and accordingly the expression 357, I. involves that number, namely *four* in the term $Vq_0\rho$, on account of the constant *quaternion* q_0 , and *five* in the other term $V\lambda\rho\mu$, each of the two *unit-vectors*, $U\lambda$ and $U\mu$, counting as *two scalars*, and the *tensor* $T\lambda\mu$ as *one more*. But a *self-*

conjugate linear and vector function, or the *self-conjugate part* $\phi_{\alpha\rho}$ of the *general* function $\phi\rho$, involves only *six* scalar constants; either because *three* disappear with the term $V\gamma\rho$ of $\phi\rho$; or because the *condition of self-conjugation*, $\Sigma V\beta a = 2\gamma = 0$ (comp. 349, XXII. and 353, XXXVI.), which arises when we take for $\phi\rho$ the form $\Sigma\beta S a\rho$ (347, XXXI.), is equivalent to a system of *three scalar equations*, connecting the *nine constants*. And for the same reason the *general quadratic* but *scalar* function, $S\rho\phi\rho$, involves in like manner only *six* scalar constants. Accordingly there enter only six such constants into the expressions 357, II., II', V., VIII., XIV.; c_1, c_2, c_3 , for instance, being *three* such, and the rectangular unit system a_1, a_2, a_3 answering to *three others*. The following *other general transformations* of $S\rho\phi\rho$ and $\phi_{\alpha\rho}$, although not quite so simple as 357, II. and XIV., involve the *same number (six)* of scalar constants, and deserve to be briefly considered: namely the forms,

$$\begin{aligned} \text{I. . . } S\rho\phi\rho &= a(Va\rho)^2 + b(S\beta\rho)^2; \\ \text{II. . . } \phi_{\alpha\rho} &= -aVa\rho + b\beta S\beta\rho; \end{aligned}$$

in which a, b are two real scalars, and a, β are two real unit-vectors. We shall merely set down the leading formulæ, leaving the reader to supply the analysis, which at this stage he cannot find difficult.

(1.) In accomplishing the reduction of the expressions,

$$S\rho\phi\rho = c_1(Sa_1\rho)^2 + c_2(Sa_2\rho)^2 + c_3(Sa_3\rho)^2, \quad 357, \text{ VIII.}$$

$$\text{and} \quad \phi\rho = c_1a_1Sa_1\rho + c_2a_2Sa_2\rho + c_3a_3Sa_3\rho, \quad 357, \text{ V.}$$

to these new forms I. and II., it is found that, if the result is to be a *real* one, $-a$ must be *that* root of the scalar cubic $M_0 = 0$, the *reciprocal* of which is algebraically *intermediate*, between the reciprocals of the other two. It is therefore convenient *here* to assume this *new condition*, respecting the *order* of the *inequalities*,

$$\text{III. . . } c_1^{-1} > c_2^{-1} > c_3^{-1};$$

which will indeed *coincide* with the arrangement 357, IX., if the three roots c_1, c_2, c_3 , be *all positive*, but will be *incompatible* with it in every *other case*.

(2.) This being laid down (or even, if we choose, the *opposite* order being taken), the (real) values of $a, b; a, \beta$ may be thus expressed :

$$\text{IV. . . } a = -c_2, \quad b = c_1 - c_2 + c_3;$$

$$\text{V. . . } a = xa_1 + xa_3, \quad \beta = x'a_1 + x'a_3;$$

in which

$$\text{VI. . . } x^2 = \frac{c_1^{-1} - c_3^{-1}}{c_1^{-1} - c_2^{-1}}, \quad x'^2 = \frac{c_2^{-1} - c_3^{-1}}{c_1^{-1} - c_2^{-1}};$$

$$\text{VII. . . } \frac{c_1x}{x'} = \frac{c_2x}{x'} = b(xx' + xx') = -bSa\beta = (aay) b';$$

$$\text{VIII.} \dots b^2 = e_1 e_2^{-1} e_3 b = c_1^2 x^2 + c_2^2 z^2; \quad \text{IX.} \dots x^2 + y^2 = a^2 + y^2 = 1;$$

$$\text{X.} \dots bx'x' = e_2 xz;$$

$$\text{XI.} \dots c_1 x^2 + c_2 z^2 = c_1 e_2^{-1} e_3 = b^{-1} b^2 = b(Sa\beta)^2, \quad c_1 c_2 = -ab(Sa\beta)^2;$$

$$\text{XII.} \dots b'\beta = -b\beta Sa\beta = c_1 x a_1 + c_2 z a_3; \quad \&c.$$

(3.) And there result the transformations:

$$\text{XIII.} \dots \phi_{2\rho} = (c_1 - c_2) a_1 S a_1 \rho + (c_2 - c_1) a_3 S a_3 \rho$$

$$= -c_2(x a_1 + z a_3) S(x a_1 + z a_3) \rho + \frac{c_2}{c_1 c_2} (x c_1 a_1 + z c_2 a_3) S(x c_1 a_1 + z c_2 a_3) \rho;$$

$$\text{XIV.} \dots \phi_{0\rho} = c_1 a_1 S a_1 \rho + c_2 a_3 S a_3 \rho + c_3 a_2 S a_2 \rho$$

$$= c_2(x a_1 + z a_3) V(x a_1 + z a_3) \rho + \frac{c_3}{c_1 c_2} (x c_1 a_1 + z c_2 a_3) S(x c_1 a_1 + z c_2 a_3) \rho;$$

$$\text{XV.} \dots S\rho\phi\rho = -c_2(V(x a_1 + z a_3) \rho)^2 + \frac{c_3}{c_1 c_2} (S(x c_1 a_1 + z c_2 a_3) \rho)^2;$$

which last, if $c_1 c_2$ be positive, gives this other real form,

$$\text{XVI.} \dots S\rho\phi\rho = \frac{c_2}{c_1 c_2} N\{S(x c_1 a_1 + z c_2 a_3) \rho + (c_1 c_2)^{\frac{1}{2}} V(x a_1 + z a_3) \rho\};$$

x^2 and z^2 being determined by the expressions VI.

(4.) Those expressions allow us to change the sign of x : x , and thereby to determine a second pair of real unit lines, a' and β' , which may be substituted for a and β in the forms I. and II.; the order of inequalities III. (or the opposite order), and the values IV. of a and b , remaining unchanged. We have therefore the double transformations:

$$\text{XVII.} \dots S\rho\phi\rho = -c_2(Va\rho)^2 + (c_1 - c_2 + c_3) (S\beta\rho)^2 = -c_2(Va'\rho)^2 + (c_1 - c_2 + c_3) (S\beta'\rho)^2;$$

$$\text{XVIII.} \dots \phi\rho = c_2 a V a \rho + (c_1 - c_2 + c_3) \beta S \beta \rho = c_2 a' V a' \rho + (c_1 - c_2 + c_3) \beta' S \beta' \rho.$$

(5.) If either of the two connected forms I. and II. had been given, we might have proposed to deduce from it the values of $c_1 c_2 c_3$, and of $a_1 a_2 a_3$, by the general method of this Section. We should thus have had the cubic,

$$\text{XIX.} \dots 0 = M_0 = (c + a) \{c^2 + (a - b)c - ab(Sa\beta)^2\};$$

and because the quadratic $(c + a)^{-1} M_0 = 0$ may be thus written,

$$\text{XX.} \dots (c^{-1} + a^{-1})^2 (Sa\beta)^2 - (c^{-1} + a^{-1}) (a^{-1} S. (a\beta)^2 + b^{-1}) + a^{-2} (Va\beta)^2 = 0,$$

it gives two real values of $c^{-1} + a^{-1}$, one positive and the other negative; if then we arrange the reciprocals of the three roots of $M_0 = 0$ in the order III., we have the expressions,

$$\text{XXI.} \dots \begin{cases} c_1 = \frac{1}{2}(b - a) + \frac{1}{2}ab \sqrt{(a^{-2} + 2a^{-1}b^{-1}S. (a\beta)^2 + b^{-2})}; & c_2 = -a; \\ c_3 = \frac{1}{2}(b - a) - \frac{1}{2}ab \sqrt{(a^{-2} + 2a^{-1}b^{-1}S. (a\beta)^2 + b^{-2})}; \end{cases}$$

the signs of the radical being determined by the condition that $(c_1 - c_2) : ab(Sa\beta)^2 = c_1^{-1} - c_2^{-1} > 0$. Accordingly these expressions for the roots agree evidently with the former results, IV. and XI., because $S. (a\beta)^2 = 2(Sa\beta)^2 - 1$.

(6.) The roots c_1, c_2, c_3 being thus known, the same general method gives for the directions of a_1, a_2, a_3 the versors of the following expressions (or of their negatives):

$$\text{XXII.} \dots \begin{cases} \psi_{1\rho} = ac_2^{-1}(c_2\alpha + b\beta Sa\beta) S(c_2\alpha + b\beta Sa\beta)\rho; \\ \psi_{2\rho} = abVa\beta Sa\beta\rho; \\ \psi_{3\rho} = ac_1^{-1}(c_1\alpha + b\beta Sa\beta) S(c_1\alpha + b\beta Sa\beta)\rho; \end{cases}$$

of which the *monomial forms* may again be noted, and which give,

$$\text{XXII}' \dots a_1 = \pm U(c_2\alpha + b\beta Sa\beta), \quad a_2 = \pm UVa\beta, \quad a_3 = \pm U(c_1\alpha + b\beta Sa\beta).$$

(7.) Accordingly the expressions in (2.), give (if we suppose $a_2a_1 = +a_2$),

$$\text{XXIII.} \dots c_2\alpha + b\beta Sa\beta = (c_2 - c_1)x\alpha_1, \quad Va\beta = (x'z - zx')a_2, \quad c_1\alpha + b\beta Sa\beta = (c_1 - c_2)x\alpha_2;$$

and as an additional verification of the *consistency* of the various parts of this whole theory, it may be observed (comp. 357, XXIV.), that

$$\text{XXIV.} \dots -ac_2^{-1}(c_2\alpha + b\beta Sa\beta)^2 = (c_2 - c_1)(c_1 - c_2), \quad ab(Va\beta)^2 = (c_2 - c_1)(c_2 - c_1), \quad -ac_1^{-1}(c_1\alpha + b\beta Sa\beta)^2 = (c_1 - c_2)(c_2 - c_1).$$

(8.) As regards the *second transformations*, XVII. and XVIII., it is easy to prove that we may write,

$$\text{XXV.} \dots (c_2 - c_1)\alpha' = b\beta a\beta - a\alpha, \quad (c_1 - c_2)\beta' = a\alpha\beta a - b\beta,$$

$$\text{XXVI.} \dots -(c_2 - c_1)^2 = (b\beta a\beta - a\alpha)^2 = (a\alpha\beta a - b\beta)^2;$$

so that we have the following equation,

$$\text{XXVII.} \dots (a(Va\rho)^2 + b(S\beta\rho)^2)(a^2 + 2abS.(a\beta)^2 + b^2) = a(V(b\beta a\beta - a\alpha)\rho)^2 + b(S(a\alpha\beta a - b\beta)\rho)^2,$$

which is true for any vector ρ , any two unit lines α, β , and any two scalars a, b .

(9.) Accordingly it is evident from (4.), that a_1, a_2 must be the bisectors of the angles made by α, α' , and also of those made by β, β' ; and the expressions XXV. may be thus written (because $b - a = c_1 + c_2$),

$$\text{XXVIII.} \dots (c_2 - c_1)\alpha' = (c_2 + c_1)\alpha + 2b\beta Sa\beta, \quad (c_1 - c_2)\beta' = (c_1 + c_2)\beta - 2aa\beta a;$$

whence, by XXIII., we may write,

$$\text{XXIX.} \dots a + \alpha' = 2xa_1, \quad a - \alpha' = 2xa_2;$$

so that a_1 bisects the internal angle, and a_2 the external angle, of the lines a, α' .

(10.) At the same time we have these other expressions,

$$\text{XXX.} \dots (c_1 - c_2)(\beta + \beta') = 2(c_1\beta - aa\beta a), \quad (c_2 - c_1)(\beta - \beta') = 2(c_2\beta - aa\beta a);$$

which can easily be reduced to the simple forms,

$$\text{XXXI.} \dots \beta + \beta' = 2x'\alpha_1, \quad \beta - \beta' = 2x''\alpha_2,$$

with the recent meanings of the coefficients x' and x'' .

(11.) And although, for the sake of obtaining *real transformations*, we have supposed (comp. III.) that

$$\text{XXXII.} \dots (c_1^{-1} - c_2^{-1})(c_2^{-1} - c_3^{-1}) > 0,$$

because the assumed relation $a = xa_1 + xa_2$ between the three unit vectors aa_1a_2 , whereof the two latter are rectangular, gives $x^2 + x'^2 = 1$, as in IX., so that each of the two expressions VI. involves the other, and their comparison gives the ratio,

$$\text{XXXIII.} \dots x^2 : x'^2 = (c_1^{-1} - c_2^{-1}) : (c_2^{-1} - c_3^{-1}),$$

yet we see that, *without* this inequality XXXII. existing, the foregoing transformations hold good in an *imaginary* (or merely *symbolical*) sense: so that we may say, in general, that the functions $S\rho\phi\rho$ and $\phi_0\rho$ can be brought to the forms I. and II. in *six distinct ways*, whereof *two* are *real*, and the *four others* are *imaginary*.

(12.) It may be added that the first equation XXII. admits of being replaced by the following,

$$\text{XXXIV.} \dots \psi_{1\rho} = -\delta\alpha^{-1}(c_1\beta - a\alpha S a\beta) S(c_1\beta - a\alpha S a\beta)\rho,$$

with a corresponding form for $\psi_{2\rho}$; and that thus, instead of XXII., we are at liberty to write the expressions,

$$\text{XXXV.} \dots a_1 = U(c_1\beta - a\alpha S a\beta), \quad a_2 = UVa\beta, \quad a_3 = U(c_2\beta - a\alpha S a\beta),$$

for the rectangular unit system, deduced from I. or II.

359. If we call, as we naturally may, the expressions

$$\text{I.} \dots \phi_0\rho = c_1 a_1 S a_{1\rho} + c_2 a_2 S a_{2\rho} + c_3 a_3 S a_{3\rho}, \quad 357, \text{ V.},$$

$$\text{and II.} \dots S\rho\phi\rho = c_1 (S a_{1\rho})^2 + c_2 (S a_{2\rho})^2 + c_3 (S a_{3\rho})^2, \quad 357, \text{ VIII.},$$

the *Rectangular Transformations* of the *Functions* $\phi_0\rho$ and $S\rho\phi\rho$, then by another *geometrical analogy*, which will be seen when we come to speak briefly of the theory of *Surfaces of the Second Order*, we may call the expressions,

$$\text{III.} \dots \phi_0\rho = g\rho + V\lambda\rho\mu, \quad 357, \text{ XIV.},$$

$$\text{and IV.} \dots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho, \quad 357, \text{ II.},$$

the *Cyclic* Transformations* of the same two functions; and may say that the two other and more recent expressions,

$$\text{V.} \dots \phi_0\rho = -a\alpha V a\rho + b\beta S\beta\rho, \quad 358, \text{ II.},$$

$$\text{and VI.} \dots S\rho\phi\rho = a(V a\rho)^2 + b(S\beta\rho)^2, \quad 358, \text{ I.},$$

are *Focal† Transformations* of the same. We have already shown (357) how to exchange *rectangular forms* with *cyclic* ones; and also (358) how to pass from *rectangular* expressions to *focal* ones, and reciprocally: but it may be worth while to consider briefly the mutual relations which exist, between *cyclic* and *focal* expressions, and the modes of passing from either to the other.

(1.) To pass from IV. to VI., or from the *cyclic* to the *focal* form, we may first accomplish the *rectangular* transformation II., with the values 357, XX., and XXI., of c_1, c_2, c_3 , and of a_1, a_2, a_3 , the order of inequality being assumed to be

* Compare the Note to Art. 357.

† It will be found that the two *real vectors* a, a' , of 358, are the *two real focal lines* of the *real or imaginary cone*, which is *asymptotic* to the *surface of the second order*, $S\rho\phi\rho = \text{const}$.

$$\text{XXII.} \dots \begin{cases} \psi_{1\rho} = ac_2^{-1}(c_2\alpha + b\beta Sa\beta) S(c_2\alpha + b\beta Sa\beta)\rho; \\ \psi_{2\rho} = abVa\beta Sa\beta\rho; \\ \psi_{3\rho} = ac_1^{-1}(c_1\alpha + b\beta Sa\beta) S(c_1\alpha + b\beta Sa\beta)\rho; \end{cases}$$

of which the monomial forms may again be noted, and which give,

$$\text{XXII}' \dots a_1 = \pm U(c_2\alpha + b\beta Sa\beta), \quad a_2 = \pm UVa\beta, \quad a_3 = \pm U(c_1\alpha + b\beta Sa\beta).$$

(7.) Accordingly the expressions in (2.), give (if we suppose $a_3a_1 = +a_2$),

$$\text{XXIII.} \dots c_2\alpha + b\beta Sa\beta = (c_2 - c_1)xa_1, \quad Va\beta = (x'z - zx')a_2, \quad c_1\alpha + b\beta Sa\beta = (c_1 - c_2)xa_3;$$

and as an additional verification of the consistency of the various parts of this whole theory, it may be observed (comp. 357, XXIV.), that

$$\text{XXIV.} \dots -ac_2^{-1}(c_2\alpha + b\beta Sa\beta)^2 = (c_2 - c_1)(c_1 - c_2), \quad ab(Va\beta)^2 = (c_2 - c_2)(c_2 - c_1), \quad -ac_1^{-1}(c_1\alpha + b\beta Sa\beta)^2 = (c_1 - c_2)(c_2 - c_2).$$

(8.) As regards the second transformations, XVII. and XVIII., it is easy to prove that we may write,

$$\text{XXV.} \dots (c_2 - c_1)\alpha' = b\beta a\beta - a\alpha, \quad (c_2 - c_1)\beta' = a\alpha\beta a - b\beta,$$

$$\text{XXVI.} \dots - (c_2 - c_1)^2 = (b\beta a\beta - a\alpha)^2 = (a\alpha\beta a - b\beta)^2;$$

so that we have the following equation,

$$\text{XXVII.} \dots (a(Va\rho)^2 + b(S\beta\rho)^2)(a^2 + 2abS.(a\beta)^2 + b^2) = a(V(b\beta a\beta - a\alpha)\rho)^2 + b(S(a\alpha\beta a - b\beta)\rho)^2,$$

which is true for any vector ρ , any two unit lines a, β , and any two scalars a, b .

(9.) Accordingly it is evident from (4.), that a_1, a_3 must be the bisectors of the angles made by a, a' , and also of those made by β, β' ; and the expressions XXV. may be thus written (because $b - a = c_1 + c_2$),

$$\text{XXVIII.} \dots (c_2 - c_1)\alpha' = (c_2 + c_1)\alpha + 2b\beta Sa\beta, \quad (c_1 - c_2)\beta' = (c_1 + c_2)\beta - 2aa\beta a;$$

whence, by XXIII., we may write,

$$\text{XXIX.} \dots a + a' = 2xa_1, \quad a - a' = 2xa_3;$$

so that a_1 bisects the internal angle, and a_3 the external angle, of the lines a, a' .

(10.) At the same time we have these other expressions,

$$\text{XXX.} \dots (c_1 - c_2)(\beta + \beta') = 2(c_1\beta - aa\beta a), \quad (c_2 - c_1)(\beta - \beta') = 2(c_2\beta - aa\beta a);$$

which can easily be reduced to the simple forms,

$$\text{XXXI.} \dots \beta + \beta' = 2\alpha'a_1, \quad \beta - \beta' = 2\alpha'a_3,$$

with the recent meanings of the coefficients α' and α'' .

(11.) And although, for the sake of obtaining real transformations, we have supposed (comp. III.) that

$$\text{XXXII.} \dots (c_1^{-1} - c_2^{-1})(c_2^{-1} - c_3^{-1}) > 0,$$

because the assumed relation $\alpha = xa_1 + xa_2$ between the three unit vectors aa_1a_2 , whereof the two latter are rectangular, gives $x^2 + x'^2 = 1$, as in IX., so that each of the two expressions VI. involves the other, and their comparison gives the ratio,

$$\text{XXXIII.} \dots x^2 : x'^2 = (c_1^{-1} - c_2^{-1}) : (c_2^{-1} - c_3^{-1}),$$

(5.) Or we may make

XX. . . $\iota = \frac{1}{2}(c_1^{-1} + c_3^{-1})U\lambda$, $\kappa = \frac{1}{2}(c_1^{-1} - c_3^{-1})U\mu$, whence $\kappa^2 - \iota^2 = c_1^{-1}c_3^{-1}$; and shall then have the transformation,

$$\text{XXI. . . } S\rho\phi\rho = N \frac{\iota\rho + \rho\kappa}{\kappa^2 - \iota^2},$$

which may be compared with the equation 281, XXIX. of the *ellipsoid*, and for the *reality* of which form, or of its two *vector constants*, ι , κ , it is necessary that the roots c of the cubic should all be *positive* as above.

(6.) It was lately shewn (in 358, (8.), &c.), how to pass from a *given* and *real focal form* to a *second* of the same kind, with its *new real unit lines* α' , β' in the *same plane* as the two *old* or *given lines*, α , β ; but we have not yet shown how to pass from a *focal form* to a *cyclic* one, although the *converse* passage has been recently discussed. Let us then now suppose that the *form* VI. is *real* and *given*, or that the two scalar constants a , b , and the two unit vectors α , β , have real and given values; and let us seek to reduce this expression VI. to the earlier form IV.

(7.) We might, for this purpose, begin by assuming that

$$\text{XXII. . . } c_1^{-1} > c_2^{-1} > c_3^{-1}, \text{ as in 358, III. ;}$$

which would give the expressions 358, XXI. and XXII., for $c_1c_2c_3$ and $a_1a_2a_3$, and so would supply the *rectangular transformation*, from which we could pass, as before, to the *cyclic* one.

(8.) But to vary a little the analysis, let us now suppose that the *given focal form* is some one of the four following (comp. (1.)) :

$$\begin{aligned} \text{XXIII. . . } S\rho\phi\rho &= (S\beta_0\rho)^2 - (V\alpha_0\rho)^2; & \text{XXIII'. . . } S\rho\phi\rho &= (V\alpha_0\rho)^2 - (S\beta_0\rho)^2; \\ \text{XXIV. . . } S\rho\phi\rho &= (S\beta_0\rho)^2 + (V\alpha_0\rho)^2; & \text{XXIV'. . . } S\rho\phi\rho &= -(V\alpha_0\rho)^2 - (S\beta_0\rho)^2; \end{aligned}$$

in each of which α_0 and β_0 are conceived to be *given* and *real vectors*, but not generally *unit lines*; and which are in fact the *four cases* included under the *general form*, $a(V\alpha\rho)^2 + b(S\beta\rho)^2$, according as the scalars a and b are positive or negative. It will be sufficient to consider the two cases, XXIII. and XXIV., from which the two others will follow at once.

(9.) For the case XXIII. we easily derive the *real cyclic transformation*,

$$\begin{aligned} \text{XXV. . . } S\rho\phi\rho &= (S\beta_0\rho)^2 - (S\alpha_0\rho)^2 + \alpha_0^2\rho^2 \\ &= S(\beta_0 + \alpha_0)\rho \cdot S(\beta_0 - \alpha_0)\rho + \alpha_0^2\rho^2 \\ &= g\rho^2 + S\lambda\rho\mu\rho = (g - S\lambda\mu)\rho^2 + 2S\lambda\mu S\mu\rho, \end{aligned}$$

where XXVI. . . $\lambda = \beta_0 + \alpha_0$, $\mu = \frac{1}{2}(\beta_0 - \alpha_0)$, $g = \frac{1}{2}(\alpha_0^2 + \beta_0^2)$;

and the equations 357, (9.) enable us to pass thence to the two *imaginary cyclic forms*.

(10.) For example, if the proposed function be (comp. XIX.),

$$\text{XXVII. . . } S\rho\phi\rho = N(\iota_0\rho + \rho\kappa_0) = (S(\iota_0 + \kappa_0)\rho)^2 - (V(\iota_0 - \kappa_0)\rho)^2,$$

we may write

$$\alpha_0 = \iota_0 - \kappa_0, \quad \beta_0 = \iota_0 + \kappa_0, \quad \lambda = 2\iota_0, \quad \mu = \kappa_0, \quad g = \iota_0^2 + \kappa_0^2;$$

and the required transformation is (comp. 356, XI.),

$$\text{XXVIII. . . } N(\iota_0\rho + \rho\kappa_0) = (\iota_0^2 + \kappa_0^2)\rho^2 + 2S\iota_0\rho\kappa_0.$$

(11.) To treat the case XXIV. by our general method, we may omit for simplicity the subindices α , and write simply (comp. V. and VI.) the expressions,

XXIX. . . $\phi\rho = -\alpha V a\rho + \beta S\beta\rho$, and **XXX.** . . $S\rho\phi\rho = (V a\rho)^2 + (S\beta\rho)^2$;
 in which however it is to be observed that α and β , though *real vectors*, are *not now*
scat lines (8.). Hence because $-\alpha V a\rho = \alpha S a\rho - \alpha^2\rho$, we easily form the expres-
 sions :

XXXI. . . $m = \alpha^2 (S a\beta)^2$, $m' = \alpha^2 (\alpha^2 - \beta^2) - (S a\beta)^2$, $m'' = \beta^2 - 2\alpha^2$;

XXXII. . .
$$\begin{cases} \psi\rho = V a\beta S\beta a\rho - \alpha^2 (\alpha V a\rho + \beta V \beta\rho) + \alpha^4\rho \\ = V a\rho\beta S a\beta + \alpha (\alpha^2 - \beta^2) S a\rho, \\ \chi\rho = -(\alpha S a\rho + \beta S\beta\rho) + (\beta^2 - \alpha^2)\rho ; \end{cases}$$

and therefore **XXXIII.** . . $M = (c - \alpha^2) (c^2 + (\beta^2 - \alpha^2)c - (S a\beta)^2)$,
 and **XXXIV.** . . $\Psi\rho = V a\rho\beta S a\beta + (\beta^2 - \alpha^2) (c\rho - \alpha S a\rho) - c (\alpha S a\rho + \beta S\beta\rho) + c^2\rho$
 $= (\alpha (\alpha^2 - \beta^2 - c) + \beta S a\beta) S a\rho + (\alpha S a\beta - c\beta) S\beta\rho + (c^2 + (\beta^2 - \alpha^2)c - (S a\beta)^2)\rho$.

(12.) Introducing then a real and positive scalar constant, r , such that

XXXV. . . $r^4 = (\alpha^2 - \beta^2)^2 + 4 (S a\beta)^2 = (\alpha^2 + \beta^2)^2 + 4 (V a\beta)^2$
 $= \alpha^4 + (\alpha\beta)^2 + (\beta\alpha)^2 + \beta^4 = \alpha^4 + 2S. (\alpha\beta)^2 + \beta^4$
 $= \alpha^2 (\alpha^2 + \beta a\beta)^2 = \beta^2 (\beta^2 + \beta a\alpha)^2 = \&c.$

in which (by 199, &c.),

$S. (\alpha\beta)^2 = (S a\beta)^2 + (V a\beta)^2 = 2 (S a\beta)^2 - \alpha^2\beta^2 = 2 (V a\beta)^2 + \alpha^2\beta^2$,

the roots of $M = 0$ admit of being expressed as follows :

XXXVI. . . $e_1 = \frac{1}{2} (\alpha^2 - \beta^2 + r^2)$, $e_2 = \alpha^2$, $e_3 = \frac{1}{2} (\alpha^2 - \beta^2 - r^2)$;

and when they are thus arranged, we have the inequalities,

XXXVII. . . $e_1 > 0 > e_3 > e_2$, $e_1^{-1} > 0 > e_3^{-1} > e_2^{-1}$.

(13.) The corresponding forms of $\Psi\rho$ are the three monomial expressions,

XXXVIII. . .
$$\begin{cases} \psi_1\rho = e_3^{-1} (\alpha e_3 + \beta S a\beta) S (\alpha e_3 + \beta S a\beta) \rho, & \psi_2\rho = V a\beta S\beta a\rho, \\ \psi_3\rho = e_1^{-1} (\alpha e_1 + \beta S a\beta) S (\alpha e_1 + \beta S a\beta) \rho ; \end{cases}$$

which may be variously transformed and verified, and give the three following rect-
 angular vector units,

XXXIX. . . $a_1 = U (\alpha e_3 + \beta S a\beta)$, $a_2 = U V a\beta$, $a_3 = U (\alpha e_1 + \beta S a\beta)$;

in connexion with which it is easy to prove that

XL. . .
$$\begin{cases} T (\alpha e_3 + \beta S a\beta) = (-e_3)^{\frac{1}{2}} (e_1 - e_3)^{\frac{1}{2}} (e_1 - e_3)^{\frac{1}{2}} = r (e_1 - e_3)^{\frac{1}{2}} (-e_3)^{\frac{1}{2}}, \\ T V a\beta = (e_1 - e_3)^{\frac{1}{2}} (e_3 - e_2)^{\frac{1}{2}} ; \\ T (\alpha e_1 + \beta S a\beta) = e_1^{\frac{1}{2}} (e_3 - e_2)^{\frac{1}{2}} (e_1 - e_3)^{\frac{1}{2}} = r (e_3 - e_2)^{\frac{1}{2}} e_1^{\frac{1}{2}} ; \end{cases}$$

the radicals being all real, by **XXXVII.**

(14.) We have thus, for the *given focal form* **XXX.**, the *rectangular transfor-
 mation*,

XLi. . . $S\rho\phi\rho = (V a\rho)^2 + (S\beta\rho)^2$
 $= \frac{e_1 (S (\alpha e_3 + \beta S a\beta) \rho)^2}{-e_3 (e_1 - e_2) r^2} + \frac{e_2 (S a\beta \rho)^2}{(e_1 - e_2) (e_3 - e_2)} + \frac{e_3 (S (\alpha e_1 + \beta S a\beta) \rho)^2}{e_1 (e_3 - e_2) r^2}$,

or briefly,

XLii. . . $S\rho\phi\rho = (V a\rho)^2 + (S\beta\rho)^2 = e_1 (S. \rho U (\alpha e_3 + \beta S a\beta) \rho)^2$
 $+ \alpha^2 (S. \rho U V a\beta)^2 + e_3 (S. \rho U (\alpha e_1 + \beta S a\beta) \rho)^2$;

in which the first term is positive, but the two others are negative, and e_1, e_3 are
 the roots of the quadratic,

XLiii. . . $0 = e^2 + (\beta^2 - \alpha^2)e - (S a\beta)^2$.

(15.) We have also the parallelisms,

$$\text{XLIV.} \dots a\alpha_2 + \beta Sa\beta \parallel \beta c_1 - aSa\beta, \quad a\alpha_1 + \beta Sa\beta \parallel \beta c_2 - aSa\beta,$$

because

$$c_1 c_2 = - (Sa\beta)^2;$$

and may therefore write,

$$\text{XLV.} \dots S\rho\phi\rho = (V a\rho)^2 + (S\beta\rho)^2 = c_1 (S.\rho U(\beta c_1 - aSa\beta))^2 \\ + a^2 (S.\rho UV a\beta)^2 + c_2 (S.\rho U(\beta c_2 - aSa\beta))^2;$$

while

$$\text{XLVI.} \dots T(\beta c_1 - aSa\beta) = r c_1^{\frac{1}{2}} (c_1 - c_2)^{\frac{1}{2}}, \quad T(\beta c_2 - aSa\beta) = r (-c_2)^{\frac{1}{2}} (c_2 - c_1)^{\frac{1}{2}},$$

and $r = (c_1 - c_2)^{\frac{1}{2}}$, with real radicals as before.

(16.) Multiplying then by $r^2 (TV a\beta)^2$, or by $(c_1 - c_2) (c_1 - c_2) (c_2 - c_1)$, we obtain this new equation,

$$\text{XLVII.} \dots (c_1 - c_2) \{ (TV a\beta)^2 \{ (V a\rho)^2 + (S\beta\rho)^2 \} - a^2 (Sa\beta\rho)^2 \} \\ = (c_2 - a^2) (c_1 S\beta\rho - Sa\beta S a\rho)^2 - (c_1 - a^2) (c_2 S\beta\rho - aSa\beta)^2;$$

which is only another way of expressing the same rectangular transformation as before, but has the advantage of being freed from *divisors*.

(17.) Developing the second member of XLVII., and dividing by $c_1 - c_2$, we obtain this new transformation:

$$\text{XLVIII.} \dots (TV a\beta)^2 S\rho\phi\rho = - (V a\beta)^2 \{ (V a\rho)^2 + (S\beta\rho)^2 \} \\ = a^2 (Sa\beta\rho)^2 - (Sa\beta)^2 (S a\rho)^2 + 2a^2 Sa\beta S a\rho S\beta\rho + C (S\beta\rho)^2;$$

in which we have written for abridgment,

$$\text{XLIX.} \dots C = c_1 c_2 - a^2 (c_1 + c_2).$$

(18.) The expressions XXXVI. for c_1, c_2 give thus,

$$\text{L.} \dots C = -a^4 - (V a\beta)^2;$$

and accordingly, when this value is substituted for C in XLVIII., that equation becomes an *identity*, or holds good for *all* values of the three vectors, α, β, ρ ; as may be proved* in various ways.

(19.) Admitting this result, we see that for the mere establishment of the equation XLVII., it is *not necessary* that c_1 and c_2 should be roots of the *particular quadratic* XLIII. It is sufficient, for *this* purpose, that they should be roots of *any* quadratic,

LI. $\dots c^2 + Ac + B = 0$, with the relation LII. $\dots Aa^2 + B + a^4 + (V a\beta)^2 = 0$, between its coefficients. But when we combine with this the *condition of rectangularity*, $a_2 \perp a_1$, or

$$\text{LIII.} \dots 0 = S.(c_1\beta - aSa\beta)(c_2\beta - aSa\beta) = A(Sa\beta)^2 + B\beta^2 + a^4(Sa\beta)^2,$$

we obtain thus a *second* relation, which gives *definitely*, for the two coefficients, the values,

$$\text{LIV.} \dots A = \beta^2 - a^2, \quad B = - (Sa\beta)^2;$$

and so conducts, in a new way, to the equation XLIII.

* Many such proofs, or verifications, as the one here alluded to, are purposely left, at this stage, as exercises, to the student.

(20.) In this manner, then, we *might* have been led to perceive the truth of the rectangular transformation XLVII., with the *quadratic* equation XLIII. of which c_1 and c_2 are roots, without having previously found the *cubic* XXXIII., of which the quadratic is a *factor*, and of which the *other* root is $c_2 = a^2$. But if we had not employed the *general method* of the present Section, which conducted us to form *first* that *cubic* equation, there would have been nothing to *suggest* the *particular form* XLVII., which could thus have only been by some sort of *chance* arrived at.

(21.) The values of $a_1 a_2 a_3$ give also (comp. 857, VII.),

LV. . . $-\rho^3 = (S. \rho U(\beta c_1 - a S a \beta))^2 + (S. \rho UV a \beta)^2 + (S. \rho U(\beta c_2 - a S a \beta))^2$;
that is, by XL. and XLVI.,

$$\text{LVI. . . } c_1 c_2 (e_1 - c_2) (\rho^2 (\nabla a \beta)^2 - (S a \beta \rho)^2) - c_2 (c_2 - a^2) (c_1 S \beta \rho - S a \beta S a \rho)^2 \\ - c_1 (c_1 - a^2) (c_2 S \beta \rho - S a \beta S a \rho)^2 ;$$

and accordingly the values XXXVI. of c_1, c_2 enable us to express each member of this last equation under the common form, $-c_1 c_2 (e_1 - c_2) (a S \beta \rho - \beta S a \rho)^2$.

(22.) Comparing the recent inequalities $c_1 > c_2 > c_3$ (XXXVII.) with the arrangement 857, IX., we see, by 857, (6.), that for the *real cyclic transformation* (6.) at present sought, the plane of λ, μ is to be perpendicular to a_3 (and not to a_2 , as in 857, (3.), &c.). We are therefore to eliminate $(e_2 S \beta \rho - S a \beta S a \rho)^2$ between the equations XLVII. and LVI., which gives (after a few reductions) the real transformation :

$$\text{LVII. . . } ((S a \beta)^2 - c_1 \beta^2) ((\nabla a \rho)^2 + (S \beta \rho)^2) - (c_1 - a^2) (S a \beta)^2 \rho^2 \\ = (c_1 S \beta \rho - S a \beta S a \rho)^2 - e_1 (S a \beta \rho)^2 \\ = S. \rho (e_1 \beta - a S a \beta + e_1 \nabla a \beta) S. \rho (e_1 \beta - a S a \beta - e_1 \nabla a \beta) ;$$

which is of the kind required.

(23.) Accordingly it will be found that the following equation,

$$\text{LVIII. . . } ((S a \beta)^2 - c \beta^2) (\nabla a \rho)^2 + (c - a^2) (c (S \beta \rho)^2 - \rho^2 S (a \beta)^2) \\ = (c S \beta \rho - S a \beta S a \rho)^2 - c (S a \beta \rho)^2,$$

is an *identity*, or that it holds good for *all values* of the scalar c , and of the vectors a, β, ρ ; since, by addition of $c(\nabla a \beta)^2 \rho^2$ on both sides, it takes this *obviously* identical form,

$$\text{LIX. . . } ((S a \beta)^2 - c \beta^2) (S a \rho)^2 + c(c - a^2) (S \beta \rho)^2 = (c S \beta \rho - S a \beta S a \rho)^2 \\ - c(a S \beta \rho - \beta S a \rho)^2 ;$$

so that if c_1 be *either* root of the quadratic XLIII., or if $c_1(c_1 - a^2) = (S a \beta)^2 - c_1 \beta^2$, the transformation LVII. is at least *symbolically valid* : but we must take, as above, the *positive* root of that quadratic for c_1 , if we wish that transformation to be a *real* one, as regards the *constants* which it employs. And if we had *happened* (comp. (20.)) to perceive this *identity* LIX., and to see its transformation LVIII., we might have been in that way led to form the *quadratic* XLIII., without having previously formed the *cubic* XXXIII.

(24.) Already, then, we see how to obtain *one* of the two *imaginary cyclic transformations* of the given focal form XXX., namely by changing c_1 to c_2 in LVII. ; and the *other* imaginary transformation is had, on principles before explained, by eliminating $(S a \beta \rho)^2$ between XLVII. and LVI. ; a process which easily conducts to the equation,

$$\text{LX.} \dots (\nabla\alpha\rho)^2 + (\mathcal{S}\beta\rho)^2 + \alpha^2\rho^2 = (c_1 - c_2)^{-1} \{ c_1^{-1}(c_2\mathcal{S}\beta\rho - \mathcal{S}\alpha\beta\mathcal{S}\alpha\rho)^2 - c_2^{-1}(c_1\mathcal{S}\beta\rho - \mathcal{S}\alpha\beta\mathcal{S}\alpha\rho)^2 \},$$

where the second member is the *sum of two squares* (c_1 being > 0 , but $c_2 < 0$), as the second expression LVII. would also become, if c_1 were replaced by c_2 . Accordingly, each member of LX. is equal to $(\mathcal{S}\alpha\rho)^2 + (\mathcal{S}\beta\rho)^2$, if c_1, c_2 be the roots of *any* quadratic LI., with only the *one* condition,

$$\text{LXI.} \dots c_1c_2 = B = -(\mathcal{S}\alpha\beta)^2;$$

which however, when *combined* with the *condition of rectangularity* LIII., suffices to give also $A = \beta^2 - \alpha^2$, as in LIV., and so to lead us back to the quadratic XLIII., which had been deduced by the *general method*, as a *factor of the cubic equation* XXXIII.

(25.) And since the values XXXVI. of c_1, c_2 reduce, as above, the second member of LX. to the simple form $(\mathcal{S}\alpha\rho)^2 + (\mathcal{S}\beta\rho)^2$, we may thus, or even without employing the *roots* c_1, c_2 at all, deduce the following expression for the last imaginary cyclic transformation :

$$\text{LXII.} \dots \mathcal{S}\rho\phi\rho = (\nabla\alpha\rho)^2 + (\mathcal{S}\beta\rho)^2 = -\alpha^2\rho^2 + \mathcal{S}(\alpha + \sqrt{-1}\beta)\rho \cdot \mathcal{S}(\alpha - \sqrt{-1}\beta)\rho,$$

where $\sqrt{-1}$ is the imaginary of algebra (comp. 214, (6.)); while the *real scalar* r^4 of XXXV. may at the same time receive the connected *imaginary form*,

$$\text{LXIII.} \dots r^4 = (\alpha^2 - \beta^2)^2 + 4(\mathcal{S}\alpha\beta)^2 = (\alpha + \sqrt{-1}\beta)^2 (\alpha - \sqrt{-1}\beta)^2.$$

(26.) Finally, as regards the passage from the *given form* XXX., to a *second real focal form* (comp. 358, (4.)), or the transformation,

$$\text{LXIV.} \dots (\nabla\alpha\rho)^2 + (\mathcal{S}\beta\rho)^2 = (\nabla\alpha'\rho)^2 + (\mathcal{S}\beta'\rho)^2,$$

in which α' and β' are real vectors, distinct from $\pm\alpha$ and $\pm\beta$, but in the same plane with them, it may be sufficient (comp. 358, (8.)), to write down the formulæ :

$$\text{LXV.} \dots r^2\alpha' = -(\alpha^2 + \beta\alpha\beta), \quad r^2\beta' = -(\beta^2 + \alpha\beta\alpha),$$

with the same real value of r^2 as before; so that (by XXXV., &c.) we have the relations,

$$\text{LXVI.} \dots T\alpha' = T\alpha, \quad T\beta' = T\beta, \quad \mathcal{S}\alpha'\beta' = \mathcal{S}\alpha\beta;$$

$$\text{LXVII.} \dots \begin{cases} r^2(\alpha + \alpha') = \alpha(r^2 - \alpha^2 + \beta^2) - 2\beta\mathcal{S}\alpha\beta = -2(\alpha c_2 + \beta\mathcal{S}\alpha\beta) \parallel \alpha_1, \\ r^2(\alpha - \alpha') = \alpha(r^2 + \alpha^2 - \beta^2) + 2\beta\mathcal{S}\alpha\beta = 2(\alpha c_1 + \beta\mathcal{S}\alpha\beta) \parallel \alpha_2; \end{cases}$$

$$\text{LXVIII.} \dots \begin{cases} r^2(\beta + \beta') = \beta(r^2 + \alpha^2 - \beta^2) - 2\alpha\mathcal{S}\alpha\beta = 2(\beta c_1 - \alpha\mathcal{S}\alpha\beta) \parallel \alpha_1, \\ r^2(\beta - \beta') = \beta(r^2 - \alpha^2 + \beta^2) + 2\alpha\mathcal{S}\alpha\beta = -2(\beta c_2 - \alpha\mathcal{S}\alpha\beta) \parallel \alpha_2. \end{cases}$$

(27.) We have then the identity,

$$\text{LXIX.} \dots (\nabla(\alpha^2 + \beta\alpha\beta)\rho)^2 + (\mathcal{S}(\beta^3 + \alpha\beta\alpha)\rho)^2 = (\alpha^4 + 2\mathcal{S}(\alpha\beta)^2 + \beta^4) ((\nabla\alpha\rho)^2 + (\mathcal{S}\beta\rho)^2);$$

with which may be combined this other of the same kind,

$$\text{LXX.} \dots -(\nabla(\alpha^2 - \beta\alpha\beta)\rho)^2 + (\mathcal{S}(\beta^3 - \alpha\beta\alpha)\rho)^2 = (\alpha^4 - 2\mathcal{S}(\alpha\beta)^2 + \beta^4) (-(\nabla\alpha\rho)^2 + (\mathcal{S}\beta\rho)^2),$$

which enables us to pass from the focal form XXIII., to a *second real focal form*, with its two new lines in the same plane as the two old ones : and it may be noted that we can pass from LXIX. to LXX., by changing α to $\alpha\sqrt{-1}$.

360. Besides the rectangular, cyclic, and focal transformations of $S\rho\phi\rho$, which have been already considered, there are others, although perhaps of less importance: but we shall here mention only two of them, as specimens, whereof one may be called the *Bifocal*, and the other the *Mixed Transformation*.

(1.) The two lines a, a' , of 859, LXV., being called *focal lines*,* an expression which shall introduce them *both* may be called on that account a *bifocal transformation*.

(2.) Retaining then the value 859, XXXV. of r^4 , and introducing a new auxiliary constant e , which shall satisfy the equation,

$$\text{I. . . } \beta^2 - a^2 = r^2 e, \text{ and therefore II. . . } 4(Sa\beta)^2 = r^4(1 - e^2),$$

so that

$$\text{III. . . } 4e^2(Sa\beta)^2 = (1 - e^2)(\beta^2 - a^2)^2,$$

the first equation 859, LXV. gives,

$$\text{IV. . . } r^2(ea - a') = 2\beta Sa\beta, \quad \text{V. . . } r^2(eSap - Sa'\rho) = 2Sa\beta S\beta\rho;$$

and therefore, with the form 859, XXX. of $S\rho\phi\rho$,

$$\begin{aligned} \text{VI. . . } (1 - e^2)S\rho\phi\rho &= (1 - e^2)((\nabla a\rho)^2 + (S\beta\rho)^2) \\ &= (1 - e^2)(\nabla a\rho)^2 + (eSap - Sa'\rho)^2 \\ &= (e^2 - 1)a^2\rho^2 + (Sap)^2 - 2eSapSa'\rho + (Sa'\rho)^2; \end{aligned}$$

in which $\rho^2 = a'^2$, by 859, LXVI., so that a and a' may be considered to enter *symmetrically* into this last transformation, which is of the *bifocal* kind above mentioned.

(3.) For the same reason, the expression last found for $S\rho\phi\rho$ involves again (comp. 858) *six* scalar constants; namely, $e, Ta(=Ta')$, and the four involved in the two unit lines, Ua, Ua' .

(4.) In all the foregoing transformations, the scalar and quadratic function $S\rho\phi\rho$ has been *evidently homogeneous*, or has been seen to involve no terms below the *second degree* in ρ . We may however also employ this *apparently heterogeneous* or *mixed form*,

$$\text{VII. . . } S\rho\phi\rho = g'(\rho - \epsilon)^2 + 2S\lambda(\rho - \zeta)S\mu(\rho - \zeta) + e;$$

in which g', λ, μ have the same significations as in 857, but e, ϵ, ζ are *three new constants*, subject to the two conditions of homogeneity,

$$\text{VIII. . . } g'\epsilon + \lambda S\mu\zeta + \mu S\lambda\zeta = 0,$$

and

$$\text{IX. . . } g'\epsilon^2 + 2S\lambda\zeta S\mu\zeta + e = 0,$$

in order that the expression VII. may admit of reduction to the form,

$$\text{X. . . } S\rho\phi\rho = g'\rho^2 + 2S\lambda\rho S\mu\rho, \text{ as in 857, II'.$$

(5.) Other general *homogeneous* transformations of $S\rho\phi\rho$, which are themselves *real*, although connected with *imaginary† cyclic forms* (comp. 857, (7.)), because

* Compare the Note to Art. 859.

† $\lambda_1 \pm \sqrt{-1} \mu_1$, and $\lambda_2 \pm \sqrt{-1} \mu_2$, may here be said to be two pairs of *imaginary cyclic normals*, of that *real surface* of the second order, of which the equation is, as before, $S\rho\phi\rho = \text{const.}$ Compare the Notes to pages 468, 474.

a sum of two squares of linear and scalar functions is, in an imaginary sense, a product of two such functions, are the two following (comp. 357, (9.)) :

$$\text{XI.} \dots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho = g_1\rho^2 + (S\lambda_1\rho)^2 + (S\mu_1\rho)^2;$$

$$\text{XII.} \dots S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho = g_2\rho^2 - (S\lambda_2\rho)^2 - (S\mu_2\rho)^2;$$

in which (comp. 357, (2.) and (8.)),

$$\text{XIII.} \dots g_1 = g + T\lambda\mu = -e_1, \quad g_2 = g - T\lambda\mu = -e_2,$$

$$\text{XIV.} \dots \lambda_1 = \sqrt{\lambda\mu} (T\lambda\mu - S\lambda\mu)^{-\frac{1}{2}}, \quad \mu_1 = (\lambda T\mu + \mu T\lambda) (T\lambda\mu - S\lambda\mu)^{-\frac{1}{2}},$$

$$\text{and XV.} \dots \lambda_2 = \sqrt{\lambda\mu} (T\lambda\mu + S\lambda\mu)^{-\frac{1}{2}}, \quad \mu_2 = (\lambda T\mu - \mu T\lambda) (T\lambda\mu + S\lambda\mu)^{-\frac{1}{2}};$$

so that g_1, λ_1, μ_1 , and g_2, λ_2, μ_2 are real, if g, λ, μ be such.

(6.) We have therefore the two new mixed transformations following :

$$\text{XVI.} \dots S\rho\phi\rho = g_1(\rho - e_1)^2 + (S\lambda_1(\rho - \zeta_1))^2 + (S\mu_1(\rho - \zeta_1))^2 + e_1;$$

$$\text{XVII.} \dots S\rho\phi\rho = g_2(\rho - e_2)^2 - (S\lambda_2(\rho - \zeta_2))^2 - (S\mu_2(\rho - \zeta_2))^2 + e_2;$$

with these two new pairs of equations, as conditions of homogeneity,

$$\text{XVIII.} \dots g_1 e_1 + \lambda_1 S\zeta_1 \lambda_1 + \mu_1 S\zeta_1 \mu_1 = 0,$$

$$\text{XIX.} \dots g_1 e_1^2 + (S\zeta_1 \lambda_1)^2 + (S\zeta_1 \mu_1)^2 + e_1 = 0,$$

and

$$\text{XX.} \dots g_2 e_2 - \lambda_2 S\zeta_2 \lambda_2 - \mu_2 S\zeta_2 \mu_2 = 0,$$

$$\text{XXI.} \dots g_2 e_2^2 - (S\zeta_2 \lambda_2)^2 - (S\zeta_2 \mu_2)^2 + e_2 = 0.$$

361. We saw, in the sub-articles to 336, that the differential, $d\phi\rho$, of a scalar function of a vector, may in general be expressed under the form,

$$\text{I.} \dots d\phi\rho = nS\nu d\rho,$$

where ν is a derived vector function, of the same variable vector ρ , and n is a scalar coefficient. And we now propose to show, that if

$$\text{II.} \dots \phi\rho = S\rho\phi\rho,$$

$\phi\rho$ still denoting the linear and vector function which has been considered in the present Section, and of which $\phi_0\rho$ is still the self-conjugate part, we shall have the equation I. with the values,

$$\text{III.} \dots n = 2, \quad \nu = \phi_0\rho;$$

so that the part $\phi_0\rho$ may thus be deduced from $\phi\rho$ by operating with $\frac{1}{2}dS.\rho$, and seeking the coefficient of $d\rho$ under the sign S . in the result: while there exist certain general relations of reciprocity (comp. 336, (6.)), between the two vectors ρ and ν , which are in this way connected, as linear functions of each other.

(1.) We have here, by the supposed linear form of $\phi\rho$, the differential equation (comp. 334, VI.),

$$\text{IV.} \dots d\phi\rho = \phi d\rho;$$

also $S(d\rho \cdot \phi\rho) = S(\phi\rho \cdot d\rho)$, and $S(\rho \cdot \phi d\rho) = S(\phi' \rho \cdot d\rho)$;
hence, by 349, XIII., we have, as asserted,

$$V. \dots dS\rho\phi\rho = S(\phi\rho + \phi'\rho)d\rho = 2S \cdot \phi\rho d\rho.$$

(2.) As an example of the employment of this formula, in the deduction of $\phi\rho$ from $\phi\rho$, let us take the expression,

$$VI. \dots \phi\rho = \Sigma\beta S a\rho, \quad 347, \text{XXXI.},$$

which gives,

$$VII. \dots f\rho = S\rho\phi\rho = \Sigma S a\rho S\beta\rho,$$

and therefore

$$VIII. \dots d f\rho = \Sigma(\beta S a\rho + a S\beta\rho) d\rho.$$

Comparing this with the general formula,

$$IX. \dots \frac{1}{2} d f\rho = S\nu d\rho = S \cdot \phi\rho d\rho,$$

we find that the form VI. of $\phi\rho$ has for its self-conjugate part,

$$X. \dots \nu = \phi\rho = \frac{1}{2} \Sigma(\beta S a\rho + a S\beta\rho);$$

and in fact we saw (347, XXXII.) that this form gives, as its conjugate, the expression,

$$XI. \dots \phi'\rho = \Sigma a S\beta\rho.$$

(3.) Supposing now, for simplicity, that the function ϕ is given, or made, self-conjugate, by taking (if necessary) the semisum of itself and its own conjugate function, we may write ϕ instead of ϕ_0 , and shall thus have, simply,

$$XII. \dots \nu = \phi\rho, \quad XIII. \dots f\rho = S\nu\rho, \quad XIV. \dots d f\rho = 2S\nu d\rho;$$

whence also (comp. 348, I. II.),

$$XV. \dots \rho = \phi^{-1}\nu = m^{-1}\psi\nu, \quad \text{and} \quad XVI. \dots S\nu d\rho = S\rho d\nu.$$

(4.) Writing, then,

$$XVII. \dots F\nu = S\nu\phi^{-1}\nu = m^{-1}S\nu\psi\nu,$$

we shall have the equations,

$$XVIII. \dots F\nu = f\rho, \quad XIX. \dots dF\nu = 2S\rho d\nu = 2S \cdot \phi^{-1}\nu d\nu;$$

so that ρ may be deduced from $F\nu$, as ν was deduced from $f\rho$; and generally, as above stated, there exists a perfect reciprocity of relations, between the vectors ρ and ν , and also between their scalar functions, $f\rho$ and $F\nu$.

(5.) As regards the deduction, or derivation, of ν from $f\rho$, and of ρ from $F\nu$, it may occasionally be convenient to denote it thus:

$$XX. \dots \nu = \frac{1}{2}(S \cdot d\rho)^{-1} d f\rho; \quad XXI. \dots \rho = \frac{1}{2}(S \cdot d\nu)^{-1} d F\nu;$$

in fact, these last may be considered as only *symbolical transformations* of the expressions,

$$XXII. \dots d f\rho = 2S(d\rho \cdot \nu), \quad d F\nu = 2S(d\nu \cdot \rho),$$

which follow immediately from XIV. and XIX.

(6.) As an example of the passage from an expression such as $f\rho$, to an equal expression of the reciprocal form $F\nu$, let us resume the cyclic form 357, II., writing thus,

$$XXIII. \dots f\rho = S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho,$$

and supposing that g , λ , and μ are real. Here, by what has been already shown (in sub-articles to 354 and 357), if $\phi\rho$ be supposed self-conjugate, as in (3.), we have,

$$\text{XXIV.} \dots \nu = \phi\rho = g\rho + \nabla\lambda\rho\mu;$$

$$\text{XXV.} \dots m = (g - S\lambda\mu) (g^2 - \lambda^2\mu^2) = -c_1c_2c_3;$$

$$\text{XXVI.} \dots \psi\nu = \nabla\lambda\nu\mu S\lambda\mu - \nabla\lambda\mu S\lambda\nu\mu - g(\lambda S\mu\nu + \mu S\lambda\nu) + g^2\nu;$$

and therefore

$$\text{XXVII.} \dots mF\nu = S\nu\psi\nu$$

$$\begin{aligned} &= S\lambda\nu\mu\nu S\lambda\mu + (S\lambda\nu\mu)^2 - 2gS\lambda\nu S\mu\nu + g^2\nu^2 \\ &= (g^2 - \lambda^2\mu^2)\nu^2 + \lambda^2(S\mu\nu)^2 + \mu^2(S\lambda\nu)^2 - 2gS\lambda\nu S\mu\nu; \end{aligned}$$

which last, when compared with 360, VI., is seen to be what we have called a *bifocal form*: its *focal lines* α, α' (360, (1.)) having here the directions of λ, μ , that is of what may be called the *cyclic lines** of the form XXIII. The *cyclic* and *bifocal transformations* are therefore *reciprocals* of each other.

(7.) As another example of this reciprocal relation between cyclic and focal lines, in the passage from $f\rho$ to $F\nu$, or conversely from the latter to the former, let us now begin with the *focal form*,

$$\text{XXVIII.} \dots f\rho = S\rho\phi\rho = (\nabla\alpha\rho)^2 + (S\beta\rho)^2, \quad 359, \text{XXX.}$$

in which α and β are supposed to be given and real vectors. We have now, by 359, (11.),

$$\text{XXIX.} \dots \begin{cases} \nu = \phi\rho = -\alpha\nabla\alpha\rho + \beta S\beta\rho, & m = \alpha^2(S\alpha\beta)^2, \\ \psi\nu = \nabla\alpha\nu\beta S\alpha\beta + \alpha(\alpha^2 - \beta^2)S\alpha\nu, \end{cases}$$

and therefore,

$$\begin{aligned} \text{XXX.} \dots mF\nu &= \alpha^2(S\alpha\beta)^2 F\nu = S\nu\psi\nu \\ &= S\alpha\nu\beta\nu S\alpha\beta + (\alpha^2 - \beta^2)(S\alpha\nu)^2 \\ &= -\nu^2(S\alpha\beta)^2 + S\alpha\nu((\alpha^2 - \beta^2)S\alpha\nu + 2S\alpha\beta S\beta\nu) \\ &= -\nu^2(S\alpha\beta)^2 + S\alpha\nu S(\alpha^2 + \beta\alpha\beta)\nu, \end{aligned}$$

an expression which is of *cyclic form*; one cyclic line of $F\nu$ being the *given* focal line α of $f\rho$; and the *other* cyclic line of $F\nu$ having the direction of $\pm(\alpha^2 + \beta\alpha\beta)$, and consequently (by 359, LXV.) of $\mp\alpha'$, where α' is the *second* real and focal line of $f\rho$.

(8.) And to verify the equation XVIII., or to show by an example that the *two functions* $f\rho$ and $F\nu$ are *equal in value*, although they are (generally) *different in form*, it is sufficient to substitute in XXX. the value XXIX. of ν ; which, after a few reductions, will exhibit the asserted equality.

362. It is often convenient to introduce a certain *scalar and symmetric function* of *two independent vectors*, ρ and ρ' , which is *linear* with respect to *each* of them, and is deduced from the *linear and self-conjugate vector function* $\phi\rho$, of a *single vector* ρ , as follows:

$$\text{I.} \dots f(\rho, \rho') = f(\rho', \rho) = S\rho'\phi\rho = S\rho\phi\rho'.$$

With this notation, we have

* They are in fact (compare the Note to page 468) the *cyclic normals*, or the normals to the *cyclic planes*, of that *surface of the second order*, which has for its equation $f\rho = \text{const.}$; while they are, as above, the *focal lines* of that *other or reciprocal surface*, of which ν is the variable vector, and the equation is $F\nu = \text{const.}$

$$\begin{aligned} \text{II.} \dots f(\rho + \rho') &= f\rho + 2f(\rho, \rho') + f\rho'; \\ \text{III.} \dots f(\rho, \rho' + \rho'') &= f(\rho, \rho') + f(\rho, \rho''); \\ \text{IV.} \dots f(\rho, \rho) &= f\rho; \quad \text{V.} \dots df\rho = 2f(\rho, d\rho); \\ \text{VI.} \dots f(x\rho, y\rho) &= xyf(\rho, \rho'), \quad \text{if } \nabla x = \nabla y = 0; \end{aligned}$$

and as a verification,

$$\text{VII.} \dots f(x\rho) = x^2 f\rho,$$

a result which might have been obtained, without introducing this new function I.

(1.) It appears to be unnecessary, at this stage, to write down proofs of the foregoing consequences, II. to VI., of the definition I.; but it may be worth remarking, that we *here* depart a little, in the formula V., from a notation (325) which was used in some early Articles of the present Chapter, although avowedly only as a *temporary* one, and adopted merely for convenience of exposition of the *principles* of Quaternion Differentials.

(2.) In *that* provisional notation (comp. 325, IX.) we should have had, for the differentiation of the recent function $f\rho$ (361, II.), the formulae,

$$df\rho = f(\rho, d\rho), \quad f(\rho, \rho') = 2S\rho'\phi\rho;$$

the numerical coefficient being thus *transferred* from one of them to the other, as compared with the recent equations, I. and V. But there is a convenience *now* in adopting these last equations V. and I., namely,

$$df\rho = 2f(\rho, d\rho), \quad f(\rho, \rho') = S\rho'\phi\rho;$$

because this *function* $S\rho'\phi\rho$, or $S\rho\phi\rho'$, occurs frequently in the applications of quaternions to surfaces of the second order, and not always with the *coefficient* 2.

(3.) Retaining then the recent notations, and treating $d\rho$ as constant, or $d^2\rho$ as null, successive differentiation of $f\rho$ gives, by IV. and V., the formulae,

$$\text{VIII.} \dots d^2 f\rho = 2f(d\rho); \quad d^3 f\rho = 0; \quad \&c.;$$

so that the theorem 342, I. is here verified, under the form,

$$\begin{aligned} \text{IX.} \dots \varepsilon^d f\rho &= (1 + d + \frac{1}{2}d^2) f\rho \\ &= f\rho + 2f(\rho, d\rho) + f d\rho; \end{aligned}$$

or briefly,

$$\text{X.} \dots \varepsilon^d f\rho = f(\rho + d\rho),$$

an equation which by II. is *rigorously exact* (comp. 339, (4.)), without any supposition whatever being made, respecting any *smallness* of the *tensor*, $Td\rho$.

363. *Linear and vector functions* of vectors, such as those considered in the present Section, although *not generally* satisfying the condition of *self-conjugation*, present themselves generally in the *differentiation* of *non-linear* but *vector functions* of vectors. In fact, if we denote for the moment *such* a non-linear function by $\omega(\rho)$, or simply by $\omega\rho$, the general *distributive property* (326) of differential expressions allows us to write,

$$\text{I.} \dots d\omega(\rho) = \phi(d\rho), \quad \text{or briefly,} \quad \text{I'.} \dots d\omega\rho = \phi d\rho;$$

where ϕ has all the properties hitherto employed, including that of *not* being generally self-conjugate, as has been just observed. There is, however, as we shall soon see, an extensive and important case, in which the property of self-conjugation *exists*, for such a function ϕ ; namely when the *differentiated function*, $\omega\rho$, is *itself* the result ν of the *differentiation* of a *scalar function* $f\rho$ of the variable vector ρ , although *not necessarily* a function of the *second dimension*, such as has been recently considered (361); or more fully, when it is the coefficient of $d\rho$, under the sign S ., in the differential (361, I.) of that scalar function $f\rho$, whether it be multiplied or not by any *scalar constant* (such as n , in the formula last referred to). And generally (comp. 346), the *inversion* of the linear and vector function ϕ in I. corresponds to the *differentiation* of the *inverse* (or *implicit*) function ω^{-1} ; in such a manner that the equation I. or I'. may be written under this other form,

$$\text{II.} \dots d\omega^{-1}\sigma = \phi^{-1}d\sigma = m^{-1}\psi d\sigma, \quad \text{if } \sigma = \omega\rho.$$

(1.) As a very simple example of a non-linear but vector function, let us take the form,

$$\text{III.} \dots \sigma = \omega(\rho) = \rho a \rho, \quad \text{where } a \text{ is a constant vector.}$$

This gives, if $d\rho = \rho'$,

$$\text{IV.} \dots \phi\rho' = \phi d\rho = d\omega\rho = \rho'a\rho + \rho a \rho' = 2V\rho a \rho';$$

$$\text{V.} \dots S\lambda\phi\rho' = 2S\lambda\rho a \rho' = S\rho'\phi\lambda;$$

$$\text{VI.} \dots \phi'\lambda = 2V\lambda\rho a = 2V a \rho\lambda, \quad \phi'\rho' = 2V a \rho\rho';$$

so that $\phi\rho'$ and $\phi'\rho'$ are unequal, and the linear function $\phi\rho'$ is *not* self-conjugate.

(2.) To find its self-conjugate part $\phi\phi\rho'$, by the method of Art. 361, we are to form the scalar expression,

$$\text{VII.} \dots \frac{1}{2}f\rho' = \frac{1}{2}S\rho'\phi\rho' = \rho'^2 S a \rho;$$

of which the differential, taken with respect to ρ' , is

$$\text{VIII.} \dots \frac{1}{2}df\rho' = S.\phi\phi\rho'd\rho' = 2S a \rho S\rho'd\rho', \quad \text{giving IX.} \dots \phi\phi\rho' = 2\rho'S a \rho;$$

and accordingly this is equal to the semisum of the two expressions, IV. and VI., for $\phi\rho'$ and its conjugate.

(3.) On the other hand, as an example of the *self-conjugation* of the linear and vector function,

$$\text{X.} \dots d\nu = d\omega\rho = \phi d\rho, \quad \text{when X'.} \dots df\rho = 2S\nu d\rho = 2S.\omega\rho d\rho,$$

even if the *scalar function* $f\rho$ be of a higher dimension than the second, let this last function have the form,

$$\text{XI.} \dots f\rho = S q \rho q' \rho q'' \rho, \quad q, q', q'' \text{ being three constant quaternions.}$$

Here

$$\text{XII.} \dots \nu = \omega\rho = \frac{1}{2}V(q\rho q' \rho q'' + q' \rho q'' \rho q + q'' \rho q \rho q');$$

$$\text{XIII.} \dots d\nu = \phi d\rho = \phi\rho' = \frac{1}{2}V(q\rho' q' \rho q'' + q' \rho q'' \rho' q) + \frac{1}{2}V(q' \rho' q'' \rho q + q'' \rho q \rho' q') + \frac{1}{2}V(q'' \rho' q \rho q' + q \rho q' \rho' q'');$$

and XIV. . . $S\lambda\phi\rho' = \frac{1}{2}S.q'\rho q''(\lambda q\rho' + \rho'q\lambda) + \&c. = S\rho'\phi\lambda;$

so that $\phi' = \phi$, as asserted.

(4.) In general, if δ be used as a *second* and *independent symbol* of differentiation, we may write (comp. 845, IV.),

$$XV. . . \delta dfq = d\delta fq,$$

where f_q may denote any function of a quaternion; in fact, each member is, by the principles of the present Chapter (comp. 844, I., and 845, IX.), an expression for the limit,*

$$XVI. . . \lim_{\substack{\pi' \rightarrow \pi \\ \pi' \rightarrow \infty}} \{f(q + \pi^{-1}dq + \pi'^{-1}\delta q) - f(q + \pi^{-1}dq) - f(q + \pi'^{-1}\delta q) + f_q\}.$$

(5.) As another statement of the same theorem, we may remark that a first differentiation of f_q , with each symbol separately taken, gives results of the forms,

$$XVII. . . dqf = f(q, dq), \quad \delta f_q = f(q, \delta q);$$

and then the assertion is, that if we differentiate the first of these with δ , and the second with d , operating only on q with each, and not on dq nor on δq , we obtain equal results, of these other forms,

$$XVIII. . . \delta dfq = f(q, dq, \delta q) = f(q, \delta q, dq) = d\delta f_q.$$

For example, if

$$XIX. . . f_q = qcq, \text{ where } c \text{ is a constant quaternion,}$$

the common value of these last expressions is,

$$XX. . . \delta dfq = d\delta f_q = \delta q.c.dq + dq.c.\delta q.$$

(6.) Writing then, by X.,

$$XXI. . . df\rho = 2S\omega\rho d\rho, \quad \delta f\rho = 2S\omega\rho\delta\rho,$$

and

$$XXII. . . \delta\omega\rho = \phi\delta\rho, \text{ with } d\omega\rho = \phi d\rho, \text{ as before,}$$

we have the general equation,

$$XXIII. . . S(d\rho.\phi\delta\rho) = S(\delta\rho.\phi d\rho),$$

in which $d\rho$ and $\delta\rho$ may represent any two vectors; the linear and vector function, ϕ , which is thus derived from a scalar function $f\rho$ by differentiation, is therefore (as above asserted and exemplified) always self-conjugate.

(7.) The equation XXIII. may be thus briefly written,

$$XXIV. . . Sd\rho\delta\nu = S\delta\rho d\nu;$$

and it will be found to be virtually equivalent to the following system of three known equations, in the calculus of partial differential coefficients,

$$XXV. . . D_2D_1 = D_1D_2, \quad D_1D_3 = D_3D_1, \quad D_3D_2 = D_2D_3.$$

364. At the commencement of the present Section, we reduced (in 347) the problem of the inversion (346) of a linear (or distributive) quaternion function of a quaternion, to the

* We may also say that each of the two symbols XV. represents the coefficient of $x^i y^j$, in the development of $f(q + xdq + y\delta q)$ according to ascending powers of x and y , when such development is possible.

corresponding problem for *vectors*; and, under this reduced or simplified *form*, have *resolved* it. Yet it may be interesting, and it will now be easy, to *resume* the *linear* and *quaternion equation*,

$$\text{I. . . } fq = r, \quad \text{with} \quad \text{II. . . } f(q + q') = fq + fq',$$

and to assign a *quaternion expression* for the *solution* of that equation, or for the *inverse quaternion function*,

$$\text{III. . . } q = f^{-1}r,$$

with the aid of notations already employed, and of results already established.

(1.) The *conjugate* of the linear and quaternion function fq being defined (comp. 347, IV.) by the equation,

$$\text{IV. . . } Spfq = Sqf'p,$$

in which p and q are arbitrary quaternions, if we set out (comp. 347, XXXI.) with the *form*,

$$\text{V. . . } fq = tqs + t'qs' + \dots = \Sigma tqs,$$

in which s, s', \dots and t, t', \dots are *arbitrary* but *constant quaternions*, and which is more than sufficiently general, we shall have (comp. 347, XXXII.) the *conjugate form*,

$$\text{VI. . . } f'p = spt + s'pt' + \dots = \Sigma spt;$$

whence

$$\text{VII. . . } f'1 = \Sigma ts, \quad \text{and} \quad \text{VIII. . . } f'1 = \Sigma st;$$

it is then possible, for each *given particular form* of the linear function fq , to assign *one scalar constant* ϵ , and *two vector constants*, ϵ, ϵ' , such that

$$\text{IX. . . } f'1 = \epsilon + \epsilon', \quad f'1 = \epsilon + \epsilon';$$

and then we shall have the general transformations (comp. 347, I.):

$$\text{X. . . } Sf'q = S.qf'1 = \epsilon Sq + S\epsilon'q;$$

$$\text{XI. . . } Vf'q = \epsilon Sq + V.f'Vq = \epsilon Sq + \phi Vq;$$

and

$$\text{XII. . . } fq = (\epsilon + \epsilon')Sq + S\epsilon'q + \phi Vq;$$

in which $S\epsilon'q = S.\epsilon'Vq$, and ϕVq or $Vf'Vq$ is a *linear* and *vector function* of Vq , of the kind already considered in this Section; being also such that, with the form V. of fq , we have

$$\text{XIII. . . } \phi\rho = \Sigma Vt\rho s.$$

(2.) As regards the number of *independent* and *scalar constants* which enter, at least implicitly, into the composition of the quaternion function fq , it may in various ways be shown to be *sixteen*; and accordingly, in the expression XII., the *scalar* ϵ is *one*; the *two vectors*, ϵ and ϵ' , count *each* as *three*; and the *linear* and *vector function*, ϕVq , counts as *nine* (comp. 347, (1.)).

(3.) Since we already know (347, &c.) how to *invert* a function of this last kind ϕ , we may in general write,

$$\text{XIV. . . } r = Sr + Vr = Sr + \phi\rho, \quad \text{where} \quad \text{XV. . . } \rho = \phi^{-1}Vr = m^{-1}\psi Vr;$$

the *scalar constant*, m , and the *auxiliary linear* and *vector function*, ψ , being deduced

from the function ϕ by methods already explained. It is required then to express q , or Sq and Vq , in terms of r , or of Sr and ρ , so as to satisfy the linear equation,

$$\text{XVI.} \dots (e + \epsilon)Sq + S\epsilon'q + \phi Vq = Sr + \phi\rho;$$

the constants ϵ , ϵ' , and the form of ϕ , being given.

(4.) Assuming for this purpose the expression,

$$\text{XVII.} \dots q = q' + \rho,$$

in which q' is a new sought quaternion, we have the new equation,

$$\text{XVIII.} \dots fq' = Sr + \phi\rho - f\rho = S(r - \epsilon'\rho);$$

whence

$$\text{XIX.} \dots q' = S(r - \epsilon'\rho) \cdot f^{-1},$$

and

$$\text{XX.} \dots q = \rho + S(r - \epsilon'\rho) \cdot f^{-1};$$

in which ρ is (by supposition) a known vector, and $S(r - \epsilon'\rho)$ is a known scalar; so that it only remains to determine the unknown but constant quaternion, f^{-1} , or to resolve the particular equation,

$$\text{XXI.} \dots fq_0 = 1, \text{ in which } \text{XXII.} \dots q_0 = e + \gamma = f^{-1},$$

e being a new and sought scalar constant, and γ being a new and sought vector constant.

(5.) Taking scalar and vector parts, the quaternion equation XXI. breaks up into the two following (comp. X. and XI.):

$$\text{XXIII.} \dots 1 = Sf(c + \gamma) = ec + S\epsilon'\gamma; \quad \text{XXIV.} \dots 0 = Vf(c + \gamma) = \epsilon c + \phi\gamma;$$

which give the required values of e and γ , namely,

$$\text{XXV.} \dots e = (e - S\epsilon'\phi^{-1}\epsilon)^{-1}, \text{ and } \text{XXVI.} \dots \gamma = -c\phi^{-1}\epsilon;$$

whence

$$\text{XXVII.} \dots f^{-1} = \frac{1 - \phi^{-1}\epsilon}{e - S\epsilon'\phi^{-1}\epsilon};$$

and accordingly we have, by XII., the equation,

$$\text{XXVIII.} \dots f(1 - \phi^{-1}\epsilon) = e - S\epsilon'\phi^{-1}\epsilon = V^{-1}0.$$

(6.) The problem of quaternion inversion is therefore reduced anew to that of vector inversion, and solved thereby; but we can now advance some steps further, in the elimination of inverse operations, and in the substitution for them of direct ones. Thus, if we observe, that $\phi^{-1} = m^{-1}\psi$, as before, and write for abridgment,

$$\text{XXIX.} \dots n = m\epsilon - S\epsilon'\psi\epsilon = f(m - \psi\epsilon),$$

so that n is a new and known scalar constant, we shall have, by XV. XX. XXVII. XXIX.,

$$\text{XXX.} \dots m\rho = \psi Vr; \quad \text{XXXI.} \dots n f^{-1} = m - \psi\epsilon;$$

and

$$\text{XXXII.} \dots maq = n\psi Vr + (mSr - S\epsilon'\psi Vr) \cdot (m - \psi\epsilon),$$

an expression from which all inverse operations have disappeared, but which still admits of being simplified, through a division by m , as follows.

(7.) Substituting (by XXIX.), in the term $n\psi Vr$ of XXXII., the value $m\epsilon - S\epsilon'\psi\epsilon$ for n , and changing (by XXX.) ψVr to $m\rho$, in the terms which are not obviously divisible by m , such a division gives,

$$\text{XXXIII.} \dots nq = (m - \psi\epsilon)Sr + e\psi Vr - S\epsilon'\psi Vr + \sigma,$$

where

$$\text{XXXIV.} \dots \sigma = -\rho S\epsilon'\psi\epsilon + \psi\epsilon S\epsilon'\rho = V \cdot \epsilon' V \rho \psi \epsilon.$$

But (by 348, VII., interchanging accents) we have the transformation,

$$\text{XXXV.} \dots V\rho\psi\epsilon = -\phi'V\epsilon\phi\rho = -\phi'V\epsilon Vr,$$

because $\phi\rho = Vr$, by XIV. or XV.; everything *inverse* therefore *again* disappears, with this new elimination of the auxiliary vector ρ , and we have this final expression,

$$\text{XXXVI.} \dots nq = nf^{-1}r = (me - S\epsilon'\psi\epsilon).f^{-1}r \\ = (m - \psi\epsilon)Sr + e\psi Vr - S\epsilon'\psi Vr - V\epsilon'\phi V\epsilon Vr,$$

in which each symbol of operation governs all that follows it, except where a point indicates the contrary, and which it appears to be impossible further to reduce, as the *formula of solution* of the linear equation I., with the *form XII.* of the *quaternion function*, fq .

(8.) Such having been the *analysis* of the problem, the *synthesis*, by which an *a posteriori* proof of the correctness of the resulting formula is to be given, may be simplified by using the *scalar* value XXIX. of $f(m - \psi\epsilon)$; and it is sufficient to show (denoting Vr by ω), that for every *vector* ω the following equation holds good, with the same form XII. of f :

$$\text{XXXVII.} \dots f(e\psi\omega - S\epsilon'\psi\omega) - fV\epsilon'\phi V\epsilon\omega = (me - S\epsilon'\psi\epsilon).\omega.$$

(9.) Accordingly, that form of f gives, with the help of the principle employed in XXXV.,

$$\text{XXXVIII.} \dots \begin{cases} ef\psi\omega = e(S\epsilon'\psi\omega + m\omega), & -fS\epsilon'\psi\omega = -(e + \epsilon)S\epsilon'\psi\omega, \\ -fV\epsilon'\phi V\epsilon\omega = -\phi V\epsilon'\phi V\epsilon\omega = V(V\epsilon\omega.\psi\epsilon') = eS\epsilon'\psi\omega - \omega S\epsilon'\psi\epsilon, \end{cases}$$

because $S\omega\psi\epsilon' = S\epsilon'\psi\omega$, &c.; and thus the equation XXXVI. is proved, by actually operating with f .

(10.) As an *example*, if we take the particular form,

$$\text{XXXIX.} \dots r = fq = pq + qp,$$

in which

$$\text{XL.} \dots p = a + a = \text{a given quaternion,}$$

we have then,

$$\text{XLI.} \dots f1 = f^{-1}1 = 2p, \quad e = 2a, \quad \epsilon = \epsilon' = 2a, \quad \phi\rho = 2ap;$$

whence by the theory of linear and *vector* functions,

$$\text{XLII.} \dots \phi'\rho = 2ap, \quad \psi\rho = 4a^2\rho, \quad m = 8a^2,$$

and therefore, XLIII. $\dots \psi\epsilon = 8a^2a$, $m - \psi\epsilon = 8a^2(a - a)$, $n = 16a^2(a^2 - a^2)$;

so that, dividing by $8a$, the formula XXXVI. becomes,

$$\text{XLIV.} \dots 2a(a^2 - a^2)q = a(a - a)Sr + a^2Vr - aS.aVr - aV.aVr,$$

or

$$\text{XLV.} \dots 2a(a + a)q = aSr + (a + a)Vr - 8ar,$$

or

$$\text{XLVI.} \dots 2pqSp = S.rKp + pVr = rSp + V(Vp.Vr),$$

or

$$\text{XLVII.} \dots 4pqSp = 2rSp + (pr - rp) = pr + rKp;$$

or finally,

$$\text{XLVIII.} \dots q = f^{-1}r = \frac{r + p^{-1}rKp}{4Sp} = \frac{r + Kp.rp^{-1}}{4Sp}.$$

Accordingly,

$$\text{XLIX.} \dots (pr + rKp) + (rp + Kp.r) = 2r(p + Kp) = 4rSp.$$

(11.) In *so simple* an example as the last, we may with advantage avail ourselves of *special methods*; for instance (comp. 846), we may use that which was employed in 882, (6.), to *differentiate the square root of a quaternion*, and which conducted there more rapidly to a formula (882, XIX.) agreeing with the recent XLVIII.

(12.) We might also have observed, in the same case XXXIX., that

L. . . $pr - rp = p^2q - qp^2 = 2V(V(p^2) \cdot Vq) = 4Sp \cdot V(Vp \cdot Vq) = 2Sp \cdot (pq - qp)$;
 whence $pq - qp$, and therefore pq and qp , can be at once deduced, with the same resulting value for q , or for $f^{-1}r$, as before : and generally it is possible to *differentiate*, on a similar plan, the n^{th} root of a quaternion.

365. We shall conclude this Section on *Linear Functions*, of the kinds above considered, by proving the general existence of a *Symbolic and Biquadratic Equation*, of the form,

$$I. . . 0 = n - n'f + n''f^2 - n'''f^3 + f^4,$$

which is thus satisfied by the Symbol (f) of *Linear and Quaternion Operation on a Quaternion*, as the *Symbolic and Cubic Equation*,

$$I'. . . 0 = m - m'\phi + m''\phi^2 - \phi^3, \quad 350, I.,$$

was satisfied by the symbol (ϕ) of *linear and vector operation on a vector* ; the four coefficients, n, n', n'', n''' , being four scalar constants, deduced from the function f in this extended or quaternion theory, as the three scalar coefficients m, m', m'' were constants deduced from ϕ , in the former or vector theory. And at the same time we shall see that there exists a *System*, of *Three Auxiliary Functions, F, G, H*, of the *Linear and Quaternion* kind, analogous to the two vector functions, ψ and χ , which have been so useful in the foregoing theory of vectors, and like them connected with each other, and with the given quaternion function f ; by several simple and useful relations.

(1.) The formula of solution, 364, XXXVI., of the linear and quaternion equation $fq = r$, being denoted briefly as follows,

$$II. . . nq = nf^{-1}r = Fr,$$

so that (comp. 348, III'.) we may write, briefly and symbolically,

$$III. . . fF = Ff = n,$$

it may next be proposed to examine the changes which the scalar n and the function Fr undergo, when fr is changed to $fr + cr$, or f to $f + c$, where c is any scalar constant ; that is, by 364, XII., when e is changed to $e + c$, and ϕ to $\phi + c$; ϕ', ψ , and m being at the same time changed, according to the laws of the earlier theory.

(2.) Writing, then,

$$IV. . . f_c = f + c, \quad e_c = e + c, \quad \phi_c = \phi + c, \quad \phi'_c = \phi' + c,$$

and

$$V. . . \psi_c = \psi + c\chi + c^2, \quad m_c = m + m'e + m''e^2 + c^3,$$

we may represent the new form of the equation 364, XXXVI. as follows :

$$VI. . . n_c f_c^{-1} r = F_c r, \quad \text{or} \quad VII. . . f_c F_c = n_c;$$

where VIII. . . $F\sigma = (m_0 - \psi_0\epsilon)Sr + e\psi_0Vr - S\epsilon'\psi_0Vr - V\epsilon'\phi_0V\epsilon Vr$,
and IX. . . $n_0 = e_0m_0 - S\epsilon'\psi_0\epsilon$.

(8.) In this manner it is seen that we may write,

$$X. . . F_c = F + cG + c^2H + c^3,$$

and

$$XI. . . n_c = n + n'c + n''c^2 + n'''c^3 + c^4;$$

where F, G, H , are three functional symbols, such that

$$XII. . . \begin{cases} Fr = (m - \psi\epsilon)Sr + e\psi Vr - S\epsilon'\psi Vr - V\epsilon'\phi'Ve Vr; \\ Gr = (m' - \chi\epsilon)Sr + (e\chi + \psi)Vr - S\epsilon'\chi Vr - V\epsilon'Ve Vr; \\ Hr = (m'' - \epsilon)Sr + (e + \chi)Vr - S\epsilon'r; \end{cases}$$

and n, n', n'', n''' are four scalar constants, namely,

$$XIII. . . \begin{cases} n = em - S\epsilon'\psi\epsilon \text{ (as in 864, XXIX);} \\ n' = m + em' - S\epsilon'\chi\epsilon; \\ n'' = m' + em'' - S\epsilon'\epsilon; \\ n''' = m'' + e. \end{cases}$$

(4.) Developing then the symbolical equation VII., with the help of X. and XI., and comparing powers of c , we obtain these new symbolical equations (comp. 850, XVI. XXI. XXIII.):

$$XIV. . . \begin{cases} H = n''' - f; \\ G = n'' - fH = n'' - n'''f + f^2; \\ F = n' - fG = n' - n''f + n'''f^2 - f^3; \end{cases}$$

and finally,

$$XV. . . n = Ff = n'f - n''f^2 + n'''f^3 - f^4,$$

which is only another way of writing the *symbolic and biquadratic equation I.*

(5.) *Other functional relations* exist, between these various symbols of operation, which we cannot here delay to develop: but we may remark that, as in the theory of linear and vector functions, these usually introduce a mixture of functions with their *conjugates* (comp. 847, XI., &c.).

(6.) This seems however to be a proper place for observing, that if we write, as temporary notations, for *any four quaternions*, p, q, r, s , the equations,

$$XVI. . . [pq] = pq - qp; \quad XVII. . . (pqr) = S.p [qr];$$

$$XVIII. . . [pqr] = (pqr) + [rq]Sp + [pr]Sq + [qr]Sr;$$

and

$$XIX. . . (pqrs) = S.p [qrs],$$

so that $[pq]$ is a vector, (pqr) and $(pqrs)$ are scalars, and $[pqr]$ is a quaternion, we shall have, in the first place, the relations:

$$XX. . . [pq] = -[qp], \quad [pp] = 0;$$

$$XXI. . . (pqr) = -(qpr) = (qrp) = \&c., \quad (ppr) = 0;$$

$$XXII. . . [pqr] = -[qpr] = [qrp] = \&c., \quad [ppr] = 0;$$

and XXIII. . . $(pqrs) = -(qprs) = (qrps) = -(qrsp) = \&c., \quad (pprs) = 0$.

(7.) In the next place, if t be any fifth quaternion, the quaternion equation,

$$XXIV. . . 0 = p(qrat) + q(ratp) + r(stpq) + s(tpqr) + t(pqrs),$$

which may also be thus written,

$$XXV. . . q(prst) = p(qrat) + r(pqst) + s(prqt) + t(pqrs),$$

and which is analogous to the *vector equation*,

$$XXVI. . . 0 = aS\beta\gamma\delta - \beta S\gamma\delta\alpha + \gamma S\delta\alpha\beta - \delta S\alpha\beta\gamma,$$

or to the continually* occurring transformation (comp. 294, XIV.),

$$\text{XXVII.} \dots \delta Sa\beta\gamma = a\delta\delta\beta\gamma + \beta Sa\delta\gamma + \gamma Sa\beta\delta,$$

is satisfied generally, because it is satisfied for the four distinct suppositions,

$$\text{XXVIII.} \dots q = p, \quad q = r, \quad q = s, \quad q = t.$$

(8.) In the third place, we have this other general quaternion equation,

$$\text{XXIX.} \dots q(prst) = [rst]Spq - [stp]Sr q + [tp r]Seq - [prs]Stq,$$

which is analogous to this other† useful vector formula (comp. 294, XV.),

$$\text{XXX.} \dots \delta Sa\beta\gamma = V\beta\gamma Sa\delta + V\gamma a\delta\beta + Va\beta S\gamma\delta;$$

because the equation XXIX. gives true results, when it is operated on by the four distinct symbols (comp. 312),

$$\text{XXXI.} \dots S.p, \quad S.r, \quad S.s, \quad S.t.$$

(9.) Assuming then any four quaternions, p, r, s, t , which are not connected by the relation,

$$\text{XXXII.} \dots (prst) = 0,$$

and deducing from them four others, p', r', s', t' , by the equations,

$$\text{XXXIII.} \dots \begin{cases} p'(prst) = f[rst], & r'(prst) = -f[stp], \\ s'(prst) = f[tp r], & t'(prst) = -f[prs], \end{cases}$$

in which f is still supposed to be a symbol of linear and quaternion operation on a quaternion, the formula XXIX. allows us to write generally, as an expression for the function $f q$, which may here be denoted by q' (because r is now otherwise used):

$$\text{XXXIV.} \dots q' = f q = p'Sp q + r'Sr q + s'Seq + t'St q;$$

and its sixteen scalar constants (comp. 364, (2.)) are now those which are involved in its four quaternion constants, p', r', s', t' .

(10.) Operating on this last equation with the four symbols,

$$\text{XXXV.} \dots S.[r's't'], \quad S.[s't'p'], \quad S.[t'p'r'], \quad S.[p'r's'],$$

we obtain the four following results:

$$\text{XXXVI.} \dots \begin{cases} (q'r's't') = (p'r's't')Spq; & (q's't'p') = (r's't'p')Sr q; \\ (q't'p'r') = (s't'p'r')Seq; & (q'p'r's') = (t'p'r's')Stq; \end{cases}$$

and when the values thus found for the four scalars,

$$\text{XXXVII.} \dots Spq, \quad Sr q, \quad Seq, \quad Stq,$$

are substituted in the formula XXIX., we have the following new formula of quaternion inversion:

$$\text{XXXVIII.} \dots (p'r's't')(prst)q = (p'r's't')(prst)f^{-1}q' \\ = [rst](q'r's't') + [stp](q's't'p') + [tp r](q't'p'r') + [prs](q'p'r's');$$

* The equations XXVII. and XXX., which had been proved under slightly different forms in the sub-articles to 294, have been in fact freely employed as transformations in the course of the present Chapter, and are supposed to be familiar to the student. Compare the Note to page 437.

† Compare the Note immediately preceding.

which shows, in a new way, how to *resolve a linear equation in quaternions*, when put under what we may call (comp. 347, (1.)) the *Standard Quadriomial Form*, XXXIV.

(11.) Accordingly, if we operate on the formula XXXVIII. with f , attending to the equations XXXIII., and dividing by $(prst)$, we get this new equation,

$$\text{XXXIX.} \dots (p'r's't')fq = p'(q'r's't') - r'(q's't'p') + s'(q't'p'r') - t'(q'p'r's'); \\ \text{whence} \quad fq = q', \text{ by XXV.}$$

(12.) It has been remarked (9.), that p, r, s, t , in recent formulae, may be any four quaternions, which do not satisfy the equation XXXII.; we may therefore assume,

$$\text{XL.} \dots p = 1, \quad r = i, \quad s = j, \quad t = k,$$

with the laws of 182, &c., for the symbols i, j, k , because those laws give here,

$$\text{XLI.} \dots (ijk) = -2;$$

and then it will be found that the equations XXXIII. give simply,

$$\text{XLII.} \dots p' = f1, \quad r' = -fi, \quad s' = -fj, \quad t' = -fk;$$

so that the *standard quadriomial form* XXXIV. becomes, with this selection of $prst$,

$$\text{XLIII.} \dots fq = f1.Sq - fi.Siq - fj.Sjq - fk.Skq,$$

and admits of an immediate verification, because any quaternion, q , may be expressed (comp. 221) by the *quadriomial*,

$$\text{XLIV.} \dots q = Sq - iSiq - jSjq - kSkq.$$

(13.) Conversely, if we set out with the expression,

$$\text{XLV.} \dots q = w + ix + jy + kz, \quad 221, \text{ III.}$$

which gives,

$$\text{XLVI.} \dots fq = wf1 + xfi + yfj + zfk,$$

or briefly,

$$\text{XLVII.} \dots e = aw + bx + cy + dz,$$

the letters $abcde$ being here used to denote five known quaternions, while $wxyz$ are four sought scalars, the problem of quaternion inversion comes to be that of the separate determination (comp. 312) of these four scalars, so as to satisfy the one equation XLVII.; and it is resolved (comp. XXV.) by the system of the four following formulæ:

$$\text{XLVIII.} \dots \begin{cases} w(abcd) = (ebcd); & x(abcd) = (accd); \\ y(abcd) = (abcd); & z(abcd) = (abce); \end{cases}$$

the notations (6.) being retained.

(14.) Finally it may be shown, as follows, that the *biquadratic equation* I., for linear functions of quaternions, includes* the cubic I', or 850, I., for vectors. Sup-

* In like manner it may be said, that the cubic equation includes a quadratic one, when we confine ourselves to the consideration of vectors in one plane; for which case $m = 0$, and also $\psi\rho = 0$, if ρ be a line in the given plane: for we have then $\phi\chi = m' - \psi = m'$, or

$$\phi^2 - m''\phi + m' = 0,$$

pose, for this purpose, that the linear and quaternion function, $f\bar{q}$, reduces itself to the last term of the general expression 864, XII., or becomes,

$$\text{XLIX.} \dots f\bar{q} = \phi \nabla q, \text{ so that } L \dots e = 0, \quad s = s' = 0, \quad f1 = f'1 = 0;$$

the coefficients n, n', n'', n''' take then, by XIII., the values,

$$\text{LI.} \dots n = 0, \quad n' = m, \quad n'' = m', \quad n''' = m'';$$

and the biquadratic I. becomes,

$$\text{LII.} \dots 0 = (-m + m'f - m''f^2 + f^3)f.$$

But $f\bar{q}$ is now a *vector*, by XLIX., and it may be any vector, ρ ; also the operation f is now equivalent to that denoted by ϕ , when the *subject* of the operation is a vector; we may therefore, in the case here considered, write this last equation LII. under the form,

$$\text{LIII.} \dots 0 = (-m + m'\phi - m''\phi^2 + \phi^3)\rho,$$

which agrees with 851, I., and reproduces the *symbolical cubic*, when the symbol of the *operand* (ρ) is suppressed.

CHAPTER III.

ON SOME ADDITIONAL APPLICATIONS OF QUATERNIONS, WITH SOME CONCLUDING REMARKS.

SECTION 1.—*Remarks Introductory to this Concluding Chapter.*

366. WHEN the *Third Book* of the present *Elements* was begun, it was hoped (277) that this Book might be made a much shorter one, than either of the two preceding. That purpose it was found impossible to accomplish, without injustice to the subject; but at least an intention was expressed (317), at the commencement of the *Second Chapter*, of rendering that Chapter the *last*: while some new *Examples of Geo-*

with this understanding as to the operand. In fact, the *cubic* gives here (because $m = 0$),

$$(\phi^3 - m''\phi + m')\phi\rho = 0;$$

and therefore

$$(\phi^2 - m''\phi + m')\sigma = 0;$$

if σ be already the result of an operation with ϕ , on any vector ρ : that is if it be, as above supposed, a line in the given plane.

metrical Applications, and some few *Specimens of Physical ones*, were promised.

367. The promise, thus referred to, has been perhaps already in part redeemed; for instance, by the investigations (315) respecting certain *tangents, normals, areas, volumes, and pressures*, which have served to illustrate certain portions of the theory of *differentials* and *integrals* of quaternions. But it may be admitted, that the six preceding Sections have treated chiefly of that *Theory of Quaternion Differentials*, including of course its *Principles* and *Rules*; and of the connected and scarcely less important *theory of Linear or Distributive Functions*, of *Vectors* and *Quaternions*: *Examples* and *Applications* having thus played hitherto a merely *subordinate* or *illustrative* part, in the progress of the present Volume.

368. Such was, indeed, *designed* from the outset to be, *upon the whole*, the result of the present undertaking: which was rather to *teach*, than to *apply*, the *Calculus of Quaternions*. Yet it still appears to be possible, without quite exceeding suitable limits, and accordingly we shall now endeavour, to condense into a short *Third Chapter* some *Additional Examples*, geometrical and physical, of the *application* of the *principles* and *rules* of that Calculus, supposed to be already *known*, and even to have become by this time *familiar** to the reader. And then, with a few general remarks, the work may be brought to its close.

SECTION 2.—On Tangents and Normal Planes to Curves in Space.

369. It was shown (100) towards the close of the First Book, that if the *equation* of a *curve in space*, whether *plane* or of *double curvature*, be *given* under the form,

$$\text{I. . . } \rho = \phi(t) = \phi t,$$

where t is a scalar variable, and ϕ is a functional sign, then the *derived vector*,

$$\text{II. . . } D\rho = D\phi t = \phi' t = \rho' = d\rho : dt,$$

* Accordingly, even *references* to former Articles will now be supplied more sparingly than before.

represents a *line* which is, or is parallel to, the *tangent* to the curve, drawn at the extremity of the variable vector ρ . If then we suppose that τ is a point situated upon the tangent thus drawn to a curve PQ at P and that u is a point in the corresponding normal plane, so that the angle τPu is right, and if we denote the vectors OP, OT, OU by ρ, τ, v , the *equations* of the *tangent line* and *normal plane* at P may now be thus expressed:

$$\text{III.} \dots V(\tau - \rho)\rho' = 0; \quad \text{IV.} \dots S(v - \rho)\rho' = 0;$$

the vector τ being treated as the *only variable* in III., and in like manner v as the only variable in IV., when once the *curve* PQ is given, and the *point* P is selected.

(1.) It is permitted, however, to express these last equations under *other forms*; for example, we may replace ρ' by $d\rho$, and thus write, for the same tangent line and normal plane,

$$\text{V.} \dots V(\tau - \rho)d\rho = 0; \quad \text{VI.} \dots S(v - \rho)d\rho = 0;$$

where the vector differential $d\rho$ may represent *any line, parallel to the tangent* to the curve at P , and is *not necessarily small* (compare again 100).

(2.) We may also write, as the equation of the tangent,

$$\text{VII.} \dots \tau = \rho + x\rho', \text{ where } x \text{ is a scalar variable;}$$

and as the equation of the normal plane,

$$\text{VIII.} \dots d_p T(v - \rho) = 0; \text{ or VIII'.} \dots dT(v - \rho) = 0, \text{ if } dv = 0;$$

because this *partial differential* of $T(v - \rho)$, or of $\overline{P\bar{U}}$, is (by 334, XII., &c.),

$$\text{IX.} \dots dT(v - \rho) = S(U(v - \rho).d\rho).$$

(3.) For the *circular locus* 314, (1.), or 337, (1.), of which the equation is,

$$\text{X.} \dots \rho = \alpha^t \beta, \text{ with } T\alpha = 1, \text{ and } S\alpha\beta = 0,$$

the equation of the tangent is, by VII., and by the value 337, VI. of ρ' ,

$$\text{XI.} \dots \tau = \rho + y\alpha\rho, \text{ where } y \text{ is a new scalar variable;}$$

the *perpendicularity* of the *tangent* to the *radius* being thus put in evidence.

(4.) For the *plane but elliptic locus*, 314, (2.), or 337, (2.), for which,

$$\text{XII.} \dots \rho = V.\alpha^t \beta, \text{ with } T\alpha = 1, \text{ but not } S\alpha\beta = 0,$$

the value 337, VIII. of ρ' shows that the tangent, at the extremity of any *one* semidiameter ρ , is *parallel* to the *conjugate* semidiameter of the curve; that is, to the one obtained by altering the *excentric anomaly* (314, (2.)), by a *quadrant*: or to the value of ρ which results, when we change t to $t + 1$.

(5.) For the *helix*, 314, (10.), of which the equation is,

$$\text{XIII.} \dots \rho = c\alpha + \alpha^t \beta, \text{ with } T\alpha = 1, \text{ and } S\alpha\beta = 0,$$

c being a scalar constant, we have the derived vector,

$$\text{XIV.} \dots \rho' = c\alpha + \frac{\pi}{2} \alpha^{t+1} \beta; \text{ whence XV.} \dots S\alpha^{-1} \rho' = c,$$

$$\text{XVI.} \dots TV\alpha^{-1} \rho' = \frac{\pi}{2} T\beta, \text{ and XVII.} \dots (TV:S)\alpha^{-1} \rho' = \frac{\pi T\beta}{2c};$$

the *tangent line* (ρ') to the *helix* is therefore inclined to the *axis* (α) of the *cylinder* whereon that curve is traced, at a *constant angle* (a), whereof the *trigonometrical tangent* ($\tan a$) is given by this formula XVII. ; and accordingly, the numerator $\pi T\beta$ of that formula represents the *semicircumference* of the *cylindric base* ; while the denominator $2c$ is an expression for *half the interval* between two *successive spires*, measured in a direction parallel to the *axis*. We may then write,

$$\text{XVIII.} \dots \pi T\beta = 2c \tan a = 2c \cot b,$$

if a thus denote the *constant inclination* of the *helix* to the *axis*, while b denotes the constant and complementary inclination of that curve to the *base*, or to the *circles* which it crosses on the *cylinder*.

(6.) In general, the *parallels* ρ' to the *tangents* to a curve of *double curvature*, which are drawn from a *fixed origin* o , have a certain *cone* for their *locus* ; and for the case of the *helix*, the *equation* of this cone is given by the formula XVII., or by any legitimate transformation thereof, such as the following,

$$\text{XIX.} \dots S U \alpha^{-1} \rho' = \pm \cos a = \pm \sin b ;$$

it is therefore, in this case, a *cone of revolution*, with its *semiangle* $= a$.

(7.) As an example of the determination of a *normal plane* to a curve of double curvature, we may observe that the equation XIII. of the *helix* gives,

$$\text{XX.} \dots \rho^2 = \beta^2 - c^2 t^2, \text{ and therefore } \text{XXI.} \dots S \rho \rho' = -c^2 t ;$$

the equation IV. becomes therefore, for the case of this curve,

$$\text{XXII.} \dots 0 = S \rho' v + c^2 t, \text{ with the value XIV. of } \rho'.$$

(8.) If then it be required to assign the point v in which the *normal plane* to the *helix* meets the *axis* of the *cylinder*, we have only to combine this equation XXII. with the condition $v \parallel \alpha$, and we find, by XIII. and XIV.,

$$\text{XXIII.} \dots o v = v = -c^2 t \alpha : S \alpha \rho' = c t \alpha, \quad \text{XXIV.} \dots S \alpha (v - \rho) = 0 ;$$

the line rv is therefore *perpendicular* to the *axis*, being in fact a *normal* to the *cylinder*.

370. *Another view of tangents and normal planes* may be proposed, which shall connect them in calculation with *Taylor's Series* adapted to quaternions (342), as follows.

(1.) Writing I. . . $\rho_t = \rho_0 + u t \rho'_0$, or briefly, I'. . . $\rho_t = \rho + u t \rho'$, the *coefficient* u_t or u will generally be a *quaternion*, but its *limiting value* will be *positive unity*, when t tends to *zero* as its limit ; or in symbols,

$$\text{II.} \dots u_0 = \lim_{t \rightarrow 0} u = 1.$$

(2.) Admitting this, which follows either from *Taylor's Series*, or (in so simple a case) from the mere *definition* of the *derived vector* ρ' , we may conceive that vector ρ' to be *constructed* by some given line PR , without yet supposing it to be known that this line is *tangential* at P to the curve PQ , of which the *variable vector* is $oQ = \rho_t$, while $oP = \rho_0 = \rho$, so that the line $rq = u t \rho'$ is a *vector chord* from P , which *diminishes* indefinitely with the *scalar variable*, t , and is *small*, if t be small.

(3.) Conceiving next that $\omega = OR$ = the vector of some new and arbitrary point R, we may let fall a perpendicular QM on the line PR, and so decompose the chord PQ into the two rectangular lines, PM and MQ; which, when divided by the same chord, give rigorously the two (generally) quaternion quotients,

$$\text{III.} \dots \frac{PM}{PQ} = \frac{S\omega\rho'(\omega - \rho)}{u\rho'(\omega - \rho)}, \quad \text{IV.} \dots \frac{MQ}{PQ} = \frac{V\omega\rho'(\omega - \rho)}{u\rho'(\omega - \rho)};$$

the variable t thus disappearing through the division, except so far as it enters into u , which tends as above to 1.

(4.) Passing then to the limits, we have these other rigorous equations,

$$\text{V.} \dots \lim. \frac{PM}{PQ} = \frac{S\rho'(\omega - \rho)}{\rho'(\omega - \rho)}, \quad \text{VI.} \dots \lim. \frac{MQ}{PQ} = \frac{V\rho'(\omega - \rho)}{\rho'(\omega - \rho)};$$

by comparing which with 369, III. and IV., we see that those two equations represent respectively, as before stated, the tangent and the normal plane to the proposed curve at P; because, if $V\rho'(\omega - \rho) = 0$, the chord PQ tends, by V. or VI., to coincide, both in length and in direction, with its projection PM on the line PR; while on the other hand, if $S\rho'(\omega - \rho) = 0$, that projection tends to vanish, even as compared with the chord PQ; which chord tends now to coincide with its other projection MQ, or with the perpendicular to the line PR, erected so as to reach the point Q: whence PR must, in this last case, be a normal to the curve at P.

(5.) We may also investigate an equation for the normal plane, by considering it as the limiting position of the plane which perpendicularly bisects the chord. If R be supposed to be a point of this last plane, then, with the recent notations, the vector $\omega = OR$ must satisfy the condition,

$$\text{VII.} \dots T(\omega - \rho) = T(\omega - \rho_0), \quad \text{or} \quad \text{VIII.} \dots (\omega - \rho - ut\rho')^2 = (\omega - \rho)^2,$$

or

$$\text{IX.} \dots 2S\omega\rho'(\omega - \rho) = t(u\rho')^2,$$

in which it may be noted that $u\rho'$ is a vector (in the direction of the chord, PQ), although u itself is generally a quaternion, as before: such then is the equation of the bisecting plane, with ω for its variable vector, and its limit,

$$\text{X.} \dots S\rho'(\omega - \rho) = 0, \quad \text{as before.}$$

(6.) The last process may also be presented under the form,

$$\text{XI.} \dots 0 = \lim. t^{-1}\{T(\omega - \rho) - T(\omega - \rho_0)\} = D_t T(\omega - \rho), \quad \text{when } t = 0;$$

and thus the equation 369, VIII. may be obtained anew.

(7.) Geometrically, if we set off on RQ a portion RS equal in length to RP, as in the annexed Figure 76, we shall have the limiting equation,

$$\text{XII.} \dots \pm \overline{SQ} : \overline{PQ} = (\overline{RQ} - \overline{RP}) : \overline{PQ} = (\text{ultimately}) - \cos RPT;$$

which agrees with 369, IX.

(8.) If then the point R be taken out of the normal plane at P, this limit of the quotient, $\overline{RQ} - \overline{RP}$ divided by \overline{PQ} , has a finite value, positive or negative; and if the chord PQ be called small of the first order, the difference of distances of its extremities from R may then be said to be small of the same (first) order. But if R be taken in the normal plane at P (and not coincident with that point P itself), this difference of dis-

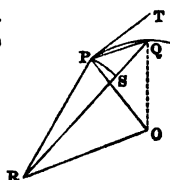


Fig. 76.

tances may then be said to be *small*, of an order higher than the first: which answers to the evanescence of the first differential of the tensor, $T(\omega - \rho)$ in XI., or $T(v - \rho)$ in 369, VIII'.

371. A *curve* may occasionally be represented in quaternions, by an equation which is not of the form, 369, I., although it must always be conceived capable of *reduction* to that form: for instance, this new equation,

$$\text{I. . . } \nabla \rho \cdot \nabla \rho a' = (\nabla a a')^2, \text{ with } TV a a' > 0,$$

is not *immediately* of the form $\rho = \phi t$, but it is reducible to that form as follows,

$$\text{II. . . } \rho = ta + t^{-1}a'.$$

An equation such as I. may therefore have its *differential* or its *derivative* taken, with respect to the scalar variable t on which ρ is thus *conceived* to depend, even if the exact *law* of such dependence be unknown: and $d\rho$, or ρ' , may then be changed to the tangential vector $\omega - \rho$ to which it is parallel, in order to form an *equation* of the *tangent*, or a *condition* which the vector ω of a point on that sought line must satisfy.

(1.) To pass from I. to II., we may first operate with the sign ∇ , which gives,

$$\text{III. . . } \rho S a a' \rho = 0, \text{ or simply, III'. . . } S a a' \rho = 0;$$

whence, t and t' being scalars, we may write,

$$\text{IV. . . } \rho = ta + t'a', \quad \nabla \rho = t' \nabla a a', \quad \nabla \rho a' = t \nabla a a', \quad t t' = 1,$$

and the required reduction is effected: while the *return* from II. to I., or the *elimination* of the scalar t , is an even easier operation.

(2.) Under the form II., it is at once seen that ρ is the vector of a *plane hyperbola*, with the origin for *centre*, and the lines a, a' for *asymptotes*; and accordingly all the properties of such a curve may be deduced from the expression II., by the rules of the present Calculus.

(3.) For example, since the derivative of that expression is,

$$\text{V. . . } \rho' = a - t^2 a',$$

the tangent may (comp. 369, VII.) have its equation thus written:

$$\text{VI. . . } \omega = (t + x)a + t^{-2}(t - x)a';$$

it intersects therefore the lines a, a' in the points of which the vectors are $2ta, 2t^{-1}a'$; so that (as is well known) the *intercept*, upon the tangent, between the *asymptotes*, is *bisected* at the point of *contact*: and the intercepted *area* is *constant*, because $V(ta \cdot t^{-1}a') = V a a'$, &c.

(4.) But we may also operate *immediately*, as above remarked, on the form I.; and thus arrive (by substitution of $\omega - \rho$ for $d\rho$, &c.) at the *equation of conjugation*,

$$\text{VII. . . } \nabla a \omega \cdot \nabla \rho a' + \nabla \rho \cdot \nabla \omega a' = 2(V a a')^2,$$

which expresses (comp. 215, (18.), &c.) that if $\rho = \text{OR}$, and $\omega = \text{OZ}$, as before, then either κ is on the *tangent* to the curve, at the point P , or at least *each* of these two points is situated on the *polar* of the other, with respect to the same hyperbola.

(5.) Again, it is frequently convenient to consider a *curve* as the *intersection* of *two surfaces*; and, in connexion with *this* conception, to represent it by a system of *two scalar equations*, not explicitly involving any *scalar variable*: in which case, *both* equations are to be differentiated, or derivated, with reference to *such* a variable understood, and $d\rho$ or ρ' deduced, or replaced by $\omega - \rho$ as before.

(6.) Thus we may substitute, for the equation I., the system of the two following (whereof the first had occurred as III.);

$$\text{VIII.} \dots Saa'\rho = 0, \quad \rho^2 Saa' - S\rho Sa'\rho = (Vaa')^2;$$

and the derivated equations corresponding are,

$$\text{IX.} \dots Saa'\rho' = 0, \quad 2Saa'S\rho\rho' - S\rho'Sa'\rho - S\rho Sa'\rho' = 0;$$

or, with the substitution of $\omega - \rho$ for ρ' , &c.,

$$\text{X.} \dots Saa'\omega = 0, \quad 2Saa'S\rho\omega - Sa\omega Sa'\rho - S\rho Sa'\omega = 2(Vaa')^2;$$

the last of which might also have been deduced from VII., by operating with S.

(7.) And it may be remarked that the two equations VIII. represent respectively in general a *plane* and an *hyperboloid*, of which the *intersection* (5.) is the *hyperbola* I. or II.; or a *plane* and an *hyperbolic cylinder*, if $Saa' = 0$.

SECTION 3.—On Normals and Tangent Planes to Surfaces.

372. It was early shown (100, (9.)), that when a *curved surface* is represented by an equation of the form,

$$\text{I.} \dots \rho = \phi(x, y),$$

in which ϕ is a functional sign, and x, y are two independent and scalar variables, then either the *two partial differentials*, or the *two partial derivatives*, of the *first order*,

$$\text{II.} \dots d_x\rho, d_y\rho, \quad \text{or} \quad \text{III.} \dots D_x\rho, D_y\rho,$$

represent *two tangential vectors*, or at least vectors *parallel* to *two tangents* to the surface, drawn at the extremity or *term* P of ρ ; so that the *plane* of these two differential vectors, or of lines parallel to them, is (or is parallel to) the *tangent plane* at that point: and the principle has been since exemplified, in 100, (11.) and (12.), and in the sub-articles to 345, &c. It follows that any *vector* ν , which is *perpendicular* to *both* of two such *non-parallel* differentials, or derivatives, must (comp. 345, (11.)) be a *normal vector* at P , or at least one having the *direction* of the normal to the surface at that point; so that each of the two vectors,

$$\text{IV.} \dots V.d_x\rho d_y\rho, \quad \text{V.} \dots V.D_x\rho D_y\rho,$$

if *actual*, represents such a normal.

(1.) As an additional example, let us take the case of the *ruled paraboloid*, on which a given *gauche quadrilateral* ABCD is *superscribed*. The expression for the vector ρ of a variable point P of this surface, considered as a function of two independent and scalar variables, x and y , may be thus written (comp. 99, (9.)):

$$\text{VI.} \dots \rho = xya + (1-x)y\beta + (1-x)(1-y)\gamma + x(1-y)\delta;$$

where the supposition $y = 1$ places the point P on the line AB; $x = 0$ places it on BC; $y = 0$, on CD; and $x = 1$, on DA.

(2.) We have here, by partial derivations,

$$\text{VII.} \dots D_x\rho = y(a - \beta) + (1-y)(\delta - \gamma); \quad D_y\rho = x(a - \delta) + (1-x)(\beta - \gamma);$$

these then represent the directions of *two distinct tangents* to the paraboloid VI, at what may be called *the point* (x, y) ; whence it is easy to deduce the *tangent plane* and the *normal* at that point, by constructions on which we cannot here delay, except to remark that if (comp. Fig. 81, Art. 98) we draw two right lines, QS and PR, through P, so as to cut the sides AB, BC, CD, DA of the quadrilateral in points Q, R, S, T, we shall have by VI. the vectors,

$$\text{VIII.} \dots \begin{cases} \text{OQ} = xa + (1-x)\beta, & \text{OR} = y\beta + (1-y)\gamma, \\ \text{OS} = x\delta + (1-x)\gamma, & \text{OT} = ya + (1-y)\delta, \end{cases}$$

and therefore, by VII.,

$$\text{IX.} \dots D_x\rho = \text{RT}, \quad D_y\rho = \text{SQ};$$

so that *these two tangents* are simply the *two generating lines* of the surface, which pass through the proposed point P.

(3.) For example, at the point $(1, 1)$, or A, the *tangents* thus found are the *sides* BA, DA, and the *tangent plane* is that of the *angle* BAD, as indeed is evident from geometry.

(4.) Again, the equation of the *screw surface* (comp. 814, XVI.),

$$\text{X.} \dots \rho = cxa + ya^2\beta, \quad \text{with } Ta = 1, \quad \text{and } Sa\beta = 0,$$

gives the two tangents,

$$\text{XI.} \dots D_x\rho = ca + \frac{\pi}{2}ya^{2+1}\beta, \quad D_y\rho = a^2\beta,$$

whereof the latter is perpendicular to the former, and to the axis a of the cylinder; so that the corresponding *normal* to the surface X. at the point (x, y) is represented by the product,

$$\text{XII.} \dots v = D_x\rho \cdot D_y\rho = ca^{2+1}\beta + \frac{\pi}{2}y\beta^2a.$$

373. Whenever a variable vector ρ is thus expressed or even *conceived* to be expressed, as a *function* of *two* scalar variables, x and y (or s and t , &c.), if we assume any *three* diplanar vectors, such as a, β, γ (or ι, κ, λ , &c.), the *three scalar expressions*, $Sa\rho, S\beta\rho, S\gamma\rho$ (or $S\iota\rho, S\kappa\rho, S\lambda\rho$, &c.) will then be functions of the same *two* scalar variables; and will therefore be *connected* with each other by some *one scalar equation*, of the form,

$$\text{I.} \dots F(Sa\rho, S\beta\rho, S\gamma\rho) = 0,$$

or briefly,

$$\text{II. . . } f\rho = C;$$

where C is a scalar constant, introduced (instead of zero) for greater generality of expression; and F, f are used as functional but scalar signs. If then (comp. 361, XIV.) we express the *first differential* of this scalar function $f\rho$ under the form,

$$\text{III. . . } d f\rho = 2S\nu d\rho,$$

in which ν is a certain *derived vector*, and is *here* considered as being (at least implicitly) a *vector function* (like ρ) of the *two scalar variables* above mentioned, we shall have the two equations,

$$\text{IV. . . } S\nu d_x\rho = 0, \quad S\nu d_y\rho = 0,$$

or these two other and corresponding ones,

$$\text{V. . . } S\nu D_x\rho = 0, \quad S\nu D_y\rho = 0;$$

from which it follows (by 372) that ν has the direction of the normal to the surface I. or II., at the point P in which the vector ρ terminates. Hence the *equation* of that *normal* (with ω for its variable vector) may, under these conditions, be thus written:

$$\text{VI. . . } V\nu(\omega - \rho) = 0;$$

and the corresponding *equation of the tangent planes* at the same point P is,

$$\text{VII. . . } S\nu(\omega - \rho) = 0.$$

(1.) For example, if we take the expression 808, XVIII., or 845, XII., namely

$$\text{VIII. . . } \rho = r k^t j^k i^s k^t, \quad \text{in which } k j^s = j^s k, \text{ \&c.,}$$

treating the scalar r as constant, but s and t as variable, we have then (comp. 845, XIV.), the equations, α denoting any unit-vector,

$$\text{IX. . . } S i\rho = r S . \alpha^{2s} S . \alpha^{2t+1}, \quad S j\rho = r S . \alpha^{2t-1} S . \alpha^{2s+1}, \quad S k\rho = r S . \alpha^{2s+2};$$

between which s and t can be eliminated, by simply adding their squares, because $(\alpha^s)^2 + (\alpha^{t-1})^2 = 1$, by 315, V., if $T\alpha = 1$. In this manner then we arrive at equations of the forms I. and II., namely (comp. 357, VII., and 808, (10.) and (18.)),

$$\text{X. . . } (S i\rho)^2 + (S j\rho)^2 + (S k\rho)^2 - r^2 = 0,$$

and

$$\text{XI. . . } f\rho = \rho^2 = -r^2 = \text{const.}, \quad \text{or } \text{XI'. . . } T\rho = r;$$

which last results had indeed been otherwise obtained before.

(2.) With this *form* XI. of $f\rho$, we have the *differential expression* of the first order,

$$\text{XII. . . } d f\rho = 2S\nu d\rho = 2S\rho d\rho, \quad \text{whence } \text{XIII. . . } \nu = \rho;$$

and if we *still conceive* that ρ is, as above, *some vector function* of *two scalar variables*, s and t , although the *particular law* VIII. of its dependence on them may now be supposed to be *unknown* (or to be forgotten), we may write also,

$$\text{XIV. . . } \frac{1}{2} d f\rho = S\nu d\rho = S\rho d\rho = S\rho(d_s + d_t)\rho = S\rho D_s\rho . ds + S\rho D_t\rho . dt;$$

if then the *function* $f\rho$ have (as above) a *value*, $= -r^2$, which is *constant*, or is *inde-*

pendent of both the variables, s and t , while their differentials are arbitrary, and are independent of each other, we shall thus have separately (comp. V., and 387, XIII., XVII.),

$$\text{XV.} \dots S\rho D_s\rho = 0, \quad S\rho D_t\rho = 0.$$

The radius ρ of the sphere XI. is therefore in this way seen to have the direction of the normal at its own extremity, because it is perpendicular to two distinct tangents, $D_s\rho$ and $D_t\rho$, at that point; which are indeed, in the present case, perpendicular to each other also (387, (8.)).

(8.) Instead of treating the two scalar variables, x and y , or s and t , &c., as both entirely arbitrary and independent, we may conceive that one is an arbitrary (but scalar) function of the other; and then the vector ν , determined by the equation III., will be seen anew to be the normal at the extremity P of ρ , because it is perpendicular to the tangent at P to an arbitrary curve upon the surface, which passes through that point: or (otherwise stated) because it is a line in an arbitrary normal plane at P , if a normal plane to a curve on a surface be called (as usual) a normal plane to that surface also.

(4.) For example, if we conceive that s in VIII. is thus an arbitrary function of t , the last expression XIV. will take the form,

$$\text{XVI.} \dots 0 = \frac{1}{2} d^2 f\rho = S. \rho (s' D_s \rho + D_t \rho) dt, \quad \text{if } ds = s' dt;$$

whence, dt being still arbitrary, we have the one scalar equation,

$$\text{XVII.} \dots S. \rho (s' D_s \rho + D_t \rho) = 0, \quad \text{or} \quad \text{XVIII.} \dots \rho \perp s' D_s \rho + D_t \rho,$$

and although, on account of the arbitrary coefficient s' , this one equation XVII. is equivalent to the system of the two equations XV., yet it immediately signifies, as in XVIII., that the directed radius ρ , of the sphere XI., is perpendicular to the arbitrary tangent, $s' D_s \rho + D_t \rho$; or to the tangent to an arbitrary spherical curve through P , the centre O and tensor $T\rho$ (or undirected radius, r) remaining as before.

(5.) As regards the logic of the subject, it may be worth while to read again the proof (381), of the validity of the rule for differentiating a function of a function; because this rule is virtually employed, when after thus reducing, or conceiving as reduced, the scalar function $f\rho$ of a vector ρ , to another scalar function such as Ft of a scalar t , by treating ρ as equal to some vector function ϕt of this last scalar, we infer that

$$\text{XIX.} \dots dFt = df\phi t = 2S. \nu d\phi t, \quad \text{if } d\rho = 2S\nu d\rho, \text{ as before.}$$

(6.) And as regards the applications of the formulæ VI. and VII., or of the equations given by them for the normal and tangent plane to a surface generally, the difficulty is only to select, out of a multitude of examples which might be given: yet it may not be useless to add a few such here, the case of the sphere having of course been only taken to illustrate the theory, because the normal property of its radii was manifest, independently of any calculation.

(7.) Taking then the equation of the ellipsoid, under the form,

$$\text{XX.} \dots T(\rho + \rho\kappa) = \kappa^2 - t^2, \quad 282, \text{ XIX.},$$

of which the first differential may (see the sub-articles to 386) be thus written,

$$\text{XXI.} \dots 0 = S. \{(\kappa - \kappa)\rho + 2(\iota S\kappa\rho + \kappa S\iota\rho)\} d\rho = S\nu d\rho,$$

and introducing an auxiliary vector, $o\mathbf{N}$ or ξ , such that

$$\text{XXII.} \dots o\mathbf{N} = \xi = -2(\kappa - \kappa)^{-1}(\iota S\kappa\rho + \kappa S\iota\rho),$$

we have $\nu \parallel \rho - \xi$, and may write, as the equation of the normal at the extremity ρ of ρ , the following,

$$\text{XXIII.} \dots V. (\xi - \rho) (\omega - \rho) = 0, \text{ or XXIV.} \dots \omega = \rho + x(\xi - \rho),$$

in which x is a scalar variable (comp. 869, VII.); making then $x = 1$, we see that ξ is the vector of the point π in which the normal intersects the plane of the two fixed lines ι, κ , supposed to be drawn from the origin, which is here the centre of the ellipsoid.

