To
Dr. Florian Cajori.

Compliments of
A. Palmer

921.

ANALYTIC GEOMETRY

ANALYTIC GEOMETRY

WITH INTRODUCTORY CHAPTER ON THE

CALCULUS

BY

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PREFACE

The object of this book is to present analytic geometry to the student in as natural and simple a manner as possible without losing mathematical rigor. The average student thinks visually instead of abstractly, and it is for the average student that this work has been written. It was prepared primarily to meet the requirements in mathematics for the second half of the first year at the Armour Institute of Technology. To make it adaptable to courses in other institutions of learning certain topics not usually taught in an engineering school have been added.

While it is useless to claim any great originality in treatment or in the selection of subject matter, the methods and illustrations have been thoroughly tested in the class room. It is believed that the topics are so presented as to bring the ideas within the grasp of students found in classes where mathematics is a required subject. No attempt has been made to be novel only; but the best ideas and treatment have been used, no matter how often they have appeared in other works on the subject.

The following points are to be especially noted:

(1) The great central idea is the passing from the geometric to the analytic and *vice versa*. This idea is held consistently throughout the book.

(2) In the beginning a broad foundation is laid in the algebraic treatment of geometric ideas. Here the student should acquire the analytic method if he is to make a success of the course.

(3) Transformation of coördinates is given early and used frequently throughout the book, not confined to a single chapter as is so frequently the case. The same may be said of polar coördinates.

(4) Fundamental concepts are dealt with in an informal as well as in a formal manner. The informal often fixes and clarifies the ideas where the formal does not.

(5) Numerous illustrative examples are worked out in order that the student may get a clear idea of the methods to be used in the solution of problems.

(6) The conic sections are treated from the starting point of the *focus* and *directrix* definition.

(7) Because of its great importance in engineering practice the empirical equation is dealt with more completely than is usual. This treatment has been made as elementary as possible, but sufficiently comprehensive to enable one to solve the average problem in empirical equations.

(8) The fundamental concepts of the calculus are presented in a very concrete manner, and a much greater use then is usual is made of the differential. The ideas are thus more readily visualized than is possible otherwise. The applications are mainly to tangents, normals, areas, and the discussion of equations.

(9) The concluding chapter gives an adequate and careful treatment of solid geometry so necessary in the study of the calculus.

(10) The exercises are numerous, carefully graded, and include many practical applications.

(11) In the introductory chapter are found various short tables and formulas, and at the end are given four place tables of logarithms and trigonometric functions.

The authors take this opportunity to express their indebtedness to their colleagues, Professors D. F. Campbell, H. R. Phalen, and W. L. Miser, for their assistance in the preparation of the text.

THE AUTHORS.

CHICAGO, ILL.,
May, 1921.

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ANALYTIC GEOMETRY

CHAPTER I

INTRODUCTION

1. Introductory remarks.—Although it is not always possible for a student to appreciate at the outset the content of a subject, it is well, however, to consider the object of the study, and to understand as far as possible its fundamental aims.

2. Algebra and geometry united.—Analytic geometry, or algebraic geometry, is a subject that unites algebra and geometry in such a manner that each clarifies and helps the other. Lagrange says: "As long as algebra and geometry travelled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection. It is to Descartes[1] that we owe the application of algebra to geometry—an application which has furnished the key to the greatest discoveries in all branches of mathematics."

3. Fundamental questions.—The fundamental questions of analytic geometry are three.

First, given a figure defined geometrically, to determine its equation, or algebraic representation.

[1] René Descartes (1596–1650) was one of the most distinguished philosophers. It was in pure mathematics, however, that he achieved the greatest and most lasting results, especially by his invention of analytic geometry. In developing this branch he had in mind the elucidation of algebra by means of geometric intuition and concepts. He introduced the present plan of representing known and unknown quantities, gave standing to the present system of exponents, and set forth the well known Descartes' Rule of Signs. His invention of analytic geometry may be said to constitute the point of departure of modern mathematics.

Second, given numbers or equations, to determine the geometric figure corresponding to them.

Third, to study the relations that exist between the geometric properties of a figure and the algebraic, or analytic, properties of the equation.

To pursue the subject of analytic geometry successfully the student should be familiar with plane and solid geometry, and should *know* algebra through quadratic equations and plane trigonometry.

While parts of analytic geometry can be applied at once to the solution of various interesting and practical problems, much of it is studied because it is used in more advanced subjects in mathematics.

Some of the more frequently used facts of algebra and trigonometry are given here for convenience of reference.

4. Algebra.—*Quadratic equations.*—The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$r_1 + r_2 = -\frac{b}{a}, \text{ and } r_1 r_2 = \frac{c}{a}.$$

These roots are

real and equal if $b^2 - 4ac = 0$,
real and unequal if $b^2 - 4ac > 0$,
imaginary if $b^2 - 4ac < 0$.

The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

Logarithms.

(1) $\log MN = \log M + \log N.$
(2) $\log (M \div N) = \log M - \log N.$
(3) $\log N^n = n \log N.$
(4) $\log \sqrt[n]{N} = \frac{1}{n} \log N.$
(5) $\log 1 = 0.$
(6) $\log_b b = 1.$

(7) $a^{\log_a N} = N.$

(8) $\log \dfrac{1}{N} = -\log N.$

(9) $\log_b N = \dfrac{1}{\log_a b} \log_a N.$

(10) $\log_b a \log_a b = 1.$

(11) $\log_e N = 2.302585 \log_{10} N.$

(12) $\log_{10} N = 0.43429 \log_e N.$

The base $e = 2.718281828459 \cdots$. $\pi = 3.141592653589 \cdots$.

5. Trigonometry.—*Formulas.*

(1) 2π radians $= 360°$, π radians $= 180°$.

(2) $1 \text{ radian} = \dfrac{180°}{\pi} = 57.29578° - = 57° \, 17' \, 44.8''.$

(3) $1° = \dfrac{\pi}{180} = 0.0174533 -$ radians.

(4) $\sin^2 \theta + \cos^2 \theta = 1.$

(5) $1 + \tan^2 \theta = \sec^2 \theta.$

(6) $1 + \cot^2 \theta = \csc^2 \theta.$

(7) $\sin \theta = \dfrac{1}{\csc \theta}$, and $\csc \theta = \dfrac{1}{\sin \theta}.$

(8) $\cos \theta = \dfrac{1}{\sec \theta}$, and $\sec \theta = \dfrac{1}{\cos \theta}.$

(9) $\tan \theta = \dfrac{1}{\cot \theta}$, and $\cot \theta = \dfrac{1}{\tan \theta}.$

(10) $\tan \theta = \dfrac{\sin \theta}{\cos \theta} = \dfrac{\sec \theta}{\csc \theta}.$

(11) $\cot \theta = \dfrac{\cos \theta}{\sin \theta} = \dfrac{\csc \theta}{\sec \theta}.$

(12) $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$

(13) $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$

(14) $\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$

(15) $\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$

(16) $\tan (\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$

(17) $\tan (\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.$

(18) $\sin 2\theta = 2 \sin \theta \cos \theta.$

(19) $\cos 2\theta = \cos^2 \theta - \sin \theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1.$

(20) $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}.$

(21) $\sin \frac{1}{2}\theta = \pm \sqrt{\frac{1 - \cos \theta}{2}}$. (22) $\cos \frac{1}{2}\theta = \pm \sqrt{\frac{1 + \cos \theta}{2}}$.

(23) $\tan \frac{1}{2}\theta = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$.

(24) $\sin \alpha + \sin \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$.

(25) $\sin \alpha - \sin \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)$.

(26) $\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)$.

(27) $\cos \alpha - \cos \beta = -2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)$.

(28) $\sin \alpha \cos \beta = \frac{1}{2} \sin (\alpha + \beta) + \frac{1}{2} \sin (\alpha - \beta)$.

(29) $\cos \alpha \sin \beta = \frac{1}{2} \sin (\alpha + \beta) - \frac{1}{2} \sin (\alpha - \beta)$.

(30) $\cos \alpha \cos \beta = \frac{1}{2} \cos (\alpha + \beta) + \frac{1}{2} \cos (\alpha - \beta)$.

(31) $\sin \alpha \sin \beta = -\frac{1}{2} \cos (\alpha + \beta) + \frac{1}{2} \cos (\alpha - \beta)$.

(32) $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$. (Sine Law.)

(33) $a^2 = b^2 + c^2 - 2bc \cos \alpha$. (Cosine Law.)

(34) $\sin (\frac{1}{2}\pi - \theta) = \cos \theta$.
$\cos (\frac{1}{2}\pi - \theta) = \sin \theta$.
$\tan (\frac{1}{2}\pi - \theta) = \cot \theta$.
$\cot (\frac{1}{2}\pi - \theta) = \tan \theta$.

(35) $\sin (\frac{1}{2}\pi + \theta) = \cos \theta$.
$\cos (\frac{1}{2}\pi + \theta) = -\sin \theta$.
$\tan (\frac{1}{2}\pi + \theta) = -\cot \theta$.
$\cot (\frac{1}{2}\pi + \theta) = -\tan \theta$.

(36) $\sin (\pi - \theta) = \sin \theta$.
$\cos (\pi - \theta) = -\cos \theta$.
$\tan (\pi - \theta) = -\tan \theta$.
$\cot (\pi - \theta) = -\cot \theta$.

(37) $\sin (\pi + \theta) = -\sin \theta$.
$\cos (\pi + \theta) = -\cos \theta$.
$\tan (\pi + \theta) = \tan \theta$.
$\cot (\pi + \theta) = \cot \theta$.

(38) $\sin (\frac{3}{2}\pi - \theta) = -\cos \theta$.
$\cos (\frac{3}{2}\pi - \theta) = -\sin \theta$.
$\tan (\frac{3}{2}\pi - \theta) = \cot \theta$.
$\cot (\frac{3}{2}\pi - \theta) = \tan \theta$.

(39) $\sin(\frac{3}{2}\pi + \theta) = -\cos\theta.$
$\cos(\frac{3}{2}\pi + \theta) = \sin\theta.$
$\tan(\frac{3}{2}\pi + \theta) = -\cot\theta.$
$\cot(\frac{3}{2}\pi + \theta) = -\tan\theta.$

(40) $\sin(2\pi - \theta) = -\sin\theta.$
$\cos(2\pi - \theta) = \cos\theta.$
$\tan(2\pi - \theta) = -\tan\theta.$
$\cot(2\pi - \theta) = -\cot\theta.$

(41) $\sin(-\theta) = -\sin\theta.$
$\cos(-\theta) = \cos\theta.$
$\tan(-\theta) = -\tan\theta.$
$\cot(-\theta) = -\cot\theta.$

6. Useful tables.

VALUES OF e^x FROM $x = 0$ TO $x = 4.9$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.00	1.11	1.22	1.35	1.49	1.65	1.82	2.01	2.23	2.46
1	2.72	3.00	3.32	3.67	4.06	4.48	4.95	5.47	6.05	6.69
2	7.39	8.17	9.03	9.97	11.0	12.2	13.5	14.9	16.4	18.2
3	20.1	22.2	24.5	27.1	30.0	33.1	36.6	40.4	44.7	49.4
4	54.6	60.3	66.7	73.7	81.5	90.0	99.5	109.9	121.5	134.3

VALUES OF e^{-x} FROM $x = 0$ TO $x = 4.9$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.00	0.90	0.82	0.74	0.67	0.61	0.55	0.50	0.45	0.41
1	0.37	0.33	0.30	0.27	0.25	0.22	0.20	0.18	0.17	0.15
2	0.14	0.12	0.11	0.10	0.09	0.08	0.07	0.07	0.06	0.06
3	0.05	0.05	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.02
4	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01

ANY FUNCTION IN TERMS OF EACH OF THE OTHERS

$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\dfrac{1}{\csc\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\dfrac{1}{\sqrt{1+\tan^2\theta}}$	$\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\dfrac{1}{\sec\theta}$	$\dfrac{\sqrt{\csc^2\theta-1}}{\csc\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\dfrac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\dfrac{1}{\sqrt{\csc^2\theta-1}}$
$\cot\theta$	$\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\dfrac{1}{\tan\theta}$	$\cot\theta$	$\dfrac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\csc^2\theta-1}$
$\sec\theta$	$\dfrac{1}{\sqrt{1-\sin^2\theta}}$	$\dfrac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\dfrac{\csc\theta}{\sqrt{\csc^2\theta-1}}$
$\csc\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{1-\cos^2\theta}}$	$\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\csc\theta$

TABLE OF FREQUENTLY USED TRIGONOMETRIC FUNCTIONS

$\theta°$	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	0	1	0	∞	1	∞
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	∞	0	∞	1
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
180°	π	0	-1	0	∞	-1	∞
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	-1	0	∞	0	∞	-1
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2
360°	2π	0	1	0	∞	1	∞

CHAPTER II

GEOMETRIC FACTS EXPRESSED ANALYTICALLY, AND CONVERSELY

7. General statement.—Geometry deals with points, lines, and figures composed of points and lines. Algebra deals with numbers and algebraic statements composed of numbers, such as the equation.

In order to study geometric relations by means of algebra, and conversely, it is necessary to be able to represent points, lines, and geometric figures by means of numbers and equations, and conversely. That is, it is necessary to be able to *translate* from the language of geometry to that of algebra, and conversely.

8. Points as numbers, and conversely.—If a point moves from A to B in a straight line, the point is said to **generate** the line segment AB, that is, the line segment AB is the **locus** of the point. If the point moves from B to A it generates the line segment BA. It is convenient to consider AB and BA as separate line segments having opposite directions. The arrow is often used to denote the positive direction.

Fig. 1.

Such line segments as AB and BA are called **directed line segments.** The point from which the moving point starts is called the **initial point,** and the point where it stops is called the **terminal point.**

It is to be noted that a line segment is read by naming the initial point first.

Let $X'X$ be a straight line of indefinite length, and choose: *first,* a unit of length; *second,* a direction of motion, which we shall call **positive** if toward the right and **negative** if toward the left; *third*, a point O called the **origin** from which to start.

Then any point P can be determined by a real number—integral, fractional, or irrational—which shows the number of units the point has moved from the origin.

The number is positive or negative according as the motion is in the positive or negative direction. The origin is designated by 0.

Conversely, any real number corresponds to a point which is distant that number of units in the proper direction from the origin.

Thus, in Fig. 2, $+6$ designates the point P_1 and -2 the point P_2; while R corresponds to $-3\frac{1}{2}$, and Q to $\sqrt{3}$.

The line $X'X$ is a directed straight line if it is thought of as generated by a point moving in the direction from X' to X or from X to X'.

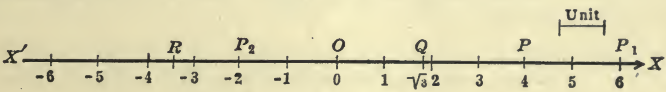


FIG. 2

9. The line segment.—The **magnitude** of a line segment is determined by the number of units in its length, that is, by the number of units a point moves in generating it.

The **value** of a line segment is determined by its length and direction, and is defined to be *the number which would represent the terminal point of the segment if the initial point were taken as origin.*

It follows from this definition that the value of a line segment read in one direction is the negative of the value if read in the opposite direction.

Thus, $AB = -BA$, or $AB + BA = 0$.

By the **numerical value** of a line segment is meant the number of units of length in it without reference to its direction.

Two line segments are equal if they have the same direction and the same length, that is, the same value.

In Fig. 3, $AB = +2$, $CD = +2$, $DN = +6$, $EC = -4$, $FA = -8$, $AB = CD$, and $AF = CN$. AC and FD are equal in numerical value.

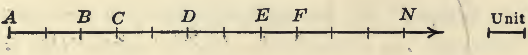


FIG. 3.

EXERCISES

1. Draw a line segment 5 in. long and take the origin at the center. Choose as a unit of measure a line $\frac{1}{4}$ in. long. What numbers designate the ends of the line? Locate the points corresponding to the numbers 9, $7\frac{1}{2}$, -4, $-3\frac{1}{2}$, $\sqrt{2}$, $-\sqrt{3}$, π.

2. Draw a line segment 20 units in length, with the origin, O, at the center. Locate the following points: A corresponding to 3, B corresponding to 8, C corresponding to -4, D corresponding to -10, E corresponding to 10. Give the values of the following line segments: AB, DA, CE, BC, EA, AC.

3. In exercise 2, how are the numbers designating the points affected if the origin is moved two units to the right? How are the values of the line segments affected?

FIG. 4.

10. Addition and subtraction of line segments.—In Fig. 4, if $A, B, C, \cdots M, N$ are any arrangement of points on a straight line, then

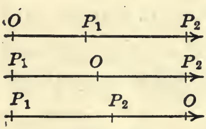


FIG. 5.

$$AB + BC + \cdots + MN + NA = 0.$$

For the moving point generates in succession the line segments AB, BC, $\cdots$ MN, NA, starting at A and returning to A. It therefore generates as much in the negative direction as in the positive. Hence the sum is zero.

A case of frequent occurrence is that of three points O, P_1, and P_2 on a straight line, Fig. 5. If O is taken as origin, then

[1] (1) $\mathbf{OP_2 = OP_1 + P_1P_2}$,

(2) $\mathbf{P_1P_2 = OP_2 - OP_1}$.

Proof. $OP_1 + P_1P_2 + P_2O = 0$. Why?

Adding $OP_2 = OP_2$, gives $OP_1 + P_1P_2 = OP_2$.

Adding $OP_2 + P_1O = OP_2 + P_1O$, gives $P_1P_2 = OP_2 + P_1O$.

$$\therefore P_1P_2 = OP_2 - OP_1.$$

11. Line segment between two points.—To find the value of the line segment between two points on a straight line, when the numbers determining these two points with reference to an origin on the same line, are known.

In Fig. 5, O is the origin and x_1 and x_2 are the numbers determining the points P_1 and P_2 respectively. It is required to find the value of the line segment P_1P_2, that is, the magnitude and direction of P_1P_2.

$$P_1P_2 = OP_2 - OP_1. \qquad \text{By [1].}$$

But $OP_1 = x_1$, $OP_2 = x_2$.

[2] $$\therefore \mathbf{P_1P_2 = x_2 - x_1}.$$

This states that *the value of the line segment between two points on a straight line is equal to the number determining the terminal point of the line segment minus the number determining its initial point, when a point on the straight line is taken as the origin.*

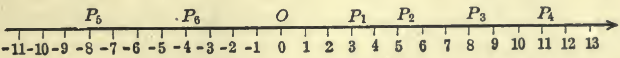


FIG. 6.

Thus, in Fig. 6, $P_1P_2 = OP_2 - OP_1 = 5 - 3 = 2$.

$$P_3P_2 = OP_2 - OP_3 = 5 - 8 = -3.$$
$$P_5P_6 = OP_6 - OP_5 = -4 - (-8) = +4.$$
$$P_6P_2 = OP_2 - OP_6 = 5 - (-4) = +9.$$
$$P_3P_5 = OP_5 - OP_3 = -8 - 8 = -16.$$

12. Geometric addition and subtraction of line segments. From the preceding article it readily follows that two line segments having the same or opposite directions can be added by placing the initial point of the second upon the terminal point of the first. *The sum of the line segments is the line segment* *having, as initial point, the initial point of the first and, as terminal point, the terminal point of the second.*

A line segment is subtracted from another by reversing its direction and adding.

Thus, in Fig. 6, $OP_1 + P_1P_3 = OP_3 = 8.$
$P_1P_3 + P_3P_5 = P_1P_5 = -11.$
$P_6P_3 - P_2P_3 = P_6P_3 + P_3P_2 = P_6P_2 = 9.$
$P_2P_4 - P_6P_4 = P_2P_4 + P_4P_6 = P_2P_6 = -9.$

EXERCISES

1. On a line with origin at O, locate the following points: A determined by 2, B by 3, C by 8, D by -5, E by -8. By the method of article 10, find the value of the line segments AB, BC, BD, AE, DE, EB, CE, CD.

2. On a line with origin at O, locate the points P_1, P_2, P_3, P_4, determined by the numbers x_1, x_2, x_3, x_4 respectively. (1) Give the values of the line segments P_1P_3, P_3P_4, P_4P_2, P_4P_1. (2) Give the line segments that have the following values: $x_4 - x_3$, $x_1 - x_4$, $x_3 - x_1$. Do the relative positions of the points make any difference in the answers?

13. Determination of a point in a plane.—It was shown in article 8 that the position of a point on a straight line can be determined by *one* number, which shows the direction and the distance that the point is from a fixed point on the straight line.

Various methods may be given for locating a point in a plane. For the purposes of analytic geometry, two of these will be chosen. They correspond to the two methods ordinarily used in locating a point on the surface of the earth.

First, a house in a city is located by giving its street and number. That is, by stating its distance and direction from each of two intersecting streets.

Second, a city may be located by giving its distance and direction from another city.

In analytic geometry, the two corresponding methods of locating a point in a plane are (1) the method by *cartesian coördinates*, and (2) the method by *polar coördinates.*

CARTESIAN COÖRDINATES

14. Coördinate axes.—(1) The lines of reference $X'X$ and $Y'Y$, Fig. 7, intersecting in the point O, are chosen. These lines are considered perpendicular to each other in this article, and will always be so taken unless otherwise stated.

The line $X'X$ is called the **axis of abscissas** or the **x-axis.** The line $Y'Y$ is called the **axis of ordinates** or the **y-axis.** Together they are called the **coördinate axes.**

When the coördinate axes are perpendicular to each other they form a **rectangular system.**

The coördinate axes divide the plane into four quadrants, numbered I, II, III, and IV as in trigonometry.

(2) A line segment of convenient length is chosen for a unit of measure. This may be of any length whatever.

(3) The direction is chosen as positive when towards the right parallel to the x-axis, or upwards parallel to the y-axis. Hence the negative direction is towards the left, or downwards.

15. Plotting a point.—A point P_1 in the plane is determined by the line segments N_1P_1 and M_1P_1, Fig. 7, drawn parallel to $X'X$ and $Y'Y$ respectively, for the values of these line segments tell how far and in what direction P_1 is from the lines of reference.

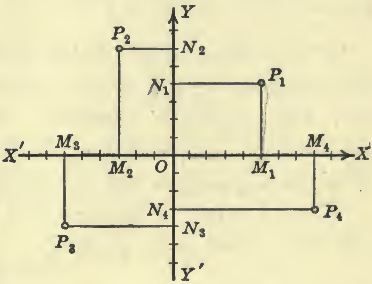


FIG. 7.

Here the line segment $N_1P_1 = +5$, and $M_1P_1 = +4$.

The point P_2 is determined by the line segments $N_2P_2 = -3$, and $M_2P_2 = +6$.

The point P_3 is determined by the line segments $N_3P_3 = -6$, and $M_3P_3 = -4$.

It is evident that any point in the plane is determined by *one pair of numbers, and only one;* and, conversely, every pair of real numbers determines *one point in the plane, and only one.*

The two numbers that determine a point in a plane are called the **coördinates** of the point. The number which is the value of the line segment parallel to the x-axis is called the **abscissa** of the point, and is usually represented by x. The number which is the value of the line segment parallel to the y-axis is called the **ordinate** of the point, and is usually represented by y.

The coördinates are written, for brevity, within parentheses and separated by a comma, the abscissa always being first, as (x, y). The letter designating the point is often written just before the parentheses.

Thus, the points in Fig. 7 are written: $P_1(5, 4)$, $P_2(-3, 6)$, $P_3(-6, -4)$, and $P_4(8, -3)$. The points M_1, M_2, N_1, N_3, and O are respectively the points $(5, 0)$, $(-3, 0)$, $(0, 4)$, $(0, -4)$, and $(0, 0)$.

It is evident that, in the first quadrant, both coördinates are positive; in the second quadrant, the abscissa is negative and the ordinate positive; in the third quadrant, both coördinates are negative; and, in the fourth quadrant, the abscissa is positive and the ordinate negative.

When a point is located in a plane by means of its coördinates it is said to be **plotted**.

The locating of points is greatly facilitated by using paper that is ruled into small squares. Such paper is called **coördinate paper**.

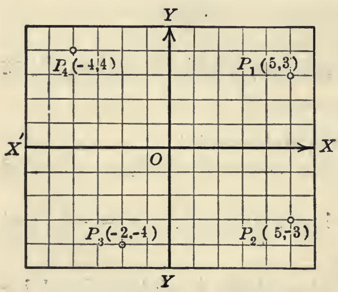


FIG. 8.

Example.—Plot the points $P_1(5,3)$, $P_2(5, -3)$, $P_3(-2, -4)$, and $P_4(-4, 4)$.

The point $P_1(5, 3)$ is plotted by counting off from O along $X'X$ a number of divisions equal to the abscissa 5, and then from the point so determined, a number of divisions on a line parallel to the y-axis, equal to the ordinate 3.

The points $(5, -3)$, $(-2, -4)$, and $(-4, 4)$ are located in a similar manner.

16. Oblique cartesian coördinates.—In determining a point in a plane, it is not necessary that the coördinate axes shall be perpendicular to each other, but they may form an angle ω. Such a set of axes is called an **oblique cartesian system.**

In Fig. 9, the abscissa of P_1 is $N_1P_1 = 3$, and its ordinate is $M_1P_1 = 5$. The coördinates of P_2 are $N_2P_2 = -4$, and $M_2P_2 = 3$.

17. Notation.—To secure clearness of statement, subscripts will be used with the letters designating points, and they will agree with the subscripts used with the coördinates of the points.

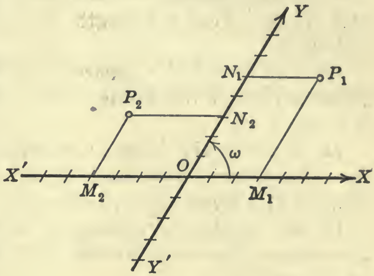


FIG. 9.

Thus, the point P_1 has coördinates (x_1, y_1), the point P_2 has coördinates (x_2, y_2), and so on.

Points designated in this manner will, in general, be fixed points, while a point that may vary in position will be designated by a letter, as P, without a subscript and have coördinates (x, y).

EXERCISES

1. Draw a pair of axes and plot the following points: (2, 3), (7, 9), (−2, 4), (−7, −2), (4, −3), (−2, −8), (0, 0), (0, 5), (−6, 0).

2. Draw the triangle whose vertices are (0, 2), (−2, −3), and (3, −2).

3. Draw the quadrilateral whose vertices are (3, 0), (0, 2), (−6, 2), and(0, −2).

4. If the ordinate of a point is 0, where is the point? Where if its abscissa is 0? Using x for the abscissa and y for the ordinate, express each as an equation.

5. What is the locus of all points that have abscissas equal to 5? Of all points having ordinates equal to 10? Use x for the abscissas and y for the ordinates and write these statements as equations.

6. The abscissas of two points are each a. How is the line joining them situated with reference to the y-axis? The ordinates of two points are each $-b$. How is the line joining them situated with reference to the x-axis? Write each of these lines as an equation.

7. Two points are placed so that the abscissa of each is equal to its ordinate. How is the line joining the points situated with reference to the coördinate axes? In what two quadrants can the points lie? Write the equation.

8. Two points are placed so that the abscissa of each is equal to the negative of the ordinate. How is the line connecting them situated with reference to the coördinate axes? In what two quadrants can the points lie? Write the equation.

9. Draw a rectangle whose vertices are $(-4, 2)$, $(-4, -5)$, $(7, -5)$, and $(7, 2)$. Find the length of its sides by differences of abscissas or ordinates.

10. The vertex of a square is at the origin, and a diagonal lies on the positive part of the x-axis. Find the coördinates of the other vertices if a side is 10.

11. What is the locus of a point which moves so that the ratio of its ordinate to its abscissa is always 1? So that this ratio is always -1? Always 2? Write the equations.

12. An equilateral triangle of side a has a vertex at the origin and one side on the x-axis at the right of the origin. Find the coördinates of its vertices.

13. A regular hexagon of side 8 is placed so that its center is at the origin and one diagonal is along the x-axis. Find the coördinates of its vertices.

18. Value of line segment parallel to an axis.—If the segment of a line is parallel to one of the coördinate axes, it has a definite direction as well as a length, that is, it has a value. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points on a line parallel to the x-axis, then

[2₁] $$\mathbf{P_1P_2 = x_2 - x_1.}$$

This follows directly from article 11, for if P_1P_2 intersects the y-axis in N_1, $P_1P_2 = N_1P_2 - N_1P_1 = x_2 - x_1$.

Likewise, if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points on a line parallel to the y-axis, then

[2₂] $$\mathbf{P_1P_2 = y_2 - y_1.}$$

The student should locate points in various positions and satisfy himself that [$\mathbf{2_1}$] and [$\mathbf{2_2}$] are true. Figure 10 shows several positions of P_1 and P_2

These facts may be stated as follows:

(1) *The value of a line segment parallel to the x-axis equals the abscissa of its terminal point minus the abscissa of its initial point.*

(2) *The value of a line segment parallel to the y-axis equals the ordinate of its terminal point minus the ordinate of its initial point.*

19. Distance between two points in rectangular coördinates.—(1) The distance between two points is the numerical value of the line segment connecting these points, that is, it is the length of the line segment connecting the two points. It follows that the distance between two points, having abscissas x_1 and x_2 on a line parallel to the x-axis, is either $x_2 - x_1$ or $x_1 - x_2$, the difference being taken positive when its numerical value can be determined.

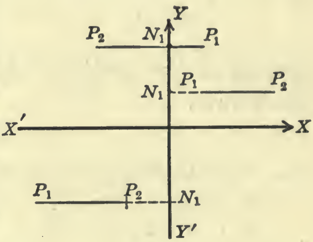


FIG. 10.

Likewise, when the two points are on a line parallel to the y-axis the distance between them is $y_2 - y_1$ or $y_1 - y_2$.

(2) Ordinarily a line segment that is not parallel to one of the coördinate axes does not have a direction assigned to it. We do not then speak of its value. The length of such a line segment is the distance between its end points.

The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula

$$[3] \qquad \mathbf{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.}$$

Proof.—Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the two points. Through P_1 and P_2 draw lines parallel, respectively, to the x-axis and y-axis to intersect in Q.

Then P_1QP_2 is a right triangle, and

$$d = P_1P_2 = \sqrt{\overline{P_1Q}^2 + \overline{QP_2}^2}.$$

But $P_1Q = x_2 - x_1$, and $QP_2 = y_2 - y_1$. By [2_1] and [2_2].

Hence $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Since $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$,

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

It should be noted that the line through P_1 could as well have been drawn parallel to the y-axis, and the line through P_2 parallel to the x-axis.

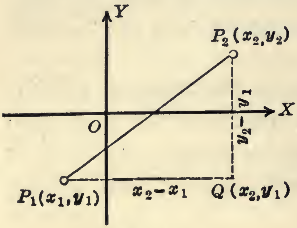


FIG. 11.

It is to be noted that the above proof is general and is made without reference to a figure. The student, however, should draw several figures locating the points in different positions and satisfy himself of the truth of [**3**]. Figure 11 shows one position of the points.

EXERCISES

1. Find the distance between each of the following pairs of points:

(1) (3, 4), (−6, −8). (3) (−1, 0), (12, −2).
(2) (−10, 4), (3, −9). (4) (6, 7), (−5, −5).

2. In Fig. 12, express each of the following line segments as the difference between two abscissas: M_1M_4, M_4M_3, M_2M_1, M_1M_3.

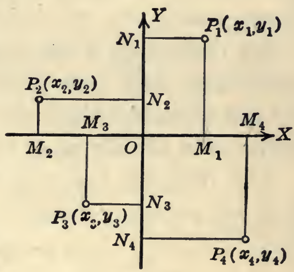


FIG. 12.

3. Express each of the following line segments as the difference between two ordinates: N_1N_2, N_3N_2, N_4N_1, N_2N_4.

4. Derive the formula for the distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, (1) when both P_1 and P_2 are in the first quadrant, (2) when P_1 is in the third and P_2 in the fourth quadrant, (3) when P_1 is in the fourth and P_2 in the second quadrant.

5. Find the lengths of the sides of the following triangles:

(1) (2, 3), (−5, 8), (−2, −4). (2) (3, −6), (0, 5), (−4, −2).

6. Show that the points (9, 12), (−3, −4), and $(5, 4 - 4\sqrt{6})$ lie on a circle whose center is at the point (3, 4).

7. Find a point whose abscissa is 3 and whose distance from (−3, 6) is 10.

Suggestion.—Let y be the ordinate of the point.

Then $\sqrt{(3+3)^2 + (y-6)^2} = 10$. Solve for y.

8. Find the center of the circle passing through the three points (6, 15), (13, 8), and (−4, −9).

Suggestion.—By the definition of a circle, if a circle passes through these three points, there must be a point (x, y) from which they are equally distant. Write the distance of each point from the point (x, y) and form two equations. Solve these equations for x and y.

9. Three vertices of a parallelogram are (−2, 4), (5, 2), and (6, 1). Find a fourth vertex. How many are there?

Suggestion.—Use the fact that the opposite sides of a parallelogram are equal.

10. Two vertices of an equilateral triangle are (2, 10) and (8, 2). Find the third vertex.

11. Find the equation which states that the point (x, y) is 5 units from the point (3, 4). What is the locus of the point (x, y)? Draw the locus.

12. Find the equation that expresses the fact that the point (x, y) is equally distant from the points (2, 3) and (7, −4). What is the locus?

13. Show that the values of line segments parallel to either axes in rectangular coördinates hold true when the axes are oblique.

14. If the axes are inclined to each other at an angle of ω, and if lines P_1Q and QP_2 of Fig. 11 are drawn parallel to the axes, then the angle P_1QP_2 equals ω or $180° - \omega$. By the cosine theorem of t igonometry show that then the distance between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega}$.

15. The angle between two oblique axes is 60°. Find the distance between the points (−2, 3) and (6, −4).

DIVISION OF A LINE SEGMENT

20. Internal and external division of a line segment.—If P_1 and P_2 are any two points on a straight line, then any third point, P_0, on the line is said to divide the line segment P_1P_2 into two parts.

FIG. 13.

The point P_0 is said to divide the line segment P_1P_2 **internally** if P_0 lies between P_1 and P_2; and **externally** if P_0 lies beyond P_2 as at P_0', or beyond P_1 as at P_0''.

When P_0 lies between P_1 and P_2 the two parts are P_1P_0 and P_0P_2. When P_0 lies at P_0' beyond P_2, the two parts are P_1P_0' and $P_0'P_2$. When P_0 lies at P'' beyond P_1, the two parts are P_1P_0'' and $P_0''P_2$. The parts are always read as here, that is, from the initial point to the division point and from the division point to the terminal point.

When the line segment P_1P_2 is divided *internally*, both P_1P_0 and P_0P_2 are read in the same direction, and therefore the ratio $\frac{P_1P_0}{P_0P_2}$ is positive, and has a small value when P_0 is near P_1, and a large value when P_0 is near P_2, that is, the value of the ratio is between 0 and $+\infty$.

When the line segment P_1P_2 is divided *externally* the two parts are read in opposite directions, and therefore the ratio $\frac{P_1P_0'}{P_0'P_2}$, or $\frac{P_1P_0''}{P_0''P_2}$, is negative. Further, when the point of division lies beyond P_2 the ratio is between $-\infty$ and -1, and when the point of division lies beyond P_1 the ratio is between -1 and 0.

It remains to express these geometric ideas analytically. This is done in the next articles.

EXERCISES

1. Upon a straight line locate two points P_1 and P_2 6 units apart. Locate a third point P_0 such that $\frac{P_1P_0}{P_0P_2} = \frac{2}{3}$. Such that $\frac{P_1P_0}{P_0P_2} = -\frac{2}{3}$.

Suggestion.—These may be determined by methods of plane geometry, or may be computed by algebra.

2. Divide a line 4 in. long into two parts that are in the ratio 3:1. In the ratio -5.

21. To find the coördinates of a point that divides a line segment in a given ratio.

Example 1. *Internal point.*—Required the coördinates of the point that divides the line segment from $P_1(-2, -4)$ to $P_2(5, 6)$ in the ratio $\frac{5}{2}$.

Solution.—Draw a pair of axes as in Fig. 14, and locate the points P_1 and P_2. Let $P_0(x_0, y_0)$ be the required point. Draw lines through these points parallel to the y-axis and cutting the x-axis in M_1, M_2, and M_0 respectively.

Then, by plane geometry, $\frac{M_1M_0}{M_0M_2} = \frac{P_1P_0}{P_0P_2}$.

But $M_1M_0 = x_0 - (-2)$ and $M_0M_2 = 5 - x_0$, by [$\mathbf{2}_1$], for the abscissas of M_1, M_0, and M_2 are respectively -2, x_0, and 5.

And it is given that $\frac{P_1P_0}{P_0P_2} = \frac{5}{2}$.

Hence $\frac{x_0 - (-2)}{5 - x_0} = \frac{5}{2}$.

Solving this equation, $x_0 = 3$.

Similarly, draw lines parallel to the x-axis cutting the y-axis in N_1 N_0, and N_2 respectively.

Then $\frac{N_1N_0}{N_0N_2} = \frac{P_1P_0}{P_0P_2}$.

But $N_1N_0 = y_0 - (-4)$ and $N_0N_2 = 6 - y_0$. [$\mathbf{2}_2$]

Hence $\frac{y_0 - (-4)}{6 - y_0} = \frac{5}{2}$.

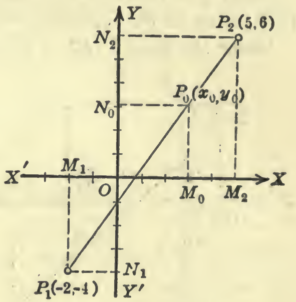


FIG. 14.

Solving this equation, $y_0 = 3\frac{1}{7}$.

Therefore the point P_0 has as coördinates $(3, 3\frac{1}{7})$.

Example 2. *External point.*—Required the coördinates of the point that divides the line segment from $P_1(-3, 5)$ to $P_2(2, -3)$ in the ratio $-\frac{5}{3}$.

Solution.—Locate P_1 and P_2 as in Fig. 15.

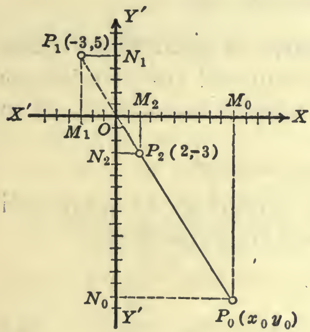


FIG. 15.

Since the ratio is $-\frac{5}{3}$, the point of division, $P_0(x_0, y_0)$ must be farther from P_1 than from P_2, and so is beyond P_2 as shown.

Draw the lines P_1M_1, P_2M_2, and P_0M_0 as in example 1.

Then $\frac{M_1M_0}{M_0M_2} = \frac{P_1P_0}{P_0P_2} = -\frac{5}{3}$.

But $M_1M_0 = x_0 - (-3)$ and $M_0M_2 = 2 - x_0$. [$\mathbf{2}_1$].

Hence $\frac{x_0 - (-3)}{2 - x_0} = -\frac{5}{3}$.

Solving this equation, $x_0 = 9\frac{1}{2}$.

Similarly, draw P_1N_1, P_2N_2, and P_0N_0.

Then $\frac{N_1N_0}{N_0N_2} = \frac{P_1P_0}{P_0P_2} = -\frac{5}{3}$.

But $N_1N_0 = y_0 - 5$ and $N_0N_2 = -3 - y_0$. [$\mathbf{2}_2$].

Hence $\frac{y_0 - 5}{-3 - y_0} = -\frac{5}{3}$.

Solving this equation, $y_0 = -15$.

Therefore the point P_0 has as coördinates $(9\frac{1}{2}, -15)$.

Example 3. *External point.*—Required the coördinates of the point that divides the line segment from $P_1(5, -2)$ to $P_2(-2, 4)$ in the ratio $-\frac{2}{5}$.

Solution.—Locate P_1 and P_2 as in Fig. 16.

Since the ratio is $-\frac{2}{5}$, the point of division $P_0(x_0, y_0)$ must lie nearer to P_1 than to P_2, and so is beyond P_1 as shown.

Draw the lines P_1M_1, P_2M_2, and P_0M_0 as in example 1.

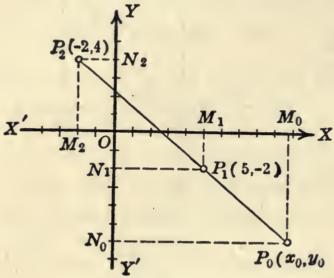


FIG. 16.

Then $\dfrac{M_1M_0}{M_0M_2} = \dfrac{P_1P_0}{P_0P_2} = -\dfrac{2}{5}$.

But $M_1M_0 = x_0 - 5$ and $M_0M_2 = -2 - x_0$. [$\mathbf{2}_1$].

Hence $\dfrac{x_0 - 5}{-2 - x_0} = -\dfrac{2}{5}$.

Solving this equation, $x_0 = 9\frac{2}{3}$.

Similarly, draw P_1N_1, P_2N_2, and P_0N_0.

Then $\dfrac{N_1N_0}{N_0N_2} = \dfrac{P_1P_0}{P_0P_2} = -\dfrac{2}{5}$.

But $N_1N_0 = y_0 - (-2)$ and $N_0N_2 = 4 - y_0$. [$\mathbf{2}_2$].

Hence $$\frac{y_0 + 2}{4 - y_0} = -\frac{2}{5}.$$

Solving this equation, $y_0 = -6$.

Therefore the point P_0 has as coördinates $(9\frac{2}{3}, -6)$.

22. Formulas for finding coördinates of point that divides a line segment in a given ratio.—Required the coördinates of the point that divides the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ in the ratio $r_1{:}r_2$

Let $P_0(x_0, y_0)$ be the required point.

Draw lines through P_1, P_2, and P_0 parallel to the y-axis and intersecting the x-axis in M_1, M_2, and M_0 respectively.

Then $\dfrac{M_1M_0}{M_0M_2} = \dfrac{P_1P_0}{P_0P_2} = \dfrac{r_1}{r_2}$.

But $M_1M_0 = x_0 - x_1$ and $M_0M_2 = x_2 - x_0$. [$\mathbf{2}_1$]

Hence $\dfrac{x_0 - x_1}{x_2 - x_0} = \dfrac{r_1}{r_2}$.

Solving for x_0, $$x_0 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}.$$

Similarly, draw lines through P_1, P_2, and P_0 parallel to the x-axis and intersecting the y-axis in N_1, N_2, and N_0 respectively.

Then $$\frac{N_1N_0}{N_0N_2} = \frac{P_1P_0}{P_0P_2} = \frac{r_1}{r_2}.$$

But $$N_1N_0 = y_0 - y_1 \text{ and } N_0N_2 = y_2 - y_0. \qquad [2_2]$$

Hence $$\frac{y_0 - y_1}{y_2 - y_0} = \frac{r_1}{r_2}.$$

Solving for y_0, $$y^0 = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}$$

Therefore the coördinates of P_0 are

[4] $$\mathbf{x}_0 = \frac{\mathbf{r}_1\mathbf{x}_2 + \mathbf{r}_2\mathbf{x}_1}{\mathbf{r}_1 + \mathbf{r}_2} \qquad \mathbf{y}_0 = \frac{\mathbf{r}_1\mathbf{y}_2 + \mathbf{r}_2\mathbf{y}_1}{\mathbf{r}_1 + \mathbf{r}_2}.$$

Special case.—It is frequently required to find the coördinates of the point bisecting a line segment. In this case the two parts are equal, and the ratio $\frac{r_1}{r_2} = 1$. Formula [4] then becomes

[5] $$\mathbf{x}_0 = \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}, \qquad \mathbf{y}_0 = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}.$$

It is readily seen that the results of the last two articles are true for oblique axes as well as for rectangular axes.

EXERCISES

In the first four exercises draw the figure, and solve without using formulas [4] and [5].

1. Find the coördinates of the point which divides the line from $(-5, -8)$ to $(-1, 4)$ in the ratio $3:1$.

2. Find the coördinates of the point which divides the line from $(-1, 4)$ to $(8, 1)$ in the ratio $1:3$.

3. Find the coördinates of the point which bisects the line from $(8, 6)$ to $(-2, -3)$.

4. Find the coördinates of the point which divides the line from $(-4, 8)$ to $(2, 6)$ in the ratio $-\frac{3}{4}$.

5. Do each of the first four exercises by the formulas.

6. Find the coördinates of the point which divides the line from $(3, -9)$ to $(-1, 5)$ in the ratio $5:3$.

7. Find the coördinates of the point which divides the line from $(-6, 8)$ to $(3, -2)$ in the ratio $3:1$. In the ratio $-2:3$.

8. Draw a triangle the coördinates of whose vertices are (1, 1), (2, −3), and (−4, −6), and find the coördinates of the middle points of its sides.

9. The coördinates of P are (2, 3) and of Q are (3, 4). Find the coördinates of R so that $PR : RQ = 3 : 4$.

10. Draw the triangle with vertices at (3, 5), (−5, −3), and (9, −7). Find the lengths of its medians.

11. Show that the line joining the middle points of two sides of the triangle, having as vertices the points (8, 6), (1, 1), and (4, −5), is equal to one-half the third side.

12. Prove that the diagonals of the parallelogram whose vertices are (10, 4), (−3, 4), (−6, −6), and (7, −6) bisect each other.

13. The middle point of a line is at (4, 6) and one extremity is at (−3, −2). Find the other extremity.

14. Find the coördinates of the points that trisect the line from (2, 2) to (−7, −4).

15. Show that the median of the trapezoid whose vertices are (0, 0), (a, 0), (b, c), and (d, c) equals one-half the sum of the parallel sides.

ANGLES FORMED BY LINES

23. The angle between two lines.—That one line forms an angle with another is a geometric idea, and does not necessarily depend upon whether or not the lines are considered as having a positive or negative sense, that is, direction. In order to express the facts analytically, we start with the following:

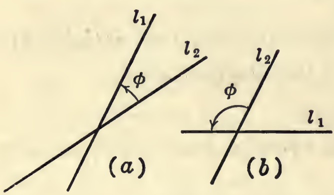


Fig. 17.

Definition.—The angle that a line l_1 makes with a line l_2 is the angle, not greater than 180°, generated by revolving l_2 in a positive direction until it coincides with l_1.

In Fig. 17, both (a) and (b), the angle φ is the angle that l_1 makes with l_2.

It follows that the angle that l_2 makes with l_1 is $\pi - \varphi$. The definition still holds when the lines do not intersect, that is, are not in the same plane, if it is understood that l_2 revolves in a plane parallel to l_1, until it is parallel to l_1.

24. Inclination and slope of a line.—An important special case of the angle that one line makes with another is the angle that a line makes with the x-axis. This angle is called the **inclination of the line.** It is always measured from the positive part of the x-axis.

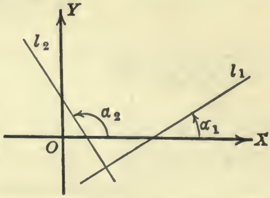


FIG. 18.

Thus, in Fig. 18, α_1 is the inclination of l_1 and α_2 the inclination of l_2.

Definition.—The tangent of the inclination of a line is called the **slope of the line.**

Thus, if m is the slope of a line and α its inclination, then $m = \tan \alpha$.

Since the inclination may be any angle in the first or second quadrant, the slope of a line may have any real value either positive or negative, including 0 and $\pm\ \infty$.

25. Analytical expression for slope of a line.

Example.—Required the slope and inclination of a line l passing through the points $P_1(2, 3)$ and $P_2(5, 6)$.

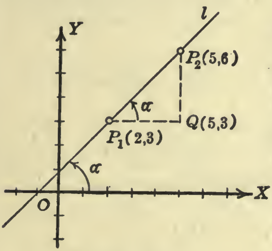


FIG. 19.

Solution.—Locate the points P_1 and P_2 and draw the line l as in Fig. 19. Through P_1 draw a line parallel to the x-axis, and through P_2 a line parallel to the y-axis. These lines meet at Q, and the angle QP_1P_2 is the inclination of l. Then $m = \tan QP_1P_2$.

But $\tan QP_1P_2 = \dfrac{QP_2}{P_1Q} = \dfrac{6-3}{5-2} = 1.$

Hence $m = 1$, and $\alpha = \tan^{-1}1 = 45°$.

It should be noted that, whatever the position of the points, the line drawn parallel to the x-axis is so drawn that an angle equal to the inclination is formed.

26. Formula for finding the slope of a line through two points.—Required to find the slope of a line l in terms of the coördinates of two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line.

Let the line be in either of the positions shown in Fig. 20.

In either case draw a line through P_1 parallel to the x-axis, forming an angle equal to α as shown; and through P_2 draw a line parallel to the y-axis meeting the first line in Q.

Then, whether the slope m of l is positive or negative,

$$m = \tan \alpha = \frac{QP_2}{P_1Q} = \frac{y_2 - y_1}{x_2 - x_1}.$$

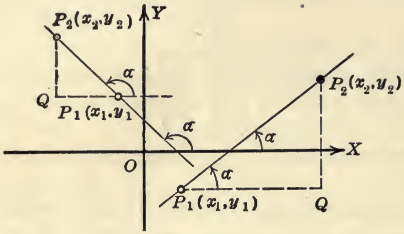


FIG. 20.

If P_1 and P_2 are interchanged, the slope is $\frac{y_1 - y_2}{x_1 - x_2}$, which equals $\frac{y_2 - y_1}{x_2 - x_1}$. Therefore, in any case, the formula is

$$[6] \quad \mathbf{m = \tan \alpha = \frac{y_1 - y_2}{x_1 - x_2}}.$$

27. The tangent of the angle that one line makes with another in terms of their slopes.—Required the tangent of the angle that line l_1, having a slope of m_1, makes with l_2, having a slope of m_2.

Let the inclinations of l_1 and l_2 be α_1 and α_2 respectively.

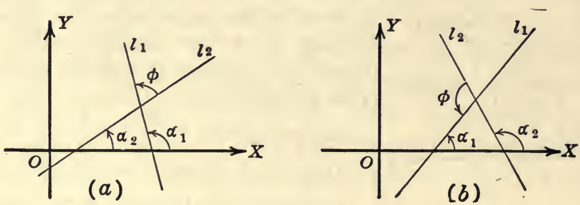


FIG. 21.

Then $\tan \alpha_1 = m_1$, and $\tan \alpha_2 = m_2$.

There are two cases: case I when $\alpha_1 > \alpha_2$, Fig. 21 (a); and case II when $\alpha_1 < \alpha_2$, Fig. 21 (b).

In each case, let φ be the angle that l_1 makes with l_2.

Then, in case I, $\alpha_1 = \alpha_2 + \varphi$, or $\varphi = \alpha_1 - \alpha_2$.

And, in case II, $\alpha_2 = \alpha_1 + (180° - \varphi)$,

or $$\varphi = 180° + (\alpha_1 - \alpha_2).$$

In either case,

$$\tan \varphi = \tan (\alpha_1 - \alpha_2) = \frac{\tan \alpha_1 - \tan \alpha_2}{1 + \tan \alpha_1 \tan \alpha_2} = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

[7] $$\mathbf{tan} \ = \varphi \, \frac{\mathbf{m_1 - m_2}}{\mathbf{1 + m_1 m_2}}.$$

28. Parallel and perpendicular lines.—*If two lines are parallel, their slopes are equal, and conversely.*

If two lines are perpendicular to each other, the slope of one is the negative reciprocal of the slope of the other, and conversely.

If line l_1 is parallel to line l_2, then $\alpha_1 = \alpha_2$, and $m_1 = m_2$.

Conversely. If $m_1 = m_2$, $\dfrac{m_1 - m_2}{1 + m_1 m_2} = 0$.

Then $\tan \varphi = 0$, and hence $\varphi = 0$.

Therefore l_1 and l_2 are parallel by **Art. 23.**

If l_1 and l_2 are perpendicular to each other, $\alpha_1 = \alpha_2 + 90°$, or $\alpha_1 = \alpha_2 - 90°$.

Then $\tan \alpha_1 = \tan (\alpha_2 \pm 90°) = -\cot \alpha_2 = -\dfrac{1}{\tan \alpha_2}$.

Therefore $m_1 = -\dfrac{1}{m_2}$, and $m_2 = -\dfrac{1}{m_1}$.

Conversely. If $m_1 = -\dfrac{1}{m_2}$, $\tan \alpha_1 = -\dfrac{1}{\tan \alpha_2} = -\cot \alpha_2$.

But $\cot \alpha_2 = \tan (90° - \alpha_2) = -\tan (\alpha_2 - 90°)$, or $\cot \alpha_2 = -\tan (90° + \alpha_2)$.

Then $\tan \alpha_1 = \tan (\alpha_2 - 90°)$, or $\tan \alpha_1 = \tan (90° + \alpha_2)$.

From this $\alpha_1 = \alpha_2 - 90°$, or $\alpha_1 = 90° + \alpha_2$.

Hence either $\alpha_2 - \alpha_1 = 90°$ or $\alpha_1 - \alpha_2 = 90°$.

Therefore $\varphi = 90°$, and l_1 and l_2 are perpendicular to each other.

The following are the important facts to remember:

[8] For parallel lines, $\mathbf{m_1 = m_2}$.

[9] For perpendicular lines, $\mathbf{m_1} = -\dfrac{1}{\mathbf{m_2}}$, and $\mathbf{m_2} = -\dfrac{1}{\mathbf{m_1}}$.

Example.—Find the angle that the line through (4, 5) and (−2, −4) makes with the line through (0, 4) and (−6, −8).

Solution.—The slope of l_1 is $m_1 = \dfrac{5 + 4}{4 + 2} = \dfrac{3}{2}.$

The slope of l_2 is $m_2 = \dfrac{4 + 8}{0 + 6} = 2.$

Substituting in [**4**], $\tan \varphi = \dfrac{\frac{3}{2} - 2}{1 + \frac{3}{2} \cdot 2} = -\dfrac{1}{8}.$

$$\therefore \varphi = \tan^{-1}\left(-\tfrac{1}{8}\right) = 172° \ 52.4'.$$

EXERCISES

1. Find the slopes of the lines through the following pairs of points:

(1) $(-4, -4)$ and $(4, 4)$.
(2) $(-4, 3)$ and $(-3, 2)$.
(3) $(5, 0)$ and $(6, \sqrt{3})$.
(4) $(-\sqrt{3}, \sqrt{2})$ and $(\sqrt{2}, \sqrt{3})$.
(5) $(-a, b)$ and (c, d).
(6) $(\sqrt{3}, 2)$ and $(\sqrt{2}, 3)$.

2. Find the inclination of each of the lines of exercise 1.

3. Find the slope of a line that is perpendicular to the line through the points $(3, 4)$ and $(-2, -3)$.

4. Show that the line through $(4, 2)$ and $(3, 7)$ is perpendicular to the line through $(8, 1)$ and $(13, 2)$.

5. Find the value of y so that the line through $(3, 7)$ and $(4, y)$ shall be perpendicular to the line through $(9, 10)$ and $(6, 8)$.

6. Prove by means of slopes that the three points $(6, -3)$, $(2, 3)$, and $(-2, 9)$ are on the same straight line.

7. Find the value of x so that the three points $(x, 6)$, $(2, 8)$, and $(4, 7)$ shall be on the same straight line.

8. Express by an equation the fact that a line passing through the points $(4, 5)$ and (x, y) has a slope of $\frac{3}{2}$.

9. A line passes through the point $(-4, 6)$ and has a slope of $-\frac{5}{3}$. Find the abscissa of the point on the line whose ordinate is -3.

10. Express by an equation the fact that a line passing through the point $(-3, -6)$ is perpendicular to the line through the points $(-2, 7)$ and $(4, 6)$.

11. Express by an equation the fact that a line passing through the point $(7, 2)$ is parallel to the line through $(-6, -2)$ and $(4, -7)$. Find the point on this line whose abscissa is -3.

12. Two lines l_1 and l_2 make $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}(-\frac{2}{3})$ respectively with the x-axis. Find the angle that l_1 makes with l_2.

13. Find the slope of the line that makes an angle of 47° with the line having a slope of 0.3674.

Suggestion.—Substitute $\varphi = 47°$ and $m_2 = 0.3674$ in [**7**] and solve for m_1.

14. Find the angle that the line through the points $(-3, 6)$ and $(4, -2)$ makes with the line through the points $(1, 1)$ and $(-7, -7)$.

15. A line passes through the point (4, 5) and is parallel to the line through the points (3, −2) and (−2, 5). Find where the line cuts the y-axis.

16. A line l makes an angle of 30° with the line through the points (2, 3) and (6, 7). Find the slope of l.

17. Show that the lines joining the middle points of the sides of a quadrilateral whose vertices are the points (5, −4), (3, 6), (−1, 4), and (−3, −2) taken in order, form a parallelogram.

18. Prove by means of the slopes of the sides that the quadrilateral whose vertices are the points (4, 2), (2, 6), (6, 8), and (8, 4) is a rectangle.

19. A point is equidistant from the points (−5, −2) and (2, −5), and the line joining the point to (4, 2) has a slope of $-\frac{1}{2}$. Find the coördinates of the point.

20. A line passes through the point (4, 5) and has a slope of 0.7236. Find the ordinate of the point on this line having as abscissa −2.

21. The vertices of a triangle are $P_1(3, 4)$, $P_2(-4, 3)$, and $P_3(-1, -4)$. Find the angle of the triangle at the vertex P_2.

POLAR COÖRDINATES

29. Location of points in a plane by polar coördinates.—Thus far only the first method mentioned in article 13 for

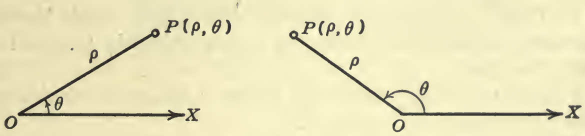


Fig. 22.

locating points, has been used. The second method, that by polar coördinates, has advantages over the cartesian system in certain cases. This method will now be explained.

In polar coördinates we locate a point in a plane by giving its distance and direction from a given fixed point in the plane. Thus, in Fig. 22, given the fixed point O in the fixed directed line OX, then any point P in the plane may be located by stating its distance $OP = \rho$ from O, and the angle θ through which OX must turn to coincide with OP.

Definitions.—The fixed point O is called the **pole** or **origin**;

the fixed line OX the **initial line, or polar axis;** the line segment $OP = \rho$ is called the **radius vector** of P; and the angle θ the **vectorial** or **directional angle** of P. Together, ρ and θ are the **polar coördinates** of P, and are written (ρ, θ).

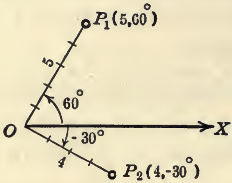


FIG. 23.

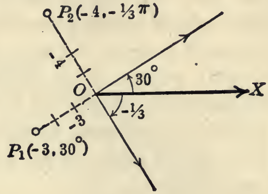


FIG. 24.

In order to use both positive and negative numbers as coördinates of points, the usual conventions of trigonometry as to positive and negative angles of any size are accepted. It is also agreed that the radius vector is positive if measured from O along the terminal side of the angle θ, and negative if measured in the opposite direction.

Thus, P_1 $(5, 60°)$ is located as shown in Fig. 23, the angle being measured counter-clockwise and the radius vector along the terminal side in the positive direction.

To plot the point $P_2(4, -30°)$, the angle is measured clockwise and the radius vector positive. (Fig. 23.)

The following points are plotted as shown in Fig. 24: $P_1(-3, 30°)$ and $P_2(-4, -\frac{1}{3}\pi)$.

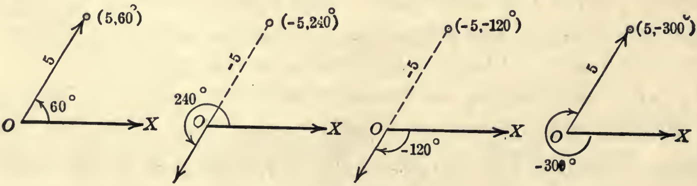


FIG. 25

From the above illustrations it is clear that one pair of polar coördinates determine one, and but one, point in the plane.

On the other hand, for a single point there are an indefinite number of pairs of polar coördinates.

Thus, if only values of θ numerically less than 360° are taken, then the four pairs of coördinates (5, 60°), (−5, 240°), (−5, −120°), and (5, −300°) all determine the same point as shown in Fig. 25.

For convenience in plotting, polar coördinate paper ruled with concentric circles and radial lines, as shown in Fig. 26, can be obtained. The following points are shown plotted in

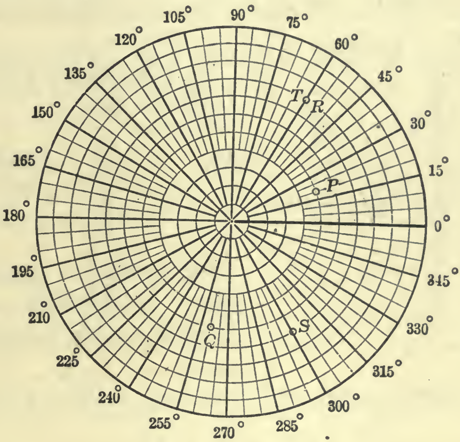


FIG. 26.

Fig. 26: $P(5,\ 20°)$, $Q(-6,\ 80°)$, $R(8,\ \frac{1}{3}\pi)$, $S(-7,\ \frac{2}{3}\pi)$, and $T(-8,\ -\frac{2}{3}\pi)$.

EXERCISES

1. Plot the following points in polar coördinates:

(1) (3, 30°).	(6) $(3,\ -\frac{2}{3}\pi)$.	(11) (3, 1).
(2) (7, 120°).	(7) $(-4,\ \pi)$.	(12) (−4, −2).
(3) (−2, 40°).	(8) $(-2,\ -\pi)$.	(13) (5, −3).
(4) (−6, 150°).	(9) (2, 0).	(14) (−6, −5).
(5) (4, −75°).	(10) $(-6\ \frac{7}{6}\pi)$.	(15) $(\pi,\ -\pi)$

2. Give three other pairs of coördinates in which θ is numerically less than 360° for each of the following points: (1) (7, 30°), (2) $(-3,\ \frac{3}{4}\pi)$.

3. The side of a square is 4 in. and the diagonal is taken as the polar axis with the pole at a vertex. Find the polar coördinates of the vertices.

4. What is the locus of all points for which $\rho = 5$? For which $\theta = \frac{1}{6}\pi$? For which $\theta = \frac{2}{3}\pi$?

30. Relations between rectangular and polar coördinates.—Let $X'X$ and $Y'Y$, Fig. 27, be a set of rectangular coördinate axes; and let the polar axis OX be taken on the positive part of the x-axis with the pole at the origin.

Let P be any point in the plane. Draw OP, and QP perpendicular to $X'X$. Then by the definitions already given, $OQ = x$, $QP = y$, $OP = \rho$, and $\angle XOP = \theta$.

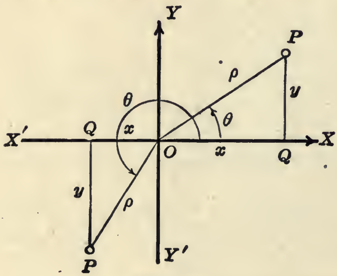


FIG. 27.

By trigonometry and geometry it follows that

[10] $$\mathbf{x = \rho \cos \theta,}$$
$$\mathbf{y = \rho \sin \theta,}$$
$$\mathbf{x^2 + y^2 = \rho^2.}$$

By means of these formulas polar coördinates can be expressed in rectangular coördinates.

Also by trigonometry and geometry it follows that

[11] $$\mathbf{\rho = \sqrt{x^2 + y^2},}$$
$$\mathbf{\theta = \tan^{-1} \frac{y}{x}.}$$

By means of these formulas rectangular coördinates can be expressed in polar coördinates.

EXERCISES

1. The origin in rectangular coördinates coincides with the pole in the polar system, and the x-axis falls upon the polar axis. Find the rectangular coördinates of the following points: $(6, \frac{1}{4}\pi)$, $(-2, \frac{1}{4}\pi)$, $(-5, \frac{1}{3}\pi)$, $(6, \frac{1}{6}\pi)$, $(3, \frac{2}{3}\pi)$, $(8, \frac{3}{4}\pi)$, $(2, \pi)$, $(6, \frac{3}{2}\pi)$.

2. Find the rectangular coördinates of the points whose polar coördinates are: (2, 40°), (3, 70°), (6.5, −30°), (1.2, 130°), (−4.5, 155°).

3. Find two pairs of polar coördinates for each of the following: $(4, 4\sqrt{3})$, $(-3, -3)$, $(3, 5)$, $(-\sqrt{2}, \sqrt{6})$.

4. By means of [10] derive $d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)}$ from [3]. Here (x_1, y_1) and (ρ_1, θ_1) are the same point, as also are (x_2, y_2) and (ρ_2, θ_2).

5. Derive the formula for the distance between two points in polar coördinates directly by means of the cosine theorem of trigonometry.

6. The polar coördinates of P are $(5, 75°)$ and of Q are $(4, 15°)$. Find the distance PQ.

TRANSFORMATION OF RECTANGULAR COÖRDINATES

31. Changing from one system of axes to another.—From the discussions already given, it is evident that the coördinate axes may be chosen at pleasure. In any particular case it is clear that they should be so chosen that they can be used to the best advantage. In order to discuss certain problems that occur in analytic geometry, it is necessary to express the coördinates of points in the plane in another system of coördinates than that in which they are already expressed.

It is of advantage to deduce formulas for making these transformations which are of two kinds.

(1) *Transformation by translation of axes,* or changing to new axes that are parallel to the original axes.

(2) *Transformation by rotation of axes,* or changing to new axes that make a certain angle with the original axes.

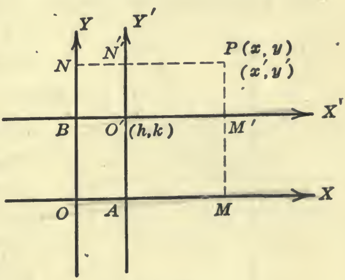


Fig. 28.

32. Translation of coördinate axes.—Let OX and OY, Fig. 28, be any system of cartesian axes; and let $O'X'$ and $O'Y'$ be another set parallel to the original. Let O', the origin of the new system, have coördinates (h, k) when referred to the original system.

Let P be any point in the plane, having coördinates (x, y) when referred to the original system and (x', y') when referred to the new system.

Draw a line through P parallel to the x-axis, intersecting OY in N and $O'Y'$ in N'. Also draw a line through P parallel to the y-axis, intersecting OX in M and $O'X'$ in M'.

Then $NP = NN' + N'P$, and $MP = MM' + M'P$. **Arts. 10** and **18.**

But $NP = x$, $NN' = h$, $N'P = x'$, $MP = y$, $MM' = k$, $M'P = y'$.

Therefore the formulas for translating the axes are:

$$[12] \qquad \mathbf{x = x' + h,} \quad \mathbf{y = y' + k.}$$

Solving these formulas for x' and y',

$$[12_1] \qquad \mathbf{x' = x - h,} \quad \mathbf{y' = y - k.}$$

In this article it is not implied that the axes are rectangular, and therefore the formulas hold for transforming from any set of cartesian coördinate axes to a parallel set.

33. Rotation of axes. Transformation to axes making an angle φ with the original.—Let OX and OY, Fig. 29, be any system of rectangular axes, and let OX' and OY' be another set of rectangular axes having the same origin as the original, but making an angle φ with OX and OY respectively.

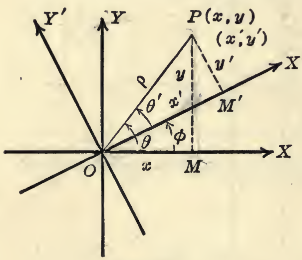


FIG. 29.

Let P be any point in the plane, having coördinates (x, y) when referred to the original system, and (x', y') when referred to the new system of axes.

Join O to P, draw MP perpendicular to OX, draw $M'P$ perpendicular to OX'. Let $\angle XOP = \theta$, and $\angle X'OP = \theta'$, and $OP = \rho$.

Then $x = \rho \cos \theta = \rho \cos (\theta' + \varphi)$

$$= \rho \cos \theta' \cos \varphi - \rho \sin \theta' \sin \varphi. \quad [10]$$

And $y = \rho \sin\theta = \rho \sin (\theta' + \varphi) =$

$$\rho \sin \theta' \cos \varphi + \rho \cos \theta' \sin \varphi. \quad [10]$$

But $x' = \rho \cos \theta'$ and $y' = \rho \sin \theta'$. [10]

Substituting these values,

[13] $$\mathbf{x = x' \cos \varphi - y' \sin \varphi,}$$
$$\mathbf{y = x' \sin \varphi + y' \cos \varphi.}$$

Solving these formulas for x' and y',

[13_1] $$\mathbf{x' = x \cos \varphi + y \sin \varphi,}$$
$$\mathbf{y' = y \cos \varphi - x \sin \varphi.}$$

Example.—The point P has coördinates $(\sqrt{2}, 2\sqrt{2})$ when referred to a certain system of rectangular axes. Find the coördinates of P when referred to a new set of rectangular axes having the same origin but making an angle of 45° with the original.

Solution.—Here we are to find x' and y' when x, y, and φ are known, and so we use formulas [13_1].

Substituting in these formulas,

$$x' = \sqrt{2} \cos 45° + 2\sqrt{2} \sin 45°$$
$$= \sqrt{2} \times \tfrac{1}{2}\sqrt{2} + 2\sqrt{2} \times \tfrac{1}{2}\sqrt{2} = 1 + 2 = 3.$$
$$y' = 2\sqrt{2} \cos 45° - \sqrt{2} \sin 45°$$
$$= 2\sqrt{2} \times \tfrac{1}{2}\sqrt{2} - \sqrt{2} \times \tfrac{1}{2}\sqrt{2} = 2 - 1 = 1.$$

Check the values by a drawing.

EXERCISES

1. Find the coördinates of the following points when referred to axes parallel to the original and with origin at the point (3, 4): (7, 8), (4, 3), (0, 0), (−2, 6), (−7, −5), (6, −8).

2. Find the coördinates of the following points when referred to axes having the same origin as the original, but making an angle of 45° with them: (2, 3), (−3, 4), (−5, −5), (7, −1).

3. The coördinates of the vertices of a triangle are $P_1(-3, -4)$, $P_2(6, -2)$, and $P_3(2, 7)$. Find the coördinates of the vertices when referred to parallel axes with origin at P_1. Plot.

4. The coördinates of a point P_1 are (3, 2). Find the coördinates of the origin of a new set of axes parallel to the old so that the coördinates of P_1 shall be (−4, −6) when referred to the new axes.

5. The coördinates of the vertices of a triangle are $P_1(0, 0)$, $P_2(2, 2\sqrt{3})$, and $P_3(-2, 4)$. Find the angle through which the axes must be rotated so that P_2 shall lie on the new x-axis. Find the coördinates of the vertices of the triangle referred to the new axes.

6. Derive formulas [**13**$_1$] from Fig. 29 without solving [**13**].

AREAS OF POLYGONS

34. Area of a triangle in rectangular coördinates.—Let $\Delta P_1P_2P_3$, Fig. 30, be any triangle having vertices $P_1(x_1,y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ referred to the axes OX and OY. Translate the axes to a new system having as origin one vertex of the triangle, say P_1 (x_1, y_1),

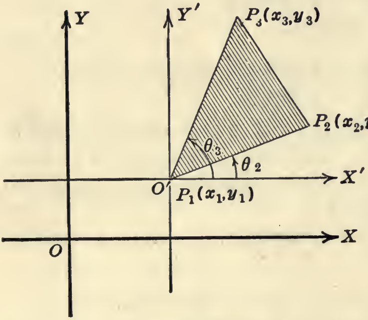


Fig. 30.

Then the coördinates of P_2 and P_3 referred to $O'X'$ and $O'Y'$ are $P_2(x_2', y_2')$ and $P_3(x_3', y_3')$ respectively, where $x_2' = x_2 - x_1$, $y_2' = y_2 - y_1$, $x_3' = x_3 - x_1$, $y_3' = y_3 - y_1$ by [**12**$_1$].

Let $\angle X'O'P_2 = \theta_2$, and $\angle X'O'P_3 = \theta_3$.

The area of $\Delta P_1P_2P_3 = \frac{1}{2}P_1P_2$ times the altitude from P_3 to P_1P_2.

Hence area $\Delta P_1P_2P_3 = \frac{1}{2}P_1P_2 \times P_1P_3 \sin (\theta_3 - \theta_2)$

$$= \tfrac{1}{2}P_1P_2 \times P_1P_3 (\sin\theta_3 \cos\theta_2 - \cos\theta_3 \sin\theta_2)$$
$$= \tfrac{1}{2}(P_1P_2 \cos\theta_2 \times P_1P_3 \sin\theta_3 - P_1P_3 \cos\theta_3 \times P_1P_2 \sin\theta_2).$$

But $P_1P_2 \cos\theta_2 = x_2' = x_2 - x_1$, $P_1P_3 \sin\theta_3 = y_3' = y_3 - y_1$,

$$P_1P_3 \cos\theta_3 = x_3' = x_3 - x_1,\ P_1P_2 \sin\theta_2 = y_2' = y_2 - y_1.$$

Substituting these values and putting A for area of $\Delta P_1P_2P_3$,

$$A = \tfrac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)].$$

Multiplying and arranging,

[**14**] $\mathbf{A = \tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)}.$

This may be written in the determinant form

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

by which it can be readily remembered. The formula can also be remembered and the computation carried out by the following:

Rule.—First, write down in a line the abscissas of the vertices taken in counter-clockwise order, repeating the first abscissa at the end; and under the abscissas write the corresponding ordinates.

Thus,
$$\begin{matrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{matrix}$$

Second, multiply each abscissa by the ordinate of the following column, and add. This gives $x_1y_2 + x_2y_3 + x_3y_1$.

Third, multiply each ordinate by the abscissa of the following column and add. This gives $y_1x_2 + y_2x_3 + y_3x_1$.

Fourth, subtract the third from the second and divide by 2. This gives the area as in formula [**14**].

It is evident that the expression $\frac{1}{2}P_1P_2 \times P_1P_3 \sin(\theta_3 - \theta_2)$ for the area is positive or negative according as sin $(\theta_3 - \theta_2)$ is positive or negative. In order then to have the area positive, $\sin(\theta_3 - \theta_2)$ must be positive. Hence $\theta_3 - \theta_2$ must be positive, and $\theta_3 > \theta_2$. That is, P_1P_2 is turned counter-clockwise to coincide with P_1P_3. This will be true only if, in passing around the triangle in the order the vertices are taken, the area is always at the left as shown in Fig. 31. That is, a point moving around the triangle must move counter-clockwise. Otherwise the area will be negative.

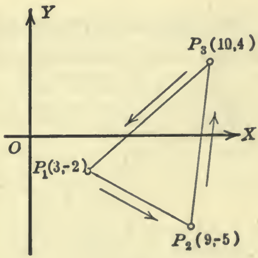


FIG. 31.

Thus, in Fig. 31, the area of the triangle, if the vertices are taken in the order P_1, P_2, P_3, is

$$A = \tfrac{1}{2}[3(-5) - 9(-2) + 9\cdot 4 - 10(-5) + 10(-2) - 3\cdot 4] = 28\tfrac{1}{2}.$$

If, however, the vertices are taken in the order P_1, P_3, P_2, the area is $A = \frac{1}{2}[3\cdot 4 - 10(-2) + 10(-5) - 9\cdot 4 + 9(-2) - 3(-5)] = -28\frac{1}{2}$.

35. Area of any polygon.—Any polygon having its vertices given in rectangular coördinates can be divided into triangles by diagonals drawn from any vertex. Its area can then be found. It can be readily shown that the area of any polygon can be found by the rule given for finding the area of a triangle.

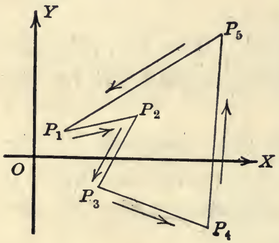


FIG. 32.

Thus, a polygon, Fig. 32, having five vertices as follows, taken in order *counter-clockwise;* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ $P_4(x_4, y_4)$, and $P_5(x_5, y_5)$, has its area represented by the scheme,

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_1 \end{matrix}$$

which evaluated by the rule gives

$$A = \tfrac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_5 + x_5y_1) - (y_1x_2 + y_2x_3 + y_3x_4 + y_4x_5 + y_5x_1)].$$

Example.—Find the area of the polygon having the following vertices: $(-3, 6)$, $(2, -4)$, $(8, 1)$, and $(4, 7)$.

Solution.—Arranging the abscissas and ordinates,

$$\begin{matrix} -3 & 2 & 8 & 4 & -3 \\ 6 & -4 & 1 & 7 & 6 \end{matrix}$$

$A = \frac{1}{2}\{[(-3)(-4) + 2\cdot 1 + 8\cdot 7 + 4\cdot 6] - [6\cdot 2 + (-4)8 + 1\cdot 4 + 7(-3)]\} = 65\frac{1}{2}$ square units.

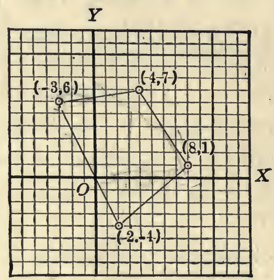


FIG. 33.

EXERCISES

1. Find the area of the triangles having the following points as vertices, in each case draw the figure:

(1) $(0, 0)$, $(10, 12)$, $(-6, 8)$.
(2) $(-4, 6)$, $(-2, 9)$, $(10, -4)$.
(3) $(17, 2)$, $(-3, 9)$, $(-6, -10)$.
(4) $(0, 7)$, $(10, -3)$, $(-4, 9)$.

2. Find the area of the quadrilateral with vertices (2, 5), (−7, 9), (−10, −3), and (6, −9).

3. Find the area of the pentagon with vertices (1, −2), (3, 1),(6, 2), (4, −4), and (2, −5).

4. Show that the area of the triangle with vertices (2, 4), (−6, −8), and (10, −4) is four times the area of the triangle formed by joining the middle points of the sides.

5. Find the area of the isosceles triangle with vertices (4, 5), (10, 13), and (4, 15). Find the altitude from the vertex at (4, 5), and find the area as one-half the product of the base and altitude. Do the two results agree?

6. Find the area of the triangle with vertices $P_1(7, 9)$, $P_2(-6, -8)$, and P_3 (4, −6). Find the point P_4 dividing P_2P_3 in the ratio 2:3, and show by areas that the triangle is divided into two triangles the areas of which are in the ratio 2:3.

7. If the vertices of a triangle in polar coördinates are P_1 (ρ_1, θ_1), $P_2(\rho_2, \theta_2)$, and $P_3(\rho_3, \theta_3)$, derive a formula for its area.

Suggestion.—In **[14]** put $x_1 = \rho_1 \cos \theta_1$, $y_1 = \rho_1 \sin \theta_1$, and similarly for P_2 and P_3. Arrange and apply the subtraction formula for sines.

8. Find the area of a triangle the vertices of which in polar coördinates are (10, 30°), (−12, 120°), and (6, 135°).

APPLICATIONS

36. Analytic methods applied to the proofs of geometric theorems.—One of the necessary conditions for the mastery of a mathematical subject is a thorough understanding of the fundamental ideas and methods. In the present chapter, stress has been laid upon the expressing of geometric ideas in an analytic form. Time will be well spent in reviewing these methods until they are fully comprehended.

As will be repeatedly found in subsequent chapters, analytic geometry gives a powerful method for treating a great variety of geometric questions. As an illustration of this a few elementary examples of the application of algebra to geometry are given in this article.

One of the great advantages of the analytic method of solving geometric problems lies in the fact that *an analytic result obtained by the simplest arrangement of the axes with* *reference to the geometric figure holds equally well for all other arrangements of the axes.*

It is well then always to make use of the simplest relations between the geometric figure and the coördinate axes.

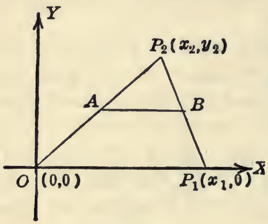


FIG. 34.

Example 1.—Prove analytically that the line segment joining the middle points of two sides of a triangle is equal to one-half the third side and is parallel to it.

Given any triangle OP_1P_2, and AB bisecting OP_2 and P_1P_2.

To prove $AB = \frac{1}{2}OP_1$, and that AB is parallel to OP_1.

Proof.—Choose the coördinate axes with origin at O and the x-axis along OP_1. Then the coördinates of O are $(0, 0)$, and P_1 and P_2 may be designated by $(x_1, 0)$ and (x_2, y_2) respectively.

Coördinates of A and B are $\left(\frac{x_2}{2}, \frac{y_2}{2}\right)$ and $\left(\frac{x_1 + x_2}{2}, \frac{y_2}{2}\right)$ respectively. [5]

$$\text{Length of } AB = \sqrt{\left(\frac{x_1 + x_2}{2} - \frac{x_2}{2}\right)^2 + \left(\frac{y_2}{2} - \frac{y_2}{2}\right)^2} = \frac{x_1}{2}. \quad [3]$$

But $$OP_1 = x_1 - 0 = x_1.$$

$$\therefore AB = \tfrac{1}{2}OP_1.$$

Also slope of $AB = 0$, and slope of $OP_1 = 0$. [6]

$$\therefore AB \text{ is parallel to } OP_1.$$

To see the desirability of this choice of the axes, the student should write out the proof when the vertices of the triangle are (x_1, y_1), (x_2, y_2), and (x_3, y_3).

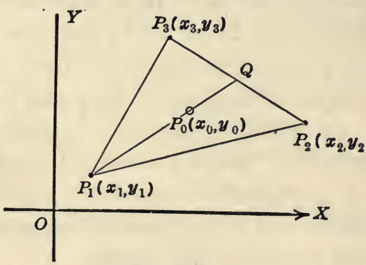


FIG. 35.

Example 2.—Derive a formula for the center of gravity of a triangle with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$; it being known that the center of gravity of a triangle is at the intersection of its medians, which is two-thirds of the length of any median from a vertex.

Solution.—Here no choice of axes can be made that will simplify the work.

Choose any median as P_1Q. Then it is required to find the coördinates (x_0, y_0) of P_0 such that $P_1P_0 : P_0Q = 2:1$.

Coördinates of Q are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$.

By [4] $\quad x_0 = \frac{x_1 + x_2 + x_3}{3}$, and $y_0 = \frac{y_1 + y_2 + y_3}{3}$.

EXERCISES

1. Use the formulas derived in example 2 and find the coördinates of the center of gravity of a triangle with vertices (2, 6), (−8, 3), and (−4, −3).

Prove the theorems in exercises 2–12.

2. The diagonals of a rectangle are equal.

3. The diagonals of a parallelogram bisect each other.

4. The medians of a triangle intersect in a point which is two-thirds the length of any median from a vertex.

5. The middle point of the hypotenuse of a right triangle is equally distant from the three vertices.

6. The diagonals of a square are perpendicular to each other.

7. If the diagonals of a parallelogram are equal, the figure is a rectangle.

8. The distance between the middle points of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.

9. The lines joining the middle points of the successive sides of any quadrilateral form a parallelogram.

10. The lines joining the middle points of the successive sides of any rectangle form a rhombus.

11. In any quadrilateral, the lines joining the middle points of the opposite sides, and the line joining the middle points of the diagonals meet in a point and bisect each other.

12. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of its diagonals.

13. Given P_1 any point in the plane of a rectangle, prove that the sum of the squares of the distances from P_1 to two opposite vertices of the rectangle is equal to the sum of the squares of the distances from P_1 to the other two vertices.

GENERAL EXERCISES

1. If the points A, B, C, D, and E are any points on the same straight line, show that:

(1) $AB + CD - CB - ED = AE.$
(2) $AE + EB + DE + EC - DB = AC.$
(3) $AC - EB + CB - AE = 0.$

2. If the coördinates of the vertices of a rectangle are (0, 0), (8, 0), (8, 6), and (0, 6), what will be the oblique coördinates of its vertices if the y-axis is the diagonal through the origin, the x-axis remaining as before?

3. What are the oblique coördinates of the vertices of the rectangle of exercise 2 if the y-axis is taken as the diagonal through the point (0, 6)?

4. What are the coördinates of the vertices of a square if a side is $4\sqrt{2}$, and its diagonals are taken as the coördinate axes?

5. A rhombus lies wholly in the first quadrant, and the angle between two of its sides is 30°. If the coördinates of two of its vertices are (0, 0) and $(a, 0)$, find the coördinates of the remaining vertices.

6. Find the coördinates of the vertices of an equilateral triangle of side a if its center is at the origin and the y-axis passes through one vertex.

7. The angle between two oblique axes is 135°. Find the distance between the points (1, 3) and (−1, −3).

8. What is the ratio in which the y-axis divides the line segment joining (−2, 3) to (5, −1)?

9. Find the coördinates of two points which divide the line segment from (2, 4) to (8, −8) internally and externally in the ratio whose numerical value is 2.

10. Find the coördinates of P_1 and P_2 where P_1 is on the positive y-axis, P_2 on the positive x-axis, and the point (2, 3) divides P_1P_2 in the ratio 2 : 1.

11. The point (−2, −2) divides the line P_1P_2 in the ratio −4 : 3. If P_1 has the coördinates (2, 6), find the coördinates of P_2.

12. If P_1 has the coördinates (1, 4) and P_2 the coördinates (5, 1), find a point P_3 on P_1P_2 such that P_1P_2 will be a mean proportional between P_1P_3 and 25.

13. Prove analytically that the diagonals of a rhombus intersect at right angles.

14. The hypotenuse of a right triangle is the line joining (−1, −2) to (6, 4). Find the coördinates of the third vertex if it lies on the x-axis.

15. One end of the line whose length is 5 is at (4, 2). The abscissa of the other end is 7, what is its ordinate?

16. The end points of a diagonal of a parallelogram are (2, −3) and (3, 2). Find the coördinates of the remaining vertices if they are on the x-axis and y-axis respectively. Why is there only one solution?

17. The coördinates of the end points of one diagonal of a rhombus are (0, 0) and (2, 4). If one side lies along the positive x-axis, find the coördinates of the end points of the other diagonal.

18. Two points P_1 and P_2 are at the same distance from the origin. If their polar coördinates are (ρ, θ_1) and (ρ, θ_2), show that the slope of the line joining them is $-\cot\frac{\theta_1 + \theta_2}{2}$.

19. What does the slope of the line joining $(-1, 3)$ to $(6, 7)$ become if the axes are rotated through an angle $\varphi = \tan^{-1} \frac{3}{4}$?

20. What does the slope of the line joining (4, 3) to $(-5, 6)$ become if the axes are rotated through 30°?

21. What is the slope of the line through the points the polar coördinates of which are (6, 30°) and (4, 60°)?

22. Find the area of a triangle the polar coördinates of the vertices of which are $(\frac{1}{2}\pi, \frac{1}{3}\pi)$, $(\pi, \frac{1}{2}\pi)$, and $(2\pi, \frac{1}{2}\pi)$.

23. Find the area of a triangle the polar coördinates of whose vertices are (1, 60°), (3, 210°), and (2, 240°).

24. A rectangle of sides 5 and 12 lies entirely in the second quadrant, with one vertex at the origin and the longest side on the negative x-axis. Find the coördinates of its vertices if the axes are revolved so that the y-axis coincides with one diagonal.

25. The coördinates of the vertices of a parallelogram are (0, 0), $(4, -3)$, (5, 0), and (1, 3). What will be the coördinates of its vertices if the axes are rotated so that the x-axis coincides with the longest side?

CHAPTER III

LOCI AND EQUATIONS

37. General statement.—In the present chapter will be considered some of the more simple cases of the first two fundamental questions of analytic geometry as stated in article 3. The locus of an equation will be considered first, and then the equation of a locus. That is, the geometric interpretation of an equation will be dealt with first.

38. Constants and variables.—A **constant** is a number that never changes, or one that does not change in the course of a discussion.

Constants that never change are definite numbers, as 2, $\frac{2}{3}$, $\sqrt{2}$, log 3, and π. Numbers that are constant during a discussion, but may be different in another discussion are represented by the letters that are assumed to have known values.

A **variable** is a number whose value changes arbitrarily, or according to some law.

