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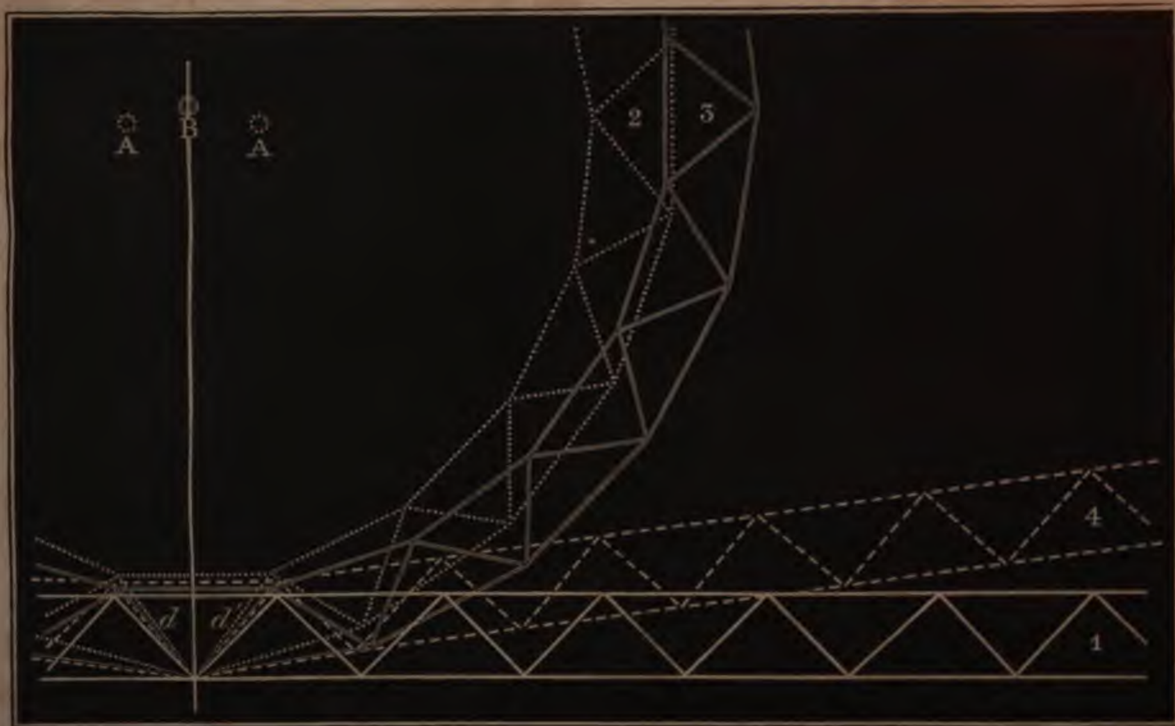
PLATE I.



THE

THEORY OF STRAINS

IN

GIRDERS AND SIMILAR STRUCTURES

WITH

OBSERVATIONS ON THE APPLICATION OF THEORY TO PRACTICE

AND

TABLES OF THE STRENGTH AND OTHER PROPERTIES OF MATERIALS.

BY

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Prius quàm incipias, consulto; et ubi consulueris, mature facto opus est.

IN TWO VOLUMES.

With numerous Illustrations engraved on Wood by Oldham.

VOL. I.

SECOND EDITION.

LONDON:
LONGMANS, GREEN, AND CO.
1869.

JOHN FALCONER, PRINTER, 53, UPPER SACKVILLE-STREET, DUBLIN.

PREFACE TO THE FIRST EDITION.

THE following pages have been written at various times during such brief intervals of leisure as the author could spare from his professional duties. They are for the most part the result of experience combined with theory; it is therefore hoped that they may supply the student with what has long been a want in Engineering literature, namely, a *Handbook on the Theory of Strains and the Strength of Materials*, giving practical methods for calculating the strains which occur in girders and similar structures. The theory of transverse strain has indeed been incidentally treated by writers on Mechanical Philosophy; their researches, however, have been confined to strains in plain girders, or to a few brief remarks on the more elementary forms of trussing, which, without further development, are of little practical use, and but too frequently afford a pretext for the ill-concealed contempt which so-called practical men sometimes entertain for theoretic knowledge.

A thorough acquaintance with the theory of strains and the strength and other properties of materials forms the basis of all sound engineering practice, and when this is wanting even natural constructive talent of a high order is frequently at fault, and the result is either excess and consequent waste of material or, what is still more disastrous, weakness in parts where strength is essential. The time has gone by when practical sagacity formed the sole qualification for high engineering success. Before the improvement of the steam engine gave rise to a new profession there were indeed some memorable names on the roll of engineers, generally self-taught mechanics, whom great natural ability had raised to pre-eminence in their profession; but practice which was formerly excusable, or even worthy of the highest commendation, would, now that knowledge has increased, be properly described as culpable waste, arising either from prejudice or ignorance.