(8.) If we look back on the sub-articles to 216 and 217, we shall see that these lines ι, κ have the directions of the *two real cyclic normals*, or of the normals to the *two (real) cyclic planes*; which planes are now represented by the two equations,

$$\text{XXV.} \dots S\iota\rho = 0, \quad S\kappa\rho = 0.$$

Accordingly the equation XX. of the ellipsoid may be put (comp. 836, 857, 859) under the *cyclic forms*,

$$\begin{aligned} \text{XXVI.} \dots S\rho\phi\rho &= (\iota^2 + \kappa^2)\rho^2 + 2S\iota\rho\kappa\rho \\ &= (\iota - \kappa)^2\rho^2 + 4S\iota\rho S\kappa\rho = (\kappa^2 - \iota^2)^2 = \text{const.}; \end{aligned}$$

hence each of the *two diametral planes* XXV. cuts the surface in a *circle*, the *common radius* of these two *circular sections* being

$$\text{XXVII.} \dots T\rho = \frac{T\iota^2 - T\kappa^2}{T(\iota - \kappa)} = b,$$

where b denotes, as in 219, (1.), the length of the *mean semiaxis* of the ellipsoid; and in fact, this value of $T\rho$ can be at once obtained from the equation XX., by making either $\iota\rho = -\rho\iota$, or $\rho\kappa = -\kappa\rho$, in virtue of XXV.

(9.) By the sub-article last cited, the *greatest* and *least semiaxes* have for their *lengths*,

$$\text{XXVIII.} \dots a = T\iota + T\kappa, \quad c = T\iota - T\kappa;$$

and the *construction* in 219, (2.) shows (by Fig. 53, annexed to 217, (4.)) that these three semiaxes a, b, c have the respective *directions* of the lines,

$$\text{XXIX.} \dots \iota T\kappa - \kappa T\iota, \quad V\iota\kappa, \quad \iota T\kappa + \kappa T\iota;$$

all which agrees with the *rectangular transformation*,

$$\begin{aligned} \text{XXX.} \dots 1 &= \frac{S\rho\phi\rho}{(\kappa^2 - \iota^2)^2} = \left(\frac{T(\iota\rho + \rho\kappa)}{\kappa^2 - \iota^2} \right)^2 \\ &= \left(\frac{S. \rho U(\iota T\kappa - \kappa T\iota)}{T\iota + T\kappa} \right)^2 + \left(\frac{T(\iota - \kappa) S. \rho UV\iota\kappa}{T\iota^2 - T\kappa^2} \right)^2 + \left(\frac{S. \rho U(\iota T\kappa + \kappa T\iota)}{T\iota - T\kappa} \right)^2, \end{aligned}$$

in deducing which (comp. 859, (1.)) from 857, VIII., by means of the formulæ 357, XX. and XXI., we employ the values (comp. XXVI.),

$$\text{XXXL.} \dots g = \iota^2 + \kappa^2, \quad \lambda = 2\iota, \quad \mu = \kappa.$$

(10.) The *fixed plane* (7.), of the *cyclic normals* ι and κ (8.), is therefore also the plane of the *extreme semiaxes*, a and c (9.), or that which may be called perhaps the *principal plane** of the ellipsoid: namely, the plane of the *generating tri-*

* This plane may also be said to be the plane of the *principal elliptic section* (219, (9.)); or it may be distinguished (comp. the Note to page 281) as the plane of the *focal hyperbola*, of which important curve we shall soon assign the equation in quaternions.

angle (218), (1.)), in that construction of the surface (217, (6.) or (7.)) which is illustrated by Fig. 53, and was deduced as an interpretation of the quaternion equation XX., or of the somewhat less simple form 217, XVI., with the value $T\epsilon^2 - T\kappa^2$ of ϵ^2 .

(11.) Let π denote the length of that portion of the normal, which is intercepted between the surface and the principal plane (10.), so that, by (7.),

$$\text{XXXII.} \dots \pi = \overline{NF} = T(\rho - \xi), \quad \pi^2 = -(\rho - \xi)^2,$$

with the value XXII. of ξ . Let $\sigma = os$ be the vector of a point s on the surface of a new or auxiliary sphere, described about the point π as centre, with a radius = π , and therefore tangential to the ellipsoid at P ; and let us inquire in what curve or curves, real or imaginary, does this sphere cut the ellipsoid.

(12.) The equations (comp. 371, (5.)) of the sought intersection are the two following,

$$\text{XXXIII.} \dots (\sigma - \xi)^2 + \pi^2 = 0, \quad \text{and} \quad \text{XXXIV.} \dots T(\iota\sigma + \sigma\kappa) = \kappa^2 - \iota^2;$$

whereof the first expresses that s is a point of the sphere, and the second that it is a point of the ellipsoid; while ρ or oP enters virtually into XXXIII., through ξ and π , but is here treated as a constant, the point P being now supposed to be a given one.

(13.) We shall remove (18) the origin to this point P of the ellipsoid, if we write,

$$\text{XXXV.} \dots \sigma = \rho + \sigma', \quad \text{or} \quad \text{XXXV'.} \dots \sigma' = \sigma - \rho = P\sigma;$$

and thus we obtain the new or transformed equations,

$$\text{XXXVI.} \dots 0 = \sigma'^2 + 2S(\rho - \xi)\sigma', \quad \text{XXXVII.} \dots 0 = N(\iota\sigma' + \sigma'\kappa) + 2S\nu\sigma';$$

in which (as in (7.), comp. also 210, XX.),

$$\text{XXXVIII.} \dots \nu = (\iota - \kappa)^2 \rho + 2(\iota S\kappa\rho + \kappa S\rho) = (\iota - \kappa)^2 (\rho - \xi),$$

and

$$\text{XXXIX.} \dots N(\iota\sigma' + \sigma'\kappa) = (\iota - \kappa)^2 \sigma'^2 + 4S\iota\sigma'S\kappa\sigma'.$$

(14.) Eliminating then σ'^2 , we obtain from the two equations XXXVI. and XXXVII. this other,

$$\text{XL.} \dots S\sigma'S\kappa\sigma' = 0;$$

which like them is of the second degree in σ' , but breaks up, as we see, into two linear and scalar factors, representing two distinct planes, parallel by XXV. to the two diametral and cyclic planes of the ellipsoid. The sought intersection consists then of a pair of (real) circles, upon that given surface; namely, two circular (but not diametral) sections, which pass through the given point P .

(15.) Conversely, because the equations XXXVII. XXXVIII. XXXIX. XL. give XXXVI. and XXXIII., with the foregoing values of ξ and π , it follows that these two plane sections of the ellipsoid at P are on one common sphere, namely that which has π for centre, and π for radius, as above; and thus we might have found, without differentials, that the line $P\pi$ is the normal at P ; or that this normal crosses the principal plane (10.), in the point determined by the formula XXII.

(16.) In general, the cyclic form of the equation of any central surface of the second order, namely the form (comp. 357, II.),

$$\text{XLI.} \dots S\rho\phi\rho = g'\rho^2 + 2S\lambda\rho S\mu\rho = C = \text{const.},$$

shows that the two circles (real or imaginary) in which that surface is cut by any two planes,

$$\text{XLII.} \dots S\lambda\rho = l, \quad S\mu\rho = m,$$

drawn *parallel* respectively to the two real *cyclic planes*, which are jointly represented (comp. XL., and 216, (7.)) by the one equation,

$$\text{XLIII.} \dots S\lambda\rho S\mu\rho = 0,$$

are *homospherical*, being *both* on that one sphere of which the equation is,

$$\text{XLIV.} \dots g'\rho^2 + 2(S\mu\rho + mS\lambda\rho) = 2lm + C.$$

(17.) But the *centre* (say \mathfrak{N}) of this *new sphere*, has for its vector (say ξ),

$$\text{XLV.} \dots o\mathfrak{N} = \xi = -g'^{-1}(l\mu + m\lambda);$$

it is therefore situated in the *plane* of the two real *cyclic normals*, λ and μ ; and if l and m in XLV. receive the values XLII., then this new ξ is the *vector of intersection* of that *plane*, with the *normal to the surface* at \mathfrak{P} : because it is (comp. 15.)) the vector of the centre of a sphere which *touches* (though also *cutting*, in the two circular sections) the surface at that point.

(18.) We can therefore thus *infer* (comp. again (15.)), *without the differential calculus*, that the line,

$$\text{XLVI.} \dots g'(\rho - \xi) = g'\rho + \lambda S\mu\rho + \mu S\lambda\rho = \phi\rho,$$

as having the direction of $\mathfrak{N}\mathfrak{P}$, is the *normal* at \mathfrak{P} to the surface XLI.; which agrees with, and may be considered as confirming (if confirmation were required), the conclusion otherwise obtained through the differential expression (861),

$$\text{XLVII.} \dots dS\rho\phi\rho = 2S\nu d\rho = 2S\phi\rho d\rho;$$

the linear function $\phi\rho$ being here supposed (comp. 861, (8.)) to be self-conjugate.

(19.) Hence, with the notation 862, I., the *equation of the tangent plane* to a central surface of the second order, at the same point \mathfrak{P} , may by VII. be thus written,

$$\text{XLVIII.} \dots f(\omega, \rho) = C, \quad \text{if } S\rho\phi\rho = C = \text{const.};$$

in which it is to be remembered, that

$$\text{XLIX.} \dots f(\omega, \rho) = f(\rho, \omega) = S\omega\phi\rho = S\rho\phi\omega.$$

(20.) And if we choose to *interpret* this equation XLVIII., which is only of the *first degree* (862) with respect to *each* separately of the two vectors, ρ and ω , or $o\mathfrak{P}$ and $o\mathfrak{N}$, and involves them *symmetrically*, without requiring that \mathfrak{P} shall be a point on the surface, we may then say (comp. 215, (18.), and 316, (31.)), that the formula in question is an *equation of conjugation*, which expresses that *each* of the two points \mathfrak{P} and \mathfrak{N} , is situated in the *polar plane* of the other.

(21.) In general, if we suppose that the *length* and *direction* of a line ν are so adjusted as to satisfy the two equations (comp. 336, XII. XIII. XIV.),

$$\text{L.} \dots S\nu\rho = 1, \quad S\nu d\rho = 0, \quad \text{and therefore also} \quad \text{LI.} \dots S\rho d\nu = 0;$$

then, because the equation VII. of the *tangent plane* to any curved surface may now be thus written,

$$\text{LII.} \dots S\nu(\omega - \nu^{-1}) = 0,$$

it follows that ν^{-1} represents, in length and direction, the *perpendicular* from o on that *tangent plane* at \mathfrak{P} ; so that ν itself represents the *reciprocal* of that *perpendicular*, or what may be called (comp. 336, (8.)) the *vector of proximity*, of the tangent plane to the origin. And we see, by LI., that the two vectors, ρ and ν , if drawn from a *common origin*, terminate on two surfaces which are, in a known and

important sense (comp. the sub-arta. to 861), *reciprocals** of one another: the line ρ^{-1} , for instance, being the perpendicular from o on the tangent plane to the *second* surface, at the extremity of the vector ν .

374. In the two preceding Articles, we have treated the symbol $d\rho$ as representing (rigorously) a *tangent to a curve on a given surface*, and therefore also to that surface *itself*; and thus the formula $S\nu d\rho = 0$ has been considered as expressing that ν has the direction of the *normal* to that *surface*, because it is *perpendicular to two tangents* (372), and therefore generally to *every tangent* (373), which can be drawn at a given point p . But without at present introducing any other† *signification* for this *symbol* $d\rho$, we may *interpret* in another way, and with a reference to *chords* rather than to *curves*, the *differential equation*,

$$I. \dots d f \rho = 2 S \nu d \rho,$$

supposed still to be a *rigorous* one (in virtue of our *definitions* of differentials, which do not require that $d\rho$ should be *small*); and may still deduce from it the *normal property* of the vector ν , but now with the help of *Taylor's Series* adapted to quaternions (comp. 342, 370). In fact, that series gives here a *differenced equation*, of the form,

$$II. \dots \Delta f \rho = 2 S \nu \Delta \rho + R;$$

where R is a scalar *remainder* (comp. again 342), having the property that

$$III. \dots \lim. (R : T \Delta \rho) = 0, \quad \text{if} \quad \lim. T \Delta \rho = 0;$$

whence $IV. \dots \lim. (\Delta f \rho : T \Delta \rho) = 2 \lim. S \nu \cup \Delta \rho,$

whatever the ultimate direction of $\Delta \rho$ may be. If then we conceive that

* Compare the Note to page 484.

† It is *permitted*, for example, by general principles above explained, to treat the *differential* $d\rho$ as denoting a *chordal vector*, or to substitute it for $\Delta \rho$, and so to represent the *differenced equation* of the surface under the form (comp. 342),

$$0 = \Delta f \rho = (\epsilon^d - 1) f \rho = d f \rho + \frac{1}{2} d^2 f \rho + \&c.;$$

but with this meaning of the symbol $d\rho$, the equation $d f \rho = 0$, or $S \nu d \rho = 0$, is no longer rigorous, and must (for rigour) be replaced by such an equation as the following,

$$0 = 2 S \nu d \rho + S d \nu d \rho + R, \quad \text{if} \quad d f \rho = 2 S \nu d \rho, \quad \text{as before};$$

the remainder R vanishing, when the surface is only of the *second order* (comp. 862, (8.)). Accordingly this last form is *useful* in some investigations, especially in those which relate to the *curvatures of normal sections*: but for the present it seems to be clearer to *adhere* to the recent signification of $d\rho$, and therefore to treat it as still denoting a *tangent*, which may or may not be *small*.

$\Delta\rho$ represents a *small* and *indefinitely decreasing chord* PQ of the *surface*, drawn from the extremity P of ρ , so that

$$V. \dots \Delta f\rho = f(\rho + \Delta\rho) - f\rho = 0, \quad \text{and} \quad \lim. T\Delta\rho = 0,$$

the equation IV. becomes simply,

$$VI. \dots \lim. S\nu U\Delta\rho = 0;$$

and thus proves, in a *new way*, that ν is *normal to the surface at the proposed point P*, by proving that it is *ultimately perpendicular to all the chords PQ from that point*, when those chords become *indefinitely small*, or *tend indefinitely to vanish*.

(1.) For example, if

$$VII. \dots f\rho = \rho^2, \quad \nu = \rho, \quad \text{then} \quad VIII. \dots R = \Delta\rho^2, \quad \text{and} \quad R : T\Delta\rho = -T\Delta\rho;$$

thus, for *every point of space*, we have *rigorously*, with this form of $f\rho$,

$$IX. \dots \Delta f\rho : T\Delta\rho = 2S\rho U\Delta\rho - T\Delta\rho;$$

and for *every point Q of the spheric surface*, $f\rho = \text{const.}$, we have with *equal rigour*,

$$X. \dots 2S\rho U\Delta\rho = T\Delta\rho, \quad \text{or} \quad XI. \dots \overline{PQ} = 2\overline{OP} \cdot \cos \text{OPQ};$$

in fact, either of these two last formulæ expresses simply, that the *projection of a diameter of a sphere, on a conterminous chord*, is *equal to that chord itself*, and of course *diminishes with it*.

(2.) Passing then to the *limit*, or conceiving the point Q of the surface to *approach* indefinitely to P , we derive the limiting equations,

$$XII. \dots \lim. S\rho U\Delta\rho = 0; \quad XIII. \dots \lim. \cos \text{OPQ} = 0;$$

either of which shows, in a *new way*, that the *radii of a sphere are its normals*; with the *analogous result for other surfaces*, that the *vector ν in I. has a normal direction*, as before: because its *projection on a chord PQ tends indefinitely to diminish with that chord*.

(3.) We may also interpret the differential equation I. as expressing, through II. and III., that the *plane 873, VII.*, which is drawn through the point P in a direction perpendicular to ν , is the *tangent plane* to the surface: because the *projection of the chord $\Delta\rho$ on the normal ν to that plane, or the perpendicular distance*,

$$XIV. \dots -S(U\nu \cdot \Delta\rho) = \frac{1}{2}R \cdot T\nu^{-1},$$

of a *near point Q from the plane* thus drawn through P , is *small of an order higher than the first* (comp. 870, (8.)), if the *chord PQ itself* be considered as *small of the first order*.

375. This occasion may be taken (comp. 374, I. II. III.), to give a *new Enunciation of Taylor's Theorem*, in a form *adapted to Quaternions*, which has some advantages over that given (342) in the preceding Chapter. We shall therefore now express that important *Theorem* as follows:—

“If none of the $m + 1$ functions,

I. . . $f q, d f q, d^2 f q, \dots d^m f q$, in which $d^2 q = 0$,
become infinite in the immediate vicinity of a given quaternion q , then the quotient,

$$\text{II. . . } Q = \left\{ f(q + dq) - f q - d f q - \frac{d^2 f q}{2} - \frac{d^3 f q}{2 \cdot 3} - \&c. \right. \\ \left. - \frac{d^m f q}{2 \cdot 3 \dots m} \right\} : \frac{d q^m}{2 \cdot 3 \dots m},$$

can be made to tend indefinitely to zero, for any ultimate value of the versor $U d q$, by indefinitely diminishing the tensor $T d q$."

(1.) The *proof* of the theorem, as thus enunciated, can easily be supplied by an attentive reader of Articles 841, 842, and their sub-articles; a few *hints* may however here be given.

(2.) We do not now suppose, as in 842, that $d^m f q$ must be *different from zero*; we only assume that it is *not infinite*: and we add, to the expression 842, VI. for $F x$, the term,

$$\text{III. . . } \frac{-x^m d^m f q}{2 \cdot 3 \dots m}$$

(3.) Hence *each* of the expressions 842, VII., for the successive *derivatives* of $F x$, receives an *additional term*; the *last* of them thus becoming,

$$\text{IV. . . } D^m F x = F^{(m)} x = d^m f(q + x dx) - d^m f q;$$

so that we have now (comp. 842, X.) the values

$$\text{V. . . } F 0 = 0, \quad F' 0 = 0, \quad F'' 0 = 0, \dots F^{(m-1)} 0 = 0, \quad F^{(m)} 0 = 0.$$

(4.) Assuming therefore now (comp. 842, XII.) the new auxiliary function,

$$\text{VI. . . } \psi x = \frac{x^m d q^m}{2 \cdot 3 \dots m}, \quad \text{with } T d q > 0,$$

which gives,

$$\text{VII. . . } \psi 0 = 0, \quad \psi' 0 = 0, \quad \psi'' 0 = 0, \dots \psi^{(m-1)} 0 = 0, \quad \psi^{(m)} 0 = d q^m,$$

we find (by 841, (8.), (9.), comp. again 842, XII.) that

$$\text{VIII. . . } \lim_{x \rightarrow 0} (F x : \psi x) = 0.$$

(5.) But these two new functions, $F x$ and ψx , are formed from the dividend and the divisor of the quotient Q in II., by changing $d q$ to $x d q$; and (comp. 842, (3.)) instead of thus *multiplying a given quaternion differential $d q$, by a small and indefinitely decreasing scalar, x , we may indefinitely diminish the tensor, $T d q$, without changing the versor, $U d q$.*

(6.) And *even if $U d q$ be changed*, while the differential $d q$ is thus made to *tend to zero*, we can always conceive that it *tends to some limit*; which *limiting* or *ultimate value* of that versor $U d q$ may then be treated as if it were a *constant one*, without affecting the *limit* of the quotient Q .

(7.) The *theorem*, as above enunciated, is therefore fully proved; and we are at liberty to *choose*, in any application, between the two forms of statement, 842 and 875, of which one is more convenient at one time, and the other at another.

SECTION 4.— *On Osculating Planes, and Absolute Normals, to Curves of Double Curvature.*

376. The variable vector ρ , of a curve in space may in general be thus expressed, with the help of Taylor's Series (comp. 370, (1.)):

$$I. \dots \rho_t = \rho + t\rho' + \frac{1}{2}t^2u\rho'', \quad \text{with } u_0 = 1;$$

ρ, ρ', ρ'', u being here abridged symbols for $\rho_0, \rho'_0, \rho''_0, u_0$; and the product $u\rho''$ being a vector, although the factor u is generally a quaternion (comp. 370, (5.)). And the different terms of this expression I. may be thus constructed (compare the annexed Figure 77):

$$II. \dots \rho = OP; \quad t\rho' = PT; \quad \frac{1}{2}t^2u\rho'' = TQ;$$

$$\text{while } III. \dots \rho_t = OQ, \quad \text{and } t\rho' + \frac{1}{2}t^2u\rho'' = PQ;$$

the line TQ , or the term $\frac{1}{2}t^2u\rho''$, being thus what may be called the *deflection* of the curve PQR , at Q , from its *tangent* PT at P , measured in a *direction* which depends on the *law* according to which ρ_t varies with t , and on the *distance* of Q from P . The *equation of the plane* of the triangle PTQ is *rigorously* (by II.) the following, with ω for its variable vector,

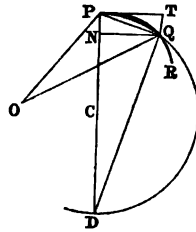


Fig. 77.

$$IV. \dots 0 = \text{Sup}'\rho'(w - \rho);$$

this *plane* IV. then *touches* the curve at P , and (generally) *cuts* it at Q ; so that if the point Q be conceived to *approach* indefinitely to P , the resulting formula,

$$V. \dots 0 = \text{Sp}'\rho'(w - \rho), \quad \text{or} \quad V'. \dots 0 = \text{Sp}'\rho''(w - \rho),$$

is the equation of the plane PTQ in that *limiting position*, in which it is called the *osculating plane*, or is said to *osculate to the curve* PQR , at the point P .

(1.) If the variable vector ρ be immediately given as a function ρ_s of a variable scalar, s , which is itself a function of the former scalar variable t , we shall then have (comp. 331) the expressions,

$$VI. \dots \rho'_t = s'D_s\rho_s, \quad \rho''_t = s''D_s\rho_s + s'^2D_s^2\rho_s, \quad \text{with } s' = D_t s, \quad s'' = D_t^2 s;$$

thus the vector ρ'' may change, even in direction, when we change the independent scalar variable; but ρ'' will always be a line, either *in* or *parallel* to the osculating plane; while ρ' will always represent a *tangent*, whatever scalar variable may be selected.

(2.) As an example, let us take the equation 314, XV., or 369, XIII., of the

helix. With the independent variable t of that equation, we have (comp. 369, XIV.) the derived expressions,

$$\text{VII.} \dots \rho' = c\alpha + \frac{\pi}{2}\alpha^{*+1}\beta, \quad \rho'' = -\left(\frac{\pi}{2}\right)^2\alpha^2\beta = \left(\frac{\pi}{2}\right)^2(c\alpha - \rho);$$

ρ'' has therefore here (comp. 369, (8.)) the direction of the *normal* to the cylinder; and consequently, the *osculating plane* to the *helix* is a *normal plane* to the *cylinder* of revolution, on which that curve is traced: a result well known, and which will soon be greatly extended.

(8.) When a curve of *double curvature* degenerates into a *plane curve*, its *osculating plane* becomes *constant*, and reciprocally. The *condition of planarity* of a *curve in space* may therefore be expressed by the equation,

$$\text{VIII.} \dots UV\rho'\rho'' = \pm \text{a constant unit line};$$

or, by 385, II., and 388, VIII.,

$$\text{IX.} \dots 0 = V \frac{V(\rho'\rho'')}{V\rho'\rho''} = V \frac{V\rho'\rho''}{V\rho'\rho''};$$

or finally,

$$\text{X.} \dots S\rho'\rho''\rho''' = 0, \quad \text{or} \quad \text{XI.} \dots \rho''' \parallel \rho', \rho''.$$

(4.) Accordingly, for a *plane curve*, if λ be a given normal to its plane, we have the three equations,

$$\text{XII.} \dots S\lambda\rho' = 0, \quad S\lambda\rho'' = 0, \quad S\lambda\rho''' = 0;$$

which conduct, by 294, (11.), to X.

(5.) For example, if we had not otherwise known that the equation 387, (2.) represented a *plane ellipse*, we might have perceived that it was the equation of *some plane curve*, because it gives the *three successive derivatives*,

$$\text{XIII.} \dots \rho' = \frac{\pi}{2} V\alpha^{*+1}\beta, \quad \rho'' = -\left(\frac{\pi}{2}\right)^2 V\alpha^2\beta, \quad \rho''' = -\left(\frac{\pi}{2}\right)^3 V\alpha^{*+1}\beta,$$

which are *coplanar lines*, the third having a direction *opposite* to the first.

(6.) And generally, the formula X. enables us to assign, on *any curve of double curvature*, for which ρ is expressed as a function of t , the *points** at which it *most resembles a plane curve*, or *approaches most closely* to its own *osculating plane*.

377. An important and *characteristic property* of the *osculating plane* to a curve of double curvature, is that the *perpendiculars* let fall on it, from points of the curve near to the point of osculation, are *small* of an order *higher* than the *second*, if their *distances* from that *point* be considered as *small* of the *first* order.

(1.) To exhibit this by quaternions, let us begin by considering an *arbitrary plane*,

* Namely, in a modern phraseology, the places of *four-point contact* with a *plane*. The equation, $V\rho'\rho'' = 0$, indicates in like manner the places, if any, at which a curve has *three-point contact* with a *right line*. For curves of double curvature, these are also called points of *simple* and *double inflexion*.

$$I. \dots S\lambda(\omega - \rho) = 0, \text{ with } T\lambda = 1,$$

drawn through a point r of the curve. Using the expression 376, I., for the vector OQ , or ρ_t , of another point Q of the same curve, we have, for the perpendicular distance of Q from the plane I., this other rigorous expression,

$$II. \dots S\lambda(\rho_t - \rho) = tS\lambda\rho' + \frac{1}{2}t^2S\lambda\omega\rho'';$$

which represents, in general, a small quantity of the first order, if t be assumed to be such.

(2.) The expression II. represents however, generally, a small quantity of the second order, if the direction of λ satisfy the condition,

$$III. \dots S\lambda\rho' = 0;$$

that is, if the plane I. touch the curve.

(3.) And if the condition,

$$IV. \dots S\lambda\rho'' = 0,$$

be also satisfied by λ , then, but not otherwise, the expression II. tends to bear an evanescent ratio to t^2 , or is small of an order higher than the second.

(4.) But the combination of the two conditions, III. and IV., conducts to the expression,

$$V. \dots \lambda = \pm UV\rho'\rho'';$$

comparing which with 376, V., we see that the property above stated is one which belongs to the osculating plane, and to no other.

378. Another remarkable property* of the osculating plane to a curve is, that it is the tangent plane to the cone of parallels to tangents (369, (6.)), which has its vertex at the point of osculation.

(1.) In general, if $\rho = \phi x$ be (comp. 369, I.) the equation of a curve in space, the equation of the cone which has its vertex at the origin, and passes through this curve, is of the form,

$$I. \dots \rho = y\phi x;$$

in which x and y are two independent and scalar variables.

(2.) We have thus the two partial derivatives,

$$II. \dots D_x\rho = y\phi'x, \quad D_y\rho = \phi x;$$

and the tangent plane along the side (x) has for equation,

$$III. \dots 0 = S(\omega \cdot \phi x \cdot \phi'x); \text{ or briefly, } III'. \dots 0 = S\omega\phi\phi'.$$

(3.) Changing then x, ϕ, ϕ', ω to $t, \rho', \rho'', \omega - \rho$, we see that the equation 376, V., of the osculating plane to the curve 376, I., is also that of the tangent plane to the cone of parallels, &c., as asserted.

379. Among all the normals to a curve, at any one point, there are two which deserve special attention; namely the one which is in

* The writer does not remember seeing this property in print; but of course it is an easy consequence from the doctrine of infinitesimals, which doctrine however it has not been thought convenient to adopt, as the basis of the present exposition.

the osculating plane, and is called the *absolute* (or *principal*) *normal*; and the one which is *perpendicular* to that plane, and which it has been lately proposed to name the *binormal*.* It is easy to assign expressions, by quaternions, for these two normals, as follows.

(1.) The *absolute normal*, as being perpendicular to ρ' , but complanar with ρ' and ρ'' , has a direction expressed by any one of the following formulæ (comp. 203, 384):

$$\text{I. . . } V\rho''\rho'.\rho'^{-1}; \text{ or II. . . } dU\rho'; \text{ or III. . . } dUd\rho.$$

(2.) There is an extensive class† of cases, for which the following equations hold good:

$$\text{IV. . . } T\rho' = \text{const.}; \quad \text{V. . . } \rho'^2 = \text{const.}; \quad \text{VI. . . } S\rho'\rho'' = 0;$$

and in all *such* cases, the expression I. reduces itself to ρ'' , which is therefore *then* a representative of the absolute normal.

(3.) For example, in the case of the *helix*, with the equation several times before employed, the conditions (2.) are satisfied; and accordingly the absolute normal to that curve coincides with the normal ρ'' to the *cylinder*, on which it is traced: the *locus of the absolute normal* being here that *screw surface* or *Helicoid*, which has been already partially considered (comp. 314, (11.), and 372, (4.)).

(4.) And as regards the *binormal*, it may be sufficient here to remark, that because it is perpendicular to the osculating plane, it has the *direction* expressed by one or other of the two symbols (comp. 377, V.),

$$\text{VII. . . } V\rho'\rho'', \text{ or VII'. . . } Vd\rho d^2\rho.$$

(5.) There exists, of course, a system of *three rectangular planes*, the *osculating plane* being *one*, which are connected with the system of the *three rectangular lines*, the *tangent*, the *absolute normal*, and the *binormal*, and of which any one who has studied the Quaternions so far can easily form the expressions.

(6.) And a *construction*‡ for the *absolute normal* may be assigned, analogous to and including that lately given (378) for the *osculating plane*, as an *interpretation* of the expression II. or III., or of the *symbol* $dU\rho'$ or $dUd\rho$. From any origin o conceive a system of unit lines ($U\rho'$ or $Ud\rho$) to be drawn, in the *directions* of the successive *tangents* to the *given curve* of double curvature; these lines will terminate

* By M. de Saint-Venant, as being perpendicular at once to *two consecutive elements* of the curve, in the infinitesimal treatment of this subject. See page 261 of the very valuable Treatise on *Analytic Geometry of Three Dimensions* (Hodges and Smith, Dublin), by the Rev. George Salmon, D. D., which has been published in the present year (1862), but not till after the printing of these *Elements of Quaternions* (begun in 1860) had been too far advanced, to allow the writer of them to profit by the study of it, so much as he would otherwise have sought to do.

† Namely, those in which the *arc of the curve*, or that arc multiplied by a scalar constant, is taken as the *independent variable*.

‡ This construction also has not been met with by the writer in print, so far as he remembers; but it may easily have escaped his notice, even in the books which he has seen.

on a certain *spherical curve*; and the *tangent*, say ss' , to this *new curve*, at the point s which *corresponds* to the point P of the *old one*, will have the direction of the absolute normal at that old point.

(7.) At the same time, the *plane* oss' of the *great circle*, which *touches* the *new curve* upon the *unit sphere*, being the *tangent plane* to the *cone of parallels* (378), has the direction of the *osculating plane* to the old curve; and the *radius* drawn to its *pole* is parallel to the *binormal*.

(8.) As an *example* of the *auxiliary* (or *spherical*) curve, constructed as in (6.), we may take again the *helix* (869, XIII., &c.) as the *given curve* of double curvature, and observe that the expression 869, XIV., namely,

$$\text{VIII.} \dots \rho' = ca + \frac{\pi}{2} a^{4+1}\beta, \text{ gives IX.} \dots \rho'^2 = -c^2 + \frac{\pi^2\beta^2}{4} = \text{const. (comp. (8.))};$$

whence $T\rho'$ is *constant* (as in IV.), and we have the equation (comp. 869, XV. XIX.),

$$\text{X.} \dots SaU\rho' = -c \left(c^2 - \frac{\pi^2\beta^2}{4} \right)^{-1} = -\cos a = \text{const.},$$

a being again the inclination of the helix to the axis of its cylinder; which shows that the *new curve* is in *this case* a *plane one*, namely a certain *small circle* of the unit sphere.

(9.) In general, if the *given curve* be conceived to be an *orbit* described by a *point*, which *moves* with a *constant velocity* taken for *unity*, the *auxiliary* or *spherical curve* becomes what we have proposed (100, (5.)) to call the *hodograph* of that *motion*.

(10.) And if the *given curve* be supposed to be described with a *variable velocity*, the *hodograph* is still *some curve* upon the *cone of parallels* to tangents.

SECTION 5.—On Geodetic Lines, and Families of Surfaces.

380. Adopting as the *definition* of a *geodetic line*, on any proposed curved surface, the property that it is one of which the *osculating plane* is always a *normal plane* to that surface, or that the *absolute normal* to the curve is also the *normal* to the surface, we have *two principal modes* of expressing by quaternions this general and *characteristic property*. For we may either write,

$$\text{I.} \dots S\nu\rho'\rho'' = 0, \text{ or II.} \dots S\nu d\rho d^2\rho = 0,$$

to express that the *normal* ν to the surface (comp. 373) is *perpendicular* to the *binormal* $V\rho'\rho''$ or $Vd\rho d^2\rho$ to the curve (comp. 379, VII. VII'); or else, at pleasure,

$$\text{III.} \dots V\nu(U\rho') = 0, \text{ or IV.} \dots V\nu dU d\rho = 0,$$

to express that the *same normal* ν has the *direction* of the *absolute normal* $(U\rho')$ or $dU d\rho$ (comp. 379, II. III.), to the same *geodetic line*. And thus it becomes easy to deduce the known *relations* of such *lines* (or *curves*) to some important *families of surfaces*, on which

they can be traced. Accordingly, after beginning for simplicity with the *sphere*, we shall proceed in the following sub-articles to determine the geodetic lines on *cylindrical* and *conical* surfaces, with *arbitrary bases*; intending afterwards to show how the corresponding lines can be investigated, upon *developable surfaces*, and surfaces of *revolution*.

(1.) On a *sphere*, with centre at the origin, we have $\nu \parallel \rho$, and the *differential equation* IV. admits of an immediate integration;* for it here becomes, $V \dots 0 = V\rho dU\rho = dV\rho Ud\rho$, whence VI. $\dots V\rho Ud\rho = \omega$, and VII. $\dots S\omega\rho = 0$, ω being some constant vector; the curve is therefore in this case a *great circle*, as being wholly contained in *one diametral plane*.

(2.) Or we may observe that the equation,

$$\text{VIII.} \dots S\rho\rho'' = 0, \quad \text{or} \quad \text{IX.} \dots S\rho d\rho^2 = 0,$$

obtained by changing ν to ρ in I. or II., has generally for a *first integral* (comp. 335, (1.)), whether $T\rho$ be constant or variable,

$$\text{X.} \dots UV\rho\rho' = UV\rho d\rho = \omega = \text{const.};$$

it expresses therefore that ρ is the vector of *some curve* (or line) *in a plane through the origin*; which curve must consequently be here a *great circle*, as before.

(3.) Accordingly, as a verification of X., if we write

$$\text{XI.} \dots \rho = ax + \beta y, \quad x \text{ and } y \text{ being scalar functions of } t,$$

where t is still some independent scalar variable, and α, β are two vector constants, we shall have the derivatives,

$$\text{XII.} \dots \rho' = ax' + \beta y', \quad \rho'' = ax'' + \beta y'' \parallel \rho, \rho';$$

so that the equation VIII. is satisfied.

(4.) For an *arbitrary cylinder*, with generating lines parallel to a fixed line α , we may write,

$$\text{XIII.} \dots S\alpha\nu = 0, \quad \text{XIV.} \dots S\alpha dU\rho = 0, \quad \text{XV.} \dots S\alpha U\rho = \text{const.};$$

a *geodetic* on a cylinder *crosses* therefore the *generating lines* at a *constant angle*, and consequently becomes a *right line* when the cylinder is *unfolded* into a *plane*: both which known properties are accordingly verified (comp. 369, (5.), and 376, (2.)) for the case of a cylinder of *revolution*, in which case the geodetic is a *helix*.

(5.) For an *arbitrary cone*, with vertex at the origin, we have the equations,

$$\text{XVI.} \dots S\nu\rho = 0, \quad \text{XVII.} \dots S\rho dU\rho = 0, \\ \text{XVIII.} \dots dS\rho U\rho = S(d\rho. U\rho) = -T d\rho;$$

multiplying the last of which equations by $2S\rho U\rho$, and observing that $-2S\rho d\rho = -d.\rho^2$, we obtain the transformations,

* We here assume as evident, that the *differential* of a *variable* cannot be *constantly zero* (comp. 335, (7.)); and we employ the principle (comp. 338, (5.)), that $V.d\rho U\rho = -VT d\rho = 0$.

XIX. . . $0 = d\{(S\rho U d\rho)^2 + \rho^2\} = d.(V\rho U d\rho)^2$, XX. . . $TV\rho U d\rho = \text{const.}$;
 the perpendicular from the vertex, on a tangent to any one geodetic upon a cone, has therefore a constant length; and all such tangents touch also a concentric sphere,* or one which has its centre at the vertex of the cone.

(6.) Conceive then that at each point P or P' of the geodetic a tangent PT or P'T' is drawn, and that the angles OPT, OT'P' are right; we shall have, by what has just been shown,

XXI. . . $\overline{OT} = \overline{OT'} = \text{const.} = \text{radius of concentric sphere}$;

and if the cone be developed (or unfolded) into a plane, this constant or common length, of the perpendiculars from O on the tangents, will remain unchanged, because the length \overline{OP} and the angle OPT are unaltered by such development; the geodetic becomes therefore some plane line, with the same property as before; and although this property would belong, not only to a right line, but also to a circle with O for centre (compare the second part of the annexed Figure 78), yet we have in this result merely an effect of the foreign factor $S\rho U d\rho$, which was introduced in (5.), in order to facilitate the integration of the differential equation XVIII., and which (by that very equation) cannot be constantly equal to zero. We are therefore to exclude the curves in which the cone is cut by spheres concentric with it: and there remain, as the sought geodetic lines, only those of which the developments are rectilinear, as in (4.).

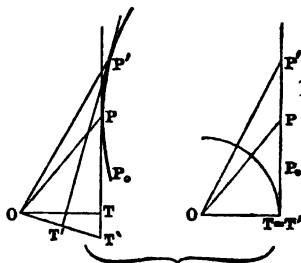


Fig. 78.

XVIII., and which (by that very equation) cannot be constantly equal to zero. We are therefore to exclude the curves in which the cone is cut by spheres concentric with it: and there remain, as the sought geodetic lines, only those of which the developments are rectilinear, as in (4.).

(7.) Another mode of interpreting, and at the same time of integrating, the equation XVIII., is connected with the interpretation of the symbol $Td\rho$; which can be proved, on the principles of the present Calculus, to represent rigorously the differential ds of the arc (s) of that curve, whatever it may be, of which ρ is the variable vector; so that we have the general and rigorous equation,

XXII. . . $Td\rho = ds$, if s thus denote the arc :

whether that are itself, or some other scalar, t , be taken as the independent variable; and whether its differential ds be small or large, provided that it be positive.

(8.) In fact if we suppose, for the sake of greater generality, that the vector ρ and the scalar s are thus both functions, ρ_t and s_t , of some one independent and scalar variable, t , our principles direct us first to take, or to conceive as taken, a submultiple, $\omega^{-1}dt$, of the finite differential dt , considered as an assumed and arbitrary increment of that independent variable, t ; to determine next the vector $\rho_{t+\omega^{-1}ds}$, and the scalar $s_{t+\omega^{-1}ds}$, which correspond to the point $P_{t+\omega^{-1}ds}$ of the curve on which ρ_t terminates in P_t , and of which s_t is the arc, $\widehat{P_0P_t}$, measured to P_t from some fixed point P_0 on the same curve; to take the differences,

* When the cone is of the second order, this becomes a case of a known theorem respecting geodetic lines on a surface of the same second order, the tangents to any one of which curves touch also a confocal surface.

$$\rho_{t:n}^{-1}ds - \rho_t, \quad \text{and} \quad s_{t:n}^{-1}ds - s_t,$$

which represent respectively the *directed chord*, and the *length*, of the arc $P_t P_{t+n}^{-1}ds$, which arc will generally be *small*, if the number n be *large*, and will *indefinitely diminish* when that number tends to *infinity*; to multiply these two decreasing differences, of ρ_t and s_t , by n ; and finally to seek the *limits* to which the *products* tend, when n thus tends to ∞ : such *limits* being, by our *definitions*, the values of the two sought and *simultaneous differentials*, $d\rho$ and ds , which answer to the assumed values of t and dt . And because the *small arc*, Δs , and the *length*, $T\Delta\rho$, of its *small chord*, in the foregoing construction, tend indefinitely to a *ratio of equality*, such must be the *rigorous ratio* of ds and $Td\rho$, which are (comp. 320) the *limits of their equimultiples*.

(9.) Admitting then the exact equality XXII. of $Td\rho$ and ds , at least when the latter like the former is taken *positively*, we have only to substitute $-ds$ for $-Td\rho$ in the equation XVIII., which then becomes immediately integrable, and gives,

$$\text{XXIII.} \dots s + S\rho Udp = s - S(\rho : Udp) = \text{const.};$$

where $S(\rho : Udp)$ denotes the *projection* \overline{TF} , of the vector ρ or OP , on the *tangent* to the geodetic at P , considered as a *positive scalar* when ρ makes an *acute angle* with $d\rho$, that is, when the *distance* $T\rho$ or \overline{OP} from the vertex is *increasing*; while s denotes, as above, the *length* of the arc P_0P of the same curve, measured from *some fixed point* P_0 thereon, and considered as a *scalar* which *changes sign*, when the *variable point* P passes through the position P_0 .

(10.) But the *length* of TP does not change (comp. (6.)), when the cone is *developed*, as before; we have therefore the equations (comp. again Fig. 78),

$$\text{XXIV.} \dots \widehat{P_0P} - \overline{TP} = \text{const.} = \widehat{P_0P'} - \overline{T'F'}, \quad \text{XXV.} \dots \widehat{PP'} = \overline{T'F'} - \overline{TF},$$

which must hold good both *before* and *after* the supposed *development* of the conical surface; and it is easy to see that this can only be, by the *geodetic* on the cone becoming a *right line*, as before. In fact, if ox' in the *plane* be supposed to intersect the tangent TP in a point T' , and if P' be conceived to approach to P , the second member of XXV. bears a limiting ratio of equality to the first member, increased or diminished by \overline{TF} ; which latter *line*, and therefore also the *angle* $TO'T$ between the perpendiculars on the two near tangents, or the angle between those tangents themselves, if existing, must bear an indefinitely decreasing ratio to the arc $\widehat{PP'}$; so that the *radius of curvature* of the *supposed curve* is *infinite*, or T' *coincides* with T , and the *development* is *rectilinear* as before.

(11.) The important and general equation, $Td\rho = ds$ (XXII.), conducts to many other consequences, and may be put under several other forms. For example, we may write generally,

$$\text{XXVI.} \dots Td\rho = 1, \quad \text{XXVII.} \dots (D\rho)^2 + 1 = 0;$$

$$\text{also} \quad \text{XXVIII.} \dots (D\rho)^2 + (Ds)^2 = 0, \quad \text{or} \quad \text{XXIX.} \dots \rho'^2 + s'^2 = 0,$$

if ρ' and s' be the first derivatives of ρ and s , taken with respect to any independent scalar variable, such as t ; whence, by continued derivation,

$$\text{XXX.} \dots S\rho\rho'' + s's'' = 0, \quad \text{XXXI.} \dots S\rho\rho''' + \rho'^3 + s's''' + s'^3 = 0, \text{ \&c.}$$

(12.) And if the arc s be *itself* taken as the independent variable, then (comp. 379, (2.)) the equations XXIX., &c., become,

$$\text{XXXII.} \dots \rho'^2 + 1 = 0, \quad S\rho\rho'' = 0, \quad S\rho\rho''' + \rho'^3 = 0, \text{ \&c.}$$

381. In general, if we conceive (comp. 372, I.) that the *vector* ρ of a *given surface* is expressed as a *given function* of *two scalar variables*, x and y , whereof *one*, suppose y , is regarded at first as an *unknown function* of the other, so that we have again,

$$I. . . \rho = \phi(x, y), \quad \text{but now with } II. . . y = fx,$$

where the *form* of ϕ is *known*, but that of f is *sought*; we may then regard ρ as being *implicitly* a function of the *single* (or *independent*) *scalar variable*, x , and may consider the equation,

$$III. . . \rho = \phi(x, fx),$$

as being that of *some curve* on the given surface, to be determined by assigned conditions. Denoting then the *unknown total derivative* $D\phi(x, fx)$ by ρ' , but the *known partial derivatives* of the same first order by $D_x\phi$ and $D_y\phi$, with analogous notations for orders higher than the first, we have (comp. 376, VI.) the expressions,

$$IV. . . \rho' = D_x\phi + y'D_y\phi, \quad \rho'' = D_x^2\phi + 2y'D_xD_y\phi + y'^2D_y^2\phi + y''D_y\phi, \text{ \&c.};$$

in which $y' = D_x y = f'x$, $y'' = D_x^2 y = f''x$, &c. Hence, writing for the *normal* ν to the *surface* the expression,

$$V. . . \nu = V(D_x\phi, D_y\phi) = V.D_x\phi D_y\phi, \quad \text{comp. 372, V.},$$

or this vector multiplied by any scalar, the *equation* 380, I. of a *geodetic line* takes this *new form*,

$$VI. . . 0 = S\nu\rho'\rho'' = S(V.D_x\phi D_y\phi.V\rho'\rho'');$$

or, by a general transformation which has been often employed already (comp. 352, XXXI., &c.),

$$VII. . . 0 = S\rho'D_y\phi.S\rho''D_x\phi - S\rho'D_x\phi.S\rho''D_y\phi;$$

and thus, by substituting the expressions IV. for ρ' and ρ'' , we obtain an *ordinary* (or *scalar*) *differential equation*, of the *second order*, in x and y , which is *satisfied by all the geodetics* on the given *surface*, and of which the *complete integral* (when found) expresses, with *two arbitrary* and *scalar constants*, the *form* of the *scalar function* f in II., or the *law* of the dependence of y on x , for the *geodetic curves* in question.

(1.) As an *example*, let us take the equation,

$$VIII. . . \rho = \phi(x, y) = y\psi x, \quad \text{comp. 378, I.},$$

of a *cone* with its vertex at the origin; which cone becomes a *known one*, when the *form* of the vector function ψ is given, that is, when we know a *guiding curve* $\rho = \psi x$, through which the *sides* of the cone all pass. We have here the partial derivatives,

IX. . . $D_x\phi = yD_x\psi x = y\psi'$, $D_y\phi = \psi x = \psi$, comp. 378, II.;
and X. . . $D_x^2\phi = yD_x^2\psi x = y\psi''$, $D_xD_y\phi = \psi'$, $D_y^2\phi = 0$;

the expressions IV. become, then,

$$\text{XI. . . } \rho' = y\psi' + y'\psi, \quad \rho'' = y\psi'' + 2y'\psi' + y''\psi;$$

and since only the *direction* of the normal is important, we may divide V. by $-y$, and write,

$$\text{XII. . . } \nu = \nabla\psi\psi'.$$

(2.) The expressions XI. and XII. give (comp. VI. and VII.) for the *geodesics* on the *cone* VIII., the differential equation of the second order,

$$\begin{aligned} \text{XIII. . . } 0 &= S(\nabla\psi\psi' \cdot \nabla\rho'\rho'') = S\rho''\psi S\phi'\psi' - S\rho''\psi' S\rho'\psi \\ &= (yS\psi\psi'' + 2y'S\psi\psi' + y''\psi^2) (y\psi'^2 + y'S\psi\psi') \\ &\quad - (yS\psi\psi'\psi'' + 2y'\psi'^2 + y''S\psi\psi') (yS\psi\psi' + y'\psi^2), \end{aligned}$$

in which ψ^2 and ψ'^2 are abridged symbols for $(\psi x)^2$ and $(\psi' x)^2$; but this equation in x and y may be greatly simplified, by some permitted suppositions.

(3.) Thus, we are allowed to suppose that the *guiding curve* (1.) is the *intersection* of the *cone* with the *concentric unit sphere*, so that

$$\text{XIV. . . } T\psi x = 1, \quad \psi^2 = -1, \quad S\psi\psi' = 0, \quad S\psi\psi'' + \psi'^2 = 0;$$

and if we further assume that the *arc* of this *spherical curve* is taken as the *independent variable*, x , we have then, by 380, (12.), combined with the last equation XIV.,

$$\text{XV. . . } T\psi'x = 1, \quad \psi'^2 = -1, \quad S\psi\psi' = 0, \quad S\psi\psi'' = -\psi'^2 = 1.$$

(4.) With these simplifications, the *differential equation* XIII. becomes,

$$\text{XVI. . . } 0 - (y - y'')(-y) - (-2y')(-y') = yy'' - 2y'^2 - y^2;$$

and its *complete integral* is found by *ordinary methods* to be,

$$\text{XVII. . . } y = b \sec(x + c),$$

in which b and c are two arbitrary but scalar constants.

(5.) To *interpret* now this *integrated* and *scalar equation* in x and y , of the *geodesics* on an *arbitrary cone*, we may observe that, by the suppositions (3.), y represents the *distance*, Op or \overline{OP} , from the vertex o , and $x + c$ represents the *angle* $\angle OPo$, in the *developed state* of cone and curve, from some *fixed line* OA in the *plane*, to the variable line OP ; the *projection* of this *new* OP on that *fixed line* OA is therefore *constant* (being $= b$, by XVII.), and the *developed geodesic* is again found to be a *right line*, as before.

382. Let $ABCDE \dots$ (see the annexed Figure 79) be any given series of points in space. Draw the successive right lines, AB, BC, CD, DE, \dots and prolong them to points B', C', D', E', \dots the lengths of these prolongations being arbitrary; join also $B'C', C'D', D'E', \dots$. We shall thus have a series of plane triangles, $B'BC', C'CD', D'DE', \dots$ all generally in different planes; so that $BCD'C'B', CDE'D'C', \dots$ are generally *gauche pentagons*, while $BCDE'D'C'B'$ is a *gauche heptagon*, &c. But we

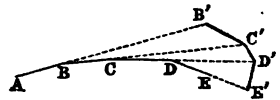


Fig. 79.

can conceive the *first triangle* $B'BC'$ to turn round its *side* BCC' , till it comes into the plane of the *second triangle*, $C'CD'$; which will transform the *first gauche pentagon* into a *plane* one, denoted still by $BCD'C'B'$. We can then conceive *this* plane figure to turn round its *side* CDD' , till it comes into the plane of the *third triangle*, $D'DE'$; whereby the *first gauche heptagon* will have become a *plane* one, denoted as before by $BCDE'D'C'B'$: and so we can proceed indefinitely. Passing then to the *limit*, at which the *points* $ABCDE\dots$ are conceived to be each *indefinitely near* to the one which precedes or follows it in the series, we conclude as usual (comp. 98, (12.)) that the *locus of the tangents* to a *curve of double curvature* is a *developable surface*: or that it admits of being *unfolded* (like a cone or cylinder) *into a plane*, without any breach of continuity. It is now proposed to *translate* these *conceptions* into the *language of quaternions*, and to draw from them some of their consequences: especially as regards the determination of the *geodetic lines*, on such a developable surface.

(1.) Let ψ , or simply ψ , denote the variable vector of a point upon the *curve*, or *cusp-edge*, or *edge of regression* of the developable, to which curve the *generating lines* of that *surface* are thus *tangents*, considered as a *function* ψ of its *arc*, x , measured from some fixed point Λ upon it; so that while the *equation* of the surface will be of the form (comp. 100, (8.)),

$$I. \dots \rho = \phi(x, y) = \psi_x + y\psi'_x = \psi + y\psi',$$

y being a second scalar variable, we shall have the relations (comp. 881, XV.),

$$II. \dots T\psi'_x = 1, \quad \psi^2 = -1, \quad S\psi\psi'' = 0, \quad S\psi'\psi''' = -\psi''^2 = z^2, \quad \text{if } z = T\psi''.$$

$$(2.) \text{ Hence } \quad III. \dots D_x\phi = \psi' + y\psi'', \quad D_y\phi = \psi';$$

$$IV. \dots \rho' = (1 + y')\psi' + y\psi'', \quad \rho'' = y''\psi' + (1 + 2y')\psi'' + y\psi''';$$

and

$$V. \dots \nu = \nabla\psi'\psi'' = \psi'\psi'', \text{ multiplied by any scalar.}$$

(8.) The differential equation of the geodetics may therefore be thus written (comp. 881, XIII.),

$$VI. \dots 0 = S(\nabla\psi'\psi'' \cdot \nabla\rho\rho'') = S\rho'\psi''S\rho''\psi' - S\rho''\psi''S\rho'\psi';$$

in which, by (1.) and (2.),

$$VII. \dots \begin{cases} S\rho'\psi'' = -yz^2, & S\rho''\psi' = -y'' + yz^2, \\ S\rho''\psi'' = -(1 + 2y')z^2 - yz z', & S\rho'\psi' = -(1 + y'); \end{cases}$$

the equation becomes therefore, after division by $-z$,

$$VIII. \dots 0 = z\{(1 + y')^2 + (yz)^2\} + (1 + y')(yz)' - y''yz,$$

or simply,

$$IX. \dots z + v' = 0, \quad \text{or } IX'. \dots Td\psi' + dv = 0, \quad \text{if } X. \dots \tan v = \frac{yz}{1 + y'} = \frac{yT\psi''}{1 + y'}.$$

(4.) To *interpret* now this very simple equation IX. or IX', we may observe that z , or $T\psi''$, or $Td\psi' : dx$, expresses the *limiting ratio*, which the *angle* between two near *tangents* ψ and $\psi + \Delta\psi$, to the *cusp-edge* (1.), bears to the *small arc* Δz

of that curve which is intercepted between their points of contact; while v is, by IV., that *other angle*, at which such a variable tangent, or *generating line* of the developable, crosses the *geodetic* on that surface; and therefore its derivative, v' or $dv : dx$, represents the limiting ratio, which the change Δv of this last angle, in passing from one generating line to another, bears to the same *small arc* Δx of the curve which those lines touch.

(5.) Referring then to Figure 79, in which, instead of two continuous curves, there were two *gauche polygons*, or at least two systems of successive right lines, connected by prolongations of the lines of the first system, we see that the recent formula IX. or IX'. is equivalent to this limiting equation,

$$\text{XI.} \dots \lim. \frac{CD'C' - BC'B'}{C'D'} = -1;$$

but these three angles remain unaltered, in the development of the surface: the bent line $B'C'D'$ for space becomes therefore ultimately a straight line in the plane, and similarly for all other portions of the original polygon, or twisted line, $B'C'D'E' \dots$, of which $B'C'D'$ was a part.

(6.) Returning then to curves and surfaces in space, the quaternion analysis (3.) is found, by this simple reasoning,* to conduct to an expression for the known and characteristic property of the geodetics on a developable: namely that they become right lines, as those on cylinders (880, (4.)), and on cones (880, (6.) and (10.)), or 881, (5.)), were lately seen to do, when the surface on which they are thus traced is unfolded into a plane.

383. This known result, respecting geodetics on developables, may be very simply verified, by means of a new determination of the absolute† normal (379) to a curve in space, as follows.

(1.) The arc s of any curve being taken for the independent variable, we may write (comp. 376, I.), by Taylor's Series, the following rigorous expressions,

$$\text{I.} \dots \rho_{-s} = \rho - s\rho' + \frac{1}{2}s^2 u_{-s} \rho'', \quad \rho_0 = \rho, \quad \rho_s = \rho + s\rho' + \frac{1}{2}s^2 u_s \rho'', \quad \text{with } u_0 = 1,$$

for the vectors of three near points, P_{-s}, P_0, P_s , on the curve, whereof the second divides the arc, $2s$, intercepted between the first and third.

(2.) If then we conceive the parallelogram $P_{-s}P_0P_sR_s$ to be completed, we shall have, for the two diagonals of this new figure these other rigorous expressions,

$$\text{II.} \dots P_{-s}P_s = \rho_s - \rho_{-s} = 2s\rho' + \frac{1}{2}s^2(u_s - u_{-s})\rho'';$$

$$\text{III.} \dots P_0R_s = \rho_s + \rho_{-s} - 2\rho_0 = \frac{1}{2}s^2(u_s + u_{-s})\rho'';$$

* In the *Lectures* (page 581), nearly the same analysis was employed, for geodetics on a developable; but the interpretation of the result was made to depend on an equation which, with the recent significations of ψ and v , may be thus written, as the integral of IX'. $v + \int Td\psi' = \text{const.}$; where $\int Td\psi'$ represents the finite angle between the extreme tangents to the finite arc $\int Td\psi$, or Δx , of the cusp-edge, when that curve is developed into a plane one.

† Called also, and perhaps more usually, the principal normal.

which give the limiting equations,

$$\text{IV.} \dots \lim_{s \rightarrow 0} s^{-1} P_0 P_s = 2\rho'; \quad \text{V.} \dots \lim_{s \rightarrow 0} s^{-2} P_0 P_s = \rho''.$$

(3.) But the length $\overline{P_0 P_s}$ of what may be called the *long diagonal*, or the *chord* of the *double arc*, $2s$, is *ultimately equal* to that double arc; we have therefore, by IV., the equation,

$$\text{VI.} \dots T\rho' = 1, \text{ if } \rho' = Ds, \text{ and if } s \text{ denote the arc,}$$

considered as the scalar variable on which the vector ρ depends: a result agreeing with what was otherwise found in 880, (12.).

(4.) At the same time, since the *ultimate direction* of the same long diagonal is evidently that of the *tangent* at P_0 , we see anew that the same *first derived vector* ρ' represents what may be called the *wait-tangent** to the curve at that point.

(5.) And because the *lengths* of the *two sides* $P_0 P_s$ and $P_0 P_s$, considered as *chords* of the two successive and *equal arcs*, s and s , are *ultimately equal* to them and to each other, it follows that the *parallelogram* (2.) is *ultimately equilateral*, and therefore that its *diagonals* are *ultimately rectangular*; but these diagonals, by IV. and V., have ultimately the directions of ρ' and ρ'' ; we find therefore anew the equation,

$$\text{VII.} \dots S\rho'' = 0, \text{ if the arc be the independent variable,}$$

which had been otherwise deduced before, in 880, (12.).

(6.) But under the same condition, we saw (879, (2.)) that the *second derived vector* ρ'' has the direction of the *absolute normal* to the curve; such then is by V. the *ultimate direction* of what we may call the *short diagonal* $P_0 P_s$, constructed as in (2.); or, *ultimately*, the direction of the *bisector* of the (obtuse) *angle* $P_0 P_s P_s$, between the *two near* and *nearly equal chords* from the point P_0 : while the *plane* of the *parallelogram* becomes *ultimately the osculating plane*.

(7.) All this is quite independent of the consideration of any *surface*, on which the *curve* may be conceived to be *traced*. But if we now conceive that this curve is formed *from a right line* $B'C'D'$. . . (comp. Fig. 79), by *wrapping* round a *developable surface* a *plane* on which the *line* had been drawn, and if the successive portions $B'C'$, $C'D'$, . . . of that line be supposed to have been *equal*, then because the *two right lines* $C'B'$ and $C'D'$ *originally* made *supplementary angles* with any *other line* $C'C$ in the *plane*, the *two chords* $C'B'$ and $C'D'$ of the *curve* on the *developable* *tend* to make *supplementary angles* with the *generatrix* $C'C$ of that surface; on which account the *bisector* (6.) of their *angle* $B'C'D'$ *tends* to be *perpendicular* to that *generating line* $C'C$, *as well as* to the *chord* $B'D'$, or *ultimately* to the *tangent* to the *curve* at C' , when chords and arcs *diminish* together. The *absolute normal* (6.) to the curve thus formed is therefore *perpendicular* to *two distinct tangents* to the surface at C' , and is consequently (comp. 372) the *normal* to that *surface* at that point; whence, by the *definition* (380), the *curve* is, as before, a *geodetic* on the *developable*.

(8.) As regards the asserted *rectangularity* (7.), of the *bisector* of the angle $B'C'D'$ to the line $C'C$, when the angles $CC'B'$ and $CC'D'$ are supposed to be *supplementary*, but *not in one plane*, a simple proof may be given by conceiving that the

* Compare the Note to page 152.

right line $B'C'$ is prolonged to C'' , in such a manner that $\overline{C'O''} = \overline{C'D'}$; for then these two *equally long* lines from O' make *equal angles* with the line $C'O'$, so that the one may be formed from the other by a *rotation* round that line as an *axis*; whence $C'O'D'$, which is evidently *parallel* to the bisector of $B'C'D'$, is also *perpendicular* to $C'O'$.

(9.) In quaternions, if a and ρ be any two vectors, and if t be any scalar, we have the equation,

$$\text{VIII. . . } S. a(a't\rho a^{-t} - \rho) = 0;$$

which is, by 308, (8.), an expression for the geometrical principle last stated.

384. The recent analysis (382) enables us to deduce with ease, by quaternions, other known and important properties of developable surfaces: for instance, the property that each such surface may be considered as the *envelopes* of a *series of planes*, involving only *one* scalar and *arbitrary constant* (or *parameter*) in their *common equation*; and that *each plane* of this series *osculates* to the *cuspidal edge* of the *developable*.

(1.) The equation of the developable *surface* being still,

$$\text{I. . . } \rho = \phi(x, y) = \psi_x + y\psi'_x = \psi + y\psi', \quad \text{as in 382, I.},$$

its *normal* ν is easily found to have as in 382, V., the *direction* of $\nabla\psi'\psi''$, whether the scalar variable x be, or be not, the *arc* of the *cuspidal edge*, of which curve the equation is,

$$\text{II. . . } \rho = \psi_x.$$

(2.) Hence, by 378, VII., the equation of the *tangent plane* takes the form,

$$\text{III. . . } S\omega\psi'\psi'' = 8\psi\psi'\psi'',$$

from which the *second* scalar variable y thus *disappears*: this *common equation*, of all the *tangent planes* to the developable, involves therefore, as above stated, only *one* variable and scalar *parameter*, namely x ; and the *envelope* of all these *planes* is the developable surface itself.

(3.) The plane III., for any *given value* of this parameter x , that is, for any *given point* of the *cuspidal edge*, *touches the surface* along the *whole extent* of the *generating line*, which is the *tangent* to this last curve.

(4.) And by comparing its equation III. with the formula 376, V., we see at once that this plane *osculates* to the same *cuspidal edge*, at the point of *contact* of that curve with the same *generatrix* of the developable.

385. If the *reciprocals* of the *perpendiculars*, let fall from a given origin, on the *tangent planes* to a developable surface, be considered as being themselves vectors from that origin, they terminate on a *curve*, which is connected with the *cuspidal edge* of the developable by some interesting relations of *reciprocity* (comp. 373, (21.)): in such a manner that if this *new curve* be made the *cuspidal edge* of a *new developable*, we can *return* from it to the *former surface*, and to its *cuspidal edge*, by a *similar process* of construction.

(1.) In general, if ψ_x and χ_x , or briefly ψ and χ , be two vector functions of a scalar variable x , such that χ may be deduced from ψ by the three scalar equations,

$$I. \dots S\psi\chi = c, \quad S\psi'\chi = 0, \quad S\psi''\chi = 0,$$

in which $S\psi\chi$ is written briefly for $S(\psi_x \cdot \chi_x)$, and c is any scalar constant, we have then this *reciprocal system* of three such equations,

$$II. \dots S\chi\psi = c, \quad S\chi'\psi = 0, \quad S\chi''\psi = 0;$$

an intermediate step being the equation,

$$III. \dots S\psi'\chi' = S\chi'\psi' = 0.$$

(2.) Hence, generally,

$$IV. \dots \text{if } \chi = \frac{cV\psi'\psi''}{S\psi\psi'\psi''} \quad \text{then} \quad V. \dots \psi = \frac{cV\chi'\chi''}{S\chi\chi'\chi''}.$$

(3.) But if ρ be the variable vector of a curve in space, and ρ' , ρ'' its first and second derivatives with respect to any scalar variable, then, by the equation 876, V. of the osculating plane to the curve, we have the general expression,

$$VI. \dots \frac{S\rho\rho'\rho''}{V\rho'\rho''} = \text{perpendicular from origin on osculating plane};$$

so that if ψ and χ be considered as the vectors of *two curves*, each vector is $c \times$ the reciprocal of the perpendicular, thus let fall from a common point, on the osculating plane to the other.

(4.) We have therefore this Theorem:—

If, from any assumed point, o, there be drawn lines equal to the reciprocals of the perpendiculars from that point, on the osculating planes to a given curve of double curvature, or to those perpendiculars multiplied by any given and constant scalar; then the locus of the extremities of the lines so drawn will be a second curve, from which we can return to the first curve by a precisely similar process.*

386. The theory of *developable surfaces*, considered as *envelopes of planes with one scalar and variable parameter* (384), may be additionally illustrated by connecting it with *Taylor's Series*, as follows.

(1.) Let α_t denote any vector function of a scalar variable t , so that

$$I. \dots \alpha_t = \alpha_0 + t\omega_1\alpha'_0 = \alpha + t\omega_1\alpha', \quad \text{with } \omega_0 = 1;$$

or, by another step in the expansion,

$$II. \dots \alpha_t = \alpha_0 + t\alpha'_0 + \frac{1}{2}t^2v_1\alpha''_0 = \alpha + t\alpha' + \frac{1}{2}t^2v_1\alpha'', \quad \omega_0 = 1;$$

where ω and v are generally *quaternions*, but $\omega\alpha'$ and $v\alpha''$ are *vectors*.

* The *two curves* may be said to be *polar reciprocals*, with respect to the (real or imaginary) *sphere*, $\rho^3 = c$; and an analogous relation of reciprocity exists generally, when the *points* of one curve are the *poles* of the *osculating planes* of the other, with respect to any *surface of the second order*: corresponding *tangents* being then *reciprocal polars*. Compare the theory of *developables reciprocal to curves*, given in Salmon's *Analytical Geometry of Three Dimensions*, page 89; see also Chapter XI. (page 224, &c.), of the same excellent work.

(2.) Then, as the *rigorous equation* of the *variable plane*, the reciprocal of the perpendicular on which from the origin is $-a$, we have either,

$$\text{III.} \dots -1 = Sa\rho = Sap + tSua'\rho,$$

or

$$\text{IV.} \dots -1 = Sap + tSa'\rho + \frac{1}{2}t^2Sua''\rho,$$

according as we adopt the expression I., or the equally but not more rigorous expression II., for the variable vector a .

(8.) Hence, by the form III., the *line of intersection* of the *two planes*, which answer to the *two values* 0 and t of the scalar variable, or *parameter*, t , is *rigorously* represented by the system of the *two scalar equations*,

$$\text{V.} \dots Sa\rho + 1 = 0, \quad Sua'\rho = 0.$$

(4.) And the *limiting position* of this *right line* V., which answers to the conceived *indefinite approach* of the *second plane* to the *first*, is given with *equal rigour* by the equations,

$$\text{VI.} \dots Sap + 1 = 0, \quad Sa'\rho = 0;$$

whereof it is seen that the *second* may be formed from the *first*, by *derivating* with respect to t , and treating ρ as *constant*: although *no such rule* of calculation had been *previously laid down*, for the comparatively *geometrical process* which is here supposed to be adopted.

(5.) The *locus* of all the *lines* VI. is evidently *some ruled surface*; to determine the *normal* v to which, at the extremity of the vector ρ , we may consider that vector to be a function (372) of *two independent* and scalar variables, whereof one is t , and the other may be called for the moment w ; and thus we shall have the *two partial derivatives*,

$$\text{VII.} \dots SaD_t\rho = 0, \quad SaD_w\rho = 0, \quad \text{giving } v \parallel a.$$

(6.) Hence the *line* a has the direction of the required *normal* v ; the *plane* $Sa\rho + 1 = 0$ *touches* the *surface* (comp. 384, (3.)) along the *whole extent* of the *limiting line* VI.; and the *locus* of all *such lines* is the *envelope* of all the *planes*, of the system recently considered.

(7.) The *line* VI. *cuts* generally the *plane* IV., in a *point* which is *rigorously* determined by the three equations,

$$\text{VIII.} \dots Sap + 1 = 0, \quad Sa'\rho = 0, \quad Sua''\rho = 0;$$

and the *limiting position* of this *intersection* is, with equal rigour, the *point* determined by this other system of equations,

$$\text{IX.} \dots Sap + 1 = 0, \quad Sa'\rho = 0, \quad Sa''\rho = 0;$$

in which it may be remarked (comp. (4.)), that the *third* is the *derivative* of the *second*, if ρ be treated as *constant*.

(8.) The *locus* of all these *points* IX. is generally *some curve* upon the *surface* (5.), which is the *locus* of the *lines* VI., and has been seen to be the *envelope* (6.) of the *planes* III. or IV.; and to find the *tangent* to this *curve*, at the *point* answering to a *given value* of t , or to a *given line* VI., we have by IX. the derived equation,

$$\text{X.} \dots Sa\rho' = 0, \quad Sa'\rho' = 0, \quad \text{whence } \rho' \parallel Va\alpha';$$

comparing which with the equations VI. we see that the *lines* VI. *touch* the *curve*, which is thus *their common envelope*.

(9.) We see then, in a new way, that the envelope of the planes III., which have one scalar parameter (t) in their common equation, and may represent any system of planes subject to this condition, is a developable surface: because it is in general (comp. 882) the locus of the tangents to a curve in space, although this curve may reduce itself to a point, as we shall shortly see.

(10.) We may add that if a_t in III. be considered as the vector of a given curve, this curve is the locus of the poles* of the tangent planes to the developable, taken with respect to the unit sphere; and conversely, that the developable surface is the envelope of the polar planes of the points of the same given curve, with respect to the same sphere.

(11.) If then it happen that this given curve, with a_t for vector, is a plane one, so that we have this new condition,

$$\text{XI.} \dots S\beta a_t + 1 = 0, \beta \text{ being any constant vector,}$$

namely the vector of the pole of the supposed plane of the given curve, the variable plane III., or $S\beta a_t + 1 = 0$, of which the surface (5.) is the envelope, passes constantly through this fixed pole; so that the developable becomes in this case a cone, with β for the vector of its vertex: the equations IX. giving now $\rho = \beta$.

(12.) The same degeneration, of a developable into a conical surface, may also be conceived to take place in another way, by the cusp-edge (or at least some finite portion thereof) tending to become indefinitely small, while yet the direction of its tangents does not tend to become constant. For example, with recent notations, the developable which is the locus of the tangents to the helix may have its equation written thus:

$$\text{XII.} \dots \rho = \phi(x, y) = c(x\alpha + \frac{2}{\pi} \tan \alpha \cdot \alpha^2 U\beta) + y\alpha(1 + \tan \alpha \cdot \alpha^2 U\beta);$$

which when the quarter-interval, c , between the spires, tends to zero, without their inclination α to the axis α being changed, tends to become a cone of revolution round that axis, with its semiangle = α .

387. So far, then, we may be said to have considered, in the present Section, and in connexion with geodetic lines, the four following families of surfaces (if the first of them may be so called). First, spherical surfaces, of which the characteristic property is expressed by the equation,

$$\text{I.} \dots V\nu(\rho - a) = 0, \text{ if } a \text{ be vector of centre;}$$

second, cylindrical surfaces, with the property,

$$\text{II.} \dots S\nu a = 0, \text{ if } a \text{ be parallel to the generating lines;}$$

third, conical surfaces, with the property,

$$\text{III.} \dots S\nu(\rho - a) = 0, \text{ if } a \text{ be vector of vertex;}$$

and fourth, developable surfaces, with the distinguishing property expressed by the more general equation,

* Compare the Note to page 525.

IV. . . $V \nu d\nu = 0$, if $d\rho$ have the *direction of a generatrix* ;
 ν being in each the *normal vector* to the surface, so that .

V. . . $S \nu d\rho = 0$, for all *tangential directions* of $d\rho$;
 and the *fourth family including the third*, which in its turn includes
 the *second*. A few additional remarks on these equations may be
 here made.

(1.) The *geometrical signification* of the equation I. (as regards the *radius*) is ob-
 vious ; but on the side of *calculation* it may be useful to remark, that *elimination* of
 ν between I. and V. gives, for *spheres*,

$$\text{VI. . . } S(\rho - a)d\rho = 0, \text{ or VII. . . } T(\rho - a) = \text{const.}$$

(2.) The equations II. and V. show that $d\rho$, and therefore $\Delta\rho$, may have the
 given direction of a ; for an *arbitrary cylinder*, then, we have the *vector equation*
 (872),

$$\text{VIII. . . } \rho = \phi(x, y) = \psi_n + \gamma a,$$

where ψ_n is an *arbitrary vector function* of x .

(8.) From VIII. we can at once infer, that

$$\text{IX. . . } S\beta\rho = S\beta\psi_n, \quad S\gamma\rho = S\gamma\psi_n, \text{ if } a = V\beta\gamma ;$$

the *scalar equation* (878) of a *cylindrical surface* is therefore generally of the form
 (comp. 871, (6.), (7.)),

$$\text{X. . . } 0 = F(S\beta\rho, S\gamma\rho) ;$$

β and γ being two constant vectors, and the generating lines being perpendicular to
 both.

(4.) The equation III. may be thus written,

$$\text{XI. . . } S\nu Ua = Ta^{-1}S\nu\rho ; \text{ whence XII. . . } S\nu Ua = 0, \text{ if } Ta = \infty ;$$

the equation for *cones* includes therefore that for *cylinders*, as was to be expected,
 and reduces itself thereto, when the vertex becomes infinitely distant.

(5.) The same equation III., when compared with V., shows that $d\rho$ may have
 the direction of $\rho - a$, and therefore that $\rho - a$ may be multiplied by any scalar ; the
vector equation of a *conical surface* is therefore of the form, *

$$\text{XIII. . . } \rho = a + y\psi_n, \quad \psi_n \text{ being an arbitrary vector function.}$$

(6.) The *scalar equation* of a cone may be said to be the result of the *elimination*
 of a scalar variable t , between two equations of the forms,

$$\text{XIV. . . } S(\rho - a)\chi_t = 0, \quad S(\rho - a)\chi'_t = 0,$$

which express that the *cone* is the *envelope* (comp. 886, (11.)) of a *variable plane*,
 which passes through a *fixed point*, and involves only *one scalar parameter* in its
 equation : with a new reduction to a *cylinder*, in a case on which we need not here
 delay.

(7.) The equation IV. implies, that for each *point* of the surface there is a *direc-
 tion* along which we may move, *without changing the tangent plane* ; and therefore
 that the surface is an *envelope of planes*, &c., as in 886, and consequently that it is
developable, in the sense of Art. 882.

(8.) The *vector equation* of a *general developable surface* may be written under
 the form,

$$\text{XV.} \dots \rho = \phi(x, y) = \psi_u + yU\psi'_u;$$

the sign of a *versor* being here introduced, for the sake of facilitating the passage, at a certain limit, to a cone (comp. 386, (12.)).

(9.) And the *scalar equation* of the same *arbitrary developable* may be represented as the result of the elimination of t , between the two equations,

$$\text{XVI.} \dots S\rho\chi_t + 1 = 0, \quad S\rho\chi'_t = 0;$$

in which χ_t is an arbitrary vector function of t .

(10.) The envelope of a *plane* with two arbitrary and scalar parameters, t and u , is generally a *curved but undevelopable surface*, which may be represented by the system of the three scalar equations,

$$\text{XVII.} \dots S\rho\chi_{t,u} + 1 = 0, \quad S\rho D_t\chi = 0, \quad S\rho D_u\chi = 0;$$

where $-\chi$ denotes the reciprocal of the perpendicular from the origin on the tangent plane to the surface, at what may be called *the point* (t, u).

388. It remains, on the plan lately stated (380), to consider briefly *surfaces of revolution*, and to investigate the *geodetic lines*, on this additional *family* of surfaces; of which the *equation*, analogous to those marked I. II. III. IV. in 387, for spheres, cylinders, cones, and developables, is of the form,

$$\text{I.} \dots S a\rho\nu = 0,$$

if a be a given line in the direction of the *axis* of revolution, supposed for simplicity to pass through the origin; but which may also be represented by either of these two other equations, not involving the normal ν ,

$$\text{II.} \dots T\rho = f(S a\rho), \quad \text{or} \quad \text{III.} \dots T V a\rho = F(S a\rho),$$

where f and F are used as characteristics of two *arbitrary* but *scalar functions*: between which $S a\rho$ may be conceived to be eliminated, and so a *third form* of the same sort obtained.

(1.) In fact, the equation I. expresses that ν ||| a , ρ , or that the *normal* to the surface intersects the *axis*; while II. expresses that the *distance* from a *fixed point* upon that axis is a *function* of its own *projection* on the same *fixed line*, or that the *sections* made by *planes perpendicular* to the axis are *circles*; and the same *circularity* of these sections is otherwise expressed by III., since that equation signifies that the *distance from the axis* depends on the *position* of the *cutting plane*, and is *constant* or *variable* with it: while the two last forms are connected with each other in *calculation*, by means of the general relation (comp. 204, XXI.),

$$\text{IV.} \dots (T a\rho)^2 = (S a\rho)^2 + (T V a\rho)^2.$$

(2.) The equation I. is *analogous*, in *quaternions*, to a *partial differential equation* of the *first order*, and either of the two other equations, II. and III., is *analogous* to the *integral* of that equation, in the *usual differential calculus of scalars*.

(3.) To accomplish the corresponding *integration in quaternions*, or to pass from the form I. to II., whence III. can be deduced by IV., we may observe that the equation I. allows us to write (because $Svd\rho = 0$),

$$V \dots v = xa + yp, \quad VI \dots xSad\rho + ySpd\rho = 0,$$

so that the two scalars Sap and Sp are *together constant*, or *together variable*, and must therefore be *functions of each other*.

(4.) Conversely, to *eliminate the arbitrary function* from the form II., quaternion *differentiation* gives,

$$VII \dots 0 = S(U\rho \cdot d\rho) + f''(Sap) \cdot Sap\rho = S.(U\rho + af''Sap)d\rho;$$

hence VIII. . . $v \parallel U\rho + af''Sap$, and IX. . . $v \parallel a, \rho$, as before;

so that we can *return* in this way to the equation I., the *functional sign f disappearing*.

(5.) We have thus the germs of a *Calculus of Partial Differentials in Quaternions*,* *analogous* to that employed by Monge, in his researches respecting *families of surfaces*: but we cannot attempt to pursue the subject farther here.

(6.) But as regards the *geodetic lines* upon a surface of revolution, we have only to substitute for v , in the recent formula I., by 380, IV., the expression $dU\rho$, which gives at once the *differential equation*,

$$X \dots 0 = SapdU\rho = d.SapU\rho \text{ (because } S(ad\rho \cdot U\rho) = -SaTd\rho = 0);$$

whence, by a first integration, c being a scalar constant,

$$XI \dots c = SapU\rho = TVap \cdot SU(Vap \cdot d\rho).$$

(7.) The characteristic property of the sought curves is, therefore, that for each of them the *perpendicular distance from the axis of revolution varies inversely as the cosine† of the angle, at which the geodetic crosses a parallel, or circular section of the surface*: because, if $Ta = 1$, the line Vap has the *length of the perpendicular* let fall from a point of the curve on the axis, and has the *direction of a tangent* to the parallel.

* The same remark was made in page 574 of the *Lectures*, in which also was given the elimination of the arbitrary function from an equation of the recent form III. It was also observed, in page 578, that *geodetics* furnish a very simple example of what may be called the *Calculus of Variations in Quaternions*; since we may write,

$$\begin{aligned} \delta \int ds &= \delta \int Td\rho = \int \delta Td\rho = -\int S(Ud\rho \cdot \delta d\rho) \\ &= -\int S(Ud\rho \cdot d\delta\rho) = -\Delta S(Ud\rho \cdot \delta\rho) + \int S(dUd\rho \cdot \delta\rho), \end{aligned}$$

and therefore $dUd\rho \parallel v$, or $VvdUd\rho = 0$, as in 380, IV., in order that the expression under the last integral sign may vanish for all variations $\delta\rho$ consistent with the *equation of the surface*: while the evanescence of the part which is *outside* that sign \int supplies the *equations of limits*, or shows that the *shortest line between two curves* on a given surface is *perpendicular to both*, as usual.

† Unless it happen that this cosine is *constantly zero*, in which case $c = 0$, and the *geodetic* is a *meridian* of the surface.

(8.) The equation XI. may also be thus written,

$$\text{XII.} \dots cT\rho' = Sapp', \text{ where } \rho' = D_t\rho;$$

and if the independent variable t be supposed to denote the *time*, while the *geodetic* is conceived to be a curve described by a *moving point*, then while $T\rho'$ evidently represents the *linear velocity* of that point, as being $= ds : dt$, if s denote the *arc* (comp. 100, (5.), and 380, (7.), (11.)), it is easy to prove that $Sapp'$ represents the *double areal velocity, projected on a plane perpendicular to the axis*; the one of these two velocities varies therefore *directly as the other*: and in fact, it is known from mechanics, that *each velocity would be constant*,* if the point were to describe the curve, subject only to the *normal reaction* of the surface, and undisturbed by any other force.

(9.) As regards the *analysis*, it is to be observed that the *differential equation* X. is satisfied, *not only* by the *geodetics* on the surface of revolution, but *also* by the *parallels* on that surface: which fact of calculation is connected with the obvious geometrical property, that *every normal plane* to such a parallel *contains the axis* of revolution.

(10.) In fact if we draw the normal plane to *any curve* on the surface, at a point where it *crosses a parallel*, this plane will *intersect the axis*, in the point where that axis is met by the *normal* to the surface, drawn at the same point of crossing; but *this construction fails to determine that normal*, if the curve *coincide* with, or even *touch* a parallel, at the point where its normal plane is drawn.

SECTION 6.—On Osculating Circles and Spheres, to Curves in Space; with some connected Constructions.

389. Resuming the expression 376, I. for ρ_n , and referring again to Fig. 77, we see that if a circle PQD be described, so as to touch a given curve PQE, or its tangent PT, at a given point P, and to cut the curve at a near point Q, and if PN be the projection of the chord PQ on the diameter PD, or on the radius CP, then because we have, rigorously,

$$\text{I.} \dots PQ = t\rho' + \frac{1}{2}t^2u\rho'', \text{ with } u = 1 \text{ for } t = 0,$$

we have also

$$\text{II.} \dots PN = \frac{1}{2}t^2 \nabla u\rho''\rho' : \rho',$$

and

$$\text{III.} \dots \frac{1}{PC} = \frac{2}{PD} = \frac{2PN}{PQ^2} = \frac{\nabla u\rho''\rho'}{(\rho' + \frac{1}{2}t^2u\rho'')^2 \rho'}$$

Conceiving then that the near point Q approaches indefinitely to the given point P, in which case the *ultimate state* or *limiting position* of

* This remark is virtually made in page 448 of Professor De Morgan's *Differential and Integral Calculus* (London, 1842), which was alluded to in page 578 of the *Lectures on Quaternions*.

the circle PQD is said to be that of the *osculating circle* to the curve at that point P, we see that while the *plane* of this last circle is the *osculating plane* (376), the *vector* κ of its centre κ , or of the *limiting position* of the point O, is rigorously expressed by the formula:

$$\text{IV.} \dots \kappa = \rho + \frac{\rho^3}{\sqrt{\rho''\rho'}};$$

which may however be in many ways *transformed*, by the rules of the present Calculus.

(1.) Thus, we may write, as *transformations* of the expression IV., the following:

$$\text{V.} \dots \kappa = \rho + \frac{\rho'}{\sqrt{\rho''\rho'^{-1}}} = \rho - \frac{T\rho'}{\sqrt{\rho''\rho'^{-1}} \cdot U\rho'} = \rho - \frac{T\rho'}{(U\rho')^2};$$

or introducing *differentials* instead of *derivatives*, but leaving still the independent variable *arbitrary*,

$$\text{VI.} \dots \kappa = \rho - \frac{d\rho^3}{\sqrt{d\rho d^2\rho}} = \rho + \frac{d\rho}{\sqrt{d^2\rho d\rho^{-1}}} = \rho - \frac{Td\rho}{dU\rho'} = \rho - \frac{ds}{dUd\rho};$$

if s be the *arc* of the curve; so that the last expression gives this very simple formula, for the *reciprocal of the radius of curvature*, or for the *ultimate value* of $1 : \text{CR}$,

$$\text{VII.} \dots (\rho - \kappa)^{-1} = D, U\rho', \quad \text{where } U\rho' = Ud\rho, \text{ as before.}$$

(2.) To *interpret* this result, we may employ again that *auxiliary* and *spherical curve*, upon the *cone of parallels to tangents*, which has already served us to *construct*, in 379, (6.) and (7.), the *osculating plane*, the *absolute normal*, and the *binormal*, to the *given curve* in space. And thus we see, that while the *semidiameter* PC has ultimately the *direction* of $dU\rho'$, and therefore that of the *absolute normal* (379, II.) at P, the *length* of the same radius is ultimately equal to the *arc* PQ (or Δs) of the *given curve*, divided by the *corresponding arc of the auxiliary curve*; or that the *radius of curvature*, or *radius of the osculating circle* at P, is equal to the *ultimate quotient* of the arc PQ, divided by the *angle between the tangents*, PR and (say) QU, to that arc PQ itself at P, and to its *prolongation* QR at Q, although these two tangents are *generally* in *different planes*, and have *no common point*, so long as PQ remains *finite*: because we suppose that the *given curve* is in *general* one of *double curvature*, although the *formulae*, and the *construction*, above given, are applicable to *plane curves* also.

(3.) For the *helix*, the formula IV. gives, by values already assigned for ρ, ρ', ρ'' , and a , the expression,

$$\text{VIII.} \dots \kappa = \text{cta} - a'\beta \cot^2 a, \quad \text{whence IX.} \dots \rho - \kappa = a'\beta \text{cosec}^2 a,$$

a being the *inclination* of the given helix to the axis; the *locus of the centre* of the osculating circle is therefore in this case a *second helix*, on the *same cylinder*, if $a = \frac{\pi}{4}$, but otherwise on a *co-axial cylinder*, of which the *radius* = the given radius

$T\beta$, multiplied by the square of the cotangent of a ; and the *radius of curvature* = $T(\rho - \kappa) = T\beta \times \text{cosec}^2 a$, so that *this radius* always *exceeds* the radius of the cylinder, and is cut *perpendicularly* (without being *prolonged*) by the axis.

(4.) In general, if $T\rho' = \text{const.}$, and therefore $S\rho'\rho'' = 0$ (comp. 879, (2.)), the expression IV. becomes,*

$$\text{X.} \dots \kappa = \rho + \frac{\rho'^2}{\rho''}; \text{ whence, XI.} \dots \kappa = \rho - \rho''^{-1}, \text{ if } T\rho' = 1,$$

that is, if the arc be taken as the independent variable (880, (12.)). Under this last condition, then, the formula VII. reduces itself to the following,

$$\text{XII.} \dots (\rho - \kappa)^{-1} = \rho'' = D_s^2\rho = \text{ultimate reciprocal of radius of } \rho;$$

so that ρ'' (for $T\rho' = 1$) may be called the *Vector of Curvature*, because its tensor $T\rho''$ is a numerical measure for what is usually called the *curvature*† at the point ρ , and its versor $U\rho''$ represents the *ultimate direction* of the *semidiameter* PC , of the circle constructed as above.

(5.) As an example of the application (2.) of the formula IV. for κ , to a *plane curve*, let us take the *ellipse*,

$$\text{XIII.} \dots \rho = Va^t\beta, \quad T\alpha = 1, \quad S\alpha\beta > 0, \quad 887, (2.),$$

considered as an *oblique section* (814, (4.)) of a *right cylinder*. The expressions 876, (5.) for the derivatives of ρ , combined with the expression XIII. for that vector itself, give here the relations,

$$\text{XIV.} \dots V\rho\rho'' = 0, \quad V\rho'\rho'' = 0;$$

and therefore comp. (888, (5.)),

$$\text{XV.} \dots V\rho\rho' = \text{const.} \ddagger = \frac{\pi}{2} \beta\gamma, \quad V\rho'\rho'' = \text{const.} = \left(\frac{\pi}{2}\right)^2 \beta\gamma, \quad \text{if } \gamma = Va\beta;$$

hence for the present curve we have by IV.,

$$\text{XVI.} \dots \kappa = \rho - \frac{\rho'^2}{V\rho'\rho''} = Va^t\beta - (Va^{t+1}\beta)^2 (\beta\gamma)^{-1}.$$

(6.) To interpret this result, we may write it as follows,

$$\text{XVII.} \dots \kappa = \rho - \frac{\rho_1^2}{V\rho\rho' \cdot \rho^{-1}}, \quad \text{where XVIII.} \dots \rho_1 = \frac{2}{\pi} \rho' = Va^{t+1}\beta;$$

so that ρ_1 is the *conjugate semidiameter* of the ellipse (comp. 869, (4.)), and $V\rho\rho' : \rho'$ is the *perpendicular from the centre of that curve on the tangent*. We recover then, by this simple analysis, the known result, that the *radius of curvature* of an ellipse is equal to the *square of the conjugate semidiameter, divided by the perpendicular*.

(7.) We may also write the equation XVI. under the form,

$$\text{XIX.} \dots \kappa = \rho - \frac{\rho_1^2}{V\rho\rho_1}, \quad \text{where XX.} \dots V\rho\rho_1 = \beta\gamma = \text{const.};$$

* The expressions X. XI. may also be easily deduced by limits, from the construction in 888, (2.).

† It may be remarked that the quantity κ , or $T\Psi''$, in the investigation (882) respecting *geodesics on a developable*, represents thus the *curvature of the cusp-edge*, for any proposed value of the arc, x , of that curve.

‡ These values XV. might have been obtained *without integrations*, but this seemed to be the readiest way.

and may interpret it as expressing, that the radius of curvature is equal to the cube of the conjugate semidiameter, divided by the constant parallelogram under any two such conjugates; or by the rectangle under the major and minor semi-axes, which are here the vectors β and γ (comp. 314, (2)).

(8.) The expression XVI. or XIX. for κ is easily seen to vanish, as it ought to do, at the limit where the ellipse becomes a circle, by the cylinder being cut perpendicularly, or by the condition $S\alpha\beta = 0$ being satisfied; and accordingly if we write,

$$\text{XXI.} \dots e = S\alpha\beta = \text{eccentricity of ellipse, or } \text{XXII.} \dots \gamma^2 = (1 - e^2)\beta^2,$$

we easily find the expressions,

$$\text{XXIII.} \dots \rho = \beta S.a^t + \gamma S.a^{t-1}, \quad \rho_1 = -\beta S.a^{t+1} + \gamma S.a^t;$$

$$\text{XXIV.} \dots \rho_1^2 = \beta^2(1 - e^2(S.a^t)^2), \quad \frac{\rho_1}{\sqrt{\rho\rho_1}} = \frac{\rho_1}{\beta\gamma} = \beta^{-2} \left(\beta S.a^t + \frac{\gamma S.a^{t+1}}{1 - e^2} \right);$$

so that the formula XIX. becomes,

$$\text{XXV.} \dots \kappa = e^2 \left(\beta(S.a^t)^3 - \frac{\gamma(S.a^{t+1})^2}{1 - e^2} \right) = e^2 \left(\beta(S.a^t)^3 - \frac{\beta^2}{\gamma}(S.a^{t+1})^2 \right),$$

thus containing e^2 as a factor.

(9.) And it may be remarked in passing, that the expression XVI., or its recent transformation XXV., for κ as a function of t , may be considered as being in quaternions the vector equation (comp. 99, L, or 369, L) of the evolute* of the ellipse, or the equation of the locus of centres of curvature of that plane curve; and that the last form gives, by elimination of t (comp. † 315, (1.), and 371, (5.)), the following system of two scalar equations for the same evolute,

$$\text{XXVI.} \dots \left(S \frac{\rho}{\beta} \right)^{\frac{2}{3}} + \left(S \frac{\gamma \rho}{\beta^2} \right)^{\frac{2}{3}} = e^{\frac{2}{3}}, \quad S\beta\gamma\kappa = 0;$$

$$\text{or } \text{XXVI.} \dots (S\beta\kappa)^{\frac{2}{3}} + (S\gamma\kappa)^{\frac{2}{3}} = (e\beta)^{\frac{2}{3}}, \text{ \&c. ;}$$

which will be found to agree with known results.

(10.) As another example of application to a plane curve, we may consider the hyperbola,

$$\text{XXVII.} \dots \rho = t\alpha + t^{-1}\beta, \quad \text{comp. 371, II.,}$$

with α and β for asymptotes, and with its centre at the origin. In this case the derived vectors are,

$$\text{XXVIII.} \dots \rho' = \alpha - t^{-2}\beta, \quad \rho'' = 2t^{-3}\beta,$$

whence

$$\text{XXIX.} \dots \sqrt{\rho''\rho'} = 2t^{-3}\sqrt{\beta\alpha} = t^{-4}\sqrt{\rho\rho'},$$

and the formula IV. becomes,

$$\text{XXX.} \dots \kappa - \rho = \frac{(t\rho')^2}{\sqrt{\rho\rho'}:\rho'} = \frac{r\tau^2}{ov},$$

where ov is the perpendicular from the centre o on the tangent to the curve at r , and $r\tau$ is the portion of that tangent, intercepted between the same point r and an asymptote (comp. (6.) and 371, (3.)).

* That is to say, of the plane evolute; for we shall soon have occasion to consider briefly those evolutes of double curvature, which have been shown by Monge to exist, even when the given curve is plane.

† In lately referring (373, (1.)) to the formula 315, V., that formula was inadvertently printed as $(a^t)^2 + (a^{t-1})^2 = 1$, the sign S . before each power being omitted.

(11.) We may also interpret the denominator in XXX. as denoting the *projection* of the *semidiameter* or on the *normal*, or as the line *NP* where *N* is the foot of the perpendicular from the curve on that normal line; if then *x* be the sought centre of the osculating circle, we have the *geometrical* equations,

$$\text{XXXI.} \dots \text{NP} \cdot \text{PK} = \text{PT}^2, \quad \text{XXXII.} \dots \angle \text{NPK} = \frac{\pi}{2};$$

whereof the last furnishes evidently an extremely simple *construction* for the *centres of curvature* of an *hyperbola*, which we shall soon find to admit of being extended, with little modification, to a *spherical conic** and its *cyclic arcs*.

(12.) The *logarithmic spiral* with its *pole* at the *origin*,

$$\text{XXXIII.} \dots \rho = a^t \beta, \quad \text{Sa} \beta = 0, \quad \text{Ta} > 1, \quad \text{comp. 814, (5.)}$$

may be taken as a *third example* of a *plane curve*, for the application of the foregoing formulæ. A first derivation gives, by 888, VII.,

$$\text{XXXIV.} \dots \rho' = (c + \gamma)\rho = \rho(c - \gamma), \quad \rho' \rho^{-1} = c + \gamma, \quad \text{if } c = 1\text{T}a, \text{ and } \gamma = \frac{\pi}{2} \text{U}a;$$

the *constant quaternion quotient*, $\rho' : \rho$, here showing that the prolonged *vector* or makes with the *tangent* *PT* a *constant angle*, κ , which is given by the formula,

$$\text{XXXV.} \dots \tan \kappa = (\text{TV} : \text{S}) (\rho' : \rho) = c^{-1} \text{T}\gamma, \quad \text{or} \quad \cot \kappa = \frac{2}{\pi} 1\text{T}a; \dagger$$

and a second derivation gives next,

$$\text{XXXVI.} \dots \rho'' = (c + \gamma)^2 \rho, \quad \text{V}\rho'' \rho' = (c^2 - \gamma^2) \rho^2 \gamma = \rho'^2 \gamma.$$

The formula IV. becomes therefore, in this case,

$$\text{XXXVII.} \dots \kappa = \rho + \rho' \gamma^{-1} = \rho c \gamma^{-1} = -c \gamma^{-1} \rho = \frac{21\text{T}a}{\pi \text{T}a} \cdot a^{t+1} \beta;$$

the *evolute* is therefore a *second spiral*, of the same kind as the first, and the *radius of curvature* *KP* subtends a *right angle* at the *common pole*. But we cannot longer here delay on *applications within the plane*, and must resume the treatment by quaternions of *curves of double curvature*.

390. When the *logic* by which the expression 389, IV. was obtained, for the *vector* κ of the centre of the osculating circle, has once been fully *understood*, the *process* may be conveniently and safely *abridged*, as follows. Referring still to Fig. 77, we may write briefly,

* It was in fact for the *spherical curve* that the *geometrical construction* alluded to was *first* perceived by the writer, soon after the invention of the quaternions, and as a consequence of calculation with them: but it has been thought that a sub-article or two might be devoted, as above, to the *plane case*, or *hyperbolic limit*, which may serve at least as a verification.

† If r be radius vector, and θ polar angle, and if we suppose for simplicity that $\text{T}\beta = 1$, the ordinary *polar equation* of the spiral becomes $r = a^\theta$, with $a = \text{Ta}^{\frac{2}{\pi}}$, and $\cot \kappa = la$, as usual.

as equations which are all *ultimately true*, or true *at the limit*, in a sense which is supposed to be now distinctly seen:

$$\text{I. . . } PT = d\rho, \quad TQ = \frac{1}{2}d^2\rho, \quad PN = (\text{part of } PQ \perp PT) = \frac{Vd^2\rho d\rho}{2d\rho},$$

by 203, &c.; whence, ultimately,

$$\text{II. . . } \kappa - \rho = PC = \frac{PQ^2}{2PN} = \frac{PT^2}{2PN} = \frac{d\rho^2}{Vd^2\rho d\rho},$$

as before: this *last* expression, in which $Vd^2\rho d\rho$ denotes briefly $V(d^2\rho \cdot d\rho)$, being *rigorous*, and permitting the choice of *any scalar*, to be used as the *independent variable*. And then, by writing,

$$\text{III. . . } d\rho = \rho' dt, \quad d^2\rho = 0, \quad d^3\rho = \rho'' d^2t,$$

the factor d^2t disappears, and we pass at once to the expression,

$$\text{IV. . . } \kappa - \rho = \frac{\rho'^2}{V\rho''\rho'}, \quad 389, \text{ IV.},$$

which had been otherwise found before.

(1.) When the *arc* of the curve is taken for the independent variable, then (comp. 380, (12.), &c.) the expression II. reduces itself to the following,

$$\text{V. . . } \kappa - \rho = \frac{d\rho^2}{d^2\rho}, \quad \text{because } Sd^2\rho d\rho = 0;$$

and accordingly the *angle* PRQ in Fig. 77 is then *ultimately right* (comp. 383, (5.)), so that we may at once write, with *this choice* of the scalar variable,

$$\text{VI. . . } \kappa - \rho = (\text{ult.}) PC = (\text{ult.}) \frac{PT^2}{2TQ} = \frac{d\rho^2}{d^2\rho}, \quad \text{as above.}$$

(2.) Suppose then that we have thus *geometrically* (and *very simply*) deduced the expression V. for $\kappa - \rho$, for this *particular choice* of the scalar variable; and let us consider how we might thence *pass*, in *calculation*, to the more *general* formula II., in which that variable is left *arbitrary*. For this purpose, we may write, by principles already stated,

$$\begin{aligned} \text{VII. . . } (\rho - \kappa)^{-1} &= \frac{d^2\rho}{(Td\rho)^2} = \frac{1}{Td\rho} d \frac{d\rho}{Td\rho} = \frac{dUd\rho}{Td\rho} = \frac{Vd^2\rho d\rho^{-1} \cdot Ud\rho}{Td\rho} \\ &= - \frac{Vd^2\rho d\rho^{-1}}{d\rho} = \frac{Vd\rho d^2\rho}{d\rho^3}; \end{aligned}$$

and the required transformation is accomplished.

(3.) And generally, if s denote the *arc* of any curve of which ρ is the variable vector, we may establish the *symbolical equations*,

$$\text{VIII. . . } D_s = \frac{1}{Td\rho} d; \quad D_s^2 = \frac{1}{Td\rho} d \frac{1}{Td\rho} d = \left(\frac{1}{Td\rho} d \right)^2; \quad \&c.$$

(4.) For example (comp. 389, XII.), the *Vector of Curvature*, $D_s^2\rho$, admits of being expressed *generally* under any one of the five last forms VII.

391. Instead of determining the vector κ of the centre of the osculating circle by *one vector expression*, such as 389, IV., or any of its transformations, we may determine it by a system of *three scalar equations*, such as the following,

$$\begin{aligned} \text{I. . . } S(\kappa - \rho)\rho' &= 0; & \text{II. . . } S(\kappa - \rho)\rho'' - \rho'^2 &= 0; \\ & & \text{III. . . } S(\kappa - \rho)\rho'\rho'' &= 0, \end{aligned}$$

of which it may be observed that the second is the *derivative* of the first, if κ be treated as constant (comp. 386, (4.)); and of which the first expresses (369, IV.) that the sought *centre* is *in the normal plane* to the curve, while the third expresses (376, V.) that it is *in the osculating plane*; and the second serves to fix its position *on the absolute normal* (379), in which those two planes intersect.

(1.) Using *differentials* instead of derivatives, but leaving still the independent variable arbitrary, we may establish this equivalent system of three equations,

$$\text{IV. . . } S(\kappa - \rho)d\rho = 0; \quad \text{V. . . } S(\kappa - \rho)\rho^2 d\rho - d\rho^3 = 0; \quad \text{VI. . . } S(\kappa - \rho)d\rho d^2\rho = 0;$$

of which the second is the differential of the first, if κ be again treated as constant.

(2.) It is also permitted (comp. 369, (2.), 376, (8.), and 380, (2.)), with the same supposition respecting κ , to write these equations under the forms,

$$\text{VII. . . } dT(\kappa - \rho) = 0; \quad \text{VIII. . . } d^2T(\kappa - \rho) = 0; \quad \text{IX. . . } dUV(\kappa - \rho)d\rho = 0;$$

and to connect them with *geometrical interpretations*.

(3.) For instance, we may say that the *centre* of the osculating circle is the point, in which the osculating *plane*, III. or VI. or IX., is *intersected* by the *axis* of that circle; namely, by the *right line* which is drawn through its centre, at right angles to its plane; and which is represented by the *two scalar equations*,

$$\text{I. and II., or IV. and V., or VII. and VIII.}$$

(4.) And we may observe (comp. 370, (8.)), that whereas for a point κ taken arbitrarily *in the normal plane* to a curve at a given point P , we can only say *in general*, that if a *chord* PQ be called *small* of the *first order*, then the *difference of distances*, $\overline{RQ} - \overline{RP}$, is small of an order *higher* than the *first*; yet, if the point κ be taken *on the axis* (3.) of the osculating circle, then this difference of distances is small, of an order *higher than the second*, in virtue of the equations VII. and VIII.

(5.) The *right line* I. II., or IV. V., or VII. VIII., as being the *locus of points* which may be called *poles* of the osculating circle, on all possible *spheres* passing through it, is also called the *Polar Axis* of the curve itself, corresponding to the given *point of osculation*.

(6.) And because the equation II. is (as above remarked) the *derivative* of I., the known theorem follows (comp. 386), that the *locus* of all such *polar axes* is a *developable surface*, namely that which is called the *Polar Developable*, or the *envelope of the normal planes* to the given curve; of which surface we shall soon have occasion to consider briefly the *cuspidal edge*.

392. The following is an entirely different method of investigating, by quaternions, not merely the *radius* or the *centre* of the *osculating circle* to a curve in space, but the *vector equation* of that circle itself: and in a way which is *applicable alike*, to *plane curves*, and to curves of *double curvature*.

(1.) In general, conceive that $or = r$ is a *given tangent* to a circle, at a given point which is for the moment taken as the origin; and let $rp' = \rho'$ represent a *variable tangent*, drawn at the extremity of the variable chord $op = \rho$: also let v be the *intersection*, of these two tangents, of these two tangents. Then the isosceles triangle oup , combined with the formula 324, XI. for the differential of a reciprocal, gives easily the equations,

$$\begin{aligned} \text{I.} \dots \rho' \parallel \rho r^{-1} \rho; \quad \text{II.} \dots \nabla r \rho^{-1} \rho' \rho^{-1} &= -(\nabla r \rho^{-1})' = 0; \\ \text{III.} \dots \nabla r \rho^{-1} &= \text{const.} = \nabla r \alpha^{-1}, \text{ as in 296, IX.} \end{aligned}$$

if α be the vector oa of any second *given point* a of the circumference.

(2.) The *vector equation* of the circle rqd (389) is therefore,

$$\text{IV.} \dots \nabla \frac{2\rho'}{\omega - \rho} = \nabla \frac{2\rho'}{\rho' - \rho} = \frac{2}{t} \nabla \cdot (1 + \frac{1}{2} t u \rho'' \rho'^{-1})^{-1} = -\nabla \cdot u \rho'' \rho'^{-1} (1 + \frac{1}{2} t u \rho'' \rho'^{-1})^{-1};$$

whence, passing to the *limit* ($t = 0, u = 1$), the analogous equation of the *osculating circle* is at once found to be,

$$\text{V.} \dots \nabla \frac{2\rho'}{\omega - \rho} = -\nabla \frac{\rho''}{\rho'}, \quad \text{or} \quad \text{VI.} \dots \nabla \left(\frac{2d\rho}{\omega - \rho} + \frac{d^2\rho}{d\rho} \right) = 0;$$

with the verification (comp. 296, (9.)), that when we suppose,

$$\text{VII.} \dots \omega - \rho = 2(\kappa - \rho) \perp \rho',$$

the vector κ of the *centre* is seen to satisfy the equation,

$$\text{VIII.} \dots \frac{\rho'}{\kappa - \rho} = -\nabla \frac{\rho''}{\rho'}, \quad \text{or} \quad \text{IX.} \dots \frac{d\rho}{\kappa - \rho} + \nabla \frac{d^2\rho}{d\rho} = 0;$$

which agrees with recent results (389, IV., &c.).

(3.) Instead of conceiving that a circle is described (389), so as to *touch* a given curve (Fig. 77) at P , and to *cut* it at *one* near point Q , we may conceive that a circle *cuts* the curve in the *given point* P , and *also* in *two* near points, Q and R , unconnected by any given *law*, but *both* tending together to *coincidence* with P : and may inquire what is the *limiting position* (if any) of the circle rqe , which thus *intersects* the curve in *three near points*, whereof *one* (P) is *given*.

(4.) In general, if α, β, ρ be *three co-initial chords*, oa, ob, op , of any circle, their *reciprocals* $\alpha^{-1}, \beta^{-1}, \rho^{-1}$, if still co-initial, are *termino-collinear* (260); applying which principle, we are led to investigate the condition for the three co-initial vectors,

$$\text{X.} \dots (\omega - \rho)^{-1}, \quad (s\rho' + \frac{1}{2}s^2u\rho'')^{-1}, \quad (t\rho' + \frac{1}{2}t^2u\rho'')^{-1},$$

with $u_0 = 1$, thus *ultimately terminating* on *one right line*; or for our having ultimately a relation of the form,

$$\text{XI.} \dots \frac{xs + yt}{\omega - \rho} = \frac{x}{\rho' + \frac{1}{2}s\rho''} + \frac{y}{\rho' + \frac{1}{2}t\rho''};$$

or XII. . .
$$\frac{(xs + yt)\rho'}{\omega - \rho} = \frac{x}{1 + \frac{1}{2}s\rho''\rho'^{-1}} + \frac{y}{1 + \frac{1}{2}t\rho''\rho'^{-1}}$$

$$= x + y - \frac{1}{2}(xs + yt)\rho''\rho'^{-1} + \&c. :$$

in which last equation, both members are generally *quaternions*.

(5.) The comparison of the *scalar parts* gives here no useful information, on account of the *arbitrary* character of the coefficients x and y ; but *these* disappear, with the two *other* scalars, s and t , in the comparison of the *vector parts*, whence follows the *determinate* and *limiting equation*,

$$\text{XIII. . . } 2V\rho'(\omega - \rho)^{-1} = -V\rho''\rho'^{-1},$$

which evidently agrees with V.

(6.) It is then found, by this little quaternion calculation, as was of course to be expected,* that the *circle* (8.), through *any three near points* of a curve in space, coincides *ultimately* with the *osculating circle*, if the *latter* be still *defined* (889) with reference to a *given tangent*, and a *near point*, which *tends* to coincide with the *given point of contact*.

393. An osculating circle to a curve of *double curvature* does not generally meet that curve *again*; but it intersects generally a *plane curve*, of the degree n , to which it osculates, in $2n - 3$ points, distinct from the point P of osculation, whereof *one* at least must be *real*, although it may happen to *coincide* with that point P: and such a circle intersects also generally a *spherical curve* of double curvature, and of the degree n , in $n - 3$ other points, namely in those where the *osculating plane* to the curve meets it again. An *example* of each of these two last cases, as treated by quaternions, may be useful.

(1.) In general, if we clear the recent equation, 392, V. or XIII., of fractions, it becomes,

$$\text{I. . . } 0 = 2\rho'^2V\rho'(\omega - \rho) + (\omega - \rho)^2V\rho''\rho';$$

in which $\rho = OP =$ the vector of the given point of osculation, and ρ', ρ'' are its first and second derivatives, taken with respect to any scalar variable t , and for the particular value (whether zero or not) of that variable, which answers to the *particular point* P; while ω denotes generally the vector of *any point* upon the circle, which osculates to the given curve at that point P.

(2.) Writing then (comp. 889, (10.)),

$$\text{II. . . } \rho = t\alpha + t^1\beta, \quad \rho' = \alpha - t^2\beta, \quad \rho'' = 2t^3\beta,$$

and

$$\text{III. . . } \omega = OQ = x\alpha + x^1\beta,$$

to express that we are seeking for the *remaining intersection* Q of a *plane hyperbola*

* This conclusion is indeed so well known, and follows so obviously from the doctrine of *infinitesimals*, that it is only deduced here as a *verification* of previous formulae, and for the sake of *practice* in the present Calculus.

with its *osculating circle* at P , the equation I. becomes, after a few easy reductions, including a division by $V\alpha\beta$, the following *biquadratic* in x ,

$$\text{IV.} \dots 0 = (x - t)^2 (t^2 \alpha^2 x - \beta^2);$$

in which the *cubic factor* is to be set aside, as answering only to the point P itself.

(8.) Substituting then, in III., the remaining value IV. of x , we find the expression,

$$\text{V.} \dots \omega = oq = \frac{(t\alpha)^2}{t'\beta} + \frac{(t'\beta)^2}{t\alpha} = \frac{1}{2} \left\{ \frac{(2t\alpha)^2}{2t'\beta} + \frac{(2t'\beta)^2}{2t\alpha} \right\};$$

comparing which with 871, (3.), we see that if the tangent to the hyperbola at the given point P intersects the asymptotes in the points A, B , then the tangent at the sought point Q meets the same lines OA, OB in points A', B' , such that

$$\text{VI.} \dots OA \cdot OA' = OB^2, \quad OB \cdot OB' = OA'^2;$$

whence Q is at once found, as the bisecting point of the line $A'B'$.

(4.) A still more simple construction, and one more obviously agreeing with known results, may be derived from the following expression for the chord PQ :

$$\begin{aligned} \text{VII.} \dots PQ = \omega - \rho &= (t^2\beta^2 - t'^2\alpha^2) (t\alpha^2\beta - t'\alpha\beta^2) \\ &= (t^2\beta^2 - t'^2\alpha^2) \alpha\rho'\beta \parallel \alpha\rho'^{-1}\beta; \end{aligned}$$

whence it follows (comp. 226) that if this chord PQ , both ways prolonged, meets the two asymptotes OB and OA in the points R and S , we have then the *inverse similitude of triangles* (118),

$$\text{VIII.} \dots \Delta ROS \sim \alpha' AOB.$$

(5.) As regards the *equality* of the *intercepts*, RP and QS , it can be verified *without specifying* the *second point* Q on the hyperbola, or the *second scalar*, x , by observing that the formula III., combined with the first equation II., conducts to the expressions,

$$\text{IX.} \dots OR = \frac{x\rho - t\omega}{x - t} = (x^{-1} + t^{-1})\beta, \quad OS = \frac{t\rho - x\omega}{t - x} = (x + t)\alpha;$$

which give, generally,

$$\text{X.} \dots RP = QS = t\alpha - x^{-1}\beta.$$

(6.) And as regards the *general reduction*, of the determination of the *osculating circle* to a spherical curve of double curvature, to the determination of the *osculating plane*, it is sufficient to observe that when we take the centre of the sphere for the origin, and therefore write (comp. 881, XIV.),

$$\text{XI.} \dots \rho^2 = \text{const.}, \quad S\rho\rho' = 0, \quad S\rho\rho'' = -\rho'^2,$$

then if we operate on the vector equation I. with the symbol $V. \rho$, and divide by $-\rho'^2$, there results the scalar equation,

$$\text{XII.} \dots 0 = 2S\rho(\omega - \rho) + (\omega - \rho)^2 = \omega^2 - \rho^2,$$

which expresses that the *circle* is entirely contained on the *same spherio* surface* as the curve; while the *other* scalar equation,

$$\text{XIII.} \dots 0 = S\rho''\rho'(\omega - \rho),$$

obtained by operating on I. with $S. \rho''$, expresses (comp. 876, V.) that the same

* This conclusion is geometrically evident, but is here drawn as above, for the sake of practice in the quaternions.

circle is in the osculating plane :* so that its centre κ is the foot of the perpendicular let fall on that plane from the origin, and we may therefore write (comp. 385, VI.),

$$\text{XIV.} \dots \text{OK} = \kappa = \frac{S\rho''\rho'\rho}{\sqrt{\rho''\rho'}}, \text{ with the relations, XV.} \dots S\frac{\omega}{\kappa} = S\frac{\rho}{\kappa} = 1;$$

and with the verification that the expression XIV. agrees with the general formula, 389, IV., because

$$\text{XVI.} \dots \rho\sqrt{\rho''\rho'} + \rho'^3 = S\rho''\rho'\rho,$$

when the conditions XI. are satisfied.

(7.) And even if the given curve be not a spherical one, yet if we retain the general expression for κ ,

$$\text{XVII.} \dots \kappa = \rho + \frac{\rho'^3}{\sqrt{\rho''\rho'}}, \quad 389, \text{IV.},$$

and operate on I. with $S.\rho''$ and $S.\rho''\rho'$, we find again the equation XIII. of the osculating plane, combined with a new scalar equation, which may after a few reductions be written thus,

$$\text{XVIII.} \dots (\omega - \kappa)^2 = (\rho - \kappa)^2;$$

and which represents a new sphere, whereon the osculating circle to the curve is a great circle.

394. To give now an example of a spherical curve of double curvature, with its osculating circle and plane for any proposed point P , and with a determination of the point Q in which these meet the curve again (393), we may consider that spherical conic, or sphero-conic, of which the equations are (comp. 357, II.),

$$\text{I.} \dots \rho^2 + r^2 = 0, \quad \text{II.} \dots g\rho^2 + S\lambda\rho\mu = 0;$$

namely the intersection of the sphere, which has its centre at the origin, and its radius = r , with a cone of the second order, which has the same origin for vertex, and has the given lines λ and μ for its two (real) cyclic normals. And thus we shall be led to some sufficiently simple spherical constructions, which include, as their plane limits, the analogous constructions recently assigned for the case of the common hyperbola.

(1.) Since $S\lambda\rho\mu\rho = 2S\lambda\rho S\mu\rho - \rho^2 S\lambda\mu$ (comp. 357, II.), the equations I. and II. allow us to write, as their first derivatives, or at least as equations consistent therewith,

$$\text{III.} \dots S\rho\rho' = 0, \quad S\lambda\rho' + S\lambda\rho = 0, \quad S\mu\rho' - S\mu\rho = 0,$$

because the independent variable is here arbitrary, so that we may conceive the first derived vector ρ' to be multiplied by any convenient scalar; in fact, it is only the

* Compare the Note immediately preceding.

direction of this tangential vector ρ' which is here important, although we must continue the derivations consistently, and so must write, as consequences of III., the equations,

$$\text{IV.} \dots S\rho\rho'' + \rho^2 = 0, \quad S\lambda\rho'' + S\lambda\rho' = 0, \quad S\mu\rho'' - S\mu\rho' = 0.$$

(2.) Introducing then the auxiliary vectors,

$$\text{V.} \dots \eta = V\lambda\mu, \quad \sigma = \lambda S\mu\rho + \mu S\lambda\rho, \quad \tau = \rho + \rho', \quad \nu = \rho - \rho',$$

whence

$$\text{VI.} \dots 0 = S\eta\sigma = S\lambda\tau = S\mu\nu, \quad S\rho\sigma = 2S\lambda\rho S\mu\rho, \quad S\mu\tau = 2S\mu\rho, \quad S\lambda\nu = 2S\lambda\rho, \\ \tau^2 = \nu^2 = \rho^2 + \rho'^2,$$

and by new derivations,

$$\text{VII.} \dots \sigma' = V\eta\rho, \quad \tau' = \rho' + \rho'', \quad \nu' = \rho' - \rho'', \quad S\lambda\tau' = S\mu\nu' = 0, \quad S\mu\tau' = S\mu\rho, \\ S\lambda\nu' = -S\lambda\nu,$$

we see first that τ and ν are the vectors OT and OV of the points in which the *rectilinear tangent* to the curve at P meets the two *cyclic planes*, perpendicular respectively to λ and μ ; and because the *radius* OP is seen to be the *perpendicular bisector* of the *linear intercept* TU between those two planes, so that

$$\text{VIII.} \dots \rho' = PT = UP \perp OP, \quad \text{we have} \quad \text{IX.} \dots UOP = POT,$$

or

$$\text{X.} \dots \angle AP = \angle PB,$$

if the *tangent arc* on the sphere, to the same conic at the same point P , meet the two *cyclic arcs* CA and CB in the points A and B : the *intercepted arc* AB being thus *bisected* at its point of *contact* P , which is a well-known property of such a curve.

(3.) Another known property of a *sphero-conic* is, that for any *one* such curve the *sum of the two spherical angles* CAB and ABC , and therefore also the *area of the spherical triangle* ABC , is *constant*. We can only here remark, in passing, that quaternions recognise this property, under the form (comp. II.),

$$\text{XI.} \dots \cos(A + B) = -S\lambda\rho\mu\rho = -g : T\lambda\mu = \text{const.}$$

(4.) The scalar equations III. and IV. give immediately the vector expressions,

$$\text{XII.} \dots \rho' = \frac{V\rho(\lambda S\mu\rho + \mu S\lambda\rho)}{S\lambda\mu\rho}, \quad \text{XIII.} \dots \rho'' = \rho - \frac{(\rho^2 + \rho'^2)V\lambda\mu}{S\lambda\mu\rho};$$

or by (2.),

$$\text{XIV.} \dots \rho' = \frac{V\rho\sigma}{S\eta\rho}, \quad \text{and} \quad \text{XV.} \dots \rho'' = \rho - \xi, \quad \text{if} \quad \text{XVI.} \dots \xi = \frac{\tau^2\eta}{S\eta\rho} \\ = \tau - \tau' = \nu + \nu',$$

the new auxiliary vector ξ being thus that of the point x , in which the osculating plane to the conic at P meets the line η of intersection of the cyclic planes: so that we have the geometrical expressions,

$$\text{XVII.} \dots \rho'' = xP, \quad \tau' = xT, \quad -\nu' = xU, \quad \text{if} \quad \xi = OX,$$

and the lines* τ' and ν' are the traces of the osculating plane on those two cyclic

* We may also consider the derived vectors τ' and ν' , or the lines xT and xU , as *corresponding tangents*, at the points T and U (2.), to the *two sections*, made by the *cyclic planes*, of that *developable surface* which is the *locus of the tangents* TP to the *spherical conic* in question.

planes, or of the latter on the former; while σ and σ' , as being perpendicular respectively to ρ' and ρ , while each $\perp \eta$, are the traces on the plane $\lambda\mu$ of the two cyclic normals, of the normal plane to the conic at the point P , and of the tangent plane to the sphere at that point: or at least these lines have the *directions* of those *traces*.

(5.) Already, from the expression XVI. for the portion OX of the radius OC (2.), or of that radius prolonged, which is cut off by the *osculating plane* at P , we can derive a simple *construction* for the position of the *spherical centre*, or *pole*, say E , of the *small circle* which osculates at that point P , to the proposed *sphero-conic*. For if we take the radius r for unity, we have the trigonometric expressions,

$$\text{XVIII.} \dots \sec CE \cos EP = (T\xi = T, r^2 : SU\eta^{-1}\rho) \sec^2 PE \sec CP;$$

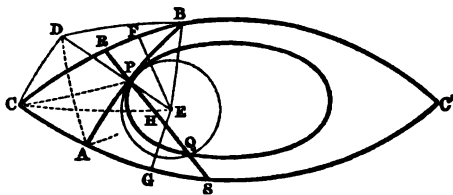


Fig. 80.

or letting fall (comp. Fig. 80) the perpendicular CD on the normal arc PE ,

$$\text{XIX.} \dots \cos DE = \cos DP \cos PE \cdot \cos PE \cos PE = \cos DB \cos BE;$$

or finally,

$$\text{XX.} \dots \angle DBE \text{ (or } \angle DAE) = \frac{\pi}{2}.$$

(6.) But although it is a perfectly *legitimate* process to *mix* thus *spherical trigonometry* with *quaternions* (since in fact the latter *include* the former), yet it may be satisfactory to deduce this last result by a more *purely* quaternionic method, which can easily be done as follows. The values (4.) of ρ' and ρ'' give,

$$\begin{aligned} \text{XXI.} \dots V\rho'\rho''S\eta\rho &= \rho S\sigma\rho'' - \sigma S\rho\rho'' = \rho S\rho\sigma + \rho^2\sigma \\ &= (r - \rho)S\sigma r + \sigma S\rho' r = rS\sigma r + Vr\rho'\sigma \quad ||| r, \quad Vr\rho'\sigma, \end{aligned}$$

in which $\rho'\sigma$ denotes a vector $\perp \rho'$ (because $S\rho'\sigma = 0$), and $||| \eta, \rho'$ (because $S\eta\rho'\rho'\sigma = 0$); this line $\rho'\sigma$ has therefore the direction of the *projection* of the line η on a plane perpendicular to ρ' , and we are thus led to draw, through the line OC of intersection of the cyclic planes, a *plane* COD perpendicular to the normal plane to the conic at P , or to let fall (as in Fig. 80) a perpendicular *arc* CD on the normal arc PD ; after which the normal to the sought osculating plane, or the *axis* OE of the osculating circle sought, as being $|| V\rho'\rho''$, will be contained in the *plane* through the trace r , or OT , or OB , which is *perpendicular* to the *plane* of r and $\rho'\sigma$, or to the plane DOB ; and therefore the *spherical angle* DBE (or DAB) will be a *right angle*, as before.