The number expressing the speed of a train as it gains headway is a variable. The price of a stock may change from day to day, and is expressed by a variable. The velocity of a falling body changes from instant to instant, and is expressed by a variable.

If two variables are so related that for every value of one there is a corresponding value of the other, then the one is said to be a **function** of the other.

Thus in the formula for the area of a circle, $A = \pi r^2$, for every value of r there is a value of A. Then A is a function of r. This is written $A = f(r)$. Likewise r may be considered a function of A.

EXERCISES

1. Give various illustrations of variables and constants.

2. In the formula, $A = \frac{4}{3}\pi r^3$, for the volume of a sphere, which are constants and which variables? Is A a function of r? Is r a function of A?

3. In the equation $x + y = 6$, can either x or y be assigned values arbitrarily? Can both be given arbitrary values at the same time? Is x a function of y? Is y a function of x?

39. The locus.—If the location of a point is determined by certain stated conditions, then the **locus of the point** is the geometric figure such that, (1) *all points of the figure satisfy the given conditions*, and (2) *all points that satisfy the given conditions are in the figure.*

In proving that a certain figure is the required locus, it is sometimes more convenient, instead of (2), to prove that *any point not in the figure does not satisfy the given conditions.*

The conditions determining a locus may be stated in the language of geometry, or may be stated by an equation.

In the more simple cases the locus can be given immediately from the conditions stated.

EXERCISES

1. What is the locus of a point that is equally distant from two fixed points?

2. What is the locus of a point in a plane and at a constant distance from a fixed point in that plane?

3. What is the locus of a point equally distant from two intersecting straight lines and in the plane determined by those lines?

4. What is the locus of a point equally distant from three fixed points and in the plane determined by the three points?

5. In rectangular coördinates, what is the locus of a point whose abscissa is 0? Whose abscissa is 5? Whose ordinate is -6?

6. What is the locus of a point whose coördinates satisfy the equation $x = 4$? Which satisfy the equation $x = y$? The equation $x + y = 0$?

40. The locus of an equation.—If an equation is the analytic statement of geometric conditions, then it follows from the definition of a locus, **Art. 39,** that the locus of an equation is the locus of all points whose coördinates satisfy the equation; and, conversely, the coördinates of all points on the locus must satisfy the equation.

While the preceding statement is general, only rectangular coördinates will be used in the present chapter.

The drawing of the locus is spoken of as **plotting the equation,** or plotting the locus of an equation. The locus is called the **graph of the equation.**

41. Plotting an equation.—The steps in plotting an equation are:

(1) *Solve the equation for one, or each, variable in terms of the other.*

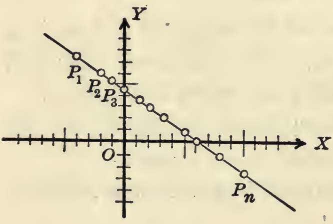


FIG. 36.

(2) *Assign convenient values to one variable and determine corresponding values of the other variable, and arrange in a table.*

(3) *Choose a suitable unit and plot the pairs of values of the variables.*

(4) *Connect these points by a smooth curve.*

The variable to which values are assigned arbitrarily is called the **independent variable.** The other is then called the **dependent variable.**

Example 1.—Plot the equation $2x + 3y = 13$.

(1) Solving for y, $$y = \frac{13 - 2x}{3}.$$

(2) Assign values to x as shown in the following table and determine the corresponding values of y.

x	-4	-2	-1	0	1	2	3	5	$6\frac{1}{2}$	8	10
y	7	$5\frac{2}{3}$	5	$4\frac{1}{3}$	$3\frac{2}{3}$	3	$2\frac{1}{3}$	1	0	-1	$-2\frac{1}{3}$

(3) Locate a pair of rectangular coördinates axes, choose a suitable unit, and plot the points $P_1(-4, 7)$, $P_2(-2, 5\frac{2}{3})$, $P_3(-1, 5)$, $\cdots$ Fig. 36 RA. as shown in Fig. 36.

(4) Draw a smooth curve through the points.

The curve is the locus of the equation and appears to be a straight line.

Example 2.—Plot the locus of the equation $x^2 + y^2 - 4x = 21$.

(1) Solving for y in terms of x and for x in terms of y,

$$y = \pm\sqrt{21 + 4x - x^2}, \text{ and } x = 2 \pm\sqrt{25 - y^2}.$$

From the first it is readily seen that y is imaginary when $x < -3$ or when $x > 7$, for then $21 + 4x - x^2 < 0$ and the square root is imaginary. Likewise, x is imaginary when $y < -5$ or when $y > 5$.

It is evident then that we should not choose values of x less than -3 nor greater than 7. And should not choose values of y less than -5 nor greater than 5.

(2) Here it is convenient to assign arbitrarily some values to x and some values to y, in each case computing the corresponding values of the other variable.

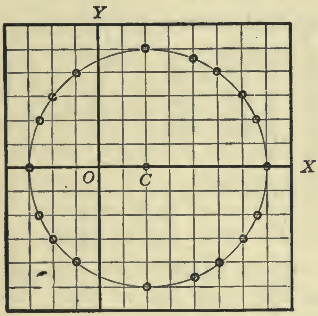


FIG. 37.

x	y	y	x
0	± 4.6	0	7 or -3
2	± 5	± 2	6.6 or -2.6
4	± 4.6	± 4	5 or -1
6	± 3		
-2	± 3		

(3) The points are plotted as shown in Fig. 37.

(4) A smooth curve is drawn connecting the points. This is the locus of the equation and appears to be a circle.

42. The imaginary number in analytic geometry.—In the plan for plotting numbers in analytic geometry no method is provided for plotting imaginary numbers. It follows then that if one, or both, of the values of the variables satisfying an equation are imaginary or complex numbers, no point can be plotted having these as coördinates. Such numbers are often said to locate **imaginary points** on a curve.

Some equations, such as $x^2 + y^2 = 0$, are satisfied by only one pair of real values for x and y. The locus of such an equation is a single point.

Thus, $x^2 + y^2 = 0$ is satisfied only by $x = 0$ and $y = 0$.

Other equations, such as $x^2 + y^2 + 4 = 0$, are satisfied by no real values for x and y. The locus of such an equation is wholly imaginary.

EXERCISES

Plot the loci of the following equations:

1. $y = 2x + 4$.
2. $2x + 3y = 8$.
3. $x - 2y - 6 = 0$.
4. $3x - 4y - 5 = 0$.
5. $16x - 3y = 42$.
6. $x^2 + y^2 = 25$.
7. $x^2 + y^2 = 18$.
8. $y^2 = 4x$.
9. $y^2 = 4x + 3$.
10. $x^2 = 8y$.
11. $x^2 - y^2 = 4$.
12. $x^2 + 3y^2 = 0$.
13. $x^2 + y^2 + 12 = 0$.
14. $x^2 + 2y^2 = 8$.
15. $x^2 + y^2 + 6x = 7$
16. $y^3 = 4x^2$.
17. $9x^2 - 4y^2 = 36$.
18. $9x^2 + 4y^2 = 36$.
19. $4y^2 - 9x^2 = 36$.
20. $y = x^3$.
21. $y = x^3 + 3$.
22. $y = x^3 - 2x^2 + 6x - 3$.

23. Plot the following equations upon the same set of axes: (1) $x^2 + y^2 = 16$, (2) $x^2 - y^2 = 16$, (3) $y^2 - x^2 = 16$, (4) $-x^2 - y^2 = 16$.

24. Plot the following equations upon the same set of axes: (1) $x^2 = 2y$, (2) $x^2 = -2y$, (3) $y^2 = 2x$, (4) $y^2 = -2x$.

DISCUSSION OF EQUATION IN RECTANGULAR COÖRDINATES

43. Geometric facts from the equation.—Since it is possible to plot but a few points of a curve, the method of determining the curve by points is sufficiently accurate only in the case of simple curves. In general, much help in learning the properties of a curve is gained by a study, or discussion, of the equation. First, it gives exact information regarding the curve; second, it furnishes a test of the accuracy of the plotting; and, third, it usually lessens the labor of plotting.

The properties of the locus of an equation that can be studied to the best advantage by analytic geometry are the following:

(1) *The intercepts of the curve.*

(2) *The symmetry of the curve.*

(3) *The extent of the curve.*

Various other properties can be studied by methods of the calculus.

44. Intercepts.—The **x-intercepts** of a curve are the abscissas of the points where the curve intersects, or meets, the x-axis. The **y-intercepts** are the ordinates of the points where the curve intersects, or meets, the y-axis. Together the x-intercepts and the y-intercepts are called the **intercepts of the curve.**

Evidently, the x-intercepts are found by putting $y = 0$ in the equation and solving for x. Likewise the y-intercepts are found by putting $x = 0$ and solving for y.

It follows that, if an equation contains no constant term, the curve passes through the origin.

Example.—Find the intercepts for the equation $16x^2 + 25y^2 = 400$

Putting $y = 0$, $\quad 16x^2 = 400$, or $x = \pm 5$.

Putting $x = 0$, $\quad 25y^2 = 400$, or $y = \pm 4$.

∴ the x-intercepts are $+5$ and -5, and the y-intercepts are $+4$ and -4.

EXERCISES

Find the intercepts for the following equations:

1. $2x - 3y = 10$.

2. $x^2 + y^2 = 36$.

3. $4x^2 + y^2 = 64$.

4. $4x^2 + y^2 - 8x - 2y + 1 = 0$.

5. $5x^2y - 15x + 4y = 0$.

6. $y = \dfrac{x(x-2)}{(x+2)(x-1)}$.

7. $y^2 = (x + 2)(x - 1)(x - 3)$.

8. $xy = 6$.

45. Symmetry, geometrical properties.—Two points are said to be **symmetrical with respect to a given point** when the given point bisects the line joining the two points. The given point is called the **center of symmetry.**

Two points are said to be **symmetrical with respect to a**

given line when the given line is the perpendicular bisector of the line joining the two points. The given line is called the **axis of symmetry.**

Thus, in Fig. 38, if Q bisects P_1P_2, P_1 and P_2 are symmetrical with respect to Q. Also, if AB is the perpendicular bisector of P_1P_2, P_1 and P_2 are symmetrical with respect to AB.

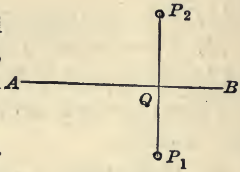


FIG. 38

If the points of a curve can be arranged in pairs which are symmetrical with respect to a line or point, then the curve itself is said to be **symmetrical** with respect to the line or point.

Thus, in Fig. 39, the curve is symmetrical with respect to each of the coördinate axes and with respect to the origin. Tell why.

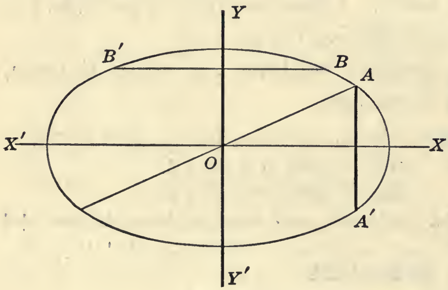


FIG. 39.

EXERCISES

1. Has a square a center of symmetry? Has a rectangle? A circle? A parallelogram? A regular hexagon?

2. How many axes of symmetry has each of the figures of exercise 1?

3. In rectangular coördinates give the point that with each of the following is symmetrical with respect to the x-axis: $(2, 4)$, $(-2, 5)$, $(-4, -2)$, $(6, -8)$, (x, y). With respect to the y-axis. With respect to the origin.

46. Symmetry, algebraic properties.—In the preceding article symmetry has been considered from the side of geometry. It remains to determine how symmetry can be seen by an inspection of the equation.

If a curve is symmetrical with respect to the x-axis, it follows that every point (x, y) on the curve has a corresponding point $(x, -y)$ on the curve. Then the coördinates of the point $(x, -y)$ must satisfy the equation; that is, if $-y$ is substituted for y, the equation reduces to the original form. It is evident that this occurs in an algebraic equation *when only even powers of y appear in the equation.* (See **Art. 120.**)

Likewise the curve is symmetrical with respect to the y-axis if, when $-x$ is substituted for x, the equation reduces to the original form. This occurs in an algebraic equation *when only even powers of x appear in the equation.*

Since the pair of points (x, y) and $(-x, -y)$ are symmetrical with respect to the origin, it follows that if, when $-x$ is substituted for x and $-y$ for y, the equation reduces to its original form, the curve is symmetrical with respect to the origin. It is evident that this occurs in an algebraic equation *if each term is of even degree, or if each term is of odd degree, in x and y.* In applying this test a constant term is considered as of even degree.

It also follows that if a curve is symmetrical to both coördinate axes it is symmetrical with respect to the origin.

EXERCISES

State for which of the following equations the curves are symmetrical with respect to the x-axis, the y-axis, and the origin.

1. $3x + y + 6 = 0.$

2. $x^2 + y^2 = 25.$

3. $3x^2 - 4y^2 = 12.$

4. $x^2 + y^2 + 2x = 16.$

5. $y^3 = 4x.$

6. $x^2y^2 = 16.$

7. $x^3 + y = 6.$

8. $x^2 + 2xy + y^2 = 9.$

9. $y^2 = (x + 1)(x - 2).$

10. $x^2y^2 + 4x^4 = 16.$

11. $x^2 + 4x + 2y + 3 = 0.$

12. $x^3 - x = y.$

47. Extent.—Under this heading we endeavor to find how the curve lies with reference to the coördinate axes by finding, first, for what values of either variable there are no points on the curve; and, second, for what values of either variable the curve extends to infinity.

To do this the equation is solved for each variable in terms of the other. First, if a radical of even index involves a variable, certain values of that variable may give imaginary values for the other variable, in which case there are no points on the curve. If no radicals of even index are involved, there will be at least, one real value of either variable for a real value of the other. In which case there are points on the curve for every value of either variable.

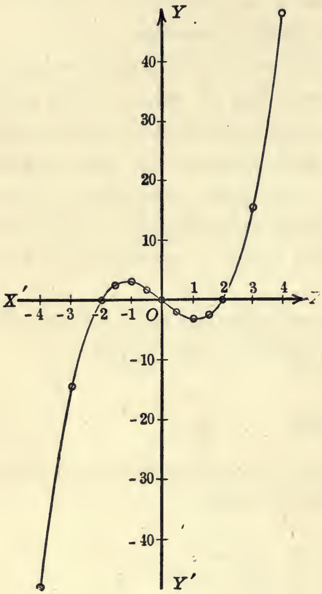


FIG. 40.

Second, if the solution for either variable gives rise to a fraction having the other variable in the denominator, then certain finite values of the second variable may make the first infinite. If no such fraction occurs both variables may become infinite at the same time.

Example 1.—Investigate $9x^2 + 4y^2 = 36$ as to extent.

Solving for x, $x = \pm \frac{2}{3}\sqrt{9 - y^2}$.

Solving for y, $y = \pm \frac{3}{2}\sqrt{4 - x^2}$.

Therefore x is imaginary when $9 - y^2 < 0$, that is, when $y < -3$, or when $y > +3$.

And y is imaginary when $4 - x^2 < 0$, that is, when $x < -2$, or when $x > +2$.

The curve is then confined to the portion of the plane in which the abscissas do not exceed 2 in numerical value, and the ordinates do not exceed 3 in numerical value.

Example 2.—Discuss the equation $x^3 - ax - y = 0$ and plot the curve.

Discussion. Intercepts.—Let $x = 0$ then $y = 0$.

Let $y = 0$ then $x^3 - ax = 0$. Solving this for x, $x = 0$ or $\pm\sqrt{a}$.

Hence the y-intercept is 0, and the x-intercepts are 0 and $\pm\sqrt{a}$.

Symmetry.—Since all terms are of odd degree in x and y, the curve is symmetrical with respect to the origin.

Extent.—Solving for y, $y = x^3 - ax$.

The letter a represents an arbitrary constant and may have any value assigned to it. But, in assigning a value, do not choose one that would cause a term to disappear. For the purposes of this discussion it is given the value 4.

Since no even root is involved, either variable has a real value for any value of the other.

Since large positive values of x make $x^3 - ax$ large and positive, for such values of x, y increases as x increases.

Likewise, for numerically large negative values of x, y decreases as x decreases.

Plotting.—Tabulating coördinates for positive values of x, the curve can be located in the first and fourth quadrants and, by symmetry, in the second and third quadrants, and is as shown in Fig. 40. The arbitrary value assigned to a is 4, and the unit on the y-axis is one-fifth of that on the x-axis.

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4
y	0	$-1\frac{7}{8}$	-3	$-2\frac{5}{8}$	0	15	48

EXERCISES

Discuss each of the following equations and plot their curves.

1. $x^2 + y^2 = 64$.

2. $x^2 - y^2 = 64$.

3. $4x^2 + 9y^2 = 36$.

4. $4x^2 - 9y^2 = 36$.

5. $y^2 = 8x$.

6. $x^2 = 8y$.

7. $x^2 = 8y - 6$.

8. $x^2 + y^2 - 4x - 20 = 0$.

9. $xy = 15$.

10. $xy + 12 = 0$.

11. $y = x^3 - 9x$.

12. $y(x^2 + 1) - 8 = 0$.

13. $9x^3 = y^2$.

14. $y^2 = (x - 2)(x + 1)(x + 3)$.

15. $y^2 = (x - 1)^2(x - 2)$.

16. $y(x - 1) = 1$.

17. $y^2 = ax^3 + x^2$.

18. $x(x - 2a)^2 - ay^2 = 0$.

48. Composite loci.—Any function of the two variables x and y may be denoted by $f(x, y)$. Then $f(x, y) = 0$ is a compact way of writing any equation in these two variables.

THEOREM.—*If the expression $f(x, y)$ can be factored into variable factors, the locus of $f(x, y) = 0$ consists of as many distinct curves as there are variable factors of $f(x, y)$.*

Proof.—Suppose $f(x, y)$ can be factored into $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$, $\cdots$.

Then $f_1(x, y) \cdot f_2(x, y) \cdot f_3(x, y) \cdots = 0$.

Now any values of x and y that will make any one of these factors equal zero will satisfy the original equation.

Hence all points on the separate loci of

$$f_1(x,y) = 0, f_2(x, y) = 0, f_3(x, y) = 0, \cdots$$

will also be points on the locus of $f(x, y) = 0$.

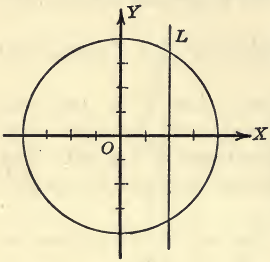


FIG. 41.

Much time is often saved when $f(x, y)$ can be factored, by plotting each of the equations $f_1(x, y) = 0$, $f_2(x, y) = 0$, $f_3(x, y) = 0$, $\cdots$ separately.

Example.—Plot the locus of the equation $x^3 + xy^2 - 2x^2 - 2y^2 - 16x + 32 = 0$.

The factors of $x^3 + xy^2 - 2x^2 - 2y^2 - 16x + 32$ are

$$(x - 2)(x^2 + y^2 - 16).$$

Equating each factor to zero, $x = 2$ and $x^2 + y^2 = 16$.

The first is a straight line and the second a circle as shown in Fig. 41

EXERCISES

Find a single equation whose locus is the combination of the loci of the separate equations in each of the following and plot.

1. $xy - 6 = 0, xy + 6 = 0$.

2. $x - 2y + 3 = 0, x - 2y - 3 = 0$.

3. $x = 0, x = 3, x = 5$.

4. $x = y, x^2 + y^2 = 16$.

5. $x^2 + y^2 = 4, xy = 6$.

Plot the locus of each of the following by first factoring $f(x, y)$, and then plotting each factor equated to zero.

6. $x^2y^2 = 16$.

7. $x^4 - y^4 = 0$.

8. $x^2 + 2xy + y^2 - 4 = 0$.

9. $x^3 - 6x^2 + 11x - 6 = 0$.

10. $x^3 + x^2y - 4x - 4y + xy^2 + y^3 = 0$.

11. Plot the locus of $(x^2 - x - 6)(y^2 + 2y - 8) = 0$, and show that the lines enclose a rectangle.

49. Intersection of two curves.—The curves of two equations are, in general, distinct, and may or may not intersect.

It follows from the definition of the locus of an equation that, if a pair of values satisfy both equations, they are the coördinates of a point of intersection. And, conversely, if the curves intersect, the coördinates of a point of intersection must satisfy both equations.

In order then to find the coördinates of the points of intersection, it is only necessary to solve the equations simultaneously. Or, in order to find values of x and y that satisfy the equations simultaneously, the equations may be plotted and the coördinates of the points of intersection determined from the figure. This is useful when the equations are such as cannot be solved simultaneously.

EXERCISES

Find the points of intersection of the curves of the following pairs of equations by solving the equations. Check the results by plotting.

1. $x^2 + y^2 = 16$, $x + y = 0$.

2. $x^2 + y^2 = 16$, $x^2 - y^2 = 9$.

3. $x^2 - 4y^2 + 7 = 0$, $2x + 3y - 12 = 0$.

4. $x^2 + y^2 = 25$, $9x^2 + 49y^2 = 441$.

5. $x^2 + y^2 = 25$, $27y^2 = 16x^3$.

6. Find the distance between the points of intersection of

$$x^2 + y^2 = 12, \text{ and } y^2 = 4x.$$

7. Solve the following equations by plotting to find the coördinates of the points of intersection: $x^2 + y = 7$, $x + y^2 = 11$.

EQUATIONS OF LOCI

50. So far in the present chapter the problem considered has been the finding of the locus when the equation was given. Here the second fundamental question is taken up, that of finding the equation of a locus when the locus is known. That is, the algebraic statement is to be found when the geometric figure or description is known.

Definition.—The **equation of a locus** is an equation such that (1) *the coördinates of every point on the locus satisfy the equation, and* (2) *every pair of values which satisfy the equation are the coördinates of a point on the locus.*

51. Derivation of the equation of a locus.—The process of deriving the equation of a locus depends largely upon the ingenuity of the individual. The following suggestions, however, will be helpful, but it is not intended that it is necessary always to take these steps in order.

(1) *From the description of the locus sketch a figure involving all the data.*

(2) *Draw a pair of coördinate axes and select* $P(x, y)$ *any point on the locus.*

Frequently the coördinate axes are determined by the data; but if they are not, they should be located so as to make the equation as simple as possible.

(3) *Write an equation between geometric magnitudes, using the conditions of the problem.*

(4) *Express the geometric magnitudes of this equation in terms of the coördinates of P and the given constants, and simplify the resulting equation.*

The final equation will, in general, contain the variables x and y, and all the constants involved.

(5) *Show that any point whose coördinates satisfy the equation, is on the locus, and thus show that the second requirement of the definition is fulfilled.*

A discussion of the equation will often give further facts concerning the locus.

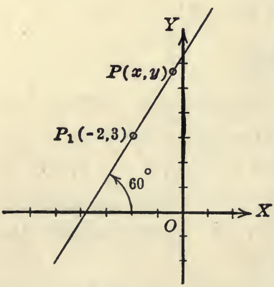


FIG. 42.

Example 1.—The locus of a point is a straight line passing through $P_1(-2, 3)$ and having an inclination of 60°. Find its equation.

Solution.—(1) Here the coördinate axes are determined by the data. In Fig. 42, OX and OY are the axes and P_1P the locus.

(2) $P(x, y)$ is any point on the locus.

(3) Slope P_1P = tan 60°.

(4) Slope $P_1P = \frac{y-3}{x+2}$ by [6].

$$\therefore \frac{y-3}{x+2} = \sqrt{3}.$$

Simplifying, $3x - \sqrt{3}y + 6 + 3\sqrt{3} = 0$.

(5) Any point $P(x, y)$ whose coördinates satisfy the equation

$$3x - \sqrt{3}y + 6 + 3\sqrt{3} = 0,$$

must also satisfy the preceding equation since this equation can be reduced to that form by reversing the steps. But the equation

$$\frac{y - 3}{x + 2} = \sqrt{3}$$

simply says that the slope of a straight line through (x, y) and $(-2, 3)$ is equal to $\sqrt{3}$, and hence its inclination is 60°.

Therefore the equation of the locus is

$$3x - \sqrt{3}y + 6 + 3\sqrt{3} = 0.$$

Example 2.—Find the equation of the locus of a point that moves at a distance 8 from the point $(3, -5)$ and remains in the plane of the coördinate axes.

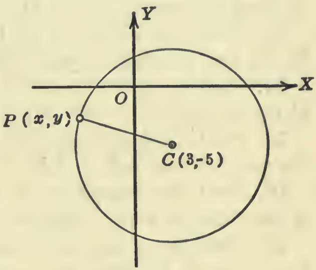


FIG. 43.

Solution.—(1) In Fig. 43, the coördinate axes are drawn and the data located.

(2) $P(x, y)$ is any point on the locus.

(3) By the conditions of the problem, $PC = 8$.

But $PC = \sqrt{(x - 3)^2 + (y + 5)^2}$ by [3].

(4) $\therefore \sqrt{(x - 3)^2 + (y + 5)^2} = 8$.

Squaring, $x^2 - 6x + 9 + y^2 + 10y + 25 = 64$.

Simplifying, $x^2 + y^2 - 6x + 10y - 30 = 0$.

This is the equation that is satisfied by the coördinates of any point on the locus. The proof of the converse is left to the student.

EXERCISES

Give orally the equations of the loci described in exercises 1—10.

1. A point moves parallel to the y-axis and 4 units to the right. Parallel to the y-axis and 6 units to the left.

2. A point moves parallel to the x-axis and 7 units above. Parallel to the x-axis and 3 units below.

3. A point moves parallel to the x-axis and 3 units above the point $(3, 6)$. Parallel to the x-axis and through the point $(-6, 4)$. Parallel to the x-axis and through the point $(0, -7)$.

4. A point moves parallel to the line $y = 4$ and 6 units above it.

5. A point moves parallel to the line $x = -3$ and 8 units to the right of it.

6. A point moves so as to bisect the angle the y-axis makes with the x-axis.

7. A point moves so as to bisect the angle the x-axis makes with the y-axis.

8. A point moves so as to keep 6 units from the origin.

9. A point moves so as to keep 8 units from the point $(2, -1)$.

10. A point moves so as to keep equidistant from the lines $y = 8$ and $y = -2$.

11. Find the equation of the locus of a point that is equidistant from the points $(5, 4)$ and $(-6, -2)$.

12. Find the equation of the locus of a point that moves at a distance 10 from the point $(-6, -8)$.

13. Find the equation of the circle having its center at the point $(3, 4)$, and passing through the point $(7, 7)$.

14. Find the equation of the circle having the extremities of a diameter at the points $(-4, -6)$ and $(2, 2)$.

15. Find the equation of the perpendicular bisector of the line joining the points $(-4, -8)$ and $(5, 2)$.

16. Find the equations of the perpendicular bisectors of the sides of the triangle whose vertices are the points $(0, 0)$, $(8, 6)$, and $(-4, 10)$.

17. Find the equation of the locus of a point that moves so as to keep four times as far from the x-axis as from the y-axis. Plot.

18. Find the equation of the locus of a point that moves so as to keep three times as far from the point $(2, 3)$ as from the point $(-6, 2)$.

19. A point moves so that its ordinate always exceeds $\frac{2}{3}$ of its abscissa by 8. Find the equation of its locus and plot.

20. A point moves so that the sum of its distances from the points $(3, 0)$ and $(-3, 0)$ is 8. Find the equation of its locus and plot.

21. A point moves so that the difference of its distances from the points $(3, 0)$ and $(-3, 0)$ is 4. Find the equation of its locus and plot.

22. A point moves so that the difference of the squares of its distances from the points $(-3, -1)$ and $(-2, -4)$ is 5. Find the equation of the locus and plot.

23. A point moves so that the slope of the line joining it to the point $(-2, 3)$ equals twice the slope of the line joining it to the point $(4, -2)$. Find the equation of the locus.

CHAPTER IV

THE STRAIGHT LINE AND THE GENERAL EQUATION OF THE FIRST DEGREE

52. Conditions determining a straight line.—In plane geometry it is found that two independent conditions determine a straight line. Just so in analytic geometry any two conditions that fix the line will determine its equation. Since the same straight line can be determined in a number of different ways, it may be expected that there will be several forms of the equation for the same straight line.

Some of the conditions that determine a straight line are the following:

(1) *A point on the line and the direction of the line.*

(2) *Two points on the line.*

(3) *The length and direction of the perpendicular from the origin to the line.*

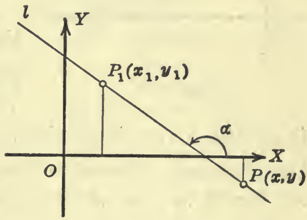


FIG. 44.

Each set of these conditions gives rise to a standard form of the equation of a straight line.

53. Point slope form of equation of the straight line.—Suppose the straight line l, Fig. 44, passes through the point $P_1(x_1, y_1)$, and that its direction is given by its slope $m = \tan \alpha$. If $P(x, y)$ is any point on l, then the slope of PP_1 must be constant and equal to m. By [6], the slope m of PP_1 is

$$m = \frac{y - y_1}{x - x_1}.$$

Clearing this equation of fractions,

[15] $$\mathbf{y - y_1 = m(x - x_1).}$$

This is the **point slope form of the equation of a straight line.**

Since $P(x, y)$ is any point on l it follows that every point on l satisfies [**15**].

In order to prove that every point which satisfies [**15**] is on line l, let $P_3(x_3, y_3)$, Fig. 45, be such a point, then

$$y_3 - y_1 = m(x_3 - x_1).$$

Dividing both sides of this equation by $x_3 - x_1$,

$$\frac{y_3 - y_1}{x_3 - x_1} = m.$$

This shows that the slope of the line $P_1P_3 = m$. Therefore P_1P_3 and l are parallel. Since P_1P_3 and l pass through the same point P_1, the line P_1P_3 and l coincide. Therefore P_3 lies on l.

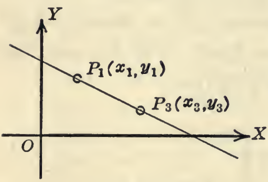


Fig. 45.

In the discussion of other forms of the equation of a straight line, the proof that every point whose coördinates satisfy the equation of the locus, is on the locus, is so similar to the proof just given that it will be omitted. Nevertheless this fact should not be lost sight of, for it is one of the essential conditions in determining the equation of a locus.

54. Lines parallel to the axes.—In article 53 it is tacitly assumed that the line whose equation is to be found is not parallel to the y-axis. If it is, α equals 90°, m is infinite, and equation [**15**] is meaningless. If the line is parallel to the y-axis, it must cut the x-axis at some point $(a, 0)$. Every point on this line has its abscissa equal to a, hence the equation of the line is

$$\mathbf{x = a.}$$

Similarly every line parallel to the x-axis cuts the y-axis at some point, say $(0, b)$. Every point on this line has its ordinate equal to b and hence the equation of the line is

$$\mathbf{y = b.}$$

Example 1.—Find the equation of a line through $(-2, 3)$ and with an inclination of 135°.

Substituting $x_1 = -2$, $y_1 = 3$, and $m = \tan 135° = -1$ in [15],

$$y - 3 = (-1)(x + 2),$$

or

$$x + y - 1 = 0.$$

Example 2.—Find the equation of a line through the point $(2, 6)$ and parallel to the line joining the points $(-3, 4)$ and $(1, 5)$.

By [6] the slope of the line joining the two points is $\frac{1}{4}$.

Therefore the slope of the required line is also $\frac{1}{4}$

Substituting $m = \frac{1}{4}$, $x_1 = 2$, and $y_1 = 6$ in [15], the equation of the required line is $y - 6 = \frac{1}{4}(x - 2)$,

or

$$x - 4y + 22 = 0.$$

EXERCISES

Find the equations of the lines determined by the following sets of conditions:

1. Through $(2, -3)$, slope $\frac{1}{2}$.
2. Through $(-2, -4)$, inclination 135°.
3. Through $(1, 5)$, inclination 120°.
4. Through $(-1, 2)$, parallel to the line joining $(7, 6)$ to $(2, 3)$.
5. Through $(-1, 2)$, perpendicular to the line joining $(7, 6)$ to $(2, 3)$.
6. Through $(3, 4)$, parallel to the y-axis.
7. Through $(3, 4)$, parallel to the x-axis.
8. Through $(-1, 2)$, inclination $= \tan^{-1} \frac{1}{2}$.
9. Through $(1, -2)$, inclination $= \sin^{-1} \frac{3}{5}$.
10. Through $(3, 2)$, inclination $= \cos^{-1} \frac{5}{13}$.
11. Find the equation of the tangent line to the curve $y = x^3 - x$, at the point whose abscissa is 2, if its slope equals 11.

Suggestion.—Find the ordinate of the point whose abscissa is 2 and substitute in [15].

12. Find the equation of the tangent line to the curve $y = 2x^2 - x + 3$ at the point whose abscissa is 2, if its slope equals 7.

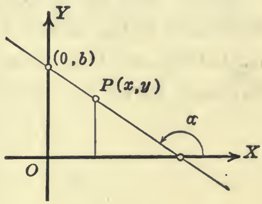


FIG. 46.

55. Slope intercept form.—In Fig. 46, let the intercept of the line on the y-axis equal b and let the slope of the line equal m. Since the y-intercept has the coördinates $(0, b)$ this problem is a special case of the point slope form.

Putting $x_1 = 0$ and $y_1 = b$ in [15], then $y - b = mx$, or

[16] $$\mathbf{y = mx + b.}$$

This is the **slope intercept form of the equation of a straight line.**

56. Two point form.—Let the two points through which the line passes be $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Since P_1 is a point on the line and m is the slope of P_1P_2, this form can be derived from [**15**] by substituting for m its value $\frac{y_1 - y_2}{x_1 - x_2}$. Equation [**15**] then becomes

[**17**] $$\mathbf{y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1).}$$

Note that this equation is not valid if $x_1 - x_2 = 0$. The line is then parallel to the y-axis and hence its equation is $x = x_1$.

This is the **two point form of the equation of a straight line.**

Since the three points P, P_1, and P_2 on the straight line through P_1P_2, always form a triangle whose area is zero, the equation of the straight line can be written in the determinant form by article 34, as follows:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

EXERCISES

Find the equations of the lines given by the following sets of conditions:

1. The y-intercept $= 3$ and the slope $= \frac{1}{2}$.

2. The y-intercept $= -2$ and the slope $= 3$.

3. The y-intercept $= \frac{1}{3}$ and the inclination $= \sin^{-1}\frac{2}{\sqrt{13}}$.

4. Passing through the points (1, 6) and (7, 2).

5. Passing through the points $(-2, 1)$ and $(3, -4)$.

6. Passing through the points $(-1, -2)$ and $(-4, -3)$.

7. What is the effect on line [**16**] if b is changed while m remains unchanged? What is the effect if m is changed while b remains unchanged?

57. Intercept form.—If the straight line cuts both axes, let its x-intercept, Fig. 47, equal a and its y-intercept equal b.

Its equation can be derived from [**17**] by replacing (x_1, y_1) by $(a, 0)$ and (x_2, y_2) by $(0, b)$.

Equation [17] then becomes $y = \frac{-b}{a}(x - a).$

Multiplying both sides of this equation by $\frac{1}{b}$, and transposing the x-term to the left hand side, it becomes

[18] $$\frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} = \mathbf{1}.$$

This is the **intercept form of the equation of a straight line.**

Care must be used in employing this form of the equation, since it is not valid if either or both intercepts are zero.

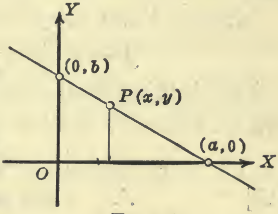


FIG. 47.

58. Normal form.—A line is completely determined if the length and direction of the perpendicular to it from the origin are known.

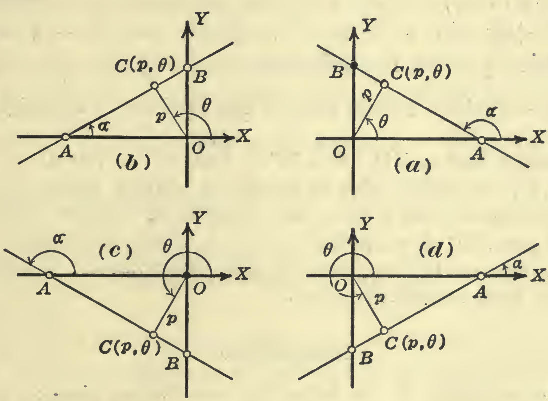


FIG. 48.

Let C, Fig. 48, be the foot of the perpendicular drawn to the line from the origin, and let (p, θ) be the polar coördinates of C.

Then $OC = p$ and angle $XOC = \theta$.

Since the line AB is perpendicular to the line OC, its slope is the negative reciprocal of the slope of OC and equals $-\cot\theta$.

Since the line AB passes through the point C, a point on AB is known. The rectangular coördinates of this point are $(p \cos \theta, p \sin \theta)$.

Hence the equation of AB can be found by substituting in [**15**], $m = -\cot \theta$, $x_1 = p \cos \theta$, and $y_1 = p \sin \theta$.

Making these substitutions, [**15**] becomes

$$y - p \sin \theta = -\cot \theta\,(x - p \cos \theta).$$

Multiplying both sides of the equation by $\sin \theta$ and transposing all terms to the left hand side, gives

$$x \cos \theta + y \sin \theta - p(\sin^2\theta + \cos^2\theta) = 0.$$

[19] $$\therefore \quad \mathbf{x \cos \theta + y \sin \theta - p = 0.}$$

This is the **normal form of the equation of a straight line.**

If $\cot \theta = \infty$, the line is parallel to the y-axis and its equation is $x = p$. But even in this case the normal form is valid, for if $\cot \theta = \infty$, $\theta = 0°$ and the normal form would read $x \cos 0° + y \sin 0° - p = 0$. Since $\cos 0° = 1$ and $\sin 0° = 0$ this equation is equivalent to $x = p$.

Example.—Find the normal form of the equation of a straight line if $\theta = 30°$ and $p = 6$.

Substituting these in [**19**], $x \cos 30° + y \sin 30° - 6 = 0$.

Since the polar coördinates of C can be written either $(6, 30°)$ or $(-6, 210°)$, the normal form of the equation of this line could also be written $x \cos 210° + y \sin 210° + 6 = 0$. That these equations are equivalent can readily be seen if the trigonometric functions are replaced by their numerical values.

EXERCISES

Find the equations of the following lines having given:

1. $a = 3, b = -2$.
2. $a = -1, b = 6$.
3. $a = \frac{1}{2}, b = \frac{2}{3}$.
4. $a = -\frac{1}{3}, b = -\frac{1}{4}$.
5. $\theta = 60°, p = 3$.
6. $\theta = 60°, p = -3$.
7. $\theta = 135°, p = 2$.
8. $\theta = 210°, p = 1$.
9. $\theta = 330°, p = -4$.
10. $\theta = 150°, p = -2$.

11. What is the effect on line [**19**] if p is changed while θ remains unchanged? What is the effect if θ is changed while p remains unchanged?

59. Linear equation. THEOREM—*Every equation of a straight line is of the first degree in one or two variables.*

Conversely. *Every equation of the first degree in one or two variables is the equation of a straight line.*

Proof.—Every straight line intersects, does not intersect, or coincides with the y-axis.

In the first case by means of article 55 its equation can be put in the form $y = mx + b$, in the second case its equation is $x = a$, and in the third case $x = 0$. Each of these equations is of the first degree in x and y.

Proof of converse.—Consider the most general equation of the first degree in two variables. This is

[20] $$\mathbf{Ax + By + C = 0.}$$

Assume that $B \neq 0$ and solve this equation for y, it becomes

$$y = -\frac{Ax}{B} - \frac{C}{B}.$$

Comparing this equation with the form $y = mx + b$ shows at once that it is the equation of a straight line whose slope $m = -\frac{A}{B}$, and whose y-intercept $b = -\frac{C}{B}$.

If $B = 0$, [20] becomes $Ax + C = 0$, or $x = -\frac{C}{A}$. This is the equation of a straight line parallel to the y-axis.

Hence every equation of the first degree is the equation of a straight line.

$Ax + By + C = 0$, the most general equation of the first degree in two variables, is called the **general equation of a straight line.**

60. Plotting linear equations.—Since every equation of the first degree represents a straight line, it is sufficient in plotting the graph of such an equation to find two points which satisfy the equation and then join these points by a straight line. Usually two such points that can be easily found are the intercepts on the x and the y-axes.

61. Comparison of standard forms.—In article 59 it was seen how the general equation could be transformed into the slope intercept form. This method of transforming one form of an equation into another is of great use in analytic geometry, since by comparing the constants in two forms of an equation of a line or curve, much information can be secured. In this particular case the transformation from the general form to the slope intercept form, enables one to read off by inspection the slope and y-intercept of the line.

For example, if the equation $3x + 4y = 12$, be solved for y it becomes $y = -\frac{3}{4}x + 3$.

Comparing this equation with $y = mx + b$, shows that the slope of the line is $-\frac{3}{4}$ and its y-intercept is 3.

Take the same equation, $3x + 4y = 12$, and divide both sides of the equation by 12, and $\frac{x}{4} + \frac{y}{3} = 1$.

Comparing this equation with $\frac{x}{a} + \frac{y}{b} = 1$ shows that the x-intercept is 4 and that the y-intercept is 3.

Here the x-intercept can be as easily found by putting $y = 0$ in the original equation; and the y-intercept by putting $x = 0$.

62. Reduction of $Ax + By + C = 0$ to the normal form.—In general A and B will not be the cosine and sine respectively of the same angle, and hence $Ax + By + C = 0$ will not be in the normal form. In order to transform it to the normal form multiply both sides of the equation by an arbitrary constant k, whose value is to be computed later.

This gives $\qquad Akx + Bky + Ck = 0.$

The quantity k is now assumed to be such a number that

$$Akx + Bky + Ck = 0$$

will be identical with

$$x \cos\theta + y \sin\theta - p = 0.$$

Comparing coefficients gives $Ak = \cos\theta$, $Bk = \sin\theta$, $Ck = -p$.

To find the value of k, square both sides of the first two equations and add. This gives

$$A^2k^2 + B^2k^2 = \cos^2\theta + \sin^2\theta = 1.$$

Solving for k, $$k = \frac{1}{\pm\sqrt{A^2 + B^2}}.$$

If k is replaced by its value, $Akx + Bky + Ck = 0$ becomes

[21] $$\frac{\mathbf{A}x}{\pm\sqrt{\mathbf{A}^2 + \mathbf{B}^2}} + \frac{\mathbf{B}y}{\pm\sqrt{\mathbf{A}^2 + \mathbf{B}^2}} + \frac{\mathbf{C}}{\pm\sqrt{\mathbf{A}^2 + \mathbf{B}^2}} = 0.$$

This is the general equation of the straight line expressed in the normal form. Either sign can be used with the radical, but of course the same sign must be used throughout the equation. Comparing this equation with the normal form gives

[22] $$\cos\theta = \frac{\mathbf{A}}{\pm\sqrt{\mathbf{A}^2 + \mathbf{B}^2}}, \quad \sin\theta = \frac{\mathbf{B}}{\pm\sqrt{\mathbf{A}^2 + \mathbf{B}^2}},$$
$$\mathbf{p} = \frac{-\mathbf{C}}{\pm\sqrt{\mathbf{A}^2 + \mathbf{B}^2}}.$$

Hence, to transform the equation $Ax + By + C = 0$ to the normal form, divide both sides of the equation by $\pm\sqrt{A^2 + B^2}$.

Example 1.—Change $3x - 4y + 6 = 0$ into the normal form. Here $A = 3$, $B = -4$, $C = 6$, and $\pm\sqrt{A^2 + B^2} = \pm\sqrt{9 + 16} = \pm 5$. Dividing the equation through by ± 5, it becomes

$$\frac{3x}{\pm 5} - \frac{4y}{\pm 5} + \frac{6}{\pm 5} = 0.$$

Either sign can be used since the two equations $\frac{3}{5}x - \frac{4}{5}y + \frac{6}{5} = 0$, and $-\frac{3}{5}x + \frac{4}{5}y - \frac{6}{5} = 0$ are equivalent.

EXERCISES

Find the slope and y-intercept of the following equations by expressing them in the slope intercept form.

1. $3x + 2y - 4 = 0$.

2. $2x - 3y + 2 = 0$.

3. $-5x + 2y - 6 = 0$.

4. $2x - 2y + 7 = 0$.

Change the following equations into normal form, and find the distance of the line from the origin.

5. $3x - 4y - 6 = 0.$

6. $-3x + 4y + 10 = 0.$

7. $5x - 12y + 26 = 0.$

8. $-5x - 12y + 39 = 0.$

9. $x + 2y - 3 = 0.$

10. $-3x + y - 6 = 0.$

11. $2x + 3 = 0.$

12. $3y - 4 = 0.$

Find the equations of the lines satisfying the following conditions:

13. Through the point (1, 2) and parallel to $3x - 4y + 6 = 0$.

14. Through the point (2, −3) and parallel to $x + 2y - 3 = 0$.

15. Through the point (6, 2) and perpendicular to $2x + y - 3 = 0$.

16. Through the point (−3, 1) and perpendicular to $x - y + 6 = 0$.

63. Distance from a point to a line.—The distance from the origin to the line $x \cos \theta + y \sin \theta - p = 0$, is the numerical value of p. Hence if d' is the distance from the origin to the line $Ax + By + C = 0$

$$d' = \frac{C}{\pm\sqrt{A^2 + B^2}}, \tag{1}$$

where the sign of the radical is chosen so as to make d' positive.

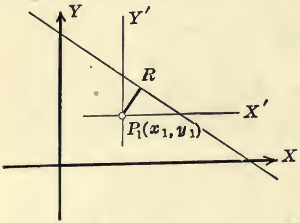


Fig. 49.

In order to find the distance d from the point $P_1(x_1, y_1)$, Fig. 49, to the line $Ax + By + C = 0$, translate the axes to the new origin $P_1(x_1, y_1)$. The equations of translation [12] are

$$x = x' + x_1,$$
$$y = y' + y_1.$$

Making these substitutions, the equation

$$Ax + By + C = 0$$

becomes $$Ax' + By' + Ax_1 + By_1 + C = 0,$$

where the new constant term is

$$Ax_1 + By_1 + C.$$

The distance $d = P_1R$ is the distance from the new origin to the line. From equation (1)

$$\text{[23]} \qquad \mathbf{d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}},$$

where the sign of the radical is chosen to make d positive.

Example 1.—Find the distance from the point (2, 1) to the line

$$3x - 4y + 6 = 0.$$

Solution.—Translate the axes so that the new origin is the point (2, 1). The equations of translation are

$$x = x' + 2,$$
$$y = y' + 1.$$

The equation of the line $3x - 4y + 6 = 0$, referred to the new origin is

$$3(x' + 2) - 4(y' + 1) + 6 = 0,$$

or

$$3x' - 4y' + 8 = 0.$$

Putting this equation into the normal form gives

$$\frac{3x' - 4y' + 8}{\pm 5} = 0.$$

Hence the distance from the new origin to the line is $d = \frac{8}{5}$, and this is the distance from the point (2, 1) to the line $3x - 4y + 6 = 0$.

This distance could be found also by substituting directly in [**23**]. Putting $A = 3$, $B = -4$, $C = 6$, $x_1 = 2$, $y_1 = 1$,

$$d = \frac{3 \cdot 2 - 4 \cdot 1 + 6}{\pm 5} = \frac{+8}{+5} = \frac{8}{5}.$$

Example 2.—Find the distance from the point (3, −2) to the line $5x + 12y - 4 = 0$.

Putting $A = 5$, $B = 12$, $C = -4$, $x_1 = 3$, and $y_1 = -2$ in [**23**],

$$d = \frac{5 \cdot 3 + 12(-2) - 4}{\pm 13} = \frac{-13}{-13} = 1.$$

This apparent inconsistency arises because both signs must first be put down and then the correct sign selected.

EXERCISES

Find the distances from the points to the lines in the following exercises:

1. Point (2, 3) to line $4x - 3y + 4 = 0$.

2. Point (−1, 2) to line $3x + 4y - 6 = 0$.

3. Point (1, 3) to line $x - y = 0$.

4. Point (2, 3) to line $x \cos 30° + y \sin 30° - 3 = 0$.

5. Point (3, −1) to line $x \cos 135° + y \sin 135° + 1 = 0$.

6. Point (1, 6) to line $y - 1 = 3(x - 4)$.

7. Find the altitudes of the triangle whose sides have the equations $y = 1$, $12x + 5y - 27 = 0$, and $3x - 4y + 9 = 0$.

8. Find the altitudes of the triangle whose vertices have the coördinates (4, 2), (−3, 1), (6, −3).

64. The bisectors of an angle.—Let the sides of an angle be formed by the lines ST and SR, Fig. 50, the equations of which are $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ respectively. Let $P(x, y)$ be any point on the bisector SP of the angle formed by these two lines.

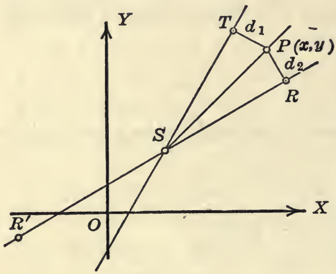


FIG. 50.

From plane geometry it is known that the bisector of an angle is the locus of points equidistant from the sides.

Hence $PT = PR$.

Expressing this fact algebraically gives

$$\frac{A_1x + B_1y + C_1}{\pm\sqrt{A_1^2 + B_1^2}} = \frac{A_2x + B_2y + C_2}{\pm\sqrt{A_2^2 + B_2^2}}.$$

The four possible combinations of signs in this equation will yield two different equations,

[24] $$\frac{\mathbf{A_1x + B_1y + C_1}}{\sqrt{\mathbf{A_1^2 + B_1^2}}} = \pm\frac{\mathbf{A_2x + B_2y + C_2}}{\sqrt{\mathbf{A_2^2 + B_2^2}}}.$$

One of these is the equation of the bisector of the angle RST, while the other is the equation of the bisector of the supplementary angle TSR'.

In order to tell which equation belongs to the bisector sought, draw the figure as accurately as possible and observe whether the slope of the required bisector is positive or negative. Since the two bisectors given by **[24]** are at right angles to each other, one has a positive slope and the other has a negative slope, so that in general it is easy to pick out the required equation. The exceptional case occurs when one bisector is very nearly parallel to the x-axis, and it is difficult to tell the sign of its slope. In this case the numerical value of its slope is small, whereas the numerical value of the slope of the other bisector is large, so that again it is easy to associate the equations with the correct bisectors.

Example.—Find the equation of the bisector of the angle which the line $l_1 \equiv 3x + 4y - 5 = 0$ makes with the line $l_2 \equiv 5x - 12y + 6 = 0$.

In Fig. 51, let l_3 be the required bisector.

By **[24]** the equations of the two bisectors are

$$\frac{3x + 4y - 5}{5} = \pm \frac{5x - 112y + 6}{13}.$$

Clearing of fractions and simplifying gives the two equations

$$14x + 112y - 95 = 0, \qquad (1)$$
$$64x - 8y - 35 = 0. \qquad (2)$$

The slope of (1) is small and negative, whereas the slope of (2) is large and positive. Since the slope of l_3 is large and positive, its equation is $64x - 8y - 35 = 0$.

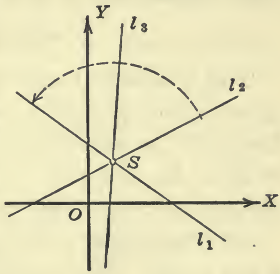


FIG. 51.

EXERCISES

Find the equations of the bisectors of the angle which the first line makes with the second in exercises 1–6.

1. $8x + y - 6 = 0$, $7x + 4y - 3 = 0$.

2. $x - 7y + 6 = 0$, $5x + 5y - 8 = 0$.

3. $11x - 2y + 12 = 0$, $2x + y - 6 = 0$.

4. $13x + y - 15 = 0$, $22x - 14y - 21 = 0$.

5. $12x + 14y - 11 = 0$, $9x - 2y + 10 = 0$.

6. $9x + 7y - 6 = 0$, $11x + 3y - 14 = 0$.

7. Find the equations of the bisectors of the angles of the triangle the equations of whose sides are $8x - y + 1 = 0$, $x + 8y + 1 = 0$, and $7x + 4y - 43 = 0$.

8. Find the equations of the bisectors of the angles of the triangle whose vertices are $(\frac{17}{2}, -\frac{11}{4})$, $(1, 1)$, and $(12, -1)$.

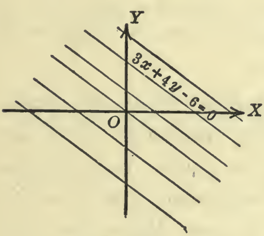


FIG. 52.

65. Systems of straight lines.—Sometimes the geometrical facts given are not sufficient to determine a straight line uniquely. In such a case not all the constants entering into the equation of the line will be determined. For instance, if the problem is to find the equation of a line that is parallel to $3x + 4y - 6 = 0$, Fig. 52, it is evident that there are an unlimited number of lines in the plane which satisfy the conditions of the problem. To find the equation of any one of these lines, substitute $m = -\frac{3}{4}$, in [**16**], which becomes

$$y = -\tfrac{3}{4}x + b, \text{ or } 3x + 4y - 4b = 0.$$

The quantity b can have any value whatsoever. If it is given some arbitrary value the equation $3x + 4y - 4b = 0$ becomes the equation of some one of the lines that are parallel to $3x + 4y - 6 = 0$. All of these parallel lines taken together are said to form a **system of lines.**

Another system of lines consists of all the lines through a given point. If the point has the coördinates (1, 2), the equation of this system of lines is $y - 1 = m(x - 2)$, by [**15**].

EXERCISES

Find the equations of the following systems of lines:

1. All the lines passing through the point $(-2, 3)$.

2. All the lines passing through the origin.

3. All the lines passing through the point (3, 4).

4. All the lines having their x-intercept equal to 3.

5. All the lines having their y-intercept equal to -4.

6. All the lines at a distance 3 from the origin.

7. All the lines at a distance 7 from the origin.

8. All the lines parallel to the line $2x + y - 3 = 0$.

9. All the lines perpendicular to the line $x - 3y + 6 = 0$.

10. All the lines such that the x-intercept of each is equal to its y-intercept.

66. Applications of systems of straight lines to problems.—Sometimes the facts determining a straight line are not such that its equation can be written down immediately. This happens if the slope m and the distance p of the line from the origin are given. In such a case there are two methods of procedure, one is to compute the constants which occur in some standard form of the equation of a straight line, by drawing the figure and applying plane geometry or trigonometry. Another method is illustrated in the following example.

Example.—Find the equation of the straight line given $m = \frac{4}{3}$, and $p = 3$.

First method.—First write down the equation of the system of lines whose slope is $\frac{4}{3}$. This is $y = \frac{4}{3}x + b$.

Next transform this line into the normal form. Its equation becomes

$$\frac{4x - 3y + 3b}{\pm 5} = 0.$$

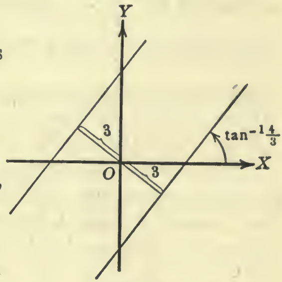


FIG. 53.

The distance of this line from the origin is $\frac{3b}{\pm 5}$, but this is p and $p = 3$.

Hence $$\frac{3b}{\pm 5} = 3, \text{ and } b = \pm 5.$$

Substitute this value of b in $y = \frac{4}{3}x + b$, and it becomes $y = \frac{4}{3}x \pm 5$,

or $$4x - 3y \pm 15 = 0.$$

There are two lines which satisfy the required conditions, and they are equally distant from the origin.

Second method.—Write down the equation of all lines distant 3 from the origin. This equation is $x \cos \theta + y \sin \theta - 3 = 0$. In order to determine θ, note that the slope of this line is $- \cot \theta$.

Therefore $- \cot \theta = \frac{4}{3}$, and θ can be in either the second or the fourth quadrants.

If θ is in the second quadrant, then $\sin \theta = \frac{3}{5}$ and $\cos \theta = -\frac{4}{5}$.

If θ is in the fourth quadrant, then $\sin \theta = -\frac{3}{5}$ and $\cos \theta = \frac{4}{5}$.

Substituting these values, the equation becomes $\pm \frac{4}{5}x \mp \frac{3}{5}y - 3 = 0$.

Multiplying both sides by ± 5, gives $4x - 3y \pm 15 = 0$.

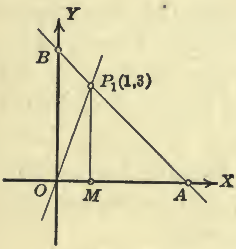


FIG. 54.

Example 2.—Find the equation of a line through the point (1, 3) and making equal intercepts on the axes.

First solution, geometric method.—In Fig. 54, let AB be the line through $P_1(1, 3)$ whose equation is to be found. Since the intercepts are equal, angle BAO = angle OBA = 45°.

Hence $a = OM + MA = OM + MP_1 = 1 + 3 = 4$, and this is also the value of b.

Therefore the required equation is $\frac{x}{4} + \frac{y}{4} = 1$, or $x + y = 4$.

Unfortunately by using the geometric method parts of the solution are liable to be overlooked. This is illustrated very well in this problem, since the line OP_1 passing through the origin and the point (1, 3), satisfies all the conditions of the problem and is therefore also a solution.

Second solution, algebraic method.—Since the line AB passes through the point (1, 3) its equation is $y - 3 = m(x - 1)$. The x-intercept of this line is $\frac{m-3}{m}$, and the y-intercept is $3 - m$.

Since these are equal, $\frac{m-3}{m} = 3 - m$.

Solving this equation gives $m = 3$ or -1.

If $m = 3$ the equation is $y - 3 = 3(x - 1)$ or $y - 3x = 0$.

If $m = -1$ the equation is $y - 3 = -(x - 1)$ or $x + y - 4 = 0$.

Third solution, algebraic method.—This differs from the preceding only in that it starts from the intercept form of the equation of a straight line, instead of from the point slope form.

Since the intercepts are equal, the intercept form of the equation is $\frac{x}{a} + \frac{y}{a} = 1$. In order to make this line pass through the point (1, 3), substitute these coördinates for x and y. This gives $\frac{1}{a} + \frac{3}{a} = 1$.

Solving, gives $a = 4$ and the required equation is $\frac{x}{4} + \frac{y}{4} = 1$,

or $$x + y = 4.$$

The question naturally arises, what happened to the solution $y - 3x = 0$? This is certainly a solution since the intercepts $a = 0$ and $b = 0$ are equal and the line passes through the point (1,3). This question can be answered by noting as stated in article 57 that the intercept form is not valid when either or both intercepts are 0. Hence the solution $y - 3x = 0$ cannot be secured from the intercept form. Whenever this form of the equation of a straight line is used the question as to whether either or both intercepts are zero must be answered independent of the equation.

EXERCISES

Find the equations of the lines determined by the following conditions:

1. The slope of the line equals $-\frac{3}{4}$ and it is distant $1\frac{1}{5}$ units from the origin.

2. The line makes equal intercepts on the axes and passes through the point (4, 2).

3. The line passes through the point $(-7, 4)$ and is tangent to a circle whose center is the origin and radius equal to 1.

4. The line passes through the point $(4, 2)$ and is tangent to a circle whose center is the origin and radius equal to 2.

5. The slope of the line is 2 and its x-intercept equals 3.

6. The slope of the line is -2 and the sum of its intercepts is 9.

7. The slope of the line is $-\frac{1}{4}$ and the sum of its intercepts is 5.

8. The line makes intercepts which are equal numerically but opposite in sign, and passes through the point $(6, 3)$.

9. The line passes through the point $(1, 3)$ and the sum of its intercepts equals 8.

10. The line passes through the point $(3, 1)$ and the portion included between the axes is bisected by this point.

11. The line passes through the point $(3, \sqrt{3})$ and the perpendicular from the origin on the line has an inclination of 60°.

12. The line is perpendicular to the line $4x + 3y - 6 = 0$ and distant 2 units from the origin.

13. The line is distant 3 units from the origin and its y-intercept equals 5.

14. The line is distant 2 units from the origin and the product of its intercepts is $\frac{25}{3}$.

15. The line passes through the point $(1, 2)$ and makes with the axes a triangle in the first quadrant whose area equals 4.

16. The line passes through the point $(1, 2)$ and makes with the axes a triangle in the second or fourth quadrants whose area equals 4.

67. Loci through the intersection of two loci.

—THEOREM. *If $f(x, y) = 0$ and $g(x, y) = 0$ are the equations of any two loci and k is any constant not zero, then $f(x, y) + kg(x, y) = 0$ is the equation of a curve which passes through all the points of intersection of $f(x, y) = 0$ and $g(x, y) = 0$, but does not intersect these curves in any other point.*

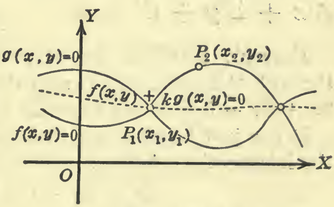


FIG. 55.

Proof.—Let $P_1(x_1, y_1)$, Fig. 55, be any point of intersection of $f(x, y) = 0$ and $g(x, y) = 0$. Since P_1 lies on both these curves its coördinates must satisfy each equation, therefore $f(x_1, y_1) = 0$ and $g(x_1, y_1) = 0$.

Substituting the coördinates of P_1 in $f(x, y) + kg(x, y) = 0$,

$$f(x_1, y_1) + kg(x_1, y_1) = 0 + k\cdot 0 = 0.$$

Therefore P_1 lies also on the curve $f(x, y) + kg(x, y) = 0$.

But P_1 was any point of intersection of $f(x, y) = 0$ and $g(x, y) = 0$, therefore every point of intersection of these curves lies on $f(x, y) + kg(x, y) = 0$.

Furthermore the curve $f(x, y) + kg(x, y) = 0$ cannot meet either $f(x, y) = 0$ or $g(x, y) = 0$ in any other point. For if it did, suppose it meets $f(x, y) = 0$ at $P_2(x_2, y_2)$, and that P_2 is not on $g(x, y) = 0$. Then $f(x_2, y_2) = 0$, but $g(x_2, y_2) = a$ where $a \neq 0$.

Substituting the coördinates of P_2 in $f(x, y) + kg(x, y) = 0$,

$$f(x_2, y_2) + kg(x_2, y_2) = 0 + ka \neq 0.$$

In like manner it can be shown that $f(x, y) + kg(x, y) = 0$ will meet $g(x, y) = 0$ only at the points of intersection of $f(x, y) = 0$ and $g(x, y) = 0$.

If $f(x, y) = 0$ is the straight line $Ax + By + C = 0$, and $g(x, y) = 0$ is the straight line $A'x + B'y + C' = 0$, then

$$f(x, y) + kg(x, y) = 0,$$

or

$$Ax + By + C + k(A'x + B'y + C') = 0$$

is the equation of a straight line through the point of intersection of the straight lines, $Ax + By + C = 0$ and $A'x + B'y + C' = 0$.

Example 1.—Find the equation of the straight line which passes through the point (4, 3) and through the intersection of the two lines $2x + 3y - 5 = 0$ and $3x - 4y + 1 = 0$.

It has just been shown that the equation of any line passing through the intersection of these two lines is of the form

$$2x + 3y - 5 + k(3x - 4y + 1) = 0.$$

Since this line passes through the point (4, 3), its equation is satisfied when $x = 4$ and $y = 3$. This gives

$$12 + k(1) = 0.$$

Therefore $k = -12$, and the required equation is

$$2x + 3y - 5 - 12(3x - 4y + 1) = 0,$$

or

$$2x - 3y + 1 = 0.$$

EXERCISES

Find the equations of the lines satisfying the following conditions.

1. Passing through the point of intersection of $2x + 3y - 3 = 0$ and $3x - y - 1 = 0$, and through the point (1, 1).

2. Passing through the point of intersection of $5x - 4y - 2 = 0$ and $2x + 4y - 15 = 0$, and through the point (2, 3).

3. Passing through the point of intersection of $3x + 2y - 6 = 0$ and $x + y = 3$, and perpendicular to $2x + y - 1 = 0$.

4. Passing through the point of intersection of $x - 6y = 3$ and $2x - y = 2$, and perpendicular to $x - 2y + 1 = 0$.

5. Passing through the intersection of $y = 6 + x$ and $3y = 4 - 2x$, and parallel to $x + 3y - 4 = 0$.

68. Plotting by factoring.—Since it is easy to plot a straight line, the theorem of article 48 gives a simple method of plotting equations which can be factored into linear factors.

Example.—Plot the equation $2x^2 + 2x + 7y = xy + 3y^2 + 4$.

First transpose all terms to the left hand side of the equation

$$2x^2 + 2x + 7y - xy - 3y^2 - 4 = 0.$$

In order to find out if this equation can be factored, regard it as a quadratic in x or y, and solve for that variable. For the sake of convenience the variable chosen this time will be x. Collecting like powers of x,

$$2x^2 + (2 - y)x - 4 + 7y - 3y^2 = 0.$$

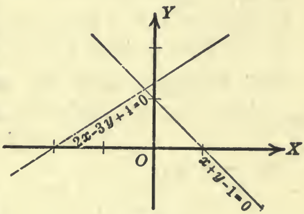


Fig. 56.

Solving for x, by means of the formula, Art. 4, where $a = 2$, $b = 2 - y$, and $c = -4 + 7y - 3y^2$,

$$x = \frac{-2 + y \pm \sqrt{25y^2 - 60y + 36}}{4} = \frac{-2 + y \pm (5y - 6)}{4}.$$

Hence $x = \dfrac{3y - 4}{2}$ or $-y + 1$, and the left hand side can be factored into $2\left(x - \dfrac{3y - 4}{2}\right)\left(x + y - 1\right)\cdot$

The equation now becomes $(2x - 3y + 4)(x + y - 1) = 0$.

Therefore the graph of $2x^2 + 2x + 7y = xy + 3y^2 + 4$ consists of the two straight lines $2x - 3y + 4 = 0$ and $x + y - 1 = 0$.

When an equation of the second degree in each of two variables, is solved for one variable in terms of the other, an expression is obtained under a radical sign. If this expression is a perfect square, the graph of the equation consists of two straight lines.

Thus, in the problem just solved, $25y^2 - 60y + 36$ is a perfect square. If it had not been a perfect square $2x^2 + 2x + 7y = xy + 3y^2 + 4$ could not have been plotted by this method.

Example 2.—Plot the curve $x^2y = y^3$.

Transposing all terms to the left hand side, $x^2y - y^3 = 0$.

Factoring the left hand side, $y(x - y)(x + y) = 0$.

The graph consists of the line $y = 0$ which is the x-axis, the line $x - y = 0$ and the line $x + y = 0$.

EXERCISES

1. Find the equation of the triangle whose sides are $x = y$, $y = 0$, and $x + y = 1$.

2. Find the equation of the square whose bounding lines are $x = 1$, $x = 2$, $y = 1$, and $y = 2$.

Plot the following curves by first factoring:

3. $x^2 - 2y^2 - xy + 3y - 1 = 0$.

4. $2y^2 = xy + x^2$.

5. $x^2 + 2x + 1 = 4y^2$.

6. $2x^2 + xy + 4x + y + 2 = y^2$.

69. Straight line in polar coördinates.—In general the equations of straight lines in polar coördinates are not as simple as those in rectangular coördinates. The simplest case is the one in which the known quantities are the polar coördinates of the foot of the perpendicular from the origin to the line. This is the same data as was given for the normal form of the equation of a straight line. If the end of the perpendicular to the line from the origin has the coördinates (ρ_1, θ_1), Fig. 57, let $P(\rho, \theta)$ be any point on the line.

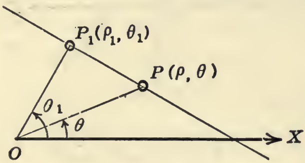


FIG. 57.

Then

$$\cos POP_1 = \frac{OP_1}{OP}, \quad \text{or} \quad \cos(\theta - \theta_1) = \frac{\rho_1}{\rho}.$$

Hence

$$\rho \cos(\theta - \theta_1) = \rho_1. \tag{1}$$

This equation can also be obtained from the normal form of the equation of a straight line by replacing p by ρ_1, θ by θ_1, x by $\rho \cos \theta$, and y by $\rho \sin \theta$.

In the special case where the line is perpendicular to the polar axis, $\theta_1 = 0$, and the polar form of the equation of the straight line takes the form $\rho \cos \theta = \rho_1$. If the straight line is parallel to the polar axis, $\theta_1 = 90°$ and the equation of the straight line becomes $\rho \sin \theta = \rho_1$.

Example.—Find the polar form of the equation of a straight line if the coördinates of the foot of the perpendicular drawn to it from the origin are (3, 60°).

Substituting in equation (1), gives $\rho \cos (\theta - 60°) = 3$.

EXERCISES

Write the equations of the following straight lines in polar coördinates, if the coördinates of the end of the perpendicular from the origin to the line are:

1. (3, 45°).
2. (−2, 60°).
3. (7, 90°).
4. (4, 180°).
5. (−4, 135°).
6. (3, 315°).

Change from rectangular to polar coördinates.

7. $x + y - 1 = \ $.
8. $x = 3$.
9. $y = -7$.
1 . $x\sqrt{3} + y = 4$.
11. $x - y\sqrt{3} + 6 = 0$.
12. $y - 2x = 0$.

Change from polar to rectangular coördinates.

13. $\rho = 3 \sec \theta$.
14. $\rho = 4 \csc \theta$.
15. $\tan \theta = 6$.
16. $\rho = \dfrac{2}{\cos \theta + \sin \theta}$.
17. $\rho = \dfrac{3(\cos \theta - \sin \theta)}{\cos 2\theta}$.
18. $\rho = \dfrac{3}{4 \cos \theta - 6 \sin \theta}$.
19. $5 \sin \theta = 3$.
20. $13 \cos \theta = -5$.
21. $\rho = \dfrac{\sqrt{2}}{\sin (\theta - 45°)}$.
22. $\rho = \dfrac{3}{\cos (\theta + 60°)}$.

70. Applications of the straight line.—Whenever two variables are related so that one varies directly as the other, or so that a change in one varies directly as the corresponding change in the other, the relation between the variables is linear, and the graph showing the relation between the variables is a straight line.

Since many of the relations in physics, mechanics, and engineering are of this nature, the straight line has a wide field of application. Oftentimes the curves representing the relation between physical quantities are within certain limits so nearly straight lines that the more complicated equation is replaced, on account of its simplicity, by the linear relation. A few specific instances are the following.

(1) The increase in velocity of a body falling under the action of gravity is proportional to the time. This is expressed by the relation

$$v - v_0 = k(t - t_0),$$

where v_0 is the velocity of the body at the time t_0 and v is the velocity of the body at any time t. If v and v_0 are expressed in feet per second, and t and t_0 are expressed in seconds, then k, the proportionality factor, is the familiar constant g. This relation is often expressed

$$v = kt + v_0,$$

where v_0 is the velocity when $t = t_0 = 0$.

(2) *Hooke's Law.*—The extension of an elastic string varies directly as the tension. This is expressed by the relation

$$l = kt + l_0,$$

where l is the length of the string under the tension t, and l_0 is the length of the string when $t = 0$.

(3) The expansion of a bar due to heat, is very nearly proportional to its increase in temperature. This is expressed by the relation

$$l - l_0 = k(t - t_0),$$

where l_0 is the length of the bar at some temperature t_0 and l is its length at any temperature t.

(4) The weight of a column of mercury in a barometer varies directly as its height. This is expressed by the relation

$$w = kh,$$

where the weight w is taken as zero when $h = 0$.

In all four cases the graph representing the linear relation between the variables is a straight line.

For further applications of the straight line see Chapter X on empirical equations.

GENERAL EXERCISES

1. Translate the following algebraic statements into words and draw their loci:

(1) $y = 4x$, (2) $y = 4x - 4$,
(3) $x = 3y + 2$, (4) $x = 5y - 2$.

2. Find the equation of the line (1) through the point $(4, -3)$ and parallel to $2x - 3y = 4$; (2) through the point $(5, 7)$ and perpendicular to $2x + 7y = 14$.

3. Find the equation of the line (1) through the point $(-2, -5)$ and parallel to $x - 7y = 3$; (2) through the point (h, k) and parallel to the line $y = mx + b$.

4. Find the length of the following perpendiculars:

(1) From $(3, 2)$ to $4x - 3y - 7 = 0$.
(2) From $(0, -3)$ to $5x - y - 6 = 0$.
(3) From $(2, 3)$ to $6x - 8y - 10 = 0$.

5. Find the lengths of the three altitudes of the triangle whose vertices are $(4, 5)$, $(-2, 2)$, and $(3, -4)$.

6. Find the distances from the line $2x + 3y - 12 = 0$ to each of the points $(4, 4)$, $(2, -3)$, $(0, 0)$, $(-3, 5)$, and $(-2, 8)$.

7. Given $4x + ky - 5 = 0$; determine the value of k for which the line will (1) pass through the point $(-4, 3)$, and (2) be parallel to $3x - 2y + 7 = 0$.

8. Find the equations of the lines through the intersection of the lines $2x + y - 16 = 0$ and $x - y + 2 = 0$ and also

(1) passing through the point $(2, 7)$,
(2) parallel to the line $7x - 2y + 6 = 0$,
(3) perpendicular to the line $3x - 4y + 2 = 0$,
(4) having the slope $-\frac{5}{2}$.

9. Given a triangle having as vertices the points $(6, 2)$, $(-3, 5)$, and $(-1, -3)$; find the equations of the perpendicular bisectors of the three sides, and the coördinates of their point of intersection.

10. Show that $15x^2 - 14xy - 8y^2 = 0$ is the equation of two straight lines intersecting at the origin.

11. Prove that, if A, B, and C are real numbers, $Ax^2 + Bxy + Cy^2 = 0$ represents two straight lines passing through the origin, and that these lines are real and distinct, real and coincident, or imaginary according as $B^2 - 4AC$ is positive, zero, or negative.

12. The perpendicular drawn from the origin to a line makes an angle of $60°$ with the x-axis and its length is 2, find the equation of the line.

13. Write the equations of the following lines:

(1) Passing through the point $(3, 5)$ and having an inclination of $45°$.
(2) Passing through $(-1, -3)$ and having a slope of 2.
(3) Passing through $(-2, 8)$ and having an inclination of $120°$.

14. Show that the following lines form a parallelogram:

(*a*) $2x + 3y = 10$, (*b*) $2x + 3y = 20$, (*c*) $x - 2y = 5$,
(*d*) $2x - 4y = 17$.

15. Write the equation of the line passing through the intersection of $x - 3y + 8 = 0$ and $3x + 2y + 2 = 0$ and making an angle whose tangent is 2 with the x-axis.

16. Find the coördinates of the point in which the perpendicular to the line $2x - y - 1 = 0$ and passing through $(-2, 3)$ intersects that line.

17. What does the equation $3x - 2y + 4 = 0$ become when the coördinate axes are turned through an angle of $45°$? Plot the locus of the equation in both cases.

18. Plot each of the following lines, translate the axes so that the new origin shall be at the point indicated and replot from the new equation.

(1) $y = 3x + 4$, $(2, 3)$. (3) $y = mx + b$, (c, d).
(2) $2y - 3x - 2 = 0$, $(-2, 3)$. (4) $y - 4x + 5 = 0$, $(\frac{3}{4}, -2)$.

19. The three vertices of a triangle are $(8, 2)$, $(4, 8)$, and $(-2, -6)$. Find the equations of the lines each of which bisects two sides of the triangle.

20. Given two straight lines each having an inclination of $45°$ and having intercepts on the y-axis of 6 and -8 respectively; find the equation of the straight line that is equidistant from the two lines.

21. Find the equation of a straight line such that the perpendicular from the origin to it equals 8 and makes an angle of $45°$ with the x-axis.

22. Find the equation of the straight line which passes through the intersection of the lines $x - 2y - 4 = 0$ and $x + 3y - 8 = 0$ and is parallel to the line $3x + 4y = 4$.

23. Find the equation of the line through the point $(2, 3)$ making an angle $\tan^{-1} \frac{2}{3}$ with the line $2x - 4y + 7 = 0$.

24. Find the equation of the line through the point $(-1, 2)$ making an angle $\sin^{-1} \frac{3}{5}$ with the line $x + 3y - 4 = 0$.

25. Find the equation of the line through the point $(6, 4)$ making an angle $\cos^{-1} (-\frac{4}{5})$ with the line $2x - y + 6 = 0$.

26. Find the equations of the two lines through the point $(-1, -3)$, which form an equilateral triangle with the line $x + y = 2$.

27. Find the equation of the line through the point (0, 6) which together with the y-axis as the other equal side forms an isosceles triangle with the line $2x - y + 4 = 0$.

28. Find the equation of the line through the point (0, 6) which together with the y-axis for the other leg forms an isosceles triangle with the line $2x + y - 4 = 0$.

29. The equations of the two equal sides of an isosceles triangle are $x - 2y + 6 = 0$ and $2x - y - 2 = 0$. Find the equation of the third side if it passes through the point (9, 4).

30. Find the slope of the line $2x + 3y - 4 = 0$ after the axes are rotated through 30°.

31. Find the slope of the line $x - 3y + 6 = 0$ after the axes are rotated through the angle θ, where $\cos\theta = -\frac{3}{5}$ and θ is in the second quadrant.

32. Find the equations of the two lines through the point $(-1, 3)$ which trisect that part of the line $2x + y - 6 = 0$ which is intercepted between the axes.

33. An equilateral triangle lies wholly in the first quadrant. If one side has its extremities at (1, 6) and (6, 1), what are the equations of the other two sides?

34. An isosceles right triangle is constructed with its hypotenuse along the line $2x + y - 6 = 0$. If its vertex is the point (3, 4), find the equations of its sides.

35. A circle is inscribed in the triangle the equations of whose sides are $x + 2y - 16 = 0$, $2x - y + 3 = 0$, and $2x + y - 7 = 0$. Find its radius and the coördinates of its center.

36. The base of an isosceles triangle is the line joining the points (1, 5) and (4, 6), its vertex is on the line $x + y - 7 = 0$. Find the coördinates of its vertex.

37. Find the locus of a point which moves so as to be always equidistant from the points (3, 5) and $(-1, 7)$.

38. Find the equation of the locus of a point which moves so that its distance from the line $7x + 4y - 6 = 0$ is twice its distance from the line $x - 8y + 3 = 0$.

39. Find the equation of the locus of a point which moves so that the difference of the squares of its distances from the points $(-2, 3)$ and (1, 6) shall be constant and equal to 2.

40. Find the equations of two lines through the point (1, 1) such that the perpendiculars let fall from the point (1, 3) on them are each of length $\frac{8}{5}$.

41. Prove that the feet of the perpendiculars let fall from the point (3, 1) on the sides of the triangle $x = 0$, $y = 0$, and $2x + y - 4 = 0$ lie in a straight line.

42. Find the equations of the straight lines through the point (3, 6) and intersecting the line $x + y - 2 = 0$ at a distance 5 from this point.

43. Prove that the perpendicular bisectors of the sides of a triangle meet in a point.

44. Find the equation of the locus of a point that is always twice as far from the origin as from the x-axis.

45. The coördinates of two points are (3, 5) and (4, 4). Find the equation of a straight line which bisects the line segment connecting these points and makes an angle of 45° with the x-axis.

46. A straight line inclined to the x-axis at an angle of 150° has an x-intercept equal to 8. Find the equation of a straight line passing through the origin and bisecting that portion of the line included between the axes.

47. Find the equations of the four sides of a square two of whose opposite vertices are (2, 3) and (3, 4).

48. A straight line moves so as to keep the sum of the reciprocals of its intercepts on the axes a constant. Show that the moving line passes through a fixed point.

49. Find the equation of the straight line passing through the point (2, 6) and making an angle of 30° with the line $x - 2y = 1$.

50. Find the equation of a straight line passing through the point $(c, 0)$ and making an angle of 45° with the line $bx - ay = ab$.

51. The equation of a straight line is $3x + 5y = 15$; find the equation of the same line referred to parallel axes whose origin is at (3, 2).

52. Find the equations of the straight lines bisecting the angles formed by the lines $12x + 5y = 8$ and $3x - 4y = 3$.

53. Show that an angle of 45° is formed by the lines represented by the equation $x^2 - xy - 6y^2 + 2x - y + 1 = 0$.

54. Given the equation $Ax + By + C = 0$. Find the relation between A, B and C, (1) so that the x- and y-intercepts shall be equal; (2) so that the inclination of the line shall be 45°; (3) so that the line shall pass through the point (1, 2).

55. Determine the angle that the first line of each of the following pairs makes with the second:

(1) $x + 2y = 5,\ 3x - 4y = 4.$
(2) $3x + 4y = 6,\ 2x - y = 2.$
(3) $\sqrt{3}x + y = 4,\ \sqrt{3}x - y + 4 = 0.$

56. Determine the value of m in $y = mx + 6$, so that it shall make an angle of 60° with $x - 2y = 3$.

57. Find the coördinates of the point through which the three lines $y - 4x = 5$, $y - 3x = 4$, and $y - 2x = 3$ pass.

58. Find the value of m so that $y = mx + 3$ shall pass through the intersection of $y - x = 1$ and $y - 2x = 2$.

59. Find the equation of the line perpendicular to $5x + 8y = 3$ and having a y-intercept equal to 6.

60. Find the angle which the line $4x - y = 8$ makes with the line $6x - y = 9$.

61. Find the equation of the locus of a point whose distance from $3x + 4y = 5$ is one-half its distance from $12x - 5y = 16$.

62. Given the two fixed points $P_1(-2, 4)$ and $P_2(1, 3)$. Find the equation of the locus of the variable point $P(x, y)$ which moves so that the area of the triangle PP_1P_2 is always equal to 10.

63. Find the equation of the locus of a point which moves so that the slope of the line joining it to the point (0, 2) is twice the slope of the line joining it to the point (0, −2).

64. If the equations of the sides of a triangle are $x + 2y - 15=0$, $2x - y + 5 = 0$, and $2x - 11y + 15=0$, find the coördinates of the point of intersection of the bisectors of the interior angles of the triangle.

65. Find the equation of a line passing at a distance $\sqrt{2}$ from the origin if the sum of its intercepts is 4.

66. If the three lines

$$A_1x + B_1y + C_1 = 0,$$
$$A_2x + B_2y + C_2 + 0,$$
$$A_3x + B_3y + C_3 = 0,$$

meet in a point, show that

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

CHAPTER V

THE CIRCLE AND CERTAIN FORMS OF THE SECOND DEGREE EQUATION

71. Introduction.—The circle affords other examples of the ease and power obtained in analytic geometry by applying algebra to geometry. Since the properties of the circle are well known from plane geometry, attention can be confined to the methods used in solving the various problems.

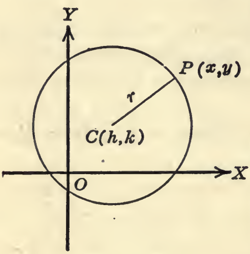


FIG. 58.

72. Equation of circle in terms of center and radius.—A circle is defined in plane geometry to be the locus of all points in a plane equidistant from a fixed point in the plane called the center of the circle.

Let the center of the circle be the fixed point, $C(h, k)$, Fig. 58, and let the constant distance, or radius, be r.

Then if $P(x, y)$ is any point on the circle, the distance $PC = r$.

But by [3], $PC = \sqrt{(x-h)^2 + (y-k)^2}$.

Then $$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

[25] $$\therefore (x-h)^2 + (y-k)^2 = r^2.$$

Furthermore, comparison of this equation with [3] shows that every equation of the form of [25] is the equation of a circle.

If the center of the circle is the origin, this equation takes the simple form

[26] $$x^2 + y^2 = r^2.$$

73. General equation of the circle.—Equation [25] when expanded becomes

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is in the form

[27] $$\mathbf{x^2 + y^2 + Dx + Ey + F = 0.}$$

This is called the **general equation of the circle.**

Conversely, every equation in the form of [27] is the equation of a circle, since after completing the squares in the x and the y-terms, it can be written in the form

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = \frac{D^2}{4} + \frac{E^2}{4} - F,$$

or $$(x + \tfrac{1}{2}D)^2 + (y + \tfrac{1}{2}E)^2 = (\tfrac{1}{2}\sqrt{D^2 + E^2 - 4F})^2.$$

Comparison of this equation with equation [25], shows that $h = -\frac{1}{2}D$, $k = -\frac{1}{2}E$, and $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

Therefore every equation in the form of [27] is the equation of a circle.

If $D^2 + E^2 - 4F > 0$, equation [27] represents the equation of a real circle.

If $D^2 + E^2 - 4F = 0$, the radius of the circle equals 0, and the locus becomes a point. Such a circle is called a **null** or **point circle.**

If $D^2 + E^2 - 4F < 0$, the radius of the circle is imaginary and the circle is called an **imaginary circle.**

74. Special form of the general equation of the second degree.—The equation of a circle is a special case of the most general equation of the second degree in two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

In order that this shall be the equation of a circle comparison with [27] shows that $B = 0$ and $A = C$, for then this equation becomes

$$Ax^2 + Ay^2 + Dx + Ey + F = 0,$$

which can be reduced to [27] by dividing by A. The quantity A cannot be zero, since if it were, this equation would become the equation of a straight line.

Example.—Find the coördinates of the center, and the radius of the circle $5x^2 + 5y^2 + 2x - 3y - 4 = 0$.

Solution.—Dividing by 5.

$$x^2 + y^2 + \tfrac{2}{5}x - \tfrac{3}{5}y - \tfrac{4}{5} = 0.$$

Completing squares

$$x^2 + \tfrac{2}{5}x + \tfrac{1}{25} + y^2 - \tfrac{3}{5}y + \tfrac{9}{100} = \tfrac{1}{25} + \tfrac{9}{100} + \tfrac{4}{5}.$$

or

$$(x + \tfrac{1}{5})^2 + (y - \tfrac{3}{10})^2 = \tfrac{93}{100}.$$

Comparing this equation with [**25**], shows that the center has the coördinates $(-\tfrac{1}{5}, \tfrac{3}{10})$ and that the radius is equal to $\sqrt{\tfrac{93}{100}} = \tfrac{1}{10}\sqrt{93}$.

This problem could also be solved by substituting the values $D = \tfrac{2}{5}$, $E = -\tfrac{3}{5}$, and $F = -\tfrac{4}{5}$, in the formulas of Art. **73**.

EXERCISES

Find the coördinates of the centers and the radii of the following circles:

1. $x^2 + y^2 - 2x - 4y - 4 = 0$.
2. $x^2 + y^2 + 4x - 6y + 12 = 0$.
3. $x^2 + y^2 + 12x + 6y + 41 = 0$.
4. $x^2 + y^2 - x - 4y + 2 = 0$.
5. $3x^2 + 3y^2 - 2x - 4y + 1 = 0$.
6. $2x^2 + 2y^2 + x + 3y - 5 = 0$.
7. $2x^2 + 2y^2 + 2x + 6y + 5 = 0$.
8. $x^2 + y^2 - 2ax - 6ay + a^2 = 0$.
9. $2x^2 + 2y^2 + 12ax + 10ay - a^2 = 0$.
10. $9x^2 + 9y^2 - 6ax + 15ay + 5a^2 = 0$.

75. Equation of a circle satisfying three conditions.—Since both equations [**25**] and [**27**] involve three arbitrary constants, the circle is determined if enough geometric or algebraic conditions are given to determine the three constants uniquely.

There are two methods of procedure. One is to compute the constants in [**25**] geometrically. That is to say, from the given conditions compute the radius of the circle and the coördinates of its center, then substitute these values in [**25**]. Another method is to set up three equations involving h, k, and r, or three equations involving D, E, and F, and solve these equations simultaneously. This method is generally more satisfactory, and is illustrated for both sets of constants in the following example.

Example 1.—Find the equation of a circle passing through the points (3, 5), (4, 4), (1, 1).

First method, geometrical.—Find the equations of the perpendicular bisectors of two of the sides of the triangle (3, 5), (4, 4), (1, 1). Solve these equations simultaneously. This gives the coördinates of the center of the circle.

Next find the distance from the center of the circle to any one of the three vertices. This gives the radius of the circle. Substituting the values of h, k and r thus found in [25] gives the desired equation.