The usual resource of the merely practical man is precedent, but the true way of benefiting by the experience of others is not by blindly following their practice, but by avoiding their errors as well as extending and improving what time and experience have proved successful. If one were asked what is the difference between an engineer and a mere craftsman, he would well reply, that the one merely executes mechanically the designs of others, or copies something which has been done before without introducing any new application of scientific principles, while the other moulds matter into new forms suited for the special object to be attained; and though experience and practical knowledge are essential for this, he lets his experience be guided and aided by theoretic knowledge, so as to arrange and proportion the various parts to the exact duty they are intended to fulfil.

Then prove we now with best endeavour
  What from our efforts yet may spring;
He justly is despised who never
  Did thought to aid his labours bring.
For this is art's true indication,
  When skill is minister to thought;
When types that are the mind's creation,
  The hand to perfect form has wrought.

The well-educated engineer should combine the qualifications of the practical man and of the physicist, and the more he blends these together, making each mould and soften what the other would seem to dictate if allowed to act alone, the more will his works be successful and attain the exact object for which they are designed. The engineer should be a physicist, who, in place of confining his operations to the laboratory or the study, exerts his energies in a wider field in developing the industrial resources of nature, and compelling mere matter to become subservient to the wants and comforts and civilization of the human race.

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CHAPTER I.

INTRODUCTORY.

1. Strain—Tension—Compression—Transverse strain—Shearing-strain—Torsion.—On the application of force all bodies change either form or volume, or both together. Forces considered with reference to the internal changes they tend to produce in any solid are termed strains, and may be classified as follows:—

Tensile strains,	producing fracture by	tearing asunder.
Compressive do.,		crushing.
Transverse do.,		breaking across.
Shearing do.,		cutting asunder.
Torsion,		twisting asunder.

This five-fold division is made for convenience merely, for the strength of any material, in whatever manner it may be employed, depends ultimately on its capability of sustaining strains which tend either to tear its parts asunder or to crush them together. It is therefore of essential importance to know the ultimate resistance to tension or compression which each material possesses, and thence deduce those strains which may be safely imposed in practice. To this end various experimenters have devoted their attention; in the United Kingdom, none with more perseverance or success than the late Eaton Hodgkinson, Esq., to whose life-long labours we are mainly indebted for the physical investigations on which calculations of the strength of structures are based.

2. Unit-strain—Inch-strain—Foot-strain.—Wherever English measures are used tensile and compressive forces are measured by the number of tons or pounds strain on the square inch. It will be convenient however to have some short expression for the strain on the unit of sectional area, irrespective of any particular measure of length or weight, and I have ventured to adopt the term *Unit-strain* to denote this quantity, and the words *Inch-strain* or *Foot-strain* to express the strain per square inch or square foot, as the case may be. The unit-strains of tension and compression are represented indifferently by the symbol f, unless it be desirable to distinguish them, in which case the unit-strain of compression is represented by the symbol f'. Thus, if F be the total strain in any bar whose area $= a$, we have

$$\mathsf{F} = af. \tag{1}$$

Ex. 1. If the crushing unit-strain of cast-iron be 42 tons per square inch, what weight will crush a short solid pillar 9 inches in diameter?

$$\text{Here,} \quad a = \frac{9 \times 9 \times \pi}{4} = 63{\cdot}6 \text{ inches,}$$
$$f = 42 \text{ tons.}$$
$$\textit{Answer.} \quad \mathsf{F} = af = 63{\cdot}6 \times 42 = 2{,}671 \text{ tons.}$$

Ex. 2. If the tearing unit-strain of beech be 11,500 pounds per square inch, what force in tons will tear asunder a tie-beam 15 inches square?

$$\text{Here,} \quad a = 15 \times 15 = 225 \text{ square inches,}$$
$$f = 11{,}500 \text{ lbs.}$$
$$\textit{Answer.} \quad \mathsf{F} = 225 \times \frac{11{,}500}{2{,}240} = 1{,}155 \text{ tons.}$$

3. Elasticity—Cubic elasticity—Linear elasticity.—Besides the strains of tension and compression another matter claims attention, namely, the alteration of length, or in other words, the *elongation* and *shortening* of the material subject to strain. *Elasticity* is the property which all bodies under the influence of external force possess to a greater or less degree of perfection of returning to their original volume or form after the force has been withdrawn. Thus we have *Cubic elasticity* or elasticity of volume, and *Linear elasticity* or elasticity of form. Fluids possess elasticity of volume, but not of form. Solids possess both, but linear elasticity alone demands our attention in questions relating to the strength of materials.

4. Elastic stiffness and Elastic flexibility are correlative terms which express the strength or weakness of the elastic reaction of the fibres of any elastic solid, whether that reaction be due to tensile or compressive strains applied separately, or in combination so as to produce flexure or torsion. Thus glass is elastically stiff, indian-rubber elastically flexible. In general, the terms *Stiffness* and *Flexibility* are not restricted to elastic solids, but express merely the relative amount of resistance to change of form, whether the material returns to its original shape or not after the force is withdrawn. In this sense copper is stiffer than lead, but neither is elastic, or but very slightly so. Elasticity should not, as in popular language, be confounded with a wide *range* of elastic flexibility. Glass, for instance, is both stiff and elastic, whereas indian-rubber, though very flexible, is less perfectly elastic than glass, that is, it returns with less exactness to its original form after being strained. Again, a thin spring of tempered steel is both elastic and flexible. In popular language however, indian-rubber is said to be more elastic than glass or steel, because its *range* of elastic flexibility exceeds that of either.