(7.) We may also observe that if X be the *centre* of the osculating circle, considered in its *own plane*, or the *foot* of the *perpendicular* on that plane from O , then by XXI.,

$$\text{XXII.} \dots \text{OK} = \kappa = \frac{S\rho\rho'\rho''}{\sqrt{\rho\rho''}} = \frac{r^2 S\rho\sigma}{\rho S\rho\sigma + \rho'^2\sigma}, \quad \text{KP} = \rho - \kappa = \frac{\rho^2 \sqrt{\rho\sigma}}{\rho S\rho\sigma + \rho'^2\sigma},$$

and therefore

$$\text{XXIII.} \dots \frac{\text{KP}}{\text{OK}} = \frac{\rho - \kappa}{\kappa} = \frac{\rho^2}{r^2} \frac{\sqrt{\rho\sigma}}{S\rho\sigma}, \quad \text{XXIV.} \dots \tan \text{KP} = \sin^2 \text{PB} \cot \text{PD},$$

which gives again the angular relation XX; the quotient XXIII. being thus a vector, as it ought by 893, XV. to be; and the trigonometric formula XXIV. being obtained from its expression, by observing that

XXV. . . $\text{Tr}'r^{-1} = \overline{\text{PT}} : \overline{\text{OT}} = \sin \text{ROT} = \sin \text{PB}$, and $(\sqrt{S})\rho\sigma = \text{Up}' \cdot \cot \text{PD}$, because $\sigma \perp \rho'\sigma$, but $\| \rho, \rho'\sigma$, or $\rho'\sigma \perp \sigma$, but $\| \rho, \sigma$.

(8.) The rectangularity of the planes of r, κ and $r, \rho'\sigma$ is also expressed by the equation,

$$\text{XXVI.} \dots 0 = S(\sqrt{\kappa r} \cdot \sqrt{\rho'\sigma r}) = S\kappa r S\rho'\sigma r - r^2 S\rho'\sigma \kappa;$$

in proving which we may employ the values,

$$\text{XXVII.} \dots S r \kappa^{-1} = 1, \quad S\rho'\sigma \kappa^{-1} = (-r^2 \rho'^2 S\eta\rho) S\rho'\sigma r^{-1}.$$

(9.) We may also interpret these equations XXVII., as expressing the system of the two relations,

$$\text{XXVIII.} \dots \kappa^{-1} - r^{-1} \perp r, \quad \kappa^{-1} - r^{-1} \perp \rho'\sigma;$$

from which it follows that κ^{-1} , and therefore also that κ , is a line in the plane so drawn through r , as to be perpendicular to the plane through r and $\rho'\sigma$, as before.

(10.) And the two relations XXVIII. are both included in the following expression,

$$\text{XXIX.} \dots \kappa^{-1} - r^{-1} = \sqrt{r^{-1}\rho'\sigma} : S\rho\sigma.$$

(11.) We may also easily deduce, from the foregoing *spherical construction*, the following *trigonometric expressions*, for the *arcual radius* $r = \text{KP}$ of the *osculating small circle* (5.), and for the *angle* $\alpha = \text{PAE} = \text{EBP}$ which it subtends at Δ or at B :

$$\text{XXX.} \dots \tan r = \sin \frac{c}{2} \tan \alpha; \quad \text{XXXI.} \dots \tan \alpha = \frac{1}{2} (\cot A + \cot B);$$

A and B here denoting, as in XL, the *base angles* of the triangle ΔBC with C for vertex, and c denoting as usual the *base* ΔB , namely the portion of the *arcual tangent* (2.) to the conic, which is intercepted between the cyclic arc.

(12.) The *osculating plane and circle* at P being thus fully and in various ways determined, we may next inquire (893) *in what point Q* do they meet the conic again. In symbols, denoting by ω the vector of this point, we have the *three scalar equations*,

$$\text{XXXII.} \dots S\kappa\omega = S\kappa\rho, \quad S\lambda\omega S\mu\omega = S\lambda\rho S\mu\rho, \quad \omega^2 = \rho^2,$$

which are all evidently satisfied by the value $\omega = \rho$, but can in general be satisfied also by one other vector value, which it is the object of the problem to assign.

(13.) We satisfy the two first of these three equations XXXII., by assuming the expression,

$$\text{XXXIII.} \dots \omega = \xi + \frac{1}{2} (x^{-1}r' - xv'),$$

in which x is any scalar; in fact we have the relations,

$$\text{XXXIV.} \dots S\kappa\xi = S\kappa\rho, \quad S\lambda v' = -2S\lambda\rho, \quad S\mu r' = 2S\mu\rho, \\ 0 = S\lambda\xi = S\mu\xi = S\lambda r' = S\mu v' = S\kappa r' = S\kappa v',$$

whence XXXIII. gives, XXXV. . . $S\lambda\omega = xS\lambda\rho, \quad S\mu\omega = x^{-1}S\mu\rho, \&c.$

And because

$$\text{XXXVI.} \dots \rho = \xi + \frac{1}{2}(r' - v'),$$

we shall satisfy also the *third* equation XXXII., if we adopt for x any root of that new scalar equation, which is obtained by equating the square of the expression XXXIII. for ω , to what that square becomes when x is changed to λ .

(14.) To facilitate the formation of this new equation, we may observe that the relations,

$$\xi = \rho - \rho'', \quad r' = \rho' + \rho'', \quad v' = \rho' - \rho'', \quad S\rho\rho' = 0, \quad S\rho\rho'' = -\rho'^2,$$

which have all occurred before, give

$$\text{XXXVII.} \dots -4S\xi r' = 8r'^2 + v'^2, \quad 4S\xi v' = r'^2 + 3v'^2;$$

the resulting equation is therefore, after a few slight reductions, the following *biquadratic* in x ,

$$\text{XXXVIII.} \dots 0 = (x - 1)^3 (v'^2 x - r'^2);$$

of which the *cubic factor* is to be rejected (comp. 393, (2.)), as answering only to the point π itself.

(15.) We have then the values,

$$\text{XXXIX.} \dots x = r'^2 v'^{-2}, \quad \text{XL.} \dots OQ = \omega = \xi + \frac{1}{2} \left(\frac{v'^2}{r'} - \frac{r'^2}{v'} \right);$$

comparing which last expression with the formulæ XVII., we see that the required point of intersection q , of the sphero-conic with its osculating circle, can be *constructed* by the following rule. On the traces (4.), of the osculating plane on the two cyclic planes, determine two points τ_1 and u_1 , by the conditions,

$$\text{XLI.} \dots x\tau. x\tau_1 = xv^2, \quad xu. xu_1 = xt^2; \quad \text{XLII.} \dots \tau_1 q = qu_1,$$

or in words, *the right line $\tau_1 u_1$ is bisected by the sought point q .*

(16.) But a still more simple or more *graphic* construction may be obtained, by investigating (comp. 393, (4.)) the *direction of the chord pq* . The *vector value* of this *rectilinear chord* is, by XXXVI. and XL.,

$$\begin{aligned} \text{XLIII.} \dots pq = \omega - \rho &= \frac{1}{2}(v'^2 - r'^2)(v'^{-1} + r'^{-1}) = \frac{1}{2}(r'^{-2} - v'^{-2})r'(r' + v')v' \\ &= \left(\frac{\rho'^2}{r'^2} - \frac{\rho'^2}{v'^2} \right) r' \rho'^{-1} v', \quad \text{because } \rho' = \frac{1}{2}(r' + v'); \end{aligned}$$

the chord pq has therefore the direction (or its opposite) of the *fourth proportional* (226) to the *three vectors*, ρ' , r' , and $-v'$, or pt , $x\tau$, and xu ; if then we conceive this chord or its prolongations to meet the traces $x\tau$, xu in two new points τ_2 , u_2 , we shall have (comp. 393, VIII.) the two *inversely similar triangles* (118),

$$\text{XLIV.} \dots \Delta \tau_2 x u_2 \alpha' u x \tau.$$

(17.) To deduce hence a *spherical construction* for q , we may conceive *four planes*, through the *axis okk* , *perpendicular* respectively to the *four following right lines* in the *osculating plane* :

$$\text{XLV.} \dots r', \quad -v', \quad \rho', \quad \omega - \rho, \quad \text{or} \quad x\tau, \quad xu, \quad pt, \quad pq;$$

which planes will cut the *sphere* in *four great circles*, whereof the *four arcs*,

$$\text{XLVI.} \dots ef, \quad eg, \quad ep, \quad eh,$$

are *parts*, if \mathcal{F} , \mathcal{G} , \mathcal{H} (see again Fig. 80) be the feet of the *three arcual perpendiculars* from the *pole π* of the osculating circle on the two cyclic arcs cb , ca , and on the arcual chord pq .

(18.) *These four arcs XLVI. are therefore connected by the same angular relation as the four lines XLV.; and we have thus the very simple formula,*

$$\text{XLVII.} \dots \text{GEH} = \text{PEF},$$

expressing an equality between *two spherical angles* at the pole E, which serves to determine the *direction* of the arc EH, and therefore also the *positions* of the *points* H and Q, by means of the relations,

$$\text{XLVIII.} \dots \text{PHE} = \frac{\pi}{2}, \quad \cap \text{PH} = \cap \text{HQ}.$$

(19.) If the arcual chord PQ, both ways prolonged, or any chord of the conic, cut the cyclic arcs CB and CA in the points E and S (Fig. 80), it is well known that there exists the *equality of intercepts* (comp. 270, (2.)),

$$\text{XLIX.} \dots \cap \text{EP} = \cap \text{QS};$$

and conversely this equation, combined with the formulæ (11.), or with the trigonometric expression,

$$\text{L.} \dots \tan \text{PE} = \tan r = \frac{1}{2} \sin \frac{c}{2} (\cot A + \cot B),$$

for the tangent of the *arcual radius* of the osculating circle, enables us to determine what may be called perhaps the *arcual chord of osculation* PQ, by determining the spherical angle EPB, or simply P, from principles of *spherical trigonometry alone*, in a way which may serve as a verification of the results above deduced from *quaternions*.

(20.) Denoting by *t* the semitransversal EH = HS, and by *s* the semichord PH = HQ, the oblique-angled triangles RPB, SPA give the equations,

$$\text{LI.} \dots \begin{cases} \cot(t-s) \sin \frac{c}{2} = \cos P \cos \frac{c}{2} + \sin P \cot B, \\ \cot(t+s) \sin \frac{c}{2} = \cos P \cos \frac{c}{2} - \sin P \cot A; \end{cases}$$

while the right angled triangle PHE gives,

$$\text{LII.} \dots \tan s = \sin P \tan r.$$

Equating then the values of $\cot 2s$, deduced from LI. and LII., we eliminate *s* and *t*, and obtain a quadratic in $\tan P$, of which one root is zero, when $\tan r$ has the value L.; such then might in this new way be inferred to be the tangent of the arcual radius of curvature of the conic, and the remaining root of the equation is then,

$$\text{LIII.} \dots \tan P = \frac{\cos \frac{c}{2} (\cot B - \cot A)}{\cot A \cot B + \cos^2 \frac{c}{2} - \tan^2 r};$$

a formula which ought to determine the inclination P, or RPB, or QPA, of the chord PQ to the tangent PA, but which does not appear at first sight to admit of any simple interpretation.*

* We might however at once see from this formula, that $P = A - B$ at the *plane limit*; which agrees with the known construction 398, (4.), for the corresponding chord PQ in the case of the *plane hyperbola*.

(21.) On the other hand, the construction (17.) (18.), to which the quaternion analysis led us, gives

$$\text{LIV. . . HEP} = \text{GEP} - \text{GEH} = \text{GEP} - \text{PEF} = \text{FEB} + \text{GEA},$$

and therefore, by the four right-angled triangles, PHE, BFE, AGE, and BFE or EPA, conducts to this other formula,

$$\begin{aligned} \text{LV. . . } \cot^{-1}(\cos r \cot P) &= \cot^{-1}\left(\cos r \cos \frac{c}{2} \tan(B + a)\right) \\ &\quad - \cot^{-1}\left(\cos r \cos \frac{c}{2} \tan(A + a)\right), \end{aligned}$$

in which a is the same auxiliary angle as in XXXI.; we ought therefore to find, as the proposed verification (19.), that this last equation LV. expresses virtually the same relation between A , B , c , and P , as the formula LIII., although there seems at first to be no connexion between them; and such agreement can accordingly be proved to exist, by a chain of ordinary trigonometric transformations, which it may be left to the reader to investigate.

(22.) A geometrical proof of the validity of the construction (17.) (18.) may be derived in the following way. The product of the sines of the arcual perpendiculars, from a point of a given sphero-conic on its two cyclic arcs, is well known to be constant; hence also the rectangle under the distances of the same variable point from the two cyclic planes is constant, and the curve is therefore the intersection of the sphere with an hyperbolic cylinder, to which those planes are asymptotic. It may then be considered to be thus geometrically evident, that the circle which osculates to the spherical curve, at any given point P , osculates also to the hyperbola, which is the section of that cylinder, made by the osculating plane at this point; and that the point Q , of recent investigations, is the point in which this hyperbola is met again, by its own osculating circle at P . But the determination 898, (4.) of such a point of intersection, although above deduced (for practice) by quaternions, is a plane problem of which the solution was known; we may then be considered to have reduced, to this known and plane problem, the corresponding spherical problem (12.); and thus the inverse similarity of the two plane triangles XLIV., although found by the quaternion analysis, may be said to be geometrically explained, or accounted for: the traces xr and xv , or r' and $-v'$, of the osculating plane to the conic on the two cyclic planes (4.), being evidently the asymptotes of the hyperbola in question.

(23.) In quaternions, the constant product of sines, &c., is expressed by this form of the equation II. of the cone,

$$\text{LVI. . . } \text{SU}\lambda\rho. \text{SU}\mu\rho = (g - S\lambda\mu) : 2T\lambda\mu = \text{const.};$$

and the scalar equation of the hyperbolic cylinder, obtained by eliminating ρ^2 between I. and II., after the first substitution (1.), is

$$\text{LVII. . . } S\lambda\rho S\mu\rho = \frac{1}{2}r^2(g - S\lambda\mu) = \text{const.};$$

while the expression XXXIII. for ω may be considered as the vector equation of the hyperbola, of which the intersection Q with the circle, or with the sphere, is determined by combining that equation with the condition $\omega^2 = \rho^2 (= -r^2)$.

(24.) In the foregoing investigation, we have treated a *sphero-conic* in connexion with its *cyclic arcs* (2.); but it would have been about equally easy to have treated the same curve, with reference to its *focal points*: or to the *focal lines* of the *cone*, of which it is the *intersection* with a concentric *sphere*. (Compare what has been called the *bifocal transformation*, in 860, (2.)).

(25.) We can however only state generally here the *result* of such an application of quaternions, as regards the construction of the osculating small circle to a spherical conic, considered relatively to its *foci*: which *construction** can indeed be also *geometrically* deduced, as a certain *polar reciprocal* of the one given above. Two focal points (not mutually opposite) being called *F* and *G*, let *PX* be the *normal arc* at *P*, which is thus *equally inclined*, by a well-known principle, to the two *vector arcs*, *FP*, *GP*; so that if the focus *G* be suitably distinguished from its own opposite, the spherical angle *FPG* is *bisected* by the arc *PX*, which is here supposed to *terminate* on the *given arc FG*. At *X* erect an arc *QXR*, perpendicular to *PX*, and terminating in *Q* and *R* on the two *vector arcs*. Perpendiculars, *QE*, *RE*, to these last arcs, will meet on the *normal arc PX*, in the *sought pole* (or *spherical centre*) *E*, of the *sought small circle*, which *osculates* to the conic at the given point *P*.

(26.) The two *focal* and *arcual chords* of curvature from *P*, which pass through *F* and *G*, and terminate on the osculating circle, are evidently *bisected* at *Q* and *R*, in virtue of the foregoing *construction*, which may therefore be thus enunciated:—

The great circle QE, which is the common bisector of the two focal and arcual chords of curvature from a given point P, intersects the normal arc PX on the fixed arc FG, connecting the two foci; that is, on the arcual major axis of the conic.

(27.) The construction (5.) fails to determine the position of the auxiliary point *D* in Fig. 80, for the case when the given point *P* is on the *minor axis* of the conic; and in fact the expressions (4.) for ρ' and ρ'' become infinite, when the denominator $S\lambda\mu\rho$ is zero. But it is easy to see that the auxiliary vector σ , which represents generally the *trace* of the *normal plane* to the curve on the plane of the two *cyclic normals*, becomes at the *limit* here considered the required *axis* of the osculating circle; and accordingly, if we assume simply (comp. (1.) and (2.)),

$$\text{LVIII.} \dots \rho' = \nabla\rho\sigma, \text{ and therefore } \rho'' = \nabla\rho'\sigma + \nabla\rho\sigma',$$

we have $\text{LIX.} \dots \sigma' = 0$, and $\nabla\rho'\rho'' \parallel \sigma$, when $S\lambda\mu\rho = 0$.

(28.) In general, if we determine three points *L*, *M*, *S* in the plane of $\lambda\mu$, by the formulæ (comp. again (2.)),

$$\text{LX.} \dots OL = \frac{\lambda\rho^2}{S\lambda\rho}, \quad OM = \frac{\mu\rho^2}{S\mu\rho}, \quad OS = \frac{\sigma\rho^2}{S\sigma\rho} = \frac{1}{2}(OL + OM),$$

then *L* and *M* will be the intersections of the *cyclic normals* λ , μ with the *tangent*

* The reader can easily draw the Figure for himself. As regards the *known rule*, lately alluded to (in 393, (4.), and 394, (22.)), for determining the *chord of intersection* of a *plane conic* with its *osculating circle*, it will be found (for instance) in page 194 of *Hamilton's Conic Sections* (in Latin, London, 1758). The two *spherical constructions*, for the *small circle* osculating to a *spherical conic*, were early deduced and published by the present writer, as consequences of quaternion calculations. Compare the first Note to page 585.

plane to the sphere at r , and the normal plane to the curve at the same point will bisect the right line LM in the point s ; we shall also have this proportion of sines,

$$\begin{aligned} \text{LXI.} \dots \sin LOS : \sin SOM &= SU\lambda\rho : SU\mu\rho \\ &= \cos LOP : \cos POM = \sin FP_1 : \sin FP_2, \end{aligned} \quad \text{comp. (23.),}$$

if FP_1, FP_2 be the arcual perpendiculars from the point P of the conic on the two cyclic arcs; and this *general rule* for determining the position of the line os , or σ , applies even to the *limiting case* (27.), when that variable line becomes the *axis* of the osculating circle, at a *minor summit* of the curve.

(29.) As an *example*, let us suppose that the constants g, λ, μ in the equation II. are connected by the relation,

$$\text{LXII.} \dots g = -S\lambda\mu, \quad \text{whence} \quad \text{LXIII.} \dots S(\sqrt{\lambda\rho} \cdot \sqrt{\mu\rho}) = 0;$$

the *cyclic normals* are therefore in this case *sides* of the cone, and the *two planes* which connect them with *any third side* are mutually *rectangular*; so that the *conic* is now the *locus of the vertex* of a *right-angled spherical triangle*, of which the *hypotenuse* is *given*. And by applying either the formula LXI., or the construction (28.) which it represents, we find that the *trigonometric tangent* of the *arcual radius* of the osculating small circle to *such a conic*, at either end of the given hypotenuse, is equal to *half* the tangent* of that *hypotenuse itself*.

(30.) It is obvious that every determination, of an *osculating circle* to a *spherical curve*, is at the same time the determination of what may be (and is) called an *osculating right cone* (or *cone of revolution*), to the *cone* which *rests* upon that curve, and has its *vertex* at the *centre* of the sphere. Applying this remark to the last example (29.), we arrive at the following theorem, which can however be otherwise deduced:—

If a cone be cut in a circle by a plane perpendicular to a side, the axis of the right cone which osculates to it along that side passes through the centre of the section.

395. When a given curve of double curvature is *not a spherical curve*, we may propose to investigate the *spheric surface* which approaches to it *most closely*, at any assigned point. An *osculating circle* has been defined (389) to be the *limit* of a circle, which *touches* a given curve, or its tangent PT , at a *given point* P , and *cuts* the same curve at a *near point* Q ; while the *tangent* PT itself had been regarded (100) as the *limit* of a *rectilinear secant*, or as the *ultimate position* of the *small chord* PQ . It is natural then to *define* the *osculating sphere*, as being the *limit of a spheric surface*, which passes *through* the *osculating circle*, at a given point P of a curve, and also *cuts* that curve in a point Q , which is supposed to *approach* indefinitely to P , and *ultimately to coincide* with it. Accordingly we shall find that this *definition* conducts by quaternions to *formulae* sufficiently sim-

* This may also be inferred by limits from the formulæ (11.); in which r and α were used, provisionally, to denote a certain spherical arc and angle.

ple; and that their geometrical *interpretations* are consistent with known results: for example, the *centre of spherical curvature*, or the *centre of the osculating sphere*, will thus be shown to be, as usual, the point in which the *polar axis* (391, (5.)) *touches the cusp-edge* of the *polar developable* (391, (6.)). It will also be seen, that whereas in general, if \mathbf{R} be a point in the *normal plane* (370, (8.)) to a given curve at \mathbf{P} , we can only say that the *difference of distances*, $\overline{\mathbf{RQ}} - \overline{\mathbf{RF}}$, is small of an order *higher than the first*, if the *chord* \mathbf{PR} be small of the *first order*; and whereas, even if \mathbf{R} be *on the polar axis* (391, (4.)), we can only say generally that this difference of distances is small, of an order *higher than the second*; yet, if \mathbf{R} be placed *at the centre* \mathbf{s} of *spherical curvature*, the difference $\overline{\mathbf{sQ}} - \overline{\mathbf{sF}}$ is small, of an order *higher than the third*: so that the *distance of a near point* \mathbf{Q} , *from the osculating sphere at the given point* \mathbf{P} , *is generally small of the fourth order*, the *chord* being still small of the *first*.

(1.) Operating with $\mathbf{S}\lambda$, where λ is an arbitrary line, on the vector equation 392, V. of the osculating circle, we obtain the scalar equation of a sphere through that circle under the form,

$$\text{I. . . } 0 = 2\mathbf{S} \frac{\lambda\rho'}{\omega - \rho} + \mathbf{S} \frac{\lambda\rho''}{\rho'};$$

which may however, by 393, (7.), be brought to this other form, better suited to our present purpose,

$$\text{II. . . } (\omega - \kappa)^2 = (\rho - \kappa)^2 + 2c\mathbf{S}\rho''\rho'(\omega - \rho);$$

c being any scalar constant, while κ is still the vector of the centre \mathbf{x} of the circle: and the vector σ of the centre \mathbf{s} of the sphere is given by the formula,

$$\text{III. . . } \sigma = \kappa + c\mathbf{V}\rho''\rho';$$

which evidently expresses that this last centre is on the polar axis.

(2.) To express now that this sphere cuts the curve in a near point \mathbf{Q} , we are to substitute for ω the expression,

$$\text{IV. . . } \omega = \rho_t = \rho + t\rho' + \frac{1}{2}t^2\rho'' + \frac{1}{6}t^3\rho''', \quad \text{with } u_0 = 1;$$

but κ has been seen (in 391) to satisfy the three equations,

$$\text{V. . . } 0 = \mathbf{S}\rho'(\kappa - \rho), \quad 0 = \mathbf{S}\rho''(\kappa - \rho) - \rho'^2, \quad 0 = \mathbf{S}\rho'''\rho'(\kappa - \rho);$$

reducing then, dividing by $\frac{1}{6}t^3$, and passing to the limit, we find for the *osculating sphere* the condition,

$$\text{VI. . . } \mathbf{S}\rho''(\rho - \kappa) + 3\mathbf{S}\rho''\rho' = c\mathbf{S}\rho'''\rho''\rho';$$

so that finally the vector σ satisfies the three scalar equations,

$$\text{VII. . . } 0 = \mathbf{S}\rho'(\sigma - \rho), \quad 0 = \mathbf{S}\rho''(\sigma - \rho) - \rho'^2, \quad 0 = \mathbf{S}\rho''(\sigma - \rho) - 3\mathbf{S}\rho''\rho''',$$

by which it is completely determined, and of which the two last are seen to be the successive derivatives of the first, while that first is the equation of the normal plane:

whence the *centre* s of this *sphere* is (by the sub-arts. to 386, comp. 391, (6.)) the point where the *polar axis* κs touches the *cusp-edge* of the polar developable.

(3.) Differentials may be substituted for derivatives in the equations VII., which may also be thus written (comp. 391, (4.)),

$$\text{VIII.} \dots 0 = dT(\rho - \sigma), \quad 0 = d^2T(\rho - \sigma), \quad 0 = d^3T(\rho - \sigma), \quad \text{if } d\sigma = 0;$$

the *distance* of a near point q of the given curve from the osculating *sphere* is therefore *small* (as above said), of an order *higher than the third*, if the *chord* PQ be small of the *first order*.

(4.) The two first equations VII., combined with V., give also

$$\text{IX.} \dots 0 = S\rho'(\sigma - \kappa), \quad 0 = S\rho''(\sigma - \kappa), \quad 0 = S(\kappa - \rho)(\sigma - \kappa);$$

which expresses that the line κs is perpendicular to the osculating plane and absolute normal at P , as it ought to be, because it is part of the polar axis.

(5.) Conceiving the *three points* P, κ, s , or their *vectors* ρ, κ, σ , to *vary together*, the equations V. and VII., combined with their own derivatives, give among other results the following:

$$\text{X.} \dots 0 = S\kappa'\rho' = S\sigma'\rho' = S\sigma'\rho'' = S\sigma'(\kappa - \rho) = S\sigma''\rho';$$

of which the geometrical interpretations are easily perceived.

(6.) Another easy combination is the following,

$$\text{XI.} \dots 0 = S\kappa'(\sigma + \rho - 2\kappa),$$

as appears by derivating the last equation IX., with attention to other relations; but $2\kappa - \rho$ is the vector of the extremity, say \mathfrak{M} , of the *diameter* of the osculating *circle*, drawn from the given point P : we have therefore this construction:—

On the tangent $\kappa\kappa'$ *to the locus of the centre of the osculating circle, let fall a perpendicular from the extremity* \mathfrak{M} *of the diameter drawn from the given point* P ; *this perpendicular prolonged will intersect the polar axis, in the centre* s *of the osculating sphere to the given curve at* P .

(7.) In general, the three scalar equations VII. conduct to the vector expression,

$$\text{XII.} \dots \sigma = \rho + \frac{8V\rho'\rho''S\rho'\rho'' + \rho^3V\rho''\rho'}{S\rho'\rho''\rho''};$$

or with differentials,

$$\text{XIII.} \dots \sigma = \rho + \frac{8Vd\rho d^2\rho Sd\rho d^2\rho + d\rho^3Vd^2\rho d\rho}{Sd\rho d^2\rho d^2\rho};$$

the scalar variable being still left arbitrary.

(8.) And if, as an *example*, we introduce the values for the *helix*,

$$\begin{aligned} \text{XIV.} \dots \rho &= cta + a^2\beta, & \rho' &= ca + \frac{\pi}{2} a^{2+1}\beta, & \rho'' &= -\left(\frac{\pi}{2}\right)^2 a^2\beta, \\ & & \rho''' &= -\left(\frac{\pi}{2}\right)^3 a^{2+1}\beta, \end{aligned}$$

whereof the three first occurred before, we find after some slight reductions the expression, in which a denotes again the constant inclination of the curve to the axis of the cylinder,

$$\text{XV.} \dots \sigma = \rho - a^2\beta \operatorname{cosec}^2 a = cta - a^2\beta \cot^2 a;$$

but this is precisely what we found for κ , in 389, VIII.; for the *helix*, then, the two *centres*, κ and s , of *absolute and spherical curvature*, coincide.

(9.) This known result is a consequence, and may serve as an illustration, of the general construction (6.); because it is easy to infer, from what was shown in 389, (8.), respecting the locus of the centre κ of the osculating circle to the helix, as being another helix on a co-axial cylinder, that the tangent $\kappa\kappa'$ to this locus is perpendicular to the radius of curvature κP , while the same tangent ($\kappa\kappa'$ or $\kappa\hat{}$) is always perpendicular (X.) to the tangent (PP' or ρ') to the curve; $\kappa\kappa'$ is therefore here at right angles to the osculating plane of the given helix, or coincides with its polar axis: so that the perpendicular on it from the extremity κ of the diameter of curvature falls at the point κ itself, with which consequently the point s in the present case coincides, as found by calculation in (8.).

(10.) In general, if we introduce the expressions 876, VI., or the following,

$$\text{XVI.} \dots \rho' = s' D_s \rho, \quad \rho'' = s'^2 D_s^2 \rho + s'' D_s \rho, \quad \rho''' = s'^3 D_s^3 \rho + 3s's'' D_s^2 \rho + s''' D_s \rho,$$

in which s denotes the arc of the curve, but the accents still indicate derivations with respect to an arbitrary scalar t ; and if we observe (comp. 880, (12.)) that the relations,

$$\text{XVII.} \dots D_s \rho^2 = -1, \quad S. D_s \rho D_s^2 \rho = 0, \quad S. D_s \rho D_s^3 \rho + D_s^2 \rho^2 = 0,$$

in which $D_s \rho^2$ and $D_s^2 \rho^2$ denote the squares of $D_s \rho$ and $D_s^2 \rho$, and $S. D_s \rho D_s^2 \rho$ denotes $S(D_s \rho \cdot D_s^2 \rho)$, &c., exist independently of the form of the curve; we find that s'' and s''' disappear from the numerator and denominator of the expression XII. for $\sigma - \rho$, and that they have s'^6 for a common factor: setting aside which, we have thus the simpler formulæ,

$$\text{XVIII.} \dots \sigma - \rho = \frac{V. D_s \rho D_s^2 \rho}{S. D_s \rho D_s^2 \rho D_s^3 \rho} = \frac{D_s \cdot D_s \rho D_s^2 \rho}{S. D_s \rho D_s^2 \rho D_s^3 \rho}.$$

And accordingly the three scalar equations VII., which determine the centre of the osculating sphere, may now be written thus,

$$\text{XIX.} \dots S(\sigma - \rho) D_s \rho = 0, \quad S(\sigma - \rho) D_s^2 \rho + 1 = 0, \quad S(\sigma - \rho) D_s^3 \rho = 0.$$

(11.) Conversely, when we have any formula involving thus the successive derivatives of the vector ρ taken with respect to the arc, s , we can always and easily generalize the expression, and introduce an arbitrary variable t , by inverting the equations XVI.; or by writing (comp. 890, VIII.),

$$\text{XX.} \dots D_s \rho = s'^{-1} \rho', \quad D_s^2 \rho = s'^{-1} (s'^{-1} \rho')' = s'^{-2} \rho'' - s'^{-3} s'' \rho', \quad \&c.$$

(12.) It may happen (comp. 879, (2.)) that the independent variable t is only proportional to s , without being equal thereto; but as we have the general relation,

$$\text{XXI.} \dots D_t \rho = s'^n D_s \rho, \quad \text{if } s' = D_t s = T \rho' = \text{const.},$$

it is nearly or quite as easy to effect the transformations (10.) and (11.) in the case here supposed, or to pass from t to s and reciprocally, as if we had $s' = 1$.

(13.) If the vector σ be treated as constant in the derivations, or if we consider for a moment the centre s of the sphere as a fixed point, and attend only to the variations of distance of a point on the curve from it, then (remembering that $T(\rho - \sigma)^2 = -(\rho - \sigma)^2$) we not only easily put (comp. VIII.) the three equations XIX. under the forms,

$$\text{XXII.} \dots 0 = D_s T(\rho - \sigma) = D_s^2 T(\rho - \sigma) = D_s^3 T(\rho - \sigma),$$

but also obtain by XVII. this fourth equation,

$$\text{XXIII.} \dots T(\rho - \sigma) D_s^4 T(\rho - \sigma) = S. (\sigma - \rho) D_s^4 \rho + D_s^2 \rho^2.$$

(14.) If then we write, for abridgment,

$$\text{XXIV.} \dots r = T(\kappa - \rho) = T D_s^2 \rho^{-1} = \text{radius of osculating circle};$$

$$\text{XXV.} \dots R = T(\sigma - \rho) = \text{radius of osculating sphere};$$

and

$$\text{XXVL.} \dots S = \frac{S(\sigma - \rho) D_s^4 \rho}{-D_s^2 \rho^2} = \frac{S \cdot D_s \rho^2 D_s^2 \rho D_s^4 \rho}{S \cdot D_s \rho D_s^2 \rho^2 D_s^2 \rho};$$

we see that *this scalar, S, must be constantly equal to unity, for every spherical curve*; but that for a curve which is *non-spherical*, the distance \overline{SQ} of a near point Q, from the centre S of the osculating sphere at P, is generally given by an expression of the form,

$$\text{XXVII.} \dots \overline{SQ} = R + \frac{(S-1) u_0 \rho^4}{24r^2 R}, \text{ with } u_0 = 1;$$

so that, at least for *near points* Q, on each side of the *given point* P, the curve lies *without* or *within* the sphere which osculates at that given point, according as the scalar, *s*, determined as above, is *greater* or *less* than unity.

(15.) In the case (12.), the formula XXVI. may be thus written,

$$\text{XXVIII.} \dots S = \frac{S \cdot \rho'^3 \rho'' \rho'''}{S \cdot \rho \rho'^2 \rho''};$$

whence, by carrying the derivations one step farther than in (8.), we find for the helix,

$$\text{XXIX.} \dots S = \text{cosec}^2 a > 1, \text{ or } \text{XXIX'}. \dots S - 1 = \cot^2 a > 0;$$

and accordingly it is easy to prove that *this curve lies wholly without* its osculating sphere, except at the point of osculation.

(16.) In general, the scalar $S - 1$, which vanishes (14.) for *all spherical curves*, and which enters as a *coefficient* into the expression XXVII. for the deviation $\overline{SQ} - \overline{SP}$ of a near point of any other curve from its own osculating sphere, may be called the *Coefficient of Non-Sphericity*; and if QR be the perpendicular from that near point Q on the tangent PR to the curve at the given point P, we have then this *limiting equation*, by which the value of that coefficient may be expressed,

$$\text{XXX.} \dots S - 1 = \lim. 3 \left(\frac{\overline{SQ}^2 - \overline{SP}^2}{Q\overline{P}^2} \right).$$

(17.) Besides the forms XVIII., other transformations of the expressions XII. XIII. for the vector σ of the centre of an osculating sphere might be assigned; but it seems sufficient here to suggest that some useful practice may be had, in proving that those expressions for σ reduce themselves generally to zero, when the condition,

$$\text{XXXI.} \dots T\rho = \text{const.}$$

is satisfied.

(18.) It may just be remarked, that as r^{-1} is often called (comp. 389, (4.)) the *absolute curvature*, or simply *the curvature*, of the curve in space which is considered, so R^{-1} is sometimes called the *spherical curvature* of that curve: while r and R are called the *radii** of those two curvatures respectively.

* We shall soon have occasion to consider *another scalar radius*, which we propose to denote by the small roman letter r , of what is not uncommonly called the *torsion*, or the *second curvature*, of the same curve in space.

396. When the *arc* (s) of the curve is made the independent variable, the *calculations* (as we have seen) become considerably simplified, while no essential *generality* is lost, because the transformations requisite for the introduction of an *arbitrary* scalar variable (t) follow a simple and uniform *law* (395, (11.), &c.). Adopting then the expression (comp. 395, IV.),

$$\text{I. . . } \rho_s = \rho + s\tau + \frac{1}{2}s^2\tau' + \frac{1}{6}s^3u_s\tau'', \quad \text{with } u_0 = 1,$$

in which

$$\text{II. . . } \tau = D_s\rho, \quad \tau' = D_s^2\rho, \quad \tau'' = D_s^3\rho,$$

and therefore

$$\text{III. . . } \tau^2 + 1 = 0, \quad S\tau\tau' = 0, \quad S\tau\tau'' + \tau'^2 = 0,$$

we shall proceed to deduce some *other affections* of the curve, besides its *spherical curvature* (395, (18.)), which do not involve the consideration of the *fourth power* of the arc (or chord). In particular, we shall determine expressions for that known *Second Curvature* (or *torsion*), which depends on the *change of the osculating plane*, and is measured by the *ultimate ratio* of that change, expressed as an *angle*, to the *arc* of the curve itself; and shall assign the quaternion equations of the known *Rectifying Plane*, and *Rectifying Line*, which are respectively the *tangent plane*, and the *generating line*, of that known *Rectifying Developable*, whereon the proposed curve is a *geodesic* (382): so that it would *become a right line*, by the *unfolding* of this last *surface* into a *plane*. But first it may be well to express, in this new notation, the principal affections or properties of the curve, which depend only on the *three first terms* of the expansion I., or on the *three initial vectors* ρ , τ , τ' , or rather on the *two last* of these; and which *include*, as we shall see, the *rectifying plane*, but *not* the *rectifying line*: nor what has been called above the *second* curvature*.

(1.) Using then first, instead of I., this less expanded but still rigorous expression (comp. 376, I.),

$$\text{IV. . . } \rho_s = \rho + s\tau + \frac{1}{2}s^2u_s\tau', \quad \text{with } u_0 = 1,$$

* In a Note to a very able and interesting Memoir, "*Sur les lignes courbes non planes*" (referred to by Dr. Salmon in the Note to page 277 of his already cited Treatise, and published in *Cahier XXX.* of the *Journal de l'Ecole Polytechnique*), M. de Saint-Venant brings forward several objections to the use of this appellation, and also to the phrases *torsion*, *flexion*, &c., instead of which he proposes to introduce the new name, "*cambrure*:" but the expression "*second curvature*" may serve us for the present, as being at least not unusual, and appearing to be sufficiently suggestive

and with the relations II. and III., we have at once the following system of *three rectangular lines*, which are conceived to be all drawn from the given point P of the curve :

$$\text{V. . . } r = \text{unit tangent}; \quad \text{VI. . . } r' = \text{vector of curvature (389, (4.))};$$

$$\text{and VII. . . } \nu = rr' = -r'r = r'r^{-1} = \text{binormal (comp. 379, (4.))};$$

r being a line drawn in the direction of a conceived *motion* along the curve, in virtue of which the *arc* (s) *increases*; while r' is directed *towards the centre of curvature*, or of the *osculating circle*, of which centre κ the vector is now,

$$\text{VIII. . . } \text{OK} = \kappa = \rho - r'^{-1} = \rho + r^2 r' = \rho + r \text{Ur}';$$

if IX. . . $r^{-1} = \text{Tr}' = \text{curvature at P}$, or IX'. . . $r = \text{Tr}'^{-1} = \text{radius of curvature}$; and the *third* line ν (which is *normal* at P to the *surface of tangents* to the curve) has the *same length* ($\text{Tr}\nu = r^{-1}$) as r' , and is *directed* so that the rotation round it from r to r' is *positive*.

(2.) At the same time, we have evidently a system of *three rectangular vector units* from the same point P, which may be called respectively the *tangent unit*, the *normal unit*, and the *binormal unit*, namely the three lines,

$$\text{X. . . } \text{Ur} = r, \quad \text{Ur}' = rr', \quad \text{U}\nu = rrr';$$

the *normal unit* being thus directed (like r') *towards the centre of curvature*.

(8.) The *vector equation* (comp. 392, (2.)) of the *circle of curvature* takes now the form,

$$\text{XI. . . } \nabla \frac{2r}{\omega - \rho} = -\nu;$$

with the verification that it is satisfied by the value,

$$\text{XII. . . } \omega = \mu = 2\kappa - \rho = \rho - 2r'^{-1},$$

in which μ (comp. 395. (6.)) is the vector OK of the extremity of the *diameter of curvature* PM .

(4.) The *normal plane*, the *rectifying plane*, and the *osculating plane*, to the curve at the given point, form a *rectangular system of planes* (comp. 379, (5.)), *perpendicular* respectively to the *three lines* (1.); so that their scalar equations are, in the present notation,

$$\text{XIII. . . } \text{Sr}(\omega - \rho) = 0; \quad \text{XIV. . . } \text{Sr}'(\omega - \rho) = 0; \quad \text{XV. . . } \text{S}\nu(\omega - \rho) = 0;$$

by *pairing* which we can represent the *tangent*, *normal*, and *binormal* to the curve, regarded as *indefinite right lines*; or by the three vector equations,

$$\text{XVI. . . } \text{Vr}(\omega - \rho) = 0; \quad \text{XVII. . . } \text{Vr}'(\omega - \rho) = 0; \quad \text{XVIII. . . } \text{V}\nu(\omega - \rho) = 0.$$

(5.) In general, if the two vector equations,

$$\text{XIX. . . } \text{V}\eta(\omega - \rho) = 0, \quad \text{and} \quad \text{XIX'. . . } \text{V}\eta_s(\omega_s - \rho_s) = 0,$$

represent *two right lines*, PH and P_sH_s , which are conceived to *emanate* according to *any given law* from *any given curve* in space, the *identical formula*,*

* It is obvious that we have thus an easy quaternion solution of the problem, *to draw a common perpendicular to any two right lines in space*.

$$\text{XX.} \dots \rho_s - \rho + \nabla \left(\nabla \eta \eta_s \cdot \nabla \frac{\rho_s - \rho}{\nabla \eta \eta_s} \right) = \frac{S \eta \eta_s (\rho_s - \rho)}{\nabla \eta \eta_s},$$

shows that the *common perpendicular* to these two emanants, which as a vector is represented by either member of this formula XX., intersects the two lines in the two points of which the vectors are,

$$\text{XXI.} \dots \omega = \rho + \eta S \frac{(\rho_s - \rho) \eta_s}{\nabla \eta \eta_s}; \quad \text{XXI'}. \dots \omega_s = \rho_s + \eta_s S \frac{(\rho_s - \rho) \eta}{\nabla \eta \eta_s}.$$

(6.) In general also, the passage of a right line from any one given position in space to any other may be conceived to be accomplished by a sort of *screw motion*, with the *common perpendicular* for the axis of the screw, and with two *proportional velocities*, of translation along, and of rotation round that axis: the *locus* of the two given and of all the intermediate positions of the line (when thus interpolated) being a *Screw Surface*, such as that of which the vector equation was assigned in 814, (11.), and was used in 872, (4.).

(7.) Again, for any quaternion, q , we have (by 816, XX. and XXIII.*) the two equations,

$$\text{XXII.} \dots lUq = \angle q \cdot UVq, \quad \text{XXII'}. \dots \nabla Uq = \sin \angle q \cdot UVq;$$

comparing which we see that

$$\text{XXIII.} \dots \nabla Uq : lUq = \sin \angle q : \angle q = (\text{very nearly}) 1,$$

if the angle of the quaternion be small; so that the *logarithm* and the *vector* of the versor of a small-angled quaternion are very nearly equal to each other, and we may write the following general approximate formula for such a versor:

$$\text{XXIV.} \dots Uq = (\varepsilon^{lUq}) \varepsilon^{\nabla Uq}, \text{ nearly, if } \angle q \text{ be small;}$$

the error of this last formula being in fact small of the *third order*, if the angle be small of the first.

(8.) And thus or otherwise (comp. 834, XIII. and XV.), we may perceive that if the quaternion q have the form (comp. (5.)),

$$\text{XXV.} \dots q = \eta_s \eta^{-1}, \text{ with } \text{XXVI.} \dots \eta_s = \eta + s\eta' + \dots,$$

and if we write for abridgment,

$$\text{XXVII.} \dots \theta = \nabla \frac{\eta'}{\eta}, \text{ and } \text{XXVIII.} \dots \lambda = S \frac{\eta'}{\eta},$$

we shall then have nearly, if s be small, the expressions,

$$\text{XXIX.} \dots Uq = U \frac{\eta_s}{\eta} = \varepsilon^{\theta}, \text{ and } \text{XXX.} \dots Tq = T \frac{\eta_s}{\eta} = 1 + s\lambda;$$

or, neglecting s^2 ,

$$\text{XXXI.} \dots \eta_s = (1 + s\lambda) \varepsilon^{\theta} \eta = \varepsilon^{\theta} \eta + s\lambda \eta,$$

in which last binomial, the *first* (or *exponential*) term alone influences the direction of the near emanant line (5.).

* Although the expression XXII'. for ∇Uq is here deduced from 816, XXIII., yet it might have been introduced at a much earlier stage of these *Elements*; for instance, in connexion with the formula 204, XIX., namely $TVUq = \sin \angle q$.

(9.) At the same time, by supposing s to tend to 0, the formula XXI. gives, as a limit,

$$\text{XXXII.} \dots \omega_{\mathbf{H}} = \omega_0 = \rho + \eta S \frac{\tau \eta}{\sqrt{\eta \eta'}} = \rho - \eta S \frac{\tau}{\theta \eta},$$

for the vector of the point, say \mathbf{H} , on the given emanant \mathbf{PH} , in which that given line is ultimately intersected by the common perpendicular (5.), or by the axis of the screw rotation (6.); but the direction of that axis is represented by the versor $\mathbf{U}\theta$, and the angular velocity of that rotation is represented by the tensor $\mathbf{T}\theta$, if the velocity of motion (1.) along the given curve be taken as unity: we may therefore say that the vector θ itself, or the factor which multiplies the arc, s , in the exponential term XXXI., if set off from the point \mathbf{H} determined by XXXII., is the Vector of Rotation of the Emanant, whatever the law (5.) of the emanation may be.

(10.) And as regards the screw translation (6.), its linear velocity is in like manner represented, in length and in direction, by the following expression (obtained by limits from XX.),

$$\text{XXXIII.} \dots \iota = \theta S \frac{\tau}{\theta} \text{ (set off from } \mathbf{H}) = \text{Vector of Translation of Emanant,} \\ = \text{projection of unit-tangent on screw-axis (or of } \tau \text{ on } \theta).$$

And the indefinite right line through the point \mathbf{H} , of which this line ι is a part, may be called the Axis of Displacement of the Emanant.

(11.) It is easy in this manner to assign what may be called the Osculating Screw Surface to the (generally gauche) Surface of Emanants, or indeed to any proposed skew surface; namely, the screw surface which has the given emanant (or other) line for one of its generatrices, and touches the skew surface in the whole extent of that right line.

(12.) It is however more important here to observe, that in the case when the surface of emanants is developable, the vector ι of translation vanishes; and that conversely this vector ι cannot be constantly zero, if that surface be undevelopable. The Condition of Developability of the Surface of Emanants is therefore expressed by the equation,

$$\text{XXXIV.} \dots \iota = 0, \text{ or } S\tau\theta = 0, \text{ or XXXIV'.} \dots S\eta\eta'\tau = 0;$$

and accordingly this condition is satisfied (as was to be expected) when $\eta = \tau$, that is, for the surface of tangents.

(13.) In the same case, of $\eta = \tau$, or $\parallel \tau$, the vector θ of rotation becomes equal (by XXVII. and VII.) to the binormal ν ; and the expression XXXII., for the vector ω_0 of the foot \mathbf{H} of the axis reduces itself to ρ ; and thus we might be led to see (what indeed is otherwise evident), that the passage from a given tangent to a near one may be approximately made, by a rotation round the binormal, through the small angle, $s\mathbf{T}\nu = sr^{-1} = \text{arc divided by radius of curvature}$.

(14.) Instead of emanating lines, we may consider a system of emanating planes, which are respectively perpendicular to those lines, and pass through the same points of the given curve. It may be sufficient here to remark, that the passage from one to another of two such near emanant planes, represented by the equations,

$$\text{XXXV.} \dots S\eta(\omega - \rho) = 0, \quad \text{XXXV'.} \dots S\eta_1(\omega - \rho) = 0,$$

may be conceived to be made by a rotation through an angle $= s\mathbf{T}\theta$, round the right line,

$$\text{XXXVI.} \dots S\eta(\omega - \rho) = 0, \quad S\eta'(\omega - \rho) - S\eta\tau = 0,$$

or

$$\text{XXXVI.} \dots V\theta(\omega - \rho) + \eta^{-1}S\eta\tau = 0,$$

in which the *plane* XXXV. touches its developable envelope, and which is parallel to the recent vector θ , or to the vector of rotation (9.) of the emanant line; so that if an equal vector be set off on this new line XXXVI., it may be said to be the *Vector Axis of Rotation of the Emanant Plane*.

(15.) For example, if we again make $\eta = \tau$, so that the equation XXXV. represents now the *normal plane* to the curve, we are led to combine the equation XIII. of that plane with its *derived* equation, and so to form the system of the two scalar equations,

$$\text{XXXVII.} \dots S\tau(\omega - \rho) = 0, \quad S\tau'(\omega - \rho) + 1 = 0,$$

whereof the second represents a plane parallel to the *rectifying plane* XIV., and drawn through the centre of curvature VIII.; and which jointly represent the *polar axis* (891, (5.)), considered as an *indefinite* right line, which is represented otherwise by the *one* vector equation,

$$\text{XXXVIII.} \dots V\nu(\omega - \kappa) = 0, \quad \text{or} \quad \text{XXXVIII.}' \dots V\nu(\omega - \rho) = -r.$$

(16.) And if, on this *indefinite line*, we set off a portion equal to the *binormal* ν , such *portion* (which may conveniently be measured from the centre κ) may be said, by (14.), to be the *Vector Axis of Rotation of the Normal Plane*; or briefly, the *Polar Axis*, considered as representing not only the *direction* but also the *velocity* of that rotation, which velocity = $T\nu = r^{-1}$ = the *curvature* (IX.) of the given curve: while another *portion* = $U\nu$ = the *binormal unit* (2.), set off on the *same axis* from the *same centre* of curvature, may be called the *Polar Unit*.

(17.) This suggests a *new way* of representing the *osculating circle* by a *vector equation* (comp. (8.), and 316), as follows:

$$\begin{aligned} \text{XXXIX.} \dots \omega_s &= \kappa + \varepsilon^{\omega}(\rho - \kappa) = \rho + (\varepsilon^{\omega} - 1)r^{-1} \\ &= \rho + s\tau + (\varepsilon^{\omega} - 1 - s\nu)r^{-1} \\ &= \rho + s\tau + \frac{1}{2}s^2r' + (\varepsilon^{\omega} - 1 - s\nu - \frac{1}{2}s^2\nu^2)r^{-1}; \end{aligned}$$

which agrees, as we see, with the expression I. or IV., if s^3 be neglected; and of which, when the expansion is continued, the *next term* is,

$$\text{XL.} \dots \frac{1}{6}s^3\nu^2r^{-1} = \frac{1}{6}s^3\nu r' = -\frac{s^3r}{6r^3}$$

(18.) The *complete expansion* of the *exponential form* XXXIX., for the variable vector of the *osculating circle*, may be briefly summed up in the following *trigonometric* (but *vector*) expression:

$$\text{XLI.} \dots \omega_s = \kappa + \left(\cos \frac{s}{r} + U\nu \cdot \sin \frac{s}{r} \right) (\rho - \kappa),$$

in which, XLII. $\dots \rho - \kappa = -r^2r'$, and $U\nu \cdot (\rho - \kappa) = r\nu r^{-1} = rr$;

so that we may also write, neglecting no power of s ,

$$\text{XLIII.} \dots \omega_s = \rho + rr \sin \frac{s}{r} + r^2r' \text{vers } \frac{s}{r};$$

and if this be subtracted from the *full* expression for the vector ρ_s , the *remainder* may be called the *deviation of the given curve in space, from its own circle of curvature*: which *deviation*, as we already see, is *small of the third order*, and will soon be de-

composed into its *two principal parts*, or *terms*, of *that order*, in the directions of the *normal* and the *binormal* respectively.

(19.) Meantime we may remark, that if we only neglect terms of the *fourth order*, the expansion I. gives, by III. and IX., for the *length* of a *small chord* PF_0 , the formula :

$$\begin{aligned} \text{XLIV. . . } \overline{PF_0} &= T(\rho_0 - \rho) = T\left(s\tau + \frac{1}{2}s^2\tau' + \frac{1}{6}s^3\tau''\right) \\ &= \sqrt{\left\{-\left(s\tau + \frac{1}{2}s^2\tau' + \frac{1}{6}s^3\tau''\right)^2\right\}} \\ &= \sqrt{\left\{s^2 + s^4\tau'^2\left(\frac{1}{2} - \frac{1}{6}\right)\right\}} \\ &= \sqrt{\left(s^2 - \frac{s^4}{12\tau'^2}\right)} = s - \frac{s^3}{24\tau'^2} = 2r \sin \frac{s}{2r}; \end{aligned}$$

this *length* then is the *same* (to this degree of approximation), as that of the *chord* of an *equally long arc* of the *osculating circle* : and although the *chord* of even a *small arc* of a *curve* is *always shorter* than that *arc itself*, yet we see that the *difference* is *generally* a *small quantity* of the *third* order*, if the *arc* be *small* of the *first*.

397. Resuming now the expression 396, I., but suppressing here the coefficient u_0 , of which the limit is unity, and therefore writing simply,

$$\text{I. . . } \rho_0 = \rho + s\tau + \frac{1}{2}s^2\tau' + \frac{1}{6}s^3\tau'',$$

with the relations,

$$\text{II. . . } \tau^2 = -1, \quad S\tau\tau' = 0, \quad S\tau\tau'' = -\tau'^2 = r'^{-2}, \quad S\tau'\tau'' = r'^{-3}r',$$

if $s = \text{arc}$, and $r^{-1} = T\tau' = \text{curvature}$, † as before, or $r = \text{radius of curvature}$ (> 0), while $r' = D_s r$; and introducing the *new scalar*,

$$\text{III. . . } r^{-1} = S \frac{\tau''}{\tau\tau'} = \tau^{-1} V \frac{\nu'}{\nu} = \text{Second} \ddagger \text{ Curvature,}$$

with $\nu = \tau\tau' = \text{binormal}$, or the *new vector*,

$$\text{IV. . . } r^{-1}\tau = \tau S \frac{\tau''}{\tau\tau'} = V \frac{\nu'}{\nu} = \text{Vector of Second Curvature,}$$

supposed to be set off tangentially from the given point P of the curve, or finally this *other* new scalar ($>$ or $<$ 0),

$$\text{V. . . } r = \left(S \frac{\tau''}{\tau\tau'}\right)^{-1} = \text{Radius of Second Curvature,}$$

* This ought to have been expressly stated in the reasoning of 383, (5.), for which it was not sufficient to observe that the *arc* and *chord* tend to bear to each other a *ratio of equality*, without showing (or at least mentioning) that their *difference* tends to vanish, even as compared with a line which is ultimately of the same order as the *square* of either.

† Whenever this word *curvature* is thus used, without any qualifying adjective, it is always to be understood as denoting the *absolute* (or *first*) curvature of the curve in space.

‡ Compare the Note to page 554.

which gives the expression,

$$\begin{aligned} \text{VI.} \dots \tau'' &= -r^2\tau - r^{-1}r'\tau' + r^{-1}\tau\tau' \\ &= -r^2U\tau + (r^{-1})'U\tau' + (r\tau)^{-1}U\nu; \end{aligned}$$

we proceed to deduce some of the chief affections of a curve in space, which depend on the *third power* of the arc or chord. In doing this, although everything *new* can be *ultimately* reduced to a dependence on the *two new scalars*, r' and r , or on the *one new vector* τ'' , or even on $\nu' = V\tau\tau'$, yet some *auxiliary symbols* will be found useful, and almost necessary. Retaining then the symbols ν , κ , σ , R , as well as τ , τ' , r , and therefore writing as before (comp. 396, VIII.),

$$\text{VII.} \dots \text{OK} = \kappa = \rho - \tau'^{-1} = \rho + rU\tau' = \rho + r^2\tau',$$

$$\text{VIII.} \dots (\rho - \kappa)^{-1} = r^{-1}U(\kappa - \rho) = \tau' = D_r\rho = \text{Vector of Curvature,}$$

we may now write also, by 395, XVIII.,

$$\text{IX.} \dots \text{os} = \sigma = \rho - \frac{\nu'}{S\tau'\nu'} = \kappa + r\tau'r\nu = \kappa + r'U\nu,$$

and

X. . . $(\rho - \sigma)^{-1} = R^{-1}U(\sigma - \rho) = \nu'^{-1}S\tau'\nu' = \text{Vector of Spherical Curvature,}$
 = projection of vector (τ') of curvature on radius (R) of osculating sphere ;
 because we have now, by VI.,

$$\text{XI.} \dots \nu' = (\tau\tau')' = V\tau\tau' = -r^{-1}\tau' - r^{-1}\tau'\nu,$$

or

$$\text{XI}'. \dots (U\nu)' = (r\nu)' = -r\tau^{-1}\tau' = -r^{-1}U\tau',$$

and

$$\text{XII.} \dots S\tau'\nu' = -S\tau\tau'\tau'' = -r^{-1}\tau'^2 = r^2r^{-1}.$$

If then we denote by p and P the *linear* and *angular elevations*, of the centre s of the osculating sphere above the osculating plane, we shall have these *two new auxiliary scalars*, which are positive or negative together, according as the linear height κs has the direction of $+\nu$ or of $-\nu$:

$$\text{XIII.} \dots p = \frac{\sigma - \kappa}{U\nu} = r'\tau; \quad \text{XIV.} \dots P = \kappa P s = \tan^{-1} \frac{p}{r} = \sin^{-1} \frac{p}{R} = \cos^{-1} \frac{r}{R};$$

$$\text{while} \quad \text{XV.} \dots R = T(\sigma - \rho) = \sqrt{(r^2 + p^2)} = \sqrt{(r^2 + r'^2 r^2)};$$

the angle P being treated as generally *acute*. Another important line, and an accompanying angle of elevation, are given by the formulæ,

$$\text{XVI.} \dots \lambda = V \frac{\tau''}{\tau} = r^2 V\tau'\tau'' = r^{-1}\tau + \tau\tau' = r^{-1}U\tau + r^{-1}U\nu$$

$$= V\nu'\nu^{-1} + \nu = \text{Rectifying Vector (set off from given point } p),$$

$$= \text{Vector of Second Curvature plus Binormal};$$

XVII. . . $H = \angle \frac{\lambda}{r} = \tan^{-1} \frac{r'}{r} = \text{Elevation of Rectifying Line } (> 0, < \pi)$,

= the angle (acute or obtuse, but here regarded as positive),

which that known and important line (396) makes with the tangent to the curve; so that (by XIII., XIV.) these two auxiliary angles,* H and P , from which (instead of deducing them from r' and r) all the affections of the curve depending on s^2 can be deduced, are connected with each other and with r' by the relation,

$$\text{XVIII. . . } \tan P = r' \tan H.$$

Many other combinations of the symbols offer themselves easily, by the rules of the present calculus; for instance, the vector σ may be determined by the three scalar equations (comp. 395, XIX.),

$$\text{XIX. . . } S\tau(\sigma - \rho) = 0, \quad S\tau'(\sigma - \rho) = -1, \quad S\tau''(\sigma - \rho) = 0,$$

whence, by XVI.,

$$\text{XX. . . } r^2 \tau'' = r^2 \nabla(\nabla\tau'\tau''). (\sigma - \rho) = \nabla\lambda(\sigma - \rho),$$

a result which also follows from the expressions,

$$\text{XXI. . . } \tau'' = \left(\nabla \frac{\tau''}{\tau'} + S \frac{\tau'''}{\tau'} \right) \tau' = (\lambda - r^2 r') \tau',$$

and

$$\text{XXII. . . } \sigma - \rho = r^2 \tau' + r p \nu = r U \tau' + p U \nu,$$

because

$$\text{XXIII. . . } r p \nabla \lambda \nu = - r p r^{-1} \tau' = - r r' \tau';$$

we may therefore replace the formula I. for the vector of the curve by the following, which is true to the same order of approximation, †

$$\text{XXIV. . . } \rho_s = \rho + s\tau + \frac{s^2}{2r^2} (\kappa - \rho) + \frac{s^2}{6r^2} \nabla\lambda(\sigma - \rho):$$

and may thus exhibit, even to the eye, the dependence of all affections connected with s^2 , on the two new lines, λ and $\sigma - \rho$, which were not required when s^2 was neglected, but can now be determined by the two scalars r and p (or r and r' , or H and P as before). The geometrical signification of the scalar p is evident from what precedes, namely, the height (κs) of the centre of the osculating sphere above that of the osculating circle, divided by the binormal unit ($U\nu$); and

* The angle H appears to have been first considered by Lancret, in connexion with his theory of rectifying lines, planes, and surfaces: but the angle here called P was virtually included in the earlier results of Monge.

† As regards the homogeneity of such expressions, if we treat the four vectors ρ, p, κ , and σ , and the five scalars s, r, R, p , and r , as being each of the first dimension, we are then to regard the dimensions of τ, r', κ', H , and P as being each zero; those of τ', ν , and λ as each equal to -1 ; and that of either r'' or ν' as being -2 .

as regards what has been called the *radius* r of *second curvature* (V.), we shall see that this is in fact the *geometrical radius* of a *second circle*, which *osculates*, at the extremity of the tangential vector rr , to the *principal normal section* of the developable *Surface of Tangents*; and thereby determines an *osculating oblique cone* to that important surface, and also an *osculating right cone** thereto, of which latter cone the *semiangle* is H , and the *rectifying line* λ is the *axis of revolution*: being also a *side* of an *osculating right cylinder*, on which is traced what is called the *osculating helix*. We shall assign the quaternion equations of these two cones, and of this cylinder, and helix; and shall show that although the *helix* has *not* generally *complete contact* of the *third order* with the *given curve*, yet it approaches *more nearly* to that curve (supposed to be of *double curvature*), than does the *osculating circle*. But an *osculating parabola* will also be assigned, namely, the parabola which osculates to the *projection* of the curve, on its *own osculating plane*: and it will be shown that this *parabola* represents or *constructs one* of the *two principal and rectangular components* (396, (18.)), of the *deviation* of the curve from its *osculating circle*, in a direction which is (ultimately) *tangential to the osculating sphere*, while the *helix constructs the other component*. An *osculating right cone* to the *cone of chords*, drawn from a *given point* of the curve, will also be assigned by quaternions: and will be shown to have in general a *smaller acute semiangle* C (or $\pi - C$), than the acute semiangle H (or $\pi - H$), of the *osculating right cone* (above mentioned) to the *surface of tangents*, or (as will be seen) to the *cone of parallels* to tangents (369, (6.), &c.): the *relation* between these *two semiangles*, of *two osculating right cones*, being rigorously expressed by the formula,

$$\text{XXV.} \dots \tan C = \frac{2}{3} \tan H.$$

A *new oblique cone* of the second order will be assigned, which has contact of the *same order* with the *cone of chords*, as the *second right cone* (C), while the latter *osculates to both* of them; and also an *osculating parabolic cylinder*, which rests upon the *osculating parabola*, and is *cut perpendicularly* in that auxiliary curve by the *osculating plane* to the given curve. And the *intersection* of these *two last surfaces* of the second order (*oblique cone and parabolic cylinder*) will

* These two *osculating cones*, oblique and right, to the *surface of tangents*, appear to have been first assigned, in the Memoir already cited, by M. de Saint Venant: the *osculating (circular) helix*, and the *osculating (circular) cylinder*, having been previously considered by M. Olivier.

be found to consist partly of the *binormal* at the given point, and partly of a certain *twisted cubic** (or *gauche curve* of the *third degree*), which latter curve has *complete contact* of the *third order* with the *given curve* in space. *Constructions* (comp. 395, (6.)) will be assigned, which will connect, more closely than before, the *tangent* to the *locus of centres of curvature*, with *other properties* or *affections* of that given curve. And finally we shall prove, by a very simple quaternion analysis, as a consequence of the formula XI', the known theorem,† that *when the ratio of the two curvatures is constant, the curve is a geodesic on a cylinder.*

(1.) The scalar expression III., for the *second curvature* of a curve in space, as defined in 396, may be deduced from the formulæ (396, (5.), &c.) of the recent theory of *emanants*, which give,

$$\text{XXVI.} \dots \theta = \nabla \nu' \nu^{-1} = r^{-1} r, \quad \omega_0 = \rho, \quad \epsilon = r, \quad \text{if } \eta = \nu,$$

while the *line of contact* (396, (14.)), of the *emanant plane* with its envelope, coincides in position with the *tangent* to the *curve*; in passing, then, from the given point P to the near point P', the *binormal* (ν) and the *osculating plane* ($\perp \nu$) have (nearly) *revolved together, round that tangent* (r) as a *common axis*, through a *small angle* $= r^{-1} \epsilon$, and therefore with a *velocity* $= r^{-1}$, if this symbol have the value assigned by III., or by the following extended expression, in which the *scalar variable* (t) is *arbitrary* (comp. 395, (11.), &c.),

$$\text{XXVII.} \dots r^{-1} = S \frac{\rho'''}{\nabla \rho' \rho''} = S \frac{d^3 \rho}{\nabla d \rho d^2 \rho} = \text{Second Curvature} :$$

while the *binormal* has at the same time been *translated* (nearly), in a direction perpendicular to the *tangent* r , through the *small interval* $\epsilon = \epsilon r$, which (in the present order of approximation) represents the *small chord* PP'.

(2.) As an *example*, if we take this *new form* of the equation of the *helix*,

XXVIII. . . $\rho \epsilon = b (\cot a + \epsilon a^2 \beta)$, with $Ta = T\beta = 1$, and $Sa\beta = 0$, which gives the derived vectors,

$$\text{XXIX.} \dots \rho \epsilon' = ba (\cot a + \epsilon a^2 \beta), \quad \rho \epsilon'' = -b \epsilon a^2 \beta, \quad \rho \epsilon''' = a \rho \epsilon'';$$

and this expression for the *arc* s (supposed to begin with t),

$$\text{XXX.} \dots s = s' t, \quad \text{where } s' = T\rho' = b \cos a = \text{const.},$$

we easily find (after a few reductions) the following values for the *two curvatures* :

* This convenient appellation (of *twisted cubic*) has been proposed by Dr. Salmon, for a curve of the kind here considered : see pages 241, &c., of his already cited Treatise. The *osculating twisted cubic* will be considered somewhat later.

† This theorem was established, on sufficient grounds, in the cited Memoir of M. de Saint Venant (page 26); but it has also been otherwise deduced by M. Serret, in the Additions to M. Liouville's Edition of Monge (Paris, 1850, page 561, &c.).

$$\text{XXXI.} \dots r^{-1} = b^{-1} \sin^2 \alpha, \quad r^{-1} = b^{-1} \sin \alpha \cos \alpha;$$

while the *common centre* (396), of the *osculating circle and sphere*, has now for its vector (comp. 369, (3.)),

$$\text{XXXII.} \dots \kappa = \sigma = \rho_1 - b \epsilon^{2/3} \cos \alpha = b \cot \alpha (\alpha - \epsilon^{2/3} \cot \alpha);$$

b being here the *radius of the cylinder*, but α denoting still the *constant inclination of the tangent* (ρ') to the *axis* (α).

(3.) The *rectifying line* (396), considered merely as to its *position*, being the *line of contact of the rectifying plane* (396, XIV.) with its own envelope, is represented by the equations,

$$\text{XXXIII.} \dots 0 = 8r'(\omega - \rho) = 8r''(\omega - \rho), \quad \text{or} \quad \text{XXXIII'}. \dots 0 = \nabla \lambda(\omega - \rho),$$

with the signification XVI. of λ ; and accordingly, if we treat the *rectifying planes as emanants*, or change η to r' , we find the value $\theta = \nabla r'' r'^{-1} = \lambda$, which shows also that in the passage from P to P_1 , the *rectifying plane turns* (nearly) *round the rectifying line, through a small angle* $= \epsilon T \lambda$, or with a *velocity of rotation* represented by the tensor,

$$\text{XXXIV.} \dots T \lambda = \sqrt{(r^{-2} + r'^2)} = r^{-1} \operatorname{cosec} H = r^{-1} \sec H;$$

so that what we have called the *rectifying vector*, λ , coincides in fact (by the general theory of emanants) with the *vector axis* (396, (14.)) of this *rotation of the rectifying plane*: as the *vector of second curvature* ($r^{-1}r$) has been seen to be, in the *same full sense* (comp. (1.)), the *vector axis of rotation of the osculating plane*, when *velocity, direction, and position* are all taken into account.

(4.) When the derivative ϵ' of the *arc* is only *constant*, without being equal to *unity* (comp. 395, (12.)), the expression XVI. may be put under this slightly more general form,

$$\text{XXXV.} \dots \lambda = \nabla \frac{\rho'''}{\rho''} = \nabla \frac{d^3 \rho}{ds^2 ds} = \text{Rectifying Vector};$$

and accordingly for the *helix* (2.) we have thus the values,

$$\text{XXXVI.} \dots \lambda = a \epsilon'^{-1} = a b^{-1} \sin \alpha = a r^{-1} \operatorname{cosec} \alpha, \quad U \lambda = \alpha;$$

the *rectifying line* is therefore, for this curve, *parallel to the axis*, and *coincides with the generating line of the cylinder*, as is otherwise evident from geometry. The value, $T \lambda = b^{-1} \sin \alpha$, of the *velocity of rotation of the rectifying plane*, which is here the *tangent plane* to the cylinder, when compared with a conceived *velocity of motion along the curve*, is also easily interpreted; and the formulae XVII., XVIII. give, for the same helix (by XXXI.), the values,

$$\text{XXXVII.} \dots r' = 0, \quad H = \alpha, \quad P = 0.$$

(5.) The *normal* (or the *radius of curvature*), as being *perpendicular to the rectifying plane*, revolves with the *same velocity*, and round a *parallel line*; to determine the *position of which new line*, or the *point H* in which it *cuts the normal*, we have only to change η to r' in the formula 396, XXXII., which then becomes,

$$\begin{aligned} \text{XXXVIII.} \dots O H &= \omega_0 = \rho - r' S \frac{r}{\lambda r} = \rho - \lambda^{-2} r' \\ &= \rho + \frac{r'^2 (\kappa - \rho)}{r^{-2} + r'^2} = \frac{r^2 \rho + r'^2 \kappa}{r^2 + r'^2} \\ &= \rho \cos^2 H + \kappa \sin^2 H; \end{aligned}$$

the *vector of rotation* (896, (9.)) of the *normal* is therefore a line \parallel and $=\lambda$, which *divides* (internally) the *radius* (r) of *curvature* into the *two segments*,*

$$\text{XXXIX.} \dots \overline{PH} = r \sin^2 H, \quad \overline{HK} = r \cos^2 H;$$

namely, into segments which are *proportional to the squares* (r^2 and r^2) of the *first* and *second curvatures*.

(6.) At the same time, what we have called generally the *vector of translation* of an emanant line becomes, for the *normal* (by 896, (10.), changing θ to λ), the line

$$\text{XL.} \dots \epsilon = \lambda S \frac{r}{\lambda} = U \lambda \cos H = -r^2 \lambda^{-1}, \text{ set off from the same point H;}$$

and the *indefinite right line*, or *axis*, through that point H,

$$\text{XLI.} \dots 0 = V \lambda (\omega - \omega_0), \quad \text{or} \quad \text{XLI'.} \dots 0 = V \lambda (\omega - \rho \cos^2 H - \epsilon \sin^2 H),$$

along which *axis* the *normal moves*, through the *small line* ϵ , while it *turns round* the *same axis* (as before) through the *small angle* $\epsilon T \lambda$, may be called (comp. again 396, (10.)) the *Axis of Displacement of the Normal* (or of the *radius of curvature*).

(7.) As a verification, for the *helix* (2.) we have thus the values,

$$\text{XLII.} \dots \overline{PH} = b, \quad \omega_0 = \rho \epsilon - b \epsilon a' \beta = b a t \cot a, \quad \epsilon = a \cos a;$$

so that the *axis of displacement* (6.) coincides with the *axis* (a) of the *cylinder*, as was of course to be expected.

(8.) When the given curve is *not* a *helix*, the values VI., XVI., XXXVIII., and XL., of r' , λ , ω_0 , and ϵ , enable us to put the expression I. for ρ , under the form,

$$\text{XLIII.} \dots \rho_s = \omega_0 + \epsilon t + \epsilon^{2\lambda} (\rho - \omega_0) - \frac{\epsilon^2 r' r'}{6r};$$

the *curve* therefore generally *deviates*, by this last *small vector* of the *third order*, namely by that *part* of the *term* $\frac{1}{6} \epsilon^2 r''$ which has the *direction* of the *normal* r' , or of $-r'$, and which depends on r' , *from the osculating helix*,

$$\text{XLIV.} \dots \omega_s = \omega_0 + \epsilon t + \epsilon^{2\lambda} (\rho - \omega_0),$$

and from the *osculating right cylinder*,

$$\text{XLV.} \dots T V \lambda (\omega - \omega_0) = \sin H,$$

whereon that *helix* is traced, and of which the *rectifying line* (XXXIII.) is a *side*, while its *axis of revolution* (comp. (7.)) is the *axis of displacement* (XLI.) of the *normal*.

(9.) *Another general transformation*, of the expression I. for the *vector* of the *curve*, is had by the substitution,

$$\text{XLVI.} \dots s = t + \frac{\epsilon^2 r'}{6r} + \frac{\epsilon^2}{6r^2}$$

in which t is a new scalar variable; for this gives the new form,

* This law of *division of a radius* of *curvature* into segments, by the *common perpendicular* to that radius and to its *consecutive*, has been otherwise deduced by M. de Saint Venant, in the *Memoir* already referred to.

$$\text{XLVII.} \dots \rho_t = \rho + t\tau + \frac{1}{2}t^2 \left(\tau' + \frac{r'\tau}{3r} \right) + \frac{1}{6}t^3 r^{-1} \nu,$$

and therefore shows that the curve deviates, by this other small vector of the third order,

$$\text{XLVIII.} \dots \frac{1}{6}t^3 r^{-1} \nu = \frac{1}{6}t^3 r^{-1} r r',$$

that is, by the part of the term $\frac{1}{6}t^3 r^{-1} \nu$ which has the direction of the binormal ν , and which depends on r , from what we propose to call the *Osculating Parabola*, namely that new auxiliary curve of which the equation is,

$$\text{XLIX.} \dots \omega_t = \rho + t\tau + \frac{1}{6}t^3 \left(\tau' + \frac{r'\tau}{3r} \right);$$

or from the parabola which osculates at the given point P , to the projection of the given curve on its own osculating plane.

(10.) And because the small deviation XLVIII. of the curve from the parabola is also the deviation of the same curve from this last plane, if we conceive that a near point Q of the curve is projected into three new points Q_1, Q_2, Q_3 , on the tangent, normal, and binormal respectively, we shall have the limiting equation,

$$\text{L.} \dots \lim. \frac{\delta P Q_3}{P Q_1 \cdot P Q_2} = r^{-1} = \text{Second Curvature};$$

the sign of this scalar quotient being determined by the rules of quaternions.

(11.) But we may also (comp. 896, (17.), (18.)) employ this third general transformation of L, analogous to the forms XLIII. and XLVII,

$$\text{LI.} \dots \rho_s = \kappa + s^w (\rho - \kappa) + \frac{s^3}{6} \nu' \tau,$$

with the value XI. of ν' ; in which the sum of the two first terms gives the vector of the point of the osculating circle, which is distant from the given point PP , by an arc of that circle equal to the arc s of the given curve; and the third term,

$$\text{LII.} \dots \frac{1}{6}s^3 \nu' \tau = \frac{1}{6}s^3 (\tau'' + r^{-2} \tau) = -\frac{1}{6}s^3 r^{-1} r' \tau' + \frac{1}{6}s^3 r^{-1} \nu,$$

which represents the deviation from the same circle, measured in a direction (comp. IX. or X.) tangential to the osculating sphere, is (as we see) the vector sum of two rectangular components, which represent respectively the deviations of the curve, from the osculating helix (8.), and from the osculating parabola (9.).

(12.) It follows, then, that although neither helix nor parabola has in general complete contact of the third order with a given curve in space, since the deviation from each is generally a small vector of that (third) order, yet each of these two auxiliary curves, one on a right cylinder XLV., and the other on the osculating plane, approaches in general more closely to the given curve, than does the osculating circle: while circle, helix, and parabola have, all three, complete contact of the second* order with the curve, and with each other.

* It appears then that we may say that the helix and parabola have each a contact with the curve in space, which is intermediate between the second and third orders: or that the exponent of the order of each contact is the fractional index, $2\frac{1}{2}$. But it must be left to mathematicians to judge, whether this phraseology can properly be adopted.

(13.) As regards the *geometrical signification* of the *new variable scalar*, t , in the equation XLIX. of the parabola, that equation gives,

$$\text{LIII.} \dots T\omega'_t = T\left\{\left(1 + \frac{r't}{8r}\right)r + tr'\right\} = 1 + \frac{r't}{8r} + \frac{t^2}{2r^2} \dots,$$

and therefore (to the present order of approximation),

LIV. . . *Arc of Osculating Parabola* (from ω_0 to ω_t)

$$= \int_0^t T\omega'_t dt = t + \frac{r't^2}{6r} + \frac{t^3}{6r^2} = s \text{ (by XLVI.)}$$

= *Arc of Curve in Space* (from ρ_0 to ρ_s);

If then an *arc* = s be thus set off upon the parabola, with the same initial point r , and the same initial direction, and if this *parabolic arc*, or its *chord* $\omega_t - \omega_0$, be obliquely projected on the *initial tangent* r , by drawing a *diameter* of the parabola through its final point, the *oblique tangential projection* so obtained will be = tr by XLIX. ; and its *length*, or the *ordinate to that diameter*, will be the scalar t .

(14.) And as regards the *direction* of the *diameter* of the osculating parabola, drawn as we may suppose from r , if we denote for a moment by D its inclination to the normal + r' , regarded as positive when towards the tangent + r , we have (by XLIX. and XVIII.) the formula,

$$\text{LV.} \dots \tan D = \frac{r'}{8} = \frac{1}{8} \tan P \cot H:$$

which is an instance of the reducibility, above mentioned, of all *affections* of the curve depending on s^2 , to a dependence on the *two angles*, H and P .

(15.) Some of these *affections*, besides the *direction* of the *rectifying line* λ , can be deduced from the angle H alone. As an example, we may observe that the vector equation of the *surface of tangents* is of the form,

$$\text{LVI.} \dots \omega_s t = \rho_s + t\rho'_s = \rho_s + tr_s,$$

in which s and t are two independent and scalar variables, and

$$\text{LVII.} \dots r_s = r + sr' + \frac{s^2}{2} r'';$$

+ terms depending on s^4 in ρ_s . If then we cut this *developable* LVI. by the *plane*,

$$\text{LVIII.} \dots Sr(\omega - \rho) = -c = \text{any given scalar constant,}$$

which is, relatively to the *surface*, a *normal plane* at the extremity of the tangential vector cr from P , while this *tangent* is also a *generating line*, we get thus a *principal* normal section*, of which the variable vector has for its approximate expression,

$$\text{LIX.} \dots \omega_s = (\rho + cr) + (cs + \dots)r' + \left(\frac{1}{2}cs^2r'' + \dots\right)v;$$

the terms suppressed being of higher orders than the terms retained, and having no influence on the *curvature* of the section. We find then thus, that the *vector* of the *centre* of the *osculating circle* to this *normal section* of the *surface of tangents* to the given curve is, *rigorously*,

* Some *general acquaintance* with the known theory of *sections of surfaces* is here supposed, although that subject will soon be briefly treated by quaternions.

$$\text{LX.} \dots \rho + cr + \frac{(cr)^2}{cs^2r^{-1}\nu} = \rho + c(\tau + r\nu) = \rho + cr\lambda;$$

so that the *locus* of all such centres is the *rectifying line* XXXIII'. And if, in particular, we make $c = r$, or cut the developable at the extremity of the tangential vector rr , the expression LX. becomes then $\rho + rr + rU\nu$; which expresses that the *radius* of the *circle of curvature* of this normal section of the surface is precisely what has been called the *Radius* (r) of *Second Curvature*, of the given curve in space. But this radius ($r = r \tan H$) depends only on the angle H , when the radius (r) of (absolute) curvature is given, or has been previously determined.

(16.) The *cone of the second order*, represented by the quaternion equation,

$$\text{LXI.} \dots 0 = 2rSr(\omega - \rho)S\nu(\omega - \rho) + (Vr(\omega - \rho))^2,$$

has its *vertex* at the given point r , and *rests* upon the *circle* last determined; it is then the *locus* of all the *circles* lately mentioned (15.), and is therefore (in a known sense) an *osculating oblique cone* to the developable *surface of tangents*: its *cyclic normals* (comp. 357, &c.) being r and $r + 2r\nu$, or r and $rr + 2rU\nu$. But, by 394, (30.), the *osculating right cone* to this cone LXI., and therefore also (in a sense likewise known) to the *surface of tangents itself*, is one which has the recent *locus of centres* (15.), namely the *rectifying line* (λ), for its *axis of revolution*, while the *tangent* (r) to the curve is one of its *sides*: its *semiangle* is therefore $= H$, and a form of the quaternion equation of this *osculating right cone* is the following (comp. XLV.),

$$\text{LXII.} \dots TVU\lambda(\omega - \rho) = \sin H.$$

(17.) The *right cone* LXII., which thus osculates to the developable *surface of tangents* LVI., along the given tangent r , osculates *also* along that tangential line to the *cone of parallels to tangents*, which has its vertex at the given point r ; as is at once seen (comp. 394, (30.)), by changing ρ' and ρ'' to r' and r'' , in the general expression $V\rho'\rho''$ (393, (6.), or 394, (6.)), for a line in the direction of the *axis* of the *osculating circle* to a curve upon a *sphere*. And the *axis* of the *right cone* thus determined, namely (again) the *rectifying line* (λ), *intersects* the *plane* of the *great circle* of the *osculating sphere*, which is *parallel* to the *osculating plane*, in a point l of which the vector is,

$$\text{LXIII.} \dots Ol = \rho + r\lambda = \rho + rr'r + r\nu\nu.$$

(18.) We have thus, in *general*, a *gauche quadrilateral*, PKSL, right-angled except at l , with the help of which *one* figure all affections of the curve, not depending on s^4 , can be geometrically represented or *constructed*: although it must be observed that when $r' = 0$, which happens for the *helix* (XXXVII.), the *osculating circle* is then *itself* a *great circle* of the *osculating sphere*, and the points r and l , like the points κ and s , *coincide*.

(19.) In the general case, it may assist the conceptions to suppose lines set off, from the given point r , on the tangent and binormal, as follows:

$$\text{LXIV.} \dots Pt = Bl = rr'r; \quad Pb = Tl = ks = r\nu\nu;$$

for thus we shall have a *right triangular prism*, with the two right-angled triangles, $rT\kappa$ and lBs , in the *osculating plane* and in the *parallel plane* (17.), for two of its faces, while the three others are the rectangles, $r\kappa Bs$, $PBlT$, κSlT , whereof the two first are situated respectively in the normal and rectifying planes.

(20.) All scalar properties of this auxiliary prism may be deduced, by our general methods, from the three scalars, r, r', r'' , or r, H, P ; and all vector properties of the same prism can in like manner be deduced from the three vectors r, r', r'' , or from r, ν, ν' , which (as we have seen) are not entirely arbitrary, but are subject to certain conditions.

(21.) As an example of such deduction (compare the annexed Figure 81), the equation of the diagonal plane SPL , which contains the radius (R) of spherical curvature and the rectifying line (λ), and the equation of the trace, say PU , of that plane on the osculating plane, which trace is evidently parallel (by the construction) to the edges LS, TK of the prism, are in the recent notations (comp. XX.),

LXV. . . $0 = Sr''(\omega - \rho)$; LXVI. . . $0 = V(r'r')(\omega - \rho)$;
with the verification that $rSr'r'' = r'Sr'r'' = r^2r'$, by II.

(22.) In general, by 204, (22.), if α and β be any two vectors, we have the expressions,

$$\begin{aligned} \text{LXVII. . . } \tan \angle \frac{\beta}{\alpha} &= \tan \angle \frac{\alpha}{\beta} = -\tan \angle \beta\alpha = -\tan \angle \alpha\beta \\ &= TV \frac{\beta}{\alpha} : S \frac{\beta}{\alpha} = \frac{TV}{S} \cdot \frac{\beta}{\alpha} = -(TV : S) \alpha\beta, \end{aligned}$$

the angles of quaternions here considered being supposed as usual (comp. 180) to be generally > 0 , but $< \pi$; for example, we have thus,

$$\text{LXVIII. . . } \tan H = \tan \angle \frac{\lambda}{r} = (TV : S) \lambda r^{-1} = (TV : S) (r^{-1} - r') = rTr' = rr^{-1},$$

as in XVII.; and in like manner we have generally, by principles already explained (comp. 196, XVI.),

$$\begin{aligned} \text{LXIX. . . } \cos \angle \frac{\beta}{\alpha} &= \cos \angle \frac{\alpha}{\beta} = -\cos \angle \beta\alpha = -\cos \angle \alpha\beta \\ &= S \frac{\beta}{\alpha} : T \frac{\beta}{\alpha} = SU \frac{\beta}{\alpha} = -SU\alpha\beta. \end{aligned}$$

(23.) Applying these principles to investigate the inclinations of the vector r'' , which is perpendicular to the diagonal plane LXV. of the prism, to the three rectangular lines r, r', ν , or the inclinations of that diagonal plane itself to the normal, rectifying, and osculating planes, with the help of the expressions deduced from VI. for the three products,* $rr', r'r'', \nu r''$, we arrive easily at the following results:

* A student, who should be inclined to pursue this subject, might find it useful to form for himself a table of all the binary products of the nine vectors,

$$r, r', r'', \nu, \nu', \lambda, \sigma - \rho, \sigma - \mu, \text{ and } \kappa',$$

considered as so many quaternions, and reduced to the common quadrimomial form, $a + br + cr' + e\nu$, in which a, b, c, e are scalars, whereof some may vanish, but which are generally functions of r, r' , and r'' .

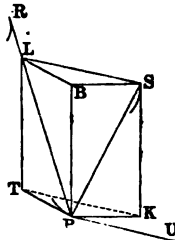


Fig. 81.

$$\text{LXX.} \dots \cos \angle \frac{r''}{r} = \frac{-r^3}{Tr''}; \quad \cos \angle \frac{r''}{r'} = -\frac{r^2 r'}{Tr''}; \quad \cos \angle \frac{r''}{\nu} = \frac{r^{-1} r^{-1}}{Tr''};$$

with the verification, that the sum of the squares of these three cosines is unity, because

$$\text{LXXI.} \dots r^2 Tr'' = \sqrt{(1 + r^2 R^2)} = \sqrt{(1 + r^2 + r^2 R^2)}; \\ \text{or} \quad \text{LXXI'}. \dots r Tr'' = \sqrt{(r^2 r'^2 + T\lambda^2)}, \quad Tr'' = \sqrt{(r^{-4} + T\nu^2)}.$$

(24.) Or we may write, on the same general plan,

$$\text{LXXII.} \dots \tan \angle \frac{r''}{r} = \frac{-R}{Tr}; \quad \tan \angle \frac{r''}{r'} = \frac{-r T\lambda}{r'}; \quad \tan \angle \frac{r''}{\nu} = \frac{r}{r} \sqrt{(1 + r^2)};$$

or

$$\text{LXXIII.} \dots \tan \angle r r'' = R Tr^{-1}; \quad \tan \angle r' r'' = r r'^{-1} T\lambda; \quad \tan \angle \nu r'' = -r^{-1} \sqrt{(1 + r^2)};$$

and may modify the expressions, by introducing the auxiliary angles H and P , with which may be combined, if we think fit, the following *angle of the prism*,

$$\text{LXXIV.} \dots PKT = BSL = \tan^{-1} r'.$$

(25.) Instead of thus comparing the *plane SPL* with the three rectangular planes (879, (5.)) of the construction, we may inquire what is the value of the *angle SPL*, which the radius (R) of spherical curvature makes with the rectifying line (λ); and we find, on the same plan, by quaternions, the following very simple expression for the cosine of this angle, which may however be deduced by spherical trigonometry also,

$$\text{LXXV.} \dots \cos \text{SPL} = -\text{SU}\lambda(\sigma - \rho) = \frac{pr^{-1}}{RT\lambda} = \sin P \sin H;$$

or

$$\text{LXXV'}. \dots \cos \text{SPL} = \cos \text{SPB} \cos \text{BPL}.$$

(26.) In general, it is easy to form, by methods already explained, the quaternion equation of a *cone* which has a *given vertex*, and *rests on a given curve* in space; and also to determine the *right cone* which *osculates* (894, (30.)) to this *general cone*, along any *given side* of it.

(27.) But if we merely wish to assign the *osculating right cone* to the *cone of chords* from P , or to the *locus of the line PP*, we may imitate a recent process: and may observe that if this *new cone* be cut by the *normal plane LVIII.*, the *vector of the section* has the following approximate expression, analogous to LIX., and like it sufficient for our purpose,

$$\text{LXXVI.} \dots \omega_s = \rho + cr + \frac{1}{2}csr' + \frac{1}{8}cs^2r^{-1}\nu;$$

from which it may be inferred (comp. (15.), (16.)), that the *axis of revolution* of the *new right cone* has for equation,

$$\text{LXXVII.} \dots 0 = V(r^{-1}r + \frac{1}{2}\nu)(\omega - \rho).$$

This *axis* is therefore situated in the *rectifying plane*, between the *rectifying line* (λ or $r^{-1}r + \nu$), and the *tangential vector* (IV.) of *second curvature* ($r^{-1}r$): while the *semiangle C* of the same *new cone* (measured like H from $+r$ towards $+\nu$) has the value already assigned by anticipation in the formula XXV., and is therefore *less* than the *semiangle H* if both be *acute*, but *greater* than H if both be *obtuse*; so that, in each case, the *new right cone (C)* is *sharper* than the *old right cone (H)*.

(28.) The same result may be otherwise obtained, by observing that an unit-

vector in the direction of the chord PP_1 , has (by 896, XLIV., and 897, I.) the approximate expression,

$$\begin{aligned} \text{LXXVIII.} \dots \chi_s = U(\rho_s - \rho) &= \left(1 + \frac{s^2}{24r^3}\right) \left(r + \frac{sr'}{2} + \frac{s^2 r''}{6}\right) \\ &= r + \frac{sr'}{2} + \frac{s^2}{6} \left(r'' + \frac{r'^2 r}{4}\right); \end{aligned}$$

whence the axis of the osculating right cone to the cone of chords (27.) has rigorously the direction of the line $V\chi'\chi''$ (for $s=0$), or of the vector,

$$\text{LXXIX.} \dots \xi = \nabla r'(r^2 r'' + \frac{1}{2} r') = \lambda - \frac{1}{2} \nu = r^{-1} r + \frac{1}{2} \nu, \text{ as before.}$$

(29.) This axis ξ makes (if we neglect s^3) the same angle C , with the chord PP_1 , as with the tangent τ ; whereas the former axis λ makes unequal angles with those two lines, within the same order (or degree) of approximation: for our methods conduct to the expression,

$$\text{LXXX.} \dots \angle \frac{\rho_s - \rho}{\lambda} = H - \frac{s^2}{24r^2},$$

from which the relation XXV., between the two right cones, may easily be deduced anew.

(30.) Neglecting only s^4 , and employing the substitution XLVI., the expression XLVII. for the vector of the given curve becomes,

$$\text{LXXXI.} \dots \rho_t = \rho + t\tau + \frac{1}{2} t^2 \nu + \frac{1}{6} t^3 r^{-1} \nu, \text{ if } \text{LXXXII.} \dots \nu = r' + \frac{r' r'}{8r};$$

where the variable scalar t denotes, by (13.), the ordinate of the osculating parabola, and the constant vector ν has the direction, by (14.), of the diameter of that parabola.

(31.) In the present order of approximation, then, the proposed curve in space may be considered to be the common intersection of the three following surfaces of the second order, all passing through the given point P :

$$\text{LXXXIII.} \dots 2(Sr'(\omega - \rho))^2 = 8rS\nu(\omega - \rho)S\nu\nu(\omega - \rho);$$

$$\text{LXXXIV.} \dots 2Sr'(\omega - \rho) = -r^2(S\nu\nu(\omega - \rho))^2;$$

$$\text{LXXXV.} \dots 8rS\nu(\omega - \rho) = -r^2Sr'(\omega - \rho)S\nu\nu(\omega - \rho);$$

whereof the first represents a new osculating oblique cone, which has a contact of the same (second) order with the cone of chords, as the osculating right cone (27.); the second represents an osculating parabolic cylinder, which is cut perpendicularly in the osculating parabola (9.), by the osculating plane to the curve; and the third represents a certain osculating hyperbolic (or ruled) paraboloid, whereof the tangent (τ) is one of the generating lines, while the diameter (ν) of the osculating parabola is another.

(32.) Each of these three surfaces (31.) has in fact generally a contact of the third order with the given curve; or has its equation satisfied, not only (as is obvious on inspection) by the point P itself, but also when we derivate successively with respect to the scalar variable t , and then substitute the values (comp. LXXXI.),

$$\text{LXXXVI.} \dots \omega = \rho_0 = \rho, \quad \omega' = \rho_0' = \tau, \quad \omega'' = \rho_0'' = \nu, \quad \omega''' = \rho_0''' = r^{-1} \nu;$$

$r, \tau, \rho, \tau', \nu$, and ν being treated as constants of the equation, or of the surface, in each of these derivations.

(83.) The cone LXXXIII., and the cylinder LXXXIV., have a common generatrix, namely the binormal* (ν); and in like manner, another generating line of the same cone, namely the tangent (τ) to the curve, has just been seen (31.) to be a line on the paraboloid LXXXV.: and although the cylinder and paraboloid have no finitely distant right line common, yet each may be said to contain the line at infinity, in the diametral plane of the cylinder, namely in the plane of ν and v , of which plane the quaternion equation is (comp. (14.)),

$$\text{LXXXVII.} \dots 0 = S\nu\nu(\omega - \rho), \quad \text{or} \quad \text{LXXXVII.} \dots 0 = S(\tau\tau'\tau' - 3\tau)(\omega - \rho);$$

or the line in which this diametral meets the parallel axial plane.

(84.) On the whole, then, it is clear, from the known theory of intersections of surfaces of the second order having a common generating line, that the given curve of double curvature (whatever it may be) has contact of the third order with the twisted cubic, † or gauche curve of the third degree, which is represented without ambiguity by the system of the two scalar equations,

$$\text{LXXXVIII.} \dots y = x^2, \quad z = x^3,$$

if we write for abridgment,

$$\text{LXXXIX.} \dots \begin{cases} x = (t =) - r^2 S\nu\nu(\omega - \rho), \\ y = (t^2 =) - 2r^2 S\tau'\tau'(\omega - \rho), \\ z = (t^3 =) - 6r^2 r S\nu\nu(\omega - \rho). \end{cases}$$

(85.) As another geometrical connexion between the elements of the present theory, it may be observed that while the osculating plane to the curve, of which plane the equation is,

$$\text{XC.} \dots S\nu(\omega - \rho) = 0, \quad \text{as in 896, XV.},$$

touches the oblique cone LXXXIII., along the tangent τ to the same curve, the diametral plane LXXXVII. touches the same cone along the binormal ν , which was lately seen (83.) to be, as well as τ , a side of that oblique cone; but these two sides of contact, τ and ν , are both in the rectifying plane (896, XIV.), and the two tangent planes corresponding intersect in the diameter v of the parabola (9.); we have therefore this theorem:—

The diameter of the osculating parabola to a curve of double curvature is the polar of the rectifying plane, with respect to the osculating oblique cone LXXXIII.; that is, with respect to a certain cone of the second order, which has been above deduced from the expression LXXXI. for the vector ρ_t of the curve, as one naturally suggested thereby; and as having a contact of the third order with the curve at P ,

* The geometrical reason, for the osculating cone LXXXIII. to the cone of chords containing the binormal (ν), is that if the expression LXXXI. for ρ_t were rigorous, and if the variable t were supposed to increase indefinitely, the ultimate direction of the chord PP_t would be perpendicular to the osculating plane. And the same binormal is a generating line of the parabolic cylinder also, because that cylinder passes through P , and all its generating lines are perpendicular to the last mentioned plane. It is sufficient however to observe, on the side of calculation, that the equations LXXXIII. and LXXXIV. are satisfied, when we suppose $\omega - \rho \parallel \nu$.

† Compare again page 241, already cited, of Dr. Salmon's Treatise; also Art. 285, in page 225 of the same work.

and therefore also a contact of the *second order* with the *cone of chords* from that point.

(86.) Conversely, *this particular cone LXXXIII. is geometrically distinguished from all other* cones of the same (second) order, which have their vertices at the given point P, and have each a contact of the same second order, with the given cone of chords from that point, or of the third order with the given curve, by the condition that it is touched (as above), along the binormal (ν), by the diametral plane (νν) of the osculating parabolic cylinder LXXXIV.*

(87.) We have already considered, in 895, (5.), the simultaneous variations of the points P and κ, or of the vectors ρ and κ. With recent notations, including the expression $\mu = 2\kappa - \rho$, we have the following among other transformations, for the *first derivative* of the latter vector, and therefore for the *tangent κκ'* to the *locus of centres of curvature*, of a given curve in space :

$$\begin{aligned} \text{XCI. } \dots \kappa\kappa' &= D\kappa = \kappa' = (\rho - r^{-1}\nu)' = \tau + r^{-1}\tau''r^{-1} \\ &= (\rho + r^2\tau') = \tau + r^2\tau'' + 2r\tau'r' \\ &= r\tau'r' + r^2r^{-1}\nu = r\tau'(r' + p^{-1}r\nu) = r^{-1}(p\tau' + r\nu) \\ &= \frac{r\tau'}{\rho - \kappa} - \frac{r\tau'}{\sigma - \kappa} = \frac{r\tau'(\sigma - \mu)}{(\sigma - \kappa)(\kappa - \rho)} = r^{-1}(\sigma - \mu)\tau \\ &= \cot H(Ur' \tan P + U\nu) = r^{-1}R(Ur' \sin P + U\nu \cos P) \\ &= r^4\nu\nu'r' = r^4r^{-1}\nu = \nu^{-1}\nu'r^{-1} = r^{-1}\nu'\nu^{-1} \\ &= r^{-1}\nu(\rho - \sigma)(\kappa - \rho) = r^{-1}(\kappa - \rho)(\rho - \sigma)\nu \\ &= r^{-1}RU(\nu(\rho - \sigma)(\kappa - \rho)) = \&c. ; \end{aligned}$$

if then we draw the *diameter of curvature* PM, and let fall a perpendicular κN from the centre κ of the osculating circle on the new radius SM of the osculating sphere (as in the annexed Figure-82), *this perpendicular will touch† the locus of the centre κ, a result which agrees with the construction in 895, (6.); and we see, at the same time, that the length of the line κκ', or the tensor Tκ', may be expressed (comp. LXXXIII.) as follows,*

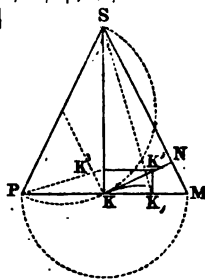


Fig. 82.

$$\text{XCII. } \dots \overline{\kappa\kappa'} = T\kappa' = RT^{-1} = r^2T\nu' = \tan \angle r\tau''.$$

(88.) If we *project* the tangent κκ', into its two rectangular components, κκ and κκ', on the diameter of curvature and the polar axis, we shall have by XCI. the expressions :

* The cone of *this system* (86.), which is *touched along the binormal by the normal plane*, and which therefore *intersects the parabolic cylinder LXXXIV. in a new twisted cubic* (comp. (84.)), having also contact of the *third order* with the *curve*, is easily found to have, for its quaternion equation, the following :

$$2r^2(Sr'(\omega - \rho))^2 = 8rSr(\omega - \rho)S\nu(\omega - \rho);$$

and with respect to *this cone* (comp. (85.)), the *polar of the rectifying plane* is the (*absolute*) *normal (r')* to the *curve*.

† Geometrically, and by infinitesimals, if we conceive κ' to be an infinitely near point of the locus of κ, and therefore in the normal plane at P, the angle Pκ's (like Pκs) will be right, and the point κ' will be on the semicircle Pκs; but the radius of this semicircle drawn to κ (comp. Fig. 82) is *parallel* to the line SM, to which line the *tangent κκ'* is therefore *perpendicular*, as above.

$$\text{XCIII.} \dots \mathbf{KK} = r r' r' = r' U r' = \frac{r r'}{\rho - \kappa} = \&c.;$$

$$\text{XCIV.} \dots \mathbf{KK}' = r^2 r^{-1} \nu = r r^{-1} U \nu = \frac{-r \nu'}{\sigma - \kappa} = \&c.;$$

these two projections then, or the vector-tangent \mathbf{KK}' itself, would suffice to determine r and r' , or H and P , and thereby all the affections of the curve which depend on s^2 , but not on s^4 .

(39.) We have also the similar triangles (see again Fig. 82),

$$\text{XCV.} \dots \Delta \mathbf{K} \mathbf{K}' \mathbf{K} \propto \mathbf{K}' \mathbf{K} \mathbf{K}' \propto \mathbf{K} \mathbf{M} \mathbf{S};$$

and the vector equations,

$$\begin{aligned} \text{XCVI.} \dots \mathbf{KK}' : \mathbf{SM} = \mathbf{KK} : \mathbf{SK} = \mathbf{KK}' : \mathbf{KM} = \mathbf{KK}' : \mathbf{PK} \\ = r^{-1} r = \text{Vector of second curvature (IV.)}; \end{aligned}$$

whence also result the scalar expressions,

$$\text{XCVII.} \dots \tan \mathbf{KSK} = \tan \mathbf{KPK}' = r^{-1} = \text{Second}^{\text{d}} \text{Curvature (III.)};$$

this last scalar being positive or negative, according as the rotation \mathbf{KSK} , (or \mathbf{KPK}') appears to be positive or negative, when seen from that side of the normal plane, towards which the conceived motion (396, (1.)) along the given curve, or the unit tangent $+r$, is directed.†

(40.) Besides the seven expressions, III., XXVII., L., and XCVII., this important scalar r^{-1} admits of many others, of which the following, numbered for reference as 8, 9, &c., and deduced from formulæ and principles already laid down, are examples: and may serve as exercises in transformation, according to the rules of the present Calculus, while some of them may also be found useful, in future geometrical applications.

(41.) We have then (among others) the transformations :

XCVIII. . . Second Curvature = r^{-1} (= seven preceding expressions)

$$\begin{aligned} &= p^{-1} r' = r^{-1} \cot H = \text{TL} \cos H = r^{-1} r' \cot P & (8, 9, 10, 11) \\ &= r^2 S \nu' r' = -S \nu' r^{-1} = -r^2 S r r' r'' = S r r'^{-1} r'' & (12, 13, 14, 15) \\ &= -r^2 S \nu r'' = S \nu^{-1} r'' = -S \nu \kappa' = S r \kappa' r' & (16, 17, 18, 19) \\ &= r \kappa' (\sigma - \mu)^{-1} = S \lambda r^{-1} = (\kappa - \rho) V \lambda \nu = -r^{-1} V \lambda \nu & (20, 21, 22, 23) \\ &= r^2 r' V \lambda \nu = r^2 S \lambda \nu r' = S \lambda r' \nu^{-1} = S \lambda r'^{-1} \nu & (24, 25, 26, 27) \\ &= r^2 S \nu' \lambda r = r^2 S \nu' \nu r = S r \nu^{-1} \nu' = r^2 S \nu' \nu^{-1} r'' & (28, 29, 30, 31) \\ &= r^2 S \nu \nu' r'' = r^{-1} V \nu' \lambda = r^2 r'^{-1} S \nu' \lambda r' = r^2 r'^{-1} S \nu \lambda r'' & (32, 33, 34, 35) \\ &= S \nu \lambda r''^{-1} = \text{Tr}^{-2} S \lambda \nu' r'' = \frac{-(r \nu)'}{r r'} = \frac{-r^2 \nu'}{\sigma - \rho} & (36, 37, 38, 39) \end{aligned}$$

* In illustration it may be observed, that if ds be treated as infinitely small, and if the line \mathbf{KK}' be supposed to represent (not the derivative κ' , but) the differential vector $d\kappa = \kappa' ds$, then the projections \mathbf{KK} , and \mathbf{KK}' become dr and $r r^{-1} ds$ (comp. XCIII. and XCIV.); while \mathbf{KPK}' (in Fig. 82) represents the infinitesimal angle $r^{-1} ds$, through which the osculating plane (comp. (1.)) revolves, round the tangent r to the curve during the change ds of the arc.

† This direction of $+r$ is to be conceived (comp. Fig. 81) to be towards the back of Fig. 82, as drawn, if the scalars r' and r (and therefore also p) be positive.

$$= \frac{-rv'}{rr' + p\nu'} = \frac{r^2 r'' + \tau}{r(\sigma - \rho)} = R^{-1} \tan \angle rrr'' = R^{-1} \tan \angle \frac{V\lambda\nu'}{r} \quad (40, 41, 42, 43)$$

$$= \frac{rr'\nu}{\sigma - \kappa} = \frac{rr'\tau'}{(\sigma - \kappa)\tau} = \frac{r'}{r} \cdot \frac{\tau(\kappa - \rho)}{\sigma - \kappa} = \frac{rr'\tau}{(\sigma - \kappa)(\rho - \kappa)} \quad (44, 45, 46, 47)$$

$$= S \frac{r\rho\lambda}{(\sigma - \kappa)(\rho - \kappa)} = S \frac{\rho + r\rho\lambda - \kappa}{(\sigma - \kappa)(\rho - \kappa)} = S \frac{KL}{KS.KP} \quad (48, 49, 50)$$

$$= S \frac{SL}{PK.KS} = \frac{-(Sar\nu)'}{r(Sar)'} = \frac{-d \cos \angle \frac{\nu}{a}}{rd \cos \angle \frac{r}{a}} \quad (51, 52, 53);$$

PKSL, in the forms 50 and 51, being points of the same *gauche quadrilateral* as in (18.); and *a*, in 52 and 53,* denoting any constant vector: while several other varieties of form may be deduced from the foregoing by very simple processes, such as the substitution of $U\nu$ for $r\nu$, &c., which gives for instance (comp. XI.), from the form 88, these others,

$$XCVIII'. \dots r^{-1} = \frac{-(U\nu)'}{rr'} = \frac{-(U\nu)'}{U\tau'} = \frac{-dU\nu}{rd\tau} \quad (54, 55, 56).$$

We may also write, with the significations (10.) of Q_1 and Q_3 , the following expression analogous to L,

$$XCVIII''. \dots r^{-1} = 6KP. \lim. \frac{PQ_3}{PQ_1^3} \quad (57),$$

which contains the law of the *inflection of the plane curve*, into which the proposed curve of *double curvature* is projected, on its own *rectifying plane*: the sign of the scalar, to which this last expression ultimately reduces itself, being determined by the rules of quaternions.

(42.) And besides the various expressions for the positive scalar r^{-2} , which are immediately obtained by *squaring* the foregoing forms, the following are a few others:

$$\begin{aligned} \text{XCIX.} \dots \text{Square of Second Curvature} &= r^{-2} = Tr^{-2} \\ &= T\lambda^2 - r^{-2} = r^2 Sr'\tau'\lambda - r^{-2} = r^2 T\nu'^2 - r^{-2}r^2 \quad (1, 2, 3) \\ &= r^2 Sr\nu'r'' - r^{-2}r^2 = r^2 T\tau''^2 - r^{-2} - r^{-2}r^2 = R^{-2}(r^4 T\tau''^2 - 1) \quad (4, 5, 6) \\ &= R^{-2}r^4 T\nu'^2 = R^{-2}T\kappa^2 = R^{-2} \tan^2 \angle r\tau'' \quad (7, 8, 9); \end{aligned}$$

while the important vector r'' , besides its two original forms VI., admits of the following among other expressions (comp. XX. XXI.):

$$\begin{aligned} \text{C.} \dots r'' &= D_r^2 \rho \quad (= \text{the two expressions VI.}) \\ &= r^{-2} \nabla \lambda (\sigma - \rho) = \lambda r' - r^{-1} r' r' = \nu' \tau - r^{-2} \tau \quad (3, 4, 5) \\ &= r \nabla \nu \lambda = r^{-2} \kappa^{-1} \tau (\sigma - \rho - \tau) = r^{-2} p + r^{-2} \lambda (\sigma - \rho) \quad (6, 7, 8) \\ &= ((\rho - \kappa)^{-1})' = r'(\kappa' - \tau)r' = -r^{-2} \tau. - \frac{r^{-1} r'}{\rho - \kappa} - \frac{r^{-1} r'}{\sigma - \kappa} \quad (9, 10, 11). \end{aligned}$$

(48.) As regards the general theory (896, (5.), &c.) of *emanant lines* (η) from curves, it might have been observed that if we write,

* This last form 53 corresponds to and contains a theorem of M. Serret, alluded to in the second Note to page 568.

$$\text{Cl.} \dots \zeta = V \frac{\tau}{\theta}, \quad \text{with} \quad \text{CIL} \dots \theta = V \frac{\eta'}{\eta}, \quad \text{as in 396, XXVII.}$$

the equation 396, XXXII. takes the simplified form,

$$\text{CIII.} \dots \text{PH} = \omega_0 - \rho = \eta \text{S}\eta^{-1}\zeta = \text{projection of vector } \zeta \text{ on emanant } \eta;$$

for example, when $\eta = \nu$, then $\theta = r^{-1}r$, and $\zeta = 0$, $\text{PH} = 0$, or $\omega_0 = \rho$, as in (1.); and when $\eta = \tau$, then $\theta = \nu$, $\zeta = r^2 r' \perp \eta$, so that the projection PH again vanishes, as in 396, (13.).

(44.) In an extensive class of applications, the emanant lines are perpendicular to the given curve ($\eta \perp \tau$); and since we have, by (43.),

$$\text{CIV.} \dots \zeta = \frac{V\tau V\eta' \eta}{\eta^2 \theta^2} = \eta^{-1} \theta^{-2} \text{Sr}\eta' = \frac{\eta^{-1} \text{S}\eta \tau'}{\text{T}\theta^2}, \quad \text{if } \text{Sr}\eta = 0,$$

we may write, for this case of normal emanation, the formula,

$$\text{CV.} \dots \text{PH} = \zeta = \frac{\text{projection of vector of curvature } (\tau') \text{ on emanant line } (\eta)}{\text{square of velocity } (\text{T}\theta) \text{ of rotation of that emanant}};$$

for example, when the emanant (η) coincides with the absolute normal (τ'), we have then $\theta = \lambda$, as in (3.), and the recent formula CV. becomes,

$$\text{CVI.} \dots \text{PH} = \omega_0 - \rho = \zeta = \tau' \text{T}\lambda^{-2} = r^2 r' \sin^2 H = (\kappa - \rho) \sin^2 H,$$

which agrees with the expression XXXVIII.

(45.) And in the corresponding case of tangential emanant planes, by making $\text{Sr}\eta = 0$ in the second equation 396, XXXVI., and passing to a second derived equation, we find for the intercept between the point τ of the curve, and the point, say B , in which the line of contact of the plane with its own envelope touches the cusp-edge of that developable surface, the expression,

$$\text{CVII.} \dots \text{PB} = \frac{-V\eta \eta' \text{S}\eta \tau'}{\text{S}\eta \eta' \eta''} = \frac{-\text{S}\eta \tau' \text{ (or } + \text{S}\eta \tau')}{\text{projection of } \eta'' \text{ on } \theta};$$

which accordingly vanishes, as it ought to do, when $\eta = \nu$, that is, when the emanant plane $\text{S}\eta(\omega - \rho) = 0$ coincides with the osculating plane XC .

(46.) Some additional light may be thrown on this whole theory, of the affections of a curve in space depending on the third power of the arc, and even on those affections which depend on higher powers of s , by that conception of an auxiliary spherical curve, which was employed in 379, (6) and (7.), to supply constructions (or geometrical representations) for the directions, not only of the tangent (ρ') to the given curve, to which indeed the unit-vector (τ) of the new curve is parallel, but also of the absolute normal, the binormal, and the osculating plane; while the same auxiliary curve served also, in 389, (2.), to furnish a measure of the curvature of the original curve, which is in fact the velocity* of motion in the new or spherical curve, if that in the old or given one be supposed to be constant, and be taken for unity.

* Accordingly the vector of velocity τ' , of this conceived motion in the auxiliary curve, is precisely what we have called (389, (4.), comp. 396, VI.) the vector of curvature of the proposed curve in space: and its tensor ($\text{T}\tau'$) is equal to the reciprocal of the radius (r) of that curvature.

(47.) We might for instance have observed, that while the *normal plane* to the curve in space is represented (in direction) by the *tangent plane* to the sphere, the *rectifying plane* (as being perpendicular to the absolute normal) is represented similarly by the *normal plane* to the spherical curve: and it is not difficult to prove that the *rectifying line* has the direction of that *new radius* of the sphere, which is drawn to the point (say L) where the *normal arc* to the auxiliary curve touches its own envelope.

(48.) The point L thus determined is the *common spherical centre* (comp. 394, (5.)) of curvature, of the auxiliary curve itself, and of that reciprocal* curve on the same sphere, of which the radii have the directions (comp. 379, (7.)) of the binormals to the original curve; the *trigonometric tangent* of the *arcual radius* of curvature of the auxiliary curve is therefore ultimately equal to a *small arc* of that curve, divided by the corresponding arc of the reciprocal curve (or rather by the latter arc with its direction reversed, if the point L fall between the two curves upon the sphere); and therefore to the *first curvature* (r^{-1}) of the given curve, divided by the *second curvature* (r): and thus we have not only a simple *geometrical interpretation* of the quaternion equation XI', but also a *geometrical proof* (which may be said to require no calculation), of the important but known relation XVII., which connects the ratio ($r : r$) of the two curvatures, with the angle (H) between the tangent (τ) and the rectifying line (λ), for any curve in space.

(49.) In whatever manner this known relation ($\tan H = r : r$) has once been established, it is geometrically evident, that if the ratio of the two curvatures be constant, then, because the curve crosses the generating lines of its own rectifying developable (396) under a constant angle (H), that developable surface must be cylindrical: or in other words, the proposed curve of double curvature must, in the case supposed, be a geodesic† on a cylinder (comp. 380, (4.)). Accordingly the point L, in the two last sub-articles, becomes then a fixed point upon the sphere, and is the common pole of two complementary small circles, to which the auxiliary spherical curve (46.), and the reciprocal curve (48.), in the case here considered, reduce themselves; so that the tangent and the binormal to the curve in space make (in the

* The reciprocity here spoken of, between these two spherical curves, is of that known kind, in which each point of one is a pole of the great-circle tangent, at the corresponding point of the other: and accordingly, with our recent symbols, we have not only $v = \nabla \tau r'$, but also, $\nabla v v' = r^{-2} \nabla v' v^{-1} = r^{-2} r^{-1} r \parallel \tau$.

† The writer has not happened to meet with the geometrical proof of this known theorem, which is attributed to M. Bertrand by M. Liouville, in page 558 of the already cited Additions to Monge; but the deduction of it as above, from the fundamental property (396) of the rectifying line, is sufficiently obvious, and appears to have suggested the method employed by M. de Saint-Venant, in the part (p. 26) of his *Memoir sur les lignes courbes non planes*, &c., before referred to, in which the result is enunciated. Another, and perhaps even a simpler method, suggested by quaternions, of geometrically establishing the same theorem, will be sketched in the present sub-article (49.); and in the following sub-article (50.), a proof by the quaternion analysis will be given, which seems to leave nothing to be desired on the side of simplicity of calculation.

same case) *constant angles*, with the *fixed radius* drawn to that point : and the *curve itself* is therefore (as before) a *geodetic line*, on *some cylindrical surface*.

(50.) By quaternions, when the *two curvatures* have thus a *constant ratio*, the equations XI'. and XVI. give,

$$\text{CVIII.} \dots (r\lambda)' = (U\nu + r r^{-1} r)' = (r r^{-1})' r = 0,$$

or

$$\text{CIX.} \dots r\lambda = \text{a constant vector};$$

the *tangent* (r) makes therefore, in this case, a *constant angle* (H) with a *constant line* ($r\lambda$): and the *curve* is thus seen again, by this very simple *analysis*, to be a *geodetic on a cylinder*. And because it is easy to prove (comp. XXXI.), that we have in the same case the expression,

$$\text{CX.} \dots r \sin^2 H = \text{radius of curvature of base,}$$

or of the *section* of the cylinder made by a plane perpendicular to the generating lines, this *other known theorem* results, with which we shall conclude the present series of sub-articles : *When both the curvatures are constant, the curve is a geodetic on a right circular cylinder (or cylinder of revolution);* or it is what has been called above, for simplicity and by eminence, a *helix*.*

398. When the *fourth power* (s^4) of the *arc* is taken into account, the expansion of the vector ρ , involves *another term*, and takes the form (comp. 397, I.),

$$\text{I.} \dots \rho_s = \rho + s\tau + \frac{1}{2}s^2\tau' + \frac{1}{6}s^3\tau'' + \frac{1}{24}s^4\tau''',$$

in which

$$\text{II.} \dots \tau''' = D_s^4 \rho, \quad \text{and} \quad \text{III.} \dots S\tau\tau''' = -3S\tau'\tau'' = -3r^3r';$$

so that the *new affections* of the curve, thus introduced, depend only on *two new scalars*, such as r' and r'' , or r' and R' , or H' and P' , &c. We must be content to offer here a very few remarks on the theory of *such affections*, and on the manner in which it may be extended by the introduction of derivatives of *higher orders*.

* In general, the expression XLIV. for the vector ω , of the *osculating helix*, in which $\iota = -r^{-1}\lambda^{-1} = r - \lambda^{-1}r'$, and $\rho - \omega_0 = \lambda^{-2}r'$, gives $T\omega'_s = 1$; so that the *derivation* (8.) may be considered (comp. (18.)) to be measured from the extremity of an *arc of the helix*, which is *equal in length* to the arc s of the *curve*, and is set off from the same initial point r , with the same initial direction: while ω_0 does not *here* denote the value of ω , answering to $s = 0$, but has a special signification assigned by the formula XXXVIII. It may also be noted that the *conception*, referred to in (46.), of an *auxiliary spherical curve*, corresponds to the ideal substitution of the *motion of a point* with a *varying velocity* upon a *sphere*, for a motion with an *uniform velocity* in *space*, in the investigation of the *general properties of curves of double curvature*: and that thus it is intimately connected (comp. 379, (9)) with the general theory of *hodographs*.

(1.) The new vector τ''' , on which everything here depends, is easily reduced to the following forms,* analogous to the expressions 397, VI. for τ'' :

$$\text{IV.} \dots \tau''' = \frac{r(r''')'}{r} + \frac{(r''')' - r''^2}{r'} - \frac{(r''r')'}{\nu} \\ = 3r''r' + (r(r''')' + \lambda^2)r' + (r''r')'\nu^2.$$

(2.) The first derivatives of the four vectors, ν' , κ' , λ' , σ' , taken in like manner with respect to the arc s of the curve, are the following:

$$\text{V.} \dots \nu'' = (V\tau\tau'')' = V\tau\tau''' + r^2\lambda \\ = r''r''r + (r''r')'\tau^{-1} + (r(r''')' - r^2)\nu; \\ \text{VI.} \dots \kappa'' = -r^{-1}r'\tau + (rr'' - r^2r''^2)r' + (r''r')'\nu; \\ \text{VII.} \dots \lambda' = (r^{-1})'\tau + (r^{-1})'\tau\nu, \text{ or VII'.} \dots (r\lambda)' = (r\tau^{-1})'\tau \text{ (comp. 397, CVIII.);} \\ \text{VIII.} \dots \sigma' = (\kappa + pr\nu)' = (p' + r\tau^{-1})r\nu = RRp^{-1}r\nu;$$

in which last the scalar derivatives p' and R' are determined, in terms of r'' and r' , by the equations,

$$\text{IX.} \dots p' = (r'r)' = r''r + r'r',$$

and $\text{X.} \dots R' = R^{-1}(pp' + rr') = p' \sin P + r' \cos P = (p' + \cot H) \sin P.$

We have also the derivatives,

$$\text{XI.} \dots H' = \frac{rr' - r'r'}{r^2 + r'^2} = \frac{r^{-1}r' - r^{-1}r'}{r\lambda^2},$$

$$\text{XII.} \dots P' = \frac{rp' - r'p}{r^2 + p^2} = \frac{(rr'' - r'^2)r + rr'r'}{R^2};$$

and the relations,

$$\text{XIII.} \dots Srr'\tau''' = S\nu\tau''' = -(r''r^{-1})';$$

$$\text{XIV.} \dots Srr''\tau''' = S\nu\tau''' = -r''r^{-2}(p' - rr\lambda^2);$$

$$\text{XV.} \dots Sr'r''\tau''' = r''S\lambda r''' = -r''(r\tau^{-1})';$$

which may be proved in various ways, and by the two first (or the two last) of which, the derivatives r' and p' , and therefore also H' and P' , can be *separately* calculated, as *scalar functions* of the *four vectors* τ , r' , r'' , r''' , or of some three of them, including the *new vector* τ''' .

(3.) We may also deduce, from either V. or VIII., the following *vector expressions*, of which the *geometrical signification* is evident from the recent theory (396, 397) of *emanant lines and planes*:

XVI. . . *Vector of Rotation of Radius (R) of Spherical Curvature*
= *Vector of Rotation of Tangent Plane to Osculating Sphere*

$$= (\text{say}) \phi = V \frac{\nu''}{\nu'} = V \frac{\sigma' - \tau}{\sigma - \rho} = R^{-2}r(\nu^{-1}\sigma' + \sigma - \rho) \quad (1, 2, 3)$$

$$= \frac{r}{R} \left(\frac{rR'}{p} + \frac{\sigma - \rho}{R} \right) = \frac{r'r}{R^2} \left(p' + \frac{r}{r} + rr' + p\nu \right) = R^{-2}r(r\lambda + p'r - pr') \quad (4, 5, 6);$$

whence follows this *tensor value* for the *common angular velocity* of these two connected *rotations*, compared still with the velocity of *motion* along the *curve*,

* In these new expressions, on the plan of the second Note to page 561, the scalars r' , p' , R' , and the vector σ' , are to be regarded as of the dimension *zero*; r'' , H' , P' , and κ'' of the dimension -1 ; λ' of the dimension -2 ; and ν'' and τ''' , as being each of the dimension -3 .