It is obvious that this method is long and hence the actual computation is not given. A shorter method is the following.

Second method, algebraic.—Make the coördinates of each of the three points satisfy the equation $(x - h)^2 + (y - k)^2 = r^2$. This gives

$$(3 - h)^2 + (5 - k)^2 = r^2,$$
$$(4 - h)^2 + (4 - k)^2 = r^2,$$
$$(1 - h)^2 + (1 - k)^2 = r^2.$$

Simplifying each of these equations gives

$$h^2 + k^2 - 6h - 10k - r^2 + 34 = 0,$$
$$h^2 + k^2 - 8h - 8k - r^2 + 32 = 0,$$
$$h^2 + k^2 - 2h - 2k - r^2 + 2 = 0.$$

Solve these equations by subtracting the second from the first and the third from the second. Then solving the two equations thus obtained gives $h = 2$, $k = 3$, $r = \sqrt{5}$.

Hence the equation of the required circle is

$$(x - 2)^2 + (y - 3)^2 = 5$$

Simplifying, this becomes

$$x^2 + y^2 - 4x - 6y + 8 = 0.$$

Third method.—Make the coördinates of the three points satisfy the equation $x^2 + y^2 + Dx + Ey + F = 0$. This gives

$$9 + 25 + 3D + 5E + F = 0,$$
$$16 + 16 + 4D + 4E + F = 0,$$
$$1 + 1 + D + E + F = 0.$$

Solving these equations simultaneously gives

$$D = -4, E = -6, F = 8.$$

Hence the equation of the circle is

$$x^2 + y^2 - 4x - 6y + 8 = 0.$$

Example 2.—Find the equation of a circle which passes through the points $(-1, 7)$ and $(7, 1)$ and is tangent to the line $x + y - 10 = 0$.

This problem illustrates how a combination of both algebraic and geometric methods may sometimes be useful.

Solution.—Use the equation $(x - h)^2 + (y - k)^2 = r^2$, and make the circle go through the points $(-1, 7)$ and $(7, 1)$. This gives the two equations

$$(-1 - h)^2 + (7 - k)^2 = r^2, \tag{1}$$

and

$$(7 - h)^2 + (1 - k)^2 = r^2. \tag{2}$$

Since the line $x + y - 10 = 0$ is tangent to the circle, the distance from the point (h, k) to the line $x + y - 10 = 0$ equals r, hence by **[23]**

$$\frac{h + k - 10}{\pm\sqrt{2}} = r. \tag{3}$$

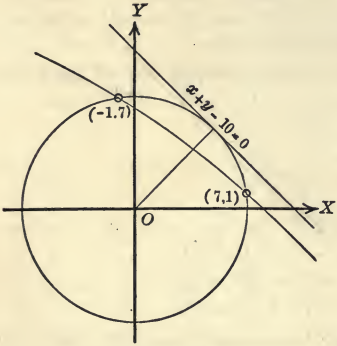


Fig. 59.

Simplifying and combining like terms in (1) and (2) gives

$$h^2 + k^2 + 2h - 14k + 50 = r^2, \tag{4}$$
$$h^2 + k^2 - 14h - 2k + 50 = r^2. \tag{5}$$

Subtracting (5) from (4) and dividing both sides of the resulting equation by 4,

$$4h - 3k = 0. \tag{6}$$

Substituting the value of r from equation (3) in (4) and simplifying,

$$h^2 - 2hk + k^2 + 24h - 8k = 0. \tag{7}$$

Substituting $h = \frac{3}{4}k$ from equation (6) in equation (7) and simplifying,

$$k^2 + 160k = 0.$$

Hence $$k = 0 \text{ or } k = -160.$$

Computing the value of h from equation (6), gives $h = 0$ or $h = -120$.

Computing the value of r from equation (3), gives

$$r = 25\sqrt{2} \text{ or } r = 145\sqrt{2}.$$

Substituting the values of h, k, and r in the general equation of the circle gives the two solutions $x^2 + y^2 = 50$,

and $$(x + 120)^2 + (y + 160)^2 = 42{,}050.$$

EXERCISES

Find the equations of the circles through the following points:

1. (5, 5), (1, 3), (2, 6).
2. (3, −2), (−1, −4), (2, −5).
3. (0, 3), (−4, 3), (−3, 4).
4. (1, 6), (4, 5), (−3, −2).
5. (−1, 4), (−4, −5), (3, 2).
6. (2, 8), (−1, −1), (−2, 6).
7. (2, 4), (2, −2), (3, 3).
8. (4, 4), (5, 3), (−3, 3).
9. (3, 7), (1, 1), (5, 3).
10. (−1, 1), (1, 5), (−5, 3).

Find the equations of the circles fulfilling the following conditions:

11. Passing through the origin, radius 5, and ordinate of center -3.

12. Passing through the origin, radius 13, and abscissa of center 12.

13. Center at origin and tangent to line $x + 2y = 10$.

14. Center at point (1, 2) and passing through the point (3, -1).

15. Center at (-1, 3) and tangent to line $3x + y - 10 = 0$.

16. Center on x-axis and passing through the points (3, 3) and (5, -1).

17. Radius 5 and passing through the points (5, 6) and (2, 7).

18. Radius 5 and tangent to the line $4x + 3y - 16 = 0$ at the point (1, 4).

19. Having the line joining (-3, 2) and (5, 6) as diameter.

20. Passing through the point (1, 1) and having the same center as $x^2 + y^2 + 4x - 6y = 0$.

21. Intercept on x-axis equals 3, and passing through the points (-1, 2) and (2, 3).

22. Tangent to x-axis, radius 4, and abscissa of center 3.

23. Tangent to y-axis, radius 2, and ordinate of center 4.

24. Center on the line $x - y + 2 = 0$, and passing through the points (3, 7) and (1, 1).

25. Center on the line $2x - y - 3 = 0$, tangent to both axes, and in the first quadrant.

26. Center on the line $2x - y - 3 = 0$, tangent to both axes, and in the fourth quadrant.

27. Center on the line $3x - y + 8 = 0$, tangent to both axes, and in the second quadrant.

28. Radius 3, tangent to both axes, and in the second quadrant.

29. Tangent to the line $3x + y + 2 = 0$ at the point (-1, 1) and passing through the point (3, 5)

30. Intercept on the y-axis 4, and tangent to the line $x + 2y + 1 = 0$ at the point (-3, 1).

31. Tangent to both axes, in the second quadrant, and also tangent to the line $3x - 4y + 30 = 0$. (Two solutions.)

32. Tangent to both axes, in the first quadrant, and also tangent to the line $3x - 4y + 30 = 0$.

33. Tangent to both axes and passing through the point (8, 1). (Two solutions.)

34. Find the equation of the diameter with slope 2 of the circle $x^2 - 4x + y^2 + 6y - 3 = 0$.

35. The point (-1, 2) bisects a chord of the circle $x^2 + y^2 = 10$. Find the equation and length of the chord.

36. A chord of the circle $x^2 + y^2 + 2x + 4y - 15 = 0$ is bisected by the point (-2, 1). Find the equation and length of the chord.

37. Find the equation of the circle inscribed in the triangle whose sides are the lines $6x + 7y = 85$, $-7x + 6y = 85$, and $2x - 9y = 85$.

38. Find the equation of the circle inscribed in the triangle whose sides are the lines $3x + 4y = 18$, $-4x + 3y = 26$, and $y + 4 = 0$.

39. Find the equation of the circle circumscribing the triangle whose sides are the lines $7x + 9y = 65$, $3x + y = 25$, and $x + 2y = 15$.

40. Prove analytically that an angle inscribed in a semicircle is a right angle.

41. Prove analytically that a line from the center of a circle bisecting a chord is perpendicular to it.

Suggestion.—Let the ends of the chord be $(r, 0)$ and (b, c).

42. Prove analytically that the length of a perpendicular from any point on the circumference of a circle to a diameter, is a mean proportional between the segments into which it divides the diameter.

43. Prove that the length of the tangent from the point (x_1, y_1) to the circle $x^2 + y^2 + Dx + Ey + F = 0$ is $x_1^2 + y_1^2 + Dx_1 + Ey_1 + F = 0$.

76. Systems of circles.—If $f_1(x, y) = 0$ and $f_2(x, y) = 0$ are the equations of any two circles, then by article 67 $f_1(x, y) + kf_2(x, y) = 0$ is the equation of a curve through all the points of intersection of $f_1(x, y) = 0$ and $f_2(x, y) = 0$. Furthermore in this case the curve will always be a circle or a straight line.

To prove that this is so, let $f_1(x, y) = 0$ stand for the equation $A_1x^2 + A_1y^2 + D_1x + E_1y + F_1 = 0$, and let $f_2(x, y) = 0$ stand for $A_2x^2 + A_2y^2 + D_2x + E_2y + F_2 = 0$.

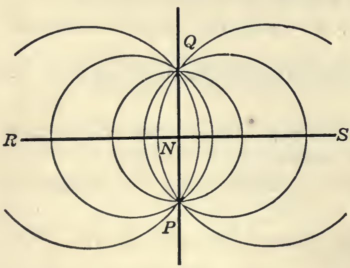


FIG. 60.

Then $f_1(x, y) + kf_2(x, y) = 0$ becomes $A_1x^2 + A_1y^2 + D_1x + E_1y + F_1 + k(A_2x^2 + A_2y^2 + D_2x + E_2y + F_2) = 0$.

Collecting like powers of x and y, this equation becomes $(A_1 + kA_2)x^2 + (A_1 + kA_2)y^2 + (D_1 + kD_2)x + (E_1 + kE_2)y + F_1 + kF_2 = 0$.

Since the coefficient of x^2 equals the coefficient of y^2 and the coefficient of xy equals 0, this is the equation of a circle. The exception occurs when $A_1 + kA_2 = 0$, in which case, this equation is of the first degree and therefore is the equation of a straight line.

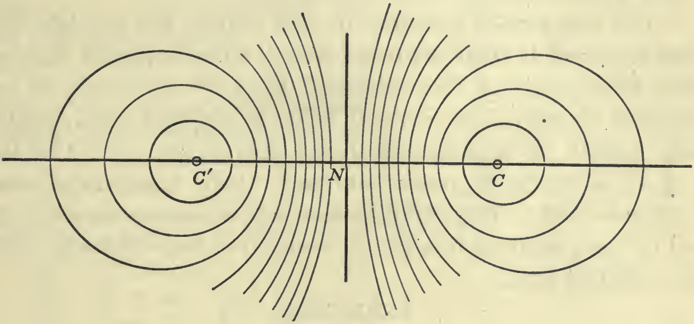


FIG. 61.

Example 1.—Find the equation of a circle through the point (1, 2) and the points of intersection of the circles $2x^2 + 2y^2 - 3x - 4y - 1 = 0$ and $3x^2 + 3y^2 - 8x - y - 4 = 0$.

Solution.—The equation of any circle through the points of intersection of these two circles is

$$2x^2 + 2y^2 - 3x - 4y - 1 + k(3x^2 + 3y^2 - 8x - y - 4) = 0.$$

Since the point (1, 2) is on this circle its coördinates must satisfy the equation of the circle, therefore

$$2 + 8 - 3 - 8 - 1 + k(3 + 12 - 8 - 2 - 4) = 0.$$

Solving for k, gives $k = 2$.

Therefore the required equation is

$$2x^2 + 2y^2 - 3x - 4y - 1 + 2(3x^2 + 3y^2 - 8x - y - 4) = 0,$$

or
$$8x^2 + 8y^2 - 19x - 6y - 9 = 0.$$

Example 2.—Find the equation of the common chord of the circles, $2x^2 + 2y^2 - 6x - 4y + 1 = 0$ and $x^2 + y^2 - 2x - y + 3 = 0$.

Solution.—The equation of any circle through the points of intersection of these two circles is

$$2x^2 + 2y^2 - 6x - 4y + 1 + k(x^2 + y^2 - 2x - y + 3) = 0.$$

In order that this equation shall be the equation of a straight line, it is necessary that the coefficient of x^2 shall vanish, hence $2 + k = 0$. This gives $k = -2$.

Making this substitution the equation becomes

$$2x^2 + 2y^2 - 6x - 4y + 1 - 2(x^2 + y^2 - 2x - y + 3) = 0,$$

or

$$2x + 2y + 5 = 0.$$

This is the equation of their common chord.

If the two circles intersect in real points, the straight line thus obtained is their common chord, since it passes through their two points of intersection. If the two circles do not intersect visually, they are still said to intersect algebraically, their points of intersection being imaginary, and the line $f_1(x, y) + kf_2(x, y)$ passes through their imaginary points of intersection. The straight line which passes through the real or imaginary points of intersection of two circles is called their **radical axis.**

EXERCISES

Find the equation of the common chord or the radical axis of the circles in exercises 1–6.

1. $x^2 + y^2 - 3x + y - 6 = 0,$
$x^2 + y^2 - 5x - 3y + 4 = 0.$

2. $x^2 + y^2 - 6x - 8y + 3 = 0,$
$x^2 + y^2 + 4x + 2y - 7 = 0.$

3. $x^2 + y^2 - 3x - 4y + 2 = 0,$
$x^2 + y^2 - 2x - 2y + 6 = 0.$

4. $2x^2 + 2y^2 - 3x - 3y + 5 = 0,$
$3x^2 + 3y^2 - 2x - 3y + 4 = 0.$

5. $4x^2 + 4y^2 - x + y - 6 = 0,$
$3x^2 + 3y^2 - 2x - 3y + 4 = 0.$

6. $3x^2 + 3y^2 - 2x - 3y + 6 = 0,$
$2x^2 + 2y^2 + x + y - 2 = 0.$

7. Find the equation of the circle through the point (1, 1) and through the points of intersection of the circles

$$x^2 + y^2 - 2x - 3y + 4 = 0,$$
$$x^2 + y^2 - 4x - 5y + 6 = 0.$$

8. Find the equation of the circle through the point (3, 4) and through the points of intersection of the circles

$$x^2 + y^2 - 7x - 3y + 10 = 0,$$
$$x^2 + y^2 - 8x + 2y - 6 = 0.$$

9. Prove that the common chords of the following circles, taken two at a time, meet in a point:

$$x^2 + y^2 - 4x - 3y + 6 = 0,$$
$$x^2 + y^2 - 2x + 5y - 2 = 0,$$
$$x^2 + y^2 + x + 2y - 4 = 0.$$

77. Locus problems involving circles.—Although the elements dealt with in plane geometry are the point, straight line and circle, nevertheless the locus problems that can readily be handled by plane geometry are only of the simplest kind. On the other hand analytic geometry lends itself easily to the solution of locus problems as is illustrated by the following example.

Example.—Find the locus of the point, which moves so that the sum of the squares of its distances from the points (0, 1) and (2, 1) is constant and equal to 20.

Solution.—Let $P(x, y)$ be any point on the locus, then

$$\overline{PS}^2 + \overline{PT}^2 = 20, \qquad (1)$$
$$\overline{PS}^2 = x^2 + (y - 1)^2,$$
$$\overline{PT}^2 = (x - 2^2 + (y - 1)^2.$$

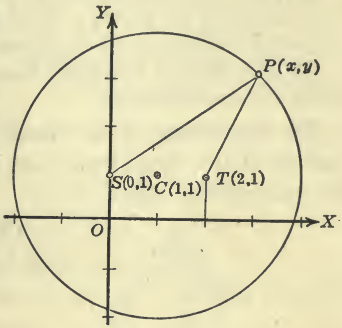


FIG. 62.

Substituting these values in equation (1)

$$x^2 + (y - 1)^2 + (x - 2)^2 + (y - 1)^2 = 20.$$

Simplifying, $x^2 + y^2 - 2x - 2y = 7$.

Completing the squares in the x and y-terms,

$$(x - 1)^2 + (y - 1)^2 = 3^2.$$

Hence the required locus is a circle whose center is the point (1, 1) and whose radius is 3.

EXERCISES

1. Find the locus of a point which moves so that the sum of the squares of its distances from $(-2, 0)$ and $(2, 0)$ is constant and equal to 26.

2. Find the locus of a point which moves so that the sum of the squares of its distances from $(-1, 2)$ and $(2, 1)$ is constant and equal to 10.

3. Find the locus of a point such that its distance from the point $(-2, 0)$ shall always be twice its distance from the point $(2, 0)$.

4. Find the locus of a point moving so that its distance from the line $3x + 4y - 5 = 0$ shall equal the square of its distance from the point $(1, 0)$.

5. Find the locus of a point such that its distance from the y-axis shall equal the square of its distance from the point $(0, 2)$. (Two solutions.)

6. In an isosceles triangle of base 6 and equal sides of length 5, a point moves so that the product of its distances from the equal sides equals the square of its distance from the base. Prove one of the loci to be a circle and find its radius.

7. Find the locus of the vertex of a right angle if its two sides always pass through the points $(-2, -4)$ and $(2, 6)$.

8. Find the locus of the vertex of an angle of 30°, whose sides pass through the points $(-2, 0)$ and $(2, 0)$. (Two solutions.)

9. Find the locus of the vertex of a triangle, if the remaining two vertices are at the points $(-3, 0)$ and $(3, 0)$ and the length of the median from the vertex $(-3, 0)$ is constant and equal to 5.

10. The ends of a straight line of length 6 rest on the axes, find the locus of its middle point.

78. Equation of a circle in polar coördinates.—Let the radius of the circle be r, and let $C(\rho_1, \theta_1)$ be the coördinates of its center, Fig. 63.

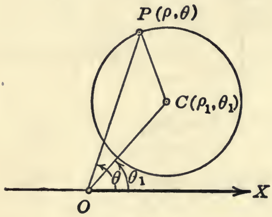


FIG. 63.

Then if $P(\rho, \theta)$ is any point on the circle, by trigonometry,

$$r^2 = \rho^2 + \rho_1^2 - 2\rho\rho_1 \cos COP.$$

Replacing angle COP by its value $(\theta - \theta_1)$,

$$r^2 = \rho^2 + \rho_1^2 - 2\rho\rho_1 \cos (\theta - \theta_1).$$

The forms of this equation which occur most frequently are those where the center is the pole or where the circle passes through the pole and the center of the circle is either on the initial line or on the line $\theta = 90°$.

If the center is the pole, $\rho_1 = 0$, and the equation becomes

$$\rho = r.$$

If the circle passes through the pole and has its center on the initial line, $\theta_1 = 0$ and $\rho_1 = \pm r$. The equation of the circle then becomes

$$\rho = 2r \cos \theta, \text{ or } \rho = -2r \cos \theta,$$

according as the center is on the initial line or the initial line produced through the pole.

If the circle passes through the pole and its center is on the line $\theta = 90°$, $\theta_1 = 90°$ and $\rho = \pm r$, and the equation becomes

$$\rho = 2r \sin \theta, \text{ or } \rho = -2r \sin \theta,$$

according as the circle lies above or below the polar axis.

EXERCISES

Find the equations of the following circles in polar coördinates:

1. The center is at the pole and the radius equals 2.

2. The center is at the point (5, 0) and the radius equals 5.

3. The center is at the point $(-4, 0)$ and the radius equals 4.

4. The center is at the point $(3, \frac{1}{2}\pi)$ and the radius equals 3.

5. The center is at the point $(-2, \frac{1}{2}\pi)$ and the radius equals 2.

6. The circle is tangent to the initial line at the pole and the radius equals 6.

7. The circle is tangent to the line $\theta = 90°$ at the pole and the radius equals 6.

8. The center is at the point $(3, \frac{1}{4}\pi)$ and the radius equals 3.

Change from rectangular to polar coördinates.

9. $x^2 + y^2 = 6.$

10. $x^2 + y^2 - 3y = 0.$

11. $2x^2 + 2y^2 + 5x = 0.$

12. $x^2 + y^2 - 6x - 8y = 0.$

Change from polar to rectangular coördinates and find the center and radius of each of the following circles.

13. $\rho + 6 \sin \theta = 0.$

14. $\rho - 4 \cos \theta = 0.$

15. $\rho = \cos \theta + \sin \theta.$

16. $\rho = 5.$

17. $\rho + 2 \cos \theta + 3 \sin \theta = 0.$

18. $\rho^2 + 3\rho \cos \theta + 4\rho \sin \theta - 6 = 0.$

19. $\rho^2 = 9 \sec^2\theta - \rho^2 \tan^2\theta.$

20. $\rho^2 = 4 \csc^2\theta - \rho^2 \cot^2\theta.$

CHAPTER VI

THE PARABOLA AND CERTAIN FORMS OF THE SECOND DEGREE EQUATION

79. General statement.—It is an interesting and useful fact that an equation of the second degree in two variables, if plotted with reference to rectangular axes, gives a *conic section,* or simply a *conic.* That is, the graph is some plane section of a right circular cone.

80. Conic sections.—When a plane intersects a circular cone there may be formed a circle, a parabola, an ellipse, an hyperbola, or, for certain positions of the plane, a point, two intersecting straight lines, or two coincident lines.

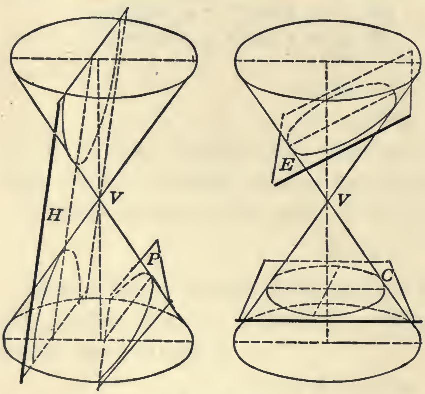


Fig. 64.

In Fig. 64, plane C is perpendicular to the axis of the cone and forms a circle; plane E is inclined to the axis but intersects only one nappe of the cone and forms an ellipse; plane P is parallel to an element of the cone and forms a parabola; plane H intersects both nappes of the cone and forms an hyperbola. The intersection is a point when a plane passes through the point V only; two intersecting straight lines are formed when the plane passes through V and intersects the nappes; and two coincident lines are formed when the plane passes through *V* and is tangent to the cone.

The conic sections were first studied by the Greeks, who discovered and discussed their properties by methods of geometry. The modern method of studying these figures is by the help of algebra, which makes the treatment much simpler. For the purposes of this method of treatment, other definitions of the conic sections are given; but it can be readily shown that these definitions agree with the definitions mentioned above.

EXERCISES

1. Explain how a conic section could be two lines inclined to each other at an angle of 45°. Could the two straight lines formed on the same cone form different angles with each other?

2. If the vertex angle of a cone is 30°, what would be the angle between the intersecting lines formed by the plane intersecting the cone?

3. In forming an hyperbola, does the plane have to be parallel to the axis of the cone? Could hyperbolas of different shapes be formed on the same cone?

4. Explain how a parabola of different widths could be formed on the same cone.

5. Explain how ellipses of different widths could be formed on the same cone. Explain the change in the shape of the ellipse formed by a plane that revolved into a position parallel to an element of the cone.

81. Conics.—A definition of a conic section, and one that can readily be translated into algebraic language, is the following: A **conic** is the locus of a point that moves in the plane of a fixed straight line and a fixed point not on the line, in such a manner that its distance from the fixed point is in a constant ratio to its distance from the fixed line.

The fixed point is called the **focus** of the conic, and the fixed line is called the **directrix.** The constant ratio is called the **eccentricity** and is usually represented by e.

The constant e is positive, and may be equal to 1, less than 1, or greater than 1.

If $e = 1$, the conic is a **parabola.**

If $e < 1$, the conic is an **ellipse.**

If $e > 1$, the conic is an **hyperbola.**

82. The equation of the parabola.—By the definition of the preceding article, the parabola is the locus of a point equidistant from the focus and the directrix.

In Fig. 65, let F be the focus and $D'D$ the directrix. Choose as x-axis the line $X'X$ through F and perpendicular to $D'D$ at R. The point O on $X'X$ midway between R and F is a point on the locus. Choose this point as origin. Then $Y'Y$ parallel to $D'D$ is the y-axis.

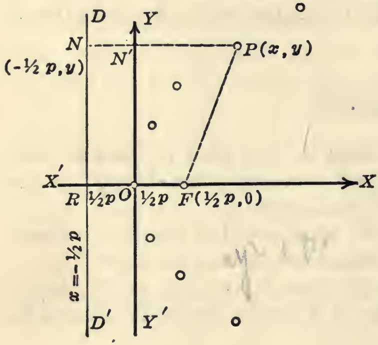


Fig. 65.

Let p represent the length and direction of RF. Then the coördinates of F are $(\frac{1}{2}\ p, 0)$, and the equation of $D'D$ is

$$x = -\tfrac{1}{2}p.$$

To derive the equation of the parabola, let $P(x, y)$ be any point on the locus, and draw FP, and NP perpendicular to $D'D$.

By definition $\qquad FP = NP.$

But $FP = \sqrt{(x - \frac{1}{2}p)^2 + y^2}$, and $NP = x + \frac{1}{2}p$.

Then $\qquad \sqrt{(x - \frac{1}{2}p)^2 + y^2} = x + \frac{1}{2}p.$

Squaring and simplifying, this becomes

[28] $$\mathbf{y^2 = 2px.}$$

The simple form of this equation is due to the choice of the coördinate axes. If they had been chosen differently, the equation would be more complicated; but the locus itself would be unaltered.

Equation [28] is the required equation. For it has been proved true for every point on the parabola; and it is not true for any point that is not on the parabola, for then FP is not equal to NP, and therefore y^2 is not equal to $2px$.

It should be remembered, that in the equation $y^2 = 2px$, p represents the *length* and *direction* of RF. Therefore, when the focus lies to the right of the directrix, p is positive; but, when the focus lies to the left of the directrix, p is negative.

83. Shape of the parabola.—The shape of the parabola and its position relative to the coördinate axes can be readily determined from the equation $y^2 = 2px$. Solving for y gives

$$y = \pm\sqrt{2px}.$$

For any positive value of p we have:

(1) When $x = 0$, $y = 0$. Hence the curve passes through the origin.

(2) For all positive values of x, y has two numerically equal values but opposite in sign. Hence the curve is symmetrical with respect to the x-axis.

(3) For any negative value of x, y is imaginary. Hence no part of the curve is at the left of the y-axis. As x increases from 0, the positive value of y increases and the negative value decreases.

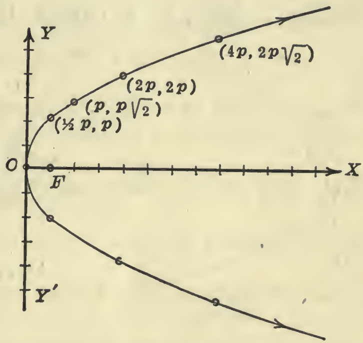


FIG. 66.

The curve can be located more precisely by the following points:

x	0	$\frac{1}{2}p$	p	$2p$	$4p$	$8p$	$50p$
y	0	$\pm p$	$\pm p\sqrt{2}$	$\pm 2p$	$\pm 2p\sqrt{2}$	$\pm 4p$	$\pm 10p$

The parabola has the shape shown in Fig. 66. It is evident that all parabolas have the same shape, the appearance depending only upon the size of the unit chosen. For a negative value of p, the parabola will be exactly the same shape but opening toward the left.

84. Definitions.—The point of the parabola midway between the focus and the directrix is called the **vertex** of the parabola.

The line through the focus and perpendicular to the directrix is called the **axis** of the parabola. As has been proved in the preceding article, the axis bisects all the chords of the parabola which are parallel to the directrix, since the axis of the parabola lies on the x-axis.

The chord of the parabola through the focus and perpendicular to the axis is called the **latus rectum.** The length of

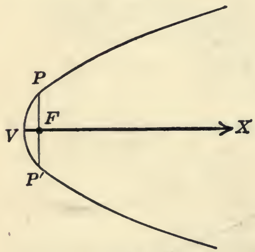
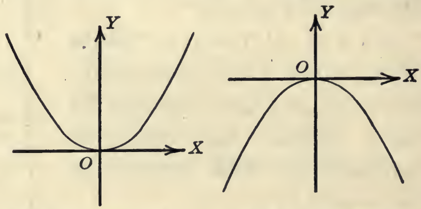


FIG. 67.



$x^2 = 2py$, p positive $x^2 = 2py$, p negative

FIG. 68.

the latus rectum is the absolute value of $2p$. For the abscissa of the focus is $\frac{1}{2}p$, and, when $x = \frac{1}{2}p$, $y = \pm p$.

In Fig. 67, V is the vertex of the parabola, VX is the axis, and $P'P$ is the latus rectum.

A parabola can be readily sketched if the position of the vertex V and focus F and the length of the latus rectum, $P'P$, are known.

85. Parabola with axis on the y-axis.—The equation of a parabola whose axis is on the y-axis and whose vertex is at the origin is obviously obtained by interchanging x and y in the work of article 82. The equation is

[29] $$\mathbf{x^2 = 2py.}$$

The focus is at the point $(0, \frac{1}{2}p)$, and is on the positive or the negative half of the y-axis according as p is positive or negative. If p is positive, the parabola, Fig. 68, is above the x-axis; and, if negative, it is below the x-axis.

It is to be remembered that the origin is at the vertex of every parabola whose equation is of the form **[28]** or **[29]**. These forms are called the **standard forms** of the equation of the parabola.

EXERCISES

1. Plot the following parabolas: $y^2 = 2x$, $y^2 = -2x$, $x^2 = 2y$, and $x^2 = -2y$.

2. Give the coördinates of the foci of the parabolas in exercise 1. Give the equations of their directrices. What are their latera recta?

3. Plot $y^2 = 4x$, using successively $\frac{1}{16}$ in., $\frac{1}{8}$ in., $\frac{1}{4}$ in., $\frac{1}{2}$ in., 1 in., and 2 in. as a unit.

4. Plot $y^2 = \frac{1}{8}x$ using 4 in. as a unit. Plot $y^2 = \frac{1}{2}x$ using 1 in. as a unit. Plot $y^2 = x$ using $\frac{1}{2}$ in. as a unit. Plot $y^2 = 4x$ using $\frac{1}{8}$ in. as a unit. Are all parabolas of the same shape?

5. Write the equation of a parabola whose vertex is at the origin and focus at (1) (3, 0), (2) (0, 6), (3) (−4, 0), (4) (0, −2).

6. Find the equations of the following parabolas, and give the latus rectum of each:

(1) Vertex at origin, axis on x-axis, and passing through the point (2, 4).

(2) Vertex at origin, axis on y-axis, and passing through the point (2, 4).

7. The cables of a suspension bridge hang in the form of a parabola. Find the equation for such a cable in a bridge 1000 ft. between supports if the distance from the lowest point of the cable to the level of the top of the piers is 50 ft.

Suggestion.—Take the origin at the lowest point of the cable. Then the point (500, 50) is on the parabola. Substitute these values in **[29]** and solve for p.

8. Derive equation **[29]** from **[28]** by revolving the coördinate axes through an angle $\varphi = -90°$.

86. Equation of parabola when axes are translated.—Transform the equation $y^2 = 2px$ by translating the axes to a new origin at the point $O'(-h, -k)$, Fig. 69.

By [**12**], $x = x' - h$ and $y = y' - k$. Substituting these values in $y^2 = 2px$ gives

$$(y' - k)^2 = 2p(x' - h).$$

This is the equation of a parabola having its vertex at the point (h, k) when referred to the new coördinate axes, that is, the x', y'-axes. If the primes are dropped, this becomes

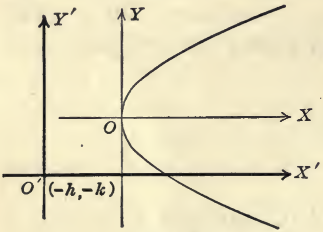


Fig. 69.

[30] $$(\mathbf{y} - \mathbf{k})^2 = \mathbf{2p}(\mathbf{x} - \mathbf{h}),$$

which is a convenient form for writing the equation of a parabola with vertex at point (h, k) and axis parallel to the x-axis. If p is positive, the parabola opens toward the right; and if negative, it opens toward the left.

Similarly, when the axis of the parabola is parallel to the y-axis, the equation is

[30₁] $$(\mathbf{x} - \mathbf{h})^2 = \mathbf{2p}(\mathbf{y} - \mathbf{k}).$$

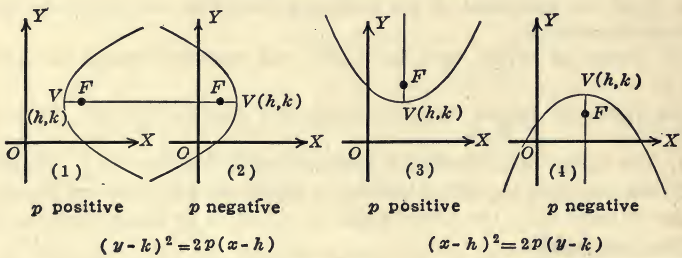


Fig. 70.

The position of these parabolas with reference to the coördinate axes is shown in Fig. 70.

Example 1.—Find the equation of a parabola with vertex at the point $(2, -3)$, axis parallel to the x-axis, and $p = 2$. Plot.

Substituting in [**30**], $(y + 3)^2 = 2 \times 2(x - 2)$.

Simplifying, $y^2 + 6y - 4x + 17 = 0$.

The curve is plotted in Fig. 71.

Example 2.—Find the equation of the parabola whose vertex is at the point $(3, -6)$, axis parallel to the y-axis, and which passes through the point $(-3, -10)$.

Solution.—The equation is of the form [**30_1**], in which $h = 3$, $k = -6$, and p is to be found.

Substituting in [**30_1**],

$$(-3 - 3)^2 = 2p(-10, +6).$$

Solving for p, $p = -4\frac{1}{2}$.

Substituting values of h, k, and p in [**30_1**] gives

$$(x - 3)^2 = 2(-4\tfrac{1}{2})(y + 6).$$

Simplifying, $x^2 - 6x + 9y + 63 = 0$, the required equation.

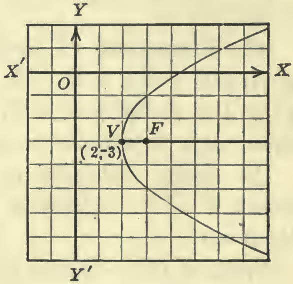


FIG. 71.

EXERCISES

1. Write the equations of the following parabolas:

(1) Vertex at $(3, 4)$, $p = 4$, and axis parallel to the x-axis.
(2) Vertex at $(2, 3)$, $p = -4$, and axis parallel to the x-axis.
(3) Vertex at $(-6, 2)$, $p = 6$, and axis parallel to the y-axis.
(4) Vertex at $(2, -3)$, $p = -3$, and axis parallel to the y-axis.

2. In each part of exercise 1, give the coördinates of the focus, equation of the directrix, and plot the parabola.

3. Write the equations of the parabolas with vertex of each at $(-4, -2)$, latus rectum of each equal to 10, and axes parallel to x-axis.

4. Write the equation of the parabola with vertex at $(3, -2)$, origin on the directrix, and axis parallel to y-axis.

5. Transform $x^2 + 8y = 12$ to new axes parallel to the old, with the new origin at the point $(2, 5)$.

6. Find the equations of the following parabolas, and sketch each curve:

(1) Vertex at $(4, 5)$ and the focus at $(6, 5)$.
(2) Vertex at $(-4, 2)$ and the focus at $(-4, 4)$.
(3) Vertex at $(-4, 2)$ and the focus at $(-6, 2)$.
(4) Vertex at $(3, -4)$ and the focus at $(3, -6)$.

7. Find the equations of each of the following parabolas, and sketch each curve:

(1) Vertex at $(2, 3)$, axis parallel to x-axis, and passing through the point $(5, 6)$.

(2) Vertex at $(3, -2)$, axis parallel to x-axis, and passing through the point $(-1, 3)$.

(3) Vertex at (2, 3), axis parallel to y-axis, and passing through the point $(-1, 1)$.

(4) Vertex at (3, -2), axis parallel to y-axis, and passing through the point $(-1, 3)$.

87. Equations of forms $y^2 + Dx + Ey + F = 0$ and $x^2 + Dx + Ey + F = 0$.—(1) *Every equation of the form $y^2 + Dx + Ey + F = 0$, where $D \neq 0$, represents a parabola whose axis is parallel to the x-axis.*

(2) *Every equation of the form $x^2 + Dx + Ey + F = 0$, where $E \neq 0$, represents a parabola whose axis is parallel to the y-axis.*

Proof of (1).—Given $y^2 + Dx + Ey + F = 0$, where $D \neq 0$.

Completing square in y, $y^2 + Ey + \frac{E^2}{4} = -Dx + \frac{E^2}{4} - F$.

Or
$$\left(y + \frac{E}{2}\right)^2 = -D\left(x - \frac{E^2 - 4F}{4D}\right).$$

This is in the form of [**30**], where $h = \frac{E^2 - 4F}{4D}$, $k = -\frac{E}{2}$, and $p = -\frac{D}{2}$.

Therefore the equation $y^2 + Dx + Ey + F = 0$ where $D \neq 0$ represents a parabola whose axis is parallel to the x-axis.

The proof of (2) is similar to that of (1).

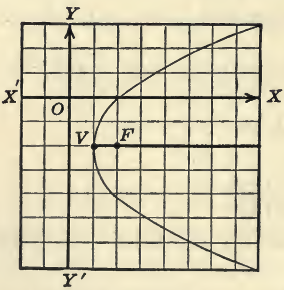


FIG. 72.

Example 1.—Transform the equation $y^2 + 4y - 4x + 8 = 0$ into the form of [**30**], give the coördinates of the vertex and focus, write the equations of the axis and directrix, and sketch the parabola.

Solution.—Completing the square in y, $y^2 + 4y + 4 = 4x - 8 + 4$.

Or $(y + 2)^2 = 4(x - 1)$.

Hence the vertex is at the point V $(1 - 2)$.

Since $2p = 4$, $p = 2$, and the focus is one unit to the right of the vertex, or at the point $(2, -2)$.

The axis is parallel to the x-axis and two units below. Hence its equation is $y = -2$.

The directrix is perpendicular to the x-axis and one unit to the left of the vertex. Hence its equation is $x = 0$.

The parabola is shown in Fig. 72.

Example 2.—Find the equation of the parabola with its axis parallel to the x-axis, which passes through the points (0, 1), (2, 3), (5, 2).

Solution.—The equation is of the form $y^2 + Dx + Ey + F = 0$. Since the parabola passes through the point (0, 1), these coördinates satisfy the equation. Substituting these coördinates gives

$$1 + E + F = 0.$$

Likewise (2, 3) give $9 + 2D + 3E + F = 0$.
And (5, 2) give

$$4 + 5D + 2E + F = 0.$$

Solving these equations for D, E, and F, $D = \frac{1}{4}$, $E = -\frac{17}{4}$, and $F = \frac{13}{4}$. Substituting these values in $y^2 + Dx + Ey + F = 0$, gives

$$y^2 + \tfrac{1}{4}x - \tfrac{17}{4}y + \tfrac{13}{4} = 0.$$

Or $\quad 4y^2 + x - 17y + 13 = 0$, the required equation.

88. The quadratic function $ax^2 + bx + c$.—The locus of the equation $y = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, is a parabola with axis parallel to the y-axis.

To see this, reduce the equation to the standard form [30_1].

Completing the square in x, $y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$.

Or
$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right).$$

This is in the form $(x - h)^2 = 2p(y - k)$, where $h = -\frac{b}{2a}$ and $k = -\frac{b^2 - 4ac}{4a}$, and is a parabola with vertex at the point $\left(-\frac{b}{2a} \cdot -\frac{b^2 - 4ac}{4a}\right)$ and axis on the line $x + \frac{b}{2a} = 0$.

Evidently the parabola opens upward if $a > 0$ and downward if $a < 0$.

89. Equation simplified by translation of coördinate axes.—It is evident that $y^2 + Dx + Ey + F = 0$ and $x^2 + Dx + Ey + F = 0$ can be transformed to the forms of

[28] and [29], respectively, by a suitable translation of the coördinate axes.

In the first equation, the term in y and the constant term can be made to vanish; and, in the second, the term in x and the constant can be made to vanish.

Example.—Translate the coördinate axes so as to transform the equation $y^2 + 6x - 4y + 10 = 0$ to the form of $y^2 = 2px$.

Solution.—Using [12], put $x = x' + h$ and $y = y' + k$, then

$$(y' + k)^2 + 6(x' + h) - 4(y' + k) + 10 = 0.$$

Or $$y'^2 + 6x' + (2k - 4)y' + (k^2 - 4k + 6h + 10) = 0.$$

In order that the y' term and the constant term shall vanish

$$2k - 4 = 0 \text{ and } k^2 - 4k + 6h + 10 = 0.$$

Solving these equations, $h = -1$ and $k = 2$.

Therefore the transformed equation is $y'^2 = -6x'$.

The transformation can also be made by completing the square in y, whence

$$y^2 - 4y + 4 = -6x - 6,$$

or $$(y - 2)^2 = -6(x + 1).$$

Put $y - 2 = y'$ and $x + 1 = x'$, and obtain $y'^2 = -6x'$, as before.

The curve is plotted in Fig. 73.

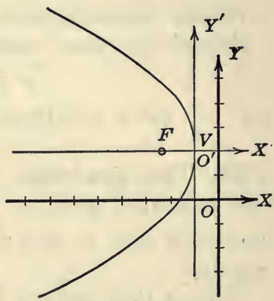


Fig. 73.

EXERCISES

1. Transform the equations of the following parabolas to the form of [30] or [30_1]; and in each case give the coördinates of the vertex and the focus, write the equations of the axis and directrix, and plot.

(1) $y^2 - 4x - 4y + 16 = 0$. (4) $2x^2 - 24x + 3y + 78 = 0$.

(2) $y^2 + 2x + 8y + 6 = 0$. (5) $3y^2 + 15x - 12y + 20 = 0$.

(3) $4x^2 + 12x - 20y + 49 = 0$. (6) $2x^2 - 18x + 15y - 21 = 0$.

2. Find the equation of the parabola with axis parallel to the y-axis, which passes through the points (2, 3), (1, 0), and (0, 2). Find the coördinates of the focus and vertex of this parabola, and its latus rectum.

3. Find the equation of the parabola which has the line $y = 4$ as axis, the line $x = -2$ as directrix, and $p = 6$.

4. Find the equation of the parabola which has its vertex at (2, −3), its axis parallel to the x-axis, and which passes through the point (5, 2).

5. Translate the coördinate axes so as to transform the following parabolas to the form of **[28]** or **[29]**. In each case plot showing both sets of axes.

(1) $y^2 - 4x - 6y + 8 = 0.$ (3) $y^2 + 8x - 4y - 4 = 0.$

(2) $x^2 - 8x + 16y = 0.$ (4) $3x^2 + 5x - 7y + 8 = 0.$

6. For each of the parabolas of exercise 5, find the equation of the directrix with reference to both sets of axes. Give the coördinates of the focus for both sets of axes, and the value of the latus rectum.

7. Plot the equation $y = ax^2 + bx + c$ discussed in article 88 for (1) $b^2 - 4ac > 0$, (2) $b^2 - 4ac = 0$, (3) $b^2 - 4ac < 0$, both when $a > 0$ and when $a < 0$.

90. Equation of a parabola when the coördinate axes are rotated.—Transform the equation $y^2 + Dx + Ey + F = 0$ by rotating the coördinate axes through an angle φ, using the formulas **[13]**.

Putting $x = x' \cos\varphi - y' \sin\varphi$, and $y = x' \sin\varphi + y' \cos\varphi$, in $y^2 + Dx + Ey + F = 0$, gives $(x' \sin\varphi + y' \cos\varphi)^2 + D(x' \cos\varphi - y' \sin\varphi) + E(x' \sin\varphi + y' \cos\varphi) + F = 0.$

Collecting terms,

$$x'^2 \sin^2\varphi + 2 \sin\varphi \cos\varphi \, x'y' + y'^2 \cos^2\varphi$$
$$+ (D \cos\varphi + E \sin\varphi)x' + (E \cos\varphi - D \sin\varphi)y' + F = 0. \quad \text{(I)}$$

A similar form is obtained from $x^2 + Dx + Ey + F = 0.$

If the angle of rotation is some multiple of 90°, then $2 \sin\varphi \cos\varphi = 0$, and the coefficient of $x'y'$ is 0. Hence, in this case, the $x'y'$-term vanishes.

If the coördinate axes are rotated through an angle φ, such that the axis of a parabola is not parallel to either coördinate axis, the equation of a parabola is of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad \text{(II)}$$

the most general form of an equation of the second degree in x and y.

It is readily seen that in equation (I), $B^2 - 4AC = 0$. It will be shown later, **Art. 121,** that the necessary and sufficient condition that any equation of the form of (II) represents a parabola is that $B^2 - 4AC = 0.$

Example 1.—Transform the equation $x^2 + 2x - 3y + 4 = 0$ by rotating the coördinate axes through an angle of 45°. Plot.

Solution.—Substituting $x = x' \cos 45° - y' \sin 45°$

and $$y = x' \sin 45° + y' \cos 45°,$$

$$(x' \cos 45° - y' \sin 45°)^2 + 2(x' \cos 45° - y' \sin 45°) - 3(x' \sin 45° + y' \cos 45°) + 4 = 0.$$

Simplifying, $x'^2 - 2x'y' + y'^2 - \sqrt{2}x' - 5\sqrt{2}y' + 8 = 0.$

Example 2.—By rotating the coördinate axes transform the equation $9x^2 - 24xy + 16y^2 - 116x - 162y + 221 = 0$, to a form which contains no term in xy.

Solution.—Putting $x = x' \cos \varphi - y' \sin \varphi$, and $y = x' \sin \varphi + y' \cos \varphi$,

$$9(x' \cos \varphi - y' \sin \varphi)^2 - 24(x' \cos \varphi - y' \sin \varphi)(x' \sin \varphi + y' \cos \varphi) + 16(x' \sin \varphi + y' \cos \varphi)^2 - 116(x' \cos \varphi - y' \sin \varphi) - 162(x' \sin \varphi + y' \cos \varphi) + 221 = 0.$$

Collecting terms, $(9 \cos^2 \varphi - 24 \sin \varphi \cos \varphi + 16 \sin^2 \varphi)x'^2 + (14 \sin \varphi \cos \varphi + 24 \sin^2 \varphi - 24 \cos^2 \varphi)x'y' + (9 \sin^2 \varphi + 24 \sin \varphi \cos \varphi + 16 \cos^2 \varphi)y'^2 - (162 \sin \varphi + 116 \cos \varphi)x' + (116 \sin \varphi - 162 \cos \varphi)y' + 221 = 0.$

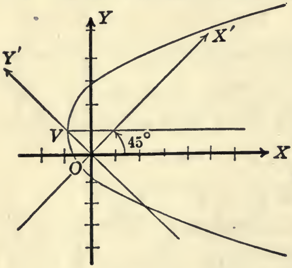


FIG. 74.

Now, in order that the $x'y'$ term shall vanish, its coefficient must be 0. Hence $24 \sin^2 \varphi - 24 \cos^2 \varphi + 14 \sin \varphi \cos \varphi = 0.$

Or $$-24 \cos 2\varphi + 7 \sin 2\varphi = 0.$$

Dividing by $\cos 2\varphi$, $7 \tan 2\varphi = 24$, or $\tan 2\varphi = \frac{24}{7}$.

From this by trigonometry, $\cos 2\varphi = \frac{7}{25}$.

Then $$\sin \varphi = \sqrt{\frac{1 - \cos 2\varphi}{2}} = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}.$$

And $$\cos \varphi = \sqrt{\frac{1 + \cos 2\varphi}{2}} = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}.$$

Substituting these values for $\sin \varphi$ and $\cos \varphi$ in the above equation and simplifying, $25y'^2 - 190x' - 60y' + 221 = 0.$

EXERCISES

1. Transform the equations $y^2 = 2px$ and $x^2 = 2py$ by rotating the coördinate axes through an angle of 90°.

2. Transform the following equations by rotating the coördinate axes through the angle given in each case:

(1) $y^2 = 4x$. $\varphi = 45°$.

(2) $x^2 + 3x - 2y + 6 = 0$. $\varphi = 30°$.

(3) $4x^2 - 4xy + y^2 + 2x - 6y - 10 = 0$. $\varphi = \sin^{-1} \frac{2}{5}\sqrt{5}$.

(4) $9x^2 + 12xy + 4y^2 + 10x - 54y - 68 = 0$. $2\varphi = \tan^{-1} \frac{12}{5}$.

3. Derive the equation of the parabola whose directrix is the line $4x + 3y + 2 = 0$, and whose focus is at the point (2, 3).

4. Simplify the following equations, and plot. First rotate the coördinate axes to free of xy-term, then translate to change to the standard form.

(1) $x^2 - 2xy + y^2 - 6x - 6y + 9 = 0$.
(2) $2x^2 + 8xy + 8y^2 + x + y + 3 = 0$.
(3) $x^2 + 2xy + y^2 - 12x + 2y - 3 = 0$.

91. Equation of parabola in polar coördinates.—Starting with the definition of article 81, the equation of parabola in polar coördinates can be easily derived.

In Fig. 75, let O be the fixed point (focus), and $D'D$ the fixed line (directrix). Choose O as pole and OX, perpendicular to $D'D$, as the polar axis. Let $P(\rho, \theta)$ be any point on the locus. Draw MP and NP perpendicular to OX and $D'D$ respectively.

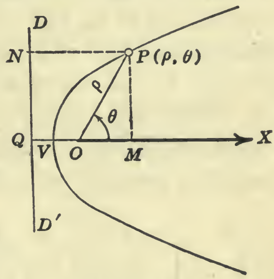


FIG. 75.

By definition, $OP = NP$.

But $OP = \rho$, and $NP = QM = p + \rho \cos \theta$.

Hence $\rho = p + \rho \cos \theta$.

Solving for ρ,

$$[31] \qquad \rho = \frac{p}{1 - \cos \theta}.$$

This is the polar equation of a parabola referred to its focus and axis.

EXERCISES

1. Given the equation $\rho = \dfrac{p}{1 - \cos \theta}$, transform it to rectangular coördinates and by translation of axes derive the equation $y^2 = 2px$.

2. By taking the focus at the left of the directrix, derive the equation of the parabola in the form $\rho = \dfrac{p}{1 + \cos \theta}$.

3. Change the following equation into polar coördinates with the pole at the origin, and the polar axis on the positive part of the x-axis: $y^2 = 2px + p^2$.

4. Show that if the vertex of the parabola is taken as pole and the axis of the parabola as polar axis, the equation of the parabola in polar coördinates is $\rho = \frac{2p \cos \theta}{\sin^2 \theta}$.

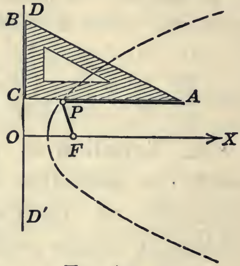


FIG. 76.

92. Construction of a parabola.—*First method.*—The directrix $D'D$ and the focus F are supposed known.

Place a right triangle, Fig. 76, with one side CB on the directrix as shown. Fasten one end of a string whose length is CA, at the focus F and the other at A. With a pencil at P, keep the string taut and move the triangle along the directrix. Then $FP = CP$, and the point P will generate a parabola. Why?

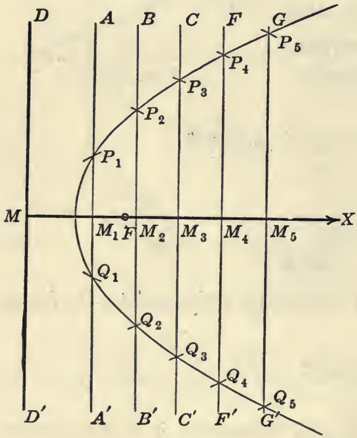


FIG. 77.

Second method.—As before, the directrix $D'D$ and the focus F are supposed known.

In Fig. 77, draw MX through F and perpendicular to $D'D$. Draw any number of lines $A'A$, $B'B$, etc., parallel to the directrix, and intersecting MX in M_1, M_2, etc. With F as center and a radius equal to MM_1, strike arcs intersecting $A'A$ in P_1 and Q_1. In like manner, with MM_2 as a radius, strike arcs intersecting $B'B$. Continue in like manner for the other lines drawn. Then the points, thus determined, lie on the parabola. Why? In this way the parabola can be located as accurately as desired.

EXERCISES

1. Construct a parabola by the second method, in which $p = 1$ in. In which $p = \frac{3}{4}$ in.

2. Construct a circle of radius 8 in., and a parabola with its vertex at the center of the circle, and its focus on the positive x-axis at the point midway between the center and circumference. Write the equation of each in the standard forms, and compute the coördinates of the points of intersection of the curves.

3. Explain how the construction shown in Fig. 78, determines a parabola.

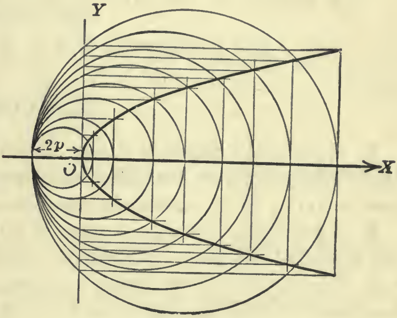


FIG. 78.

APPLICATIONS

93. Parabolic arch.—The cable of a suspension bridge hangs in the form of an inverted parabolic arch. Arches for bridges, when the weight is uniformly distributed, are properly constructed in the form of a parabola. In metal-arch bridges the loading is practically uniform on the horizontal, and so such bridge structures are in the form of parabolic arches. The arches of concrete bridges are seldom if ever built in the form of a parabola, for, in such structures, the loading cannot be uniformly distributed on the horizontal.

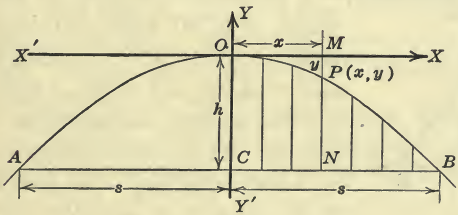


FIG. 79.

In the parabolic arch, Fig. 79, $AB = 2s$ is the span, and $CO = h$ is the height. If the origin is taken at the vertex of the parabola, and the axis along the y-axis, the equation is of the form $x^2 = 2py$.

To find the value of p, we know that the point $B(s, -h)$ is on the parabola. Substituting these coördinates in $x^2 = 2py$, gives $s^2 = -2ph$, and $p = -\frac{s^2}{2h}$. Hence the equation of the parabola is $x^2 = -\frac{s^2}{h}y$, and from this $y = -\frac{hx^2}{s^2}$.

The height of the arch at any distance x from the center is

$$NP = NM + MP = h + y = h - \frac{hx^2}{s^2}.$$

EXERCISES

1. A parabolic arch has a span of 120 ft. and a height of 25 ft. Derive the equation of the parabola, and compute the heights of the arch at points 10 ft., 20 ft., and 40 ft. from the center.

2. A parabolic arch has a span of 40 ft. and a height of 15 ft. Find the height of the arch at intervals of 5 ft. from the center.

3. The distance between the supports on the river span of the Brooklyn suspension bridge is about 1600 ft., and the vertex of the curve of the cables is 140 ft. below the suspension points. Find the equation of the curve if the lowest point is taken as origin.

4. The towers supporting a suspension bridge are 320 ft. apart and rise 80 ft. above the roadbed. The lowest point of the parabola formed by the cables is 20 ft. above the roadbed. Find the equation of the curve of the cables using as origin the point in the roadbed below the vertex of the parabola.

94. The path of a projectile.—A projectile starting at the origin, Fig. 80, with an initial velocity of v ft. per second, and making an angle α with the horizontal, would after t seconds have the position $x = v \cos \alpha \cdot t$ and $y = v \sin \alpha \cdot t$, if the action of gravity and the resistance of the air were not considered. If the action of gravity is considered, y is decreased by $\frac{1}{2}gt^2$ ft. in t seconds. Then the coördinates of the projectile at time t are

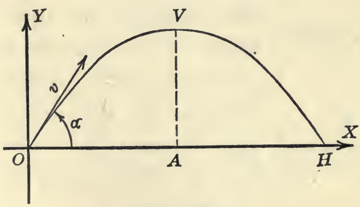


FIG. 80.

$$x = v \cos \alpha \cdot t, \text{ and } y = v \sin \alpha \cdot t - \tfrac{1}{2}gt^2. \qquad \text{(I)}$$

The equation of the path of the projectile in rectangular coördinates is found by eliminating t between these equations and is

$$y = \tan \alpha \cdot x - \frac{g}{2v^2 \cos^2 \alpha} x^2. \qquad \text{(II)}$$

EXERCISES

1. Eliminate t between the equations (I) and derive equation (II).

2. Show that equation (II) is a parabola with its vertex at the point

$$\left(\frac{v^2 \sin 2\alpha}{2g}, \frac{v^2 \sin^2 \alpha}{2g}\right), \text{ and } p = -\frac{v^2 \cos^2 \alpha}{g}.$$

3. Find the x-intercept of (II), and thus find the range on the horizontal to be $\frac{v^2 \sin 2\alpha}{g}$.

4. Find the height of the projectile when at a horizontal distance equal to one-fourth the range.

5. Find the horizontal range when $v = 2000$ ft. per second and (1) $\alpha = 45°$, (2) $\alpha = 30°$, (3) $\alpha = 60°$. Use $g = 32$.

6. Show that a projectile with a given velocity and at an angle of 60°, rises three times as high as it would if the angle were 30°.

7. What must be the initial velocity v of a projectile, if with an angle of elevation of 20°, it is to strike an object 80 ft. above the horizontal plane of the starting point, and at a horizontal distance of 1000 yd.?

GENERAL EXERCISES

1. The formula for the height of a bullet shot vertically upward with a velocity of 2000 ft. per second is $s = 2000t - 16t^2$. Find the coördinates of the vertex, and plot the curve from which the height s at any time t may be read.

2. When one variable varies directly as the square of another, the equation connecting the two variables will represent a parabola. The length of a pendulum varies as the square of the time of a beat. This gives the formula $t^2 = \frac{\pi^2}{g^2} l$, where t is time in seconds, g is 32, and l is length in feet. Plot a curve from which can be read the time of a beat for lengths up to 20 ft.

3. In a parabolic reflector, such as used for an automobile headlight, the source of light is placed at the focus of the parabola that is a section of the reflector. Find the position of the source of light in a reflector 10 in. in diameter and 5 in. deep.

4. Find the coördinates of the points of intersection of the parabola $x^2 = 8y$ and the line $3x - 2y - 8 = 0$.

5. Find the equation of the straight line passing through the focus of the parabola $y^2 = 8x$ and making an angle of 45° with the axis of the parabola.

6. What value must be given to k if the line $3x + 2y + k = 0$ is to be tangent to the parabola $x^2 = -6y$? Plot.

Suggestion.—Eliminate y between the two equations. Since a tangent meets the curve in two coincident points, the two values of x in the resulting equation must be equal. Hence put the discriminant of this quadratic equation equal to zero and solve for values of k.

7. Find the points of intersection of the following curves: $x - 3y = 0$ and $y^2 - 3x - 6y + 14 = 0$.

8. For what values of m is the straight line $y = mx + 2$ tangent to the parabola $x^2 - 6x + 8y + 41 = 0$?

9. One end of a chord through the focus of a parabola is at the point (10, 10). Find the coördinates of the other end if the parabola has its vertex at the origin and its axis on the positive part of the x-axis.

10. Transform the following equations in polar coördinates into rectangular coördinates and simplify:

(1) $\rho = \dfrac{2}{1 + \cos\theta}$. (2) $\rho = \dfrac{6}{5 - 5\cos\theta}$. (3) $\rho = \dfrac{12}{1 + \cos\theta}$.

11. Plot the following curves given in polar coördinates and find the coördinates of their points of intersection:

(1) $\rho\cos\theta = 4,\quad \rho = \dfrac{8}{1 - \cos\theta}$. (2) $\rho = 4,\quad \rho = \dfrac{4}{1 + \cos\theta}$.

12. Show that the equation $\rho = 8\sec^2\frac{1}{2}\theta$ is that of a parabola, and sketch the curve.

13. Find the equation of the circle circumscribing the segment of the parabola $y^2 = 2px$, cut off by the latus rectum.

14. An equilateral triangle having one vertex at the origin is inscribed in the parabola $y^2 = 2px$. Find the length of a side of the triangle.

15. Show that $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is the equation of a parabola. Sketch the curve.

16. Find the equation of the parabola with $x + y = 0$ as directrix, and focus at $(\frac{1}{2}a, \frac{1}{2}a)$. Express in the form given in the previous exercise.

CHAPTER VII

THE ELLIPSE AND CERTAIN FORMS OF THE SECOND DEGREE EQUATION

95. The equation of the ellipse.—By the definition of article 81, the ellipse is the locus of a point whose distance from a fixed point, the focus, is to its distance from a fixed straight line, the directrix, in a constant ratio e, less than 1.

In Fig. 81, let F be the focus and $D'D$ the directrix. Choose as x-axis the line $X'X$ through F and perpendicular to $D'D$ at R.

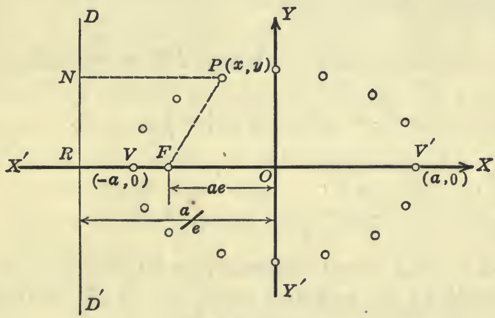


FIG. 81.

Since $e<1$, there are two points V and V' on $X'X$ such that $\frac{VF}{RV} = e$ and $\frac{FV'}{RV'} = e$. Hence the points V and V' are on the locus.

Choose O, the point midway between V and V', as origin, and $Y'Y$ through O, parallel to $D'D$, as y-axis.

Let the length of $VV' = 2a$.

Then $VO = OV' = a$.

It is necessary first to find the equation of the directrix and the coördinates of the focus.

From the definition of the ellipse,

$$VF = e \cdot RV, \text{ or } a - FO = e(RO - a), \qquad (1)$$

and $$FV' = e \cdot RV', \text{ or } a + FO = e(RO + a). \qquad (2)$$

Adding equations (1) and (2),

$$2a = 2eRO, \text{ or } RO = \frac{a}{e}.$$

Then the equation of the directrix is $x = -\frac{a}{e}$.

Subtracting equation (1) from equation (2),

$$2FO = 2ae, \text{ or } FO = ae.$$

Then the coördinates of the focus F are $(-ae, 0)$.

Now to derive the equation, let $P(x, y)$ be any point on the locus, and draw FP, and NP perpendicular to $D'D$.

By definition, $FP = e \cdot NP$.

But $FP = \sqrt{(x + ae)^2 + y^2}$, and $NP = \frac{a}{e} + x$.

Then $\sqrt{(x + ae)^2 + y^2} = e\left(\frac{a}{e} + x\right)$.

Squaring and arranging, this becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

Since $e < 1$, $a^2(1 - e^2)$ is positive and less than a^2. Let it be represented by b^2 and the equation of the ellipse is

[32] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is a standard form of the equation of the ellipse, and is the form in which the equation of the ellipse is usually written. Its simple form is due to the choice of the coördinate axes. A different choice of axes would give a less simple form of the equation, but the locus itself would be unaltered.

Since $b^2 = a^2(1 - e^2)$, $e = \frac{\sqrt{a^2 - b^2}}{a}$.

Equation [32] is the required equation of the ellipse. For it has been proved true for every point on the ellipse, and it can be readily proved that it is not true for any point that is not on the locus. The proof of this is left as an exercise.

96. Shape of the Ellipse.—The shape of the ellipse and its position relative to the coördinate axes can be readily determined from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solving for x, $x = \pm\frac{a}{b}\sqrt{b^2 - y^2}$.

Solving for y, $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$.

(1) For all values of y such that $b^2 - y^2 > 0$, x has two real values, numerically equal but opposite in sign. When $y^2 = b^2$, $x = 0$. For all values of x such that $a^2 - x^2 > 0$, y has two real values, numerically equal but opposite in sign. When $x^2 = a^2$, $y = 0$. Hence the curve is symmetrical with respect to both coördinate axes and the origin, and its intercepts are a and $-a$ on the x-axis, and b and $-b$ on the y-axis.

(2) For all values of y such that $b^2 - y^2 < 0$, x is imaginary; and for all values of x such that $a^2 - x^2 < 0$, y is imaginary. Hence no part of the curve lies outside of the rectangle bounded by the four lines $x = \pm a$ and $y = \pm b$.

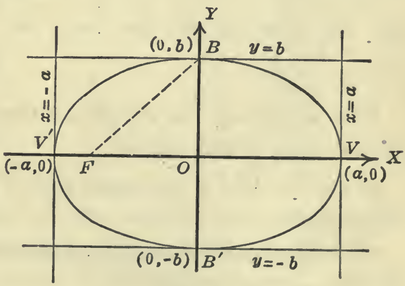


Fig. 82.

(3) As x increases from $-a$ to 0, the positive value of y increases from 0 to b, and the negative value of y decreases from 0 to $-b$. As x increases from 0 to a, the positive value of y decreases from b to 0, and the negative value of y increases from $-b$ to 0.

The ellipse has the shape shown in Fig. 82.

The formula $b^2 = a^2(1 - e^2)$ can now be readily interpreted geometrically. For in the right triangle FOB, Fig. 82, $FO = ae$ and $OB = b$.

Then $$\overline{FB}^2 = (ae)^2 + b^2.$$

But from $b^2 = a^2(1 - e^2)$, $a^2 = (ae)^2 + b^2$.

Hence $$a^2 = \overline{FB}^2, \text{ or } a = FB.$$

97. Definitions.—The center of symmetry of the ellipse is called the **center** of the ellipse.

The chord through the focus and center of an ellipse is called the **major axis.** Its length is $2a$. One-half of the major axis is called the **semimajor axis.**

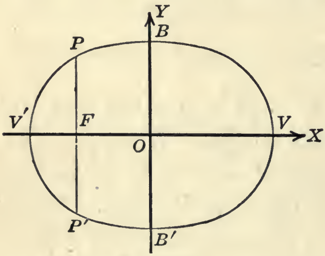


FIG. 83.

The chord through the center of the ellipse and perpendicular to the major axis is called the **minor axis.** Its length is $2b$. One-half of the minor axis is called the **semiminor axis.**

The chord of the ellipse through the focus and perpendicular to the major axis is called the **latus rectum.** Its length is $\frac{2b^2}{a}$, for the abscissa of the focus is $-ae$, and when $x = -ae$, $y = \pm \frac{b^2}{a}$.

The points on the ellipse at the ends of the major axis are the **vertices** of the ellipse.

In Fig. 83, $V'V$ is the major axis, $B'B$ the minor axis, and $P'P$ the latus rectum.

An ellipse can be readily sketched if the position and lengths of the axes are known.

98. Second focus and second directrix. THEOREM.—*An ellipse has two foci and two directrices.*

In Fig. 84, on OV' take $OF' = FO$ and $OR' = RO$. Draw $E'E$ parallel to $D'D$. Then F' is also a focus and $E'E$ the corresponding directrix of the ellipse.

Proof.—Let P be any point of the ellipse. Through P draw PN parallel to the x-axis and intersecting $D'D$ in N. Because of the symmetry of the ellipse, PN intersects the ellipse at a second point P' and the line $E'E$ at N'. Draw PF and $P'F'$.

From the symmetry of the figure, $FP = F'P'$, and $NP = P'N'$.

But $\frac{FP}{NP} = e$. $\therefore \frac{F'P'}{P'N'} = e$.

Then the ellipse is also the locus of a point P' whose distance from F' divided by its distance from $E'E$ is e.

Therefore F' is a focus and $E'E$ is the corresponding directrix of the ellipse, and the ellipse has two foci and two directrices.

The coördinates of the foci are $(\pm ae, 0)$, and the equations of the directrices are $x = \pm\frac{a}{e}$.

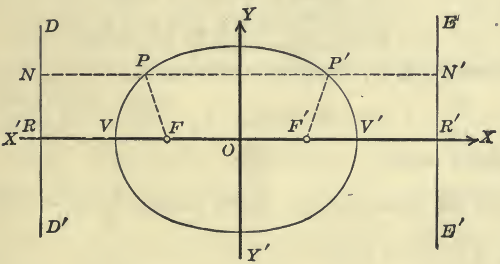


FIG. 84.

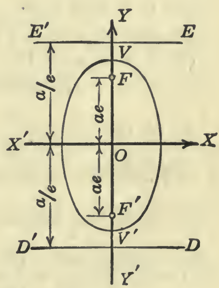


FIG. 85.

99. Ellipse with major axis on the y-axis.—The equation of an ellipse whose major axis is on the y-axis, and whose center is at the origin is obviously obtained by interchanging x and y in the work of article 95. The equation then is

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1. \qquad [33]$$

Here the major axis is $2a$ as before; the minor axis is $2b$; the coördinates of the vertices are $(0, \pm a)$; the coördinates of the foci are $(0, \pm ae)$; and the equations of the directrices are $y = \pm\frac{a}{e}\cdot$

EXERCISES

1. In each of the following ellipses find the semimajor axis, the semiminor axis, the eccentricity, the coördinates of the foci, and the equations of the directrices. Sketch each ellipse.

(1) $\frac{x^2}{25} + \frac{y^2}{16} = 1.$ (4) $\frac{x^2}{16} + \frac{y^2}{9} = 1.$

(2) $\frac{x^2}{36} + \frac{y^2}{100} = 1.$ (5) $\frac{x^2}{8} + \frac{y^2}{5} = 1.$

(3) $4x^2 + 9y^2 = 36.$ (6) $6x^2 + 9y^2 = 54.$

2. Find the distance from the foci to the ends of the minor axis in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

3. Write the equation of an ellipse with center at the origin, and major axis on the x-axis, having given:

(1) $a = 6, b = 4.$ (4) Focus at $(5, 0)$, $e = \frac{2}{3}.$

(2) $a = 4, e = \frac{1}{2}\sqrt{3}.$ (5) Directrix is $x = 7$, $e = \frac{3}{4}.$

(3) $b = 3, e = \frac{2}{3}.$ (6) Latus rectum $= 4$, $a = 8.$

(7) Focus at $(\sqrt{3}, 0)$, directrix is $x = 3\sqrt{3}.$

4. In the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, find the values of y when $x = 2$, when $x = 4$, when $x = 5$, when $x = 6$.

5. Find the length of the latus rectum in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1.$ In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

6. Find the equation of the ellipse with center at the origin, axes on the coördinate axes, and passing through the points $(1, \frac{2}{3}\sqrt{3})$ and $(\frac{1}{2}\sqrt{6}, 1)$.

7. Derive equation **[33]** from **[32]** by rotating the coördinate axes through an angle $\varphi = 90°$.

8. Find the semi-axes, eccentricity, and the latus rectum of each of the following ellipses:

(1) $6y^2 = 30 - 5x^2.$

(2) $2x^2 + y^2 = 2m, m > 0.$

(3) $x^2 + qy^2 = s, q > 1$, and $s > 0.$

(4) $px^2 + qy^2 = pq, p > 0, q > 0$, and $q > p.$

9. Find the distances from the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ to a point on the ellipse, whose abscissa is 2.

10. The minor axis of an ellipse is 24, and the foci and origin divide the major axis into four equal parts. Find the equation of the ellipse.

11. Assume the equation of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and show that the sum of the distances of any point on it from its foci is $2a$.

12. Regard the circle as an ellipse with $a = b$, and find its foci, directrices, and eccentricity.

13. Find the coördinates of the points of intersection of the ellipse $2x^2 + 3y^2 = 14$ and the parabola $y^2 = 4x$.

14. Find the locus of the vertex of a triangle if the base is $2a$, and the product of the tangents of the angles at the base is $\frac{b^2}{c^2}$.

Suggestion.—Take the x-axis on the base and the origin at the center.

15. Find the locus of the vertex of a triangle if its base is $2b$ and the sum of the other sides is $2a$. Take the x-axis on the base and the origin at the midpoint.

16. Discuss the equations $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$. The first of these is the equation of a **point ellipse** and the second is that of an **imaginary ellipse.**

100. Equation of ellipse when axes are translated.— Transform the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by translating the coördinate axes to a new origin at a point $O'(-h, -k)$, using [**12**], and we have

$$\frac{(x' - h)^2}{a^2} + \frac{(y' - k)^2}{b^2} = 1.$$

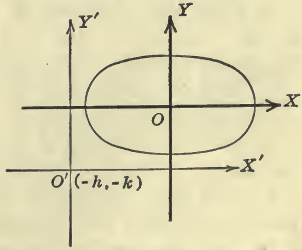


FIG. 86.

This is the equation of an ellipse having its center at the point (h, k) referred to the new coördinate axes, and having its axes parallel respectively to the x'-axis and the y'-axis, as shown in Fig. 86.

If the primes are dropped, this equation becomes

[34] $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

which is a second standard form of the equation of the ellipse, and is a convenient form for writing the equation of an ellipse with center at the point (h, k) and major axis parallel to the x-axis.

Similarly, the equation of an ellipse with center at (h, k) and major axis parallel to the y-axis is of the form

[34_1] $$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1.$$

Example.—Find the equation of an ellipse with semimajor axis 5, semiminor axis 4, center at point $(3, -2)$, and major axis parallel to the x-axis.

Substituting in [34], $\frac{(x-3)^2}{5^2} + \frac{(y+2)^2}{4^2} = 1.$

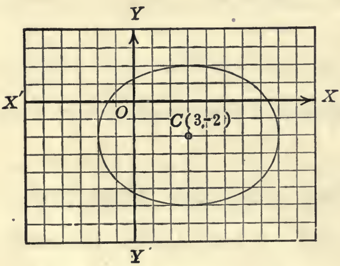


FIG. 87.

Simplifying,

$$16x^2 + 25y^2 - 96x + 100y - 156 = 0.$$

The ellipse is shown in Fig. 87.

EXERCISES

1. Write the equations of the following ellipses, and plot:

(1) Center at $(3, 4)$, $a = 5$, $b = 3$, and major axis parallel to x-axis.

(2) Center at $(-3, -7)$, $a = 6$, $b = \frac{1}{2}\sqrt{3}$, and major axis parallel to y-axis.

2. Find the equations of the two ellipses, each having its center at $(5, -4)$, $a = 6$, and $b = 4$; one having its major axis parallel to the x-axis and the other having its major axis parallel to the y-axis.

3. Find the coördinates of the foci and the equations of the directrices of each ellipse of exercise 1.

4. Find the equation of the ellipse with center at $(10, 2)$, one directrix the line $x = 2$, and eccentricity $\frac{3}{4}$.

5. Find the equation of the ellipse with center at $(3, 4)$, major axis parallel to x-axis, and passing through the points $(-2, 4)$ and $(3, 0)$.

6. Find the equation of the ellipse having its center at $(4, 2)$, major axis parallel to the y-axis, semimajor axis 6, and passing through the point $(8, 4)$.

101. Equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$. *Every equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A and C have like signs but different values, represents an ellipse with axes parallel to the coördinate axes.*

Proof.—Given $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

Completing the squares in x and in y,

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}.$$

Dividing by the second member of this equation,

$$\frac{\left(x + \frac{D}{2A}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4A^2C}} + \frac{\left(y + \frac{E}{2C}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4AC^2}} = 1.$$

This is in the form of [**34**] if $A < C$ where

$$h = -\frac{D}{2A},\ k = -\frac{E}{2C},\ a^2 = \frac{CD^2 + AE^2 - 4ACF}{4A^2C},\ \text{and}$$

$$b^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC^2},$$

and therefore represents an ellipse with axes parallel to the coördinate axes if A and C have like signs so that $4A^2C$ and $4AC^2$ have like signs. It is of the form of [34_1] if $A < C$.

From the preceding, it follows that the equation of an ellipse in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ can be transformed into one of the forms [**32**] or [**33**] by a suitable translation of the coördinate axes, the new origin being at the point $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

Example 1.—Transform to the second standard form, the equation of the ellipse $24x^2 + 49y^2 - 96x + 294y - 639 = 0$, find the coördinates of the center, foci, and vertices, the length of the semimajor and semiminor axes, and the equations of its directrices. Plot.

Solution.—Completing the squares in x and in y,

$$24(x^2 - 4x + 4) + 49(y^2 + 6y + 9) = 639 + 96 + 441,$$

or

$$24(x - 2)^2 + 49(y + 3)^2 = 1176.$$

Dividing by 1176 and putting in the form of [**34**],

$$\frac{(x - 2)^2}{49} + \frac{(y + 3)^2}{24} = 1.$$

This is an ellipse (Fig. 88) with center at the point $C(2, -3)$ and axes parallel to the coördinate axes. The semi axes are $a = 7$, and $b = 2\sqrt{6}$.

The eccentricity $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{5}{7}$.

The distance from the center to the foci is $ae = 5$, and the foci are $F(7, -3)$ and $F'(-3, -3)$.

The vertices are $V(9, -3)$ and $V'(-5, -3)$.

The distance from the center to the directrices is $\dfrac{a}{e} = \frac{49}{5}$, and the equations of the directrices are $x = \frac{59}{5}$ and $x = -\frac{39}{5}$.

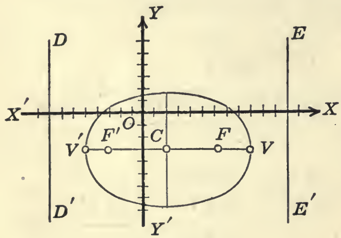


FIG. 88.

The ellipse is as shown in the figure.

Example 2.—Find the equation of the ellipse whose axes are parallel to the coördinate axes and which passes through the points $(-2, 7)$, $(2, 4)$, $(-2, 1)$, and $(-6, 4)$.

Solution.—The required equation is of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

If this is divided by A the equation is of the form

$$x^2 + C'y^2 + D'x + E'y + F' = 0,$$

and therefore contains only four arbitrary constants, which can be found from four equations.

Dropping the primes and substituting the coördinates of the four given points,

$$\begin{aligned} 4 + 49C - 2D + 7E + F &= 0, \\ 4 + 16C + 2D + 4E + F &= 0, \\ 4 + C - 2D + E + F &= 0, \\ 36 + 16C - 6D + 4E + F &= 0. \end{aligned}$$

Solving, $C = \frac{16}{9}$, $D = 4$, $E = -\frac{128}{9}$, $F = \frac{148}{9}$.

The required equation is

$$x^2 + \tfrac{16}{9}y^2 + 4x - \tfrac{128}{9}y + \tfrac{148}{9} = 0,$$

or

$$9x^2 + 16y^2 + 36x - 128y + 148 = 0.$$

Example 3.—Translate the coördinate axes so that the equation of the ellipse $4x^2 + 9y^2 - 24x - 36y + 36 = 0$ is in the form [**32**].

Solution.—Completing the squares in x and in y,

$$4(x^2 - 6x + 9) + 9(y^2 - 4y + 4) = -36 + 36 + 36.$$

Whence $\dfrac{(x-3)^2}{9} + \dfrac{(y-2)^2}{4} = 1.$

Putting $x - 3 = x'$, or $x = x' + 3$ and $y - 2 = y'$, or $y = y' + 2$,

$$\frac{x'^2}{9} + \frac{y'^2}{4} = 1.$$

This is of the form [**32**], and is an ellipse referred to coördinate axes that are parallel to the old coördinate axes, and with the new origin at the point (3, 2).

The ellipse is as shown in Fig. 89.

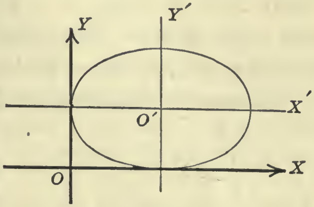


FIG. 89.

The transformation could evidently be made by substituting $x = x' + h$ and $y = y' + k$, and proceeding as in the example of article 89.

EXERCISES

1. Express the equations of the following ellipses in the form [**34**] or [**34**$_1$], find the coördinates of the centers, foci, and vertices, the lengths of the semimajor and semiminor axes, and the equations of the directrices. Plot each.

(1) $7x^2 + 16y^2 + 14x - 64y - 41 = 0$.
(2) $8x^2 + 4y^2 - 64x - 8y + 68 = 0$.
(3) $4x^2 + 9y^2 - 8x + 18y + 12 = 0$.
(4) $8x^2 + 9y^2 + 16x - 54y - 1 = 0$.

2. Transform $6x^2 + 7y^2 - 36x + 14y + 53 = 0$ to new axes parallel respectively to the old axes, with the new origin at $(3, -1)$.

3. Transform each of the ellipses of exercise 1 to the form [**32**] or [**33**], find the coördinates of the foci, and the equations of the directrices referred to the new coördinate axes.

4. Find the equation of the ellipse with major axis parallel to the x-axis, and center at the point $(-3, 4)$, eccentricity $\frac{4}{5}$, and passing through the point (6, 9).

5. Find the equation of the ellipse with one focus at the point (6, 2), corresponding directrix the line $x = 12$, and eccentricity $\frac{1}{2}$.

6. Transform the following equation to one in which there are no x and y terms, and plot: $9x^2 + 12y^2 - 18x - 72y + 9 = 0$.

7. Find the equation of the ellipse with eccentricity $\frac{1}{2}$, a focus at the point (2, 0), and the corresponding directrix the line $x + 2 = 0$.

102. Equation of ellipse when axes are rotated.—In article 90 it was seen that when the coördinate axes were rotated through an angle φ, a term in xy appeared in the equation of the parabola. Likewise if the equation of an ellipse with axes parallel to the coördinate axes, $Ax^2 + Cy^2 + Dx + Ey + F = 0$, is transformed by using formulas **[13]**, the equation takes the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. This is the most general form of an equation of the second degree in x and y, where $B^2 - 4AC < 0$. (See **Art. 122.**)

Conversely, starting with an equation containing an xy-term, rotation through a properly chosen angle will cause the xy-term to disappear by having its coefficient zero.

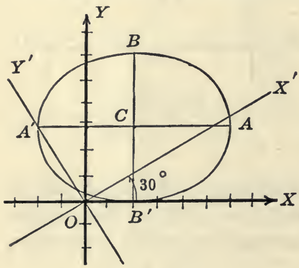


FIG. 90.

Example 1.—Transform the equation $9x^2 + 16y^2 - 36x - 96y + 36 = 0$, by rotating the coördinate axes through an angle of 30°. Sketch the ellipse.

Solution.—Using formulas **[13]**,

$$x = x' \cos 30° - y' \sin 30° = \tfrac{1}{2}\sqrt{3}x' - \tfrac{1}{2}y' = \tfrac{1}{2}(\sqrt{3}x' - y'),$$

and $y = x' \sin 30° + y' \cos 30° = \tfrac{1}{2}x' + \tfrac{1}{2}\sqrt{3}y' = \tfrac{1}{2}(x' + \sqrt{3}y')$.

Substituting these values in the given equation and simplifying,

$$43x'^2 + 14\sqrt{3}x'y' + 57y'^2 - 24(3\sqrt{3} + 8)x' - 24(8\sqrt{3} - 3)y' + 144 = 0.$$

The ellipse and the two sets of coördinate axes are sketched in Fig. 90

EXERCISES

1. Transform the following equations by rotating the coördinate axes through the angle given in each case:

(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1. \quad \varphi = 45°.$

(2) $16x^2 + 9y^2 = 144. \quad \varphi = 60°.$

(3) $36x^2 + 4y^2 = 144. \quad \varphi = 90°.$

(4) $2x^2 + 3y^2 - 4x + 3y - 10 = 0. \quad \varphi = 30°.$

(5) $x^2 + xy + y^2 + 2x + y + 2 = 0. \quad \varphi = 45°.$

(6) $6x^2 + 4xy + 6y^2 + 5x - 8y = 0. \quad \varphi = \tan^{-1} \tfrac{1}{2}.$

2. Transform the following equation to the standard form by rotating the axes: $29x^2 + 16xy + 41y^2 - 45 = 0$. Sketch the ellipse with both sets of axes.

3. Simplify the following equation by first translating the axes to remove the x-term and the y-term, then by rotating through an angle that will remove the xy-term. Sketch the curve and the three sets of coördinate axes: $x^2 + xy + y^2 + 2x + 3y - 3 = 0$.

103. Equation of ellipse in polar coördinates.—In a manner similar to that of article 91, the equation of the ellipse in polar coördinates may be derived.

EXERCISES

1. Derive the equation of an ellipse with the pole at the focus to the right of its corresponding directrix, and the polar axis perpendicular to the directrix. Also derive the equation when the focus is taken at the left of its corresponding directrix. Let p equal the distance from the focus to the directrix.

2. Transform the results of exercise 1 to rectangular coördinates, and change to the standard form by translation of axes.

3. Derive the polar equation of an ellipse, the pole being at a focus, by starting with the equation $\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$, and then putting $x = \rho \cos \theta$, $y = \rho \sin \theta$, $c = ae$, and $b^2 = (1 - e^2)a^2$; finally solving the quadratic equation for ρ.

4. Derive the polar equation of an ellipse, the pole being at the center and the polar axis along the major axis.

5. Show that pe in exercise 1 is one-half the latus rectum.

104. Construction of an ellipse.—*First method.*—The length of the major axis $2a$ and the foci F and F' are supposed to be known.

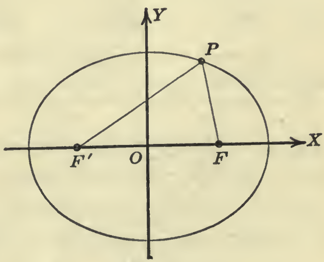


FIG. 91.

On a drawing board fasten the ends of a string of length $2a$ at F and F', Fig. 91. Place a pencil point, P, in the string and move it about keeping the string taut. Then the point P will generate an ellipse.

This construction depends upon the following:

THEOREM.—*The sum of the distances from any point on an ellipse to its foci is constant and equal to the major axis.*

This may be proved as follows: In Fig. 92, from the definition of an ellipse,

$$PF' = e \cdot N'P = e\left(\frac{a}{e} + x\right) = a + ex,$$

and $$PF = e \cdot PN = e\left(\frac{a}{e} - x\right) = a - ex.$$

Adding, $PF' + PF = 2a =$ major axis.

Second method.—The major axis, $2a$, and the minor axis, $2b$, are supposed to be known, as well as the position of the center and direction of axes.

With the center O of the ellipse as a center describe two circles of radii a and b respectively, Fig. 93. Draw any radius intersecting the inner circle in R and the outer circle in Q. Through R draw a line parallel to the major axis, and through Q a line parallel to the minor axis. Then the point P where these lines intersect is a point on the ellipse. In this manner any number of points on the ellipse can be determined.

That the point P is on the ellipse with its major axis on the x-axis can be proved as follows:

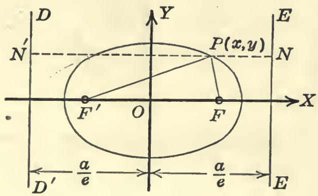


FIG. 92.

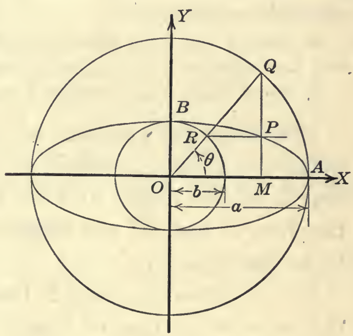


FIG. 93.

Equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

If θ is the angle XOQ, the coördinates of P are

$$x = OM = a \cos \theta,$$

and $$y = MP = b \sin \theta,$$

Substituting in the equation of the ellipse,

$$\frac{a^2 \cos^2 \theta}{a^2} + \frac{b^2 \sin^2 \theta}{b^2} = \cos^2 \theta + \sin^2 \theta = 1.$$

Hence the equation is satisfied and the point P is on the ellipse.

EXERCISES

1. By the second method construct an ellipse having semiminor and semimajor axes 1 in. and $1\frac{1}{2}$ in . respectively.

2. Prove that the projection of a circle upon a plane making an acute angle with the plane of the circle is an ellipse.

APPLICATIONS

105. Uses of the ellipse.—The ellipse is involved in many practical considerations, as well as being frequently used in mathematics and its applications.

It was believed by the ancient Greeks that the sun was the center of the universe in which we live. Kepler (1571–1630) stated that the orbits of the planets are ellipses. Newton (1642–1727) showed that the law of gravitation determines the orbits to be ellipses.