5. Toughness—Brittleness.—*Toughness* consists in the union of tenacity with ductility. *Brittleness* is incapability of sustaining rapid changes of form without fracture, and is opposed to toughness. Low-Moor iron, for instance, is tough; a bar of it can be twisted into a knot without breaking. But highly tempered steel is brittle; though more tenacious than iron, it breaks short without any sensible change of length; it is not ductile; it will not stretch under strain. Sealing-wax also is brittle; though more ductile than iron under prolonged pressure, it is not tenacious and will not bear a sudden change of shape without fracture. Accurately speaking we may doubt if there is such a thing as a perfect elastic solid, for Mr. Hodgkinson's investigations seem to prove that there is no strain, however slight, which will not after its removal leave a permanent, though perhaps to ordinary tests an inappreciable, alteration of length in any of the materials on which be experimented. In other words, every material is more or less ductile.*

6. Set.—When the unit-strain is considerable the defect of elasticity becomes very apparent, especially in metals, for they do not return to their original length when released from strain, but remain sensibly elongated or shortened, as the case may be, by a certain amount which varies according to the nature of the material and the force applied. This residual elongation or shortening is called the *Set*, and is not sensibly increased by subsequent applications of the same unit-strain which first produced it. It should be observed however, that the ultimate set is not instantaneously produced on the application of force. Iron, and probably all materials, take time more or less prolonged to adapt themselves to new conditions of strain. Hence a rapidly applied force may snap a brittle bar without producing any very perceptible change in its length.

7. Law of elasticity—Limit of elasticity.—It is evident that the elastic reaction of any material is equal to the force producing extension or compression, and it has been proved by experiment that the following law of uniform elastic reaction, expressed by Hooke in the phrase "ut tensio sic vis," and generally known as the *Law of elasticity*, though perhaps not accurately true of any solid, is practically true of the materials used in construction. *When any material is strained either by a tensile or a compressive force, the elastic reaction of the fibres (equal to the applied force) is proportional to their increment or decrement of length*, provided the alteration of length does not exceed a certain limit beyond which the above stated law ceases to apply, and the change of length, no longer regular, increases for each additional unit of strain more rapidly than the reaction due to the elasticity of the fibres; this produces set and ultimately rupture. Experience has proved that the safe working strain of any material does not exceed its sensible limit of uniform elastic reaction, generally called the *Limit of elasticity*; indeed it generally lies considerably within it.

* *Report of the Commissioners appointed to inquire into the application of Iron to Railway Structures*, 1849, App. A, p. 1. Also, *Experimental Researches on the Strength and other Properties of Cast-iron*, by E. Hodgkinson, pp. 381, 409.

8. Coefficient of elasticity E—Table of coefficients.—The coefficient of elastic reaction, or the *Coefficient of elasticity*,* is generally represented by the symbol E, and is the weight (in lbs.) requisite to elongate or shorten a bar whose transverse section equals a superficial unit (one square inch) by an amount equal to its length, on the imaginary hypothesis that the law of elasticity holds good for so great a range. In assuming that the coefficient of elasticity is the same for compression and extension I have followed Navier,† but some writers on the strength of materials seem to overlook the fact that, if the law of elasticity be rigidly exact, a given force of compression will shorten any material by the same proportion of its original length that an equal tensile force will extend it. In practice the coefficient of elastic compression will generally be found to differ slightly from that of elastic tension.

If a bar whose length $= l$ be extended or compressed within the limits of elasticity by a strain of f lbs. per square inch, the increment or decrement of length λ is expressed by the following relation,

$$\frac{\lambda}{l} = \frac{f}{\mathsf{E}}$$

whence,

$$\mathsf{E} = \frac{fl}{\lambda} \qquad (2)$$

Ex. How much will an inch-strain of 5 tons stretch a bar of wrought-iron whose length equals 10 feet?

Here (see table following), $\mathsf{E} = 24{,}000{,}000$ lbs.,

$f = 5$ tons,

$l = 10$ feet.

Answer. $\lambda = \frac{fl}{\mathsf{E}} = \frac{5 \times 2{,}240 \times 10 \times 12}{24{,}000{,}000} = \cdot 056$ inches.

It is obvious that the coefficient of elasticity should be deduced from experiments in which the applied unit-strain lies within the limit of elastic reaction. It should also be noted whether the material has been previously *stretched;* otherwise the results will be anomalous (**412**). The following table contains the coefficients of elasticity of various materials:—

* Called also the *Modulus of elasticity.*

† *Résumé des Leçons données à l'Ecole des Ponts et Chaussées*, p. 41.