XVII. . . *Velocity of Rotation of Radius (R), or of Tangent Plane to Sphere,*

$$= T\phi = TV \frac{v'}{v} = R^{-1} \sqrt{(1 + R^2 \cot^2 P)} = R^{-1} \sqrt{\{1 + (p' + \cot H)^2 \cos^2 P\}};$$

with the verifications, for the case of the *helix*, for which $p = 0$, $p' = 0$, $P = 0$, and $R = r$, that these expressions XVI. and XVII. become,

$$\text{XVI}' \dots \phi = \lambda, \text{ and } \text{XVII}' \dots T\phi = T\lambda = r^{-1} \operatorname{cosec} H,$$

which agree with those found before, for the vector and velocity of rotation of the radius (r) of *absolute curvature*.

(4.) As another verification, we have $R' = 0$ for every *spherical curve*, and the general expressions take then the forms,

$$\text{XVI}'' \dots \phi = \frac{-r}{\sigma - \rho}, \text{ and } \text{XVII}'' \dots T\phi = R^{-1},$$

of which the interpretation is easy.

(5.) In general, the formula XVII. may also be thus written,

$$\text{XVIII.} \dots R^2 \phi^2 + 1 = -R^2 \cot^2 P = R^2 - p^2 R^2 R^2 = R^2 + \sigma^2 = \sigma^2 \cos^2 P;$$

or thus, XIX. . . $RT\phi = \sqrt{(1 + T\sigma^2 \cos^2 P)} = \sqrt{(1 + T\sigma^2 - R^2)}$;

or finally, XX. . . $R^2 T\phi = \sqrt{(R^2 - r^2 \sigma^2)} = \sqrt{(R^2 + r^2 T\sigma^2)}$;

so that the *small angle*, $sT\phi$, between the two near radii of *spherical curvature*, R and R_n , is ultimately equal to the *square root of the sum of the squares of the two small angles*, in two rectangular planes, sR^{-1} and $rsR^{-1}T\sigma'$, or sRP , and sP_n , which are subtended, respectively, at the centre s of the *osculating sphere* by the *small arc* s of the given curve, and at the given point P by the *small corresponding arc* $sT\sigma'$ of the locus of centres s of *spherical curvature*, or of the *cusp-edge* (895, (2.)) of the *polar developable*; exactly* as the *small angle* $sT\lambda$, between two near radii (897, (5.)) of *absolute curvature*, r and r_n , is ultimately the square root of the sum of the squares of the two other small angles, sr^{-1} and sr^{-1} , or PKP , and PK_n , which are likewise situated in two rectangular planes, and are subtended at the centre κ of the *osculating circle* by the small arc s of the curve, and at the given point P by the corresponding arc $sT\kappa'$ of the locus of the centre κ (comp. 397, XXXIV., XCIV.).

(6.) The point, say v , in which the radius R of the *osculating sphere* at P approaches most nearly to the near radius R_n from P_n , is ultimately determined (comp. 897, CV. and X.) by the formula,

$$\text{XXI.} \dots Pv = \zeta = \frac{\text{Vector of Spherical Curvature}}{\text{Square of Angular Velocity of Radius (R)}}$$

$$= (\rho - \sigma)^{-1} T\phi^{-2} = \frac{\sigma - \rho}{1 + R^2 \cot^2 P} = \frac{\sigma - \rho}{1 + p^2 r^2 R^2};$$

the vector of this point v (in its ultimate position) is therefore

$$\text{XXII.} \dots ov = \rho + \zeta = \frac{r^2 R^2 \rho + p^2 \sigma}{r^2 R^2 + p^2} = \frac{r^2 R^2 \rho + r^2 r^2 \sigma}{r^2 R^2 + r^2 r^2};$$

with the verification, that (by X., comp. XVII.) the scalar $p^{-1} r R$ or $R' \cot P$ re-

* It will soon be seen that these two results, and others connected with them, depend geometrically on one common principle, which extends to all systems of normal emanants (897, (44.)).

duces itself to $\cot H$, or to r^{-1} , for the case $p = 0, p' = 0, P = 0$ (comp. (8.)): and that thus the expression 897, XXXVIII., for the vector OH of the point of nearest approach, of a radius (r) of absolute curvature to a consecutive* radius of the same kind, is reproduced.

(7.) In general, if we introduce a new auxiliary angle, J , determined by the formula,

$$\text{XXIII.} \dots \cot J = p^{-1}rR' = R' \cot P = (p' + \cot H) \cos P = R(r^{-1} + P'),$$

the expression XXII. takes the simplified form (comp. again 897, XXXVIII.),

$$\text{XXIV.} \dots \overline{ov} = \rho + \zeta = \rho \cos^2 J + \sigma \sin^2 J;$$

and the segments, into which the point v divides (internally) the radius R of the sphere, have the values (comp. 897, XXXIX.),

$$\text{XXV.} \dots \overline{pv} = R \sin^2 J, \quad \overline{vs} = R \cos^2 J.$$

(8.) A geometrical signification may be assigned for this new angle J , which is analogous to the known signification of the angle H (897, XVII.). In fact, the tangent plane to the osculating sphere at P touches its own developable envelope along a new right line, of which the scalar equations are,

$$\text{XXVI.} \dots S(\sigma - \rho)(\omega - \rho) = 0, \quad S(\sigma' - \tau)(\omega - \rho) = 0;$$

and because the developable locus of all such lines can be shown to be circumscribed, along the given curve, to the locus of the osculating circle, which is at the same time the envelope of the osculating sphere, we shall briefly call this locus of the line XXVI. the Circumscribed Developable. And the inclination of the generatrix of this new developable surface, to the tangent to the given curve at P , if suitably measured in the tangent plane to the sphere, is precisely the angle which has been above denoted by J .

(9.) To render this conception more completely clear, let us suppose that a finite right line PJ is set off from the given point P , on the indefinite line XXVI., so as to represent, by its length and direction, the velocity of the rotation of the tangent plane to the osculating sphere; and so to be, in the phraseology (896, (14.)) of the general theory of emanants, the vector-axis of that rotation. We shall then have the values,

$$\text{XXVII.} \dots PJ = \phi (= \text{the six expressions XVI.})$$

$$= R^{-1}\tau(\cot J + U(\sigma - \rho)) = R^{-1} \operatorname{cosec} J (\tau \cos J + \tau U(\sigma - \rho) \sin J) \quad (7, 8);$$

the angle J being determined by the formula XXIII., and a new expression, $T\phi = R^{-1} \operatorname{cosec} J$, being thus obtained for the velocity XVII.

(10.) Hence the new angle J , if conceived to be included (like H) between the limits 0 and π , may be considered to be measured from τ to ϕ , or from the unit-tangent to the curve at P , to the generating line PJ of the circumscribed developable (8.), in the direction from τ to $\tau(\sigma - \rho)$: which last tangent to the osculating sphere

* This usual expression, consecutive, is obviously borrowed here from the language of infinitesimals, but is supposed to be interpreted, like those used in other parts of the present series of Articles, by a reference to the conception of limits.

makes generally, like the tangent ϕ or PJ itself, an *acute* angle with the positive binormal ν , as appears from the common *sign* of the scalar *coefficients* of that vector, in their developed expressions.

(11.) It may also be remarked, as an additional point of analogy, and as serving to verify some formulæ, that while the older angle H becomes *right*, when the given curve is *plane*, so the new angle $J = \frac{\pi}{2}$, for every *spherical curve*.

(12.) As another geometrical illustration of the properties of the angle J , and of some other results of recent sub-articles, which may serve to connect them, still more closely, with the general theory of *normal emanants* from curves (897, (44.)), let us conceive that AB, BC, CD are three successive right lines, perpendicular each to each; let us denote by a and b the angles BCA and CBD , and by c the inclination of the line AD to BC : and let us suppose that these two lines are intersected by their common perpendicular in the points G and H respectively.

(13.) Then, by completing the rectangle $BCDE$, and letting fall the perpendicular BF on the hypotenuse of the right-angled triangle ABE , we obtain the projections, AE and FB , of the two lines AD and GH , on the plane through B perpendicular to BC ; and hence, by elementary reasonings, we can infer the relations:

$$\text{XXVIII.} \dots \tan^2 c = \tan^2 ADE = \tan^2 a + \tan^2 b;$$

$$\text{and} \quad \text{XXIX.} \dots \frac{BH}{BC} = \frac{AG}{AD} = \frac{AF}{AE} = \frac{AB^2}{AE^2} = \sin^2 AEB,$$

$$\text{or} \quad \text{XXIX'.} \dots BH = BC \sin^2 j, \quad \text{if} \quad \tan j = \tan a \cot b;$$

nothing here being supposed to be *small*. It may also be observed, that the two *rectilinear angles*, BCA and CBD , or a and b , represent respectively the inclinations of the *plane* ACD to the *plane* BCD , and of the *plane* ABD to the *plane* ABC .

(14.) Conceive next that PQ and P_1Q_1 are two near normal emanants, touching the polar developable in the points Q and Q_1 , whereof Q is thus on the given polar axis Ks , and Q_1 is on the near polar axis K_1Q_1 ; and let the second emanant be cut, in the points P' and Q' , by planes through P and Q , perpendicular to the first emanant PQ . The line PP' will then be very nearly tangential to the given curve at P ; and the line QQ' will be very nearly situated in the corresponding normal plane to that curve: so that these two new lines will be very nearly perpendicular to each other, and the gauche quadrilateral $P'PQQ'$ will ultimately have the properties of the recently considered quadrilateral $ABCD$.

(15.) This being perceived, if we denote by e the length of the emanant line PQ , the small angle a is very nearly $= e^{-1}e$; and if the small angle b be put under the form $b'j$, then the new coefficient b' is ultimately equal (by XXIX') to $e^{-1} \cot j$: where j is an auxiliary angle, not generally small, and is such that we have ultimately $P'H = PQ \cdot \sin^2 j$, if H be the point in which the given normal emanant PQ approaches most closely to the consecutive emanant P_1Q_1 .

$$\begin{aligned} \text{(16.) We have then the ultimate equation,} \\ \text{XXX.} \dots \cot j = eb' = PQ \times \text{lim.} (e^{-1} \cdot PQ_1) \\ = \text{length of emanant line } (PQ) \\ \times \text{angular velocity of the tangential plane } (P'PQ) \text{ containing it;} \end{aligned}$$

this latter plane being here conceived as *turning*, for a moment, round the tangent to the given curve at P , and the velocity of *motion* along that curve being still taken for unity.

(17.) Accordingly, when we change ϵ to r , b' to r^{-1} , and j to H , we recover in this way the fundamental value $\cot H = rr^{-1}$ (397, XVII.), for the cotangent of the *older* angle H ; and when, on the other hand, we treat the radius of *spherical* curvature as the normal emanant, supposing q to coincide with s , and therefore changing ϵ to R , and b' to $r^{-1} + P'$, we recover the last of the expressions XXIII. for the cotangent of the *new* but analogous angle J , namely $\cot J = R(r^{-1} + P')$, together with an *interpretation*, which may not have at first seemed obvious: although that expression *itself* was deducible, in the following among other ways, from equations previously established,

$$\text{XXXI.} \dots R^{-1} \cot J - r^{-1} = \frac{rR'}{pR} - \frac{r'}{p} = -\frac{R}{p} \left(\frac{r}{R} \right)' = -\frac{(\cos P)'}{\sin P} = P'.$$

(18.) As regards the *angular velocity*, say v , of the emanant *line* PQ , or the ultimate quotient of the *angle* between two such near lines, divided by the small *arc* s of the given curve, we see by XXVIII. (comp. (5.)) that this *small angle* vs is ultimately equal to the *square root of the sum of the squares* of the two other *small angles*, above denoted by a and b , and found to be equal, nearly, to ϵ^{-1} s and ϵ^{-1} s $\cot j$ respectively: we may then establish the general formula,

$$\text{XXXII.} \dots \text{Angular Velocity of Normal Emanant} = v = \epsilon^{-1} \text{cosec } j;$$

which reproduces the values, $r^{-1} \text{cosec } H$, and $R^{-1} \text{cosec } J$, already found for the angular velocities of the two radii, r and R .

(19.) And if we observe that the *projection of the vector of curvature*, κP^{-1} , on the emanant PQ , is easily proved to be $= QP^{-1} = \epsilon^{-2} \cdot PQ$, we see by XXXII. that if this *projection be divided by the square of the angular velocity* (v) of the line PQ , the *quotient* is the line $PQ \cdot \sin^2 j$, or PH (15.): which reproduces the general result, 897, CV., for all *systems of normal emanants*, together with a geometrical *interpretation*.

(20.) As still *another geometrical illustration* of the properties of the new angle J , we may observe that in the construction (12.) and (13.) the corresponding auxiliary angle j was equal to $\angle AEB$, or to $\angle ABF$, and that the line BF ($= HG$) was *perpendicular to both* BC and AD , although not *intersecting* the latter. Substituting then, as in (14.), the quadrilateral $P'PQQ'$ for $ABCD$, and passing to the limit, we may say that if a new line PJ be a *common perpendicular*, at the given point p , to two consecutive* *normal emanants*, PQ and $P'Q'$, the *general auxiliary angle* j is simply the *inclination* $P'PJ$, of that *common perpendicular* PJ , to the *tangent* PP' to the curve.

(21.) And if, instead of *normally emanating lines* PQ , we consider a system of *tangential emanant planes* (as in 897, (45.)), to which those lines are *perpendicular*, we may then (comp. 896, (14.)) consider the recent line PJ as being a *generating line* of the *developable surface*, which is the *envelope* of all the *planes* of the system; the *auxiliary angle*, $\dagger j$, is therefore *generally* by (20.) the *inclination* of this *gene-*

* Compare the Note to page 581.

† In these *geometrical illustrations*, the angle j has been treated, for simplicity, as being both *positive* and *acute*; although the *general formulæ*, which involve the corresponding angles H and J , permit and require that we should occasionally attribute to them *obtuse* (but still *positive*) values: while those angles may also become *right*, in some particular cases (comp. (11.)).

matrix to the *tangent*: a result which agrees with, and includes, the known and fundamental property (397, XVII.) of the angle H , in connexion with the *Rectifying Developable* (396); and also the *analogous* property of the *newer* angle J , connected (8.) with what it has been above proposed to call the *Circumscribed Developable*.

(22.) We shall soon return briefly on the theory of that *new developable surface* (8.), and of the *new locus* (of the *osculating circle*, or *envelope* of the *osculating sphere*) to which it has been said to be *circumscribed*: but may here observe, that if we write for abridgment (comp. VIII. and XXIII.),

$$\text{XXXIII.} \dots \alpha = \frac{\sigma'}{r\nu} = \frac{RK}{p} = p' + \cot H = \cot J \sec P,$$

then what has been called the *coefficient of non-sphericity* (comp. 395, (14.) and (16.)) is easily seen to have by XIV. the values,

$$\text{XXXIV.} \dots S-1 = \frac{8r^2r''r''''}{8rr'^2r'''} - 1 = -r^2rS\nu'r'''' - 1 \quad (1, 2)$$

$$= \frac{r}{r'}(p' - r\lambda^2) - 1 = \frac{r}{r'}\left(p' + \frac{r}{r'}\right) = nr\tau^{-1} \quad (3, 4, 5)$$

$$= \frac{\sigma'}{r\nu} = \cot H \cot J \sec P = \frac{rRR'}{pr} \quad (6, 7, 8);$$

whence also the *deviation* of a near point r_s of the curve, from the *osculating sphere* at r , is ultimately (by 395, XXVII.)

$$\text{XXXV.} \dots \overline{sr}_s - \overline{sr} = \frac{(S-1)s^4}{24r^2R} = \frac{ns^4}{24r\tau R} = \frac{R's^4}{24r\tau p};$$

and accordingly, the square of the vector $\rho_s - \sigma$ is given now (comp. I.) by the expression,

$$(\rho_s - \sigma)^2 = (\rho - \sigma)^2 - \frac{s^4}{12r^2} \{r^2S(\sigma - \rho)r'''' - 1\},$$

in which $r^2S(\sigma - \rho)r'''' = S = 1 + nr\tau^{-1} = \&c.$, as above.

(23.) The same auxiliary scalar α enters into the following expressions for the *arc*, and for the *scalar radii* of the *first* and *second curvatures*, of the *locus of the centre* α of the *osculating sphere*, or of the *cuspid-edge* of the *polar developable* (comp. 391, (6.), and 395, (2.)):

$$\text{XXXVI.} \dots \pm \int \alpha ds = \text{Arc of that Cusp-Edge (or of locus of } \alpha \text{)};$$

$$\text{XXXVI.} \dots r_1 = nr = r + p'r = \frac{RR'}{r'} = (\text{Scalar}) \text{ Radius of Curvature of same edge};$$

$$\text{XXXVI.} \dots r_2 = nr = \sigma'\nu^{-1} = (\text{Scalar}) \text{ Radius of Second Curvature of same curve};$$

these two latter being here called *scalar radii*, because the *first* as well as the *second* (comp. 397, V.) is conceived to have an algebraic *sign*. In fact, if we denote by κ_1 the *centre of the osculating circle* to the *cuspid-edge* in question, its *vector* is (by the general formula 389, IV.),

$$\text{XXXVII.} \dots \text{OK}_1 = \kappa_1 = \sigma + \frac{\sigma^3}{V\sigma''\sigma'} = \sigma - nr\tau r' = \rho - p'\tau r' + pr\nu = \sigma - r_1 r r',$$

with the signification XXXVI. of r_1 ; because by XXXIII. (comp. 397, XI.),

$$\text{XXXVIII.} \dots \sigma' = nr\nu, \quad \sigma'' = n'r\nu + \alpha(r\nu)' = n'r\nu - nr\tau^{-1}r',$$

and therefore

$$\text{XXXIX.} \dots \sigma^2 = -n^2, \quad V\sigma'\sigma'' = n^2r^{-1}r'.$$

We may also observe that the relation $\sigma' \parallel \nu$ gives (by 397, IV.),

$$\text{XL.} \dots V \frac{\sigma''}{\sigma'} = V \frac{\nu'}{\nu} = r^{-1} r = \text{Vector of Second Curvature of given curve};$$

and that we have the equation,

$$\text{XLI.} \dots \frac{\kappa_1 s}{PK} = \frac{\sigma - \kappa_1}{\kappa - \rho} = \frac{r_1}{r}, \text{ with } r > 0, \text{ but } r_1 > \text{ or } < 0,$$

according as the *cuspid-edge* turns its *concavity* or its *convexity* towards the *given curve* at P.

(24.) The *radius of (first) curvature* of that *cuspid-edge*, when regarded as a *positive* quantity, is therefore represented by the *tensor*,

$$\text{XLII.} \dots \sqrt{r_1^2} = \pm r_1 = Tr_1 = RT \frac{R'}{r} = \pm \frac{RdR}{dr} (> 0);$$

and as regards the *scalar radius XXXVI''* of *second curvature* of the same *cuspid-edge*, its expression follows by XXXVIII. from the general formula 397, XXVII., which gives here,

$$\text{XLIII.} \dots r_1^{-1} = S \frac{\sigma'''}{V\sigma'\sigma'} = \frac{1}{nr} S \frac{\nu''}{V\nu\nu'} = n^{-1} r^{-1}, \text{ because XLIII'.} \dots S \frac{\nu''}{V\nu\nu'} = 1;$$

the *two scalar derivatives*, n' and n'' , which would have introduced the derived vectors r'' and r' , or $D_s^2 p$ and $D_s p$, of the *fifth* and *sixth orders*, thus *disappearing from the expressions of the two curvatures* of the *locus of the centre s* of the *osculating sphere*, as was to be expected from geometrical* considerations.

(25.) For the *helix*, the formula XXXVII. gives $\kappa_1 = \rho$, or $\kappa_1 = P$; we have then thus, as a verification, the known result, that the *given point P* of *this curve* is *itself the centre of curvature* κ_1 of that *other helix* (comp. 389, (3.), and 395, (8.)), which is in this case the *common locus* of the *two coincident centres*, κ and s . It is scarcely necessary to observe that for the *helix* we have also $J = H$.

(26.) In general, the *rectifying plane* of the *locus of s* is *parallel* to the *rectifying plane* of the *given curve*, because the *radii* of their *osculating circles* are *parallel*; the *rectifying lines* for these two curves are therefore not only *parallel* but *equal*; and accordingly we have here the formula,

$$\text{XLIV.} \dots \lambda_1 = V \frac{r_1''}{r_1} = V \frac{r''}{r} = \lambda \text{ (by 397, XVI.)},$$

which will be found to agree with this other expression (comp. 397, XVII.),

$$\text{XLV.} \dots \tan H_1 = \frac{r_1}{Tr_1} = \frac{r}{r} Ur_1 = \pm \cot H,$$

the upper or lower sign being taken, according as the *new curve* is *concave* (as in Figs. 81, 82), or is *convex* at s (comp. (23.)), towards the *old* (or *given*) curve at P: and the *new angle* H_1 being measured in the *new rectifying plane*, from the *new*

* In fact, n represents here the velocity of motion of the point s along its own locus, while r^{-1} and r^{-1} represent respectively the velocities of rotation of the tangent and binormal to that curve: so that nr and nr must be, as above, the radii of its two curvatures.

tangent σ' or $\sigma r v$, to the new rectifying line λ_1 , and in the direction from that new tangent to the new binormal v_1 , or (comp. XL.) to a line from s which is equal to the vector of second curvature $r^{-1}r$ of the given curve, multiplied by a positive scalar, namely by $T\pi^{-1}$, or by the coefficient π^{-1} taken positively.

(27.) The former rectifying line λ touches the cusp-edge of the rectifying developable (396) of the given curve, in a new point κ (comp. Fig. 81), of which by 397, (45.), and by XV., the vector from the given point is, generally,

$$\text{XLVI.} \dots PR = -\frac{Vr^{-2}r''}{Sr'r''r'''} = \frac{r^{-2}\lambda}{S\lambda r''''} = -\frac{r\lambda}{(rT^{-1})'} = \frac{U\lambda \sin H}{H'}$$

with the verification that this expression becomes infinite (comp. 397, (49.), (50.)), when the curve is a geodetic on a cylinder.

(28.) In general, the vector OR of the point of contact R , which vector we shall here denote by v , may be thus expressed,

$$\text{XLVII.} \dots v = OR = \rho + lU\lambda, \quad \text{if} \quad \text{XLVIII.} \dots l = \frac{\sin H}{H'} = \frac{-rT\lambda}{(rT^{-1})'}$$

and because $(r\lambda)' = (rT^{-1})'r$, by VII., its first derivative is,

$$\text{XLIX.} \dots v' = r\lambda \left(\frac{v - \rho}{r\lambda} \right)' = U\lambda \operatorname{cosec} H (l \sin H)' = U\lambda (l' + \cos H);$$

in which however the new derived scalar l' involves H'' , and so depends on r'' : while the scalar coefficient l itself represents the portion $(\pm \overline{FR})$ of the rectifying line, intercepted between the given curve, and the cusp-edge (27.) of the rectifying developable, and considered as positive when the direction of this intercept FR coincides with that of the line $+\lambda$, but as negative in the contrary case.

(29.) For abridgment of discourse, the cusp-edge last considered, namely that of the rectifying developable, as being the locus of a point which we have denoted by the letter R , may be called simply "the curve (R);" while the former cusp-edge (23.), or that of the polar developable, may be called in like manner "the curve (s);" the locus of the centre κ of (absolute) curvature may be called "the curve (κ):" and the given curve itself (comp. again Figs. 81, 82) may be called, on the same plan, "the curve (P)."

(30.) The arc RR_1 , of the curve (R), is (by XLIX., comp. XXXVI.),

$$\text{L.} \dots \pm \int_0^s T v' ds = l_2 - l + \int_0^s \cos H ds;$$

this arc being treated as positive, when the direction of motion along it coincides with that of $+\lambda$.

(31.) The expression VII. for λ' , combined with the former expression 397, XVI. for λ , gives easily by the general formula 389, IV.,

LI. . . Vector of Centre of Curvature of the Curve (R)

$$= v + \frac{v'}{Vv''v^{-1}} = v + \frac{v'}{V\lambda'\lambda^{-1}} = v + \frac{v'}{H'} U r';$$

whence LII. . . Radius of Curvature of Curve (R) = $T \frac{v'}{H'} = T \frac{dv}{dH}$,

the scalar variable being here arbitrary.

(32.) We see, at the same time, that the angular velocity of the rectifying line λ , or of the tangent to this curve (R), is represented by $\pm H'$; or that the small

angle* between two such near lines, λ and λ_s , is nearly equal to sH' , or to $H_s - H$: while the vector axis ($\nabla\lambda\lambda^{-1}$) of rotation of the rectifying line, set off from the point \mathfrak{R} , has $-H'U\tau$, or $-H'r\tau'$, for its expression.

(33.) As regards the second curvature of the same curve (\mathfrak{R}), we may observe that the expression (comp. VII. and LI.),

$$\text{LIII.} \dots \lambda'' = (r^{-1})''\tau + (r^{-1})'\tau\nu + r^{-1}(r\tau^{-1})'\tau' = (r^{-1})''\tau + (r^{-1})'\tau\nu + \nabla\lambda\lambda',$$

combined with the parallelism (XLIX.) of ν to λ , gives, by the general formula 397, XXVII.,

LIV. . . Radius of Second Curvature of Curve (\mathfrak{R})

$$= \left(S \frac{\nu''}{\nabla\nu'\nu''} \right)^{-1} = \frac{\nu'}{\lambda} \left(S \frac{\lambda''}{\nabla\lambda\lambda'} \right)^{-1} = \frac{\nu'}{\lambda} = \frac{l' + \cos H}{T\lambda};$$

with the verification, that while $l' + \cos H$ represents, by (30.), the velocity of motion along this curve (\mathfrak{R}), $T\lambda$ represents, by 397, (8.); the velocity of rotation of its osculating plane, namely the rectifying plane of the given curve (\mathfrak{P}): and it is worth observing, that although each of these two radii of curvature, LII. and LIV., depends on r'' through l' (28.), yet neither of them depends on r' (comp. (24.)). As another verification, it can be shown that the plane of the two lines λ and τ' from \mathfrak{P} , namely the plane,

$$\text{LIV'.} \dots S\tau'\lambda(\omega - \rho) = 0,$$

which is the normal plane to the rectifying developable along the rectifying line, and contains the absolute normal to the given curve (\mathfrak{P}), touches its own developable envelope along the line \mathfrak{RH} , if \mathfrak{H} be the point determined by the formula 397, XXXVIII., or the point of nearest approach of a radius of curvature (r) of that given curve to its consecutive (comp. (6.)); this line \mathfrak{RH} must therefore be the rectifying line of the curve (\mathfrak{R}): and accordingly (comp. 397, XVII.), the trigonometric tangent of its inclination to the tangent \mathfrak{RP} to this last curve has for expression (abstracting from sign),

$$\text{LIV''.} \dots \tan \mathfrak{PRH} = \overline{\mathfrak{PH}} = \pm l^{-1/2} \sin^2 H = \pm rH' \sin H = T\lambda^{-1}H'$$

$$= \frac{\text{Radius (LIV.) of Second Curvature of Curve } (\mathfrak{R})}{\text{Radius (LII.) of First Curvature of same Curve}}$$

(84.) Without even introducing r'' , we can assign as follows a twisted cubic (comp. 397, (84.)), which shall have contact of the fourth order with the given curve at \mathfrak{P} ; or rather an indefinite variety of such cubics, or gauche curves of the third degree. Writing, for abridgment,

$$\text{LV.} \dots x = -S\tau(\omega - \rho), \quad y = -Srr'(\omega - \rho), \quad z = -Srv(\omega - \rho),$$

so that

$$\text{LVI.} \dots \omega = \rho + xr + yrr' + zrv,$$

the scalar equation,

$$\text{LVII.} \dots \left(\frac{2ry}{r} \right)^2 = 6 \left(\frac{r}{r} \right)^3 xz + \left(\frac{r^2}{r^2} \right)' yz + ez^2,$$

* A result substantially equivalent to this is deduced, by an entirely different analysis, in the above cited Memoir of M. de Saint-Venant, and is illustrated by geometrical considerations: which also lead to expressions for the two curvatures (or, as he calls them, the *courbure* and *cambrure*), of the cusp-edge of the rectifying developable; and to a determination of the rectifying line of that cusp-edge.

in which ϵ is an arbitrary but scalar constant, represents evidently, by its *form*, a *cone of the second order*, with its *vertex* at the given point P ; and *this cone* can be proved to have *contact of the fourth order* with the *curve** at that point: or of the *third order* with the *cone of chords* from it (comp. 397, (81.), (82.)). In fact the coefficients will be found to have been so determined, that the difference of the two members of this equation LVII. contains s^5 as a factor, when we change ω to ρ , as given by the formula I., or when we substitute for xyz their approximate values for the curve, as functions of the arc s ; namely, by the expressions IV. for r'' , and 397, VI. for r'' ,

$$\text{LVIII.} \dots \begin{cases} x_s = s - \frac{s^3}{6r^2} + \frac{r's^4}{8r^3} \\ y_s = \frac{s^2}{2r} - \frac{r's^2}{6r^2} - \frac{s^4}{24} ((r^{-2}r') + r^{-3} + r^{-1}r^{-2}), \\ z_s = \frac{s^3}{6r} + \frac{r's^4}{24} (r^{-2}r^{-1}); \end{cases}$$

where the terms set down are more than sufficient for the purpose of the proof. It may be added that the coefficient of $\frac{-s^4}{24}$ in y_s , which is the only one at all complex here, may be transformed as follows:

$$\text{LVIII}' \dots Srr'r'' = -(r^{-1})'' - r^{-1}\lambda^2 = r^{-3}S + p(r^{-2}r^{-1});$$

S being that scalar for which (or more immediately for its excess over unity) several expressions† have lately been assigned (22.), and which had occurred in an earlier investigation (395, (14.), &c.).

(85.) With the same significations LV. of the three scalars xyz , this other equation,

$$\text{LIX.} \dots 18ry - (3x - r'y)^2 = (9 + r'^2 - 8rr'' - 8r^2r^{-2})y^2,$$

or

$$\text{LIX}' \dots 2ry - (x - \frac{1}{2}r'y)^2 = (1 - \frac{1}{2}r'^2 (r^{-2})'' - \frac{1}{2}r^2r^{-2})y^2,$$

will be found to be satisfied when we substitute for x and y the values LVIII. of x_s and y_s , and neglect or suppress s^5 ; it therefore represents an *elliptic* (or *hyperbolic*) *cylinder*, which is *cut perpendicularly*, by the *osculating plane* to the given curve at P , in an *ellipse* (or *hyperbola*), having *contact of the fourth order* with the *projection* (comp. 397, (9.)), of that given *curve* upon that *osculating plane*: and the *cylinder itself* has *contact of the same (fourth) order* with the *curve in space*, at the

* In the language of infinitesimals, the cone LVII. contains *five consecutive points* of the *curve*, or has *five-point contact* therewith: but it contains only *four consecutive sides* of the *cone of chords* from the given point, or has only *four-side contact* with *that cone*, except for one particular *value* of the *constant*, ϵ , which we shall presently assign. It may be observed that xyz form *here* a (scalar) system of *three rectangular co-ordinates*, of the usual kind, with their *origin* at the point P of the curve, and with their *positive semiaxes* in the directions of the *tangent* r , the *vector of curvature* r' , and the *binormal* v .

† It might have been observed, in addition to the eight forms XXXIV., that we have also,

$$\text{XXXIV}' \dots S - 1 = Rr^{-1} \cot J = s \cot H \quad (9, 10).$$

same given point \mathbf{r} , so that we may call it (comp. 897, (81.)) the *Osculating Elliptic (or Hyperbolic) Cylinder, perpendicular to the osculating plane*.

(36.) As a verification, if we suppress the second member of either LIX. or LX'., we obtain, under a new form, the equation of what has been already called the *Osculating Parabolic Cylinder* (897, LXXXIV.); and as another verification, the coefficient of y^3 in that second member *vanishes*, as it ought to do, when the given curve is supposed to be a *parabola*: that plane curve, in fact, satisfying the differential equation of the second order,

$$\text{LX.} \dots 3rr'' - r'^2 = 9, \quad \text{or} \quad \text{LX'}. \dots r^{\frac{4}{3}} (r^{\frac{2}{3}})' = 2,$$

$$\text{or} \quad \text{LX''}. \dots r^{\frac{4}{3}} \left(\left(\frac{dr}{3ds} \right)^2 + 1 \right) = \text{const.} = p^3,$$

if r be still the *radius of curvature*, considered as a function of the arc, s , while p is here the *semiparameter*.

(37.) The *binormal* ν is, by the construction, a *generating line* of the cylinder LIX.; and although this line is *not generally a side* of the cone LVII., yet we can *make* it such, by assigning the particular value *zero* to the arbitrary constant, ϵ , in its equation, or by suppressing the term, ϵz^3 . And when this is done, the cone LVII. will *intersect* the cylinder LX., not only in this *common side* ν (comp. 897, (88.)), but also in a certain *twisted cubic*, which will have *contact* of the *fourth order* with the *given curve* at \mathbf{r} , as stated at the commencement of (34.).

(38.) But, as was also stated there, *infinitely many* such cubics can be described, which shall have contact of the *same (fourth) order*, with the *same curve*, at the *same point*. For we may assume any point \mathbf{z} of space, or any vector (comp. LVI.),

$$\text{LXI.} \dots \mathbf{oz} = \epsilon = \rho + a\mathbf{r} + b\mathbf{r}' + c\mathbf{r}\nu,$$

in which a, b, c are any three scalar constants; and then the vector equation,

$$\text{LXII.} \dots \omega = \rho_s + t(\epsilon - \rho),$$

in which t is a new scalar variable, will represent a *cylindric surface*, not generally of the second order, but passing through the given curve, and having the line \mathbf{PE} for a *generatrix*. We can then cut (generally) this new cylinder by the *osculating plane* to the curve at \mathbf{r} , and so obtain (generally) a new and *oblique projection* of the curve upon that plane; the x and y of which new projected curve will depend on the arc s of the original curve by the relations,

$$\text{LXIII.} \dots x = x_s - ac^{-1}z_s, \quad y = y_s - bc^{-1}z_s;$$

with the approximate expressions LVIII. for x_s, y_s, z_s . And if we then determine two new scalar constants, B and C , by the condition that the substitution of these last expressions LXIII. for x and y shall satisfy this new equation,

$$\text{LXIV.} \dots 2ry = x^2 + 2Bxy + Cy^2,$$

if only s^2 be neglected (comp. (85.)), or by equating the coefficients of s^3 and s^4 , in the result of such substitution, then, on restoring the significations LV. of xyz , and writing for abridgment,

$$\text{LXV.} \dots X = x - ac^{-1}z, \quad Y = y - bc^{-1}z,$$

the equation of the second degree,

$$\text{LXVI.} \dots 2rY = X^2 + 2BXY + CY^2,$$

will represent generally an *oblique osculating elliptic* (or hyperbolic) *cylinder*, which has contact of the *fourth order* with the given curve at P, and contains the assumed line PE. If then we determine finally the constant e in LVII., by the result of the substitution of abc for xyz , or by the condition,

$$\text{LXVII.} \dots \left(\frac{2rb}{r}\right)^2 = 6\left(\frac{r}{r}\right)^3 ac + \left(\frac{r^2}{r^2}\right) bc + ec^2,$$

the *cone* LVII., and the *cylinder* LXVI., will have that line PE for a *common side*; and will *intersect* each other, not only in *that line*, but *also* (as before) in a *twisted cubic*, although now a *new one*, which will have the *required (fourth) order of contact*, with the given curve at the given point.

(39.) If, after the substitution (38.) in LXIV., we equate the coefficients of the *three powers*, s^3 , s^4 , s^5 , and then eliminate B and C , we are conducted to an *equation of condition*, which is found to be of the *form*,

$$\text{LXVIII.} \dots ab^3 + bb^2c + cbc^2 + ec^3 = ac(bg + ch);$$

in which the ratios of abc still serve to determine the direction of the generating line PE, while the coefficients a, b, c, e, g, h are *assignable functions* of r, r', r'', r''', r'''' , and r'''' , depending on the vector r'''' : and when this *condition* LXVIII. is satisfied, the *cylinder* LXVI. has *contact of the fifth order* with the given curve at P.

(40.) Again, if we *improve* the approximate expressions LVIII. for the three scalars x, y, z , by taking account of s^5 , or by introducing the *new term* $\frac{s^5 r^{17}}{120}$ (comp. I.) of ρ_s , and if we substitute the expressions so improved, instead of x, y, z , in the equation of the *cone* LVII. and then equate to zero (comp. (34.)) the coefficient of s^5 in the difference of the two members of that equation, we obtain a *definite expression* for the *constant*, e , which had been *arbitrary* before, but becomes now a *given function* of $rrr'r''''$ and r'''' (not involving r''''), namely the following

$$\text{LXIX.} \dots e = \frac{r^4}{\delta} \left(\frac{9}{r^4} - \frac{21}{r^2 r^2} + \frac{r^2}{r^4} - \frac{3r''}{r^3} + \frac{3r'r'}{r^2 r} - \frac{27r''^2}{4r^2 r^2} + \frac{9r''}{r^2 r} \right);$$

and when the constant e receives *this value*,* the *cone* has contact of the *fifth order* with the *curve* at the given point.

(41.) Finally, if we multiply the equation LXVII. by $bg + ch$, we can at once eliminate a by LXVIII., and so obtain a *cubic equation* in $b : c$, which has *at least one real root*, answering to a *real system of ratios* a, b, c , and therefore to a *real direction* of the line PE in (38.). It is therefore possible to assign *at least one real cylinder* of the second order (39.), which shall have *contact of the fifth order* with the curve at P, and shall at the same time have *one side PE common* with the *cone* of the second order (40.), which has contact of the *same (fifth) order* with the *curve* (or of the *fourth order* with the *cone of chords*): and consequently it is possible in this way to assign, as the *intersection* of this cylinder with this cone, at least *one real*

* Compare the first Note to page 588.

twisted cubic, which has *contact of the fifth** order with the given curve of double curvature, at the given point thereof. And *such* a cubic curve may be called, by eminence, an *Osculating† Twisted Cubic*.

(42.) Not intending to return, in these *Elements*, on the subject of such *cubic curves*, we may take this occasion to remark, that the very simple *vector equation*,‡

$$\text{LXX.} \dots V\alpha\rho = \rho V\beta\rho,$$

represents a *curve of this kind*, if α and β be any two constant and non-parallel vectors. In fact, if we operate on this equation by the symbol $S\lambda$, in which λ is an *arbitrary* but constant vector, the *scalar equation* so obtained, namely,

$$\text{LXXI.} \dots S\lambda\rho\alpha = S\lambda\rho S\beta\rho - \rho^2 S\beta\lambda,$$

represents a *surface of the second order*, on which the *curve* is wholly contained; making then successively $\lambda = \alpha$ and $\lambda = \beta$, we get, in particular, the *two equations*,

$$\text{LXXII.} \dots S(V\alpha\rho.V\beta\rho) = 0, \text{ and } \text{LXXIII.} \dots (V\beta\rho)^2 + S\alpha\beta\rho = 0,$$

representing respectively a *cone* and *cylinder* of that order, with the vector β from the origin as a *common side*: and the *remaining part* of the intersection of these two surfaces, is precisely the *curve* LXX., which therefore is a *twisted cubic*, in the known sense already referred to.

(43.) *Other surfaces* of the same order, containing the same curve, would be obtained by assigning other values to λ ; for example (comp. 897, (81.)), we should get generally an *hyperbolic paraboloid* from the form LXXI., by taking $\lambda \perp \beta$. But it may be more important here to observe, that *without* supposing any acquaintance with the theory of *curved surfaces*, the *vector equation* LXX. can be shown, by

* Accordingly, it is known (see page 242 of Dr. Salmon's Treatise, already cited), that a *twisted cubic* can generally be described through any *six given points*; and also (page 248), that *three quadric cylinders* (or cylinders of the second order or degree) can be described, containing a *given cubic curve*, their edges being parallel to the *three* (real or imaginary) *asymptotes*.

† Compare the first Note to page 563.

‡ This example was given in pages 679, &c., of the *Lectures*, with some connected transformations, the equation having been found as a certain condition for the inscription of a *gauche quadrilateral*, or other *even-sided polygon*, in a *given spheric surface* (comp. the sub-articles to 296): the *2n successive sides* of the figure being obliged to pass through the same *even number* of given points of space. It was shown that the *curve* might be said to intersect the *unit-sphere* ($\rho^2 = -1$) in *two imaginary points at infinity*, and also in *two real* and *two imaginary points*, situated on *two real right lines*, which were *reciprocal polars* relatively to the sphere, and might be called *chords of solution*, with respect to the proposed *problem of inscription* of the polygon; and that *analogous results* existed for *even-sided polygons in ellipsoide*, and *other surfaces* of the *second order*: whereas the corresponding problem, of the inscription of an *odd-sided polygon* in such a *surface*, conducted only to the assignment of a *single chord of solution*, as happens in the known and analogous theory of *polygons in conics*, whether the number of sides be (in *that theory*) *even* or *odd*. But we cannot here pursue the subject, which has been treated at some length in the *Lectures*, and in the *Appendices* to them.

quaternions, to represent a *curve of the third degree*, in the sense that it is *cut*, by an *arbitrary plane*, in *three points* (real or imaginary). In fact, we may write the equation as follows,

$$\text{LXXIV.} \dots Vqp = -a, \text{ if } \text{LXXV.} \dots q = g + \beta,$$

q being here a *quaternion*, of which the *vector part* β is *given*, but the *scalar part* g is *arbitrary*; and then, by *resolving* (comp. 347) this *linear equation* LXXIV., we may still further transform it as follows,

$$\text{LXXVI.} \dots g(g^2 - \beta^2)\rho = \beta S\beta a + gV\beta a - g^2 a,$$

which conducts to a *cubic equation* in g , when combined with the equation,

$$\text{LXXVII.} \dots S\rho = e,$$

of any proposed *secant plane*.

(44.) The vector equation LXX., however, is *not sufficiently general*, to represent an *arbitrary twisted cubic*, through an *assumed point* taken as *origin*; for which purpose, *ten scalar constants* ought to be disposable, in order to allow of the curve being made to pass through *five* other arbitrary points*: whereas the equation referred to involves *only five* such constants, namely the *four* included in Ua and $U\beta$, and the *one* quotient of tensors $T\beta : Ta$ (comp. 358).

(45.) It is easy, however, to accomplish the *generalization* thus required, with the help of that theory of *linear and vector functions* ($\phi\rho$) of *vectors*, which was assigned in the Sixth Section of the preceding Chapter (Arts. 347, &c.) We have only to write, instead of the equation LXX., this other but analogous *form* which includes it,

$$\text{LXXVIII.} \dots Vap + V\rho\phi\rho = 0, \text{ or } \text{LXXVIII.}' \dots \phi\rho + c\rho = a,$$

and which gives, by principles and methods already explained (comp. 354, (1.)), the transformation,

$$\text{LXXIX.} \dots \rho = (\phi + c)^{-1}a = \frac{\psi a + c\chi a + c^2 a}{m + m'c + m''c^2 + c^3};$$

a , ψa , and χa being here *fixed vectors*, and m , m' , m'' being *fixed scalars*, but c being an *arbitrary and variable scalar*, which may receive *any value*, without the expression LXXIX. ceasing to satisfy the equation LXXVIII.

* Compare the first Note to page 591. In general, when a *curve in space* is supposed to be represented (comp. 371, (5.)) by *two scalar equations*, each *new arbitrary point*, through which it is required to *pass*, introduces a necessity for *two new disposable constants*, of the *scalar* kind: and accordingly *each new order*, say the n^{th} , of *contact* with such a *curve*, has been seen to introduce a *new vector*, $D^n\rho$, or $r^{(n-1)}$, subject to a *condition* resulting from the general equation $TD^n\rho = 1$, or $r^2 = -1$ (comp. 380, XXVI., and 396, III.), but involving *virtually two new scalar constants*. Thus, besides the *four* such constants, which enter through r and r' into the determination of the *directions* of the *rectangular system* of lines, *tangent*, *normal*, and *binormal* (comp. 379, (5.), or 396, (2.)), and of the *length* of the *radius* of (*first*) *curvature*, r , the *three successive derivatives*, r' , r'' , r''' , of that radius, and the radius r of *second curvature*, with its *two first derivatives*, r' and r'' , have been seen to enter, through the *three other vectors*, r'' , r''' , r^{1v} , into the determination (41.) of the *osculating twisted cubic*.

(46.) The curve LXXVIII. is therefore cut (comp. (43.)) by the plane LXXVII. in three points (real or imaginary), answering to and determined by the three roots of the cubic in c , which is formed by substituting the expression LXXIX. for ρ in the equation of that secant plane; and consequently it is a curve of the third degree, the three (real or imaginary) asymptotes to which have directions corresponding to the three values of c , obtained by equating to zero the denominator of that expression LXXIX., or by making $M=0$, in a notation formerly employed: so that they have the directions of the three lines β , which satisfy this other vector equation (comp. 354, I.),

$$\text{LXXX.} \dots V\beta\phi\beta = 0.$$

(47.) Accordingly, if β be such a line, and if γ be any vector in the plane of α and β , the curve LXXVIII. is a part of the intersection of the two surfaces of the second order,

$$\text{LXXXI.} \dots S\alpha\rho\phi = 0, \text{ and } \text{LXXXII.} \dots S\gamma\alpha\rho + S\gamma\rho\phi = 0,$$

whereof the first is a cone, and which have the line β from the origin for a common side (comp. (42.)) : the curve is therefore found anew to be a twisted cubic.

(48.) And as regards the number of the scalar constants, which are to be conceived as entering into its vector equation LXXVIII., when we take for $\phi\rho$ the form $Vq_0\rho + V\lambda\rho\mu$ assigned in 357, I., in which q_0 is an arbitrary but constant quaternion, such as $g + \gamma$, and λ, μ are constant vectors, the term $g\rho$ of $\phi\rho$ disappears under the symbol of operation $V.$, and the equation (45.) of the curve becomes,

$$\text{LXXXIII.} \dots V\alpha\rho + \rho V\gamma\rho + V\rho V\lambda\rho\mu = 0;$$

in which the four versors, $U\alpha, U\gamma, U\lambda, U\mu$, introduce each two scalar constants, while the two tensor quotients, $T\gamma : T\alpha$ and $T\lambda\mu : T\alpha$, count as two others: so that the required number of ten such constants (44.) is exactly made up, the curve being still supposed to pass through an assumed origin, and therefore to have one point given. It is scarcely worth observing, that we can at once remove this last restriction, by merely adding a new constant vector to ρ , in the last equation, LXXXIII.

(49.) Although, for the determination of the osculating twisted cubic (41.), to a given curve of double curvature, it was necessary (comp. (40.)) to employ the vector r^{IV} or $D_s^4\rho$, or to take account of s^4 in the vector ρ_s , or in the connected scalars x, y, z , of (34.), and therefore to improve the expressions LVIII., by carrying in each of them (or at least in the two latter) the approximation one step farther, yet there are many other problems relating to curves in space, besides some that have been already considered, for which those scalar expressions LVIII. are sufficiently approximate: or for which the vector expression I. suffices.

(50.) Resuming, for instance, the questions considered in (22.) and (23.), we may throw some additional light on the law of the deviation of a near point P_s of the curve, from the osculating sphere at P , as follows. Eliminating α by XXXVI' from XXXV., we find this new expression,

$$\text{LXXXIV.} \dots \overline{8P_s} - \overline{8P} = \frac{r_1 s^4}{24r r^2 R};$$

the direction of this deviation from the sphere (R) depends therefore on the sign of the scalar radius r_1 (23.) of curvature of the cusp-edge (s) of the polar developable: and it is outward or inward (comp. 395, (14.)), according as that cusp-edge turns its concavity (comp. XLI.) or its convexity, at the centre s of the oscu-

lating sphere, towards the point P of the *given curve*, that is, towards the *point of osculation*.

(51.) Again, if we only take account of s^3 , the deviation of P_s from the *osculating circle* at P has been seen to be a vector *tangential to the osculating sphere*, which may be thus expressed (comp. 397, IX., LII.),

$$\text{LXXXV.} \dots C_s P_s = \frac{s^3}{6} \nu' r = \frac{s^3 r (\sigma - \rho)}{6r^2 r},$$

if C_s be the point on the circle, which is distant from the given point P by an arc of that circle = s , with the same initial direction of motion, or of departure from P , represented by the common unit tangent r ; the quantity of this deviation is therefore expressed by the scalar $\frac{s^3 R}{6r^2 r}$: that is, by the deviation $\frac{s^3}{6r}$ (comp. 397, (9.), (10.)) from the *osculating plane** at P , multiplied by the *secant* ($r^{-1}R$) of the inclination (P) of the radius (R) of *spherical curvature*, to the radius (r) of *absolute curvature*, and *positive* when this last deviation has the direction of the *binormal* ν .

(52.) On the other hand (comp. (5.)) the *small angle*, which the *small arc* ss , of the *cuspid-edge* (s) of the *polar developable* subtends at the point P , is ultimately expressed by the scalar,

$$\text{LXXXVI.} \dots s p s = (\overline{P s_s} - \overline{P s}) \cdot R^{-1} \cot P = \frac{r R^2 s}{p R} = \frac{\nu r s}{R^2} \text{ (by XXXIII.),}$$

this angle being treated as *positive*, when the corresponding *rotation*† round $+r$ from

* Besides the nine expressions in 397, (42.) for the *square* r^{-2} of the *second curvature*, the following may be remarked, as containing the *law of the regression* of the *projection of a curve* of double curvature on its own normal plane:

$$r^{-2} = \frac{9}{2KP} \cdot \lim. \frac{PQ_3^2}{PQ_3^2}, \quad 397, \text{XCIX.}, (10);$$

κ being still the centre of the osculating circle, and Q_1, Q_2, Q_3 being still (as in 397, (10.)) the projections of a *near point* Q (or P_s), on the *tangent*, the *absolute normal* (or inward radius of curvature PK), and the *binormal* at P . In fact, the *principal terms* of the *three vector projections* corresponding, of the *small chord* PQ (or PP_s), are (comp. LVIII.):

$$PQ_1 = sr; \quad PQ_2 = (\frac{1}{2}s^2 r') = \frac{s^2}{2r} U r'; \quad PQ_3 = (\frac{1}{6}s^3 r^{-1} \nu) = \frac{s^3}{6r^2} U \nu;$$

whence, ultimately,

$$\frac{9}{2} \cdot \frac{PQ_3^2}{PQ_3^2} = -r^{-2} r U r' = r^{-2} \cdot KP.$$

† Considered as a *rotation*, this small angle may be represented by the *small vector*, $rp^{-1}R'R^{-1}sr$; and if the *vector deviation* LXXXV. from the *osculating circle* be multiplied by *this*, the *quarter of the product* is (comp. XXXV.) the *vector deviation* from the *osculating sphere*, under the form,

$$\frac{s^4(\rho - \sigma)}{24R} \cdot \frac{R'}{rrp}.$$

rs to ρs , is positive: and if we multiply *this* scalar, by that which has just been assigned (51.), as an expression for the deviation $c_s P_s$, from the osculating circle, we get, by XXXV., the product,

$$\text{LXXXVII.} \dots \frac{s^2 R}{6r^2 r} \cdot \frac{r R' s}{p R} = \frac{R' s^4}{6r p} = 4 (\overline{sP_s} - \overline{SP}).$$

(53.) Combining then the recent results (50.), (51.), (52.), we arrive at the following *Theorem*:

The deviation of a near point P_s , of a curve in space, from the osculating sphere at the given point P , is ultimately equal to the quarter of the deviation of the same near point from the osculating circle at P , multiplied by the sine of the small angle which the arc ss_s , of the locus of centres of spherical curvature (s), or of the cusp-edge of the polar developable, subtends at the same point P ; and this deviation ($\overline{sP_s} - \overline{SP}$) from the sphere has an outward or an inward direction, according as the same arc ss_s is concave or convex towards the same given point.

(54.) The vector of the centre s_s , of the near osculating sphere at P_s , is (in the same order of approximation, comp. I.),

$$\text{LXXXVIII.} \dots os_s = \sigma_s = \sigma + \sigma' + \frac{1}{2} s^2 \sigma'' + \frac{1}{6} s^3 \sigma''' + \frac{1}{24} s^4 \sigma^{IV};$$

and although $\sigma - \rho$ is already a function (by 397, IX., &c.) of r, r', r'' , so that σ' is (as in (2.) or (22.)) a function of r', r'', r''' , and $\sigma'', \sigma''', \sigma^{IV}$ introduce respectively the new derived vectors r^{IV}, r', r'' , or $D_s^4 \rho, D_s^3 \rho, D_s^2 \rho$, which we are not at present employing (49.), yet we have seen, in (28.) and (24.), that some useful combinations of σ'' and σ''' can be expressed without r^{IV}, r' : and the following is another remarkable example of the same species of reduction, involving not only σ'' and σ''' but also σ^{IV} , but still admitting, like the former, of a simple geometrical interpretation.

(55.) Remembering (comp. (22.), and 397, XV.) that

$$\text{LXXXIX.} \dots (\sigma - \rho)^2 + R^2 = 0, \quad \text{XC.} \dots S r''' (\sigma - \rho) = r^2 S = r^2 + nr^{-1} r^{-1},$$

and reducing the successive derivatives of LXXXIX. with the help of the equations 397, XIX., and of their derivatives, we are conducted easily to the following system of equations, into which the derived vectors $r, r',$ &c. do not expressly enter, but which involve $\sigma', \sigma'', \sigma''', \sigma^{IV}$, and R', R'', R''', R^{IV} :

$$\text{XCI.} \dots S \sigma' (\sigma - \rho) + R R' = 0; \quad \text{XCII.} \dots S \sigma' \sigma'' (\sigma - \rho) = 0;$$

$$\text{XCIII.} \dots S \sigma'' (\sigma - \rho) + \sigma'^2 + (R R')' = 0;$$

$$\text{XCIV.} \dots S \sigma''' (\sigma - \rho) + 3 S \sigma' \sigma'' + (R R'')' = 0;$$

$$\text{XCV.} \dots S \sigma^{IV} (\sigma - \rho) + 4 S \sigma' \sigma''' + 3 \sigma'^2 + (R R''')' = - \frac{R R''}{r p} = - \frac{n}{r};$$

auxiliary equations being,

$$\text{XCVI.} \dots S \sigma' r = 0, \quad S \sigma'' r' = 0, \quad S \sigma'' r = 0, \quad \text{comp. 395, X.}$$

$$\text{and XCVII.} \dots S \sigma''' r = - S \sigma'' r' = S \sigma' r'' = S r r'' - S (\sigma - \rho) r''' \\ = - r^2 (S - 1) = - nr^{-1} r^{-1}.$$

(56.) But, if R_s denote the radius of the near sphere, and if we still neglect s^2 , we have,

$$\text{XCVIII.} \dots \overline{P_s s_s^2} = - (\sigma_s - \rho_s)^2 = R_s^2 \\ = R^2 + 2s R R' + s^2 (R R')' + \frac{s^3}{3} (R R'')' + \frac{s^4}{12} (R R''')'';$$

whence follows, by LXXXVIII., and by the recent equations, this very simple expression, from which (comp. (24.)) *everything depending on* r'' , r' , r'' *has disappeared,*

$$\text{XCIX.} \dots (\sigma_s - \rho)^2 + R_s^2 = \frac{-RR's^4}{12rrp};$$

and which gives (within the same order of approximation, attending to XXXV.) the *geometrical relation,*

$$\text{C.} \dots \overline{P_2S_2} - \overline{P_1S_2} = T(\sigma_s - \rho) - R_s = \frac{R's^4}{24rrp} = \frac{ns^4}{24rrR} = \overline{SP_2} - \overline{SP};$$

or

$$\text{C'.} \dots \overline{S_2P} - \overline{SP_2} = \overline{S_2P_2} - \overline{SP} = R_s - R.$$

(57.) This result might have been *foreseen*, from the following very simple consideration. When the *coefficient* $S-1$ of *non-sphericity* (395, (16.)), or of the deviation of a curve from a sphere, is *positive*, so that a *near point* P_2 of the curve is *exterior* to (what we may call) the *given sphere*, which *osculates* to that curve at P , by an amount which is ultimately proportional to the *fourth power* of the arc, s , of the curve, then the *given point* P must be, for the same reason, *exterior to the near sphere*, which osculates at the point P_2 ; and the *two deviations*, $\overline{P_2S_2} - \overline{P_1S_2}$, and $\overline{SP_2} - \overline{SP}$, which have been found by *calculation* to be *equal* (C.), if s^5 be neglected, must in fact bear to each other an *ultimate ratio of equality*, because the *two arcs*, $+s$ and $-s$, from P to P_2 , and from P_2 back to P , are *equally long*, although *oppositely directed*; or because $(+s)^4 = (-s)^4$. And precisely the same reasoning applies, when the coefficient $S-1$ is *negative*, so that the *deviations*, equated in the formula C., are *both inwards*.

(58.) As regards the deviation (51.) of the near point P_2 of the curve from the *osculating circle* at P , we may generalize and render more exact the expression LXXXV., by considering a point OC_2 of that circle, which is distant by a *circular arc* $=t$ from the given point P ; and of which the vector is, *rigorously*, by 896, (18.),

$$\text{CI.} \dots OC_2 = \omega_2 = \rho + rr \sin \frac{t}{r} + r^2 r' \text{ vers } \frac{t}{r};$$

or if we only neglect t^5 ,

$$\text{CII.} \dots OC_2 = \omega_2 = \rho + r \left(t - \frac{t^3}{6r^3} \right) + rr' \left(\frac{t^2}{2r} - \frac{t^4}{24r^3} \right).$$

(59.) In this way we shall have (comp. (84.)) the *vector deviation*,

$$\text{CIII.} \dots C_2P_2 = \rho_s - \omega_2 = Xr + Yrr' + Zrv,$$

with the *scalar coefficients*,

$$\text{CIV.} \dots X = x_s - r \sin \frac{t}{r}, \quad Y = y_s - r \text{ vers } \frac{t}{r}, \quad Z = z_s;$$

or, neglecting s^5 and t^5 , and attending to the expressions LVIII. and LVIII',

$$\text{CV.} \dots \begin{cases} X = s - t - \frac{s^3 - t^3}{6r^3} + \frac{r's^4}{8r^3}; \\ Y = \frac{s^2 - t^2}{2r} - \frac{p}{r} Z - \frac{s^4 - t^4}{24r^3} - \frac{ns^4}{24r^3}; \\ Z = \frac{s^3}{6r} + \frac{rs^4}{24} (r^{-3} r^{-1}); \end{cases}$$

in which r , r' , r , p , and n have the same significations as before.

(60.) Assuming then for the *circular arc* t the value,

$$\text{CVI.} \dots t = s + \frac{r's^4}{8r^3},$$

which differs (as we see) by only a quantity of the *fourth order* from the arc s of the *curve*, we shall have, to the same order of approximation, the expressions,

$$\text{CVII.} \dots X = 0, \quad Y = \frac{-p}{r} Z - \frac{ns^4}{24r^2r}, \quad Z = z_s \text{ \&c., as before,}$$

the *deviation* at P_s from the *circle* being here measured in a direction *parallel to the normal plane* at P ; and if s^4 be neglected (although the expressions enable us to take account of it), this deviation is also *parallel* (as before) to the *tangent* $r(\sigma - \rho)$ to the *osculating sphere* in that plane: while it is represented in quantity by $Rr^{-1}z_s$, which agrees with the result in (51.).

(61.) The expressions CVII. give also, *without neglecting* s^4 ,

$$\text{CVIII.} \dots \frac{rY + pZ}{R} = -\frac{ns^4}{24rrR} = \overline{SF} - \overline{SF}_s;$$

such then is the *component of the deviation* from the *osculating circle*, which is *parallel to the normal* rs to the *sphere* at P ; and we see that it only differs in *sign* (because it is *positive* when its *direction* is that of the *inward normal*, or *inward radius* rs), from the expression XXXV. (comp. C.), for the *outward deviation* $\overline{SF} - \overline{SF}_s$ of the *near point* P_s , from the same *osculating sphere* at the given point P .

(62.) This *latter component* (61.) is *small*, even as *compared* with the *former* small component (60.); and the *small quotient*, of the latter divided by the former, is ultimately (by LXXXVI.),

$$\text{CIX.} \dots \frac{rY + pZ}{rZ - pY} = \frac{-nr s}{4R^2} = -\frac{1}{4}sr s_s;$$

where the *small angle* $sr s_s$ is *positive or negative*, according to the rule stated in (52.), and may be replaced by its *sine*, or by its *tangent*.

(63.) Instead of cutting the given *osculating circle*, as in (60.), by a plane which is *parallel to the given normal plane* at P , we may propose to *cut that circle by the near normal plane* at P_s , or to satisfy this *new condition*,

$$\text{CX.} \dots 0 = Sr_s(\rho_s - \omega_t), \quad \text{or} \quad \text{CX'.} \dots 0 = X S r r_s + Y S r r' r_s + Z S r r r_s;$$

which is easily found to give by CV. the values (s and t being still supposed to be small, and s^2 being still neglected):

$$\text{CXI.} \dots t = s - \frac{r's^4}{24r^3}, \quad \text{and} \quad \text{CXII.} \dots X = \frac{r's^4}{6r^3}, \quad Y = \&c., \quad Z = \&c., \quad \text{as in CVII.};$$

so that in passing to this *new near point* C_t of the *circle*, we only change X from zero to a small quantity of the *fourth order*, and make *no change* in the values of Y and Z .

(64.) The *new deviation* $C_t P_s$ from the *given circle* may be decomposed into *two partial deviations*, in the *near normal plane*, of which *one* has the *direction of the unit-tangent* $R_s^{-1}r_s(\sigma_s - \rho_s)$ to the *near sphere* at P_s , and the other has that of the *unit-normal* $R_s^{-1}(\sigma_s - \rho_s)$ to the same *sphere* at the same point (or the opposites of these two directions); and the *scalar coefficients* of these *two vector units*, if we attend only to *principal terms*, are easily found to be,

$$\text{CXIII.} \dots \frac{rZ - pY}{R} = \frac{Rs^3}{6r^2r}, \quad \text{and} \quad \text{CXIV.} \dots \frac{rY + (p + ns)Z}{R} = \frac{ns^4}{8rrR}.$$

(65.) We may then write :

$$\begin{aligned} \text{CXV.} \dots \textit{Deviation of near point } P_s \textit{ from given osculating circle,} \\ \textit{measured in the near normal plane to the curve at } P_n \\ = \text{new } C_t P_s = \frac{Rs^3}{6r^2r} U r_s(\sigma_s - \rho_s) + \frac{ns^4}{8rrR} U(\sigma_s - \rho_s); \end{aligned}$$

in which it may be observed, that the *second scalar coefficient* is equal to *three times the scalar deviation* $\overline{SP}_s - \overline{SP}$ (XXXV. or C.), of the near point P_s of the curve, from the given osculating sphere (at P).

(66.) But we may also interpret the *new coefficient* last mentioned, as representing a *new deviation*; namely, that of the point C_t of the given circle, from the near osculating sphere at P_n , considered as *positive* when that new point C_t is *exterior* to that near sphere; or as denoting the *difference of distances*, $s_s C_t - s_s P_s$. We have therefore (comp. (56.)) this *new geometrical relation*, of an extremely simple kind :

$$\text{CXVI.} \dots \overline{s_s C_t} - \overline{s_s P_s} = 3(\overline{SP}_s - \overline{SP}) = 3(\overline{s_s P} - \overline{s_s P_s});$$

or

$$\text{CXVI.} \dots \overline{s_s C_t} = 3\overline{s_s P} - 2\overline{s_s P_s}.$$

(67.) Supposing, then, at first, that the *coefficient of non-sphericity* $S - 1$ is *positive* (comp. 395, (16.)), if we conceive a point to move *backwards*, upon the curve, from P_s to P, and then *forwards*, upon the circle which osculates at P, to the new point C_t (63.), we see that it will *first* attain (at P) a position *exterior* to the sphere which osculates at P_n , or will have an amount, determined in (56.), of *outward deviation*, with respect to that near osculating sphere; and that it will *afterwards* attain (at the new point C_t) a *deviation of the same character* (namely *outwards*, if $S > 1$), from the same near sphere, but one of which the amount will be *threefold* the former: this last relation holding also when $S < 1$, or when *both deviations* are *inwards*.

(68.) It is easy also to infer from (65.), (comp. (57.)), that if we go *back* from P_n on the near circle which osculates at that near point, through an arc (t) of that circle, which will only differ by a small quantity of the *fourth order* (comp. (60.)) from the arc (s) of the curve, so as to arrive at a point, which for the moment we shall simply denote by c, and in which (as well as in *another point of section*, not necessary here to be considered) the near osculating circle is cut by the given normal plane at P, the *vector deviation* of this new point c of the new circle, from the given point P of the curve, must be, nearly :

$$\text{CXVII.} \dots PC = \frac{Rs^3}{6r^2r} U r(\sigma - \rho) - \frac{ns^4}{8rrR} U(\sigma - \rho);$$

the coefficients being formed from those of the formula CXV., by first changing s to $-s$, and then changing the signs of the results: while the relation CXVI. or CXVII. takes now the form,

$$\text{CXVIII.} \dots \overline{cC} - \overline{cP} = 3(\overline{SP}_s - \overline{SP}), \quad \text{or} \quad \text{CXVIII.} \dots \overline{cC} = 3\overline{SP}_s - 2\overline{SP}.$$

(69.) Accordingly if, after going from P to P_s along the curve, we go *forward* or *backward*, through any *positive* or *negative arc*, t , of the circle which osculates at that point P_n , we shall arrive at a point which we may here denote by C_n, t ; and the *vector* (comp. again 396, (18.)) of this near point (*more general* than any of those hitherto considered) will be, *rigorously*,

$$\text{CXIX.} \dots \omega_{n,t} = \text{OC}_{n,t} = \rho_s + r_s r_s \sin \frac{t}{r_s} + r_s^2 r'_s \text{ vers } \frac{t}{r_s}$$

And if we *develope* this new expression to the accuracy of the *fourth order* inclusive, we find that we satisfy the *new condition* (comp. (68.)),

$$\text{CXX.} \dots \text{Sr}(\omega_{n,t} - \rho) = 0, \text{ when CXXI.} \dots t = -s - \frac{r'_s s^4}{24r^3};$$

and that then the expression CXIX. agrees with CXVII., within the order of approximation here considered.

(70.) A *geometrical connexion* can be shown to exist, between the *two equivalents* which have been found above, one for the *quadruple* (LXXXVII., comp. (53.)), and the other for the *triple* (CXVIII.), of the *deviation* $\overline{SP}_s - \overline{SP}$ of a *near point* P_s of the *curve*, from the *sphere* which *osculates* at the *given point* P : in such a manner that if *either* of those two expressions be regarded as *known*, the *other* can be *inferred* from it.

(71.) In fact if we draw, in the normal plane, perpendiculars PD and PE to the lines PS and PS_n , and determine points D and E upon them by drawing a parallel to PS through the point C of (68.), letting fall also a perpendicular CF on PS_n , the *two small lines* PD and DC will ultimately represent the *two terms* or components CXVII. of PC ; and the *small angle* DPC will ultimately be equal to *three quarters* of the small angle SPS_n , and will correspond to the *same direction* of rotation round τ , because

$$\text{CXXII.} \dots \frac{DC}{PD} = \frac{3}{4} \cdot \frac{nrst}{R^2} = \frac{3}{4} V \frac{\sigma's}{\sigma - \rho},$$

or

$$\text{CXXIII.} \dots DPC = \frac{3}{4} SPS_n = \frac{3}{4} DPE;$$

so that we shall have the *ultimate ratios* (comp. the annexed Fig. 83*):

$$\text{CXXIV.} \dots DC : DE : CE \text{ (or } FP) = 3 : 4 : 1.$$

But the line CF is ultimately the *trace*, on the given normal plane, of the tangent plane at c to the near osculating sphere; the small line FP (or CE) represents therefore the *deviation* $\overline{s_n P} - \overline{s P}$, of the given point P from that near sphere, or the *equal deviation* (57.), $\overline{SP}_s - \overline{SP}$; its ultimate *quadruple*, DE , represents the *product* mentioned in (52.); and the ultimate *triple*, DC , of the same small line CE , is a geometrical representation of that *other deviation* $\overline{SC} - \overline{SP}$, which has been more recently considered.

(72.) When the *two scalars*, s and t , are supposed capable of receiving *any values*, the point $C_{n,t}$ in (69.) may be *any point of the Locus* (8.) of the *Osculating Circle* to the given curve of double curvature; and if we seek the direction of the *normal* to this superficial *locus*, at this point, on the plan of Art. 372, writing first the equation of the surface under the slightly simplified, but equally rigorous form,

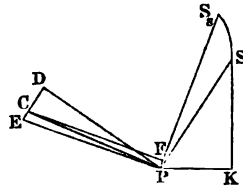


Fig. 83.

* In Figs. 81, 82, the little arc near s is to be conceived as *terminating there*, or as being a *preceding arc* of the curve which is the locus of s , if r' , r , n , and therefore also p and r_1 , be *positive* (comp. the second Note to page 574). In the new Figure 83, the triangle PDE is to be conceived as being in fact *much smaller* than PKS , though *magnified* to exhibit angular and other relations.

$$\text{CXXV.} \dots \omega_{n,u} = \rho_s + r_s r_s \sin u + r_s^2 r_s' \text{ vers } u,$$

with

$$\text{CXXVI.} \dots u = r_s^{-1} t = P_s K_s C_s t,$$

so that u is here a new scalar variable, representing the *angle subtended at the centre* K_s , of the osculating circle at P_s , by the arc, t , of that circle, we are led, after a few reductions, to the expression,

$$\text{CXXVII.} \dots V(D_u \omega_{n,u} \dots D_s \omega_{n,u}) = r_s r_s^{-1} (\omega_{n,u} - \sigma_s) \text{ vers } u;$$

which proves, by quaternions, what was to be expected from geometrical²⁸ considerations, that the *locus of the osculating circle* is also (as stated in (8.) and (22.)) the *Envelope of the Osculating Sphere*.

(73.) The normal to this locus, at any proposed point $C_s t$ of any one osculating circle, is thus the radius of the sphere to which that circle belongs, or which has the same point of osculation P_s , with the given curve, whether the arc (s) of that curve, and the arc (t) of the circle, be small or large. We must therefore consider the tangent plane to the locus, at the given point P of the curve, as coinciding with the tangent plane to the osculating sphere at that point; and in fact, while this latter plane ($\perp r_s$) contains the tangent r to the curve, which is at the same time a tangent to the locus, it contains also the tangent $r(\sigma - \rho)$ to the sphere, which is by CXVII. another tangent to the locus, as being the tangent at P to the section of that surface, which is made by the normal plane to the curve.

(74.) But when we come to examine, with the help of the same equation CXVII., what is the law of the deviation \overline{DC} (comp. Fig. 83) of that normal section of the locus, considered as a new curve (c), from its own tangent \overline{PD} , we find that this law is ultimately expressed (comp. (71.)) by the formula,

$$\text{CXXVIII.} \dots \frac{\overline{DC}^3}{\overline{PD}^4} = \frac{81}{82} \cdot \frac{\pi^2 r^2 (\sigma - \rho)}{R^3} = \text{const.};$$

hence \overline{DC} varies ultimately as the power of \overline{PD} , which has the fraction $\frac{4}{3}$ for its exponent; the limit of $\overline{PD}^2 : \overline{DC}$ is therefore null, and the curvature of the section is infinite at P .

(75.) It follows that this point P is a singular point of the curve (c), in which the locus (8.) is cut (73.), by the normal plane to the given curve at that point; but it is not a cusp on that section, because the tangential component \overline{PD} of the vector chord \overline{PC} is ultimately proportional to an odd power (namely to the cube, by CXVII., comp. (71.)) of the scalar variable, s , and therefore has its direction reversed, when that variable changes sign: whereas the normal component \overline{DC} of the same chord \overline{PC} is proportional to an even power (namely the fourth, by the same equation CXVII.) of the same arc, s , of the given curve, and therefore retains its direction unchanged, when we pass from a near point P_s , on one side of the given point P , to a near point P_s , on the other side of it.

(76.) To illustrate this by a contrasted case, let G be the point in which the tangent to the given curve at P , is cut by the normal plane at P ; or a point of the section, by that plane, of the developable surface of tangents. We shall then have

* In the language of infinitesimals, two consecutive osculating spheres, to any curve in space, intersect each other in an osculating circle to that curve.

the sufficiently approximate expressions,

$$\text{CXXIX.} \dots \text{PG} = \rho_s - \rho - \left(s + \frac{s^2}{8r^2} \right) r_s = \frac{-s^2 r'}{2} - \frac{s^2 v}{8r} = -\text{PQ}_2 - 2\text{PQ}_3,$$

with the significations 397, (10.) of Q_2 and Q_3 ; hence the point P of the curve is (as is well known) a *cusp* of the section (G) of the developable surface of tangents (comp. 397, (15.)), because the tangential component ($-\text{PQ}_2$) of the vector chord (PG) has here a fixed direction, namely that of the outward radius (XP prolonged) of the circle of curvature at P: while it is now the normal component (-2PQ_3) which changes direction, when the arc s of the curve changes sign. At the same time we see* that the equation of this last section (G) may ultimately be thus expressed :

$$\text{CXXX.} \dots \frac{(-2\text{PQ}_3)^2}{(-\text{PQ}_2)^2} = \frac{8\text{PK}}{9r^2} = \text{const.};$$

comparing which with the equation CXXVIII., we see that although, in each case, the curvatures of the section is infinite, at the point P of the curve, yet the normal component (or co-ordinate) varies (ultimately) as the power $\frac{2}{3}$ of the tangential component, for the section (G) of the Surface of Tangents: whereas the former component varies by (74.) as the power $\frac{1}{3}$ of the latter, for the corresponding section (O) of the Locus of the Osculating Circle.

(77.) It follows also that the curve (P) itself, although it is not a cusp-edge of the last-mentioned locus (8.), while it is such on the surface of tangents, is yet a Singular Line upon that locus likewise: the nature and origin of which line will perhaps be seen more clearly, by reverting to the view (8.), (22.), (72.), according to which that Locus of a Circle is at the same time the Envelope of a Sphere.

(78.) In general, if we suppose that σ and R are any two real functions, of the vector and scalar kinds, of any one real and scalar variable, t , and that σ' , R' , and σ'' , R'' , &c. denote their successive derivatives, taken with respect to it, then σ may be conceived to be the variable vector of a point s of a curve in space, and R to be the variable radius of a sphere, which has its centre at that point s , but alters generally its magnitude, at the same time that it alters its position, by the motion of its centre along the curve (s).

(79.) Passing from one such sphere, with centre s and radius R , considered as given, and represented by the scalar equation,†

$$(\sigma - \rho)^2 + R^2 = 0, \qquad \text{LXXXIX.},$$

in which ρ is now conceived to be the vector of a variable point P upon its surface, to a near sphere of the same system, for which σ , s , and R are replaced by σ_s , s_s , and R_s , where t is supposed to be small, we easily infer (comp. 886, (4.)) that the equation,

$$8\sigma'(\sigma - \rho) + RR' = 0, \qquad \text{XCI.},$$

which is formed from LXXXIX. by once derivating σ and R with respect to t , but

* Compare the first Note to page 594.

† This equation, and a few others which we shall require, occurred before in this series, but in a connexion so different, that it appears convenient to repeat them here.

treating ρ as constant, represents the *real plane* (comp. 282, (12.)) of the (*real or imaginary*) *circle*, which is the *ultimate intersection* of the near sphere with the given one; the *radius* of this *circle*, which we shall call r , being found by the following formula,

$$\text{CXXXI.} \dots r^2\sigma^2 = R^2(R^2 + \sigma^2), \text{ or } \text{CXXXI}'. \dots r^2T\sigma^2 = R^2(T\sigma^2 - R^2),$$

and being therefore *real* when

$$\text{CXXXII.} \dots R^2 + \sigma^2 < 0, \text{ or } \text{CXXXII}'. \dots R^2 < T\sigma^2;$$

while the *centre*, say κ , of the *circle* is *always real*, and its *vector* is,

$$\text{CXXXI}'. \dots \sigma\kappa = \kappa = \sigma + RR'\sigma^{-1};$$

and the *plane* XCI. of the same *circle* is *parallel* to the *normal plane* of the *curve* (8.).

(80.) With the *condition* CXXXII., the *two scalar equations*, LXXXIX. and XCI., represent then *jointly a real circle*; and the *locus* of all such *circles* (comp. 386, (6.)) is easily proved to be also the *envelope* of all the *spheres*, of which one is represented by the equation LXXXIX. *alone*; each such *sphere touching this locus*, in the *whole extent* of the corresponding *circle* of the system.

(81.) The *plane* XCI., considered as *varying* with t , has a *developable surface* for its *envelope*; and the *real right line*, or *generatrix*, along which one touches the other, is represented (comp. again 386, (6.)) by the system of the *two scalar equations*, XCI. and

$$S\sigma''(\sigma - \rho) + \sigma'^2 + (RR')' = 0, \quad \text{XCIII.};$$

where ρ is now the *variable vector* of the *line of contact*, although it has been treated as *constant* (comp. 386, (4.)), in the process by which we are here conceived to pass, by a *second derivation*, from LXXXIX. through XCI. to XCIII.

(82.) This *real right line* (81.) *meets* generally the *sphere*, and also the *circle* (as being in its plane), in *two* (real or imaginary) *points*, say P_1, P_2 ; and the *curvilinear locus* of all such *points* forms generally a species of *singular line*,* upon the *superficial locus* (or *envelope*) recently considered (80.); or rather it forms in general *two branches* (real or imaginary) of such a *line*: which generally *two-branched line* (or *curve*) is the (real or imaginary) *envelope* (comp. 386, (8.)), of all the *circles* of the system.

(83.) The equation,

$$S\sigma'\sigma''(\sigma - \rho) = 0, \quad \text{XCII.},$$

which now represents (comp. 376, V.) the *osculating plane* to the *curve* (8.), shows

* Called by Monge an *arête de rebroussement*, except in the case to which we shall next proceed, when its *two branches coincide*. The *envelope* (80.) of a *varying sphere* has been considered in two distinct Sections, § XXII. and § XXVI., of the *Application de l'Analyse à la Géométrie*; but the author of that great work does not appear to have perceived the *interpretation* which will soon be pointed out, of the *condition* of such *coincidence*. Meantime it may be mentioned, in passing, that quaternions are found to confirm the geometrical result, that when the *two branches* (P_1) (P_2) are *distinct*, then each is a *cusp-edge* of the *surface*; but that when they are *coincident*, the *singular line* (P) in which they merge has then a *different character*.

that this plane through the centre s of the sphere is perpendicular to the right line (81.), and consequently contains the perpendicular let fall from that centre on that line: the foot P of this last perpendicular is therefore found by combining the three linear and scalar equations, XCI., XCII., XCIII., and its vector is,

$$\text{CXXXIII.} \dots OP = \rho = \sigma + \frac{g\sigma' + RR'\sigma''}{\sqrt{\sigma^2\sigma''}},$$

if $\text{CXXXIV.} \dots g = -\sigma^2 - R^2 - RR' = T\sigma^2 - (RR')$.

(84.) The condition of contact of the right line (81.) with the sphere (78.), or with the circle (79.), or the condition of contact between two consecutive* circles of the system (80.), or finally the condition of coincidence of the two branches (82.) of that singular line upon the surface which is touched by all those circles, is at the same time the condition of coexistence of the four scalar equations, LXXXIX., XCI., XCII., XCIII.; it is therefore expressed by the equation (comp. CXXXIII.),

$$\text{CXXXV.} \dots R^2(\sqrt{\sigma^2\sigma''})^2 = (g\sigma' + RR'\sigma'')^2;$$

which may also be thus written,†

$$\text{CXXXVI.} \dots (R\sigma'\sigma'' - R'g)^2 = (R^2 + \sigma^2)(R^2\sigma'^2 + g^2),$$

or thus, $\text{CXXXVII.} \dots R^2(R^2 + \sigma^2)(\sqrt{\sigma^2\sigma''})^2 = (g\sigma^2 + RR'S\sigma'\sigma'')^2;$

the scalar variable t (78.), with respect to which the derivations are performed, remaining still entirely arbitrary, but the point P , which is determined by the formula CXXXIII., being now situated on both the sphere and the circle: and its curvilinear locus, which we may call the curve (P), being now the singular line itself, in its re-

* Compare the Note to page 581.

† In page 372 of Liouville's Edition already cited, or in page 325 of the Fourth Edition (Paris, 1809), of the *Application de l'Analyse*, &c., it will be found that this condition is assigned by Monge, as that of the evanescence of a certain radical, under the form (an accidentally omitted exponent of π'' in the second part of the first member being here restored):

$$[\alpha(\phi'\phi'' + \psi'\psi'' + \pi'\pi'') - h^2]^2 + h^2[\alpha^2(\phi'^2 + \psi'^2 + \pi'^2) - h^4] = 0;$$

in which he writes, for abridgment,

$$h^2 = 1 - \phi'^2 - \psi'^2 - \pi'^2,$$

and ϕ, ψ, π are the three rectangular co-ordinates of the centre of a moving sphere, considered as functions of its radius a . Accordingly, if we change R to a , and σ to $i\phi + j\psi + k\pi$, supposing also that $R' = a' = 1$, and $R'' = a'' = 0$, whereby g is changed to $-h^2$, and $R^2 + \sigma^2$ to h^2 , in the condition CXXXVI., that condition takes, by the rules of quaternions, the exact form of the equation cited in this Note: which, for the sake of reference, we shall call, for the present, the *Equation of Monge*, although it does not appear to have been either interpreted or integrated by that illustrious author. Indeed, if Monge had not hastened over this case of coincident branches, on which he seems to have designed to return in a subsequent Memoir (unhappily not written, or not published), he would scarcely have chosen such a symbol as h^2 (instead of $-h^2$), to denote a quantity which is essentially negative, whenever (as here) the envelope of the sphere is real.

duced and one-branched state. And the last form CXXXVII. shows, what was to be expected from geometry, that when this condition of coincidence is satisfied, the earlier condition of reality CXXXII. is satisfied also: together with this other equality,

$$\text{CXXXVIII.} \dots R^2 \sigma'^2 + g^2 < 0,$$

which then results from the form CXXXVI.

(85.) The equations CXXXI., CXXXIV., and the general formula 889, IV., give the expressions,

$$\text{CXXXIX.} \dots \frac{rr'}{RK} = \frac{g\sigma'^2 + RK'S\sigma'\sigma''}{-\sigma'^4}; \quad \text{CXL.} \dots r_1^{-2} = \frac{(V\sigma'\sigma'')^2}{\sigma'^6};$$

where r is still the radius of the circle of contact of the sphere with its envelope, and r_1 is the radius of curvature of the locus of the centre s of the same variable sphere; whence it is easy to infer, that the condition CXXXV. may be reduced to the following very simple form (comp. XXXVI'. and XLII.):

$$\text{CXLI.} \dots (r'r_1)^2 = (RK)^2; \quad \text{or} \quad \text{CXLI'}. \dots r_1 dr = \pm R dR;$$

the independent variable being still arbitrary.

(86.) If the arc of the curve (s) be taken as that variable t , the form CXXXVI. of the same condition is easily reduced to the following,

$$\text{CXLII.} \dots R^2 = (RK)^2 + g^2 r_1^2, \quad \text{with} \quad \text{CXLIII.} \dots g = 1 - (RK)';$$

derivating then, and dividing by $2g$, we have this new differential equation, which is of linear form with respect to RK' , whereas the condition itself may be considered as a differential equation of the second degree, as well as of the second order,*

CXLIV. $\dots RK' = r_1(g r_1)'$; or CXLV. $\dots r_1^2 u'' + r_1 r_1' (u' - 1) + u = 0$,
if CXLVI. $\dots u = RK' = R D_t R$, and therefore CXLVII. $\dots u^2 = R^2 - r^2$,
by CXXXI. or CXXXI', because we have now,

$$\text{CXLVIII.} \dots \sigma'^2 = -1, \quad \text{or} \quad T\sigma' = 1, \quad \text{or} \quad dt = T d\sigma;$$

so that the new scalar variable, RK' , or u , with respect to which the linear equation CXLIV. or CXLV. is only of the second order, represents the perpendicular height† of the centre s of the sphere, above the plane of the circle, considered as a function of the arc (t) of the curve (s), and as positive when the radius R of the sphere increases, for positive motion along that curve, or for an increasing value of its arc.

(87.) If the curve (s) be given, or even if we only know the law according to which its radius of curvature (r_1) depends on its arc (t), the coefficients of the linear equation CXLV. are known; and if we succeed in integrating that equation, so as to

* We shall soon assign the complete integral of the differential equation in quaternions (84.), and also that of the corresponding Equation of Monge, cited in the preceding Note.

† It will be found that this new scalar u , if we abstract from sign, corresponds precisely to the p of earlier sub-articles, although presenting itself in a different connexion: for the sphere (78.), and the circle (79.), under the condition (84.), will soon be shown to be the osculating sphere and circle to the recent curve (P), or to the singular line (84.) upon the surface at present considered, that is, on the locus or envelope (80.).

find an expression for the *perpendicular* w as a function of that *arc* t , we shall then be able to express also, as functions of the *same arc*, the *radii* R and r of the *sphere* and *circle*, by the formulae,

CXLIX. . . $\pm r = gr_1 = r_1(1 - u')$, and CL. . . $R^2 = 2 \int u dt = u^2 + r_1^2(1 - u')^2$; the *third scalar constant*, which the integral $2 \int u dt$ would otherwise introduce into the expression for R^2 , being in this manner *determined*, by means of the *other two*, which arise from the integration of the *equation* above mentioned.

(88.) For example, it may happen that the locus of the centre s of the sphere has a *constant curvature*, or that $r_1 = \text{const.}$; and then the *complete integral* of the linear equation CXLV. is at once seen to be of the form,

$$\text{CLI. . . } u = a \sin(r_1^{-1}t + b),$$

a and b being *two arbitrary* (but *scalar*) constants; after which we may write, by (87.),

$$\text{CLII. . . } \pm r = r_1 - a \cos(r_1^{-1}t + b); \quad \text{CLIII. . . } R^2 = r_1^2 - 2ar_1 \cos(r_1^{-1}t + b) + a^2;$$

so that, in *this case*, both the *radii*, r and R , of *circle* and *sphere*, are *periodical functions* of the *arc* of the curve (s).

(89.) In general, if that *curve* (s) be *completely given*, so that the *vector* σ is a *known function* of a *scalar variable*, and if an expression have been *found* (or *given*) for the *scalar* R which satisfies any one of the forms of the *condition* (84.), we can then determine *also* the *vector* ρ , by the formula CXXXIII., as a function of the *same variable*; and so can assign the *point* P of the *singular line* (84.), which *corresponds* to any *given position* of the centre s of the *sphere*. For this purpose we have, when the *arc* of the curve (s) is taken, as in (86.), for the independent variable t , the formula,

$$\text{CLIV. . . } \rho = \sigma - u\sigma' - (1 - u')\sigma''^{-1} = \kappa_1 - u\sigma' - r_1^2 u' \sigma'',$$

if κ_1 be the *vector* of the centre, say κ_1 , of the *osculating circle* at s to that *given curve*, so that (comp. 389, XI.) it has the value,

$$\text{CLV. . . } \sigma \kappa_1 = \kappa_1 = \sigma - \sigma''^{-1} = \sigma + r_1^2 \sigma'', \quad \text{with CLV'. . . } \sigma''^2 + r_1^{-2} = 0.$$

If then we denote by v the *distance* of the point P from *this* centre κ_1 , and attend to the *linear equation* CXLV., we see that

$$\text{CLVI. . . } v = \overline{\kappa_1 P} = T(\rho - \kappa_1) = \sqrt{(u^2 + r_1^2 u'^2)},$$

and $\text{CLVI'. . . } v\sigma' = r_1 r_1' u_1'$, with $T\sigma' = 1$;

or more generally, $\text{CLVII. . . } v\sigma' s_1' = r_1 r_1' u_1'$,

if $\text{CLVII'. . . } u = RR's_1'^{-1}$, and $\text{CLVII''. . . } s_1 = \int Td\sigma$,

while $\text{CLVI''. . . } v^2 = u^2 + r_1^2 u'^2 s_1'^{-2}$;

so that s_1 denotes the *arc* of the *curve* (s), when the independent variable t is again left arbitrary. *This distance*, v , is therefore *constant* ($= a$) in the *case* (88.), namely when the *radius of curvature* r_1 of that curve is *itself* a constant quantity.

(90.) When $s_1' = T\sigma' = 1$, as in CXLVIII., the part $\sigma - u\sigma'$ of the first expression CLIV. for ρ becomes κ , by CXXXI. and CXLVI.; attending then to CLV., we have the *scalar quotient*,

$$\text{CLVIII. . . } \frac{\kappa - \rho}{\sigma - \kappa_1} = 1 - u';$$

whence generally,

$$\text{CLVIII.} \dots \frac{\kappa - \rho}{\sigma - \kappa_1} = 1 - \frac{1}{s_1'} \left(\frac{RR'}{s_1'} \right)' = 1 - \left(\frac{d}{ds_1} \right)^2 \left(\frac{R^2}{2} \right),$$

the independent variable t being again arbitrary. Accordingly, if we combine the general expression CXXXIII. for ρ , with the expression CXXXI'. for κ , and with the following for κ_1 (comp. 389, IV.),

$$\text{CLIX.} \dots \kappa_1 = \sigma + \frac{\sigma^2}{\sqrt{\sigma'\sigma'}}, \text{ for an arbitrary scalar variable,}$$

we easily deduce this new form of the scalar quotient,

$$\text{CLIX.} \dots \frac{\kappa - \rho}{\sigma - \kappa_1} = 1 + ((RR')' - RR'S\sigma^{-1}\sigma'')\sigma'^{-2};$$

which agrees with CLVIII', because $-\sigma'^2 = s_1'^2$, and $S \frac{\sigma''}{\sigma'} = \frac{s_1''}{s_1'}$.

(91.) It has then been fully shown, how to determine the vector ρ as a function of the scalar t , when σ and R are two known functions of that variable, which satisfy any one of the forms of the condition (84.). It must then be possible to determine also the derived vectors, ρ' , ρ'' , &c., as functions of the same variable; and accordingly this can be done, by derivating any three of the four scalar equations, LXXXIX. XCI. XCII. XCIII., of which that condition (84.) expresses the coexistence. Now if we derivate a first time the two first of these, and then reduce by the second and fourth, we get the equations,

$$\text{CLX.} \dots S\rho'(\sigma - \rho) = 0, \quad S\rho'\sigma' = 0, \quad \text{whence} \quad \text{CLX'.} \dots \rho' \parallel V\sigma'(\sigma - \rho);$$

and although this last formula only determines the direction of the tangent to the singular line at P , namely that of the common tangent at that point to two consecutive circles (84.), yet it enables us to infer, by the remaining equation XCII., that

$$\text{CLXI.} \dots \rho' \perp \sigma'', \quad \rho' \parallel V\sigma'\sigma'', \quad \text{and} \quad \text{CLXI'.} \dots S\rho'\sigma'' = 0;$$

reducing by which the derivative of XCIII., we find,

$$S\sigma'''(\sigma - \rho) + 3S\sigma'\sigma'' + (RR'')'' = 0, \quad \text{XCIV.},$$

the scalar variable being still arbitrary. And conversely, the system* of the four equations LXXXIX. XCI. XCIII. XCIV. gives the three equations CLX. CLXI', and so conducts to the equation XCII., and thence to the condition (84.); unless we suppose that ρ is a constant vector a , or that the variable sphere passes through a fixed point Δ , a case which we do not here consider, because in it the singular line (P) would reduce itself to that one point.

(92.) Derivating the two equations CLX., and reducing with the help of CLXI', we find these new equations,

$$\text{CLXII.} \dots S\rho''(\sigma - \rho) - \rho'^2 = 0, \quad S\rho''\sigma' = 0;$$

whence

$$\text{CLXIII.} \dots S\rho'''(\sigma - \rho) - 3S\rho'\rho'' = 0.$$

* In the language of infinitesimals, this system of equations expresses that four consecutive spheres intersect, in one common point P . When that point happens to be a fixed one, the condition (84.) requires that we should have the relation $S\sigma'\sigma''(\sigma - a) = 0$; or geometrically, that the curve (s) should be in a plane through the fixed point, which is then a singular point of the envelope.

We are led then, by *elimination of the derivatives of σ* , to the system of the three equations 895, VII. ; and we conclude, that *the point s is the centre*, and the *radius R is the radius, of the osculating sphere* to the singular line (\mathcal{F})*: whence it is easy to infer also, that the *plane of contact (79.) of the spheres with its envelope is the osculating plane*, and that the *circle of contact (80.) is the osculating circle (comp. (72.)), to the same curve (\mathcal{F}), at the point where two consecutive circles touch one another (84.)*.

(98.) *In general, and even without the condition (84.), the tangent to a branch (82.) of the curvilinear envelope of the circles of the system, at any point P_1 of that branch, has the direction represented by the vector $V\sigma'(\sigma - \rho_1)$, of the tangent to the circle at that point; but when that condition is satisfied, so that the two branches of the singular line coincide, the point P of that line is in the osculating plane (88.) to the curve (s): and then the equation XCII. shows that the tangent ρ' , or $V\sigma'(\sigma - \rho)$, to the line, is perpendicular to σ'' , or parallel to $V\sigma'\sigma''$ (comp. CLXI.), and therefore that the singular line crosses that plane at right angles.*

(94.) It follows that, with the condition (84.), the singular line (\mathcal{F}) is an *orthogonal trajectory to the system of osculating planes to the curve (s)*; and whereas, when this last curve is given, there ought to be *one* such trajectory for every point of a given osculating plane, this circumstance is analytically represented, in our recent calculations, by the *biordinal form of the differential equation CXLV.*, of which the *complete integral* must be conceived (87.) to involve generally, as in the case (88.), *two arbitrary constants*.

(95.) It follows also that, with the same condition of *coincidence of branches*, the *singular line (\mathcal{F}) must have the curve (s) for the cusp-edge of its polar developable*; or that the *sphere, with s for centre, and with R for radius, must be the osculating sphere to the curve (\mathcal{F})*, as otherwise found by calculation in (92.): while the *circle (80.) must be, as before, the osculating circle to that curve*.

(96.) Accordingly, *all equations, and inequalities, which have been stated in the recent sub-articles (79.), &c., respecting the envelope of a moving sphere with variable radius, under that condition (84.), and without any special selection of the independent variable, admit of being verified, by means of the earlier formulæ for the osculating circle and sphere to a curve (\mathcal{F}) treated as a given one, when the arc (s) of that curve is taken as such a variable*.

(97.) For example, we had lately the *two inequalities, $R^2 + \sigma'^2 < 0$, CXXXII., and $R^2\sigma'^2 + g^2 < 0$, CXXXVIII.* And accordingly the earlier sub-articles (22.), (23.) give, for those two combinations, the *essentially negative values*,

$$\text{CLXIV.} \dots R^2 + \sigma'^2 = -p^2 r^2 R'^2; \quad \text{CLXV.} \dots R^2 \sigma'^2 + g^2 = -((nr)')^2;$$

* In the language of infinitesimals (comp. the preceding Note), if every four consecutive spheres of a system intersect in one point of a curve, then each sphere passes through four consecutive points of that curve. Simple as this geometrical reasoning is, the writer is not aware that it has been anticipated; and indeed he is at present led to suppose that this whole theory, of the *Locus of the Osculating Circle*, as the *Envelope of the Osculating Sphere*, is new. Monge had however considered, but rejected (page 374 of Liouville's Edition), the case of a *system of circles having each a simple contact with a curve in space*.

in obtaining which last, the following transformations have been employed :

$$\text{CLXVI.} \dots \sigma'^2 = -\kappa^2 - \kappa^2 r^{-2}; \quad \text{CLXVII.} \dots g = -\kappa' p + \kappa r^{-1}.$$

(98.) As regards the verification of the equations, it may be sufficient to give one example; and we shall take for it the *last general form* CLVII. of the differential equation of condition (84.). For this purpose we may now write, by (22.) and (23.),

$$\text{CLXVIII.} \dots s_1' = \pm \kappa, \quad \kappa = \pm p, \quad \kappa' = \pm p', \quad r_1 s_1' s_1^{-1} = p' r_1 \kappa^{-1} = p' r;$$

and have only to observe that

$$\text{CLXIX.} \dots \frac{1}{2}(p^2 + p'^2 r^2) = p' r (r + p' r)', \quad \text{because } p = r' r.$$

(99.) If we denote by c_1, c_2, c_3 the first members of the equations XCI., XCIII., XCIV., then besides the equation LXXXIX., which may be regarded as a mere *definition* of the radius R , we have $c_1 = 0$ for the *whole* of the superficial locus or envelope (80.); but we have not *also* $c_2 = 0$, except for a point on one or other of the *two* (generally distinct) *branches* of the singular line (82.) upon that locus. And if, at any *other* and *ordinary point*, we cut the surface by a *plane* perpendicular to the circle at that point, we find, by a process of the same kind as some which have been already employed, expressions for the *tangential* and *normal components* of the vector chord, whereof the *principal terms* involve the scalar c_2 as a *factor*, while the latter varies (ultimately) as the *square* of the former, so that the *curvature of the section* is *finite* and known, but *tends* to become *infinite* when c_2 tends to zero.

(100.) If the *condition of coincidence* (84.) be *not satisfied*, so that the two branches of the singular line (82.) remain *distinct*, and that thus $c_2 = 0$, but *not* $c_3 = 0$ (comp. (91.)), for any ordinary point on *one* of those two branches, then if we cut the surface at that point by a *plane* perpendicular to the *branch*, or to the *circle* which *touches* it there, we find an ultimate expression for the vector chord which involves the scalar c_3 as a *factor*, and of which the *normal component* varies as the *sesquiquilate power* of the *tangential one*: so that we have here the case of a *semi-cubical cusp*, and each *branch* of the *singular line* is a *cusp-edge** of the surface, exactly in the *same known sense* (comp. (76.)) as that in which a *curve* of double curvature is *generally such*, on the *developable locus* of its *tangents*.

(101.) But when the condition (84.) is satisfied, so that the two branches *coincide*, and that thus (comp. again (91.)) we have at once the *three equations*,

$$\text{CLXX.} \dots c_1 = 0, \quad c_2 = 0, \quad c_3 = 0,$$

then the *terms*, which were lately the *principal ones* (100.), *disappear*: and a *new expression* arises, for the vector chord of a section of the surface, made by a *plane* perpendicular to the singular line, which (when we take $t = s$, as in (96.)) is found to admit of being identified with the formula CXVII., and of course conducts to precisely the same system of consequences; the *tangential component* *now* varying ultimately as the *cube*, and the *normal component* as the *fourth power* of a small variable, so that the *cuspidal property* of the point P of the *section* no longer exists although the *curvature* at that point is still *infinite*, as in (74.): and the *Singular Line*, reduced now to a *single branch*, to which all the *circles* of the system *osculate*,

* Compare the Note to page 602.

(92.), (95.), is not a cusp-edge of the Surface, as had been otherwise found before (77.), but a line of a different character,* which may thus be regarded, with reference to a more general Envelope (80.), as the result of a Fusion (84.) of Two Cusp-Edges.

(102.) The condition of such fusion (or coincidence) has been seen (84.) to be expressible by the differential equation of the second order, and second degree,

$$(RS\sigma'\sigma'' - R'g)^2 = (R^2 + \sigma'^2)(R^2\sigma'^2 + g^2), \quad \text{CXXXVI.}$$

with
$$g = -\sigma'^2 - (RR)', \quad \text{CXXXIV.}$$

and with the independent variable arbitrary. And we are now prepared to assign the complete general integral† of this differential equation; namely the system of the two following equations (comp. 895, (7.) and (14.)), of the vector and scalar kinds,

$$\text{CLXXI.} \dots \sigma = \rho + \frac{3V\rho'\rho''S\rho'\rho'' + V\rho'''\rho'^3}{8\rho'\rho''\rho'''}, \quad \text{and} \quad \text{CLXXII.} \dots R = T(\sigma - \rho);$$

in which ρ is an arbitrary vector function of any scalar variable, t , and which express, when geometrically interpreted, that σ is the variable vector of the centre s , and that R is the variable radius, of the osculating sphere to an arbitrary curve (\mathcal{F}), of which the variable vector of a point \mathcal{F} is ρ .

(103.) In fact, if we met the cited equation of condition CXXXVI., g representing therein the expression CXXXIV., without any previous knowledge of its meaning or origin, we might first, by the rules of quaternions, and as a mere affair of calculation, transform it to the equation CXXXV.; which would evidently allow the assumption of the formula CXXXIII., ρ being treated as an auxiliary vector, which satisfies (in virtue of the supposed condition) the system of the four scalar equations, LXXXIX., XCI., XCII., XCIII.; whence derivating and combining, as in (91.) and (92.), we are led to a new system‡ of four scalar equations, whereof one

* Compare the Note to page 602. Monge (in page 372 of Liouville's Edition) has the remark, that (when a certain radical vanishes) "les deux branches de la courbe touchée par toutes les caractéristiques se confondent en une seule: et cette courbe, sans cesser d'être une ligne singulière de la surface, n'est plus une arête de rebroussement, elle est une ligne de striction." The propriety of this last name, "line of striction," appears to the present writer questionable: although he has confirmed, as above, by calculations with quaternions, the result that, in the case referred to, the singular line is not a cusp-edge. Monge does not seem to have perceived that, in the same case of fusion, the curved line in question is not merely touched, but osculated, by all the circles of the system.

† Compare the first Note to page 604. We say here, general integral, because a less general one, although involving one arbitrary function (of the scalar kind), will soon be pointed out.

‡ The Equation of Monge (comp. the second Note to page 603) may be considered as the condition of coexistence of the four following equations, in which ϕ , ψ , π are supposed to be functions of α , and to be differentiated or derivated as such:

is again the equation LXXXIX., and may be written under the form CLXXII.; while the *three others* are those formerly numbered as 395, VII., and conduct (except in a particular case which we shall presently consider) to the *vector expression* CLXXI., which conversely is *sufficient* to represent them, all *derivatives of σ* and of *R* being thus *eliminated*.

(104.) The case just now alluded to, in which the *general integral* (102.) is replaced by a *less general form*, is the case (91.) when the *variable sphere* passes through a *fixed point A*, to which *point*, in that case, the *singular line* reduces itself. And the *integral equations*,* which then replace CLXXI. and CLXXII., may be thus written:

$$\text{CLXXIII.} \dots \sigma = \alpha + t\beta + u\gamma, \text{ with } u = F(t), \text{ and CLXXIV.} \dots R = T(t\beta + u\gamma);$$

$$(1) \dots (x - \phi)^2 + (y - \psi)^2 + (z - \pi)^2 = \alpha^2;$$

$$(2) \dots (x - \phi)\phi' + (y - \psi)\psi' + (z - \pi)\pi' + \alpha = 0;$$

$$(3) \dots (x - \phi)\phi'' + (y - \psi)\psi'' + (z - \pi)\pi'' + 1 - \phi'^2 - \psi'^2 - \pi'^2 = 0;$$

$$(4) \dots (x - \phi)(\psi'\pi'' - \pi'\psi'') + (y - \psi)(\pi'\phi'' - \phi'\pi'') + (z - \pi)(\phi'\psi'' - \psi'\phi'') = 0;$$

whereof the first *three* have been employed by Monge himself, but the *fourth* does not seem to have been perceived by him, the condition of *evanescence of a radical* having been used in its stead. And by a *translation* of quaternion results, above deduced, into the usual language of analysis, it is found that the *complete and general integral*, of the *non-linear differential equation* of the *second order*, which is obtained by the elimination of *x, y, z* between these four, is expressed by a *new system* of four equations, the equation (1) being *one* of them; and the *three others*, in which *x, y, z* are now treated as *arbitrary functions* of *a*, and are derivated as such, being the following:

$$(5) \dots (x - \phi)x' + (y - \psi)y' + (z - \pi)z' = 0;$$

$$(6) \dots (x - \phi)x'' + (y - \psi)y'' + (z - \pi)z'' + x'^2 + y'^2 + z'^2 = 0;$$

$$(7) \dots (x - \phi)x''' + (y - \psi)y''' + (z - \pi)z''' + 3(x'x'' + y'y'' + z'z'') = 0.$$

By treating *a* as a function of some *other* independent variable, *t*, the terms *+ a* and *+ 1*, in (2) and (3), come to be replaced by *+ aa'* and *+ aa'' + a^2*; and the *slightly more general form*, which Monge's Equation thus assumes, has *still* its *complete general integral* assigned by the system (1) (5) (6) (7), if *x, y, z* (as well as *a*) be now regarded as arbitrary functions of the *new variable t*, in the place of which it is permitted (for instance) to take *x*, and so to write *x' = 1, x'' = 0*: only *two arbitrary functions* thus entering, in the last analysis, into the *general solution*, as was to be expected from the form of the equation.

* The *particular integral* corresponding, of the *Equation of Monge*, is expressed by the following system:

$$\phi = a + et + lu, \quad \psi = b + ft + mu, \quad \pi = c + gt + nu, \\ (et + lu)^2 + (ft + mu)^2 + (gt + nu)^2 = a^2;$$

abcdelfylms being *nine arbitrary constants*, while *t* and *u* are *two functions* of *a*, whereof *one* is *arbitrary*, but the *other* is algebraically deduced from it, by means of the fourth equation. The writer is not aware that either of these integrals has been assigned before.

the *second scalar coefficient*, u , being here an *arbitrary function* of the *first scalar coefficient*, or of the independent variable t , and α, β, γ being *three arbitrary but constant vectors*: so that the curve (s) is now obliged to lie in *some one plane** through the *fixed point A*, but remains in other respects *arbitrary*. Accordingly it will be found that this *last integral system*, although *less general* than the former system (102.), and not properly *included* in it, *satisfies* the differential equation CXXXVI.; whereof the two members acquire, by the substitutions indicated, this *common value*,

$$\text{CLXXV.} \dots (RS\sigma\sigma'' - R'g)^2 = \&c. = R^{-2}t^2(tu' - u)^2 u''^2 (\nabla\beta\gamma)^4.$$

(105.) Other problems might be proposed and resolved, with the help of formulæ† already given, respecting the properties or affections of curves in space which depend on the *fourth power* (s^4) of the *arc*, or on the *fourth derivative* $D_s^4\rho$ or τ'''' of the vector ρ_s ; but it is time to conclude this series of sub-articles, which has extended to a much greater length than was designed, by observing that, in virtue of the vector form 896, XI. for the equation of a *circle of curvature*, the *Locus* (8.) of the *Osculating Circle* may be concisely but sufficiently represented by the *Vector Equation*,

$$\text{CLXXVI.} \dots \nabla \frac{2r_s}{\omega - \rho_s} + \nu_s = 0,$$

* Compare the Note to page 606.

† We might for example employ the formula VI. for κ'' , in conjunction with one of the expressions 897, XCI. for κ' , to determine, by the general formula 889, IV., the vector (say ξ) of the *centre of curvature* of the curve (κ), and therefore also the *radius of curvature* of that curve, which is the *locus of the centres of curvature* of the *given curve* (ρ), supposed to be in *general one of double curvature*. After a few reductions, with the help of XII., we should thus find the equations,

$$\text{CLXXVII.} \dots \nabla \frac{\kappa''}{\kappa'} = \frac{-r'\tau}{r\kappa} + (r^{-1} - P')\tau,$$

$$\text{CLXXVIII.} \dots \xi = \kappa + \frac{\kappa'}{\nabla \frac{\kappa''}{\kappa'}} = \kappa + \frac{\sigma - 2\kappa + \rho}{1 - \frac{rdP}{ds} + \frac{pd_s}{rd\kappa}},$$

in which last the denominator is a quaternion, and the scalar variable is arbitrary: whence also,

CLXXIX. . . *Radius of curvature of curve* (κ),
or of *locus of centres of osculating circles to a given curve* (ρ) *in space*,

$$\begin{aligned} &= T(\xi - \kappa) = R \left\{ \left(1 - \frac{rdP}{ds} \right)^2 + \left(\frac{pr}{Rr} \right)^2 \right\}^{-\frac{1}{2}} \\ &= \pm \frac{Rdr}{pd_s} \left\{ \left(\frac{1}{r} - \frac{dP}{ds} \right)^2 + \left(\frac{p}{Rr} \right)^2 \right\}^{-\frac{1}{2}}; \end{aligned}$$

with the verification, that for the case of a *plane curve* (ρ), for which therefore

$\frac{R}{p} = 1$, and $\frac{1}{r} = 0 = \frac{dP}{ds}$, we have thus the elementary expression,

$$\text{CLXXX.} \dots \text{Radius of Curvature of Plane Evolute} = \pm \frac{rd_r}{ds},$$

r being still the radius of curvature, and s the *arc*, of the *given curve*.

which apparently involves only one scalar variable, s , namely, the arc of the curve (\mathcal{P}), the other scalar variable, such as t , which corresponds (69.) to the arc of the circle, disappearing under the sign ∇ : and that the surface, which was called in (8.) the *Circumscribed Developable*, is now seen to be in fact circumscribed to that *Locus*, or *Envelope*, in a certain singular (or eminent) sense, as touching it along its *Singular Line*.

399. When we take account of the fifth power (s^5) of the arc, the expression for ρ , receives a new term, and becomes (comp. 398, I.),

$$I. \dots \rho_s = \rho + s\tau + \frac{1}{2}s^2\tau' + \frac{1}{6}s^3\tau'' + \frac{1}{24}s^4\tau''' + \frac{1}{120}s^5\tau^{IV};$$

and although some of the consequences of such an expression have been already considered, especially as regards the general determination of what has been above called the *Osculating Twisted Cubic* to a curve of double curvature, or the *gauche curve of the third degree* which has contact of the fifth order with a given curve in space, yet, without repeating any calculations already made, some additional light may be thrown on the subject as follows.

(1.) As regards the successive deduction of the derived vectors in the formula I., it may be remarked that if we write (comp. 398, LVI., LXL),

$$II. \dots D^{n+1}\rho = r^{(n)} = a_n\tau + b_n\tau' + c_n\tau'',$$

we shall have, generally,

$$III. \dots a_{n+1} = a'_n - r^{-1}b_n, \quad b_{n+1} = b'_n + r^{-1}a_n - r^{-1}c_n, \quad c_{n+1} = c'_n + r^{-1}b_n,$$

with the initial values,

$$IV. \dots a_0 = 1, \quad b_0 = 0, \quad c_0 = 0, \quad \text{or} \quad IV'. \dots a_1 = 0, \quad b_1 = r^{-1}, \quad c_1 = 0;$$

whence $V. \dots \begin{cases} a_2 = -r^{-2}, & b_2 = (r^{-1})', & c_2 = r^{-1}r^{-1}, \\ a_3 = 3r^{-3}r', & b_3 = (r^{-1})'' - r^{-2} - r^{-1}r^{-2}, & c_3 = r(r^{-2}r^{-1})', \end{cases}$

as in the expressions 397, VI. for r'' , and 398, IV. for r''' ; the corresponding coefficients of r^{IV} being in like manner found to be,

$$VI. \dots \begin{cases} a_4 = -2(r^{-2})'' + ((r^{-1})')^2 + r^{-2}(r^{-2} + r^{-2}); \\ b_4 = (r^{-1})''' - 2(r^{-2})' - 3(r^{-1}r^{-1})'r^{-1}; \\ c_4 = r^{-1}(r^{-1})'' + 3((r^{-1})'r^{-1})' - r^{-1}r^{-1}(r^{-2} + r^{-2}); \end{cases}$$

and being sufficient for the investigation of all affections or properties of a curve in space, which depend only on the fifth power of the arc s .

(2.) For the *helix* the two curvatures are constant, so that all the derivatives of the two radii r and r vanish; the expressions become therefore greatly simplified, and a law is easily perceived, allowing us to sum the infinite series for ρ_s , and so to obtain the following rigorous expressions for the co-ordinates* x_n, y_n, z_n , of this

* We have here, and in this whole investigation, an instance of the facility with which quaternions can be combined with co-ordinates, whenever the geometrical as-

particular curve, instead of those which were developed generally in 898, LVIII., but only as far as s^4 inclusive:

$$\text{VII.} \dots x_s = l^2(r^2t + r^2 \sin t); \quad y_s = l^2r^{-1} \text{vers } t; \quad z_s = l^2r^{-1}r^{-1}(t - \sin t);$$

where l and t are an auxiliary constant and variable, namely,

$$\text{VIII.} \dots l = (r^2 + r'^2)^{-\frac{1}{2}} = r \sin H, \quad t = l^{-1}s,$$

l being thus what was denoted in earlier formulæ by $T\lambda^{-1}$, and t being the angle between two axial planes; while the origin is still placed at the point P of the curve, and the tangent, normal, and binormal are still made the axes of xyz .

(8.) The cone of the second order, 898, (40.), which has generally a contact of the fifth order with a proposed curve in space, at a point P taken for vertex, has in this case of the helix the equation (comp. 898, LVII.* and LXIX.),

$$\text{IX.} \dots y^2 = \frac{8r}{2r} \left\{ x + \left(\frac{8r}{10r} - \frac{7r}{10r} \right) z \right\} z.$$

Accordingly it can be shown, by elementary methods, that if we write, for a moment,

$$\text{X.} \dots f(t) = 8(t - \sin t)(8t + 7 \sin t) - 20 \text{vers}^2 t,$$

we have the eight evanescent values,

$$\text{XI.} \dots f_0 = f'_0 = f''_0 = f'''_0 = f^{IV}_0 = f^{V}_0 = f^{VI}_0 = f^{VII}_0 = 0;$$

whence it is easy to infer that this cone IX. has (in the present example, although not generally) a contact as high as the sixth order† with the curve, of which the co-ordinates have here the expressions VII.; and consequently that the cone in question must wholly contain the osculating twisted cubic to that curve.

ture of a question may render it convenient so to combine them, by offering to our notice any obvious planes of reference. If it be thought useful to pass to a system connected more immediately with the right cylinder than with the helix, we may write,

$$\text{VII.} \dots \begin{cases} x_s = l(r^{-1}x_s - r^{-1}z_s) = l^2r^{-1} \sin t, \\ y_s = l^2r^{-1} - y_s = l^2r^{-1} \cos t, \\ z_s = l(r^{-1}x_s + r^{-1}z_s) = l^2r^{-1}t, \end{cases}$$

where $l^2r^{-1} = r \sin^2 H$ is the radius of the cylinder, with converse formulæ easily assigned.

* In the corresponding equation 898, LXVII., the coefficient of $6ac$ ought to have been printed as $\left(\frac{r}{r'}\right)^3$, like the coefficient of $6xz$ in the equation LVII.

† Or in modern language, *seven-point contact*, in the sense that the cone passes, in this case, through seven consecutive points of the curve. It may be remarked that the gauche curve of the fourth degree, or the quartic curve, in which this cone cuts the cylinder of revolution whereon the helix is traced (cutting also in it a certain other cylinder of the second order), and which has the point P for a double point, crosses the helix by one of its two branches at that point, while it has seven-point contact with the same helix by its other branch; and that thus the fact of calculation, expressed by the formula XI., is geometrically accounted for.

(4.) In general, to find a *second locus* for such a *cubic curve*, the method of recent sub-articles (398, (88.) &c.) leads us to form the equation (398, L.XVI.) of a *cylinder of the second order*, or briefly of a *quadric* cylinder*, which like the *quadric cone* (8.) shall have *contact of the fifth order* with the proposed curve in space, at the given point P; the ratios of *abc*, which determine the direction of a generating line PE, being obliged for this purpose to satisfy a certain *equation of condition* (398, L.XVIII.), of which the form indicates that the locus of this line PE is generally a certain *cubic cone*, having the tangent (say PT) to the curve for a *nodal side*: along which side it is touched, not only (like the quadric cone) by the *osculating plane* ($z = 0$) to that given curve, but also by a *second plane*, whereof the equation ($gy + hx = 0$, or after reductions $y - \frac{1}{3}r'x = 0$) shows that the *second branch* of the cubic cone crosses the *first branch*, or the quadric cone, or the osculating plane to the curve, at an angle of which the trigonometric *cotangent* is equal to half the differential of the radius (r) of *second curvature*, divided by the differential of the arc (s); so that this *second tangent plane* to the cone coincides with the *rectifying plane* to the curve, when the second curvature happens to be constant. The tangent PT therefore counts as three of the six common sides of the two cones with P for vertex: and the three other common sides, for the assigning of which it has been shown (in 398, (41.)) how to form a cubic equation in $b : c$, are the parallels from that point P to the three real or imaginary asymptotes† of the twisted cubic, and are generating lines PE of three quadric cylinders, whereof one at least is necessarily real, and contains, as a second locus, that sought *osculating gauche curve* of the third degree.

(5.) In applying this general method to the case of the helix, it is found that the cubic cone breaks up, in this example, into a system of a new quadric cone, which touches the former quadric cone IX. along the tangent PT to the curve (the two other common sides of these two cones being imaginary), and of a plane ($y = 0$), namely the rectifying plane (comp. (4.)) of the helix, or the tangent plane to the cylinder of revolution on which that given curve is traced: and that this last plane cuts the first quadric cone in two real right lines, the tangent being again one of them, and the other having the sought direction of a real asymptote to the sought *osculating twisted cubic*. Without entering here into details of calculation, the resulting equation of the real‡ quadric cylinder, on which that sought gauche curve is situated, may be at once stated to be (with the present system of co-ordinates),

* So called by Dr. Salmon, in his Treatise already cited. Compare the first Note to page 591 of these Elements.

† Compare again the Note last referred to.

‡ As regards the two imaginary quadric cylinders, their equations can be formed by the same general method, employing as generating lines the two imaginary common sides (5.) of the cone IX., and of that other quadric cone above referred to, which is here a separable part of the general cubic locus, and has for equation,

$$\text{IX'.} \dots \frac{20}{9}y^2 = 5\frac{r}{r'}xz + \left(8\frac{r^3}{r'^3} - 2\right)x^2.$$

It seems sufficient here to remark, that by taking the sum and difference of the equations of those two imaginary cylinders, two new real quadric surfaces are obtained.

$$\text{XII.} \dots 2ry = \left\{ x + \left(\frac{8r}{10r} - \frac{7r}{10r} \right) z \right\}^2 + \frac{8}{5} \left(1 + \frac{r^2}{r^2} \right) y^2;$$

in such a manner that if we set aside the right line,

$$\text{XIII.} \dots y = 0, \quad x + \left(\frac{8r}{10r} - \frac{7r}{10r} \right) z = 0,$$

which is a common side of the cone IX. and of the cylinder XII., the curve, which is the remaining part of their complete intersection, is the twisted cubic sought. As an elementary verification of the fact, that this gauche curve of intersection IX. XII. has contact of the fifth order with the helix at the point P, it may be observed that if we change the co-ordinates xyz in XII. to the expressions VII., and write for abridgment,

$$\text{XIV.} \dots F(t) = (8t + 7t \sin t)^2 - 200 \text{ vers } t + 60 \text{ vers}^2 t,$$

we have then (comp. X. XI.) the six evanescent values,

$$\text{XV.} \dots F0 = F'0 = F''0 = F'''0 = F^{IV}0 = F^V0 = 0.$$

(6.) As another verification, which is at the same time a sufficient proof, of the *a posteriori* kind, that the gauche curve IX. XII. has in fact contact of the fifth order with the helix, it can be shown that while the co-ordinates y_s and z_s of the latter may (by VII., writing simply x for x_s , and neglecting x^7) be thus developed,

$$\text{XVI.} \dots \begin{cases} y_s = \frac{x^2}{2r} + \frac{x^4}{24r} \left(\frac{8}{r^2} - \frac{1}{r^2} \right) + \frac{x^6}{720r} \left(\frac{45}{r^4} - \frac{24}{r^2 r^2} + \frac{1}{r^4} \right), \\ z_s = \frac{x^3}{6rt} + \frac{x^5}{120rt} \left(\frac{9}{r^2} - \frac{1}{r^2} \right), \end{cases}$$

the corresponding co-ordinates y and z of the former, that is, of the curvilinear part of the intersection of the cone IX. with the cylinder XII., have (in the same order of approximation) developments which may be thus abridged,

$$\text{XVII.} \dots y = y_s - \frac{(r^2 + r^2)^2 x^6}{800r}, \quad z = z_s.$$

(7.) The deviation of the helix from the gauche curve IX. XII. is therefore of the sixth order (with respect to x , or s), and it has an inward direction, or in other words, the osculating twisted cubic deviates outwardly from the helix, with respect to the right cylinder; the ultimate (or initial) amount of this deviation, or the law according to which it tends to vary, being represented by the formula,

$$\text{XVII'.} \dots y_s - y = \frac{(r^2 + r^2)^2 s^6}{800r} = \frac{t^6 y_s}{400};$$

which also contains the osculating twisted cubic, and intersect each other in that gauche curve: namely two hyperbolic paraboloids, which have a common side at infinity, and of which the equations can be otherwise deduced (by way of verification), without imaginaries, through easy algebraical combinations of the two real equations IX. and XII.

where t denotes as in (2.) the *angle*, which a *plane* drawn through a near point P , and through the axis of the *right cylinder*,*

$$\text{XVIII.} \dots 2ry = \left(x - \frac{r}{r}z\right)^2 + \left(1 + \frac{r^2}{r^2}\right)y^2,$$

whereon the helix is traced, makes with the plane drawn through the same *axis* of *revolution*, or through the right line,

$$\text{XIX.} \dots x = \frac{r}{r}z, \quad y = r^{-1}(r_0^2 + r^2)^{-1} = t^2 r^{-1},$$

and through the given point P : while y , is still the (inward) distance of the same near point P , from the tangent plane to the same cylinder at the same given point P .