In architecture, because of the beauty of its form, the elliptic arch is frequently used. Some noted structures were built in the form of an ellipse. The Colosseum at Rome was of this form.

In bridge structures, many of the most noted stone-arch bridges of the world are elliptical.

In machinery, elliptical gears are often used where changeable rates of motion are desired, as in shapers, planers, and slotters where the cutting speed is less than the return motion.

In the study of electricity and mechanics, the ellipse is frequently used.

EXERCISES

1. The Colosseum at Rome is in the form of an ellipse 615 ft. long and 510 ft. wide. Find the equation of the ellipse and the position of the foci.

2. A stone-arch of a bridge has a span of 200 ft. and a height of 42 ft. The arch is in the form of a semi-ellipse. Find the equation of the ellipse and the position of the foci.

3. In exercise 2, find the heights of points 50 ft. and 25 ft. from one end of the arch.

4. In considering equipotential surfaces in electricity, the equation $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$ is used. If $a > b$ and λ denotes an arbitrary constant, such that $\lambda > -b^2$ show that the equation represents a system of ellipses having the same foci.

5. An arch is in the form of a semi-ellipse with major axis horizontal. The span is 80 ft. and the height is 30 ft. Find the distance of the arch below the level of its top for each 10 ft. of the span.

6. The earth's orbit is an ellipse with the sun at one focus. The major axis is 185.8 million miles and the eccentricity is about $\frac{1}{60}$. Find the difference between the greatest and the least distance from the earth to the sun.

7. Show that, if two equal elliptical gears turn on mountings at corresponding foci, they are always in contact.

8. If two equal elliptical gears have major axes and minor axes of 12 in. and 8 in. respectively, and revolve once in 10 seconds, find the greatest and the least linear speed of a point on the driving ellipse.

Suggestion.—Use the greatest and the least radius on which a point is turning.

The driving gear has uniform angular velocity, and the mountings are at corresponding foci.

GENERAL EXERCISES

1. Find the equation of an ellipse in the form of **[32]** having the sum of its axes 20, and the difference 4.

2. Find the equation of an ellipse in the form of **[33]** if its major axis is 24, and its minor axis is equal to the distance between the foci.

3. Find the equation of an ellipse in the form of **[32]** if the minor axis is 12, and the distance between the foci is 12.

4. Find the equation of the ellipse in the form of **[33]** in which $a = 8$, and the foci bisect the semimajor axes.

Find the semi-axes, coördinates of foci, eccentricity, and the equations of the directrices of each of the following ellipses:

5. $16x^2 + 9y^2 = 144$.

6. $24x^2 + 36y^2 = 864$.

7. $16x^2 + 25y^2 - 64x + 100y = 236$.

Transform each of the following equations to axes parallel respectively to the old, the new origin being at the point given in each case. Plot the curve and both sets of axes.

8. $9x^2 + 4y^2 + 36x - 24y + 36 = 0.$ $(-2, 3).$

9. $25x^2 + 16y^2 + 50x + 32y - 359 = 0.$ $(-1, -1).$

10. Derive an equation that will represent all ellipses having foci at the points (3, 0) and (−3, 0).

11. Derive the equation of the ellipse with a focus at (3, 1), eccentricity equal to $\frac{2}{3}$, and with $3x - 4y + 6 = 0$ as directrix.

12. Show that the latus rectum of an ellipse is a third proportional to the two axes. Find the latus rectum of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ by this method.

CHAPTER VIII

THE HYPERBOLA AND CERTAIN FORMS OF THE SECOND DEGREE EQUATION

106. The equation of the hyperbola.—By the definition of article 81, the hyperbola is the locus of a point whose distance from a fixed point, the focus, is to its distance from a fixed straight line, the directrix, in a constant ratio e, greater than 1.

The method used in deriving the equation is exactly the same as that for the ellipse, **Art. 95.** In Fig. 94, let F be the focus and $D'D$ the directrix. Choose as x-axis the line $X'X$ through F and perpendicular to $D'D$ at R.

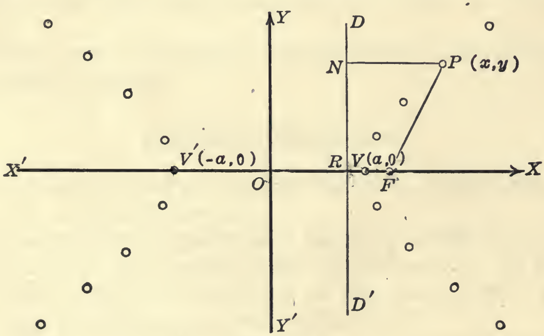


Fig. 94.

Since $e > 1$ there are two points V and V' on $X'X$ such that $\dfrac{VF}{RV} = e$ and $\dfrac{V'F}{V'R} = e$. Hence the points V and V' are on the locus.

Choose O, the point midway between V and V', as origin, and $Y'Y$, parallel to $D'D$, as y-axis.

Let the length of $V'V = 2a$. Then $V'O = OV = a$.

As with the ellipse it is necessary to find the equation of the directrix and the coördinates of the focus.

From the definition of the hyperbola,

$$VF = e \cdot RV, \text{ or } OF - a = e(a - OR), \tag{1}$$

and

$$V'F = e \cdot V'R, \text{ or } OF + a = e(a + OR). \tag{2}$$

Subtracting equation (1) from equation (2),

$$2a = 2e \cdot OR, \quad \text{or} \quad OR = \frac{a}{e}.$$

Then the equation of the directrix is $x = \frac{a}{e}$.

Adding equations (1) and (2),

$$2OF = 2ae, \text{ or } OF = ae.$$

Then the coördinates of F are $(ae, 0)$.

To derive the equation of the hyperbola, let $P(x, y)$ be any point on the locus, join F and P, and draw NP perpendicular to $D'D$.

By definition, $FP = e \cdot NP$.

But $FP = \sqrt{(x - ae)^2 + y^2}$, and $NP = x - \frac{a}{e}$.

Then $\sqrt{(x - ae)^2 + y^2} = e\left(x - \frac{a}{e}\right)$.

Squaring and arranging, this equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

Since $e > 1$, $a^2(e^2 - 1)$ is positive. Let it be represented by b^2 and the equation of the hyperbola is

[35] $$\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1}.$$

This is a standard form of the equation of the hyperbola, and is the form in which the hyperbola is usually written. Its simple form is due to the choice of the coördinate axes. A different choice of axes would give a less simple form of the equation, but the locus would be unaltered.

Since $b^2 = a^2(e^2 - 1)$, $e = \frac{\sqrt{a^2 + b^2}}{a}$.

Equation [**35**] is the required equation for it has been proved true for every point on the hyperbola, and it can be readily proved that it is not true for any point that is not on the locus.

107. Shape of the hyperbola.—The shape of the hyperbola and its position relative to the coördinate axes can be readily determined from the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving for x, $\quad x = \pm \frac{a}{b}\sqrt{b^2 + y^2}$.

Solving for y, $\quad y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$.

(1) For all values of y, x has two real values, numerically equal but opposite in sign. For all values of x such that

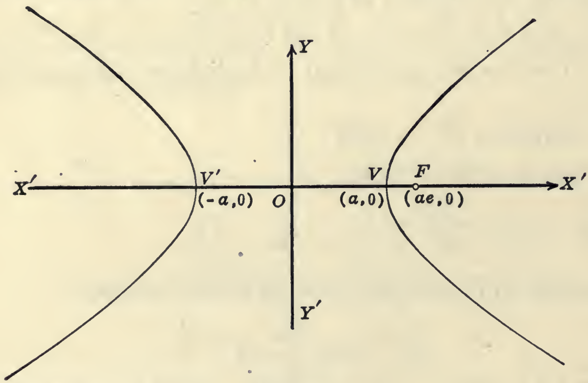


FIG. 95.

$x^2 - a^2 > 0$, y has two real values, numerically equal but opposite in sign. When $x^2 = a^2$, $y = 0$. Hence the curve is symmetrical with respect to both coördinate axes and the origin, and its intercepts on the x-axis are a and $-a$.

(2) For all values of x such that $x^2 - a^2 < 0$, y is imaginary, but no value of y will make x imaginary.

(3) As x increases from $+a$ or decreases from $-a$, the positive values of y increase and the negative values of y decrease.

The hyperbola has the shape shown in Fig 95.

108. Definitions.—The center of symmetry of the hyperbola is called the **center** of the hyperbola.

The line through the focus and perpendicular to the directrix is called the **principal axis** of the hyperbola.

The points in which the hyperbola intersects the principal axis are called the **vertices** of the hyperbola.

The portion of the principal axis lying between the vertices is called the **transverse axis** of the hyperbola. Its length is $2a$.

The **conjugate axis** of the hyperbola has a length $2b$, is perpendicular to the principal axis, is bisected by it, and passes through the center.

The chord of the hyperbola through the focus and perpendicular to the principal axis is called the **latus rectum.** Its length is $\frac{2b^2}{a}$, for the abscissa of the focus is ae, and when $x = ae$, $y = \pm \frac{b^2}{a}$.

109. Second focus and second directrix.—The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has a second focus at the point $(-ae, 0)$, and a second directrix which is the line $x = -\frac{a}{e}$. The proof is similar to that of article 98 for the ellipse and is left as an exercise for the student.

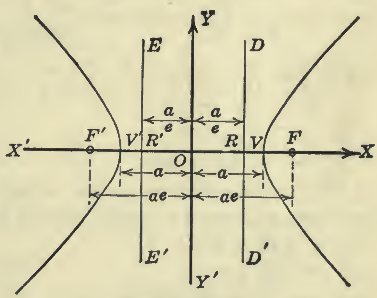


FIG. 96.

In Fig. 96, F and F' are the foci, and the lines $D'D$ and $E'E$ are the directrices.

110. Hyperbola with transverse axis on the y-axis.—The equation of an hyperbola whose transverse axis is on the y-axis and whose center is at the origin is obtained by interchanging x and y in the work of article 106. The equation is then

[36] $$\frac{\mathbf{y}^2}{\mathbf{a}^2} - \frac{\mathbf{x}^2}{\mathbf{b}^2} = \mathbf{1}.$$

Here the transverse axis is $2a$; the conjugate axis is $2b$; the coördinates of the vertices are $(0, \pm a)$; the coördinates of the foci are $(0, \pm ae)$; and the equations of the directrices are $y = \pm\frac{a}{e}$. (See Fig. 97.)

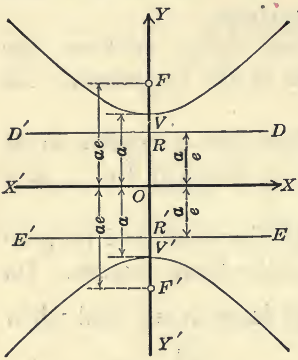


FIG. 97.

EXERCISES

1. In each of the following hyperbolas, find the length of the transverse axis and the conjugate axis, the coördinates of the foci, and the equations of the directrices. Sketch each hyperbola.

(1) $\frac{x^2}{25} - \frac{y^2}{16} = 1.$

(2) $\frac{x^2}{36} - \frac{y^2}{100} = 1.$

(3) $16x^2 - 9y^2 = 144.$

(4) $\frac{y^2}{64} - \frac{x^2}{36} = 1.$

(5) $\frac{y^2}{2} - \frac{x^2}{3} = 1.$

(6) $9y^2 - 25x^2 = 225.$

2. Write the equation of the hyperbola with center at the origin, and transverse axis on the x-axis, having given:

(1) $a = 6,\ b = 4.$

(2) $a = 4,\ e = 2.$

(3) $b = 8,\ ae = \frac{24}{5}\sqrt{5}.$

(4) $b = 3,\ ae = 5.$

(5) $a = 9,\ e = \frac{4}{3}.$

(6) $b = 6,\ ae = \sqrt{85}.$

3. In the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find the value of y when $x = 3$, when $x = 5$, when $x = 2$.

4. Find the length of the latus rectum of the hyperbola $\frac{x^2}{36} - \frac{y^2}{16} = 1.$

Of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$

5. Find the equation of an hyperbola with transverse axis on the x-axis, center at origin, and passing through the points $(6, 4)$ and $(-3, 1)$.

6. Find the equation of the locus of a point moving so that the difference of its distances from the points $(\pm 6, 0)$ is 8.

7. Derive equation [**36**] from [**35**] by rotating the coördinate axes through an angle $\varphi = 90°$.

8. Find the semi-axes, eccentricity, and the latus rectum of each of the following hyperbolas:

(1) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1.$ (4) $\dfrac{x^2}{2} - y^2 = m.$

(2) $4x^2 - 3y^2 = 24.$ (5) $px^2 - qy^2 = pq.$

(3) $16x^2 - y^2 = 16.$ (6) $x^2 - qy^2 = s.$

9. Find the semi-axes, coördinates of foci, eccentricity, and equations of directrices of each of the following hyperbolas:

(1) $16x^2 - 9y^2 = 144.$ (2) $24x^2 - 36y^2 = 864.$

10. Find the equation of an hyperbola with transverse axis on the y-axis, center at the origin, eccentricity equal to 2, and passing through the point (3, 2).

11. Assume the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ of the hyperbola, and show that the difference of the distances of any point on it from the foci is $2a$.

12. Show that the latus rectum of an hyperbola is a third proportional to the two axes.

13. What does the equation $x^2 - y^2 = 16$ become when the coördinate axes are rotated through an angle $\varphi = -45°$?

14. Find the equation of an hyperbola if its center is at the origin, transverse axis is 24, and the distance between its foci is 32.

15. Find the equation of an hyperbola if its center is at the origin, transverse axis is 24, and its conjugate axis equals one-half the distance between its foci.

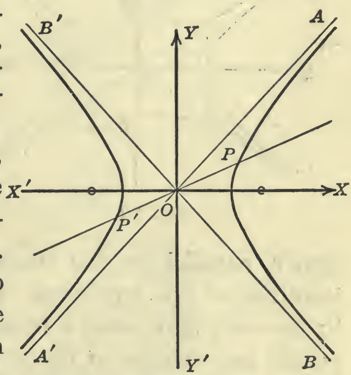


Fig. 98.

111. Asymptotes.

—In Fig. 98, $P'P$ is a line passing through the center of the hyperbola and intersecting the curve in P' and P. If P is made to move off to infinity along the curve, the line $P'P$, continually passing through the center, will turn about O and will approach one of the two lines $A'A$ or $B'B$. These lines are called the **asymptotes**[1] of the hyperbola.

[1] This is not the general definition for asymptotes, but is true for the hyperbola.

The equation of any line $P'P$ through the origin is $y = mx$. The coördinates of its intersection with the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are found by solving the equations as simultaneous equations.

Solving for x,

$$x = \pm \frac{ab}{\sqrt{b^2 - a^2m^2}}.$$

Now as P moves off to infinity along the curve x becomes infinite. Therefore the denominator of the fraction must approach 0.

This gives $b^2 - a^2m^2 = 0$, or $m = \pm \frac{b}{a}$.

Hence the equations of the asymptotes are

[37] $$\mathbf{y = \frac{b}{a}x, \quad \text{and} \quad y = -\frac{b}{a}x.}$$

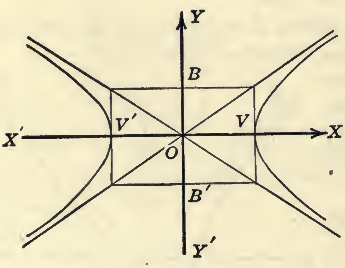


Fig. 99.

These equations can be combined into the single equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

The conjugate axis $B'B$, Fig. 99, can now be brought into a closer relation to the hyperbola. If through the extremities of $B'B$ lines are drawn parallel to the transverse axis, and through the extremities of the transverse axis $V'V$ lines are drawn parallel to the conjugate axis, a rectangle is formed with its diagonals on the asymptotes of the hyperbola.

It can readily be shown that if the transverse axis of the hyperbola is on the y-axis, the equations of the asymptotes are

$$y = \frac{a}{b}x, \quad \text{and} \quad y = -\frac{a}{b}x.$$

By the help of the asymptotes, a simple and fairly accurate method for sketching an hyperbola is as follows: Locate the vertices and draw the asymptotes, then draw the hyperbola so that the curve continually approaches the asymptotes as it moves off toward infinity.

Example.—Sketch the hyperbola $16x^2 - 25y^2 = 400$.

First put $16x^2 - 25y^2 = 400$ in the form $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

Then $a = 5$, $b = 4$, the foci are at the points $(5, 0)$ and $(-5, 0)$, and the equations of the asymptotes are $y = \frac{4}{5}x$ and $y = -\frac{4}{5}x$.

The curve is as shown in Fig. 100.

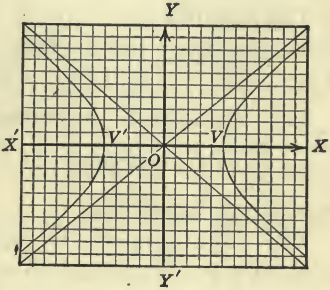


FIG. 100.

112. Conjugate hyperbolas.—Two hyperbolas that are so related that the transverse axis of each is the conjugate axis

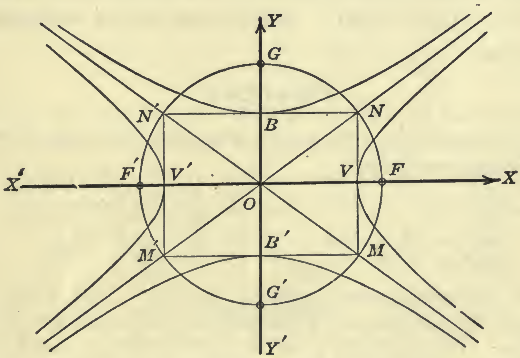


FIG. 101.

of the other, both in magnitude and in position, are called **conjugate hyperbolas.**

Thus, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, Fig. 101, are conjugate hyperbolas.

From article 111, it is seen that the asymptotes of each are $y = \pm \frac{b}{a}x$. Therefore two conjugate hyperbolas have the same asymptotes.

The formula $b^2 = a^2(e^2 - 1)$ can now be readily interpreted geometrically. For in right triangle OVN, Fig. 101, $OV = a$, $VN = OB = b$, and $ON = OF = ae$.

113. Equilateral hyperbola.—If $a = b$, the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ becomes $x^2 - y^2 = a^2$. This is called an **equilateral hyperbola.**

The equations of its asymptotes are $y = \pm x$, and are evidently perpendicular to each other and make angles of 45° with the axes of the hyperbola.

An equilateral hyperbola is also called a **rectangular hyperbola.**

It may be noted that the equilateral hyperbola is the simplest of hyperbolas, just as the circle is the simplest of ellipses, being the ellipse in which the major axis and minor axis are equal.

EXERCISES

1. Find the equations of the asymptotes and sketch the curve for each of the following hyperbolas:

(1) $\frac{x^2}{18} - \frac{y^2}{12} = 1$.

(2) $9x^2 - 18y^2 = 16$.

(3) $9x^2 - 18y^2 = -16$.

(4) $\frac{y^2}{25} - \frac{x^2}{16} = 1$.

(5) $x^2 - y^2 = 12$.

(6) $x^2 - y^2 = -12$.

2. One of two conjugate hyperbolas is $12x^2 - 16y^2 = 192$, find the other. Find the coördinates of the foci and the equations of the directrices of each.

3. Show that the four foci of two conjugate hyperbolas, and the four points of intersection of the tangents at their vertices, all lie on a circle whose center is at the common center of the two hyperbolas.

4. Show that the eccentricity of an equilateral hyperbola is $\sqrt{2}$.

5. Transform the equation of the equilateral hyperbola $x^2 - y^2 = a^2$, by rotating the coördinate axes through an angle $\varphi = -45°$. This refers the hyperbola to its asymptotes as axes.

6. Find the equation of the hyperbola whose vertices are at $(\pm 4, 0)$ and the angle between whose asymptotes is 60°.

7. If e_1 and e_2 respectively are the eccentricities of two conjugate hyperbolas, show that $ae_1 = be_2$ and that $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$.

8. Plot the equilateral hyperbolas $x^2 - y^2 = a^2$ and $y^2 - x^2 = a^2$ and locate their foci. With the same coördinate axes plot the circle $x^2 + y^2 = 2a^2$. Also plot $x^2 - y^2 = 0$ on the same set of axes.

9. Prove that in any hyperbola the distance from a focus to an asymptote equals the semi-conjugate axis.

10. Prove that in any hyperbola the distance from the center to the foot of the perpendicular from a focus to an asymptote equals the semi-transverse axis.

11. Find the value of b in order that the line $y = 2x + b$ shall be tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

12. Find the value of m in order that the line $y = mx + 2$ shall be tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

114. Equation of hyperbola when axes are translated.— By a method identical to that of article 100 for the ellipse, the equation

[38] $$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

is found for the hyperbola with its center at the point (h, k), and whose transverse axis is parallel to the x-axis. This is a second standard form of the equation of the hyperbola.

If the transverse axis is parallel to the y-axis the equation is

[38_1] $$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1.$$

EXERCISES

1. Write the equations of the following hyperbolas:

(1) Center at $(4, -3)$, $a = 5$, $b = 3$, and transverse axis parallel to x-axis.

(2) Center at $(-6, -2)$, $a = 2$, $b = 4$, and transverse axis parallel to y-axis.

2. Find the coördinates of the vertices and the foci, and the equations of the directrices of each hyperbola of exercise 1.

3. Find the equation of the hyperbola with center at $(-2, 7)$, one directrix the line $y = 5$, and eccentricity equal to $\frac{5}{4}$.

4. Find the equations of the hyperbolas that are conjugate hyperbolas with those of exercise 1.

5. Find the equations of the asymptotes of the hyperbolas of exercise 1.

115. Equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$. *Every equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A and C have unlike signs, represents an hyperbola with axes parallel to the coördinate axes.*

Proof.—In a manner identical to that of article **101** the equation takes the form

$$\frac{\left(x + \frac{D}{2A}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4A^2C}} + \frac{\left(y + \frac{E}{2C}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4AC^2}} = 1.$$

This is of the form of [**38**] or [**38$_1$**] for the denominators have unlike signs since A and C are unlike in sign and therefore $4A^2C$ and $4AC^2$ are unlike in sign.

If the second denominator is negative, the transverse axis is parallel to the x-axis. If the first denominator is negative, the transverse axis is parallel to the y-axis.

From the preceding proof it follows that the equation of an hyperbola in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ can be transformed into the standard forms, [**35**] or [**36**], by a suitable translation of the coördinate axes, the new origin being at the point $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

Example.—Express the hyperbola $36x^2 - 25y^2 + 216x + 100y - 676 = 0$ in the form of [**38**]. What are the coördinates of its center, foci, and vertices; the lengths of the semi-axes; and the equations of its directrices and asymptotes? Plot. Finally, translate the coördinate axes so as to change to the form [**35**] and answer the same questions with reference to the new axes.

Solution.—Completing the squares in x and in y,

$$36(x^2 + 6x + 9) - 25(y^2 - 4y + 4) = 676 + 324 - 100,$$

or

$$36(x + 3)^2 - 25(y - 2)^2 = 900.$$

Dividing by 900 and putting in the form [**38**],

$$\frac{(x + 3)^2}{25} - \frac{(y - 2)^2}{36} = 1.$$

This is an hyperbola with its center at the point $C(-3, 2)$ and transverse axis parallel to the x-axis.

The semi-axes are 5 and 6.

The eccentricity $e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{61}}{5}$.

The distance from the center to the foci is $ae = \sqrt{61}$, and the foci are $F(-3+\sqrt{61}, 2)$ and $F'(-3-\sqrt{61}, 2)$.

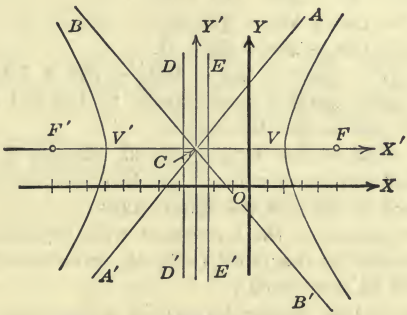


FIG. 102.

The vertices are $V(2, 2)$ and $V'(-8, 2)$.

The distance from the center to the directrices is $\frac{a}{e} = \frac{5}{\sqrt{61}}$, and the equations of the directrices are $x = -3 + \frac{5}{\sqrt{61}}$, and $x = -3 - \frac{5}{\sqrt{61}}$.

The asymptotes have slopes of $\frac{6}{5}$ and $-\frac{6}{5}$ respectively, and pass through $C(-3, 2)$. Their equations are by [15],

$$y - 2 = \tfrac{6}{5}(x+3), \text{ and } y - 2 = -\tfrac{6}{5}(x+3),$$

or

$$6x - 5y + 28 = 0, \text{ and } 6x + 5y + 8 = 0.$$

The hyperbola is as shown in Fig. 102.

To change to the form of [35], put $x + 3 = x'$ and $y - 2 = y'$. Then the equation $\frac{(x+3)^2}{25} - \frac{(y-2)^2}{36} = 1$ becomes $\frac{x'^2}{25} - \frac{y'^2}{36} = 1$, referred to CX' and CY' as axes. The center is $C(0, 0)$; foci are $F(\sqrt{61}, 0)$ and $F'(-\sqrt{61}, 0)$; vertices are $V(5, 0)$ and $V'(-5, 0)$; equations of directrices are $x = \frac{5}{\sqrt{61}}$ and $x = -\frac{5}{\sqrt{61}}$; and asymptotes are $6x - 5y = 0$ and $6x + 5y = 0$.

EXERCISES

1. Express the equations of the following hyperbolas in the form of **[38]** or **[38_1]**. Find the coördinates of the centers, foci, and vertices; the lengths of the semi-axes; and the equations of the directrices and asymptotes. Sketch each curve.

(1) $9x^2 - 16y^2 - 108x + 96y + 36 = 0$.

(2) $16y^2 - x^2 - 6x - 80y + 75 = 0$.

(3) $8x^2 - 28y^2 - 8x - 28y - 61 = 0$.

(4) $8x^2 - 9y^2 - 16x + 54y - 1 = 0$.

(5) $3y^2 - 4x^2 - 16x - 24y - 52 = 0$.

2. Transform $9x^2 - 25y^2 + 54x + 100y + 206 = 0$ by translating to new coördinate axes parallel respectively to the old axes, with new origin at $(-3, 2)$, and sketch the curve.

3. Transform each of the hyperbolas of exercise 1 to the form of **[35]** or **[36]**. Find the coördinates of the foci, and the equations of the directrices referred to the new coördinate axes.

4. Find the equation of the hyperbola with conjugate axis parallel to the x-axis, center at the point $(-3, 4)$, eccentricity $\frac{5}{3}$, and passing through the point $(9, 4 + 8\sqrt{3})$.

5. Find the equation of the hyperbola whose axes are parallel to the coördinate axes and which passes through the points $(3, 4)$, $(-7, 4)$, $(8, 4 + 4\sqrt{3})$, and $(-12, 4 - 4\sqrt{3})$.

6. Find the equation of the hyperbola having a focus at $(6, 2)$, a directrix the line $x - 12 = 0$, and $e = 2$.

116. Equation of hyperbola when axes are rotated.—In like manner to that for the parabola (**Art. 90**) and the ellipse (**Art. 102**), the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$, which is that of an hyperbola with axes parallel to the coördinate axes, is transformed by using equations **[13]** to the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. This is the most general form of the second degree equation in x and y, where $B^2 - 4AC > 0$. (See **Art. 121.**)

Conversely, starting with an equation containing an xy-term, rotation through a properly chosen angle will cause the xy-term to disappear by having its coefficient zero.

EXERCISES

1. Transform the following equations by rotating the coördinate axes through the angle given in each case:

(1) $x^2 - y^2 = 16.$ $\varphi = -45°.$ (2) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$ $\varphi = 90°.$

(3) $xy = 8.$ $\varphi = 45°.$ (4) $9y^2 - 16x^2 = 144.$ $\varphi = 60°.$

(5) $x^2 - 4xy + y^2 + 9 = 0.$ $\varphi = 45°.$

(6) $x^2 - 4xy + y^2 + 4\sqrt{2}x - 2\sqrt{2}y + 11 = 0.$ $\varphi = \tan^{-1} 1.$

2. Transform the following equations into the standard form rotating the axes through a proper angle, and sketch the curve in each case:

(1) $x^2 + 4xy + y^2 = 16.$

(2) $9x^2 + 14\sqrt{3}xy - 5y^2 - 48 = 0.$

3. Simplify the following equation by first translating the axes to remove the x-term and the y-term, then by rotating through an angle that will remove the xy-term. Sketch the curve and the three sets of coördinate axes.

$x^2 + 2xy - y^2 + 8x + 4y - 8 = 0.$

Suggestion.—Find $\varphi = 22\frac{1}{2}°$. Then use $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 - \sqrt{2}}$ and $\cos 22\frac{1}{2}° - \frac{1}{2}\sqrt{2 + \sqrt{2}}$.

4. In the hyperbola of exercise 3 find the coördinates of the center and the foci, and the equations of the transverse and conjugate axes, and asymptotes, referred to the original axes.

117. Equation of hyperbola in polar coördinates.—Here the procedure is similar to that for the parabola and ellipse, and the equations of these three conics should be compared.

EXERCISES

1. Derive the equation in polar coördinates of the hyperbola with the pole at the left hand focus and polar axis along the transverse axis. Let p equal the distance from the focus to the directrix.

2. Plot the following hyperbolas and draw the asymptotes:

(1) $\rho = \dfrac{ep}{1 - e \sin \theta}.$ (2) $\rho = \dfrac{ep}{1 + e \sin \theta}.$

3. Transform $x^2 + y^2 = e^2(x + p)^2$ into polar coördinates.

4. Show that in the equation of the hyperbola, $\rho = \dfrac{ep}{1 - e \cos \theta}$, the inclination of the asymptotes is $\cos^{-1}\left(\pm\dfrac{1}{e}\right)$.

5. Find the polar intercepts of the conic $\rho = \dfrac{ep}{1 - e \cos \theta}$, and show that the transverse axis of the hyperbola is $\dfrac{2ep}{e^2 - 1}$, and the major axis of the ellipse is $\dfrac{2ep}{1 - e^2}$.

6. Transform the polar equation $\rho^2 \cos 2\theta = a^2$ into rectangular coördinates, having the origin at the pole and the x-axis along the polar axis.

7. Show that $\rho^2 = \dfrac{b^2}{1 - e^2 \cos^2 \theta}$ is an ellipse if $e < 1$; and that $\rho^2 = \dfrac{-b^2}{1 - e^2 \cos^2 \theta}$ is an hyperbola if $e > 1$. Sketch each curve.

118. Construction of an hyperbola.—*First method.*—The length of the transverse axis, $2a$, and the foci F and F' are supposed known. On a drawing board place two tacks at F and F', respectively, Fig. 103. Tie a pencil firmly at point P near the middle of a string. Pass one part of the string

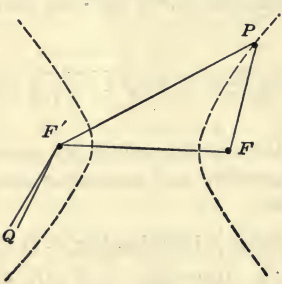


FIG. 103.

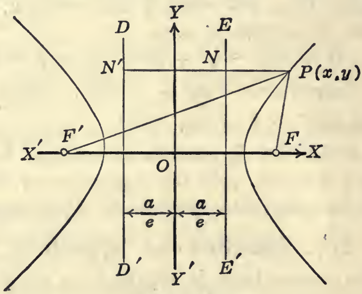


FIG. 104.

under the tack at F and over the tack at F', and the other part over the tack at F'. Adjust the string so that $PF' - PF = 2a$. Hold the parts of the string firmly together at Q and pull downward. The point P will generate an arc of an hyperbola. By arranging the string properly other arcs of the hyperbola may be generated.

This construction depends upon the following.

THEOREM.—*The difference of the distances from any point on an hyperbola to its two foci is constant and equal to the transverse axis.*

This may be proved as follows: In Fig. 104, from the definition of an hyperbola,

$$PF' = e \cdot N'P = e\left(x + \frac{a}{e}\right) = ex + a,$$

and

$$PF = e \cdot NP = e\left(x - \frac{a}{e}\right) = ex - a.$$

Subtracting, $PF' - PF = 2a =$ transverse axis.

Second method.—A focus, the corresponding directrix, and the value of e are supposed known.

In Fig. 105, let F be the focus, $D'D$ the directrix, $X'X$ the axis through the focus and intersecting $D'D$ in A, and let the lines QR and TS be drawn through A with inclinations respectively equal to $\tan^{-1}(\pm e)$. Also draw a series of lines parallel to $D'D$.

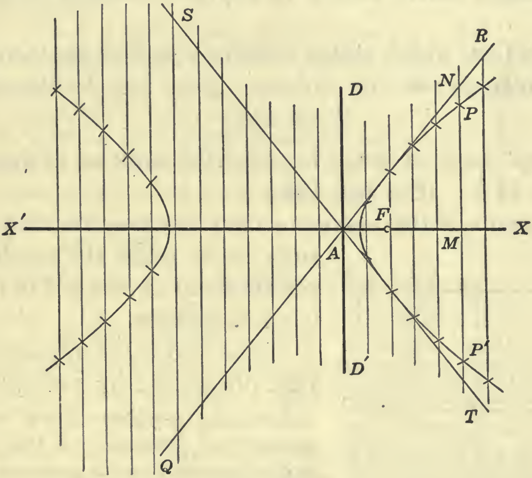


FIG. 105.

Then the points P and P' of the curve, on any one of these parallels, are found by striking arcs with the focus as center intersecting the parallel lines and using as a radius the length MN of that particular parallel. Show why this is so.

EXERCISES

1. Locate a directrix and a focus and construct an hyperbola with $e = \frac{3}{2}$. With $e = \frac{5}{4}$. With $e = \sqrt{3}$.

2. Construct a parabola by the same method.

3. Using the same method, construct an ellipse with (1) $e = \frac{1}{2}$, (2) $e = \frac{1}{3}\sqrt{3}$, (3) $e = \frac{4}{5}$.

4. The difference of the distances of a point on an hyperbola from the foci is 4; and the foci are at the points (3, 0) and (−3, 0). Use the theorem of **Art. 118** and derive the equation of the hyperbola.

APPLICATIONS

119. Uses of the hyperbola.—Whenever the law connecting two variables is an inverse variation it gives rise to the equation $xy = k$, where x and y are the variables and k is a constant. This relation often occurs in physics, chemistry, and engineering.

Boyle's Law which states that for a perfect gas the pressure varies inversely as the volume, gives rise to the equation

$$pv = k.$$

This is not used so much in practical work as is some slight variation of it. (See **Art. 124.**)

Then again, if the law governing the location of a point is such as to fulfill the conditions of the theorem of article 118 the locus is an hyperbola.

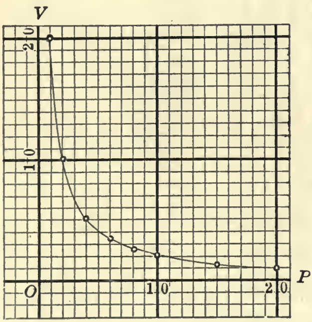


FIG. 106.

These and other applications are best illustrated by examples.

Example 1.—Given 20 c.c. of air at 1 atmosphere pressure. If the volume v varies inversely as the pressure p, derive the equation showing the relation between the volume and the pressure. Plot the curve for values of p from 1 atmosphere to 20 atmospheres.

Solution.—Since v varies inversely as the pressure, $pv = k$.

When $p = 1$, $v = 20$, hence $1 \cdot 20 = k$, or $k = 20$.

Therefore the equation showing the relation between p and v is

$$pv = 20.$$

This is the equation of an equilateral hyperbola referred to its asymptotes as axes, and can be plotted, as accurately as desired, by points. It is plotted in the first quadrant only because both volume and pressure must be positive. (See Fig. 106.)

p	1	2	4	6	8	10	15	20
v	20	10	5	3.3	2.5	2	1.3	1

Example 2.—Instruments for recording sound are placed at two points A and B 500 ft. apart, Fig. 107. The report of a cannon is recorded 0.25 second earlier at A than at B. Find the equation of the locus of the position of the cannon, and plot. (Sound travels 1120 ft. per second.)

Solution.—Since sound travels 1120 ft. per second, the cannon is 0.25×1120 ft. $= 280$ ft. nearer A than B.

Choose as origin the point midway between A and B, with x-axis through these points. Let $P(x, y)$ be any position of the cannon.

The coördinates of A and B are respectively $(250, 0)$ and $(-250, 0)$.

Then $BP - AP = 280$.

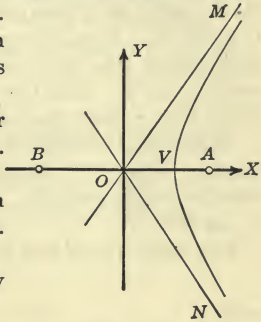


FIG. 107.

Or $\sqrt{(-250 - x)^2 + y^2} - \sqrt{(250 - x)^2 + y^2} = 280$.

Simplifying, $\dfrac{x^2}{19{,}600} - \dfrac{y^2}{42{,}900} = 1$.

This is an hyperbola with $a = 140$, and $b = 207.1$. The conditions require, however, that the locus of the position of the cannon shall be the branch of the hyperbola nearer to A.

NOTE.—The above example illustrates the principle made use of in the most accurate instruments used by the Allies in the Great War for locating hidden guns. Near the close of the war they were locating guns 10 miles distant within a radius of 50 ft.

EXERCISES

1. Given 10 c.c. of air at atmospheric pressure. If the volume v varies inversely as the pressure p derive the equation expressing the relation between the volume and the pressure. Plot for pressures from $\frac{1}{2}$ atmosphere to 10 atmospheres.

2. Given an oak beam 10 ft. long of such dimensions that it supports 2 tons at its midpoint when resting at each end upon a support. If the weight w such a beam will support varies inversely as its length l derive the equation expressing the relation between w and l. Plot for values of l from 1 ft. to 20 ft.

3. Find the locus of the center of a circle tangent externally to two given circles.

4. Find the locus of the center of a circle having one of two given circles tangent to it internally and the other tangent to it externally.

5. The base of a triangle is fixed, and the difference of the angles at the base is $\frac{1}{2}\pi$. Find the locus of the vertex opposite the base.

6. Three instruments for recording sound are located at three points, A, B, and C, in a straight line. From A to B is 300 ft. and from B to C is 500 ft. A sound, such as the report of a cannon, is recorded at B 0.05 second after it is recorded at A, and at C 0.35 second after it is recorded at B. Find the location of the source of the sound in distance and direction from the point midway between A and B.

Suggestion.—Choose the origin of coördinates at the point midway between A and B. Derive the equations of the hyperbolas and solve as simultaneous equations. Note that only one branch of each hyperbola is possible.

The equations of the hyperbolas will be found to be

$$27.7x^2 - y^2 = 21716,$$

and

$$(x - 400)^2 - 1.6y^2 = 38416.$$

The solution of these equations gives

$$x = -70 \text{ and } y = 337.7.$$

GENERAL EXERCISES

1. Find the semi-axes, the eccentricity, and the coördinates of the foci of the hyperbola $2x^2 - 3y^2 = 12$. Also find the equation of the hyperbola that is conjugate with this.

2. Find the coördinates of the points of intersection of the hyperbola $2x^2 - 3y^2 = 12$ and the circle $x^2 + y^2 = 16$.

3. Find the semi-axes, coördinates of foci, eccentricity, and equations of directrices of the hyperbola $9x^2 - 4y^2 - 54x + 16y + 209 = 0$.

4. Show that the following equation represents two straight lines parallel respectively to the coördinate axes: $12xy + 8x - 27y - 18 = 0$.

Transform the following equations as indicated, illustrating each by a drawing:

5. $x^2 - 10xy + y^2 + x + y + 1 = 0$ to $32x^2 - 48y^2 = 9$.

6. $x^2 - 2xy - y^2 - 2 = 0$ to $x^2 - y^2 + \sqrt{2} = 0$.

7. Find the equation of the locus of a point that moves so that the difference of its distances from $(-4, 2)$ and $(4, 2)$ is always equal to 8.

8. Given the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$, find the coördinates of the point on the hyperbola, with abscissa double the ordinate.

9. Find the distances from the foci of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ to a point on the hyperbola, with abscissa 10.

10. Find the equation of an hyperbola whose axes are parallel respectively to the coördinate axes and which passes through the points (0, 0), (1, 1), (−2, −1), and (−2, 2).

11. The lines $x - 2y = 0$ and $x + 2y = 0$ are the asymptotes of an hyperbola that passes through the point (−5, 3). Find its equation.

12. Prove that for all values of α the point ($a \sec \alpha$, $b \tan \alpha$) is on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

13. Prove that $\sec \alpha$ is the eccentricity of an hyperbola with asymptotes including an angle 2α.

14. Prove that the portion of an asymptote of an hyperbola, which is intercepted between the directrices is equal to the transverse axis.

CHAPTER IX

OTHER LOCI AND EQUATIONS

120. General statement.—In the previous chapters, for the most part, equations of the first and second degree in two variables, and their loci are considered. In the present chapter a consideration will be made of other equations also and their loci, where they are of importance in the study of more advanced mathematics, or are of use in immediate applications to science and engineering.

Such equations and loci are of infinite variety and form. They may be divided into two classes, (1) **algebraic** and (2) **transcendental.**

Algebraic curves the degree of whose equations is higher than the second, and all transcendental curves that lie wholly in a plane, are often called **higher plane curves.**

In Cartesian coördinates an equation that can be expressed in a finite number of terms of the form Qx^ny^m, in which the variables are affected by constant exponents and Q is a constant, is called **algebraic,** all others are called **transcendental.**

121. Summary for second degree equations.—The most general equation of the second degree in two variables may be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Theorem.—*In rectangular coördinates, the equation of the second degree in two variables represents a conic section.*

To prove this it is only necessary to show that, by a suitable change of the coördinate axes, the equation reduces to a form already discussed.

Given $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$

From [13] substituting $x = x' \cos\varphi - y' \sin\varphi$,

and $$y = x' \sin\varphi + y' \cos\varphi,$$

$$A(x' \cos\varphi - y' \sin\varphi)^2 + B(x' \cos\varphi - y' \sin\varphi)(x' \sin\varphi + y' \cos\varphi) + C(x' \sin\varphi + y' \cos\varphi)^2 + D(x' \cos\varphi - y' \sin\varphi) + E(x' \sin\varphi + y' \cos\varphi) + F = 0.$$

Expanding and collecting terms,

$$x'^2(A \cos^2\varphi + B \sin\varphi \cos\varphi + C \sin^2\varphi) + x'y'(-2A \sin\varphi \cos\varphi + 2C \sin\varphi \cos\varphi - B \sin^2\varphi + B \cos^2\varphi) + y'^2(A \sin^2\varphi - B \sin\varphi \cos\varphi + C \cos^2\varphi) + x'(D \cos\varphi + E \sin\varphi) + y'(E \cos\varphi - D \sin\varphi) + F = 0.$$

In this equation the $x'y'$ term will vanish if

$$-2A \sin\varphi \cos\varphi + 2C \sin\varphi \cos\varphi - B \sin^2\varphi + B \cos^2\varphi = 0.$$

Or if $B(\cos^2\varphi - \sin^2\varphi) = (A - C)2 \sin\varphi \cos\varphi$.

By trigonometry, this becomes $B \cos 2\varphi = (A - C) \sin 2\varphi$.

[39] $$\therefore \tan 2\varphi = \frac{B}{A - C}.$$

Since the tangent of an angle may have any value from $-\infty$ to $+\infty$, it is always possible to rotate the coördinate axes through such an angle that the $x'y'$-term will vanish.

Further, since the smallest positive value of 2φ is less than 180°, φ is an acute angle. This value of φ can always be chosen for the rotation.

The general equation then reduces to the form

$$A'x^2 + C'y^2 + D'x + E'y + F' = 0.$$

From the considerations of the previous chapters, this equation represents one of the conic sections as follows:

(1) A circle if $A' = C'$.

(2) A parabola if $A' \neq 0$, $E' \neq 0$, and $C' = 0$,
or if $C' \neq 0$, $D' \neq 0$, and $A' = 0$.

(3) An ellipse if A' and C' are of like signs and unequal.

(4) An hyperbola if A' and C' have unlike signs.

THEOREM.—*The general equation of the second degree in x and y, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, represents a* *parabola, an ellipse, or an hyperbola according as* $B^2 - 4AC$ *equals zero, is less than zero, or is greater than zero.*

Proof.—Using the values of A', B', and C',

$$A' = A\cos^2\varphi + B\sin\varphi\cos\varphi + C\sin^2\varphi, \quad (1)$$

$$B' = B(\cos^2\varphi - \sin^2\varphi) - (A - C)2\sin\varphi\cos\varphi, \quad (2)$$

$$C' = A\sin^2\varphi - B\sin\varphi\cos\varphi + C\cos^2\varphi. \quad (3)$$

Adding (1) and (3),

$$A' + C' = A + C. \quad (4)$$

Subtracting (3) from (1),

$$A' - C' = (A - C)\cos 2\varphi + B\sin 2\varphi. \quad (5)$$

Squaring (2) and (5) and adding,

$$B'^2 + (A' - C')^2 = B^2 + (A - C)^2. \quad (6)$$

Squaring (4) and subtracting from (6),

$$B'^2 - 4A'C' = B^2 - 4AC. \quad (7)$$

But, if φ is chosen so that $\tan 2\varphi = \dfrac{B}{A - C}$, $B' = 0$.

Hence $B^2 - 4AC = -4A'C'$.

From this it follows that

(*a*) $B^2 - 4AC = 0$ if either $A' = 0$ or $C' = 0$,

(*b*) $B^2 - 4AC < 0$ if A' and C' have like signs,

(*c*) $B^2 - 4AC > 0$ if A' and C' have unlike signs.

These are respectively the conditions necessary for a parabola, an ellipse, or an hyperbola.

Example.—Given $x^2 + 24xy - 6y^2 + 4x + 48y + 34 = 0$. (1) Determine whether it represents a parabola, an ellipse, or an hyperbola; (2) transform so as to free of the xy-term; (3) reduce to the standard form; (4) plot and show the three sets of axes.

Solution.—(1) $B^2 - 4AC = 24^2 - 4\cdot1(-6) = 600$.

Therefore the equation represents an hyperbola.

(2) $\tan 2\varphi = \dfrac{B}{A - C} = \frac{24}{7}$. $\cos 2\varphi = \frac{7}{25}$.

$\sin\varphi = \sqrt{\frac{1}{2}(1 - \cos 2\varphi)} = \sqrt{\frac{1}{2}(1 - \frac{7}{25})} = \frac{3}{5}$.

$\cos\varphi = \sqrt{\frac{1}{2}(1 + \cos 2\varphi)} = \sqrt{\frac{1}{2}(1 + \frac{7}{25})} = \frac{4}{5}$.

Then the formulas [**13**] become $x = \frac{4}{5}x' - \frac{3}{5}y'$, and $y = \frac{3}{5}x' + \frac{4}{5}y'$.

Substituting in the equation, $(\frac{4}{5}x' - \frac{3}{5}y')^2 + 24(\frac{4}{5}x' - \frac{3}{5}y')(\frac{3}{5}x' + \frac{4}{5}y') - 6(\frac{3}{5}x' + \frac{4}{5}y')^2 + 4(\frac{4}{5}x' - \frac{3}{5}y') + 48(\frac{3}{5}x' + \frac{4}{5}y') + 34 = 0$.

Simplifying, $10x'^2 - 15y'^2 + 32x' + 36y' + 34 = 0$.

(3) Putting $x' = x'' + h$ and $y' = y'' + k$, and simplifying,

$$10x''^2 - 15y''^2 + (20h + 32)x'' - (30k - 36)y'' + 10h^2 - 15k^2 + 32h + 36k + 34 = 0.$$

Equating coefficients of x'' and y'' to 0, and solving,

$$20h + 32 = 0,\ 30k - 36 = 0.\quad \therefore h = -\tfrac{8}{5}, \text{ and } k = \tfrac{6}{5}.$$

Substituting these values and simplifying,

$$2x''^2 - 3y''^2 + 6 = 0, \text{ or } \frac{y''^2}{2} - \frac{x''^2}{3} = 1.$$

This is an hyperbola with its center at the origin and its transverse axis along the y''-axis.

(4) The three sets of coördinate axes and the curve are as shown in Fig. 108.

Remark.—The values of h and k could have been found by completing the squares in x and y. In the solution given above, the rotation of axes was made first; but the work would have been shortened somewhat if the axes had been translated first.

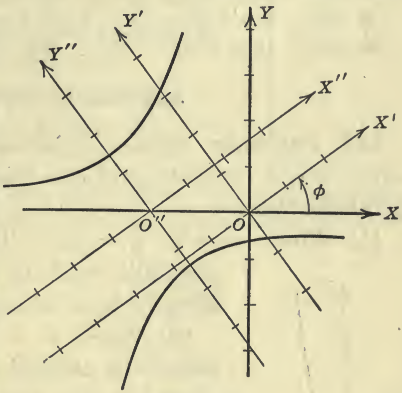


Fig. 108.

122. Suggestions for simplifying second degree equations.—If the equation is that of an ellipse or hyperbola, first translate the axes to remove the terms of the first degree, and then rotate the axes to remove the xy-term.

If the equation is that of a parabola, first rotate the axes to remove the xy-term, and then translate to remove the constant term and one of the terms of first degree.

It may be that the locus is a point, that it is composed of straight lines, or is imaginary. These forms are often called **degenerate forms** and are best discovered from the simplified equation.

EXERCISES

Test each of the following equations as to whether it is a parabola, an ellipse, or an hyperbola. Simplify each and plot showing all sets of coördinate axes.

1. $6x^2 + 24xy - y^2 + 50y - 55 = 0.$
2. $25x^2 - 14xy + 25y^2 + 142x - 178y + 121 = 0.$
3. $x^2 + xy + y^2 - 3y + 6 = 0.$
4. $32x^2 - 48xy + 18y^2 + 35x - 120y + 200 = 0.$
5. $13x^2 - 6\sqrt{3}xy + 7y^2 - 64 = 0.$
6. $x^2 - 2\sqrt{3}xy + 3y^2 - 6\sqrt{3}x - 6y = 0.$
7. $2x^2 + 6xy + 10y^2 - 2x - 6y + 19 = 0.$
8. $6x^2 + 13xy + 6y^2 - 8x - 7y + 2 = 0.$
9. $4x^2 + 4xy + y^2 + 4x - 3y + 4 = 0.$
10. $9x^2 - 12xy + 4y^2 - 20x - 30y - 50 = 0.$

ALGEBRAIC EQUATIONS

123. Parabolic type.—Equations of the form $y = ax^n$, where a is a constant and n is positive, are said to be of the **parabolic type.**

(1) When $n = 2$, $y = ax^2$. The locus is the ordinary parabola with its axis on the y-axis, and has already been discussed.

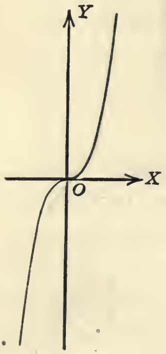


Fig. 109.

(2) When $n = 3$, $y = ax^3$. The locus is called the **cubical parabola.** It has the form shown in Fig. 109, for $a = 1$.

Discussion.—When $x = 0$, $y = 0$, and the curve passes through the origin. It is not symmetrical with respect to either coördinate axis, but is symmetrical with respect to the origin. Why?

For any positive value of x, y is positive; and for any negative value of x, y is negative. Hence the curve lies wholly in the first and third quadrants.

This information together with a few points makes it possible to sketch the curve with considerable accuracy.

(3) When $n = \frac{3}{2}$, $y = ax^{\frac{3}{2}}$. The locus is called the **semi-cubical parabola.** It has the form shown in Fig. 110, for $a = 1$.

Discussion.—When $x = 0$, $y = 0$, and the curve passes through the origin. Writing $y = x^{\frac{3}{2}}$ in the form $y^2 = x^3$, it is seen that the curve is symmetrical with respect to the x-axis.

For any positive value of x, y has two values numerically equal but opposite in sign. For any negative value of x, y is imaginary. Hence the curve lies wholly in the first and fourth quadrants.

124. Hyperbolic type.—Equations of the form $y = ax^n$, where a is a constant and n is negative, are said to be of the hyperbolic type.

(1) When $n = -1$, $y = ax^{-1}$, or $xy = a$. The locus is the ordinary equilateral hyperbola lying in the first and third quadrants.

(2) When $n = -2$, $y = ax^{-2}$, or $x^2y = a$. The locus has the form shown in Fig. 111, for $a = 1$.

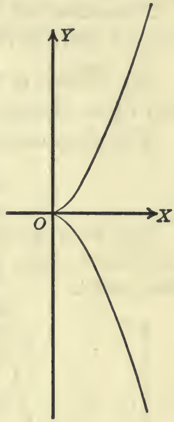


FIG. 110.

Discussion.—No finite value of x will make $y = 0$, and no finite value of y will make $x = 0$. Hence the curve does not meet either of the coördinate axes.

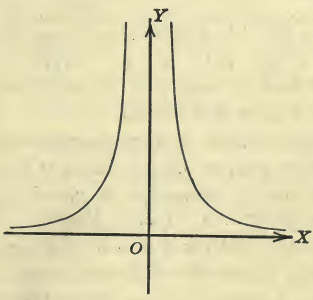


FIG. 111.

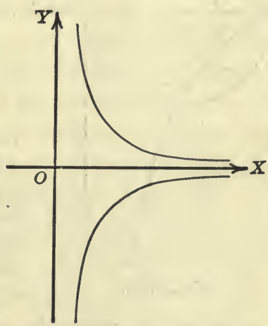


FIG. 112.

Since x is affected only by an even exponent and y only by an odd exponent, the curve is symmetric only to the y-axis.

For all positive finite values of y, x has two finite values equal numerically but opposite in sign. For all negative values of y, x is imaginary. As y becomes large positively, x approaches zero both from the positive and the negative side. As x becomes large either positively or negatively, y approaches zero from the positive side.

The curve lies in the first and second quadrants, and is asymptotic to both coördinate axes.

(3) When $n = -\frac{3}{2}$, $y = ax^{-\frac{3}{2}}$, or $x^{\frac{3}{2}}y = a$. The locus has the form shown in Fig. 112, for $a = 1$.

The discussion is left as an exercise.

EXERCISES

Plot each group of the following equations upon the same set of coördinate axes, by first discussing the equation and then finding a few points.

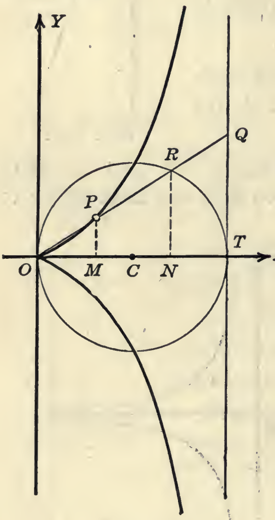


Fig. 113.

1. (1) $y = x^2$, (2) $y = x^4$, (3) $y = x^6$.
2. (1) $y = x$, (2) $y = x^3$, (3) $y = x^5$.
3. (1) $y = x^{\frac{1}{2}}$, (2) $y = x^{\frac{3}{2}}$, (3) $y = x^{\frac{5}{2}}$.
4. (1) $y = x^{\frac{1}{3}}$, (2) $y = x^{\frac{2}{3}}$.
5. (1) $y = x^{-1}$, (2) $y = x^{-3}$.
6. (1) $y = x^{-2}$, (2) $y = x^{-4}$.
7. (1) $y = x^{-\frac{1}{2}}$, (2) $y = x^{-\frac{3}{2}}$.
8. (1) $y = x^{-\frac{1}{3}}$, (2) $y = x^{-\frac{2}{3}}$.

9. If p is the pressure and t the absolute temperature of a gas in adiabatic expansion, $p = kt^{\frac{\gamma}{\gamma - 1}}$, where k is a constant and $\gamma = 1.41$ for air. If $p = 2700$ when $t = 300$, find k, and plot the equation for values of t from 200 to 400.

10. In a mixture in a gas engine expanding without gain or loss of heat, it is found that the law of expansion is given by the equation $pv^{1.37} = c$. Given that $p = 188.2$ when $v = 11$, find the value of the constant c, and plot the curve of the equation using this value of the constant. Consider values of v from 10 to 25.

125. The cissoid of Diocles.—In Fig. 113, OT is the diameter of a fixed circle. At T a tangent is drawn, while about O a secant revolves meeting the tangent in Q and the circle in R. The point P on the line OQ is taken so that $OP = RQ$. The locus of the point P is the **cissoid of Diocles.**

To derive the equation of the cissoid, choose O as origin and the x-axis along OT.

Draw MP and NR perpendicular to OT.

Denote the coördinates of P by (x, y).

Let the radius of the circle be a.

From the definition of the cissoid, $OP = RQ$.

And evidently $OM = NT$.

But $OM = x$, hence $NT = x$ and $ON = 2a - x$.

Also NR is a mean proportional between ON and NT.

Hence $NR = \sqrt{x(2a - x)}$.

By similar triangles, $OM : ON = MP : NR$.

Substituting values, $x : 2a - x = y : \sqrt{x(2a - x)}$.

From this $y^2 = \dfrac{x^3}{2a - x}$, the equation required.

The curve may be plotted from the definition given above, or from the equation.

Note.—By means of the cissoid the problem of the duplication of the cube can be solved. This problem, to find a cube that is double a given cube, was one of the famous problems of antiquity.

126. Other algebraic equations.—An unlimited number of definitions of loci could be given that would result in algebraic equations. There are many such curves that are of more or less historical importance as well as of value in mathematics and other sciences. Also there are an unlimited number of algebraic equations that may be discussed and their curves plotted. It should be remembered that an algebraic equation as truly defines a locus in terms of rectangular coördinates, as does the definition of the preceding article define the cissoid.

Example 1.—Discuss and plot the equation $y = \dfrac{8a^3}{x^2 + 4a^2}$.

Intercepts.—When $x = 0$, $y = 2a$. The curve does not meet the x-axis, since no finite value of x will make $y = 0$.

Symmetry.—Since only an even power of x occurs the curve is symmetrical with respect to the y-axis.

Extent.—If a is a positive number, y is positive for all real values of x, and the largest value of y is $2a$ when $x = 0$. As x becomes very large in absolute value, y becomes very small but always positive. Hence the curve is asymptotic to the x-axis in both directions.

Or, solving for x, $x = \pm 2a\sqrt{\frac{2a - y}{y}}$.

Hence x is imaginary when $\frac{2a - y}{y} < 0$.

This is true when $y < 0$ or when $y > 2a$.

Hence the curve lies in the first and second quadrants, is symmetrical with respect to the y-axis, and lies between the x-axis and the line $y = 2a$.

Points on the curve and in the first quadrant can be found by choosing positive values for x.

x	0	a	$2a$	$3a$	$4a$	$6a$
y	$2a$	$\frac{8}{5}a$	a	$\frac{8}{13}a$	$\frac{2}{5}a$	$\frac{1}{5}a$

The curve is as shown in Fig. 114, and is known as the **witch of Agnesi.**

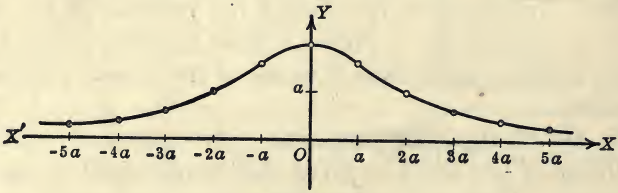


FIG. 114.

EXERCISES

Discuss the equations and plot the curves in exercises 1–16.

1. $y = \frac{8}{x - 2}$.

2. $y = \frac{x + 3}{x - 3}$.

3. $y = x(x - 1)(x - 2)$.

4. $y^2 = (x - 1)(x - 4)(x - 6)$.

5. $y = \frac{x + 3}{(x - 2)(x + 1)}$.

6. $y = \frac{(x - 4)(x + 3)}{(x + 1)(x - 2)}$.

7. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

9. $x^2y^2 = (y + 2)^2(16 - y^2)$. (Example of Conchoid of Nichomedes).

10. $xy^2 = (x - a)^2(2a - x)$.

11. $9y^2 = (x + 7)(x + 4)^2$.

12. $y = x^3 + x - 3$.

13. $y = x^3 + 6x^2 + 10x - 2$.

14. $y = x^4 - 10x^2 - 4x + 8$.

15. $x = \frac{a^3}{y^2 + a^2}$.

16. $(\frac{1}{4}x)^2 + (\frac{1}{3}y)^{\frac{2}{3}} = 1$.

17. Two fixed points F' and F are $2a$ units apart. Choose the origin at the center of the line joining F' and F, and the x-axis along this line. Find the equations of the loci of the point $P(x, y)$ when

(1) $\frac{FP}{F'P}$ = a constant not unity,

(2) $FP + F'P$ = a constant,

(3) $FP - F'P$ = a constant,

(4) $FP \times F'P$ = a constant, k.

In (4) the locus is called a **Cassinian oval,** and its equation is $(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = k^2 - a^4$.

18. Sketch the loci of (4) of the preceding exercise when $a = 1$, and k has successively the values 0, 1, and 2.

19. Write the equation of (4) of exercise 17 when $k = a^2$, and plot the curve. This curve is called a **lemniscate.**

20. Express the lemniscate in polar coördinates, using the positive part of the x-axis as polar axis.

21. A uniform beam of length l, fixed in position by being held at one end, supports a weight at the other end. The deflection y at any distance x from the fixed end is given by the equation $y = k(\frac{1}{2}lx^2 - \frac{1}{6}x^3)$. Find k for a beam 12 ft. long if the weight deflects the outer end 18 in., and plot a curve showing the shape of the beam for its entire length. Choose the fixed end as the origin and consider y positive when measured downward.

TRANSCENDENTAL EQUATIONS

127. Exponential equations.—An equation of the form $y = b^x$, where b is any positive constant, is called an **exponential equation.** If the exponent is fractional and involves even roots of b, only the positive values of these roots are used.

Example 1.—Discuss the equation $y = b^x$ when $b > 1$. Plot the curve when $b = 1.5$.

Intercepts.—When $x = 0$, $y = b^0 = 1$. This shows that the curve passes through the point (0, 1) for *any* value of b.

If $y = 0$, $b^x = 0$, which is impossible for any finite value of x. This shows that the curve neither meets nor crosses the x-axis. However, for sufficiently large negative values of x, the value of b^x can be made to become as near zero as desired. The curve is then asymptotic to the x-axis in the negative direction.

Symmetry.—Since changing x to $-x$ or y to $-y$ changes the equation, the curve is not symmetrical with respect to either coördinate axis.

Extent.—Since no integral value of x can make y negative, and since only positive values of b^x are to be taken when x is a fraction, the curve is wholly above the x-axis.

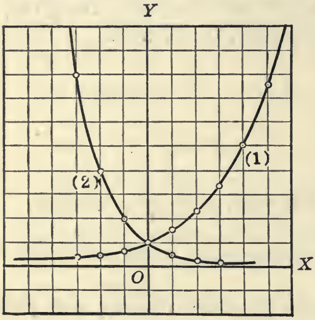


FIG. 115.

Further, since y is not imaginary for any value of x, and increases as x increases, the curve lies in the first and second quadrants, exists for all values of x, and continually rises from left to right.

Plotting.—The curve of $y = 1.5^x$ can be plotted as accurately as desired by finding points. Taking logarithms of both sides of the equation,

$$\log y = x \log 1.5 = 0.1761x.$$

The following points are readily found, and the curve is as shown in (1) of Fig. 115.

x	-3	-2	-1	0	1	2	3	4	5
$\log y$	$\bar{1}.4717$	$\bar{1}.6478$	$\bar{1}.8239$	0	0.1761	0.3522	0.5283	0.7044	0.8805
y	0.296	0.444	0.667	1	1.5	2.25	3.375	5.063	7.595

Example 2.—Discuss the equation $y = b^x$ when $b<1$. Plot the curve when $b = \frac{1}{2}$.

The discussion is similar to that of example 1. It is to be noted that y decreases as x increases, and the curve is asymptotic to the x-axis in the positive direction.

Plotting.—Points for plotting $y = (\frac{1}{2})^x$ are found and the curve is as shown in (2) of Fig. 115.

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	32	16	8	4	2	1	0.5	0.25	0.125	0.0625

128. Applications.—The most important case of the exponential equation is the case where the base is e, which is the base of the natural system of logarithms and equals $2.71828 \cdots$. It usually occurs in the form $y = ae^{kx}$, where

a and k are constants that may be determined in particular applications. This function is often called the "law of organic growth," or the "compound interest law," and is a function where the rate of increase or decrease at any instant is directly proportional to the value of the function at that instant. Just what the applications are cannot well be shown here, but the following uses are suggestive:

(1) To express the pressure of the atmosphere at any height.

(2) In physics and electricity, it is used in considering damped vibrations.

(3) In medicine and surgery, to express the progress of the healing of a wound.

(4) In biology, to determine the number of bacteria in a culture at any given time.

(5) In chemistry, to express the progress of a chemical action.

(6) In mechanics, in connection with the slipping of a belt on a pulley.

Numerous applications will be discovered by the student as he progresses in his studies.

Because of its frequent occurrence in problems involving conditions in nature, the base e is sometimes called "a constant of nature."

129. Logarithmic equations.—The logarithmic equation is of the form $y = \log_b x$, where b is a positive number different from 1.

By the definition of the logarithm of a number, the equation $y = \log_b x$ can be written in the exponential form $x = b^y$. This is the same as the equation of article 127 with x and y interchanged.

It is evident then that the discussion of the logarithmic equation $y = \log_b x$ follows that of the exponential equation $x = b^y$, and gives the following when $b > 1$:

The x-intercept is at the point (1, 0).

There is no y-intercept for as x approaches 0, y becomes $-\infty$, that is, the curve is asymptotic to the y-axis in the negative direction.

The curve is not symmetrical with respect to either axis. When $x > 1$, $y > 0$, and as x becomes ∞, y becomes ∞ also. When $x < 1$, $y < 0$, and there is no value of y which will make x negative.

Example 1.—Plot the curve of $y = \log_{10} x$.

The following points are found, and the curve is (1) of Fig. 116. The unit on the y-axis is taken twice that on the x-axis.

x	0.001	0.01	0.1	0.5	1	2	3	4	5	7	10	15	50	100
y	−3	−2	−1	−0.301	0	0.301	0.477	0.602	0.699	0.845	1	1.176	1.699	2

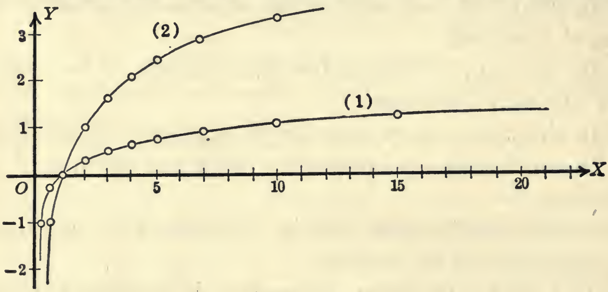


FIG. 116.

Example 2.—Plot the curve of $y = \log_2 x$.

The points can be readily found from the relation

$$\log_2 x = \frac{1}{\log_{10} 2} \times \log_{10} x = 3.322 \log_{10} x.$$

The curve is (2) of Fig. 116.

x	0.001	0.01	0.1	0.5	1	2	3	4	5	7	10	15	100
y	−9.97	−6.64	−3.32	−1	0	1	1.58	2	2.32	2.81	3.32	3.91	6.64

EXERCISES

1. Plot the following exponential equations:

(1) $y = 2^x$. (2) $y = 3^x$. (3) $y = e^x$, where $e = 2.718$.

(4) $y = (0.75)^x$. (5) $y = (0.4)^x$.

2. Discuss the effect upon the curve of $y = b^x$ when $b < 1$ and increases to 1.

3. Discuss the effect upon the curve of $y = b^x$ when $b > 1$ and increases from 1.

4. Plot the curves of the following:

(1) $y = 1^{-x}$. (2) $y = 2^{-x}$. (3) $y = x^x$, $x > 0$. (4) $y = e^{-x}$.

5. Plot the curves of the following:

(1) $y = \log_e x$, where $e = 2.718$. (2) $y = \log_4 x$. (3) $y = 2 \log_3 x$.

6. Discuss and plot the curve of $y = \dfrac{e^x + e^{-x}}{2}$.

Suggestion.—First plot $y_1 = \frac{1}{2}e^x$ and $y_2 = \frac{1}{2}e^{-x}$. Then plot $y = \dfrac{e^x + e^{-x}}{2}$ by adding the ordinates y_1 and y_2 to find y for the dfferent values of x.

7. Discuss and plot the curve of $y = \frac{1}{2}a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$. This is the equation of the catenary, the curve assumed by a flexible cord suspended between two points.

8. A wire, weighing 0.2 lb. per foot, is suspended from two points in a horizontal line 50 ft. apart. The horizontal tension at each end is 10 lb. Plot the catenary formed by the wire. The constant a in the formula, $y = \frac{1}{2}a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, is found by dividing the horizontal tension by the weight per unit length of the wire.

9. Plot the curve of $i = be^{-at}$, where i and t are the variables. Choose $b = 1.5$ and $a = 0.4$.

10. If a body is heated to a temperature T_1 above the surrounding bodies, and suspended in air, its excess of temperature T above the surrounding bodies at any time, t seconds thereafter, is given by Newton's law of cooling expressed by the equation $T = T_1e^{-at}$, where a is a constant that can be determined by experiment. Given $T_1 = 20$ and $a = 0.014$, plot a curve showing the temperature at any time t up to 100 seconds.

11. The dying away of the current on the sudden removal of the electro-motive force from a circuit containing resistance and self-induction, is expressed by the equation, $i = Ie^{-\frac{Rt}{L}}$, where i is the current at any time, t seconds, after the e.m.f. is removed, R is the resistance, and L the coefficient of self-induction. Plot a curve to show the current at any time from $t = 0$ to $t = 0.2$, if $I = 10$ amperes, $R = 0.1$ ohm, and $L = 0.01$ henry.

TRIGONOMETRIC EQUATIONS

130. The sine curve.—Discuss the equation $y = \sin x$, and plot the curve.

Intercepts.—When $x = 0$, $y = 0$. Hence the curve passes through the origin. When $y = 0$, $\sin x = 0$, and $x = n\pi$ radians, where n is any integer either positive or negative.

Symmetry.—Putting $-y$ for y or $-x$ for x, changes the equation. Hence the curve is not symmetrical with respect to either axis. But putting $-y$ for y and $-x$ for x, does not change the equation. Hence the curve is symmetrical with respect to the origin.

Extent.—Since there is a sine of any angle, the curve extends indefinitely in both the positive and negative directions.

Since the sine of an angle is not greater than 1 nor less than -1, the curve does not extend above the line $y = 1$ nor below the line $y = -1$.

Plotting.—Any length can be chosen as a unit on the coördinate axes. What may be called the **proper sine curve** is

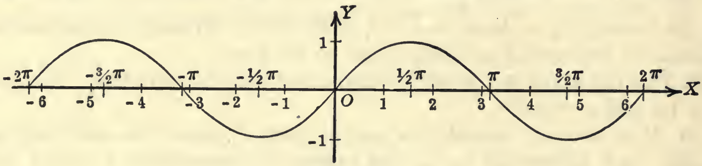


FIG. 117.

plotted by choosing as a unit on the y-axis the same length that is chosen to represent one radian on the x-axis. The curve is shown in Fig. 117.

x	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{5}{4}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{7}{4}\pi$	$\frac{11}{6}\pi$	2π
y	0	.5	.707	.866	1	.866	.707	.5	0	−.5	−.707	−.866	−1	−.866	−.707	−.5	0

From 2π radians to 4π radians or from -2π radians to 0, these values repeat. They also repeat for each interval of 2π radians in both directions.

131. Periodic functions.—A curve that repeats in form as illustrated by the sine curve is called a **periodic curve.** The function that gives rise to a periodic curve is called a **periodic function.** The least repeating part of a periodic curve is called a **cycle** of the curve. The change in the value of the variable necessary for a **cycle** is called the period of the function. The greatest absolute value of the ordinates of a periodic function is called the **amplitude** of the function.

In engineering and other practical applications of mathematics, there are many phenomena that repeat. It is for this reason that the periodic functions are of great importance. By a suitable choice of periodic functions almost any periodic phenomenon can be represented by a function.

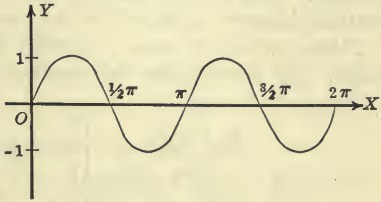


Fig. 118.

132. Period and amplitude of a function.

Example 1.—Find the period of sin nx, and plot $y = \sin 2x$.

Since, in finding the value of sin nx, the angle x is multiplied by n before finding the sine, the period is $\frac{2\pi}{n}$.

The curve for $y = \sin 2x$ is shown in Fig. 118. The period of the function is π radians, and there are two cycles of the curve in 2π radians.

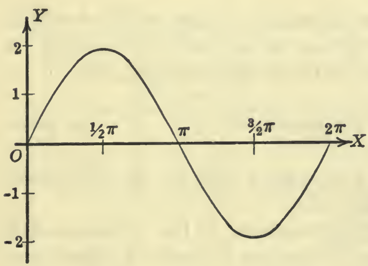


Fig. 119.

Definition.—The number n in sin nx is called the **periodicity factor.**

Example 2.—Find the amplitude of $b \sin x$, and plot $y = 2 \sin x$.

Since, in finding the value of $b \sin x$, $\sin x$ is found and then multiplied by b, the amplitude of the function is b, for the greatest value of $\sin x$ is 1.

The curve for $y = 2 \sin x$ is shown in Fig. 119. The amplitude is 2.

Definition.—The number b in $b \sin x$ is sometimes called the **amplitude factor.**

By a proper choice of a periodicity factor and an amplitude factor a function of any amplitude and any period desired can be found.

133. Projection of a point having uniform circular motion. Simple harmonic motion.

Example 1.—A point P, Fig. 120, moves around a vertical circle of radius 3 inches in a counter-clockwise direction. It starts with the point at A and moves with an angular velocity of 1 revolution in 10 seconds. Plot a curve showing the distance the projection of P on the vertical diameter is from O at any time t, and find its equation.

Plotting.—Let OP be any position of the radius drawn to the moving point. OP starts from the position OA and at the end of 1 second

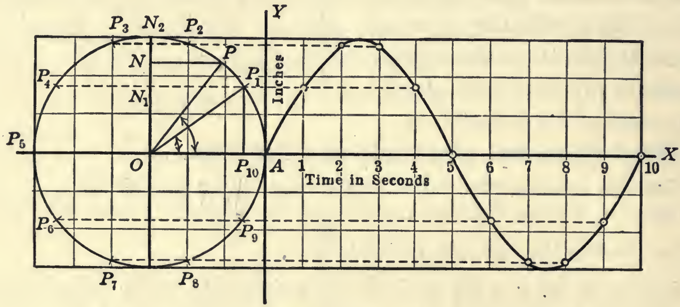


Fig. 120.

is in the position OP_1, having turned through an angle of $36° = 0.6283$ radians. At the end of 2 seconds it has turned to OP_2, through an angle of $72° = 1.2566$ radians, and so on to the positions $OP_3, OP_4, \cdots, OP_{10}$.

The points $N_1, N_2, \cdots$ are the projections of $P_1, P_2, \cdots$ respectively, on the vertical diameter.

Produce the horizontal diameter OA through A, and lay off the seconds on this to some scale, taking the origin at A.

For each second plot a point whose ordinate is the corresponding distance of N from O. These points determine a curve of which any ordinate y is the distance from the center O of the projection of P upon the vertical diameter at the time t represented by the abscissa of the point.

It is evident that for the second and each successive revolution, the curve repeats, that is, it is a periodic curve.

Since the radius OP turns through 0.6283 radians per second, angle $AOP = 0.6283t$ radians, and $ON = OP \cdot \sin 0.6283t$. Or $y = 3 \sin 0.6283t$, the equation of the curve.

In general, then, it is readily seen that if a straight line of length r starts in a horizontal position when time, $t = 0$, and revolves in a vertical plane around one end at a uniform angular velocity ω per unit of time, the projection y of the moving end upon a vertical straight line has a motion represented by the equation

$$y = r \sin \omega t.$$

Similarly, the projection of the moving point upon the horizontal is given by the ordinates of the curve whose equation is

$$y = r \cos \omega t.$$

The motion of the point N is a **simple harmonic motion.** If the time is counted from some other instant than that from which the above is counted, then the motion is represented by

$$y = r \sin (\omega t + \alpha),$$

where α is the angle that OP makes with the line OA at the instant from which t is counted. As an illustration of this consider the following:

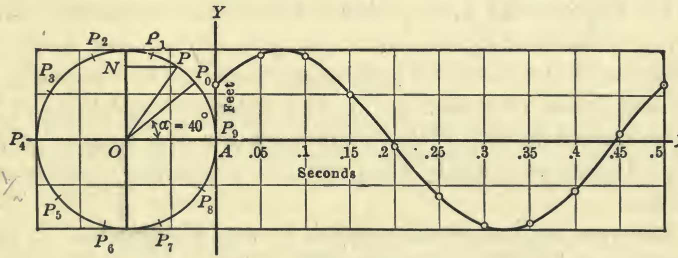


FIG. 121.

Example 2.—A crank OP, Fig. 121, of length 2 ft. starts from a position making an angle $\alpha = 40° = \frac{2}{9}\pi$ radians with the horizontal line OA when $t = 0$. It rotates in the positive direction at the rate of 2 revolutions per second. Plot the curve showing the projection of P upon a vertical diameter, and write the equation.

Plotting.—The axes are chosen as before, and points are found for each 0.05 second. The curve is as shown in Fig. 121.

The equation is $y = 2 \sin (4\pi t + \frac{2}{9}\pi)$.

Definitions.—The number of cycles of a periodic curve in a unit of time is called the **frequency.**

It is evident that

$$f = \frac{1}{T},$$

where f is the frequency and T is the period.

In $$y = r \sin(\omega t + \alpha),\ f = \frac{\omega}{2\pi} \text{ and } T = \frac{2\pi}{\omega}.$$

The angle α is called the **angle of lag.**

134. Other applications of periodic functions.—The illustrations already given are by no means the only uses of periodic functions. Many uses occur in connection with sound, light, and electricity. Periodic curves are traced mechanically on smoked glass in experiments in sound and electricity. Such curves are also traced by instruments for recording heartbeats, breathing movements, and tides.

Any periodic motion can be represented exactly, or can be closely approximated, by functions involving sines and cosines.

135. Exponential and periodic functions combined.—The curve represented by the equation, $y = be^{ax} \sin(nx + \alpha)$, is important in the theory of alternating currents, in representing the oscillations of a stiff spring, the damped oscillations of a galvanometer needle, or the oscillations of a disk suspended in a liquid, such as is used to compare the viscosities of different liquids.

The curve is most readily plotted by first plotting the curves represented by the exponential function and the periodic function separately on the same set of axes, and then finding the ordinates for various values of x by multiplying together the ordinates for these values of x in the exponential and periodic functions.

It will be noted that the curve is periodic, and that the amplitude of the successive waves gets less and less while the wave length remains the same.

Example.—Plot the curve showing the values of y for any value of x from $x = -\frac{7}{8}\pi$ to $x = 2\pi$ for the equation, $y = e^{-0.5x} \sin (2x + \frac{1}{4}\pi)$.

The curve is readily plotted by first plotting $y_1 = e^{-0.05x}$ and $y_2 = \sin (2x + \frac{1}{4}\pi)$, and then finding various values of y from the relation $y = y_1 y_2$. In Fig. 122, (1) is the exponential curve, (2) the sine curve, and (3) the final curve. Note that (3) and (2) intersect the x-axis at the same points.

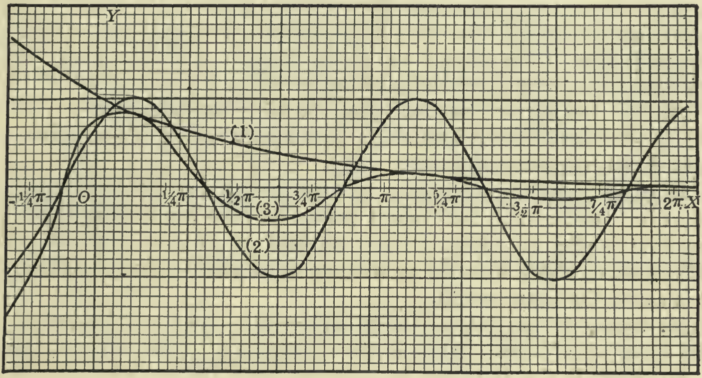


Fig. 122.

EXERCISES

1. Plot $y = \sin x$, using several different lengths on the x-axis as units.

2. Discuss and plot $y = \cos x$. Give its period.

3. Discuss, and plot $y = \tan x$, and $y = \cot x$ on the same set of axes. Give the period of each.

4. Plot $y = \sin x + \cos x$.

Suggestion.—Plot $y_1 = \sin x$ and $y_2 = \cos x$ on the same set of axes. Then find y from $y = y_1 + y_2$, by adding the ordinates for various values of x.

5. Plot $y = \sin^2 x$ and $y = \cos^2 x$ on the same set of axes.

6. Plot $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

7. Plot $y = \sec x$ and $y = \csc x$, and give the period of each.

8. Plot $y = \sin \frac{1}{2}x$, $y = \sin x$, $y = \sin 2x$, and $y = \sin \frac{3}{2}x$ on the same set of axes.

9. Plot $y = \frac{1}{2} \sin x$, $y = \sin x$, $y = 2 \sin x$, and $y = \frac{3}{2} \sin x$ on the same set of axes.

10. Plot $y = \sin 2x + 2 \cos x$, and give the period.

11. Plot $y = \sin x + x$. Is this periodic?

12. A crank 18 in. long starts from a horizontal position and rotates in the positive direction in a vertical plane at the rate of $\frac{1}{4}\pi$ radians per second. The projection of the moving end of the crank upon a vertical line oscillates with a simple harmonic motion. Construct a curve that represents this motion, and write its equation.

13. A crank 8 in. long starts from a position making an angle of 55° with the horizontal, and rotates in a vertical plane in the positive direction at the rate of one revolution in 3 seconds. Construct a curve showing the projection of the moving end of the crank in a vertical line. Write the equation of the curve and give the period and the frequency.

14. Plot the curves that represent the following motions: (1) $y = 12 \sin (1.88t + 0.44)$, (2) $y = 2.5 \sin (\frac{1}{8}\pi t + \frac{1}{12}\pi)$. Give the period and frequency of each.

15. Plot $y = r \sin \frac{1}{2}\pi t$ and $y = r \sin (\frac{1}{2}\pi t + \frac{1}{4}\pi)$ on the same set of axes. Notice that the highest points on each are separated by the constant angle $\frac{1}{4}\pi$. Such curves are said to be out of phase. The difference in phase is stated in time or as an angle. In the latter case it is called the **phase angle.**

16. Plot $y = r \sin \frac{1}{4}\pi t$, $y = r \sin (\frac{1}{4}\pi t - \frac{1}{4}\pi)$, and $y = r \cos \frac{1}{4}\pi t$ all on the same set of axes. What is the difference in phase between these?

17. What is the difference in phase between the curves of $y = \sin x$ and $y = \cos x$? Between $y = \cos x$ and $y = \sin (x + \frac{1}{2}\pi)$?

18. Plot the curve $y = e^{-x} \sin x$ for values of x from 0 to 2π.

19. Plot the curve $i = e^{-\frac{1}{3}t} \sin (2t + \frac{1}{2}\pi)$ for values of t from -2 to 8.

20. In an oscillatory discharge of a condenser under certain conditions, the charge q at any time t is represented by the equation, $q = 0.00224e^{-4000t} \sin (8000t + \tan^{-1} 2)$, where q is in coulombs and t in seconds. Plot the curve showing values of q for values of t from 0 to 0.0012 second. What is the period?

Suggestion.—Choose 0.0001 second as a unit on the t-axis, and 0.001 coulomb as a unit on the q-axis; and let the length representing a unit on the q-axis be about twice that for the unit on the t-axis. Plot the exponential curve first, and then the sine curve choosing as a unit on the q-axis the length representing 0.001 coulomb.

EQUATIONS IN POLAR COÖRDINATES

136. Discussion of the equation.—As in the case of equations in rectangular coördinates, in polar coördinates the discussion of an equation helps greatly in learning the properties of the curve. The discussion is exactly similar to that in rectangular coördinates.

(1) *Intercepts.*—(*a*) The intercepts on the polar axis are found by putting $\theta = 0°, 180°, 360°, \cdots n180°$. (*b*) The intercepts on the 90°-line are found by putting $\theta = 90°, 270°$, etc. (*c*) Putting $\rho = 0$ and solving for θ, gives the values of θ for which the curve passes through the pole.

(2) *Symmetry.*—(*a*) If the form of the equation does not change when $-\rho$ is substituted for ρ, the curve is symmetrical with respect to the pole. (*b*) If it does not change when $-\theta$ is substituted for θ, the curve is symmetrical with respect to the polar axis. (*c*) If it does not change when $\pi - \theta$ is substituted for θ, the curve is symmetrical with respect to the 90°-line.

Show why each of these is true. Are their converses true? (See **Art. 138.**)

(3) *Extent.*—If the equation is solved for ρ in terms of θ, the following can be determined: (*a*) Values of θ for which ρ has maximum or minimum values. In general this can be done readily when trigonometric functions are involved. (*b*) Values of θ for which ρ becomes infinite. These values determine the direction in which the curve extends to infinity. (*c*) Values of θ for which ρ is imaginary, that is, for which there is no curve.

137. Loci of polar equations.—Since some of the conditions of the previous article are sufficient but not necessary, care must be taken in determining symmetry and extent of curves. On the whole, however, the plotting is very similar to that in rectangular coördinates, and is best illustrated by examples. It will be found convenient to use polar coördinate paper.

Example 1.—Discuss and plot $\rho = 1 + 2 \sin \theta$.

Discussion.—(1) Intercepts on polar axis, $\theta = 0$, $\rho = 1$; $\theta = 180°$, $\rho = 1$. Intercepts on 90°-line, $\theta = 90°$, $\rho = 3$; $\theta = 270°$, $\rho = -1$.

When $\rho = 0$, $\sin \theta = -\frac{1}{2}$, and $\theta = 210°$ or $330°$.

(2) Condition for symmetry with respect to the polar axis does not hold; but the curve is symmetrical with respect to the 90°-line since $\sin(\pi - \theta) = \sin\theta$.

(3) Since $\rho = 1 + 2\sin\theta$, the maximum value of ρ will occur when $\sin\theta = 1$, or $\theta = 90°$; and the minimum value of ρ will occur when $\sin\theta = -1$, $\theta = 270°$. No value of θ makes ρ imaginary.

Plotting.—On account of the symmetry it is only necessary to find points for values of θ from 0° to 90° and from 270° to 360°. The curve is shown in Fig. 123.

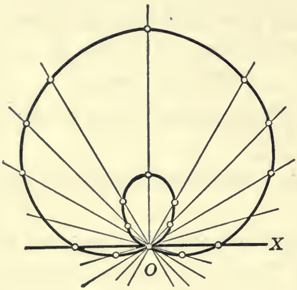


FIG. 123.

θ	$\sin\theta$	ρ
0°	0.0	1.00
30°	0.50	2.00
45°	0.707	2.41
60°	0.87	2.73
90°	1.00	3.00
270°	−1.00	−1.00
300°	−0.87	−0.73
315°	−0.707	−0.41
330°	−0.50	0.0
345°	−0.26	0.48
360°	0.0	1.00

Example 2.—Discuss and plot $\rho = a\cos 2\theta$. The four-leafed rose.

Discussion.—(1) Intercepts on polar axis, $\theta = 0$, $\rho = a$; $\theta = 180°$, $\rho = a$. Intercepts on 90°-line, $\theta = 90°$, $\rho = -a$; $\theta = 270°$, $\rho = -a$. When $\rho = 0$, $\theta = 45°, 135°, 225°, 315°$.