Description of Material.	Coefficient of Elasticity in lbs. per square inch. E	Authority.
Metals.		
Brass (cast),	8,930,000	Tredgold.
Gun metal (copper 8, tin 1),	9,873,000	do.
Iron (cast),	18,400,000	do.
Do.,	12,000,000	Hodgkinson.
Do. (wrought),	24,000,000	do.
Do. (annealed bar),	27,700,000	do.
Lead (cast),	720,000	Tredgold.
Steel,	29,000,000	Young.
Tin (cast),	4,608,000	Tredgold.
Zinc (cast),	13,680,000	do.
Timber.		
Acacia (English growth),	1,152,000	Barlow.
Ash,	1,644,800	do.
Beech,	1,353,600	do.
Birch (American black),	1,477,000	do.
Do. (common),	1,644,800	do.
Box (Australia),	2,155,200	Trickett.
Elm,	699,840	Barlow.
Fir (Mar Forest),	645,360	do.
Do. (do., another specimen),	869,600	do.
Do. (New England),	2,191,200	do.
Do. (Riga),	1,328,800	do.
Do. (do., another specimen),	990,400	do.
Greenheart,	2,656,400	do.
Iron bark (Australia),	1,669,600	Trickett.
Larch,	616,320	Barlow.
Do. (another specimen),	1,052,800	do.
Mahogany (Honduras),	1,596,000	Tredgold.
Norway spar,	1,457,600	Barlow.
Oak (Adriatic),	974,400	do.
Do. (African),	2,305,400	do.
Do. (Canadian),	2,148,800	do.
Do. (Dantzic),	1,191,200	do.
Do. (English),	1,451,200	do.
Do. (do., inferior),	873,600	do.
Pine (Pitch),	1,225,600	do.
Do. (Red),	1,840,000	do.
Do. (do.),	1,200,000	Clark.
Do. (American yellow),	1,600,000	Tredgold.
Poon,	1,689,600	Barlow.

Description of Material.	Coefficient of Elasticity in lbs. per square inch. E	Authority.
TIMBER—*continued.*		
Spotted gum (Australia), ...	1,942,000	Trickett.
Stringy bark (do.), ...	1,375,600	do.
Teak, ...	2,414,400	Barlow.
Whalebone, ...	820,000	Tredgold.
STONES.		
Marble (White), ...	2,520,000	Tredgold.
Quartz Rock (Holyhead), across lamination, .	4,598,000	Mallet.
Do., do., parallel to lamination,	545,000	do.
Slate (Welsh), ...	15,800,000	Tredgold.
Do. (Westmoreland), ...	12,900,000	do.
Do. (Scotch), ...	15,790,000	do.
Do. (Portland), ...	1,533,000	do.

Barlow, see *Barlow on the Strength of Materials*—Clark, see *Clark on the Britannia and Conway Tubular Bridges*, p. 463—Hodgkinson, see *Report of Commissioners appointed to inquire into the application of Iron to Railway Structures*, 1849, pp. 108, 172—Mallet, see *Philosophical Transactions*, 1862, p. 671—Tredgold, see *Tredgold on the Strength of Cast-iron*—Young, see *Do.*

9. Mechanical laws—Resolution of forces.—The investigation of transverse strains may be reduced to the three following fundamental laws in mechanics:—

If three forces acting at the same point balance (are in equilibrium), three lines parallel to their directions will form a triangle the sides of which are proportional to the forces. Also, *If two out of three forces which balance meet, the third passes through their point of intersection.*

Hence it follows that, if we know the magnitude and direction of two intersecting forces, we can find both the magnitude and direction of their resultant; and if the directions of any two components into which a single known force is resolved be given, the amount of these components can be found. Thus the weight W, Fig. 1, is supported by an oblique tie and a horizontal strut. The weight and the strains in the tie and strut meet at A,

Fig. 1.

and may be represented by the triangle $h\,t\,s$. Let the sides of the triangle be as the numbers 3, 4 and 5; then, if $\mathsf{W}=3$ tons, t will sustain a tension of 5 tons, and s a thrust or compression of 4 tons. Calling the angle the tie makes with the vertical line θ, the relation between these three forces may be algebraically expressed as follows:—

$$t = \mathsf{W}\sec\theta \qquad\qquad s = \mathsf{W}\tan\theta$$

10. The Lever.—*If a weight rest upon a beam supported by two props at its extremities, these props react with two upward pressures whose sum is equal to the weight, and by the principle of the lever, the portion of the weight sustained by either prop is to the whole weight as the remote segment is to the whole beam.*

Fig. 2.

Thus in Fig. 2, if $\mathsf{W} =$ 10 tons, and the segments are as 3 : 2, the reaction of the left abutment $\mathsf{R} = 4$ tons; that of the right $\mathsf{R}' = 6$ tons. Calling the segments m and n, these relations may be algebraically expressed as follows:—

$$\mathsf{R} + \mathsf{R}' = \mathsf{W}, \qquad \mathsf{R} = \frac{n}{m+n}\mathsf{W}, \qquad \mathsf{R}' = \frac{m}{m+n}\mathsf{W}.$$

It is obvious that this principle is not affected by any bracing of the beam within itself, provided it merely rests on the points of support.