(8.) If we cut the cone IX., and the cylinder XII., by any plane,

$$\text{XX.} \dots 2ry = w \left\{ x + \left(\frac{8r}{10r} - \frac{7r}{10r} \right) z \right\},$$

drawn through their common side XIII., we obtain two other sides, one for each of these two quadric surfaces; and these two new right lines, in this plane XX., intersect each other in a new point, † of which the co-ordinates xyz are given, as functions of the new variable w , by the *three fractional expressions*, ‡

$$\text{XXI.} \dots x = \frac{w + \left(\frac{7}{r^2} - \frac{8}{r^2} \right) \frac{w^3}{60}}{1 + \frac{8}{20} \frac{w^2}{r^2}}; \quad 2ry = \frac{w^3}{1 + \frac{8}{20} \frac{w^2}{r^2}}; \quad 6rtz = \frac{w^3}{1 + \frac{8}{20} \frac{w^2}{r^2}};$$

while the *twisted cubic*, which *osculates* (as above) to the helix at P , is the *locus* of all the *points of intersection* thus determined. Accordingly, if we develop xyz by XXI., in ascending powers of w , neglecting w^7 (or x^7), we are conducted, by elimination of w , to expressions for y and z in terms of x , which agree with those found in (6.), and thereby establish in a new way the existence of the required contact of the *fifth order*, between the two curves of double curvature.

* With the co-ordinates VII'. of a recent Note (to page 612), the equation of this cylinder would be,

$$\text{XVIII}'. \dots x^2 + y^2 = t^4 r^2.$$

† The plane XX., as containing the line XIII., is parallel to an asymptote, and therefore meets the cubic at infinity; it also passes through the given point P : and therefore it can only cut the twisted cubic in *one other point*, of which the position is expressed by the equations XXI.

‡ *Quaternions* suggest such fractional expressions, through the formula 398, LXXIX. for the vector $(\phi + c)^{-1}a$; but it is proper to state that expressions of *fractional form*, for the co-ordinates of a curve in *space* of the *third order* (or degree) were given by Möbius, who appears to have been the first to discover the existence of such *gauche* curves, and who published several of their principal properties in his *Barycentric Calculus* (der barycentrische Calcul, Leipzig, 1827). Compare the Notes to pages 23 and 85, and Note B at the end of these Elements.

(9.) The *real asymptote* to the cubic curve is found by supposing the auxiliary variable w to tend to infinity in the expressions XXI.; it is therefore the right line (comp. XX.),

$$\text{XXII.} \dots y = \frac{10r^2}{3r}, \quad x + \left(\frac{8r}{10r} - \frac{7r}{10r} \right) z = 0,$$

namely the *second side* in which the *elliptic cylinder* XII. is cut by a normal plane through the side XIII.; and by comparing the value of its y with the equation XIX., we see that the *least distance between the real asymptote to the osculating twisted cubic, and the axis of revolution of the cylinder on which the helix is traced, is equal to seven-thirds of the radius of that right cylinder.*

(10.) As regards the *two imaginary asymptotes*, they correspond to the two imaginary values of w , which cause the *common denominator* of the expressions XXI. to vanish; but it may be sufficient here to observe, that because those expressions give, generally,

$$\text{XXIII.} \dots x + \left(\frac{6r}{5r} + \frac{1r}{5r} \right) z = w,$$

the two imaginary lines in question are to be considered as being contained in *two imaginary planes*, which are both *parallel to the real plane** through P ,

$$\text{XXIV.} \dots x + \left(\frac{6r}{5r} + \frac{1r}{5r} \right) z = 0;$$

namely to a certain *common normal plane* to the *two real cylinders* XII. and XVIII., or to the *elliptic and right cylinders* already mentioned.

(11.) *In general*, instead of seeking to determine, as above, a *cylinder* of the second order, which shall have *contact of the fifth order* with *any given curve* of double curvature, at a given point P , we may propose to find a *second cone* of the same (second) order, which shall have *such contact* with that curve at that point, its *vertex* being at *some other point* of space (abc). Writing (comp. 398, LXVI.) the equation of such a *cone* under the form,

$$\text{XXV.} \dots 2r(cy - bz)(e - z) = (cx - az)^2 + 2B(cx - az)(cy - bz) + C(cy - bz)^2;$$

substituting for xyz the co-ordinates $x_3y_3z_3$ of the curve, under the forms (comp. 398, LVIII.),

$$\text{XXVI.} \dots \begin{cases} x_3 = s - \frac{s^3}{6r^2} + \frac{a_3s^4}{24} + \frac{a_4s^5}{120}, \\ y_3 = \frac{s^2}{2r} - \frac{r's^3}{6r^2} + \frac{b_3s^4}{24} + \frac{b_4s^5}{120}, \\ z_3 = \frac{s^2}{6r} + \frac{c_3s^4}{24} + \frac{c_4s^5}{120}, \end{cases}$$

in which the coefficients $a_3b_3c_3$ and $a_4b_4c_4$ have the values assigned in (1.); developing according to powers of s , neglecting s^6 , and comparing coefficients of s^3, s^4, s^5 ; we find first the expressions,

* The *right line at infinity*, in this plane XXIV., is the *common side* of the *two hyperbolic paraboloids* mentioned in the third Note to page 614, as each containing the whole twisted cubic.

$$\text{XXVII.} \dots B = -\frac{1}{3} \left(r' + \frac{b}{c} r \right), \quad C = -\frac{4}{9} \left(r' + \frac{b}{c} r \right)^2 + \frac{4}{8} \left(1 + \frac{ar}{c, r} \right) + \frac{r^2}{3} \left(b_3 - \frac{b}{c} c_3 \right),$$

which are the same as for cylinder: and then are led to the new equation of condition,

$$\text{XXVIII.} \dots \frac{r}{\delta} \left(b_4 - \frac{b}{c} c_4 \right) = a_3 - \frac{a}{c} c_3 + \frac{2}{crr} + B \left(b_3 - \frac{b}{c} c_3 - \frac{2}{r^2} - \frac{2a}{cr^2r} \right) - 2C \left(\frac{r'}{r^2} + \frac{b}{cr^2r} \right),$$

which differs from the corresponding equation for the determination of a cylinder having the same (fifth) order of contact with the curve, but only by the one term $\frac{2}{crr}$ in the second member, which term vanishes when the co-ordinate c of the vertex is infinite.

(12.) Eliminating B and C , and substituting for $a_3 b_3 c_3$ and $a_4 b_4 c_4$ their values V. and VI., we find that the condition XXVIII. may be thus expressed (comp. 398, LXVIII.):

$$\text{XXIX.} \dots ac \left(b - \frac{r'}{2} c \right) - rc^2 = ab^3 + b^2c + bc^2 + ec^3;$$

in which we have written, for abridgment,

$$\text{XXX.} \dots \begin{cases} a = \frac{4r}{9r}; & b = \frac{r'}{8} - \frac{r}{r^2}; \\ c = \frac{1}{80} (6r''r - 3rr'' - 2r^{-1}r'^2r - 6r'r' + 6rr^{-1}r'^2 - 18r^{-1}r + 12rr^{-1}); \\ e = \frac{1}{90} (9r''r^2 - 9r^{-1}r'r''r^2 + 4r^{-2}r'^2r^2 + 86r^{-2}r'r^2 + 18r' - 27rr^{-1}r'). \end{cases}$$

The locus of the vertex of the sought quadric cone XXV. is therefore that cubic surface, or surface of the third order, which is represented by the equation XXIX. in abc ; this surface, then, is a second locus (comp. (4.)) for the osculating twisted cubic, whatever the given curve in space may be: a first locus for that cubic curve being still the quadric cone (comp. (3.)), of which the equation in abc is (by 398, LXVII.* and LXIX.),

$$\text{XXXI.} \dots 4 \left(\frac{r}{r} \right) b^2 = 6 \left(\frac{r}{r} \right)_3 ac + \left(\frac{r^2}{r^2} \right)' bc + \frac{r^4}{5} \left(\frac{9}{r^4} - \frac{21}{r^2r^2} + \frac{r^2}{r^4} - \frac{3r''}{r^2} + \frac{3r'r'}{r^2r} - \frac{27r'^2}{4r^2r^2} + \frac{9r''}{r^2r} \right) c^2,$$

and which has contact of the fifth order with the curve, while its vertex is at the given point P of osculation.

* After making the correction indicated in a former Note (to page 613), so as to bring the cited equation into agreement with the earlier formula 398, LVII. The quadric cone XXXI. may be said to have five-side contact with the cone of chords of the given curve (compare the first Note to page 588).

(13.) Instead of thus introducing, as *data*, the *derivatives* of the two radii of curvature, r and r , taken with respect to the arc, s , it may be more convenient in many applications to treat the two co-ordinates y and z of the curve as functions of the third co-ordinate x , assumed as the independent variable: and so to write (comp. (6.)) these new developments,

$$\text{XXXII.} \dots y_x = \frac{x^3}{2r} + \frac{y''x^2}{6} + \frac{y''x^4}{24} + \frac{y''x^5}{120}, \quad z_x = \frac{x^3}{6r} + \frac{x''x^4}{24} + \frac{z''x^5}{120};$$

and then the equation of the quadric cone XXXI. will be found to become (in xyz),

$$\text{XXXIII.} \dots y^2 = \frac{8r}{2r}xz + 2gyz + hx^2,$$

with the coefficients,

$$\text{XXXIV.} \dots g = r\tau \left(y'' - \frac{8}{8} r^2 x'' \right), \quad h = \frac{8}{2} r^2 \left(y'' - \frac{8}{10} r^2 x'' \right) - r^2 r^2 \left(y''^2 + \frac{8}{4} r^2 x'' y'' - \frac{9}{16} r^2 x''^2 \right);$$

while the cubic surface XXIX. will also come to be represented by an equation of the *same form* as before, namely (in xyz) by the following,

$$\text{XXXV.} \dots xz(y + hz) - rz^2 = ay^3 + by^2z + cyz^2 + ez^3,$$

in which the coefficients are,

$$\text{XXXVI.} \dots \begin{cases} a = \frac{4r}{9r} \text{ (as before); } & b = -\frac{4}{8} r^2 y'' + \frac{r^2 r}{2} z''; & h = -r\tau y'' + \frac{1}{2} r^2 z''; \\ c = \frac{4}{9} r^2 y''^2 - \frac{1}{2} r^2 r^2 y'' z'' - \frac{1}{2} r^2 r y''^2 + \frac{1}{16} r^2 r^2 z''; \\ e = -\frac{4}{9} r^2 r^2 y''^2 + \frac{1}{2} r^2 r^2 y'' z'' - \frac{1}{16} r^2 r^2 z''^2. \end{cases}$$

(14.) Whichever set of expressions for the *coefficients* we may adopt, some general consequences may be drawn from the mere *forms* of the equations, XXXI. and XXIX., or XXXIII. and XXXV., of the *quadric cone* and *cubic surface*, considered as *two loci* (12.) of the *osculating twisted cubic* to a given curve of double curvature. Thus, if we eliminate ac (comp. 398, (41.)) from XXIX. by XXXI., or xz by XXXIII. from XXXV., we get an equation between b, c , or between y, z , which rises *no higher* than the *third degree*, and is of the *form*,

$$\text{XXXVII.} \dots 2rz^2 = ay^3 + by^2z + cyz^2 + ez^3,$$

with the same value of a as before; such then is the equation of the *projection of the twisted cubic, on the normal plane* to the curve; and we see that, as was to be expected, the *plane cubic* thus obtained has a *cusp* at the given point P , which (when we neglect s^2 or x^2) *coincides* with the *corresponding cusp** of the projection of the given curve of double curvature *itself*, on the same normal plane.

(15.) The equation XXXVII. may also be considered as representing a *cubic cylinder*, which is a *third locus* of the twisted cubic; and on which the *tangent* PR

* Compare the first formula of the first Note to page 594.

to the curve is a *cuspid-edge*, in such a manner that an arbitrary plane through this line, suppose the plane

$$\text{XXXVIII.} \dots 8rz = \nu y,$$

where ν is any assumed constant, cuts the cylinder in that line twice, and a third time in a real and parallel right line, which intersects the quadric cone in a point at infinity (because the tangent PT is a side of that cone), and in another real point, which is on the twisted cubic, and may be made to be any point of that sought curve, by a suitable value of ν : in fact, the plane XXXVIII. touches both curves at P , and therefore intersects the cubic curve in one other real point. And thus may fractional expressions (comp. (8.)) for the co-ordinates of the osculating cubic be found generally, which we shall not here delay to write down.

(16.) Without introducing the cubic cylinder XXXVII., it is easy to see that any plane, such as XXXVIII., which is tangential to the given curve at P , cuts the cubic surface XXXV. in a section which may be said to consist of the tangent twice taken, and of a certain other right line, which varies with the direction of this secant plane, so that the locus XXXV. or XXIX. is a Ruled Cubic Surface, with the given tangent PT for a singular* line, which is intersected by all the other right lines on that surface, determined as above: and if we set aside this line, the remaining part of the complete intersection of that cubic surface with the quadric cone XXXIII. or XXXI. is the twisted cubic sought. We may then consider ourselves to have completely and generally determined the Osculating Twisted Cubic to a curve of double curvature, without requiring (as in 398, (41.)), the solution of any cubic or other equation.†

(17.) As illustrations and verifications, it may be added that the general ruled cubic surface, and cubic cylinder, lately considered, take for the case of the helix (2.), the particular forms,‡

* If the cubic surface be cut by a plane perpendicular to the tangent PT , at any point τ distinct from the point P itself, the section is a plane cubic, which has τ for a double point; and this point counts for three of the six common points, or points of intersection, of the plane cubic just mentioned with the plane conic in which the quadric cone is cut by the same secant plane, because one branch, or one tangent, of the plane cubic at τ touches the plane conic at that point, in the osculating plane to the given curve at P , while the other branch, or the other tangent, cuts that plane conic there.

† It may be remarked that, by equating the second member of XXXVII. to zero, and changing y, z to b, c , we obtain generally the cubic equation, referred to in 398, (41.); and that by suppressing the term $-rc^2$ in XXIX., or the term $-r^2$ in XXXV., we pass, in like manner generally, from the cubic surface of recent sub-articles, to the earlier cubic cone (4.).

‡ By suppressing the term $-rz^2$, dividing by $\frac{ry}{br}$, and transposing, we pass for the case of the helix from the equation XXXIX. of the cubic locus, to the equation IX. in the last Note to page 614; namely to the equation of that quadric cone which forms (in this example) a separable part of the general cubic cone, the other part being here the tangent plane ($y = 0$) to the right cylinder.

$$\text{XXXIX.} \dots xyz - rz^2 = \frac{4r}{9r}y^3 + \left(\frac{2r}{5r} - \frac{8r}{6r}\right)yz^2,$$

and

$$\text{XL.} \dots rz^2 = \frac{2r}{9r}y^3 + \frac{8}{10}\left(\frac{r}{r} + \frac{r}{r}\right)yz^2;$$

and that accordingly these two last equations are satisfied, independently of w , when the fractional expressions XXI. are substituted for xyz .

400. The general theory* of *evolutes of curves in space* may be briefly treated by quaternions, as follows: a *second curve* (in space, or in one plane) being *defined* to bear to a *first curve* the relation of *evolute* to *involute*, when the *first cuts the tangents to the second at right angles*.

(1.) Let ρ and σ be corresponding vectors, $o\rho$ and $o\sigma$, of involute and evolute, and let ρ' , σ' , ρ'' , σ'' denote their first and second derivatives, taken with respect to a scalar variable t , on which they are both conceived to depend. Then the two fundamental equations, which express the relation between the two curves, as above defined, are the following:

$$\text{I.} \dots S(\sigma - \rho)\rho' = 0; \quad \text{II.} \dots V(\sigma - \rho)\sigma' = 0;$$

which express, respectively, that the point s is in the normal plane to the involute at P , and that the latter point is on the tangent to the evolute at s : so that the locus of P (the involute) is a *rectangular trajectory* to all such tangents to the locus of s (the evolute).

(2.) Eliminating $\sigma - \rho$ between the two preceding equations, and taking their derivatives, we find,

$$\text{III.} \dots S\rho'\sigma' = 0, \quad \text{IV.} \dots S(\sigma - \rho)\rho'' - \rho'^2 = 0, \quad \text{V.} \dots V(\sigma - \rho)\sigma'' - V\rho'\sigma' = 0;$$

whence also, $\text{VI.} \dots S\rho'\sigma'\sigma'' = 0.$

(3.) Interpreting these results, we see *first*, by IV. combined with I. (comp. 391, (5.)), that the point s of the evolute is on the polar axis of the involute at P , and therefore that the evolute itself is some curve on the polar developable of the involute; and *second*, by VI. (comp. 380, I.), that this curve is a *geodetic line* on that polar surface, because the osculating plane to the evolute at s contains the tangent to the involute at P , and therefore also the (parallel) normal to the locus of *evolutes*.

(4.) The locus of centres of curvature (395, (8.)) of a curve in space is not generally an *evolute* of that curve, because the tangents† $\kappa\kappa'$ to that locus do not generally intersect the curve at all; but a given plane involute has always the locus just

* Invented by Monge.

† It might have been remarked, in connexion with a recent series of sub-articles (397), that this tangent $\kappa\kappa'$ or κ' is inclined to the rectifying line λ , at an angle of which the cosine is,

$$-SU\kappa'\lambda = \pm R^{-1}T\lambda^{-1} = \pm \sin H \cos P;$$

upper or lower signs being taken, according as the second curvature r^{-1} is positive or negative, because $S\kappa\lambda = -r^{-1}$.

mentioned for one of its evolutes; and has, besides, indefinitely many others,* which are all *geodesics on the cylinder* which rests perpendicularly on that one *plane evolute* as its base.

(5.) An easy combination of the foregoing equations gives,

$$\text{VII.} \dots (T(\sigma - \rho))^2 = -S(U(\sigma - \rho) \cdot (\sigma' - \rho')) = \mp S\sigma'U\sigma' = \pm T\sigma',$$

or with differentials, VIII. $\dots dT(\sigma - \rho) = \pm Td\sigma$;

whence by an immediate integration (comp. 380, XXII. and 397, LIV.),

$$\text{IX.} \dots \Delta T(\sigma - \rho) = \pm \int Td\sigma = \pm \text{arc of the evolute} :$$

this arc then, between two points such as s and s_1 of the latter curve, is equal to the difference between the lengths of the two lines, Ps and P_1s_1 , intercepted between the two curves themselves.

(6.) Another quaternion combination of the same equations gives, after a few steps of reduction, the differential formula (comp. 385, VI.),

$$\text{X.} \dots d \cos ops = -dSU \frac{\sigma - \rho}{\rho} = \frac{dT\rho}{T(\sigma - \rho)} \cdot S \frac{\sigma}{\rho};$$

if then the involute be a curve on a given sphere, with its centre at the origin o , so that the evolute is a geodesic on a concentric cone, this differential X . vanishes, and we have the integrated equation,

$$\text{XI.} \dots \cos ops = \text{const.}, \text{ or simply, XI'.} \dots ops = \text{const.} ;$$

the tangents ps to the evolute being thus inclined (in the case here considered) at a constant angle, † to the radii op of the sphere.

(7.) In general, if we denote by R the interval \overline{ps} between two corresponding points of involute and evolute, we shall have the equation,

$$\text{XII.} \dots (\sigma - \rho)^2 + R^2 = 0, \text{ or XII'.} \dots T(\sigma - \rho) = R;$$

and the formula VII. may be replaced by the following,

$$\text{XIII.} \dots R^2 + \sigma^2 = 0, \text{ or XIII'.} \dots D_t R = \pm TD_t \sigma,$$

in which the independent variable t is still left arbitrary.

(8.) But if we take for that variable the arc sos_1 of the evolute, measured from some fixed point of that curve, we may then write,

$$\text{XIV.} \dots t = \int Td\sigma, \quad \text{XV.} \dots dR_t = \pm dt, \quad \text{XVI.} \dots D_t R_t = \pm 1;$$

* Compare the first Note to page 534; from the formulæ of which page it now appears, that if the involute be an ellipse, with $\beta = ob$ and $\gamma = oc$ for its major and minor semiaxes, and therefore with the scalar equations,

$$(S\beta^{-1}\rho)^2 + (S\gamma^{-1}\rho)^2 = 1, \quad S\beta\gamma\rho = 0,$$

the evolutes are geodesics on the cylinder of which the corresponding equation is,

$$(S\beta\sigma)^2 + (S\gamma\sigma)^2 = (\beta^2 - \gamma^2)^2.$$

† This property of the evolutes of a spherical curve was deduced by Professor De Morgan, in a Paper *On the Connexion of Involute and Evolute in Space* (Cambridge and Dublin Mathematical Journal for November, 1851); in which also a definition of involute and evolute was proposed, substantially the same as that above adopted.

whence

$$\text{XVII.} \dots D_t(R_t \mp t) = 0, \text{ and } \text{XVIII.} \dots R_t \mp t = \text{const.} = R_0,$$

the integral IX. being thus under a new form reproduced.

(9.) In this last mode of obtaining the result,

$$\text{XIX.} \dots \Delta \overline{P\delta} = R_t - R_0 = \pm t = \pm \text{arc } \widehat{S_0 S_t} \text{ of evolute,}$$

no use is made of *infinitesimals*,* or even of *small differentials*. We only infer, as in XVIII. (comp. 880, (9.)), that the quantity $R_t \mp t$ is *constant*,† because its *derivative is null*: it having been previously proved (880, (8.)), as a consequence of our *definition of differentials* (820, 824) that if s be the arc and ρ the vector of any curve, then the equation $ds = Td\rho$ (880, XXII.) is *rigorously satisfied*, whatever the *independent variable* t may be, and whether the two connected and *simultaneous differentials* be *small or large*.

(10.) But when we employ the *notation of integrals*, and introduce, as above, the symbol $\int Tds$, we are then led to *interpret* that *symbol* as denoting the *limit of a sum* (comp. 845, (12.)); or to write, generally,

$$\text{XX.} \dots \int Td\rho = \lim. \Sigma T\Delta\rho, \text{ if } \lim. \Delta\rho = 0,$$

with analogous formulæ for other cases of *integration in quaternions*. Geometrically, the equation,

$$\text{XXI.} \dots \int Td\rho = \Delta s, \text{ or } \text{XXI'.} \dots \int Td\sigma = \Delta t,$$

if s and t denote *arcs* of curves of which ρ and σ are *vectors*, comes thus to be interpreted as an expression of the well-known principle, that the *perimeter of any curve* (or of any *part* thereof) is the *limit of the perimeter* of an *inscribed polygon* (or of the corresponding *portion* of that polygon), when the *number of the sides* is indefinitely *increased*, and when their *lengths* are *diminished* indefinitely.

(11.) The equations I. and XII. give,

$$\text{XXII.} \dots S\sigma'(\sigma - \rho) + RR' = 0,$$

the independent variable t being again arbitrary; but these equations XII. and XXII. coincide with the formulæ 398, LXXXIX. and XCI.; we may then, by 398, (79.) and (80.), consider the *locus of the point P* as the *envelope of a variable sphere*, namely of the sphere which has s for centre and R for radius, and is represented by the recent equation XII., if $\rho = OP$ be the vector of a *variable point* thereon.

(12.) But whereas *such* an envelope has been seen to be *generally a surface*, which is *real or imaginary* (398. (79.)) according as $R^2 + \sigma'^2 < \text{or} > 0$, we have *here* by XIII. the *intermediate or limiting case* (comp. 398, CXXXI.), for which the *circles*

* In general, it may have been observed that we have hitherto *abstained*, at least in the *text* of this whole Chapter of *Applications*, from making any use of *infinitesimals*, although they have been often referred to in these *Notes*, and employed therein to assist the *geometrical investigation* or *enunciation* of results. But as regards the *mechanism of calculation*, it is at least as easy to use *infinitesimals in quaternions* as in any other system: as will perhaps be shown by a few examples, farther on.

† Compare the Note to page 516.

of the system become *points*, and the surface itself degenerates into a *curve*, which is here the *involute* (ρ) above considered. The *involute* of a given curve (s) are therefore *included*, as a *limit*, in that general system of envelopes which was considered in the lately cited sub-articles, and in others immediately following.

(13.) The *equation of condition*, 398, CXXXVI., is in this case satisfied by XIII., both members vanishing; but we cannot now put it under the form 398, CXLL, because in the passage to that form, in 398, (85.), there was tacitly effected a *division by r^2* , which is not now allowed, the radius r of the circle on the envelope being in the present case equal to zero. For a similar reason, we cannot now *divide by g* , as was done in 398, (86.); and because, in virtue of II., the two equations 398, CLX. reduce themselves to *one*, they no longer conduct to the formulæ 398, CLX'. CLXI. CLXI'. CLXIII. XCIV.; nor to the second equation 398, CLXII.

(14.) The general geometrical relations of the curves (ρ) and (s), which were investigated in the sub-articles to 398 for the case when the *condition** above referred to is satisfied, are therefore only very *partially* applicable to a system of *involute* and *evolute* in space: at least if we still consider the *former* curve (the involute) as being a *rectangular trajectory* to the *tangents* to the *latter* (the evolute), instead of being, like the curve (ρ) previously considered, a *rectangular trajectory* (398, (94.)) to the *osculating planes*† of the curve (s).

* If, without thinking of *evolutes*, we merely suppose that the *condition* 398, CXXXVI. is satisfied, as lately in (13.), by our having the relation $R^2 + \sigma^2 = 0$, it will be found (comp. the symbolical expression 274, XX. for $0i$, and the imaginary solution in 358, (18.) of the system $Sy\rho = 0$, $\rho^2 = 0$), that the *envelope* of the *sphere* $(\sigma - \rho)^2 + R^2 = 0$, or the *locus* of the (null) *circles* in which such spheres are (conceived to be) *cut* by the (tangent) *planes*, $S\sigma'(\sigma - \rho) + RR' = 0$, may be said to be *generally* the system of all those *imaginary points*, of which the vectors (or the *bivectors*, comp. 214, (6.)) are assigned by the formula,

$$\rho = \sigma - RR'^{-1}\sigma' + (U\sigma' + \sqrt{-1})V\sigma'\mu;$$

where μ is an *arbitrary vector*, and $\sqrt{-1}$ is the *old imaginary* of algebra. By making $\mu = 0$ we reduce this expression for ρ to the *real vector form*,

$$\rho = \sigma - RR'^{-1}\sigma' = \sigma + RR'\sigma'^{-1},$$

= the κ of 398, CXXXI."; and thus the *curve* (ρ), which is here the *locus* of the *centres* of the *null circles* of *contact*, and coincides with the *involute* in the present series of sub-articles, may still be called a *Singular Line* upon the *Envelope of the Sphere* (with *One Variable Parameter*), as being in the present case the *only real part* of that *elsewhere imaginary surface*.

† The curve to the *osculating planes* of which another curve is thus an *orthogonal trajectory*, and which is therefore (398, (95.)) the *cusp-edge* of the *polar developable* of the latter curve, was called by Lancret its *evolute by the plane* (developpée par le plan); whereas the curve (s) of the *present series* (400) of sub-articles, to whose *tangents* the corresponding curve (ρ) is an orthogonal trajectory, has been called by way of distinction the *evolute by the thread* (developpée par le fil) of this last curve. It would be improper to delay here on subjects so well known to geometers: but the student may be invited to read again, in connexion with them, the sub-articles (88.) and (89.) to Art. 398.

(15.) If the arc of the evolute be again taken for the independent variable t , and if the positive direction of motion along that arc be always towards the involute, we may write,

$$\text{XXIII.} \dots \rho = \dot{\sigma} + R\sigma', \quad R' = -1, \quad \sigma'^2 = -1, \text{ \&c. ;}$$

whence

$$\text{XXIV.} \dots \rho' = R\sigma'', \quad \rho'' = R\sigma''' - \sigma'', \quad V\rho''\rho' = R^2V\sigma''\sigma';$$

if then $\kappa = \text{OK}$ be the vector of the centre κ of the circle which osculates to the involute at P , the general formula 889, IV. gives, after a few reductions,* the expression (comp. 897, XVI. XXXIV., and XCVIII. (15)),

$$\begin{aligned} \text{XXV.} \dots \kappa &= \rho + \frac{\rho^2}{V\rho''\rho'} = \sigma + R \left(\sigma' + \frac{\sigma'^2}{V\sigma''\sigma'} \right) \\ &= \sigma + \frac{R\sigma'\sigma''\sigma'''}{V\sigma''\sigma'''} = \sigma - \frac{R\sigma'\sigma'^{-1}\sigma'''}{V\sigma''\sigma''^{-1}} \\ &= \sigma - Rr_1^{-1}\lambda_1^{-1} = \sigma + U\lambda_1. R \cos H_1, \end{aligned}$$

if r_1 , H_1 , and λ_1 be what r , H , and λ in 897 become, when we pass from the curve (P) to the curve (s), with the present relations between those two curves; this centre of curvature κ is therefore the foot of the perpendicular let fall from the point P of the involute, on the rectifying line λ_1 of the evolute: as indeed is evident from geometrical considerations, because by (8.) this rectifying line of the curve (s) is the polar axis of the curve (P).

(16.) If we conceive (comp. 889, (2.)) an auxiliary spherical curve to be described, of which the variable unit-vector shall be,

$$\text{XXVI.} \dots \sigma\tau = \tau = \sigma' = U(\rho - \sigma) = R^{-1}(\rho - \sigma),$$

and suppose that v is the vector OU of the centre of curvature of this new curve, at the point τ which corresponds to the point s of the evolute, we shall then have by XXV. the expression,

$$\text{XXVII.} \dots \tau v = v - \tau = \frac{r^2}{Vr''r'} = \frac{\sigma'^2}{V\sigma''\sigma''} = \frac{\kappa - \rho}{R} = \text{PK} : \overline{\text{PS}};$$

we have therefore this theorem, that the inward radius of curvature of the hodograph of the evolute (conceived to be an orbit described, as in 879, (9.), with a constant velocity taken for unity) is equal to the inward radius of curvature of the involute, divided by the interval R between the two curves (P) and (s): and that these two radii of curvature, τv and PK , have one common direction, at least if the direction of motion on the evolute be supposed, as in (15.), to be towards the involute.

(17.) The following is a simpler enunciation of the theorem† just stated:—If $\text{P}, \text{P}_1, \text{P}_2, \dots$ and $\text{s}, \text{s}_1, \text{s}_2, \dots$ be corresponding points of involute and evo-

* Especially by observing that $V\sigma'V\sigma''\sigma''' = -\sigma'^3$, because $S\sigma'\sigma'' = 0$, and $S\sigma'\sigma''' = -\sigma'^2$.

† Some additional light may be thrown on this theorem, by comparing it with the construction in 897, (48.); and by observing that the equations 897, XVI. XXXIV. give generally, in the notations of the Article referred to, for the vector of the centre of curvature of the hodograph of any curve, the transformations,

$$r + \frac{r'}{Vr''r'^{-1}} = r + \frac{r'}{\lambda} = -r^{-1}\lambda^{-1} = U\lambda. \cos H.$$

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lute, and if we draw lines $ST_1 \parallel S_1P_1$, $ST_2 \parallel S_2P_2$, .. with a common length = \overline{SP} , the spherical curve PT_1T_2 .. will then have contact of the second order with the curve FP_1P_2 .., that is with the involute at P.

401. The fundamental formula 389, IV., for the vector of the centre of the osculating circle to a curve in space, namely the formula,

$$\text{I. . . } \kappa = \rho + \frac{\rho^3}{\sqrt{\rho'^2 \rho''}}, \quad \text{or} \quad \text{II. . . } \kappa = \rho + \frac{d\rho^3}{Vd^2\rho d\rho},$$

which has been so extensively employed throughout the present Section, has hitherto been established and used in connexion with *derivatives* and *differentials* of vectors, rather than with *differences*, great or small. We may however establish, in another way, an essentially *equivalent* formula, into which *differences* enter by their *limits* (or rather by their *limiting relations*), namely, the following,

$$\text{III. . . } \kappa = \rho + \lim. \frac{\Delta\rho^3}{V\Delta^2\rho\Delta\rho}, \quad \text{if } \lim. \Delta\rho = 0, \quad \text{and } \lim. \frac{\Delta^2\rho}{\Delta\rho} = 0,$$

the denominator $V\Delta^2\rho\Delta\rho$ being understood to signify the same thing as $V(\Delta^2\rho \cdot \Delta\rho)$; and then may, if we think fit, *interpret* the *differential expression* II. as if $d\rho$ and $d^2\rho$ in it denoted *infinitesimals*,* of the *first and second orders*: with similar interpretations in other but analogous investigations.

(1.) If in the second expression 316, L., for the perpendicular from o on the line AB, we change α and β to their reciprocals (comp. Figs. 58, 64) and then take the reciprocal of the result, we obtain this new expression,

$$\text{IV. . . } OD = \delta = \frac{\alpha^{-1} - \beta^{-1}}{V\beta^{-1}\alpha^{-1}} = \frac{\alpha(\beta - \alpha)\beta}{V\beta\alpha} = \frac{OA \cdot AB \cdot OB}{V(OB \cdot OA)},$$

in the denominator of which, OB may be replaced by AB, or by AO + AB, for the diameter OD of the circle OAB; so that if o be the centre of this circle, its vector $\gamma = OC = \frac{1}{2}OD = \frac{1}{2}\delta = \&c$. Supposing then that P, Q, R are any three points of any given curve in space, while O is as usual an arbitrary origin, and writing

$$\text{V. . . } OP = \rho, \quad OQ = \rho + \Delta\rho, \quad OR = \rho + 2\Delta\rho + \Delta^2\rho,$$

and therefore

$$\text{VI. . . } PQ = \Delta\rho, \quad QR = \Delta\rho + \Delta^2\rho, \quad \frac{1}{2}PR = \Delta\rho + \frac{1}{2}\Delta^2\rho,$$

the centre c of the circle PQR has the following *rigorous expression* for its vector:

$$\text{VII. . . } OC = \gamma = \rho + \frac{\Delta\rho(\Delta\rho + \Delta^2\rho)(\Delta\rho + \frac{1}{2}\Delta^2\rho)}{V(\Delta^2\rho \cdot \Delta\rho)};$$

* Compare 345, (17.), and the first Note to page 623.

whence passing to the *limit*, we obtain successively the expressions III. and II. for the vector κ of the *centres of curvature* to the curve PQR at P; the two other points, Q and R, being *both* supposed to *approach indefinitely* to the given point P, according to *any law* (comp. 892, (6.)), which allows the two successive vector chords, PQ and QR, to bear to each other an *ultimate ratio of equality*.

(2.) Instead of thus *first* forming a *rigorous* expression, such as VII., involving the *differences* $\Delta\rho$ and $\Delta^2\rho$; then *simplifying* the formula so found, by the rejection of *terms*, which become *indefinitely small*, with respect to the terms retained; and finally changing differences to *differentials* (comp. 844, (2.)); namely $\Delta\rho$ to $d\rho$, and $\Delta^2\rho$ to $d^2\rho$, in the *homogeneous* expression which results, and of which the *limit* is to be taken: we may *abridge* the calculation, by *at once writing* the *differential symbols*, in place of *differences*, and *at once suppressing* any terms, of which we *foresee* that they must *disappear from the final result*. Thus, in the recent example, when we have perceived, by quaternions, that if κ be the centre of the circle PQR, the equation

$$\text{VIII.} \dots \text{PK} = \frac{\text{PQ} \cdot \text{QR} \cdot \frac{1}{2}(\text{PQ} + \text{QR})}{\text{V}\{(\text{QR} - \text{PQ})\text{PQ}\}}$$

is *rigorous*, we may *at once* change each of the *three factors* of the *numerator* to $d\rho$, while the factor $\text{QR} - \text{PQ}$ in the *denominator* is to be changed to $d^2\rho$; and thus the *differential expression* II., for the *inward vector-radius of curvature* $\kappa - \rho$, is at once obtained.

(8.) It is scarcely necessary to observe, that this expression for that *radius*, as a *vector*, agrees with and *includes* the known expressions for the same radius of curvature of a curve in space, considered as a (positive) *scalar*, which has been denoted in the present Section by the italic letter r (because the more usual symbol ρ would have *here* caused confusion). Thus, while the formula II. gives *immediately* (because $\text{T}d\rho = ds$) the equation,

$$\text{IX.} \dots r^{-1}ds^3 = \text{T}Vd\rho d^2\rho,$$

it gives also (because $d\rho^2 = -ds^2$, and $\text{S}d\rho d^2\rho = -dsd^2s$) the transformed equation,

$$\text{X.} \dots r^{-1}ds^2 = \text{V}(\text{T}d^2\rho^2 - d^2s^2);$$

and it conducts (by 889, VI.) to this still simpler formula (comp. the equation $r^{-1} = \text{T}r'$, 896, IX.),

$$\text{XI.} \dots r^{-1}ds = \text{T}dUd\rho.$$

(4.) Accordingly, if we employ the *standard trinomial form* (295, I.) for a *vector*,

$$\text{XII.} \dots \rho = ix + jy + kz,$$

which gives, by the laws of the symbols ijk (182, 183),

$$\text{XIII.} \dots \begin{cases} d\rho = idx + jdy + kdz, & ds = \text{T}d\rho = \text{V}(dx^2 + dy^2 + dz^2), \\ d^2\rho = id^2x + jd^2y + kd^2z, & \text{T}d^2\rho = \text{V}(d^2x^2 + d^2y^2 + d^2z^2), \\ \text{V}d\rho d^2\rho = i(dydz - dzdy) + j(dzdx - dxdz) + k(dxdy - dydx), \\ \text{U}d\rho = i\frac{dx}{ds} + j\frac{dy}{ds} + k\frac{dz}{ds}, & dUd\rho = id\frac{dx}{ds} + \dots \end{cases}$$

the recent equations IX. X. XI. take these *known forms*:

$$\text{IX'.} \dots r^{-1}ds^3 = \text{V}((dydz^2 - dzdy^2) + \dots);$$

$$\text{X'.} \dots r^{-1}ds^2 = \text{V}(d^2x^2 + d^2y^2 + d^2z^2 - d^2s^2);$$

$$\text{XI'.} \dots r^{-1}ds = \sqrt{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2}.$$

(5.) The formula IV., which lately served us to determine a diameter of a circle through three given points, may be more symmetrically written as follows. If AD be a diameter of the circle ABC , then

$$\text{XIV.} \dots AD \cdot V(AB \cdot BC) = AB \cdot BC \cdot CA;$$

an equation* in which $V(AB \cdot BC)$ may be changed to $V(AB \cdot AC)$, &c., and in which it may be remarked that each member is an expression (comp. 296, V.) for a vector AT , which touches at A the segment ABC : while its length is at once a representation of the product of the lengths of the sides of the triangle ABC , and also of the double area of that triangle (comp. 281, XIII.), multiplied by the diameter of the circumscribed circle.

(6.) In general, if $PQRS$ be any four concircular points, they satisfy (by 260, IX., comp. 296, (8.)) the condition of concircularity,

$$\text{XV.} \dots V\left(\frac{PS}{SQ} \cdot \frac{QR}{RP}\right) = 0,$$

which may be thus transformed:†

$$\text{XVI.} \dots V\left(\frac{PQ}{PS} + \frac{QP + QR}{PR}\right) = V\left(\frac{1}{PS} \cdot PQ \cdot \frac{QP + QR}{PR}\right).$$

Writing then (comp. VI., and the remarks in (2.)),

$$\text{XVII.} \dots PS = \omega - \rho, \quad PQ = d\rho, \quad PR = 2d\rho + d^2\rho, \quad QP + QR = d^2\rho,$$

the second member is seen to be, on the present plan, an infinitesimal of the second order, which is therefore to be suppressed, because the first member is only of the first order; and thus we obtain at once the following vector equation of the osculating circle to the curve PQR at P ,

* A student might find it useful practice to verify, that if we write in like manner,

$$\text{XIV'.} \dots BE \cdot V(BC \cdot CA) = BC \cdot CA \cdot AB,$$

so that BE is a second diameter, then $AB = ED$, or $ABDE$ is a parallelogram. He may employ the principles, that $a\beta\gamma = \gamma\beta a$, if $Sa\beta\gamma = 0$, and that $\beta\gamma - \gamma\beta = 2V\beta\gamma$; in virtue of which, after subtracting XIV' from XIV., and dividing by $V(BC \cdot CA)$, or by its equal $V(AB \cdot BC)$, the equation $AD - BE = 2AB$ is obtained, and proves the relation mentioned. It is easy also to prove that

$$\text{XIV''.} \dots BD \cdot V(BC \cdot CA) = AB \cdot S(BC \cdot CA),$$

and therefore that $ABDE$ is a rectangle.

† Without having recourse to this transformation XVI., we might treat the condition XV. by infinitesimals, as follows:

$$\text{XVII'.} \dots \begin{cases} \frac{PS}{QS} = 1 + \frac{PQ}{QS} = 1 + \frac{d\rho}{\omega - \rho - d\rho} = 1 + \frac{d\rho}{\omega - \rho}; \\ \frac{2QR}{PR} = 1 + \frac{QP + QR}{PR} = 1 + \frac{d^2\rho}{2d\rho + d^2\rho} = 1 + \frac{d^2\rho}{2d\rho} \end{cases}$$

equating then to zero the vector part of the product of these two expressions, and suppressing the infinitesimal of the second order, the equation XVIII. of the osculating circle is obtained anew.

$$\text{XVIII.} \dots V \left(\frac{d\rho}{\omega - \rho} + \frac{d^2\rho}{2d\rho} \right) = 0;$$

which agrees with the equation 892, VI., although deduced in a quite different manner, and conducts anew to the expression II. for $\kappa - \rho$, under the form,

$$\text{XIX.} \dots \frac{d\rho}{\kappa - \rho} + V \frac{d^2\rho}{d\rho}, \text{ as in 892, VIII.}$$

(7.) Again, if $OD = \delta$ be the diameter from the origin, of any sphere through that point o , which passes also through any three other given points A, B, C , with $OA = a$, &c., we have by 296, XXVI. the formula,

$$\text{XX.} \dots \delta Sa\beta\gamma = Va(\beta - a)(\gamma - \beta)\gamma;$$

writing then (comp. XVII.),

$$\text{XXI.} \dots a = d\rho, \quad \beta - a = d\rho + d^2\rho, \quad \gamma - \beta = d\rho + 2d^2\rho + d^3\rho,$$

and $\text{XXII.} \dots \delta = 2rs = 2(\sigma - \rho),$

where σ is (as in 895, &c.) the vector os (from an arbitrary origin o) of the centre of the osculating sphere to a curve of double curvature at r , we have by infinitesimals, suppressing terms which are of the seventh and higher orders, because the first member is only of the sixth order, and reducing* by the rules of quaternions,

$$\text{XXIII.} \dots (\sigma - \rho)Sd\rho d^2\rho d^3\rho = \frac{1}{2}Vd\rho(d\rho + d^2\rho)(d\rho + 2d^2\rho + d^3\rho)(8d\rho + 8d^2\rho + d^3\rho) \\ = 8Vd\rho d^2\rho Sd\rho d^3\rho + d\rho^2 Vd^3\rho d\rho;$$

which agrees precisely with the formula 895, XIII., although obtained by a process so different.

(8.) Finally as regards the osculating plane, and the second curvature, of a curve in space, infinitesimals give at once for that plane the equation,

$$\text{XXIV.} \dots S(\omega - \rho)d\rho d^2\rho = 0, \text{ agreeing with 876, V. ;}$$

and if three consecutive elements of the curve be represented (comp. XXI.) by the differential expressions,

$$\text{XXV.} \dots r\rho = d\rho, \quad q\rho = d\rho + d^2\rho, \quad r\rho = d\rho + 2d^2\rho + d^3\rho,$$

the second curvature r^{-1} , defined as in 896, is easily seen to be connected as follows with the angle of a certain auxiliary quaternion q , which differs infinitely little from unity :

$$\text{XXVI.} \dots r^{-1}ds = \angle q, \quad \text{if} \quad \text{XXVII.} \dots q = \frac{V(q\rho \cdot r\rho)}{V(r\rho \cdot q\rho)} = 1 + \frac{Vd\rho d^2\rho}{Vd\rho d^2\rho};$$

* Of the eighteen terms which would follow the sign of operation $\frac{1}{2}V$, if the second member of XXIII. were fully developed, one is of the fourth order, but is a scalar; three are of the fifth order, but have a scalar sum; nine are of orders higher than the sixth; and two terms of the sixth order are scalars, so that there remain only three terms of that order to be considered. In this manner it is found that the second member in question reduces itself to the sum of the two vector parts,

$$\frac{1}{2}V.(d\rho d^2\rho)^2 = 8Vd\rho d^2\rho \cdot Sd\rho d^2\rho,$$

and $\frac{1}{2}d\rho^2 V(d\rho d^2\rho + 8d^2\rho d\rho) = d\rho^2 Vd^3\rho d\rho;$

and thus the third member of XXIII. is obtained.

we have then the expression,

$$\text{XXVIII. . Second Curvature} = r^{-1} = \frac{Vq}{d\rho} = S \frac{d^2\rho}{\sqrt{d\rho d^2\rho}},$$

which agrees with the formula 897, XXVII., and has been illustrated, in the sub-articles to 897 and 898, by numerous geometrical applications.

(9.) On the whole, then, it appears that although the *logic of derived vectors*, and of *differentials of vectors* considered as *finite lines, proportional to such derivatives*, is perhaps a *little clearer* than that of *infinitesimals*, because it shows more *evidently* when combined with *Taylor's Series adapted to Quaternions*, 842, 875) that *nothing is neglected*, yet it is perfectly *possible to combine** quaternions, *in practice*, with methods founded on the *more usual notion of Differentials*, as *infinitely small differences*: and that when this combination is judiciously made, *abridgments of calculation arise, without any ultimate error*.

SECTION 7.—On Surfaces of the Second Order; and on Curvatures of Surfaces.

402. As early as in the First Book of these *Elements*, some specimens were given of the treatment or expression of *Surfaces of the Second Order by Vectors*; or by *Anharmonic Equations* which were *derived* from the theory of *vectors*, without any introduction, at that stage, of *Quaternions* properly so called. Thus it was shown, in the sub-articles to 98, that a very simple *anharmonic equation* ($xz = yw$) might represent either a *ruled paraboloid*, or a *ruled hyperboloid*, according as a certain *condition* ($ac = bd$) was or was not satisfied, by the *constants* of the surface. Again, in the sub-articles to 99, *two* examples were given, of *vector expressions* for *cones* of the *second order* (and *one* such expression for a *cone* of the *third order*, with a *conjugate ray* (99, (5.)); while an expression of the same sort, namely,

$$\text{I. . . } \rho = xa + y\beta + z\gamma, \quad \text{with } x^2 + y^2 + z^2 = 1,$$

was assigned (99, (2.)) as representing *generally* an *ellipsoid*,† with a , β , γ , or OA , OB , OC , for three *conjugate semidiameters*. And finally,

* Compare the first Note to page 623. It will however be of course necessary, in any *future applications* of quaternions, to specify in *which* of these *two senses*, as a *finite differential*, or as an *infinitesimal*, such a *symbol* as $d\rho$ is employed.

† In like manner the expression,

$$\text{II. . . } \rho = xa + y\beta + z\gamma, \quad \text{with } x^2 + y^2 - z^2 = 1, \text{ or } = -1,$$

represents a *general hyperboloid*, of *one sheet*, or of *two*, with $a\beta\gamma$ for conjugate *semidiameters*: while, with the *scalar equation* $x^2 + y^2 - z^2 = 0$, the *same vector expression* represents their *common asymptotic cone* (not generally of *revolution*).

in the sub-articles (11.) and (12.) to Art. 100, an instance was furnished of the determination of a *tangential plane to a cone*, by means of *partial derived vectors*.

403. In the Second Book, a much greater *range of expression* was attained, in consequence of the introduction of the *peculiar symbols*, or *characteristics of operation*, which belong to the present Calculus; but still with that *limitation* which was caused, by the *conception* and *notation* of a *Quaternion* being confined, in that Book, to *Quotients of Vectors* (112, 116, comp. 307, (5.)), without yet admitting *Products* or *Powers of Directed Lines in Space*: although *vectors*, *tensors*, and even *norms** of such *vectors* were already introduced (156, 185, 273).

(1.) The *Sphere*,† for instance, which has its *centre* at the *origin*, and has the *vector* OA , or a , with a *length* $Ta = a$, for one of its *radii*, admitted of being represented, not only (comp. 402, I.) by the *vector expression*,

$$I. \dots \rho = xa + y\beta + z\gamma, \quad x^2 + y^2 + z^2 = 1,$$

with

$$I'. \dots Ta = T\beta = T\gamma = a, \quad \text{and} \quad I''. \dots S \frac{\beta}{a} = S \frac{\gamma}{a} = S \frac{\gamma}{\beta} = 0,$$

but also by any one of the following *equations*, in which it is permitted to change a to $-a$:

$$II. \dots \frac{\alpha}{\rho} = K \frac{\rho}{\alpha}; \quad III. \dots \frac{\rho}{\alpha} K \frac{\rho}{\alpha} = 1; \quad IV. \dots N \frac{\rho}{\alpha} = 1; \quad 145, (8.), (12.)$$

$$V. \dots T\rho = a; \quad VI. \dots T\rho = Ta; \quad VII. \dots T \frac{\rho}{\alpha} = 1; \quad \begin{matrix} 186, (2.), \\ 187, (1.) \end{matrix}$$

$$VIII. \dots S \frac{\rho - a}{\rho + a} = 0; \quad IX. \dots N \frac{\rho}{\alpha} = N \frac{a}{\alpha}; \quad X. \dots N\rho = Na; \quad \begin{matrix} 200, (11.), \\ 215, (10.), \\ 273, (1.) \end{matrix}$$

$$XI. \dots \left(S \frac{\rho}{\alpha} \right)^2 - \left(V \frac{\rho}{\alpha} \right)^2 = 1; \quad XII. \dots NS \frac{\rho}{\alpha} + NV \frac{\rho}{\alpha} = 1; \quad 204, (6.), XXV., XXVI.$$

$$XIII. \dots N \left(S \frac{\rho}{\alpha} + V \frac{\rho}{\alpha} \right) = 1; \quad XIV. \dots T \left(S \frac{\rho}{\alpha} + V \frac{\rho}{\alpha} \right) = 1; \quad 204, (9.)$$

or by the *system of equations*,

$$XV. \dots S \frac{\rho}{\alpha} = x, \quad \left(V \frac{\rho}{\alpha} \right)^2 = x^2 - 1 (\leq 0), \quad 204, (4.)$$

representing a *system of circles*, with the *spheric surface* for their *locus*.

* The notation Na , for $(Ta)^2$, although not formally introduced before Art. 273, had been used by anticipation in 200, (8.), page 188.

† That is to say, the *spheric surface* through A , with o for centre. Compare the Note to page 197.

(2.) *Other forms of equation, for the same spheric surface, may on the same principles be assigned; for example we may write,*

$$\begin{array}{lll} \text{XVI.} \dots \frac{\rho}{\alpha} = \frac{\kappa}{\rho} \alpha; & \text{XVII.} \dots N \frac{\alpha}{\rho} = 1; & \text{XVIII.} \dots T \frac{\alpha}{\rho} = 1; \\ \text{XIX.} \dots \angle \frac{\rho - \alpha}{\rho + \alpha} = \frac{\pi}{2}; & \text{XX.} \dots S \frac{2\alpha}{\rho + \alpha} = 1; & \text{XXI.} \dots S \frac{2\rho}{\rho + \alpha} = 1; \end{array}$$

or (comp. 186, (5.), and 200, (3.)),

$$\text{XXII.} \dots T(\rho - \alpha) = T(c\rho - \alpha), \quad c^2 > 1;$$

under which *last form, the sphere may be considered to be generated by the revolution of the circle, which has been already spoken of as the Apollonian* Locus.*

(3.) And from *any one to any other, of all these various forms, it is possible, and easy to pass, by general Rules of Transformation,†* which were established in the Second Book: while *each of them is capable of receiving, on the principles of the same Book, a Geometrical Interpretation.*

(4.) But we could not, on the principles of the Second Book *alone, advance to such subsequent equations of the same sphere, as*

$$\text{XXIII.} \dots \rho^2 = a^2, \quad \text{or} \quad \text{XXIV.} \dots \rho^2 + a^2 = 0, \quad 282, \text{VII. XIII.}$$

whereof the latter includes (282, (9.)) the important equation $\rho^2 + 1 = 0$, or $\rho^2 = -1$, of what we have called the *Unit-Sphere* (128); nor to such an *exponential expression for the variable vector ρ of the same spheric surface, as*

$$\text{XXV.} \dots \rho = ak^t j^s k^j \cdot k^{-t}, \quad 808, \text{XVIII.}$$

in which j and k belong to the fundamental system ijk of *three rectangular unit-lines* (295), connected by the fundamental Formula A of Art. 183, namely,

$$i^2 = j^2 = k^2 = ijk = -1, \quad (\text{A})$$

while s and t are *two arbitrary and scalar variables, with simple geometrical‡ significations*: because we were not *then* prepared to introduce any *symbol, such as ρ^2 , or k^t , which should represent a square or other power of a vector.*§ And similar re-

* Compare the first Note to page 128.

† This *richness of transformation, of quaternion expressions or equations, has been noticed, by some friendly critics, as a characteristic of the present Calculus.* In the preceding parts of this work, the reader may compare pages 128, 140, 183, 573, 574, 575; in the two last of which, the *variety of the expressions for the second curvature (r^{-1}) of a curve in space may be considered worthy of remark.* On the other hand, it may be thought remarkable that, in this Calculus, a *single expression, such as that given by the first formula (889, IV.) of page 582, adapts itself with equal ease to the determination of the vector (ε) of the centre of the osculating circle, to a plane curve, and to a curve of double curvature, as has been sufficiently exemplified in the foregoing Section.*

‡ Compare the second Note to page 365.

§ It is true that the formula A was established in the course of the Second Book (page 160); but it is to be remembered that the symbols ijk were *there* treated as denoting a system of *three right versors, in three mutually rectangular planes* (181):

marks apply to the representation, by quaternions, of *other surfaces of the second order*.

404. A brief review, or *recapitulation*, of some of the chief expressions connected with the *Ellipsoid*, for example, which have been already established in these *Elements*, with *references* to a few others, may not be useless here.

(1.) Besides the *vector expression* $\rho = \alpha a + \beta b + \gamma c$, with the *scalar relation* $x^2 + y^2 + z^2 = 1$, and with *arbitrary vector values* of the constants α, β, γ , which was lately cited (402) from the First Book, or the equations 403, I., *without the conditions* 403, I', II'. which are peculiar to the *sphere*, there were given in the Second Book (204, (18.), (14.)) equations which differed from those lately numbered as 403, XI. XII. XIII. XIV. XV., only by the substitution of $V \frac{\rho}{\beta}$ for $V \frac{\rho}{\alpha}$; for instance, there was the equation,

$$I \dots \left(S \frac{\rho}{\alpha} \right)^2 - \left(V \frac{\rho}{\beta} \right)^2 = 1, \quad 204, (14.)$$

analogous to 403, XI., and representing generally* an *ellipsoid*, regarded as the *locus* of a certain system of *ellipses*, which were thus substituted for the *circles*† (403, XV.) of the *sphere*, by a species of *geometrical deformation*, which led to the establishment of certain *homologies* (developed in the sub-articles to 274).

although it has since been found possible and useful, in this Third Book, to *identify* those right *vectors* with their own *indices* or *axes* (295), and so to treat them as a system of *three rectangular lines*, as above.

* In the case of *parallelism* of the two vector constants ($\beta \parallel \alpha$), the equation I. represents generally a *Spheroid of revolution*, with its *axis* in the direction of α ; while in the contrary case of *perpendicularity* ($\beta \perp \alpha$), the same equation I. represents an *elliptic Cylinder*, with its *generating lines* in the direction of β . Compare 204, (10.), (11.), and the Note to page 224.

† The equation I. might also have been thus written, on the principles of the Second Book,

$$I' \dots \left(S \frac{\rho}{\alpha} + S \frac{\rho}{\beta} \right) \left(S \frac{\rho}{\alpha} - S \frac{\rho}{\beta} \right) + \left(T \frac{\rho}{\beta} \right)^2 = 1;$$

whence it would have followed at once (comp. 216, (7.)), that the *ellipsoid* I. is cut in two circles, with a common radius = $T\beta$, by the two *diametral planes*,

$$I'' \dots S \frac{\rho}{\alpha} + S \frac{\rho}{\beta} = 0, \quad S \frac{\rho}{\alpha} - S \frac{\rho}{\beta} = 0.$$

In fact, this equation I'. is what was called in 359 a *cyclic form*, while I. itself is what was there called a *focal form*, of the equation of the surface; the lines $\alpha^{-1} \pm \beta^{-1}$ being, by the Third Book, the two (real) *cyclic normals*, while β is one of the two (real) *focal lines* of the (imaginary) *asymptotic cone*. Compare the Note to page 474.

(2.) Employing still *only quotients of vectors*, but introducing *two other pairs of vector-constants*, γ, δ and ϵ, κ , instead of the pair α, β in the equation I., which were however connected with that pair and with each other by certain assigned relations, that equation was transformed successively to

$$\text{II.} \dots T \left(\frac{\rho}{\gamma} + K \frac{\rho}{\delta} \right) = 1, \quad 216, \text{X.}$$

and to a form which may be written thus (comp. 217, (5.)),

$$\text{III.} \dots T \left(\epsilon + K \frac{\kappa}{\rho} \cdot \rho \right) T \rho = T \epsilon^2 - T \kappa^2; \quad 217, \text{XVI.}$$

and this last form was interpreted, so as to lead to a *Rule of Construction** (217, (6.), (7.)), which was illustrated by a *Diagram* (Fig. 58), and from which many *geometrical properties* of that surface were deduced (218, 219) in a very simple manner, and were confirmed by calculation with quaternions: the equation and construction being also modified afterwards, by the introduction (220) of a *new pair of vector-constants*, i' and κ' , which were shown to admit of being substituted for ϵ and κ , in the recent form III.

(8.) And although the *Equation of Conjugation*,

$$\text{IV.} \dots S \frac{\lambda}{\alpha} S \frac{\mu}{\alpha} - S \left(V \frac{\lambda}{\beta} \cdot V \frac{\mu}{\beta} \right) = 1, \quad 316, \text{LXIII.}$$

which connects the vectors λ, μ of any two points L, M , whereof one is on the *polar plane* of the other, with respect to the ellipsoid I., was not assigned till near the end of the First Chapter of the present Book, yet it was there deduced by principles and processes of the Second Book *alone*: which thus were *adequate*, although not in the most practically convenient way, to the treatment of questions respecting *tangent planes* and *normals to an ellipsoid*, and similarly for *other surfaces*† of the same second order.

* This *Construction of the Ellipsoid*, by means of a *Generating Triangle* and a *Diacentric Sphere* (page 227), is believed to have been new, when it was deduced by the writer in 1846, and was in that year stated to the Royal Irish Academy (see its *Proceedings*, vol. iii. pp. 288, 289), as a result of the *Method of Quaternions*, which had been previously communicated by him to that Academy (in the year 1848).

† The following are a few other references, on this subject, to the Second Book. Expressions for a *Right Cone* (or for a single *sheet* of such a cone) have been given in pages 119, 179, 220, 221. In page 179 the equation $S \frac{\rho}{\alpha} S \frac{\beta}{\rho} = 1$, has been assigned, with a transformation in page 180, to represent generally a *Cyclic Cone*, or a cone of the *second order*, with its vertex at the origin; and to exhibit its *cyclic planes*, and *subcontrary sections* (pp. 181, 182). *Right Cylinders* have occurred in pages 198, 196, 197, 198, 199, 218. A case of an *Elliptic Cylinder* has been already mentioned (the case when $\beta \perp \alpha$ in I.); and a transformation of the equation III. of the *Ellipsoid*, by means of *reciprocals* and *norms of vectors*, was assigned in page 298. And several expressions (comp. 408), for a *Sphere* of which the ori-

(4.) But in this Third Book we have been able to write the equation III. under the simpler form,*

$$V. \dots T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \quad 282, \text{XXIX.}$$

which has again admitted of numerous transformations; for instance, of all those which are obtained by equating $(\kappa^2 - \iota^2)^2$ to any one of the expressions 386, (5.), for the square of this last tensor in V., or for the norm of the quaternion $\iota\rho + \rho\kappa$; cyclic forms† of equation thus arising, which are easily converted into focal forms (359); while a rectangular transformation (378, XXX.) has subsequently been assigned, whereby the lengths (abc), and also the directions, of the three semiaxes of the surface, are expressed in terms of the two vector-constants, ι, κ : the results thus obtained by calculation being found to agree with those previously deduced, from the geometrical construction (2.) in the Second Book.

(5.) The equation V. has also been differentiated (386), and a normal vector $\nu = \phi\rho$ has thus been deduced, such that, for the ellipsoid in question,

$$VI. \dots S\nu d\rho = 0, \quad \text{and} \quad VII. \dots S\nu\rho = 1;$$

a process which has since been extended (361), and appears to furnish one of the best general methods of treating surfaces‡ of the second order by quaternions: especially when combined with that theory of linear and vector functions ($\phi\rho$) of vectors, which was developed in the Sixth Section§ of the Second Chapter of the present Book.

gia was not the centre, occurred in pages 164, 179, 189, and perhaps elsewhere, without any employment of products of vectors.

* Mentioned by anticipation in the Note to page 233.

† Compare the second Note to page 633. The vectors ι and κ are here the cyclic normals, and $\iota - \kappa$ is one of the focal lines; the other being the line $\iota' - \kappa'$ of page 232.

‡ The following are a few additional references to preceding parts of this Third Book, which has extended to a much greater length than was designed (page 802). In the First Chapter, the reader may consult pages 805, 806, 307, for some other forms of equation of the ellipsoid and the sphere. In the Second Chapter, pages 416, 417 contain some useful practice, above alluded to, in the differentiation and transformation of the equation $r^2 = T(\iota\rho + \rho\kappa)$. As regards the Sixth Section of that Chapter, which we are about to use (405), as one supposed to be familiar to the reader, it may be sufficient here to mention Arts. 357-362, and the Notes (or some of them) to pages 464, 466, 468, 474, 481, 484. In this Third Chapter, the sub-articles (7.)-(21.) to 373 (pages 504, &c.) might be re-perused; and perhaps the investigations respecting cones and sphero-conics, in 394 and its sub-articles (pages 541, &c.), including remarks on an hyperbolic cylinder, and its asymptotic planes (in page 547). Finally, in a few longer and later series of sub-articles, to Arts. 397, &c., a certain degree of familiarity with some of the chief properties of surfaces of the second order has been assumed; as in pages 571, 588, 591, and generally in the recent investigations respecting the osculating twisted cubic (pages 591, 620, &c.), to a helix, or other curve in space.

§ It appears that this Section may be conveniently referred to, as III. ii. 6; and similarly in other cases.

405. Dismissing then, at least for the present, the *special* consideration of the *ellipsoid*, but still confining ourselves, for the moment, to *Central Surfaces* of the *Second Order*, and using freely the principles of this Third Book, but especially those of the Section (III. ii. 6) last referred to, we may denote any such *central* and *non-conical surface* by the scalar equation (comp. 361),

$$I. \dots f\rho = S\rho\phi\rho = 1;$$

the *asymptotic cone* (real or imaginary) being represented by the connected equation,

$$II. \dots f\rho = S\rho\phi\rho = 0;$$

and the *equation of conjugation*, between the vectors ρ , ρ' of any two points P, P', which are *conjugate* relatively to this surface I. (comp. 362, and 404, (3.)), see also 373, (20.)), being,

$$III. \dots f(\rho, \rho') = f(\rho', \rho) = S\rho\phi\rho' = S\rho'\phi\rho = 1;$$

while the *differential equation* of the surface is of the form (361),

$$IV. \dots 0 = df\rho = 2S\nu d\rho, \text{ with } V. \dots \nu = \phi\rho;$$

this *vector-function* $\phi\rho$, which represents the *normal* ν to the surface, being at once *linear* and *self-conjugate* (361, (3.)); and the *surface itself* being the *locus* of all the points P which are *conjugate to themselves*, so that its *equation* I. may be thus written,

$$I'. \dots f(\rho, \rho) = 1, \text{ because } f(\rho, \rho) = f\rho, \text{ by 362, IV.}$$

(1.) Such being the *form* of $\phi\rho$, it has been seen that there are always *three real* and *rectangular unit-lines*, a_1, a_2, a_3 , and *three real scalars*, c_1, c_2, c_3 , such as to satisfy (comp. 357, III.) the *three vector equations*,

$$VI. \dots \phi a_1 = -c_1 a_1, \quad \phi a_2 = -c_2 a_2, \quad \phi a_3 = -c_3 a_3;$$

whence also these *three scalar equations* are satisfied,

$$VII. f a_1 = c_1, \quad f a_2 = c_2, \quad f a_3 = c_3;$$

and therefore (comp. 362, VII.),

$$VIII. \dots f(c_1^{-1} a_1) = f(c_2^{-1} a_2) = f(c_3^{-1} a_3) = 1.$$

(2.) It follows then that the *three* (real or imaginary) *rectangular lines*,

$$IX. \dots \beta_1 = c_1^{-1} a_1, \quad \beta_2 = c_2^{-1} a_2, \quad \beta_3 = c_3^{-1} a_3,$$

are the *three* (real or imaginary) *vector semiaxes* of the surface I.; and that the *three* (positive or negative) *scalars*, c_1, c_2, c_3 , namely the *three roots* of the *scalar and cubic equation** $M = 0$ (comp. 357, (1.)), are the (always real) *inverse squares* of the *three* (real or imaginary) *scalar semiaxes*, of the same central surface of the second order.

* It is unnecessary *here* to write $M_0 = 0$, as in page 462, &c., because the function ϕ is here supposed to be *self-conjugate*; its constants being also *real*.

(8.) For the *reality* of that *surface I.*, it is necessary and sufficient that *one at least* of the three scalars c_1, c_2, c_3 should be *positive*; if *all* be such, the surface is an *ellipsoid*; if *two*, but not the third, it is a *single-sheeted hyperboloid*; and if only *one*, it is a *double-sheeted hyperboloid*: those scalars being here supposed to be each *finite*, and different from zero.

(4.) We have already seen (357, (2.)) how to obtain the *rectangular transformation*,

$$\text{X.} \dots f\rho = c_1(Sa_1\rho)^2 + c_2(Sa_2\rho)^2 + c_3(Sa_3\rho)^2,$$

which may now, by IX., be thus written,

$$\text{XI.} \dots f\rho = (S\beta_1^{-1}\rho)^2 + (S\beta_2^{-1}\rho)^2 + (S\beta_3^{-1}\rho)^2;$$

but it is to be remembered that, by (2.) and (8.), *one* or even *two* of these three vectors $\beta_1\beta_2\beta_3$ may become *imaginary*, without the *surface* ceasing to be *real*.

(5.) We had also the *cyclic transformation* (357, II. II.),

$$\text{XII.} \dots f\rho = g\rho^2 + S\lambda\rho\mu\rho = \rho^2(g - S\lambda\mu) + 2S\lambda\rho S\mu\rho,$$

in which the scalar g and the vector λ, μ are *real*, and the latter have the directions of the two (real) cyclic normals;* in fact it is obvious on inspection, that the surface is cut in circles, by planes perpendicular to these two last lines.

(6.) It has been proved that the *four real scalars*, $c_1c_2c_3g$, and the *five real vectors*, $a_1a_2a_3\lambda\mu$, are connected by the relations† (357, XX. and XXI.),

$$\text{XIII.} \dots c_1 = -g - T\lambda\mu, \quad c_2 = -g + S\lambda\mu, \quad c_3 = -g + T\lambda\mu;$$

$$\text{XIV.} \dots a_1 = U(\lambda T\mu - \mu T\lambda), \quad a_2 = UV\lambda\mu, \quad a_3 = U(\lambda T\mu + \mu T\lambda);$$

at least if the *three roots* $c_1c_2c_3$ of the cubic $M=0$ be arranged in *algebraically ascending order* (357, IX.), so that $c_1 < c_2 < c_3$.

(7.) It may happen (comp. (3.)), that *one* of these three roots *vanishes*; and in that case (comp. (2.)), *one* of the three *semiaxes* becomes *infinite*, and the *surface I.* becomes a *cylinder*.

(8.) Thus, in particular, if $c_1 = 0$, or $g = -T\lambda\mu$, so that the *two other roots* are both *positive*, the equation takes (by XII., comp. 357, XXII.) a form which may be thus written,

$$\text{XV.} \dots (S\lambda\mu\rho)^2 + (S\lambda\rho T\mu + S\mu\rho T\lambda)^2 = T\lambda\mu - S\lambda\mu > 0;$$

and it represents an *elliptic cylinder*.

(9.) Again, if $c_2 = 0$, or $g = S\lambda\mu$, the equation becomes,

$$\text{XVI.} \dots 2S\lambda\rho S\mu\rho = 1,$$

and represents an *hyperbolic cylinder*: the root c_1 being in this case *negative*, while the remaining root c_3 is *positive*.

* Compare the Note to page 468; see also the proof by quaternions, in 378, (16.), &c., of the known theorem, that any two *subcontrary* circular sections are *homospherical*, with the equation (378, XLIV.) of their *common sphere*, which is found to have its *centre* in the *diametral plane* of the two *cyclic normals* λ, μ .

† These relations and a few others mentioned are so useful that, although they occurred in an earlier part of the work, it seems convenient to restate them here.

(10.) But if we suppose that $c_3 = 0$, or $g = T\lambda\mu$, so that c_1 and c_2 are both negative, the equation may (by 857, XXIII.) be reduced to the form,

$$\text{XVII.} \dots (S\lambda\mu\rho)^2 + (S\lambda\rho T\mu - S\mu\rho T\lambda)^2 = -T\lambda\mu - S\lambda\mu < 0;$$

it represents therefore, in *this case*, nothing real, although it may be said to be, in the same case, the equation of an *imaginary* elliptic cylinder*.

(11.) It is scarcely worth while to remark, that we have here supposed each of the two vectors λ and μ to be not only real but actual (Art. 1); for if either of them were to vanish, the equation of the surface would take by XII. the form,

$$\text{XVIII.} \dots \rho^2 = g^{-1}, \text{ or XVIII'.} \dots T\rho = (-g)^{\frac{1}{2}},$$

and would represent a real or imaginary sphere, according as the scalar constant g was negative or positive: λ and μ have also distinct directions, except in the case of surfaces of revolution.

(12.) In general, it results from the relations (6.), that the plane of the two (real) cyclic normals, λ, μ , is perpendicular to the (real) direction of that (real or imaginary) semiaxis, of which, when considered as a scalar (2.), the inverse square c_3 is algebraically intermediate between the inverse squares c_1, c_2 of the other two; or that the two diametral and cyclic planes ($S\lambda\rho = 0, S\mu\rho = 0$) intersect in that real line ($V\lambda\mu$) which has the direction of the real unit-vector a_3 (1.), corresponding to the mean root c_3 of the cubic equation $M = 0$: all which agrees with known results, respecting the circular sections of the (real) ellipsoid, and of the two hyperboloids.

406. Some additional light may be thrown on the theory of the central surface 405, I., by the consideration of its asymptotic cone 405, II.; of which cone, by 405, XII., the equation may be thus written,

$$\text{I.} \dots f\rho = g\rho^2 + S\lambda\rho\mu\rho = \rho^2(g - S\lambda\mu) + 2S\lambda\rho S\mu\rho = 0;$$

and which is real or imaginary, according as we have the inequality,

$$\text{II.} \dots g^2 < \lambda^2\mu^2, \text{ or III.} \dots g^2 > \lambda^2\mu^2;$$

that is, by 405, (6.), according as the product c_1c_2 of the extreme roots of the cubic $M = 0$ is negative or positive; or finally, according as the surface $f\rho = 1$ is a (real) hyperboloid, or an ellipsoid (real or imaginary†).

* In the Section (III. ii. 6) above referred to, many symbolical results have been established, respecting imaginary cyclic normals, or focal lines, &c., on which it is unnecessary to return. But it may be remarked that as, when the scalar function $f\rho$ admits of changing sign, for a change of direction of the real vector ρ , so as to be positive for some such directions, and negative for others, although $f(-\rho) = f(+\rho)$, the two equations, $f\rho = +1, f\rho = -1$, represent then two real and conjugate hyperboloids, of different species: so, when the function $f\rho$ is either essentially positive, or else essentially negative, for real values of ρ , the equations $f\rho = 1$ and $f\rho = -1$ may then be said to represent two conjugate ellipsoids, one real, and the other imaginary.

† Compare the Note immediately preceding; also the second Note to page 474.

(1.) As regards the asserted *reality* of the cone I., when the *condition* II. is satisfied, it may suffice to observe that if we cut the cone by the *plane*,

$$\text{IV.} \dots S\lambda(\rho - \mu) = -g,$$

the *section* is a *circle* of the *real* and *diacentric* sphere,

$$\text{V.} \dots \rho^2 = 2S\mu\rho, \text{ or } \text{V}'. \dots (\rho - \mu)^2 = \mu^2;$$

and a *real circle*, because it is on the *real cylinder of revolution*,

$$\text{VI.} \dots \text{TV}(\rho - \mu)\text{U}\lambda = (\text{T}\mu^2 - g^2\text{T}\lambda^{-2})^{\frac{1}{2}},$$

so that its *radius* is equal to this last *real radical*.

(2.) For example, the cone

$$\text{VII.} \dots S \frac{\rho}{a} S \frac{\beta}{\rho} = 1, \text{ or } \text{VII}'. \dots 2(\text{S}\alpha\rho\text{S}\beta\rho - \alpha^2\rho^2) = 0,$$

which under the form VII. occurred as early as 196, (8.), and for which $\lambda = \alpha$, $\mu = \beta$, $g = \text{S}\alpha\beta - 2\alpha^2$, and therefore $\text{T}\lambda\mu + g > 0$, the *condition* II. reduces itself to $\text{T}\lambda\mu - g > 0$; or after division by $2\text{T}\alpha^2$, &c., to the form (comp. 199, XII.),

$$\text{VIII.} \dots \frac{1}{2}(\text{T} + \text{S}) \frac{\beta}{\alpha} > 1, \text{ or } \text{VIII}'. \dots \text{S} \sqrt{\frac{\beta}{\alpha}} > 1;$$

and accordingly, when either of these two last inequalities exists, it will be found that the *sphere* $\text{S} \frac{\beta}{\rho} = 1$ is cut by the *plane* $\text{S} \frac{\rho}{a} = 1$ in a *real circle*, the base of a *real cone* VII.

(3.) As an example of the *variety of processes* by which problems in this Calculus may be treated, we might propose to determine, by the general formula 389, IV., the vector κ of the *centre* of the *osculating circle* to the *curve* IV. V., considered merely as an *intersection* of two surfaces. The first derivatives of the equations would allow us to assume $\rho' = \nabla\lambda(\rho - \mu)$, and therefore $\rho'' = \lambda\rho'$; whence, by the formula, we have

$$\text{IX.} \dots \kappa = \rho + \frac{\rho'^2}{\nabla\rho''\rho'} = \rho + \frac{\rho'}{\lambda} = \frac{\text{S}\rho\lambda + \nabla\mu\lambda}{\lambda} = \mu - g\lambda^{-1};$$

the *section* is therefore a *circle*, because its *centre of curvature* is *constant*; and its *radius* is,

$$\text{X.} \dots r = \text{T}(\rho - \kappa) = \text{T}(\rho - \mu + g\lambda^{-1}) = (\text{T}\mu^2 - g^2\text{T}\lambda^{-2})^{\frac{1}{2}},$$

= the *radius* of the *cylinder* VI.

(4.) When the *opposite inequality* III. exists, the *radius* X., the *cylinder* VI., the *circle* IV. V., and the *cone* I., become all four *imaginary*; the *plane* IV. being then wholly *external* to the *sphere* V., as happens, for instance, with the *plane* and *sphere* in (2.), when the *condition* VIII. or VIII'. is *reversed*.

(5.) In the *intermediate case*, when

$$\text{XI.} \dots g^2 = \lambda^2\mu^2, \text{ or } \text{XI}'. \dots g = \mp\text{T}\lambda\mu,$$

the *radius* r in X. *vanishes*; the *right cylinder* VI. reduces itself to its *axis*; and the *circle* IV. V. becomes a *point*, in which the *sphere* is *touch*ed by the *plane*. In this case, then, the *cone* I. is reduced to a *single (real*) right line*, which has

* It may however be said, that in this case the *cone* consists of a *pair of imaginary planes*, which *intersect* in a *real right line*.

(compare the equations of the *elliptic cylinders*, 405, XV. XVII.) the direction of $\lambda T\mu - \mu T\lambda$, if $g = -T\lambda\mu$, but the perpendicular direction of $\lambda T\mu + \mu T\lambda$, if $g = +T\lambda\mu$.

(6.) In general (comp. 405, X.), the equation of the cone I. admits of the *rectangular transformation*,

$$\text{XII.} \dots f\rho = c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2 = 0;$$

and the two *sub-cases* last considered (5.) correspond respectively (by 405, (6.)) to the *evanescence* of the roots c_1, c_3 of the cubic $M = 0$, with the resulting *directions* α_1, α_3 of the only *real side* of the cone. An analogous but intermediate case (comp. 405, (9.)) is that when $c_3 = 0$, or $g = S\lambda\mu$; in which case, the cone I. reduces itself to the *pair of (real) planes*,

$$\text{XIII.} \dots S\lambda\rho \cdot S\mu\rho = 0,$$

namely to the *asymptotic planes* of the *hyperbolic cylinder* 405, XVI., or to those which are usually the *two cyclic* planes* of the cone.

(7.) The case (comp. 394, (29.)),

$$\text{XIV.} \dots g = -S\lambda\mu, \text{ or } \text{XIV'.} \dots \epsilon_1 - \epsilon_2 + \epsilon_3 = 0,$$

for which the equation I. of the cone becomes,

$$\text{XV.} \dots 0 = f\rho = 2(S\lambda\rho S\mu\rho - \rho^2 S\lambda\mu) = 2S(\sqrt{\lambda}\rho \cdot \sqrt{\mu}\rho),$$

may deserve a moment's attention. In this case, the *two planes*, of $\lambda\rho$ and $\mu\rho$, which connect the *two cyclic normals* λ and μ with an *arbitrary side* ρ of the cone, are always *rectangular* to each other; and these two normals to the cyclic planes are at the same time *sides* of the cone, which thus is *cut in circles*, by planes *perpendicular to those two sides*. And because the equation of the cone may (in the same case) be thus written,

$$\text{XVI.} \dots TV(\lambda + \mu)\rho = TV(\lambda - \mu)\rho,$$

while the *lengths* of λ and μ may *vary*, if their *product* $T\lambda\mu$ be left unchanged, so that $\lambda + \mu$ and $\lambda - \mu$ may represent *any two lines* from the vertex, in the *plane* of the *two cyclic normals*, and *harmonically conjugate with respect to them*, it follows that, for this cone XV., the *sines of the inclinations of an arbitrary side* ρ , to these *two new lines*, have a *constant ratio* to each other.

(8.) In general, the second form I. of $f\rho$ shows (comp. 394, (23.)), that the *constant product of the sines of the inclinations*, of a *side* ρ of the cone to the *two cyclic planes*, has for expression,

$$\text{XVII.} \dots \cos \angle \frac{\rho}{\lambda} \cdot \cos \angle \frac{\rho}{\mu} = \frac{1}{2} \left(\frac{g}{T\lambda\mu} + \cos \angle \frac{\mu}{\lambda} \right);$$

while the first form I. of the same function $f\rho$ reproduces the *condition of reality* II., by showing that $g : T\lambda\mu$ is (for a *real cone*) the *cosine of a real angle*, namely, that of the quaternion product $\lambda\rho\mu\rho$, since it gives the relation,

$$\text{XVIII.} \dots \frac{g}{T\lambda\mu} = SU\lambda\rho\mu\rho = \cos \angle \lambda\rho\mu\rho = \cos \angle \frac{\rho\mu^{-1}\rho}{\lambda}$$

* The cones and surfaces which have a common centre, and common values of the vectors λ and μ , but different values of the scalar g , may thus be said, in a known phraseology, to be *biconcyclic*.

(9.) We may also observe that in the case of reality II., with exclusion of the sub-case (6.), if a_3 have the direction of the internal axis of the cone, so that

$$\text{XIX.} \dots c_1 < 0, \quad c_2 < 0, \quad c_3 > 0, \quad \text{or} \quad \text{XIX'.} \dots g > S\lambda\mu, \quad g < T\lambda\mu,$$

the two sides (of one sheet) in the plane of $\lambda\mu$ have the directions,

$$\text{XX.} \dots \rho_1 = c_3^{-1}a_3 + (-c_1)^{-1}a_1, \quad \rho_2 = c_3^{-1}a_3 - (-c_1)^{-1}a_1;$$

if then their mutual inclination, or the angle of the cone in the plane of the cyclic normals, be denoted by $2b$, we have the values,

$$\text{XXI.} \dots \tan^2 b = \frac{c_3}{-c_1}, \quad \text{XXI'.} \dots \cos 2b = \frac{-c_1 - c_3}{-c_1 + c_3} = \frac{g}{T\lambda\mu};$$

the angle of the quaternion $\lambda\rho\rho\mu$ is therefore (by XVIII.), equal to this angle $2b$, namely to the arcual minor axis of the sphero-conic, in which the cone is cut by the concentric unit-sphere.

(10.) The same condition of reality II. may be obtained in a quite different way, as that of the reality of the reciprocal cone, which is the locus of the normal vector,

$$\text{XXII.} \dots v = \phi\rho = g\rho + V\lambda\rho\mu.$$

Inverting this linear function ϕ , by the method of the Section III. ii. 6, we find first the expression (comp. 854, (12.), and 361, (6.)^{*}),

$$\text{XXIII.} \dots m\rho = \psi v = \mu^2\lambda S\lambda v + \lambda^2\mu S\mu v - g(\lambda S\mu v + \mu S\lambda v) + (g^2 - \lambda^2\mu^2)v,$$

in which $\text{XXIV.} \dots m = (g - S\lambda\mu)(g^2 - \lambda^2\mu^2) = -c_1c_2c_3;$

and next the reciprocal equation (comp. 861, XXVII.),

$$\text{XXV.} \dots 0 = S v \psi v = \mu^2(S\lambda v)^2 + \lambda^2(S\mu v)^2 - 2gS\lambda v S\mu v + (g^2 - \lambda^2\mu^2)v^2,$$

which may be put under the form,

$$\text{XXVI.} \dots \cos \left(\angle \frac{v}{\lambda} + \angle \frac{v}{\mu} \right) = \frac{-g}{T\lambda\mu};$$

the quotient $g : T\lambda\mu$ thus presenting itself anew as a cosine, namely as that of the supplement of the sum of the inclinations of the normal v (to the cone I.), to the two cyclic normals λ, μ (of that cone); or as the cosine† of $\pi - A - B$, if A and B denote (comp. Fig. 80) the two spherical angles, which the tangent arc to the sphero-conic (9.) makes with the two cyclic arcs: so that by comparison of XXI'. and XXVI. we have the relation,

$$\text{XXVII.} \dots A + B = \angle \frac{v}{\lambda} + \angle \frac{v}{\mu} = \pi - 2b.$$

(11.) Comparing the expression XXI'. for $\cos 2b$, with the last expression

* In the expression 861, XXVI. for ψv , the second term ought to have been printed as $-V\lambda\mu S\lambda v\mu$; or else the sign should have been changed.

† This relation was mentioned by anticipation in 894, (8.); and the relation in XXVII. may easily be verified, by conceiving the point of contact P in Fig. 80 (page 548) to tend towards a minor summit of the conic, or the tangent arc APB to tend to pass through the two points c, c' , in which the cyclic arcs intersect.

XVIII. for $g: T\lambda\mu$, we derive the following construction for a *sphero-conic*, which may easily be verified by geometry:*

Having assumed two points (L, M) on a sphere, and having described a small circle round one of them (say L), bisect the arcs (MQ) which are drawn to its circumference from the other point; the locus of the bisecting points (P) will be a *sphero-conic*, with the two fixed points for its two cyclic poles (or for the poles of its cyclic arcs), and with an arcual minor axis ($2b$) equal to the arcual radius of the small circle.

(12.) As regards the arcual major axis (say $2a$) of the same sphero-conic, it is (with the conditions XIX.) the angle between the two sides (comp. XX.),

$$\text{XXVIII.} \dots \rho_3 = c_3^{-1}a_3 + (-c_2)^{-1}a_2, \quad \rho_4 = c_3^{-1}a_3 - (-c_2)^{-1}a_2;$$

whence (comp. XXI.),

$$\text{XXIX.} \dots \tan^2 a = \frac{c_3}{-c_2}, \quad \text{or} \quad \text{XXIX'.} \dots \cos 2a = \frac{-c_2 - c_3}{-c_2 + c_3} = (\text{say}) e,$$

and therefore, a few easy reductions being made,

$$\text{XXX.} \dots \frac{\sin b}{\sin a} = \sqrt{\left\{ \frac{1}{2} \left(1 + 8U \frac{\mu}{\lambda} \right) \right\}} = \cos \frac{1}{2} \angle \frac{\mu}{\lambda};$$

from which we can at once infer, that if a *focus* of the conic be determined, by drawing from a minor summit to the major axis an arc equal to the major semiaxis a , the minor axis subtends at this focus (or at the other) a spherical angle equal to the angle between the two cyclic arcs.

(13.) For the two real *unifocal transformations* of the equation of the cone, or the forms,

$$\text{XXXI.} \dots a(Va\rho)^2 + b(S\beta\rho)^2 = 0, \quad \text{and} \quad \text{XXXI'.} \dots a(Va'\rho)^2 + b(S\beta'\rho)^2 = 0,$$

with one common set of real values of the scalar coefficients, a and b , but with two real focal unit lines a, a' , and two real directive normals β, β' corresponding, it may be sufficient here to refer to the sub-articles to 358; except that it should be noticed, that if the cone be *real*, and if the line a_3 have the direction of its *internal axis*, so that the inequalities XIX. are satisfied, and therefore also (by 405, (6.)),

$$\text{XXXII.} \dots c_3^{-1} > 0 > c_1^{-1} > c_2^{-1},$$

instead of the inequalities 358, III., or 359, XXXVII., we are now to change, in the earlier formulæ referred to, the symbols $c_1c_2c_3a_1a_2a_3$ to $c_3c_1c_2a_3a_1a_2$, so that we have now the values,

$$\text{XXXIII.} \dots a = -c_1, \quad b = c_3 - c_1 + c_2, \quad \text{if} \quad T\beta = T\beta' = 1.$$

(14.) And as regards the *interpretation* of the *unifocal form* XXXI., with these last values, it is evidently contained in this other equation,

$$\text{XXXIV.} \dots \sin \angle \frac{\rho}{a} \cdot \sec \angle \frac{\rho}{\beta} = \frac{TVa\rho}{-S\beta\rho} = \left(\frac{c_3 - c_1 + c_2}{-c_1} \right) = \text{const.};$$

the sines of the inclinations of an arbitrary side (ρ) of the cone, to a focal line (a),

* In fact, the bisecting radii OP are parallel to the supplementary chords $M'Q$, if MM' be a diameter of the sphere; and the locus of all such chords is a cyclic cone, resting on the small circle as its base.

and to the corresponding *director plane* ($\perp \beta$), thus bearing to each other (as is well known) a *constant ratio*, which remains unchanged when we pass to the *other* (real) *focal line* (α'), and at the same time to the *other* (real) *director plane* ($\perp \beta'$): and the *focal plane* of these *two lines* (α, α') being *perpendicular to that one of the three axes*, which corresponds to the *root* (here e_1 , by XXXII.) of the *cubic*, of which the *reciprocal* is algebraically *intermediates* between the reciprocals of the other two.

(15.) It is, however, more *symmetric* to employ the *bifocal transformation* (comp. 360, VI.*),

$$\text{XXXV.} \dots 0 = (Sap)^2 - 2eSapSa'\rho + (Sa'\rho)^2 + (1 - e^2)\rho^2;$$

in which the scalar constant e has the value (comp. XXIX'),

$$\text{XXXVI.} \dots e = \cos 2a;$$

and α, α' are the *two†* real and *focal unit lines*, recently considered (13.).

(16.) The equation XXXV., for the case of a *real cone*, may be thus written (comp. XXVI. XXXVI.),

$$\text{XXXVII.} \dots \angle \frac{\rho}{\alpha} + \angle \frac{\rho}{\alpha'} = \cos^{-1} e = 2a;$$

the *sum‡* of the *inclinations* of the *side* ρ to the *two focal lines* α, α' being thus *constant*, and equal (as is well known) to the *major axis* of the *spherical conic*: and although, when $e > 1$, the *cone* becomes *imaginary*, yet it is then *asymptotic* to a *real ellipsoid*, as we shall shortly see.

407. The *bifocal form* (406, XXXV.) of the equation of a *cone* may suggest the corresponding *form*,

$$\text{I.} \dots C = C\rho = (Sap)^2 - 2eSapSa'\rho + (Sa'\rho)^2 + (1 - e^2)\rho^2,$$

in which a and α' are given and generally non-parallel unit-lines, while e and C are scalar constants, as capable of representing *generally* (comp. 360, (2.), (3.)) a *central* but *non-conical surface* ($f\rho = 1$) of the *second order*. And we shall find that if, in passing from *one* such surface to *another*, we suppose a and α' to remain *unchanged*, but e and C to *vary together*, so as to be always connected by the relation,

$$\text{II.} \dots C = (e^2 - 1) (e + Saa')^2,$$

in which l is some real, positive, and *given scalar*, then all the sur-

* It is to be remembered that, in the formula here cited, the symbols α, α' did not denote unit-vectors.

† When these two vectors α, α' remain *constant*, but the scalar e *changes*, there arises a system of *biconfocal cones*: or, by their intersections with a concentric sphere, a system of *biconfocal sphero-conics*. Compare the Note to page 640.

‡ Or the *difference*, according to the choice between two opposite directions, for one of the two focal lines. The *angular transformation* XXXVII. may be accomplished, by *resolving* the equation XXXV. as a *quadratic* in e , and then interpreting the result.

faces I. so deduced, or in other words the surfaces represented by the common equation,

$$\text{III.} \dots f^2 = P^2 f \rho = \frac{(S a \rho)^2 - 2 e S a \rho S a' \rho + (S a' \rho)^2 + (1 - e^2) \rho^2}{(e^2 - 1)(e + S a a')},$$

with e for the only variable parameter, compose a *Confocal System*.

(1.) The scalar form III. of $f \rho$ gives the connected vector form,

$$\text{IV.} \dots P \nu = P \phi \rho = \frac{a S(a - e a') \rho + a' S(a' - e a) \rho + (1 - e^2) \rho}{(e^2 - 1)(e + S a a')},$$

which may also be thus written, with the value II. of C ,

$$\text{V.} \dots C \nu = C \phi \rho = (a - e a') S a \rho + (a' - e a) S a' \rho + (1 - e^2) \rho,$$

so that the function ϕ is *self-conjugate*, as it ought to be.

(2.) And because we have thus,

$$\text{VI.} \dots (e^2 - 1) P \phi a = a' - e a, \quad (e^2 - 1) P \phi a' = a - e a',$$

if we write, for abridgment,

$$\text{VII.} \dots a^2 = (e + 1) P, \quad b^2 = (e + S a a') P, \quad c^2 = (e - 1) P,$$

we shall have the values,

$$\text{VIII.} \dots \begin{cases} \phi(a + a') = -a^2(a + a'), \\ \phi V a a' = -b^2 V a a', \\ \phi(a - a') = -c^2(a - a'); \end{cases}$$

comparing which with 405, (1.), (2.), we see that the three (real or imaginary) lines,

$$\text{IX.} \dots a U(a + a'), \quad b U V a a', \quad e U(a - a'),$$

of any one of which the direction may be reversed, are the *three vector semi-axes* of the surface $f \rho = 1$; and therefore, by VII., that the *system* III. is one of *confocals*, as asserted.

(3.) The *rectangular transformations*, scalar and vector, are now (comp. 405, X., and 857, V. VIII.):

$$\text{X.} \dots f^2 = P^2 f \rho = \frac{(S \rho U(a + a'))^2}{e + 1} + \frac{(S \rho U V a a')^2}{e + S a a'} + \frac{(S \rho U(a - a'))^2}{e - 1};$$

$$\text{XI.} \dots P \nu = P \phi \rho = \frac{U(a + a') \cdot S \rho U(a + a')}{e + 1} + \frac{U V a a' \cdot S \rho U V a a'}{e + S a a'} + \frac{U(a - a') \cdot S \rho U(a - a')}{e - 1};$$

which can both be established, by the rules of the present Calculus, in several other ways, and from the first of which it follows that (as is well known) *through any proposed point* P of space there can in general be drawn *three confocal surfaces*, of a given *system* III.; one an *ellipsoid*, for which $e > 1$, and therefore $a^2 > b^2 > c^2 > 0$; another a *single-sheeted hyperboloid*, for which $e < 1$, $e > -S a a'$, $a^2 > b^2 > 0 > c^2$; and the third a *double-sheeted hyperboloid*, for which $e < -S a a'$, $e > -1$, $a^2 > 0 > b^2 > c^2$.

(4.) From the *other* rectangular transformation XI. it follows, that if we denote by $\nu_1 = \phi_1\rho$ what the normal vector $\nu = \phi\rho$ becomes, when ρ remains the same, but e is changed to a *second root* e_1 of the equation III. or X. of the *surface*, considered as a *cubic* in e , then

$$\text{XII.} \dots \frac{\nu_1 - \nu}{e_1 - e} = \rho^2 \phi \nu_1 = \rho^2 \phi_1 \nu = \rho^2 \phi_1 \phi \rho = \rho^2 \phi \phi_1 \rho;$$

but

$$\text{XIII.} \dots S\rho\nu_1 = S\rho\nu = f_1\rho = f\rho = 1,$$

$f_1\rho$ being formed from $f\rho$, by the substitution of e_1 for e ; therefore,

$$\text{XIV.} \dots 0 = S\rho\phi\nu_1 = S\nu_1\phi\rho = S\nu_1\nu,$$

and the known theorem results, that *confocal surfaces cut each other orthogonally*.*

(5.) It follows, from V. and VI., that the *inverse function* $\phi^{-1}\rho$ can be expressed as follows:

$$\text{XV.} \dots \phi^{-1}\rho = \rho^2(aS\alpha'\rho + a'S\alpha\rho) - b^2\rho;$$

or that ρ may be deduced from ν by the formula,

$$\text{XVI.} \dots \rho = \phi^{-1}\nu = \rho^2(aS\alpha'\nu + a'S\alpha\nu) - b^2\nu,$$

which can easily be otherwise established. Hence (comp. 861, (4.)), the equation of the *surface reciprocal* to the surface I. or III., or of that *new surface* which has ν (instead of ρ) for its *variable vector*, is

$$\text{XVII.} \dots 1 = F\nu = S\nu\phi^{-1}\nu = 2\rho^2S\alpha\nu S\alpha'\nu - b^2\nu^2;$$

the *fixed focal lines* a, a' of the *confocal system* III., or of the corresponding system of the *asymptotic cones*, becoming thus (in agreement with known results) the *fixed cyclic normals* (or *cyclic lines*, comp. 861, (6.)) of the *reciprocal system* XVII.

(6.) In thus deducing the equation XVII. from III., *no use* has been made of the *rectangular transformations* X. XI., of the functions $f\rho$ and $\phi\rho$. *Without* the transformations last referred to, we could therefore have inferred, by a slight modification of the form XVII., that the *reciprocal surface* ($F\nu = 1$) with ν for its *variable vector*, which has the *same rectangular system* of *directions* for its three *semiaxes* as the *original surface* ($f\rho = 1$), but with *inverse squares* (the roots of its cubic) equal to the *direct squares* of the *original semiaxes*, has for equation (comp. 405, XII.),

$$\text{XVIII.} \dots 1 = F\nu = \rho^2(S\alpha\nu a'\nu - e\nu^2) = S\lambda\nu\mu\nu + g\nu^2,$$

if

$$\text{XIX.} \dots \lambda = la, \quad \mu = la', \quad g = -e\rho^2 = -eT\lambda\mu;$$

the values VII. of a^2, b^2, c^2 being thus deduced *anew*, but by a process quite different from that employed in (2.), under the forms (comp. 405, XIII.),

$$\text{XX.} \dots a^2 = c_2 = -g + T\lambda\mu; \quad b^2 = c_2 = -g + S\lambda\mu; \quad c^2 = c_1 = -g - T\lambda\mu;$$

while the *directions* IX. of the corresponding *semiaxes* may be deduced as those of $\alpha_2, \alpha_2, \alpha_1$, from the formulæ 405, XIV.

(7.) If the symbol $\omega(\nu)$, or simply $\omega\nu$, be used to denote a *new linear and self-conjugate vector function* of ν , defined by the equation,

$$\text{XXI.} \dots \omega\nu = \rho S\rho\nu - \rho^2(aS\alpha'\nu + a'S\alpha\nu),$$

* We shall soon see that the *same formula* XII., by expressing that ν, ν_1 , and $\phi\nu_1$ or $\phi_1\nu$ are *coplanar*, contains this *other part* of the known theorem referred to, that the *intersection* is a *line of curvature*, on each of the two *confocals*.

with ρ here treated as a vector constant, then (because $S\rho\nu = 1$) the equation XVI. may be thus written (comp. 854, &c.),

$$\text{XXII.} \dots (\omega + b^2)\nu = 0;$$

the three rectangular directions, of the three normals ν, ν_1, ν_2 to the three confocals through P , are therefore those which satisfy (comp. again 354) the vector quadratic equation,

$$\text{XXIII.} \dots V\nu\omega\nu = 0;$$

and they are the directions of the axes of this new surface of the second order (comp. 857, &c.),

$$\text{XXIV.} \dots S\nu\nu\nu = (S\rho\nu)^2 - 2P^2S\nu\nu S\nu\nu = 1,$$

in which ρ is still treated as a constant vector, but ν as a variable one.

(8.) The inverse squares of the scalar semi-axes of this new surface ($S\nu\nu\nu = 1$), are the direct squares b^2, b_1^2, b_2^2 of what may be called the mean semi-axes of the three confocals; these latter squares must therefore be the roots of this new cubic,

$$\text{XXV.} \dots 0 = m + m'b^2 + m''(b^2)^2 + (b^2)^3,$$

in which the coefficients m, m', m'' , deduced here from the new function ω , as they were deduced from ϕ in the Section III. ii. 6, have the values,

$$\text{XXVI.} \dots \begin{cases} m = I^4(Saa'\rho)^2; \\ m' = I^4(Vaa')^2 + 2P^2S(Vap.Va'\rho); \\ m'' = \rho^2 - 2P^2Saa'. \end{cases}$$

Accordingly, if we observe that (because $Ta = Ta' = 1$) we have among others the transformation,

$$\text{XXVII.} \dots (Saa'\rho)^2 = \rho^2(Vaa')^2 - (Sap)^2 - 2Saa'SapSa'\rho - (Sa'\rho)^2,$$

we can express this last cubic equation XXV., with these values XXVI. of its coefficients, under the form,

$$\text{XXVIII.} \dots 0 = (b^2 + \rho^2) \{ (b^2 - P^2Saa')^2 - I^4 \} \\ + 2P^2(b^2 - P^2Saa')SapSa'\rho - I^4((Sap)^2 + (Sa'\rho)^2);$$

which, when we change b^2 by VII. to its value $P^2(e + Saa')$, and divide by I^4 , becomes the cubic in e , or the equation III. under the form,

$$\text{XXIX.} \dots 0 = (e^2 - 1) \{ P^2(e + Saa') + \rho^2 \} + 2eSapSa'\rho - (Sap)^2 - (Sa'\rho)^2.$$

(9.) As an additional test of the consistency of this whole theory and method, the directions of the three axes of the new surface XXIV., or those of the three normals (7.) to the confocals, or the three vector roots (854) of the equation XXIII., ought to admit of being assigned by three expressions of the forms,

$$\text{XXX.} \dots \begin{cases} n\nu = \psi\sigma + b^2\chi\sigma + b^4\sigma, \\ n_1\nu_1 = \psi\sigma_1 + b_1^2\chi\sigma_1 + b_1^4\sigma_1, \\ n_2\nu_2 = \psi\sigma_2 + b_2^2\chi\sigma_2 + b_2^4\sigma_2; \end{cases}$$

in which b^2, b_1^2, b_2^2 are the three scalar roots of the cubic XXV. or XXVIII., while $\sigma, \sigma_1, \sigma_2$ are three arbitrary vectors; n, n_1, n_2 are three scalar coefficients, which can be determined by the conditions $S\rho\nu = S\rho\nu_1 = S\rho\nu_2 = 1$ (comp. XIII.); and ψ, χ are two new auxiliary linear and vector functions, to be deduced here from the function ω , in the same manner as they were deduced from ϕ in the Section lately referred to.

(10.) Accordingly, by the method of that Section, taking for convenience the given* vector ρ (instead of the arbitrary vectors $\sigma, \sigma_1, \sigma_2$) as the subject of the operations ψ and χ , we find the expressions,

$$\text{XXXI.} \dots \psi\rho = l^2 \nabla a a' S a a' \rho, \quad \chi\rho = l^2 (a S a' \rho + a' S a \rho - 2\rho S a a');$$

whence, after a few reductions, with elimination of s by the relation $S\rho\nu = 1$, and by the cubic in b^2 , the first equation XXX. becomes:

$$\text{XXXII.} \dots 0 = (b^2\nu + \rho) \{ (b^2 - l^2 S a a')^2 - l^4 \} \\ + l^2 (b^2 - l^2 S a a') (a S a' \rho + a' S a \rho) - l^4 (a S a \rho + a' S a' \rho);$$

which is in fact a form of the relation between ν and ρ , for any one of the confocals, as appears (for instance) by again changing b^2 to $l^2(e + S a a')$, and comparing with the equation IV.

(11.) Another and a more interesting auxiliary surface, of which the axes have still the directions of the normals ν , is found by inverting the new linear function ω , or by forming from XXII. the inverse equation,

$$\text{XXXIII.} \dots (\omega^{-1} + b^{-2})\nu = 0;$$

in which,

$$\text{XXXIV.} \dots \omega^{-1}\nu \cdot (S a a' \rho)^2 = \nabla a a' S a a' \nu + l^2 (\nabla a \rho S a' \rho \nu + \nabla a' \rho S a \rho \nu);$$

and from which it follows that the normals ν to the confocals through P have the directions of the axes of this new cone,

$$\text{XXXV.} \dots S \nu \omega^{-1} \nu = 0, \quad \text{or} \quad \text{XXXVI.} \dots 0 = l^2 (S a a' \nu)^2 + 2 S a \rho \nu S a' \rho \nu,$$

with ρ treated as a constant, as before.

(12.) The vertex of this auxiliary cone being placed at the given point P , of intersection of the three confocals, we may inquire in what curve is the cone cut, by the plane of the given focal lines, a, a' , drawn through the common centre O of all the surfaces III. Denoting by $\sigma = t a + t' a'$ the vector of a point s of this sought section, and writing

$$\text{XXXVII.} \dots \nu = \sigma - \rho = t a + t' a' - \rho,$$

the equation XXXVI. gives the relation,

$$\text{XXXVIII.} \dots t t' = \frac{l^2}{2} = \frac{a^2 - c^2}{4} = \text{const.};$$

the section is therefore an hyperbola, which is independent of the point P , and has the focal lines of the system for its asymptotes. And because its vector equation may be thus written (comp. 871, II.),

$$\text{XXXIX.} \dots \sigma = t a + \frac{1}{2} l^2 t^{-1} a',$$

or what may be called its quaternion equation as follows (comp. 871, I.),

$$\text{XL.} \dots 2 \nabla a \sigma \cdot \nabla a' \sigma = l^2 (\nabla a a')^2,$$

it satisfies the two scalar equations,

$$\text{XLI.} \dots m = 0, \quad m' = 0,$$

with the significations XXVI. of m and m' ; it is therefore that important curve, which is known by the name of the Focal Hyperbola :† namely the limit to which

* The general expressions for $\psi\sigma$ and $\chi\sigma$ include terms, which vanish when $\sigma = \rho$.

† Compare the Notes to pages 231, 505.

the section of the confocal surface by the plane of its extreme* axes tends, when the mean axis ($2b$) tends to vanish. We are then led thus to the known theorem, that if, with any assumed point r for vertex, and with the focal hyperbola† for base, a cone be constructed, the axes of this focal cone have the directions of the normals to the confocals through r .

(13.) As regards the Focal Ellipse, its two scalar equations may be deduced from the rectangular form X., by equating to zero both the numerator and the denominator of its last term; they are therefore,

$$\text{XLII.} \dots S(a - a')\rho = 0, \quad 2l^2 = (SpU(a + a'))^2 + \left(\frac{SpUVaa'}{SVaa'} \right)^2;$$

the curve being thus given as a perpendicular section of an elliptic cylinder, with $l\sqrt{2}$ and $l\sqrt{1 + Saa'}$, or $(a^2 - c^2)^{\frac{1}{2}}$ and $(b^2 - c^2)^{\frac{1}{2}}$, for the semiaxes of its base, or of the ellipse itself.

(14.) The same curve may also be represented by the equations,

$$\text{XLIII.} \dots Sap = Sa'\rho, \quad TVa\rho = (b^2 - c^2)^{\frac{1}{2}},$$

or

$$\text{XLIII'.} \dots Sa'\rho = Sap, \quad TVa'\rho = (b^2 - c^2)^{\frac{1}{2}};$$

which express that it is the common intersection of its own plane ($\perp a - a'$) with two right cylinders,‡ which have the two focal lines a, a' of the system for their axes of revolution, and have equal radii, denoted each by the radical last written.

(15.) In general, the unifocal form (comp. 406, (13.)) of the equation III., namely,

$$\text{XLIV.} \dots 0 = (1 - e^2)((Vap)^2 + b^2) + (S(a' - ea)\rho)^2,$$

in which a and a' may be interchanged, shows that the two equal right cylinders,

$$\text{XLV.} \dots (Vap)^2 + b^2 = 0, \quad \text{XLV'.} \dots (Va'\rho)^2 + b^2 = 0,$$

or

$$\text{XLVI.} \dots TVa\rho = b, \quad \text{XLVI'.} \dots TVa'\rho = b,$$

which are real if their common radius b be such, that is, if the confocal (e) be either an ellipsoid (supposed to be real), or else a single-sheeted hyperboloid, and which have the focal lines a, a' of the system for their axes of revolution, envelopes§ that confocal surface; the planes of the two ellipses of contact (which again are real curves, if b be real) being given by the equations,

$$\text{XLVII.} \dots S(a' - ea)\rho = 0, \quad \text{XLVII'.} \dots S(a - ea')\rho = 0;$$

so that they pass through the centre o of the surface (or of the system), and are the (real) director planes (comp. 406, (14.)) of the asymptotic cone (real or imaginary), to the particular confocal (e).

* Namely, those two of which the squares algebraically include between them that of the third; this latter being, for the same reason, considered here as the mean.

† We shall soon see that quaternions give, with equal ease, a more general known theorem, in which this is included as a limit.

‡ The reader may consult page 513 of the Lectures, for the case of this theorem which answers to a given ellipsoid. The focal ellipse may also be represented generally by the expression (comp. page 382 of these Elements),

$$\rho = (a^2 - c^2)^{\frac{1}{2}} V.a'U(a + a');$$

or by the same expression, with a and a' interchanged.

§ Compare pages 199, 228, 233, 299.

(16.) Whether the mean *semiaxis* (b) be real or imaginary, the *surface* III. (supposed to be itself *real*) is always, by the form XLIV. of its equation, the *locus* of a system of *real ellipses* (comp. 404, (1.)), in planes *parallel* to the *director plane* XLVII., which have their *centres on the focal line* a , and are *orthogonally projected into circles* on a plane *perpendicular to that line*.

(17.) The *same surface* is also the *locus* of a *second system* of such ellipses, related similarly to the *second focal line* a' , and to the *second director plane* XLVII'. ; and it appears that *these two systems* of *elliptic sections* of a *surface* of the *second order*, which from some points of view are nearly as interesting as the *circular sections*, may conveniently be called its *Centro-Focal Ellipses*.

(18.) For example, when the *first quaternion form* (204, (14.)), or 404, I. of the equation of the *ellipsoid* is employed, *one system* of such ellipses coincides with the *system* (204, (18.)) of which, in the *first generation** of the *surface*, the *ellipsoid*

* Besides that *first generation* (I) of the *ellipsoid*, which was a *double one*, in the sense that a *second system* (17.) of *generating ellipses* might be employed, and which served to connect the *surface* with a *concentric sphere*, by certain relations of *homology* (274); and the *second double generation* or *construction* (II), by means of either of *two diacentric spheres* (217, (4.), (6.), (7.), and 220, (3.)), which was illustrated by Fig. 58 (page 226): several *other generations* of the same important *surface* were deduced from *quaternions* in the *Lectures*, to which it is only possible here to *refer*. A reader, then, who happens to have a copy of that earlier work, may consult page 499 for a *generation* (III) of a *system* of *two reciprocal ellipsoids*, with a *common mean axis* ($2b$), by means of a *moving sphere*, of which the *radius* ($=b$) is *given*, but of which the *centre* has the *original ellipsoid* for its *locus*; while the *corresponding point* on the *reciprocal surface*, and also the *normals* at the two points, are easily deduced from the construction. In page 502, he will find another and perhaps a *simpler generation* (IV), of the *same pair* of *reciprocal ellipsoids*, by means of *quadrilaterals inscribed in a fixed sphere* (the *common mean sphere*, comp. 216, (10.)); the *directions* of the *four sides* of such a *quadrilateral* being *given*, and *one pair* of *opposite sides* intersecting in a point of *one surface*, while the *other pair* have for their intersection the *corresponding point* of the *other* (or *reciprocal*) *ellipsoid*. In the page last cited, and in the following page, there is given a *new double generation* (V) of any *one ellipsoid*; its *circular sections* (of either system) being constructed as *intersections* of *two equal spheres* (or *spheric surfaces*), of which the *line of centres* retains a *fixed direction*, while the *spheres slide* within *two equal and right cylinders*, whose *axes* intersect each other (in the *centre* of the generated *surface*), and of which the *common radius* is the *mean semiaxis* (b). Finally, in page 699 of the same volume, there will be found a *new generation* (VI) of the *original ellipsoid* (abc), *analogous* to the *generation* (IV) by the *fixed (mean) sphere*, but with *new directions* of the *sides* of the *quadrilaterals*, which are also (in this *last generation*) *inscribed in the circles* of a certain *mean ellipsoid* (or *prolate spheroid*) of *revolution*, which has the *mean axis* ($2b$) for its *major axis*, and has *two medial foci* on that axis, whose *common distance* from the *centre* is represented by the expression,

$$\frac{\sqrt{(a^2 - b^2)} \sqrt{(b^2 - c^2)}}{\sqrt{(a^2 - b^2 + c^2)}};$$

was treated as the *locus*; and an analogous generation of the two hyperboloids, by geometrical deformation of two corresponding surfaces of revolution, with certain resulting homologies (comp. sub-arts. to 274), through substitution of (*centre-focal*) ellipses for circles, conducts to equations of those hyperboloids of the same *unifocal form*; namely, if α and β have significations analogous to those in the cited equation of the ellipsoid (so that β and not α is here a focal line),

$$\text{XLVIII.} \dots \left(S \frac{\rho}{\alpha} \right)^2 + \left(V \frac{\rho}{\beta} \right)^2 = \mp 1;$$

the upper or the lower sign being taken, according as the surface consists of one sheet or of two.

(19.) It may also be remarked that as, by changing β to α in the corresponding equation of the ellipsoid, we could return (comp. 404, (1.)) to a form (403, XL) of the equation of the sphere, so the same change in XLVIII. conducts to equations of the equilateral hyperboloids of revolution, of one sheet and of two, under the very simple forms* (comp. 210. XI.),

$$\text{XLIX.} \dots S \left(\frac{\rho}{\alpha} \right)^2 = -1, \quad \text{and} \quad \text{L.} \dots S \left(\frac{\rho}{\alpha} \right)^2 = +1;$$

in which it seems unnecessary to insert *points* after the signs S, and of which the geometrical interpretations become obvious when then they are written thus (comp. 199, V.),

$$\text{LI.} \dots T \frac{\rho}{\alpha} = \sqrt{\sec 2 \left(\frac{\pi}{2} - \angle \frac{\rho}{\alpha} \right)}, \quad \text{LII.} \dots T \frac{\rho}{\alpha} = \sqrt{\sec 2 \angle \frac{\rho}{\alpha}};$$

where $T \frac{\rho}{\alpha} = \overline{OP} : \overline{OA}$, while $\angle \frac{\rho}{\alpha}$ is the inclination $\angle OP$ of the *semidiameter* or to the axis of revolution OA , and $\frac{\pi}{2} - \angle \frac{\rho}{\alpha}$ is the inclination of the same semidiameter to a plane perpendicular to that axis.

(20.) The *real cyclic forms* of the equation of the surface III. might be deduced from the *unifocal form* XLIV., by the general method of the subarticles to 359; but since we have ready the *rectangular form* X., it is simpler to obtain them from that form, with the help of the identity,

$$\text{LIII.} \dots -\rho^2 = (SpU(\alpha + \alpha'))^2 + (SpUV\alpha\alpha')^2 + (SpU(\alpha - \alpha'))^2,$$

by eliminating the *first* of these three terms for the case of a *single-sheeted hyperbo-*

the common tangent planes, to this mean (or medial) ellipsoid, and to the given (or generated) ellipsoid (*abc*), which are parallel to their common axis (2*b*), being parallel also to the two umbilical diameters of the latter surface.

* The same forms, but with σ for ρ , and β for α , may be deduced from XLVIII. on the plan of 274, (2.), (4.), by assuming an auxiliary vector σ such that $S \frac{\sigma}{\beta} = \pm S \frac{\rho}{\alpha}$, and $V \frac{\sigma}{\beta} = V \frac{\rho}{\alpha}$; the homologies, above alluded to, between the general hyperboloid of either species, and the equilateral hyperboloid of revolution of the same species, admitting also thus of being easily exhibited.

loid (for which $b^{-2} > a^{-2} > 0 > c^{-2}$); the second for an ellipsoid ($c^{-2} > b^{-2} > a^{-2} > 0$); and the third for a double-sheeted hyperboloid ($a^{-2} > 0 > c^{-2} > b^{-2}$).

(21.) Whatever the species of the surface III. may be, we can always derive from the unifocal form XLIV. of its equation what may be called an *Exponential Transformation*; namely the vector expression,

$$\text{LIV.} \dots \rho = xa + yVa\beta, \text{ with LV.} \dots x^2fa + y^2fUVaa' = 1;$$

the scalar exponent, t , remaining arbitrary, but the two scalar coefficients, x and y , being connected by this last equation of the second degree: provided that the new constant vector β be derived from a , a' , and ϵ , by the formula,

$$\text{LVI.} \dots \beta = \frac{(a' - \epsilon a) UVaa'}{\epsilon + Sa'a'}$$

which gives after a few reductions (comp. the expression 315, III. for a' , when $Ta = 1$),

$$\text{LVII.} \dots Va\beta = UVaa', \quad S(a' - \epsilon a)\beta = 0, \quad Saa'\beta = 0;$$

$$\text{LVIII.} \dots Va\beta = \beta S.a^t + UVaa'.S.a^{t-1}; \quad \text{LIX.} \dots V.aVa'\beta = a'UVaa' = T^{-1};$$

$$\text{LX.} \dots S(a' - \epsilon a)\rho = x(\epsilon + Sa'a'), \quad Vap = ya'UVaa';$$

$$\text{while LXI.} \dots fa = a^{-2}b^2c^{-2}, \quad \text{and LXII.} \dots f\beta = fUVaa' = b^{-2}.$$

(22.) If we treat the exponent, t , as the only variable in the expression LIV. for ρ , then (comp. 314, (2.)) that exponential expression represents what we have called (17.) a *centro-focal ellipse*; the distance of its centre (or of its plane) from the centre of the surface, measured along the focal line a , being represented by the coefficient x ; and the radius of the right cylinder, of which the ellipse is a section, or the radius of the circle (16.) into which that ellipse is projected, on a plane $\perp a$, being represented by the other coefficient, y : while $\frac{1}{2}t\pi$ is the excentric anomaly.

(23.) If, on the contrary, we treat the exponent t as given, but the coefficients x and y as varying together, so as to satisfy the equation LV. of the second degree, the expression LIV. then represents a different section of the surface III., made by a plane through the line a , which makes with the focal plane (of a , a') an angle $= \frac{t\pi}{2}$; this latter section (like the former) being always real, if the surface itself be such: but being an ellipse for an ellipsoid, and a hyperbola for either hyperboloid, because

$$\text{LXIII.} \dots fa.fUVaa' = a^{-2}c^{-2} \text{ by LXI. and LXII.}$$

(24.) And it is scarcely necessary to remark, that by interchanging a and a' we obtain a *Second Exponential Transformation*, connected with the second system (17.) of centro-focal ellipses, as the first exponential transformation LIV. is connected with the first system (16.).

(25.) The asymptotic cone $f\rho = 0$ has likewise its two systems of centro-focal ellipses, and its equation admits in like manner of two exponential transformations, of the form LIV.; the only difference being, that the equation LV. is replaced by the following,

$$\text{LXIV.} \dots x^2fa + y^2fUVaa' = 0,$$

in which, for a real cone, the coefficients of x^2 and y^2 have opposite signs by (23.).

(26.) Finally, as regards the confocal relation of the surfaces III., which may represent any confocal system of surfaces of the second order, it may be perceived

from (4.) that an *essential character* of such a relation is expressed by the equation,

$$\text{LXV.} \dots \nabla \nu, \phi \nu, = \nabla \nu \phi, \nu ;$$

which may perhaps be called, on that account, the *Equation of Confocals*.

(27.) It is understood that the two *confocal surfaces* here considered, are represented by the two scalar equations,

$$\text{LXVI.} \dots S\rho\phi\rho = 1, \quad S\rho\phi, \rho = 1, \quad \text{or} \quad \text{LXVI'.} \dots f\rho = 1, \quad f, \rho = 1 ;$$

and that the two *linear and vector functions*, ν and ν, ρ , of an *arbitrary vector* ρ , which represent *normals* to the two *concentric and similar and similarly posited surfaces*,

$$\text{LXVII.} \dots f\rho = \text{const.}, \quad f, \rho = \text{const.},$$

passing through any proposed point P , are expressed as follows,

$$\text{LXVIII.} \dots \nu = \phi\rho, \quad \nu, = \phi, \rho.$$

(28.) It is understood also, that the two surfaces LXVI. or LXVI'. are not only *concentric*, as their equations show, but also *coaxial*, so far as the *directions of their axes* are concerned: or that the two *vector quadratics* (comp. 254),

$$\text{LXIX.} \dots \nabla\rho\phi\rho = 0, \quad \text{and} \quad \text{LXX.} \dots \nabla\rho\phi, \rho = 0,$$

are satisfied by one common *system of three rectangular unit lines*. And with these understandings, it will be found that the equation LXV., which has been called above the *Equation of Confocals*, is not only *necessary* but *sufficient*, for the establishment of the relation required.

(29.) It is worth while however to observe, before closing the present series of subarticles, that the equations XII., and those formed from them by introducing e_2 and ν_2 , give the following among other relations:

$$\text{LXXI.} \dots fU\nu_1 = (b^2 - b_1^2)^{-1} = -f_1U\nu; \quad f_1U\nu_2 = (b_1^2 - b_2^2)^{-1} = -f_2U\nu_1; \quad \&c. ;$$

$$\text{and} \quad \text{LXXII.} \dots f(\nu_1, \nu_2) = f_1(\nu_2, \nu) = f_2(\nu, \nu_1) = 0 ;$$

and therefore,

$$\text{LXXIII.} \dots f_1\{(b_2^2 - b_1^2)^{\frac{1}{2}}U\nu_2 \pm (b_1^2 - b^2)^{\frac{1}{2}}U\nu\} = 0 ;$$

whence it is easy to see that the two *vectors* under the functional sign f_1 in this last expression have the directions of the *generating lines* of the *single-sheeted hyperboloid* (e_1) through P , if we suppose that $b_2^2 > b_1^2 > 0 > b^2$, so that the confocal (e_2) is here an *ellipsoid*, and (e) a *double-sheeted hyperboloid*.

(30.) But if σ be taken to denote the variable vector of the *auxiliary surface* XXIV., the equation of that surface may by (7.) and (8.) be brought to the following rectangular form, with the meaning XXI. of ω ,

$$\text{LXXIV.} \dots 1 = S\omega\omega = (S\rho\sigma)^2 - 2f^2Sa\sigma Sa'\sigma = b^2(S\sigma U\nu)^2 + b_1^2(S\sigma U\nu_1)^2 + b_2^2(S\sigma U\nu_2)^2 ;$$

hence, with the inequalities (29.), its *cyclic normals*, or those of its *asymptotic cone* $S\omega\omega = 0$, or the *focal lines* of the *reciprocal cone* $S\omega^{-1}\sigma = 0$, that is of the cone XXXVI., or finally the *focal lines of the focal cone* (12.), which rests on the *focal hyperbola*, have the directions of the lines LXXIII.; those *focal lines* are therefore

* A more general known theorem, including this, will soon be proved by quaternions.

(by what has just been seen) the generating lines of the hyperboloid (σ_1), which passes through the given point r .

(81.) And for an arbitrary σ we have the transformation,

$$\text{LXXV.} \dots t^2(S\rho\sigma)^2 - S\sigma\sigma'\sigma = \varepsilon(S\sigma U\nu)^2 + \varepsilon_1(S\sigma U\nu_1)^2 + \varepsilon_2(S\sigma U\nu_2)^2.$$

408. The general equation* of conjugation,

$$\text{I.} \dots (f\rho, \rho') = 1, \quad 405, \text{ III.}$$

connecting the vectors ρ, ρ' of any two points P, P' which are conjugate with respect to the central but non-conical surface $f\rho = 1$, may be called for that reason the Equation of Conjugate Points; while the analogous equation,

$$\text{II.} \dots f(\rho, \rho') = 0,$$

which replaces the former for the case of the asymptotic cone $f\rho = 0$, may be called by contrast the Equation of Conjugate Directions: in fact, it is satisfied by any two conjugate semidiameters, as may be at once inferred from the differential equation $f(\rho, d\rho) = 0$ of the surface $f\rho = \text{const.}$ (comp. 362). Each of these two formulæ admits of numerous applications, among which we shall here consider the deduction, and some of the transformations, of the Equation of a Circumscribed Cone,

$$\text{III.} \dots (f(\rho, \rho') - 1)^2 = (f\rho - 1)(f\rho' - 1);$$

which may also be considered as the Condition of Contact, of the right line PP' with the surface $f\rho = 1$.