(2) Symmetrical with respect to the polar axis, and the 90°-line.

(3) Since $\rho = a\cos 2\theta$, the maximum values of ρ occur when $\cos 2\theta = 1$, or when $\theta = 0°$ and 180°. The minimum values occur when $\theta = 90°$ and 270°.

Plotting.—On account of symmetry find points for values of θ in the first quadrant. The curve is as shown in Fig. 124. The arrow heads indicate direction in which the curve is traced as θ increases from 0° to 360°.

θ	$\cos 2\theta$	ρ
0°	1.0	a
10°	0.94	$0.94a$
20°	0.77	$0.77a$
30°	0.50	$0.50a$
40°	0.17	$0.17a$
45°	0.0	0.0
50°	−0.17	$-0.17a$
60°	−0.50	$-0.50a$
70°	−0.77	$-0.77a$
80°	−0.94	$-0.94a$
90°	−1.0	$-a$

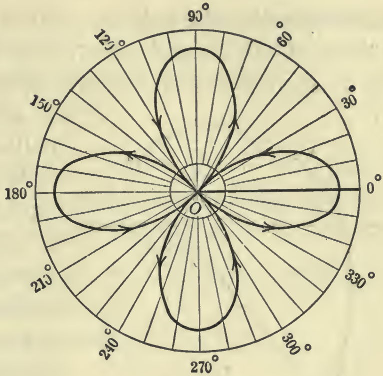


FIG. 124.

138. Remarks on loci of polar equations.—By definition, the locus of an equation requires that, (1) if the coördinates of any point satisfy the equation, the point is on the locus; (2) if any point is on the locus, the coördinates of this point satisfy the equation.

In rectangular coördinates, there is no trouble in seeing that these conditions are fulfilled. This is because, in rectangular coördinates, there is one and only one point for every pair of coördinates; and, conversely, to every point there is just one pair of coördinates.

In polar coördinates, trouble may arise since there is an ambiguity because a point has an indefinite number of pairs of coördinates determining it.

Thus, in example 2 of the preceding article, it is seen that the point determined by the pairs of coördinates $(\frac{1}{2}a, 60°)$, $(-\frac{1}{2}a, 240°)$, $(-\frac{1}{2}a, -120°)$, and $(\frac{1}{2}a, -300°)$ is on the locus; but only $(-\frac{1}{2}a, 240°)$ and $(-\frac{1}{2}a, -120°)$ satisfy the equation.

For a like reason a curve may be symmetrical with respect to the polar axis even though $(\rho, -\theta)$ when substituted for (ρ, θ) changes the equation. In this case, some of the other pairs of coördinates of the point $(\rho, -\theta)$ would not change the equation.

139. Spirals.—*Definition.*—The locus of a point that revolves about a fixed point, and, at the same time, recedes from or approaches this point according to some law, is called a **spiral.** The fixed point is called the **center** of the spiral.

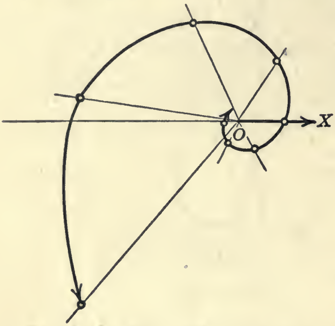


FIG. 125.

When an angle is used in an equation and is not involved in a trigonometric function it is considered to be expressed in radians.

Example.—Discuss and plot the equation $\rho = a^\theta$, $\rho = a > 1$. This is the logarithmic spiral.

Discussion.—(1) When $\theta = 0$, $\rho = 1$. (2) There is no symmetry. (3) As θ increases toward $+\infty$, ρ increases toward $+\infty$. As θ decreases toward $-\infty$, ρ approaches 0.

Plotting.—The curve is readily plotted from a series of points. For $a = 1.5$ it is as shown in Fig. 125.

θ	-3	-2	-1	0	1	2	3	4	5	6
ρ	0.296	0.444	0.667	1	1.5	2.25	3.38	5.06	7.59	11.39

140. Polar equation of a locus.—The equation of a locus may often be found with greater ease in polar than in rectangular coördinates. The method is similar to that for finding the equation in rectangular coördinates, and has already been applied to the straight line and the conic sections. (See **Arts. 69, 78, 91, 103, 117.**)

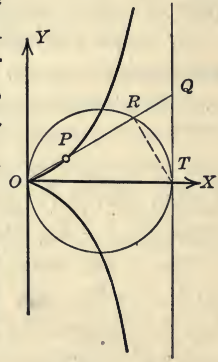


FIG. 126.

Example.—In Fig. 126, OT is the diameter of a fixed circle. At T a tangent is drawn, while about O a secant revolves meeting the tangent in Q and the circle in R. The point P on the line OQ is taken so that $OP = RQ$. Find the equation of the locus of P.

Solution.—Choose O as pole, OT as polar axis, and let $OT = 2a$. Let the polar coördinates of P be (ρ, θ).

Then since ORT is a right angle, $PQ = OR = 2a \cos \theta$.

Since OTQ is a right angle, $OQ = \dfrac{2a}{\cos \theta}$.

Hence $\rho = OP = OQ - PQ = \dfrac{2a}{\cos \theta} - 2a \cos \theta$, or $\rho = \dfrac{2a \sin^2 \theta}{\cos \theta}$.

This may be transformed to rectangular coördinates and obtain $y^2 = \dfrac{x^3}{2a - x}$, the equation of the cissoid of Diocles derived in article 125.

Compare the derivations of the equation by the two methods.

EXERCISES

Discuss and plot the following equations:

1. $\rho = \dfrac{2a}{1 - \cos \theta}$. A parabola.
2. $\rho \sin \theta \tan \theta = 4a$. A parabola.
3. $\rho^2 \cos 2\theta = a^2$. An equilateral hyperbola.
4. $\rho = 3 \cos \theta + 2$. Transform to rectangular coördinates.
5. $\rho = a \tan^2 \theta \sec \theta$. Semi-cubical parabola.
6. $\rho = a \cot^2 \theta \csc \theta$. Semi-cubical parabola.
7. Transform equations of exercises 5 and 6 to rectangular coördinates and compare with article 123.
8. $\rho = a - b \sin \theta$ when $a < b$, when $a = b$, and when $a > b$. Limaçons of Pascal.
9. $\rho^2 = a^2 \cos 3\theta$. Is the curve symmetrical with respect to the 90°-line? Does the test apply?

Sketch the following roses by first drawing the radial lines corresponding to values of θ which make $\rho = 0$, and for values of θ which make ρ maximum in numerical value; and then determining the changes in the values of ρ between these successive values of θ.

10. $\rho = a \sin 2\theta$. 11. $\rho = a \sin 3\theta$. 12. $\rho = a \sin 4\theta$.
13. $\rho = a \cos 3\theta$. 14. $\rho = a \cos 4\theta$. 15. $\rho = a \cos 5\theta$.

In plotting the curves of the following equations, it should be noted that in polar coördinates it is sometimes necessary to carry the angle beyond 360° in order to secure the complete locus.

16. $\rho = a \sin^3 \tfrac{1}{3}\theta$. 17. $\rho = a \sin \tfrac{1}{2}\theta$. 18. $\rho^2 \cos \theta = a^2 \sin 3\theta$.
19. $\rho = a(\sin 2\theta + \cos 2\theta)$. 20. $\rho^2 = a^2 \sin \tfrac{1}{2}\theta$.
21. $\rho = a(1 \pm \cos \theta)$. The cardioids.

Discuss and plot the following spirals:

22. $\rho\theta = a$. Hyperbolic or reciprocal spiral.
23. $\rho = a\theta$. Spiral of Archimedes.
24. $\rho^2 = a\theta$. Parabolic spiral.

25. $\rho^2\theta = a$. The lituus or trumpet.

26. Derive the equation of the locus of a point such that:

(1) Its radius vector is inversely proportional to its vectorial angle. *Ans.* The hyperbolic spiral.

(2) Its radius vector is directly proportional to its vectorial angle. *Ans.* The spiral of Archimedes.

(3) The square of its radius vector is directly proportional to its vectorial angle. *Ans.* The parabolic spiral.

(4) The square of its radius vector is inversely proportional to its vectorial angle. *Ans.* The lituus.

(5) The logarithm of its radius vector is directly proportional to its vectorial angle. *Ans.* The logarithmic spiral.

27. Find the equation of the locus of the midpoints of the chords of the circle $\rho = 2r \cos \theta$, and passing through the pole.

28. Chords of the circle $\rho = 2r \cos \theta$ and passing through the pole are extended a distance $2b$. Find the equation of the locus of the extremities.

PARAMETRIC EQUATIONS OF LOCI

141. Parametric equations.—When the coördinates of points on a locus are expressed separately as functions of a third variable, these equations are called the **parametric equations** of the locus.

The new variable introduced in finding the parametric equations is called a **parameter.**

The parameter may be introduced either for convenience or as a necessity, since in some cases it is easier to obtain the coördinates of points on a locus as functions of a third variable than it is to obtain a single equation connecting the coördinates of the points; and frequently two equations using the parameter can be obtained where it is not possible to obtain a single equation connecting the two variables.

As will be seen, the parameter can be chosen in a great variety of ways, but it is usually chosen because of some simple geometric relation, or it is the time during which the point tracing the curve has been in motion.

Example 1.—The parametric equations $x = x_1 + nt$ and $y = y_1 + mt$ represent the straight line which passes through the point (x_1, y_1) and has the slope $\frac{m}{n}$.

That this is so can be seen by assigning values to t and plotting the values of x and y, or by eliminating t and obtaining the equation

$$y - y_1 = \frac{m}{n}(x - x_1),$$

which is the equation of a straight line.

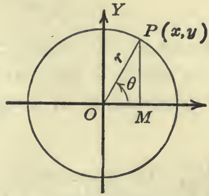


FIG. 127.

Example 2.—Consider a circle with center at the origin and radius r as generated by a point P starting on the x-axis and moving counter-clockwise. Then it is evident from Fig. 127 and the definitions of the sine and cosine, that the parametric equations

$$x = r \cos \theta \text{ and } y = r \sin \theta,$$

where θ is the angle generated by the radius to the point P, represent the circle.

Also, squaring and adding the equations, $x^2 + y^2 = r^2$.

Example 3.—The equations $x = t^2$ and $y = 2t$ are parametric equations of the parabola $y^2 = 4x$, as can be seen by eliminating t from the two equations. The curve can be plotted by assigning values to t and computing the corresponding values of x and y.

t	-4	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4
x	16	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9	16
y	-8	-6	-4	-2	-1	0	1	2	4	6	8

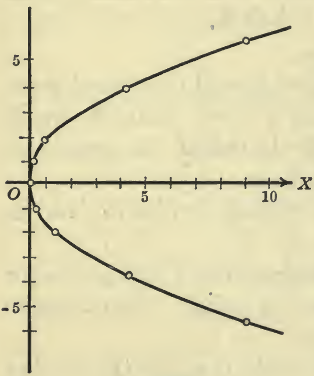


FIG. 128.

The values of x and y are plotted, and the curve is as shown in Fig. 128. It is observed that as t varies from $-\infty$ to $+\infty$ the corresponding point will trace out the curve, coming from ∞ on the lower half and going to ∞ on the upper half of the parabola.

Example 4.—The equations $x = a \cos \theta$ and $y = b \sin \theta$ are parametric equations of the ellipse as is shown in article 104.

Example 5.—The equations $x = a \sec \theta$ and $y = b \tan \theta$ are parametric equations of the hyperbola; for dividing the first by a, the second by b, squaring, and subtracting the second from the first gives

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 \theta - \tan^2 \theta = 1.$$

Example 6.—Equations (1) given in article 94, $x = v \cos \alpha t$ and $y = v \sin \alpha t - \frac{1}{2}gt^2$, are parametric equations of a parabola. Here t is the number of seconds the point has been moving.

EXERCISES

1. Write parametric equations of the straight line through $(-3, 2)$ and having a slope of 2. Plot the line from these equations.

2. Write parametric equations of the circle with center at $(2, 3)$ and radius 5.

3. Represent the parabola $y^2 = 4x$ by several pairs of parametric equations.

Suggestion.—Either x or y can be represented at pleasure, but the other must be determined in accordance with this. For instance, if $x = t^2 + 1$, $y = 2\sqrt{t^2 + 1}$.

Plot the following parametric equations. In each case eliminate the parameter and find a single equation representing the same curve.

4. $x = 4 - t^2$, $y = t - 1$.

5. $x = 5 \cos \theta$, $y = 3 \sin \theta$.

6. $x = 2 + \sin \theta$, $y = 2 \cos \theta$.

7. $x = t + t^2$, $y = t - t^2$.

8. $x = 5 + 2 \cos \theta$, $y = 4 + 3 \sin \theta$.

9. $x = 1 - \cos \theta$, $y = \frac{1}{2} \sin \frac{1}{2}\theta$.

10. $x = \cos \theta$, $y = \cos 2\theta$.

11. $x = a \sin \theta + b \cos \theta$, $y = a \cos \theta - b \sin \theta$.

12. $x = a \cos^3 \theta$, $y = b \sin^3 \theta$.

142. The cycloid.—The plane curve traced by a fixed point on a circle as the circle rolls along a fixed straight line is called a **cycloid.** The rolling circle is called the **generator circle** and the fixed straight line the **base.**

The parametric equations of the cycloid can be derived as follows:

In Fig. 129, let OX be the fixed straight line, C the generator circle of radius a, and $P(x, y)$ the tracing point. Also suppose the circle is rolling towards the right.

Choose OX as the x-axis and the origin O where the tracing point is in contact with the fixed line. Also choose as parameter the angle θ, through which the radius to the tracing point turns. Draw the lines shown in the figure.

Then $x = ON = OM - NM = OM - PQ$,
and $y = NP = MC - QC$.

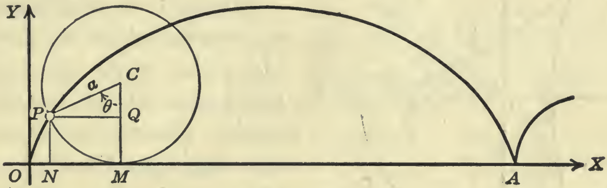


FIG. 129.

But $OM =$ arc $MP = a\theta$, $PQ = a \sin \theta$, $MC = a$, and $QC = a \cos \theta$.

Substituting these values gives

[40] $$\mathbf{x = a(\theta - \sin \theta),}$$
$$\mathbf{y = a(1 - \cos \theta).}$$

These are the forms of the equations most frequently used in dealing with the cycloid. If θ is eliminated the equation in x and y is

$$x = a \operatorname{vers}^{-1} \frac{y}{a} - \sqrt{2ay - y^2},$$

a form that is seldom used.

EXERCISES

1. Plot the cycloid from the parametric equations. Is the curve periodic?

2. Construct a figure in which $90° < \theta < 180°$, and derive the equation of the cycloid from it.

3. Derive parametric equations for the locus traced by a point on a fixed radius and at a distance b from the center of the circle rolling as in generating the cycloid. First, suppose $b < a$; second, suppose $b > a$.

4. Plot the curves of exercise 3. Such curves are called **trochoids.**

143. The hypocycloid.—The plane curve traced by a fixed point on a circle as the circle rolls along a fixed circle internally is called an **hypocycloid.**

The derivation of the parametric equations is as follows: In Fig. 130, let O be the fixed circle with radius a, and C the generator circle with radius b. Let $P(x, y)$ be the tracing point.

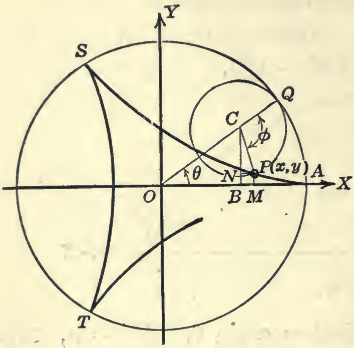


Fig. 130.

Choose O as origin and OX as x-axis. Also let the tracing point start at A where the x-axis intersects the fixed circle. Choose as parameters the angle θ, through which the line of centers of the two circles turns, and the angle φ, through which the radius of the generator circle turns. Draw the lines shown in the figure.

Then $x = OM = OB + BM = OB + NP,$

and $y = MP = BN = BC - NC.$

But $OB = OC \cos \theta = (a - b) \cos \theta,$

and $NP = CP \sin PCN = b \cos (\varphi - \theta).$

Also $BC = OC \sin \theta = (a - b) \sin \theta,$

and $NC = CP \cos PCN = b \sin (\varphi - \theta).$

Substituting these values gives

$$x = (a - b) \cos \theta + b \cos (\varphi - \theta),$$

and
$$y = (a - b) \sin \theta - b \sin (\varphi - \theta).$$

To eliminate the parameter φ, notice that

$$\text{arc } AQ = \text{arc } PQ, \text{ or } a\theta = b\varphi, \text{ and hence } \varphi = \frac{a\theta}{b}.$$

Substituting the value of φ in the above equations,

[41]
$$\mathbf{x} = (\mathbf{a} - \mathbf{b}) \cos \boldsymbol{\theta} + \mathbf{b} \cos \left(\frac{\mathbf{a} - \mathbf{b}}{\mathbf{b}}\right) \boldsymbol{\theta},$$
$$\mathbf{y} = (\mathbf{a} - \mathbf{b}) \sin \boldsymbol{\theta} - \mathbf{b} \sin \left(\frac{\mathbf{a} - \mathbf{b}}{\mathbf{b}}\right) \boldsymbol{\theta}.$$

The hypocycloid is a closed curve only when the diameters of the two circles are commensurable.

If $a = 2b$, equations [**41**] become $x = a \cos\theta$ and $y = 0$. Therefore when the radius of the generator circle is one-half the radius of the fixed circle, the tracing point moves in a straight line.

The most important special case of the hypocycloid is the four-cusped hypocycloid, in which $a = 4b$. The curve is shown in Fig. 131. Here the parameter can be eliminated and a single equation in x and y obtained.

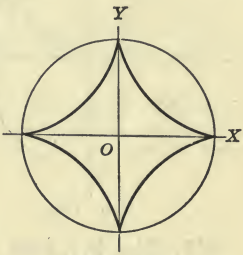


FIG. 131.

Putting $b = \frac{1}{4}a$ in equations [**41**],

$$x = \tfrac{3}{4}a \cos\theta + \tfrac{1}{4}a \cos 3\theta,$$

and

$$y = \tfrac{3}{4}a \sin\theta - \tfrac{1}{4}a \sin 3\theta.$$

But from trigonometry

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

and

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Substituting and simplifying, $x = a\cos^3\theta$ and $y = a\sin^3\theta$.

Affecting by the exponent $\frac{2}{3}$ and adding, gives the equation in x and y,

[42] $$\mathbf{x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.}$$

144. The epicycloid.—The plane curve traced by a fixed point on a circle as the circle rolls along a fixed circle externally is called an **epicycloid.**

Using a and b as the radii of the fixed circle and the generator circle respectively, and θ and φ as shown in Fig. 132, the equations of the epicycloid are

[43] $$\mathbf{x = (a + b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta,}$$
$$\mathbf{y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta.}$$

An important special case of the epicycloid is the **cardioid,** in which $a = b$. The curve is as shown in Fig. 133. The equations here become

$$x = 2a \cos \theta - a \cos 2\theta,$$

and
$$y = 2a \sin \theta - a \sin 2\theta.$$

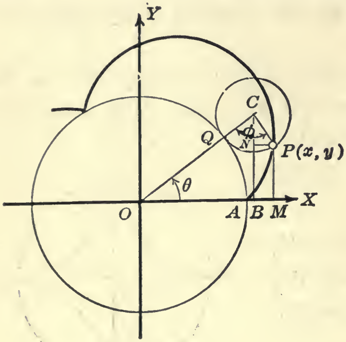


FIG. 132.

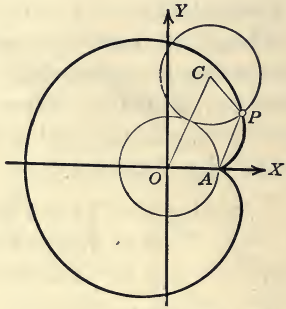


FIG. 133.

145. The involute of a circle.—If a string is wound around a circle, the curve in the plane of the circle, traced by a point on the string as it is unwound and kept taut, is called the **involute of the circle.**

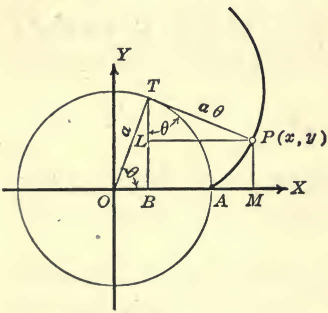


FIG. 134.

The parametric equations may be derived as follows:

Choose the x-axis through the point where the tracing point is in contact with the circle, and the origin at its center. The parameter θ, Fig. 134, is the angle through which the radius to the point of tangency of the string has turned.

Then from the figure,

$$x = OM = OB + BM = OB + LP,$$

and
$$y = MP = BL = BT - LT$$

But $OB = a \cos \theta$, $LP = TP \sin \theta = a\theta \sin \theta$,

and $BT = a \sin \theta$, $LT = TP \cos \theta = a\theta \cos \theta$.

Substituting these values gives

[44]
$$x = a\cos\theta + a\theta\sin\theta$$
$$y = a\sin\theta - a\theta\cos\theta.$$

EXERCISES

1. Derive the equation of the four-cusped hypocycloid.

2. Derive the equation of the epicycloid.

3. Derive the polar form of the equation of the cardioid from the parametric equations given in article 144.

Suggestion.—In the polar form of the equation the pole is at A, Fig. 133. Notice that $\angle XAP = \angle XOC$, and hence the parameter θ is equal to the polar coördinate θ.

First, square and add the equations of Article 144, then translate to new origin at A (a, O), finally, transform to polar coördinates and derive the equation $\rho = 2a(1 - \cos\theta)$.

CHAPTER X

EMPIRICAL LOCI AND EQUATIONS

146. General statement.—In common every day affairs, in business, in the sciences as physics, chemistry, and biology, and in engineering, questions often arise involving the relations of variables. Values of these variables can be plotted according to some system of coördinates, and, in this manner, curves obtained that give valuable information. Often the desired facts can be discovered directly from the curve; but frequently, especially in the sciences and in engineering, it is of the utmost importance to find a mathematical equation representing the curve more or less accurately.

The determination of the equation may be a comparatively simple matter, but often it is very laborious and involves methods beyond the scope of this text.

A curve that is plotted from observed values of the related variables is called an **empirical curve** or **locus.**

The equation of an empirical curve is an **empirical equation.**

Usually the empirical equation represents a curve that only approximates the empirical curve more or less accurately.

147. Empirical curves.—Innumerable examples of empirical curves could be given. For many of these there may be no necessity nor reason for finding equations representing them.

The rise and fall in the price of a certain stock may be represented graphically by using the price each day as the ordinate of a point of which the date is the abscissa. The curve drawn through these points will show at a glance the fluctuations of this particular stock.

If the weight of a child is taken from month to month, a curve can be plotted by using the weights as ordinates and the corresponding dates as abscissas of points.

Empirical curves are often traced mechanically by instruments designed for that particular purpose. In this manner, at a weather bureau station, a curve is traced showing the relation between the temperature and the time. In Fig. 135, is a similar curve that shows the per cent of carbon dioxide

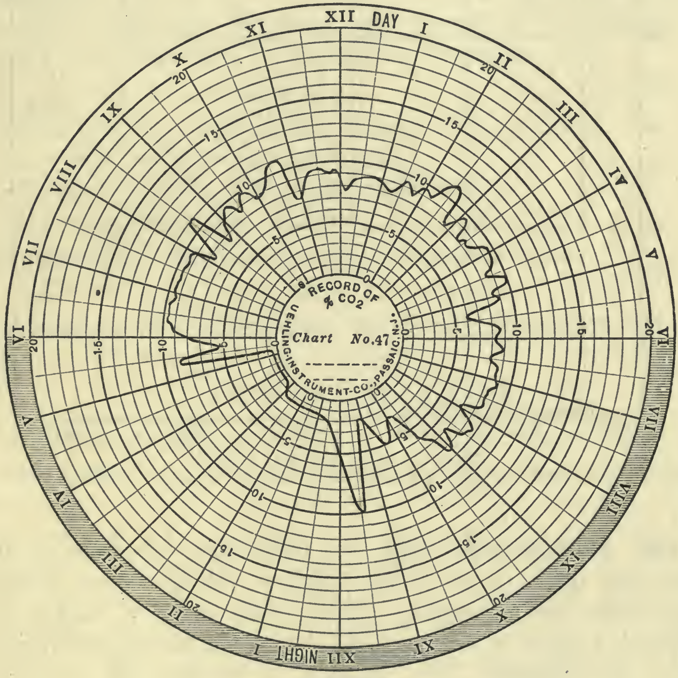
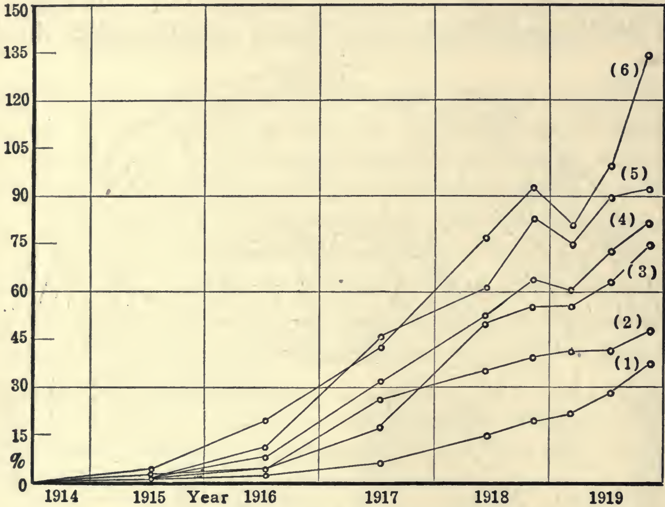


FIG. 135.

in the flue gas from a power plant. The variables are the time and the per cent of carbon dioxide. The system of coördinates is apparent.

In Fig. 136, are plotted several curves showing the changes in the cost of living from July, 1914 to November, 1919. The data was taken from the Research Report issued by the National Industrial Conference Board.

In such curves as these the information desired is gained directly from the curve, and no attempt would be made to derive an equation.



Change in prices from 1914 to 1919.
(1) Shelter; (2) Heat and light; (3) Sundries; (4) Cost of living; (5) Food; (6) Clothing.
FIG. 136.

148. Experimental data.—In laboratory experiments and practical tests, pairs of simultaneous values of two varying quantities are measured. When these pairs of values are plotted, a curve is determined from which useful information may be obtained. The problem of finding the empirical equations representing such curves will now be considered.

All data that is a result of measurements must be assumed to be subject to some degree of error, hence the endeavor will always be to approximate as closely as possible, both in the curve and in the equation.

Sometimes in a problem of this kind the general form of the equation of the curve is known beforehand, and sometimes nothing at all is known but the coördinates measured in the experiment. If the general form of the equation is known, the computations for finding the definite equation can be made at once; but, if the general form of the equation is not known, the points are plotted so as to discover the general form if possible.

149. General forms of equations.—The forms of equations frequently used are the following. Most of these have been studied in previous chapters, and should be reviewed, if necessary, so that their forms may be clearly in mind.

(1) $y = mx + b$, straight line.
(2) $y = cx^n$, $n > 0$, parabolic type.
(3) $y = cx^n$, $n < 0$, hyperbolic type.
(4) $y = ab^x$ or $y = ae^{kx}$, exponential type.
(5) $y = a + bx + cx^2 + dx^3 + \cdots + qx^n$.

For the parabolic type it is often necessary to use

$$y - k = c(x - h)^n, \ n > 0,$$

where the vertex is at the point (h, k); and for the hyperbolic type

$$y - k = c(x - h)^n, \ n < 0.$$

150. Straight line, y = mx + b.—This is the form of the empirical equation when it is known that the relation between the variables is that of a direct variation. Since in the equation $y = mx + b$ there are but two arbitrary constants, two pairs of measured values would be sufficient to determine the equation completely provided the values could be measured accurately. Since this is not possible, a larger number of pairs of values are measured, and from these an equation is determined that represents the straight line lying *most nearly* to all the points. The method used in the following example for securing the equation is called the **method of least squares.** The theory underlying the method is too difficult to be given here.

Example.—Find the equation of the straight line that is in the form $y = mx + b$, lying most nearly to the points determined by the following measured values of x and y:

x	40	50	62.4	70	80.5	90	97
y	8.7	7.5	6.5	5.85	5.05	4.25	3.75

Solution.—Here the type of the equation is given so there is no need of plotting the points.

Substituting each pair of values successively in the equation $y = mx + b$ gives the seven equations:

$$\begin{aligned} 8.7 &= 40m + b, \\ 7.5 &= 50m + b, \\ 6.5 &= 62.4m + b, \\ 5.85 &= 70m + b, \\ 5.05 &= 80.5m + b, \\ 4.25 &= 90m + b, \\ 3.75 &= 97m + b. \end{aligned}$$

Multiplying each of these by the coefficient of m in that equation and adding the seven resulting equations, gives

$$2690.875 = 36883.01m + 489.9b. \tag{1}$$

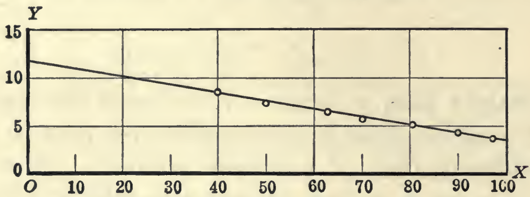


FIG. 137.

Multiplying each of the seven equations by the coefficient of b in that equation and adding the results, gives

$$41.6 = 489.9m + 7b. \tag{2}$$

Solving (1) and (2) for m and b, gives

$$m = -0.085, \text{ and } b = 11.89.$$

Substituting these values in $y = mx + b$, gives

$$y = -0.085x + 11.89. \tag{3}$$

This is taken as the equation of the straight line lying most nearly to all the points.

In Fig. 137 are plotted equation (3) and the points whose coördinates are the observed values of x and y.

151. Method of least squares.—The method of least squares given in the previous article becomes tedious when there are many observations and the numbers are large. A sufficiently accurate result may be obtained by plotting the points, and obtaining the arbitrary constants of the equation by using two points that lie on the straight line that appears to be the best. If none of the plotted points lie on this line, use coördinates of points that do lie on the line.

The method of least squares is quite mechanical, while the best straight line if determined by plotting is a matter of judgment and a good eye.

The method by least squares for finding the empirical equation is stated in the following:

RULE.—*First, substitute each pair of observed values of the variables in the general equation.*

Second, if there are just as many equations as there are constants to be found, solve these equations for the constants. If there are more equations than there are constants, multiply each equation by the coefficient of the first constant in that equation, and add the resulting equations to form one equation. Proceed likewise for each other constant, and thus find as many equations as there are constants.

Third, solve these equations for the constants.

Fourth, substitute the constants, thus found, in the general equation and obtain the required empirical equation.

EXERCISES

1. A wire under tension is found by experiment to stretch an amount l, in inches, under a tension T, in pounds, as given in the following table. Assume the relation $l = kT$ (Hooke's law) and find the equation which best represents the relation between l and T.

T	5	10	20	30	40	50
l	0.003	0.009	0.019	0.030	0.040	0.053

2. Find the empirical equation in the form of $y = mx + b$ best representing the relations between the values given in the following table:

x	12	15.3	17.8	19
y	24.4	29	32.6	34.2

3. Find the equation of the straight line lying most nearly to the points determined by the following pairs of measured values:

x	12	15	18	21	24
y	24.4	28.6	32.7	37.1	41.2

4. Find the empirical equation connecting R and t from the following table of experimental values. R is in ohms and t in degrees centigrade. The equation is assumed to be in the form $R = mt + b$.

t	10.1	15	21	26.8	33.1	40.4
R	9.907	9.923	9.940	9.959	9.979	10.002

5. Find the empirical equation giving H in terms of t, from the data of the following table. H is the total heat in a pound of saturated steam at t degrees centigrade. The general form is $H = mt + b$.

t	65	85	100	110	120
H	626.3	632.4	637	640.9	643.1

6. In an experiment with a Weston differential pulley block, the effort E, in pounds, required to lift a weight W, in pounds, was found to be as follows:

W	10	20	30	40	50	70	90	100
E	3.25	4.875	6.25	7.5	9	12.25	15	16.5

Find the empirical equation in the form $E = mW + b$.

7. Plot the data given in exercise 6, and draw a line that, in your judgment, lies most nearly to all the points. Select two points that lie as nearly on the line as any, and determine the equation from the coördinates of these points. Compare the result with that of exercise 6.

8. In the following table, W is the weight of potassium bromide which will dissolve in 100 grams of water at t degrees centigrade. Find the empirical formula in the form $W = mt + b$, connecting W and t.

t	0	20	40	60	80
W	53.4	64.6	74.6	84.7	93.5

152. Parabolic type, $y = cx^n$, $n > 0$.—If it is not known that the general form of the equation is of some particular type, it is well to plot the data on rectangular coördinate paper and judge the type from the curve. After the general form is selected, it is often difficult to determine whether or not it actually represents the observed values with sufficient accuracy for the purposes of the problem. A device that is of great assistance in determining whether to retain or reject the type selected is to transform the general equation into a linear equation, and see if the data plots as a straight line. This is done as follows when the equation is of the parabolic type:

Given equation, $y = cx^n$.

Taking logarithms of both sides,

$$\log y = \log c + n \log x,$$

which is a linear equation in $\log x$ and $\log y$. If the points with coördinates $(\log x, \log y)$, where x and y for each point are a pair of observed values, are plotted, and these points are found to lie approximately on a straight line, then the general form of the equation is suitable to the problem.

The values of the constants, $\log c$ and n, can be determined by the method of least squares as in article 151.

Of course, if the general form of the equation to be used is known to be of the parabolic type, the plotting is not necessary.

Example.—Q is the quantity of water, in cubic feet per second, that flows through a right isosceles triangular notch when the surface of the still water is at a height H feet above the bottom of the notch. The values of H and Q in the following table are measured. Find the equation connecting H and Q.

H	1	1.5	2	2.5	3	4
Q	2.63	7.25	15	26	41	84.4

Solution.—The values of H and Q are plotted in Fig. 138, and connected with the curve, which appears to be of the parabolic type.

Assume the general form $Q = cH^n$. (1)

Taking logarithms, $\log Q = \log c + n \log H$. (2)

Put $\log H = x$, $\log Q = y$ and $\log c = b$, and the equation becomes

$$y = nx + b.$$

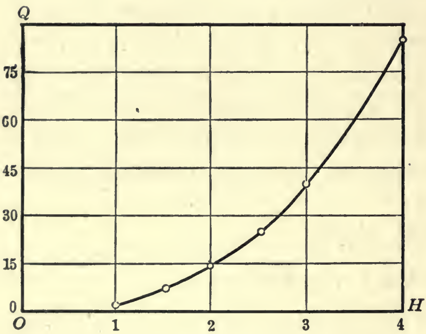


FIG. 138.

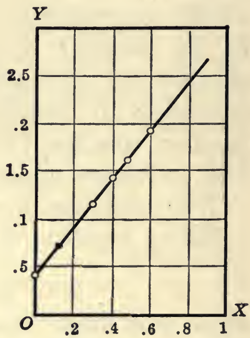


FIG. 139.

The values of x and y are found and plotted in Fig. 139. The points lie approximately in a straight line.

log H	0	0.1761	0.3010	0.3979	0.4771	0.6021
log Q	0.4200	0.8603	1.1761	1.4150	1.6128	1.9263

Substituting the values for x and y in the equation $y = nx + b$, the following equations are obtained:

$$\begin{aligned} 0.4200 &= 0n + b, \\ 0.8603 &= 0.1761n + b, \\ 1.1761 &= 0.3010n + b, \\ 1.4150 &= 0.3979n + b, \\ 1.6128 &= 0.4771n + b, \\ 1.9263 &= 0.6021n + b. \end{aligned}$$

Solving these equations by the method of least squares,

$$n = 2.5 \text{ and } b = 0.4208.$$

Substituting in equation (2), $\log Q = 0.4208 + 2.5 \log H$.

Then $\qquad \log Q = \log 2.635 + \log H^{2.5}$.

Or $\qquad \log Q = \log (2.635H^{2.5})$.

$$\therefore Q = 2.635H^{2.5}, \text{ the required equation.}$$

The equation can be tested by computing values of Q for the several observed values of H, and comparing with the observed values of Q.

153. Hyperbolic type $y = cx^n$, $n < 0$.—Data that are known to give an equation of this type can be handled in precisely the same manner as the parabolic type. The only difference that will arise will be that the value of n is negative.

154. Exponential type, $y = ab^x$ or $y = ae^{kx}$.—The data from certain experiments, such as those involving friction, give rise to exponential equations. As with the other types the data can be plotted on rectangular coördinate paper and the general form of the equation determined. If it is thought to be of the exponential type, it can be tested by taking the logarithms of both sides of the equation and plotting on rectangular coördinate paper. If the points lie on a straight line, the assumed equation is correct.

In order to express the form $y = ab^x$ in the form $y = ae^{kx}$ it is only necessary to put $b = e^k$, whence $\log b = k \log e$, or

$$k = \frac{\log b}{\log e} = 2.3026 \log b.$$

Example.—From the following data determine the relation between W and θ.

θ	1.57	3.14	4.71	6.28	7.85	9.42	11
W	5.35	7.15	9.55	12.8	17.12	22.9	30.8

Solution.—First, plot the data given and determine the form of the equation to be used. The plotting is shown in Fig. 140, and the equation assumed is

$$W = ab^{\theta}. \tag{1}$$

Second, to test this, take the logarithms of both sides of $W = ab^{\theta}$.

This gives $\log W = \log a + \theta \log b$.

Put $\log W = y$, $\log a = B$, and $\log b = m$.

This gives $y = m\theta + B$. (2)

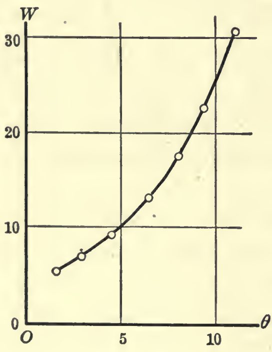


Fig. 140.

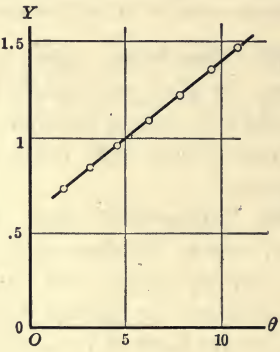


Fig. 141.

Arranging these values in a table and plotting, gives approximately a straight line as shown in Fig. 141.

θ	1.57	3.14	4.71	6.28	7.85	9.42	11
y	0.728	0.854	0.980	1.107	1.234	1.360	1.489

Substituting the pairs of values of θ and y in equation (2),

$$\begin{aligned} 0.728 &= 1.57m + B, \\ 0.854 &= 3.14m + B, \\ 0.980 &= 4.71m + B, \\ 1.107 &= 6.28m + B, \\ 1.234 &= 7.85m + B, \\ 1.360 &= 9.42m + B, \\ 1.489 &= 11m + B. \end{aligned}$$

Solving these by the method of least squares, gives

$$m = 0.0807 \text{ and } B = 0.6005.$$

Then $$a = 3.985 \quad \text{and} \quad b = 1.204.$$

$$\therefore W = 3.985 \times 1.204^{\theta}, \text{ the required equation.}$$

This expressed in the form $W = ae^{k\theta}$ gives,

$$W = 3.985e^{0.1858\theta}.$$

155. Probability Curve, $y = ae^{-bx^2}$.—The curve that is perhaps the most widely used of any in dealing with experimental data is one variously called "the probability curve," "the error curve," and "the normal distribution curve." It is represented by the equation

$$y = ae^{-bx^2},$$

where a and b are constants to be determined from the data. It is evidently symmetrical with respect to the y-axis. While

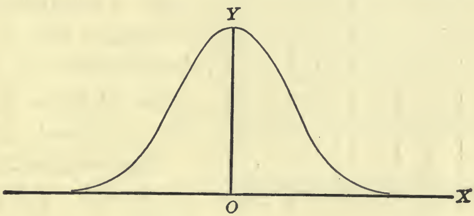


FIG. 142.

definite uses of this curve are beyond the scope of this chapter, it may be stated that it is used wherever a most probable correct value is to be determined from a large number of independent measurements or observations. It is used in the study of statistics, in astronomy, biology, and chemistry, and in the study of theory of measurements.

The form of the curve is shown in Fig. 142.

156. Logarithmic paper.—Because of the frequent occurrence of formulas of the parabolic and hyperbolic types, considerable use is made in engineering practice of **logarithmic paper,** that is, paper that is ruled in lines whose distances, horizontally and vertically, are proportional to the logarithms of the numbers 1, 2, 3, etc.

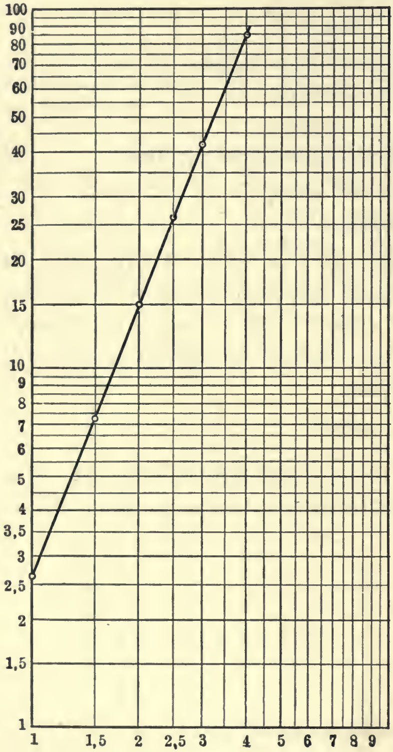


Fig. 143.

Logarithmic paper can be used instead of actually looking up the logarithms of the numbers as was done in the example of article 152. For if the values of H and Q are plotted as shown in Fig. 143, a straight line is determined just as when the logarithms of H and Q were plotted on rectangular coördinate paper.

Semi-logarithmic paper is ruled uniformly the same as ordinary coördinate paper in one direction, and in lines spaced as on logarithmic paper in the other direction. Semi-logarithmic paper may be used to advantage when testing an exponential type. In Fig. 144, the values of θ and W of the example of article 154 are plotted into a straight line.

EXERCISES

1. Solve the equations of article 152 by the method of least squares and check the results given.

2. Determine the equation of the hyperbolic type connecting x and y from the following pairs of values:

x	1.5	2.8	5.6	8.3
y	0.573	0.243	0.094	0.055

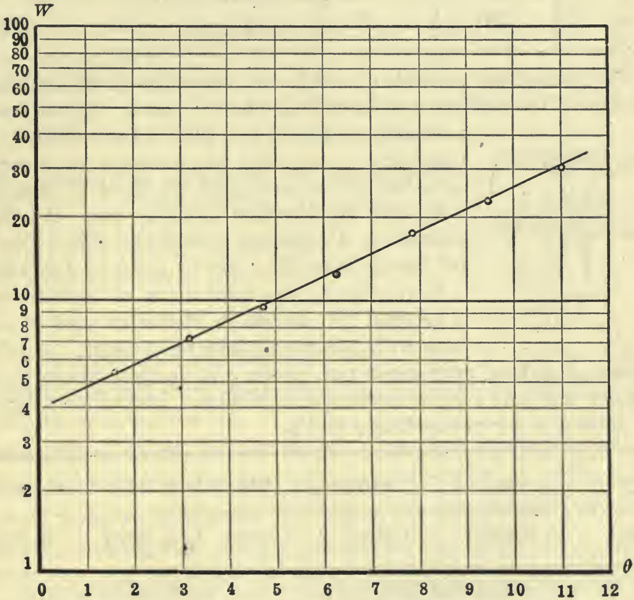


Fig. 144.

3. In propelling a ship of a certain class at 10 knots, the following pairs of values of D and H are measured, where D is the displacement in tons and H is the indicated horse-power. Find a formula of the parabolic type connecting D and H.

D	1100	1530	1820	2500	3130
H	440	550	620	770	890

Compute H when $D = 2000$.

4. For different heights, h in feet above the surface of the earth, the reading, p in inches, of the barometer are taken as given in the following table. Determine a formula of the form $p = ae^{kh}$ connecting p and h.

h	0	886	2753	4763	6942
p	30	29	27	25	23

5. The data of the example of article 154 was taken in an experiment to determine the coefficient of friction μ, when a cord is wrapped around a cylindrical shaft, Fig. 145. In performing the experiment, the cord has a weight of 2 pounds attached to one end, and a pull of W pounds at the other end induces slipping when the arc of contact is θ radians. Determine the value of μ for the equation $W = ae^{\mu\theta}$. $\mu = k$ of **Art. 154.**

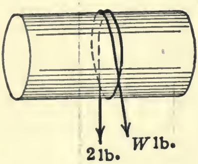


FIG. 145.

6. In testing the lubrication of certain oils in a bearing, $4\frac{1}{2}$ inches in diameter and 8 inches long with 250 revolutions per minute, the following pairs of values were measured, where p is the pressure in pounds per square inch and μ is the coefficient of friction. Determine a formula of the form $\mu = ap^n$ connecting p and μ.

p	65	115	215	315	465
μ	0.0090	0.0056	0.0036	0.0028	0.0025

Plot the values showing that the curve is of the hyperbolic type.

7. In the same experiment as in exercise 6, but using another oil, the following values were obtained. Determine a formula connecting p and μ.

p	65	115	215	315	415	515
μ	0.00788	0.00528	0.00338	0.00267	0.00235	0.00215

8. In the following table are given the measurements taken in an experiment on friction, where μ is the coefficient of friction in a certain bearing running at a velocity of V feet per minute. Determine a formula of the form $\mu = aV^n$ connecting V and μ.

V	105	157	209	262	314	366	419	471
μ	0.0018	0.0021	0.0025	0.0028	0.003	0.0033	0.0036	0.004

9. In a mixture in a cylinder of a gas-engine, under adiabatic expansion, the following pairs of values are measured. Determine a formula in the form $pv^n = C$ connecting v and p.

v	0.8	2	4	6	9
p	200	57	22	12.6	7.2

157. Empirical formulas of the type $y = a + bx + cx^2 + dx^3 + \cdot \cdot \cdot + qx^n$.—When a given set of corresponding pairs of values will not satisfy, in a satisfactory manner, any of the type equations already considered, the general equation

$$y = a + bx + cx^2 + dx^3 + \cdot \cdot \cdot + qx^n$$

may be assumed. By substituting pairs of values in this equation, enough equations can be obtained to determine the constants $a, b, c, \cdot \cdot \cdot$.

Since there must be, at least, as many equations as constants, no more terms can be assumed than the number of pairs of values measured. If there are more pairs of values than the number of terms assumed, the equations can be solved by the method of least squares. A less accurate method, but one more easily carried out, is to select as many of the equations as there are constants, and solve these for the constants. The equation thus found can be tested by substituting the pairs of values not used in the equations that are solved for the constants.

If the points when plotted suggest a parabola, only three terms need be used. If the arrangement of the points is more irregular, more terms must be assumed.

Example.—The following measurements at different depths were made to determine the rate of flow in a river, where x is the fractional part of the depth from the surface and y is the rate of flow. Determine a formula of the form $y = a + bx + cx^2$ connecting x and y.

x	0	0.2	0.3	0.4	0.6	0.8	0.9
y	3.195	3.253	3.261	3.252	3.181	3.059	2.976

Solution.—Substituting the pairs of values in $y = a + bx + cx^2$,

$$\begin{aligned}
3.195 &= a + 0b + 0c,\\
3.253 &= a + 0.2b + 0.04c,\\
3.261 &= a + 0.3b + 0.09c,\\
3.252 &= a + 0.4b + 0.16c,\\
3.181 &= a + 0.6b + 0.36c,\\
3.059 &= a + 0.8b + 0.64c,\\
2.976 &= a + 0.9b + 0.81c.
\end{aligned}$$

These are solved by the method of least squares as follows:

Multiplying each by its coefficient of a and adding the seven resulting equations, gives

$$22.177 = 7a + 3.2b + 2.1c. \qquad (1)$$

Multiplying each by its coefficient of b and adding, gives

$$9.9639 = 3.2a + 2.1b + 1.556c. \qquad (2)$$

Multiplying each by its coefficient of c and adding, gives

$$6.45741 = 2.1a + 1.556b + 1.2306c. \qquad (3)$$

Solving equations (1), (2), and (3) for a, b, and c,

$$a = 3.196,\ b = 0.438,\ c = -0.7608.$$

Substituting these values in $y = a + bx + cx^2$, gives

$$y = 3.196 + 0.438x - 0.7608x^2,$$

which is the required equation.

EXERCISES

1. Plot the corresponding pairs of values of x and y given in the example of article 157, and draw a smooth curve lying as near as possible to all the points. Select the three points lying most nearly on the curve, and use the coördinates of these to find the values of a, b, and c. Compare with the result given in the solution by the method of least squares.

2. Determine an equation of the form $y = a + bx + cx^2 + dx^3$ from the following experimental values. Solve both by the method of least squares and by using the coördinates of four points.

x	0.4	0.6	0.8	1.0	1.2	1.4	1.6
y	0.89	1.35	1.96	2.72	3.62	4.63	5.76

3. The melting point of an alloy of lead and tin containing x per cent of lead is t degrees centigrade. From the following table of measured values, find a formula in the form $t = a + bx + cx^2$, giving the melting point of an alloy containing any known per cent of lead from 90 per cent to 35 per cent.

x	87.5	84	77.8	63.7	46.7	36.9
t	292	283	270	235	197	181

For a further discussion of the subject of this chapter, the following works may be consulted: Merriman, *Method of Least Squares;* Weld, *Theory of Errors and Least Squares;* Johnson, *Theory of Errors and Method of Least Squares;* Palmer, *Theory of Measurements;* Steinmetz, *Engineering Mathematics;* Running, *Empirical Formulas;* Lipka, *Graphical and Mechanical Computation.*

CHAPTER XI

POLES, POLARS, AND DIAMETERS

158. Harmonic ratio.—If two points A and B divide a line segment MN externally and internally in ratios that have the same numerical values, then A and B are said to divide MN **harmonically.** A and B are called **harmonic conjugates** with respect to the line segment MN.

THEOREM.—*If the points A and B, divide the line segment MN harmonically, then the points M and N divide the line segment AB harmonically.*

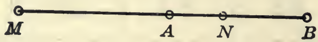


FIG. 146.

Proof.—By hypothesis, $\frac{MA}{AN} = -\frac{MB}{BN}$.

Taking this proportion by alternation, $\frac{MA}{MB} = -\frac{AN}{BN}$.

Multiplying both sides of this equation by -1, and replacing $-MA$ by AM and $-BN$ by NB gives the required proportion

$$\frac{AM}{MB} = -\frac{AN}{NB}.$$

159. Poles and polars.—*Definition.*—If a line drawn through some point P_1 is allowed to rotate about P_1 while cutting a conic in the variable points M and N, then the locus of all points harmonically conjugate to P_1 with respect to M and N is called the **polar** of P_1 with respect to the conic, and P_1 is called the **pole** of the locus.

To find the equation of the polar of P_1 with respect to an ellipse, suppose the ellipse in Fig. 147 is given in the standard form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If P_2 is the conjugate of P_1 with respect to the ellipse and P_1P_2 is any line cutting the ellipse in the points M and N, then M and N are harmonic conjugates with respect to the line P_1P_2.

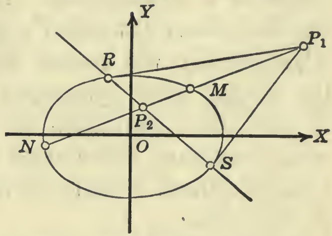


FIG. 147.

If the coördinates of P_1 are (x_1, y_1) and the coördinates of P_2 are (x_2, y_2); then by [4] the coördinates of M are

$$\left(\frac{r_2x_1 + r_1x_2}{r_1 + r_2}, \quad \frac{r_2y_1 + r_1y_2}{r_1 + r_2}\right),$$

and of N are

$$\left(\frac{r_2x_1 - r_1x_2}{r_1 - r_2}, \quad \frac{r_2y_1 - r_1y_2}{r_1 - r_2}\right).$$

Since M and N are points on the ellipse, their coördinates must satisfy the equation of the ellipse, therefore

$$\frac{\left(\dfrac{r_2x_1 + r_1x_2}{r_1 + r_2}\right)^2}{a^2} + \frac{\left(\dfrac{r_2y_1 + r_1y_2}{r_1 + r_2}\right)^2}{b^2} = 1,$$

and

$$\frac{\left(\dfrac{r_2x_1 - r_1x_2}{r_1 - r_2}\right)^2}{a^2} + \frac{\left(\dfrac{r_2y_1 - r_1y_2}{r_1 - r_2}\right)^2}{b^2} = 1.$$

Clearing each equation of fractions and subtracting the second from the first gives the equation

$$4b^2r_1r_2x_1x_2 + 4a^2r_1r_2y_1y_2 = 4r_1r_2a^2b^2.$$

Dividing both sides of the equation by $4r_1r_2$ and dropping the subscripts for the coördinates of the point P_2 gives the equation of the polar of P_1

$$b^2x_1x + a^2y_1y = a^2b^2.$$

$$\therefore \frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

This shows that the polar of a point with respect to an ellipse is a straight line. If the point P_1 is outside the ellipse the line drawn through P_1 will not always intersect the ellipse. Algebraically the points of intersection of such a line and the ellipse have imaginary coördinates, but the coördinates of the point conjugate to P_1 with respect to these points with imaginary coördinates are real. Hence that part of the locus obtained outside of the ellipse is also included as part of the locus.

In like manner it can be shown that the polar of a point P_1 with respect to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1.$$

Also the polar of P_1 with respect to the parabola

$$y^2 = 2px$$

is

$$y_1y = px + px_1.$$

Likewise the polar of a point P_1 with respect to the general conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

[45] $$\mathbf{Ax_1x + \frac{B}{2}x_1y + \frac{B}{2}xy_1 + Cy_1y + \frac{D}{2}x + \frac{D}{2}x_1 + \frac{E}{2}y + \frac{E}{2}y_1 + F = 0.}$$

The similarity should be noticed between this equation and the general equation of the conic written

$$Axx + \frac{B}{2}xy + \frac{B}{2}xy + Cyy + \frac{D}{2}x + \frac{D}{2}x + \frac{E}{2}y + \frac{E}{2}y + F = 0.$$

These equations show that the polar of a point P_1 with respect to any conic is a straight line.

160. Properties of poles and polars.—THEOREM 1.—*If two points are so situated that one lies on the polar of the second, the second lies on the polar of the first.*

Suppose the conic is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the point is $P_1(x_1, y_1)$.

Then the polar of P_1 is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$.

If P_2 lies on the polar of P_1 its coördinates will satisfy that equation, hence

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 1.$$

But this is precisely the condition that P_1 shall satisfy the equation

$$\frac{x_2x}{a^2} + \frac{y_2y}{b^2} = 1,$$

which is the equation of the polar of P_2.

This proof can easily be extended to the general equation of the second degree.

THEOREM 2.—*If tangents can be drawn from a point to a conic, the polar of this point passes through the points of contact of the tangents.*

In Fig. 147, the tangent P_1R meets the conic in two coincident points at R. Since the conjugate to P_1 lies between these two coincident points, it must coincide with R. Likewise the polar must pass through S.

THEOREM 3.—*Tangents to a conic at the points where a line cuts the conic pass through the pole of the line.*

This follows at once from theorem 2 in conjunction with the assumption that only one tangent line can be drawn to a conic at a given point.

Example.—Find the pole of the line $x + 2y = 1$ with respect to the conic $3x^2 + 4y^2 = 6$.

The polar of the point (x_1, y_1) with respect to the ellipse $3x^2 + 4y^2 = 6$ is $3x_1x + 4y_1y = 6$.

Since this equation and $x + 2y = 1$ are equations of the same line the coefficients of x, y, and the constant term must be proportional, then

$$\frac{3x_1}{1} = \frac{6}{1}, \quad \text{and} \quad \frac{4y_1}{2} = \frac{6}{1}.$$

Hence $x_1 = 2$, $y_1 = 3$, and the required pole is the point (2, 3).

EXERCISES

Find the equations of the polars of the points in exercises 1–8 with respect to the conics following.

1. (2, 3) $3x^2 + 4y^2 = 6.$
2. (−1, 6) $2x^2 + y^2 = -3.$
3. (−1, 2) $3x^2 - 2y^2 = 1.$
4. (1, −3) $2x^2 - 4y^2 = -5.$
5. (1, 2) $y^2 = 6x.$
6. (−3, −2) $x^2 = 4y.$
7. (1, 2) $x^2 - xy + y^2 - 6x - 3y + 2 = 0.$
8. (3, −4) $xy + 3y^2 + 3x + 7y + 1 = 0.$

Find the coördinates of the poles of the lines in exercises 9–16 with respect to the conics following.

9. $2x + 4y = 1,$ $6x^2 + 4y^2 = 3.$
10. $2x + 2y - 1 = 0,$ $2x^2 + 5y^2 = 5.$
11. $2x - 3y - 6 = 0,$ $4x^2 - 3y^2 = 12.$
12. $x - 2y + 4 = 0,$ $y^2 - 2x = 0.$
13. $x + y + 1 = 0,$ $x^2 + 6y = 0.$
14. $4x + 5y = 2,$ $x^2 + xy + y^2 = 3.$
15. $3x + 5y + 2 = 0,$ $x^2 + 2y^2 + x + y = 0.$
16. $2x + 1 = 0,$ $x^2 + 2xy + 2y - 2 = 0.$

17. Find a point which with (2, 4) divides the line joining (1, 1) to (4, 10) harmonically.

18. Prove that in any conic, the polar of the focus is the directrix.

161. Diameters of an ellipse.—*Definition.*—The locus of the middle points of a set of parallel chords of a conic is called a **diameter of the conic.**

To find the diameter of an ellipse, let its equation be given in the form of **[32]** and suppose that the slope of the parallel chords is m_1. Unless m_1 is infinite, the equations of these chords have the form $y = m_1x + c$, where m_1 is constant for any one system of parallel chords, but c will have different values for different chords of the system.

Suppose MN, Fig. 148, is one of these chords, the coördinates of M and N can be found by solving simultaneously the equations

$$y = m_1x + c$$

and
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

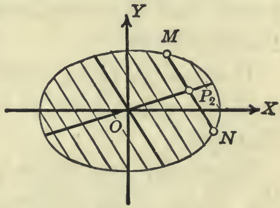


FIG. 148.

Eliminating y between these equations gives

$$(b^2 + a^2m_1^2)x^2 + 2a^2cm_1x + a^2c^2 - a^2b^2 = 0.$$

The two roots of this equation are the abscissas of the points M and N. Half their sum is the abscissa of $P_2(x_2,y_2)$, the middle point of MN.

By a well-known theorem, **Art. 4,** the sum of the roots of the quadratic equation

$$Ax^2 + Bx + C = 0$$

is equal to $-\frac{B}{A}$.

Hence
$$x_2 = -\frac{a^2cm_1}{b^2 + a^2m_1^2}.$$

To find y_2, substitute this value of x_2 in the equation

$$y = m_1x + c.$$

Then
$$y_2 = \frac{b^2c}{b^2 + a^2m_1^2}.$$

The relation between x_2 and y_2 for any one, and therefore for every one, of these parallel chords must be independent of c. Hence eliminate c by dividing y_2 by x_2. This gives

$$\frac{y_2}{x_2} = -\frac{b^2}{a^2m_1}.$$

Dropping the subscripts for P_2 gives the following equation of the diameter which bisects all chords of slope m_1:

$$y = -\frac{b^2}{a^2m_1}x.$$

This is the equation of a straight line passing through the center of the ellipse.

If m_1 is infinite, the parallel chords are all parallel to the y-axis and the symmetry of the ellipse shows the x-axis to be the diameter.

If $m_1 = 0$, the parallel chords are all parallel to the x-axis, and the symmetry of the ellipse shows the y-axis to be the diameter.

Since m_1 can have any value, any line passing through the center of the ellipse is a diameter.

The **length of a diameter** of an ellipse is the distance between the points where the diameter cuts the ellipse.

162. Conjugate diameters of an ellipse.—The slope of the diameter bisecting all chords parallel to the diameter

$$y = -\frac{b^2}{a^2m_1}x \tag{1}$$

is

$$\frac{-b^2}{a^2\left(\frac{-b^2}{a^2m_1}\right)} = m_1,$$

and its equation is

$$y = m_1x \tag{2}$$

But the diameter (2) is the diameter of the ellipse parallel to the set of parallel chords of article 161. Hence the diameter (1) bisects all chords parallel to diameter (2), and the diameter (2) bisects all chords parallel to diameter (1).

Two diameters such that each bisects all chords parallel to the other are called **conjugate diameters.** Hence diameters (1) and (2) are conjugate diameters.

If m_2 is the slope of (1)

$$m_2 = -\frac{b^2}{a^2m_1},$$

or

$$m_1m_2 = -\frac{b^2}{a^2}.$$

163. Diameters and conjugate diameters of an hyperbola.—Methods exactly similar to those in articles 161 and 162 show that all diameters of an hyperbola pass through its center. The diameter of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which bisects all chords of slope m_1, Fig. 149, is

$$y = \frac{b^2}{a^2 m_1} x.$$

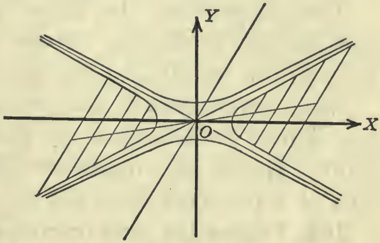


FIG. 149.

The slopes of two conjugate diameters of an hyperbola are connected by the relations

$$m_1 m_2 = \frac{b^2}{a^2}.$$

The **length of a diameter** of an hyperbola when the diameter meets the hyperbola is the distance between the points where the diameter cuts the hyperbola. If the diameter does not cut the hyperbola, its length is defined as the distance between the points where it cuts the conjugate hyperbola.

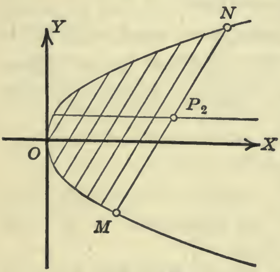


FIG. 150.

164. Diameters and conjugate diameters of a parabola.—Let the slope of the parallel chords be m_1, Fig. 150, and let their equations be $y = m_1 x + c$, where c will have different values for different chords.

The ordinates of the points of intersection of the parabola $y^2 = 2px$, and these parallel chords are given by the equation

$$m_1 y^2 - 2py + 2pc = 0.$$

If y_2 is the ordinate of any one of their middle points

$$y_2 = \frac{p}{m_1}.$$

Since this equation is independent of c, it is the condition that all points on the diameter must satisfy. Hence dropping subscripts, the equation of the diameter of a parabola is

$$y = \frac{p}{m_1}.$$

This shows that the diameter of a parabola is a straight line parallel to its axis. Since m_1 may have any value except 0, any line parallel to the axis of a parabola is a diameter.

As m_1 approaches 0, the system of parallel chords approaches parallelism to the axis of the parabola, and y increases without limit. Hence the diameter bisecting chords parallel to the axis of a parabola does not lie in the finite part of the plane.

165. Diameters and conjugate diameters of the general conic.—Since the slope of a line remains unchanged by translation of axes, all the results obtained so far hold good after translation for conics whose axes are parallel to the coördinate axes, providing that in the ellipse and hyperbola the major and the transverse axis respectively, and in the parabola the axis of the parabola are parallel to the x-axis.

Formulas obtained for conjugate diameters, and equations of diameters do not hold true for rotation of axes unless account is taken in m_1 and m_2 of the change made by the rotation.

EXERCISES

1. Find the equation of the diameter of the ellipse $3x^2 + 4y^2 = 6$, which bisects chords of slope 3. Chords of slope $-\frac{1}{2}$.

2. Find the equation of the diameter of the hyperbola $2x^2 - 4y^2 = 1$, which bisects chords of slope 3. Chords of slope $-\frac{1}{2}$.

3. Find the equation of the diameter of the parabola $y^2 = 4x$, which bisects chords of slope 3. Chords of slope $-\frac{1}{2}$.

4. Find the equation of the diameter which bisects chords of slope 3, for the ellipse $2x^2 + 3y^2 - 4x - 12y + 2 = 0$.

Suggestion.—Translate axes to center of conic, and then translate back to the original axes.

5. Find the equation of the diameter which bisects chords of slope 3, for the hyperbola $2x^2 - 3y^2 - 4x + 12y - 22 = 0$.

6. If (2, 1) is one extremity of a diameter of the ellipse $4x^2 + 9y^2 = 25$, find the coördinates of the extremities of the conjugate diameter.

7. If the point (1, 2) is one extremity of a diameter of the hyperbola $25x^2 - 4y^2 = 9$, find the coördinates of the extremities of the conjugate diameter.

8. If (x_1, y_1) is an extremity of a diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, what are the coördinates of the extremities of the conjugate diameter?

9. Prove that the sum of the squares of any two semi-conjugate diameters of an ellipse is constant and equal to $a^2 + b^2$.

10. If (x_1, y_1) is an extremity of a diameter of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, what are the coördinates of the extremities of the conjugate diameter?

11. Prove that the difference of the squares of any two semi-conjugate diameters of an hyperbola is constant and equal to $a^2 - b^2$.

12. Find the equation of the chord of the hyperbola $2x^2 - 3y^2 = 6$, through the point (4, 1) which is bisected by the diameter $y = 4x$.

13. Find the equation of the chord of the ellipse $x^2 + 2y^2 = 4$, through the point (6, 3) which is bisected by the diameter $3y + x = 0$.

14. Find the equation of the chord of the parabola $y^2 = 4x$ through the point (1, 6) which is bisected by the diameter $y = 3$.

15. Prove that the polar of any point $P_1(x_1, y_1)$ on a diameter of an ellipse is parallel to the conjugate diameter.

16. Two lines connecting a point on an ellipse with the ends of a diameter are called **supplemental chords.** Prove that supplemental chords are always parallel to a pair of conjugate diameters.

17. Prove that if a parallelogram is inscribed in an ellipse its sides are parallel to conjugate diameters.

18. Find the locus of the middle points of chords which connect the ends of pairs of conjugate diameters of a fixed ellipse.

CHAPTER XII

ELEMENTS OF CALCULUS

166. Introductory remarks.—As has been stated, the discovery of the methods of analytic geometry during the first half of the seventeenth century gave the first great start in the development of modern mathematics. During the latter half of the same century Newton and Leibniz, building upon the writing and teaching of Isaac Barrow and others, discovered the method of the *infinitesimal calculus*. In this subject are studied very powerful methods of investigating functions and problems concerning variables. It is in the calculus that we find the greatest development of mathematical analysis and its applications in almost every field of science and engineering. Some of these methods and applications will now be considered.

Here, as is always the case in the study of mathematics, it is necessary to understand clearly *what* is under consideration and *how* it is represented in mathematical symbols.

167. Functions, variables, increments.—*Example* 1.—If a suspended coiled wire spring has a weight attached to its lower end, the spring will be stretched. The amount of stretching will depend upon the weight, the greater the weight the greater the elongation. The elongation is then a function of the weight. If the weight is not so great that the elastic limit of the spring is exceeded, *the elongation varies directly as the weight.* The law connecting the variables is then stated by the linear equation

$$y = kx,$$

where y is the elongation, x the weight, and k a constant.

That is, y is a function of x, and a change in the variable x produces a corresponding change in y.

A change in the weight is called an *increment of the weight*, or an increment of x, and is represented by the symbol Δx (read "increment of x" or "delta x"). A corresponding change in the elongation is called an increment of the elongation, or an increment of y, and is represented by Δy.

Here x represents the independent variable and y the dependent variable.

It is evident that for every Δx there is a Δy. Their relation may be shown as follows:

For any particular value of x as x_1, $y_1 = kx_1$. (1)

If $x = x_1 + \Delta x$, $\quad y_1 + \Delta y = k(x_1 + \Delta x)$. (2)

Subtracting (1) from (2), $\quad \Delta y = k\Delta x$.

That is, Δy varies directly as Δx, and is independent of the value of x.

This is shown graphically in Fig. 151. The locus of $y = kx$ is a straight line with slope k. P_1 is a point on the line with coördinates (x_1, y_1). $MN = P_1Q = \Delta x$, and $QR = \Delta y$.

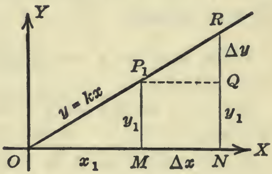


FIG. 151.

No matter what the magnitude of Δx,

$$\Delta y = \Delta x \tan QP_1R = k\Delta x.$$

Example 2.—The distance s that a heavy body near the earth's surface falls from rest in time t is given by the formula

$$s = \tfrac{1}{2}gt^2.$$

If $t = t_1$, $\quad s_1 = \frac{1}{2}gt_1^2$. (1)

If $t = t_1 + \Delta t$, $\quad s_1 + \Delta s = \frac{1}{2}g(t_1 + \Delta t)^2$. (2)

Subtracting (1) from (2), $\Delta s = \frac{1}{2}g(2t_1\Delta t + \Delta t^2)$.

That is, the value of Δs depends upon both t and Δt.

This is shown graphically in Fig. 152. The locus of $s = \frac{1}{2}gt^2$ is a parabola. The point P_1 has coördinates (t_1, s_1), and

Δt and Δs are as shown in the figure. It is evident from the figure that Δs depends upon both t and Δt.

Definitions and notation.—If y is a function of x it may be written $y = f(x)$, which is to be read "y equals a function of x" or "y equals f of x." For convenience other symbols may be used for functions, as $F(x)$, $\varphi(x)$, $f'(x)$, etc.

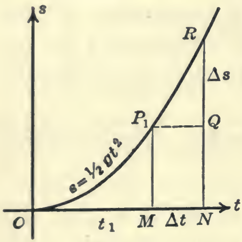


FIG. 152.

In the equation $y = f(x)$, that is, when the equation expresses y explicitly in terms of x, y is an **explicit function** of x.

If two variables are involved in an equation in such a manner that it is necessary to solve the equation in order to express either explicitly in terms of the other, then either variable is said to be an **implicit function** of the other.

Thus, in $x^2 + y^2 = r^2$, y is an implicit function of x and x is an implicit function of y.

If this is solved for y, $y = \pm\sqrt{r^2 - x^2}$, in which y is an explicit function of x.

If solved for x, $x = \pm\sqrt{r^2 - y^2}$, in which x is an explicit function of y.

Implicit functions of x and y may be written $f(x, y)$, $F(x, y)$, $\varphi(x, y)$, etc.

In the same discussion or problem the same functional symbol is used to represent the same function.

Thus, if $f(x) = 2x^2 + 3x + 1$,
then $f(a) = 2a^2 + 3a + 1$,
and $f(3) = 2 \cdot 3^2 + 3 \cdot 3 + 1 = 28$.
If $f(x, y) = 3x^2 + 4xy - y$,
then $f(2, 3) = 3 \cdot 2^2 + 4 \cdot 2 \cdot 3 - 3 = 33$,
and $f(y, x) = 3y^2 + 4xy - x$.

EXERCISES

1. If $y = 10x$ and x_1 is any particular value of x, find Δy when x takes the increment Δx. Find Δy when $x_1 = 4$ and $\Delta x = 2$. Find Δy for any other value of x and $\Delta x = 2$. Plot so as to show these graphically.

2. If $y = 2x^2 + 1$ and $x_1 = 2$, find Δy when $\Delta x = 0.5$. Find Δy when $\Delta x = 0.01$. Plot.

3. Express the area A of a square as a function of its side x. Find ΔA for $x = 6$ and $\Delta x = 1$. Illustrate by means of a square.

4. Express the area A of a circle as a function of its radius x. Find ΔA for $x = 10$ and $\Delta x = 0.5$. Illustrate by means of a circle.

5. Express the area of a square as a function of its diagonal. Express its diagonal as a function of its area.

6. Express the circumference of a circle as a function of its area. Express the surface of a sphere as a function of its volume.

7. Express the volume of a right circular cylinder as a function of its radius and altitude. Express the altitude as a function of its volume and radius. Express its lateral area as a function of its volume and diameter.

8. If $f(x) = x^3 + 3x^2 - 2x - 4$, find $f(0), f(3), f(-4)$.

9. If $F(x) = \sqrt{x^2 + 1}$, find $F(0)$, $F(-3)$, $F(4\sqrt{3})$.

10. If $\varphi(x) = \log_{10} x$, find $\varphi(100)$, $\varphi(47.62)$, $\varphi(0.012)$.

11. If $f(x) = \cos x$, find $f(30°)$, $f(\frac{1}{2}\pi)$, $f(240°)$.

12. If $f(x, y) = 3x^2y + 4xy^2 - 2y^2$, find $f(-x, y), f(x, -y), f(-x, -y)$.

13. If $f(y) = 3^y$, prove $f(x) \cdot f(y) = f(x + y)$.

14. If $f(x) = \sin x$ and $F(x) = \cos x$, prove that

$$f(x + y) = f(x)F(y) + F(x)f(y).$$

15. If $y = \sin x$, express x explicitly in terms of y. If $y = 2^x$, express x explicitly in terms of y.

In each of the following equations express each variable explicitly in terms of the other, if it can be done by methods previously studied.

16. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

17. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$

18. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

19. $x(x - 2a)^2 - ay^2 = 0.$

20. $x^2y^2 + 4x^4 = 16.$

21. $4x^2 + y^2 - 8x - 2y + 1 = 0.$

22. $\frac{x^2y^2}{16 - y^2} = (y + 2)^2.$

23. $\frac{i}{\sin (2t + \frac{1}{2}\pi)} = e^{-\frac{1}{2}t}.$

24. $\varphi^2 \cos 2\theta = a^2.$

25. $\varphi \sin \theta \tan \theta = 4a.$

26. $\varphi^2 \cos \theta = a^2 \sin 3\theta.$

27. $\log_{10} x - \log_{10} y + 3 \log_{10} a = 0.$

28. $\sin^{-1} x - \sin^{-1} y = 45°.$

29. $\frac{x^2y^2}{x^3 - 1} = \frac{x - 1}{x + y}.$

30. If $s = 16t^2$ and $t_1 = 2$, find Δs and $\frac{\Delta s}{\Delta t}$ when $\Delta t = 1$; when $\Delta t = 0.1$; when $\Delta t = 0.01$; when $\Delta t = 0.001$. What value does $\frac{\Delta s}{\Delta t}$ seem to be approaching as Δt becomes smaller?

31. If $y = x^3$ and $x_1 = 1$, find Δy and $\frac{\Delta y}{\Delta x}$ when $\Delta x = 10$; when $\Delta x = 1$;

when $\Delta x = 0.1$; when $\Delta x = 0.01$; when $\Delta x = 0.001$. What value does $\frac{\Delta y}{\Delta x}$ approach? What then is the slope of the tangent at the point where $x = 1$ of the curve of $y = x^3$? Plot.

LIMITS

168. Illustrations and definitions.—Considerable use has been made of limits in elementary geometry, trigonometry, and algebra, but much greater use is necessary in the study of calculus. The following are simple examples of limits:

(1) The variable which takes the successive values 1.3, 1.33, 1.333, · · · has as a limit $1\frac{1}{3}$. That is, the more figures there are taken, the more nearly the number approaches $1\frac{1}{3}$.

(2) The number $\sqrt{2}$ is the limit of the successive values 1.4, 1.41, 1.414, 1.4142, · · · . The diagonal of a unit square is the limit of the line lengths represented by this series of numbers.

(3) If a point starts at the end A of the line AB, Fig. 153, and during the first second moves half the length of the line to C; during the next second, half of the remaining distance to D; continuing in this way to move half the remaining distance during each successive second, then the distance that the point is from A is a variable of which AB is the limit.

Fig. 153.

(4) If $y = \dfrac{12}{x + 2}$ and x is a variable approaching 2 as a limit, then evidently y is a variable approaching 3 as a limit.

Definitions.—When a variable changes in such a manner that its successive values approach a constant so nearly that the difference between the constant and the variable *becomes* and *remains* less, in absolute value, than any assigned positive number, however small, the constant is the **limit of the variable.**

The variable is also said to *approach the constant as a limit.* If the variable is represented by x and the constant by a, then the statement "x approaches a as a limit" is written thus, $x \doteq a$.

The form $\lim\limits_{x \doteq a} [f(x)] = A$ is read "the limit of $f(x)$ as x approaches a as a limit is A."

When a variable changes in such a manner that it becomes and remains greater than any assigned positive number, however great, it is said to **increase without limit** or to **become infinite.**

The notation to represent this is $x = \infty$, which is read "x increases without limit" or "x becomes infinite."

The form $\lim\limits_{x = \infty} [f(x)] = A$ is read "the limit of $f(x)$ as x becomes infinite is A."

169. Elementary theorems of limits.—The following theorems will be found useful in dealing with limits. They are given here without proof.

(1) *If two variables that approach limits are equal for all their successive values, their limits are equal.*

(2) *The limit of the sum of a constant and a variable that approaches a limit is the sum of the constant and the limit of the variable.*

(3) *The limit of the product of a constant and a variable that approaches a limit is the product of the constant and the limit of the variable.*

(4) *If each of a finite number of variables approaches a limit, the limit of their sum is the sum of their respective limits.*

(5) *If each of a finite number of variables approaches a limit, the limit of their product is the product of their respective limits.*

(6) *If each of two variables approaches a limit, the limit of their quotient is the quotient of their limits, except when the limit of the divisor is zero.*

If the limit of the divisor is zero the limit of the quotient may have a definite finite value or the quotient may become infinite, but it is not determined by finding the *quotient of* *the limits* of the two variables. The calculus determines such limits as these exceptional cases.

170. Derivatives.—The fundamental conception of differential calculus, and one that is of the greatest importance in mathematics, is the *derivative* of a function. Using the notation of this chapter the **derivative** is defined to be the limit approached by the quotient $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

If the curve, Fig. 154, represents the function $y = f(x)$, the quotient $\frac{\Delta y}{\Delta x}$ is the slope of the secant line P_1P.

If P_1 remains fixed and Δx approaches zero as a limit, the point P moves along the curve and approaches P_1 as a limit, and the secant P_1P turns about P_1 to the limiting position QR, which is defined to be the **tangent to the curve** at the point P_1.

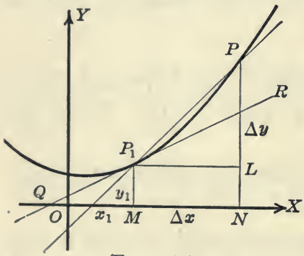


Fig. 154.

Hence, *the slope of the tangent is precisely the quantity called the derivative.*

It is evident that the value of the derivative depends upon the position of P_1 on the curve.

Definition.—The **slope of a curve** at any point is the slope of the tangent to the curve at that point.

The notation for the derivative is $\frac{dy}{dx}$, read "the derivative of y with respect to x." Then by definition

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \lim_{\Delta \mathbf{x} \doteq 0} \left[\frac{\Delta \mathbf{y}}{\Delta \mathbf{x}}\right].$$

Of course, the independent variable and the function may be represented by other letters. Thus, $\frac{du}{dt} = \lim_{\Delta t \doteq 0} \left[\frac{\Delta u}{\Delta t}\right]$.

The notation $\left.\frac{dy}{dx}\right|_{x=x_1}$ is used to indicate the value of $\frac{dy}{dx}$ for the particular value x_1 of x.

Example.—Given $y = x^2$, find $\left.\frac{dy}{dx}\right|_{x=2}$, and thus find the slope of the tangent to the parabola at the point (2, 4). Also find the equation of this tangent and plot.

Solution.—(1) Given $y = x^2$.

(2) When $x = 2$, $y = 4$.

(3) If x takes an increment Δx, $y + \Delta y = (2 + \Delta x)^2 = 4 + 4\Delta x + \overline{\Delta x}^2$.

(4) Subtracting (2) from (3), $\Delta y = 4\Delta x + \overline{\Delta x}^2$.

(5) Dividing by Δx, $\frac{\Delta y}{\Delta x} = 4 + \Delta x$.

(6) Letting $\Delta x = 0$, $\left.\frac{dy}{dx}\right|_{x=2} = 4$.

Hence the slope of the tangent to the parabola at the point (2, 4) is 4.

The equation of this tangent by **[15]** is

$$y - 4 = 4(x - 2), \text{ or } 4x - y = 4.$$

The plotting is shown in Fig. 155.

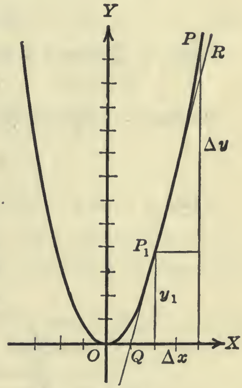


FIG. 155.

171. Tangents and normals.—It follows from the preceding article and **[15]** that the equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) is

[46] $$\mathbf{y - y_1 = \left.\frac{dy}{dx}\right|_{x=x_1}(x - x_1).}$$

Definition.—The **normal to a curve** at any point is the line perpendicular to the tangent to the curve at that point.

Then by **[9]** and **[15]** the equation of the normal to the curve $y = f(x)$ at the point (x_1, y_1) is

[47] $$\mathbf{y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_1}}(x - x_1).}$$

Example.—Find the equations of the tangent and normal to the ellipse $4x^2 + 9y^2 = 36$ at the point (x_1, y_1). Also find these equations when $x_1 = 2$. Plot.

Solution.—(1) Given $4x^2 + 9y^2 = 36$.

(2) Let $x = x_1$ and $y = y_1$. $4x_1^2 + 9y_1^2 = 36$.

If x takes the increment Δx, y will have the increment Δy, and

(3) $$4(x_1 + \Delta x)^2 + 9(y_1 + \Delta y)^2 = 36,$$

or $$4x_1^2 + 8x_1\Delta x + 4\overline{\Delta x}^2 + 9y_1^2 + 18y_1\Delta y + 9\overline{\Delta y}^2 = 36.$$

(4) Subtracting (2) from (3), $8x_1\Delta x + 4\overline{\Delta x}^2 + 18y_1\Delta y + 9\overline{\Delta y}^2 = 0.$

(5) Transposing and arranging, $\frac{\Delta y}{\Delta x} = -\frac{8x_1 + 4\Delta x}{18y_1 + 9\Delta y}.$

Passing to the limits and noticing that $\Delta y \doteq 0$ as $\Delta x \doteq 0$,

$$\left.\frac{dy}{dx}\right|_{x=x_1} = -\frac{4x_1}{9y_1}.$$

Substituting in [**46**], the equation of the tangent is

$$y - y_1 = -\frac{4x_1}{9y_1}(x - x_1),$$

or

$$4x_1x + 9y_1y = 4x_1^2 + 9y_1^2.$$

Since by (2) $4x_1^2 + 9y_1^2 = 36$ the equation of the tangent is

$$4x_1x + 9y_1y = 36.$$

Similarly the equation of the normal is

$$y - y_1 = \frac{9y_1}{4x_1}(x - x_1).$$

When $x = 2$, $y = \pm\frac{2}{3}\sqrt{5}$.

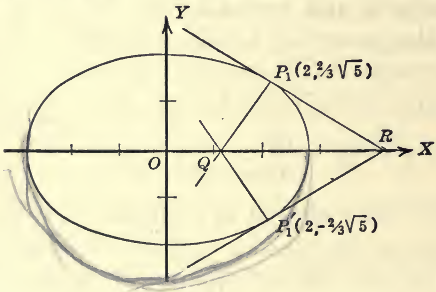


FIG. 156.

Substituting these values for x_1 and y_1 in the equation of the tangent, the equation of the tangent at $(2, \frac{2}{3}\sqrt{5})$ is

$$4\cdot 2x + 9\cdot\tfrac{2}{3}\sqrt{5}y = 36, \text{ or } 4x + 3\sqrt{5}y = 18.$$

And the equation of the tangent at the point $(2, -\frac{2}{3}\sqrt{5})$ is

$$4\cdot 2x + 9(-\tfrac{2}{3}\sqrt{5})y = 36,$$

or

$$4x - 3\sqrt{5}y = 18.$$

Likewise the equations of the normals are, by [**47**]:
at the point $(2, \frac{2}{3}\sqrt{5})$, $9\sqrt{5}x - 12y = 10\sqrt{5}$;
and at the point $(2, -\frac{2}{3}\sqrt{5})$, $9\sqrt{5}x + 12y = 10\sqrt{5}$.

The plotting is shown in Fig. 156.

EXERCISES

1. Given $y = x^3$, compute the values of Δy and $\dfrac{\Delta y}{\Delta x}$ when $x = 0.5$ and $\Delta x = 1$, 0.1, 0.01, and 0.001 respectively.

2. Find the slope of the tangent and normal to $y = 4x^2$ at the point where $x = 0.5$.

3. Given $y = x^2 + 2$, find $\left.\dfrac{dy}{dx}\right|_{x=1}$ and write the equation of the tangent and normal at this point. Check the result by plotting.

Find $\dfrac{dy}{dx}$ at the point (x_1, y_1) for each of the following:

4. $y = 3x^2 - 1$.

5. $y = x^3 + 4$.

6. $y = 2x + 5$.

7. $xy = 4$.

8. $y^2 = 2px$.

9. $y = \dfrac{1}{x - 1}$.

10. $y = x^3 + 2x^2 + 5$.

11. $y = 3x^3 - 4x^2 + 6x$.

12. $x^2 + 2y^2 = 16$.

13. $4x^2 - 9y^2 = 36$.

14. $y^2 = 4x + 8$.

15. $y = \dfrac{x + 1}{x - 1}$.

16. Given the parabola $y^2 = 2px$, find the equations of the lines tangent to the parabola at the extremities of the latus rectum, and show that they meet on the directrix.

17. Find the slope of the circle $x^2 + y^2 = 25$ where $x = 2$, (*a*) when the point is in the first quadrant, and (*b*) when the point is in the fourth quadrant.

18. Find the angle that the line $3x - 4y + 7 = 0$ makes with the circle $x^2 + y^2 = 25$ at their point of intersection in the first quadrant.

19. At what angle does the circle $x^2 + y^2 = 16$ intersect the circle $x^2 + y^2 = 8x$ at their point of intersection in the first quadrant?

ALGEBRAIC FUNCTIONS

172. Differentiation by rules.—The process of finding the derivative of a function is called **differentiation.**

The method used in the preceding articles in finding the derivative is called the **fundamental method** since it is based directly upon the definition of a derivative. The derivative of any function can be found by this method, but the work can be greatly shortened by using rules or formulas which can be established by fundamental methods or otherwise. The rules needed in differentiating algebraic functions will be considered first, and later some of those necessary to differentiate trigonometric, exponential, and logarithmic functions.

In the formulas, x, y, u, and v denote variables, which, of course, may be functions of variables, and a, c, and n denote constants.

173. The derivative when f(x) is x.—Since the equation $y = x$ represents a straight line with slope equal to 1, and by article 170, $\frac{dy}{dx}$ is the slope of the curve at any point, it follows that

$$\text{I.} \qquad \frac{\mathbf{dx}}{\mathbf{dx}} = \mathbf{1}.$$

In general, *the derivative of a variable with respect to itself is unity.*

174. The derivative when f(x) is c.—Since $y = c$ is the equation of a straight line with slope equal to 0, it follows that

$$\text{II.} \qquad \frac{\mathbf{dc}}{\mathbf{dx}} = \mathbf{0}.$$

In general, *the derivative of a constant is zero.*

175. The derivative of the sum of functions.—Given $y = u + v$, where u and v are functions of x, and let Δy, Δu, and Δv be the increments of y, u, and v, respectively, corresponding to the increment Δx.

Let $x = x_1$, then $y_1 = u_1 + v_1$.

Let $x = x_1 + \Delta x$, then $y_1 + \Delta y = u_1 + \Delta u + v_1 + \Delta v$.

Subtracting, $\Delta y = \Delta u + \Delta v$.