11. Equality of moments.—*When any number of forces acting in the same plane on a rigid body balance (are in equilibrium), the sum of the moments of the forces tending to turn it in one direction round any given point is equal to the sum of the moments of those tending to turn it in the opposite direction.* Also, *when any number of forces acting in the same plane have a single resultant, the sum of the moments of each force round a given point is equal to the moment of their resultant.**

Thus in Fig. 2, taking moments round the right abutment,

* The moment of a force round a given point is the product of the force by the perpendicular let fall on its direction from the point.

$R \times \overline{m + n} = W n$; the moment of R' vanishes, since R' passes through the point round which the moments are taken.

On these three mechanical laws—the *Resolution of Forces*, the law of the *Lever*, and the *Equality of Moments*—are founded all the following investigations of the strength of materials when subject to transverse strain.

12. Beam—Girder—Semi-girder.—The term *Beam* is generally applied to any piece of material of considerable scantling, whether subject to transverse strain or not; as for example, "Collar-beam," "Tie-beam," "Bressummer-beam;" the two former being subject to longitudinal strains of compression and tension respectively, and the latter to transverse strain. The term *Girder* is, however, restricted to beams subject to transverse strain and exerting a vertical pressure merely on their points of support. This term was originally applied to the main beams of floors, but has now become universally adopted by engineers. A *Semi-girder* is a cantilever, that is, a beam fixed at one extremity only and subject to transverse strain. In addition to its vertical pressure it exerts a tendency to overthrow the wall or other structure to which it is attached.

13. Flanged girder—Single-webbed girder—Double-webbed or Tubular girder—Tubular bridge.—In the term *Flanged girder* are included not only cast-iron girders of the ordinary I form, but also all girders which consist of one or two flanges united to a vertical web, whether the latter be continuous as in plate girders, or open work as in lattice and bowstring girders. Flanged girders are again subdivided into *Single-webbed* and *Double-webbed* or *Tubular*. A single-webbed girder is one whose flanges are connected by a single vertical web. Thus we have "Single-webbed cast-iron girders," "Single-webbed plate girders," "Single-webbed lattice," and "Single-webbed bowstring girders," &c. A *Double-webbed* and *Tubular girder* is one whose flanges are connected by a double vertical web, continuous or open-work as the case may be. A *Tubular bridge* is merely a tubular girder of such large dimensions that the roadway passes through the tube.

In the following theoretic investigations all girders are assumed to be horizontal and without weight, unless otherwise stated.

CHAPTER II.

FLANGED GIRDERS WITH BRACED OR THIN CONTINUOUS WEBS.

14. Transverse-strain — Shearing-strain. — The formulæ investigated in this chapter are, unless otherwise expressed, applicable to all flanged girders whose webs are formed of bracing, or if continuous, yet so thin that the transverse strength of the web as an independent rectangular girder may be neglected without sensible error. Our knowledge of the strains in this vertical web when continuous is still imperfect. Analogy indeed leads us to conclude that they follow laws similar to those which hold good in braced girders, but in the absence of experimental proof this is to a certain degree conjecture—a conjecture however, which I feel confident my readers will share after they have had the patience to read through this book.

The mode in which a load affects a girder may be thus analysed. From experience we learn that the load bends the girder downwards and develops longitudinal strains of tension and compression in the flanges. If the semi-girder, represented in Fig. 3, be supposed divided into vertical slices or transverse sections of small thickness, the weight tends to shear or separate the section on which it immediately rests from the adjoining one. The lateral connexion of the sections however prevents this separation, and the second section is drawn down by a vertical force equal to the weight which tends to shear it from the third section, and so on. Thus *a vertical force equal to the weight is transmitted from section to section as far as the point of support.* This vertical strain has been aptly named the *Shearing-strain;* but few writers, until the last few years, have noticed the practical results which follow from the fact that this force can be communicated from section to section only through the medium of some diagonal strain. Respecting the exact directions of the strains which this shearing force develops in a continuous web we know nothing positively; it is probable that they assume various directions crossing each other like close lattice-work, some vertical, some diagonal, perhaps some curved. However this may be, we know that certain of them must be diagonal, since the weight, which is a vertical force, produces strains in the flanges, which are longitudinal, through the medium of the web, which in fact fulfils the part of bracing in a lattice girder. The reader will perceive that we have really three sets of forces to deal with, namely, horizontal, vertical, and diagonal forces. The latter, however, may be resolved into horizontal and vertical components, and thus we have at present only horizontal and shearing forces to consider, recollecting that *the shearing-strain of any transverse section of a girder means the total vertical strain transmitted through that section, including in the term the vertical components of diagonal strains.*