(1.) In this last view, the equation III. may be at once deduced, as the condition of equal roots in the scalar and quadratic equation (comp. 216, (2.), and 816, (80.)),

$$\text{IV.} \dots 0 = f(x\rho + x'\rho') - (x + x')^2,$$

$$\text{or V.} \dots 0 = x^2(f\rho - 1) + 2xx'(f(\rho, \rho') - 1) + x'^2(f\rho' - 1);$$

which gives in general the two vectors of intersection, as the two values of the expression $\frac{x\rho + x'\rho'}{x + x'}$.

(2.) If we treat the point r' as given, and denote the two secants drawn from it in any given direction r by $t_1^{-1}r$ and $t_2^{-1}r$, then t_1 and t_2 are the roots of this other quadratic, $f(\rho' + t^{-1}r) = 1$, or

$$\text{VI.} \dots 0 = f(t\rho' + r) - t^2 = t^2(f\rho' - 1) + 2tf(\rho', r) + fr;$$

denoting then by $t_0^{-1}r$ the harmonic mean of these two secants, so that $2t_0 = t_1 + t_2$, and writing $\rho = \rho' + t_0^{-1}r$, we have

$$\text{VII.} \dots t_0(1 - f\rho) = f(\rho', r), \quad f(\rho, \rho') = 1;$$

* For the notation used, Art. 362 may be again referred to.

we are then led in this way to the formula I., as the *Equation of the Polar Plane* of the point r' , if that plane be here supposed to be defined by its well-known *harmonic property* (comp. 215, (16.), and 316, (31.), (32.)).

(3.) At the same time we obtain this *other form of the condition of contact* III., as that of equal roots in VI.,

$$\text{VIII.} \dots f(\rho', \tau)^2 = f\tau \cdot (f\rho' - 1),$$

the first member being an abridgment of $(f(\rho', \tau))^2$: and because this last equation VIII. is *homogeneous with respect to τ* , it represents a cone, namely the *Cone of Tangents* (τ) to the given surface $f\rho = 1$, from the given point r' . Accordingly it is easy to prove that the equation III. may be thus written,

$$\text{IX.} \dots f(\rho', \rho - \rho')^2 = f(\rho - \rho') \cdot (f\rho' - 1),$$

under which last form it is seen to be homogeneous with respect to $\rho - \rho'$.

(4.) Without expressly introducing τ , the transformation IX. shows that the equation III. represents *some cone*, with the given point r' for its vertex; and because the *intersection of this cone with the given surface* is expressed by the *squares of the equation I. of the polar plane* of that point, the cone must be (as above stated) *circumscribed to the surface $f\rho = 1$* , touching it *along the curve* (real or imaginary) in which that surface is cut by that *plane I.*

(5.) Another important transformation, or set of transformations, of the equation III. may be obtained as follows. In general, for *any two vectors* ρ and ρ' , if the scalar constant m , the vector function ψ , and the scalar function F , be derived from the linear and vector function ϕ , which is here *self-conjugate* (405), by the method of the Section III. il. 6, we have successively,

$$\begin{aligned} \text{X.} \dots f(\rho, \rho')^2 - f\rho \cdot f\rho' &= S\rho\phi\rho' \cdot S\rho'\phi\rho - S\rho\phi\rho \cdot S\rho'\phi\rho' = S(\sqrt{\rho\rho'} \cdot \sqrt{\phi\rho\phi\rho'}) \\ &= S \cdot \rho\rho' \psi \sqrt{\rho\rho'} = mS \cdot \rho\rho' \phi^{-1} \sqrt{\rho\rho'} = mF\sqrt{\rho\rho'}; \end{aligned}$$

and thus the equation III. of the *circumscribed cone* becomes,

$$\text{XI.} \dots mF\sqrt{\rho\rho'} + f(\rho - \rho') = 0, \quad \text{or} \quad \text{XII.} \dots mF\sqrt{\tau\rho'} + f\tau = 0,$$

if $\tau = \rho - \rho'$ be a *tangent* from r' . Or because $\phi\psi = m$, and $m = -c_1c_2c_3 = -a^{-2}b^2c^2$, by 406, XXIV., we may write (with $\tau = \rho - \rho'$) either

$$\text{XIII.} \dots 0 = S\tau\psi^{-1}\tau + S\psi\phi^{-1}v, \quad \text{if} \quad v = \sqrt{\tau\rho'} = \sqrt{\rho\rho'},$$

or

$$\text{XIV.} \dots F\sqrt{\rho\rho'} = a^2b^2c^2f(\rho - \rho'),$$

as the *condition of contact of the line rr' with the surface $f\rho = 1$* .

(6.) A *geometrical interpretation*, of this last form XIV. of that *condition*, can easily be assigned as follows. Supposing at first for simplicity that the surface is an ellipsoid, let r be the point of contact, so that $f\rho = 1$, $f(\rho, \tau) = 0$; and let the tangent rr' be taken equal to the parallel semidiameter or , so that $f\tau = f(\rho - \rho') = 1$. Then, with the signification XIII. of v , the equation XIV. becomes,

$$\text{XV.} \dots \sqrt{Fv} = Tv \cdot \sqrt{Fv} = abc;$$

in which the factor Tv represents the area of the parallelogram under the conjugate semidiameters or , or' of the given surface $f\rho = 1$; while the other factor \sqrt{Fv} represents the reciprocal of the semidiameter of the reciprocal surface $Fv = 1$, which is perpendicular to their plane ror ; or the perpendicular distance between that plane, and a parallel plane which touches the given ellipsoid: so that their product \sqrt{Fv} is equal, by elementary principles, to the product of the three semiaxes, as stated in the formula XV. And the result may easily be extended by squaring, to other central surfaces.

(7.) It may be remarked in passing, that if ρ, σ, τ be any three conjugate semi-diameters of any central surface $f\rho = 1$, so that

$$\text{XVI.} \dots f\rho = f\sigma = f\tau = 1, \text{ and XVII.} \dots f(\rho, \sigma) = f(\sigma, \tau) = f(\tau, \rho) = 0,$$

and if $x\rho + y\sigma + z\tau$ be any other semidiameter of the same surface, we have then the scalar equation,

$$\text{XVIII.} \dots f(x\rho + y\sigma + z\tau) = x^2 + y^2 + z^2 = 1;$$

a relation between the coefficients, x, y, z , which has been already noticed for the ellipsoid in 99, (2.), and in 402, I., and is indeed deducible for that surface, from principles of real scalars and real vectors alone: but in extending which to the hyperboloids, one at least of those three coefficients becomes imaginary, as well as one at least of the three vectors ρ, σ, τ .

(8.) Under the same conditions XVI. XVII., we have also,

$$\text{XIX.} \dots V\rho\sigma = \pm abc\phi\tau = \pm (-m)^{-1}\phi\tau;$$

$$\text{XX.} \dots \tau = \pm (-m)\phi^{-1}V\rho\sigma = \mp (-m)^{-1}V\phi\rho\phi\sigma;$$

$$\text{XXI.} \dots S\rho\sigma\tau = \pm abc = \pm (-m)^{-1};$$

together with this very simple relation,

$$\text{XXII.} \dots S\rho\sigma\tau.S\phi\rho\phi\sigma\phi\tau = -1.$$

(9.) Under the same conditions, if $x\rho + y\sigma + z\tau$ and $x'\rho + y'\sigma + z'\tau$ have only conjugate directions, that is, if they have the directions of any two conjugate semi-diameters, the six scalar coefficients must satisfy (comp. II.) the equation,

$$\text{XXIII.} \dots xx' + yy' + zz' = 0.$$

(10.) The equation VIII., with ρ for ρ' , may be written under the form,

$$\text{XXIV.} \dots 0 = S\sigma\tau = S\tau\omega\tau, \text{ if XXV.} \dots \sigma = \omega\tau = \phi\rho S\phi\phi\tau + \phi\tau(1 - f\rho),$$

= a new linear and vector function, which represents a normal to the cone of tangents from P , to the surface $f\rho = 1$. Inverting this last function, we find

$$\text{XXVI.} \dots \tau = \omega^{-1}\sigma = \frac{\phi^{-1}\sigma - \rho S\sigma\sigma}{1 - f\rho};$$

the equation in σ of the reciprocal cone, or of the cone of normals to the circumscribed cone from P , is therefore,

$$\text{XXVII.} \dots S\sigma\omega^{-1}\sigma = 0, \text{ or XXVIII.} \dots F\sigma = (S\rho\sigma)^2, \text{ or finally}$$

$$\text{XXVIII.}' \dots F(\sigma : S\rho\sigma) = 1;$$

a remarkably simple form, which admits also of a simple interpretation. In fact, the line $\sigma : S\rho\sigma$ is the reciprocal of the perpendicular, from the centre o , on a tangent plane to the cone, which is also a tangent plane to the surface; it is therefore one of the values of the vector ν (comp. (6.), and 378, (21.)), and consequently it is a semidiameter of the reciprocal surface $F\nu = 1$.

(11.) As an application of the equation XXVIII., let the surface be the confocal (ϵ), represented by the equation 407, III. or X., of which the reciprocal is represented by 407, XVII. or XVIII. Substituting for $F\sigma$ its value thus deduced, the equation of the reciprocal cone (10.), with σ for a side, becomes,*

$$\text{XXIX.} \dots 2b^2 S\sigma\epsilon S\sigma\epsilon - (S\rho\sigma)^2 = b^2\sigma^2, \text{ or XXIX.}' \dots S\sigma\epsilon\sigma\epsilon - t^2(S\rho\sigma)^2 = \sigma^2;$$

if then the vertex P be fixed, but the confocal vary, by a change of ϵ , or of b^2 which

* It may be observed that, when $b = 0$, this equation XXIX. represents the asymptotic cone to the auxiliary surface 407, XXIV.; and at the same time the reciprocal of that focal cone, 407, XXXVI., which rests on the focal hyperbola.

varies with it, the cone XXIX. will also vary, but will belong to a biconcyclic system; whence it follows that the (direct or) circumscribed cones from a given point are all biconfocal: and also, by 407, (30.), that their common focal lines are the generating lines of the confocal hyperboloid² of one sheet, which passes through their common vertex.

(12.) Changing e to e_1 in XXIX., and using the transformation 407, LXXV., with the identity (comp. 407, LIII.),

$$-\sigma^2 = (S\sigma U\nu)^2 + (S\sigma U\nu_1)^2 + (S\sigma U\nu_2)^2,$$

we find that if σ be a normal to the cone of tangents from P to (e_1) , it satisfies the equation,

$$\text{XXX.} \dots 0 = (e - e_1)(S\sigma U\nu)^2 + (e_1 - e_2)(S\sigma U\nu_1)^2 + (e_2 - e_3)(S\sigma U\nu_2)^2;$$

and therefore that if τ be a tangent from the same point P , to the same confocal (e_1) , it satisfies this other condition,

$$\text{XXXI.} \dots 0 = (e - e_1)^{-1}(S\tau U\nu)^2 + (e_1 - e_2)^{-1}(S\tau U\nu_1)^2 + (e_2 - e_3)^{-1}(S\tau U\nu_2)^2,$$

which thus is a form of the equation of the circumscribed cone to (e_1) , with its vertex at a given point P : the confocal character (11.) of all such cones being hereby exhibited anew.

(13.) It follows also from XXXI., that the axes of every cone thus circumscribed have the directions of the normals ν, ν_1, ν_2 to the three confocals through P ; and this known theorem† may be otherwise deduced, from the Equation of Confocals (407, LXV.), by our general method, as follows. That equation gives

$$\nu, -\nu \parallel \phi, \nu \text{ (because } \phi\nu = \phi, \nu), \text{ and therefore,}$$

$$\text{XXXII.} \dots (\nu, -\nu) S\nu\nu = \phi, \nu(f, \rho - 1), \quad V\nu\nu, S\nu\nu, + V\nu\phi, \nu(1 - f, \rho) = 0;$$

changing then V to S , and ν to τ , we see that ν, ν_1, ν_2 , as being the roots (354) of this last vector quadratic XXXII., have the directions of the axes of the cone, with r for side,

$$\text{XXXIII.} \dots f, (\rho, \tau)^2 + f, \tau, (1 - f, \rho) = 0;$$

that is, by VIII., the directions of the axes of the cone of tangents, from P to (e) .

(14.) As an application of the formula XIV., with the abridged symbols r and v of (5.) for $\rho - \rho'$ and Vpp' , the condition of contact of the line PP' with the confocal (e) becomes, by the expressions 407, III., XVIII., and VII. for the functions f, F , and the squares a^2, b^2, c^2 , the following quadratic in e :

$$\text{XXXIV.} \dots (Sa'r)^2 - 2eSa'rSa'r + (Sa'r)^2 + (1 - e^2)r^2 = t^2(Sava'v - ev^2);$$

there are therefore in general (as is known) two confocals, say (e) and (e_1) , of a given system, which touch a given right line; and their parameters, † e and e_1 , are the two roots of the last equation: for instance, their sum is given by the formula,

$$\text{XXXV.} \dots (e + e_1)r^2 = t^2v^2 - 2Sa'rSa'r.$$

* This theorem (which includes that of 407, (30.)) is cited from Jacobi, and is proved, in page 148 of Dr. Salmon's Treatise, referred to in several former Notes.

† Compare the second Note to page 648.

‡ This name of parameter is here given, as in 407, to the arbitrary constant $e = \frac{a^2 + c^2}{a^2 - e^2}$ of which the value distinguishes one confocal (e) of a system from another.

(15.) Conceive then that ρ is a given semidiameter of a given confocal (e), and that $d\rho$ is a tangent, given in direction, at its extremity; the equation XXXIV. will then of course be satisfied,* if we change r to $d\rho$, and v to $\sqrt{\rho d\rho}$, retaining the given value of e ; but it will also be satisfied, for the same ρ and $d\rho$ (or for the same r and v), when we change e to this new parameter,

$$\text{XXXVI.} \dots e = -e + 2SaUd\rho.8aUd\rho - l^2(V\rho Ud\rho)^2;$$

that is to say, the new confocal (e), with a parameter determined by this last formula, will touch the given tangent to the given confocal (e).

(16.) If we at once make $l^2 = 0$ in the equation 407, III. of a Confocal System of Central Surfaces, leaving the parameter e finite, we fall back on the system 406, XXXV. of Biconfocal Cones; but if we conceive that l^2 only tends to zero, and that e at the same time tends to positive infinity, in such a manner that their product tends to a finite limit, r^2 , or that

$$\text{XXXVII.} \dots \lim. l = 0, \quad \lim. e = \infty, \quad \lim. el^2 = r^2,$$

then the equation of the surface (e) tends to this limiting form,

$$\text{XXXVIII.} \dots \rho^2 + r^2 = 0, \quad \text{or} \quad \text{XXXVIII.} \dots T\rho = r;$$

a system of biconfocal cones is therefore to be combined with a system of concentric spheres, in order to make up a complete confocal system.

(17.) Accordingly, any given right line PF' is in general touched by only one cone of the system just referred to, namely by that particular cone (e), for which (comp. XXXIV.) we have the value,

$$\text{XXXIX.} \dots e = Sava'v^{-1}, \quad \text{or} \quad \text{XXXIX.} \dots e + Saa' = 2SavSa'v^{-1},$$

with $v = \sqrt{\rho\rho'}$, as before, so that v is perpendicular to the given plane OPF' , which contains the vertex and the line; in fact, the reciprocals of the biconfocal cones 406, XXXV., when a, a' are treated as given unit lines, but e as a variable parameter, compose the biconcyclic† system (comp. 407, XVIII.),

$$\text{XL.} \dots Sava'v = ev^2.$$

But, besides the tangent cone thus found, there is a tangent sphere with the same centre o ; of which, by passing to the limits XXXVII., the radius r may be found from the same formula XXXIV. to be,

$$\text{XLI.} \dots r = T \frac{v}{r} = T \frac{\sqrt{\rho\rho'}}{\rho - \rho'};$$

and such is in fact an expression (comp. 316, L.) for the length of the perpendicular from the origin on the given line PF' .

(18.) In general, the equation XXXIV. is a form of the equation of the cone, with ρ for its variable vector, which has a given vertex F' , and is circumscribed to a given confocal (e). Accordingly, by making $e = -Saa'$ in that formula, we are

* In fact it follows easily from the transformations (5.), that

$$f\rho \cdot f\delta\rho - a^2b^2c^2FV\rho d\rho = f(\rho, d\rho)^2.$$

† The bifocal form of the equation of this reciprocal system of cones XL. was given in 406, XXV., but with other constants (λ, μ, g), connected with the cyclic form (406, I.) of the equation of the given system.

led (after a few reductions, comp. 407, XXVII.) to an equation which may be thus written,

$$\text{XLII.} \dots 0 = P^2(Sa\alpha'r)^2 + 2Sa\rho'rSa'p'r,$$

with the variable side $r = \rho - \rho'$, as before; and which differs only by the substitution of ρ' and r for ρ and ν , from the equation 407, XXXVI. for that *focal cone*, which rests on the *focal hyperbola*. The *other (real) focal cone* which has the same arbitrary vertex F' , but rests on the *focal ellipse*, has for equation,

$$\text{XLIII.} \dots P^2(S(a - a')r)^2 = Sava'v - v^2,$$

as is found by changing e to 1 in the same formula XXXIV.

(19.) It is however simpler, or at least it gives more symmetric results, to change e , in XXXI. to $-Sa\alpha'$ for the focal hyperbola, and to $+1$ for the focal ellipse, in order to obtain the *two real focal cones* with F for vertex, which rest on those two curves; while that *third and wholly imaginary focal cone*, which has the same vertex, but rests on the known *imaginary focal curve*, in the plane of b and c , is found by changing e , to -1 . This imaginary focal cone, and the two real ones which rest as above on the hyperbola and ellipse respectively, may thus be represented by the three equations,

$$\text{XLIV.} \dots 0 = a^{-2}(SrU\nu)^2 + a_1^{-2}(SrU\nu_1)^2 + a_2^{-2}(SrU\nu_2)^2;$$

$$\text{XLV.} \dots 0 = b^{-2}(SrU\nu)^2 + b_1^{-2}(SrU\nu_1)^2 + b_2^{-2}(SrU\nu_2)^2;$$

$$\text{XLVI.} \dots 0 = c^{-2}(SrU\nu)^2 + c_1^{-2}(SrU\nu_1)^2 + c_2^{-2}(SrU\nu_2)^2;$$

r being in each case a side of the cone, and ν, ν_1, ν_2 having the same significations as before.

(20.) On the other hand, if we place the *vertex* of a circumscribed cone at a point F of a *focal curve*, real or imaginary, the *enveloped surface* being the *confocal* (e), we find first, by XXX., for the *reciprocal cones*, or *cones of normals* σ , with the same order of succession as in (19.), the three equations,

$$\text{XLVII.} \dots a^2(SU\nu\sigma)^2 = a^2;$$

$$\text{XLVIII.} \dots b^2(SU\nu\sigma)^2 = b^2;$$

$$\text{XLIX.} \dots c^2(SU\nu\sigma)^2 = c^2;$$

and next, for the *circumscribed cones* themselves, or *cones of tangents* r , the connected equations:

$$\text{L.} \dots a^2(VU\nu r)^2 + a^2 = 0;$$

$$\text{LI.} \dots b^2(VU\nu r)^2 + b^2 = 0;$$

$$\text{LII.} \dots c^2(VU\nu r)^2 + c^2 = 0;$$

all which have the *forms* of equations of *cones of revolution*, but on the *geometrical meanings* of the three last of which it may be worth while to say a few words.

(21.) The cone L. has an *imaginary vertex*, and is always *itself* imaginary; but the *two other cones*, LI. and LII., have each a *real vertex* F , with $b^2 > 0$ for the first, and $c^2 < 0$ for the second; b being the mean semiaxis of the *ellipsoid*, which passes through a given point of the *focal hyperbola*, and c^2 being the negative and algebraically least square of a scalar semiaxis of the *double-sheeted hyperboloid*, which passes through a given point of the *focal ellipse*: while, in each case, ν has the direction of the *normal* to the *surface*, which is also the *tangent* to the *curve* at that point, and is at the same time the *axis* of revolution of the *cone*.

(22.) The *semiaxes* of the two last cones, LI. and LII., have for their respective *sines* the two quotients,

LIII. . . $b : b$, and LIV. . . $(-c^2)^{\frac{1}{2}} : (-c^2)^{\frac{1}{2}}$;

each of those two cones is therefore real, if circumscribed to a single-sheeted hyperboloid, because, for such an enveloped surface (e), b is real, and less than the b of any confocal ellipsoid, while c , is imaginary, and its square is algebraically greater (or nearer to zero) than the square of the imaginary semiaxis c of every double-sheeted hyperboloid, of the same confocal system; but the cone LI. is imaginary, if the enveloped surface (e) be either an hyperboloid of two sheets (b , imaginary), or an exterior ellipsoid ($b > b$); and the other cone LII. is imaginary, if the surface (e) be either any ellipsoid (c , real), or else an exterior and double-sheeted hyperboloid ($a^2 < a^2, c^2 < c^2, -c^2 > -c^2$). Accordingly it is known that the focal hyperbola, which is the locus of the vertex of the cone LL, lies entirely inside every double-sheeted hyperboloid of the system; while the focal ellipse, which is in like manner the locus of the vertex of the cone LII., is interior to every ellipsoid: and real tangents to a single-sheeted hyperboloid can be drawn, from every real point of space.

(28.) The twelve points (whereof only four at most can be real), in which a surface (e) or (abc) is cut by the three focal curves, are called the Umbilics of that surface; the vectors, say ω, ω, ω , of three such umbilics, in the respective planes of ca, ab, bc , are:

LV. . . $\omega = \frac{a}{2}(a + a') + \frac{c}{2}(a - a')$;

LVI. . . $\omega = \frac{a(a + a')}{1 - Saa'} + \frac{\sqrt{-1}b\sqrt{Vaa'}}{1 - Saa'}$;

LVII. . . $\omega = \frac{c(a - a')}{1 + Saa'} - \frac{\sqrt{-1}b\sqrt{Vaa'}}{1 + Saa'}$;

and the others can be formed from these, by changing the signs of the terms, or of some of them. The four real umbilics of an ellipsoid are given by the formula LV., and those of a double-sheeted hyperboloid by LVI., with the changes of sign just mentioned.

(24.) In transforming expressions of this sort, it is useful to observe that the expressions for the squares of the semiaxes,

$a^2 = b^2(e + 1), \quad b^2 = b^2(e + Saa'), \quad c^2 = b^2(e - 1), \quad 407, VII.$

combined with $Ta = Ta' = 1$, give not only $a^2 - c^2 = 2b^2$, but also,

LVIII. . . $T \frac{a + a'}{2} = \sqrt{\frac{1 - Saa'}{2}} = \cos \frac{1}{2} \angle \frac{a'}{a} = \left(\frac{a^2 - b^2}{a^2 - c^2}\right)^{\frac{1}{2}}$;

LIX. . . $T \frac{a - a'}{2} = \sqrt{\frac{1 + Saa'}{2}} = \sin \frac{1}{2} \angle \frac{a'}{a} = \left(\frac{b^2 - c^2}{a^2 - c^2}\right)^{\frac{1}{2}}$;

and LX. . . $TVaa' = \sqrt{1 - (Saa')^2} = \sin \angle \frac{a'}{a} = l^{-2}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}}$,

with the verification, that because

LXI. . . $(a - a')(a + a') = 2Vaa'$,

therefore LXI'. . . $T(a - a').T(a + a') = 2TVaa'$.

We have also the relations,

LXII. . . $T(a + a')^{-2} + T(a - a')^{-2} = (TVaa')^{-2}$;

LXIII. . . $T(a + a')^{-2} - T(a - a')^{-2} = Saa'.(TVaa')^{-2}$;

with others easily deduced.

(25.) The expression LV. conducts to the following among other consequences, which all admit of elementary verifications,* and may be illustrated by the annexed Fig. 84. Let u, u' be the two real points in which an ellipsoid (abc) is cut by one branch of the focal hyperbola, with κ for summit, and with \mathfrak{F} for its interior focus; the adjacent major summit of the surface being κ , and $\mathfrak{R}, \mathfrak{R}'$ being (as in the Figure) the adjacent points of intersection of the same surface with the focal lines α, α' , that is, with the asymptotes to the hyperbola. Let also v, τ be the points in which the same asymptotes α, α' meet the tangent to the hyperbola at u , or the normal to the ellipsoid at that real umbilic, of which we may suppose that the vector ou is the ω of the formula LV.; and let s be the foot of the perpendicular on this normal to the surface, or tangent tv to the curve, let fall from the centre o . Then, besides the obvious values,

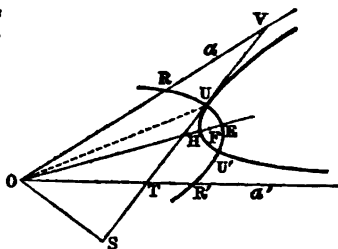


Fig. 84.

LXIV. . . $\overline{OE} = a$, $\overline{OF} = (a^2 - c^2)^{\frac{1}{2}}$, $\overline{OH} = (a^2 - b^2)^{\frac{1}{2}}$, and the obvious relations, that the intercept tv is bisected at u , and that the point F is at once a summit of the focal ellipse, and a focus of that other ellipse in which the surface is cut by the plane (ac) of the figure, we shall have these vector expressions (comp. 371, (8.), and 407, VIII. LXI.):

$$\text{LXV. . . } \overline{ov} = (a + c) \alpha, \quad \overline{ot} = (a - c) \alpha', \quad \overline{tv} = a(\alpha - \alpha') + c(\alpha + \alpha');$$

$$\text{LXVI. . . } \overline{su}^{-1} = \phi \omega = -\frac{a^{-1}}{2} (\alpha + \alpha') - \frac{c^{-1}}{2} (\alpha - \alpha'), \quad \overline{su} = -ac : \tau u;$$

$$\text{LXVII. . . } \overline{OR} = \frac{a}{\sqrt{fa}} = ab^{-1}ca, \quad \overline{OR'} = \frac{a'}{\sqrt{fa'}} = ab^{-1}ca';$$

whence follow by (24.) these other values,

$$\text{LXVIII. . . } \overline{OV} = a + c, \quad \overline{OT} = a - c, \quad \overline{TV} = 2b;$$

$$\text{LXIX. . . } \overline{TU} = \overline{UV} = b, \quad \overline{SU} = \overline{OR} = \overline{OR'} = ab^{-1}c;$$

$$\text{LXX. . . } \overline{OU} = T\omega = (a^2 - b^2 + c^2)^{\frac{1}{2}};$$

$$\text{LXXI. . . } \overline{OS} = (a^2 - b^2 + c^2 - a^2b^{-2}c^2)^{\frac{1}{2}} = b^{-1}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}}.$$

(26.) It follows that the lengths of the sides ov, ot, tv of the umbilicar triangle tov are equal to the sum and difference ($a \pm c$) of the extreme semiaxes, and to the mean axis ($2b$) of the ellipsoid; while the area of that triangle = $\overline{os} \cdot \overline{tu} = (a^2 - b^2)^{\frac{1}{2}} (b^2 - c^2)^{\frac{1}{2}}$ = the rectangle under the two semiaxes of the hyperbola, if both be treated as real. The length $(T\phi\omega)^{-1}$, or \overline{su} , of the perpendicular from the centre o , on the tangent plane at an umbilic u , is $ab^{-1}c$; and the sphere concentric with the ellipsoid, which touches the four umbilicar tangent planes, passes through the points $\mathfrak{R}, \mathfrak{R}'$ of intersection of that ellipsoid with the focal lines α, α' ; that is, as before, with the

* Some such verifications were given in the *Lectures*, pages 691, 692, in connexion with Fig. 102 of that former volume, which answered in several respects to the present Fig. 84.

asymptotes to the hyperbola; or, by (21.)(22.), with the axes of the two circumscribed right cylinders.* And finally the length, say u , of the umbilicar semidiameter ou , is given by the formula,

$$\text{LXXII.} \dots u^2 = a^2 - b^2 + c^2;$$

all which agrees (25.) with known results.

(27.) An umbilic of a surface of the second order may be otherwise defined (comp. (28.)), as a real or imaginary point at which the tangent plane is parallel to a cyclic plane; and accordingly it is easy to prove (comp. 407, (20.)) that the umbilicar normal $\phi\omega$ in LXVI. has the direction of a cyclic normal. To employ this known property in verification of the recent expressions (25.), (26.), for the lengths of ou and su , it is only necessary to observe that the common radius of the diametral and circular sections of the ellipsoid is the mean semiaxis b (comp. 216, (7.) (9.), &c.); and that, by a slight extension of the analysis in (7.), (8.), (9.), it can be shown that if ρ, σ, τ and ρ', σ', τ' be any two systems of three conjugate semidiameters of any central surface, $f\rho = 1$, then

$$\text{LXXIII.} \dots \rho'^2 + \sigma'^2 + \tau'^2 = \rho^2 + \sigma^2 + \tau^2, \text{ and } \text{LXXIV.} \dots (S\rho'\sigma'\tau')^2 = (S\rho\sigma\tau)^2.$$

(28.) A less elementary verification of the value LXXII. of u^2 , but one which is useful for other purposes, may be obtained from either the cubic in b^2 , or that in e , assigned in 407, (8.). For if b_0^2, b_1^2, b_2^2 be the roots of the former cubic, and e_0, e_1, e_2 the roots of the latter, inspection of those equations shows at once that we have generally,

$$\text{LXXV.} \dots -\rho^2 = b_0^2 + b_1^2 + b_2^2 - 2f^2 S a a' = f^2 (e_0 + e_1 + e_2 + S a a');$$

$$\text{or } \text{LXXVI.} \dots \overline{OP}^2 = T\rho^2 = a_0^2 + b_1^2 + c_2^2 = b_0^2 + c_1^2 + a_2^2 = \&c.,$$

where the semiaxes a_0, b_1, c_2 belong to the three confocals through any proposed point P . Making then,

$$\text{LXXVII.} \dots a_0^2 = a^2, \quad b_1^2 = 0, \quad c_2^2 = c^2 - b^2,$$

we recover the expression assigned above, for the square of the length u of an umbilicar semidiameter of an ellipsoid.

(29.) For any central surface, the principle (27.) shows that if λ, μ be, as in 405, (5.), &c., the two real cyclic normals, and if g be the real scalar associated with them as before, then the vectors of the four real umbilics (if such exist) must admit of being thus expressed:

$$\text{LXXVIII.} \dots \pm \phi^{-1}\lambda : \sqrt{F\lambda} = \pm abc (gU\lambda + \mu T\lambda);$$

$$\text{LXXIX.} \dots \pm \phi^{-1}\mu : \sqrt{F\mu} = \pm abc (gU\mu + \lambda T\mu);$$

and thus we see anew, that an hyperboloid with one sheet has (as is well known) no

* Compare 218, (5.), and 220, (4.); in which the points κ, κ' (comp. also Fig. 53, page 226) may now be conceived to coincide with the points κ, κ' of the new Figure 84. It is obvious that the theory of circumscribed cylinders is included in that of circumscribed cones; so that the cylinder circumscribed to the confocal (e), with its generating lines parallel to a given (real or imaginary) semidiameter γ of that surface ($f\gamma = 1$), may be represented (comp. III. XIV.) by the equation,

$$\text{III'.} \dots f(\rho, \gamma)^2 = f\rho - 1; \text{ or } \text{XIV'.} \dots FV\gamma\rho = a^2 b^2 c^2;$$

with interpretations easily deduced, from principles already established.

real umbilic, because for that surface the product abc of the semi-axes is imaginary; or because it has no real tangent plane parallel to either of its two real planes of circular section.

(30.) Of whatever species the surface may be, the three umbilical vectors (23.), of which only one at most can be real, with the particular signs there given, but which have the forms of lines in the three principal planes, must be conceived, in virtue of their expressions LV. LVI. LVII., to terminate on an imaginary right line, of which the vector equation is,

$$\text{LXXX.} \dots \rho = \frac{-a(e'+1)}{a+a'} - \sqrt{-1} \frac{b(e'+Saa')}{Vaa'} + \frac{c(e'-1)}{a-a'};$$

e' being a scalar variable, which receives the three values, $-Saa'$, $+1$, and -1 , when ρ comes to coincide with ω_1 , ω_2 , and ω_3 , respectively. And such an imaginary right line, which is easily proved to satisfy, for all values of the variable e' , both the rectangular and the bifocal forms of the equation of the surface (e), or to be (in an imaginary sense) wholly contained upon that surface, may be called an *Umbilical Generatrix*.

(31.) There are in general eight such generatrices of any central surface of the second order, whereof each connects three umbilics, in the three principal planes, two passing through each of the twelve umbilical points (28.); and because e^2 disappears from the square of the expression LXXX. for ρ , which square reduces itself to the following,

$$\text{LXXXI.} \dots \rho^2 = -I^2(2e' + e + Saa') = -b^2 - 2I^2e',$$

they may be said to be the eight generating lines through the four imaginary points, in which the surface meets the circle at infinity.

(32.) In general, from the cubics in e and in b^2 , or from either of them, it may be without difficulty inferred (comp. (28.)), that the eight intersections (real or imaginary) of any three confocals (e_0) (e_1) (e_2) have their vectors ρ represented by the formula:

$$\text{LXXXII.} \dots \rho = \frac{\pm a_0 a_1 a_2}{I^2(a+a')} \pm \frac{\sqrt{-1} b_0 b_1 b_2}{I^2 Vaa'} \pm \frac{e_0 e_1 e_2}{I^2(a-a')};$$

comparing which with the vector expression LXXX., we see that the three confocals, through the point determined by that former expression, for any given value of e' , are (e), (e'), and (e') again; and therefore that two of the three confocal surfaces through any point of an umbilical generatrix (30.) coincide: a result which gives in a new way (comp. LXXV.) the expression LXXXI. for ρ^2 .

(33.) The locus of all such generatrices, for all the confocals (e) of the system, is a certain ruled surface, of which the doubly variable vector may be thus expressed, as a function of the two scalar variables, e and e' :

$$\text{LXXXIII.} \dots \rho_{e,e'} = \frac{\pm I(e+1)^{\frac{1}{2}}(e'+1)}{a+a'} \pm \frac{\sqrt{-1}I(e+Saa')^{\frac{1}{2}}(e'+Saa')}{Vaa'} \\ \pm \frac{I(e-1)^{\frac{1}{2}}(e'-1)}{a-a'};$$

and because we have thus, for any one set of signs, the differential relation,

$$\text{LXXXIV.} \dots D^2 \rho_{e,e'} = \frac{1}{2} D_{e'} \rho_{e,e'},$$

it follows that *this ruled locus is a Developable Surface*: its *edge of regression* being that wholly *imaginary curve*, of which the *vector* is $\rho_{n, \epsilon}$, and which is therefore by (82.) the *locus* of all the *imaginary points*, through each of which pass *three coincident confocals*.

(84.) The *only real part* of this *imaginary developable* consists of the *two real focal curves*, which are *double lines* upon it, as are also the *imaginary focal*, and the *circle at infinity* (81.); and the *scalar equation* of the same *imaginary surface*, obtained by elimination of the two arbitrary scalars ϵ and ϵ' , is found to be of the *eighth degree*, namely the following:

$$\text{LXXXV.} \dots \begin{cases} 0 = \Sigma m^2 x^6 + 2 \Sigma m(m-n)x^4 y^2 + \Sigma (p^2 - 6mn)x^2 y^4 \\ + 2 \Sigma (8m^2 - np)x^2 y^2 z^2 + 2 \Sigma m^2(n-p)x^6 + 2 \Sigma m(mp - 8n^2)x^4 y^2 \\ + 2(m-n)(n-p)(p-m)x^2 y^2 z^2 + \Sigma m^3(m^2 - 6np)x^4 \\ + 2 \Sigma mn(mn - 8p^2)x^2 y^2 + 2 \Sigma m^2 np(p-n)x^2 + m^2 n^2 p^2; \end{cases}$$

in which we have written, for abridgment,

$$\text{LXXXVI.} \dots x = -\text{Sp}U(a+a'), \quad y = -\text{Sp}UVaa', \quad z = -\text{Sp}U(a-a'),$$

$$\text{and LXXXVII.} \dots m = b^2 - c^2, \quad n = c^2 - a^2, \quad p = a^2 - b^2,$$

$$\text{so that LXXXVIII.} \dots m + n + p = 0;$$

while each sign Σ indicates a sum of three or of six terms, obtained by cyclical or binary* interchanges.

(85.) From the manner in which the equation of this *imaginary surface* (83.) or (84.) has been deduced, we easily see by (82.) that it has the double property: I.st of being (comp. (20.)) the *locus* of the *vertices* of all the (real or imaginary) *right cones*, which can be *circumscribed* to the *confocals* of the system; and II.nd of being at the same time the *common envelope* of all those *confocals*: which *envelope* accordingly is known to be a *developable† surface*.

(86.) The *eight imaginary lines* (81.) will come to be mentioned again, in connexion with the *lines of curvature* of a surface of the second order; and before closing the present series of subarticles, it may be remarked that the equation in (15.), for the determination of the *second confocal* (ϵ), which *touches a given tangent*, $d\rho$ or $\rho\rho'$, to a *given surface* (ϵ) of the same system, will soon appear under a new form, in connexion with that theory of *geodesic lines*, on surfaces of the second order, to which we next proceed.

* When xyz and abc are *cyclically* changed to yzx and bca , then mnp are similarly changed to npm ; but when, for instance, retaining x and a unchanged, we make only *binary interchanges* of y, z , and of b, c , we then change m, n , and p , to $-m, -p$, and $-n$ respectively.

† This theorem is given, for instance, in page 157 of the several times already cited Treatise by Dr. Salmon, who also mentions the *double lines* &c. upon the surface; but the present writer does not yet know whether the theory above given, of the *eight umbilical generatrices*, has been anticipated: the *locus* (83.) of which *imaginary right lines* (80.) is here represented by the *vector equation* LXXXIII., from which the *scalar equation* LXXXV. has been above deduced (84.), and ought to be found to agree (notation excepted) with the known co-ordinate equation of the *developable envelope* (85.) of a *confocal system*.

409. A general theory of *geodetic lines*, as treated by quaternions, was given in the Fifth Section (III. iii. 5) of the present Chapter; and was illustrated by applications to several different families of surfaces. We can only here spare room for applying the same theory to the deduction, in a new way, of a few known but principal properties of geodetics on central surfaces of the second order; the differential equation employed being one of those formerly used, namely (comp. 380, IV.),

$$\text{I.} \dots \nabla v d^2 \rho = 0, \quad \text{if} \quad \text{II.} \dots T d \rho = \text{const.};$$

that is, if the *arc* of the geodetic be made the independent variable.

(1.) In general, for any surface, of which v is a normal vector, so that the first differential equation of the surface is $S v d \rho = 0$, the second differential equation $d S v d \rho = 0$ gives, by I., for a geodetic on that surface, the expression,

$$\text{III.} \dots d^2 \rho = -v^{-1} S v d v d \rho.$$

(2.) Again, the surface $f \rho = \text{const.}$ being still quite general, if we write (comp. 868, X', 878, III., &c.),

$$\text{IV.} \dots d f \rho = 2 S v d \rho = 2 S \phi \rho d \rho, \quad \text{we shall have} \quad \text{V.} \dots d f d \rho = 2 S (\phi d \rho \cdot d^2 \rho);$$

and therefore, by III., for a geodetic,

$$\text{VI.} \dots \frac{d f d \rho}{S \phi \rho d \phi \rho} + 2 S \frac{\phi d \rho}{\phi \rho} = 0.$$

(3.) For a central surface of the second order, $\phi \rho$ is a linear function, and we may write (comp. 861, IV.),

$$\text{VII.} \dots \phi d \rho = d \phi \rho = d v, \quad S d \rho d \phi \rho = S d \rho \phi d \rho = f d \rho;$$

the general differential equation VI. becomes therefore here,

$$\text{VIII.} \dots \frac{d f d \rho}{f d \rho} + 2 S \frac{d v}{v} = 0;$$

and gives, by a first integration, with the condition II.,

$$\text{IX.} \dots v^2 f d \rho = h d \rho^2, \quad \text{or} \quad \text{IX'.} \dots T v^2 f U d \rho = h = \text{const.};$$

or

$$\text{X.} \dots P^{-2} D^{-2} = h, \quad \text{or} \quad \text{X'.} \dots P \cdot D = h^{-1} = \text{const.};$$

where

$$P = T v^{-1} = \text{perpendicular from centre on tangent plane,}$$

and

$$D = (f U d \rho)^{-1} = \text{semidiameter parallel to tangent};$$

these two last quantities being treated as scalars, whereof the latter may be real or imaginary,* together with the last scalar constant h^{-1} .

* For the case of the *ellipsoid*, for which the product $P \cdot D$ is necessarily real, the foregoing deduction, by quaternions, of Joachimstal's celebrated first integral, $P \cdot D = \text{const.}$, was given (in substance) in page 580 of the *Lectures*.

(4.) The following is a quite different way of accomplishing a first integration, which conducts to another known result of not less interest, although rather of a *graphic* than of a *metric* kind. Operating on the equation 407, XVI. by $S.d\rho$, and remembering that $S\rho\nu = 1$, and $S\nu d\rho = 0$, we obtain the differential equation,

$$\text{XI.} \dots S\rho\nu S\rho d\rho = R^2(Sa'\nu Sa d\rho + Sa\nu Sa'd\rho);$$

that is, by I. and II.,

$$\text{XII.} \dots S\rho d\rho \cdot S\rho d^2\rho - \rho^2 S d\rho^2 \rho = R^2 d(Sa d\rho \cdot Sa'd\rho),$$

in which the first member, like the second, is an exact differential, because

$$\text{XIII.} \dots S(\nabla\rho d\rho \cdot \nabla\rho d^2\rho) = \frac{1}{2} d(\nabla\rho d\rho)^2;$$

hence, for the geodetic,

$$\text{XIV.} \dots l^{-2}(\nabla\rho d\rho)^2 - 2Sa d\rho Sa'd\rho = h'd\rho^2,$$

or

$$\text{XV.} \dots 2Sa U d\rho \cdot Sa' U d\rho - l^{-2}(\nabla\rho U d\rho)^2 = h',$$

h' being a new scalar constant.

(5.) Comparing this last equation with the formula 408, XXXVI., we find that the new constant h' is the *sum*, $e + e_0$, of what have been above called the *parameters*,* of the *given surface* (e) on which the geodetic is traced, and of the *confocal* (e_0) which *touches* a given *tangent* to that curve: whence follows the known† theorem, that the *tangents to a geodetic, on any central surface of the second order, all touch one common confocal*.‡

(6.) The new constant $e_0 (= h' - e)$ may, by 407, LXXV. and 408, LXXY. (with e for e_0), be thus transformed:

$$\begin{aligned} \text{XVI.} \dots e_0 &= e_1(TV U \nu_1 d\rho)^2 + e_2(TV U \nu_2 d\rho)^2 \\ &= e_1(SU \nu_2 d\rho)^2 + e_2(SU \nu_1 d\rho)^2 = \text{const.}; \end{aligned}$$

where e_1, e_2 are the parameters of the two confocals through the point P of the geodetic on (e), and ν_1, ν_2 are as before the normals at that point, to those two surfaces (e_1), (e_2).

(7.) In fact, the two equations last cited give the *general transformation*,

$$\begin{aligned} \text{XVII.} \dots l^{-2}(\nabla\rho\sigma)^2 - 2Sa\sigma Sa'\sigma \\ = e(\nabla\sigma U\nu)^2 + e_1(\nabla\sigma U\nu_1)^2 + e_2(\nabla\sigma U\nu_2)^2; \end{aligned}$$

σ being an *arbitrary vector*, which may for instance be replaced by $d\rho$. Equating then this last expression to $(e + e_0)\sigma^2$, or to $e(\nabla\sigma U\nu)^2 - e_0 T\sigma^2$, since $S\nu\sigma = 0$, we obtain the first and therefore also the second transformation XVI., because the three normals ν, ν_1, ν_2 compose a rectangular system (comp. 407, (4.), &c.).

(8.) It is, however, simpler to deduce the second expression XVI. from the equation 408, XXXI. of the cone of tangents from P to (e), by changing r to $Ud\rho$; and then if we write

$$\text{XVIII.} \dots v_1 = l \frac{d\rho}{\nu_1},$$

* Compare the last Note to page 656.

† Discovered by M. Chasles.

‡ This touched confocal becomes a *sphere*, when the given confocal is a *cons*. Compare 380, (5.), and 408, (16.), (17.); also the Note to page 517.

so that ν_1 denotes the angle at which the geodetic crosses the normal ν_1 to (e_1) , considered as a tangent to the given surface (e) , the first integral XVI. takes the form,*

$$\text{XIX.} \dots e_1 = e_1 \sin^2 \nu_1 + e_2 \cos^2 \nu_1,$$

or

$$\text{XX.} \dots a_1^2 = a_1^2 \sin^2 \nu_1 + a_2^2 \cos^2 \nu_1, \text{ \&c. ;}$$

in which the constant a , is the primary semiaxis of the touched confocal (5.).

(9.) Without supposing that $Td\rho$ is constant, we may investigate as follows the differential of the real scalar λ in IX. or X., or of the product $P^{-2} \cdot D^{-2}$, for any curve on a central surface of the second order. Leaving at first the surface arbitrary, as in (1.) and (2.), and resolving $d^2\rho$ in the three rectangular directions of ν , $d\rho$, and $\nu d\rho$, we get the general expression,

$$\text{XXI.} \dots d^2\rho = -\nu^{-1} Sd\nu d\rho + d\rho^{-1} Sd\rho d^2\rho + (\nu d\rho)^{-1} S\nu d\rho d^2\rho;$$

of which, under the conditions I. and II., the two last terms vanish, as in III. Without assuming those conditions, if we now introduce the relations VII. which belong to a central surface of the second order, we have by V. and IX. the expression,†

$$\text{XXII.} \dots \frac{1}{2} d\lambda \cdot d\rho^2 = \nu^2 Sd\nu d^2\rho + S\nu d\nu Sd\nu d\rho - \lambda Sd\rho d^2\rho = S\nu d\nu d\rho^{-1} \cdot S\nu d\rho d^2\rho,$$

$$\text{or} \quad \text{XXIII.} \dots d\lambda = d \cdot \nu^2 Sd\nu d\rho^{-1} = d \cdot P^{-2} D^{-2} = 2S\nu d\nu d\rho^{-1} S\nu d\rho^{-1} d^2\rho;$$

or finally,

$$\text{XXIV.} \dots d\lambda \cdot d\rho^2 = 2S\nu d\nu d\rho \cdot S\nu d\rho d^2\rho,$$

the scalar variable with respect to which the differentiations are performed being here entirely arbitrary.

(10.) For a geodetic line on any surface, referred thus to any scalar variable, we have by 880, II. the differential equation,

$$\text{XXV.} \dots S\nu d\rho d^2\rho = 0;$$

and therefore by XXIV., for such a line on a central surface of the second order, we have again, as in (8.),

$$\text{XXVI.} \dots d\lambda = 0, \quad \text{or} \quad \text{XXVI'.} \dots \lambda = \text{const.},$$

with $\lambda = P^{-2} D^{-2}$ as in X.

(11.) But we now see, by XXIV., that for such a surface the condition XXVI. is satisfied, not only by this differential equation of the second order XXV. but also by this other differential equation,

$$\text{XXVII.} \dots S\nu d\nu d\rho = 0;$$

the product $P^{-2} D^{-2}$ (or PD itself) is therefore constant, not only as in (8.) for every

* Under this form XX., the integral is easily seen to coincide with that of M. Liouville,

$$\mu^2 \cos^2 i + \nu^2 \sin^2 i = \mu'^2 = \text{const.},$$

cited in page 290 of Dr. Salmon's Treatise.

† In deducing this expression, it is to be remembered that

$$dS\nu d\rho = d^2\nu d\rho = 2Sd\nu d^2\rho;$$

in fact, the linear and self-conjugate form of $\nu = \phi\rho$ gives,

$$Sd\rho d^2\nu = f(d\rho, d^2\rho) = Sd\nu d^2\rho.$$

geodesic on the surface, but *also* for every curve of another set,* represented by this last equation XXVII., which is only of the *first order*, and the geometrical meaning of which we next propose to consider.

410. In general, if ν and $\nu + \Delta\nu$ have the directions of the normals to any surface, at the extremities of the vectors ρ and $\rho + \Delta\rho$, the condition of intersection (or parallelism) of these two normals is, rigorously,

$$\text{I. . . } S\nu\Delta\nu\Delta\rho = 0;$$

the differential equation† of what are called the *Lines of Curvature*, on an arbitrary surface, is therefore (comp. 409, XXVII.),

$$\text{II. . . } S\nu d\nu d\rho = 0;$$

from which we are now to deduce a few *general* consequences, together with some that are peculiar to surfaces of the *second order*.

(1.) The differential equation of the surface being, as usual,

$$\text{III. . . } S\nu d\rho = 0,$$

the normal vector ν is generally some function of ρ , although *not* generally linear, because the surface is as yet *arbitrary*: its differential $d\nu$ is therefore generally some function of ρ and $d\rho$, which is linear relatively to the latter. And if, attending only to the dependence of $d\nu$ on $d\rho$, we write

$$\text{IV. . . } d\nu = \phi d\rho,$$

it results from what has been already proved (363), that *this linear* and *vector function* ϕ is at the same time *self-conjugate*.

(2.) Denoting then by r a tangent‡ rr to a line of curvature, drawn at the given extremity R of ρ , we see that the vector r must satisfy the two following scalar equations, in which ν is supposed to be given,

* Namely, the *lines of curvature*, as is known, and as will presently be proved by quaternions.

† In this equation II., $d\rho$ and $d\nu$ are two simultaneous differentials, which may (according to the theory of the present Chapter, and of the one preceding it) be at pleasure regarded, either as two finite right lines, whereof $d\rho$ is (rigorously) *tangent* to the surface, and to the line of curvature; or else as two infinitely small vectors, $d\rho$ being, on this latter plan, an *infinitesimal chord* $\Delta\rho$. (Compare pages 99, 392, 497, 626, and the first Notes to pages 628, 630.) The treatment of the equations is the same, in these two views, whereof one may appear clearer to some readers, and the other view to others.

‡ This symbol r is used here partly for abridgment, and partly that the reader may not be obliged to interpret $d\rho$ as denoting a finite tangent, although the principles of this work allow him so to interpret it.

$$V. \dots S\nu r = 0, \text{ and VI.} \dots S\nu r \phi = 0;$$

this tangent r admits therefore (355) of *two real and rectangular directions*, but not in general of *more*: *opposite directions being not here counted as distinct*. Hence, as is indeed well known, *through each point of any surface there pass generally two lines of curvature*: and *these two curves intersect each other at right angles*.

(8.) A *construction* for the two rectangular directions of r can easily be assigned as follows. Assuming, as we may, that the *length* of the tangent r varies with its *direction*, according to the law,

$$VII. \dots S r \phi r = 1,$$

which gives

$$VIII. \dots S(\phi r \cdot d r) = 0, \text{ or briefly VIII'.} \dots S \phi r d r = 0,$$

by the properties above mentioned of ϕ ; and remembering that ν is treated as a constant in V., so that we may write,

$$IX. \dots S \nu d r = 0, \text{ and therefore (by VI.), X.} \dots S r d r = 0;$$

we see that, under the conditions of the question, the above mentioned *length* Tr , of this tangential vector r , is a *maximum* or *minimum*: and therefore that the *two directions* sought are those of the *two axes* of the *plane conic* V. VII., which has its *centre at the given point* P of the surface, and is *in the tangent plane* at that point.

(4.) This plane conic V. VII. may be called the *Index Curve*, for the given surface at the given point P ; in fact it is easily proved to coincide, if we abstract from mere dimensions, with the known *indicatrix* (la courbe indicatrice) of Dupin,* who first pointed out the coincidence (8.) of the directions of its *axes*, with those of the lines of curvature; and also established a more general relation of *conjugation* between *two tangents* to a surface at one point, which exists when they have the directions of any *two conjugate semidiameters* of that curve: so that the lines of curvature are distinguished by this *characteristic property*, that the *tangent to each is perpendicular to its conjugate*.

(5.) In our notations, this relation of conjugation between two tangents r, r' , which satisfy as such the equations,

$$V. \dots S \nu r = 0, \text{ and V'.} \dots S \nu r' = 0,$$

is expressed by the formula,

$$XI. \dots S r \phi r' = 0, \text{ or XI'.} \dots S r' \phi r = 0;$$

we have therefore the parallelisms,†

$$XII. \dots r \parallel V \nu \phi r', \text{ XII'.} \dots r' \parallel V \nu \phi r;$$

so that the equation VI. may be written under the very simple form,

$$XIII. \dots S r r' = 0,$$

which gives at once the *rectangularity* lately mentioned.

* *Développements de Géométrie* (Paris, 1813), pages 48, 145, &c.

† The *conjugate character* of these two parallelisms, or the relation,

$$V. \nu \phi V \nu \phi r \parallel r, \text{ if } S \nu r = 0,$$

may easily be deduced from the *self-conjugate* property of ϕ , with the help of the formula 348, VII., in page 440.

(6.) The parallelism XII' may be otherwise expressed by saying (comp. (4.)) that

$$\text{XIV.} \dots d\rho \text{ and } V\nu d\nu$$

have the directions of *conjugate tangents*; or that the two vectors,

$$\text{XV.} \dots \Delta\rho \text{ and } V\nu\Delta\nu,$$

have *ultimately* such directions, when $T\Delta\rho$ diminishes indefinitely. But whatever may be this length of the chord $\Delta\rho$, the vector $V\nu\Delta\nu$ has the direction of the line of intersection of the two tangent planes to the surface, drawn at its two extremities: another theorem of Dupin* is therefore reproduced, namely, that *if a developable be circumscribed to any surface, along any proposed curve thereon, the generating lines of this developable are everywhere conjugate, as tangents to the surface, to the corresponding tangents to the curve, with the recent definition (4.) of such conjugation.*

(7.) The following is a very simple mode of proving by quaternions, that if a tangent r satisfies the equation VI., then the *rectangular tangent*,

$$\text{XVI.} \dots r' = \nu r,$$

satisfies the same equation. For this purpose we have only to observe, that the *self-conjugate* property of ϕ gives, by VI. and XVI.,

$$\text{XVII.} \dots 0 = Sr'\phi r = Sr\phi r' = \nu^{-2} S\nu r' \phi r'.$$

(8.) Another way of exhibiting, by quaternions, the mutual rectangularity of the lines of curvature, is by employing (comp. 357, I.) the *self-conjugate form*,

$$\text{XVIII.} \dots \phi r = gr + V\lambda r\mu;$$

in which the vectors λ , μ , and the scalar g , depend only on the surface and the point, and are independent of the direction of the tangent. The equation VI. then becomes by V.,

$$\text{XIX.} \dots 0 = S\nu r\lambda r\mu = S\nu r\lambda S\mu r + S\nu r\mu S\lambda r;$$

assuming then the expression,

$$\text{XX.} \dots r = xV\nu\lambda + yV\nu\mu,$$

we easily find that

$$\text{XXI.} \dots y^2(V\nu\mu)^2 = x^2(V\nu\lambda)^2, \text{ or } \text{XXI}'. \dots yTV\nu\mu = \pm xTV\nu\lambda;$$

the two directions of r are therefore those of the two lines,

$$\text{XXII.} \dots UV\nu\lambda \pm UV\nu\mu,$$

which are evidently perpendicular† to each other.

* Dupin proved first (*Dév. de Géométrie*, pp. 43, 44, &c.), that two such tangents as are described in the text have a relation of *reciprocity* to each other, on which account he called them "*tangentes conjuguées*:" and afterwards he gave a sort of *image*, or *construction*, of this relation and of others connected with it, by means of the curve which he named "*l'indicatrice*" (in his already cited page 43, &c.).

† This mode, however, of determining generally the directions of the lines of curvature, gives only an illusory result, when the normal ν has the direction of either λ or μ , which happens at an *umbilic* of the surface. Compare 408, (27.), (29.), and the first Note to page 466.

(9.) An interpretation, of some interest, may be given to this last expression XXII., by the introduction of a certain auxiliary surface of the second order, which may be called the *Index Surface*, because the *index curve* (4.) is the *diametral section* of this new surface, made by the *tangent plane* to the given one. With the recent signification of ϕ , this *index surface* is represented by the equation VII., if r be now supposed (comp. (2.)) to represent a line PR drawn in any direction from the given point P , and therefore not now obliged to satisfy the condition V. of tangency. Or if, for greater clearness, we denote by $\rho + \rho'$ the vector from the origin o to a point of the index surface, the equation to be satisfied is, by the form XVIII. of ϕ (comp. §57, II.),

$$\text{XXIII.} \dots 1 = S\rho'\phi\rho' = g\rho'^2 + S\lambda\rho'\mu\rho';$$

the centre of this auxiliary surface being thus at P , and its two (real) cyclic normals being the lines λ and μ : so that $V\nu\lambda$ and $V\nu\mu$ have the directions of the traces of its two cyclic planes, on that diametral plane ($S\nu\rho' = 0$) which touches the given surface. We have therefore, by XXII., this general theorem, that the bisectors of the angle formed by these two traces are the tangents to the two lines of curvature, whatever the form of the given surface may be.

(10.) Supposing now that the given surface is itself one of the second order, and that its centre is at the origin o , so that it may be represented (comp. 405, XII.) by the equation,

$$\text{XXIV.} \dots 1 = S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho,$$

with constant values of λ , μ , and g , which will reproduce with those values the form XVIII. of ϕ , we see that the *index surface* (9.) becomes in this case simply that given one, with its centre transported from o to P ; and therefore with a tangent plane at the origin, which is parallel to the given tangent plane. And thus the traces (9.), of the cyclic planes on the diametral plane of the index surface, become here the tangents to the circular sections of the given surface. We recover then, as a case of the general theorem in (9.), this known but less general theorem: that the angles formed by the two circular sections, at any point of a surface of the second order, are bisected by the lines of curvature, which pass through the same point.

(11.) And because the tangents to these latter lines coincide generally, by (3.) (4.) (9.), with the axes of the diametral section of the index surface, made by the tangent plane to the given surface, they are parallel, in the case (10.), as indeed is well known, to the axes of the parallel section of a given surface of the second order.*

(12.) And if we now look back to the Equation of Confocals in 407, (26.), and to the earlier formulæ of 407, (4.), we shall see that because the vector ν_1 , in the last cited sub-article, represents a tangent to the given surface $S\rho\phi\rho = 1$, *complanar** with the normal ν and the derived vector $\phi\nu_1$, so that it satisfies (comp. 407, XII. XIV., and the recent formulæ V. VI.) the two scalar equations,

$$\text{XXV.} \dots S\nu\nu_1 = 0, \quad \text{and} \quad \text{XXVI.} \dots S\nu\nu_1\phi\nu_1 = 0,$$

which are likewise satisfied (comp. (7.)) when we change ν_1 to the rectangular tan-

* Compare the Note to page 645.

gent ν_2 , it follows that these two vectors, ν_1 and ν_2 , which are the normals to the two confocals to (e) through P , are also the tangents to the two lines of curvature on that given surface of the second order at that point: whence follows this other theorem* of Dupin, that the curve of orthogonal intersection (407, (4.)), of two confocal surfaces, is a line of curvature on each.

(13.) And by combining this known theorem, with what was lately shown respecting the umbilicar generatrices (in 408, (80.), (82.), comp. also (85.), (86.)), we may see that while, on the one hand, the lines of curvature on a central surface of the second order have no real envelope, yet on the other hand, in an imaginary sense, they have for their common envelope† the system of the eight imaginary right lines (408, (81.)), which connect the twelve (real or imaginary) umbilics of the surface, three by three, and are at once generating lines of the surface itself, and also of the known developable envelope of the confocal system.

(14.) It may be added, as another curious property of these eight imaginary right lines, that each is, in an imaginary sense, itself a line of curvature upon the surface: or rather, each represents two coincident lines of that kind. In fact, if we denote the variable vector 408, LXXX. of such a generatrix by the expression,

$$\text{XXVII.} \dots \rho = e'\sigma + \sigma',$$

in which e' is a variable scalar, but σ, σ' are two given or constant but imaginary vectors, such that

$$\text{XXVIII.} \dots \sigma^2 = 0, \quad S\sigma\sigma' = -D, \quad \sigma'^2 = -D^2,$$

and $\text{XXIX.} \dots f\sigma = S\sigma\phi\sigma = 0, \quad f(\sigma, \sigma') = S\sigma'\phi\sigma = 0, \quad f\sigma' = 1,$

we have the imaginary normal ν , with (for the case of a real umbilic) a real tensor,

$$\text{XXX.} \dots \nu = e'\phi\sigma + \phi\sigma' \perp \sigma, \quad \text{XXXI.} \dots T\nu = \pm \frac{(e - e')D^2}{abc};$$

* *Dév. de Géométrie*, page 271, &c.

† The writer is not aware that this theorem, to which he was conducted by quaternions, has been enunciated before; but it has evidently an intimate connexion with a result of Professor Michael Roberts, cited in page 290 of Dr. Salmon's Treatise, respecting the imaginary geodetic tangents to a line of curvature, drawn from an umbilicar point, which are analogous to the imaginary tangents to a plane conic, drawn from a focus of that curve. An illustration, which is almost a visible representation, of the theorem (13.) is supplied by Plate II. to Liouville's Monge (and by the corresponding plate in an earlier edition), in which the prolonged and dotted parts of certain ellipses, answering to the real projections of imaginary portions of the lines of curvature of the ellipsoid, are seen to touch a system of four real right lines, namely the projections (on the same plane of the greatest and least axes), of the four real umbilicar tangent planes, and therefore also of what have been above called (408, (80.), (81.)) the eight (imaginary) umbilicar generatrices of the surface. Accordingly Monge observes (page 150 of Liouville's edition), that "toutes les ellipses, projections des lignes de courbure, seront inscrites dans ce parallélogramme dont chacune d'elles touchera les quatre côtés:" with a similar remark in his explanation of the corresponding Figure (page 160).

and we find, after reductions, the imaginary expression,

$$\text{XXXII.} \dots \nu\sigma = \pm \sqrt{-1} \sigma T\nu, \text{ whence XXXIII.} \dots S\nu\sigma = 0, S\nu\sigma\phi\sigma = 0.$$

The differential equations V. VI. of a line of curvature are therefore *symbolically satisfied*, when we substitute, for the tangential vector τ , either the imaginary line σ itself; or the apparently *perpendicular* but in an imaginary sense *coincident** vector $\nu\sigma$; and the recent assertions are justified.

(15.) As regards the *real* lines of curvature, on a central surface of the second order, we see by comparing the *general* differential equation II. with the expression 409, XXIII. for the differential of h , or of P^2D^2 , that this latter product, or the product $P.D$ itself, is *constant*† for a *line of curvature*, as well as for a *geodesic line*, on such a surface, as indeed it is well known to be: although this *last* constant ($P.D$) may become *imaginary*, for the case of a *single-sheeted*‡ *hyperboloid*, and must be such for a line of curvature on an hyperboloid of *two sheets*.

(16.) And as regards the *general theory* of the *index surface* (9.), it is to be observed that this auxiliary surface depends *primarily* on the *scalar function* f , in the equation $f\rho = 1$, or generally $f\rho = \text{const.}$, of the *given surface*; and that it is *not* *entirely determined* by means of *that surface alone*. For if we write, for instance,

$$\text{XXXIV.} \dots f\rho = f1, \text{ with } d\rho = 2S\nu d\rho \text{ as before,}$$

we shall have, as the new first differential equation of the same given surface, instead of III.,

$$\text{XXXV.} \dots 0 = df\rho = 2S\nu d\rho, \text{ with XXXVI.} \dots n = f'\rho;$$

and if we then write, by analogy to IV.,

$$\text{XXXVII.} \dots d.n\nu = \phi d\rho = n\phi d\rho + n'\nu S\nu d\rho, \text{ with XXXVIII.} \dots n' = 2f'\rho,$$

the *new index surface*, constructed on the plan (9.), will have for its equation, analogous to XXIII., the following:

$$\text{XXXIX.} \dots S\rho'\phi\rho' = nS\rho'\phi\rho' + n'(S\nu\rho)^2 = \text{const.}$$

* As regards the paradox, of the *imaginary vector* σ being thus apparently *perpendicular to itself*, a similar one had occurred before, in the investigation 353, (17.), (18.), (19.); and it is explained, on the principles of modern geometry, by observing that this imaginary vector is *directed to the circle at infinity*. Compare 408, (81.), and the Note to page 459.

† Compare the first Note to page 667.

‡ Although the writer has been content to employ, in the present work, some of these usual but rather long appellations, he feels the elegance of Dupin's phraseology, adopted also by Möbius, and by some other authors, according to which the two central hyperboloids are distinguished, as *elliptic* (for the case of *two sheets*), and *hyperbolic* (for the case of *one*). The phrase "*quadric*," for the *general surface of the second order* (or *second degree*), employed by Dr. Salmon and Mr. Cayley, is also very convenient. It may be here remarked, that Dupin was perfectly aware of, or rather appears to have first discovered, the existence of what have since his time come to be called the *focal conics*; which important curves were considered by him, as being at once *limits of confocal surfaces*, and also *loci of umbilics*. *Comp. Dée. de Géométrie*, pages 270, 277, 278, 279; see also page 390 of the *Aperçu Historique*, &c., by M. Chasles (Brussels, 1837).

(17.) But if we take this last constant = n , the two index surfaces, XXXIII. and XXXIX., will have a common diametral section, made by the given tangent plane, namely the index curve (4.); and they will touch each other, in the whole extent of that curve. And it will be found that the construction (9.), for the directions of the lines of curvature, applies equally well to the one as to the other, of these two auxiliary surfaces: in fact, it is evident that the differential equation II., namely $Sv\delta v d\rho = 0$, receives no real alteration, when v is multiplied by any scalar, n , even if that scalar should be variable.

(18.) And instead of supposing that the variable vector ρ is thus obliged, as in 378, to satisfy a given scalar equation, of the form*

$$f\rho = \text{const.},$$

* If $\rho = ix + jy + kz$, and $v = f\rho = F(x, y, z)$, and if we write,

$$\begin{aligned} dv &= pdx + qdy + rdx, & dp &= p'dx + r'dy + q''dx, \\ dq &= q'dy + p''dx + r'dx, & dr &= r'dx + q''dx + p''dy, \end{aligned}$$

we may then write also, on the present plan, which gives $\delta f\rho = 2Sv\delta\rho$,

$$\begin{aligned} d\rho &= idx + jdy + kdz, & v &= -\frac{1}{2}(ip + jq + kr), \\ dv &= -\frac{1}{2}(idp + jdq + kdr), & S\delta\rho dv &= \frac{1}{2}(dxdp + dydq + dzdr); \end{aligned}$$

and the index surfaces, constructed as in (9.), and with ρ' changed to $\Delta\rho = i\Delta x + j\Delta y + k\Delta z$, will thus have the equation,

$$(a) \dots \frac{1}{2}p'\Delta x^2 + \frac{1}{2}q'\Delta y^2 + \frac{1}{2}r'\Delta z^2 + p''\Delta y\Delta x + q''\Delta x\Delta x + r''\Delta x\Delta y = 1,$$

or more generally = const.; so that it may be made in this way to depend upon, and be entirely determined by, the six partial differential coefficients of the second order, $p' \dots p'' \dots$, of the function v or $f\rho$, taken with respect to the three rectangular coordinates, xyz . And by comparing this equation (a) with the following equation of the same auxiliary surface, which results more directly from the principles employed in the text (comp. XVIII. XXIII.),

$$(b) \dots S\Delta\rho\phi\Delta\rho = g\Delta\rho^2 + S\lambda\Delta\rho\mu\Delta\rho = 1,$$

we can easily deduce expressions for those six partial coefficients, in terms of g, λ, μ . Thus, for example,

$$\frac{1}{2}D_x^2v = \frac{1}{2}p' = -g + S\lambda\mu i = S\lambda\mu - g + 2S\lambda S\mu i;$$

but $S\lambda S\mu i + Sj\lambda Sj\mu + S\lambda S\lambda\mu = -S\lambda\mu$; therefore,

$$(c) \dots \frac{1}{2}(D_x^2v + D_y^2v + D_z^2v) = S\lambda\mu - 3g = c_1 + c_2 + c_3 = -m'',$$

if c_1, c_2, c_3 be the roots and m'' a coefficient of a certain cubic (854, III.), deduced from the linear and vector function $dv = \phi d\rho$, on a plan already explained. If then the function v satisfy, as in several physical questions, the partial differential equation,

$$(d) \dots D_x^2v + D_y^2v + D_z^2v = 0,$$

the sum of these three roots, c_1, c_2, c_3 , will vanish: and consequently, the asymptotic cone to the index-surfaces, found by changing 1 to 0 in the second member of (a), is real, and has (comp. 406, XXI., XXIX.) the property that

$$(e) \dots \cot^2 a + \cot^2 b = 1,$$

if a, b denote its two extreme semiangles. An entirely different method of trans-

we may suppose, as in 372, that ρ is a *given vector function of two scalar variables*, x and y , between which there will then arise, by the same fundamental formula II., a *differential equation of the first order and second degree*, to be integrated (when possible) by known methods. For example, if we write,

$$\text{XLI.} \dots \rho = ix + jy + kz, \quad dz = pdx + qdy,$$

we shall satisfy the equation III. by assuming (with a constant factor understood),

$$\text{XLII.} \dots \nu = ip + jq - k, \quad \text{whence} \quad \text{XLIII.} \dots d\nu = idp + jdq;$$

and thus the *general equation* II., for the lines of curvature on an arbitrary surface, receives (by the laws of ijk) the form,

$$\text{XLIII.} \dots dp(dy + qdx) = dq(dx + pdz);$$

which last form has accordingly been assigned, and in several important questions employed, by Monge*: but which is now seen to be *included* in the still more concise (and more easily deduced and interpreted) *quaternion equation*,

$$S\nu d\nu d\rho = 0.$$

411. For a *central surface of the second order*, we have as usual $\nu = \phi\rho$, $\Delta\nu = \phi\Delta\rho$, and therefore (by 347, 348, and by the self-conjugate form of ϕ),

$$\text{I.} \dots \nabla\nu\Delta\nu = \nabla\phi\rho\phi\Delta\rho = \psi\nabla\rho\Delta\rho = m\phi^{-1}\nabla\rho\Delta\rho;$$

the *general condition of intersection* 410, I. of two normals, at the extremities of a *finite chord* $\Delta\rho$, and the *general differential equation* 410, II. of the *lines of curvature*, may therefore for *such* a surface receive these *new and special forms* :

forming, by quaternions, the well known equation (d), occurred early to the present writer, and will be briefly mentioned somewhat farther on. In the mean time it may be remarked, that because $m'' = 0$ by (c), when the equation (d) is satisfied, we have then, by the general theory III. ii. 6 of linear and vector functions, and especially by the subarticles to 350, remembering that ϕ is here self-conjugate, the formula,

$$(f) \dots d\nu + \chi d\rho = 0, \quad \text{and} \quad (g) \dots \psi\sigma - \phi^2\sigma = m'\sigma,$$

χ , ψ being auxiliary functions, and m' another coefficient of the cubic, while σ is an arbitrary vector. For the same reason, and under the same condition (d), the function ϕ itself has the properties expressed by the equations,

$$(h) \dots \phi\nabla\iota\kappa = \kappa\phi\iota - \iota\phi\kappa, \quad \text{and} \quad (i) \dots \phi^2\nabla\iota\kappa = \nabla\phi\iota\phi\kappa - m'\nabla\iota\kappa;$$

in which the *two vectors* ι , κ are *arbitrary*, and m' is the same *scalar coefficient* as before.

* See the enunciation of the formula here numbered as XLIII., in page 133 of Liouville's Monge: compare also the applications of it, in pages 274, 303, 305, 357. (The corresponding pages of the Fourth Edition are, 115, 240, 265, 267, 312.) The quaternion equation, $S\nu d\nu d\rho = 0$, was published by the present writer, in a communication to the Philosophical Magazine, for the month of October, 1847 (page 289). See also the Supplement to the same Volume xxxi. (Third Series); and the Proceedings of the Royal Irish Academy for July, 1846.

$$\text{II.} \dots S\Delta\rho\phi^{-1}\nabla\rho\Delta\rho = 0, \quad \text{or} \quad \text{II}'. \dots S\rho\Delta\rho\phi^{-1}\Delta\rho = 0;$$

$$\text{III.} \dots Sd\rho\phi^{-1}\nabla\rho d\rho = 0, \quad \text{or} \quad \text{III}'. \dots S\rho d\rho\phi^{-1}d\rho = 0;$$

which admit of geometrical interpretations, and conduct to some new theorems, especially when they are transformed as follows:

$$\text{IV.} \dots S\lambda\Delta\rho.S\rho\Delta\rho\phi^{-1}\mu + S\mu\Delta\rho.S\rho\Delta\rho\phi^{-1}\lambda = 0,$$

$$\text{V.} \dots S\lambda d\rho.S\rho d\rho\phi^{-1}\mu + S\mu d\rho.S\rho d\rho\phi^{-1}\lambda = 0,$$

λ and μ being (as in 405, (5.), &c.) the two real cyclic normals of the surface: while the same equations may also be written under the still more simple forms,

$$\text{VI.} \dots Sa\Delta\rho.Sa'\rho\Delta\rho + Sa'\Delta\rho.Sa\rho\Delta\rho = 0,$$

$$\text{VII.} \dots Sa d\rho.Sa'\rho d\rho + Sa'\Delta\rho.Sa\rho d\rho = 0,$$

a, a' being, as in several recent investigations, the two real focal unit lines, which are common to a whole confocal system.

(1.) The vector $\phi^{-1}\nabla\rho\Delta\rho$ in II. has by I. the direction of $\nabla\nu\Delta\nu$; whence, by 410, (6.), the interpretation of the recent equation II., or (for the present purpose) of the more general equation 410, I., is that the chord PP' is perpendicular to its own polar, if the normals at its extremities intersect. Accordingly, if their point of intersection be called x , the polar of PP' is perpendicular at once to Px and $P'x$, and therefore to PP' itself.

(2.) The equation II'. may be interpreted as expressing, that when the normals at P and P' thus intersect in a point x , there exists a point x'' in the diametral plane OPx' , at which the normal $x''x''$ is parallel to the chord PP' : a result which may be otherwise deduced, from elementary principles of the geometry of surfaces of the second order.

(3.) It is unnecessary to dwell on the converse propositions, that when either of these conditions is satisfied, there is intersection (or parallelism) of the two normals at P and P' : or on the corresponding but limiting results, expressed by the equations III. and III'.

(4.) In order, however, to make any use in calculation of these new forms II., III., we must select some suitable expression for the self-conjugate function ϕ , and deduce a corresponding expression for the inverse function ϕ^{-1} . The form,*

$$\text{VIII.} \dots \phi\rho = g\rho + \nabla\lambda\rho\mu,$$

which has already several times occurred, has also been more than once inverted: but the following new inverse† form,

* The vector form VIII. occurred, for instance, in pages 463, 469, 474, 484, 641, 669; and the connected scalar form,

$$f\rho = g\rho^2 + S\lambda\rho\mu\rho, \quad 357, \text{II.}$$

has likewise been frequently employed.

† Inverse forms, for $\phi^{-1}\rho$ or $\kappa^{-1}\psi\rho$, have occurred in pages 463, 484, 641 (the

$$\text{IX.} \dots (g - S\lambda\mu) \cdot \phi^{-1}\rho = \rho - \lambda S\rho\phi^{-1}\mu - \mu S\rho\phi^{-1}\lambda,$$

has an advantage, for our present purpose, over those assigned before. In fact, this form IX. gives at once the equation,

$$\text{X.} \dots (g - S\lambda\mu) \cdot \phi^{-1}\nabla\rho\Delta\rho = \nabla\rho\Delta\phi - \lambda S\rho\Delta\rho\phi^{-1}\mu - \mu S\rho\Delta\rho\phi^{-1}\lambda;$$

and so conducts immediately from II. to IV., or from III. to V. as a limit.

(5.) The equation IV. expresses generally, that the chord $\Delta\rho$, or FR' , is a side of a certain cone of the second order, which has its vertex at the point r of the given surface, and passes through all the points r' for which the normals to that surface intersect the given normal at r ; and the equation V. expresses generally, that the two sides of this last cone, in which it is cut by the given tangent plane at the same point r , are the tangents to the lines of curvature.

(6.) But if the surface be an ellipsoid, or a double-sheeted hyperboloid, then (comp. 408, (29.)) the always real vectors,* $\phi^{-1}\lambda$ and $\phi^{-1}\mu$, have the directions of semidiameters drawn to two of the four real umbilics; supposing then that ρ is such a semikiameter, and that it has the direction of $\pm\phi^{-1}\lambda$, the second term of the first member of the equation IV. vanishes, and the cone IV. breaks up into a pair of planes, of which the equations in ρ' are,

$$\text{XI.} \dots S\lambda(\rho' - \rho) = 0, \quad \text{and} \quad \text{XII.} \dots S\rho'\phi^{-1}\lambda\phi^{-1}\mu = 0;$$

whereof the former represents the tangent plane at the umbilic r , and the latter represents the plane of the four real umbilics.

(7.) It follows, then, that the normal at the real umbilic r is not intersected by any real normal to the surface, except those which are drawn at points r' of that principal section, on which all the real umbilics are situated: but that the same real umbilic normal rN is, in an imaginary sense, intersected by all the imaginary normals, which are drawn from the imaginary points r' of either of the two imaginary generatrices through r .

(8.) In fact, the locus of the point r' , under the condition of intersection of its normal $\text{r}'\text{N}'$ with a given normal rN , is generally a quartic curve, namely the intersection of the given surface with the cone IV.; but when this cone breaks up, as in (6.), into two planes, whereof one is normal, and the other tangential to the surface, the general quartic is likewise decomposed, and becomes a system of a real conic, namely the principal section (7.), and a pair of imaginary right lines, namely the two umbilic generatrices at r .

(9.) We see, at the same time, in a new way (comp. 410, (14.)), that each such generatrix is (in an imaginary sense) a line of curvature: because the (imaginary) normals to the surface, at all the points of that generatrix, are situated by (7.) in one common (imaginary) normal plane.

(10.) Hence through a real umbilic, on a surface of the second order, there pass

correction in a Note to which last page should be attended to). In comparing these with the form IX., it will easily be seen (comp. page 661) that

$$\phi^{-1}\lambda = \frac{g\lambda - \lambda^2\mu}{g^2 - \lambda^2\mu^2}, \quad \phi^{-1}\mu = \frac{g\mu - \mu^2\lambda}{g^2 - \lambda^2\mu^2}.$$

* Compare the Note immediately preceding.

three lines of curvature: whereof one is a real conic (8.), and the two others are imaginary right lines, namely, the umbilicar generatrices as before.

(11.) If we prefer differentials to differences, and therefore use the equation V. of the lines of curvature, we find that this equation takes the form $0 = 0$, if the point P be an umbilic; and that if the normal at that point be parallel to λ , the differential of the equation V. breaks up into two factors, namely,

$$\text{XIII.} \dots S\lambda d^2\rho = 0, \quad \text{and} \quad \text{XIV.} \dots S\rho\phi^{-1}\lambda\phi^{-1}\mu = 0;$$

whereof the former gives two imaginary directions, and the latter gives one real direction, coinciding precisely with the three directions (10.).

(12.) And if ρ , instead of being the vector of an umbilic, be only the vector of a point on a generatrix corresponding, we shall still satisfy the differential equation V., by supposing that $d\rho$ belongs to the same imaginary right line: because we shall then have, as at the umbilic itself,

$$\text{XV.} \dots S\lambda d\rho = 0, \quad S\rho d\rho\phi^{-1}\lambda = 0.$$

An umbilicar generatrix is therefore proved anew (comp. (9.)) to be, in its whole extent, a line of curvature.

(13.) The recent reasonings and calculations apply (6.), not only to an ellipsoid, but also to a double-sheeted hyperboloid, four umbilics for each of these two surfaces being real. But if for a moment we now consider specially the case of an ellipsoid, and if we denote for abridgment the real quotient $\frac{a-c}{a+c}$ by h , we may then substitute in IV. and V. for λ , μ , $\phi^{-1}\lambda$, $\phi^{-1}\mu$ the expressions,

$$\text{XVI.} \dots \alpha - h\alpha' = \frac{2bU\lambda}{a+c}; \quad h\alpha - \alpha' = \frac{2bU\mu}{a+c};$$

$$\text{XVII.} \dots \alpha + h\alpha' = \frac{-2b\phi^{-1}U\lambda}{ac(a+c)}; \quad -h\alpha - \alpha' = \frac{-2b\phi^{-1}U\mu}{ac(a+c)};$$

and then, after division by $h^2 - 1$, there remain only the two vector constants α , α' , the equation IV. reducing itself to VI., and V. to VII.

(14.) The simplified equations thus obtained are not however peculiar to ellipsoids, but extend to a whole confocal system. To prove this, we have only to combine the equations II. and III. with the inverse form,

$$\text{XVIII.} \dots l^2\phi^{-1}\rho = aSa'\rho + a'Sa\rho - \rho(e + Saa'),$$

which follows from 407, XV., and gives at once the equations VI. and VII., whatever the species of the surface may be.

(15.) The differential equation VII. must then be satisfied by the three rectangular directions of $d\rho$, or of a tangent to a line of curvature, which answer to the orthogonal intersections (410, (12.)) of the three confocals through a given point P; it ought therefore, as a verification, to be satisfied also, when we substitute ν for $d\rho$, ν being a normal to a confocal through that point: that is, we ought to have the equation,

$$\text{XIX.} \dots Sa\nu Sa'\rho\nu + Sa'\nu Sa\rho\nu = 0.$$

And accordingly this is at once obtained from 407, XVI., by operating with $S.\rho\nu$; so that the three normals ν are all sides of this cone XIX., or of the cone VII. with $d\rho$ for a side, with which the cone V. is found to coincide (13.).

(16.) And because this last equation XIX., like VI. and VII., involves only the two focal lines α , α' as its constants, we may infer from it this theorem: "If inde-

finite many surfaces of the second order have only their asymptotic cones biconfocal,* and pass through a given point, their normals at that point have a cone of the second order for their locus;† which latter cone is also the locus of the tangents, at the same point, to all the lines of curvature which pass through it, when different values are successively assigned to the scalar constant $a^2 - c^2$ (or $2F$): that is, when the asymptotes a, a' to the focal hyperbola remain unchanged in position, but the semiaxes $(a^2 - b^2)^{\frac{1}{2}}, (b^2 - c^2)^{\frac{1}{2}}$ of that curve (here treated as both real) vary together.

(17.) The equation VI. of the cone of chords (b.) introduces the fixed focal lines a, a' by their directions only. But if we suppose that the lengths of those two lines are equal, without being here obliged to assume that each of those lengths is unity, we shall then have (comp. 407, (2.), (3.)), the following rectangular system of unit lines, in the directions of the axes of the system,

$$\text{XX.} \dots \text{U}(a + a'), \text{UV}aa', \text{U}(a - a'),$$

which obey in all respects the laws of ijk , and may often be conveniently denoted by those symbols, in investigations such as the present. And then, by decomposing the semidiameter ρ , and the chord $\Delta\rho$, in these three directions XX., we easily find the following rectangular transformation† of the foregoing equation VI.,

$$\text{XXI.} \dots \frac{\text{S}(a + a')^{-1}\rho}{\text{S}(a + a')\Delta\rho} + \frac{\text{S}(a - a')^{-1}\rho}{\text{S}(a - a')\Delta\rho} = \frac{\text{S}(\text{V}aa')^{-1}\rho}{\text{S}.\text{U}aa'\Delta\rho};$$

in which it is permitted to change $\Delta\rho$ to $d\rho$, in order to obtain a new form of the differential equation of the lines of curvature; or else at pleasure to v , and so to find, in a new way, a condition satisfied by the three normals, to the three confocals through \mathbf{P} .

(18.) The cone, VI. or XXI., is generally the locus of a system of three rectangular lines; each plane through the vertex, which is perpendicular to any real side, cutting it in a real pair of mutually rectangular sides: while, for the same reason, the section of the same cone, by any plane which does not pass through its vertex \mathbf{P} , but cuts any side perpendicularly, is generally an equilateral hyperbola.

(19.) If, however, the point \mathbf{P} be situated in any one of the three principal planes, perpendicular to the three lines XX., then the cone XXI. (as its equation shows) breaks up (comp. (6.)) into a pair of planes, of which one is that principal

* That is, if the surfaces (supposed to have a common centre) be cut by the plane at infinity in biconfocal conics, real or imaginary.

† The corresponding form, in rectangular co-ordinates, of the condition of intersection, of normals at two points (xyz) and $(x'y'z')$, to the surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is the equation (probably a known one, although the writer has not happened to meet with it),

$$\frac{(b^2 - c^2)x'}{x - x'} + \frac{(c^2 - a^2)y'}{y - y'} + \frac{(a^2 - b^2)z'}{z - z'} = 0;$$

in which it is evident that xyz and $x'y'z'$ may be interchanged.

plane itself, while the *other* is *perpendicular* thereto. And while the *former* plane cuts the surface in a *principal section*, which is *always* a *line of curvature* through P, the *latter* plane usually cuts the surface in *another conic*, which *crosses* the former section at *right angles*, and gives the *direction* of the second line of curvature.

(20.) But if we further suppose, as in (6.), that the point P is an *umbilic*, then (as has been seen) the *second plane* is a *tangent plane*; and the *second conic* (19.) is itself *decomposed*, into a *pair of imaginary right lines*: namely, as before, the *two umbilicar generatrices* through the point, which have been shown to be, in an *imaginary sense*, both *lines of curvature themselves*, and also a *portion of the envelope of all the others*.

(21.) We shall only here add, as *another transformation* of the general equation VI. of the *cone of chords*, which does not even assume $Ta = Ta'$, the following:

$$\text{XXII.} \dots S(a + a')\Delta\rho \cdot S(a + a')\rho\Delta\rho = S(a - a')\Delta\rho \cdot S(a - a')\rho\Delta\rho;$$

where the *directions* of the *two new lines*, $a + a'$ and $a - a'$, are only obliged to be *harmonically conjugate* with respect to the *directions of the fixed focal lines* of the system: or in other words, are those of any *two conjugate semidiameters* of the *focal hyperbola*.

412. The subject of *Lines of Curvature* receives of course an additional illustration, when it is combined with the known conception of the corresponding *Centres of Curvature*. Without yet entering on the *general theory* of the *curvatures of sections* of an arbitrary surface, we may at least consider here the curvatures of those *normal sections*, which *touch* at any given point the *lines of curvature*. Denoting then by σ the vector of the *centre s* of curvature of *such a section*, and by R the *radius* Ps, considered as a *scalar* which is positive when it has the direction of $+\nu$, it is easy to see that we have the *two fundamental equations*:

$$\text{I.} \dots \sigma = \rho + RU\nu; \quad \text{II.} \dots R^{-1}d\rho + dU\nu = 0;$$

whence follows this *new form* of the general differential equation 410, II. of the *lines of curvature*,

$$\text{III.} \dots Vd\rho dU\nu = 0;$$

with several other combinations or transformations, among which the following may be noticed here:

$$\text{IV.} \dots \frac{T\nu}{R} + S \frac{d\nu}{d\rho} = 0.$$

(1.) The equation I. requires no proof; and from it the equation II. is obtained by merely differentiating* as if σ and R were constant: after which the formula III. follows at once, and IV. is easily deduced.

* To students who are accustomed to *infinitesimals*, the *easiest* way is here to

(2.) To obtain from this last equation a more developed expression for R , we may assume for $d\nu$, considered as a linear and self-conjugate function of $d\rho$ (410, (1.)), the general form (comp. 410, XVIII.),

$$V. \dots d\nu = g d\rho + \nabla \lambda d\rho \mu,$$

in which g, λ, μ are independent of $d\rho$; and then, while the tangent $d\rho$ has (by 410, XXII.) one or other of the two directions,

$$VI. \dots d\rho \parallel UV\nu\lambda \pm UV\nu\mu,$$

the curvature R^{-1} receives one or other of the two values corresponding,

$$VII. \dots R^{-1} = -T\nu^{-1}(g + S\lambda U\nu.S\mu U\nu \pm T\nu\lambda U\nu.T\nu\mu U\nu).$$

(3.) One mode of arriving at this last transformation, or of showing that it (comp. again 410, XXII.) we assume,

$$VIII. \dots \tau = (\text{or } \parallel) UV\lambda\nu \pm UV\mu\nu,$$

then

$$IX. \dots S\lambda r\mu r^{-1} = S\lambda U\nu.S\mu U\nu \pm T\nu\lambda U\nu.T\nu\mu U\nu,$$

or

$$X. \dots 2S\lambda r.S\mu r^{-1} = S(\nabla\lambda U\nu.\nabla\mu U\nu) \pm T\nu\lambda U\nu.T\nu\mu U\nu,$$

or finally, XI. $\dots 2SU\lambda r.SU\mu r^{-1} = S(\nabla U\lambda\nu.\nabla U\mu\nu) \pm TVU\lambda\nu.TVU\mu\nu,$

is to introduce the auxiliary quaternion,

$$XII. \dots q = \nabla U\lambda\nu.\nabla U\mu\nu;$$

and to prove that, with the value (or direction) VIII. of τ , we have thus the equation (in which Vq^2 , as usual, represents the square of Vq),

$$XIII. \dots 2SU\lambda r.SU\mu r^{-1} = Sq \pm Tq = \frac{Vq^2}{Sq \mp Tq}.$$

(4.) And this may be done, by simply observing that we have thus (with the value VIII.) the expressions,

$$XIV. \dots SrU\lambda = \frac{\pm SU\lambda\mu\nu}{TVU\mu\nu}, \quad SrU\mu = \frac{-SU\lambda\mu\nu}{TVU\lambda\nu};$$

$$XV. \dots SrU\lambda.SrU\mu = \frac{\mp (SU\lambda\mu\nu)^2}{TVU\lambda\nu.TVU\mu\nu} = \frac{\pm Vq^2}{Tq},$$

because

$$XVI. \dots Vq = -U\nu.SU\lambda\mu\nu;$$

and

$$XVII. \dots r^2 = -2 \pm 2SUq = \pm \frac{2(Sq \mp Tq)}{Tq}$$

(5.) Admitting then the expression VII., for the curvature R^{-1} , we easily see that it may be thus transformed:

$$XVIII. \dots R^{-1} = -T\nu^{-1} \left(g + T\lambda\mu \cdot \cos \left(\angle \frac{\nu}{\lambda} \mp \angle \frac{\nu}{\mu} \right) \right);$$

and that the difference of the two (principal) curvatures, of normal sections of an arbitrary surface, answering generally to the two (rectangular) directions of the

conceive the differentials to be such. But it has already been abundantly shown, that this view of the latter is by no means necessary, in the treatment of them by quaternions. (Compare the second Note to page 667.)

lines of curvature through the particular point considered, *vanishes* when the *normal* ν has the *direction* of either of the two *cyclic normals*, λ , μ , of the *index surface* (410, (9.)); that is, when the *index curve* (410, (4.)), considered as a *section* of that *index surface*, is a *circle*: or finally, when the point in question is, in a received sense, an *umbilic** of the given surface.

(6.) That surface, although considered to be a *given* one, has hitherto (in these last sub-articles) been treated as quite *general*. But if we now suppose it to be a *central surface* of the *second order*, and to be represented by the equation,

$$\text{XIX.} \dots f\rho = g\rho^2 + S\lambda\rho\mu\rho = 1,$$

which has already several times occurred, we see at once, from the formula VII. or XVIII. (comp. 410, (10.)), that the *difference of curvatures*, of the two principal normal sections of any such surface, *varies* proportionally to the *perpendicular* ($T\nu^{-1}$ or P) from the centre on the tangent plane, multiplied by the *product of the sines of the inclinations* of that plane, to the two *cyclic planes* of the surface.

(7.) In general (comp. 409, (8.)), it is easy to see that

$$\text{XX.} \dots S \frac{d\nu}{d\rho} = Sr^{-1}\phi r = -D^{-2},$$

if D denote the (scalar) *semidiameter* of the *index surface*, in the direction of $d\rho$ or of r ; but for the two directions of the *lines of curvature*, these semidiameters become (410, (3.)), (4.) the *semiaxes* of the *index curve*. Denoting then by a_1 and a_2 these last semiaxes, the *two principal radii of curvature* of any surface come by IV. to be thus expressed:

$$\text{XXI.} \dots R_1 = a_1^2 T\nu; \quad R_2 = a_2^2 T\nu.$$

And if the surface be a *central one*, of the *second order*, then a_1 , a_2 are the *semiaxes of the diametral section*, parallel to the *tangent plane*; while $T\nu$ is (comp. again 409, (8.)) the *reciprocal* P^{-1} of the *perpendicular*, let fall on that plane from the centre. Accordingly (comp. (6.)), and 219, (4.)), it is known that the *difference of the inverse squares* of those *semiaxes* varies proportionally to the *product of the sines of the inclinations*, of the plane of the section to the two *cyclic planes*.

(8.) And as regards the *squares themselves*, it follows from 407, LXXI., that they may be thus expressed, in terms of the *principal semiaxes* of the *confocal surfaces*, and in agreement with known results:

$$\text{XXII.} \dots a_1^2 = a^2 - a_1'^2; \quad a_2^2 = a^2 - a_2'^2;$$

being thus *both positive* for the case of an *ellipsoid*; *both negative*, for that of a *double-sheeted hyperboloid*; and *one positive*, but the *other negative*, for the case of an *hyperboloid of one sheet* (comp. 410, (15.)).

(9.) In all these cases, the *normal* ν is drawn towards the *same side* of the *tangent plane*, as that on which the *centre* o of the *surface* is situated (because $S\nu\rho = 1$); hence (by I. and XXI.) *both the radii* of curvature R_1 , R_2 are drawn in *this direction*, or towards *this side*, for the *ellipsoid*; but *one* such radius for the *single-sheeted hyperboloid*, and *both radii* for the *hyperboloid of two sheets*, are directed towards the *opposite side*, as indeed is evident from the forms of these surfaces.

* Compare the second Note to page 669.

(10.) The following is another method of deducing *generally* the two principal curvatures of a surface, from the *self-conjugate function*,

$$\text{XXIII.} \dots d\nu = \phi d\rho, \quad 410, \text{IV.}$$

which affords some good practice in the processes of the present Calculus. Writing, for abridgment,

$$\text{XXIV.} \dots r = \frac{\nu}{\sigma - \rho} = R^{-1}T\nu = -S \frac{d\nu}{d\rho} = -Sr^{-1}\phi r,$$

where r is still a tangent to a line of curvature, the equation II. is easily brought to the form,

$$\text{XXV.} \dots -rr = \nu^{-1}V\nu\phi r = \phi r - \nu^{-1}Sr\phi\nu = \Phi r,$$

where Φ denotes a *new linear and vector function*, which however is *not* in general *self-conjugate*, because we have not generally $\phi\nu \parallel \nu$. Treating then this *new function* on the plan of the Section III. ii. 6, we derive from it a *new cubic equation*, of the form,

$$\text{XXVI.} \dots 0 = M + M'r + M''r^2 + r^3,$$

and with the coefficients,

$$\text{XXVII.} \dots M = 0, \quad M' = S\nu^{-1}\psi\nu, \quad M'' = m'' - S\nu^{-1}\phi\nu;$$

ψ being a certain *auxiliary function* ($= m\phi^{-1}$), and m'' being the *coefficients** analogous to M'' , in the cubic derived from the function ϕ *itself*. The root $r = 0$ is foreign to the present inquiry; but the *two curvatures*, R_1^{-1} , R_2^{-1} , are the *two roots* of the following *quadratic* in R^{-1} , obtained from the equation XXVI. by the rejection of that foreign root:

$$\text{XXVIII.} \dots 0 = (R^{-1}T\nu)^2 + M''R^{-1}T\nu + M'.$$

(11.) As a first application of this general equation XXVIII., let ϕr have again, as in V., the form $gr + V\lambda r\mu$; we shall then have the values,

$$\text{XXIX.} \dots M'' = 2(g + S\lambda U\nu \cdot S\mu U\nu),$$

and

$$\text{XXX.} \dots M' = (g + S\lambda U\nu \cdot S\mu U\nu)^2 - (V\lambda U\nu)^2 (V\mu U\nu)^2,$$

= a great variety of transformed expressions; and the two resulting curvatures agree with those assigned by VII.

(12.) As a second application, let the surface be central of the second order, with abc for its scalar semiaxes (real or imaginary); then the *symbolical cubic* (350) in ϕ becomes,

$$\text{XXXI.} \dots 0 = \phi^3 - m''\phi^2 + m'\phi - m = (\phi + a^{-2})(\phi + b^{-2})(\phi + c^{-2});$$

and the coefficients of the quadratic XXVIII. in R^{-1} take the values, in which N denotes the semidiameter of the surface in the direction of the normal:

$$\text{XXXII.} \dots R_1^{-1} + R_2^{-1} = -M''T\nu^{-1} = -(m'' + fU\nu)P = (a^{-2} + b^{-2} + c^{-2} - N^2)P;$$

* Compare the Note to page 673, continued in page 674. The reason of the evanescence of the coefficient M' , or of the occurrence of a null root of the cubic, is that we have here $\Phi\phi^{-1}\nu = 0$, so that the symbol $\Phi^{-1}0$ may represent an actual vector (comp. 851). Geometrically, this corresponds to the circumstance that when we pass, along a semidiameter prolonged, from a surface of the second order to another surface of the same kind, concentric, similar, and similarly placed, the direction of the normal does not change.

$$\text{XXXIII.} \dots R_1^{-1}R_2^{-1} = M^{-1}T\nu^{-2} = -m\nu^{-4} = a^{-2}b^{-2}c^{-2}P^4;$$

both of which agree with known results, and admit of elementary verifications.*

(18.) *In general*, if we observe that $m^{-1} - \phi = \chi$ (350, XVI.), we shall see that the quadratic XXVIII. in r (or in $R^{-1}T\nu$) may be thus written:

$$\text{XXXIV.} \dots 0 = S\nu^{-1}(r^2\nu + r\chi\nu + \psi\nu);$$

or thus more briefly (comp. 898, LXXIX.),

$$\text{XXXV.} \dots 0 = S\nu^{-1}(\phi + r)^{-1}\nu.$$

(14.) Accordingly, the formula XXV. gives the expression,

$$\text{XXXVI.} \dots \nu^2r = (\phi + r)^{-1}\nu.Sr\phi\nu;$$

from which, under the condition $S\nu r = 0$, the equation XXXV. follows at once.

(15.) We have therefore *generally*, for the *product* of the two principal curvatures of sections of any surface at any point, the expression:

$$\text{XXXVII.} \dots R_1^{-1}R_2^{-1} = r_1r_2T\nu^{-2} = -\nu^{-4}S\nu\psi\nu = -S\frac{1}{\nu}\psi\frac{1}{\nu};$$

which contains an important theorem of Gauss, whereto we shall presently proceed.

(16.) Meanwhile we may remark that the recent analysis shows, that the squares a_1^2, a_2^2 (7.) of the semiaxes of the index-curve are *generally* the roots of the following equation,

$$\text{XXXVIII.} \dots 0 = S\nu(\phi + a^2)^{-1}\nu,$$

when developed as a quadratic in a^2 .

(17.) And that the same quadratic assigns the squares of the semiaxes of a diametral section, made by a plane $\perp \nu$, of the central surface of the second order which has $S\rho\phi\rho = 1$ for its equation.

(18.) Accordingly, $\nabla\rho\phi\rho$ has the direction of a tangent to this surface, which is perpendicular to ρ at its extremity; and therefore the vector,

$$\text{XXXIX.} \dots \sigma = \rho^{-1}\nabla\rho\phi\rho = \phi\rho - \rho^{-1} = (\phi - \rho^{-2})\rho,$$

is perpendicular to the plane of the diametral section, which has the *semidiameter* ρ for a *semiaxis*: so that it is perpendicular also to ρ itself. The equation,

$$\text{XL.} \dots S\sigma(\phi - \rho^{-2})^{-1}\sigma = 0,$$

assigns therefore the values of the squares ($-\rho^2$) of the scalar semiaxes of the central section $\perp \sigma$; which agrees with the formula XXXVIII.

(19.) If then a surface be derived from a given central surface of the second order, as the locus of the extremities of normals (erected at the centre) to the diametral sections of the given surface, each such normal (when real) having the length of one of the semiaxes of that section, the equation of this new surface† (or locus) will admit of being written thus:

$$\text{XLI.} \dots S\rho(\phi - \rho^{-2})^{-1}\rho = 0.$$

* As an easy verification by quaternions of the expression XXXII., it may be remarked (comp. 408, (27.)), that if a, β, γ be any three rectangular unit lines, then

$$f\alpha + f\beta + f\gamma = \text{const.} = c_1 + c_2 + c_3 = a^{-2} + b^{-2} + c^{-2}.$$

† When the given surface is an *ellipsoid*, this *derived* surface XLI. is therefore the celebrated *Wave Surface* of Fresnel, which will be briefly mentioned somewhat farther on.

(20.) The first of the values XXIV., for the auxiliary scalar r , gives the expression (if $\nu = \phi\rho$, as it is for a central surface of the second order),

$$\text{XLII.} \dots \sigma = \rho + r^{-1}\nu = (1 + r^{-1}\phi)\rho = r^{-1}(\phi + r)\rho;$$

whence, by inversion, and operation with ϕ ,

$$\text{XLIII.} \dots \rho = r(\phi + r)^{-1}\sigma; \quad \text{XLIV.} \dots \nu = r(\phi + r)^{-1}\phi\sigma;$$

and therefore, because $S\rho\nu = 1$,

$$\text{XLV.} \dots r^2 = S((\phi + r)^{-1}\sigma \cdot (\phi + r)^{-1}\phi\sigma) = S.\sigma(\phi + r)^{-2}\phi\sigma.$$

(21.) The following is a quite different way of arriving at this result, which is also useful for other purposes. Considering σ as the vector os of a point s on the *Surface of Centres*, that is, on the *locus* of all the centres of curvature of principal normal sections, the vector (say ν) of the *Reciprocal Surface* is connected with σ (comp. 373, (21.)) by the *equations of reciprocity*,*

$$\text{XLVI.} \dots S\sigma\nu = S\nu\sigma = 1; \quad \text{XLVII.} \dots S\sigma\delta\sigma = 0; \quad \text{XLVIII.} \dots S\sigma\delta\nu = 0;$$

which are all satisfied by the vector expression,

$$\text{XLIX.} \dots \nu = \frac{r}{S\rho r},$$

where r is, as before, a tangent to the line of curvature: so that, if ω denote the variable vector of the normal plane to this last curve, the equation of that plane (comp. 369, IV.) may be thus written,

$$\text{L.} \dots S\nu(\omega - \rho) = 0.$$

This *normal plane*, to the *line of curvature* at r , is therefore at the same time the *tangent plane* to the *surface of centres* at s , as indeed it is known to be, from simple geometrical considerations, independently of the *form* of the *given surface*, which remains here entirely arbitrary.

(22.) The expression XLIX. for ν gives *generally* the relation,

$$\text{LI.} \dots S\rho\nu = 1;$$

giving also, by 410, V. and VI., these two other equations,

* It is understood that $d\sigma$ and $d\nu$, in the differential equations XLVII., XLVIII., are in general only obliged to have directions *tangential* to the surface of centres, and to its reciprocal, at corresponding points: so that the equations might be in some respects more clearly written thus, $S\nu d\sigma = 0$, $S\sigma d\nu = 0$, the mark d being reserved to indicate changes which arise from motion along a *given line* of curvature, while δ should have a more general signification. Accordingly if, in particular, we write $\delta\rho = \nu d\rho$, for a variation answering to motion along the *other line*, and denote the two radii of curvature for the two directions $d\rho$ and $\delta\rho$ by R_1 and R_2 , we shall have by II., $R_1^{-1}d\rho + dU\nu = 0$, $R_2^{-1}\delta\rho + \delta U\nu = 0$, and therefore by I.,

$$d\sigma = dR_1.U\nu, \quad \delta\sigma = \delta\rho + \delta(R_1U\nu) = (1 - R_1R_2^{-1})\nu d\rho + \delta R_1.U\nu;$$

so that we have *both* $Sd\rho d\sigma = 0$, and $Sd\rho \delta\sigma = 0$, and therefore the *tangent* $d\rho$ or r to the *given line* of curvature has the direction of the *normal* ν to the corresponding *sheet* of the surface of centres, as is otherwise visible from geometry. And when we have thus found an equation of the form $t\nu = r$, operation with $S.\sigma$ gives by XLVI. the value $t = S\rho r$, as in XLIX., because $\sigma - \rho \parallel \nu \perp r$.

LII . . $S\nu\nu = 0$, and LIII . . $S\nu\nu\phi\nu = 0$,

which are still independent of the form of the given surface.

(23.) But if that surface be a *central quadric*,* then the equation LI. may be thus written,

LIV . . $1 = S\nu\phi^{-1}\nu = S\nu\phi^{-1}\nu$;

combining which with LII. and LIII., we derive the expressions :

LV . . $\nu = \frac{v^2\phi\nu - vfv}{v^4 - fv.Fv}$; LVI . . $\rho = \phi^{-1}\nu = \frac{v^2 - \phi^{-1}vfv}{v^4 - fv.Fv}$;

wherein $fv = S\nu\phi\nu$, and $Fv = S\nu\phi^{-1}\nu$, as usual.

(24.) Operating with $S.\nu$ on this last expression for ρ , and attending to LII. and LIV., we find the following *quaternion forms* of the *Equation of the Reciprocal of the Surface of Centres* :

LVII . . $1 = (S\nu\rho) = \frac{-fv}{v^4 - fv.Fv}$; or LVIII . . $v^4 = (Fv - 1)fv$;

or

LIX . . $1 = (Fv - 1)f\frac{1}{\nu}$; or LX . . $Fv - \frac{1}{f\frac{1}{\nu}} = 1$; &c.,

whereof the second, when translated into co-ordinates, is found to agree perfectly with a known† equation of the same reciprocal surface.

(25.) Differentiating the form LX., and observing that

LXI . . $\left(f\frac{1}{\nu}\right)^{-1} = \frac{v^4}{fv}$, $d.v^4 = 4S\nu^2dv$, $dfv = 2S\phi\nu dv$, $dFv = 2S\phi^{-1}\nu dv$,

we find, by comparison with XLVI. and XLVIII., the expression :

LXII . . $\sigma = \phi^{-1}\nu - \frac{2v^2}{fv} + \frac{v^4\phi\nu}{(fv)^2}$; or LXIII . . $\sigma = \phi^{-1}\nu + \frac{2\nu}{fU\nu} + \frac{\phi\nu}{(fU\nu)^2}$;

or finally by XLIX., with the recent signification XXIV. of r ,

LXIV . . $\sigma = r^{-2}(\phi + r)^2\phi^{-1}\nu$, because LXV . . $r = fU\nu = fU\nu$;

and, for the same reason, the equation LX. of the *reciprocal surface* may be thus briefly written,

LXVI . . $Fv + r^{-1}\nu^2 = 1$, while LXVII . . $fv + r\nu^2 = 0$.

(26.) Inverting the last form for σ , and using again the relation XLVI., we first find for ν the expression,

LXVIII . . $\nu = r^2(\phi + r)^{-2}\phi\sigma$;

and then are conducted anew to the equation XLV., or to the following,

LXIX . . $1 = S.\sigma(1 + r^{-1}\phi)^{-2}\phi\sigma$.

* Compare the last note to page 672; see also the use made of this known name "quadric," for a surface of the second order (or degree), in the sub-articles to 899 (pages 614, &c.).

† The equation alluded to, which is one of the *fourth degree*, appears to have been first assigned by Dr. Booth, in a Tract on *Tangential Co-ordinates* (1840), cited in page 168 of Dr. Salmon's *Treatise*. See also the Abstract of a Paper by Dr. Booth, in the *Proceedings of the Royal Society* for April, 1856.

(27.) This last equation may also be thus written,

$$\text{LXIX.} \dots 1 = S.\sigma(1+r^{-1}\phi)^{-2}(\phi+r^{-1}\phi^2)\sigma;$$

but by combining XLIII. LI. LXVII. we have,

$$\text{LXX.} \dots 1 = (S\rho\nu =) S.\sigma(1+r^{-1}\phi)^{-2}\phi\sigma;$$

hence

$$\text{LXXI.} \dots 0 = S.\sigma(1+r^{-1}\phi)^{-2}\phi^2\sigma,$$

a result which may be otherwise and more directly deduced, under the form $S\nu\nu = 0$ (LI.), from the expressions XLIV. LXVII. for ν and ν .

(28.) If we write,

$$\text{LXXII.} \dots r = U d\rho, \quad r' = U(\nu d\rho), \quad \text{and therefore} \quad \text{LXXIII.} \dots r r' = U\nu,$$

r and r' being thus *unit-tangents* to the lines of curvature, the equation III. gives, generally,

$$\text{LXXIV.} \dots 0 = \nabla r d(r r') = -dr' + r S r' d r, \quad \text{whence} \quad \text{LXXIV'.} \dots dr' \parallel r;$$

of which *general parallelism* of dr' to r , the geometrical reason is (comp. again III.) that a *line of curvature* on an *arbitrary surface* is, at the same time, a line of curvature on the *developable normal surface* which rests upon that line, and to which the vectors r' or $\nu d\rho$ are *normals*.

(29.) The same substitution LXXIII. for $U\nu$ gives by II., if we denote by s the arc of a line of curvature, measured from any fixed point thereof, so that (by §80. (7.), &c.),

$$\text{LXXV.} \dots T d\rho = ds, \quad d\rho = r ds, \quad D_s \rho = r,$$

the following *general expression* for the curvature of the given surface, in the direction r of the given line, which by LXXIV'. is also that of dr' :

$$\text{LXXVI.} \dots R^{-1} = S.r D_s(r r') = -S.r r' D_s r = S(U\nu^{-1}. D_s^2 \rho);$$

but $D_s^2 \rho$ is (by §89. (4.)) what we have called the *vector of curvature* of the line of curvature, considered as a *curve in space*, and $R^{-1}U\nu$ is the corresponding vector of curvature of the *normal section* of the given surface, which has the same tangent r at the given point: hence the *latter vector of curvature* is (generally) the *projection of the former, on the normal ν to the given surface*.

(30.) In like manner, if we denote for a moment by R_s^{-1} the curvature of the developable normal surface (28.), for the same direction r , the general formula II. gives, by LXXIV.,

$$\text{LXXVII.} \dots R_s^{-1} = r D_s r' = -S r' D_s r = S.r^{-1} D_s^2 \rho;$$

the vector $R_s^{-1}r'$ of this *new curvature* is therefore the *projection on the new normal r'* , of the *vector of curvature* $D_s^2 \rho$ of the *given line* of curvature. But we shall soon see that these two last results are included in one more general,* respecting *all plane sections* of an arbitrary surface.

(31.) The general parallelism LXXIV'. conducts easily, for the case of a central quadric, to a known and important theorem, which may be thus investigated. Writing, for such a surface,

$$\text{LXXVIII.} \dots r = fr, \quad r' = fr',$$

* Namely in Meusnier's Theorem, which can be proved *generally* by quaternions with about the same ease as the two foregoing cases of it.

so that r retains here its recent signification L XV., and r' is the analogous scalar for the other direction of curvature, we have by L XXIV. the differential,

$$\text{LXXIX.} \dots dr' = 2S\phi r' dr' = 2Sr\phi r' Sr' dr = 0,$$

because $Sr\phi r' = 0$, by 410, XI.

(82.) We have then the relation,

$$\text{LXXX.} \dots fU(vd\rho) = fr' = r' = \text{const.};$$

that is to say, the square (r'^{-1}) of the scalar *semidiameter* (D') of the surface, which is *parallel to the second tangent* (r'), is *constant for any one line of curvature* (r); and accordingly (comp. XXII., and the expression 407, LXXI. for fUv_1), the value of this square is,

$$\text{LXXXI.} \dots (fUvd\rho)^{-1} = r'^{-1} = a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2,$$

if a' , b' , c' be the scalar semiaxes of the confocal, which cuts the given quadric (abc) along the line of curvature, whereof the variable tangent is r .

(83.) This constancy of $fUvd\rho$ may be proved in other ways; for instance, the general equation $Svdvd\rho = 0$ gives, for a line of curvature on an *arbitrary surface*,

$$\text{LXXXII.} \dots d\nu = \nu Sv^{-1}d\nu + d\rho S \frac{d\nu}{d\rho}; \quad \text{LXXXIII.} \dots Vvdvd\rho = vd\rho Sv^{-1}d\nu;$$

and $\text{LXXXIV.} \dots S.d\rho\phi(vd\rho) = 0$, because $d\nu = \phi d\rho$;

while for a *central quadric* ($f\rho = 1$, $\phi\rho = \nu$) it is easy to show that we have also,

$$\text{LXXXV.} \dots \phi(vd\rho) = V\rho d\rho f(vUd\rho);$$

hence, for such a surface, if we suppose for simplicity that ds or $Td\rho$ is constant, which gives $Vvd^2\rho \parallel d\rho$, we have,

$$\text{LXXXVI.} \dots df(vd\rho) = 2S(\phi(vd\rho).d(vd\rho)) = 2Sv^{-1}d\nu.f(vd\rho),$$

a differential equation of the *second order*, of which a *first integral* is evidently,

$$\text{LXXXVII.} \dots f(vd\rho) = Cv^2d\rho^2, \quad \text{or} \quad \text{LXXXVII.} \dots fU(vd\rho) = C = \text{const.}$$

(84.) But we see that the *lines of curvature* on a central quadric are thus included in a *more general system* of curves on the same surface, represented by the differential equation LXXXVI., of which the *complete integral* would involve *two constants*: and which expresses that the *semidiameters* parallel to those *tangents* to the *surface*, which *cross any one such curve at right angles*, have a *common square*, and therefore (if real) a *common length*, so that (in this case) they *terminate on a sphero-conic*.*

(85.) Admitting however, as a *case* of this property, the *constancy* LXXX. of the scalar lately called r' , namely the *second root* of the quadratic XXXIV. or XXXV., of which the coefficients and the first root r vary, in passing from one point to another of what we may call for the moment a *line of first curvature*, we have only to conceive r and v to be accented in the equations LXVI. LXVI', in order to perceive this *theorem*, which perhaps is new:

* Compare the sub-articles (6.) (7.) (8.) to 219, in page 231.

The Curves* on the Reciprocal (24.) of the Surface of Centres of curvature of a central quadric, which answers to the second curvature of that given surface for all the points of a given line of first curvature, or which is itself in a known sense the reciprocal (with respect to the given centre) of the developable normal surface (28.) which rests upon that line, is the intersection of two quadrics; whereof one (LXVI.) is a cone, concyclic with the given surface ($fp = 1$); while the other (LXVL) is a surface concyclic with the reciprocal of that given quadric ($Fv = 1$).

(86.) Again, the scalar Equation of the Surface of Centres (21.) may be said to be the result of the elimination of r^{-1} between the equations LXVIII. and LXXI., whereof the latter is the derivative† of the former with respect to that scalar; we have therefore this theorem:

An Auxiliary Quadric (LXVIII. or XLV.) touches the Second Sheet of the Surface of Centres of a given quadric, along a Quartic Curve, which is the locus of the centres of Second Curvature for all the points of a Line of First Curvature (35.); and (for the same reason) the same auxiliary quadric is circumscribed, along the same quartic, by the Developable Normal Surface (28.), which rests on that first line: with permission, of course, to interchange the words first and second, in this enunciation.

(87.) When the arbitrary constant r is thus allowed to take successively all values, corresponding to both systems of lines of curvature, the Surface of Centres is therefore at once the Envelope‡ of the Auxiliary Quadric LXVIII., and the Locus of the Quartic Curve (36.), in which one or other of its two sheets is touched, by that auxiliary quadric in one of its successive states, and also by one of the developable surfaces of normals to the given surface.

(88.) To obtain the vector equation of that envelope or locus, we may proceed

* The variable vector of this curve is easily seen (comp. XLIX.) to be,

$$v' = \frac{r'}{Sr'\rho} = \frac{vr}{Svr\rho};$$

and the reciprocal surface (21.) or (24.) is by (25.) the locus of this quartic (35.).

† The analogous relation, between the co-ordinate forms of the equations, was perhaps thought too obvious to be mentioned, in page 161 of Dr. Salmon's Treatise; or possibly it may have escaped notice, since the quartic curve (36.) is only mentioned there as an intersection of two quadrics, which is on the surface of centres, and answers to points of a line of curvature upon the given surface. But as regards the possible novelty, even in part, of any such geometrical deductions as those given in the text from the quaternion analysis employed, the writer wishes to be understood as expressing himself with the utmost diffidence, and as most willing to be corrected, if necessary. The power of derivating (or differentiating) any symbolical expression of the form LXVIII., or of any analogous form, with respect to any scalar which it involves explicitly, as if the expression were algebraical, is an important but an easy consequence from the principles of the Section III. ii. 6, which has been so often referred to.

‡ Compare the Note immediately preceding.

as follows, using a new expression for σ , in terms of ν or of ρ , which may then be transformed into a function of two independent and scalar variables. Denoting (comp. 82.) by a_1, b_1, c_1 the semiaxes of the confocal which cuts the given surface in the given line of curvature, and by a_2, b_2, c_2 those of the other confocal, so that the normals ν_1, ν_2 to these two confocals have the directions of the tangents r', r lately considered, we have not only the expressions LXXXI. for r^{-1} , with $a'b'\sigma'$ changed to a_1, b_1, c_1 , but also the analogous expressions (comp. 407, LXXI.),

$$\text{LXXXVIII.} \dots r^{-1} = a^2 - a_2^2 = b^2 - b_2^2 = c^2 - c_2^2.$$

We have therefore by XLII., combined with 407, XVI., this very simple expression for σ :

$$\text{LXXXIX.} \dots \sigma = (\phi^{-1} + r^{-1})\nu = \phi_2^{-1}\nu = \phi_2^{-1}\phi\rho;$$

containing, in the present notation, and as a result of the present analysis, a known and interesting theorem,* on which however we cannot here delay.

(39.) It follows from this last value of σ , combined with the expression 408, LXXXII. for ρ , that we may write,

$$\text{XC.} \dots \sigma = l^{-2} \left(\frac{a^{-1}a_1a_2^3}{a + a'} + \frac{\sqrt{-1}b^{-1}b_1b_2^3}{\sqrt{aa'}} + \frac{c^{-1}c_1c_2^3}{a - a'} \right),$$

as the sought *Vector Equation of the Surface of Centres of curvature of a given quadric (abc)*; ambiguous signs being virtually included in these three terms, because in the subsequent eliminations† the semiaxes enter only by their squares: while l, α, α' are constants, as in 407, &c., for the whole confocal system, and abc are also constant here, but $a^2 - a_1^2$ and $a^2 - a_2^2$, or r^{-1} and r^{-1} (38.), are variable, and may be considered to be the two independent scalars of which σ is a vector function.

413. Some brief remarks may here be made, on the connexion of the general formula,

$$\text{I.} \dots S\nu^{-1}(\phi + r)^{-1}\nu = 0, \quad 412, \text{XXXV.}$$

in which $r = R^{-1}T\nu$ (412, XXIV.), and which when developed by the rules of the Section III. ii. 6 takes (comp. 398, LXXIX.) the form of the quadratic,

* Namely Dr. Salmon's theorem (page 161 of his Treatise), that the centres of curvature of a given quadric at a given point are the poles of the tangent plane, with respect to the two confocals. The connected theorem (page 186), respecting the rectilinear locus of the poles of a given plane, with respect to the surfaces of a confocal system, is at once deducible from the quaternion expression 407, XVI. for $\phi^{-1}\nu$, although the theorem did not happen to be known to the present writer, or at least remembered by him, when he investigated that formula of inversion for other applications, of which some have been already given.

† The corresponding elimination in co-ordinates was first effected by Dr. Salmon, who thus determined the equation of the surface of centres of curvature of a quadric to be one of the twelfth degrees. (Compare pages 161, 162 of his already cited Treatise.)

$$\text{II.} \dots r^2 + rS\nu^{-1}\chi\nu + S\nu^{-1}\psi\nu = 0, \quad 412, \text{XXXIV.}$$

with Gauss's* theory of the *Measure of Curvature of a Surface*; and especially with his fundamental result, that this *measure* is equal to the *product of the two principal curvatures of sections* of that surface: a relation which, in our notations, may be thus expressed,

$$\text{III.} \dots V.dU\nu \delta U\nu = R_1^{-1}R_2^{-1}Vd\rho\delta\rho.$$

(1.) As regards the deduction, by quaternions, of the equation III., in which d and δ may be regarded as two† distinct symbols of differentiation, performed with respect to two independent scalar variables, we may observe that, by principles and rules already established,

$$\text{IV.} \dots dU\nu = V\frac{d\nu}{\nu}.U\nu, \quad \delta U\nu = V\frac{\delta\nu}{\nu}.U\nu = -U\nu.V\frac{\delta\nu}{\nu};$$

and that therefore the first member of III. may be thus transformed:

$$\text{V.} \dots V.dU\nu \delta U\nu = V\left(V\frac{d\nu}{\nu}.V\frac{\delta\nu}{\nu}\right) = -\nu^{-1}S\nu^{-1}d\nu\delta\nu.$$

(2.) Again, since we have $d\nu = \phi d\rho$ (410, IV., &c.), and in like manner $\delta\nu = \phi\delta\rho$, the relations $S\nu d\rho = 0$, $S\nu\delta\rho = 0$, and the self-conjugate property of ϕ , allow us to write,

$$\text{VI.} \dots Vd\nu\delta\nu = \psi Vd\rho\delta\rho, \quad \text{and} \quad \text{VII.} \dots Vd\rho\delta\rho = \nu^{-1}S\nu d\rho\delta\rho;$$

whence follows at once by V. the formula III., if we remember the general expression, deduced from the quadratic II.,

$$\text{VIII.} \dots R_1^{-1}R_2^{-1} = -\nu^{-2}r_1r_2 = -S\frac{1}{\nu}\psi\frac{1}{\nu}. \quad 412, \text{XXXVII.}$$

(3.) If then we suppose that P, P_1, P_2 are any three near points on an arbitrary surface, and that R, R_1, R_2 are three near and corresponding points on the unit sphere, determined by the condition of parallelism of the radii OR, OR_1, OR_2 to the normals PN, P_1N_1, P_2N_2 , the two small triangles thus formed will bear to each other the *ultimate ratio*,

$$\text{IX.} \dots \lim. \frac{\Delta RR_1R_2}{\Delta PP_1P_2} = R_1^{-1}R_2^{-1};$$

a result which justifies (although by an entirely new analysis) the adoption by Gauss

* The reader is referred to the Additions to Liouville's *Monge* (pages 505, &c.), in which the beautiful Memoir by Gauss, entitled: *Disquisitiones generales circa superficies curvas*, is with great good taste reprinted in the Latin, from the *Commentationes recentiores* of the Royal Society of Göttingen. He is also supposed to look back, if necessary, to the Section III. ii. 6 of these *Elements* (pages 485, &c.), and especially to the deduction in page 487 of ψ from ϕ , remembering that the latter function (and therefore also the former) is here self-conjugate.