Dividing by Δx, $\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$.

Let $\Delta x \doteq 0$, then $\left.\frac{dy}{dx}\right|_{x=x_1} = \left.\frac{du}{dx}\right|_{x=x_1} + \left.\frac{dv}{dx}\right|_{x=x_1}$.

It is evident that any number of functions can be treated in a similar manner, then

$$\text{III.} \qquad \frac{\mathbf{d(u + v + w + \cdots)}}{\mathbf{dx}} = \frac{\mathbf{du}}{\mathbf{dx}} + \frac{\mathbf{dv}}{\mathbf{dx}} + \frac{\mathbf{dw}}{\mathbf{dx}} + \cdots.$$

Or, *the derivative of the sum of any number of functions is equal to the sum of their derivatives.*

Example.—If $y = x^3 + 3x^2 - 4x + 3$,

$$\frac{dy}{dx} = \frac{d(x^3)}{dx} - \frac{d(3x^2)}{dx} - \frac{d(4x)}{dx} - \frac{d(3)}{dx}.$$

176. The Derivative of the product of two functions.—With the notation as in the previous article, given $y = uv$.

Let $x = x_1$, then $y_1 = u_1v_1$.

Let $x = x_1 + \Delta x$, then $y_1 + \Delta y = (u_1 + \Delta u)(v_1 + \Delta v)$.

Subtracting, $\Delta y = u_1\Delta v + v_1\Delta u + \Delta u \cdot \Delta v$.

Dividing by Δx, $\dfrac{\Delta y}{\Delta x} = u_1 \dfrac{\Delta v}{\Delta x} + v_1 \dfrac{\Delta u}{\Delta x} + \Delta u \dfrac{\Delta v}{\Delta x}$.

Let $\Delta x \doteq 0$ and notice that $\Delta u\dfrac{\Delta v}{\Delta x}$ also approaches zero as a limit,

then $$\left.\frac{dy}{dx}\right|_{x=x_1} = u_1 \left.\frac{dv}{dx}\right|_{x=x_1} + v_1 \left.\frac{du}{dx}\right|_{x=x_1}.$$

IV. $$\therefore \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Or, *the derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.*

Example.—If $y = (x - 2)(x^2 + 1)$,

$$\frac{dy}{dx} = (x - 2)\frac{d(x^2 + 1)}{dx} + (x^2 + 1)\frac{d(x - 1)}{dx}.$$

177. The derivative of the product of a constant and a function.—Given $y = cu$, where c is a constant. By the previous article

$$\frac{dy}{dx} = c\frac{du}{dx} + u\frac{dc}{dx}.$$

But $\dfrac{dc}{dx} = 0.$ By II.

V. $$\therefore \frac{d(cu)}{dx} = c\frac{du}{dx}.$$

Or, *the derivative of the product of a constant and a function is equal to the constant times the derivative of the function.*

Examples.—If $y = 4(x - 2)$, $\frac{dy}{dx} = 4\frac{d(x-2)}{dx}$.

If $y = \frac{u}{a}$, $\frac{dy}{dx} = \frac{1}{a}\frac{du}{dx}$.

178. The derivative of the quotient of two functions.—Given $y = \frac{u}{v}$.

Let $x = x_1$, then $y_1 = \frac{u_1}{v_1}$.

Let $x = x_1 + \Delta x$, then $y_1 + \Delta y = \frac{u_1 + \Delta u}{v_1 + \Delta v}$.

Subtracting, $\Delta y = \frac{u_1 + \Delta u}{v_1 + \Delta v} - \frac{u_1}{v_1} = \frac{v_1 \Delta u - u_1 \Delta v}{v_1(v_1 + \Delta v)}$.

Dividing by Δx, $\frac{\Delta y}{\Delta x} = \frac{v_1\frac{\Delta u}{\Delta x} - u_1\frac{\Delta v}{\Delta x}}{v_1(v_1 + \Delta v)}$.

Let $\Delta x \doteq 0$, then $\left.\frac{dy}{dx}\right|_{x=x_1} = \frac{v_1\left.\frac{du}{dx}\right|_{x=x_1} - u_1\left.\frac{dv}{dx}\right|_{x=x_1}}{{v_1}^2}$.

VI. $$\therefore \frac{\mathbf{d}\left(\frac{\mathbf{u}}{\mathbf{v}}\right)}{\mathbf{dx}} = \frac{\mathbf{v}\frac{\mathbf{du}}{\mathbf{dx}} - \mathbf{u}\frac{\mathbf{dv}}{\mathbf{dx}}}{\mathbf{v}^2}.$$

Or, *the derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Example.—If $y = \frac{x-1}{2x}$, $\frac{dy}{dx} = \frac{2x\frac{d(x-1)}{dx} - (x-1)\frac{d(2x)}{dx}}{4x^2}$.

179. The derivative of the power of a function.—Given $y = u^n$.

(*a*) *When n is a positive integer.*

Writing as a product $y = u \cdot u^{n-1}$.

Then $\frac{dy}{dx} = u^{n-1}\frac{du}{dx} + u\frac{d(u^{n-1})}{dx}$. By IV.

Writing u^{n-1} as the product $u \cdot u^{n-2}$,

$$\frac{dy}{dx} = u^{n-1}\frac{du}{dx} + u\left[u^{n-2}\frac{du}{dx} + u\frac{d(u^{n-2})}{dx}\right]$$

$$= 2u^{n-1}\frac{du}{dx} + u^2\frac{d(u^{n-2})}{dx}.$$

When this process is performed n times the last term will contain $\frac{d(u^{n-n})}{dx}$, which is zero by II.

VII. $$\therefore \quad \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

(b) *When n is a fraction,* $\frac{p}{q}$, *where p and q are positive integers.*

Given $$y = u^{\frac{p}{q}}$$

Raising both sides of the equation to the qth power,

$$y^q = u^p.$$

Then $$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}. \quad \text{By } (a) \text{ of this article.}$$

Solving for $\frac{dy}{dx}$, $$\frac{dy}{dx} = \frac{pu^{p-1}}{qy^{q-1}}\frac{du}{dx}$$

$$= \frac{pu^{p-1}}{qu^{\frac{p}{q}(q-1)}}\frac{du}{dx} = \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}.$$

$$\therefore \quad \frac{d(u^{\frac{p}{q}})}{dx} = \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}.$$

(c) *When n is negative, either integral or fractional.*

Let $n = -m$.

Then $$y = u^{-m} = \frac{1}{u^m}.$$

Clearing of fractions, $yu^m = 1$.

Then $myu^{m-1}\frac{du}{dx} + u^m\frac{dy}{dx} = 0$. By IV, VII, and II.

Solving for $\frac{dy}{dx}$, $\frac{dy}{dx} = -\frac{myu^{m-1}}{u^m}\frac{du}{dx}$

$$= -myu^{-1}\frac{du}{dx} = -mu^{-m-1}\frac{du}{dx}.$$

$$\therefore \quad \frac{d(u^{-m})}{dx} = -mu^{-m-1}\frac{du}{dx}.$$

Therefore formula **VII** is established when the exponent is a positive or negative integer or fraction. It is expressed in the following rule:

The derivative of a function affected by an exponent n is equal to n times the function affected by the exponent n − 1, *times the derivative of the function.*

Examples.—If $y = (x^2 + x + 1)^4$, $\frac{dy}{dx} = 4(x^2 + x + 1)^3\frac{d(x^2 + x + 1)}{dx}$.

If $y = (x + 1)^{\frac{1}{2}}$, $\frac{dy}{dx} = \frac{1}{2}(x + 1)^{-\frac{1}{2}}\frac{d(x + 1)}{dx}$.

If $y = (x^2 - 1)^{-3}$, $\frac{dy}{dx} = -3(x^2 - 1)^{-4}\frac{d(x^2 - 1)}{dx}$.

180. Summary of formulas for algebraic functions.—The formulas here summarized enable one to differentiate algebraic functions.

I. $\frac{dx}{dx} = 1.$

II. $\frac{dc}{dx} = 0.$

III. $\frac{d(u + v + w + \cdots)}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \cdots.$

IV. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$

V. $\frac{d(cu)}{dx} = c\frac{du}{dx}.$

VI. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

VII. $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$

181. Examples of Differentiation.—If the formulas and rules of differentiation are well learned, their application is one of the easiest processes in mathematics.

Example 1.—Given $y = 7x^3$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d(7x^3)}{dx} = 7\frac{d(x^3)}{dx} = 7 \cdot 3x^2 \frac{dx}{dx} = 21x^2. \quad \text{By V, VII, and I.}$$

Example 2.—Given $y = x^3 + 2x^2 - 5x + 6$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d(x^3)}{dx} + \frac{d(2x^2)}{dx} - \frac{d(5x)}{dx} + \frac{d(6)}{dx} \qquad \text{By III.}$$

$$= 3x^2 + 4x - 5. \qquad \text{By VII, V, II, and I.}$$

Example 3.—Given $y = (x^2 + 2x)(3x - 2)$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = (x^2 + 2x)\frac{d(3x - 2)}{dx} + (3x - 2)\frac{d(x^2 + 2x)}{dx} \qquad \text{By IV.}$$

$$= (x^2 + 2x)3 + (3x - 2)(2x + 2) = 9x^2 + 8x - 4.$$

By I, II, III, V, VII.

Example 4.—Given $y = \frac{x^2 + 2}{3x - 1}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{(3x - 1)\frac{d(x^2 + 2)}{dx} - (x^2 + 2)\frac{d(3x - 1)}{dx}}{(3x - 1)^2} \qquad \text{By VI.}$$

$$= \frac{(3x - 1)(2x) - (x^2 + 2)3}{(3x - 1)^2} = \frac{3x^2 - 2x - 6}{(3x - 1)^2}.$$

Example 5.—Given $y = \sqrt[3]{x^2 + 3x}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d\sqrt[3]{x^2 + 3x}}{dx} = \frac{d(x^2 + 3x)^{\frac{1}{3}}}{dx} = \tfrac{1}{3}(x^2 + 3x)^{-\frac{2}{3}}\frac{d(x^2 + 3x)}{dx}$$

$$= \tfrac{1}{3}(x^2 + 3x)^{-\frac{2}{3}}(2x + 3) = \frac{2x + 3}{3\sqrt[3]{(x^2 + 3x)^2}}.$$

EXERCISES

In the following find the derivative of the function with respect to the independent variable.

1. $y = 3x^2$.
2. $y = 5x^4$.
3. $y = 7x^{\frac{4}{3}}$.
4. $y = ax^{\frac{2}{3}}$.
5. $y = \frac{3}{4}x^{\frac{3}{4}}$.
6. $y = 4\sqrt{x}$.
7. $y = 3\sqrt[3]{x}$.
8. $y = \sqrt[5]{x^3}$.
9. $y = 3x^{-\frac{2}{3}}$.
10. $y = -4x^{\frac{2}{3}}$.
11. $y = -17\sqrt{x^3}$.
12. $y = -2\sqrt[5]{x^4}$.
13. $s = \frac{1}{2}gt^2$.
14. $s = 4t^{\frac{3}{5}}$.
15. $s = \frac{1}{2}t^{\frac{3}{2}}$.
16. $y = x^4 + 3x^2 + 2$.
17. $y = 3x^2 - 2x + 6$.

18. $y = x^3 - x^{\frac{3}{2}} + 3x.$

19. $y = x^{\frac{3}{2}} - x^{-\frac{1}{2}} + 4.$

20. $y = x^{\frac{5}{2}} - 3x^{-3} + 2.$

21. $y = (2x + 1)^3 - 3.$

22. $y = (3x^2 + 2)^4 - 2x.$

23. $y = (2x + 3)^{\frac{1}{2}} - 3x.$

24. $y = \sqrt{2x^2 - 7x}.$

25. $y = \sqrt[3]{x^2 + 7x - 2}.$

26. $y = \frac{1}{x^2}.$

27. $y = \frac{1}{x^3}.$

28. $y = \frac{3}{2x^2}.$

29. $y = \frac{2}{x + 1}.$

30. $y = \frac{1}{\sqrt{x + 1}}.$

31. $y = \frac{5}{(x^2 - 1)^2}.$

32. $y = \frac{x - 1}{x + 1}.$

33. $y = \frac{x + 2}{x - 3}.$

34. $y = \frac{x}{x^2 + 1}.$

35. $y = 3x^7 - 4x^6 + 3x^4 - 3.$

36. $y = \sqrt{x + 1} - \sqrt{x - 1}.$

37. $y = \sqrt{3x^3 + 7x^2 - 3x + 2}.$

38. $y = \sqrt{ax^2 + bx + c} - \sqrt{x + d}.$

39. $y = x^2(x^3 + 5)^{\frac{2}{3}}.$

40. $y = \frac{2x - 1}{(x - 1)^2}.$

41. $y = \sqrt{\frac{x - 1}{x + 1}}.$

42. $y = \frac{x}{\sqrt{x^2 - a^2}}.$

43. $s = \sqrt{t + 1} + \sqrt[3]{2t - 3}.$

44. $s = t^{\frac{1}{2}} + 2t^{-\frac{1}{2}} + 3t^4.$

45. $y = (x^2 + 1)(x^3 - 2x + 1).$

46. $y = (x + a)^n(x - b)^m.$

47. $y = (x + 1)^5(2x - 1)^3.$

48. $y = \frac{2x^2 - 1}{(x - 1)^2}.$

49. $s = \frac{t^n}{(1 + t)^n}.$

50. $s = \sqrt{\frac{t^2 - 1}{t^2 + 1}}.$

51. Find the slope of the tangent line to the curve $y = x^3$ at the point where $x = 0$. At the point where $x = 1$. Where $x = 2$.

52. In exercise 51, what is the slope of the curve at each of the points? How many times faster is y increasing than x at each of the points?

53. If a point is moving from the origin along the curve $y = 2x^2$ in the first quadrant, what is the relative rate of increase of x and y when $x = 1$, 2, and 4, respectively?

54. Find the equations of the tangent and the normal to the curve $y = x^3 + 4x^2 + x - 6$ at the point $(0, -6)$. At the point $(2, 20)$.

55. In the curve of exercise 54, where is the tangent line parallel to the x-axis?

56. Find the equations of the tangent and the normal to the curve $y = x + \frac{1}{x^2}$ at the point (x_1, y_1).

57. Find the point on the curve $y = x^3 + 3x^2 - 4x - 12$ at which the tangent has a slope of -7. What is the equation of the tangent at this point? Plot the curve.

58. At what angle does the line $y = x - 1$ intersect the parabola $y^2 + 4x = 4$?

59. Show that the parabola $y^2 = 4ax$ and the cissoid $y = \frac{x^3}{2a - x}$ intersect at right angles at the origin.

60. The heat H, required to raise a unit weight of water from 0°C. to a temperature $t°$, is given by the formula

$$H = t + 0.00002t^2 + 0.0000003t^3.$$

Find $\frac{dH}{dt}$ and compute the value of $\frac{dH}{dt}$ where $t° = 35°$C.

182. Differentiation of implicit functions.—In the previous exercises, the dependent variable in each was expressed as an explicit function of the independent variable. Often it is either not convenient or not possible to express one variable as an explicit function of the other. In such a case the usual rules for finding the derivative can be applied and the desired derivative found as an implicit function of the variables involved. The method can be best illustrated by examples.

Example 1.—Given $x^2 + y^2 = 25$, find $\frac{dy}{dx}$ as an implicit function of x and y.

Since y is a function of x, the left hand member is the sum of two functions of x.

Differentiating, $$2x + 2y\frac{dy}{dx} = 0.$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}.$$

Example 2.—Find the equation of the tangent line to the curve $x^5 - y^5 + x^3 - y = 0$ at the point (1, 1).

Solution.—Differentiating, $5x^4 - 5y^4\frac{dy}{dx} + 3x^2 - \frac{dy}{dx} = 0.$

Solving for $\frac{dy}{dx}$, $$\frac{dy}{dx} = \frac{5x^4 + 3x^2}{5y^4 + 1}.$$

When $x = 1$ and $y = 1$, $$\frac{dy}{dx} = \frac{4}{3}.$$

Then the slope of the tangent at (1, 1) is $\frac{4}{3}$.

Hence the equation of the tangent is $y - 1 = \frac{4}{3}(x - 1)$,

or $$4x - 3y - 1 = 0.$$

EXERCISES

In the following find the derivatives as implicit functions.

1. $x^3 + y^3 = a^3$, find $\dfrac{dy}{dx}$.

2. $y^3 + y = x^3 + x$, find $\dfrac{dy}{dx}$.

3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, find $\dfrac{dy}{dx}$.

4. $pv = c$, find $\dfrac{dp}{dv}$ and $\dfrac{dv}{dp}$.

5. $x^4 - 4x^2y^2 + y^3 = 0$, find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$.

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$.

7. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$, find $\dfrac{dy}{dx}$.

8. $(x + y)^{\frac{3}{2}} + (x - y)^{\frac{3}{2}} = a$, find $\dfrac{dy}{dx}$.

9. $\left(p + \dfrac{a}{v^2}\right)(v - b) = k$, find $\dfrac{dp}{dv}$ and $\dfrac{dv}{dp}$.

10. Find the equations of the tangent and the normal to the circle $x^2 + y^2 = 25$ at the point (3, 4).

11. Find the equations of the tangent and the normal to the circle $x^2 + y^2 - 4x + 6y - 24 = 0$ at the point (1, 3).

12. Find the equations of the tangent and normal to the ellipse $16x^2 + 25y^2 = 144$ at the point in the first quadrant where $x = 2$.

Show that the tangents to the following curves at the point (x_1, y_1) are as given.

	Equation of curve	*Equation of tangent*
13.	$x^2 + y^2 = r^2$.	$x_1x + y_1y = r^2$.
14.	$y^2 = 2px$.	$y_1y = p(x + x_1)$.
15.	$x^2 = 2py$.	$x_1x = p(y + y_1)$.
16.	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.	$\dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1$.
17.	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.	$\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$.
18.	$xy = c$.	$x_1y + y_1x = 2c$.

19. Find the equations of the tangent and normal to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ at the point (x_1, y_1).

FURTHER USES OF THE DERIVATIVE

183. Discussion.—By methods of analytic geometry the properties of the locus that are most conveniently discussed are the *intercepts*, *symmetry*, and *extent*. (See **Art. 43**). By means of the derivative other properties may be discussed. Some of these will be considered in the following articles.

The discussion will be confined to equations (1) whose curves have no break, at least in the part of the curve considered; and (2) where for each value of the independent variable there is but one point on the curve. Such curves, as well as the functions giving rise to them, are said to be **continuous** and **single-valued.**

184. Properties of a curve and its function.—If the curve, Fig. 157, is thought of as traced by a moving point passing from left to right, the following properties may be noted:

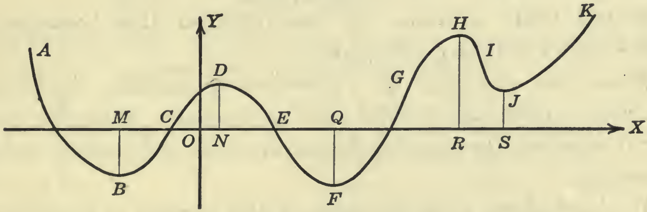


Fig. 157.

(1) The curve is *falling* from A to B, from D to F, and from H to J; and the corresponding function is *decreasing*.

(2) The curve is *rising* from B to D, from F to H, and from J to K; and the corresponding function is *increasing*.

(3) If the curve rises to a certain position and then falls, such a position is called a **maximum point** of the curve. D and H are such points. The ordinate, that is, the value of the function, at such a point is called a **maximum ordinate** or **maximum value of the function.**

(4) If the curve falls to a certain position and then rises, such a position is called a **minimum point** of the curve. B, F, and

J are such points. The ordinate, that is, the value of the function, at such a point is called a **minimum ordinate** or a **minimum value of the function.**

(5) The curve is *concave upward* between A and C, E and G, and I and K. It is *concave downward* between C and E, and G and I.

(6) Points C, E, G, and I where the concavity changes, are called **points of inflection.**

Curves may have other peculiarities, but these will not be considered here.

185. Curves rising or falling, functions increasing or decreasing.—Since by definition, **Art. 170,** the slope of a curve at any point is the same as the slope of the tangent at that point, it follows that when the slope is positive the curve is rising, and when the slope is negative the curve is falling. This is, of course, when passing from left to right.

Stated with reference to the function this becomes the following very useful principle:

When the derivative of a function is positive, the function increases as the independent variable increases; when the derivative is negative, the function decreases as the independent variable increases.

It also follows that the ratio of the change of the function at any point to that of the variable is equal to the value of the derivative of the function with respect to the variable, for that point.

Example 1.—For what values of x is the curve $y = x^2$ rising and for what values falling?

Solution.—Given $y = x^2$.

Then $$\frac{dy}{dx} = 2x.$$

Now $2x$ is positive when x is positive, and negative when x is negative. Hence the curve is rising when $x>0$, and falling when $x<0$.

Example 2.—For what values of x is the function $y = x^3$ increasing and for what values decreasing?

Here $\dfrac{dy}{dx} = 3x^2$, which is not negative for any value of x.

Hence the function is never decreasing.

Is it always increasing?

Example 3.—For what values of x is the curve $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x$ rising and for what values falling? For what values of x is y increasing 6 times as fast as x?

Solution.—Given $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x$.

Then $$\frac{dy}{dx} = x^2 - x - 6.$$

Factoring, $$\frac{dy}{dx} = (x + 2)(x - 3).$$

Then $\frac{dy}{dx}$ is positive when $x < -2$ and when $x > 3$, and negative when $-2 < x < 3$.

Hence the curve is rising when $x < -2$ and when $x > 3$, and falling when $-2 < x < 3$.

The values of x for which y is increasing 6 times as fast as x can be found by putting $x^2 - x - 6 = 6$, and solving for x.

This gives $x = 4$ or -3.

EXERCISES

Passing from left to right, for what values of x are the loci of the following equations rising and for what values falling?

1. $y = 3x - 6$.

2. $y = 4x^2 + 16x - 7$.

3. $y = \sqrt{4x}$.

4. $y^3 = 8x^2$.

5. $y = x^3 + 3$.

6. $xy = 15$.

7. $y = x^3 - 9x$.

8. $y = x^3 - x^2 - 2x$.

9. $y = x^3 - 2x^2 + x - 3$.

10. $y(1 + x^2) = x$.

11. $y(x^2 - 1)^2 = x^3$.

12. $6y = 2x^3 - 3x^2 - 12x - 6$.

13. $y = x^4 - 6x^2 + 8x + 6$.

14. $y = (x^2 - 1)^4$.

15. In exercise 8, how many times as rapidly as x is y increasing when $x = 10$? When $x = 3$? When $x = -1$? When $x = 0$?

16. In exercise 9, for what values of x is y increasing 7 times as rapidly as x? For what values of x is y decreasing 4 times as rapidly as x is increasing?

186. Maximum and minimum.—From the definitions of article 184, it is clear that if a curve is plotted in rectangular coördinates, the curve is *rising* at nearby points on the left of a maximum point, and *falling* at nearby points on the right. For a minimum point the curve is *falling* for nearby points on the left and *rising* on the right. The student can readily state this with reference to the function.

It is evident that at a maximum point or a minimum point like those shown in Fig. 157, the tangent line is parallel to the x-axis, that is, its slope is zero.

It follows that these points can be determined from the function as follows:

(1) Equate $\frac{dy}{dx}$ to zero and solve for x.

(2) Determine whether $\frac{dy}{dx}$ is positive or negative for nearby points on the left and right.

A point where $\frac{dy}{dx} = 0$ is a maximum point if $\frac{dy}{dx} > 0$ for nearby points on the left and $\frac{dy}{dx} < 0$ for nearby points on the right.

A point where $\frac{dy}{dx} = 0$ is a minimum point if $\frac{dy}{dx} < 0$ for nearby points on the left and $\frac{dy}{dx} > 0$ for nearby points on the right.

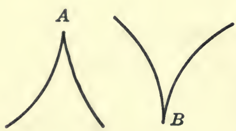


Fig. 158.

It is distinctly understood that these tests determine only such points as are illustrated in Fig. 157. For cusp maximum and minimum points as shown in Fig. 158, the tangent is perpendicular to the x-axis and hence $\frac{dy}{dx} = \infty$.

Example.—Determine the maximum and minimum points of the function $y = x^3 - 3x^2 + 4$ and plot the curve.

Solution.—Given $y = x^3 - 3x^2 + 4$.

$$\frac{dy}{dx} = 3x^2 - 6x = 3x(x - 2).$$

$$\therefore \frac{dy}{dx} = 0 \text{ for } x = 0, \text{ and } x = 2.$$

When $x < 0$ but near 0, $\frac{dy}{dx} > 0$ and the curve is rising.

When $x > 0$ but near 0, $\frac{dy}{dx} < 0$ and the curve is falling.

$\therefore$ the curve has a maximum point when $x = 0$.

When $x < 2$ but near 2, $\frac{dy}{dx} < 0$ and the curve is falling.

When $x > 2$ but near 2, $\frac{dy}{dx} > 0$ and the curve is rising.

$\therefore$ the curve has a minimum point when $x = 2$.

Plotting.—When $x = 0$, $y = 4$. $\therefore$ (0, 4) is a maximum point.

When $x = 2$, $y = 0$. $\therefore$ (2, 0) is a minimum point.

Factoring, $y = (x + 1)(x - 2)(x - 2)$.

$\therefore$ the x-intercepts are -1, 2, and 2.

A few other points will make the plotting fairly accurate. See Fig. 159.

x	1	3	4	-2
y	2	4	20	-16

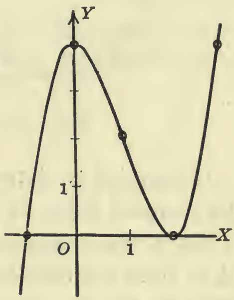


FIG. 159.

EXERCISES

Determine the maximum and minimum points of the following curves and plot.

1. $y = x^2$.
2. $2y = x^2 - 4x + 6$.
3. $y = 6x - x^2 + 4$.
4. $4x^2 + 9y^2 = 36$.
5. $y = (x + 4)(x - 2)(x - 4)$.
6. $y = x^3 - 7x^2 + 36$.
7. $16y = x^2 - 32x$.
8. $y = x^4 - 4x^3$.
9. By finding the maximum point of the curve, find the coördinates of the vertex of the parabola $2x^2 - 18x + 15y - 21 = 0$.
10. The equation of the path of a projectile is

$$y = \tan \alpha \cdot x - \frac{g}{2v^2 \cos^2 \alpha} x^2. \quad \text{(See **Art. 94.**)}$$

Find the maximum height to which the projectile rises.

187. Concavity and points of inflection.—It is evident from an inspection of a curve that is *concave upward* that the tangent line turns *counter-clockwise* in passing along a curve from left to right; that is, the slope of the tangent *increases.*

Likewise, if the curve is *concave downward*, the tangent line turns *clockwise*, that is, the slope of the tangent is *decreasing.*

Thus, in Fig. 160, the tangent line turns counter-clockwise in passing from A to D, and the slope increases from a negative value at A to a positive value at D.

Likewise, in Fig. 161, the tangent turns clockwise in passing from A to D, and the slope decreases from a positive value at A to a negative value at D.

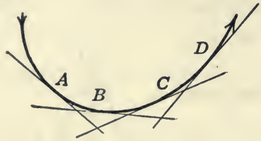


FIG. 160.

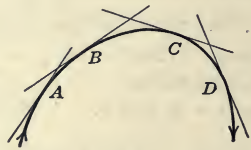


FIG. 161.

It remains to determine how the concavity of a curve can be determined from its function.

Since the derivative of a function of x is itself a function of x, it is evident that the derivative of this first derivative may be found. It is called the **second derivative** of y with respect to x.

If $y = f(x)$, the second derivative is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ and is represented by the symbol $\frac{d^2y}{dx^2}$.

Thus, if $y = x^3 - 6x^2 + 12x - 3$.

$$\frac{dy}{dx} = 3x^2 - 12x + 12,$$

and

$$\frac{d^2y}{dx^2} = 6x - 12.$$

From the foregoing, it is evident that when $\frac{d^2y}{dx^2}$ is positive, $\frac{dy}{dx}$ is increasing; and when $\frac{d^2y}{dx^2}$ is negative, $\frac{dy}{dx}$ is decreasing.

Or, if $y = f(x)$ is the equation of a curve, the slope of the tangent is increasing when passing from left to right and the curve is concave upward for the values of x that make $\frac{d^2y}{dx^2}$ *positive.*

Likewise, the curve is concave downward when $\frac{d^2y}{dx^2}$ is *negative.*

From (6) of article 184, it is evident that a point of inflection is a point on a curve at which the concavity changes from upward to downward or *vice versa.* A point of inflection can be determined by finding the values of x for which $\frac{d^2y}{dx^2}$ changes sign, providing the function is finite for that value of x.

Example.—Investigate $y = x^3 - 3x^2 + x + 2$ for concavity and points of inflection.

Solution.—Given $y = x^3 - 3x^2 + x + 2.$

$$\frac{dy}{dx} = 3x^2 - 6x + 1.$$

$$\frac{d^2y}{dx^2} = 6x - 6 = 6(x - 1).$$

Since when $x<1$, $6(x - 1)$ is negative; and when $x>1$, $6(x - 1)$ is positive, the curve is concave downward at the left of $x = 1$, and concave upward at the right of $x = 1$. Therefore, it has a point of inflection at the point (1, 1).

EXERCISES

In exercises 1–10 investigate for concavity and points of inflection.

1. $y = x^3$.

2. $y = x^4$.

3. $y = x^5$.

4. $y = 3x - x^3$.

5. $y = x^4 - 6x^2$.

6. $y = (x + 2)(x - 2)(x - 3)$.

7. $y = 3x^4 - 4x^3 - 1$.

8. $y = x^3 - 4x^2 + 4x - 1$.

9. $y = x^4 - 2x^2 + 40$.

10. $y = 3x^4 - 16x^3 - 6x^2 + 48x + 17$.

11. In the example, **Art. 187,** find the slope of the tangent to the curve at the point of inflection, find the maximum and minimum points, and plot the curve.

12. In the example referred to in exercise 11, if the curve is being traced by a point moving from left to right, for what values of x does y increase at the same rate as x? How rapidly is the curve rising when $x = 3$ if x is increasing at the rate of 2 inches per second?

13. Investigate the greatest possible number of points of inflection of the curves of

(1) $y = ax^2 + bx + c$,

(2) $y = ax^3 + bx^2 + cx + d$,

(3) $y = ax^4 + bx^3 + cx^2 + dx + e$.

In exercises 14–19 plot the curves showing the values of y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$, using the same set of axes for the three curves of each. What facts can be read from these curves?

14. $y = 4x^3$.
15. $y = 3x^4$.
16. $y = 3x - x^3$.
17. $y = (x + 2)(x - 2)(x - 3)$.
18. $y = x^3 - 12x + 7$.
19. $y = x^4 - 2x^2 - 8$.

DIFFERENTIALS

188. Relations between increments.—When two variables are so related that the ratio of their corresponding increments is *constant*, either variable is said to change **uniformly** with respect to the other.

When the variables are related by an equation of the first degree, as $y = mx + b$, where $\Delta y = m\Delta x$, then $\frac{\Delta y}{\Delta x} = m$. That is, either variable changes uniformly with respect to the other.

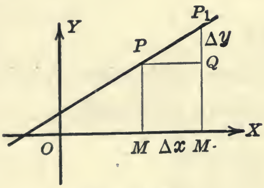


FIG. 162.

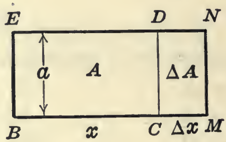


FIG. 163.

This is also evident from Fig. 162, in which $y = mx + b$ is the equation of the line PP_1 with slope m. P is any point on this line and $\frac{\Delta y}{\Delta x} = m$.

In Fig. 163, $BCDE$ is a rectangle having a constant altitude a and a variable base x. When x takes an increment Δx, the area A will take an increment $a\Delta x$.

$$\therefore \quad \Delta A = a\Delta x \quad \text{or} \quad \frac{\Delta A}{\Delta x} = a.$$

When two variables are so related that the ratio of their corresponding increments is *variable*, either variable is said to change **non-uniformly** with respect to the other.

If the variables s and t are related by the equation $s = \frac{1}{2}gt^2$, then $\Delta s = \frac{1}{2}g(2t\Delta t + \overline{\Delta t}^2)$. See **Art. 167,** example 2.

$$\therefore \quad \frac{\Delta s}{\Delta t} = \frac{1}{2}g(2t + \Delta t).$$

Here $\frac{\Delta s}{\Delta t}$ is a variable for it varies with t, that is, different values of t give different values of $\frac{\Delta s}{\Delta t}$, and the change is non-uniform.

189. Differentials.—If two variables are so related that one is dependent and the other is independent, then for corresponding values of the variables:

(1) The **differential of the independent variable** is the value of its increment.

(2) The **differential of the dependent variable** is what would be its increment, if at the corresponding values considered, its change *became* and *remained* uniform with respect to the independent variable.

The differential of a variable is denoted by writing d before it.

Thus, differential x is denoted by dx. Also dy, $d(x^3)$, $d(x^2 + 2x + 1)$, and $df(x)$ denote the differentials of y, x^3, $x^2 + 2x + 1$, and $f(x)$, respectively.

190. Illustrations.—It follows from the definitions that the differentials of variables that change *uniformly* with respect to each other, are their corresponding increments.

Thus, if $y = mx + b$, $dx = \Delta x$ and $dy = \Delta y$, for y changes *uniformly* with respect to x.

It should be noted that $dy = \Delta y$ when, and only when, the graph of $y = f(x)$ is a straight line.

If the rectangle of constant altitude, Fig. 163, is increased in area by increasing the base by the length CM, the area is increased by the rectangle $CMND$. Here evidently the area,

A, is a function of the base x. Since A and x change *uniformly* with respect to each other, $CM = dx$ and the rectangle $CMND = dA$.

Consider the curve $y = f(x)$, Fig. 164, as being traced by a point starting from the origin and moving to the right and upward. The direction that the tracing point is moving at any point is along the tangent line at that point.

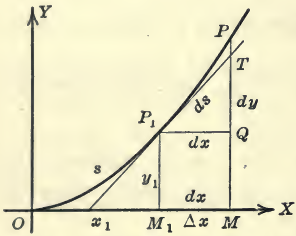


FIG. 164.

Let (x, y) be the coördinates of the moving point.

Evidently, y is changing *non-uniformly* with respect to x.

Suppose the moving point has reached P_1. Here y is evidently changing at the same rate it would if the point were moving along the tangent line at P_1. If then the change in y is to *become* and *remain* uniform with respect to x, the point must move along the tangent.

It follows that at the point P_1, if the increment of x is $\Delta x = M_1M$, $dx = \Delta x$, and $dy = QT$.

It is to be noted that the corresponding increment of y is $\Delta y = QP$.

Further, if the slope of the tangent, $\frac{dy}{dx}$, that is, the derivative, is represented by $f'(x)$,

$$\mathbf{dy = f'(x)dx.}$$

Since dy and dx are finite quantities, dividing by dx,

$$\frac{dy}{dx} = f'(x).$$

This is an extremely important and useful relation, for it states that *the derivative and the ratio of the differentials can be used interchangeably.*

Again, referring to Fig. 164, if s is the length of the curve traced, then corresponding to dx and dy, the change in s, if this change *becomes* and *remains* uniform, is $ds = P_1T$, and

$$\mathbf{ds^2 = dx^2 + dy^2.}$$

The triangle P_1QT is called the **differential triangle.**

Example.—A point is moving along the parabola $y = 3x^2$. When it has reached the point whose abscissa is 2, find dy and ds corresponding to $dx = 0.1$.

Solution.—First find the derivative of y with respect to x.

Given equation $y = 3x^2$.

$$\frac{dy}{dx} = 6x.$$

$$\therefore \quad dy = 6x \cdot dx, \text{ for any value of } x.$$

When $x = 2$ and $dx = 0.1$, $dy = 6 \cdot 2 \cdot 0.1 = 1.2$.
And $ds = \sqrt{dx^2 + dy^2} = \sqrt{0.1^2 + 1.2^2} = 1.2042-$.

EXERCISES

1. The right triangle, Fig. 165, is being generated by the altitude moving uniformly to the right. If the variable base is x and the area A, show that dA corresponding to dx is the rectangle M_1MQP_1.

2. The area of the upper half of the area of the parabola $y^2 = 4x$ is being generated by the ordinate moving toward the right. If A is the variable area, show that $dA = 2\sqrt{x}\,dx$ for any value of x. Draw the figure.

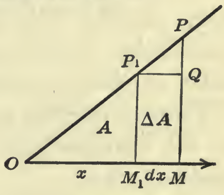


FIG. 165.

3. If the upper half of the area of the circle $x^2 + y^2 = r^2$ is being generated by the ordinate moving uniformly toward the right, show that $dA = \sqrt{r^2 - x^2}\,dx$.

4. The area above the x-axis between $y = \sin x$ and the x-axis is being generated by its ordinate. Show that $dA = \sin x\,dx$. For the part below the x-axis show that $dA = -\sin x\,dx$.

5. A point is moving on the circle $x^2 + y^2 = 25$. Find dy and ds corresponding to a change in x of $dx = 0.2$ at the point in the first quadrant where $x = 3$.

6. A point is moving on the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$. Find dy corresponding to $dx = 0.4$ at the point in the first quadrant where $x = 2$. In the second quadrant where $x = -2$.

In the following find dy for any x.

7. $y = 3x^2 + 2x - 1.$

8. $y = x^3 + 4x + 2.$

9. $y = x^4 - 3x^3 + 2x^2.$

10. $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

11. $y = \sqrt{x^2 + 4}.$

12. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4.$

13. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2.$

14. $y = \frac{1}{\sqrt{x^2 + 5}}.$

15. The distance s that a body will fall in t seconds is given by the formula $s = \frac{1}{2}gt^2$. Find ds for any value of t. Find ds when $t = 2$ and corresponding to $dt = 1$. (Use $g = 32$.)

INTEGRATION

191. The inverse of differentiation.—Just as division is the operation that is the inverse of multiplication, and the extraction of a root is the inverse of raising to a power, so differentiation has its inverse operation. Here, as usual, the inverse operation is the more difficult. In fact, it is frequently impossible to do the inverse of a differentiation except approximately.

The process of doing the inverse of a differentiation is called **integration.** The result obtained is called an **integral.**

The methods of integrating can be dealt with here to only a very limited extent. In general, an integral is found by knowledge acquired from differentiation, by reversing the rules of differentiation, or by reference to a table of integrals.

Integration has very many applications to problems arising in the sciences and in engineering as well as to problems in mathematics.

The symbol, $\int$, indicates that the differential before which it is written is to be integrated.

Thus, $\int 2x dx$ indicates that a function of x is to be found whose differential is $2xdx$. The function is evidently $x^2 + C$, where C is any constant, for

$$d(x^2 + C) = \frac{d(x^2 + C)}{dx} dx = 2x\,dx.$$

Since the differential of any constant is zero, the function sought when integrating may contain a constant no indication of which appears in the given differential. For this reason the integral of a different'al is, in general, indefinite, and is called an **indefinite integral.**

The constant C that is supplied when integrating is called the **constant of integration.**

192. Determination of the constant of integration.—The constant of integration is determined by having some fact about the function given besides its differential. This can be best illustrated by examples.

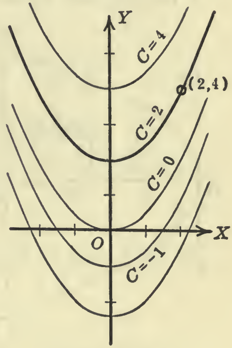


FIG. 166.

Example 1.—Find the equation of a curve such that the slope of its tangent line at any point shall equal to the abscissa of the point if, further, it is given that the curve passes through the point (2, 4).

Solution.—Since $\frac{dy}{dx}$ = slope of tangent, and x = abscissa of point of tangency,

$$\frac{dy}{dx} = x.$$

Considering $\frac{dy}{dx}$ as the ratio of dy to dx, and multiplying by dx,

$$dy = xdx$$

Then $\int dy = \int xdx$, and $y = \frac{1}{2}x^2 + C.$

Here C is any constant, and the equation represents all parabolas having their axes on the y-axis and opening upward. Some of these are represented in Fig. 166.

It is evident that one such parabola can pass through any particular point of the plane. The one sought passes through (2, 4), and therefore these values must satisfy the equation $y = \frac{1}{2}x^2 + C.$

Substituting (2, 4) in this equation,

$$4 = \tfrac{1}{2}.2^2 + C. \quad \therefore C = 2.$$

The equation of the curve satisfying both conditions is then

$$y = \tfrac{1}{2}x^2 + 2.$$

Example 2.—Find the area enclosed by the parabola $y^2 = 4x$ and the double ordinate corresponding to $x = 8$.

Solution.—The parabola $y^2 = 4x$ is shown in Fig. 167, and is symmetrical with respect to the x-axis. Then one-half of the area is above the x-axis and is the area OMC.

Consider the area A as generated by an ordinate moving from the origin toward the right. When it has advanced any distance x

$$dA = y\,dx.$$

But $y = +2\sqrt{x}$ since y is positive.
Then $dA = 2\sqrt{x}\,dx$.
Integrating, $A = \frac{4}{3}x^{\frac{3}{2}} + C$. (1)
A further fact about the area A is that $A = 0$. when $x = 0$

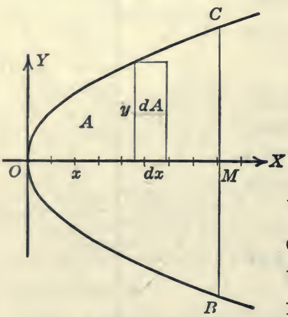


Fig. 167.

Substituting these values in (1) gives

$$0 = 0 + C. \quad \therefore C = 0.$$

Hence for any value of x, $A = \frac{4}{3}x^{\frac{3}{2}} + 0$.
And for $x=8$, $A = \frac{4}{3}\cdot 8^{\frac{3}{2}} = \frac{64}{3}\sqrt{2} = 30.17-$.
$\therefore$ the total area $= 2A = 60.34-$ square units.

193. Methods of integrating.—While a knowledge of differentiation enables one to write at once the integrals of many differentials, the following formulas will help in integrating forms that occur frequently.

(1) $$\int \mathbf{u}^n \mathbf{du} = \frac{\mathbf{u}^{n+1}}{\mathbf{n+1}} + \mathbf{C}.$$

Here u may be any function of which du is the differential, and n is not equal to -1.

That (1) is true can be readily proved by finding the differential of $\dfrac{u^{n+1}}{n+1} + C$.

Example 1.—Find $\int x^4 dx$.
Here $x = u$, $dx = du$, and $n = 4$.

$$\therefore \int x^4 dx = \tfrac{1}{5}x^5 + C.$$

Example 2.—Find $\int (x^3 + x^2)^2(3x^2 + 2x)dx$.
Here $x^3 + x^2 = u$, $(3x^2 + 2x)dx = du$, and $n = 2$.

$$\therefore \int (x^3 + x^2)^2(3x^2 + 2x)dx = \int u^2 du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(x^3 + x^2)^3 + C.$$

Example 3.—Find $\int \sqrt{x^2 - 1}\, 2x dx$.
Here $x^2 - 1 = u$, $2xdx = du$, and $n = \frac{1}{2}$.

$$\therefore \int \sqrt{x^2 - 1}\, 2xdx = \int u^{\frac{1}{2}} du = \tfrac{2}{3}u^{\frac{3}{2}} + C = \tfrac{2}{3}(x^2 - 1)^{\frac{3}{2}} + C.$$

(2) $$\int \mathbf{c\,du} = \mathbf{c}\int \mathbf{du}.$$

This states that a constant can be written either before or after a sign of integration.

Proof.—Since $d(cu) = cdu$. By differentiation.
Then $cu = \int cdu$. By definition of an integral.
But $\int du = u$. By (1) where $n = 0$.
And $c\int du = cu$. Equating values of cu
$\therefore \int cdu = c\int du$.

Example.— $\int 6x^3dx = 6\int x^3dx = 6 \cdot \frac{1}{4}x^4 + C = \frac{3}{2}x^4 + C$.

(3) $$\int (du + dv) = \int du + \int dv.$$

Proof.— $d(u + v) = du + dv$. By differentiation.
Then $\int (du + dv) = u + v$. By definition of an integral.
But $u = \int du$ and $v = \int dv$. By (1).
$\therefore \int (du + dv) = \int du + \int dv$.

This can readily be extended to the integral of the sum of any number of differentials.

Example.—$\int (x^3 + 3x^2 - x + 1)dx$
$= \int x^3dx + \int 3x^2dx - \int xdx + \int dx$ By (3).
$= \frac{1}{4}x^4 + x^3 - \frac{1}{2}x^2 + x + C$. By (1) and (2).

Here C is the sum of the several constants of integration.

EXERCISES

Find the indefinite integrals in exercises 1–10, and check by differentiation.

1. $dy = 4xdx$.
2. $dy = x^2dx$.
3. $dy = 4x^3dx$.
4. $dy = x^{\frac{2}{3}}dx$.
5. $dy = x^ndx$.
6. $dy = (2x + 1)dx$.
7. $dy = (2x^2 + x + 2)dx$.
8. $dy = (x - 1)(x + 1)dx$.
9. $dy = (x + 1)^3dx$.
10. $dy = (x + 1)^{\frac{3}{2}}dx$.

11. Find the equation of the curve whose slope at any point is equal to three times the abscissa of that point, and which passes through the point (2, 6).

12. Find the equation of the curve whose slope at any point is equal to the square of its abscissa at that point, and which passes through the point (1, 1).

13. Find the equation of the curve whose slope at any point is equal to the square root of its abscissa at that point, and which passes through the point (2, 4).

14. Find the area enclosed by the parabola $y^2 = 2x$ and the double ordinate corresponding to $x = 4$.

15. Find the area enclosed by the parabola $y^2 = 3x$, the x-axis, and the ordinates corresponding to $x = 2$ and $x = 8$.

16. Find the area between the curve $y = 2x$ and the x-axis from the origin to the ordinate corresponding to $x = 10$. Check by finding the area considered as a triangle.

17. Find the area between the curve $y = x^3$ and the x-axis from the origin to the ordinate corresponding to $x = 4$.

18. Find the area between the curve $y = x^3$ and the x-axis from the ordinate corresponding to $x = -3$ to the origin.

19. Find the area enclosed by the semi-cubical parabola $y = x^{\frac{3}{2}}$ and the double ordinate corresponding to $x = 4$.

20. Find the area enclosed by the curve $y = x^{-\frac{1}{2}}$, the x-axis, and the ordinates corresponding to $x = \frac{1}{2}$ and $x = 8$.

21. Find the area that is below the x-axis and is enclosed by the parabola $y = x^2 - 4x + 3$ and the x-axis.

22. Find the area that is below the x-axis and is enclosed by the curve $y = x^3 - 4x^2 + 3x$ and the x-axis.

TRIGONOMETRIC FUNCTIONS

194. So far in the calculus a study has been made of algebraic functions only. The trigonometric functions will now be considered to a limited extent. The sine and cosine will receive the chief attention, the formulas of the others will be given for completeness only.

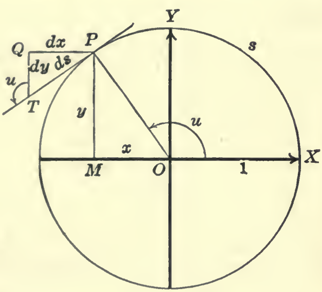


FIG. 168.

195. Derivatives of sin u and cos u.—Let O be a unit circle generated by the point $P(x, y)$ moving in the positive direction, Fig. 168.

Let u be the measure of the angle XOP in radians, and let s be the measure of the arc XP in linear units.

Then $u = s$, $x = \cos u$, and $y = \sin u$.

Differentiating, $du = ds$, $dx = d(\cos u)$, and $dy = d(\sin u)$.

In the differential triangle PQT,

$$dx = PQ,\ dy = QT, \text{ and } ds = PT.$$

Also the angle at T through which the tangent has turned is equal to u.

Now $dy = ds \cdot \cos u$, $\cos u$ and dy having the same sign.

But $dy = d(\sin u)$ and $ds = du$.

$\therefore\quad d(\sin u) = \cos u\, du.$

Dividing by dx gives the derivative formula:

VIII. $$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}.$$

Also $dx = -ds \sin u$, $\sin u$ and dx having opposite signs.

But $dx = d(\cos u)$ and $ds = du$.

$\therefore\quad d(\cos u) = -\sin u du.$

Dividing by dx gives the derivative formula:

IX. $$\frac{d(\cos n)}{dx} = -\sin u \frac{du}{dx}.$$

It is to be noted that the derivation of VIII and IX requires that the angle shall be in radians.

Example 1.—Given $y = \sin(3x^2 + 4x - 1)$, find $\frac{dy}{dx}$.

Solution.— $$\frac{dy}{dx} = \frac{d \sin(3x^2 + 4x - 1)}{dx} =$$

$$= \cos(3x^2 + 4x - 1)\frac{d(3x^2 + 4x - 1)}{dx}. \quad \text{By VIII.}$$

But $\frac{d(3x^2 + 4x - 1)}{dx} = \frac{d(3x^2)}{dx} + \frac{d(4x)}{dx} - \frac{d1}{dx} = 6x + 4.$

$\therefore\quad \frac{dy}{dx} = (6x + 4)\cos(3x^2 + 4x - 1).$

Example 2.—Find the maximum and minimum points of the curve $y = \cos x$.

Solution.— $\frac{dy}{dx} = -\sin x.$

Putting $\frac{dy}{dx} = 0$ gives $-\sin x = 0$.

$\therefore\quad x = n\pi$, where $n = 0, \pm 1, \pm 2, \cdots$.

For values of x near an even number of times π but less than π, $-\sin x$ is positive; and for values of x near an even number of times π, but greater than π, $-\sin x$ is negative. Hence the curve is rising before and falling after $x = 2n\pi$.

$\therefore$ maximum points are the points for which $x = 2n\pi$.

Likewise minimum points are the points for which $x = (2n + 1)\pi$.

Example 3.—Find the area enclosed by an arch of the curve $y = \sin x$ and the x-axis.

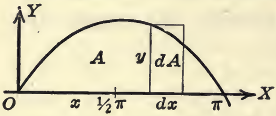


FIG. 169.

Solution.—The curve $y = \sin x$ is shown in Fig. 169. The area sought extends from $x = 0$ to $x = \pi$.

Consider this area A as being generated by an ordinate moving toward the right.

Then $dA = ydx = \sin x\, dx$.

And $\int dA = \int \sin x\, dx$.

$\therefore A = -\cos x + C$. By the inverse of differentiation.

When $x = 0$, $A = 0$.

$\therefore 0 = -\cos 0 + C$, or $C = 1$.

When $x = \pi$, $A = -\cos \pi + 1 = 2$ = number of square units in area.

196. Derivatives of other trigonometric functions.—The following formulas are stated for completeness. Their derivation is not difficult and may be performed as exercises.

X. $$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}.$$

XI. $$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}.$$

XII. $$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}.$$

XIII. $$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}.$$

XIV. $$\frac{d(\text{vers } u)}{dx} = \sin u \frac{du}{dx}.$$

XV. $$\frac{d(\sin^{-1} u)}{dx} = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

XVI. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

XVII. $$\frac{d(\tan^{-1} u)}{dx} = \frac{\frac{du}{dx}}{1 + u^2}.$$

XVIII. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{\frac{du}{dx}}{1 + u^2}.$$

XIX. $$\frac{d(\sec^{-1} u)}{dx} = \frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}}.$$

XX. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}}.$$

XXI. $$\frac{d(\text{vers}^{-1} u)}{dx} = \frac{\frac{du}{dx}}{\sqrt{2u - u^2}}.$$

197. $\int \sin u\, du$ and $\int \cos u\, du$.—Many integrals involving trigonometric functions occur in the applications of calculus, but here attention will be confined to $\int \sin u\, du$ and $\int \cos u\, du$.

$$\int \sin u\, du = -\cos u + C.$$

This is readily proved, for

$d(-\cos u + C) = d(-\cos u) = \sin u\, du.$

$$\int \cos u\, du = \sin u + C$$

For $d(\sin u + C) = d(\sin u) = \cos u\, du.$

Example.—Find $\int \sin(2x + 1)dx$.

If $2x + 1 = u$, $du = 2\, dx$.

Then write $\int \sin(2x + 1)dx$ in the form $\frac{1}{2}\int \sin(2x + 1)2dx$ and it is in the form of $\int \sin u\, du$.

$\therefore \int \sin(2x + 1)dx = \frac{1}{2}\int \sin(2x + 1)2dx = -\frac{1}{2}\cos(2x + 1) + C$

EXERCISES

In exercises 1–20 find the derivatives.

1. $y = \sin 3x$.

2. $y = \sin^2 x$.

3. $y = \cos(2x + 1)$.

4. $y = \sin x \cos x$.

5. $y = \sin 3x \cos 2x$.

6. $y = \tan 3x$.

7. $y = \dfrac{\sin x}{\cos x}$.

8. $y = \tan^3 5x$.

9. $y = x \sin x$.

10. $y = \sin (x^3 + x^2)$.

11. $y = \sin (x^2 + 3x - 4)$.

12. $y = \cos^2 (3x + 2)$.

13. $y = \sqrt{\sin 3x}$.

14. $y = \sin^2 x \cos^2 x$.

15. $y = \sin^2 x\sqrt{\sec x}$.

16. $y = m \cot^n qx$.

17. $y = \dfrac{1 - \cos x}{1 + \cos x}$.

18. $\rho = \tan 3\theta + \sec 3\theta$.

19. $\rho = \frac{1}{3} \tan^3 \theta - \tan \theta + \theta$.

20. $y = 3 \sin x - 4 \sin^3 x$.

21. Given $x = a(\theta - \sin \theta)$ and $y = a(1 - \cos \theta)$, find dx and dy, then by division find $\dfrac{dy}{dx}$.

22. Find the area enclosed by one arch of the curve $y = \cos x$ and the x-axis.

23. Find the slope of the tangent to the curve $y = \sin x$ at the point where $x = \frac{1}{4}\pi$. Where $x = 2$.

24. Find the slope of the tangent to the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ at the point where $\theta = \frac{1}{2}\pi$. Where $\theta = \pi$. Where $\theta = 0$.

25. Find the maximum and minimum points, and the points of inflection of the curve $y = \sin x$.

Find the indefinite integrals in exercises 26–33.

26. $\int \sin 3x \, dx$.

27. $\int \sin (3x - 1)dx$.

28. $\int \cos 4x \, dx$.

29. $\int \cos (4x - 2)dx$.

30. $\int \sin x \cos x \, dx$.

31. $\int \sin^3 x \cos x \, dx$.

32. $\int \cos^2 x \sin x \, dx$.

33. $\int \sin^n x \cos x \, dx$.

34. Find the equation of the curve passing through the point $(\pi, 0)$, if the slope of the tangent at any point is equal to the cosine of the abscissa of that point.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

198. Derivative of $\log_e u$.—We will first find $\dfrac{dy}{dx}$ when $y = \log_e x$.*

Let $P(x_1, y_1)$ be a point on the curve.

Then $\qquad y_1 = \log_e x_1$ or $x_1 = e^{y_1}$.

Let $x = x_1 + \Delta x$, and $x_1 + \Delta x = e^{y_1 + \Delta y}$.

Subtracting, $\Delta x = e^{y_1 + \Delta y} - e^{y_1} = e^{y_1}(e^{\Delta y} - 1)$.

* In $\log_e x$, $e = 2.71828 \cdots$, the base of the natural system of logarithms.

Dividing by Δy, $\dfrac{\Delta x}{\Delta y} = \dfrac{e^{\Delta y} - 1}{\Delta y} = x_1 \cdot \dfrac{e^{\Delta y} - 1}{\Delta y}.$

Or $$\frac{\Delta y}{\Delta x} = \frac{1}{x_1} \cdot \frac{\Delta y}{e^{\Delta y} - 1}.$$

Then $\lim\limits_{\Delta x \doteq 0} \left[\dfrac{\Delta y}{\Delta x}\right] = \dfrac{1}{x_1} \lim\limits_{\Delta y \doteq 0} \left[\dfrac{\Delta y}{e^{\Delta y} - 1}\right]$, since $\Delta y = 0$ when $\Delta x = 0$.

Or $$\frac{dy}{dx} = \frac{1}{x_1} \lim_{\Delta y \doteq 0} \left[\frac{\Delta y}{e^{\Delta y} - 1}\right].$$

But it can be shown that $\lim\limits_{\Delta y \doteq 0} \left[\dfrac{\Delta y}{e^{\Delta y} - 1}\right] = 1.$

Then, dropping the subscripts, $\dfrac{dy}{dx} = \dfrac{1}{x}$, or $dy = \dfrac{1}{x} dx.$

Evidently, if $y = \log_e u$, then $dy = \dfrac{1}{u} du.$

Dividing by dx, $\dfrac{dy}{dx} = \dfrac{1}{u} \dfrac{du}{dx}.$

XXII. $$\therefore \quad \frac{d(\log_e u)}{dx} = \frac{1}{u} \frac{du}{dx}.$$

199. Derivative of $\log_a u$.—Let a be any base. Since by a theorem of logarithms, $\log_a u = \log_e u \cdot \log_a e$.

Then $$\frac{d(\log_a u)}{dx} = \frac{d(\log_e u)}{dx} \log_a e. \qquad \text{By V.}$$

XXIII. $$\therefore \quad \frac{d(\log_a u)}{dx} = \frac{1}{u} \frac{du}{dx} \log_a e. \qquad \text{By XXII.}$$

If, in XXIII, $a = 10$, $\log_{10} u$ expresses the common logarithm of u, and

$$\frac{d(\log_{10} u)}{dx} = \frac{1}{u} \frac{du}{dx} \cdot M,$$

where $M = \log_{10} e = 0.4343-$.

200. Derivative of a^u and e^u.—Let $y = a^u$.

Then $$\log_e y = u \log_e a.$$

Taking the derivative of each side of this equation by XXII and V,

$$\frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}\log_e a.$$

Or
$$\frac{dy}{dx} = y\frac{du}{dx}\log_e a.$$

XXIV.
$$\therefore \frac{d(a^u)}{dx} = a^u\frac{du}{dx}\log_e a.$$

If a is put equal to e, and noting that $\log_e e = 1$, XXIV becomes

XXV.
$$\frac{d(e^u)}{dx} = e^u\frac{du}{dx}.$$

201. Derivative of u^v.—Let $y = u^v$, where u and v are functions of x.

Then
$$\log_e y = v\log_e u.$$

Taking the derivative of each side of this equation by XXII and IV,

$$\frac{1}{y}\frac{dy}{dx} = \frac{v}{u}\frac{du}{dx} + \frac{dv}{dx}\log_e u.$$

$$\frac{dy}{dx} = y\cdot\frac{v}{u}\frac{du}{dx} + y\cdot\frac{dv}{dx}\log_e u$$

$$= u^v\frac{v}{u}\frac{du}{dx} + u^v\frac{dv}{dx}\log_e u.$$

XXVI
$$\therefore \frac{d(u^v)}{dx} = vu^{v-1}\frac{du}{dx} + u^v\frac{dv}{dx}\log_e u.$$

The application of formulas VII, XXIV, and XXVI should be carefully distinguished. Formula VII is used when a variable is affected by a constant exponent; XXIV is used when a constant is affected by a variable exponent; and XXVI is used when a variable is affected by a variable exponent.

It is customary in calculus to omit the base when writing logarithms to the base e, and to express the base when it is not e.

Thus, $\log 5$ means $\log_e 5$.

202. Illustrative Examples.

Example 1.—Given $y = \log (x^2 + 3x)$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{x^2 + 3x} \frac{d(x^2 + 3x)}{dx}, \qquad \text{By XXII.}$$

$$= \frac{1}{x^2 + 3x} (2x + 3). \qquad \text{By III, VII, V, I.}$$

$$\therefore \frac{dy}{dx} = \frac{2x + 3}{x^2 + 3x}.$$

Example 2.—Given $y = \log_{10}(1 + 3x)$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{1 + 3x} \frac{d(1 + 3x)}{dx} \log_{10}e, \qquad \text{By XXIII.}$$

$$= \frac{1}{1 + 3x} \cdot 3 \log_{10}e. \qquad \text{By III, V, I.}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1 + 3x} \log_{10}e.$$

Example 3.—Given $y = e^{x^2+x}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = e^{x^2+x} \frac{d(x^2 + x)}{dx}, \qquad \text{By XXV.}$$

$$= e^{x^2+x} (2x + 1). \qquad \text{By III, VII, I.}$$

$$\therefore \frac{dy}{dx} = (2x + 1)e^{x^2+x}.$$

Example 4.—Given $y = \log \sin^2 x$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{\sin^2 x} \frac{d(\sin^2 x)}{dx}, \qquad \text{By XXII.}$$

$$= \frac{1}{\sin^2 x} 2 \sin x \cos x. \qquad \text{By VII, VIII, I.}$$

$$\therefore \frac{dy}{dx} = 2 \cot x.$$

Example 5.—Given the catenary $y = \frac{1}{2}a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, (see exercise 8, page 167), find the slope of the curve at the point whose abscissa is 0.

Solution.—Given $y = \frac{1}{2}a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

$$\frac{dy}{dx} = \frac{1}{2}a\left[e^{\frac{x}{a}}\frac{d\left(\frac{x}{a}\right)}{dx} + e^{-\frac{x}{a}}\frac{d\left(-\frac{x}{a}\right)}{dx}\right], \qquad \text{By III, XXV.}$$

$$= \frac{1}{2}a\left(e^{\frac{x}{a}}\cdot\frac{1}{a} - e^{-\frac{x}{a}}\cdot\frac{1}{a}\right). \qquad \text{By V, I.}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}}).$$

And $\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{2}(e^0 - e^0) = 0.$

$\therefore$ the slope of the catenary at the point where $x = 0$ is 0.

203. $\int\frac{du}{u}$, $\int e^u du$, and $\int a^u du$.—These integrals are readily evaluated and occur frequently.

$$\int\frac{du}{u} = \log_e u + C.$$

For $d(log_e u + C) = \frac{du}{u}$. By XXII.

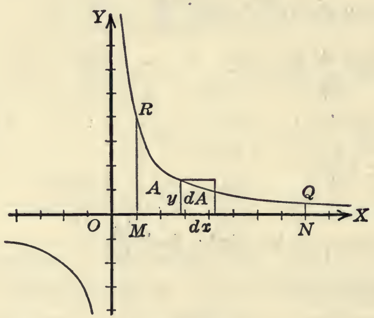


Fig. 170.

$$\int e^u du = e^u + C.$$

For $d(e^u + C) = e^u du$. By XXV.

$$\int a^u du = \frac{a^u}{\log a} + C.$$

For $d\left(\frac{a^u}{\log a} + C\right) = a^u du$. By XXIV.

Example.—Find the area bounded by the equilateral hyperbola $xy = 4$, the x-axis, and the ordinates corresponding to $x = 1$ and $x = 8$.

Solution.—The hyperbola is shown in Fig. 170, and the area sought is $MNQR$.

Consider the area A as generated by the ordinate moving toward the right.

Then $\quad dA = ydx.$

And $\quad \int dA = \int y\,dx = \int \frac{4}{x}dx = 4\int \frac{dx}{x}.$

$\therefore \quad A = 4 \log x + C.$

When $x = 1, A = 0. \quad \therefore 0 = 4 \log 1 + C$, or $C = -4 \log 1$.

When $x = 8, A = 4 \log 8 - 4 \log 1$.

$\therefore \quad A = 4 \times 2.079 - 4 \times 0 = 8.316.$

Therefore the area sought is 8.316 square units.

EXERCISES

In exercises 1–20 find the derivatives of the dependent variables with respect to the independent variables, and the differentials of the dependent variables.

1. $y = \log (x^2 + 7x)$.

2. $y = \log_{10} x^2$.

3. $y = \log \frac{1}{x}$.

4. $y = \log_{10} x^{-2}$.

5. $y = e^{2x}$.

6. $y = e^{x^2}$.

7. $y = e^{3x^2+4}$.

8. $y = e^x \sin x$.

9. $y = a^{2x}$.

10. $y = x \cdot 10^{2x+3}$.

11. $y = (3x - 2)^x$.

12. $y = \frac{1}{2}(e^x + e^{-x})$.

13. $i = be^{-at}$. (See Ex. 9, page 167.)

14. $i = Ie^{-\frac{Rt}{L}}$. (See Ex. 11, page 167.)

15. $y = e^{-x} \sin x$. (See Ex. 18, page 174.)

16. $i = e^{-\frac{1}{2}t} \sin (2t + \frac{1}{2}\pi)$. (See Ex. 19, page 174.)

17. $y = x + \log (1 + x^2)$.

18. $y = (2x + \log x)^2$.

19. $y = (3x + 2)e^{-x^2}$.

20. $y = (x^2 + 1)^{2x+3}$.

21. Find the slope of the tangent to the curve $y = e^x$ at the point where $x = 0$. Where $x = 2$.

22. Find the slope of the tangent to the curve $y = \log_{10} x$ at the point where $x = 1$. Where $x = 10$.

23. Find the minimum point of the curve $y = \log (x^2 - 2x + 3)$.

24. Find the maximum and minimum points of the curve whose equation is $y = 2x^2 - \log x$.

25. Show that the rate of change of y with respect to x for any point on the curve $y = ae^{kx}$ is proportional to y. (See **Art. 128.**)

26. Find the area bounded by the equilateral hyperbola $xy = 1$, the x-axis, and the ordinates corresponding to $x = 1$ and $x = 10$.

27. Find the area bounded by the curve $y = x + \frac{1}{x}$, the x-axis, and the ordinates corresponding $-x = 2$ and $x = 4$.

Find the indefinite integrals in exercises 28 to 37.

28. $\int \frac{dx}{x - 1}$.

29. $\int \frac{3}{x} dx$.

30. $\int \frac{x + 1}{x} dx$.

31. $\int \frac{\cos x dx}{\sin x}$.

32. $\int e^{2x} dx$.

33. $\int \frac{x^3 - 2x^2 + x}{x^2} dx$.

34. $\int (1 - x^{-1})(1 - x^{-2}) dx$.

35. $\int a^{3x} dx$.

36. $\int (e^x + 4)e^{-x} dx$.

37. $\int (e^{2x+1} + x) dx$.

38. Find the equation of the curve passing through the point (0, 1) if the slope of any point of the curve is proportional to the ordinate of the of that point.

Suggestion.— $\frac{dy}{dx} = ky$. $\therefore \frac{dy}{y} = kdx$.

39. Find the equation of the curve passing through the point (0, 1) if the slope of any point of the curve is equal to xy.

Suggestion.— $\frac{dy}{dx} = xy$. $\therefore \frac{dy}{y} = xdx$.

CHAPTER XIII

SOLID ANALYTIC GEOMETRY

204. Introduction.—In plane analytic geometry, all the points and lines are confined to one plane. In solid analytic geometry this restriction is removed, points and lines are considered as anywhere in space. In addition a new element is introduced, a surface of which the plane is a particular instance.

Since plane analytic geometry is a special case of solid analytic geometry, it is expected that the formulas obtained for plane analytic geometry can be obtained as special cases of the formulas for solid analytic geometry. Such reductions and resemblances should be constantly sought.

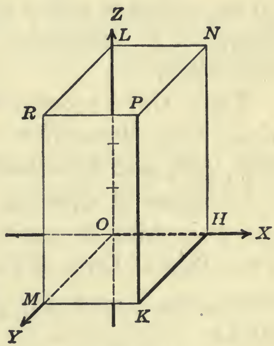


FIG. 171.

205. Rectangular coördinates in space.—If at the origin of the coördinate system in plane analytic geometry a line is erected perpendicular to the plane of the axes, this line will serve as a third axis for a space coördinate system, and is called the z-axis. It is customary in a space depiction to draw the x-axis, Fig. 171, horizontal, the z-axis vertical and the y-axis as coming toward the observer. In order to give space perspective to the figure, the positive y-axis is drawn so as to make an angle of 135° with the positive z-axis, and the unit on the y-axis is taken equal to half the diagonal of a square whose side is a unit on the x-axis.

The three axes determine three **coördinate planes**, the

xy-plane, xz-plane and **yz-plane.** These three coördinate planes are mutually perpendicular to each other and all pass through the origin O.

If through a point P in space, Fig. 171, planes are drawn perpendicular to the x, y, and z-axes, respectively, these three planes will form with the coördinate planes a rectangular parallelepiped. The edge RP perpendicular to the yz-plane and parallel to the x-axis is called the x coördinate of P. It is considered positive if measured to the right, and negative if measured to the left.

The edge NP perpendicular to the xz-plane and parallel to the y-axis is called the y coördinate of P. It is considered positive if measured toward the observer, and negative if measured away from the observer.

The edge KP perpendicular to the xy-plane and parallel to the z-axis is called the z coördinate of P. It is considered positive if measured upward and negative if measured downward.

These three coördinate lines uniquely determine a point P, since they determine three mutually perpendicular planes, MP, HP, and LP which intersect in one point P.

In place of drawing a rectangular parallelepiped to represent a point in space it is customary to draw a broken line consisting of three of its edges.

Thus, the point P, Fig. 171, would be represented by the broken line $OHKP$.

The three coördinates of a point are written $(x, y, z.)$

Thus if P, Fig. 171, is the point (2, 3, 4), its coördinates are

$$x = OH = 2,\ y = HK = 3, \text{ and } z = KP = 4.$$

The three coördinate planes divide all space into eight octants. The octant in which the point lies is denoted by the sequence of signs for the three coördinates.

Thus, the $(+, +, -)$ octant is the octant to the right of the yz-plane, in front of the xz-plane, and below the xy-plane.

EXERCISES

1. If P, in Fig. 171, has the coördinates (2, 3, 4) what are the coördinates of H, K, M, R, L and N?

2. Plot the points (1, 1, 1), (−1, 2, 3), (2, −3, 1), (−2, −1, −3).

3. Draw the triangle whose vertices have the coördinates (2, 1, 4), (−1, 3, 2), (2, −1, −3).

4. Where are all the points for which $x = 0$? $y = 0$? $z = 0$?

5. Where are all the points for which $x = -2$? $y = 3$? $z = -2$?

6. From the point (x_1, y_1, z_1), perpendiculars are drawn to the coördinate planes. Find the coördinates of the feet of these perpendiculars.

206. Geometrical methods of finding the coördinates of a point in space.—Since any point P in space can be regarded as the vertex of a rectangular parallelepiped which has the opposite vertex at the origin, the coördinates of P can be found geometrically in a number of different ways of which the following are the most useful.

(1) From P draw a line, Fig. 171, perpendicular to the xy-plane and meeting it in K. From K draw a line perpendicular to the x-axis and meeting it in H.

Then $OH = x$, $HK = y$, and $KP = z$.

(2) From P draw planes perpendicular to the x, y, and z-axes, respectively, and let the axes intersect these planes in the points H, M, and L, respectively.

Then $OH = x$, $OM = y$, and $OL = z$.

(3) From P draw lines perpendicular to the x, y, and z-axes meeting them in the points H, M, and L respectively.

Then $OH = x$, $OM = y$, and $OL = z$.

207. Distance between two points.—The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is found by constructing a rectangular parallelepiped, Fig. 172, having as opposite vertices P_1 and P_2, and whose edges are parallel to the coördinate axes.

Then P_1P_2 is the length of a diagonal of this rectangular parallelepiped.

Since P_1SP_2 is a right triangle, $\overline{P_1P_2}^2 = \overline{P_1S}^2 + \overline{SP_2}^2$.

Since P_1RS is a right triangle, $\overline{P_1S}^2 = \overline{P_1R}^2 + \overline{RS}^2$.

Therefore $\overline{P_1P_2}^2 = \overline{P_1R}^2 + \overline{RS}^2 + \overline{SP_2}^2$.

Substituting $P_1P_2 = d$, $P_1R = x_2 - x_1$, $RS = y_2 - y_1$, and $SP_2 = z_2 - z_1$, and extracting the square root of both sides of the equation, gives the distance formula

[48] $$\mathbf{d} = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{y}_1 - \mathbf{y}_2)^2 + (\mathbf{z}_1 - \mathbf{z}_2)^2}.$$

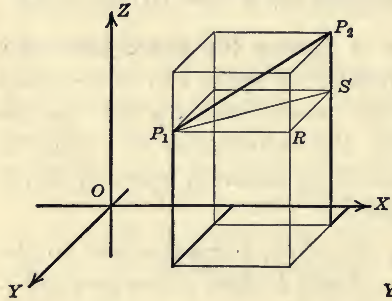


FIG. 172.

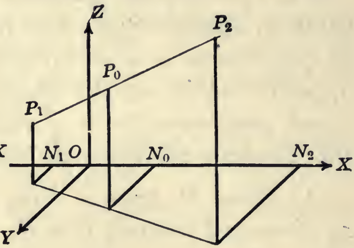


FIG. 173.

208. Coördinates of a point dividing a line segment in the ratio $\mathbf{r_1}$ to $\mathbf{r_2}$.—As in plane analytic geometry, the ratio $\dfrac{r_1}{r_2} = \dfrac{P_1P_0}{P_0P_2}$ will be considered positive for internal division and negative for external division. Let P_1 and P_2, Fig. 173, be the end points of the segment and let P_0 be the point of division. Through P_1, P_2, and P_0 draw planes perpendicular to the x-axis meeting it in the points N_1, N_2, and N_0 respectively. By a familiar theorem in solid geometry,

$$\frac{P_1P_0}{P_0P_2} = \frac{N_1N_0}{N_0N_2}.$$

But $N_1N_0 = ON_0 - ON_1$, $ON_0 = x_0$, and $ON_1 = x_1$, **Art. 206.**

Hence $N_1N_0 = x_0 - x_1$.

Similarly $N_0N_2 = x_2 - x_0$.

Substituting these values and replacing $\frac{P_1P_0}{P_0P_2}$ by $\frac{r_1}{r_2}$, gives

$$\frac{r_1}{r_2} = \frac{x_0 - x_1}{x_2 - x_0}.$$

Solving for x_0,

$$x_0 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}.$$

By drawing planes perpendicular to the y-axis and the z-axis similar formulas are obtained for y_0 and z_0.

Therefore the coördinates of P_0 dividing the line P_1P_2 in the ratio r_1 to r_2 are

$$[49] \quad \mathbf{x}_0 = \frac{\mathbf{r}_1\mathbf{x}_2 + \mathbf{r}_2\mathbf{x}_1}{\mathbf{r}_1 + \mathbf{r}_2}, \ \mathbf{y}_0 = \frac{\mathbf{r}_1\mathbf{y}_2 + \mathbf{r}_2\mathbf{y}_1}{\mathbf{r}_1 + \mathbf{r}_2}, \ \mathbf{z}_0 = \frac{\mathbf{r}_1\mathbf{z}_2 + \mathbf{r}_2\mathbf{z}_1}{\mathbf{r}_1 + \mathbf{r}_2}.$$

EXERCISES

1. Find the distance between the points (−1, 3, 7) and (1, 9, 16).

2. Find the distance between the points (3, −2, 4) and (6, −6, −8).

3. Show that the points (1, −2, 3), (7, 0, 6), (4, 6, 8) form a right triangle.

4. Show that the points (3, −1, 4), (4, 1, 7), (1, 4, 6) form a right triangle.

5. Show that the points (1, 7, 6), (2, 2, 11), (2, 8, 13) form an isosceles triangle.

6. Show that the points (−4, 2, 5), (−1, 5, 2), (−3, 3, 0) form an isosceles triangle.

7. Show that the points (7, −1, 2), (4, 2, 2), (4, −1, 5), (3, −2, 1) are the vertices of a regular tetrahedron.

8. Find the lengths of the medians of the triangle whose vertices are (−1, 7, 4), (3, −5, −2), (−5, 1, 6).

9. Find the coördinates of the point which divides the line joining (7, 2, 6) to (−3, 7, −9) in the ratio of 2:3.

10. Find the coördinates of the point which divides the line joining (7, −6, 2) to (−3, 4, −5) in the ratio −3:4.

11. In what ratio is the line joining (2, −6, 3) to (4, −3, −6) divided by the xz-plane?

12. The beginning of the line segment which is divided in the ratio 4:3 by the point (1, 2, −6) is the point (−1, 6, −2). Find the coördinates of the other extremity.

13. Prove analytically that the straight lines joining the mid-points of the opposite edges of a tetrahedron pass through a common point and are bisected by it.

14. Prove analytically that the straight lines joining the mid-points of the opposite sides of any space quadrilateral pass through a common point, and are bisected by it.

209. Orthogonal projections of line segments.—In general two lines in space will not intersect. If parallels to these lines are drawn through any point, the angle made by these intersecting lines is defined as the angle made by the non-intersecting lines.

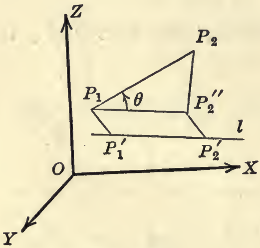


FIG. 174.

If through a point P in space a plane is constructed perpendicular to a given line, the point P' where the plane meets the line is defined as the **orthogonal projection of the point** P on the line.

If the orthogonal projection of the end points P_1 and P_2 of a line segment, Fig. 174, on a line l are P_1' and P_2', then the line segment $P_1'P_2'$ is said to be the **orthogonal projection of the line segment** P_1P_2 on l.

With these definitions it is easy to derive formulas for the projection of a line segment on a given line.

Let P_1P_2, Fig. 174, be a line segment of length d, let $P_1'P_2'$ be its projection on the line l, and let θ be the angle between P_1P_2 and $P_1'P_2'$.

Through P_1 draw a line parallel to l and meeting the plane passing through P_2, perpendicular to line l in the point P_2''. Join P_2P_2'', $P_2''P_2'$, and P_1P_1'. Then θ is the angle $P_2''P_1P_2$, and

$$P_1'P_2' = P_1P_2'' = P_1P_2 \cos\theta = d\cos\theta.$$

This gives:

THEOREM 1.—*The projection of a directed line segment on a given line is equal to its length multiplied by the cosine of the angle between the lines.*

Another theorem which is useful in solid analytic geometry is the following:

THEOREM 2.—*The projection on any line of the straight line joining any two points is equal to the algebraic sum of the projection of any broken line joining these points.*

Proof.—Let P_1P_2, Fig. 175, be the straight line joining P_1 and P_2, let l be the line on which P_1P_2 is to be projected, and let $P_1P_3P_4P_5P_2$ be any broken line joining P_1 to P_2. If the points P_1', P_3', P_4', P_5', P_2' are the projections of the points P_1, P_3, P_4, P_5, P_2, respectively, then

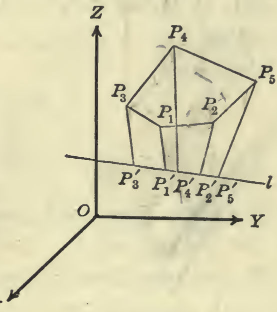


FIG. 175.

$$\text{Proj. } P_1P_3 + \text{proj. } P_3P_4 + \text{proj. } P_4P_5 + \text{proj. } P_5P_2 = P_1'P_3' + P_3'P_4' + P_4'P_5' + P_5'P_2' = P_1'P_2'.$$

But $$\text{proj. } P_1P_2 = P_1'P_2'.$$

This proves the theorem.

210. Direction cosines of a line.—Let the angles which any line in space makes with the positive x, y, and z-axes be respectively, α, β, and γ. These angles are called the **direction angles** of the line. Their cosines, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the **direction cosines** of the line.

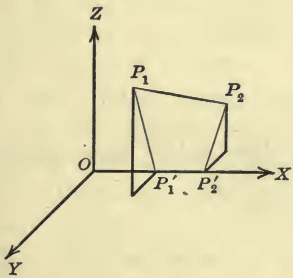


FIG. 176.

If P_1 and P_2, Fig. 176, are any two points on a line and d is the distance P_1P_2, the direction cosines are given by the formulas

$$[50] \quad \cos\alpha = \frac{x_2 - x_1}{d}, \cos\beta = \frac{y_2 - y_1}{d}, \cos\gamma = \frac{z_2 - z_1}{d}.$$

To prove this, let the projection of P_1P_2 on the x-axis be $P_1'P_2'$, then

$$P_1'P_2' = d \cos \alpha.$$

But $P_1'P_2' = OP_2' - OP_1'$ and $OP_2' = x_2$ and $OP_1' = x_1$. **Art. 206.**

Therefore

$$x_2 - x_1 = d \cos \alpha,$$

or

$$\cos \alpha = \frac{x_2 - x_1}{d}.$$

The remaining two formulas are found by projecting P_1P_2 on the y-axis and the z-axis respectively.

These three direction cosines are not independent, for squaring each equation and adding gives

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}{d^2} = \frac{d^2}{d^2} = 1.$$

Therefore

[51] $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example.—Find the direction cosines of a line if they are proportional to the numbers 2, −9, 6.

Solution.—Since $\cos \alpha : \cos \beta : \cos \gamma = 2 : -9 : 6$,

$$\cos \alpha = 2k,$$
$$\cos \beta = -9k,$$
$$\cos \gamma = 6k.$$

But by **[51]**, the sum of the squares of these cosines equals unity, therefore

$$4k^2 + 81k^2 + 36k^2 = 1.$$

$$\therefore k = \pm \frac{1}{11}.$$

Substituting, $$\cos \alpha = \frac{2}{11}, \quad \cos \beta = -\frac{9}{11}, \quad \cos \gamma = \frac{6}{11}.$$

Or $$\cos \alpha = -\frac{2}{11}, \cos \beta = \frac{9}{11}, \cos \gamma = -\frac{6}{11}.$$

EXERCISES

Find the direction cosines of the lines joining the points in exercise 1–3, and the projections of these lines on the three axes.

1. (4, 2, 3) to (5, 3, 4). **3.** (−5, 1, 4) to (−3, 4, −2).

2. (−2, 1, 7) to (5, −3, 2).

Find the direction cosines which are proportional to the numbers in exercises 4–7.

4. −6, 2, −3. **6.** 4, 3, −12.

5. 6, −7, 6. **7.** −10, −6, 15.

8. Find the orthogonal projection of the line joining (7, 6, −2) to (5, −3, 4) on the x-axis; on the y-axis; on the z-axis.

9. What are the direction cosines of the x-axis? Of the y-axis? Of the z-axis?

10. What are the direction cosines of a line parallel to the x-axis? Perpendicular to the x-axis?

11. Where do all the lines lie for which (a) $\cos\alpha = \frac{1}{2}$, (b) $\cos\beta = \frac{1}{2}$, (c) $\cos\alpha = \frac{1}{2}$ and $\cos\beta = \frac{1}{2}$, (d) $\cos\alpha = 0$, (e) $\cos\alpha = 1$?

12. A line makes an angle of 60° with both the x and the y-axis, what angle does it make with the z-axis?

13. A line makes an angle of 75° with the x-axis, and 45° with the y-axis, what angle does it make with the z-axis?

14. The equal acute angles which a line makes with the x-axis and the y-axis, are each one-half the angle which it makes with the z-axis. Find the direction cosines of the line.

15. The angles not greater than 90° which a line makes with the x, y, and z-axes are proportional to 1, 2, and 3. Find the direction cosines of the line.

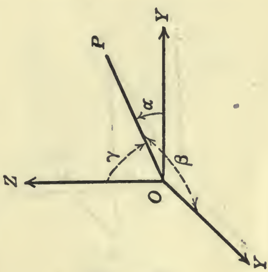


FIG. 177.

211. Polar coördinates of a point.—If the distance OP, Fig. 177, of a point P from the origin is called ρ, and if the direction angles of OP are α, β, and γ, then $(\rho, \alpha, \beta, \gamma)$ are called the **polar coördinates** of P. The relations between the polar coördinates of P and its rectangular coördinates are obtained by replacing (x_1, y_1, z_1) of article 210 by (0, 0, 0) and (x_2, y_2, z_2) by (x, y, z).

Since $d = \sqrt{x^2 + y^2 + z^2}$, formula [**50**] gives the polar coördinates in terms of the rectangular coördinates.

[**52**]
$$\rho = \pm\sqrt{x^2 + y^2 + z^2},$$
$$\cos\alpha = \frac{x}{\pm\sqrt{x^2 + y^2 + z^2}},$$
$$\cos\beta = \frac{y}{\pm\sqrt{x^2 + y^2 + z^2}},$$
$$\cos\gamma = \frac{z}{\pm\sqrt{x^2 + y^2 + z^2}}.$$

Note that the radicals must be taken either all positive or all negative.

Replacing $\sqrt{x^2 + y^2 + z^2}$ by its value ρ, and clearing of fractions gives the rectangular coördinates in terms of the polar coördinates.

[**53**]
$$x = \rho\cos\alpha,$$
$$y = \rho\cos\beta,$$
$$z = \rho\cos\gamma.$$

Note that the direction cosines are not independent but are connected by the equation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

212. Spherical coördinates.—Another method of locating a point in space is by means of spherical coördinates. From P, Fig. 178, drop a line perpendicular to the xy-plane meeting it in M. Join O, called the **pole**, to P, and O to M. Then the spherical coördinates of P are ρ, θ, and ϕ, which are written (ρ, θ, ϕ), where $\rho = OP$ is the distance of P from the origin; θ = angle NOM is the angle through which the positive x-axis would have to rotate to coincide with OM; and ϕ = angle ZOP is the angle which OP makes with the positive z-axis.

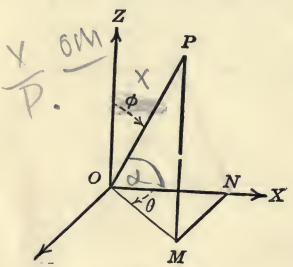


Fig. 178.

The quantity ρ is taken positive if measured along the radius vector, and negative if measured along the radius vector produced through the origin. The angle θ can have any value from 0° to 360°. The angle ϕ is restricted to values from 0° to 180°.

The relations between spherical and rectangular coördinates are then

[54]
$$\mathbf{x} = \rho \sin \varphi \cos \theta, \quad \mathbf{y} = \rho \sin \varphi \sin \theta, \quad \mathbf{z} = \rho \cos \varphi$$

[54₁]
$$\rho = \pm\sqrt{\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2},$$
$$\theta = \tan^{-1}\frac{\mathbf{y}}{\mathbf{x}} = \sin^{-1}\frac{\mathbf{y}}{\pm\sqrt{\mathbf{x}^2 + \mathbf{y}^2}},$$
$$\varphi = \cos^{-1}\frac{\mathbf{z}}{\pm\sqrt{\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2}}.$$

The convention with regard to signs is that either all the upper signs must be used, or else all the lower.

If the pole of a spherical coördinate system were taken at the center of the earth, the z-axis passing through the north pole, and the xz-plane passing through the meridian of Greenwich, then the spherical coördinates of a point in the northern hemisphere can be so chosen that ρ will give the distance of the point from the center of the earth, θ its longitude and φ its co-latitude.

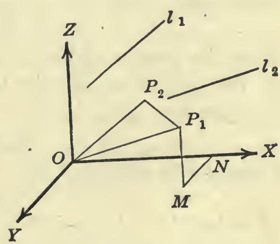


Fig 179.

213. Angle between two lines.—Let the two lines be l_1 and l_2 Fig. 179, with direction angles $\alpha_1, \beta_1, \gamma_1$, and $\alpha_2, \beta_2, \gamma_2$, respectively, and let θ be the angle between l_1 and l_2.

In order to find θ, draw two lines OP_1 and OP_2 through O parallel to l_1 and l_2 respectively, also draw ON, NM, MP_1, the coördinates of P_1, and let $OP_1 = \rho_1$.

By article 209, the angle between OP_1 and OP_2 equals θ.

Project OP_1 and the broken line $ONMP_1$ on OP_2.