15. Horizontal strains in open-work or thin continuous webs may be neglected.—When the vertical web of a girder with horizontal flanges is open-work like latticing, the shearing-strain is altogether transmitted through the bracing, the flanges being capable of conveying strains in the direction of their length only; but when the web is continuous as in a plate-girder, it is probable that a small amount of shearing-force acts upon the flanges also, so small however that we may practically neglect it. If however, one or both flanges are curved, the whole or a considerable portion of the shearing-strain is conveyed through that part of the flange which is sloped, the amount depending upon the angle of inclination. In this case the web has less duty to perform than if the flanges were horizontal and its sectional area may therefore be reduced. It will also be observed that the diagonal strains developed by the shearing force in a continuous web have horizontal components within the web itself, and consequently a continuous web aids the flanges to a certain extent, for those parts of the web which adjoin the flanges share the horizontal strains in the latter, and this flange action of the web is greater the thicker the web is. When, however, the web is very thin, the total amount of this flange action of the web is small compared with the strain in the flanges themselves, and may therefore be neglected without introducing any serious error. In this chapter all horizontal strains in the web are neglected.

CASE I.—FLANGED SEMI-GIRDER LOADED AT THE EXTREMITY.

Fig. 3.

16. Flanges—At any cross section the horizontal components of strain in the flanges are equal and of opposite kinds—Strength of flanged girders vary directly as the depth and inversely as the length.

Let **W** = the weight,

l = the distance of any cross section **A B** from **W**,

d = the depth of the girder at this cross section,

T = the horizontal strain of tension in the top flange at **A**,

C = the horizontal strain of compression in the bottom flange at **B**.*

The segment **A B W** is held in equilibrium by the weight **W**, the horizontal forces of tension and compression in the flanges at **A** and **B**, and the shearing and horizontal strains in the web at **A B**. Since these forces balance, the sum of the moments of those which tend to turn **A B W** round any point in one direction is equal to the sum of those which tend to turn it round the same point in the opposite direction (**11**). If the point lie in the cross section **A B**, the moment of the shearing force will be cipher, since its direction passes through this point. Neglecting the horizontal strain in the web when continuous, and taking moments round **A** and **B** successively, we obtain the following relations:—

* When the flanges are oblique, **T** and **C** represent the horizontal components of the longitudinal strains. Their vertical components are a portion of the shearing-strain.

$$Wl = Td = Cd \qquad (3)$$

whence,

$$T = C \qquad (4)$$

that is, *at any cross section the horizontal component of tension in one flange is equal to the horizontal component of compression in the other.*

If F represent the horizontal strain in either flange indifferently, we have from eq. 3

$$W = \frac{Fd}{l} \qquad (5)$$

$$F = \frac{Wl}{d} \qquad (6)$$

Eq. 5 proves that *the weight which a flanged girder is capable of supporting varies directly as the depth and inversely as the length.*

When both flanges are horizontal, we have from eq. 4

$$af = a'f'$$

where a and f represent the sectional area and unit-strain of the upper flange, and a' and f' those of the lower flange. Hence, when both flanges are horizontal, *the unit-strains in the flanges are to each other inversely as the areas.*

Ex. 1. A semi-girder, 9 inches deep, supports 7 tons at its extremity; what is the strain in each flange at 12 feet from the load?

Here, $W = 7$ tons,
$l = 12$ feet,
$d = 9$ inches.

Answer (Eq. 6). $F = \frac{Wl}{d} = \frac{7 \times 12 \times 12}{9} = 112$ tons.

Ex. 2. If the flange be 15 inches wide and $1\frac{1}{2}$ inches deep, what will be the inch-strain?

Here, $a = 22{\cdot}5$ square inches,
$F = 112$ tons.

Answer. $f = \frac{F}{a} = \frac{112}{22{\cdot}5} = 5$ tons inch-strain nearly.

Ex. 3. A wrought-iron semi-girder is 7 feet long and 11 inches deep, with each flange 4 inches wide and $\frac{1}{2}$ an inch thick. What weight at the end will break it across, the tearing inch-strain of wrought-iron being 20 tons?

Here, $F = af = 4 \times {\cdot}5 \times 20 = 40$ tons,
$d = 11$ inches,
$l = 7$ feet.

Answer (Eq. 5). $W = \frac{Fd}{l} = \frac{40 \times 11}{7 \times 12} = 5{\cdot}24$ tons.

17. Girder of greatest strength—Areas of horizontal flanges should be to each other in the inverse ratio of their ultimate unit-strains.—The distribution of a given amount of material in the flanges so as to produce the girder of greatest strength occurs when both flanges are simultaneously on the point of rupture, for if either flange contains more material than is required to sustain its proper strain when the other gives way, it can spare some of the surplus material to strengthen the other. When both flanges are on the point of rupture, f and f' are the ultimate unit-strains of tension and compression, and since $\frac{a}{a'} = \frac{f'}{f}$, it follows that, to ensure the greatest strength with a given amount of material in a girder with horizontal flanges, *the sectional areas of the flanges should be to each other inversely as their ultimate unit-strains*—a result amply confirmed by experience.