† Compare page 487, and the Note to page 684.

of this *product** of curvatures of sections, as the *measure* of the curvature of the surface, with his signification of the phrase.

(4.) As another form of this important product or measure, if we conceive that the vector ρ of the surface is expressed as a function (372) of two independent scalars, t and u , and if we write for abridgment,

$$\text{X.} \dots D_t \rho = \rho', \quad D_u \rho = \rho'', \quad D_t^2 \rho = \rho''', \quad D_t D_u \rho = \rho''', \quad D_u^2 \rho = \rho''''$$

which will allow us (comp. 372, V.) to assume for the normal vector ν the expression,

$$\text{XI.} \dots \nu = V \rho' \rho''$$

it is easy to prove† that we have generally,

$$\text{XII.} \dots R_1^{-1} R_2^{-1} = S \frac{\rho'''}{\nu} S \frac{\rho''}{\nu} - \left(S \frac{\rho'}{\nu} \right)^2;$$

which takes as a verification the well-known form,

$$\text{XIII.} \dots R_1^{-1} R_2^{-1} = \frac{rt - s^2}{(1 + p^2 + q^2)^2}$$

when we write (comp. 410, (18.)),

$$\text{XIV.} \dots \rho = ix + jy + kz, \quad \rho' = D_t \rho = i + kp, \quad \rho'' = D_u \rho = j + kq;$$

$$\text{XV.} \dots \nu = V \rho' \rho'' = k - ip - jq, \quad \rho''' = kr, \quad \rho'''' = ks, \quad \rho'''' = kt.$$

(5.) In general, the equation XII. may be thus transformed,

$$\text{XVI.} \dots \nu^4 R_1^{-1} R_2^{-1} = S(V \nu \rho''', V \nu \rho'') - (V \nu \rho')^2 + \nu^2 (S \rho'' \rho'' - \rho''^2);$$

also $\text{XVII.} \dots T d\rho^2 = edt^2 + 2fdtdu + gdu^2,$

if $\text{XVIII.} \dots e = -\rho'^2, \quad f = -S \rho' \rho'', \quad g = -\rho''^2,$ whence $\text{XIX.} \dots \nu^2 = f^2 - eg$

and if we still denote, as in X., derivations relatively to t and u by upper and lower accents, we may substitute in the quadruple of the equation XVI. the values,

$$\text{XX.} \dots 2V \nu \rho''' = (e - 2f') \rho' + e' \rho'', \quad 2V \nu \rho'' = -g' \rho' + e \rho'', \quad 2V \nu \rho' = -g \rho' + (2f' - g') \rho''$$

and $\text{XXI.} \dots 2(S \rho'' \rho'' - \rho''^2) = e'' - 2f'' + g'';$

hence the measure of curvature is an explicit function of the ten scalars,

$$\text{XXII.} \dots e, f, g; \quad e', f', g'; \quad e'', f'', g''; \quad \text{and} \quad e'' - 2f'' + g'';$$

and therefore, as was otherwise proved by Gauss, this measure depends only‡ on the

* If it be supposed to be in any manner known that a limit such as IX. exists, or that the quotient of the two vector areas in III. is a scalar independent of the directions of FP_1, FP_2 , or of $d\rho, \delta\rho$, we have only to assume that these are the directions of the lines of curvature, in order to obtain at once, by 412, II., the product $R_1^{-1} R_2^{-1}$ as the value of this quotient or limit.

† The quadratic in R^{-1} may be formed by operating on 412, II. with $S. \rho'$ and $S. \rho''$, and then eliminating $dt : du$.

‡ The proof by quaternions, above given, of this exclusive dependence, is perhaps as simple as the subject will allow, and is somewhat shorter than the corresponding proof in the Lectures : in page 605 of which is given however the equation,

expression (XVII.) of the square of a linear element, in terms of two independent scalars (t, u), and of their differentials (dt, du).

(6.) Hence follow also these two other theorems* of Gauss:—

If a surface be considered as an infinitely thin solid, and supposed to be flexible but inextensible, then every deformation of it, as such, will leave unaltered, 1st, the Measure of Curvature at any Point, and 2nd, the Total Curvature of any Area; that is, the area of the corresponding portion of the unit sphere, determined as in (3.) by radii parallel to normals.

(7.) Supposing now that t and u are geodesic co-ordinates, whereof the former represents the length of a geodesic ΔP from a fixed point Δ of the surface, and the latter represents the angle ΔAP which this variable geodesic makes at Δ with a fixed geodesic ΔB , it is easy to see that the general expression XVII. takes the shorter form.

$$\text{XXIII.} \dots Td\rho^2 = dt^2 + u^2 du^2, \quad \text{in which} \quad \text{XXIV.} \dots u = T\rho, = T\nu;$$

so that we have now the values,

$$\text{XXV.} \dots e = 1, \quad f = 0, \quad g = u^2, \quad g' = 2uu', \quad g'' = 2uu'' + 2u'^2,$$

and the derivatives of e and f all vanish. And thus the general expression XII. for the measure of curvature reduces itself by (5.) to the very simple form,

$$\text{XXVI.} \dots R_1^{-1} R_2^{-1} = -\pi^{-1} \pi'' = -\pi^{-1} D^2 \pi;$$

in which π is generally a function of both t and u , although here twice derivated with respect to the former only.

(8.) The point P being denoted by the symbol (t, u), and any other point P' of the surface by ($t + \Delta t, u + \Delta u$), we may consider the two connected points P_1, P_2 , of which the corresponding symbols are ($t + \Delta t, u$) and ($t, u + \Delta u$); and then the quadrilateral PP_1P_2P' , bounded by two portions PP_1, P_2P' of geodesic lines from Δ , and (as we may suppose) by two arcs PP_2, P_1P' of geodesic circles round the same fixed point, will have its area ultimately $= \pi \Delta t \Delta u$ (by XXIII.), and therefore (by XXVI., comp. (3.), (6.)) its total curvature ultimately $= -\pi'' \Delta t \Delta u$, or $= -\Delta \pi' \cdot \Delta u$, when Δt and Δu diminish together, by an approach of P' to P .

(9.) Again, in the immediate neighbourhood of Δ , we have $u = t, u' = 1$; changing then $-\Delta t \pi'$ to $-d_t \pi'$, and integrating with respect to t from $t = 0$, we obtain $1 - \pi'$ as the coefficient of Δu in the result, and are thus conducted to the expression:

$$\text{XXVII.} \dots \text{Total Curvature of Triangle } \Delta PP' = (1 - \pi') \Delta u, \text{ ultimately,}$$

if $\Delta P, \Delta P'$ be any two geodesic lines, making with each other a small angle $= \Delta u$, and if PP' be any small arc (geodesic or not) on the same surface.

$$\begin{aligned} 4(eg - f^2)^2 R_1^{-1} R_2^{-1} &= e(g'' - 2g'f' + g'e_1) \\ &\quad + f(e'g' - e_1g' - 2e_1f' - 2g'f'' + 4f''f_1) \\ &\quad + g(e_1^2 - 2e_1f' + e_1'g') - 2(eg - f^2)(e_{11} - 2f_1' + g''), \end{aligned}$$

which may now be deduced at sight from XVI., by the substitutions XIX. XX. XXI., and differs only in notation from the equation of Gauss (Liouville's Monge, page 523, or Salmon, page 809).

* See page 524 of Liouville's Monge.

(10.) Conceive then that PQ is a *finite arc of any curve* upon the surface, for which therefore t , and consequently n' , may be conceived to be a function of u ; we shall have this other expression of the same kind,

$$\text{XXVIII.} \dots \text{Total Curvature of Area } \Delta PQ = \int (1 - n') du = \Delta u - \int n' du;$$

the area here considered being bounded by the two geodetic lines AP, AQ, which make with each other the finite angle Δu , and by the arc PQ of the arbitrary curve.

(11.) If this curve be *itself a geodetic*, and if we treat its co-ordinates t , u , and its vector ρ , as functions of its arc, s , then the second differential of ρ , namely,

$$\text{XXIX.} \dots d^2\rho = \rho' d^2t + \rho, d^2u + \rho'' dt^2 + 2\rho' dt du + \rho, du^2,$$

must be normal to the surface at P, and consequently perpendicular to ρ' and ρ . Operating* therefore with $S.\rho'$, and attending to the relations XVIII. and XXV., which give

$$\text{XXX.} \dots \rho'^2 = -1, \quad S\rho'\rho, = S\rho'\rho'' = S\rho'\rho', = 0, \quad S\rho'\rho,, = -S\rho\rho', = nn',$$

we obtain the differential equation,

$$\text{XXXI.} \dots d^2t = nn' du^2, \quad \text{or} \quad \text{XXXII.} \dots dv = -n' du,$$

if we observe that we may write,

XXXIII. $\dots dt = \cos v ds, \quad n du = \sin v ds,$ because XXXIV. $\dots dt^2 + n^2 du^2 = ds^2;$ v being here the variable angle, which the geodetic PQ makes at P with AP prolonged.

(12.) Substituting then for $-n' du$, in XXVIII., its value dv given by XXXII., the integration becomes possible, and the result is $\Delta u + \Delta v$; where Δu is still the angle at A, and $\pi + \Delta v = (\pi - v) + (v + \Delta v)$ is the sum of the angles at P and Q, in the geodetic triangle APQ.

(13.) Writing then B and C instead of P and Q, we thus arrive at another most remarkable Theorem† of Gauss, which may be expressed by the formula :

$$\text{XXXV.} \dots \text{Total Curvature of a Geodetic Triangle } ABC = A + B + C - \pi,$$

= what may be called the *Spheroidal Excess*; A, B, C, in the second member, being used to denote the *three angles* of the triangle: and the *total surface* of the unit sphere (= 4π) being represented by 720° , when the *part* corresponding to the geodetic triangle is thus represented by the *angular excess*, $A + B + C - 180^\circ$.

(14.) And it is easy to perceive, on the one hand, how this theorem admits of being *extended*, as it was by Gauss, to *all geodetic polygons*: and on the other hand, how it may require to be *modified*, as it was by the same eminent geometer, so as to give what would on the same plan be called a *spheroidal defect*, when the *measure of curvature* is *negative*, as it is for surfaces (or parts of surfaces) of which the principal sections have their curvatures *oppositely directed*.

* To operate with $S.\rho$, would give a result not quite so simple, but reducible to the form XXXI., with the help of $d^2s = 0$.

† The enunciation of this theorem, respecting which its illustrious discoverer justly says, "Hoc theoremata, quod, ni fallimur, ad elegantissima in theoria superficierum curvarum referendum esse videtur,"... is given in page 588 of the Additions to

414. The only sections of a surface, of which the curvatures have been above determined, are the *two principal normal sections* at any proposed point; but the general expressions of III. iii. 6 may be applied to find the curvature of *any plane section*, normal or oblique, and therefore also of *any curve* on a *given surface*, when only its *osculating plane* is known. Denoting (as in 389, &c.) by ρ and κ the vectors of the given point P , and of the *centre* K of the *osculating circle* at that point, and by s the *arc* of the curve, we have generally (by 389, XII. and VI.),

$$I. \dots \text{Vector of Curvature of Curve} = \kappa\rho^{-1} = (\rho - \kappa)^{-1} = D_s^2\rho = \frac{1}{d\rho} \nabla \frac{d^2\rho}{d\rho};$$

the independent variable in the last expression being arbitrary. And if we denote by σ and ξ the vectors of the points s and x , in which the *axis* of the osculating circle meets respectively the *normal* and the *tangent plane* to the given surface, we shall have also, by the right-angled triangles, the general decomposition, $\kappa\rho^{-1} = s\rho^{-1} + x\rho^{-1}$ (as vectors), or

$$II. \dots D_s^2\rho = (\rho - \kappa)^{-1} = (\rho - \sigma)^{-1} + (\rho - \xi)^{-1};$$

where the two components admit of being transformed as follows:

$$III. \dots \text{Normal Component of Vector of Curvature of Curve (or Section)} = (\rho - \sigma)^{-1} = \nu^{-1}S \frac{d\nu}{d\rho} = (\rho - \sigma_1)^{-1} \cos^2 \nu + (\rho - \sigma_2)^{-1} \sin^2 \nu \\ = \text{Vector of Normal Curvature of Surface for the direction of the given tangent};$$

σ_1, σ_2 being the vectors of the centres s_1, s_2 (comp. 412) of the *two principal curvatures*, and ν being the *angle* at which the curve (or its tangent $d\rho$) crosses the *first line of curvature* (or its tangent τ_1), while σ is the vector of the centre s of the *sphere* which is said to *osculate* to the *surface*, in the *given direction* (of $d\rho$); and

$$IV. \dots \text{Tangential Component of Vector of Curvature} \\ = (\rho - \xi)^{-1} = \nu^{-1}d\rho^{-1}S\nu d\rho^{-1}d^2\rho \\ = \text{Vector of Geodetic Curvature of Curve (or Section)};$$

this latter *vector* being here so called, because in fact its *tensor* re-

Liouville's Monge. A proof by quaternions was published in the *Lectures* (pages 606-609, see also the few preceding pages), but the writer conceives that the one given above will be found to be not only shorter, but more clear.

presents what is known by the name of the *geodetic* curvature* of a *curve upon a surface*: the independent variable being still arbitrary.

(1.) As regards the decomposition II., if α, β be any two rectangular vectors OA, OB , and if $\gamma = OC$ = the perpendicular from o on AB , then (comp. 816, L, and 408, XLI.),

$$V. \dots \gamma^{-1} = \frac{\beta}{\sqrt{\alpha\beta}} + \frac{\alpha}{\sqrt{\beta\alpha}} = \alpha^{-1} + \beta^{-1}.$$

(2.) To prove the first transformation III., we have, by I. and II., observing that $dSv d\rho = 0$,

$$VI. \dots \frac{v}{\rho - \sigma} = S \frac{v}{\rho - \kappa} = S. \frac{v}{d\rho} V \frac{d^2\rho}{d\rho} = \frac{-Sv d^2\rho}{d\rho^3} = \frac{Sdvd\rho}{d\rho^3} = S \frac{dv}{d\rho}.$$

(3.) Hence, by 412, (7.), if we denote the vector III. of normal curvature by $R^{-1}Uv$, we have the general expressions (comp. 412, I. XXI.),

$$VII. \dots \sigma = \rho + RUv, \quad R = D^2.Tv, \quad \text{with} \quad VIII. \dots Tv = P^{-1}$$

for the case of a *central quadric*; D being generally the *semidiameter* of the *index surface* (410, (9.), &c.), or for a quadric the *semidiameter* of that surface itself, which has the direction of the *tangent* (or of $d\rho$): and P being, for the latter surface, the *perpendicular* from the centre on the *tangent plane*, as in some earlier formula.

(4.) To deduce the second transformation III., which contains a theorem of Euler, let τ, r_1, r_2 denote unit tangents to the section and the two lines of curvature, so that

$$IX. \dots \tau = r_1 \cos v + r_2 \sin v, \quad \text{and} \quad r^2 = r_1^2 + r_2^2 = -1;$$

we may then write generally (comp. 412, IV.),

$$X. \dots R^{-1}Tv = \frac{v}{\sigma - \rho} = -S \frac{dv}{d\rho} = -Sr^{-1}\phi\tau = Sr\phi\tau,$$

and shall have the values (comp. 410, XI.),

$$XI. \dots Sr_1\phi r_1 = R_1^{-1}Tv, \quad Sr_2\phi r_2 = R_2^{-1}Tv, \quad Sr_1\phi r_2 = Sr_2\phi r_1 = 0;$$

whence

$$XII. \dots R^{-1} = R_1^{-1} \cos^2 v + R_2^{-1} \sin^2 v,$$

and the required transformation is accomplished.

(5.) The theorem of Meusnier may be considered to be a result of the *elimination* (2.) of $d^2\rho$ from the expressions for the *normal component* III. of what we may call the *Vector $D^2\rho$ of Oblique Curvature*; and it may be expressed by the equation,

$$XIII. \dots S \frac{\rho - \sigma}{\rho - \kappa} = 1, \quad \text{or} \quad XIII'. \dots S \frac{\sigma - \kappa}{\rho - \kappa} = 0, \quad \text{which gives} \quad XIII''. \dots \rho \kappa s = \frac{\pi}{2},$$

if it be now understood that the point s , of which σ is the vector, is the *centre* of the

* The name, "*courbure géodésique*," was introduced by M. Liouville, and has been adopted by several other mathematical writers. Compare pages 568, 575, &c. of his *Additions to Monge*.

circle which *osculates* to the normal section; or of the sphere which *osculates* in the same direction to the surface, as will be more clearly seen by what follows.

(6.) In general, if $\rho + \Delta\rho$ be the vector of any second point \mathcal{F}' of the given surface, the equation

$$\text{XIV.} \dots S \frac{\nu}{\omega - \rho} = S \frac{\nu}{\Delta\rho}, \text{ with } \omega \text{ for a variable vector,}$$

represents rigorously the sphere which *touches* the surface at the given point \mathcal{F} , and passes *through* the second point \mathcal{F}' ; conceiving then that the latter point *approaches* to the former, and observing that the development* by Taylor's Series of the equation $f\rho = \text{const.}$ gives (if $df\rho = 2S\nu d\rho$, and $d\nu = \phi d\rho$),

$$\text{XV.} \dots 0 = \Delta\rho^2 \Delta f\rho = 2S \frac{\nu}{\Delta\rho} + S \frac{\phi \Delta\rho}{\Delta\rho} + \text{terms which vanish generally with } \Delta\rho,$$

even if they be not *always* null, we are conducted in a new way, by the known conception of the *Osculating Sphere* for a given direction to a surface, to the same centre s , and radius R , as before: the equation of this sphere being,

$$\text{XVI.} \dots S \frac{2\nu}{\omega - \rho} = \left(\lim. S \frac{2\nu}{\Delta\rho} = - \lim. S \frac{\phi \Delta\rho}{\Delta\rho} \right) - S \frac{d\nu}{d\rho}.$$

(7.) Conversely, if we assume a radius R , such that R^{-1} is algebraically intermediate between R_1^{-1} and R_2^{-1} , the *tangent sphere*,

$$\text{XVII.} \dots S \frac{2\nu}{\omega - \rho} = \frac{T\nu}{R}, \text{ or XVII'.} \dots S \frac{2U\nu}{\omega - \rho} = R^{-1},$$

will cut the surface in two directions of osculation, assigned by the formula XII.; but if R^{-1} be outside those limits, there will be only contact, and not any (real) intersection, at least in the vicinity of \mathcal{F} .

(8.) If \mathcal{F}' be again, as in (6.), any second point of the surface, and if we denote for a moment by (Π) and (Σ) the normal plane $\mathcal{F}\mathcal{N}\mathcal{F}'$ and the normal section corresponding, we may suppose that \mathcal{x} is the point in which the normals to the plane curve (Σ) at \mathcal{F} and \mathcal{F}' intersect; and if we then erect a perpendicular at \mathcal{x} to the plane (Π) , it will be crossed by every perpendicular at \mathcal{F}' to the tangent $\mathcal{F}'\mathcal{T}'$ to the section, and therefore in particular by the normal at \mathcal{F}' to the surface, in a point which we may call \mathcal{N}' : so that the line $\mathcal{F}'\mathcal{N}'$ is the projection, on the plane $\mathcal{F}\mathcal{F}'\mathcal{N}$, of this second normal $\mathcal{F}'\mathcal{N}'$ to the surface. Conceiving then the plane (Π) to be fixed, but the point \mathcal{F}' to approach indefinitely to \mathcal{F} , we see that the centre s of curvature of the normal section (Σ) , which is also by (6.) the centre of the *osculating sphere* to the surface for the same direction, is the limiting position of the point \mathcal{x} , in which

* Compare Art. 874, and the Second Note to page 508. The occasional use there mentioned, of the differential symbol $d\rho$ as signifying a finite and *chordal* vector, in the development of $f(\rho + d\rho)$, has appeared obscure, in the *Lectures*, to some friends of the writer; and he has therefore aimed, for the sake of clearness, in at least the text of these *Elements*, and especially in the geometrical applications, to confine that symbol to its first signification (100, 869, 873, &c.), as denoting a *tangential vector* (finite or infinitely small, and to a curve or surface): ρ itself being generally regarded as a *vector function*, and not as an independent variable (comp. 862, (3)).

the given normal at P is intersected by the projection* of the near normal P'x', on the given normal plane.

(9.) The two components III and IV. are included in the binomial expression,

XVIII . . Vector of Oblique Curvature (or of Curvature of Oblique Section)

$$= (\rho - \kappa)^{-1} = \nu^{-1} S \nu d\nu \rho^{-1} + \nu^{-1} d\rho^{-1} S \nu d\rho^{-1} d^2 \rho,$$

which is obtained by substituting in I. the general equivalent 409, XXI. for $d^2 \rho$, and in which (as before) the independent variable is arbitrary; and the tangential component IV. may be otherwise found by observing that, by I. and II.,

$$\text{XIX. . . } \frac{\nu d\rho}{\rho - \xi} = S \frac{\nu d\rho}{\rho - \kappa} = S \frac{\nu d^2 \rho}{d\rho} = -S \nu d\rho^{-1} d^2 \rho,$$

and that $-(\nu d\rho)^{-1} = \nu^{-1} d\rho^{-1}$, because $S \nu d\rho = 0$.

(10.) Another way of deducing the same component IV., is to resolve the following system of three scalar equations, which by the geometrical definition of the point x the vector ξ must satisfy:

$$\text{XX. . . } S(\xi - \rho)\nu = 0; \quad S(\xi - \rho)d\rho = 0; \quad S(\xi - \rho)d^2 \rho = d\rho^2;$$

and which give,

$$\text{XXI. . . } \xi - \rho = \frac{\nu d\rho^2}{S \nu d\rho d^2 \rho} = \frac{\nu d\rho}{S \nu d\rho^{-1} d^2 \rho},$$

or $(\rho - \xi)^{-1} = \&c.$, as before. We have also the transformations,

$$\text{XXII. . . Vector of Geodetic Curvature} = (\rho - \xi)^{-1} \\ = (\nu d\rho)^{-1} S(\nu U d\rho. dU d\rho) = -\nu d\rho S \frac{d\rho^{-2} d^2 \rho}{\nu d\rho} = \&c.$$

(11.) The definition of the point x shows also easily, that if a developable surface (D) be circumscribed to a given surface (s), along a given curve (c), and if, in the unfolding of the former surface, the point x be carried with the tangent plane, originally drawn to the latter surface at P, it will become the centre of curvature, at the new point (P), to the new or plane curve (c') obtained by this development: so that the radius (PX) of geodetic curvature is equal, as indeed it is known† to be, to the radius of plane curvature of the developed curve.

(12.) This plane curve (c') is therefore a circle‡ (or part of one) if the condition,

$$\text{XXIII. . . } \overline{PX} = T(\xi - \rho) = \text{const.},$$

* The reader may compare the calculations and constructions, in pages 600, 601 of the Lectures. In the language of infinitesimals, an infinitely near normal P'x' intersects the axis of the osculating circle, to the given normal section.

† Compare page 576 of the Additions to Liouville's Monge.

‡ The curves on any given surface, which thus become circles by development, have also the isoperimetrical property expressed in quaternions (comp. the first Note to page 580) by the formula,

$$\text{XXVI. . . } \delta \int S(U\nu. d\rho \delta \rho) + e \delta \int T d\rho = 0,$$

which conducts to the differential equation,

$$\text{XXVII. . . } e^{-1} d\rho = V. U \nu dU d\rho \text{ (comp. 380, IV.)},$$

be satisfied; but it degenerates into a *right line*, if this radius of *geodetic curvature* be *infinite*, that is, if

$$\text{XXIV.} \dots T(\rho - \xi)^{-1} = 0, \quad \text{or} \quad \text{XXV.} \dots S \nu d \rho d^2 \rho = 0,$$

or finally (by 880, II., comp. 409, XXV.), if the original curve (c) be a *geodetic line* on the *given surface* (s), and therefore also on the *developable* (D): which agrees with the fundamental property (382, 383) of geodetics on a developable surface.

(18.) Accordingly it may be here observed that the general formula IV., combined with the notations and calculations of 382, conducts to the expression $(x + \nu) T \rho^{-1}$, or $\frac{x dx + \nu d\nu}{ds}$, for the geodetic curvature of *any curve* on a developable surface, whereof the element ds crosses a generating line at the variable angle ν , while $x dx$ is the angle between two such consecutive lines: a result easily confirmed by geometrical considerations, and agreeing with the differential equation $x + \nu = 0$ (382, IX.) of *geodetics* on a developable.

415. We shall conclude the present Section with a few supplementary remarks, including a new and simplified proof of an important *theorem* (354), which we have had frequent occasion to employ for purposes of *geometry*, and which presents itself often in *physical applications* of quaternions also: namely, that *if the linear and vector function ϕ be self-conjugate*, then the *Vector Quadratic*,

$$\text{I.} \dots V \rho \phi \rho = 0, \quad 354, \text{I.}$$

represents generally a *System of Three Real and Rectangular Directions*; and that these (comp. 405, (1.), (2.), &c.) are the directions of the *Axes of the Central Surfaces of the Second Order*, which are represented by the scalar equation,

$$\text{II.} \dots S \rho \phi \rho = \text{const.};$$

or more generally,

$$\text{III.} \dots S \rho \phi \rho = C \rho^2 + C', \quad \text{where } C \text{ and } C' \text{ are any two scalar constants.}$$

(1.) It is an easy consequence of the theory (850) of the *symbolic and cubic equation* in ϕ , that if c be a root of the derived *algebraical cubic* $M = 0$ (354), and if we write $\Phi = \phi + c$ (as in that Article), the *new linear and vector function* $\Phi \rho$ must be reducible to the *binomial form* (851),

and in which the scalar constant c can be shown to have the value,

$$\text{XXVIII.} \dots c = (\xi - \rho) U. \nu d \rho = \pm T(\xi - \rho) = \text{Radius of Geodetic Curvature,}$$

= radius of developed circle; and each such curve includes, by XXVI., on the given surface, a *maximum area* with a *given perimeter*: on which account, and in allusion to a well-known classical story, the writer ventured to propose, in page 582 of the *Lectures*, the name "*Didonia*" for a curve of this kind, while acknowledging that the *curves themselves* had been discovered and discussed by M. Delaunay.

IV. . . $\Phi\rho = \phi\rho + c\rho = \beta Sa\rho + \beta' Sa'\rho$, with V. . . $\nabla\beta\alpha + \nabla\beta'a' = 0$,
as the condition (358, XXXVI.) of self-conjugation. With this condition we may then write,

$$\text{VI. . . } \beta = A\alpha + Ba', \quad \beta' = A'\alpha' + Ba;$$

and it is easy to see that no essential generality is lost, by supposing that α and α' are two rectangular vector units, which may be turned about in their own plane, if β and β' are suitably modified: so that we may assume,

VII. . . $\alpha^2 = \alpha'^2 = -1$, $Saa' = 0$; whence VIII. . . $\Phi\alpha = -\beta$, $\Phi\alpha' = -\beta'$,
and IX. . . $\nabla\beta'a' = Baa' = -\nabla\beta\alpha$, $\nabla\beta'a' = Aa\alpha'$, $\nabla\beta'a' = -A'\alpha\alpha'$.

(2.) The equation I., under the form,

X. . . $\nabla\rho\Phi\rho = 0$, is satisfied by XI. . . $\Phi\rho = 0$, or XII. . . $\nabla\alpha\alpha'\rho = 0$;
and it cannot be satisfied otherwise, unless we suppose,

$$\text{XIII. . . } \rho = x\alpha + x'\alpha', \quad \text{and XIV. . . } \nabla(x\beta + x'\beta')(x\alpha + x'\alpha') = 0;$$

that is, by IX.,

$$\text{XV. . . } B(x^2 - x'^2) + (A - A')xx' = 0:$$

while conversely the expression XIII. will satisfy I., under this condition XV. But this quadratic in $x':x$, of which the coefficients B and $A - A'$ do not generally vanish, has necessarily *two real roots*, with a product $= -1$; hence there *always exists*, as asserted, a system of *three real and rectangular directions*, such as the following,

$$\text{XVI. . . } x\alpha + x'\alpha', \quad x'\alpha - x\alpha', \quad \text{and } \alpha\alpha' \text{ (or } \nabla\alpha\alpha'),$$

which satisfy the equation I.; and this system is *generally definite*: which proves the *first part* of the Theorem.

(3.) The lines α , α' may be made by (1.) to *turn* in their own plane, till they coincide with the two first directions XVI.; which will give,

$$\text{XVII. . . } B = 0, \quad \beta = A\alpha, \quad \beta' = A'\alpha';$$

and therefore,

$$\begin{aligned} \text{XVIII. . . } \phi\rho &= -c\rho + AaSa\rho + A'\alpha'Sa'\rho \\ &= (c + A)aSa\rho + (c + A')\alpha'Sa'\rho + ca\alpha'Saa'\rho; \end{aligned}$$

and thus the scalar equation II. will take the form,

$$\text{XIX. . . } Sp\phi\rho = (c + A)(Sa\rho)^2 + (c + A')(Sa'\rho)^2 + c(Saa'\rho)^2 = \text{const.},$$

which represents generally a *central surface* of the *second order*, with its *three axes* in the *three directions* α , α' , $\alpha\alpha'$ of ρ ; and does not cease to represent such a surface, and with such axes, when for $Sp\phi\rho$ we substitute, as in III., this new expression:

$$\text{XX. . . } Sp\phi\rho - C\rho^2 = Sp\phi\rho + C((Sa\rho)^2 + (Sa'\rho)^2 + (Saa'\rho)^2) = C' = \text{const.};$$

the *second surface* being in fact *concyelic* (or having the same cyclic planes) with the first, and the *new term*, $-C\rho$, in $\phi\rho$, disappearing under the sign $\nabla\rho$: so that the *second part* of the Theorem is proved anew.

(4.) It would be useless to dwell here on the cases, in which the surfaces XIX., XX. come to be of *revolution*, or even to be *spheres*, and when consequently the *directions* of their axes, or of ρ in I., become partially or even wholly *indeterminate*. But as an example of the *reduction* of an equation in quaternions to the form I.,

without its at first presenting itself under that form, we may take the very simple equation,

$$\text{XXI.} \dots \rho\iota\rho\kappa = \iota\rho\kappa\rho, \text{ with } \kappa \text{ not } \parallel \iota,$$

which may be reduced (comp. 354, (12.)) to

$$\text{XXII.} \dots \nabla. \rho \nabla \iota \rho \kappa = 0;$$

and which is accordingly satisfied (comp. 378, XXIX.) by the three rectangular directions,

$$\text{XXIII.} \dots U_\iota - U_\kappa, \quad V_\iota \kappa, \quad U_\iota + U_\kappa,$$

of the axes (*abc*) of the ellipsoid,

$$\text{XXIV.} \dots T(\varphi + \rho\kappa) = \kappa^2 - \iota^2, \quad 282, \text{ XIX.}$$

which is one of the surfaces of the coneyclic system (comp. III.),

$$\text{XXV.} \dots S\rho\rho = C\rho^2 + C,$$

as appears from the transformations 336, XI., &c.

(5.) In applying the theorem thus recently proved anew, we have on several occasions used the expression,

$$\text{XXVI.} \dots d\nu = \phi d\rho, \quad 410, \text{ IV.}$$

in which ν is a vector normal to a surface whereof ρ is the variable vector, and the function ϕ is treated as *self-conjugate* (363).

(6.) It is, however, important to remark that, in order to justify the assertion of this last property, the following expression of integral form,

$$\text{XXVII.} \dots \int S\nu d\rho,$$

must admit of being equated to some scalar function of ρ , such as $\frac{1}{2}f\rho + \text{const.}$, without its being assumed that ρ itself is a function, of any determinate form, of a scalar variable, ι . The self-conjugation of the linear and vector function ϕ in XXVI., is the condition of the existence of the integral XXVII., considered as representing such a scalar function (comp. again 368).

(7.) There are indeed several investigations, in which it is sufficient to regard ν as denoting some normal vector, of which only the direction is important, and which may therefore be multiplied by an arbitrary scalar coefficient, constant or variable, without any change in the results (comp. the calculations respecting geodesic lines, in the Section III. iii. 5, and many others which have already occurred).

(8.) And there have been other general investigations, such as those regarding the lines of curvature on an arbitrary surface, in which $d\nu$ was treated as a self-conjugate function of $d\rho$, while yet (comp. 410, (17.)) the fundamental differential equation $S\nu d\nu d\rho = 0$ was not affected by any such multiplication of ν by κ .

(9.) But there are questions in which a factor of this sort may be introduced, with advantage for some purposes, while yet it is inconsistent with the self-conjugation above mentioned, unless the multiplier κ be such as to render the new expression $S\nu d\nu d\rho$ (comp. XXVII.) an exact differential of some scalar function of ρ .

(10.) For example, in the theory of Reciprocal Surfaces (comp. 412, (21.)), it is convenient to employ the system of the three connected equations,

$$\text{XXVIII.} \dots S\nu\rho = 1, \quad S\nu d\rho = 0, \quad S\rho d\nu = 0; \quad 378, \text{ L. LL.}$$

but when the length of ν is determined so as to satisfy the first of these equations, ν^{-1} being then the vector perpendicular from the origin on the tangent plane to the

given but arbitrary surface of which ρ is the vector, while ρ^{-1} is the corresponding perpendicular for the reciprocal surface with ν for vector, the differential $d\nu$ loses generally its self-conjugate character, as a linear and vector function of $d\rho$: although it retains that character if the scalar function $f\rho$ be homogeneous, in the equation $f\rho = \text{const.}$ of the original surface, as it is for the case of a central quadric,* for which $\nu = f\rho$, $d\nu = f d\rho$, &c., as in former Articles.

(11.) In fact, the introduction of the first equation XXVIII. is equivalent to the multiplication of ν by the factor $\kappa = (S\nu\rho)^{-1}$; and if we write (comp. 410, (16.)),

$$\text{XXIX.} \dots d f\rho = 2S\nu d\rho, \quad d\nu = \phi d\rho, \quad d\kappa = S\sigma d\rho,$$

we shall have this new pair of conjugate linear and vector functions,

$$\text{XXX.} \dots d.\kappa\nu = \phi d\rho + \nu S\sigma d\rho, \quad \text{XXXI.} \dots \phi' d\rho = \kappa \phi d\rho + \sigma S\nu d\rho;$$

and these will not be equal generally, because we shall not in general have $\sigma \parallel \nu$. But this last parallelism exists in the case of homogeneity (10.), because we have then the relations,

$$\text{XXXII.} \dots 2S\nu\rho = r f\rho, \quad d.\kappa^{-1} = dS\nu\rho = r S\nu d\rho,$$

if r be the number which represents the dimension of $f\rho$ (supposed to be whole).

(12.) On the other hand it may happen, that the differential equation $S\nu d\rho = 0$ represents a surface, or rather a set of surfaces, without the expression $S\nu d\rho$ being an exact differential, as in (6.); and then there necessarily exists a scalar factor, or multiplier, κ , which renders it such a differential.

(13.) For example the differential equation,

XXXIII. $\dots S\gamma\rho d\rho = S\nu d\rho = 0$, with XXXIV. $\dots \nu = V\gamma\rho$, $d\nu = V\gamma d\rho = \phi d\rho$, represents an arbitrary plane (or a set of planes), drawn through a given line γ ; but the expression $S\gamma\rho d\rho$ itself is not an exact differential, and the integral XXXVII. represents no scalar function of ρ , with the present form of ν , of which the differential $d\nu$ is accordingly a linear function $\phi d\rho$, which is not conjugate to itself, but to its opposite (comp. 349, (4.)), so that we have here $\phi' d\rho = -\phi d\rho$.

(14.) But if we multiply ν by the factor,

XXXV. $\dots \kappa = \nu^{-2} = (V\gamma\rho)^{-2}$, which gives XXXVI. $\dots d\kappa = S\sigma d\rho$, $\sigma = 2\kappa^2 V\gamma\rho$, and therefore $S\gamma\sigma = 0$, $S\rho\sigma = -2\kappa$, then the new normal vector $\kappa\nu$, or ν^{-1} , is found to have the self-conjugate differential,

$$\text{XXXVII.} \dots d.\kappa\nu = d.\nu^{-1} = -\nu^{-1} V\gamma d\rho. \nu^{-1} = \phi d\rho = \phi' d\rho;$$

and accordingly the new expression,

$$\text{XXXVIII.} \dots S\kappa\nu d\rho = S\nu^{-1} d\rho = S \frac{d\rho}{V\gamma\rho}, \quad \text{with } \gamma \text{ constant,}$$

is easily seen to be an exact differential, namely (if $T\gamma = 1$), that of the angle which the plane of γ and ρ makes with a fixed plane through γ : so that, when ν is thus

* It was for this reason that the symbol $T\nu$ was not interpreted generally as denoting the reciprocal, P^{-1} , of the length of the perpendicular from the origin on the tangent plane, in the formulæ of 410, 412, 414: although, in several of those formulæ, as in an equation of 409, (3.), that symbol was so interpreted, for the case of a central surface of the second order.

changed to ν , the integral in XXVII. acquires a *geometrical signification*, which is often useful in *physical applications*, since it then represents the *change of this angle*, in passing from one position of ρ to another; or the angle through which the *variable plane of $\gamma\rho$ has revolved*.

(15.) In fact, the general formula 385, XV. for the *differential of the angle of a quaternion* gives, if we write

$$\text{XXXIX.} \dots q = \frac{V\gamma\rho}{V\gamma\rho_0}, \quad \gamma = \text{const.}, \quad \rho_0 = \text{const.}, \quad T\gamma = 1,$$

the two connected expressions :

$$\text{XL.} \dots d \angle q = \pm S \frac{d\rho}{V\gamma\rho}; \quad \text{XLI.} \dots \int S \frac{d\rho}{V\gamma\rho} = \pm \Delta \angle (V\gamma\rho : V\gamma\rho_0);$$

which contain the above-stated result, and can easily be otherwise established.

(16.) *In general*, if the linear and vector function $d\nu = \phi d\rho$ be not self-conjugate, and if the function $d.\nu\nu = \phi d\rho$ be formed from it as in (11.), it results from that sub-article, and from 849, (4.), that we may write,

$$\text{XLII.} \dots (\phi - \phi')d\rho = 2V\gamma d\rho, \quad (\phi - \phi')d\rho = 2V\gamma.d\rho,$$

with the relation,

$$\text{XLIII.} \dots 2\gamma = 2\nu\gamma + V\nu\sigma;$$

where γ, γ , are independent of $d\rho$, although they may depend on ρ itself. If then the new linear function $\phi d\rho$ is to be self-conjugate, so that $\gamma = 0$, we must have

$$\text{XLIV.} \dots 2\nu\gamma + V\nu\sigma = 0, \quad \text{and therefore} \quad \text{XLV.} \dots S\gamma\nu = 0;$$

which latter very simple equation, not involving either α or σ , is thus a form, in quaternions, of the *Condition of Integrability** of the differential equation $S\nu d\rho = 0$, if the vector γ be deduced from ν as above.

(17.) The *Bifocal Transformation* of $S\rho\phi\rho$, in 860, (2.), has been sufficiently considered in the present Section (III. iii. 7); but it may be useful to remark here, that the *Three Mixed Transformations* of the same scalar function $f\rho$, in the same series of sub-articles, include virtually the whole known theory of the *Modular and Umbilicar Generations of Surfaces of the Second Order*.

(18.) Thus, in the formulæ of 860, (4.), if we make $\epsilon = 1$, ϵ is the vector of an *Umbilicar Focus* of the surface $f\rho = 1$, and ζ is the vector of a point on the *Umbilicar Directrix* corresponding; whence the *umbilicar focal conic* and *dirigent cylinder* (real or imaginary) can be deduced, as the *loci of this point and line*.

(19.) Again, by making ϵ_1 and ϵ_2 each = 1, in the formulæ of 860, (6.), we obtain *Two Modular Transformations* of the equation of the same surface; ϵ_1, ϵ_2 being

* If the proposed equation be

$$S\nu d\rho = p dx + q dy + r dz = 0, \quad \text{so that} \quad \nu = -(i p + j q + k r),$$

we easily find that $2\gamma = i P + j Q + k R$, where

$$P = D_x q - D_y p, \quad Q = D_x r - D_z p, \quad R = D_x p - D_z q;$$

the condition of integrability XLV. becomes therefore here,

$$p P + q Q + r R = 0, \quad \text{which agrees with known results.}$$

vectors of *Modular Foci*, in two distinct planes, and ζ_1, ζ_3 being vectors of points upon the *Modular Directrices* corresponding: whence the *modular focal conics*, and *dirigent cylinders* (real or imaginary), are found by easy eliminations.

(20.) Thus, by assuming that either

$$\text{XLVI.} \dots S\lambda(\rho - \zeta_1) = 0, \quad S\lambda(\rho - \zeta_3) = 0,$$

or $\text{XLVII.} \dots S\mu(\rho - \zeta_1) = 0, \quad S\mu(\rho - \zeta_3) = 0,$

the equations 360, XVI., XVII. may be brought to the forms,

$$\text{XLVIII.} \dots (\rho - \epsilon_1)^2 = m_1^2(\rho - \zeta_1)^2, \quad \text{XLIX.} \dots (\rho - \epsilon_3)^2 = m_3^2(\rho - \zeta_3)^2,$$

with the values,

$$\text{L.} \dots m_1^2 = 1 - \frac{c_3}{c_1}, \quad \text{and} \quad \text{LI.} \dots m_3^2 = 1 - \frac{c_2}{c_3};$$

in which c_1, c_2, c_3 are the *three roots of a certain cubic* ($M=0$), or the *inverse squares* of the three scalar *semiaxes* (real or imaginary) of the surface, arranged in algebraically ascending order (357, IX., XX.; 405, (6.), &c.): and m_1, m_3 are the *two* (real or imaginary) *Moduli*, or represent the *modular ratios*, in the *two modes of Modular Generation** corresponding.

(21.) It is obvious that an equation of the form,

$$\text{LII.} \dots T\phi\rho = C = \text{const.},$$

represents a *central quadric*, if $\phi\rho$ be any *linear*† and vector function of ρ , of the

* Mac Cullagh's rule of modular generation, which includes both those modes, was expressed in page 487 of the *Lectures* by an equation of the form,

$$T(\rho - a) = TV.\gamma V\beta\rho;$$

in which the origin is on a directrix, β is the vector of another point of that right line, a is the vector of the corresponding focus, γ is perpendicular to a directive (that is, generally, to a cyclic) plane, ρ is the vector of any point P of the surface, and $\pm S\beta\gamma$ is the constant modular ratio, of the distance \overline{AP} of P from the focus, to the distance of the same point P from the directrix oa , measured parallel to the directive plane. The new forms (360), above referred to, are however much better adapted to the working out of the various consequences of the construction; but it cannot be necessary, at this stage, to enter into any details of the quaternion transformations: still less need we here pause to give references on a subject so interesting, but by this time so well known to geometers, as that of the modular and umbilicar generations of surfaces of the second order. But it may just be noted, in order to facilitate the applications of the formulæ L. and LI., that if we write, as usual, for *all* the central quadrics, $a^2 > b^2 > c^2$, whether b^2 and c^2 be positive or negative, then the roots e_1, e_2, e_3 coincide, for the *ellipsoid*, with a^{-2}, b^{-2}, c^{-2} ; for the *single-sheeted hyperboloid*, with c^{-2}, a^{-2}, b^{-2} ; and for the *double-sheeted hyperboloid* with b^{-2}, c^{-2}, a^{-2} , (comp. page 651).

† In page 664 the notation,

$$d\rho = 2Svd\rho = 2S\phi\rho d\rho, \quad 409, \text{IV.}$$

was employed for an *arbitrary surface*; but with the understanding that *this function* $\phi\rho$ (comp. 863) was *generally non-linear*. It may be better, however, as a

kind considered in the Section III. § 6, whether self-conjugate or not; but it requires a little more attention to perceive, that an equation of this *other form*,

$$\text{LIII.} \dots T(\rho - V.\beta V\gamma\alpha) = T(\alpha - V.\gamma V\beta\rho),$$

represents *such* a surface, whatever the *three vector constants* α, β, γ may be. The discussion of this last *form* would present some circumstances of interest, and might be considered to supply a *new mode of generation*, on which however we cannot enter here.

(22.) The surfaces of the second order, considered hitherto in the present Section, have all had the *origin for centre*. But if, retaining the significations of ϕ, f , and F , we compare the two equations,

$$\text{LIV.} \dots f(\rho - \kappa) = C, \quad \text{and} \quad \text{LV.} \dots f\rho - 2S\varepsilon\rho = C',$$

we shall see (by 362, &c.) that the constants are connected by the two relations,

$$\text{LVI.} \dots \varepsilon = \phi\kappa, \quad C' = C - f\kappa = C - S\varepsilon\kappa = C - F\varepsilon;$$

so that the equation,

$$\text{LVII.} \dots f\rho - 2S\varepsilon\rho = f(\rho - \phi^{-1}\varepsilon) - F\varepsilon,$$

is an *identity*.

(23.) If then we meet an equation of the form LV., in which (as has been usual) we have still $f\rho = S\rho\phi\rho$ = a scalar and *homogeneous* function of ρ , of the *second* dimension, we shall know that it represents *generally* a surface of *that order*, with the expression (comp. 347, IX., &c.),

$$\text{LVIII.} \dots \kappa = \phi^{-1}\varepsilon = m^{-1}\psi\varepsilon = \text{Vector of Centre.}$$

(24.) It may happen, however, that the two relations,

$$\text{LIX.} \dots m = 0, \quad T\psi\varepsilon > 0,$$

exist together; and then the *centre* may be said to be at an *infinite distance*, but in a *definite direction*: and the surface becomes a *Paraboloid*, elliptic or hyperbolic, according to conditions which are easy consequences from what has been already shown.

(25.) On the other hand it may happen that the two equations,

$$\text{LX.} \dots m = 0, \quad \psi\varepsilon = 0,$$

are satisfied together; and then the vector κ of the centre acquires, by LVIII., an *indeterminate value*, and the surface becomes a *Cylinder*, as has been already sufficiently exemplified.

(26.) It would be tedious to dwell here on such details; but it may be worth

general rule, to *avoid* writing $\nu = \phi\rho$, *except* for central quadrics; and to confine ourselves to the notation $d\nu = \phi d\rho$, as in some recent and several earlier sub-articles, when we wish, for the sake of *association* with other investigations and results, to treat the *function* ϕ as *linear* (or *distributive*); because we shall thus be at liberty to treat the *surface* as *general*, notwithstanding this property of ϕ . As regards the methods of *generating* a quadric, it may be worth while to look back at the Note to page 649, respecting the *Six Generations of the Ellipsoid*, which were given by the writer in the *Lectures*, with suggestions of a few others, as interpretations of quaternion equations.

while to observe, that the *general equation of a Surface of the Third Degree* may be thus written :

$$\text{LXI.} \dots Sqq'q''\rho + S\rho\phi\rho + S\gamma\rho + C = 0;$$

C and γ being any scalar and vector constants; $\phi\rho$ any linear, vector, and self-conjugate function; and q, q', q'' any three constant quaternions: while ρ is, as usual, the variable vector of the surface.

(27.) In fact, besides the *one scalar constant, C*, three are included in the vector γ , and *six others* in the function ϕ (comp. 358); and of the *ten* which remain to be introduced, for the expression of a scalar and *homogeneous* function of ρ , of the *third degree*, the *three versors* Uq, Uq', Uq'' supply *nine* (comp. 312), and the *tensor* $T.qq'q''$ is the *tenth*.

(28.) And for the same reason the *monomial equation*,

$$\text{LXII.} \dots S\rho\rho'q''\rho = 0,$$

with the same significations of q, q', q'' , represents the *general Cone of the Third Degree*, or *Cubic Cone*, which has its vertex at the origin of vectors.

(29.) If then we combine this last equation with that of a *secant plane*, such as $S\rho + 1 = 0$, we shall get a quaternion expression for a *Plane Cubic*, or *plane curve of the third degree*: and if we combine it with the equation $\rho^3 + 1 = 0$ of the *unit-sphere*, we shall obtain a corresponding expression for a *Spherical Cubic*,* or for a curve upon a spheric surface, which is cut by an arbitrary great circle in *three pairs of opposite points*, real or imaginary.

(30.) Finally, as an example of *sections of surfaces*, represented by *transcendental equations*, let us consider the *Screw Surface*, or *Helicoid*,† of which the vector equation may be thus written (comp. the sub-arts. to 314):

$$\text{LXIII.} \dots \rho = c(x+a)a + ya^x\gamma, \text{ with } T\alpha = 1, \gamma = \sqrt{a}\beta, \text{ and } y > 0;$$

α being the *unit axis*, while β, γ are two other constant vectors, a, c two scalar constants, and x, y two variable scalars.

(31.) Cutting this surface by the plane of $\beta\gamma$, or supposing that

$$\text{LXIV.} \dots 0 = S\gamma\beta\rho = \beta^2S\alpha\rho - S\alpha\beta S\beta\rho, \text{ and writing } \text{LXV.} \dots c = bS\alpha\beta,$$

we easily find that the scalar and vector equations of what we may call the *Screw Section* may be thus written :

$$\text{LXVI.} \dots b(x+a) = yS.a^x; \quad \text{LXVII.} \dots \rho = y(\gamma S.a^x - \beta S.a^{x-1}).$$

(32.) Derivating these with respect to x , and eliminating β and y' , we arrive at the equation,

$$\text{LXVIII.} \dots \rho = (x+a)\rho' + z\gamma, \text{ if } \text{LXIX.} \dots 2bx = \pi y^2;$$

* Compare the Note to page 48; see also the *theorem* in that page, which contains perhaps a new mode of *generation* of cubic curves in a given plane: or, by an easy modification, of the corresponding curves upon a sphere.

† Already mentioned in pages 383, 502, 514, 557. The condition $y > 0$ answers to the supposition that, in the generation of the surface, the perpendiculars from a given helix on the axis of the cylinder are not prolonged beyond that axis.

but zy in LXVIII. is the vector of the point, say o , in which the *tangent to the section* at the point (x, y) , or r , intersects the given line γ , namely the line in the plane of that section which is *perpendicular to the axis* a : we see then, by LXIX., that *this point of intersection depends only on the constant, b , and on the variable, y , being independent of the constant, a , and of the variable, x .*

(38.) To interpret this result of calculation, which might have been otherwise found with the help of the expression 372, XII. (with β changed to γ) for the *normal* v to a screw-surface, we may observe, first, that the equation LXVII., which may be written as follows,

$$\text{LXX.} \dots \rho = yV.a^{2+1}\beta, \text{ and gives } \text{LXXI.} \dots TV\rho = yT\gamma,$$

would represent an *ellipse*, if the coefficient y were treated as *constant*; namely, the section of the right *cylinder* LXXI. by the *plane* LXIV.; the *vector semiaxes* (major and minor) of this ellipse being $y\beta$ and $y\gamma$ (comp. 314, (2.)).

(34.) By assigning a new value to the constant a , we pass to a *new screw surface* (30.), which differs only in *position* from the former, and may be conceived to be formed from it by *sliding* along the axis a ; while the value of x , corresponding to a given y , will vary by LXVI., and thus we shall have a *new screw section* (31.), which will *cross the ellipse* (33.) in a *new point* Q : but the *tangent to the section* at this point will *intersect* by (32.) the *minor axis* of the ellipse in the *same point* o as before.

(35.) We shall thus have a *Figure** such as the following (Fig. 85); in which if F be a *focus* of the ellipse BC , and o (as above) the *point of convergence* of the *tangents* to the *screw sections* at the points P, Q , &c., of that ellipse, it is easy to prove, by pursuing the same analysis a little farther, 1st, that the *angle* (g), subtended at this focus F by the minor semiaxis oc , which is also a *radius* (r) of the *cylinder* LXXI., is equal to the *inclination* of the axis (a) of that cylinder to the *plane* of the ellipse, as may indeed be inferred from elementary principles; and 2nd, what is less obvious, that the *other angle* (h), subtended at the same focus (F) by the interval og , or by what may be called (with reference to the present construction, in which it is supposed that $b < 0$, or that the angles made by D_{xp} and β with a are either *both acute*, or *both obtuse*) the *Depression* (s) of the *Skew Centre* (o), is equal to the inclination of the same axis (a) to the *helix* on the same cylinder, which is obtained (comp. 314, (10.)) by treating y as constant, in the equation LXIII. of the *Screw Surface*.

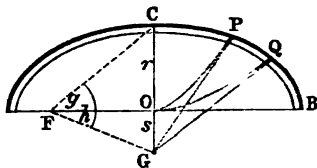


Fig. 85.

* Those who are acquainted, even slightly, with the theory of *Oblique Arches* (or *skew bridges*), will at once see that this Figure 85 may be taken as representing rudely such an arch: and it will be found that the *construction* above deduced agrees with the celebrated *Rule of the Focal Excentricity*, discovered practically by the late Mr. Buck. This application of Quaternions was alluded to, in page 620 of the *Lectures*.

SECTION 8.—*On a few Specimens of Physical Application of Quaternions, with some Concluding Remarks.*

416. It remains to give, according to promise (368), before concluding this work, some examples* of *physical applications* of the present Calculus: and as a first specimen, we shall take the *Statics of a Rigid Body*.

(1.) Let $\alpha_1, \dots, \alpha_n$ be n *Vectors of Application*, and let β_1, \dots, β_n be n corresponding *Vectors of Force*, in the sense that n forces are applied at the points A_1, \dots, A_n of a *free but rigid system*, and are represented as usual by so many right lines from those points, to which lines the vectors OB_1, \dots, OB_n are equal, though drawn from a common origin; and let $\gamma (= oc)$ be the vector of an *arbitrary point* c of space. Then the *Equation† of Equilibrium* of the system or body, under the action of these n applied forces, may be thus written:

$$I. \dots \Sigma V(\alpha - \gamma)\beta = 0; \quad \text{or thus,} \quad I'. \dots V\gamma \Sigma \beta = \Sigma V\alpha\beta.$$

(2.) The supposed *arbitrariness* (1.) of γ enables us to break up the formula I. or I', into the *two* vector equations:

$$II. \dots \Sigma \beta = 0; \quad III. \dots \Sigma V\alpha\beta = 0;$$

of each of which it is easy to assign, as follows, the *physical signification*.

(3.) The equation II. expresses that if the forces, which are applied at the points A_1, \dots of the body, were all *transported* to the origin o , their *statical resultant*, or *vector sum*, would be zero.

(4.) The equation III. expresses that the resultant of all the *couples*, produced in the usual way by such a transference of the applied forces to the assumed origin, is *null*.

(5.) And the equation I., which as above includes both II. and III., expresses that if all the given forces be transported to *any common point* c , the *couples* hence arising will *balance each other*: which is a sufficient condition of equilibrium of the system.

(6.) When we have only the relation,

$$IV. \dots S(\Sigma \beta \cdot \Sigma V\alpha\beta) = 0,$$

without $\Sigma \beta$ vanishing, the applied forces have then an *Unique Resultant* = $\Sigma \beta$, acting along the line of which I. or I'. is the equation, with γ for its variable vector.

* The reader may compare the remarks on *hydrostatic pressure*, in pages 434, 435.

† We say here, "*equation*:" because the *single quaternion formula*, I. or I', contains virtually the *six usual scalar equations*, or conditions, of the equilibrium at present considered.

(7.) And the *physical interpretation* of this condition IV. is, that when the forces are transported to o , as in (8.) and (4.) the *resultant force* is in the *plane* of the *resultant couple*.

(8.) When the equation II., but not III., is satisfied, the applied forces compound themselves into *One Couple*, of which the *Axis* $= \Sigma V a \beta$, whatever may be the position of the *origin*.

(9.) When *neither* II. nor III. is satisfied, we may still propose so to place the *auxiliary point* c , that when the *given forces* are transferred to it, as in (5.), the *resultant force* $\Sigma \beta$ may have the *direction* of the *axis* $\Sigma V(a - \gamma)\beta$ of the *resultant couple*, or else the *opposite* of that direction; so that, in each case, the condition,*

$$V. \dots V \frac{\Sigma V(a - \gamma)\beta}{\Sigma \beta} = 0,$$

shall be satisfied by a suitable limitation of the auxiliary vector γ .

(10.) This last equation V. represents therefore the *Central Axis* of the given system of applied forces, with γ for the variable vector of that right line: or the *axis* of the *screw-motion* which those forces *tend* to produce, when they are *not in balance*, as in (1.), and neither tend to produce *translation alone*, as in (6.), nor *rotation alone*, as in (8.).

(11.) In general, if q be an *auxiliary quaternion*, such that

$$VI. \dots q \Sigma \beta = \Sigma V a \beta,$$

its *vector part*, Vq , is equal by (V.) to the *Vector-Perpendicular*, let fall from the origin on the *central axis*; while its *scalar part*, Sq , is easily proved to be the *quotient*, of what may be called the *Central Moment*, divided by the *Total Force*: so that $Vq = 0$ when the *central axis* passes *through the origin*, and $Sq = 0$ when there exists an *unique resultant*.

(12.) When the total force $\Sigma \beta$ does not vanish, let Q be a *new auxiliary quaternion*, such that

$$VII. \dots Q = \frac{\Sigma a \beta}{\Sigma \beta} = q + \frac{\Sigma S a \beta}{\Sigma \beta},$$

with VIII. $\dots c = S Q = S q$, and IX. $\dots \gamma = o c = V Q$,

for its scalar and vector parts; then $c \Sigma \beta$ represents, both in quantity and in direction, the *Axis of the Central Couple* (9.), and γ is the vector of a point c which is on the *central axis* (10.), considered as a right line having *situation in space*: while the *position* of this point on this line depends only on the *given system* of applied forces, and does not vary with the assumed *origin* o .

(13.) Under the same conditions, we have the transformations,

$$\begin{aligned} X. \dots \Sigma a \beta &= (c + \gamma) \Sigma \beta; & XI. \dots T \Sigma a \beta &= (c^2 - \gamma^2)^{\frac{1}{2}} T \Sigma \beta; \\ XII. \dots \Sigma V a \beta &= c \Sigma \beta + V \gamma \Sigma \beta; & XIII. \dots (\Sigma V a \beta)^2 &= c^2 (\Sigma \beta)^2 + (V \gamma \Sigma \beta)^2; \end{aligned}$$

* The equation V. may also be obtained from the condition,

$$V. \dots T \Sigma V(a - \gamma)\beta = \text{a minimum,}$$

when γ is treated as the only variable vector; which answers to a known property of the *Central Moment*.

whereof XII. contains the known law, according to which the axis of the couple (4.), obtained by transferring all the forces to an assumed point o, varies generally in quantity and in direction with the position of that point: while XIII. expresses the known corollary from that law, in virtue of which the quantity alone, or the energy ($TEVa\beta$) of the couple here considered, is the same for all the points o of any one right cylinder, which has the central axis of the system for its axis of revolution.

(14.) If we agree to call the quaternion product $PA. AA'$ the quaternion moment, or simply the *Moment*, of the applied force AA' at A, with respect to the Point P, the quaternion sum $\Sigma a\beta$ in X. may then be said to be the *Total Moment* of the given system of forces, with respect to the assumed origin o; and the formula XI. expresses that the *tensor* of this sum, or what may be called the *quantity* of this total moment, is constant for all points o which are situated on any one spheric surface, with the point c determined in (12.) for its centre: being also a minimum when o is placed at that point c itself, and being then equal to what has been already called the *central moment*, or the energy of the central couple.

(15.) For these and other reasons, it appears not improper to call generally the point c, above determined, the *Central Point*, or simply the *Centre*, of the given system of applied forces, when the total force does not vanish; and accordingly in the particular but important case, when all those forces are parallel, without their sum being zero; so that we may write,

$$XIV. \dots \beta_1 = b_1\beta, \dots \beta_n = b_n\beta, \quad TE\beta > 0,$$

the scalar c in (12.) vanishes, and the vector γ becomes (comp. Art. 97 on *bary-centres*),

$$XV. \dots oc = \gamma = \frac{b_1a_1 + \dots + b_na_n}{b_1 + \dots + b_n} = \frac{\Sigma ba}{\Sigma b};$$

so that the point c, thus determined, is independent of the common direction β , and coincides with what is usually called the *Centre of Parallel Forces*.

(16.) The conditions of equilibrium (1.), which have been already expressed by the formula I., may also be included in this other quaternion equation,

$$XVI. \dots \text{Total Moment} = \Sigma a\beta = a \text{ scalar constant,}$$

of which the value is independent of the origin; and which, with its sign changed, represents what may perhaps be called the *Total Tension* of the system.

(17.) Any infinitely small change, in the position of a rigid body, is equivalent to the alteration of each of its vectors a to another of the form,

$$XVII. \dots a + \delta a = a + \epsilon + V\iota a,$$

ϵ and ι being two arbitrary but infinitesimal vectors, which do not vary in the passage from one point A of the body to another: and thus the conditions of equilibrium (1.) may be expressed by this other formula,

$$XVIII. \dots \Sigma S\beta\delta a = 0,$$

which contains, for the case here considered, the *Principle of Virtual Velocities*, and admits of being extended easily to other cases of Statics.

417. The general Equation of Dynamics may be thus written,

$$I. \dots \Sigma mS(D_t^2 u - \xi)\delta a = 0,$$

with significations of the symbols which will soon be stated; but as we only propose (416) to give here some *specimens* of physical application, we shall aim chiefly, in the following sub-articles, at the deduction of a few formulæ and theorems, respecting *Axes* and *Moments of Inertia*, and subjects therewith connected.

(1.) In the formula I., a is the vector of position, at the time t , of an element m of the system; δa is any variation of that vector, geometrically compatible with the mutual connexions between the parts of that system; the vector $m\xi$ represents a moving force, or ξ an accelerating force, which acts on the element m of mass; D and S are marks, as usual, of derivating and taking the scalar; and the summation denoted by Σ extends to all the elements, and is generally equivalent to a triple integration, or to an addition of triple integrals in space. And the formula is obtained (comp. 416, (17.)), by a combination of D'Alembert's principle with the principle of virtual velocities, which is analogous to that employed in the *Mécanique Analytique* by Lagrange.

(2.) For the case of a *free* but *rigid* body, we may substitute for δa the expression $\epsilon + V\iota a$, assigned by 416, XVII.; and then, on account of the arbitrariness of the two infinitesimal vectors ϵ and ι , the formula I. breaks up into the two following,

$$\text{II.} \dots \Sigma m(D_t^2 a - \xi) = 0; \quad \text{III.} \dots \Sigma m V a (D_t^2 a - \xi) = 0;$$

which correspond to the two statical equations 416, II. and III., and contain respectively the law of motion of the centre of gravity, and the law of description of areas.

(3.) If the body have a *fixed point*, which we may take for the origin o , we eliminate the reaction at that point, by attending only to the equation III.; and may then express the connexions between the elements m by the formula,

$$\text{IV.} \dots D_t a = V\iota a, \quad \text{whence} \quad V \dots D_t^2 a = \iota V\iota a - V a D_t \iota;$$

ι being the *Vector-Axis of instantaneous Rotation* of the body, in the sense that its *versor* U_t represents the *direction* of the *axis*, and that its *tensor* T_t represents the *angular velocity*, of such rotation at the time t .

(4.) By V ., the equation III. becomes,

$$\text{VI.} \dots \Sigma m a V a D_t \iota = \Sigma m (V \iota a S \iota a - V a \xi);$$

and other easy combinations give the laws of areas and living force, under the forms,

$$\text{VII.} \dots \Sigma m a D_t a - \Sigma m V \int a \xi dt = \gamma = \text{a constant vector};$$

$$\text{VIII.} \dots \frac{1}{2} \Sigma m (D_t a)^2 - \Sigma m S \int \iota a \xi dt = c = \text{a constant scalar}.$$

(5.) When the applied forces vanish, or balance each other, or more generally when they compound themselves into a single force acting at the fixed point, so that in each case the condition

$$\text{IX.} \dots \Sigma m V a \xi = 0$$

is satisfied, the equations (4.) are simplified; and if we introduce a linear, vector, and self-conjugate function ϕ , such that

$$\text{X.} \dots \phi \iota = \Sigma m a V a \iota = \iota \Sigma m a^2 - \Sigma m a S a \iota,$$

and write h^2 for $-2c$, they take the forms,

$$\text{XI.} \dots \phi D\alpha + V_t \phi_t = 0; \quad \text{XII.} \dots \phi_t + \gamma = 0; \quad \text{XIII.} \dots S_t \phi_t = h^2;$$

γ and h being two real constants, of the vector and scalar kinds, connected with each other and with t by the relation,

$$\text{XIV.} \dots S_t \gamma + h^2 = 0; \quad \text{also} \quad \text{XV.} \dots \phi D\alpha = V_t \gamma.$$

It may be added that γ is now the *vector sum* of the doubled *areal velocities* of all the elements of the body, multiplied each by the mass m of that element, and each represented by a *right line* $\alpha D_t \alpha$ perpendicular to the plane of the area described round the fixed point o in the time dt ; and that h^2 is the *living force*, or *vis viva* of the body, namely the *positive sum* of all the products obtained by multiplying each element m by the *square* of its *linear velocity*, regarded as a *scalar* ($TD_t \alpha$).

(6.) When t is regarded as a variable vector, the equation XIII. represents an *ellipsoid*, which is *fixed in the body*, but *moveable with it*; and the equation XIV. represents a *tangent plane* to this ellipsoid, which plane is *fixed in space*, but *changes in general its position* relatively to the *body*. And thus the *motion* of that body may generally be conceived, as was shown by Poinso't, to be performed by the *rolling* (*without gliding*) of an *ellipsoid upon a plane*; the former *carrying the body* with it, while its *centre* o remains *fixed*: and the *semidiameter* (t) of *contact* being the *vector-axis* (S_t) of *instantaneous rotation*.

(7.) The ellipsoid XIII. may be called, perhaps, the *Ellipsoid of Living Force*, on account of the signification (5.) of the constant h^2 in its equation; and the fixed plane XIV., on which it rolls, is parallel to what may be called the *Plane of Areas* ($S_t \gamma = 0$): no use whatever having hitherto been made, in this investigation, of any *axes or moments of inertia*. But if we here admit the usual definition of such a moment, we may say that the *Moment of Inertia* of the *body*, with respect to any *axis* t through the fixed point, is equal to the *living force* h^2 divided by the *square** of the *semidiameter* T_t of the ellipsoid XIII.; because this moment is,

$$\text{XVI.} \dots \Sigma m (TV \alpha U_t)^2 = t^{-2} \Sigma m (V_t \alpha)^2 = -S_t^{-1} \phi_t = h^2 T_t^{-2}.$$

(8.) The equations XII. and XIII. give,

$$\text{XVII.} \dots \phi^2 = \gamma^2 S_t \phi_t - h^2 (\phi_t)^2 = S_t \nu, \quad \text{if} \quad \text{XVIII.} \dots \nu = \gamma^2 \phi_t - h^2 \phi_t^2;$$

and this equation XVII. represents a *cone* of the second degree, fixed in the body (comp. (6.)), but *moveable with it*, of which the *axis* t is always a *side*, and to which the *normal*, at any point of that side, has the direction of the line ν . But it follows

* Hence it may easily be inferred, with the help of the general construction of an *ellipsoid* (217, (6.)), illustrated by Figure 58 in page 226, that for *any solid body*, and *any given point* A thereof, there can always be found (indeed in more ways than one) *two other points*, B and C , which are likewise *fixed in the body*, and are such that the *square-root* of the *moment of inertia*, round any *axis* AD , is geometrically constructed by the line BD , if the point D be determined on the *axis*, by the condition that A and D shall be *equally distant* from C . This theorem, with some others here reproduced, was given in the Abstract of a Paper read before the Royal Irish Academy on the 10th of January, 1848, and was published in the *Proceedings* of that date.

from XI., or from XII. XV., and from the properties of the function ϕ , that D_{μ} is perpendicular to both ϕt and $\phi^2 t$, and therefore also by XVIII. to ν ; the cone XVII. is therefore *touchèd*, along the side t , by that *other cone*, which is the *locus in space* of the *instantaneous axis* of rotation. We are then led, by this simple quaternion analysis, to a *second representation* of the *motion of the body*, which also was proposed by Poinso: namely, as the *rolling of one cone on another*.

(9.) To treat briefly by quaternions some of Mac Cullagh's results on this subject, it may be noted that the *line* γ , though *fixed in space*, describes in *the body* a *cone* of the second degree, of which the equation is, by what precedes,

$$\text{XIX.} \dots g^2 S \gamma \phi^{-1} \gamma + h^2 \gamma^2 = 0, \text{ if } \text{XX.} \dots g = T \gamma, \text{ or } \text{XXI.} \dots \gamma^2 + g^2 = 0;$$

while, if we write $\gamma = oc$, the *point c* is indeed *fixed in space*, but describes a *sphero-conic* in the *body*, which is part of the common intersection of the cone XIX., the *sphere* XXI., and the *reciprocal ellipsoid* (comp. XIII.),

$$\text{XXII.} \dots S \gamma \phi^{-1} \gamma = h^2.$$

(10.) Also, the *normal* to the *new cone* (9.), at any point of the side γ , has the direction of $g^2 \phi^{-1} \gamma + h^2 \gamma$, or of $t + h^2 \gamma^{-1}$ (comp. XIV.); and if a line in this direction be drawn through the fixed point o , it will be the *side of contact* of the *plane of areas* (7.), with the *cone of normals* at o to the cone XIX.; which *last* (or *reciprocal*) *cone rolls on that plane of areas*.

(11.) As regards the *Axes of Inertia*, it may be sufficient here to observe that if the body revolve round a *permanent axis*, and with a *constant velocity*, the *vector axis* t is constant; and must therefore satisfy the equation,

$$\text{XXIII.} \dots V_1 \phi t = 0, \text{ because } \text{XXIV.} \dots D_{\mu} = 0;$$

it has therefore in general (comp. 415) one or other of *Three Real and Rectangular Directions*, determined by the condition XXIII.: namely, those of the *Axes of Figure* of either of the two *Reciprocal Ellipsoids*, XIII. XXII.

(12.) And the *Three Principal Moments*, say A, B, C , corresponding to those *three principal axes*, are by XVI. the three scalar values of $-t^{-1} \phi t$; so that the *symbolical cubic* (850) in ϕ may be thus written,

$$\text{XXV.} \dots (\phi + A) (\phi + B) (\phi + C) = 0.$$

(13.) Forming then this symbolical cubic by the general method of the Section III. ii. 6, we find that the *three moments* A, B, C , are the *three roots* (always *real*, by this analysis) of the *algebraic and cubic equation*,

$$\text{XXVI.} \dots A^3 - 2n^2 A^2 + (n^4 + n'^2) A - (n^2 n'^2 - n''^2) = 0;$$

in which, n^2, n'^2, n''^2 are three positive scalars, namely,

$$\text{XXVII.} \dots n^2 = -\Sigma m a^2; \quad n'^2 = -\Sigma m m' (V a a')^2; \quad n''^2 = \Sigma m m' m'' (S a a' a'')^2;$$

and the combination $n^2 n'^2 - n''^2$ is another positive scalar, of which the value may be thus expressed,

$$\text{XXVIII.} \dots ABC = n^2 n'^2 - n''^2 = \Sigma m^2 m' a^2 (V a a')^2 \\ + 2 \Sigma m m' m'' (T a a' T a' a'' T a'' a + S a a' S a' a'' S a'' a),$$

if a, a', a'' , &c. be the vectors of the mass-elements $m, m', m'',$ &c.

(14.) And because the equation XXV. gives this other symbolical result,

$$\text{XXIX.} \dots -ABC\phi^{-1} = \phi^2 + (A + B + C)\phi + BC + CA + AB,$$

it follows that

$$\text{XXX.} \dots \phi^{-10} = 0;$$

and therefore, by XV., &c., that if a body, with a fixed point, &c., begin to revolve round one of its three principal axes of inertia, it will continue to revolve round that axis, with an unchanged velocity of rotation.

(15.) It has hitherto been supposed, that all the moments of inertia are referred to axes passing through one point *o* of the body; but it is easy to remove this restriction. For example, if we denote the moment XVI. by I_0 , and if I_ω be the corresponding moment for an axis parallel to ϵ , but drawn through a new point Ω , of which the vector is ω , then

$$\begin{aligned} \text{XXXI.} \dots I_\omega &= \epsilon^{-2} \Sigma m (V_\epsilon(\alpha - \omega))^2 \\ &= I_0 + 2 \Sigma m \cdot S(\omega \epsilon^{-1} V_\epsilon \kappa) + p^2 \Sigma m, \end{aligned}$$

if $\text{XXXII.} \dots \kappa \Sigma m = \Sigma m \alpha$, and $\text{XXXIII.} \dots p = TV\omega U\epsilon$,

so that κ is the vector of the centre of inertia (or of gravity) of the body, and p is the distance between the two parallel axes.

(16.) If then we suppose that the condition

$$\text{XXXIV.} \dots V_\epsilon \kappa = 0$$

is satisfied, that is, if the axis ϵ pass through the centre of inertia, we shall have the very simple relation,

$$\text{XXXV.} \dots I_\omega = I_0 + p^2 \Sigma m;$$

which agrees with known results.

418. As a third specimen of physical applications of quaternions, we propose to consider briefly the motions of a System of Bodies, m, m', m'', \dots regarded as free material points, of which the variable vectors are a, a', a'', \dots and which are supposed to attract each other according to the law of the inverse square: the fundamental formula employed being the following,

$$\text{I.} \dots \Sigma m S D_i^2 \hat{a} \hat{a} + \delta P = 0, \quad \text{if} \quad \text{II.} \dots P = \Sigma \frac{mm'}{T(a - a')}:$$

P thus denoting the Potential (or force-function) of the system, and the variations \hat{a}, \hat{a}', \dots being infinitesimal, but otherwise arbitrary.

(1.) To deduce the formula I., with the signification II. of P , from the general equation 417, I. of dynamics, we have first, for the case of two bodies, the following expressions for the accelerating forces,

$$\text{III.} \dots \xi = \frac{m'}{(a - a')r}, \quad \xi' = \frac{m}{(a' - a)r}, \quad \text{if} \quad r = T(a - a'); \quad \bullet$$

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whence follows the transformation,*

$$\text{IV.} \dots S(m\xi\delta a + m'\xi'\delta a') = \frac{-mm'}{r} S \frac{\delta(a-a')}{a-a'} = \delta \frac{mm'}{r};$$

a result easily extended, as above. If the law of attraction were supposed different, there would be no difficulty in modifying the expression for the potential accordingly.

(2.) In general, when a scalar, f (as here P), is a function of one or more vectors, a, a', \dots its variation (or differential) can be expressed as a linear and scalar function of their variations (or differentials), of the form $S\beta\delta a + S\beta'\delta a' + \dots$ (or $\Sigma S\beta\delta a$); in which $\beta, \beta' \dots$ are certain new and finite vectors, and are themselves generally functions of a, a', \dots , derived from the given scalar function f . And we shall find it convenient to extend the Notation† of Derivatives, so as to denote these derived vectors $\beta, \beta', \&c.$, by the symbols, $D_a f, D_a' f, \&c.$ In this manner we shall be able to write,

$$\text{V.} \dots \delta P = \Sigma S(D_a P, \delta a);$$

and the differential equations of motion of the bodies m, m', m'', \dots will take by I. the forms:

$$\text{VI.} \dots mD_t^2 a + D_a P = 0, \quad m'D_t^2 a' + D_a' P = 0, \&c.;$$

or more fully,

$$\text{VII.} \dots D_t^2 a = \frac{m'}{(a-a')T(a-a')} + \frac{m''}{(a-a'')T(a-a'')} + \dots; \&c.$$

(3.) The laws of the centre of gravity, of areas, and of living force, result immediately from these equations, under the forms,

$$\text{VIII.} \dots \Sigma m D_t \alpha = \beta; \quad \text{IX.} \dots \Sigma m V_a D_t \alpha = \gamma;$$

and

$$\text{X.} \dots T = -\frac{1}{2} \Sigma m (D_t \alpha)^2 = P + H;$$

in which β, γ are constant vectors, H is a constant scalar, and $2T$ is the living force of the system (comp. 417, (5)).

(4.) One mode (comp. 417, (2.)) of deducing the three equations, of which these are the first integrals, is the following. To obtain VIII., change every variation δa in I. to one common but arbitrary infinitesimal vector, ϵ . For IX., change δa to $V_t \alpha$, $\delta a'$ to $V_t \alpha'$, &c.; ϵ being another arbitrary and infinitesimal vector. Finally, to arrive at X., change variations to differentials (δa to da , &c.), and integrate once, as for the two former equations, with respect to the time t .

(5.) The formula I. admits of being integrated by parts, without any restriction on the variations δa , by means of the general transformation,

$$\text{XI.} \dots S(D_t^2 a, \delta a) = D_t S(D_t a, \delta a) - \frac{1}{2} \delta \cdot (D_t a)^2,$$

combined with the introduction of the following definite integral (comp. X.),

$$\text{XII.} \dots F = \int_0^t (P + T) dt.$$

* It may not be useless here to compare the expression in page 417, for the differential of a proximity.

† In this extended notation, such a formula as $dfp = 2S\nu dp$ would give,

• $\nu = \frac{1}{2} D_p f p.$

(6.) In fact, if we denote by a_0, a'_0, \dots the *initial values* of the vectors a, a', \dots or their values when $t = 0$, and by D_0a, D_0a', \dots the corresponding values of $D_t a, D_t a', \dots$, we shall thus have, as a first integral of the equation I., the formula,

$$\text{XIII.} \dots \Sigma m S (D_t a \cdot \delta a - D_0 a \cdot \delta a_0) + \delta F = 0;$$

in which no variation δt is assigned to t , and which conducts to important consequences.

(7.) To draw from it some of these, we may observe that if the masses m, m', \dots be treated as constant and known, the complete integrals of the equations VI. or VII. must be conceived to give what may be called the *final vectors of position* a, a', \dots and of *velocity* $D_t a, D_t a', \dots$ in terms of the *initial vectors* $a_0, a'_0, \dots, D_0 a, D_0 a', \dots$ and of the *time*, t : whence, conversely, we may conceive the initial vectors of velocity to be expressible as functions of the initial and final vectors of position, and of the time. In this way, then, we are led to consider $P, T,$ and F as being *scalar functions* (whether we are or are not prepared to express them as such), of $a, a', \dots, a_0, a'_0, \dots$ and t ; and thus, by (2.), the recent formula XIII. breaks up into the two following systems of equations:

$$\text{XIV.} \dots m D_t a + D_a F = 0, \quad m' D_t a' + D_{a'} F = 0, \quad \&c.;$$

$$\text{and} \quad \text{XV.} \dots -m D_0 a + D_{a_0} F = 0, \quad -m' D_0 a' + D_{a'_0} F = 0, \quad \&c.;$$

whereof the former may be said to be *intermediate integrals*, and the latter to be *final integrals*, of the differential equations of motion of the system, which are included in the formula I.

(8.) In fact, the equations XIV. do not involve the final vectors of acceleration $D_t^2 a, \dots$ as the differential equations VI. or VII. had done; and the equations XV. express, at least theoretically, the dependence of the final vectors of position a, \dots on the time, t , and on the initial vectors of position a_0, \dots and of velocity $D_0 a, \dots$ as by (7.) the complete integrals ought to do. And on account of these and other important properties, the function here denoted by F may be called the *Principal* Function of Motion of the System*.

(9.) If the initial vectors a_0, \dots and $D_0 a, \dots$ be given, that is, if we consider the actual progress in space of the mutually attracting system of masses m, \dots from one set of positions to another, then the function F depends upon the time alone; and by its definition XII., its rate or velocity of increase, or its total derivative with respect to t , is thus expressed,

$$\text{XVI.} \dots D_t F = P + T.$$

(10.) But we may inquire what is the *partial derivative*, say $(D_t F)$, of the same definite integral F , when regarded (7.) as a function of the final and initial vectors of position a, \dots, a_0, \dots which involves also the time explicitly, and is now to be derived with respect only to that variable t , as if the final vectors a, \dots were constant: whereas in fact those vectors alter with the time, in the course of any actual motions of the system.

* This function was in fact so called, in two Essays by the present writer, "On a General Method in Dynamics," published in the *Philosophical Transactions* (London), for the years 1834 and 1835; although of course *coordinates*, and not *quaternions*, were then employed, the latter not having been discovered until 1843: and the notation S , since adopted for *scalar*, was then used instead of F .

(11.) For this purpose, it is sufficient to observe that the *part* of the *total derivative* $D_t F$, which arises from the last mentioned changes of α, \dots is (by XIV. and X.),

$$\text{XVII.} \dots \Sigma S(D_\alpha F \cdot D_t \alpha) = 2T;$$

and therefore (by XVI. and X.), that the *remaining part* must be,

$$\text{XVIII.} \dots (D_t F) = P - T = -H.$$

(12.) The *complete variation* of the function F is therefore (comp. XIII.), when t as well as α, \dots and α_0, \dots is treated as *varying*,

$$\text{XIX.} \dots \delta F = -H \delta t - \Sigma m S D_t \alpha \delta \alpha + \Sigma m S D_0 \alpha \delta \alpha_0.$$

(13.) And hence, with the help of the equations X. XIV. XV., it is easy to infer that the principal function F must satisfy the two following *Partial Differential Equations in Quaternions* :

$$\text{XX.} \dots (D_t F) - \frac{1}{2} \Sigma m^{-1} (D_\alpha F)^2 = P;$$

$$\text{XXI.} \dots (D_t F) - \frac{1}{2} \Sigma m^{-1} (D_{\alpha_0} F)^2 = P_0;$$

in which P_0 denotes the initial value of the potential P .

(14.) If we write

$$\text{XXII.} \dots V = \int_0^t 2T dt,$$

so that V represents what is called the *Action*, or the accumulated living force, of the system during the time t , then by X. and XII. the two definite integrals F and V are connected by the very simple relation,

$$\text{XXIII.} \dots V = F + tH;$$

whence by XIX. the *complete variation* of V , considered as a function of the final and initial vectors of position, and of the constant H of living force, which does not explicitly involve the time, may be thus expressed,

$$\text{XXIV.} \dots \delta V = t \delta H - \Sigma m S D_t \alpha \delta \alpha + \Sigma m S D_0 \alpha \delta \alpha_0.$$

(15.) The *partial derivatives* of this new function V , which is for *some purposes* more useful than F , and may be called, by way of distinction from it, the *Characteristic* Function* of the motion of the system, are therefore,

$$\text{XXV.} \dots D_\alpha V = -m D_t \alpha, \text{ \&c. ;}$$

$$\text{XXVI.} \dots D_{\alpha_0} V = +m D_0 \alpha, \text{ \&c. ;}$$

$$\text{and XXVII.} \dots D_H V = t.$$

(16.) The *intermediate integrals* (7.) of the differential equations of motion, which were before expressed by the formulæ XIV., may now, somewhat less simply, be regarded as the result of the elimination of H between the formulæ XXV. XXVII.; and the *final integrals* of those equations VI. or VII., which were expressed by XV., are now to be obtained by eliminating the same constant H between the recent equations XXVI. XXVII.

* The *Action*, V , was in fact so called, in the two Essays mentioned in the preceding Note. The properties of this *Characteristic Function* had been perceived by the writer, before those of that which he came afterwards to call the *Principal Function*, as above.

(17.) The *Characteristic Function*, V , is obliged (comp. (18.)) to satisfy the two following *partial differential equations*,

$$\text{XXVIII.} \dots \frac{1}{2} \Sigma m^{-1} (D_{\alpha} V)^2 + P + H = 0;$$

$$\text{XXIX.} \dots \frac{1}{2} \Sigma m^{-1} (D_{\alpha_0} V)^2 + P_0 + H = 0;$$

it varies, like F , when $t=0$, at which epoch $\alpha = \alpha_0$, $\alpha' = \alpha'_0$, &c.; each of these two functions, F and V , depends *symmetrically* on the initial and final vectors of position: and each does so, only by depending on the mutual *configuration* of all these initial and final *positions*.

(18.) It follows (comp. (4.), see also 416, (17.), and 417, (2.)), that the function F must satisfy the two conditions,

$$\text{XXX.} \dots \Sigma (D_{\alpha} F + D_{\alpha_0} F) = 0; \quad \text{XXXI.} \dots \Sigma V (\alpha D_{\alpha} F + \alpha_0 D_{\alpha_0} F) = 0;$$

which accordingly are forms, by XIV. XV., of the equations VIII. and IX., and therefore are expressions for the law of motion of the centre of gravity, and the law of description of areas. And, in like manner, the function V is obliged to satisfy these two analogous conditions,

$$\text{XXXII.} \dots \Sigma (D_{\alpha} V + D_{\alpha_0} V) = 0; \quad \text{XXXIII.} \dots \Sigma V (\alpha D_{\alpha} V + \alpha_0 D_{\alpha_0} V) = 0;$$

which accordingly, by XXV. XXVI., are new forms of the same equations VIII. IX., and consequently are new expressions of the same two laws.

(19.) All the foregoing conditions are satisfied when t is *small*, that is, when the *time* of motion of the system is *short*, by the following *approximate expressions* for the functions F and V , with the respectively derived and mutually connected expressions for H and t :

$$\text{XXXIV.} \dots F = \frac{t}{2} (P + P_0) + \frac{s^2}{2t};$$

$$\text{XXXV.} \dots V = s(P + P_0 + 2H)^{\frac{1}{2}};$$

$$\text{XXXVI.} \dots H = -(D_t F) = -\frac{1}{2} (P + P_0) + \frac{s^2}{2t^2};$$

$$\text{XXXVII.} \dots t = D_H V = s(P + P_0 + 2H)^{\frac{1}{2}};$$

in which s denotes a real and positive scalar, such that

$$\text{XXXVIII.} \dots s^2 = -\Sigma m (\alpha - \alpha_0)^2, \quad \text{or} \quad \text{XXXIX.} \dots s = \sqrt{\Sigma m T (\alpha - \alpha_0)^2}.$$

419. As a *fourth specimen*, we shall take the case of a free point or particle, attracted to a fixed centre* o , from which its variable vector is α , with an accelerating force $= Mr^{-2}$, if $r = T\alpha =$ the distance

* When two free masses, m and m' , with variable vectors α and α' , attract each other according to the law of the inverse square, the differential equation of the relative motion of m about m' is, by 418, VII.,

$$I' \dots D^2 (\alpha - \alpha') = (m + m') (\alpha - \alpha')^{-1} r^{-1}, \quad \text{if} \quad r = T(\alpha - \alpha');$$

and this equation I' reduces itself to I., when we write α for $\alpha - \alpha'$, and M for $m + m'$.

of the point from the centre, while M is the attracting mass: the differential equation of the motion being,

$$\text{I. . . } D^2a = Ma^{-1}r^{-1},$$

if D (abridged from D) be the sign of derivation, with respect to the time t .

(1.) Operating on I. with $V.a$, and integrating, we obtain immediately the equation (comp. 338, (5.)),

$$\text{II. . . } VaDa = \beta = \text{const.};$$

which expresses at once that the orbit is plane, and also that the area described in it is proportional to the time; $U\beta$ being the fixed unit-normal to the plane, round which the point, in its angular motion, revolves positively; and $T\beta$ representing in quantity the double areal velocity, which is often denoted by c .

(2.) And it is important to remark, that these conclusions (1.) would have been obtained by the same analysis, if r^{-1} in I. had been replaced by any other scalar function, $f(r)$, of the distance; that is, for any other law of central force, instead of the law of the inverse square.

(3.) In general, we have the transformation,

$$\text{III. . . } a^{-1}Ta^{-1} = dUa : Vada,$$

because, by 334, XV., &c., we have,

$$\text{IV. . . } dUa = V(da.a^{-1}).Ua = a^{-2}Ua.Vada = a^{-1}Ta^{-1}.Vada;$$

the equation I. may therefore by II. be transformed as follows,

$$\text{V. . . } D^2a = \gamma DUa, \quad \text{if VI. . . } \gamma = -M\beta^{-1};$$

and thus it gives, by an immediate integration,

$$\text{VII. . . } Da = \gamma(Ua - \varepsilon), \quad \text{or VII'. . . } Da = (\varepsilon - Ua)\gamma,$$

ε being a new constant vector, but one situated in the plane of the orbit, to which plane β and γ are perpendicular.

(4.) But a , Da , D^2a are here (comp. 100, (5.) (6.) (7.)) the vectors of position, velocity, and acceleration of the moving point; and it has been defined (100, (5.)) that if, for any motion of a point, the vectors of velocity be set off from any common origin, the curve on which they terminate is the Hodograph* of that motion.

(5.) Hence a and Da , if the latter like the former be drawn from the fixed point o , are the vectors of corresponding points of orbit and hodograph; and because the formula VII. gives,

$$\text{VIII. . . } S\gamma Da = 0, \quad \text{and IX. . . } (Da + \gamma\varepsilon)^2 = \gamma^2,$$

it follows that the hodograph is, in the present question, a Circle, in the plane of the

* Compare Fig. 32, p. 98; see also pages 100, 515, 578, from the two latter of which it may be perceived, that the conception of the hodograph admits of some purely geometrical applications.

orbit, with $-\gamma\epsilon$ (or $+\epsilon\gamma$) for the vector of its centre, and with $T\gamma = MT\beta^{-1}$ for its radius, which radius we shall also denote by h .

(6.) The *Law of the Circular* Hodograph* is therefore a mathematical consequence of the *Law of the Inverse Square*; and conversely it will soon be proved, that no other law of central force would allow generally the hodograph to be a circle.

(7.) For the law of nature, the *Radius (h) of the Hodograph* is equal, by (1.) and (5.), to the quotient of the attracting mass (M), divided by the double areal velocity ($T\beta$ or c) in the orbit; and if we write

$$X. \dots \epsilon = T\epsilon,$$

this positive scalar ϵ may be called the *Excentricity* of the hodograph, regarded as a circle excentrically situated, with respect to the fixed centre of force, o .

(8.) Thus, if $\epsilon < 1$, the fixed point o is interior to the hodographic circle; if $\epsilon = 1$, the point o is on the circumference; and if $\epsilon > 1$, the centre o of force is then exterior to the hodograph, being however, in all these cases, situated in its plane.

(9.) The equation VII. gives,

$$XI. \dots \epsilon - U\alpha = -\gamma^{-1}D\alpha = D\alpha \cdot \gamma^{-1};$$

operating then on this with $S.\alpha$, and writing for abridgment,

$$XII. \dots p = \beta\gamma^{-1} = M^{-1}T\beta^2 = c^2M^{-1}, \text{ and } XIII. \dots SU\alpha\epsilon = \cos v,$$

so that p is a constant and positive scalar, while v is the inclination of α to $-\epsilon$, we find,

$$XIV. \dots r + Sa\epsilon = p; \text{ or } XV. \dots r = \frac{p}{1 + \epsilon \cos v};$$

the orbit is therefore a plane conic, with the centre of force o for a focus, having ϵ for its excentricity, and p for its semiparameter.

(10.) And we see, by XII., that if this semiparameter p be multiplied by the attracting mass M , the product is the square of the double areal velocity c ; so that this constant c may be denoted by $(Mp)^{\frac{1}{2}}$, which agrees with known results.

(11.) If, on the other hand, we divide the mass (M) by the semiparameter (p), the quotient is by XII. the square of the radius ($MT\beta^{-1}$ or h) of the hodograph.

(12.) And if we multiply the same semiparameter p by this radius $MT\beta^{-1}$ of the hodograph, the product is then, by the same formula XII., the constant $T\beta$ or c of double areal velocity in the orbit, so that $h = Mc^{-1} = cp^{-1}$.

(13.) If we had operated with $V.\alpha$ on VII., we should have found,

$$XVI. \dots \beta = V.\alpha(\epsilon - U\alpha)\gamma = (Sa\epsilon + r)\gamma;$$

which would have conducted to the same equations XIV. XV. as before.

* This law of the circular hodograph was deduced geometrically, in a paper read before the Royal Irish Academy, by the present author, on the 14th of December, 1846; but it was virtually contained in a quaternion formula, equivalent to the recent equation VII., which had formed part of an earlier communication, in July, 1845. (See the *Proceedings* for those dates; and especially pages 345, 347, and xxxix., xlix., of Vol. III.)

(14.) If we operate on VII. with $S.a$, we find this other equation,

$$\text{XVII.} \dots -rDr = SaDa = \gamma Va\epsilon;$$

but $\text{XVIII.} \dots -\gamma^2 = h^2 = \frac{M}{p}$ (by VI. and XII., comp. (11.)),

and $\text{XIX.} \dots -(\gamma a\epsilon)^2 = \epsilon^2 r^2 - (p-r)^2 = p(2r-p-r^2 a^{-1})$,

if we write $\text{XX.} \dots a = \frac{p}{1-\epsilon^2}$;

hence squaring XVII., and dividing by r^2 , we obtain the equation,

$$\text{XXI.} \dots \left(\frac{dr}{dt}\right)^2 = M \left(\frac{2}{r} - \frac{1}{a} - \frac{p}{r^2}\right).$$

(15.) It is obvious that this last equation, XXI., connects the *distance*, r , with the *time*, t , as the formula XV. connects the same distance r with the *true anomaly*, v ; that is, with the *angular elongation* in the orbit, from the *position of least distance*. But it would be improper here to delay on any of the elementary consequences of these two known equations: although it seemed useful to show, as above, how the equations themselves might easily be deduced by *quaternions*, and be connected with the theory of the *hodograph*.

(16.) The equation II. may be interpreted as expressing, that the *parallelogram* (comp. Fig. 82) under the vectors a and Da of position and velocity, or under any two *corresponding vectors* (5.) of the *orbit* and *hodograph*, has a *constant plane and area*, represented by the constant vector β , which is *perpendicular* (1.) to that plane. But it is to be observed that, by (2.), these *constancies*, and this *representation*, are *not peculiar* to the law of the *inverse square*, but exist for *all other laws* of *central force*.

(17.) In general, if any scalar function R (instead of Mr^{-2}) represent the accelerating force of attraction, at the distance r from the fixed centre o , the differential equation of motion will be (instead of I.),

$$\text{XXII.} \dots D^2a = Rra^{-1} = -RUa;$$

and if we still write $VaDa = \beta$, as in II., the formula IV. will give,

$$\text{XXIII.} \dots D^3a = -DR.Ua - Rr^{-2}\beta Ua, \quad \text{and} \quad \text{XXIV.} \dots V \frac{D^3a}{D^2a} = r^{-2}\beta;$$

in which $\beta = cU\beta$, if $c = T\beta$, as before.

(18.) Applying then the general formula 414, I., we have, for *any Law** of *force*, the expressions,

$$\text{XXV.} \dots \text{Vector of Curvature of Hodograph} = \frac{1}{D^2a} V \frac{D^3a}{D^2a} = \frac{c}{Rr^2} Ua\beta;$$

$$\text{XXVI.} \dots \text{Radius } (h) \text{ of Curvature of Hodograph} = Rr^2 c^{-1}$$

$$= \frac{\text{Force} \times \text{Square of Distance}}{\text{Double Areal Velocity in Orbit}};$$

* The general value XXVI., of the radius of curvature of the *hodograph*, was *geometrically* deduced in the Paper of 1846, referred to in a recent Note.

of which the last not only conducts, in a new way, for the *law of nature*, to the constant value (7.), $h = Mc^{-1}$, but also proves, as stated in (6.), that for *any other law* of central force the *hodograph cannot be a circle*, unless indeed the *orbit* happens to be such, and to have moreover the centre of force at *its* centre.

(19.) Confining ourselves however at present to the law of the inverse square, and writing for abridgment (comp. (5.)),

$$\text{XXVII.} \dots \kappa = oH = \epsilon\gamma = \text{Vector of Centre } H \text{ of Hodograph,}$$

which gives, by (6.) and (7.),

$$\text{XXVIII.} \dots T\kappa = ch,$$

the origin o of vectors being still the centre of *force*, we see by the properties of the *circle*, that the *product* of any two *opposite velocities* in the *orbit* is *constant*; and that this constant product* may be expressed as follows,

$$\text{XXIX.} \dots (\epsilon - 1) hU\kappa.(\epsilon + 1) hU\kappa = h^2(1 - \epsilon^2) = Ma^{-1},$$

by XVIII. and XX.

(20.) The expression XXIX. may be otherwise written as $\kappa^2 - \gamma^2$; and if v be the vector of any point u *external* to the circle, but in its plane, and κ the length of a *tangent* uT from that point, we have the analogous formula,

$$\text{XXX.} \dots \kappa^2 = \gamma^2 - (v - \kappa)^2 = T(v - \kappa)^2 - h^2.$$

(21.) Let r and r' be the vectors oT , oT' of the two points of contact of tangents thus drawn to the hodograph, from an external point u in its plane; then *each* must satisfy the system of the *three* following scalar equations,

$$\text{XXXI.} \dots S\gamma r = 0; \text{ XXXII.} \dots (r - \kappa)^2 = \gamma^2; \text{ XXXIII.} \dots S(r - \kappa)(v - \kappa) = \gamma^2;$$

whereof the *first alone* represents the *plane*; the *two first* jointly represent (comp. (5.)) the *circle*; and the *third* expresses the *condition of conjugation* of the points r and v , and may be regarded as the *scalar equation of the polar* of the latter point. It is understood that $S\gamma v = 0$, as well as $S\gamma\kappa = 0$, &c., because γ is perpendicular (8.) to the plane.

(22.) Solving this system of equations (21.), we find the two expressions,

$$\text{XXXIV.} \dots r = \kappa + \gamma(\gamma + \kappa)(v - \kappa)^{-1}; \text{ XXXIV'}. \dots r' = \kappa + \gamma(\gamma - \kappa)(v - \kappa)^{-1};$$

in which the scalar κ has the same value as in (20.). As a verification, these expressions give, by what precedes,

* In strictness, it is only for a *closed orbit*, that is, for the case (8.) of the centre of force being *interior* to the *hodograph* ($\epsilon < 1$), that two velocities can be *opposite*; their *vectors* having then, by the fundamental rules of quaternions, a *scalar* and *positive product*, which is here found to be $= Ma^{-1}$, by XXIX., in consistency with the known theory of *elliptic motion*. The result however admits of an *interpretation*, in other cases also. It is obvious that when the centre o of force is *exterior* to the *hodograph*, the *polar* of that point divides the *circle* into *two parts*, whereof one is *concave*, and the other *convex*, towards o ; and there is no difficulty in seeing, that the *former* part corresponds to the *branch* of an *hyperbolic orbit*, which can be described under the influence of an *attracting force*: while the *latter* part answers to that *other branch* of the same complete hyperbola, whereof the description would require the force to be *repulsive*.

$$\text{XXXV.} \dots S(\tau - \kappa)(\tau - \nu) = 0; \quad \text{XXXV'.} \dots S(\tau' - \kappa)(\tau' - \nu) = 0;$$

and $\text{XXXVI.} \dots (\tau - \nu)^2 = (\tau' - \nu)^2 = -\omega^2.$

In fact it is found that

$$\text{XXXVII.} \dots \tau - \nu = \omega(\omega + \gamma)(\nu - \kappa)^{-1}; \quad \text{XXXVIII.} \dots T(\omega + \gamma) = T(\nu - \kappa);$$

and $\text{XXXIX.} \dots (\tau - \nu)(\tau - \kappa) = \omega\gamma;$

$\omega + \gamma$ being here a quaternion.

(23.) If ν' be the vector OU' of any point U' , on the polar of the point ν with respect to the circle, then changing τ to ν' , and ω to z , in XXXIV., we find this vector form (comp. (21.)) of the equation of that polar,

$$\text{XL.} \dots \nu' = \kappa + \gamma(\gamma + z)(\nu - \kappa)^{-1},$$

or, by an easy transformation,

$$\text{XLI.} \dots (\lambda^2 + \omega^2)\nu' = \lambda^2\nu + \omega^2\kappa + z\gamma(\kappa - \nu),$$

in which z is an arbitrary scalar.

(24.) If then we suppose that ν' is the intersection of the chord $\tau\tau'$ with the right line OU , the condition

$$\text{XLII.} \dots \nabla \nu' \nu = 0 \quad \text{will give} \quad \text{XLIII.} \dots z\gamma = \frac{\omega^2 \nabla \kappa \nu}{\nu^2 - S\kappa \nu};$$

but $\text{XLIV.} \dots \nabla \kappa \nu \cdot (\kappa - \nu) = \kappa S(\kappa \nu - \nu^2) + \omega S(\kappa \nu - \kappa^2);$

the coefficient then of κ , in the expanded expression for ν' , disappears as it ought to do: and we find, after a few reductions,

$$\text{XLV.} \dots \nu' = \nu \left(1 + \frac{\omega^2}{\nu^2 - S\kappa \nu} \right) = \frac{\gamma^2 - \kappa^2 + S\kappa \nu}{\nu - \nu^{-1} S\kappa \nu},$$

a result which might have been otherwise obtained, by eliminating a new scalar y between the two equations,

$$\text{XLVI.} \dots \nu' = y\nu, \quad S(y\nu - \kappa)(\nu - \kappa) = \gamma^2.$$

(25.) Introducing then two auxiliary vectors, λ , μ , such that

$$\text{XLVII.} \dots \lambda = \nu^{-1} S\kappa \nu, \quad \text{or} \quad S\kappa \nu = \nu \lambda = \lambda \nu,$$

and therefore $\text{XLVII'.} \dots \lambda - \kappa = \nu^{-1} \nabla \kappa \nu$, $S\kappa \lambda = \lambda^2$, $(\lambda - \kappa)^2 = \kappa^2 - \lambda^2$,

and $\text{XLVIII.} \dots \mu = \lambda \left(1 + \left(1 + \frac{\gamma^2 - \kappa^2}{\lambda^2} \right)^{\frac{1}{2}} \right)$, whence $\mu \parallel \lambda$, $(\mu - \kappa)^2 = \gamma^2$,

we have the very simple relation,

$$\text{XLIX.} \dots (\nu - \lambda)(\nu' - \lambda) = (\mu - \lambda)^2, \quad \text{or} \quad \text{L.} \dots LU \cdot L\nu' = LM^2,$$

if $\lambda = OL$, and $\mu = OM$. Accordingly, the point L is the foot of the perpendicular let fall from the centre π on the right line OU , while M is one of the two points m, m' of intersection of that line with the circle; so that the equation L . expresses, that the points ν, ν' are harmonically conjugate, with respect to the chord mm' , of which L is the middle point, as is otherwise evident from geometry.

(26.) The vector a of the orbit (or of position), which corresponds to the vector $r (= Da)$ of the hodograph (or of velocity), and of which the length is $Ta = r$ the distance, may be deduced from r by the equations,

$$\text{LI.} \dots a = r(\kappa - r)\gamma^{-1}, \quad \text{and} \quad \text{LII.} \dots \nabla r a = -\beta = M\gamma^{-1};$$

whence follow the expressions,

$$\text{LIII.} \dots \text{Potential} = Mr^{-1} = (\text{say}) P = Sr(\kappa - r) = Sv(\kappa - r);$$

the second expression for P being deduced from the first, by means of the relation XXXV.

(27.) The first expression LIII for P shows that the potential is equal, 1st, to the rectangle under the radius of the hodograph, and the perpendicular from the centre o of forces, on the tangent at τ to that circle; and 2nd, to the square of the tangent from the same point τ of the hodograph, to what may be called the Circle of Excentricity, namely to that new circle which has OH for a diameter. And the first of these values of the potential may be otherwise deduced from the equality (7.) of the mass M , to the product hc of the radius h of the hodograph, multiplied by the constant c of double areal velocity, or by the constant parallelogram (16.) under any two corresponding vectors.

(28.) The second expression LIII. for the potential P , corresponding to the point τ of the hodograph, may (by XXXIV., &c.) be thus transformed, with the help of a few reductions of the same kind as those recently employed:

$$\text{LIV.} \dots P = \frac{M}{r} = \frac{h^2 Sq + u\gamma Vq}{h^2 + u^2}, \quad \text{if} \quad \text{LV.} \dots q = v(\kappa - v),$$

q being thus an auxiliary quaternion; and in like manner, for the other point τ' lately considered, we have the analogous value,

$$\text{LVI.} \dots P' = \frac{M}{r'} = \frac{h^2 Sq - u\gamma Vq}{h^2 + u^2};$$

whence

$$\text{LVII.} \dots P.P' = \frac{h^2 (Sq^2 + u^2 v^2)}{h^2 + u^2};$$

and therefore,

$$\text{LVIII.} \dots \frac{r}{M} = P^{-1} = \frac{Sq + u\gamma^{-1}Vq}{Sq^2 + u^2 v^2};$$

$$\text{LIX.} \dots \frac{r'}{M} = P'^{-1} = \frac{Sq - u\gamma^{-1}Vq}{Sq^2 + u^2 v^2};$$

and finally,

$$\text{LX.} \dots \frac{2M}{r+r'} = \frac{2PP'}{P+P'} = Sq + \frac{u^2 v^2}{Sq} = v(\lambda - v) = ou.v'L.$$

(29.) In fact, the same second expression LIII. shows, that if v and v' be the feet of perpendiculars from τ and τ' on HL , then the potentials are,

$$\text{LXI.} \dots P = ou.v\tau, \quad \text{and} \quad P' = ou.v'\tau';$$

and it is easy to prove, geometrically, that the segment $v'L$ is the harmonic mean between what may be called the ordinates, τv , $\tau'v'$, to the hodographic axis HL .

(30.) If we suppose the point v to take any new but near position u , in the plane, the polar chord $\tau\tau'$, and (in general) the length u of the tangent $u\tau$, will change; and we shall have the differential relations:

$$\text{LXII.} \dots d\tau = (\tau - v)^{-1} S(\tau - \kappa) dv;$$

$$\text{LXIII.} \dots d\tau' = (\tau' - v)^{-1} S(\tau' - \kappa) dv;$$

and

$$\text{LXIII.} \dots du = u^{-1} S(\kappa - v) dv.$$

(31.) Conceiving next that u moves along the line ou , or LU , so that we may write,

$$\text{LXIV.} \dots v = (x - e')(\mu - \lambda), \quad \text{if} \quad x = \frac{LU}{LM} = \frac{LM}{LU'}, \quad \text{and} \quad e' = \frac{LO}{LM},$$

we shall have,

LXV. . . $dv = (\mu - \lambda) dx = v(x - e')^{-1} dx$, with $x > 1 > e'$,

if u be on LM prolonged, and if o be on the concave side of the arc TM' ; and thus, by LIII., the differential expressions (30.) become,

LXVI. . . $dr = (v - r)^{-1} P(x - e')^{-1} dx$; $dr' = (v - r')^{-1} P'(x - e')^{-1} dx$;

and LXVII. . . $dw = u^{-1} Sq.(x - e')^{-1} dx$, with $Sq = v(\lambda - v)$;

so that LXVIII. . . $Tdr = \frac{Pdx}{u(x - e')}$, $Tdr' = \frac{P'dx}{u(x - e')}$, if $dx > 0$.

Such then are the *lengths* of the two *elementary arcs* rr' , and $r'T'_i$ of the hodograph, intercepted between two near secants rrt' and rr',t' drawn from the pole n of the chord mm' , and having u and u' for their own poles; and we see that these *arcs* are proportional to the *potentials*, P and P' , or by LXI. to the *ordinates*, rv , $r'v'$, or finally to the lines nr , nr' : and accordingly we have the *inverse similarity* (comp. 118), of the two small triangles with n for vertex,

LXIX. . . $\Delta nr\epsilon, \alpha' nr', \epsilon'$,

as appears on inspection of the annexed Figure 86.

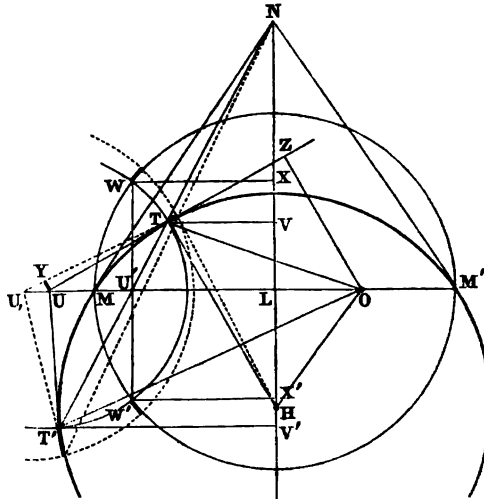


Fig. 86.

(32.) For any motion of a point, however complex, the *element* dt of time which corresponds to a given element dDa of the *hodograph*, is found by *dividing* the latter element by the *vector* D^2a of *accelerating force*; if then we denote by dt and dt' the times corresponding to the elements dr and dr' (31.), we have the expressions,

$$\text{LXX. . . } dt = M \cdot P^{-2} \cdot Tdr = \frac{Mdx}{Pu(x - e')} = \frac{rdx}{u(x - e')}$$

$$\text{LXX'. . . } dt' = M \cdot P'^{-2} \cdot Tdr' = \frac{Mdx}{P'u(x - e')} = \frac{r'dx}{u(x - e')}$$

because, for the motion here considered, the measure or quantity of the force is, by I. and LIII.,

$$\text{LXXI.} \dots TD^2\alpha = Mr^{-2} = M^{-1}P^2.$$

(33.) The *times of hodographically describing the two small circular arcs, τ, τ' and τ', τ* , are therefore *inversely* proportional to the *potentials*, or *directly* proportional to the *distances in the orbit*; and their *sum* is,

$$\text{LXXII.} \dots dt + dt' = \left(\frac{M}{P} + \frac{M}{P'} \right) \frac{u^{-1}dx}{x - e'} = \frac{(r + r') dx}{u(x - e')};$$

that is, by LX. and LXIV.,

$$\text{LXXIII.} \dots dt + dt' = \frac{2Mxdx}{u(x - e')^2 g^2} \quad \text{if} \quad \text{LXXIV.} \dots g = T(\mu - \lambda) = \overline{LM}.$$

(34.) We have also the relations,

$$\text{LXXV.} \dots u = (x^2 - 1)^{\frac{1}{2}} g, \quad \text{and} \quad \text{LXXVI.} \dots \frac{M}{a} = (1 - e'^2) g^2;$$

so that the sum of the two small times may be thus expressed,

$$\text{LXXVII.} \dots dt + dt' = \frac{2(a(1 - e'^2))^{\frac{3}{2}}}{M^{\frac{1}{2}}} \cdot \frac{(1 - e'x^{-1})^{-2} dx}{x(x^2 - 1)^{\frac{1}{2}}},$$

or finally,

$$\text{LXXVIII.} \dots dt + dt' = 2 \left(\frac{a^2(1 - e'^2)^2}{M} \right)^{\frac{1}{2}} \cdot \frac{dax}{(1 - e' \cos w)^2},$$

if $\text{LXXIX.} \dots x = \sec w$, or $w = \angle MLW$ in Fig. 86,

in which Figure $U'w$ is an ordinate of a semicircle, with the chord MM' of the hodograph for its diameter.

(35.) The two near secants (81.), from the pole π of that chord, have been here supposed to cut the half chord LM itself, as in the cited Figure 86; but if they were to cut the *other* half chord LM' , it is easy to prove that the formulæ LXXVIII. LXXIX. would still hold good, the only difference being that the angle w , or MLW , would be now *obtuse*, and its secant $x < -1$.

(36.) A circle, with u for centre, and u for radius, *cuts* the hodograph *orthogonally* in the points τ and τ' ; and in like manner a *near circle*, with u , for centre, and $u + du$ for radius, is *another orthogonal*, cutting the same hodograph in the near points τ , and τ' (31.). And by conceiving a *series* of such orthogonals, and observing that the differential expression LXXVIII. depends only on the *four scalars*, $M^{-1}a^2$, e' , w , and dax , which are all known when the *mass M* and the *five points O , L , M , u , u* , are given, so that they do not change when we retain that mass and those points, but alter the radius h of the hodograph, or the perpendicular HL let fall from its centre H on the fixed chord MM' , we see that the *sum of the times* (comp. (33.)), of *hodographically describing any two circular arcs*, such as τ, τ' and τ', τ , even if they be *not small*, but intercepted between *any two secants* from the *pole π* of the *fixed chord*, is *independent of the radius (h)*, or of the *height HL* of the centre H of the hodograph.

(37.) If then two circular hodographs, such as the two in Fig. 86, having a *common chord MM'* , which passes through, or tends towards, a *common centre of force O* , with a *common mass M* there situated, be *cut by any two common orthogonals*, the *sum of the two times* of *hodographically describing* (33.) the *two intercepted arcs* (small or large) will be the *same* for those two hodographs.

(38.) And as a case of this general result, we have the following *Theorem* of Hodographic Isochronism* (or *Synchronism*):

"If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal."

For example, in Fig. 86, we have the equation,

LXXX. . . Time of TMT' = time of WMW' .

(39.) The time of thus describing the arc TMT' (Fig. 86), if this arc be throughout concave† towards o (so that $x > 1 > e'$, as in LXV.), is expressed (comp. LXXVIII.) by the definite integral,

$$\text{LXXXI. . . Time of } TMT' = 2 \left(\frac{a^3(1-e'^2)^3}{M} \right)^{\frac{1}{2}} \int_0^{\omega} \frac{d\omega}{(1-e' \cos \omega)^2};$$

and the time of describing the remainder of the hodographic circle, if this remaining arc $T'M'T$ be throughout concave towards the centre o of force, is expressed by this other integral,

$$\text{LXXXII. . . Time of } T'M'T = 2 \left(\frac{a^3(1-e'^2)^3}{M} \right)^{\frac{1}{2}} \int_{\omega}^{\pi} \frac{d\omega}{(1-e' \cos \omega)^2}.$$

(40.) Hence, for the case of a closed orbit ($e'^2 < 1$, $e < 1$, $a > 0$), if n denote the mean angular velocity, we have the formula,

$$\begin{aligned} \text{LXXXIII. . . Periodic Time} &= \frac{2\pi}{n} = 2 \left(\frac{a^3}{M} \right)^{\frac{1}{2}} (1-e'^2)^{\frac{3}{2}} \int_0^{\pi} \frac{d\omega}{(1-e' \cos \omega)^2} \\ &= 2\pi \left(\frac{a^3}{M} \right)^{\frac{1}{2}}; \end{aligned}$$

or

$$\text{LXXXIV. . . } M = a^3 n^2, \text{ as usual.}$$

The same result, for the same case of *elliptic motion*, may be more rapidly obtained, by conceiving the chord mm' through o to be perpendicular to OH ; for, in this position of that chord, its middle point L coincides with o , and $e' = 0$ by LXIV.

(41.) In general, by LXXVI., we are at liberty to make the substitution,

$$\text{LXXXV. . . } \left(\frac{a^3(1-e'^2)^3}{M} \right)^{\frac{1}{2}} = \frac{M}{g^3}, \text{ with } g = \text{half chord of the hodograph};$$

supposing then that $e' = -1$, or placing o at the extremity m' of the chord, we have by LXXXI.,

$$\text{LXXXVI. . . Parabolic time of } TMT' = \frac{2M}{g^3} \int_0^{\omega} \frac{d\omega}{(1 + \cos \omega)^2};$$

for, when the centre of force is thus situated on the circumference of the hodographic circle, we have by (8.) the eccentricity $e = 1$, and the orbit becomes by XV. a para-

* This Theorem, in which it is understood that the common centre of force (o) is occupied by a common mass (M), was communicated to the Royal Irish Academy on the 16th of March, 1847. (See the *Proceedings* of that date, Vol. III., page 417.) It has since been treated as a subject of investigation by several able writers, to whom the author cannot hope to do justice on this subject, within the very short space which now remains at his disposal.

† Compare the Note to page 721.

without the substitution LXXXV.) is to be employed if $e' < -1$, that is, if o be on LM' prolonged; and the formula LXXXII., if $e' > 1$, $e' < \sec \omega$, that is, if o be situated between m and v .

(42.) For any law of central force, if P, P' be the points of the orbit which correspond to the points τ, τ' of the hodograph, and if Q be the point of meeting of the tangents to the orbit at P, P' , as in the annexed Figure 87, while the tangents to the orbit at τ, τ' meet as before in U , we shall have the parallelisms,

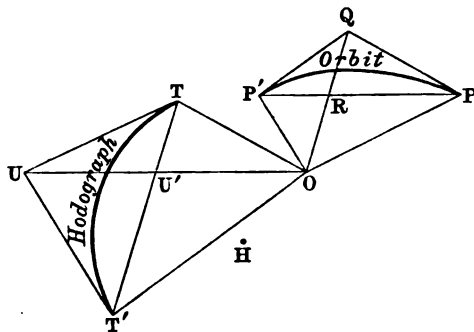


Fig. 87.

LXXXVII. . . $OP \parallel UT, OP' \parallel T'U, PQ \parallel OT, QP' \parallel OT'$;

writing then,

LXXXVIII. . . $OP = \alpha, OP' = \alpha', OT = D\alpha = \tau, OT' = D\alpha' = \tau', OU = v, OQ = \omega$,

most of which notations have occurred before, we have the equations,

LXXXIX. . . $0 = Va(r-v) = Va'(v-r) = Vr(\omega-\alpha) = Vr'(\alpha'-\omega)$;

thus XC. . . $Vav = Var = \beta = Va'r' = Va'v, \alpha' - \alpha \parallel v, PP' \parallel ou$,

and XCI. . . $Vr\omega = Vr\alpha = -\beta = Vr'\alpha' = Vr'\omega', \tau - \tau' \parallel \omega, T'T \parallel oq$.

Geometrically, the constant parallelogram (16.) under OP, OT , or under OP', OT' , is equal, by LXXXVII., to each of the four following parallelograms: I. under OP, OU ; II. under OP', OU ; III. under OQ, OT ; and IV. under OQ, OT' ; whence $PP' \parallel ou$, and $T'T \parallel oq$, as before.

(43.) The parallelism XC. may be otherwise deduced from the law of the inverse square, with recent notations, from the quaternion formulæ,

$$\text{XCII.} \dots \frac{\alpha' - \alpha}{r + r'} = \frac{u}{\lambda - v} = \frac{v - v'}{u}, \text{ in which, } \text{XCII}' \dots v' = \frac{rr' + r'r'}{r + r'},$$

and which may be obtained in various ways; whence it may also be inferred, that if s denote the length $T(\alpha' - \alpha)$ of the chord PP' of the orbit, then (comp. Fig. 86),

$$\text{XCIII.} \dots \frac{s}{r + r'} = \frac{u}{T(\lambda - v)} = \overline{UT} : \overline{UL} = \&c. = \sin w;$$

w being the same auxiliary angle as in (34.), &c.

(44.) It is easy to prove that

$$\text{XCIV.} \dots \lambda - \tau = \left(1 + \frac{u}{\gamma}\right) \frac{P}{v}, \quad \lambda - \tau' = \left(1 - \frac{u}{\gamma}\right) \frac{P}{v},$$

whence

$$\text{XCV.} \dots T \frac{r' - \lambda}{r - \lambda} = \frac{P'}{P} = \frac{r'}{r}, \quad \text{and} \quad \text{XCVI.} \dots P'^{-1}(r' - \lambda)v = K.P^{-1}(r - \lambda)v;$$

the lines LT , LT' are therefore in *length* proportional to the *potentials*, P , P' ; and their *directions* are *equally inclined* to that of ou , but at *opposite sides* of it, so that the line LU *bisects the angle* $\tau L \tau'$. Accordingly (see Fig. 86), the three points τ , L , τ' are on the *circle* (not drawn in the Figure) which has LU for diameter; so that the angles ULT' , TLU are equal to each other, as being respectively equal to the angles UTT' , $\tau\tau'U$, which the chord $\tau\tau'$ of the hodograph makes with the tangents at its extremities: the triangles TLV , $\tau'LV'$ are therefore similar, and \overline{LT} is to $\overline{LT'}$ as τv to $\tau'v'$, that is, by LXI., as P to P' , or as r to r' .

(45.) Again, calculation with quaternions gives,

$$\text{XCVII.} \dots \frac{(v - \tau)(\lambda - \tau)}{v' - \tau'} = \frac{(v - \tau')(\lambda - \tau')}{v' - \tau'} = (v - \kappa)(v - \lambda)(v - \kappa)^{-1},$$

whence

$$\text{XCVIII.} \dots T \frac{v' - \tau}{\lambda - \tau} = T \frac{v' - \tau'}{\lambda - \tau'} = T \frac{\tau - v}{\lambda - v} = \overline{UT} : \overline{UL} = \sin \omega;$$

such then is the *common ratio*, of the *segments* \overline{TV} , $\overline{V'T'}$ of the *base* $\tau\tau'$ of the triangle $\tau L \tau'$, to the adjacent *sides* \overline{LT} , $\overline{LT'}$, which are to *each other* as r to r' (44.); and because this ratio is also that of s to $r + r'$, by (48.), we have the proportion,

$$\text{XCIX.} \dots \overline{OP} : \overline{OP'} : \overline{PP'} = r : r' : s = \overline{LT} : \overline{LT'} : \overline{TT'},$$

and the formula of inverse similarity (118),

$$\text{C.} \dots \Delta LT'T \alpha' OP'.$$

Accordingly (comp. the two last Figures), the base angles $OP'F$, $OP'F'$ of the second triangle are respectively equal, by the parallelisms (42.), to the angles τOL , $\tau' OL$, and therefore, by the circle (44.), to the base angles $\tau\tau'L$, $\tau'\tau'L$, of the first triangle: but the two rotations, round O from P to P' , and round L from τ to τ' , are oppositely directed.