By theorem 2, **Art. 209**,

$$\text{proj. } OP_1 = \text{proj. } ON + \text{proj. } NM + \text{proj. } MP_1.$$

By theorem 1, **Art. 209**,

$$\begin{aligned} \text{proj. } OP_1 \text{ on } OP_2 &= \rho_1 \cos\theta, \\ \text{proj. } ON \text{ on } OP_2 &= x_1 \cos\alpha_2, \\ \text{proj. } NM \text{ on } OP_2 &= y_1 \cos\beta_2, \\ \text{proj. } MP_1 \text{ on } OP_2 &= z_1 \cos\gamma_2. \end{aligned}$$

Therefore $\rho_1 \cos\theta = x_1 \cos\alpha_2 + y_1 \cos\beta_2 + z_1 \cos\gamma_2$.

Replacing x_1, y_1, z_1 by their equivalents, [**53**], and dividing both sides of the equation by ρ_1, gives the required expression for θ,

[**55**] $\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2.$

If the two lines are perpendicular to each other,

$$\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2 = 0.$$

If the two lines are parallel to each other, it is evident that either $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$, or $\alpha_1 = 180° - \alpha_2$, $\beta_1 = 180° - \beta_2$, and $\gamma_1 = 180° - \gamma_2$.

Example 1.—Find the polar coördinates of the point $(1, -1, -\sqrt{2})$.

From [**52**], $\rho = \sqrt{4} = 2$, $\cos\alpha = \frac{1}{2}$, $\cos\beta = -\frac{1}{2}$, $\cos\gamma = -\frac{1}{2}\sqrt{2}$

$$\therefore \alpha = 60°, \beta = 120°, \gamma = 135°.$$

Then the polar coördinates of $(1, -1, -\sqrt{2})$ are $(2, 60°, 120°, 135°)$.

If the negative sign is taken with ρ the polar coördinates are $(-2, 120°, 60°, 45°)$.

Example 2.—Find the spherical coördinates of $(1, -1, -\sqrt{2})$.

From [**54**], $\rho = \sqrt{4} = 2$,

$$\theta = \tan^{-1}(-1) = \sin^{-1}\frac{-1}{\sqrt{2}} = 315°, \text{ and } \varphi = \cos^{-1}\frac{-\sqrt{2}}{2} = 135°.$$

Then the spherical coördinates of $(1, -1, -\sqrt{2})$ are $(2, 315°, 135°)$.

If the negative sign is taken with each of the radicals the spherical coördinates are $(-2, 135°, 45°)$.

Example 3.—Find the direction cosines of a line which is perpendicular to two lines having direction cosines proportional to -1, 2, 6 and 1, 4, 3 respectively.

If $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the required direction cosines, [**55**] and [**51**] give the three equations:

$$\begin{aligned} -\cos \alpha + 2 \cos \beta + 6 \cos \gamma &= 0, \\ \cos \alpha + 4 \cos \beta + 3 \cos \gamma &= 0, \\ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= 1. \end{aligned}$$

Solving these equations, gives $\cos \alpha = \frac{6}{7}$, $\cos \beta = -\frac{3}{7}$, $\cos \gamma = \frac{2}{7}$, or $\cos \alpha = -\frac{6}{7}$, $\cos \beta = \frac{3}{7}$, $\cos \gamma = -\frac{2}{7}$.

Example 4.—Find the projection of the line segment l_1 joining the points $(-1, 3, 6)$ and $(3, 7, -1)$ on the line l_2 joining the points $(3, 1, -2)$ and $(6, 7, 0)$.

From [**50**] the direction cosines of l_1 are $\frac{4}{9}$, $\frac{4}{9}$, $-\frac{7}{9}$ and of l_2 are $\frac{3}{7}$, $\frac{6}{7}$, $\frac{2}{7}$.

If θ is the angle between the two lines, by [**55**],

$$\cos \theta = \frac{12 + 24 - 14}{63} = \frac{22}{63}.$$

The length of l_1 by [**48**] is $d = 9$, and the projection of l_1 on l_2 by theorem 1, **Art. 209,** is equal to $d \cos \theta = \frac{9 \times 22}{63} = \frac{22}{7}$.

EXERCISES

Find the polar coördinates of the points in exercises 1–3, if their rectangular coördinates are:

1. $(1, \sqrt{2}, -1)$. **2.** $(4, -4, 4\sqrt{2})$. **3.** $(1, 1, 1)$.

4. If the polar coördinates of a point are $(3, 60°, 60°, \gamma)$, find γ.

Find the spherical coördinates of the points in exercises 5–7, if their rectangular coördinates are:

5. $(2, 2\sqrt{3}, 4\sqrt{3})$. **6.** $(-3, -\sqrt{3}, -2)$. **7.** $(-\sqrt{6}, \sqrt{6}, 2)$.

8. Find the acute angle between the two lines having direction cosines proportional to 11, −10, 2 and −5, 2, 14.

9. Find the direction cosines of a line which is perpendicular to two lines having direction cosines proportional to 2, 4, −3 and −1, 4, 3, respectively.

10. Find the projection of the line segment joining $(3, -1, 4)$ to $(4, 1, 6)$ on the line joining $(4, 2, -5)$ to $(-2, 4, -2)$.

11. Find the projection of the line segment joining $(7, 2, -3)$ to $(2, 4, 3)$ on the line joining $(1, -4, 3)$ to $(7, -11, -3)$.

12. Find the projection of the line segment joining $(2, 1, -3)$ to $(-2, 3, 1)$ on the line joining $(3, -10, 4)$ to $(12, 8, -2)$.

13. Verify the conventions used with regard to signs in article 212.

SURFACES

214. Locus in space.—If an equation in two variables is given in plane analytic geometry, values can be assigned at pleasure to one of these variables, and then the other is determined. The locus of all points satisfying such an equation is found in general to be a curve.

On the other hand, in solid analytic geometry, if an equation in three variables is given, values can be assigned at pleasure to two of the variables and then the third variable is determined. For instance, in the equation $z = x^2 + y^2$, to every pair of values of x and y there corresponds a value of z. Hence for every point in the xy-plane there will be a corresponding point in space for the locus of $z = x^2 + y^2$. If these points are thought of as a whole, it is obvious that they all lie on a surface. In general then the locus of a single equation in space is a surface. Sometimes one or even two variables may be missing in an equation, in which case such an equation will give rise to a special surface.

215. Equations in one variable. Planes parallel to the axes.—The equation $x = a$, is satisfied by all values of y and z, since these variables can be regarded as entering into the equation $x = a$ with zero coefficents.

Hence all the points satisfying $x = a$ will lie in a plane parallel to the yz-plane and cutting the x-axis at the point $x = a$.

If the equation has the form $f(x) = 0$, the locus will consist of a series of planes, all parallel to the yz-plane and cutting the x-axis at points whose abscissas are the roots of $f(x) = 0$.

Like considerations hold for equations which contain only the coördinate y, or only the coördinate z.

216. Equations in two variables. Cylindrical surfaces.—A cylindrical surface is generated by a straight line which moves so as to be always parallel to some fixed line, while intersecting a fixed curve. The fixed curve is called the **directrix** of the cylindrical surface, and the moving line in any one of its positions on the surface is called an **element** of the cylindrical surface.

A plane can be regarded as a particular case of a cylindrical surface whose directrix is a straight line.

Consider the equation $x^2 + y^2 = 25$, Fig. 180. In two dimensional space this is the equation of a circle with center at the origin and radius equal to 5. In three dimensional space, the coördinate z can be regarded as entering the equation with a zero coefficient. Hence with any value of x and y which satisfies the equation, say $x = 3$ and $y = 4$, there can be associated any value of z. Thus, the points $(3, 4, -1)$, $(3, 4, 0)$, $(3, 4, 2)$, and, in general, $(3, 4, z)$ where z has any value, will all be points on the surface. These particular points all lie on a line perpendicular to the xy-plane and passing through the point $x=3$, $y=4$ in the xy-plane.

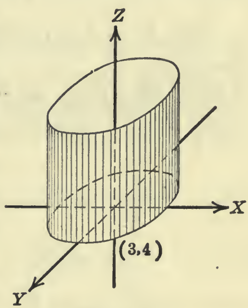


FIG. 180.

In like manner through every point on the circle $x^2 + y^2 = 25$ in the xy-plane there passes a line perpendicular to the xy-plane, and every point of this line satisfies the equation $x^2 + y^2 = 25$. Hence the locus of the equation $x^2 + y^2 = 25$ is a cylindrical surface with elements perpendicular to the xy-plane, in other words parallel to the z-axis, and having the circle $x^2 + y^2 = 25$ as directrix.

Another illustration is the surface $z^2 = x$. Its elements are parallel to the y-axis and its directrix is the parabola $z^2 = x$ in the xz-plane. This is called a parabolic cylindrical surface.

In general an equation $f(x, y) = 0$ represents in space a cylindrical surface whose elements are parallel to the z-axis, and whose directrix is the curve $f(x, y) = 0$ in the xy-plane.

The equations $f(y, z) = 0$ and $f(x, z) = 0$ represent cylindrical surfaces similarly situated with reference to the x and the y-axes, respectively.

217. Spheres.—Let $C(h, k, l)$ be the center of a sphere of radius r. Since every point P on the sphere is at the constant distance r from its center

$$CP = r.$$

Or $$\sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2} = r.$$

[56] $$\therefore \mathbf{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2}.$$

This is the equation of the sphere with center at C and radius r.

218. Surfaces of revolution.—A surface formed by revolving a curve about a line in its plane is called a surface of revolution. The simplest cases are those where the curve is revolved about one of the coördinate axes. Suppose it is desired to revolve the parabola $y^2 = x$ about the x-axis. Let P, Fig. 181, be any point on the parabola.

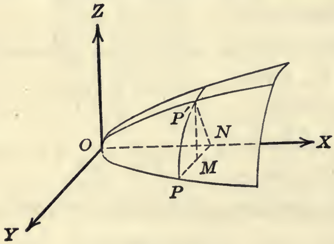


Fig. 181.

As the curve revolves about the x-axis, P describes a circle of radius NP. When P is in the xy-plane, $\overline{NP}^2 = x$.

When P takes another position, say P', then $NP' = NP$ and therefore $\overline{NP'}^2 = x$, but $NP' = \sqrt{\overline{NM}^2 + \overline{MP'}^2} = \sqrt{y^2 + z^2}$.

Replacing NP' by its value gives $(\sqrt{y^2 + z^2})^2 = x$, or $y^2 + z^2 = x$.

Since P' can be any point on the surface, this is the equation of the surface of revolution.

This equation was obtained by replacing y by $\sqrt{y^2 + z^2}$.

If $f(x, y) = 0$ is the equation of any curve in the xy-plane which is to be revolved about the x-axis, the same method of reasoning shows that $f(x, \sqrt{y^2 + z^2}) = 0$ is the equation of the surface of revolution. In like manner the equation of the surface of revolution obtained by revolving $f(x, y) = 0$ about the y-axis is $f(\sqrt{x^2 + z^2}, y) = 0$.

Thus if $y^2 = x$ is revolved about the y-axis, the equation of the surface of revolution is $y^2 = \sqrt{x^2 + z^2}$, or $y^4 = x^2 + z^2$. Similar formulas hold true if the curve is given in one of the other coördinate planes and revolved about the corresponding coördinate axes. For instance if the curve $f(y, z) = 0$ is revolved about the z-axis, the equation of the surface of revolution is $f(\sqrt{x^2 + y^2}, z) = 0$.

If a circle and a line are in the same plane, and the line does not intersect the circle, the surface formed when the circle revolves about the line is called an **anchor ring** or **torus.**

EXERCISES

1. What is the equation of the plane parallel to the xy-plane and 3 units above it? 4 units below it?

2. What is the equation of the xy-plane? Of the xz-plane? Of the yz-plane?

3. What is the equation of the locus of a point distant 3 units from the x-axis? 4 units from the z-axis?

Find the equation of the locus of a point determined by the conditions in exercises 4–9.

4. Equidistant from the points $(-1, 2, 3)$ and $(3, 4, -2)$.

5. Equidistant from the xy-plane and the xz-plane.

6. Equidistant from the x-axis and the y-axis.

7. Equidistant from the x-axis and the yz-plane.

8. Equidistant from the point $(2, -4, 3)$ and the x-axis.

9. The sum of the squares of its distances from the point $(1, 1, 1)$ and $(2, -1, 3)$ is constant and equal to 17.

10. Find the equation of a sphere with center on the x-axis, radius equal to 9, and which passes through the point $(2, 4, -8)$.

11. Find the equation of a sphere with center in the xy-plane, radius equal to 7, and which passes through the points $(3, 4, 6)$ and $(7, 3, 3)$.

12. Find the equation of a sphere passing through the points $(2, 3, -6)$, $(5, 3, -5)$, $(5, -2, 10)$, and $(-3, 6, 6)$.

Find the equations of the surfaces of revolution obtained by revolving the curves in exercises 13–24 about the axes as indicated.

13. $y = x$, about the x-axis.

14. $y = z$, about the y-axis.

15. $y = x^2$, about the x-axis.

16. $z^2 = x$, about the x-axis.

17. $x^2 = 2z$, about the x-axis.

18. $x^2 = 2z$, about the z-axis.

19. $x^2 - 2z + z^2 = 0$, about the x-axis.

20. $x^2 - 2z + z^2 = 0$, about the z-axis.

21. $y = \sin z$, about the z-axis.

22. $y = \sin z$, about the y-axis.

23. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, about its major axis. This surface is called a **prolate spheroid.**

24. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, about its minor axis. This surface is called an **oblate spheroid.**

25. What is the equation of the anchor ring obtained by revolving the circle lying in the xy-plane, with center at the point (0, 4) and radius 2, about the x-axis.

Sketch and describe the following surfaces.

26. $x - y = 0$.

27. $x^2 + y^2 = 4$.

28. $x^2 - 2x + y^2 = 0$.

29. $y^2 - 2y + z^2 = 0$.

30. $y^2 - 2z = 0$.

31. $x^2 + 4z^2 = 4$.

32. $x^2 - 4z^2 = 4$.

33. $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

34. $x^2 - 3x + 2 = 0$.

35. $y^2 - 1 = 0$.

CURVES IN SPACE

219. Equations of curves.—Since a single equation in three dimensional space is the equation of a surface, two equations will be satisfied simultaneously by all the points lying on the intersections of the two surfaces. In other words, *it takes two equations in solid analytics to define a curve.*

Thus, $y = 0$ is not sufficient to define the equation of the x-axis, for every point in the xz-plane satisfies this equation. Neither is $z = 0$ sufficient, for this is satisfied by every point in the xy-plane, but $y = 0$ and $z = 0$ are satisfied only by those points common to the xz-plane and the xy-plane, namely, the x-axis. Therefore, $y = 0$ and $z = 0$ are the equations of the x-axis.

Similarly $x = 0$ and $y = 0$ are the equations of the z-axis; and $x = 0$ and $z = 0$ are the equations of the y-axis.

Since it is evident geometrically that an unlimited number of surfaces can be passed through any curve, and that any two of these surfaces will be sufficient to define the curve, any space curve can be represented by an unlimited number of pairs of equations. Thus a circle in space has for its pair of equations, the equations of any two spheres passing through it, or the equation of any sphere and the equation of the plane in which the circle lies. Even this does not exhaust all the possibilities of representing a circle, since any two surfaces passing through the circle will define it.

220. Sections of a surface by planes parallel to the coördinate planes.—The curve in which a surface is cut by a coördinate plane is called the **trace of the surface** in the coördinate plane. Thus, the sphere $x^2 + y^2 + z^2 = 25$ and the plane $z = 0$ define a curve, the circle formed by the intersection of the sphere and the xy-plane. If z is put equal to zero in the equation of the sphere, it becomes $x^2 + y^2 = 25$. This is the equation of the trace of the sphere in the xy-plane.

By putting $y = 0$ or $x = 0$ the trace of the sphere in the xz-plane or the yz-plane is obtained.

Consider the two equations $x^2 + y^2 + z^2 = 25$ and $z = 3$. The curve AB, Fig. 182, common to these two surfaces is known from solid geometry to be a circle. If $z = 3$ is substituted in $x^2 + y^2 + z^2 = 25$, it becomes $x^2 + y^2 = 16$. This is the equation of a circular cylinder.[1] Since every point satisfying the equation of the sphere and the plane satisfies the equation of the cylinder, this cylinder must pass through the circle AB. Hence substituting $z = 3$ in the equation of the sphere $x^2 + y^2 + z^2 = 25$, gives the equation of a cylinder passing through the intersection of the plane $z = 3$ and the sphere $x^2 + y^2 + z^2 = 25$.

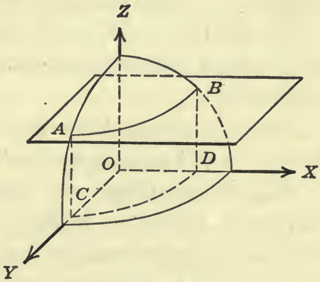


FIG. 182.

[1] For brevity the words *cylindrical surface* are often replaced by the word *cylinder*.

The meaning of this substitution can be regarded from another standpoint. The equation $x^2 + y^2 = 16$ is the equation of the circle CD in which the cylinder cuts the xy-plane. All the elements of the cylinder are perpendicular to the xy-plane and pass through the circle AB. Hence the circle CD is the projection of the circle AB in the xy-plane. In other words, substituting $z = 3$ in the equation of the sphere $x^2 + y^2 + z^2 = 25$, gives the equation of the projection on the xy-plane of the curve common to the plane $z = 3$ and the sphere $x^2 + y^2 + z^2 = 25$.

In general, the substitution $z = c$, where c is some constant, in the equation of a surface can be regarded either as giving the equation of a cylinder passing through the intersection of $z = c$ and the surface or as giving the equation of the projection on the xy-plane, of the curve of intersection of the plane $z = c$ and the surface.

By giving c different values, the shape of different cross sections of the surface in planes parallel to the xy-plane are obtained.

Like considerations hold for the substitution of $x = a$ or $y = b$ in the equation of a surface.

221. Projections of curves on the coördinate planes.—When a curve is defined in space by two equations, it is desirable sometimes to know what are the equations of its projections in the three coördinate planes.

Consider the curve defined by the equations

$$x^2 + y^2 + z^2 = 49, \quad (1)$$
$$x^2 + 3y^2 - z^2 = 39. \quad (2)$$

If z is eliminated between these two equations, the resulting equation

$$2x^2 + 4y^2 = 88,$$

or

$$x^2 + 2y^2 = 44 \quad (3)$$

represents an elliptical cylinder. Furthermore any point whose coördinates satisfy equation (1) and (2) will also satisfy equation (3), therefore the cylinder (3) passes through the curve defined by the equations (1) and (2).

From another standpoint $x^2 + 2y^2 = 44$ is the equation of the directrix of the cylinder $x^2 + 2y^2 = 44$, and since the elements of this cylinder are perpendicular to the xy-plane, the equation $x^2 + 2y^2 = 44$ is the equation of the projection on the xy-plane of the curve defined by equations (1) and (2).

In general, *to find the equation of the projection on the xy-plane of the curve defined by the equation* $f_1(x, y, z) = 0$ *and* $f_2(x, y, z) = 0$, *elminate z between these two equations. The resulting equation* $g(x, y) = 0$ *is the equation of the projection on the xy-plane of the curve that is defined by the equations* $f_1(x, y, z) = 0$ *and* $f_2(x, y, z) = 0$.

Proof.—Every point which satisfies simultaneously the equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$ will also satisfy $g(x, y) = 0$ and therefore $g(x, y) = 0$ will pass through the intersections of these two surfaces. But $g(x, y) = 0$ is the equation of a cylinder whose elements are perpendicular to the xy-plane. At the same time $g(x, y) = 0$ is the equation of the trace of this cylinder in the xy-plane. Therefore, $g(x, y) = 0$ is the equation of the projection on the xy-plane of the curve defined by the equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$.

In like manner it can be shown that to find the equation of the projection on the xz-plane of a curve defined by the equations $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$, eliminate y between these two equations; and to find the projection on the yz-plane eliminate x.

For example, the projection on the xy-plane of the curve defined by equations (1) and (2) is the ellipse $x^2 + 2z^2 = 54$, and projection on the yz-plane is the equilateral hyperbola $z^2 - y^2 = 5$.

EXERCISES

Discuss and draw the traces on the three coördinate planes of the surfaces in exercises 1–3.

1. $x^2 + 2y^2 + 3z^2 = 6$. **2.** $x^2 + xy + z = 0$. **3.** $x^2 + y^2 - z = 1$.

Find the equations of the projections on each of the three coördinate planes of the curves in problems 4–9.

4. $x^2 - y^2 + z^2 = 4,$
$2x^2 + y^2 - 3z^2 = 6.$

5. $x^2 - y = 0,$
$2x^2 + y - z^2 = 0.$

6. $x^2 + y^2 + z^2 = a^2,$
$x^2 + y^2 = ax.$

7. $x^2 + y^2 = a^2,$
$x^2 + z^2 = a^2.$

8. $x^2 + y^2 = a^2,$
$z = mx.$

9. $y^2 + z^2 = 4ax,$
$y^2 = ax.$

10. Show that sections of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$ are hyperbolas if perpendicular to the z-axis, but parabolas if perpendicular to the x-axis or y-axis.

DISCUSSION OF EQUATIONS OF SURFACES

222. Surfaces in space.—It is much more difficult to visualize a surface in solid analytic geometry from its equation than to visualize a curve in plane analytic geometry from its equation. The following discussion similar to the one in the plane case is helpful.

(1) *Symmetry.*

(2) *Intercepts on the axes.*

(3) *Traces on the coördinate planes.*

(4) *Sections of the surface by planes parallel to the coördinate planes.* See **Art. 220.**

(1) *Symmetry.*—To test the symmetry of a surface with respect to the coördinate planes,

(*a*) replace x by $-x$,

(*b*) replace y by $-y$,

(*c*) replace z by $-z$.

If the equation of the surface remains unchanged in case (*a*) it is symmetrical with respect to the yz-plane, in case (*b*) with respect to the xz-plane, in case (*c*) with respect to the xy-plane.

To test for symmetry with respect to the axes

(*a*) replace y by $-y$ and z by $-z$,

(*b*) replace z by $-z$ and x by $-x$,

(*c*) replace x by $-x$ and y by $-y$.

If its equation remains unchanged in case (*a*) it is symmetrical with respect to the x-axis, in case (*b*) with respect to the y-axis, in case (*c*) with respect to the z-axis.

To test for symmetry with respect to the origin replace

$$x \text{ by } -x,\ y \text{ by } -y,\ z \text{ by } -z.$$

If its equation remains unchanged, its surface is symmetrical with respect to the origin.

(2) *Intercepts on the axes.*—To get the intercepts on the x-axis, set both y and z equal to zero in the equation of the surface and solve the resulting equation for x. The solutions of this equation are the intercepts on the x-axis. Similar considerations hold true for the y-axis and the z-axis.

(3) *Traces on the coördinate planes.*—To get the trace of a surface in the xy-plane set $z = 0$. The resulting equation is the equation of the trace of the surface in the xy-plane. Similar considerations hold true for the traces in the xz-plane and the yz-plane. See **Art. 220.**

Example.—Discuss and draw the locus of the equation $x^2 + 2y^2 = z$.

(1) This surface is symmetrical to the yz-plane, the xz-plane and the z-axis.

(2) Its intercepts on the three axes are 0.

(3) Its traces are as follows:

In the xy-plane, the point ellipse $x^2 + 2y^2 = 0$.

In the xz-plane, the parabola $x^2 = z$.

In the yz-plane, the parabola $2y^2 = z$.

(4) Taking sections by planes $z = c$ shows the projections of these sections to be the ellipses

$$\frac{x^2}{c} + \frac{y^2}{\frac{1}{2}c} = 1.$$

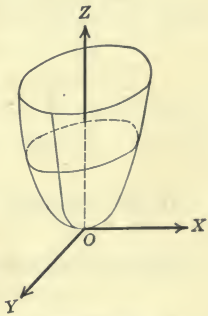


FIG. 183.

If $c<0$, these ellipses are all imaginary, hence no part of the surface lies below the xy-plane.

If $c = 0$, the equation $x^2 + 2y^2 = 0$, shows the section to be a point ellipse.

As c increases from 0 without limit, the ellipses increase in size without limit, the semimajor axis being $\sqrt{c}$ and the semiminor axis $\frac{1}{2}\sqrt{2c}$, hence the surface is as pictured in Fig. 183. In this case it is not necessary to take sections parallel to the other coördinate planes.

QUADRIC SURFACES OR CONICOIDS

223. General equation of second degree.—The locus of the general equation of the second degree,

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + L = 0,$$

is called a **quadric surface.** It is also called a **conicoid** because every section of a quadric surface by a plane is a conic. By rotation and translation of axes it can be shown that this equation has for its real locus, five distinct types of surfaces besides cylinders, cones and degenerate forms like planes, lines and points. These five types will now be considered.

224. Ellipsoid. $\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} + \frac{\mathbf{z}^2}{\mathbf{c}^2} = \mathbf{1}.$

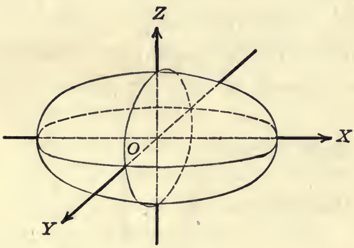


FIG. 184.

(1) This surface, Fig. 184, is symmetrical to all the coördinate planes, all the coördinate axes, and the origin.

(2) Its intercepts on the axes are $x = \pm a$, $y = \pm b$, $z = \pm c$.

(3) Its traces are as follows:

In the xy-plane, the ellipse $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

In the xz-plane, the ellipse $\quad \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1.$

In the yz-plane, the ellipse $\quad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

(4) Sections of the ellipsoid by the planes $z = k$ are the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, \quad z = k,$$

or

$$\frac{x^2}{\frac{a^2}{c^2}(c^2 - k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2 - k^2)} = 1, \quad z = k.$$

These ellipses have their centers on the z-axis, semimajor axes equal to $\frac{a}{c}\sqrt{c^2 - k^2}$, and semiminor axes equal to $\frac{b}{c}\sqrt{c^2 - k^2}$. Here it is assumed that $a > b$. If $a < b$, the axes are interchanged.

As k increases numerically from 0 to c, the axes decrease from a to 0, and from b to 0, respectively. When k is numerically greater than c, the ellipses become imaginary. Hence the ellipsoid is contained between the planes $z = -c$ and $z = c$.

A similar discussion for the other axes shows that sections parallel to the other coördinate planes are ellipses, and that the ellipsoid is contained between the planes $y = -b$ and $y = b$, and between the planes $x = -a$ and $x = a$.

The surface can be thought of as generated by a variable ellipse moving parallel to the xy-plane, with its center always on the z-axis, and the end points of its axes always on the ellipses

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \text{ and } \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Special forms of the ellipsoid are the **prolate spheroid** when $b = c$ and $a > b$, and the **oblate spheroid** when $b = c$ and $a < b$.

225. The hyperboloid of one sheet. $\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} - \frac{\mathbf{z}^2}{\mathbf{c}^2} = \mathbf{1}.$

(1) This surface, Fig. 185, is symmetrical to all the coördinate planes, all the coördinate axes and the origin.

(2) Its intercepts on the axes are $x = \pm a$, $y = \pm b$.

(3) Its traces are as follows:

In the xy-plane, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

In the xz-plane, the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$

In the yz-plane, the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

(4) Sections of the surface by the planes $z = k$ are the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \quad z = k.$$

or

$$\frac{x^2}{\frac{a^2}{c^2}(c^2 + k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2 + k^2)} = 1, \quad z = k,$$

These ellipses are real for all values of k, increasing in magnitude as k increases numerically from 0 to ∞. The smallest ellipse is the one for which $k = 0$, and this is the trace in the xy-plane. The intersections in planes parallel to the other axes are hyperbolas.

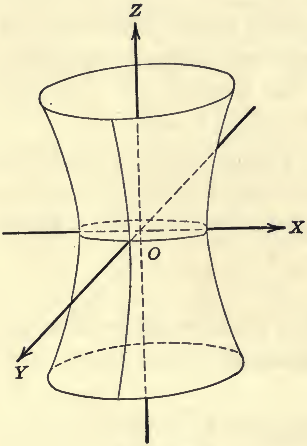


Fig. 185.

This surface can be thought of as generated by a variable ellipse moving parallel to the xy-plane, with its center on the z-axis, and the end points of its axes on the hyperbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \text{ and } \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The hyperboloid of one sheet has the property that through every point on its surface there can be drawn two lines which lie wholly in the surface. The surface can be covered with a net work of two sets of lines. No two lines of the same set intersect each other, but any line of either set intersects every line of the other set. This surface can be generated by a line moving in such a way that it always intersects three other non-intersecting lines in space. It is called a ruled surface, because through every point on its surface, there can be drawn at least one line which lies wholly on the surface.

226. The hyperboloid of two sheets. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

(1) This surface, Fig. 186, is symmetrical to all the coördinate planes, all the coördinate axes, and the origin.

(2) Its intercepts on the x-axis are $\pm a$. The intercepts on the y-axis and the z-axis are imaginary.

(3) Its traces are as follows:

In the xy-plane, the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

In the xz-plane, the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$

In the yz-plane, the imaginary ellipse $\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$

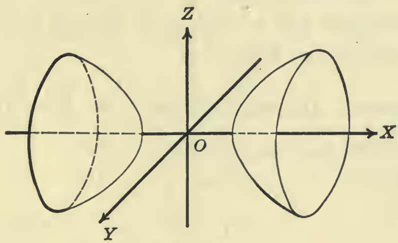


FIG. 186.

(4) Although the trace in the yz-plane is imaginary the form of the equation suggests that sections parallel to the yz-plane might be ellipses. Since it is easy to picture a surface in terms of increasing or decreasing ellipses, sections will be taken parallel to the yz-plane. Sections of this surface by such planes parallel to the yz-plane as $x = k$, are the ellipses

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1, \quad x = k,$$

or

$$\frac{y^2}{\frac{b^2}{a^2}(k^2 - a^2)} + \frac{z^2}{\frac{c^2}{a^2}(k^2 - a^2)} = 1, \quad x = k.$$

These ellipses are imaginary if $-a<k<a$. Hence there is no surface between the planes $x = -a$ and $x = a$. As k increases numerically from a to ∞, the ellipses increase indefinitely in magnitude.

Sections by planes parallel to the other axes are hyperbolas. The surface can be thought of as generated by a variable ellipse moving parallel to the yz-plane, with its center always on the x-axis, and the end points of its axes on the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ and } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$$

227. Elliptic paraboloid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z.$

(1) This surface, Fig. 187, is symmetrical to the yz-plane, the xz-plane, and the z-axis.

(2) Its intercepts are $x = 0$, $y = 0$, and $z = 0$.

(3) Its traces are as follows:

In the xy-plane, the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$.

In the xz-plane, the parabola $x^2 = a^2z$.

In the yz-plane, the parabola $y^2 = b^2z$.

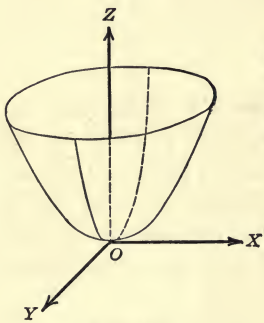


FIG. 187.

(4) The trace in the xy-plane suggests that sections parallel to this plane might be ellipses, in fact, the sections of this surface by the planes, $z = k$ are the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k, \quad z = k,$$

or

$$\frac{x^2}{a^2k} + \frac{y^2}{b^2k} = 1, \quad z = k.$$

If $k < 0$, the ellipses have an imaginary locus, hence no part of the surface lies below the xy-plane. As k increases from 0 to ∞, the ellipses increase in size indefinitely. The surface can be thought of as generated by a variable ellipse moving parallel to the xy-plane whose center is on the z-axis and the end points of whose major and minor axes are on the parabolas $x^2 = a^2z$ and $y^2 = b^2z$.

228. Hyperbolic paraboloid. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = z.$

(1) This surface, Fig. 188, is symmetrical to the yz-plane, the xz-plane, and the z-axis.

(2) Its intercepts are $x = 0$, $y = 0$, and $z = 0$.

(3) Its traces are as follows:

In the xy-plane, the two lines $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

In the xz-plane, the parabola $x^2 = a^2z$.

In the yz-plane, the parabola $y^2 = -b^2z$.

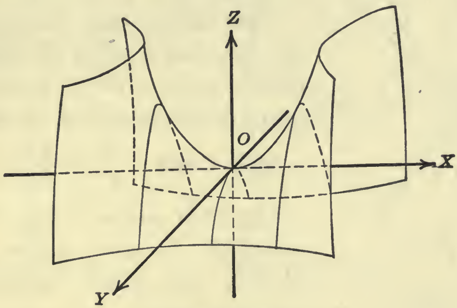


FIG. 188.

(4) Since no trace suggests an ellipse, and since it is easier to think in terms of moving parabolas instead of moving hyperbolas, sections are taken by planes parallel to the yz-plane. Sections of the surface by the planes $x = k$ are the parabolas

$$y^2 = -b^2\left(z - \frac{k^2}{a^2}\right), \quad x = k.$$

These are parabolas, symmetrical to the xz-plane, opening downward, and with vertices $\left(k,\ 0,\ \frac{k^2}{a^2}\right)$ lying on the trace $x^2 = a^2z$ of the hyperbolic paraboloid in the xz-plane. All of these parabolas are congruent. Hence the hyperbolic paraboloid may be thought of as generated by a parabola opening downward of latus rectum b^2, moving with its vertex on $x^2 = a^2z$ so that its plane is always parallel to the yz-plane.

Sections parallel to the xz-plane are parabolas opening upward, and sections parallel to the xy-plane are hyperbolas.

This hyperbolic paraboloid has the property that through every point on its surface there can be drawn two lines which lie wholly in the surface. The surface can be covered with a network of two sets of lines. No two lines of the same set intersect each other, but any line of either set meets every line of the other set. Hence the hyperbolic paraboloid is also a ruled surface.

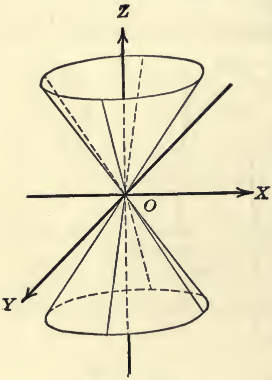


FIG. 189.

This surface can be generated by a line moving always parallel to a fixed plane, while always intersecting two non-intersecting lines in space.

229. Cone. $\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} - \frac{\mathbf{z}^2}{\mathbf{c}^2} = 0.$

(1) This surface, Fig. 189, is symmetrical to the three coördinate planes, the three coördinate axes, and the origin.

(2) Its intercepts on the axes are $x = 0$, $y = 0$, and $z = 0$.

(3) Its traces are as follows:

In the xy-plane, the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$.

In the xz-plane, the two lines $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$.

In the yz-plane, the two lines $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

(4) Sections of the cone by the planes $z = k$ are the ellipses

$$\frac{x^2}{\frac{a^2k^2}{c^2}} + \frac{y^2}{\frac{b^2k^2}{c^2}} = 1.$$

As the numerical value of k increases from 0 to ∞, the ellipses increase in magnitude indefinitely. Hence the surface can be thought of as generated by an ellipse moving parallel to the xy-plane, with the ends of its axes in the lines $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$, and the lines $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$. This cone is also a ruled surface, but it is covered by a single set only of lines, all of which pass through the origin.

EXERCISES

Discuss and draw the surfaces in exercises 1–15.

1. $x^2 + 4y^2 + 4z^2 = 9$.

2. $x^2 - 4y^2 + 4z^2 = 9$.

3. $x^2 - 4y^2 - 4z^2 = 9$.

4. $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} = 1$.

5. $\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{16} = 1$.

6. $\frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{16} = 1$.

7. $\frac{x^2}{9} - \frac{y^2}{4} - \frac{z^2}{16} = 0$.

8. $x^2 + y^2 = 4z$.

9. $x^2 - y^2 = 4z$.

10. $x^2 + y^2 - z^2 = 0$.

11. $xy + xz + yz = 0$.

12. $\cos \alpha = 0$.

13. $\cos \beta = \frac{1}{2}$.

14. $\cos \theta = \frac{1}{2}$.

15. $\cos \phi = \frac{1}{2}$.

Discuss and draw the curves or straight lines in exercises 16–20.

16. $x = 3, y = -2$.

17. $\cos \alpha = \cos \beta = 0$.

18. $y = x, x^2 + y^2 = 4$.

19. $x = y = z$.

20. $\cos \theta = \frac{1}{2}, \cos \phi = \frac{1}{2}$.

21. Find the equation of the locus of a point which moves so that the sum of the squares of its distances from the x and the y-axis equals 4. Discuss and draw the locus.

22. A point moves so that the sum of the squares of its distances from two fixed points is constant. Prove the locus to be an ellipsoid.

Suggestion.—Take the line through the two points to be the x-axis, and a point midway between them as the origin.

23. A point moves so that the difference of its distances from two fixed points is constant. Prove the locus to be an hyperboloid.

24. Find the locus of a point equidistant from the point $(p, 0, 0)$ and the xz-plane.

THE PLANE IN SPACE

230. Equation of a plane.—Some of the conditions that determine a plane in solid geometry are three points in the plane, or a point and a line in the plane. Unlike the straight line in two dimensional space, these simple conditions do not lend themselves readily to deriving the equation of a plane in three dimensional space.* Rather, one of the simplest ways of deriving the equation of a plane is by using the length of the perpendicular from the origin to the plane and the direction cosines of this perpendicular. This perpendicular is called the **normal to the plane.**

231. General equation of a plane.—Every equation of the first degree in x, y, and z as

[57] $$\mathbf{Ax + By + Cz + D = 0}$$

represents a plane.

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two points whose coördinates satisfy [57].

Then $$Ax_1 + By_1 + Cz_1 + D = 0, \quad (1)$$

and $$Ax_2 + By_2 + Cz_2 + D = 0. \quad (2)$$

Take any two constants r_1 and r_2, multiply equation (1) by $\frac{r_2}{r_1 + r_2}$, multiply equation (2) by $\frac{r_1}{r_1 + r_2}$, and add,

$$A\frac{r_1x_2 + r_2x_1}{r_1 + r_2} + B\frac{r_1y_2 + r_2y_1}{r_1 + r_2} + C\frac{r_1z_2 + r_2z_1}{r_1 + r_2} + D = 0.$$

This shows that any point on the line joining P_1P_2 also satisfies equation [57]. Since P_1 and P_2 are any two points on the surface [57], this shows that every line joining two

* The equation of a plane through three points can be expressed in determinant form. If the three points are P_1, P_2, and P_3, the equation is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

points on the surface lies wholly in the surface, and since this property is characteristic of the plane alone

$$Ax + By + Cz + D = 0$$

is the equation of a plane.

232. Normal form of the equation of a plane.—Let the length of the perpendicular OR, Fig. 190, from the origin to the plane be p, and let its direction angles be α, β, γ. If P is any point in the plane, the projection of OP on OR will be constant and equal to p. By theorem 2, **Art. 209,** the projection of OP equals the sum of the projections of the broken line $ONMP$ on OR.

Therefore proj. ON + proj. NM + proj. $MP = p$.

But proj. ON on $OR = x \cos \alpha$,
proj. NM on $OR = y \cos \beta$,
proj. MP on $OR = z \cos \gamma$.

Substituting these gives

[58] $$\mathbf{x \cos \alpha + y \cos \beta + z \cos \gamma = p.}$$

This is called the **normal form** of the equation of a plane.

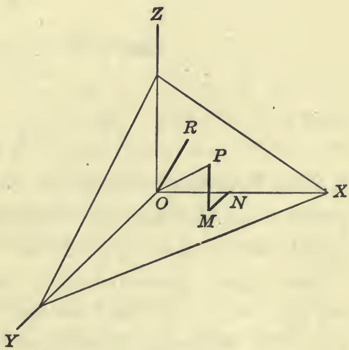


FIG. 190.

In article 231, it was shown that every equation of the first degree in x, y and z is the equation of a plane. This article proves the converse of that theorem, namely, that every equation of a plane is of the first degree in x, y, and z.

233. Reduction of the equation of a plane to the normal form.—The equations

$$Ax + By + Cz + D = 0 \text{ and}$$
$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$$

will be the equations of the same plane if they differ only by a constant factor. Suppose that k is such a factor, then

$$kAx + kBy + kCz + kD = 0. \qquad (1)$$

And therefore

$$kA = \cos\alpha,$$
$$kB = \cos\beta,$$
$$kC = \cos\gamma.$$

Squaring each equation and adding gives

$$k^2(A^2 + B^2 + C^2) = 1.$$

Or

$$k = \pm\frac{1}{\sqrt{A^2 + B^2 + C^2}}.$$

Substituting this value of k in equation (1), gives

$$\frac{Ax + By + Cz + D}{\pm\sqrt{A^2 + B^2 + C^2}} = 0,$$

where

[59]

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$\cos\beta = \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$\cos\gamma = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}},$$

$$p = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

234. Intercept form of the equation of a plane.—Let the plane cut the x-axis in the point where $x = a$, the y-axis in the point where $y = b$, and the z-axis in the point where $z = c$. These three quantities are called the intercepts of the plane on the axes. If they are given and none of them is zero, the plane is uniquely determined, for this is equivalent to giving the three points on the plane $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. To find the equation of the plane, substitute these three coördinates in succession in the general equation

$$Ax + By + Cz + D = 0. \qquad (1)$$

This gives the three equations:

$$Aa + D = 0,$$
$$Bb + D = 0,$$
$$Cc + D = 0.$$

From which

$$A = -\frac{D}{a}, \quad B = -\frac{D}{b}, \quad C = -\frac{D}{c}.$$

Substituting these values in (1),

$$-\frac{Dx}{a} - \frac{Dy}{b} - \frac{Dz}{c} + D = 0. \qquad (2)$$

Dividing by $-D$,

[60] $$\frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} + \frac{\mathbf{z}}{\mathbf{c}} = \mathbf{1}.$$

This is called the **intercept form** of the equation of a plane.

Note that this form is not valid if any of the intercepts are 0, that is if the plane passes through the origin.

235. Equation of a plane determined by three conditions. The general equation of a plane,

$$Ax + By + Cz + D = 0,$$

involves four constants, A, B, C, and D. Any three conditions that determine a plane give three relations between these four constants. These three equations can be solved for three of the constants in terms of the fourth providing that the fourth is not zero. Then after substitution the equations can be divided through by the fourth constant as in equation (2), **Art. 234.**

If the fourth constant should be zero, the three equations will turn out to be inconsistent. In such a case solve the equations in terms of another constant.

236. Angle between two planes.—Let the two planes be

$$A_1x + B_1y + C_1z + D_1 = 0,$$

and $$A_2x + B_2y + C_2z + D_2 = 0.$$

The angle θ between these two planes is the angle between their normals. Hence by [**55**] and [**59**],

$$[61] \quad \cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

The two planes are perpendicular to each other if

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

The two planes are parallel if their normals have the same direction cosines, that is, if $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}$.

237. Distance from a point to a plane.—Let $P_1(x_1, y_1, z_1)$ be the given point and $Ax + By + Cz + D = 0$ be the given plane. Pass a plane through P_1 parallel to the given plane, and find the difference between the normals to the planes. It is then found that the distance d is given by the formula

$$[62] \qquad d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}},$$

where the sign is chosen to make d positive.

Example 1.—Find the equation of a plane passing through the points (2, 1, 7) and (4, −1, −2) at a distance 2 from the origin.

Use the normal equation of a plane,

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0.$$

Since the distance of the plane from the origin equals 2,

$$p = 2.$$

Since the plane passes through the points (2, 1, 7) and (4, −1, −2).

$$2\cos\alpha + \cos\beta + 7\cos\gamma - 2 = 0,$$
$$4\cos\alpha - \cos\beta - 2\cos\gamma - 2 = 0.$$

Solving these equations with the identity

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

gives $\quad \cos\alpha = \frac{3}{7}, \quad \cos\beta = -\frac{6}{7}, \quad \cos\gamma = \frac{2}{7},$

or $\quad \cos\alpha = \frac{21}{31}, \quad \cos\beta = \frac{114}{155}, \quad \cos\gamma = -\frac{2}{155}.$

Therefore, there are two solutions to this problem and they are

$$3x - 6y + 2z - 14 = 0,$$

and

$$105x + 114y - 2z - 310 = 0.$$

Example 2.—Find the equation of the plane bisecting the angle between the planes $3x - y + 2z = 4$, and $2x + 3y - z = 4$.

If $P(x, y)$ is any point on the bisecting plane, its distance from each of the two planes is the same. Equating these distances gives

$$\frac{3x - y + 2z - 4}{\pm\sqrt{14}} = \frac{2x + 3y - z - 4}{\pm\sqrt{14}}.$$

This gives the two planes

$$x - 4y + 3z = 0,$$

and

$$5x + 2y + z - 8 = 0.$$

EXERCISES

Write the equations of the planes in exercises 1–4 in the normal form and the intercept form.

1. $x - 2y - 2z = 4$.
2. $2x + y - 2z = 9$.
3. $4x + 7y - 4z + 3 = 0$.
4. $12x - y + 12z = 18$.

Find the equations of the planes which satisfy the conditions of the exercises 5 to 16.

5. Passing through the points (1, 1, 1), (−3, 3, 8), (−2, −3, −2).
6. Passing through the points (2, −1, 0), (4, −2, 4), (−1, 3, −1).
7. $p = 5$, $\cos\alpha = \frac{1}{3}$, $\cos\beta = -\frac{2}{3}$.
8. $a = \frac{1}{3}$, $b = -\frac{2}{3}$, $c = 2$.
9. Passing through the points (4, 0, −1), (6, 3, 3) at a distance 2 from the origin.
10. Passing through the point (1, −2, 1) and parallel to the plane $y - 3x + 4z - 5 = 0$.
11. Passing through the points (1, 1, 1), (2, −1, 2) and perpendicular to the plane $3x + 4y - 7z + 10 = 0$.
12. Passing through the point (−2, −1, 3) and perpendicular to each of the planes $2x - 2y - 7z + 3 = 0$, and $4x + y - 4z - 1 = 0$.
13. Passing through the point (1, 1, 2) and perpendicular to the line joining (3, −4, 2) to (4, −6, 3).
14. Perpendicular to the line joining (7, −6, 3) to (1, 2, −5) at its middle point.
15. Parallel to the x-axis and passing through the points (2, 1, 2) and (−3, 5, 5).
16. Having the foot of the normal from the origin at the point (−3, 4, −2).
17. Find the distance from the point (3, −4, 2) to the plane $5x - 2y - 14z + 15 = 0$.

Find the angles between the planes in exercises 18–20.

18. $x - y + z = 7$ and $x + y + 5z = 3$.

19. $2x + y - z = 5$ and $4x - 2y - 2z = 3$.

20. $x + 2y - z = 7$ and $2x - y + 7z = 10$.

Find the equations of the planes bisecting the angle between the planes in exercises 21–23.

21. $2x + y - 2z = 1$ and $3x + 6y - 2z = 7$.

22. $x + y + z = 4$ and $5x - y - z = 2$.

23. $2x - y - z = 3$ and $5x - 5y + 2z = 4$.

24. Determine k so that $kx + 6y - 7z - 22 = 0$ shall be two units from the origin.

25. Find the point of intersection of the planes

$$\begin{aligned} 3x + y - z &= 3, \\ x + 5y + 7z &= 11, \\ 4x + 10y - 3z &= -8. \end{aligned}$$

26. At what acute angle does the plane $2x + 3y + 6z = 3$ cut each of the coördinate planes?

27. At what acute angle does the plane $2x + 3y + 6z = 3$ cut each coördinate axis?

28. Prove that the planes

$$\begin{aligned} 2x - y + 3z &= 4, \\ x + 6y - 6z &= 5, \\ 8x + 9y - 3z &= 22, \end{aligned}$$

have a line in common.

THE LINE IN SPACE

238. Two plane equation of a straight line.—In article 219, it was seen that it takes two equations in three dimensional space to define a curve. Hence the two equations

$$A_1x + B_1y + C_1z + D_1 = 0, \quad (1)$$
$$A_2x + B_2y + C_2z + D_2 = 0, \quad (2)$$

are the equations of a straight line. Equations (1) and (2) are the equations of any two planes through the line.

239. Projection form of the equation of a straight line.—By eliminating in turn z, y, and x between equations (1) and (2), **Art. 238,** the equations of the projections of the straight line on the xy, xz, and yz-planes are obtained. If these equations are

$$\begin{aligned} l_1x + l_2y + l_3 &= 0, \\ m_1x + m_2z + m_3 &= 0, \\ n_1y + n_2z + n_3 &= 0, \end{aligned}$$

any two of these equations are the equations of the straight line, and any two of these equations are called the **projection form** of the equations of a straight line.

240. Point direction form of the equation of a straight line. Symmetric form.

Case I.—*The line is not parallel to any coördinate plane.*

Let $P_1(x_1, y_1, z_1)$ be the point and let the direction of the straight line be given by its direction cosines, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Then, if $P(x, y, z)$ is any point on the straight line, and d is the distance from P_1 to P, by [**50**]

$$\cos\alpha = \frac{x - x_1}{d}, \ \cos\beta = \frac{y - y_1}{d}, \ \cos\gamma = \frac{z - z_1}{d}. \tag{1}$$

Solving each equation for d and equating the results,

[$\mathbf{63_1}$] $$\frac{\mathbf{x - x_1}}{\cos\alpha} = \frac{\mathbf{y - y_1}}{\cos\beta} = \frac{\mathbf{z - z_1}}{\cos\gamma}.$$

If $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are replaced by any quantities l, m, n proportional to them, the equation can be written

[$\mathbf{63_2}$] $$\frac{\mathbf{x - x_1}}{\mathbf{l}} = \frac{\mathbf{y - y_1}}{\mathbf{m}} = \frac{\mathbf{z - z_1}}{\mathbf{n}}.$$

Case II.—*The line is parallel to one or two coördinate planes.* Suppose the line is parallel to one of the coördinate planes, say the yz-plane, but is not parallel to one of the coördinate axes, then $\cos\alpha = 0$, $\cos\beta \neq 0$, $\cos\gamma \neq 0$. Equations [$\mathbf{63_1}$] and [$\mathbf{63_2}$] are not valid, but equation (1) can be written

$$0 = \frac{x - x_1}{d}, \ \cos\beta = \frac{y - y_1}{d}, \ \cos\gamma = \frac{z - z_1}{d},$$

giving the equations of the line to be

$$x - x_1 = 0,$$

and $$\frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}.$$

If the line is parallel to two coördinate planes, it is parallel to one of the coördinate axes. If this is the z-axis, then $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$. Equation (1) can then be written

$$0 = \frac{x - x_1}{d},\ 0 = \frac{y - y_1}{d},\ \cos\gamma = \frac{z - z_1}{d},$$

giving the equations of the line to be

$$x - x_1 = 0,$$
$$y - y_1 = 0.$$

Like considerations hold if the line is parallel to any of the other coördinate axes or planes.

241. Two point form of the equation of a straight line.

CASE I.—*The straight line is not parallel to any coördinate plane.* Let the two points through which the line passes be $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Since the direction cosines of this line are proportional to $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$, the quantities l, m, and n of [63_2] can be so chosen that

$$l = x_2 - x_1,\ m = y_2 - y_1,\ n = z_2 - z_1,$$

which gives

[64] $$\frac{\mathbf{x} - \mathbf{x}_1}{\mathbf{x}_2 - \mathbf{x}_1} = \frac{\mathbf{y} - \mathbf{y}_1}{\mathbf{y}_2 - \mathbf{y}_1} = \frac{\mathbf{z} - \mathbf{z}_1}{\mathbf{z}_2 - \mathbf{z}_1}.$$

CASE II.—*The line is parallel to one or two coördinate planes.* The discussion is similar to that given in article 240.

Example 1.—Reduce the equations that define the straight line, $3x + 2y - 2z + 2 = 0$, and $6x + 7y - 6z - 3 = 0$ to the symmetric form.

Solution.—Reduce these equations to the projection form by first eliminating x and then z giving

$$3y - 2z - 7 = 0,$$
$$3x - y + 9 = 0.$$

Solving each for y and equating,

$$3x + 9 = y = \frac{2z + 7}{3}.$$

This can be written in the form

$$\frac{x + 3}{\frac{1}{3}} = y = \frac{z + \frac{7}{2}}{\frac{3}{2}}.$$

In order that the denominators shall be direction cosines, multiply each equation by $\sqrt{\frac{1}{9} + 1 + \frac{9}{4}} = \frac{11}{6}$, and the equation becomes

$$\frac{x+3}{\frac{2}{11}} = \frac{y}{\frac{6}{11}} = \frac{z+\frac{7}{2}}{\frac{9}{11}}.$$

This shows that the line passes through the point $(-3, 0, -\frac{7}{2})$ with direction cosines $\frac{2}{11}, \frac{6}{11}, \frac{9}{11}$. If desired, the point $(-3, 0, -\frac{7}{2})$ can be replaced by any other point on the line, say $(-2, 3, 1)$, in which case the equation of the line takes the form

$$\frac{x+2}{\frac{2}{11}} = \frac{y-3}{\frac{6}{11}} = \frac{z-1}{\frac{9}{11}}.$$

Example 2.—Find the equation of a plane passing through the line $\frac{2x-3}{4} = \frac{y-6}{5} = \frac{z+2}{3}$, and the point $(-1, -1, -6)$.

Solution.—This equation is equivalent to the two equations

$$\frac{2x-3}{4} = \frac{y-6}{5}, \text{ and } \frac{y-6}{5} = \frac{z+2}{3}.$$

Simplifying

$$10x - 4y + 9 = 0, \text{ and } 3y - 5z - 28 = 0.$$

These are the equations of two planes passing through the given line. The equation of any plane through the line of intersection of these two planes, and hence through the given line, is evidently

$$10x - 4y + 9 + k(3y - 5z - 28) = 0.$$

To make this plane pass through the point $(-1, -1, -6)$, substitute these coördinates and solve for k. The result is

$$k = 3.$$

Hence the required plane is

$$10x - 4y + 9 + 3(3y - 5z - 28) = 0,$$

or

$$2x + y - 3z - 15 = 0.$$

EXERCISES

Find where the lines in exercises 1–5 intersect the three coördinate planes.

1. $4x + y + z - 5 = 0,\ 2x - y + z - 1 = 0.$

2. $x - y + z = 5,\ 5x - 6y + 4z = 28.$

3. $4x + y - 6z = 10,\ 7x + 3y - 8z - 15 = 0.$

4. $4x + 3y + 2z = 2,\ -3x + 4y + z - 6 = 0.$

5. $\frac{x-4}{2} = \frac{y+1}{-1} = \frac{z-3}{2}.$

6. Reduce the equations in exercises 1 and 2 to the projection form, the projecting planes being perpendicular to the xy and xz-planes.

7. Reduce the equations in exercises 3 and 4 to the projection form, the projecting planes being perpendicular to the xy and yz-planes.

8. Reduce the equations in exercises 1–4 to the symmetric form.

Find the equations in projection form of the straight lines in exercises 9–18. The projecting planes are to be taken perpendicular to the xy and xz-planes whenever possible.

9. Passing through the points $(3, -6, 4)$ and $(-2, 5, 1)$.

10. Passing through the points $(-2, 1, 2)$ and $(3, -1, 4)$.

11. Passing through the points $(2, 1, -3)$ and $(2, 3, -4)$.

12. Passing through the points $(2, 5, 6)$ and $(2, 5, 7)$.

13. Passing through the point $(1, -3, 4)$ with direction cosines in the ratio $3 : -1 : 2$.

14. Passing through the point $(3, -1, 2)$ and parallel to the z-axis.

15. Passing through the point $(3, -1, 2)$ and perpendicular to the z-axis.

16. Passing through the point $(3, -1, 2)$ and parallel to the line of exercise 1.

17. Passing through the point $(3, -1, 2)$ and making right angles with the plane $x - 2y + z = 3$.

18. Passing through the origin and perpendicular to the lines

$$x - 1 = \frac{y - 2}{2} = \frac{z + 4}{-2}, \text{ and } \frac{x - 2}{-12} = \frac{y + 1}{4} = \frac{z - 3}{3}.$$

19. Find the cosine of the acute angle between the lines

$$\frac{x - 3}{4} = \frac{y + 2}{-8} = \frac{z - 6}{1}, \text{ and } \frac{x - 2}{4} = \frac{y - 5}{4} = \frac{z}{7}.$$

20. Find the cosine of the angle between the line $2x - 7y - 7z = -8$, $x - 2y - z = 5$, and the line $12x - 15y - 2z = 70$, $5x - 5y - z = 24$.

21. Prove that the two lines $x + y + z = 0$, $2x - y + 3z = 7$, and $3x + 4y + 2z = -3$, $-6x + 2y + 10z = 0$ meet in a point.

22. Prove that the planes $2x + 2y + z + 4 = 0$, $4x + y - z - 7 = 0$, and $2x + 3y + 2z + 9 = 0$, meet in a straight line and find its direction cosines.

Find the equations of the planes that satisfy the conditions of exercises 23–26.

23. Passing through the point $(2, 1, 3)$ and the line

$$3x + 5y - 6z + 9 = 0,$$
$$2x + 2y - 2z + 1 = 0.$$

24. Passing through the point $(-1, -2, -3)$ and the line

$$\frac{x - 1}{3} = \frac{y + 1}{2} = \frac{z + 5}{3}$$

25. Passing through the parallel lines

$$\frac{x-1}{1} = \frac{y-3}{-7} = \frac{z+1}{6}, \text{ and } \frac{x-4}{1} = \frac{y-2}{-7} = \frac{z+3}{6}.$$

26. Passing through the intersecting lines

$$\frac{x}{-1} = \frac{y+5}{7} = \frac{z+4}{9}, \text{ and } \frac{-7x-1}{2} = \frac{y}{1} = \frac{7z-9}{11}.$$

27. Find the equation of a plane through the line

$$x + y - 2z + 2 = 0,$$
$$3x + 8y - 6z + 4 = 0,$$

and perpendicular to the plane $7x + 2y + 2z - 10 = 0$.

28. Find the equation of a line lying in the plane $2x - 2y + z + 11 = 0$, passing through the point $(-3, 2, -1)$, and parallel to the plane $2x + 3y - 4z + 5 = 0$.

SUMMARY OF FORMULAS

[1] (1) $OP_2 = OP_1 + P_1P_2$. (2) $P_1P_2 = OP_2 - OP_1$.

[2] $P_1P_2 = x_2 - x_1$.

[2₁] $P_1P_2 = x_2 - x_1$. [2₂] $P_1P_2 = y_2 - y_1$.

[3] $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

[4] $x_0 = \dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}, \quad y_0 = \dfrac{r_1y_2 + r_2y_1}{r_1 + r_2}$.

[5] $x_0 = \dfrac{x_1 + x_2}{2}, \quad y_0 = \dfrac{y_1 + y_2}{2}$.

[6] $m = \tan\alpha = \dfrac{y_1 - y_2}{x_1 - x_2}$.

[7] $\tan\varphi = \dfrac{m_1 - m_2}{1 + m_1m_2}$.

[8] For parallel lines, $m_1 = m_2$.

[9] For perpendicular lines, $m_1 = -\dfrac{1}{m_2}$, and $m_2 = -\dfrac{1}{m_1}$.

[10] $x = \rho\cos\theta,\ y = \rho\sin\theta,\ x^2 + y^2 = \rho^2$.

[11] $\rho = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\dfrac{y}{x}$.

[12] $x = x' + h, \quad y = y' + k$.

[12₁] $x' = x - h, \quad y' = y - k$.

[13] $x = x'\cos\varphi - y'\sin\varphi, \quad y = x'\sin\varphi + y'\cos\varphi$.

[13₁] $x' = x\cos\varphi + y\sin\varphi, \quad y' = y\cos\varphi - x\sin\varphi$.

[14] $A = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$

[15] $y - y_1 = m(x - x_1).$

[16] $y = mx + b.$

[17] $y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1).$

[18] $\frac{x}{a} + \frac{y}{b} = 1.$

[19] $x \cos \theta + y \sin \theta - p = 0.$

[20] $Ax + By + C = 0.$

[21] $\frac{Ax}{\pm\sqrt{A^2 + B^2}} + \frac{By}{\pm\sqrt{A^2 + B^2}} + \frac{C}{\pm\sqrt{A^2 + B^2}} = 0.$

[22] $\cos \theta = \frac{A}{\pm\sqrt{A^2 + B^2}}, \quad \sin \theta = \frac{B}{\pm\sqrt{A^2 + B^2}},$

$$p = \frac{-C}{\pm\sqrt{A^2 + B^2}}.$$

[23] $d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}.$

[24] $\frac{A_1x + B_1y + C_1}{\sqrt{A_1{}^2 + B_1{}^2}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2{}^2 + B_2{}^2}}.$

[25] $(x - h)^2 + (y - k)^2 = r^2.$

[26] $x^2 + y^2 = r^2.$

[27] $x^2 + y^2 + Dx + Ey + F = 0.$

[28] $y^2 = 2px.$

[29] $x^2 = 2py.$

[30] $(y - k)^2 = 2p(x - h).$

[30_1] $(x - h)^2 = 2p(y - k).$

[31] $\rho = \frac{p}{1 - \cos \theta}.$

[32] $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

[33] $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$

[34] $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$

[34₁] $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1.$

[35] $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

[36] $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$

[37] $y = \frac{b}{a}x, \; y = -\frac{b}{a}x.$

[38] $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$

[38₁] $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$

[39] $\tan 2\varphi = \frac{B}{A-C}.$

[40] $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$

[41] $\begin{cases} x = (a-b)\cos\theta + b\cos\frac{(a-b)}{b}\theta, \\ y = (a-b)\sin\theta - b\sin\frac{(a-b)}{b}\theta. \end{cases}$

[42] $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

[43] $\begin{cases} x = (a+b)\cos\theta - b\cos\frac{(a+b)}{b}\theta, \\ y = (a+b)\sin\theta - b\sin\frac{(a+b)}{b}\theta. \end{cases}$

[44] $\begin{cases} x = a\cos\theta + a\,\theta\sin\theta, \\ y = a\sin\theta - a\,\theta\cos\theta. \end{cases}$

[45] $Ax_1x + \frac{1}{2}Bx_1y + \frac{1}{2}Bxy_1 + Cy_1y + \frac{1}{2}Dx + \frac{1}{2}Dx_1 + \frac{1}{2}Ey + \frac{1}{2}Ey_1 + F = 0.$

[46] $y - y_1 = \left.\frac{dy}{dx}\right|_{x=x_1}(x - x_1).$

[47] $y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_1}}(x - x_1).$

[48] $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$

[49] $x_0 = \dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}, \quad y_0 = \dfrac{r_1y_2 + r_2y_1}{r_1 + r_2}, \quad z_0 = \dfrac{r_1z_2 + r_2z_1}{r_1 + r_2}.$

[50] $\cos\alpha = \dfrac{x_2 - x_1}{d}, \quad \cos\beta = \dfrac{y_2 - y_1}{d}, \quad \cos\gamma = \dfrac{z_2 - z_1}{d}.$

[51] $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

[52] $\rho = \pm\sqrt{x^2 + y^2 + z^2}, \quad \cos\alpha = \dfrac{x}{\pm\sqrt{x^2 + y^2 + z^2}},$

$$\cos\beta = \frac{y}{\pm\sqrt{x^2 + y^2 + z^2}}, \quad \cos\gamma = \frac{z}{\pm\sqrt{x^2 + y^2 + z^2}}.$$

[53] $x = \rho\cos\alpha, \quad y = \rho\cos\beta, \quad z = \rho\cos\gamma.$

[54] $x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi.$

[54₁] $\rho = \pm\sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\dfrac{y}{x},$

$$\theta = \sin^{-1}\frac{y}{\pm\sqrt{x^2 + y^2}}, \quad \varphi = \cos^{-1}\frac{z}{\pm\sqrt{x^2 + y^2 + z^2}}.$$

[55] $\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$

[56] $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$

[57] $Ax + By + Cz + D = 0.$

[58] $x\cos\alpha + y\cos\beta + z\cos\gamma = p.$

[59] $\cos\alpha = \dfrac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad \cos\beta = \dfrac{B}{\pm\sqrt{A^2 + B^2 + C^2}},$

$$\cos\gamma = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad p = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

[60] $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1.$

[61] $\cos\theta = \dfrac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}.$

[62] $d = \dfrac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}.$

[63₁] $\dfrac{x - x_1}{\cos\alpha} = \dfrac{y - y_1}{\cos\beta} = \dfrac{z - z_1}{\cos\gamma}.$

[63₂] $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}.$

[64] $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}.$

TABLES

I. Four-place Table of Logarithms.

II. Table of Natural and Logarithmic Sines, Cosines, Tangents, and Cotangents of Angles Differing by Ten Minutes.

TABLE I.—COMMON LOGARITHMS

N.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N.	0	1	2	3	4	5	6	7	8	9

TABLE I.—COMMON LOGARITHMS.—*Continued*

N.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N.	0	1	2	3	4	5	6	7	8	9

TABLE II.—TRIGONOMETRIC FUNCTIONS

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
0° 00′	.0000	∞	1 0000	0.0000	.0000	∞	∞	∞	90° 00′
10	.0029	7.4637	1.0000	0000	.0029	7.4637	343.77	2.5363	50
20	.0058	7648	1.0000	0000	.0058	7648	171.89	2352	40
30	.0087	9408	1.0000	0000	.0087	9409	114.59	0591	30
40	.0116	8.0658	.9999	0000	.0116	8.0658	85.940	1.9342	20
50	.0145	1627	.9999	0000	.0145	1627	68.750	8373	10
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.290	1.7581	89° 00′
10	.0204	3088	.9998	9999	.0204	3089	49.104	6911	50
20	.0233	3668	.9997	9999	.0233	3669	42.964	6331	40
30	.0262	4179	.9997	9999	.0262	4181	38.188	5819	30
40	.0291	4637	.9996	9998	.0291	4638	34.368	5362	20
50	.0320	5050	.9995	9998	.0320	5053	31.242	4947	10
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.636	1.4569	88° 00′
10	.0378	5776	.9993	9997	.0378	5779	26.432	4221	50
20	.0407	6097	.9992	9996	.0407	6101	24.542	3899	40
30	.0436	6397	.9990	9996	.0437	6401	22.904	3599	30
40	.0465	6677	.9989	9995	.0466	6682	21.470	3318	20
50	.0494	6940	.9988	9995	.0495	6945	20.206	3055	10
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.081	1.2806	87° 00′
10	.0552	7423	.9985	9993	.0553	7429	18.075	2571	50
20	.0581	7645	.9983	9993	.0582	7652	17.169	2348	40
30	.0610	7857	.9981	9992	.0612	7865	16.350	2135	30
40	.0640	8059	.9980	9991	.0641	8067	15.605	1933	20
50	.0669	8251	.9978	9990	.0670	8261	14.924	1739	10
4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.301	1.1554	86° 00′
10	.0727	8613	.9974	9989	.0729	8624	13.727	1376	50
20	.0756	8783	.9971	9988	.0758	8795	13.197	1205	40
30	.0785	8946	.9969	9987	.0787	8960	12.706	1040	30
40	.0814	9104	.9967	9986	.0816	9118	12.251	0882	20
50	.0843	9256	.9964	9985	.0846	9272	11.826	0728	10
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.430	1.0580	85° 00′
10	.0901	9545	.9959	9982	.0904	9563	11.059	0437	50
20	.0929	9682	.9957	9981	.0934	9701	10.712	0299	40
30	.0958	9816	.9954	9980	.0963	9836	10.385	0164	30
40	.0987	9945	.9951	9979	.0992	9966	10.078	0034	20
50	.1016	9.0070	.9948	9977	.1022	9.0093	9.7882	0.9907	10
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.5144	0.9784	84° 00′
10	.1074	0311	.9942	9975	.1080	0336	9.2553	9664	50
20	.1103	0426	.9939	9973	.1110	0453	9.0098	9547	40
30	.1132	0539	.9936	9972	.1139	0567	8.7769	9433	30
40	.1161	0648	.9932	9971	.1169	0678	8.5555	9322	20
50	.1190	0755	.9929	9969	.1198	0786	8.3450	9214	10
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.1443	0.9109	83° 00′
10	.1248	0961	.9922	9966	.1257	0995	7.9530	9005	50
20	.1276	1060	.9918	9964	.1287	1096	7.7704	8904	40
30	.1305	1157	.9914	9963	.1317	1194	7.5958	8806	30
40	.1334	1252	.9911	9961	.1346	1291	7.4287	8709	20
50	.1363	1345	.9907	9959	.1376	1385	7.2687	8615	10
8° 00,	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.1154	0.8522	82° 00′
10	.1421	1525	.9899	9956	.1435	1569	6.9682	8431	50
20	.1449	1612	.9894	9954	.1465	1658	6.8269	8342	40
30	.1478	1697	.9890	9952	.1495	1745	6.6912	8255	30
40	.1507	1781	.9886	9950	.1524	1831	6.5606	8169	20
50	.1536	1863	.9881	9948	.1554	1915	6.4348	8085	10
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS—*Continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
10	.1593	2022	.9872	9944	.1614	2078	6.1970	7922	50
20	.1622	2100	.9868	9942	.1644	2158	6.0844	7842	40
30	.1650	2176	.9863	9940	.1673	2236	5.9758	7764	30
40	.1679	2251	.9858	9938	.1703	2313	5.8708	7687	20
50	.1708	2324	.9853	9936	.1733	2389	5.7694	7611	10
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.6713	0.7537	80° 00′
10	.1765	2468	.9843	9931	.1793	2536	5.5764	7464	50
20	.1794	2538	.9838	9929	.1823	2609	5.4845	7391	40
30	.1822	2606	.9833	9927	.1853	2680	5.3955	7320	30
40	.1851	2674	.9827	9924	.1883	2750	5.3093	7250	20
50	.1880	2740	.9822	9922	.1914	2819	5.2257	7181	10
11° 00′	.1908	9.2806	.9816	9.9919	.1944	9.2887	5.1446	0.7113	79° 00′
10	.1937	2870	.9811	9917	.1974	2953	5.0658	7047	50
20	.1965	2934	.9805	9914	.2004	3020	4.9894	6980	40
30	.1994	2997	.9799	9912	.2035	3085	4.9152	6915	30
40	.2022	3058	.9793	9909	.2065	3149	4.8430	6851	20
50	.2051	3119	.9787	9907	.2095	3212	4.7729	6788	10
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.7046	0.6725	78° 00′
10	.2108	3238	.9775	9901	.2156	3336	4.6382	6664	50
20	.2136	3296	.9769	9899	.2186	3397	4.5736	6603	40
30	.2164	3353	.9763	9896	.2217	3458	4.5107	6542	30
40	.2193	3410	.9757	9893	.2247	3517	4.4494	6483	20
50	.2221	3466	.9750	9890	.2278	3576	4.3897	6424	10
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.3315	0.6366	77° 00′
10	.2278	3575	.9737	9884	.2339	3691	4.2747	6309	50
20	.2306	3629	.9730	9881	.2370	3748	4.2193	6252	40
30	.2334	3682	.9724	9878	.2401	3804	4.1653	6196	30
40	.2363	3734	.9717	9875	.2432	3859	4.1126	6141	20
50	.2391	3786	.9710	9872	.2462	3914	4.0611	6086	10
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.0108	0.6032	76° 00′
10	.2447	3887	.9696	9866	.2524	4021	3.9617	5979	50
20	.2476	3937	.9689	9863	.2555	4074	3.9136	5926	40
30	.2504	3986	.9681	9859	.2586	4127	3.8667	5873	30
40	.2532	4035	.9674	9856	.2617	4178	3.8208	5822	20
50	.2560	4083	.9667	9853	.2648	4230	3.7760	5770	10
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.7321	0.5719	75° 00′
10	.2616	4177	.9652	9846	.2711	4331	3.6891	5669	50
20	.2644	4223	.9644	9843	.2742	4381	3.6470	5619	40
30	.2672	4269	.9636	9839	.2773	4430	3.6059	5570	30
40	.2700	4314	.9628	9836	.2805	4479	3.5656	5521	20
50	.2728	4359	.9621	9832	.2836	4527	3.5261	5473	10
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.4874	0.5425	74° 00′
10	.2784	4447	.9605	9825	.2899	4622	3.4495	5378	50
20	.2812	4491	.9596	9821	.2931	4669	3.4124	5331	40
30	.2840	4533	.9588	9817	.2962	4716	3.3759	5284	30
40	.2868	4576	.9580	9814	.2994	4762	3.3402	5238	20
50	.2896	4618	.9572	9810	.3026	4808	3.3052	5192	10
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.2709	0.5147	73° 00′
10	.2952	4700	.9555	9802	.3089	4898	3.2371	5102	50
20	.2979	4741	.9546	9798	.3121	4943	3.2041	5057	40
30	.3007	4781	.9537	9794	.3153	4987	3.1716	5013	30
40	.3035	4821	.9528	9790	.3185	5031	3.1397	4969	20
50	.3062	4861	.9520	9786	.3217	5075	3.1084	4925	10
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS—*Continued*

Angles	Sines Nat.	Sines Log.	Cosines Nat.	Cosines Log.	Tangents Nat.	Tangents Log.	Cotangents Nat.	Cotangents Log.	Angles
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
10	.3118	4939	.9502	9778	.3281	5161	3.0475	4839	50
20	.3145	4977	.9492	9774	.3314	5203	3.0178	4797	40
30	.3173	5015	.9483	9770	.3346	5245	2.9887	4755	30
40	.3201	5052	.9474	9765	.3378	5287	2.9600	4713	20
50	.3228	5090	.9465	9761	.3411	5329	2.9319	4671	10
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.9042	0.4630	71° 00′
10	.3283	5163	.9446	9752	.3476	5411	2.8770	4589	50
20	.3311	5199	.9436	9748	.3508	5451	2.8502	4549	40
30	.3338	5235	.9426	9743	.3541	5491	2.8239	4509	30
40	.3365	5270	.9417	9739	.3574	5531	2.7980	4469	20
50	.3393	5306	.9407	9734	.3607	5571	2.7725	4429	10
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.7475	0.4389	70° 00′
10	.3448	5375	.9387	9725	.3673	5650	2.7228	4350	50
20	.3475	5409	.9377	9721	.3706	5689	2.6985	4311	40
30	.3502	5443	.9367	9716	.3739	5727	2.6746	4273	30
40	.3529	5477	.9356	9711	.3772	5766	2.6511	4234	20
50	.3557	5510	.9346	9706	.3805	5804	2.6279	4196	10
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.6051	0.4158	69° 00′
10	.3611	5576	.9325	9697	.3872	5879	2.5826	4121	50
20	.3638	5609	.9315	9692	.3906	5917	2.5605	4083	40
30	.3665	5641	.9304	9687	.3939	5954	2.5386	4046	30
40	.3692	5673	.9293	9682	.3973	5991	2.5172	4009	20
50	.3719	5704	.9283	9677	.4006	6028	2.4960	3972	10
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.4751	0.3936	68° 00′
10	.3773	5767	.9261	9667	.4074	6100	2.4545	3900	50
20	.3800	5798	.9250	9661	.4108	6136	2.4342	3864	40
30	.3827	5828	.9239	9656	.4142	6172	2.4142	3828	30
40	.3854	5859	.9228	9651	.4176	6208	2.3945	3792	20
50	.3881	5889	.9216	9646	.4210	6243	2.3750	3757	10
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.3559	0.3721	67° 00′
10	.3934	5948	.9194	9635	.4279	6314	2.3369	3686	50
20	.3961	5978	.9182	9629	.4314	6348	2.3183	3652	40
30	.3987	6007	.9171	9624	.4348	6383	2.2998	3617	30
40	.4014	6036	.9159	9618	.4383	6417	2.2817	3583	20
50	.4041	6065	.9147	9613	.4417	6452	2.2637	3548	10
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.2460	0.3514	66° 00′
10	.4094	6121	.9124	9602	.4487	6520	2.2286	3480	50
20	.4120	6149	.9112	9596	.4522	6553	2.2113	3447	40
30	.4147	6177	.9100	9590	.4557	6587	2.1943	3413	30
40	.4173	6205	.9088	9584	.4592	6620	2.1775	3380	20
50	.4200	6232	.9075	9579	.4628	6654	2.1609	3346	10
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.1445	0.3313	65° 00′
10	.4253	6286	.9051	9567	.4699	6720	2.1283	3280	50
20	.4279	6313	.9038	9561	.4734	6752	2.1123	3248	40
30	.4305	6340	.9026	9555	.4770	6785	2.0965	3215	30
40	.4331	6366	.9013	9549	.4806	6817	2.0809	3183	20
50	.4358	6392	.9001	9543	.4841	6850	2.0655	3150	10
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.0503	0.3118	64° 00′
10	.4410	6444	.8975	9530	.4913	6914	2.0353	3086	50
20	.4436	6470	.8962	9524	.4950	6946	2.0204	3054	40
30	.4462	6495	.8949	9518	.4986	6977	2.0057	3023	30′
40	.4488	6521	.8936	9512	.5022	7009	1.9912	2991	20
50	.4514	6546	.8923	9505	.5059	7040	1.9768	2960	10
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
Angles	Cosines Nat.	Cosines Log.	Sines Nat.	Sines Log.	Cotangents Nat.	Cotangents Log.	Tangents Nat.	Tangents Log.	Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS—*Continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
10	.4566	6595	.8897	9492	.5132	7103	1.9486	2897	50
20	.4592	6620	.8884	9486	.5169	7134	1.9347	2866	40
30	.4617	6644	.8870	9479	.5206	7165	1.9210	2835	30
40	.4643	6668	.8857	9473	.5243	7196	1.9074	2804	20
50	.4669	6692	.8843	9466	.5280	7226	1.8940	2774	10
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	62° 00′
10	.4720	6740	.8816	9453	.5354	7287	1.8676	2713	50
20	.4746	6763	.8802	9446	.5392	7317	1.8546	2683	40
30	.4772	6787	.8788	9439	.5430	7348	1.8418	2652	30
40	.4797	6810	.8774	9432	.5467	7378	1.8291	2622	20
50	.4823	6833	.8760	9425	.5505	7408	1.8165	2592	10
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	61° 00′
10	.4874	6878	.8732	9411	.5581	7467	1.7917	2533	50
20	.4899	6901	.8718	9404	.5619	7497	1.7796	2503	40
30	.4924	6923	.8704	9397	.5658	7526	1.7675	2474	30
40	.4950	6946	.8689	9390	.5696	7556	1.7556	2444	20
50	.4975	6968	.8675	9383	.5735	7585	1.7437	2415	10
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	60° 00′
10	.5025	7012	.8646	9368	.5812	7644	1.7205	2356	50
20	.5050	7033	.8631	9361	.5851	7673	1.7090	2327	40
30	.5075	7055	.8616	9353	.5890	7701	1.6977	2299	30
40	.5100	7076	.8601	9346	.5930	7730	1.6864	2270	20
50	.5125	7097	.8587	9338	.5969	7759	1.6753	2241	10
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	59° 00′
10	.5175	7139	.8557	9323	.6048	7816	1.6534	2184	50
20	.5200	7160	.8542	9315	.6088	7845	1.6426	2155	40
30	.5225	7181	.8526	9308	.6128	7873	1.6319	2127	30
40	.5250	7201	.8511	9300	.6168	7902	1.6212	2098	20
50	.5275	7222	.8496	9292	.6208	7930	1.6107	2070	10
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	58° 00′
10	.5324	7262	.8465	9276	.6289	7986	1.5900	2014	50
20	.5348	7282	.8450	9268	.6330	8014	1.5798	1986	40
30	.5373	7302	.8434	9260	.6371	8042	1.5697	1958	30
40	.5398	7322	.8418	9252	.6412	8070	1.5597	1930	20
50	.5422	7342	.8403	9244	.6453	8097	1.5497	1903	10
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	57° 00′
10	.5471	7380	.8371	9228	6536	8153	1.5301	1847	50
20	.5495	7400	.8355	9219	.6577	8180	1.5204	1820	40
30	.5519	7419	.8339	9211	.6619	8208	1.5108	1792	30
40	.5544	7438	.8323	9203	.6661	8235	1.5013	1765	20
50	.5568	7457	.8307	9194	.6703	8263	1.4919	1737	10
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	56° 00′
10	.5616	7494	.8274	9177	.6787	8317	1.4733	1683	50
20	.5640	7513	.8258	9169	.6830	8344	1.4641	1656	40
30	.5664	7531	.8241	9160	.6873	8371	1.4550	1629	30
40	.5688	7550	.8225	9151	.6916	8398	1.4460	1602	20
50	.5712	7568	.8208	9142	.6959	8425	1.4370	1575	10
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	55° 00′
10	.5760	7604	.8175	9125	.7046	8479	1.4193	1521	50
20	.5783	7622	.8158	9116	.7089	8506	1.4106	1494	40
30	.5807	7640	.8141	9107	.7133	8533	1.4019	1467	30
40	.5831	7657	.8124	9098	.7177	8559	1.3934	1441	20
50	.5854	7675	.8107	9089	.7221	8586	1.3848	1414	10
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS—*Continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
10	.5901	7710	.8073	9070	.7310	8639	1.3680	1361	50
20	.5925	7727	.8056	9061	.7355	8666	1.3597	1334	40
30	.5948	7744	.8039	9052	.7400	8692	1.3514	1308	30
40	.5972	7761	.8021	9042	.7445	8718	1.3432	1282	20
50	.5995	7778	.8004	9033	.7490	8745	1.3351	1255	10
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.3270	0.1229	53° 00′
10	.6041	7811	.7969	9014	.7581	8797	1.3190	1203	50
20	.6065	7828	.7951	9004	.7627	8824	1.3111	1176	40
30	.6088	7844	.7934	8995	.7673	8850	1.3032	1150	30
40	.6111	7861	.7916	8985	.7720	8876	1.2954	1124	20
50	.6134	7877	.7898	8975	.7766	8902	1.2876	1098	10
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.2799	0.1072	52° 00′
10	.6180	7910	.7862	8955	.7860	8954	1.2723	1046	50
20	.6202	7926	.7844	8945	.7907	8980	1.2647	1020	40
30	.6225	7941	.7826	8935	.7954	9006	1 2572	0994	30
40	.6248	7957	.7808	8925	.8002	9032	1.2497	0968	20
50	.6271	7973	.7790	8915	.8050	9058	1.2423	0942	10
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.2349	0.0916	51° 00′
10	.6316	8004	.7753	8895	.8146	9110	1.2276	0890	50
20	.6338	8020	.7735	8884	.8195	9135	1.2203	0865	40
30	.6361	8035	.7716	8874	.8243	9161	1.2131	0839	30
40	.6383	8050	.7698	8864	.8292	9187	1.2059	0813	20
50	.6406	8066	.7679	8853	.8342	9212	1.1988	0788	10
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.1918	0.0762	50° 00′
10	.6450	8096	.7642	8832	.8441	9264	1.1847	0736	50
20	.6472	8111	.7623	8821	.8491	9289	1.1778	0711	40
30	.6494	8125	.7604	8810	.8541	9315	1.1708	0685	30
40	.6517	8140	.7585	8800	.8591	9341	1.1640	0659	20
50	.6539	8155	.7566	8789	.8642	9366	1.1571	0634	10
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.1504	0.0608	49° 00′
10	.6583	8184	.7528	8767	.8744	9417	1.1436	0583	50
20	.6604	8198	.7509	8756	.8796	9443	1.1369	0557	40
30	.6626	8213	.7490	8745	.8847	9468	1.1303	0532	30
40	.6648	8227	.7470	8733	.8899	9494	1.1237	0506	20
50	.6670	8241	.7451	8722	.8952	9519	1.1171	0481	10
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.1106	0.0456	48° 00′
10	.6713	8269	.7412	8699	.9057	9570	1.1041	0430	50
20	.6734	8283	.7392	8688	.9110	9595	1.0977	0405	40
30	.6756	8297	.7373	8676	.9163	9621	1.0913	0379	30
40	.6777	8311	.7353	8665	.9217	9646	1.0850	0354	20
50	.6799	8324	.7333	8653	.9271	9671	1.0786	0329	10
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.0724	0.0303	47° 00′
10	.6841	8351	.7294	8629	.9380	9722	1.0661	0278	50
20	.6862	8365	.7274	8618	.9435	9747	1.0599	0253	40
30	.6884	8378	.7254	8606	.9490	9772	1.0538	0228	30
40	.6905	8391	.7234	8594	.9545	9798	1.0477	0202	20
50	.6926	8405	.7214	8582	.9601	9823	1.0416	0177	10
44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.0355	0.0152	46° 00′
10	.6967	8431	.7173	8557	.9713	9874	1.0295	0126	50
20	.6988	8444	.7153	8545	.9770	9899	1.0235	0101	40
30	.7009	8457	.7133	8532	.9827	9924	1.0176	0076	30
40	.7030	8469	.7112	8520	.9884	9949	1.0117	0051	20
50	.7050	8482	.7092	8507	.9942	9975	1.0058	0025	10
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	45° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

ANSWERS

Page 12. Art. 12.

1. 1, 5, -8, -10, -3, 11, -16, -13.

Pages 15, 16. Art. 17.

10. $(5\sqrt{2}, 5\sqrt{2})$, $(10\sqrt{2}, 0)$, $(5\sqrt{2}, -5\sqrt{2})$.
11. $x - y = 0$, $x + y = 0$, $2x - y = 0$.
12. $(0, 0)$, $(a, 0)$, $(\frac{1}{2}a, \pm\frac{1}{2}a\sqrt{3})$.
13. $(8, 0)$, $(4, 4\sqrt{3})$, $(-4, 4\sqrt{3})$, $(-8, 0)$, $(-4, -4\sqrt{3})$, $(4, -4\sqrt{3})$.

Pages 18, 19. Art. 19.

1. (1) 15, (2) 18.385$-$, (3) 13.153$-$, (4) 16.279$-$.
5. (1) 8.602$+$, 8.062$+$, 12.369$+$. (2) 11.402$-$, 8.062$+$, 8.062$+$.
7. $(3, -2)$ or $(3, 14)$. **8.** $(1, 3)$. **9.** $(-1, 3)$, $(-3, 5)$, or $(13, -1)$.
10. $(5 + 4\sqrt{3}, 6 + 3\sqrt{3})$ or $(5 - 4\sqrt{3}, 6 - 3\sqrt{3})$.
11. $x^2 + y^2 - 6x - 8y = 0$.
12. $5x - 7y - 26 = 0$. **15.** 7.550$-$.

Page 20. Art. 20.

1. $2\frac{2}{5}$ units to the right of P_1. 12 units to the left of P_1.
2. Division point between two points and 3 in. from first. Division point beyond second point, 5 in. from first.

Pages 23, 24. Art. 22.

1. $(-2, 1)$. **2.** $(1\frac{1}{4}, 3\frac{1}{4})$. **3.** $(3, 1\frac{1}{2})$. **4.** $(-22, 14)$. **6.** $(\frac{1}{2}, -\frac{1}{4})$.
7. $(\frac{3}{4}, \frac{1}{2})$, $(-24, 28)$. **8.** $(1\frac{1}{2}, -1)$, $(-1\frac{1}{2}, -2\frac{1}{2})$, $(-1, -4\frac{1}{2})$.
9. $(2\frac{3}{7}, 3\frac{3}{7})$. **10.** 10.050$-$, 11.180$+$, 12.806$+$. **13.** $(11, 14)$.
14. $(-1, 0)$, $(-4, -2)$.

Pages 28, 29. Art. 28.

1. (1) 1, (2) -1, (3) 1.732, (4) 0.1010, (5) $\frac{d - b}{a + c}$, (6) -3.1463.
2. (1) 45°, (2) 135°, (3) 60°, (4) 5° 46′, (5) $\tan^{-1}\frac{d - b}{a + c}$, (6) 107° 38′.
3. $-\frac{5}{7}$. **5.** $5\frac{1}{2}$. **7.** 6. **8.** $3x - 2y - 2 = 0$. **9.** $1\frac{2}{5}$.

10. $6x - y + 12 = 0$. **11.** $x + 2y - 11 = 0$, $(-3, 7)$. **12.** $60° 15'$.
13. 2.375. **14.** $86° 11'$. **15.** $10\frac{3}{5}$. **16.** 3.732.
19. $(1\frac{7}{17}, 3\frac{5}{17})$. **20.** 0.6584. **21.** $74° 56'$.