18. The web should contain no more material than is requisite to convey the shearing-strain—The quantity of material in the web of girders with parallel flanges is theoretically independent of their depth.—*The shearing-strain is the same at each vertical section of the semi-girder and equals* W. If the flanges are parallel this strain is transmitted from section to section through the web (**15**), which should therefore have the same sectional area throughout, and be sufficiently strong to transmit the shearing-strain to the wall or point of support. *The web should also for economical reasons contain no more material than is requisite to transmit the shearing-strain;* for any surplus material, if placed in the flanges, would increase the strength of the girder more than if it were to remain in the web, since its leverage to sustain horizontal strains would be thereby increased. This will appear clearer when the reader has perused the succeeding chapters. From these considerations it follows that *the quantity of material in the web of a girder with parallel flanges is theoretically independent of the depth.*

19. Girder of uniform strength—Economical distribution of material.—A girder of uniform strength is one in which all parts, both flanges and web, are duly proportioned to the strain which they have to bear, *i.e.*, are equally capable of sustaining the particular strain which is transmitted through them. If such a girder were perfect, there is no reason why any one part should fail before another, since the strain is the same sub-multiple in each part of the ultimate or breaking-strain of that part. The girder of uniform strength is obviously the most economical also in its proportions, for no part has a wasteful excess of material; *the unit-strain is constant throughout the entire length of each flange respectively, and the shearing-strain in each section of the web is the same as in every other section.*

20. Plan of semi-girder of uniform strength, depth constant.—From eq. 6 we have when both flanges are horizontal,

$$f = \frac{Wl}{ad} \tag{8}$$

where f and a express the unit-strain and sectional area of either flange indifferently at a distance l from the extremity.

In a girder of uniform strength f is constant for all values of l, and the quantity $\frac{l}{a}$, to which f is proportional (since by hypothesis the depth d is uniform), will be constant for every value of l: consequently a, that is, the area of each flange will vary as l, and if the depth of the flange be uniform its breadth will vary as l, and the plan of the flange will be triangular as in Fig. 4.

Fig. 4.—Plan.

21. Elevation of semi-girder of uniform strength, breadth constant.—If, however, one flange be sloped, f and a in eq. 8 apply to the horizontal flange only; hence, if its section and unit-strain remain uniform, d will vary directly as l, and the side elevation of the girder will be triangular as in Fig. 5. The strain in the oblique flange exceeds that in the horizontal flange in the ratio of their lengths (**9**). This is due to the

Fig. 5.—Elevation.

shearing-strain, which is entirely transmitted through the oblique flange in addition to a horizontal strain of the same amount as that in the horizontal flange, and the longitudinal strain in the oblique flange is their resultant. In this case the web has no duty to perform and may therefore be omitted, the girder becoming the simplest form of truss, viz., a triangle.

CASE II.—FLANGED SEMI-GIRDER LOADED UNIFORMLY.

Fig. 6.

22. Flanges.—Let w = the load per unit of length,

l = the distance of any cross section **A B** from the end of the girder,

d = the depth of the girder at this cross section,

W = wl = the load on **A C**,

F = the total horizontal strain exerted by either flange at **A** or **B**, that is, the horizontal component of the longitudinal strain if the flange is oblique.

The forces which keep **A B C** in equilibrium are the weights uniformly distributed along **A C**, the horizontal strains of tension and compression in the flanges at **A** and **B**, and the shearing and horizontal strains in the web at the plane of section **A B**. If the web be continuous and very thin, we may, as in the previous case, neglect the moments of the horizontal strains in the web as insignificant compared with those of the other horizontal forces. The sum of the moments round **A** or **B** of each weight in the length l is equal to the sum of the weights multiplied by the distance of their centre of gravity from A or B (**11**), that is, their collective moments $= wl\frac{l}{2}$. Equating this to the amount of the horizontal strain in either flange round A or B we obtain the following relations:—

$$w\frac{l^2}{2} = \mathsf{F}d \qquad (9)$$

$$\mathsf{W} = wl = \frac{2\mathsf{F}d}{l} \qquad (10)$$

$$\mathsf{F} = \frac{wl^2}{2d} = \frac{\mathsf{W}l}{2d} \qquad (11)$$

Ex. 1. A cast-iron semi-girder, 8 feet long and 13 inches deep, supports a uniform load of 1 ton per running foot; what area should the top flange have at the abutment in order that its inch-strain may not exceed 1·5 tons?

Here, $w = 1$ ton per foot,
$l = 8$ feet,
$d = 13$ inches,
$f = 1{\cdot}5$ tons.

From eq. 11, $\mathsf{F} = \frac{wl^2}{2d} = \frac{1 \times 8 \times 8 \times 12}{2 \times 13} = 29{\cdot}5$ tons.

Answer (eq. 1). $a = \frac{\mathsf{F}}{f} = \frac{29{\cdot}5}{1{\cdot}5} = 19{\cdot}7$ inches.