(46.) The investigations of the three last subarticles have not assumed any knowledge of the *form* of the *orbit* (as *elliptic*, &c.), but only the *law of attraction* according to the *inverse square*, or by (6.) the *Law of the Circular Hodograph*. And the same general principles give not only the expression LXXVI. for the constant $M\alpha^{-1}$, but also (by LX. LXIV. LXXIV. LXXIX.) this other expression,

$$\text{CI.} \dots \frac{2M}{r + r'} = (1 - e' \cos \omega) g^2; \quad \text{whence} \quad \text{CII.} \dots \frac{r + r'}{2a} = \frac{1 - e'^2}{1 - e' \cos \omega},$$

which last may be considered as a quadratic in e' , assigning two values (real or imaginary) for that scalar, when the first member of CII. and the angle ω are given; the sine of this latter angle being already expressed by XCIII.

(47.) Abstracting, then, from any *ambiguity** of solution, we see, by the definite

* That there *ought* to be some such ambiguity is evident from the consideration, that when a *focus* O , and *two points* P , P' of an *elliptic orbit* are given, it is sti-

integrals in (89.), that the *time of describing an arc PP' of an orbit*, with the law of the *inverse square*, is a *function* (comp. (86.)) *of the three ratios*,

$$\text{CIII.} \dots \frac{a^3}{M}, \quad \frac{r+r'}{a}, \quad \frac{s}{r+r'};$$

which is a form of *Lambert's Theorem*, but presents itself *here* as *deduced* from the recently stated *Theorem of Hodographic Isochronism* (88.), *without the employment of any property of conic sections.*

(48.) The differential equation I. of the present relative motion may be thus written (comp. 418, I., and generally the preceding Series 418):

$$\text{CIV.} \dots S \cdot D^2 a \delta a + \delta P = 0, \quad \text{whence} \quad \text{CV.} \dots T = P + H,$$

as in 418, X., if we now write,

$$\text{CVI.} \dots T = -\frac{1}{2} D a^2 = -\frac{1}{2} r^2, \quad \text{and} \quad \text{CVII.} \dots H = \frac{M}{2a};$$

in fact (by LIII., comp. (20.) (21.)),

$$\text{CVIII.} \dots -2H = 2(P - T) = 2P + r^2 = \kappa^2 - \gamma^2 = \frac{M}{a}.$$

(49.) Integrating CIV. by parts, &c., and writing (as in 418, XII. XXII.),

$$\text{CIX.} \dots F = \int_0^t (T + P) dt, \quad \text{and} \quad \text{CX.} \dots V = \int_0^t 2T dt,$$

so that *F* may again be called the *Principal Function* and *V* the *Characteristic Function* of the motion, we have the variations,

$$\text{CXI.} \dots \delta F = S r \delta a - S r' \delta a' - H \delta t; \quad \text{CXII.} \dots \delta V = S r \delta a - S r' \delta a' + t \delta H;$$

in which *a*, *a'* (instead of *a*₀, *a*) denote now what may be called the *initial* and *final vectors* (or, or') of the *orbit*; whence follow the *partial derivatives*,

$$\text{CXIII.} \dots D_a F = D_a V = r; \quad \text{CXIII'.} \dots D_{a'} F = D_{a'} V = -r';$$

$$\text{CXIV.} \dots (D_t F) = -H; \quad \text{and} \quad \text{CXV.} \dots D_H V = t;$$

F being here a scalar function of *a*, *a'*, *t*, while *V* is a scalar function of *a*, *a'*, *H*, if *M* be treated as given.

(50.) The two *vectors* *a*, *a'* can enter into these two *scalar functions*, only through their dependent scalars *r*, *r'*, *s* (comp. 418, (17.)); but

$$\text{CXVI.} \dots \delta r = -r^{-1} S a \delta a, \quad \delta r' = -r'^{-1} S a' \delta a', \quad \delta s = -s^{-1} S (a' - a) (\delta a' - \delta a);$$

confining ourselves then, for the moment, to the function *V*, and observing that we have by CXII. the formula,

$$\text{CXVII.} \dots S (r \delta a - r' \delta a') = D_r V \cdot \delta r + D_{r'} V \cdot \delta r' + D_s V \cdot \delta s,$$

in which the variations δa , $\delta a'$ are arbitrary, we find the expressions,

$$\text{CXVIII.} \dots r = -a r^{-1} D_r V + (a' - a) s^{-1} D_s V;$$

$$\text{CXVIII'.} \dots r' = +a' r'^{-1} D_{r'} V + (a' - a) s^{-1} D_s V;$$

permitted to conceive the motion to be performed along *either* of the *two elliptic arcs*, *PP'*, *P'P*, which together make up the whole periphery. But into details of this kind we cannot enter here.

which give these others,

$$\text{CXIX.} \dots D_r V = rV(\alpha' - \alpha) r : V\alpha\alpha';$$

$$\text{CXIX}'. \dots D_r' V = r'V(\alpha - \alpha') r' : V\alpha\alpha';$$

and

$$\text{CXX.} \dots D_s V = s\beta : V\alpha\alpha',$$

because

$$V\alpha r = V\alpha' r' = \beta.$$

(51.) But, by XCII.,

$$\text{CXXI.} \dots rr + r'r' = (r+r')v' \parallel v \parallel \alpha' - \alpha,$$

the chord rr' of the hodograph, in Figures 86, 87, being divided at v' into segments rv' , $v'r'$, which are inversely as the distances r , r' , or as the lines or , or' in the orbit; we have therefore the partial differential equation,

$$\text{CXXII.} \dots D_r V = D_r' V, \quad \text{and similarly,} \quad \text{CXXIII.} \dots D_r F = D_r' F;$$

so that *each* of the two functions, F and V , depends on the *distances* r , r' , only by depending on their *sum*, $r + r'$.

(52.) Hence, if for greater generality we now treat M as *variable*, the *Principal Function* F , and therefore by CXIV. its partial derivative $H = - (D_t F)$, are functions of the *four* scalars,

$$\text{CXXIV.} \dots r + r', \quad s, \quad t, \quad \text{and} \quad M.$$

(53.) And in like manner, the *Characteristic Function* (or *Action-Function*) V , and its partial derivative (by CXV.) the *Time*, $t = D_H V$, may be considered as functions of this *other system* of four scalars (comp. (47.)),

$$\text{CXXV.} \dots r + r', \quad s, \quad H, \quad \text{and} \quad M;$$

no knowledge whatever being here assumed, of the form or properties of the *orbit*, but only of the *law* of attraction.

(54.) But this dependence of the *time*, t , on the four scalars CXXV., is a new form of *Lambert's Theorem* (47.); which celebrated theorem is thus obtained in a new way, by the foregoing *quaternion analysis*.

(55.) Squaring the equations CXVIII. CXVIII', attending to the relation CXXII., and changing signs, we get these new partial differential equations,

$$\text{CXXVI.} \dots 2P + 2H = (D_r V)^2 + (D_s V)^2 + \frac{r^2 - r'^2 + s^2}{rs} D_r V. D_s V;$$

$$\text{CXXVI}'. \dots 2P' + 2H = (D_r' V)^2 + (D_s V)^2 + \frac{r'^2 - r^2 + s^2}{r's} D_r' V. D_s V;$$

because CXXVII. $\dots \alpha^2 = -r^2$, $\alpha'^2 = -r'^2$, $(\alpha' - \alpha)^2 = -s^2$.

Hence, by merely algebraical combinations (because $P = Mr^{-1}$, and $P' = Mr'^{-1}$), we find:

$$\text{CXXVIII.} \dots \frac{1}{2} ((D_r V)^2 + (D_s V)^2) = H + \frac{M}{r + r' + s} + \frac{M}{r + r' - s};$$

$$\text{CXXIX.} \dots D_r V. D_s V = \frac{M}{r + r' + s} - \frac{M}{r + r' - s};$$

$$\text{CXXX.} \dots (D_r V + D_s V)^2 = 2H + \frac{4M}{r + r' + s} = M \left(\frac{4}{r + r' + s} - \frac{1}{a} \right);$$

$$\text{CXXX}'. \dots (D_r' V - D_s V)^2 = 2H + \frac{4M}{r + r' - s} = M \left(\frac{4}{r + r' - s} - \frac{1}{a} \right).$$

(56.) But, by CXII. CXVII. CXXII., we have the variation,

$$\text{CXXXI.} \dots \delta V - t \delta H = \frac{1}{2} (D_r V + D_s V) \delta (r + r' + s) + \frac{1}{2} (D_r V - D_s V) \delta (r + r' - s);$$

and the function V vanishes with t , and therefore with s , at least at the commencement of the motion; whence it is easy to infer the expressions,*

$$\text{CXXXII.} \dots V = \int_{-s}^s \left(\frac{M}{r + r' + s} + \frac{H}{2} \right)^{\frac{1}{2}} ds = \int_{-s}^s \left(\frac{M}{r + r' + s} - \frac{M}{4a} \right)^{\frac{1}{2}} ds;$$

$$\text{CXXXIII.} \dots t = \frac{1}{2} \int_{-s}^s \left(\frac{M}{r + r' + s} + \frac{H}{2} \right)^{-\frac{1}{2}} ds = \frac{1}{2} \int_{-s}^s \left(\frac{4M}{r + r' + s} - \frac{M}{a} \right)^{-\frac{1}{2}} ds.$$

As a verification,† when t and s are small, and therefore r' nearly = r , we have thus the approximate values,

$$\text{CXXXIV.} \dots V = (2P + 2H)^{\frac{1}{2}} s = (2T)^{\frac{1}{2}} s = 2Tt;$$

$$\text{CXXXV.} \dots t = (2P + 2H)^{-\frac{1}{2}} s = (2T)^{-\frac{1}{2}} s;$$

in which s may be considered to be a *small arc* of the orbit, and $(2T)^{\frac{1}{2}}$ the *velocity* with which that arc is described.

(57.) Some not inelegant constructions, deduced from the theory of the hodograph, might be assigned for the case of a *closed orbit*, to represent the *eccentric* and *mean anomalies*; but whether the orbit be closed or not, the arc TM' of the *hodographic circle*, in Fig. 86, represents the arc of *true anomaly* described: for it subtends at the hodographic centre H an angle TH' , which is equal to the *angular motion* PO' in the orbit.

(58.) We may add that, whatever the *special form* of the orbit may be, the equations CXVIII. CXVIII'. give, by CXXII.,

$$\text{CXXXVI.} \dots r' - r = (U\alpha' + U\alpha) D_r V;$$

from which it follows that the *chord* TM' of the hodograph is *parallel* to the *bisector* of the angle PO' in the orbit: and therefore, by XCI., that this angle is bisected by OQ in Fig. 87, so that the *segments* PE , RP' , in that Figure, of the *chord* PP' of the orbit, are *inversely proportional* to the segments TU' , $\text{U}'\text{T}'$ of the *chord* TT' of the *hodograph*.

(59.) We arrive then thus, in a new way, and as a new verification, at this known theorem: that *if two tangents* (QP , QP') *to a conic section be drawn from*

* Expressions by definite integrals equivalent to these, for the *action* and *time* in the relative motion of a binary system, were deduced by the present writer, but by an entirely different analysis, in the *First Essay*, &c., already cited, and will be found in the *Phil. Trans.* for 1834, Part II., pages 285, 286. It is supposed that the radical in CXXXIII. does not become infinite within the extent of the integration; if it did so become, transformations would be required, on which we cannot enter here.

† An analogous verification may be applied to the definite integral LXXXI.; in which however it is to be observed that *both* $r + r'$ and s *vary*, along with the variable w : whereas, in the recent integrals CXXXII. CXXXIII., $r + r'$ is treated as *constant*.

any common point (q), they subtend equal angles at a focus (o), whatever the special form of the conic may be.

(60.) And although, in several of the preceding sub-articles, *geometrical constructions* have been used only to *illustrate* (and so to *confirm*, if confirmation were needed) *results* derived through *calculation with quaternions*; yet the eminently *suggestive* nature of the present Calculus enables us, in this as in many other questions, to *dispense with its own processes*, when once they have indicated a definite train of *geometrical investigation*, to serve as their substitute.

(61.) Thus, after having in any manner been led to perceive that, for the motion above considered, the *hodograph* is a *circle** (5.), of which the *radius* HT is equal (7.) to the attracting *mass* M , divided by the constant *parallelogram* (16.) under the vectors OP , OT of position and velocity, in the recent Figures 86 and 87, which parallelogram is equal to the *rectangle* under the distance OP in the orbit, and the *perpendicular* OZ let fall from the centre o of force on the tangent UT to the hodograph, we see *geometrically* that the *potential* P , or the mass divided by the distance, for the point P of the orbit corresponding to the point T of the hodograph, is equal (as in (27.)) to the rectangle under HT and OZ , and therefore, by the similar triangles HTV , VOZ , to the rectangle under OU and TV (as in (29.)).

(62.) In like manner, the three potentials corresponding to the second point T' of the first hodograph, and to the points w and w' of the second hodograph, in Fig. 86, are respectively equal to the rectangles under the same line OU , and the three other perpendiculars $T'v'$, wx , $w'x'$, on what we have called (29.) the *hodographic axis*, HL ; so that, for these *two pairs of points*, in which the *two circular hodographs*, with a *common chord* MM' , are cut by a *common orthogonal* with U for centre, the *four potentials* are directly proportional to the *four hodographic ordinates* (29.).

(63.) And because the force (Mr^{-2}) is equal to the *square of the potential* (Mr^{-1}), divided by the *mass* (M), the *four forces* are directly as the *squares of the four ordinates* corresponding; each force, when divided by the square of the corresponding hodographic ordinate, giving the constant or *common quotient*,

$$CXXXVII. . . \overline{OU}^2 : M.$$

(64.) It has been already seen (31.) to be a geometrical consequence of the two pairs of similar triangles, NRT , $NT'T'$, and NTV , $NT'v'$, that the *two small arcs* of the *first hodograph*, near T and T' , intercepted between two near secants from the pole π of the *fixed chord* MM' , or between two near orthogonal circles, with U and U' for centres, are proportional to the *two ordinates*, TV , $T'v'$.

(65.) Accordingly, if we draw, as in Fig. 86, the *near radius* (represented by a

* This follows, among other ways, from the general value XXVI. for the *radius of curvature* of the hodograph, with *any law of central force*; which value was *geometrically deduced*, as stated in the Note to page 720, compare the Note to page 719, by the present writer, in a Paper read before the Royal Irish Academy in 1846, and published in their *Proceedings*. In fact, that *general expression* for the radius of hodographic curvature may be obtained with great facility, by dividing the *element fit* of the hodograph (in which f denotes the force), by the corresponding element $cr^{-2}dt$ of angular motion in the orbit.

dotted line from H) of the first hodograph, and also the *small perpendicular* UX , erected at the centre U of the first orthogonal to the tangent UT , and terminated in X by the tangent from the near centre U , the two new pairs of similar triangles, THT , UTX , and THV , UVX , give the proportion,

$$\text{CXXXVIII.} \dots \overline{TT} : \overline{TV} = \overline{UU} : \overline{UT};$$

which not merely confirms what has just been stated (64.), for the case of the *first* hodograph, but proves that the *four small arcs*, of the *two circular hodographs* in Fig. 86, intercepted between the two near orthogonals, are directly proportional to the *four ordinates* already mentioned.

(66.) But the *time* of describing any small hodographic arc is the quotient (82.) of that *arc* divided by the *force*; and therefore, by (68.), (65.), the *four small times* are *inversely proportional* to the *four ordinates*. And the *harmonic mean* $U'L$ between the *two ordinates* TV , $T'V'$ of the *first* hodograph, does not vary when we pass to the *second*, or to *any other hodograph*, with the *same fixed chord* MM' , and the *same orthogonal circles*; it follows then, *geometrically*, that the *sum* (33.) of the *two small times* is the *same*, in any *one* hodograph as in any *other*, under the conditions above supposed: and that this sum is equal to the expression,

$$\text{CXXXIX.} \dots \frac{2M \cdot \overline{UU'}}{\overline{OU}^2 \cdot \overline{UT} \cdot \overline{U'L}} = \frac{2M \cdot \overline{UU'} \cdot \overline{UL}}{\overline{OU}^2 \cdot \overline{LM}^2 \cdot \overline{UT}}$$

which agrees with the formula LXXIII.

(67.) On the whole, then, it is found that the *Theorem of Hodographic Isochronism* (38.) admits of being *geometrically* proved*, although by processes suggested (60.) by quaternions: and sufficient *hints* have been already given, in connexion with Figure 87, as regards the *geometrical passage* from *that* theorem to the well-known *Theorem of Lambert*, without necessarily employing any property of *conic sections*.

420. As a *fifth specimen*, we shall deduce by quaternions an equation, which is adapted to assist in the determination of the *distance* of a *comet*, or *new planet*, from the *earth*.

(1.) Let M be the mass of the sun, or (somewhat more exactly) the sum of the masses of sun and earth; and let a and ω be the heliocentric vectors of earth and comet. Write also,

$$I \dots T\alpha = r, \quad T\omega = w, \quad T(\omega - \alpha) = z, \quad U(\omega - \alpha) = \rho,$$

so that r and w are the distances of earth and comet from the sun, while z is their distance from each other, and ρ is the unit-vector, directed from earth to comet. Then (comp. 419, I.),

* It appears from an unprinted memorandum, to have been nearly thus that the author orally deduced the theorem, in his communication of March, 1847, to the Royal Irish Academy; although, as usually happens in cases of invention, his own previous processes of investigation had involved principles and methods, of a much less simple character.

$$\begin{aligned} & \text{II. . . } D^2 a = -Mr^{-3}a, \quad D^2 \omega = -Mw^{-3}\omega, \\ \text{and} \quad & \text{III. . . } D^2 x\rho = D^2(\omega - a) = M(r^{-3} - w^{-3})a - Mxw^{-3}\rho, \\ \text{with} \quad & \text{IV. . . } w^2 = -(\alpha + x\rho)^2 = r^2 + x^2 - 2zS\alpha\rho. \end{aligned}$$

(2.) The vector α , with its tensor r , and the mass M , are given by the theory of the earth (or sun); and ρ , $D\rho$, $D^2\rho$ are deduced from three (or more) near observations of the comet; operating then on III. with $S.\rho D\rho$, we arrive at the formula,

$$\text{V. . . } \frac{S\rho D\rho D^2\rho}{S\rho D\rho U\alpha} = \frac{r}{z} \left(\frac{M}{r^3} - \frac{M}{w^3} \right);$$

which becomes by IV., when cleared of fractions and radicals, and divided by z , an algebraical equation of the seventh degree, whereof one root is the sought distance* x , of the comet (or planet) from the earth.

421. As a *sixth specimen*, we shall indicate a method, suggested by quaternions, of developing and geometrically decomposing the disturbing force of the sun on the moon, or of a relatively superior on a relatively inferior planet.

(1.) Let a , σ be the geocentric vectors of moon and sun; r , s their geocentric distances ($r = T\alpha$, $s = T\sigma$); M the sum of the masses of earth and moon; and S the mass of the sun; then the differential equation of motion of the moon about the earth may be thus written (comp. 418, 419),

$$\text{I. . . } D^2 a = M.\phi a + S.(\phi\sigma - \phi(\sigma - a)),$$

if D be still the mark of derivation relatively to the time, and

$$\text{II. . . } \phi a = \phi(a) = a^{-1}T a^{-1};$$

so that ϕa is here a *vector-function* of a , but not a *linear* one.

(2.) If we confine ourselves to the term $M\phi a$, in the second member of I., we fall back on the equation 419, I., and so are conducted anew to the laws of *undisturbed relative elliptic motion*.

(3.) If we denote the *remainder* of that second member by η , then η may be called the *Vector of Disturbing Force*; and we propose now to *develop* this vector, according to *descending powers* of $T(\sigma : a)$, or according to *ascending powers* of the *quotient* $r : s$, of the *distances* of moon and sun from the earth.

(4.) The expression for that vector may be thus transformed :

$$\begin{aligned} \text{III. . } \text{Vector of Disturbing Force} &= \eta = D^2 a - M\phi a \\ &= Ss^{-1}\sigma^{-1} \{ 1 - (1 - a\sigma^{-1})^{-1} T(1 - a\sigma^{-1})^{-1} \} \\ &= Ss^{-1}\sigma^{-1} \{ 1 - (1 - a\sigma^{-1})^{-\frac{3}{2}} (1 - \sigma^{-1}a)^{-\frac{1}{2}} \} \\ &= Ss^{-1}\sigma^{-1} \left\{ 1 - \left(1 + \frac{3}{2} a\sigma^{-1} + \frac{3.5}{2.4} (a\sigma^{-1})^2 + \dots \right) \left(1 + \frac{1}{2} \sigma^{-1}a + \frac{1.3}{2.4} (\sigma^{-1}a)^2 + \dots \right) \right\}; \end{aligned}$$

* Compare the equation in the *Mécanique Céleste* (Tom. I., p. 241, new edition, Paris, 1848). Laplace's rule for determining, by inspection of a globe, which of the two bodies is the nearer to the sun, results at once from the formula V.

that is,

$$\text{IV.} \dots \eta = \eta_1 + \eta_2 + \eta_3 + \&c.,$$

$$\text{if} \quad \text{V.} \dots \eta_1 = -S\sigma^{-1}\sigma^{-1} \left(\frac{1}{3}\sigma^{-1}\alpha + \frac{2}{3}\alpha\sigma^{-1} \right) = \frac{S}{2\sigma^2} (\alpha + 3\sigma\alpha\sigma^{-1}) = \eta_{1,1} + \eta_{1,2};$$

$$\text{VI.} \dots \eta_2 = \frac{3Sr^2}{8\sigma^5} (\alpha\sigma\alpha^{-1} + 2\sigma + 5\sigma\alpha\sigma^{-1}\sigma^{-1}) = \eta_{2,1} + \eta_{2,2} + \eta_{2,3}; \&c.$$

the *general term** of this development being easily assigned.

(5.) We have thus a *first group* of two component and disturbing forces, which are of the same order as $\frac{Sr}{\sigma^2}$; a *second group* of three such forces, of the same order as $\frac{Sr^2}{\sigma^4}$; a *third group* of four forces, and so on.

(6.) The *first component* of the *first group* has the following tensor and versor,

$$\text{VII.} \dots T\eta_{1,1} = \frac{Sr}{2\sigma^2},$$

$$U\eta_{1,1} = U\alpha;$$

it is therefore a purely *ablative force* MN, acting along the moon's geocentric vector EM prolonged, as in the annexed Figure 88.

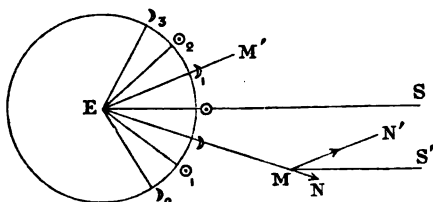


Fig. 88.

(7.) The *second component* MN', of the same first group, has an exactly *triple intensity*, $\overline{MN'} = 3\overline{MN}$; and its *direction* is such that the *angle* MNM', between these two forces of the first group, is *bisected* by a line MS' from the moon, which is *parallel* to the sun's geocentric vector ES.

(8.) If then we conceive a line EM' from the earth, having the same direction as the last force MN', this new line will meet the heavens in what may be called for the moment a *fictitious moon* D₁, such that the arc DD₁ of a *great circle*, connecting it with the *true moon* D in the heavens, shall be *bisected* by the sun ☉, as represented in Fig. 88.

(9.) Proceeding to the *second group* (5.), we have by VI. for the *first component* of this group,

$$\text{VIII.} \dots T\eta_{2,1} = \frac{3Sr^2}{8\sigma^4}, \quad U\eta_{2,1} = U\alpha\sigma\alpha^{-1} = \frac{\alpha U\sigma}{\alpha};$$

a line from the earth, parallel to this new force, meets therefore the heavens in what may be called a *first fictitious sun*, ☉₁, such that the arc of a great circle, ☉☉₁, connecting it with the true sun, is *bisected* by the moon D, as in the same Fig. 88.

* Such a general term was in fact assigned and interpreted in a communication of June 14, 1847, to the Royal Irish Academy (*Proceedings*, Vol. III., p. 514); and in the *Lectures*, page 616. The development may also be obtained, although less easily, by *Taylor's Series* adapted to quaternions. Compare pp. 427, 428, 480, 431 of the present work; and see page 382, &c., for the interpretation of such symbols as $\sigma\alpha\sigma^{-1}$, $\alpha\sigma\alpha^{-1}$,

(10.) The *second* component force, of the same second group, has an *intensity* exactly *double* that of the *first* ($T\eta_{2,2} = 2T\eta_{2,1}$); and in *direction* it is parallel to the sun's geocentric vector zs , so that a line drawn in its direction from the earth would meet the heavens in the place of the sun \odot .

(11.) The *third* component of the present group has an *intensity* which is precisely *five-fold* that of the *first* component ($T\eta_{2,3} = 5T\eta_{2,1}$); and a line drawn in its direction from the earth meets the heavens in a *second fictitious sun* \odot_2 , such that the arc $\odot_1 \odot_2$, connecting these *two* fictitious suns, is *bisected* by the *true sun* \odot .

(12.) There is no difficulty in extending this analysis, and this interpretation, to *subsequent groups* of component disturbing forces, which forces *increase in number*, and *diminish in intensity*, in passing from any one group to the next; their *intensities*, for each *separate* group, bearing *numerical ratios* to each other, and their *directions* being connected by simple *angular relations*.

(13.) For example, the *third* group consists (5.) of *four small forces*, $\eta_{3,1} \dots \eta_{3,4}$, of which the *intensities* are represented by $\frac{Sr^3}{16s^2}$, multiplied respectively by the four

whole numbers, 5, 9, 15, and 35; and which have *directions* respectively parallel to lines drawn from the earth, towards a *second fictitious moon* \mathcal{D}_2 , the true moon, the *first fictitious moon* \mathcal{D}_1 (8.), and a *third fictitious moon* \mathcal{D}_3 ; these *three* fictitious moons, like the *two* fictitious suns lately considered, being all situated in the *momentary plane* of the *three bodies* \mathcal{M} , \mathcal{M} , s : and the *three celestial arcs*, $\mathcal{D}_2\mathcal{D}_1$, $\mathcal{D}_1\mathcal{D}_3$, being each equal to double the arc $\mathcal{D}\odot$ of apparent *elongation* of sun from moon in the heavens, as indicated in the above cited Fig. 88.

(14.) An exactly similar method may be employed to develop or decompose the disturbing force of one *planet* on another, which is nearer than it to the sun; and it is important to observe that no supposition is here made, respecting any *smallness* of *eccentricities* or *inclinations*.

422. As a *seventh specimen* of the physical application of quaternions, we shall investigate briefly the construction and some of the properties of *Fresnel's Wave Surface*, as deductions from his principles or hypotheses* respecting light.

(1.) Let ρ be a *Vector of Ray-Velocity*, and μ the corresponding *Vector of Wave-Slowness* (or *Index-Vector*), for propagation of light from an origin o , within a biaxial crystal; so that

$$\text{I.} \dots S\mu\rho = -1; \quad \text{II.} \dots S\mu\delta\rho = 0; \quad \text{and therefore} \quad \text{III.} \dots S\rho\delta\mu = 0,$$

* The present writer desires to be understood as not expressing any opinion of his own, respecting these or any rival hypotheses. In the next Series (423), as an *eighth specimen* of application, he proposes to deduce, from a *quite different set of physical principles respecting light*, expressed however still in the language of the present Calculus, Mac Cullagh's Theorem of the *Polar Plane*; intending then, as a *ninth and final specimen*, to give briefly a quaternion transformation of a celebrated equation in partial differential coefficients, of the first order and second degree, which occurs in the theory of *heat*, and in that of the *attraction of spheroids*.

if $\delta\rho$ and $\delta\mu$ be any infinitesimal variations of the vectors ρ and μ , consistent with the scalar equations (supposed to be as yet unknown), of the *Wave-Surface* and its *Reciprocal* (with respect to the unit-sphere round o), namely the *Surface of Wave-Slowness*, or (as it has been otherwise called) the *Index*-Surface*: the velocity of light in a vacuum being here represented by unity.

(2.) The variation $\delta\rho$ being next conceived to represent a small *displacement*, tangential to the wave, of a particle of ether in the crystal, it was supposed by Fresnel that such a displacement $\delta\rho$ gave rise to an *elastic force*, say $\delta\varepsilon$, *not generally* in a direction exactly *opposite* to that displacement, but still a *function* thereof, which function is of the kind called by us (in the Sections III. ii. 6, and III. iii. 7) *linear*, *vector*, and *self-conjugate*; and which there will be a convenience (on account of its connexion with certain *optical constants*, a , b , c) in denoting here by $\phi^{-1}\delta\rho$ (instead of $\phi\delta\rho$): so that we shall have the two converse formulæ,

$$\text{IV.} \dots \delta\rho = \phi\delta\varepsilon; \quad \text{V.} \dots \delta\varepsilon = \phi^{-1}\delta\rho.$$

(3.) The *ether* being treated as *incompressible*, in the theory here considered, so that the *normal component* $\mu^{-1}\mathcal{S}\mu\delta\varepsilon$ of the elastic force may be neglected, or rather suppressed, there remains only the *tangential component*,

$$\text{VI.} \dots \mu^{-1}\mathcal{V}\mu\delta\varepsilon = \delta\varepsilon - \mu^{-1}\mathcal{S}\mu\delta\varepsilon,$$

as regulating the *motion*, tangential to the wave, of a disturbed and *vibrating particle*.

(4.) If then it be admitted that, for the propagation of a *rectilinear vibration*, tangential to a wave of which the velocity is $T\mu^{-1}$, the *tangential force* (3.) must be *exactly opposite* in direction to the *displacement* $\delta\rho$, and *equal* in quantity to that displacement multiplied by the *square* ($T\mu^{-2}$) of the *wave-velocity*, we have, by V. and VI., the equation,

$$\text{VII.} \dots \phi^{-1}\delta\rho - \mu^{-1}\mathcal{S}\mu\delta\varepsilon = \mu^{-2}\delta\rho, \quad \text{or} \quad \text{VIII.} \dots \delta\rho = (\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}\mathcal{S}\mu\delta\varepsilon;$$

combining which with II., we obtain at once this *Symbolical Form* of the scalar equation of the *Index Surface*,

$$\text{IX.} \dots 0 = \mathcal{S}\mu^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1};$$

or by an easy transformation,

$$\text{X.} \dots 1 = \mathcal{S}\mu\phi^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1};$$

or finally,

$$\text{XI.} \dots 1 = \mathcal{S}\mu(\mu^2 - \phi)^{-1}\mu;$$

* This brief and expressive name was proposed by the late Prof. Mac Cullagh (Trans. R. I. A., Vol. XVIII., Part I., page 38), for that *reciprocal* of the wave-surface which the present writer had previously called the *Surface of Components of Wave-Slowness*, and had employed for various purposes: for instance, to pass from the *conical cusps* to the *circular ridges* of the *Wave*, and so to establish a geometrical connexion between the theories of the *two conical refractions*, *internal* and *external*, to which his own methods had conducted him (Trans. R. I. A., Vol. XVII., Part I., pages 125-144). He afterwards found that the same Surface had been otherwise employed by M. Cauchy (*Exercices de Mathématiques*, 1830 p. 36), who did not seem however to have perceived its *reciprocal relation* to the *Wave*.

while the *direction* of the vibration $\delta\rho$, for any given tangent plane to the wave, is determined *generally* by the formula VIII.

(5.) That formula for the displacement, combined with the expression V. for the elastic force resulting, gives

$$\text{XII.} \dots \delta\rho = -\phi v S\mu\delta\epsilon, \text{ and } \text{XIII.} \dots \delta\epsilon = -v S\mu\delta\epsilon,$$

$$\text{if } \text{XIV.} \dots (\phi - \mu^2)v = \mu, \text{ or } \text{XV.} \dots v = (\phi - \mu^2)^{-1}\mu,$$

v being thus an auxiliary vector; and because the equation XI. of the index surface gives,

$$\text{XVI.} \dots S\mu v = -1, \text{ while } \text{XVII.} \dots \nabla v\delta\epsilon = 0, \text{ by XIII.},$$

it follows that the vector v , if drawn like ρ and μ from o , *terminates on the tangent plane to the wave*, and is *parallel to the direction of the elastic force*.

(6.) The equations XIV. XVI. give,

XVIII. $\dots \mu^2 v^2 - Sv\phi v = 1$, whence XIX. $\dots v^2 S\mu\delta\mu = S\mu\delta v = -Sv\delta\mu$, because $\delta S\mu v = 0$, by XVI., and $\delta Sv\phi v = 2S(\phi v.\delta v)$, by the self-conjugate property of ϕ ; comparing then XIX. with III., we see that $\pm\rho$ (as being \perp every $\delta\mu$) has the direction of $\mu + v^{-1}$, and therefore, by I. and XVI., that we may write,

$$\text{XX.} \dots \rho^{-1} = -\mu - v^{-1}; \quad \text{XXI.} \dots \rho^{-2} = \mu^2 - v^{-2}; \quad \text{XXII.} \dots S\rho v = 0;$$

which last equation shows, by (5.), that *the ray is perpendicular* (on Fresnel's principles) *to the elastic force* $\delta\epsilon$, produced by the displacement $\delta\rho$.

(7.) The equations XX. and XXI. show by XIV. that

$$\text{XXIII.} \dots (\rho^{-2} - \phi)v = \rho^{-1}, \text{ whence } \text{XXIV.} \dots v = (\rho^{-2} - \phi)^{-1}\rho^{-1};$$

we have therefore, by XXII., the following *Symbolical Form* (comp. (4.)) *of the Equation of the Wave Surface*,

$$\text{XXV.} \dots 0 = S\rho^{-1}(\phi - \rho^{-2})^{-1}\rho^{-1};$$

or, by transformations analogous to X. and XI.,

$$\text{XXVI.} \dots 1 = S\rho\phi(\phi - \rho^{-2})^{-1}\rho^{-1}; \quad \text{XXVII.} \dots 1 = S\rho(\rho^2 - \phi^{-1})^{-1}\rho;$$

and we see that we can *return* from each *equation of the wave*, to the corresponding *equation of the index surface*, by merely changing ρ to μ , and ϕ to ϕ^{-1} : but this result will soon be seen to be included in one more general, which may be called the *Rule of the Interchanges*.

(8.) The equation XXV. may also be thus written,

$$\text{XXVIII.} \dots S\rho(\phi - \rho^{-2})^{-1}\rho = 0;$$

but under this last form it coincides with the equation 412, XLI.; hence, by 412, (19.), the *Wave Surface* may be *derived from the auxiliary or Generating Ellipsoid*,

$$\text{XXIX.} \dots S\rho\phi\rho = 1,$$

by the following *Construction*, which was in fact assigned by Fresnel* himself, but as the result of far more complex calculations:—*Cut the ellipsoid* (abc) *by an arbitrary plane through its centre, and at that centre erect perpendiculars to that plane, which shall have the lengths of the semi-axes of the section; the locus of the extremities of the perpendiculars so erected will be the sought wave surface.*

* See Sir John F. W. Herschel's *Treatise on Light*, in the *Encyclopædia Metropolitana*, page 645, Art. 1017.

(9.) And we see, by IX., that the *Index Surface* may be derived, by an exactly similar construction, from that *Reciprocal Ellipsoid*, of which the equation is, on the same plan,

$$\text{XXX.} \dots S\rho\phi^{-1}\rho = 1.$$

(10.) If the scalar equations, XXVII. and XI., of the wave and index surface, be denoted by the abridged forms,

$$\text{XXXI.} \dots f\rho = 1, \text{ and } \text{XXXII.} \dots F\mu = 1,$$

then the relations I. II. III. enable us to infer the expressions (comp. the notation in 418, (2.)),

$$\text{XXXIII.} \dots \mu = \frac{-D\rho f\rho}{S\rho D\rho f\rho}; \quad \text{XXXIV.} \dots \rho = \frac{-D\mu F\mu}{S\mu D\mu F\mu};$$

in which (comp. 412, (86.), and the Note to that sub-article),

$$\begin{aligned} \text{XXXV.} \dots \frac{1}{2}D\rho f\rho &= (\rho^2 - \phi^{-1})^{-1}\rho - \rho S\rho(\rho^2 - \phi^{-1})^{-2}\rho = -\omega - \omega^2\rho, \\ \text{and } \text{XXXVI.} \dots \frac{1}{2}D\mu F\mu &= (\mu^2 - \phi)^{-1}\mu - \mu S\mu(\mu^2 - \phi)^{-2}\mu = -v - v^2\mu; \end{aligned}$$

v being the same auxiliary vector XV. as before, and ω being a new auxiliary vector, such that (by XXIV. XXVII. and IX. XV.),

$$\begin{aligned} \text{XXXVII.} \dots \omega &= (\phi^{-1} - \rho^2)^{-1}\rho = \phi v; & \text{XXXVIII.} \dots S\rho\omega &= -1; \\ \text{XXXIX.} \dots S\mu\omega &= 0; \end{aligned}$$

whence also $\omega \parallel \delta\rho$ by XII., so that (comp. (5.)) if ω be drawn (like ρ , μ , and v) from the point o , *this new vector terminates on the tangent plane to the index surface, and is parallel to the displacement on the wave*; also $\delta\rho : \delta\epsilon = \omega : v$.

(11.) Hence, by XXXIII. XXXV. XXXVIII.,

$$\text{XL.} \dots \mu = \frac{\omega + \omega^2\rho}{1 - \omega^2\rho^2} = \frac{\omega^{-1} + \rho}{\omega^{-2} - \rho^2} = -(\omega^{-1} + \rho)^{-1}, \text{ or } \text{XLI.} \dots -\mu^{-1} = \rho + \omega^{-1};$$

and similarly, by XXXIV. XXXVI. and XVI.,

$$\text{XLII.} \dots \rho = \frac{v + v^2\mu}{1 - v^2\mu^2} = \frac{v^{-1} + \mu}{v^{-2} - \mu^2} = -(v^{-1} + \mu)^{-1}, \text{ or } -\rho^{-1} = \mu + v^{-1}, \text{ as in XX.};$$

so that, with the help of the expressions XV. and XXXVII. for v and ω , the *ray-vector* ρ and the *index-vector* μ are expressed as *functions of each other*: which functions are *generally definite*, although we shall soon see *cases*, in which one or other becomes partially *indeterminate*.

(12.) It is *easy* now to enunciate the *rule of the interchanges*, alluded to in (7.), as follows:—*In any formula involving the vectors* ρ , μ , v , ω , *and the functional symbol* ϕ , *or some of them, it is permitted to exchange* ρ *with* μ , v *with* ω , *and* ϕ *with* ϕ^{-1} ; *provided that we at the same time interchange* $\delta\rho$ *with* $\delta\epsilon$ (but *not** *generally with* $\delta\mu$), *when either* $\delta\rho$ *or* $\delta\epsilon$ *occurs.*

* It is true that, in passing from II. to III. (instead of passing to XLIII.), we may be said to have exchanged not only ρ with μ , but also $\delta\rho$ with $\delta\mu$. But *usually*, in the present investigation, $\delta\rho$ represents a small *displacement* (2.), which is conceived to have a *definite direction*, tangential to the wave; whereas $\delta\mu$

For example, we pass thus from XX. to XLI., and conversely from the latter to the former; from II. we pass by the same rule, to the formula,

$$\text{XLIII.} \dots Sp\delta\varepsilon = 0, \text{ which agrees by XVII. with XXII.};$$

and, as other verifications, the following equations may be noticed,

$$\text{XLIV.} \dots \delta\rho = \mu\nabla\mu\delta\varepsilon; \quad \text{XLV.} \dots \delta\varepsilon = \rho\nabla\rho\delta\rho; \quad \text{XLVI.} \dots S\mu\delta\varepsilon = Sp\delta\rho.$$

(18.) The relations between the vectors may be illustrated by the annexed Figure 89; in which,

$$\text{XLVII.} \dots OP = \rho, \quad OQ = \mu,$$

$$OU = v, \quad OW = \omega,$$

$$\text{and XLVIII.} \dots OP' = -\rho^{-1},$$

$$OQ' = -\mu^{-1}, \quad OU' = -v^{-1}, \quad OW' = -\omega^{-1};$$

in fact it is evident on inspection, that

$$\text{XLIX.} \dots OP \cdot OP' = OQ \cdot OQ'$$

$$= OU \cdot OU' = OW \cdot OW';$$

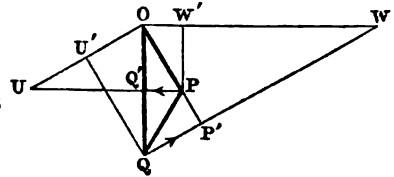


Fig. 89.

and the common value of these four scalar products is here taken as negative unity.

(14.) As examples of such illustration, the equation XX. becomes $F'O = QU'$; XLI. becomes $OQ' = W'P'$; XXIII. may be written as $\omega + \rho^{-1} = \rho^{-2}v$, or as $F'W : OU = F'O : OP$; &c. And because the lines $PQ'U$ and $Q'P'W$ are sections of the tangent planes, to the wave at the extremity P of the ray, and to the index surface at the extremity Q of the index vector, made by the plane of those two vectors ρ and μ , while $\delta\rho$ and $\delta\varepsilon$ (as being parallel to ω and v) have the directions of PQ' and $Q'P'$; we see that the displacement (or vibration) has generally, in Fresnel's theory, the direction of the projection of the ray on the tangent plane to the wave; and that the elastic force resulting has the direction of the projection of the index vector on the tangent plane to the index surface: results which might however have been otherwise deduced, from the formulæ alone.

(15.) It may be added, as regards the reciprocal deduction of the two vectors μ and ρ from each other, that (by XLI. XXXVIII., and XX. XVI.) we have the expressions,

$$\text{L.} \dots -\mu^{-1} = \omega^{-1}\nabla\omega\rho, \quad \text{and LI.} \dots -\rho^{-1} = v^{-1}\nabla v\mu;$$

which answer in Fig. 89 to the relations, that OQ' is the part (or component) of ρ , perpendicular to OW ; and that OP' is, in like manner, the part of $OQ \perp OU$.

(16.) We have also the expressions,

$$\text{LII.} \dots -\mu^{-1} = \omega^{-1}\nabla\omega v, \quad \text{and LIII.} \dots -\rho^{-1} = v^{-1}\nabla v\omega,$$

which may be similarly interpreted; and which conduct to the relations,

$$\text{LIV.} \dots -(\nabla v\omega)^2 = v^2\rho^{-2} = \omega^2\mu^{-2} = Sv\omega.$$

Hence, the Locus of each of the two Auxiliary Points v and ω , in Fig. 89, is a Surface of the Fourth Degree; the scalar equations of these two loci being,

$$\text{LV.} \dots (\nabla v\phi v)^2 + Sv\phi v = 0, \quad \text{and LVI.} \dots (\nabla\omega\phi^{-1}\omega)^2 + S\omega\phi^{-1}\omega = 0;$$

continues, as in (1.) to represent any infinitesimal tangent to the index surface, while $\delta\varepsilon$ still denotes the elastic force (2.), resulting from the displacement $\delta\rho$.

from which it would be easy to deduce *constructions* for those surfaces, with the help of the two *reciprocal ellipsoids*, XXIX. and XXX.

(17.) The equations XII. XXII., combined with the self-conjugate property of ϕ , give

$$\text{LVII.} \dots 0 = S(\phi^{-1}\rho \cdot \delta\rho), \text{ or LVIII.} \dots 0 = \delta S\rho\phi^{-1}\rho;$$

hence (between suitable limits of the constant), every *ellipsoid* of the form,

$$\text{LIX.} \dots S\rho\phi^{-1}\rho = h^4 = \text{const.},$$

which is thus *concentric and coaxial with the reciprocal ellipsoid XXX.*, being also *similar* to it, and *similarly placed*, contains upon its surface what may be called a *Line of Vibration* on the Wave*; the *intersection* of this *new ellipsoid LIX.* with the *wave surface* being generally such, that the *tangent* at each point of that line (or curve) has the *direction* of Fresnel's vibration.

(18.) The fundamental connexion (2.) of the *function* ϕ with the *optical constants*, a, b, c , of the crystal, is expressed by the *symbolical cubic* (comp. 350, I., and 417, XXV.),

$$\text{LX.} \dots (\phi + a^2)(\phi + b^2)(\phi + c^2) = 0;$$

from which it is easy to infer, by methods already explained, that if e be any *scalar*, and if we write,

$$\text{LXI.} \dots E = (e - a^2)(e - b^2)(e - c^2),$$

we have then this *formula of inversion*,

$$\text{LXII.} \dots E(\phi + e)^{-1} = e^2 - e(\phi + a^2 + b^2 + c^2) - a^2b^2c^2\phi^{-1}.$$

(19.) Changing then e to $-\rho^{-2}$, the equation XXVIII. of the wave becomes,

$$\text{LXIII.} \dots 0 = \rho^{-2} + a^2 + b^2 + c^2 + S\rho^{-1}\phi\rho - a^2b^2c^2S\rho\phi^{-1}\rho;$$

the *Wave* is therefore (as is otherwise known) a *Surface of the Fourth Degree*: and (as is likewise well known), the *Index Surface* is of the *same degree*, its equation (found by changing ρ, ϕ, a, b, c to $\mu, \phi^{-1}, a^{-1}, b^{-1}, c^{-1}$) being, on the same plan,

$$\text{LXIV.} \dots 0 = \mu^{-2} + a^2 + b^2 + c^2 + S\mu^{-1}\phi^{-1}\mu - a^2b^2c^2S\mu\phi\mu.$$

(20.) These equations may be variously transformed, with the help of the cubic LX. in ϕ , which gives the analogous cubic in ϕ^{-1} ,

$$\text{LXV.} \dots (\phi^{-1} + a^2)(\phi^{-1} + b^2)(\phi^{-1} + c^2) = 0;$$

for instance, another form of the equation of the wave is,

$$\text{LXVI.} \dots 0 = S\rho\phi^{-2}\rho + (\rho^2 + a^2 + b^2 + c^2)S\rho\phi^{-1}\rho - a^2b^2c^2;$$

in which it may be remarked that $S\rho\phi^{-2}\rho = (\phi^{-1}\rho)^2 < 0$, whereas $S\rho\phi^{-1}\rho > 0$.

(21.) Substituting then, for $S\rho\phi^{-1}\rho$ in LXVI., its value h^4 from (17.), we find that this *second variable ellipsoid*, with h for an arbitrary constant or parameter,

$$\text{LXVII.} \dots 0 = (\phi^{-1}\rho)^2 + h^4(\rho^2 + a^2 + b^2 + c^2) - a^2b^2c^2,$$

contains upon its surface the *same line of vibration* as the first variable ellipsoid LIX., which involves the same arbitrary constant h ; and therefore that the *line* in

* Such *lines of vibration* were discussed by the present writer, but by means of a quite different analysis, in his *Memoir of 1832 (Third Supplement on Systems of Rays)*, which was published in the following year, in the *Transactions of the Royal Irish Academy*. See reference in the Note to page 737.

question is a *quartic curve*, or *Curve of the Fourth Degree*, as being the intersection of these two variable but connected *ellipsoids*: and that the *wave* itself is the *locus* of all such quartic curves.

(22.) The *Generating Ellipsoid* ($S\rho\rho = 1$) has a, b, c for its semiaxes ($a > b > c > 0$); and for any vector ρ , in the plane of bc , we have the *symbolical quadratic* (comp. 353, (9.)),

$$\text{LXVIII.} \dots (\phi + b^{-2})(\phi + c^{-2}) = 0, \quad \text{or} \quad \text{LXIX.} \dots -b^{-2}c^{-2}\phi^{-1} = \phi + b^{-2} + c^{-2};$$

making then this last substitution for $\phi + b^{-2} + c^{-2}$ in LXIII., we find, for the section of the wave by this principal plane of the ellipsoid XXIX., an equation which breaks up into the two factors,

$$\text{LXX.} \dots \rho^{-2} + a^{-2} = 0, \quad \text{and} \quad \text{LXXI.} \dots 1 - b^{-2}c^{-2}S\rho\phi^{-1}\rho = 0;$$

whereof the *first* represents (the plane being understood) a *circle*, with *radius* = a , which we may call briefly *the circle* (a); while the *second* represents (with the same understanding) an *ellipse*, which may by analogy be called here *the ellipse* (a): its two semiaxes having the lengths of c and b , but in the directions of b and c , for which directions $\phi + b^{-2} = 0$ and $\phi + c^{-2} = 0$, respectively, so that this ellipse (a) is merely the *elliptic section*, (bc) of the *ellipsoid* (abc), turned through a *right angle* in its own plane, as by the construction (8.) it evidently ought to be. And an exactly similar analysis shows, what indeed is otherwise known, that the plane of ca cuts the wave in the system of a *circle* (b), and an *ellipse* (b); and that the plane of ab cuts the same wave surface, in a *circle* (c), and an *ellipse* (c).

(23.) The circle (a) is entirely *exterior* to the ellipse (a); and the circle (c) is wholly *interior* to the ellipse (c); but the circle (b) cuts the ellipse (b), in *four real points*, which are therefore (in a sense to be soon more fully examined) *cusps* (or *nodal points*) on the wave surface, or briefly *Wave-Cusps*: and the vectors ρ , say $\pm \rho_0$ and $\pm \rho_1$, which are drawn from the centre o to these *four cusps*, may be called *Lines of Single Ray-Velocity*, or briefly *Cusp-Rays*.

(24.) It is clear, from the construction (8.), that these lines or rays must have the directions of the *cyclic normals* of the ellipsoid (abc); which suggests our using here the *cyclic forms*,

$$\text{LXXII.} \dots \phi\rho = g\rho + \nabla\lambda\rho\lambda', \quad \text{and} \quad \text{LXXIII.} \dots S\rho\rho\rho = g\rho^2 + S\lambda\rho\lambda'\rho = 1,$$

for the function ϕ , and the generating ellipsoid (8.); λ' being written, to avoid confusion, instead of the μ of 357, &c., to represent the second cyclic normal.

(25.) Changing then μ to λ' , ν to ρ , and g to $g - \rho^2$, in the expression 361, XXVII. for $F\nu$ or $S\nu\phi^{-1}\nu$; equating the result to zero, and resolving the equation so obtained, as a quadratic in g ; we find this *new form* of the Equation XXVIII. of the *Wave*,

$$\text{LXXIV.} \dots g\rho^2 = 1 + S\lambda\rho S\lambda'\rho \pm T\nabla\lambda\rho T\nabla\lambda'\rho;$$

the upper sign belonging to *one sheet*, and the lower sign to the *other sheet*, of that wave surface. The new equation may also be thus written, as an expression for the *inverse square* of the *ray-velocity* $T\rho$, or of the *radius-vector*, say r , of the wave,

$$\text{LXXV.} \dots r^{-2} = T\rho^{-2} = \frac{a^{-2} + c^{-2}}{2} + \frac{a^{-2} - c^{-2}}{2} \cos\left(\angle \frac{\rho}{\lambda} \mp \angle \frac{\rho}{\lambda'}\right),$$

because, by 405, (2.), (6.), &c.,

$$\text{LXXVI.} \dots a^{-2} = -g - T\lambda\lambda', \quad b^{-2} = -g + S\lambda\lambda', \quad c^{-2} = -g + T\lambda\lambda';$$

LXXXVII. . . $r^2 = b^2$, or $r = T\rho = b$, if $\rho \parallel \lambda$ or λ' .
 (26.) If we write (comp. XXXI.),

$$\text{LXXXVIII. . . } f\rho = -\rho^3(1 + S\rho\phi\rho) + a^2b^2c^2S\rho\phi^{-1}\rho,$$

the equation LXXXIII. of the wave takes the form,

$$\text{LXXXIX. . . } f\rho = a^2 + b^2 + c^2 = \text{const.};$$

and we have the partial derivative (comp. XXXV.),

$$\begin{aligned} \text{LXXX. . . } \frac{1}{2}D_\rho f\rho &= \rho^3(1 + S\rho\phi\rho) - \rho^3\phi\rho + a^2b^2c^2\phi^{-1}\rho \\ &= \rho^3(1 - V\rho\phi\rho) + a^2b^2c^2\phi^{-1}\rho; \end{aligned}$$

which gives by XXXIII. the expression,

$$\text{LXXXI. . . } \mu = \frac{\rho^3(V\rho\phi\rho - 1) - a^2b^2c^2\phi^{-1}}{\rho^2 + a^2b^2c^2S\rho\phi^{-1}\rho},$$

and therefore a *generally definite value* (comp. (11.)) for the *index vector* μ , when the *ray* ρ is given.

(27.) If the ray be in the *plane* of ac , then (comp. LXIX.),

$$\text{LXXXII. . . } \phi\rho + (a^2 + c^2)\rho + a^2c^2\phi^{-1}\rho = 0,$$

whence LXXXIII. . . $V\rho\phi\rho = -a^2c^2V\rho\phi^{-1}\rho = a^2c^2(S\rho\phi^{-1}\rho - \rho\phi^{-1}\rho)$;

and therefore by LXXXI.,

$$\text{LXXXIV. . . } \mu = \frac{\rho^3(S\rho\phi^{-1}\rho - a^2c^2) - (\rho^2 + b^2)\phi^{-1}\rho}{b^2(S\rho\phi^{-1}\rho - a^2c^2) + (\rho^2 + b^2)a^2c^2};$$

an expression which gives, *definitely*,

$$\text{LXXXV. . . } \mu = -\rho^{-1}, \text{ if } \text{LXXXVI. . . } \rho^2 + b^2 = 0,$$

but *not*

$$\text{LXXXVII. . . } S\rho\phi^{-1}\rho = a^2c^2,$$

that is (comp. (22.)), if the ray terminate on the *circle* (b), at any point which is *not also on the ellipse* (b); and with equal *definiteness*,

LXXXVIII. . . $\mu = -a^2c^2\phi^{-1}\rho$, if LXXXVII. but *not* LXXXVI. hold good, that is, if the ray terminate on the *ellipse* (b), at any point which is *not also on the circle*.

(28.) The *normal* then to the *wave*, in each of the two cases last mentioned, coincides with the *normal* to the *section*, made by the *plane* of ac ; and if we abstract for a moment from the *cusps* (23.), we see that the *wave* is touched, *along the circle* (b), by the concentric *sphere* LXXXVI. with radius = b , which we may call the *sphere* (b); and *along the ellipse* (b) by the concentric *ellipsoid* LXXXVII. which may on the same plan be called the *ellipsoid* (b).

(29.) An exactly similar analysis shows that the *wave* is touched along the *circles* (a) and (c), by two other concentric *spheres*, with radii a and c , which may be briefly called the *spheres* (a) and (c); and along the *ellipses* (a) and (c) by two other concentric and similar *ellipsoids*, which may by analogy be called the *ellipsoids* (a) and (c). And by comparing the equation LXXXVII. of the *ellipsoid* (b) with the form LIX., we see that the *three elliptic sections* (a) (b) (c) of the *wave*, made by the *three principal planes* of the generating *ellipsoid* (abc), are *lines of vibration* (17.); the constant h^4 receiving the three values, b^2c^2 , c^2a^2 , a^2b^2 , for these three *ellipses* respectively.

(80.) But at a cusp the two equations LXXXVI. and LXXXVII. coexist, and the expression LXXXIV. for μ takes the indeterminate form $\frac{0}{0}$; in fact, there is in this case no reason for preferring either to the other of the two values, within the plane of ac ,

$$\text{LXXXIX.} \dots \mu = -\rho_0^{-1}, \quad \text{XC.} \dots \mu = \mu_0, \quad \text{if} \quad \text{XCI.} \dots \mu_0 = -\alpha^2 c^2 \phi^{-1} \rho_0;$$

in which ρ_0 is the cusp-ray (28.), and the first value of μ corresponds to the circle, but the second to the ellipse (b).

(81.) The indetermination of μ , at a wave-cusp, is however even greater than this. For, if we observe that the equations LXXIX. and LXXX. give, for this case, by LXXXII. LXXXVI. LXXXVII.,

$$\text{XCII.} \dots \rho_0 = \alpha^2 + b^2 + c^2, \quad \text{and} \quad \text{XCIII.} \dots D_\rho \rho = 0, \quad \text{for} \quad \rho = \rho_0,$$

we shall see that if ρ be changed to $\rho_0 + \rho'$ in the expression LXXVIII. for ρ , and only terms which are of the second dimension in ρ' retained, the result equated to zero will represent a cone of tangents ρ' , or a Tangent Cone to the Wave at the Cusp: which cone is of the second degree, and every normal μ to which, if limited by the condition I., is here to be considered as one value of the vector μ , corresponding to the value ρ_0 of ρ .

(82.) And it is evident, by the law (12.) of transition from the wave to the index surface, that if $\pm v_0, \pm v_1$ be the Lines of Single Normal Slowness, or the four values of μ which are analogous* to the four cusp-rays $\pm \rho_0, \pm \rho_1$ (28.), then, at the end of each such new line, there must be a Conical Cusp on the Index Surface, analogous to the Conical Cusp (81.) on the Wave, which is in like manner one of four such cusps.

(83.) In forming and applying the equation above indicated (81.), of the tangent cone to the wave at a cusp, the following transformations are useful:

$$\begin{aligned} \text{XCIV.} \dots & -(\rho + \rho')^{-2} = -\rho^{-2} (1 + \rho^{-1} \rho')^{-1} (1 + \rho' \rho^{-1})^{-1} \\ & = -\rho^{-2} + 2\rho^{-2} S\rho' \rho^{-1} + \rho^{-4} \rho'^2 - 4\rho^{-2} (S\rho \rho')^2 + \&c., \end{aligned}$$

the terms not written being of the third and higher dimensions in ρ' , and ρ, ρ' being any two vectors such that $T\rho' < T\rho$ (comp. 421, (4.)); also, without neglecting any terms, the self-conjugate property of ϕ gives (comp. 362),

$$\text{XCV.} \dots S(\rho + \rho') \phi(\rho + \rho') = S\rho \phi \rho + 2S\rho' \phi \rho + S\rho' \phi \rho',$$

with an analogous transformation for the corresponding expression in ϕ^{-1} ; while the cubic LX. in ϕ , or LXV. in ϕ^{-1} , gives for an arbitrary ρ ,

$$\text{XCVI.} \dots \phi(\phi + \alpha^2)(\phi + c^2)\rho = -b^2(\phi + \alpha^2)(\phi + c^2)\rho,$$

$$\text{XCVII.} \dots \phi^{-1}(\phi + \alpha^2)(\phi + c^2)\rho = -b^2(\phi + \alpha^2)(\phi + c^2)\rho;$$

and therefore, among other transformations of the same kind,

$$\text{XCVIII.} \dots (\phi + \alpha^2)^2(\phi + c^2)^2\rho = (\alpha^2 - b^2)(c^2 - b^2)(\phi + \alpha^2)(\phi + b^2)\rho.$$

* This word "analogous" is here more proper than "corresponding"; in fact, the cusps on each of the two surfaces will soon be seen to correspond to circles on the other, in virtue of the law of reciprocity.

We have also for a *cusp*, the values,

$$\text{XCIX.} \dots \phi \rho_0 = \mu_0 - (\alpha^2 + \epsilon^2) \rho_0; \quad \text{XCIX}'. \dots 1 + S \rho_0 \phi \rho_0 = (\alpha^2 + \epsilon^2) b^2$$

$$\text{C.} \dots \mu_0^2 = \alpha^4 \epsilon^4 S \rho_0 \phi^2 \rho_0 = \alpha^2 b^2 \epsilon^2 - (\alpha^2 + \epsilon^2).$$

(84.) In this way the *equation of the tangent cone* is easily found to take the form,

$$\text{CI.} \dots 0 = b^4 S \rho' (\phi + \alpha^2) (\phi + \epsilon^2) \rho' - 4 S \rho' \rho_0 S \rho' \mu_0$$

and to give, by operating with $D_{\rho'}$ (comp. (10.) (26.) (81.)),

$$\text{CII.} \dots \alpha \mu = b^4 (\phi + \alpha^2) (\phi + \epsilon^2) \rho' - 2 \rho_0 S \rho' \mu_0 - 2 \mu_0 S \rho' \rho_0,$$

the scalar coefficient α being determined, for each direction of the tangent ρ' to the wave at the cusp, by the condition I., which here becomes (81.),

$$\text{CIII.} \dots S \mu \rho_0 = S \mu_0 \rho_0 = -1;$$

also, by CII., &c., we have after some slight reductions,

$$\text{CIV.} \dots \alpha S \mu \rho_0 = 2 (b^2 S \rho' \mu_0 + S \rho' \rho_0);$$

$$\text{CV.} \dots \alpha S \mu \mu_0 = 2 (S \rho' \mu_0 - \mu_0^2 S \rho' \rho_0);$$

$$\text{CVI.} \dots x^2 \mu^2 = 4 (b^2 \mu_0^2 + 1) S \rho' \rho_0 S \rho' \mu_0 + 4 (\rho_0 S \rho' \mu_0 + \mu_0 S \rho' \rho_0)^2$$

$$= -4 b^2 (S \rho' \mu_0)^2 + 4 (b^2 \mu_0^2 - 1) S \rho' \rho_0 S \rho' \mu_0 + 4 \mu_0^2 (S \rho' \rho_0)^2;$$

but this last expression is equal, by CIV. CV., to $-x^2 S \mu \rho_0 S \mu \mu_0$; the *equation of the cone of perpendiculars*, let fall from the *wave-centre* O on the *tangent planes at the cusp*, takes then this very simple form,

$$\text{CVII.} \dots \mu^2 + S \mu \rho_0 S \mu \mu_0 = 0;$$

so that *this cone* of the second degree has the two vectors ρ_0 and μ_0 at once for *sides* and *cyclic normals* (comp. 406, (7.)); and it is cut, by the *plane* CIII., in a *circle*, of which the *diameter* is,

$$\text{CVIII.} \dots T(\mu_0 + \rho_0^{-1}) = (T \mu_0^2 - b^2)^{\frac{1}{2}} = b (b^2 - \alpha^2)^{\frac{1}{2}} (\epsilon^2 - b^2)^{\frac{1}{2}};$$

and therefore *subtends*, at the centre O , and in the plane of ac , the *angle*,

$$\text{CIX.} \dots \angle \frac{\mu_0}{\rho_0} = \tan^{-1} . b^2 (b^2 - \alpha^2)^{\frac{1}{2}} (\epsilon^2 - b^2)^{\frac{1}{2}}.$$

(85.) And by combining the equations CIII. CVII., we see that this circle (84.) is a *small circle of the sphere*,

$$\text{CX.} \dots \mu^2 = S \mu \mu_0, \quad \text{or} \quad \text{CX}'. \dots S \mu^{-1} \mu_0 = 1$$

which passes through the *wave-centre*, and has the vector μ_0 for a *diameter*, passing also through the extremity of the vector $-\rho_0^{-1}$.

(86.) This circle is, by III., a *curve of contact* of the *plane* CIII. with the *surface* of which μ is the vector, because *every vector* μ of the *curve* corresponds, by (31.), to the *one vector* ρ_0 of the *wave*; it is therefore one of *Four Circular Ridges on the Index Surface*, the three others having *equal diameters*, and corresponding to the three remaining *cusp-rays*, $-\rho_0, \rho_1, -\rho_1$ (28.); and there are, in like manner, *Four Circular Ridges on the Wave*, along which it is *touched by the four planes*,

$$\text{CXI.} \dots S \rho \nu_0 = -1, \quad S \rho \nu_0 = +1, \quad S \rho \nu_1 = -1, \quad S \rho \nu_1 = +1,$$

$\pm \nu_0, \pm \nu_1$ being the four lines introduced in (82.); also the *common length* of the *diameters*, of these four *circles on the wave*, is (comp. CVIII.),

$$\text{CXII.} \dots T(\sigma_0 + \nu_0^{-1}) = (T \sigma_0^2 - b^2)^{\frac{1}{2}} = b^{-1} (a^2 - b^2)^{\frac{1}{2}} (b^2 - c^2)^{\frac{1}{2}},$$

where CXIII. $\dots \sigma_0 = -a^2 c^2 \phi \nu_0$, CXIV. $\dots T \nu_0 = b^{-1}$, and CXV. $\dots S \nu_0 \sigma_0 = -1$;

finally, $-v_0^{-1}$ and σ_0 are the two values* of ρ , in the plane of ac , for the first of the four new circles: and the angle between these two vectors, or the angle which the diameter of the circle, in the same plane, subtends at the wave-centre, is (comp. CIX.),

$$\text{CXVI.} \dots \angle \frac{\sigma_0}{v_0} = \tan^{-1} \cdot b^{-2} (a^2 - b^2)^{\frac{1}{2}} (\delta^2 - c^2)^{\frac{1}{2}}.$$

(37.) In the recent calculations (88.) (84.), the circle of contact (86.) on the index surface was deduced from the tangent cone at a wave-cusp, as a section of a certain cone of normals CVII. to that tangent cone CI., made by the plane CIII.; but the following is a simpler, and perhaps more elegant method, of deducing and representing the same circle by means of its vector equation (comp. 392, IX. &c.), and without assuming any previous knowledge of the character, or even the existence, of that conical wave-cusp.

(38.) In general, by eliminating the auxiliary vector v between XX. and XXIII., we arrive at the following equation,

$$\text{CXVII.} \dots (\phi - \rho^{-2}) (\mu + \rho^{-1})^{-1} = \rho^{-1};$$

which holds good for every pair of corresponding vectors ρ and μ , of the wave and index surface. And in general, this relation is sufficient, to determine the index-vector μ , when the ray-vector ρ is given: because $(\phi + e)^{-1}0$ is generally = 0.

(39.) But when e is a root of the equation $E=0$, with the signification LXI. of E , then, by the formula of inversion LXII., the symbol $(\phi + e)^{-1}0$ takes the indeterminate form $\frac{0}{0}$; and therefore, for every point of each of the three circles (a) (b) (c) of the wave, the formula CXVII. fails to determine μ : although it is only at a cusp (28.), that the value of μ becomes in fact indeterminate (comp. (27.) (28.) (29.) (30.) (31.)).

(40.) At such a cusp ($\rho = \rho_0$), the equation CXVII. takes the symbolical form,

$$\text{CXVIII.} \dots (\mu + \rho_0^{-1})^{-1} = (\phi + b^{-2})^{-1} \rho_0^{-1} = (\mu_0 + \rho_0^{-1})^{-1} + (\phi + b^{-2})^{-1} 0;$$

μ_0 retaining its recent signification XCI., and the symbol $(\phi + b^{-2})^{-1}0$ denoting any vector of the form $y\beta$, if β be the mean vector semiaxis of the generating ellipsoid XXIX., so that

$$\text{CXIX.} \dots S\beta\phi\beta = 1, \quad (\phi + b^{-2})\beta = 0, \quad T\beta = \delta.$$

(41.) Writing then for abridgment (comp. XX.),

$$\text{CXX.} \dots v_0 = -(\mu_0 + \rho_0^{-1})^{-1},$$

the Vector Equation of the Index Ridge (86.) is obtained under the sufficiently simple form,

$$\text{CXXI.} \dots V\beta(\mu + \rho_0^{-1})^{-1} + V\beta v_0 = 0;$$

and this equation does in fact represent a Circle (comp. 296, (7.)), which is easily

* It is not difficult to show that these are the vectors of two points, in which the circle and ellipsoid (b), wherein the wave is cut by the plane of ac , are touched by a common tangent.

proved to be the same as the circular section (34.), of the cone CVII. by the plane CIII.; its diameter CVIII. being thus found anew under the form,

$$\text{CXXII.} \dots T\nu_0^{-1} = bTV\lambda\lambda' = b(b^2 - a^2)^{\frac{1}{2}}(c^2 - b^2)^{\frac{1}{2}},$$

with the significations (24.) (25.) of λ, λ' ; in fact we have now the expressions,

$$\text{CXXIII.} \dots \rho_0 = bU\lambda, \quad \nu_0 = \rho_0^{-1}(V\lambda\lambda')^{-1},$$

with the verification, that

$$\text{CXXIV.} \dots (\phi + b^2)v_0 = \lambda S\lambda'v_0 + \lambda^2 S\lambda v_0 = b^{-1}U\lambda = -\rho_0^{-1}.$$

(42.) And by a precisely similar analysis, we have first the new general relation (comp. CXVII.), for any two corresponding vectors, ρ and μ ,

$$\text{CXXV.} \dots (\phi^{-1} - \mu^{-2})(\rho + \mu^{-1})^{-1} = \mu^{-1};$$

and then in particular (comp. CXVIII.), for $\mu = \nu_0$,

$$\text{CXXVI.} \dots (\rho + \nu_0^{-1})^{-1} = (\phi^{-1} + b^2)^{-1}\nu_0^{-1} = (\sigma_0 + \nu_0^{-1})^{-1} + (\phi^{-1} + b^2)^{-1}0;$$

so that finally, if we write for abridgment (comp. XLI. CXX.),

$$\text{CXXVII.} \dots \omega_0 = -(\sigma_0 + \nu_0^{-1})^{-1},$$

the Vector Equation of a Wave-Ridge is found (comp. CXXI.) to be,

$$\text{CXXVIII.} \dots V\beta(\rho + \nu_0^{-1})^{-1} + V\beta\omega_0 = 0,$$

β being still (as in CXIX.) the mean vector semiaxis of the generating ellipsoid ($S\rho\phi\rho = 1$): and the diameter CXII., of this circle of contact of the wave with the first plane CXI., is thus found anew (comp. CXXII.), without any reference to cusps (37.), as the value of $T\omega_0^{-1}$.

(43.) Several of the foregoing results may be illustrated, by a new use of the last diagram (13.). Thus if we suppose, in that Fig. 89, that we have the values,

$$\text{CXXIX.} \dots OP = \rho_0, \quad OQ = \mu_0, \quad OU = \nu_0, \quad \text{whence} \quad \text{CXXX.} \dots OP' = -\rho_0^{-1}, \text{ \&c.,}$$

then the index-ridge (36.), corresponding to the wave-cusp P (28.), will be the circle which has P'Q for diameter, in a plane perpendicular to the plane of the Figure, which is here the plane of ac ; the cone of normals μ (34.), to the tangent cone to the wave at P, has the wave-centre O for its vertex, and rests on the last-mentioned circle, having also for a subcontrary section that second circle which has RQ' for diameter, and has its plane in like manner at right angles to the plane of POQ ; also if κ and s be any two points on the second and first circles, such that ors is a right line, namely a side μ of the cone here considered, then the chord PR of the second circle is perpendicular to this last line, and has the direction of the vibration $\delta\rho$, which answers here to the two vectors $\rho (= \rho_0)$ and μ : because (comp. (14.)) this chord is perpendicular to μ , but coplanar with ρ and μ .

(44.) Again, to illustrate the theory of the wave-ridge (36.), which corresponds to a cusp (32.) on the index-surface, we may suppose that this cusp is at the point q in Fig. 89, writing now (instead of CXXIX. CXXX.),

$$\text{CXXXI.} \dots OQ = \nu_0, \quad OP = \sigma_0, \quad OW = \omega_0, \quad OQ' = -\nu_0^{-1}, \text{ \&c.;}$$

for then the ridge (or circle of contact) on the wave will coincide with the second circle (43.), and the cone of rays ρ from O, which rests upon this circle, will have the first circle (43.) for a sub-contrary section: also the vibration, at any point κ of the wave-

ridge, will have the direction of the chord nq' , for reasons of the same kind as before.

(45.) Let κ and κ' denote the bisecting points of the lines rq' and qr' , in the same Fig. 89; then κ' is the centre of the index ridge, in the case (48.); while, in the case (44.), κ is the centre of the wave-ridge.

(46.) In the first of these two cases, the point κ is not the centre of any ridge, on either wave or index-surface; but it is the centre of a certain subcontrary and circular section (48.), of the cone with o for vertex which rests upon an index-ridge; and each of its chords PK has the direction (48.) of a vibration $\delta\rho_0$, at the wave-cusp P corresponding: so that this cusp-vibration revolves, in the plane of the circle last mentioned, with exactly half the angular velocity of the revolving radius κR .

(47.) And every one of those cusp-vibrations $\delta\rho_0$, which (as we have seen) are all situated in one plane, namely in the tangent plane at the cusp P to the ellipsoid (b) of (28.), has (as by (14.) it ought to have) the direction of the projection of the cusp-ray ρ_0 , on some tangent plane to the tangent cone to the wave, at that point P : to the determination of which last cone, by some new methods, we purpose shortly to return.

(48.) In the second of the two cases (45.), namely in the case (44.), rq' is a diameter of a wave-ridge, with κ for the centre of that circle, and with a plane (perpendicular to that of the Figure) which touches the wave at every point of the same circular ridge; and the vibration, at any such point κ , has been seen to have the direction of the chord nq' , which is in fact the projection (14.) of the ray OR upon the tangent plane at κ to the wave.

(49.) And we see that, in passing from one point to another of this wave-ridge, the vibration nq' revolves (comp. (46.)) round the fixed point q' of that circle, namely round the foot of the perpendicular from o on the ridge-plane, with (again) half the angular velocity of the revolving radius κR .

(50.) These laws of the two sets of vibrations, at a cusp and at a ridge upon the wave, are intimately connected with the two conical polarizations, which accompany the two conical refractions,* external and internal, in a biaxial crystal; because, on the one hand, the theoretical deduction of those two refractions is associated with, and was in fact accomplished by, the consideration of those cusps and ridges: while, on the other hand, in the theory of Fresnel, the vibration is always perpendicular

* The writer's anticipation, from theory, of the two Conical Refractions, was announced at a general meeting of the Royal Irish Academy, on the 22nd of October, 1832, in the course of a final reading of that *Third Supplement on Systems of Rays*, which has been cited in a former Note (p. 737). The very elegant experiments, by which his friend, the Rev. Humphrey Lloyd, succeeded shortly afterwards in exhibiting the expected results, are detailed in a Paper *On the Phenomena presented by Light, in its passage along the Axes of Biaxial Crystals*, which was read before the same Academy on the 28th of January, 1833, and is published in the same First Part of Volume XVII. of their Transactions. Dr. Lloyd has also given an account of the same phenomena, in a separate work since published, under the title of an *Elementary Treatise on the Wave Theory of Light* (London, Longman and Co., 1857, Chapter XI.).

to the *plane of polarization*. But into the details of such investigations, we cannot enter here.

(51.) It is not difficult to show, by decomposing ρ' into two other vectors, ρ_1' and ρ_2' , perpendicular and parallel to the plane of ac , that we have the *general transformation*, for any vector ρ' ,

$$\text{CXXXII.} \dots b^4 S \rho' (\phi + a^2) (\phi + c^2) \rho' = (S \mu_0 \rho_0 \rho')^2;$$

the equation CI. of the *tangent cone* at a *wave-cusp* may therefore be thus more briefly written,

$$\text{CXXXIII.} \dots (S \mu_0 \rho_0 \rho')^2 = 4 S \rho_0 \rho' S \mu_0 \rho';$$

and under this form, the *cone* in question is easily proved to be the *locus* of the *normals* from the *cusp*, to that *other cone* CVII., which has μ for a side, and the *wave-centre* o for its vertex: while the same cone CVII. is now seen, more easily than in (34.), to be reciprocally the locus of the perpendiculars from o on the *tangent planes* to the wave at the cusp, in virtue of the new equation CXXXIII., of the *tangent cone* at that point.

(52.) Another form of the equation of the cusp-cone may be obtained as follows. The equation LXXIV. of the *wave* may be thus modified (comp. LXXVI.), by the introduction of the two non-opposite cusp-rays, $\rho_0 = \delta U \lambda$ (CXXXIII.), and $\rho_1 = \delta U \lambda'$:

$$\begin{aligned} \text{CXXXIV.} \dots 2a^2 b^2 c^2 + (a^2 + c^2) b^2 \rho^2 + (a^2 - c^2) S \rho_0 \rho' \cdot S \rho_1 \rho' \\ = \mp (a^2 - c^2) TV \rho_0 \rho' \cdot TV \rho_1 \rho'; \end{aligned}$$

where it will be found that the first member vanishes, as well as the second, at the cusp for which $\rho = \rho_0$.

(53.) Changing then ρ to $\rho_0 + \rho'$, and retaining only terms of *first dimension* in ρ' (comp. (81.)), we find an equation of *unifocal form* (comp. 859, &c.),

$$\text{CXXXV.} \dots S \beta_0 \rho' = \mp TV \alpha_0 \rho', \text{ or } \text{CXXXV'.} \dots (TV \alpha_0 \rho')^2 + (S \beta_0 \rho')^2 = 0;$$

with the two constant vectors,

$$\text{CXXXVI.} \dots \alpha_0 = (b^2 - a^2)^{\frac{1}{2}} (c^2 - b^2)^{\frac{1}{2}} \rho_0; \quad \text{CXXXVI'.} \dots \beta_0 = \mu_0 - \rho_0^{-1};$$

and this equation CXXXV. or CXXXV'. represents the *tangent cone*, with ρ' for side, $S \beta_0 \rho'$ being positive for one sheet, but negative for the other.

(54.) As regards the calculations which conduct to the recent expressions for α_0 , β_0 , it may be sufficient here to observe that those expressions are found to give the equations,

$$\text{CXXXVII.} \dots 2a^2 b^2 c^2 \alpha_0 = (a^2 - c^2) \rho_0 TV \rho_0 \rho_1;$$

$$\text{CXXXVII'.} \dots 2a^2 b^2 c^2 \beta_0 = 2(a^2 + c^2) b^2 \rho_0 + (a^2 - c^2) (\rho_0 S \rho_0 \rho_1 - b^2 \rho_1);$$

and that, in deducing these, we employ the values,

$$\text{CXXXVIII.} \dots S \rho_0 \rho_1 = \frac{b^2 S \lambda \lambda'}{T \lambda \lambda'}, \quad TV \rho_0 \rho_1 = \frac{b^2 TV \lambda \lambda''}{T \lambda \lambda'};$$

together with the formula XCIX., and the following,

$$\text{CXXXIX.} \dots \phi (\rho_0 - \rho_1) = -a^2 (\rho_0 - \rho_1); \quad \phi (\rho_0 + \rho_1) = -c^2 (\rho_0 + \rho_1).$$

(55.) It is not difficult to show that the equation CXXXV. or CXXXV'. of the *tangent cone* at a cusp, can be transformed into the equation CXXXIII.; but it

may be more interesting to assign here a *geometrical interpretation*, or *construction*, of the *unifocal form* last found (58.).

(56.) Retaining then, for a moment, the use made in (48.) of Fig. 89, as serving to illustrate the case of a wave-cusp at P , with the signification (45.) of the new point κ' as bisecting the line $P'Q$, or as being the centre of the index-ridge; and conceiving a *parallel cone*, with O instead of P for *vertex*, and with a variable *side* $OR = \rho'$; then the *cusp-ray* $OP (= \rho_0 \parallel a_0)$ is a *focal line* of the new cone, and the line $OK' (= \frac{1}{2}(\mu_0 - \rho_0^{-1}) = \frac{1}{2}\beta_0)$ is the *directive normal*, or the normal to the *director plane* corresponding; and the formula CXXXV. is found to conduct to the following,

$$\text{CXL.} \dots \cos \kappa'OR = \sin \rho OK' \sin \rho OR,$$

which may be called a *Geometrical Equation of the Cusp-Cone*: or (more immediately) of that *Parallel Cone*, which has (as above) its vertex removed to the wave-centre O .

(57.) Verifications of CXL. may be obtained, by supposing the side OR to be one of the two right lines, ρ_1', ρ_2' , in which the cone is cut by the *plane* of the figure (or of ac); that is, by assuming either

$$\text{CXLI.} \dots OR = \rho_1' = \mu_0 + \rho_0^{-1} \parallel OU, \quad \text{or} \quad \text{CXLI.} \dots OR = \rho_2' = \rho_0 + \mu_0^{-1} \parallel OW;$$

and it is easy to show, not only that these *two sides*, OU , OW , make (as in Fig. 89) an *obtuse angle* with each other, but also that they belong to one *common sheet*, of the cone here considered, because each makes an *acute angle* with the *directive normal* OK' .

(58.) Another way of arriving at this result, is to observe that the equation CXXXIII. takes easily the *rectangular form*,

$$\text{CXLI.} \dots (Sp'(U\mu_0 + U\rho_0))^2 = (Sp'(U\mu_0 - U\rho_0))^2 + T\mu_0\rho_0(Sp'U\mu_0\rho_0)^2;$$

the *internal axis* of the *cusp-cone* has therefore the direction of $U\mu_0 + U\rho_0$, that is, of the *internal bisector* of the angle POQ , while the *external bisector* of the same angle is one of the two *external axes*, and the *third axis* is perpendicular to the *plane* of ρ_0, μ_0 ; but $Sp'(U\mu_0 + U\rho_0) < 0$, whether $\rho' = \rho_1'$, or ρ_2' : and therefore these *two sides*, ρ_1' and ρ_2' , belong (as above) to *one sheet*, because each is inclined at an acute angle to the *internal axis* $U\mu_0 + U\rho_0$.

(59.) It is easy to see that the *second focal line* of the *parallel cone* (56.) is μ_0 or OQ ; and that the *second directive normal* corresponding is the line OK (45.), in the same Fig. 89; whence may be derived (comp. CXL.) this *second geometrical equation* of the cone at O ,

$$\text{CXLIII.} \dots \cos \kappa OR = \sin \kappa OQ \sin \rho OR; \quad \text{with} \quad \kappa OQ = \rho OK'.$$

(60.) And finally, as a *bifocal* but still *geometrical form* of the equation of the *cusp-cone*, with its vertex thus *transferred* to O , we may write,

$$\text{CXLIV.} \dots \angle \rho OR + \angle \rho OR = \text{const.} = \angle \rho OQ.$$

(61.) Any *legitimate form* of any one of the *four functions*, $\phi\rho$, $\phi^{-1}\rho$, $Sp\phi\rho$, $Sp\phi^{-1}\rho$, when treated by rules of the present Calculus which have been already stated and exemplified, not only conducts to the connected forms of the *three other functions* of the group, but also gives the corresponding forms of equation, of the *Wave* and the *Index-Surface*.

(62.) For instance, with the significations (32.) of ν_0 and ν_1 , the scalar function $S\rho\phi^{-1}\rho$, which is = 1 in the equation XXX. of the *Reciprocal Ellipsoid* (9.), may be expressed by the following *cyclic form*, with ν_0, ν_1 for the *cyclic normals* of that ellipsoid,

$$\text{CXLV.} \dots S\rho\phi^{-1}\rho = -b^2\rho^2 + (a^2 - c^2)b^2S\nu_0\rho S\nu_1\rho;$$

reciprocating which (comp. 361), we are led to a *bifocal form* of the function $S\rho\phi\rho$, which function was made = 1 in the equation XXIX. of the *Generating Ellipsoid* (8.), and is now expressed by this other equation (comp. 360, 407),

$$\text{CXLVI.} \dots \frac{4a^2c^2}{(a^2 - c^2)^2} (S\rho\phi\rho + b^2\rho^2) = (S\nu_0\rho)^2 + (S\nu_1\rho)^2 - 2 \frac{a^2 + c^2}{a^2 - c^2} S\nu_0\rho S\nu_1\rho;$$

ν_0, ν_1 being here the two (real) *focal lines* of the same ellipsoid (8.), or of its (imaginary) asymptotic cone.

(63.) Substituting then these forms (62.), of $S\rho\phi\rho$ and $S\rho\phi^{-1}\rho$, in the equation LXIII., we find (after a few reductions) this *new form* of the *Equation of the Wave* :

$$\text{CXLVII.} \dots (2\rho^2 - (a^2 - c^2)S\nu_0\rho S\nu_1\rho + a^2 + c^2)^2 = (a^2 - c^2)^2 \{ 1 - (S\nu_0\rho)^2 \} \{ 1 - (S\nu_1\rho)^2 \};$$

whence it follows at once, that *each of the four planes CXI. touches the wave, along the circle in which it cuts the quadric*, with ν_0, ν_1 for cyclic normals, which is found by equating to zero the expression squared in the first member of CXLVII. For example, the *first plane CXI. touches the wave along that circle, or wave-ridge, of which on this plan the equations are,*

$$\text{CXLVIII.} \dots S\nu_0\rho + 1 = 0, \quad 2\rho^2 + (a^2 - c^2)S\nu_1\rho - (a^2 + c^2)S\nu_0\rho = 0;$$

and because

$$\text{CXLIX.} \dots \phi(\nu_0 + \nu_1) = -a^2(\nu_0 + \nu_1), \quad \phi(\nu_0 - \nu_1) = -c^2(\nu_0 - \nu_1),$$

and therefore, with the value CXIII. of σ_0 ,

$$\text{CL.} \dots \sigma_0 = -a^2c^2\phi\nu_0 = \frac{1}{2}((a^2 + c^2)\nu_0 - (a^2 - c^2)\nu_1),$$

the second equation CXLVIII. represents (comp. CX.) the *diacentric sphere*,

$$\text{CLI.} \dots \rho^2 = S\sigma_0\rho, \quad \text{or} \quad \text{CLI'.} \dots S\sigma_0\rho^{-1} = 1,$$

which passes through the *wave-centre* o, and of which the *ridge* here considered is a *section*. The *diameter* of that ridge may thus be shown again to have the value CXII.; and it may be observed that the circle is a section also of the *cone*,

$$\text{CLII.} \dots S\nu_0\rho S\sigma_0\rho = -\rho^2, \quad \text{or} \quad \text{CLII'.} \dots S\nu_0\rho S\sigma_0\rho^{-1} = -1.$$

(64.) It was shown in (17.) that the *vibration* $\delta\rho$, at *any point* of the *wave-surface*, or at the end of *any ray* ρ , is perpendicular to $\phi^{-1}\rho$, as well as to μ by II.; and is therefore *tangential* to the variable *ellipsoid* LIX., as well as to the *wave* itself. Hence it is easy to infer, that this vibration must have generally the direction of the auxiliary vector ω , because not only $S\mu\omega = 0$, by XXXIX., but also $S\omega\phi^{-1}\rho = S\rho\phi^{-1}\omega = S\rho\nu = 0$, by XXII. and XXXVII. Indeed, this parallelism of $\delta\rho$ to ω results at once by XXXVII. from XII.

(65.) If then we denote by $\delta'\rho$ an infinitesimal vector, such as $\mu\delta\rho$, which is *tangential to the wave*, but *perpendicular to the vibration* $\delta\rho$, the parallelism $\delta\rho \parallel \omega$ will give,

CLIII. . . $\delta\rho = \mu\delta\rho \parallel \mu\omega \perp \rho$, because CLIII'. . . $\text{Sp}\mu\omega = 0$;

whence CLIV. . . $\text{Sp}\delta\rho = 0$, $\delta T\rho = 0$, or CLV. . . $T\rho = r = \text{const.}$,

for this *new direction* $\delta\rho$ of motion upon the wave.

(66.) And thus (or otherwise) it may be shown, that the *Orthogonal Trajectories to the Lines of Vibration* (17.) are the curves in which the *Wave* is cut by *Concentric Spheres*, such as CLV.; that is, by the spheres $\rho^2 + r^2 = 0$, in which the radius r is *constant* for any one, but *varies* in passing from one to another.

(67.) The *spherical curves* (r), which are thus *orthogonal* to what we have called the *lines* (h) of vibration, are *sphero-conics* on the wave; either because each such curve (r) is, by XXVIII., situated on a *concentric and quadric cone*, namely,

$$\text{CLVI. . . } 0 = \text{Sp}(\phi + r^2)^{-1}\rho;$$

or because, by XXVII., it is on this *other concentric quadric*,

$$\text{CLVII. . . } -1 = \text{Sp}(\phi^{-1} + r^2)^{-1}\rho.$$

(68.) It is easy to prove (comp. LXXV.) that, for any *real* point of the wave, r^2 cannot be less than a^2 , nor greater than a^2 ; and that the squares of the scalar semi-axes of the new quadric CLVII. are, in algebraically ascending order, $r^2 - a^2$, $r^2 - b^2$, $r^2 - c^2$; so that this surface is generally an *hyperboloid*, with *one sheet* or with *two*, according as $r > 0$ or < 0 .

(69.) And we see, at the same time, that the *conjugate hyperboloid*,

$$\text{CLVIII. . . } +1 = \text{Sp}(\phi^{-1} + r^2)^{-1}\rho,$$

which has *two sheets* or *one*, in the same two cases, $r > b$, $r < b$, and has (in descending order) the values,

$$\text{CLIX. . . } a^2 - r^2, \quad b^2 - r^2, \quad c^2 - r^2,$$

for the squares of its scalar semi-axes, is *confocal* with the *generating ellipsoid* XXIX.: so that the quadric CLVII. itself is the *conjugate of such a confocal*.

(70.) To form a distinct conception (comp. (67.)) of the *course* of a curve (r) upon the wave, it may be convenient to distinguish the *five* following cases:

CLX. . . (a) . . $r = a$; (β) . . $r < a, > b$; (γ) . . $r = b$; (δ) . . $r < b, > c$; (ϵ) . . $r = c$.

(71.) In each of the *three* cases (a) (γ) (ϵ), the *conic* (r) becomes a *circle*, in one or other of the three principal planes: namely the circle (a), for the case (a); (b) for (γ); and (c) for (ϵ).

(72.) In the case (β), the *curve* (r) is one of *double curvature*, and consists of *two closed ovals*, *opposite* to each other on the *wave*, and *separated* by the *plane* (a), which plane is *not* (really) *met*, in *any point*, by the complete *sphero-conic* (r); and *each* separate oval *crosses* the *plane* (b) *perpendicularly*, in *two* (real) *points* of the *ellipse* (b), which are *external* to the *circle* (b): while the *same oval* *crosses* *also* the *plane* (c) *at right angles*, in *some two* real points of the *ellipse* (c).

(73.) Finally, in the remaining case (δ), the *ovals* are *separated* by the *plane* (c), and *each* *crosses* the *plane* (b) *at right angles*, in *two* points of the *ellipse* (b), which are *interior* to the *circle* (b); *crossing* *also* *perpendicularly* the *plane* (a), in *two* points of the *ellipse* (a).

(74.) Analogous remarks apply to the *lines of vibration* (A); which are either the *ellipses* (a) (b) (c), or else *orthogonals* to the *circles* (a) (b) (c), and generally to the *sphero-conics* (r), as appears easily from foregoing results.

(75.) It may be here observed, that when we only know the *direction* ($U\mu$), but not the *length* ($T\mu$), of an *index-vector* μ , so that we have *two parallel tangent planes* to the wave, at one *common side* of the centre, the *directions* of the vibrations $\delta\rho$ differ generally for these *two planes*, according to a law which it is easy to assign as follows.

(76.) The *second values* of μ and $\delta\rho$ being denoted by μ , and $\delta\rho$, we have, by the equation IX. of the index-surface, these two other equations :

$$\text{CLXI.} \dots 0 = S\mu(\phi^{-1} - \mu^{-2})^{-1}\mu; \quad \text{CLXI'}. \dots 0 = S\mu(\phi^{-1} - \mu_r^{-2})^{-1}\mu;$$

of which the difference gives, suppressing the factor $\mu_r^{-2} - \mu^{-2}$,

$$\text{CLXII.} \dots 0 = S\mu(\phi^{-1} - \mu_r^{-2})^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu;$$

or

$$\text{CLXII'}. \dots 0 = S(\phi^{-1} - \mu^{-2})^{-1}\mu(\phi^{-1} - \mu_r^{-2})^{-1}\mu,$$

because $(\phi^{-1} - \mu_r^{-2})^{-1}$, as a functional operator, is *self-conjugate*, so that μ may be transferred from one side of it to the other; just as, if $\nu = \phi\rho$ be such a self-conjugate function of ρ , then $\nu^2 = S\nu\phi\rho = S\rho\phi\nu = S\rho\phi^2\rho$, &c.

(77.) But, by VIII., we have the parallelisms,

$$\text{CLXIII.} \dots \delta\rho \parallel (\phi^{-1} - \mu^{-2})^{-1}\mu; \quad \text{CLXIII'}. \dots \delta\rho_r \parallel (\phi^{-1} - \mu_r^{-2})^{-1}\mu;$$

hence, by CLXII., we have the very simple relation,

$$\text{CLXIV.} \dots S\delta\rho\delta\rho_r = 0;$$

that is, *the two vibrations*, in the *two parallel planes*, are *mutually rectangular*.

(78.) The following quite different method has however the advantage of not only proving anew this *known relation of rectangularity*, but also of assigning *quaternion expressions* for the *two directions separately*: and, at the same time, that of leading easily to what appears to be a *new* and elegant *Geometrical Construction*, simpler in some respects than the *known* one, which can indeed be deduced from it.

(79.) By the first principles of Fresnel's theory (comp. (3.)), the *vibration* ($\delta\rho$), on any *one* tangent plane to the wave, is situated in the *normal plane* (through μ), which contains the direction ($\delta\varepsilon$) of the *elastic force*; that is to say, we have the *Equation of Complianarity*,

$$\text{CLXV.} \dots S\mu\delta\rho\delta\varepsilon = 0.$$

(80.) We have then, by II. and V., the system of the two equations,

$$\text{CLXVI.} \dots S\mu\delta\rho = 0, \quad S\mu\delta\rho\phi^{-1}\delta\rho = 0;$$

comparing which with the equations of the same form,

$$S\nu r = 0, \quad S\nu r\phi r = 0, \quad 410, \text{ V. VI.}$$

we derive at once the following *Construction*, which may also be expressed as a *Theorem* :—

“At either of the two points Q of the *Reciprocal Ellipsoid* XXX., the *tangent plane* at which is parallel to that at the given point P of the *Wave*, the *tangents* to the *Lines of Curvature* on the *Ellipsoid* are parallel to the *tangents* to the *Lines of Vibration* on the *Wave* ;” namely, to one at that given point P itself, and to another at the other point P' , on the same side of the centre, at which the *tangent plane* is parallel to each of the two others above mentioned.

(81.) Thus for each of the two points P, P' the line of *vibration* is parallel to one of the lines of *curvature* at Q ; and it is evident, from what precedes, that the other of these last lines has the direction of the corresponding *Orthogonal* (66.) at P or P' : nor is there any danger of confusion.

(82.) As regards *quaternion expressions*, for the two vibrations on a given wave-front, the sub-article, 410, (8.), with notations suitably modified, shows by its formulæ XIX. XXII. that we have here the equations,

$$\begin{aligned} \text{CLXVII.} \dots 0 &= S\mu \delta\rho \nu_0 \delta\rho \nu_1 \\ &= S\mu \delta\rho \nu_0 S\nu_1 \delta\rho + S\mu \delta\rho \nu_1 S\nu_0 \delta\rho, \end{aligned}$$

and

$$\text{CXVIII.} \dots \delta\rho \parallel UV\mu\nu_0 \pm UV\mu\nu_1,$$

if ν_0, ν_1 be, as in earlier formulæ of the present Series 422, the *cyclic normals* of the *reciprocal ellipsoid*, which are often called the *Optic Axes* of the *Crystal*.

(83.) And hence may be deduced the *known* construction, namely, that "for any given *direction of wave-front*, the two planes of *polarization*, perpendicular respectively to the two vibrations in Fresnel's theory, *bisect* the two *supplementary and dihedral angles*, which the two *optic axes subtend at the normal to the front*:" or that these planes of polarization *bisect, internally and externally*, the angle between the two planes, $\mu\nu_0$ and $\mu\nu_1$.

(84.) It may not be irrelevant here to remark, that if μ and μ , be any two index-vectors, which have (as in (76.)) the *same direction*, but *not the same length*, the equation LXIV. enables us to establish the two converse relations:

$$\text{CLXIX.} \dots abcT\mu = (S\mu\phi\mu)^{-1}; \quad \text{CLXIX'}. \dots abcT\mu = (S\mu, \phi\mu)^{-1}.$$

(85.) Either by changing a, b, c, ϕ, μ to $a^{-2}, b^{-2}, c^{-2}, \phi^{-1}, \rho$, or by treating the form LXIII., in (19.), of the *Equation of the Wave*, as we have just treated the form LXIV., of the equation of *Index Surface*, in the same sub-article (19.), we see that if ρ and ρ , be any two *directional rays* ($U\rho = U\rho$), then,

$$\begin{aligned} \text{CLXX.} \dots (abc)^{-1}T\rho &= (S\rho\phi^{-1}\rho)^{-1}, \quad \text{or,} \quad abcT\rho^{-1} = (S\rho\phi^{-1}\rho)^{-1}; \\ \text{and CLXX'}. \dots (abc)^{-1}T\rho &= S(\rho, \phi^{-1}\rho)^{-1}, \quad \text{or,} \quad abcT\rho^{-1} = (S\rho, \phi^{-1}\rho)^{-1}. \end{aligned}$$

(86.) A somewhat interesting geometrical consequence may be deduced from these last formulæ, when combined with the equation LIX. of that *variable ellipsoid*, $S\rho\phi^{-1}\rho = h^4$, which *cuts the wave* in a *line of vibration* (h). For if we introduce this symbol h^4 for $S\rho\phi^{-1}\rho$, and write r , instead of $T\rho$, to denote the *length* of the *second ray* ρ , the first equation CLXX. will take this simple form,

$$\text{CLXXI.} \dots r = abch^{-2},$$

which shows at once that r , and h are *together constant*, or *together variable*; and therefore, that "a *Line of Vibration on one Sheet of the Wave* is projected into an *Orthogonal Trajectory to all such Lines on the other Sheet*, and conversely the latter into the former, by the *Vectors* ρ of the *Wave*:" so that one of these two curves would appear to be *superposed* upon the other, to an eye placed at the *Wave-Centre* o .

(87.) The *visual cone*, here conceived, is represented by the equation CLVI., with some constant value of r ; and as being a surface of the *second degree*, it ought to *cut the wave*, which is one of the *fourth*, in some curve of the *eighth degree*; or in some *system of curves*, which have the *product* of their dimensions equal to *eight*.

Accordingly we now see that the *complete intersection*, of the cone CLVI. with the wave, consists of *two curves*, each of the *fourth degree*; one of these being, as in (67.), a *complete sphero-conic* (r), and the other a *complete line of vibration* (h): a new geometrical *connexion* being thus established between these *two quartic curves*.

(88.) As additional verifications, we may regard the *three principal planes*, as *limits of the cutting cones*; for then, in the plane (a) for instance, the *circle* (a) and the *ellipse* (a), which are (in a sense) *projections* of each other, and of which the latter has been seen to be a *line of vibration*, are represented respectively by the two equations,

$$\text{CLXXII.} \dots r = a, \text{ and } \text{CLXXII}'. \dots bc = h^2,$$

in agreement with CLXXI.; and similarly for the two other planes.

(89.) It was an early result of the quaternions, that an ellipsoid with its centre at the origin might be adequately represented by the equation (comp. 281, XXIX., or 282, XIX.),

$$\text{CLXXIII.} \dots T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \text{ if } T\iota > T\kappa;$$

or, without any restriction on the two vector constants, ι , κ , by this other equation,*

$$\text{CLXXIII}'. \dots T(\iota\rho + \rho\kappa)^2 = (\kappa^2 - \iota^2)^2.$$

(90.) Comparing this with $S\rho\phi\rho = 1$, as the equation XXIX. of the *Generating Ellipsoid*, we see that we are to satisfy, *independently of* ρ , or as an *identity*, the relation (comp. 336):

$$\begin{aligned} \text{CLXXIV.} \dots (\kappa^2 - \iota^2)^2 S\rho\phi\rho &= (\iota\rho + \rho\kappa) (\rho\iota + \kappa\rho) \\ &= (\iota^2 + \kappa^2) \rho + 2S\iota\rho\kappa\rho; \end{aligned}$$

which is done by assuming (comp. again 336) this *cyclic form* for ϕ ,

$$\begin{aligned} \text{CLXXV.} \dots (\kappa^2 - \iota^2)^2 \phi\rho &= (\iota^2 + \kappa^2) \rho + 2V\kappa\rho\iota \\ &= (\iota - \kappa)^2 \rho + 2\iota S\kappa\rho + 2\kappa S\iota\rho; \end{aligned}$$

or as in (24.) comp. 359, III. IV.,

$$\phi\rho = g\rho + V\lambda\rho\lambda', \quad S\rho\phi\rho = g\rho^2 + S\lambda\rho\lambda'\rho = 1; \quad \text{LXXII. LXXIII.}$$

* This equation, CLXXIII'. or CLXXII., which had been assigned by the author as a form of the equation of an ellipsoid, has been selected by his friend Professor Peter Guthrie Tait, now of Edinburgh, as the basis of an admirable Paper, entitled: "Quaternion Investigations connected with Fresnel's Wave-Surface," which appeared in the May number for 1865, of the *Quarterly Journal of Pure and Applied Mathematics*; and which the present writer can strongly recommend to the careful perusal of all quaternion students. Indeed, Professor Tait, who has already published tracts on other applications of Quaternions, mathematical and physical, including some on Electro-Dynamics, appears to the writer eminently fitted to carry on, happily and usefully, this new branch of mathematical science: and likely to become in it, if the expression may be allowed, one of the chief successors to its inventor.

with expressions for the constants g, λ, λ' , which give, by LXXVI., the following values for the scalar semiaxes,*

$$\text{CLXXVI.} \dots a = T\iota + T\kappa; \quad b = \frac{\kappa^2 - \iota^2}{T(\iota - \kappa)}; \quad c = T\iota - T\kappa;$$

whence conversely,

$$\text{CLXXVII.} \dots T\iota = \frac{a+c}{2} \quad T\kappa = \frac{a-c}{2}; \quad T(\iota - \kappa) = \frac{ac}{b}; \quad \&c.$$

(91.) Knowing thus the form CLXXV. of the function ϕ , which answers in the present case to the given equation CLXXIII. of the generating ellipsoid, there would be no difficulty in carrying on the calculations, so as to reproduce, in connexion with the *two* constants ι, κ , all the preceding theorems and formulæ of the present Series, respecting the Wave and the Index-Surface. But it may be more useful to show briefly, before we conclude the Series, how we can *pass* from *Quaternions* to *Cartesian Co-ordinates*, in any question or formula, of the kind lately considered.

(92.) The *three italic letters, ijk*, conceived to be connected by the *four fundamental relations*,

$$i^2 = j^2 = k^2 = ijk = -1, \quad (\text{A}), \quad 183,$$

were *originally* the *only peculiar symbols* of the present Calculus; and although they are not *now* so much used, as in the *early practice* of quaternions, because certain general *signs of operation*, such as S, V, T, U, K, have since been introduced, yet they (the symbols *ijk*) may be supposed to be *still familiar* to a student, as *links* between *quaternions* and *co-ordinates*.

(93.) We shall therefore merely write down here some leading expressions, of which the meaning and utility seem likely to be at once perceived, especially after the Calculations above performed in this Series.

(94.) The vector semiaxes of the generating ellipsoid being called α, β, γ (comp. (40.) (42.)), we may write,

$$\text{CLXXVIII.} \dots \alpha = ia, \quad \beta = jb, \quad \gamma = kc;$$

$$\text{CLXXIX.} \dots \phi\rho = \alpha^{-1}S\alpha^{-1}\rho + \beta^{-1}S\beta^{-1}\rho + \gamma^{-1}S\gamma^{-1}\rho = \Sigma\alpha^{-1}S\alpha^{-1}\rho = -\Sigma i\alpha^{-2}\rho;$$

$$\text{CLXXX.} \dots S\rho\phi\rho = \Sigma (S\alpha^{-1}\rho)^2 = \Sigma\alpha^{-2}\rho^2; \quad \text{CLXXXI.} \dots S\rho\phi^{-1}\rho = \Sigma\alpha^2\rho^2;$$

$$\text{CLXXXII.} \dots (\phi + e)\rho = \Sigma\alpha(\alpha^{-2} + e)S\alpha^{-1}\rho;$$

* The reader, at this stage, might perhaps usefully turn back to that *Construction of the Ellipsoid*, illustrated by Fig. 53 (p. 226), with the Remarks thereon, which were given in the few last Series of the Section II. i. 13, pages 223-233. It will be seen there that the *three vectors, \iota, \kappa, \iota - \kappa*, of which the lengths are expressed by CLXXVII., are the *three sides, CB, CA, AB*, of what may be called the *Generating Triangle ABC* in the Figure; and that the deduction CLXXVI., of the *three semiaxes, abc*, from the *two vector constants, \iota, \kappa*, with many connected results, can be very simply exhibited by *Geometry*. The whole subject, of the equation $T(\iota\rho + \kappa\rho) = \kappa^2 - \iota^2$ of the ellipsoid, was very fully treated in the *Lectures*; and the calculations may be made more general, by the transformations assigned in the long but important Section III. ii. 6 of the present *Elements*, so that it seems unnecessary to dwell more on it in this place.

CLXXXIII. . . $(\phi + e)^{-1}\rho = \Sigma a(a^2 + e)^{-1} S a^{-1}\rho$;

CLXXXIV. . . if $r^2 = T\rho^2 = \Sigma x^2$, then $v = r^{-2}(\phi + r^2)^{-1}\rho$
 $= r^{-2}\Sigma \frac{a S a^{-1}\rho}{r^2 - a^2} = -\Sigma \frac{ia^2x}{r^2 - a^2}$;

CLXXXV. . . for Wave, $0 = S\rho v = \Sigma \frac{a^2x^2}{r^2 - a^2} = \frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2}$;

or CLXXXVI. . . $1 = -S\rho\omega = -S\rho\phi v = -Sv\phi\rho$
 $= \Sigma \frac{x^2}{r^2 - a^2} = \frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2}$;

and the *Index-Surface* may be treated similarly, or obtained from the *Wave* by changing *abc* to their reciprocals.

423. As an *eighth specimen* of physical application we shall investigate, by quaternions, Mac Cullagh's *Theorem of the Polar Plane*,* and some things therewith connected, for an important case of incidence of polarized light on a biaxial crystal: namely, for what was called by him the case of *uniradial vibrations*.

(1.) Let homogeneous light in air (or in a vacuum), with a velocity† taken for unity, fall on a plane face of a doubly refracting crystal, with such a polarization that only *one* refracted ray shall result; let ρ, ρ', ρ'' denote the *vectors of ray-velocity* of the incident, refracted, and reflected lights respectively, ρ having the direction of the *incident ray*, prolonged *within the crystal*, but ρ'' that of the *reflected ray outside*; and let μ' be the *vector of wave-slowness*, or the *index-vector* (comp. 422, (1.)), for the refracted light: these *four* vectors being all drawn from a given point of incidence *o*, and μ' , like ρ' , being *within* the crystal.

(2.) Then, by *all‡ wave theories of light*, translated into the present notation, we have the equations,

I. . . $\rho^2 = S\mu'\rho' = \rho'^2 = -1$;

II. . . $\rho'' = -\nu\rho\nu^{-1}$, with II'. . . $\nu = \mu' - \rho$,

where ν is a *normal to the face*; whence also,

III. . . $\rho'' = \rho S \frac{\mu' + \rho}{\mu' - \rho} - 2\mu' S \frac{\rho}{\mu' - \rho}$;

IV. . . $\rho'' + \rho = 2\mu'$, if IV'. . . $\mu' = \nu^{-1}V\mu'\rho = \nu^{-1}V\nu\rho$;

and V. . . $\rho'' - \rho = -2\nu S\rho\nu^{-1} = -2\nu^{-1}S\rho\nu$;

* See pp. 39, 40 of the Paper by that great mathematical and physical philosopher, "On the Laws of Crystalline Reflexion and Refraction," already referred to in the Note to page 737 (Trans. R. I. A., Vol. XVIII., Part I.).

† Of course, by a suitable choice of the *units* of time and space, the *velocities* and *slownesses*, here spoken of, may be represented by *lines* as short as may be thought convenient.

‡ These equations may be deduced, for example, from the principles of Huyghens, as stated in his *Tractatus de Lumine* (Opera reliqua, Amst., 1728).

so that the *three vectors*, ρ , μ' , ρ'' , terminate on *one right line*, which is *perpendicular to the face of the crystal*: and the *bisector of the angle* between the *first and third* of them, or between the incident and reflected rays, is the *intersection* ι of the *plane of incidence* with the same plane face.

(3.) Let τ , τ' , τ'' be the *vectors of vibration* for the *three rays* ρ , ρ' , ρ'' , conceived to be drawn from their respective extremities; then, by *all** theories of *tangential vibration*, we have the equations,

$$\text{VI.} \dots Spr = 0; \quad \text{VII.} \dots S\mu'r' = 0; \quad \text{VIII.} \dots Sp''r'' = 0;$$

to which Mac Cullagh *adds* the supposition (a), that the vibration in the crystal is *perpendicular to the refracted ray*: or, with the present symbols, that

$$\text{IX.} \dots Sp'r' = 0; \quad \text{whence} \quad \text{X.} \dots \tau' \parallel V\mu'\rho',$$

the *direction of the refracted vibration* τ' being thus in general *determined*, when those of the vectors ρ' and μ' are given.

(4.) To deduce from τ' the *two other vibrations*, τ and τ'' , Mac Cullagh assumes, (b), the *Principle of Equivalent Vibrations*, expressed here by the formula,

$$\text{XI.} \dots \tau - \tau' + \tau'' = 0,$$

in virtue of which the *three vibrations* are *parallel* to one *common plane*, and the *refracted vibration* is the *vector sum* (or *resultant*) of the other two; (c), the *Principle of the Vis Viva*, by which the *reflected and refracted lights* are *together equal* to the *incident light*, which is conceived to have *caused* them; and (d), the *Principle of Constant Density of the Ether*, whereby the *masses of ether*, disturbed by the *three lights*, are simply *proportional* to their *volumes*: the *two last hypotheses*† being here jointly expressed by the equation,

$$\text{XII.} \dots Sv(\rho r^2 - \rho' r'^2 + \rho'' r''^2) = 0.$$

(5.) Eliminating ρ'' and τ'' from XII. by V. and XI., τ^2 goes off; and we find, with the help of I. and II', the following *linear equation* in τ ,

$$\text{XIII.} \dots 2S\frac{\tau}{r} = 1 + \frac{Sv\rho'}{Sv\rho} = \frac{Sp\rho'}{Spv}, \quad \text{if} \quad \text{XIII'.} \dots v' = \mu' - \rho';$$

a *second* such equation is obtained by eliminating ρ'' and τ'' by III. and XI. from VIII., and attending to I. VI. VII., namely,

$$\text{XIV.} \dots 2SpvS\mu'r = (\rho^2 - \mu'^2)Spr' = -S\mu'v'Spr';$$

and a *third* linear equation in τ is given immediately by VI.

* The equations VI. VII. VIII. hold good, for instance, on Fresnel's principles; but Fresnel's tangential vibration in the crystal has a direction *perpendicular* to that adopted by Mac Cullagh.

† In the concluding Note (p. 74) to this Paper, Professor Mac Cullagh refers to an elaborate Memoir by Professor Neumann, published in 1837 (in the Berlin Transactions for 1835), as containing precisely the *same system* of hypothetical principles respecting Light. But there was evidently a complete mutual independence, in the researches of those two eminent men. Some remarks on this subject will be found in the *Proceedings* of the R. I. A., Vol. I., pp. 232, 374, and Vol. II., p. 96.

(6.) Solving then for τ , by the rules of the present Calculus, this system of the three linear and scalar equations VI. XIII. XIV., we find for the *incident vibration* the following *vector expression*,*

$$\text{XV.} \dots \tau = \frac{V\rho\nu'r'}{2S\rho\nu}; \text{ or } \text{XV'.} \dots 2rS\rho\nu = r'S\rho\nu' - \nu'S\rho r';$$

and accordingly it may be verified by mere inspection, with the help of VII. and IX., that this vector value of τ satisfies the three scalar equations (5.). And when the *incident vibration* has been thus deduced from the *refracted vibration* r' , the *reflected vibration* r'' is at once given by the formula XI., or by the expression,

$$\text{XVI.} \dots r'' = r' - \tau;$$

(7.) The relation XV'. gives at once the *equation of complanarity*,

$$\text{XVII.} \dots S\nu'r r' = 0, \text{ or the formula } \text{XVIII.} \dots \mu' - \rho' \parallel r, r';$$

if then a *plane* be anywhere so drawn, as to be *parallel* (4.) to the *three vibrations* r, r', r'' , it will be *parallel also* to the line $\mu' - \rho'$, which *connects two corresponding points*, on the *wave* and *index surface* in the crystal: but this is one form of enunciation of Professor Mac Cullagh's *Theorem of the Polar Plane*, which theorem is thus deduced with great simplicity by quaternions, from the principles above supposed.

(8.) For example, if we suppose that *or* and *oq*, in Fig. 89, represent the *refracted ray* ρ' , and the *index vector* μ' corresponding, and if we draw through the line *rq* a plane perpendicular to the plane of the Figure, then the plane so drawn will *contain* (on the principles here considered) the *refracted vibration* r' , and will be *parallel* to both the *incident vibration* r and the *reflected vibration* r'' ; whence the *directions* of the two latter vibrations may be in general determined, as being also *perpendicular* respectively to the *incident* and *reflected rays*, ρ and ρ' : and then the *relative intensities* (Tr^2, Tr'^2, Tr''^2) of the *three lights* may be deduced from the *relative amplitudes* (Tr, Tr', Tr'') of the *three vibrations*, which may themselves be found from the *three complanar directions*, by a simple *resolution* of one line r' into two others, of which it is the *vector sum*, as if the vibrations were *forces*.

(9.) The equations II'. IV'. V. and XIII'. enable us to express the four vectors, $\mu' (= \rho + \nu)$, $\iota (= \rho - \nu^{-1}S\nu\rho)$, $\rho'' (= \rho - 2\nu^{-1}S\nu\rho)$, and $\rho' (= \rho + \tau - \nu')$, in terms of the three vectors ρ, ν, ν' , which are connected with each other by the relation,

$$\text{XIX.} \dots \iota (= \rho - \nu^{-1}S\nu\rho), \quad \rho'' (= \rho - 2\nu^{-1}S\nu\rho), \quad \text{and} \quad \rho' (= \rho + \nu - \nu').$$

$$\text{XIX.} \dots \nu^2 + 2S\nu\rho = S\nu'(\rho + \nu), \quad \text{because} \quad \text{XIX'.} \dots S\nu\rho' = S(\nu' - \nu)\rho,$$

* The expressions XV. XVI. enable us to determine, not only the *directions* Ur, Ur'' of the *incident* and *reflected vibrations*, but also their *amplitudes* Tr, Tr'' , or the *intensities* Tr^2, Tr''^2 of the incident and reflected lights, for any given or assumed amplitude Tr' of the *refracted vibration*, or intensity Tr'^2 of the *refracted light*, after having determined the *direction* Ur' of the *refracted vibration* by means of the formula X.

as in XIII., or because $\mu^2 - \rho^2 = S\mu'\nu'$ by I. and XIII'.; and with which r' is connected (VII. and IX.), by the two equations,

$$\text{XX.} \dots S(\rho + \nu)r' = 0, \text{ and } \text{XXI.} \dots S\nu'r' = 0;$$

while r and r'' are connected with the same three vectors, and with r' , by the relations VI. VIII. XI. XIII., which conduct, by elimination of r'' , to the following system (comp. (5.)) of three linear and scalar equations in r ,

$$\text{XXII.} \dots S\rho r = 0; \quad 2S\nu\rho S\nu r = S\nu'(\rho + \nu)S\nu r'; \quad 2S\nu\rho S r'^{-1}r = S\nu'\rho;$$

and therefore to the vector expression,

$$2rS\nu\rho = V\rho\nu'r', \text{ as in XV.}$$

(10.) By these or other transformations, there is no difficulty in deducing this new equation, in which ω may be any vector,

$$\text{XXIII.} \dots V\nu V\{(\rho - \omega)r - (\rho' - \omega)r' + (\rho'' - \omega)r''\}r' = 0;$$

and conversely, when ω is thus treated as *arbitrary*, the formula XXIII., with the relations (9.) between the vectors ρ , ρ' , ρ'' , ν , ν' , μ' , but *without* any restriction (*except itself*) on r , r' , r'' , is *sufficient* to give the *two* vector equations,

$$\text{XI.} \dots r - r' + r'' = 0, \text{ and } \text{XXIV.} \dots \rho r - \rho'r' + \rho''r'' = \alpha\nu^{-1} + y,$$

in which

XXV. $\dots \alpha = S\nu(\rho r - \rho'r' + \rho''r'') = S\nu\nu'r'$, and XXI. $\dots y = S(\rho r - \rho'r'\nu + \rho''r'')$; and which conduct to the *two* scalar equations (among others),

$$\text{XXVII.} \dots S\kappa(\rho r - \rho'r' + \rho''r'') = 0, \text{ if } \text{XXVII'.} \dots S\kappa\nu = 0,$$

and

$$\text{XXVIII.} \dots S\nu\rho(S\rho r - S\rho''r'') = S\nu\rho'S\mu'r';$$

so that if we *now* suppose the equations VI. VIII. IX. to be *given*, the equation VII. will *follow*, by XXVIII.; while, as a *case* of XXVII., and with the signification IV. or IV'. of ι , we have the equation,

$$\text{XXIX.} \dots S\iota(\rho r - \rho'r' + \rho''r'') = 0.$$

(11.) And thus (or otherwise) it may be shown, that the *three* scalar equations VI. VIII. IX., combined with the *one* vector formula XXIII., which (on account of the arbitrary ω) is equivalent to *five* scalar equations, are sufficient to give the *same* *direction* of r' , and the *same* *dependencies* of r and r'' thereon, as those expressed by the equations X. XV. XVI.; and therefore (among other consequences), to the formulæ XII. and XVII.

(12.) But the equations VI. VIII. IX. contain what may be called the *Principle of Rectangular Vibrations* (or of *vibrations rectangular to rays*); and the formula XXIII. is easily interpreted (416.), as expressing what may be termed the *Principle of the Resultant Couple*: namely the theorem, that *if the three vibrations* (or displacements), r , r' , r'' , *be regarded as three forces*, $R\tau$, $R'\tau'$, $R''\tau''$, *acting at the ends of the three rays*, ρ , ρ' , ρ'' , *or OR, OR', OR''* (drawn in the directions (1.) from the point of incidence o), *then this other system of three forces*, $R\tau$, $-R'\tau'$, $R''\tau''$ (conceived as applied to a solid body), *is equivalent to a single couple, of which the plane is parallel* (or the axis perpendicular) *to the face of the crystal.*

(13.) It follows then, by (10.) and (11.), that from these two principles,* (I.) and (II.), we can infer *all* the following :

(III.) the *Principle of Tangential Vibrations* (or of vibrations *tangential to the waves*);

(IV.) the *Principle of Equivalent Vibrations* (4.);

(V.) the *Principle of the Vis Viva*, as expressed (in conjunction with that of the *Constant Density of the Ether*) by the equation XII.;

(VI.) the *Principle* (or *Theorem*) *of the Polar Plane*;

And (VII.) what may be called the *Principle of Equivalent Moments*,† namely

* The word "Principle" is here employed with the usual latitude, as representing *either an hypothesis assumed, or a theorem deduced*, but made a *ground of subsequent deduction*. The *principle* (I.) of *rectangular vibrations coincides*, for the case of an *ordinary medium*, with the principle (III.) of *tangential vibrations*; but, for an *extraordinary medium*, except for the case (not here considered) of *ordinary rays* in an *uniaxial crystal*, these two principles are *distinct*, although *both* were assumed by Mac Cullagh and Neumann. The present writer has already disclaimed (in the Note to page 736) any responsibility for the *physical hypotheses*; so that the results given above are offered merely as instances of *mathematical deduction* and *generalization* attained through the Calculus of Quaternions.

† In a very clear and able Memoir, by Arthur Cayley, Esq. (now Professor Cayley), "On Professor Mac Cullagh's Theorem of the Polar Plane," which was read before the Royal Irish Academy on the 23rd of February, 1857, and has been printed in Vol. VI. of the *Proceedings* of that Academy (pages 481-491), this name "principle of equivalent moments," is given to a statement (p. 489), that "the moment of $R't$ round the axis AH , is equal to the sum of the moments of Rt and $R't'$ round the same axis"; the line AH being (p. 487) the intersection of the plane of incidence with the plane of separation of the two media, that is, with the *face* of the crystal: while Rt , $R't$, $R't'$ are lines representing (p. 488) the *three vibrations* (incident, refracted, and reflected), at the ends of the *three rays* AR , AR' , AR'' , which are drawn from the point of incidence A , so as to lie, *all three* (p. 487), *within the crystal*. And in fact, if this statement be modified, either by *changing the sign* of the moment of $R't'$ (p. 491), or by drawing the *reflected ray* AR'' , like the line OR'' of the present investigation *in the air* (or *in vacuo*), *instead of prolonging it backwards within the biaxial crystal*, it agrees with the case XXIX. of the more general formula XXVII., which is itself *included* in what has been called above the *Principle of the Resultant Couple*. In venturing thus to point out, as the subject obliged him to do, what seemed to him to be a slight inadvertence in a Paper of such interest and value, the present writer hopes that he will not be supposed to be deficient in the admiration (long since publicly expressed by him), which is due to the vast attainments of a mathematician so eminent as Professor Cayley.

Since the preceding Series 423, including its Notes (so far), was copied and sent to the printers, the writer's attention has been drawn to a *later Paper* by Mac Cullagh (read December 9th, 1839, and published in Vol. XXI., Part I., of the *Transactions* of the Royal Irish Academy, pp. 17-50), entitled "*An Essay towards a Dynamical Theory of crystalline Reflexion and Refraction*;" in which there is given at p. 43) a *theorem* essentially equivalent to the above-stated "Principle of the

theorem that *the Moment of the Refracted Vibration ($\mathfrak{E}'\tau'$) is equal to the Sum of the Moments of the Incident and Reflected Vibrations ($\mathfrak{E}\tau$ and $\mathfrak{E}''\tau''$), with respect to any line, which is on, or parallel to, the Face of the Crystal.*

* * * * *

[It appears by the Table of Initial Pages (see p. lix.), that the Author had intended to complete the work by the addition of Seven Articles.]

Resultant Couple," but expressed so as to include the case where the vibrations are not uniaxial, so that the double refraction of the crystal is allowed to manifest itself. Mac Cullagh speaks, in his enunciation of the theorem, of measuring each ray, in the direction of propagation: which agrees with, but of course anticipates, the direction of the reflected ray, adopted in the preceding investigation. The writer believes that subsequent experiments, by Jamin and others, are considered to diminish much the physical value of the theory above discussed.

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