Pages 32, 33. Art. 30.

1. $(3\sqrt{2}, 3\sqrt{2},)$ $(-\sqrt{2}, -\sqrt{2})$, $(-\frac{5}{2}, -\frac{5}{2}\sqrt{3})$, $(3\sqrt{3}, 3)$, $(-\frac{3}{2}, \frac{3}{2}\sqrt{3})$, $(-4\sqrt{2}, 4\sqrt{2})$, $(-2, 0)$, $(0, -6)$.

2. (1.532, 1.286), (1.026, 2.819), (5.629, − 3.250), (−0.7714, 0.9192), (4.078, −1.902).

3. $(8, 60°)$, $(-8, 240°)$; $(3\sqrt{2}, 225°)$, $(-3\sqrt{2}, 45°)$; $(\sqrt{34}, 59° 2')$, $(-\sqrt{34}, 239° 2')$; $(2\sqrt{2}, 120°)$, $(-2\sqrt{2}, 300°)$. **6.** 4.58.

Pages 35, 36. Art. 33.

2. $(\frac{5}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, $(\frac{1}{2}\sqrt{2}, \frac{7}{2}\sqrt{2})$, $(-5\sqrt{2}, 0)$, $(3\sqrt{2}, -4\sqrt{2})$.
3. (0, 0), (9, 2), (5, 11). **4.** (7, 8).
5. $60°$; (0, 0), (4, 0), $(2\sqrt{3}-1, \sqrt{3}+2)$.

Pages 38, 39. Art. 35.

1. (1) 76, (2) 31, (3) $200\frac{1}{2}$, (4) 10. **2.** 160. **3.** 18. **6.** 72.
7. $\frac{1}{2}[\rho_1\rho_2 \sin(\theta_2 - \theta_1) + \rho_2\rho_3 \sin(\theta_3 - \theta_2) + \rho_3\rho_1 \sin(\theta_1 - \theta_3)]$.
8. 98.29.

Page 41. Art. 36.

1. $(-3\frac{1}{3}, 2)$.

Pages 41–43. General Exercises.

2. (0, 0), (8, 0), (0, 10), (−8, 10). **3.** (−8, 0), (0, 0), (8, 10), (0, 10).
4. (4, 0), (0, 4), (−4, 0), (0, −4). **5.** $(a + \frac{1}{2}a\sqrt{3}, \frac{1}{2}a)$, $(\frac{1}{2}a\sqrt{3}, \frac{1}{2}a)$.
6. $(0, -\frac{1}{3}a\sqrt{3})$, $(\frac{1}{2}a, \frac{1}{6}a\sqrt{3})$, $(-\frac{1}{2}a, \frac{1}{6}a\sqrt{3})$ or $(0, \frac{1}{3}a\sqrt{3})$, $(\frac{1}{2}a, -\frac{1}{6}a\sqrt{3})$, $(-\frac{1}{2}a, -\frac{1}{6}a\sqrt{3})$. **7.** 4.799. **8.** 2:5.
9. (6, −4), (14, −20). **10.** (0, 9), (3, 0). **11.** (1, 4).
12. $(\frac{9}{5}, \frac{17}{5})$. **14.** (7, 0) or (−2, 0). **15.** 6 or −2.
16. (5, 0), (0, −1). **17.** (5, 0), (−3, 4). **19.** $-\frac{1}{8}$. **20.** −1.128.
21. −0.145. **22.** 1.2337. **23.** 2.25. **24.** (0, 0), $(4\frac{8}{13}, 1\frac{12}{13})$, (0, 13), $(-4\frac{8}{13}, 11\frac{1}{13})$. **25.** (0, 0), (5,0), (4, 3), (−1, 3).

Page 49. Art. 44.

1. 5, $-3\frac{1}{3}$. **2.** ±6, ±6. **3.** ±4, ±8. **4.** $1 \pm \frac{1}{2}\sqrt{3}$, 1. **5.** 0, 0.
6. 0, 2 and 0. **7.** −2, 1, 3 and $\pm\sqrt{6}$. **8.** None.

Page 54. Art. 48.

1. $x^2y^2 - 36 = 0$. **2.** $x^2 - 4xy + 4y^2 - 9 = 0$.
3. $x^3 - 8x^2 + 15x = 0$. **4.** $x^3 - x^2y + xy^2 - y^3 - 16x + 16y = 0$.
5. $x^3y - 6x^2 - 4xy + xy^3 - 6y^2 + 24 = 0$.

Page 55. Art. 49.

1. $(2\sqrt{2}, -2\sqrt{2})$, $(-2\sqrt{2}, 2\sqrt{2})$. **2.** $(\frac{5}{2}\sqrt{2}, \frac{1}{2}\sqrt{14})$, $(\frac{5}{2}\sqrt{2}, -\frac{1}{2}\sqrt{14})$. $(-\frac{5}{2}\sqrt{2}, \frac{1}{2}\sqrt{14})$, $(-\frac{5}{2}\sqrt{2}, \frac{1}{2}\sqrt{14})$. **3.** $(24\frac{3}{7}, -12\frac{2}{7})$, $(3, 2)$.
4. $(\frac{7}{5}\sqrt{10}, \frac{3}{5}\sqrt{15})$, $(\frac{7}{5}\sqrt{10}, -\frac{3}{5}\sqrt{15})$, $(-\frac{7}{5}\sqrt{10}, \frac{3}{5}\sqrt{15})$, $(-\frac{7}{5}\sqrt{10}, -\frac{3}{5}\sqrt{15})$.
5. $(3, 4)$, $(3, -4)$. **6.** $4\sqrt{2}$.

Pages 57, 58. Art. 51.

11. $22x + 12y - 1 = 0$. **12.** $x^2 + y^2 + 12x + 16y = 0$.
13. $x^2 + y^2 - 6x - 8y = 0$. **14.** $x^2 + y^2 + 2x + 4y - 20 = 0$.
15. $18x + 20y + 51 = 0$.
16. $4x + 3y - 25 = 0$, $2x - 5y + 29 = 0$, $3x - y + 2 = 0$.
17. $4x - y = 0$. **18.** $8x^2 + 8y^2 + 112x - 30y + 347 = 0$.
19. $2x - 3y + 24 = 0$. **20.** $7x^2 + 16y^2 - 112 = 0$.
21. $5x^2 - 4y^2 - 20 = 0$. **22.** $2x - 6y - 5 = 0$, $2x - 6y - 15 = 0$.
23. $xy + 7x + 8y - 4 = 0$.

Page 61. Art. 54.

1. $x - 2y - 8 = 0$. **2.** $x + y + 6 = 0$.
3. $\sqrt{3}x + y - \sqrt{3} - 5 = 0$. **4.** $3x - 5y + 13 = 0$.
5. $5x + 3y - 1 = 0$. **6.** $x - 3 = 0$.
7. $y - 4 = 0$. **8.** $x - 2y + 5 = 0$.
9. $3x - 4y - 11 = 0$; $3x + 4y + 5 = 0$. **10.** $12x - 5y - 26 = 0$.
11. $11x - y - 16 = 0$. **12.** $7x - y - 5 = 0$.

Page 62. Art. 56.

1. $x - 2y + 6 = 0$. **2.** $3x - y - 2 = 0$.
3. $2x - 3y + 1 = 0$; $2x + 3y - 1 = 0$. **4.** $2x + 3y - 20 = 0$.
5. $x + y + 1 = 0$. **6.** $x - 3y - 5 = 0$.

Page 64. Art. 58.

1. $2x - 3y - 6 = 0$. **2.** $6x - y + 6 = 0$.
3. $4x + 3y - 2 = 0$. **4.** $3x + 4y + 1 = 0$.
5. $x + \sqrt{3}y - 6 = 0$. **6.** $x + \sqrt{3}y + 6 = 0$.
7. $x - y + 2\sqrt{2} = 0$. **8.** $\sqrt{3}x + y + 2 = 0$.
9. $\sqrt{3}x - y + 8 = 0$. **10.** $\sqrt{3}x - y - 4 = 0$.

Pages 67, 68. Art. 62.

1. $-\frac{3}{2}, 2$. **2.** $\frac{2}{3}, \frac{2}{3}$. **3.** $\frac{5}{2}, 3$. **4.** $1, \frac{7}{2}$. **5.** $\frac{6}{5}$. **6.** 2. **7.** 2. **8.** 3.
9. $\frac{3\sqrt{5}}{5}$. **10.** $\frac{3\sqrt{10}}{5}$. **11.** $x \cos 0° + y \sin 0° + \frac{3}{2} = 0, \frac{3}{2}$.
12. $x \cos 90° + y \sin 90° - \frac{4}{3} = 0, \frac{4}{3}$. **13.** $3x - 4y + 5 = 0$.
14. $x + 2y + 4 = 0$. **15.** $x - 2y - 2 = 0$. **16.** $x + y + 2 = 0$.

Page 69. Art. 63.

1. $\frac{3}{5}$. **2.** $\frac{1}{5}$. **3.** 1.4142. **4.** 0.232. **5.** 1.828. **6.** 4.427.
7. $2, \frac{42}{13}, \frac{21}{10}$. **8.** 5.233, 6.871, 3.757.

Page 71. Art. 64.

1. $x - 3y - 3 = 0$.
2. $2x + 6y - 7 = 0$.
3. $x - 7y + 42 = 0$.
4. $16x - 4y - 17 = 0$.
5. $30x + 10y + 9 = 0$.
6. $2x + y - 2 = 0$.
7. $7x - 9y = 0, 4x + 6y - 21 = 0, 5x + y - 14 = 0$.
8. $x + 3y - 4 = 0, x - 7y - 19 = 0, 2x - 17 = 0$.

Page 72. Art. 65.

1. $mx - y + 2m + 3 = 0$.
2. $y - mx = 0$.
3. $mx - y - 3m + 4 = 0$.
4. $\frac{x}{3} + \frac{y}{b} = 1$.
5. $mx - y - 4 = 0$.
6. $x \cos \theta + y \sin \theta - 3 = 0$.
7. $x \cos \theta + y \sin \theta - 7 = 0$.
8. $2x + y - b = 0$.
9. $3x + y - b = 0$.
10. $x + y - b = 0$ or $y = mx$.

Pages 74, 75. Art. 66.

1. $3x + 4y \pm 6 = 0$. **2.** $x - 2y = 0, x + y - 6 = 0$.
3. $3x + 4y + 5 = 0, 5x + 12y - 13 = 0$.
4. $y - 2 = 0, 4x - 3y - 10 = 0$. **5.** $2x - y - 6 = 0$.
6. $2x + y - 6 = 0$. **7.** $x + 4y - 4 = 0$. **8.** $x - y - 3 = 0$.
9. $x + y - 4 = 0, 3x + y - 6 = 0$. **10.** $x + 3y - 6 = 0$.
11. $x + \sqrt{3}y - 6 = 0$. **12.** $3x - 4y \pm 10 = 0$.
13. $4x - 3y + 15 = 0, 4x + 3y - 15 = 0$.
14. $3x + 4y \pm 10 = 0, 4x + 3y \pm 10 = 0$.
15. $2x + y - 4 = 0$.
16. $2x - (3 - 2\sqrt{2})y + 4 - 4\sqrt{2} = 0$,
$2x - (3 + 2\sqrt{2})y + 4 + 4\sqrt{2} = 0$.

Page 77. Art. 67.

1. $4x - 5y + 1 = 0$. **2.** $13x + 12y - 62 = 0$. **3.** $x - 2y + 6 = 0$.
4. $22x + 11y - 14 = 0$. **5.** $5x + 15y - 34 = 0$.

Page 78. Art. 68.

1. $x^2y - y^3 - xy + y^2 = 0$.
2. $x^2y^2 - 3x^2y - 3xy^2 + 2x^2 + 9xy + 2y^2 - 6x - 6y + 4 = 0$.

Page 79. Art. 69.

1. $\rho \cos (\theta - 45°) = 3$.
2. $\rho \cos (\theta - 60°) = -2$.
3. $\rho \sin \theta = 7$.
4. $\rho \cos \theta = -4$.
5. $\rho \cos (\theta - 135°) = -4$.
6. $\rho \cos (\theta - 315°) = 3$.
7. $\rho \cos (\theta - 45°) = \frac{1}{2}\sqrt{2}$.
8. $\rho \cos \theta = 3$.
9. $\rho \sin \theta = -7$.
10. $\rho \cos (\theta - 30°) = 2$.
11. $\rho \cos (\theta - 300°) = -3$.
12. $\tan \theta = 2$.
13. $x - 3 = 0$.
14. $y - 4 = 0$.
15. $y - 6x = 0$.
16. $x + y - 2 = 0$.
17. $x + y - 3 = 0$.
18. $4x - 6y - 3 = 0$.
19. $3x \pm 4y = 0$.
20. $12x \pm 5y = 0$.
21. $x - y + 2 = 0$.
22. $x - \sqrt{3}y - 6 = 0$.

Pages 81–85. General Exercises.

2. (1) $2x - 3y - 17 = 0$.
(2) $7x - 2y - 21 = 0$.
3. (1) $x - 7y - 33 = 0$.
(2) $y = mx + k - mh$.
4. (1) $\frac{1}{5}$.
(2) 0.5883.
(3) $\frac{11}{5}$.
5. $\frac{51}{82}\sqrt{82}$, $\frac{51}{61}\sqrt{61}$, $\frac{17}{5}\sqrt{5}$.
6. $\frac{8}{13}\sqrt{13}$, $\frac{17}{13}\sqrt{13}$, $\frac{12}{13}\sqrt{13}$, $\frac{3}{13}\sqrt{13}$, $\frac{8}{13}\sqrt{13}$.
7. (1) 7.
(2) $-\frac{8}{3}$.
8. (1) $x + 8y - 58 = 0$.
(2) $21x - 6y - 58 = 0$.
(3) $12x + 9y - 116 = 0$.
(4) $15x + 6y - 110 = 0$.
9. $3x - y - 1 = 0$, $7x + 5y - 15 = 0$, $x - 4y + 6 = 0$, $(\frac{10}{11}, \frac{19}{11})$.
12. $x + \sqrt{3}y - 4 = 0$. **13.** (1) $x - y + 2 = 0$.
(2) $2x - y - 1 = 0$.
(3) $\sqrt{3}x + y - 8 + 2\sqrt{3} = 0$.
15. $2x - y + 6 = 0$. **16.** $(\frac{6}{5}, \frac{7}{5})$. **17.** $x' - 5y' + 4\sqrt{2} = 0$.
19. $4x - 5y + 1 = 0$, $7x - 3y - 27 = 0$, $3x + 2y - 5 = 0$.

20. $x - y - 1 = 0$. **21.** $x + y - 8\sqrt{2} = 0$.
22. $3x + 4y - 20 = 0$. **23.** $7x - 4y - 2 = 0$.
24. $x - 3y + 7 = 0$, $13x + 9y - 5 = 0$. **25.** $x - 2y + 2 = 0$.
26. $x - (2 - \sqrt{3})y - 1 + \sqrt{3} = 0$, $x - (2 + \sqrt{3})y - 1 - \sqrt{3} = 0$.
27. $3x - 4y + 24 = 0$. **28.** $3x + 4y - 24 = 0$.
29. $x - y - 5 = 0$, $x + y - 13 = 0$. **30.** -2.0225.
31. 3. **32.** $x - 2y + 7 = 0$, $x + 3y - 8 = 0$.
33. $x-(2+\sqrt{3})y+11+6\sqrt{3}=0$, $x-(2-\sqrt{3})y-4-\sqrt{3} = 0$.
34. $x + 3y - 15 = 0$, $3x - y - 5 = 0$.
35. $\frac{1}{3}\sqrt{5}$, $(1, \frac{20}{3})$. **36.** (3, 4). **37.** $2x - y + 4 = 0$.
38. $3x - 4y = 0$. **39.** $3x + 3y - 13 = 0$, $3x + 3y - 11 = 0$.
40. $3x - 4y + 1 = 0$, $3x + 4y - 7 = 0$.
42. $4x - 3y + 6 = 0$, $3x - 4y + 15 = 0$.
44. $x - \sqrt{3}y = 0$, $x + \sqrt{3}y = 0$. **45.** $x - y + 1 = 0$.
46. $x - \sqrt{3}y = 0$. **47.** $y = 4$, $y = 3$, $x = 2$, $x = 3$.
48. $\left(\frac{1}{k}, \frac{1}{k}\right)$. **49.** $x + (8 - 5\sqrt{3})y - 50 + 30\sqrt{3} = 0$.
50. $(a + b)x - (a - b)y - bc - ac = 0$.
51. $3x' + 5y' + 4 = 0$.
52. $21x + 77y - 1 = 0$, $99x - 27y - 79 = 0$.
54. (1) $A = B$ or $C = 0$.
(2) $A = -B$.
(3) $A + 2B + C = 0$.
55. (1) 116° 34′.
(2) 79° 42′.
(3) 60°.
56. $8 + 5\sqrt{3}$. **57.** $(-1, 1)$. **58.** 3.
59. $8x - 5y + 30 = 0$. **60.** 175° 26′.
61. $18x + 129y - 50 = 0$, $138x + 79y - 210 = 0$.
62. $x + 3y - 30 = 0$, $x + 3y + 10 = 0$. **63.** $y + 6 = 0$.
64. (2, 4). **65.** $x + y - 2 = 0$, $x - (2 + \sqrt{3})y - 2 - 2\sqrt{3} = 0$,
$x - (2 - \sqrt{3})y - 2 + 2\sqrt{3} = 0$.

Page 88. Art. 74.

1. (1, 2); 3. **2.** $(-2, 3)$; 1. **3.** $(-6, -3)$; 2. **4.** $(\frac{1}{2}, 2)$; $\frac{3}{2}$.
5. $(\frac{1}{3}, \frac{2}{3})$; $\frac{1}{3}\sqrt{2}$. **6.** $(-\frac{1}{4}, -\frac{3}{4})$; $\frac{5}{4}\sqrt{2}$.
7. $(-\frac{1}{2}, -\frac{3}{2})$; 0. **8.** $(a, 3a)$; $3a$.
9. $(-3a, -\frac{5}{2}a)$; $\frac{3}{2}\sqrt{7}a$. **10.** $(\frac{1}{3}a, -\frac{5}{6}a)$; $\frac{1}{2}a$.

Pages 90–92. Art. 75.

1. $x^2 + y^2 - 6x - 8y + 20 = 0$. **2.** $x^2 + y^2 - 2x + 6y + 5 = 0$.
3. $x^2 + y^2 + 4x - 4y + 3 = 0$. **4.** $x^2 + y^2 - 2x - 2y - 23 = 0$.

5. $x^2 + y^2 + 2x + 2y - 23 = 0.$ **6.** $x^2 + y^2 - 4x - 6y - 12 = 0.$
7. $x^2 + y^2 - 2y - 12 = 0.$ **8.** $x^2 + y^2 - 2x - 24 = 0.$
9. $x^2 + y^2 - 4x - 8y + 10 = 0.$ **10.** $x^2 + y^2 + 4x - 8y + 10 = 0.$
11. $x^2 + y^2 \pm 8x + 6y = 0.$ **12.** $x^2 + y^2 - 24x \pm 10y = 0.$
13. $x^2 + y^2 = 20.$ **14.** $x^2 + y^2 - 2x - 4y - 8 = 0.$
15. $x^2 + y^2 + 2x - 6y = 0.$ **16.** $x^2 + y^2 - 4x - 6 = 0.$
17. $x^2 + y^2 - 4x - 4y - 17 = 0,\ x^2 + y^2 - 10x - 22y + 121 = 0.$
18. $x^2 + y^2 + 6x - 2y - 15 = 0,\ x^2 + y^2 - 10x - 14y + 49 = 0.$
19. $x^2 + y^2 - 2x - 8y - 3 = 0.$ **20.** $x^2 + y^2 + 4x - 6y = 0.$
21. $x^2 + y^2 - 2x - 2y - 3 = 0.$ **22.** $x^2 + y^2 - 6x \pm 8y + 9 = 0.$
23. $x^2 + y^2 \pm 4x - 8y + 16 = 0.$ **24.** $x^2 + y^2 - 4x - 8y + 10 = 0.$
25. $x^2 + y^2 - 6x - 6y + 9 = 0.$ **26.** $x^2 + y^2 - 2x + 2y + 1 = 0.$
27. $x^2 + y^2 + 4x - 4y + 4 = 0.$ **28.** $x^2 + y^2 + 6x - 6y + 9 = 0.$
29. $x^2 + y^2 - 4x - 4y - 2 = 0.$ **30.** $x^2 + y^2 + 4x - 6y + 8 = 0.$
31. $4x^2 + 4y^2 + 20x - 20y + 25 = 0,$
$x^2 + y^2 + 30x - 30y + 225 = 0.$
32. $x^2 + y^2 - 10x - 10y + 25 = 0.$
33. $x^2 + y^2 - 10x - 10y + 25 = 0,$
$x^2 + y^2 - 26x - 26y + 169 = 0.$
34. $2x - y - 7 = 0.$ **35.** $x - 2y + 5 = 0;\ 2\sqrt{5}.$
36. $x - 3y + 5 = 0;\ 2\sqrt{10}.$ **37.** $x^2 + y^2 - 85 = 0.$
38. $36x^2 + 36y^2 + 84x - 12y - 575 = 0.$
39. $x^2 + y^2 = 65.$

Page 94. Art. 76.

1. $x + 2y - 5 = 0.$ **2.** $x + y - 1 = 0.$ **3.** $x + 2y + 4 = 0.$
4. $5x + 3y - 7 = 0.$ **5.** $5x + 15y - 34 = 0.$ **6.** $7x + 9y - 18 = 0.$
7. $x^2 + y^2 - 3x - 4y + 5 = 0.$ **8.** $x^2 + y^2 - 5x - 13y + 42 = 0.$

Pages 95, 96. Art. 77.

1. Circle, center at origin, $r = 3.$
2. Circle, center $(\frac{1}{2}, \frac{3}{2})$, $r = \frac{1}{2}\sqrt{10}.$
3. Circle, center $(\frac{10}{3}, 0)$, $r = \frac{8}{3}.$
4. Circle, center $(\frac{7}{10}, -\frac{2}{5})$, $r = \frac{1}{10}\sqrt{65}.$
5. Two circles, centers $(\pm\frac{1}{2}, 2)$, $r = \frac{1}{2}.$ **6.** $\frac{15}{4}.$
7. Circle, center $(0, 1)$, $r = \sqrt{29}.$
8. Two circles, centers $(0, \pm 2\sqrt{3})$, $r = 4.$
9. Circle, center $(-9, 0)$, $r = 10.$
10. Circle, center at origin, $r = 3.$

Page 97. Art. 78.

1 $\rho = 2$. **2.** $\rho = 10 \cos \theta$. **3.** $\rho = -8 \cos \theta$. **4.** $\rho = 6 \sin \theta$.
5. $\rho = -4 \sin \theta$. **6.** $\rho = \pm 12 \sin \theta$. **7.** $\rho = \pm 12 \cos \theta$.
8. $\rho = 6 \cos (\theta - \frac{\pi}{4})$. **9.** $\rho = \sqrt{6}$. **10.** $\rho = 3 \sin \theta$.
11. $\rho = -\frac{5}{2} \cos \theta$.
12. $\rho = 6 \cos \theta + 8 \sin \theta$.
13. $x^2 + y^2 + 6y = 0$; $(0, -3)$; 3.
14. $x^2 + y^2 - 4x = 0$; $(2, 0)$; 2.
15. $x^2 + y^2 - x - y = 0$; $(\frac{1}{2}, \frac{1}{2})$; $\frac{1}{2}\sqrt{2}$.
16. $x^2 + y^2 = 25$; $(0, 0)$; 5.
17. $x^2 + y^2 + 2x + 3y = 0$; $(-1, -\frac{3}{2})$; $\frac{1}{2}\sqrt{13}$.
18. $x^2 + y^2 + 3x + 4y - 6 = 0$; $(-\frac{3}{2}, -2)$; $\frac{7}{2}$.
19. $x^2 + y^2 - 9 = 0$; $(0, 0)$; 3.
20. $x^2 + y^2 - 4 = 0$; $(0, 0)$; 2.

Page 103. Art. 85.

2. $(\frac{1}{2}, 0)$, $(-\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(0, -\frac{1}{2})$; $2x + 1 = 0$, $2x - 1 = 0$, $2y + 1 = 0$, $2y - 1 = 0$; 2. **5.** (1) $y^2 = 12x$; (2) $x^2 = 24y$; (3) $y^2 = -16x$; (4) $x^2 = -8y$. **6.** (1) $y^2 = 8x$; (2) $x^2 = y$. **7.** $x^2 = 5000y$.

Pages 105, 106. Art. 86.

1. (1) $y^2 - 8x - 8y + 40 = 0$; (2) $y^2 + 8x - 6y - 7 = 0$; (3) $x^2 + 12x - 12y + 60 = 0$; (4) $x^2 - 4x + 6y + 22 = 0$.
2. (1) $(5, 4)$, $x - 1 = 0$; (2) $(0, 3)$, $x - 4 = 0$; (3) $(-6, 5)$, $y + 1 = 0$; (4) $(2, -4\frac{1}{2})$, $2y + 3 = 0$.
3. $y^2 - 10x + 4y - 36 = 0$, $y^2 + 10x + 4y + 44 = 0$.
4. $x^2 - 6x + 8y + 25 = 0$. **5.** $x'^2 + 4x' + 8y' + 32 = 0$.
6. (1) $y^2 - 8x - 10y + 57 = 0$; (2) $x^2 + 8x - 8y + 32 = 0$; (3) $y^2 + 8x - 4y + 36 = 0$; (4) $x^2 - 6x + 8y + 41 = 0$.
7. (1) $y^2 - 3x - 6y + 15 = 0$; (2) $4y^2 + 25x - 16y - 59 = 0$; (3) $2x^2 - 8x + 9y - 19 = 0$; (4) $5x^2 - 30x - 16y + 13 = 0$.

Pages 108, 109. Art. 89.

1. (1) $(3, 2)$, $(4, 2)$, $y - 2 = 0$, $x - 2 = 0$; (2) $(5, -4)$, $(4\frac{1}{2}, -4)$, $y + 4 = 0$, $2x - 11 = 0$; (3) $(-1\frac{1}{2}, 2)$, $(-1\frac{1}{2}, 3\frac{1}{4})$, $2x + 3 = 0$, $4y - 3 = 0$; (4) $(6, -2)$, $(6, -2\frac{3}{8})$, $x - 6 = 0$, $8y + 13 = 0$; (5) $(-\frac{8}{15}, 2)$, $(-1\frac{47}{60}, 2)$, $y - 2 = 0$, $6x - 43 = 0$; (6) $(\frac{9}{2}, \frac{41}{10})$, $(\frac{9}{2}, \frac{89}{40})$, $2x - 9 = 0$, $40y - 239 = 0$.
2. $5x^2 - 9x - 2y + 4 = 0$, $(\frac{9}{10}, \frac{3}{40})$, $(\frac{9}{10}, -\frac{1}{40})$, $\frac{2}{5}$.

3. $y^2 - 12x - 8y + 28 = 0$. **4.** $3y^2 - 25x + 18y + 77 = 0$.

5. (1) $y'^2 = 4x'$; (2) $x'^2 = -16y'$; (3) $y'^2 = -8x'$; (4) $x'^2 = \frac{7}{3}y'$.

6. (1) $4x - 3 = 0$, $x' + 1 = 0$, $(\frac{5}{4}, 3)$, $(1, 0)$, 4;
(2) $y - 5 = 0$, $y' - 4 = 0$, $(4, -3)$, $(0, -4)$, 16;
(3) $x - 3 = 0$, $x' - 2 = 0$, $(-1, 2)$, $(-2, 0)$, 8;
(4) $42y - 11 = 0$, $12y' + 7 = 0$, $(-\frac{5}{6}, \frac{10}{7})$, $(0, \frac{7}{12})$, $\frac{7}{3}$.

Pages 110, 111. Art. 90.

1. $x'^2 = -2py'$, $y'^2 = 2px'$.

2. (1) $x'^2 + 2x'y' + y'^2 - 4\sqrt{2}x' + 4\sqrt{2}y' = 0$;
(2) $3x'^2 - 2\sqrt{3}x'y' + y'^2 + (6\sqrt{3} - 4)x' - (4\sqrt{3} + 6)y' + 24 = 0$;
(3) $5y'^2 - 2\sqrt{5}x' - 2\sqrt{5}y' - 10 = 0$;
(4) $13x'^2 - 6\sqrt{13}x' - 14\sqrt{13}y' - 68 = 0$.

3. $9x^2 - 24xy + 16y^2 - 116x - 162y + 321 = 0$.

4. (1) $y''^2 = 3\sqrt{2}x''$; (2) $x''^2 = \frac{1}{50}\sqrt{5}y''$; (3) $x''^2 = -\frac{7}{2}\sqrt{2}y''$.

Pages 111, 112. Art. 91.

3. $\dfrac{\mp p}{1 \pm \cos\theta}$.

Page 113. Art. 92.

2. (3. 31, ±7.28).

Page 114. Art. 93.

1. $x^2 = -144y$, 24.31 ft., 22.22 ft., 13.89 ft.

2. 14′ $\frac{3}{4}$″, 11′ 3″, 6′ $6\frac{3}{4}$″. **3.** $x^2 = \frac{32000}{7}y$.

4. $x^2 = \frac{1280}{3}(y - 20)$.

Page 115. Art. 94.

4. $\dfrac{3v^2 \sin^2\alpha}{8g}$.

5. (1) 23.67 mi.; (2) 20.50 mi.; (3) 20.50 mi.

7. 401.5 ft. per sec.

Page 115, 116. General Exercises.

1. $(62\frac{1}{2}, 62500)$. **3.** $1\frac{1}{4}$ in. from back of reflector.

4. $(4, 2)$, $(8, 8)$. **5.** $x - y - 2 = 0$. **6.** $-6\frac{3}{4}$. **7.** $(3, 1)$, $(42, 14)$.

8. $\dfrac{3 \pm \sqrt{57}}{4}$. **9.** $(\frac{5}{8}, -2\frac{1}{2})$. **10.** (1) $y^2 + 4x - 4 = 0$;
(2) $25y^2 - 60x - 36 = 0$; (3) $y^2 + 24x - 144 = 0$.

11. (1) (12, 70° 32′), (12, 289° 28′); (2) (4, 90°), (4, 270°).

13. $2x^2 + 2y^2 - 5px = 0$. **14.** $4\sqrt{3}p$.

Pages 122, 123. Art. 99.

1. (1) $5, 4, \frac{3}{5}, (\pm 3, 0), 3x \pm 25 = 0;$
(2) $10, 6, \frac{4}{5}, (0, \pm 8), 2y \pm 25 = 0;$
(3) $3, 2, \frac{1}{3}\sqrt{5}, (\pm\sqrt{5}, 0), 5x \pm 9\sqrt{5} = 0;$
(4) $4, 3, \frac{1}{4}\sqrt{7}, (\pm\sqrt{7}, 0), 7x \pm 16\sqrt{7} = 0;$
(5) $2\sqrt{2}, \sqrt{5}, \frac{1}{4}\sqrt{6}, (\pm\sqrt{3}, 0), 3x \pm 8\sqrt{3} = 0.$
(6) $3, \sqrt{6}, \frac{1}{3}\sqrt{3}, (\pm\sqrt{3}, 0), x \pm 3\sqrt{3} = 0.$

2. a. **3.** (1) $4x^2 + 9y^2 = 144$; (2) $x^2 + 4y^2 = 16$; (3) $5x^2 + 9y^2 = 81$; (4) $20x^2 + 36y^2 = 1125$; (5) $112x^2 + 256y^2 = 3087$; (6) $x^2 + 4y^2 = 64$; (7) $2x^2 + 3y^2 = 18$.

4. $\pm \frac{4}{5}\sqrt{21}, \pm 2\frac{2}{5}, 0, \pm \frac{4}{5}\sqrt{-11}$. **6.** $2x^2 + 3y^2 = 6$.

8. (1) $\sqrt{6}, \sqrt{5}, \frac{1}{6}\sqrt{6}, \frac{5}{3}\sqrt{6}$; (2) $\sqrt{2m}, \sqrt{m}, \frac{1}{2}\sqrt{2}, \sqrt{2m}$; (3) $\sqrt{s}, \frac{1}{q}\sqrt{qs}, \frac{1}{q}\sqrt{q(q-1)}, \frac{2}{q}\sqrt{s}$; (4) $\sqrt{q}, \sqrt{p}, \frac{1}{q}\sqrt{q(q-p)}, \frac{2p}{p}\sqrt{q}$.

9. 3.8, 6.2. **10.** $3x^2 + 4y^2 = 576$. **12.** $(0, 0), x = \pm \infty, 0$.
13. $(1, \pm 2)$. **14.** $b^2x^2 + c^2y^2 = a^2b^2$.
15. $(a^2 - b^2)x^2 + a^2y^2 = a^2(a^2 - b^2)$.

Page 124. Art. 100.

1. (1) $9x^2 + 25y^2 - 54x - 200y + 256 = 0.$
(2) $48x^2 + y^2 + 288x + 14y + 445 = 0.$
2. $4x^2 + 9y^2 - 40x + 72y + 100 = 0,$
$9x^2 + 4y^2 - 90x + 32y + 145 = 0.$
3. (1) $(-1, 4), (7, 4), 4x + 13 = 0, 4x - 37 = 0;$
(2) $(-3, -7 \pm \frac{1}{2}\sqrt{141}), y = -7 \pm \frac{24}{47}\sqrt{141}.$
4. $7x^2 + 16y^2 - 140x - 64y + 512 = 0.$
5. $16x^2 + 25y^2 - 96x - 200y + 144 = 0.$
6. $2x^2 + y^2 - 16x - 4y = 0.$

Page 127. Art. 101.

1. (1) $\frac{(x+1)^2}{16} + \frac{(y-2)^2}{7} = 1$; $(-1, 2)$; $(-4, 2), (2, 2)$; $(-5, 2), (3, 2)$; $a = 4, b = \sqrt{7}$; $3x + 19 = 0, 3x - 13 = 0$.

(2) $\frac{(x-4)^2}{8} + \frac{(y-1)^2}{16} = 1$; $(4, 1)$; $(4, 1 \pm 2\sqrt{2})$; $(4, -3), (4, 5)$; $a = 4, b = 2\sqrt{2}$; $y - 1 \pm 4\sqrt{2} = 0$.

(3) $\frac{(x-1)^2}{\frac{1}{4}} + \frac{(y+1)^2}{\frac{1}{9}} = 1$; $(1 - 1)$; $(1 \pm \frac{1}{6}\sqrt{5}, -1)$; $(\frac{1}{2}, -1), (\frac{3}{2}, -1)$; $a = \frac{1}{2}, b = \frac{1}{3}$; $x - 1 \pm \frac{3}{10}\sqrt{5} = 0$.

(4) $\frac{(x+1)^2}{\frac{45}{4}} + \frac{(y-3)^2}{10} = 1$; $(-1, 3)$; $(-1 \pm \frac{1}{2}\sqrt{5}, 3)$; $(-1 \pm \frac{3}{2}\sqrt{5}, 3)$; $a = \frac{3}{2}\sqrt{5}$, $b = \sqrt{10}$; $x + 1 \pm \frac{9}{2}\sqrt{5} = 0$.

2. $6x'^2 + 7y'^2 = 8$.

3. (1) $\frac{x'^2}{16} + \frac{y'^2}{7} = 1$, $(\pm 3, 0)$, $3x' \pm 16 = 0$.

(2) $\frac{y'^2}{16} + \frac{x'^2}{8} = 1$, $(0, \pm 2\sqrt{2})$, $y' \pm 4\sqrt{2} = 0$.

(3) $\frac{x'^2}{\frac{1}{4}} + \frac{y'^2}{\frac{1}{9}} = 1$, $(\pm \frac{1}{6}\sqrt{5}, 0)$, $10x' \pm 3\sqrt{5} = 0$.

(4) $\frac{x'^2}{\frac{45}{4}} + \frac{y'^2}{10} = 1$, $(\pm \frac{1}{2}\sqrt{5}, 0)$, $2x' \pm 9\sqrt{5} = 0$.

4. $9x^2 + 25y^2 + 54x - 200y - 873 = 0$.

5. $3x^2 + 4y^2 - 24x - 16y + 16 = 0$.

6. $3x'^2 + 4y'^2 = 36$. **7.** $3x^2 + 4y^2 - 20x + 12 = 0$.

Pages 128, 129. Art. 102.

1. (1) $(a^2 + b^2)x'^2 + 2(a^2 - b^2)x'y' + (a^2+b^2)y'^2 - 2a^2b^2 = 0$,

(2) $43x'^2 - 14\sqrt{3}x'y' + 57y'^2 - 576 = 0$;

(3) $x'^2 + 9y'^2 - 36 = 0$;

(4) $9x'^2 + 2\sqrt{3}x'y' + 11y'^2 + (6 - 8\sqrt{3})x' + (6\sqrt{3} + 8)y' - 40 = 0$;

(5) $3x'^2 + y'^2 + 3\sqrt{2}x' - \sqrt{2}y' + 4 = 0$;

(6) $38x'^2 + 12x'y' + 22y'^2 + 2\sqrt{5}x' - 21\sqrt{5}y' = 0$.

2. $\frac{x'^2}{1} + \frac{y'^2}{\frac{9}{5}} = 1$. **3.** $9x''^2 + 3y''^2 - 32 = 0$.

Page 129. Art. 103.

1. $\rho = \frac{ep}{1 - e\cos\theta}$, $\rho = \frac{ep}{1 + e\cos\theta}$. **4.** $\rho^2 = \frac{a^2(1-e^2)}{1 - e^2\cos^2\theta}$.

Pages 131, 132. Art. 105.

1. $\frac{x^2}{(\frac{615}{2})^2} + \frac{y^2}{255^2} = 1$, $(\pm 171.8, 0)$. **2.** $\frac{x^2}{100^2} + \frac{y^2}{42^2} = 1$, $(\pm 90.75, 0)$.

3. 36.37 ft., 27.78 ft. **5.** 0.95+ ft., 4.02− ft., 10.16− ft.

6. 13,000 mi.

8. 45.1 in. per sec. 0.14 in. per sec.

Pages 132, 133. General Exercises.

1. $\frac{x^2}{36} + \frac{y^2}{16} = 1$. **2.** $\frac{x^2}{72} + \frac{y^2}{144} = 1$. **3.** $\frac{x^2}{72} + \frac{y^2}{36} = 1$.

4. $\frac{x^2}{48} + \frac{y^2}{64} = 1$. **5.** 4, 3; $(0, \pm\sqrt{7})$; $\frac{1}{4}\sqrt{7}$; $y \pm \frac{16}{7}\sqrt{7} = 0$.

6. 6, $2\sqrt{6}$; $(\pm 2\sqrt{3}, 0)$; $\frac{1}{3}\sqrt{3}$; $x \pm 6\sqrt{3} = 0$.
7. 5, 4; $(-1, -2)$, $(5, -2)$; $\frac{3}{5}$; $3x + 19 = 0$, $3x - 31 = 0$.
8. $9x'^2 + 4y'^2 = 36$. **9.** $25x'^2 + 16y'^2 = 400$.
10. $\frac{x^2}{a^2} + \frac{y^2}{a^2 - 9} = 1$. $a > 3$.
11. $189x^2 + 96xy + 161y^2 - 1494x - 258y + 2106 = 0$.
12. 6.4.

Pages 138, 139. Art. 110.

1. (1) 10, 8, $(\pm\sqrt{41}, 0)$, $41x \pm 25\sqrt{41} = 0$; (2) 12, 20, $(\pm 2\sqrt{34}, 0)$, $17x \pm 9\sqrt{34} = 0$; (3) 6, 8, $(\pm 5, 0)$, $5x \pm 9 = 0$; (4) 16, 12, $(0, \pm 10)$, $5y \pm 32 = 0$; (5) $2\sqrt{2}$, $2\sqrt{3}$, $(0, \pm\sqrt{5})$, $5y \pm 2\sqrt{5} = 0$; (6) 10, 6, $(0, \pm\sqrt{34})$, $34y \pm 25\sqrt{34} = 0$.

2. (1) $\frac{x^2}{36} - \frac{y^2}{16} = 1$; (2) $\frac{x^2}{16} - \frac{y^2}{48} = 1$; (3) $5x^2 - 4y^2 = 256$; (4) $\frac{x^2}{16} - \frac{y^2}{9} = 1$; (5) $\frac{x^2}{81} - \frac{y^2}{63} = 1$; (6) $\frac{x^2}{49} - \frac{y^2}{36} = 1$.

3. 0, $\pm 5\frac{1}{3}$, $\pm \frac{4}{3}\sqrt{-5}$. **4.** $5\frac{1}{3}$, $\frac{2b^2}{a}$.

5. $5x^2 - 9y^2 = 36$. **6.** $5x^2 - 4y^2 = 80$.

8. (1) 4, 3, $\frac{5}{4}$, $\frac{9}{2}$; (2) $\sqrt{6}$, $2\sqrt{2}$, $\frac{1}{3}\sqrt{21}$, $\frac{8}{3}\sqrt{6}$; (3) 1, 4, $\sqrt{17}$, 32; (4) $\sqrt{2m}$, $\sqrt{m}$, $\frac{1}{2}\sqrt{6}$, $\sqrt{2m}$; (5) $\sqrt{q}$, $\sqrt{p}$, $\frac{1}{q}\sqrt{pq + q^2}$, $\frac{2p}{q}\sqrt{q}$; (6) $\sqrt{s}$, $\frac{1}{q}\sqrt{qs}$, $\frac{1}{q}\sqrt{q^2 + q}$, $\frac{2\sqrt{s}}{q}$.

9. (1) 3, 4; $(\pm 5, 0)$; $\frac{5}{3}$; $5x \pm 9 = 0$.
(2) 6, $2\sqrt{6}$; $(\pm 2\sqrt{15}, 0)$; $(\frac{1}{3}\sqrt{15}$; $5x \pm 6\sqrt{15} = 0$.
10. $x^2 - 3y^2 + 3 = 0$. **13.** $x'y' = 8$. **14.** $7x^2 - 9y^2 = 1008$.
15. $x^2 - 3y^2 = 144$.

Pages 142, 143. Art. 113.

1. (1) $2x \pm \sqrt{6}y = 0$; (2) $x \pm \sqrt{2}y = 0$; (3) $x \pm \sqrt{2}y = 0$; (4) $5x \pm 4y = 0$; (5) $x \pm y = 0$; (6) $x \pm y = 0$.
2. $3x^2 - 4y^2 + 48 = 0$, $(\pm 2\sqrt{7}, 0)$, $(0, \pm 2\sqrt{7})$, $7x \pm 8\sqrt{7} = 0$, $7y \pm 6\sqrt{7} = 0$.
5. $2x'y' = a^2$. **6.** $x^2 - 3y^2 = 16$.
11. $\pm 4\sqrt{2}$. **12.** ± 0.9014.

Pages 143, 144. Art 114.

1. (1) $9x^2 - 25y^2 - 72x - 150y - 306 = 0$;
(2) $x^2 - 4y^2 + 12x - 16y + 36 = 0$.

2. (1) $(-1, -3)$, $(9, -3)$; $(4 \pm \sqrt{34}, -3)$;
$34x - 136 \pm 25\sqrt{34} = 0$.
(2) $(-6, -4)$, $(-6, 0)$; $(-6, -2 \pm 2\sqrt{5})$;
$5y + 10 \pm 2\sqrt{5} = 0$.

3. $64x^2 - 36y^2 + 256x + 504y - 1283 = 0$.

4. (1) $9x^2 - 25y^2 - 72x - 150y + 144 = 0$;
(2) $x^2 - 4y^2 + 12x - 16y + 4 = 0$.

5. (1) $3x - 5y - 27 = 0$, $3x + 5y + 3 = 0$;
(2) $x - 2y + 2 = 0$, $x + 2y + 10 = 0$.

Page 146. Art. 115.

1. (1) $\frac{(x-6)^2}{16} - \frac{(y-3)^2}{9} = 1$; $(6, 3)$; $(1, 3)$, $(11, 3)$; $(2, 3)$, $(10, 3)$; $a = 4$, $b = 3$; $5x - 30 \pm 16 = 0$; $3x - 4y - 6 = 0$, $3x + 4y - 30 = 0$.
(2) $\frac{(y-\frac{5}{2})^2}{1} - \frac{(x+3)^2}{16} = 1$; $(-3, \frac{5}{2})$; $(-3, \frac{5}{2} \pm \sqrt{17})$; $(-3, \frac{3}{2})$, $(-3, \frac{7}{2})$; $a = 1$, $b = 4$; $34y - 85 \pm 2\sqrt{17} = 0$; $x - 4y + 13 = 0$, $x + 4y - 7 = 0$. (3) $\frac{(x-\frac{1}{2})^2}{7} - \frac{(y+\frac{1}{2})^2}{2} = 1$; $(\frac{1}{2} - \frac{1}{2})$; $(-\frac{5}{2}, -\frac{1}{2})$, $(\frac{7}{2}, -\frac{1}{2})$; $(\frac{1}{2} \pm \sqrt{7}, -\frac{1}{2})$; $a = \sqrt{7}$, $b = \sqrt{2}$; $6x - 17 = 0$, $6x + 11 = 0$; $14y + 7 = \pm 2\sqrt{14}x \mp \sqrt{14}$. (4) $\frac{(y-3)^2}{8} - \frac{(x-1)^2}{9} = 1$; $(1, 3)$; $(1, 3 \pm \sqrt{17})$; $(1, 3 \pm 2\sqrt{2})$; $a = 2\sqrt{2}$, $b = 3$; $17y - 51 \pm 8\sqrt{17} = 0$; $4x - 3\sqrt{2}y - 4 + 9\sqrt{2} = 0$, $4x + 3\sqrt{2}y - 4 - 9\sqrt{2} = 0$.
(5) $\frac{(y-4)^2}{28} - \frac{(x+2)^2}{21} = 1$; $(-2, 4)$; $(-2, -3)$, $(-2, 11)$; $(-2, 4 \pm 2\sqrt{7})$; $a = 2\sqrt{7}$, $b = \sqrt{21}$; $y = 0$, $y - 8 = 0$; $2x - \sqrt{3}y + 4 + 4\sqrt{3} = 0$, $2x + \sqrt{3}y + 4 - 4\sqrt{3} = 0$.

2. $9x'^2 - 25y'^2 + 225 = 0$.

3. (1) $\frac{x'^2}{16} - \frac{y'^2}{9} = 1$, $(\pm 5, 0)$, $5x' \pm 16 = 0$;
(2) $\frac{y'^2}{1} - \frac{x'^2}{16} = 1$, $(0, \pm\sqrt{17})$, $17y' \pm \sqrt{17} = 0$;
(3) $\frac{x'^2}{7} - \frac{y'^2}{2} = 1$, $(\pm 3, 0)$, $3x' \pm 7 = 0$;

(4) $\frac{y'^2}{8} - \frac{x'^2}{9} = 1$, $(0, \pm\sqrt{17})$, $17y' \pm 8\sqrt{17} = 0$.

(5) $\frac{y'^2}{28} - \frac{x'^2}{21} = 1$, $(0, \pm 7)$, $y \pm 4 = 0$.

4. $9x^2 - 16y^2 + 54x + 128y + 1601 = 0$.

5. $16x^2 - 25y^2 + 64x + 200y - 736 = 0$.

6. $3x^2 - y^2 - 84x + 4y + 536 = 0$.

Pages 146, 147. Art. 116.

1. (1) $x'y' = 8$;

(2) $\frac{y'^2}{a^2} - \frac{x'^2}{b^2} = 1$;

(3) $x'^2 - y'^2 = 16$;

(4) $11x'^2 + 50\sqrt{3}x'y' - 39y'^2 = 576$;

(5) $\frac{x'^2}{9} - \frac{y'^2}{3} = 1$;

(6) $x'^2 - 3y'^2 - 2x' + 6y' - 11 = 0$.

2. (1) $\varphi = 45°$, $\frac{x'^2}{\frac{16}{3}} - \frac{y'^2}{16} = 1$;

(2) $\varphi = 30°$, $\frac{x'^2}{3} - \frac{y'^2}{4} = 1$.

3. $x''^2 - y''^2 = 11\sqrt{2}$.

4. $(-3, -1)$; $(2.15, 1.13)$, $(-8.15, -3.13)$; $x - (1 + \sqrt{2})y + 2 - \sqrt{2} = 0$, $x + (\sqrt{2} - 1)y + 2 + \sqrt{2} = 0$; $x - (\sqrt{2} - 1)y + 4 - \sqrt{2} = 0$, $x + (\sqrt{2} + 1)y + 4 + \sqrt{2} = 0$.

Pages 147, 148. Art. 117.

1. $\rho = \frac{ep}{1 + e\cos\theta}$. **3.** $\rho = \frac{\pm ep}{1 \mp e\cos\theta}$.

5. $\left(\frac{ep}{1 - e}, 0°\right)$, $\left(\frac{ep}{1 + e}, 180°\right)$. **6.** $x^2 - y^2 = a^2$.

Pages 149, 150. Art. 118.

4. $5x^2 - 4y^2 = 20$.

Pages 151, 152. Art. 119.

1. $pv = 10$. **2.** $wl = 20$. **3.** One branch of an hyperbola with foci at centers of circles and transverse axis equal to the difference of radii. **4.** Hyperbola with foci at centers of circles and transverse axis equal to the sum of radii. **5.** An equilateral hyperbola with the ends of the base as vertices. **6.** 345 ft. at an angle of 11° 43′ with the perpendicular to AB.

Pages 152, 153. General Exercises.

1. $a = \sqrt{6}$, $b = 2$; $\frac{1}{3}\sqrt{15}$; $(\pm\sqrt{10}, 0)$; $3y^2 - 2x^2 = 12$.
2. $(\pm 2\sqrt{3}, 2), (\pm 2\sqrt{3}, -2)$.
3. $a = 6$, $b = 4$; $(3, 2 \pm 2\sqrt{13})$; $\frac{1}{3}\sqrt{13}$; $13y - 26 \pm 18\sqrt{13} = 0$.
7. $(y - 2)^2 = 0$.
8. $(\frac{40}{39}\sqrt{39}, \frac{20}{39}\sqrt{39})$, $(-\frac{40}{39}\sqrt{39}, -\frac{20}{39}\sqrt{39})$.
9. 7.806+, 17.806+.
10. $x^2 - 3y^2 - x + 3y = 0$.
11. $4y^2 - x^2 = 11$.

Page 158. Art. 122.

1. Hyperbola, $3x''^2 - 2y''^2 = 6$. **2.** Ellipse, $9x''^2 + 16y''^2 = 144$.
3. Ellipse, $3x''^2 + y''^2 + 6 = 0$. **4.** Parabola, $2y''^2 = 3x''$.
5. Ellipse, $x'^2 + 4y'^2 = 16$. **6.** Parabola, $y'^2 = 3x'$.
7. Imaginary ellipse, $121x''^2 + 11y''^2 + 199 = 0$.
8. Two lines, $25x''^2 - y''^2 = 0$. **9.** Parabola, $x''^2 = \frac{2}{5}\sqrt{5}y''$.
10. Parabola, $y''^2 = \frac{10}{13}\sqrt{13}x''$.

Page 160. Art. 124.

9. $k = 0.0000082$, $p = 0.0000082t^{3.439}$. **10.** $c = 5028$, $pv^{1.37} = 5028$.

Page 162, 163. Art. 126.

17. (1) $x^2 + y^2 - 2a\dfrac{1 + k^2}{1 - k^2}x + a^2 = 0$;

(2) $\dfrac{x^2}{\frac{1}{4}k^2} + \dfrac{y^2}{\frac{1}{4}(k^2 - 4a^2)} = 1$;

(3) $\dfrac{x^2}{\frac{1}{4}k^2} - \dfrac{y^2}{\frac{1}{4}(4a^2 - k^2)} = 1$.

20. $\rho^2 = 2a^2 \cos 2\theta$.

21. $\frac{1}{384}$, $y = \dfrac{x^2}{64} - \dfrac{x^3}{2304}$.

Pages 173, 174. Art. 135.

10. 2π. **11.** No. **12.** $y = 18 \sin \frac{1}{4}\pi t$. **13.** $y = 8 \sin(t \cdot 120° + 55°)$, 3, $\frac{1}{3}$.
14. (1) 3.34, 0.299; (2) 16, $\frac{1}{16}$. **16.** $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ or 1 sec., 2 sec. **17.** $\frac{1}{2}\pi$, 0.
20. 0.0007854.

Pages 179, 180. Art. 140.

27. $\rho = r \cos \theta$. **28.** $\rho = 2(b + r \cos \theta)$.

Page 182. Art. 141.

1. $x = -3 + t, y = 2 + 2t$. **2.** $x - 2 = 5 \cos \theta, y - 3 = 5 \sin \theta$.

4. $y^2 + x + 2y - 3 = 0$. **5.** $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

6. $4x^2 + y^2 - 16x + 12 = 0$. **7.** $x^2 + 2xy + y^2 - 2x + 2y = 0$.

8. $9x^2 + 4y^2 - 90x - 32y + 253 = 0$.

9. $y^2 = \frac{1}{8}x$. **10.** $x^2 = \frac{1}{2}(y + 1)$.

11. $x^2 + y^2 = a^2 + b^2$. **12.** $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.

Page 183. Art. 142.

3. $x = a\theta - b \sin \theta, y = a - b \cos \theta$.

Pages 193–195. Art. 151.

1. $l = 0.00102T$. **2.** $y = 1.405x + 7.527$.

3. $y = 1.403x + 7.54$. **4.** $R = 0.00313t + 9.8753$.

5. $H = 03119t + 606.00$. **6.** $E = 0.1470W + 1.7957$.

8. $W = 0.5015t + 54.10$.

Pages 201–203. Art. 156.

2. $y = 0.9975x^{-1.37}$ or, very nearly, $y = x^{-1.37}$.

3. $H = 3.867D^{0.676}$, 659. **4.** $p = 30e^{-0.000038h}$.

6. $\mu = 0.1374p^{-0.667}$. **8.** $\mu = 0.00014V^{0.54}$.

9. $pv^{1.37} = 147$.

Pages 204, 205. Art. 157.

2. $y = 0.5 + 0.02x + 2.5x^2 - 0.3x^3$.

3. $t = 132 + 0.875x + 0.01125x^2$.

Page 210. Art. 160.

1. $x + 2y - 1 = 0$. **2.** $2x - 6y - 3 = 0$.

3. $3x + 4y + 1 = 0$. **4.** $2x + 12y + 5 = 0$.

5. $3x - 2y + 6 = 0$. **6.** $3x + 2y - 4 = 0$.

7. $3x + 4 = 0$. **8.** $x + 14y + 17 = 0$.

9. (1, 3). **10.** (5, 2). **11.** (1, 2). **12.** (4, 2). **13.** (3, 1).

14. (3, 6). **15.** (1, 1). **16.** (−1, 3). **17.** (−2, −8).

Pages 214, 215. Art. 165.

1. $x + 4y = 0, 3x - 2y = 0$. **2.** $x + y = 0, x - 6y = 0$.

3. $3y - 2 = 0, y + 4 = 0$. **4.** $2x + 9y - 20 = 0$.

5. $2x - 9y + 16 = 0.$ 6. $(\frac{3}{2}, -\frac{4}{3}), (-\frac{3}{2}, \frac{4}{3}).$

7. $(\frac{4}{5}, \frac{5}{2}), (-\frac{4}{5}, -\frac{5}{2}).$ 8. $\left(\frac{ay_1}{b}, -\frac{bx_1}{a}\right), \left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$

10. $\left(\frac{ay_1}{b}, \frac{bx_1}{a}\right), \left(-\frac{ay_1}{b}, -\frac{bx_1}{a}\right).$ 12. $x - 6y + 2 = 0.$

13. $3x - 2y - 12 = 0.$ 14. $2x - 3y + 16 = 0.$

18. Ellipse concentric with original ellipse, major axis $= \sqrt{2}a$, minor axis $= \sqrt{2}b$.

Pages 218–220. Art. 167.

2. 4.5, 0.0802. 3. $A = x^2$, 13. 4. $A = \pi x^2$, 10.25π.

5. $A = \frac{1}{2}d^2$, $d = \sqrt{2A}$. 6. $C = 2\sqrt{\pi A}$, $S = \sqrt[3]{36\pi V^2}$.

7. $V = \pi r^2 h$, $h = \frac{V}{\pi r^2}$, $S = \frac{4V}{d}$. 8. -4, 44, -12.

9. 1, $\sqrt{10}$, 7. 10. 2, 1.6778, -1.9208. 11. $\frac{1}{2}\sqrt{3}$, 0, $-\frac{1}{2}$.

12. $3x^2y - 4xy^2 - 2y^2$, $4xy^2 - 3x^2y - 2y^2$, $-2y^2 - 3x^2y - 4xy^2$.

15. $x = \sin^{-1} y$, $x = \frac{\log y}{\log 2}$.

16. $x = \pm \frac{a}{b}\sqrt{b^2 - y^2}$, $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$.

17. $x = (a^{\frac{1}{2}} - y^{\frac{1}{4}})^2$, $y = (a^{\frac{1}{2}} - x^{\frac{1}{2}})^2$.

18. $x = (a^{\frac{2}{3}} - y^{\frac{2}{3}})^{\frac{3}{2}}$, $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$.

19. $y = \pm \frac{x - 2a}{a}\sqrt{ax}$.

20. $x = \pm \frac{1}{4}\sqrt{-2y^2 \pm 2\sqrt{y^4 + 256}}$, $y = \pm \frac{2}{x}\sqrt{4 - x^4}$.

21. $x = 1 \pm \frac{1}{2}\sqrt{3 + 2y - y^2}$, $y = 1 \pm 2\sqrt{2x - x^2}$.

22. $x = \pm\frac{y + 2}{y}\sqrt{16 - y^2}$. 23. $i = e^{-\frac{1}{3}t} \cos 2t$.

24. $\varphi = \pm \frac{a}{\sqrt{\cos 2\theta}}$, $\theta = \frac{1}{2} \cos^{-1} \frac{a^2}{\varphi^2}$.

25. $\varphi = \frac{4a \cos \theta}{\sin^2 \theta}$, $\theta = \cos^{-1} \left(\frac{-2a \pm \sqrt{4a^2 + \varphi^2}}{\varphi}\right)$.

26. $\varphi = \pm a\sqrt{\tan \theta(3 - 4 \sin^2 \theta)}$. 27. $x = \frac{y}{a^3}$, $y = a^3x$.

28. $x = \frac{1}{2}\sqrt{2}(y + \sqrt{1 - y^2})$, $y = \frac{1}{2}\sqrt{2}(x - \sqrt{1 - x^2})$.

Page 225. Art. 171.

2. 4, $-\frac{1}{4}$. 3. 2, $2x - y + 1 = 0$, $x + 2y - 7 = 0$.

4. $6x_1$. 5. $3x_1^2$. 6. 2. 7. $-\frac{4}{x_1^2}$. 8. $\frac{p}{y_1}$. 9. $-\frac{1}{(x_1 - 1)^2}$

10. $3x_1^2 + 4x_1$. **11.** $9x_1^2 - 8x_1 + 6$. **12.** $-\frac{x_1}{2y_1}$. **13.** $\frac{4x_1}{9y_1}$. **14.** $\frac{2}{y_1}$. **15.** $-\frac{2}{(x_1 - 1)^2}$ **16.** $2x - 2y + p = 0$, $2x + 2y + p = 0$. **17.** -0.4364, 0.4364. **18.** $73° 44.4'$. **19.** $120°$.

Pages 231–233. Art. 181.

1. $\frac{dy}{dx} = 6x$.

2. $\frac{dy}{dx} = 20x^3$.

3. $\frac{dy}{dx} = \frac{28}{3} x^{\frac{4}{3}}$.

4. $\frac{dy}{dx} = \frac{2a}{3x^{\frac{1}{3}}}$.

5. $\frac{dy}{dx} = \frac{9}{16x^{\frac{1}{4}}}$.

6. $\frac{dy}{dx} = \frac{2}{x^{\frac{1}{2}}}$.

7. $\frac{dy}{dx} = \frac{1}{x^{\frac{2}{3}}}$.

8. $\frac{dy}{dx} = \frac{3}{5x^{\frac{2}{5}}}$.

9. $\frac{dy}{dx} = -\frac{2}{x^{\frac{5}{3}}}$.

10. $\frac{dy}{dx} = -\frac{8}{3x^{\frac{1}{3}}}$.

11. $\frac{dy}{dx} = -\frac{51}{2} x^{\frac{1}{2}}$.

12. $\frac{dy}{dx} = -\frac{8}{5x^{\frac{1}{5}}}$.

13. $\frac{ds}{dt} = gt$.

14. $\frac{ds}{dt} = \frac{12}{5t^{\frac{2}{5}}}$.

15. $\frac{ds}{dt} = \frac{3}{4}t^{\frac{1}{2}}$.

16. $\frac{dy}{dx} = 4x^3 + 6x$.

17. $\frac{dy}{dx} = 6x - 2$.

18. $\frac{dy}{dx} = 3x^2 - \frac{3}{2}x^{\frac{1}{2}} + 3$.

19. $\frac{dy}{dx} = \frac{3x^2 + 1}{2x^{\frac{3}{2}}}$.

20. $\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} + \frac{9}{x^4}$.

21. $\frac{dy}{dx} = 6(2x + 1)^2$.

22. $\frac{dy}{dx} = 24x(3x^2 + 2)^3 - 2$.

23. $\frac{dy}{dx} = \frac{1}{\sqrt{2x + 3}} - 3$.

24. $\frac{dy}{dx} = \frac{4x - 7}{2\sqrt{2x^2 - 7x}}$.

25. $\frac{dy}{dx} = \frac{2x + 7}{3\sqrt[3]{(x^2 + 7x - 2)^2}}$.

26. $\frac{dy}{dx} = -\frac{2}{x^3}$.

27. $\frac{dy}{dx} = -\frac{3}{x^4}$.

28. $\frac{dy}{dx} = -\frac{3}{x^3}$.

29. $\frac{dy}{dx} = -\frac{2}{(x + 1)^2}$.

30. $\frac{dy}{dx} = -\frac{1}{2\sqrt{(x + 1)^3}}$.

31. $\frac{dy}{dx} = -\frac{20x}{(x - 1)^3}$.

32. $\frac{dy}{dx} = \frac{2}{(x + 1)^2}$.

33. $\frac{dy}{dx} = -\frac{5}{(x-3)^2}$.

34. $\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2}$.

35. $\frac{dy}{dx} = 21x^6 - 24x^5 + 12x^3$.

36. $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x-1}}$.

37. $\frac{dy}{dx} = \frac{9x^2+14x-3}{2\sqrt{3x^3+7x^2-3x+2}}$.

38. $\frac{dy}{dx} = \frac{2ax+b}{2\sqrt{ax^2+bx+c}} - \frac{1}{2\sqrt{x+d}}$.

39. $\frac{dy}{dx} = \frac{4x^4+10x}{\sqrt[3]{x^3+5}}$.

40. $\frac{dy}{dx} = -\frac{2x}{(x-1)^3}$

41. $\frac{dy}{dx} = \frac{1}{(x+1)\sqrt{x^2-1}}$.

42. $\frac{dy}{dx} = -\frac{a^2}{\sqrt{(x^2-a^2)^3}}$.

43. $\frac{ds}{dt} = \frac{1}{2\sqrt{t+1}} + \frac{2}{3\sqrt[3]{2t-3)^2}}$.

44. $\frac{ds}{dt} = \frac{1}{2t^{\frac{1}{2}}} - \frac{1}{t^2} + 12t^3$.

45. $\frac{dy}{dx} = 5x^4 - 3x^2 + 2x - 2$.

46. $\frac{dy}{dx} = (x+a)^{n-1}(x-b)^{m-1}(mx+nx+am-bn)$.

47. $\frac{dy}{dx} = (x+1)^4(2x-1)^2(16x+1)$.

48. $\frac{dy}{dx} = \frac{2-4x}{(x-1)^3}$.

49. $\frac{ds}{dt} = \frac{nt^{n-1}}{(1+t)^{n+1}}$.

50. $\frac{ds}{dt} = \frac{2t}{(t^2+1)\sqrt{t^4-1}}$.

51. 0, 3, 12.

53. 1:4, 1:8, 1:16.

54. $x - y - 6 = 0$, $x + y + 6 = 0$;
$29x - y - 38 = 0$, $x + 29y - 582 = 0$.

55. At the points whose abscissas are $\frac{-4 \pm \sqrt{13}}{3}$.

56. $y - y_1 = \frac{x_1^2 - 1}{x_1^2}(x - x_1)$, $y - y_1 = \frac{x_1^2}{1 - x_1^2}(x - x_1)$.

57. $(-1, -6)$, $7x + y + 13 = 0$.

58. At (1, 0) at 135°, at (−3, −4) at 18° 26′.

60. 1.0025025.

Page 234. Art. 182.

1. $\frac{dy}{dx} = -\frac{x^2}{y^2}$.

2. $\frac{dy}{dx} = \frac{3x^2+1}{3y^2+1}$.

3. $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$.

4. $\frac{dp}{dv} = -\frac{p}{v}$, $\frac{dv}{dp} = -\frac{v}{p}$.

5. $\frac{dy}{dx} = \frac{4x^3 - 8xy^2}{8x^2y - 3y^2}, \frac{dx}{dy} = \frac{8x^2y - 3y^2}{4x^3 - 8xy^2}.$

6. $\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}, \frac{dx}{dy} = -\frac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}.$ 7. $\frac{dy}{dx} = -\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}.$

8. $\frac{dy}{dx} = -\frac{x + \sqrt{x^2 - y^2}}{y}.$

9. $\frac{dp}{dv} = \frac{av - 2ab - pv^3}{v^3(v - b)}, \frac{dv}{dp} = \frac{v^3(v - b)}{av - 2ab - pv^3}.$

10. $3x + 4y - 25 = 0, 4x - 3y = 0.$

11. $x - 6y + 17 = 0, 6x + y - 9 = 0.$

12. $8x + 5\sqrt{5}y - 36 = 0, 25x - 8\sqrt{5}y - 18 = 0.$

19. $x\sqrt{y_1} + y\sqrt{x_1} - \sqrt{ax_1y_1} = 0,$
$x\sqrt{x_1} - y\sqrt{y_1} - x_1\sqrt{x_1} + y_1\sqrt{y_1} = 0.$

Page 237. Art. 185.

1. Rising for all values of x. **2.** Rising for $x > -2$, falling for $x < -2$, **3.** Rising for $x > 0$, never falling. **4.** Rising for $x > 0$, falling for $x < 0$. **5.** Rising for all values of x, except $x = 0$. **6.** Falling for all values of x except $x = 0$. **7.** Rising for $x > \sqrt{3}$, and $x < -\sqrt{3}$, falling for $-\sqrt{3} < x < \sqrt{3}$. **8.** Rising for $x > \frac{1 + \sqrt{7}}{3}$ and $x < \frac{1 - \sqrt{7}}{3}$, falling for $\frac{1 - \sqrt{7}}{3} < x < \frac{1 + \sqrt{7}}{3}$. **9.** Rising for $x > 1$ and $x < \frac{1}{3}$, falling for $\frac{1}{3} < x < 1$. **10.** Rising for $-1 < x < 1$, falling for $x > 1$ and $x < -1$. **11.** Rising for $-1 < x < 1$, falling for $x > 1$ and $x < -1$. **12.** Rising for $x > 2$ and $x < -1$, falling for $-1 < x < 2$. **13.** Rising for $x > -2$, falling for $x < -2$. **14.** Rising for $x > 1$ and $-1 < x < 0$, falling for $x < -1$ and $0 < x < 1$. **15.** 278, 19, 3, y decreasing twice as rapidly as x is increasing.

16. $\frac{2 \pm \sqrt{22}}{3}$, no real values of x.

Page 239. Art. 186.

1. Min. at $x = 0$. **2.** Min. at $x = 2$. **3.** Max. at $x = 3$. **4.** Max. at (0, 2), Min. at (0, -2). **5.** Max. at $x = \frac{2}{3}(1 - \sqrt{13})$, Min. at $x = \frac{2}{3}(1 + \sqrt{13})$. **6.** Max. at $x = 0$, Min. at $x = 4\frac{2}{3}$. **7.** Min. at $x = 16$. **8.** Min. at $x = 3$. **9.** (4.5, 4.1). **10.** $\frac{v^2 \sin^2 \alpha}{2g}$.

Pages 241, 242. Art. 187.

1. Upward $x > 0$, downward $x < 0$, Infl. at $x = 0$. **2.** Upward for all values. **3.** Upward $x > 0$, downward $x < 0$, Infl. at $x = 0$. **4.** Upward

$x<0$, downward $x>0$, Infl. at $x = 0$. **5.** Upward $x>1$ and $x<-1$, downward $-1<x<1$, Infl. at $x = \pm 1$. **6.** Upward $x>1$, downward $x<1$, Infl. at $x = 1$. **7.** Upward $x>\frac{2}{3}$ and $x<0$, downward $0<x<\frac{2}{3}$, Infl. at $x = 0$ and $x = \frac{2}{3}$. **8.** Upward $x>\frac{4}{3}$, downward $x<\frac{4}{3}$, Infl. at $x = \frac{4}{3}$. **9.** Upward $x>\frac{1}{3}\sqrt{3}$ and $x<-\frac{1}{3}\sqrt{3}$, downward $-\frac{1}{3}\sqrt{3} < x < \frac{1}{3}\sqrt{3}$, Infl. at $x = \pm\frac{1}{3}\sqrt{3}$. **10.** Upward $x> \frac{4+\sqrt{19}}{3}$ and $x < \frac{4-\sqrt{19}}{3}$, downward $\frac{4-\sqrt{19}}{3} < x < \frac{4+\sqrt{19}}{3}$, Infl. at $x = \frac{4 \pm \sqrt{19}}{3}$. **11.** -2, Max. at $x = 1 - \frac{1}{3}\sqrt{6}$, Min. at $x = 1 + \frac{1}{3}\sqrt{6}$. **12.** 0, 2, 20 in. per sec. **13.** No points, 1, 2.

Pages 245, 246. Art. 190.

5. -0.15, 0.25. **6.** -0.09428, 0.09428. **7.** $(6x + 2)dx$.

8. $(3x^2 + 4)dx$. **9.** $(4x^3 - 9x^2 + 4x)dx$. **10.** $\frac{9xdx}{16y}$.

11. $\frac{xdx}{\sqrt{x^2+4}}$. **12.** $-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\,dx$. **13.** $-\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}\,dx$.

14. $-\frac{xdx}{\sqrt{(x^2+5)^3}}$. **15.** $gtdt$, 64.

Pages 249, 250. Art. 193.

11. $x^2 = \frac{2}{3}y$. **12.** $x^3 - 3y + 2 = 0$.

13. $2x^{\frac{3}{2}} - 3y + 4(3 - \sqrt{2}) = 0$. **14.** $\frac{32}{3}\sqrt{2}$ square units.

15. $\frac{28}{3}\sqrt{6}$ square units. **17.** 64 square units.

18. $20\frac{1}{4}$ square units. **19.** $25\frac{3}{5}$ square units.

20. $3\sqrt{2}$ square units. **21.** $\frac{4}{3}$ square units.

22. $2\frac{2}{3}$ square units.

Pages 253, 254. Art. 197.

1. $\frac{dy}{dx} = 3\cos 3x$. **2.** $\frac{dy}{dx} = \sin 2x$.

3. $\frac{dy}{dx} = -2\sin(2x + 1)$. **4.** $\frac{dy}{dx} = \cos 2x$.

5. $\frac{dy}{dx} = 3\cos 3x \cos 2x - 2\sin 3x \sin 2x$.

6. $\frac{dy}{dx} = 3\sec^2 3x$. **7.** $\frac{dy}{dx} = \sec^2 x$.

8. $\frac{dy}{dx} = 15\tan^2 5x \sec^2 5x$. **9.** $\frac{dy}{dx} = x\cos x + \sin x$.

10. $\frac{dy}{dx} = (3x^2 + 2x) \cos (x^3 + x^2)$.

11. $\frac{dy}{dx} = (2x + 3) \cos (x^2 + 3x - 4)$.

12. $\frac{dy}{dx} = -3 \sin (6x + 4)$. 13. $\frac{dy}{dx} = \frac{3}{2} \cot 3x\sqrt{\sin 3x}$.

14. $\frac{dy}{dx} = \frac{1}{2} \sin 4x$. 15. $\frac{dy}{dx} = \frac{1}{2} \tan x\sqrt{\sec x}(3 \cos^2 x + 1)$.

16. $\frac{dy}{dx} = -mnq(\cot^{n+1} qx + \cot^{n-1} qx)$.

17. $\frac{dy}{dx} = \frac{2 \sin x.}{(1 + \cos x)^2}$.

18. $\frac{d\rho}{d\theta} = \frac{3}{1 - \sin 3\theta}$. 19. $\frac{d\rho}{d\theta} = \tan^4 \theta$.

20. $\frac{dy}{dx} = 3 \cos x(1 - 4 \sin^2 x)$. 21. $\frac{dy}{dx} = \cot \frac{1}{2}\theta$.

22. 2 square units. 23. 0.7071, −0.4161.

24. 1, 0, ∞.

25. Max. at $x = (4n + 1)\frac{\pi}{2}$, Min. at $x = (4x + 3)\frac{\pi}{2}$, Infl. at $x = n\pi$.

26. $-\frac{1}{3} \cos 3x + C$. 27. $-\frac{1}{3} \cos (3x - 1) + C$.

28. $\frac{1}{4} \sin 4x + C$. 29. $\frac{1}{4} \sin (4x - 2) + C$. 30. $\frac{1}{2} \sin^2 x + C$.

31. $\frac{1}{4} \sin^4 x + C$. 32. $-\frac{1}{3} \cos^3 x + C$.

33. $\frac{1}{n+1} \sin^{n+1} x + C$. 34. $y = \sin x$.

Pages 259, 260. Art. 203.

1. $\frac{dy}{dx} = \frac{2x + 7}{x^2 + 7x}$, $dy = \frac{(2x + 7)dx}{x^2 + 7x}$.

2. $\frac{dy}{dx} = \frac{0.8686}{x}$, $dy = \frac{0.8686dx}{x}$.

3. $\frac{dy}{dx} = -\frac{1}{x}$, $dy = -\frac{dx}{x}$.

4. $\frac{dy}{dx} = -\frac{0.8686}{x}$, $dy = -\frac{0.8686dx}{x}$.

5. $\frac{dy}{dx} = 2e^{2x}$, $dy = 2e^{2x}dx$.

6. $\frac{dy}{dx} = 2xe^{x^2}$, $dy = 2xe^{x^2}dx$.

7. $\frac{dy}{dx} = 6xe^{3x^2+4}$, $dy = 6xe^{3x^2+4}dx$.

8. $\frac{dy}{dx} = e^x(\sin x + \cos x)$, $dy = e^x(\sin x + \cos x)dx$.

9. $\frac{dy}{dx} = 2a^{2x} \log_e a$, $dy = 2a^{2x} \log_e a dx$.

10. $\frac{dy}{dx} = 4.6052x \cdot 10^{2x+3}$, $dy = 4.6052x \cdot 10^{2x+3}dx$.

11. $\frac{dy}{dx} = 3x(3x - 2)^{x-1} + (3x - 2)^x \log (3x - 2)$,
$dy = [3x(3x - 2)^{x-1} + (3x - 2)^x \log (3x - 2)]dx$.

12. $\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$, $dy = \frac{1}{2}(e^x - e^{-x})dx$.

13. $\frac{di}{dt} = -\alpha be^{-\alpha t}$, $di = -\alpha be^{-\alpha t}dt$.

14. $\frac{di}{dt} = -\frac{RI}{L} e^{-\frac{Rt}{L}}$, $di = -\frac{RI}{L} e^{-\frac{Rt}{L}} dt$.

15. $\frac{dy}{dx} = e^{-x}(\cos x - \sin x)$, $dy = e^{-x}(\cos x - \sin x)dx$.

16. $\frac{di}{dt} = -\frac{1}{3}e^{-\frac{1}{3}t}(6 \sin 2t + \cos 2t)$,
$di = -\frac{1}{3}e^{-\frac{1}{3}t}(6 \sin 2t + \cos 2t)dt$.

17. $\frac{dy}{dx} = \frac{(x + 1)^2}{x^2 + 1}$, $dy = \frac{(x + 1)^2dx}{x^2 + 1}$.

18. $\frac{dy}{dx} = \left(4 + \frac{2}{x}\right) (2x + \log x)$, $dy = \left(4 + \frac{2}{x}\right) (2x + \log x)dx$.

19. $\frac{dy}{dx} = (3 - 4x - 6x^2)e^{-x^2}$, $dy = (3 - 4x - 6x^2)e^{-x^2}dx$.

20. $\frac{dy}{dx} = 2(x^2 + 1)^{2x+3}\left[\frac{2x^2 + 3x}{x^2 + 1} + \log (x^2 + 1)\right]$,
$dy = 2(x^2 + 1)^{2x+3}\left[\frac{2x^2 + 3x}{x^2 + 1} + \log (x^2 + 1)\right]dx$.

21. 1, 7.39. **22.** 0.4343, 0.0434. **23.** (1, 0.6931).

24. No Max. point, Min. ($\frac{1}{2}$, 1.193).

26. 2.3026 square units.

27. 6.693 square units. **28.** $\log C(x - 1)$.

29. $3 \log Cx$. **30.** $x + \log Cx$.

31. $\log C \sin x$. **32.** $\frac{1}{2}e^{2x} + C$.

33. $\frac{1}{2}x^2 - 2x + \log Cx$. **34.** $x - \log Cx + \frac{1}{x} - \frac{1}{2x^2}$.

35. $\frac{a^{3x}}{3 \log a} + C$. **36.** $x - 4e^{-x} + C$.

37. $\frac{1}{2}(e^{2x+1} + x^2) + C$. **38.** $y = e^{kx}$.

39. $y^2 = e^{x^2}$.

22

Pages 265, 266. Art. 208.

1. 11. **2.** 13. **8.** $\sqrt{61}$, $\sqrt{85}$, $\sqrt{166}$.
9. (3, 4, 0). **10.** (37, -36, 23). **11.** $2:-1$.
12. $(\frac{5}{2}, -1, -9)$.

Page 269. Art. 210

1. $\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}$, or $-\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}$; 1, 1, 1.
2. $\frac{7}{30}\sqrt{10}, -\frac{2}{15}\sqrt{10}, -\frac{1}{6}\sqrt{10}$, or $-\frac{7}{30}\sqrt{10}, \frac{2}{15}\sqrt{10}$ $\frac{1}{6}\sqrt{10}$; 7, -4, -5.
3. $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$, or $-\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$; 2, 3, -6.
4. $-\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}$, or $\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}$.
5. $\frac{6}{11}, -\frac{7}{11}, \frac{6}{11}$, or $-\frac{6}{11}, \frac{7}{11}, -\frac{6}{11}$.
6. $\frac{4}{13}, \frac{3}{13}, -\frac{12}{13}$, or $-\frac{4}{13}, -\frac{3}{13}, \frac{12}{13}$.
7. $\frac{10}{19}, \frac{6}{19}, -\frac{15}{19}$, or $-\frac{10}{19}, -\frac{6}{19}, \frac{15}{19}$.
8. $-2, -9, 6$. **9.** (1, 0, 0), (0, 1, 0), (0, 0, 1).
10. (1, 0, 0,) (0, $\cos\beta$, $\cos\gamma$). **12.** 45° or 135°.
13. 48° 51′ or 131° 9′. **14.** $\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 0$.
15. $\frac{1}{2}\sqrt{3}, \frac{1}{2}, 0$.

Page 273. Art. 213.

1. (2, 60°, 45°, 120°) or (-2, 120°, 135°, 60°).
2. (8, 60°, 120°, 45°) or (-8, 120°, 60°, 135°).
3. ($\sqrt{3}$, 54° 44′, 54° 44′, 54° 44′) or
($-\sqrt{3}$, 125° 16′, 125° 16′, 125° 16′).
4. 45° or 135°. **5.** (8, 60°, 30°) or (-8, 240°, 150°).
6. (4, 210°, 120°) or (-4, 30°, 60°).
7. (4, 135°, 60°) or (-4, 315°, 120°).
8. 77° 56.6′. **9.** $\frac{8}{9}, -\frac{1}{9}, \frac{4}{9}$, or $-\frac{8}{9}, \frac{1}{9}, -\frac{4}{9}$.
10. $\frac{4}{7}$. **11.** $-\frac{80}{11}$. **12.** $-\frac{8}{7}$.

Pages 277, 278. Art. 218.

1. $z = 3$. $z = -4$. **2.** $z = 0$. $y = 0$. $x = 0$.
3. $y^2 + z^2 = 9$. $x^2 + y^2 = 16$. **4.** $8x + 4y - 10z - 15 = 0$.
5. $y - z = 0$. **6.** $x^2 - y^2 = 0$.
7. $x^2 - y^2 - z^2 = 0$. **8.** $x^2 - 4x + 8y - 6z + 29 = 0$.
9. $x^2 + y^2 + z^2 - 3x - 4z = 0$.
10. $x^2 + y^2 + z^2 - 6x - 72 = 0$, $x^2 + y^2 + z^2 - 2x - 80 = 0$.
11. $17x^2 + 17y^2 + 17z^2 - 90x - 258y + 265 = 0$.
$x^2 + y^2 + z^2 - 2x - 2y - 47 = 0$.
12. $x^2 + y^2 + z^2 - 2x + 2y - 4z - 75 = 0$.
13. $x^2 - y^2 - z^2 = 0$. **14.** $x^2 - y^2 + z^2 = 0$.

15. $x^4 - y^2 - z^2 = 0.$ **16.** $x - y^2 - z^2 = 0.$

17. $x^4 - 4y^2 - 4z^2 = 0.$ **18.** $x^2 + y^2 - 2z = 0.$

19. $x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - 4y^2 - 4z^2 = 0.$

20. $x^2 + y^2 + z^2 - 2z = 0.$ **21.** $x^2 + y^2 - \sin^2 z = 0.$

22. $x^2 - (\sin^{-1} y)^2 + z^2 = 0.$

23. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1.$ **24.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1.$

25. $(x^2 + y^2 + z^2 + 12)^2 - 64(y^2 + z^2) = 0.$

Pages 281, 282. Art. 221.

4. $3y^2 - 5z^2 + 2 = 0,\ 3x^2 - 2z^2 - 10 = 0,\ 5x^2 - 2y^2 - 18 = 0.$

5. $3y - z^2 = 0,\ 3x^2 - z^2 = 0,\ x^2 - y = 0.$

6. $z^4 + a^2y^2 - a^2z^2 = 0,\ z^2 + ax - a^2 = 0,\ x^2 + y^2 - ax = 0.$

7. $y^2 - z^2 = 0,\ x^2 + z^2 - a^2 = 0,\ x^2 + y^2 - a^2 = 0.$

8. $m^2y^2 + z^2 - a^2m^2 = 0,\ z - mx = 0,\ x^2 + y^2 - a^2 = 0.$

9. $z^2 - 3y^2 = 0,\ z^2 - 3ax = 0,\ y^2 - ax = 0.$

Pages 291. Art. 229.

21. $x^2 + y^2 + 2z^2 = 4.$ **24.** $x^2 + z^2 - 2px + p^2 = 0.$

Page 297, 298. Art. 237.

1. $\frac{1}{3}x - \frac{2}{3}y - \frac{2}{3}z - \frac{4}{3} = 0,\ \frac{x}{4} + \frac{y}{-2} + \frac{z}{-2} = 1.$

2. $\frac{2}{3}x + \frac{1}{3}y - \frac{2}{3}z - 3 = 0,\ \frac{x}{\frac{9}{2}} + \frac{y}{9} + \frac{z}{-\frac{9}{2}} = 1.$

3. $\frac{4}{9}x + \frac{7}{9}y - \frac{4}{9}z + \frac{1}{3} = 0,\ \frac{x}{-\frac{3}{4}} + \frac{y}{-\frac{3}{7}} + \frac{z}{\frac{3}{4}} = 1.$

4. $\frac{12}{17}x - \frac{1}{17}y + \frac{12}{17}z - \frac{18}{17} = 0,\ \frac{x}{\frac{3}{2}} + \frac{y}{-18} + \frac{z}{\frac{3}{2}} = 1.$

5. $2x - 3y + 2z - 1 = 0.$ **6.** $3x + 2y - z - 4 = 0.$

7. $x - 2y \pm 2z - 15 = 0.$ **8.** $6x - 3y + z - 2 = 0.$

9. $x + 2y - 2z - 6 = 0,\ 91x - 122y + 46z - 318 = 0.$

10. $3x - y - 4z - 1 = 0.$ **11.** $x + y + z - 3 = 0.$

12. $3x - 4y + 2z - 4 = 0.$ **13.** $x - 2y + z - 1 = 0.$

14. $3x - 4y + 4z - 16 = 0.$ **15.** $3y - 4z + 5 = 0.$

16. $3x - 4y + 2z + 29 = 0.$ **17.** $\frac{2}{3}.$

18. 56° 15′ or 123° 45′. **19.** 48° 11′ or 131° 49′.

20. 67° 7′ or 112° 53′.

21. $5x - 11y - 8z + 14 = 0,\ 23x + 25y - 20z - 28 = 0.$

22. $x - 2y - 2z + 5 = 0,\ 4x + y + z - 7 = 0.$

23. $x + 2y - 5z - 5 = 0,\ 11x - 8y - z - 13 = 0.$

24. $\pm 6.$ **25.** $(2, -1, 2).$ **26.** 31° 1′, 64° 37′, 73° 24′.

27. 16° 36′, 25° 23′, 58° 59′.

Pages 301–303. Art. 241.

1. (1, 1, 0), (2, 0, -3), (0, 2, 3).
2. (2, -3, 0), (8, 0, -3), (0, -4, 1).
3. (3, -2, 0), (1, 0, -1), (0, 1, $-\frac{3}{2}$).
4. ($-\frac{2}{5}$, $\frac{6}{5}$, 0), (-1, 0, 3), (0, 2, -2).
5. (1, $\frac{1}{2}$, 0), (2, 0, 1), (0, 1, -1).
6. (1) $x + y - 2 = 0, 3x + z - 3 = 0.$
(2) $x - 2y - 8 = 0, x + 2z - 2 = 0.$
7. (3) $x + y - 1 = 0, y + 2z + 2 = 0.$
(4) $2x - y + 2 = 0, 5y + 2z - 6 = 0.$
8. (1) $\frac{x}{-1} = \frac{y-2}{1} = \frac{z-3}{3}.$
(2) $\frac{x}{2} = \frac{y+4}{1} = \frac{z-1}{-1}.$
(3) $\frac{x-1}{2} = \frac{y}{-2} = \frac{z+1}{1}.$
(4) $\frac{x+1}{1} = \frac{y}{2} = \frac{z-3}{-5}.$
9. $11x + 5y - 3 = 0, 3x - 5z + 11 = 0.$
10. $2x + 5y - 1 = 0, 2x - 5z + 14 = 0.$
11. $x - 2 = 0, y + 2z + 5 = 0.$
12. $x - 2 = 0, y - 5 = 0.$
13. $x + 3y + 8 = 0, 2x - 3z + 10 = 0.$
14. $x - 3 = 0, y + 1 = 0.$ **15.** $z - 2 = 0, x + 3y = 0.$
16. $x + y - 2 = 0, 3x + z - 11 = 0.$
17. $2x + y - 5 = 0, x - z - 1 = 0.$
18. $3x - 2y = 0, 2x - z = 0.$ **19.** $\frac{1}{9}$. **20.** 0. **22.** $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.
23. $x - y + 2z - 7 = 0.$ **24.** $7x - 12y + z - 14 = 0.$
25. $x + y + z - 3 = 0.$ **26.** $2x - y + z - 1 = 0.$
27. $2x - 3y - 4z + 6 = 0.$ **28.** $\frac{x+3}{1} = \frac{y-2}{2} = \frac{z+1}{2}$

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