Ex. 2. The lattice-bridge at the Boyne Viaduct is in three spans. Each side span is 140 feet 11 inches long and 22 feet 3 inches deep. The permanent load supported by one main girder of a side span equals 0·68 tons per running foot, and the sectional area of its lower flange over the centre pier is 127 inches. On one occasion an extraordinary load in the centre span depressed it to such an extent as to raise the ends of the side spans off the abutments, thus forming each side span into a semi-girder. What was the compressive inch-strain in the lower flange at the pier?

Here, $w = {\cdot}68$ tons per foot,
$l = 140{\cdot}92$ feet,
$d = 22{\cdot}25$ feet,
$a = 127$ inches.

Answer (eq 11). $f = \frac{\mathsf{F}}{a} = \frac{wl^2}{2ad} = \frac{{\cdot}68 \times 140{\cdot}92 \times 140{\cdot}92}{2 \times 127 \times 22{\cdot}25} = 2{\cdot}4$ tons inch strain.

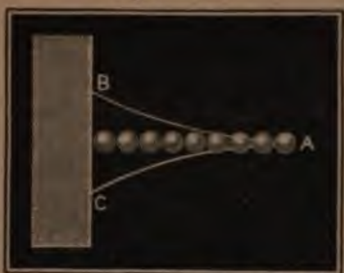
23. Web—Shearing-strain.—When a semi-girder is uniformly loaded the shearing-strain at any cross section is equal to the sum of the weights between it and the extremity of the girder, since this is the pressure transmitted through that section to the wall (**14**). *The shearing-strain therefore equals wl, and varies directly in proportion to the distance from the extremity of the girder, that is, directly as* *the ordinates of a triangle.* When the flanges are parallel all the shearing-strain passes through the web, and its sectional area should for economical reasons vary in this ratio also; for any excess of material in the web beyond that required to transmit the shearing-strain is valuable only for horizontal strains and would act with greater leverage, and therefore with greater effect, if placed in the flanges.

24. Plan of semi-girder of uniform strength, depth constant.—From eq. 11 we have when both flanges are horizontal

$$f = \frac{wl^2}{2ad} = \frac{\mathsf{W}l}{2ad} \qquad (12)$$

where a and f represent the area and unit-strain of either flange indifferently at a distance l from the extremity. If the girder be of uniform strength the unit-strain in each flange will be uniform throughout its length, and the quantity $\frac{l^2}{a}$, to which f is proportional, will be constant; that is, the sectional area of each flange will vary as l^2. Hence, if the depth of the flange be uniform, its breadth will vary as l^2, and the plan of the flange will if symmetrical be bounded by two parabolas whose common vertex is at **A**, Fig. 7, with the axis perpendicular to the length of the girder.

Fig. 7.—Plan.



25. Elevation of semi-girder of uniform strength, breadth constant.—If one flange be horizontal and the other curved, f and a in eq. 12 apply to the horizontal flange only; hence, if its sectional area be constant and if the girder be of uniform strength, d will vary as l^2, and the side elevation of the girder will be bounded by a parabola whose vertex is at **A**, Fig. 8, with its axis vertical. In this case it may be shown that the whole shearing-strain passes through the curved flange and the web has no duty to perform unless the load

Fig. 8.—Elevation.

rest upon the horizontal flange, in which case pillars, represented by vertical lines (or suspension rods if Fig. 8 be inverted with the weights beneath), are requisite for conveying the pressure of each successive weight to the curved flange.

26. Strain in curved flange.—The longitudinal strain in the curved flange is the resultant of the shearing-strain and a horizontal compression, the latter being equal to the tension in the horizontal flange. If therefore the lines **A** 1, **A** 2, **A** 3, &c., Fig. 9, represent the shearing-strains at different points, and if the horizontal line **A B** represent **F** (or the uniform horizontal compression), then the sloped lines **B** 1, **B** 2, **B** 3, &c., will represent the longitudinal strains in the curved flange at these several points (**9**).

Fig. 9.

27. Semi-girder loaded uniformly and at the extremity also—Shearing-strain.—If in addition to a uniformly distributed load the semi-girder support a weight **W′** at its extremity, the shearing-strain at any section will equal $W' + wl$. Consequently, when the flanges are parallel, the area of the web should increase in arithmetical ratio as it approaches the wall, and may be represented by the ordinates of a truncated triangle. If, for instance, the line **A B**, Fig. 10, represent the length of a uniformly loaded semi-girder, and if **A C** represent the whole distributed load, that is, the shearing-strain at the wall, then the ordinates of the triangle **A B C** will represent the shearing strains at each point. Now, let an additional weight **W′** be suspended from the end of the girder at **B**, then, if **B E** represent this weight, the ordinates of the rectangle **A D E B** will represent the shearing-strain produced by it alone; and when the girder supports both it and the uniform load the collective shearing-strain is represented by the ordinates of the trapezium **C D E B**.

Fig. 10.

28. Plan of semi-girder of uniform strength loaded uniformly and at the end, depth constant.—When both flanges are horizontal and the semi-girder supports a uniformly distributed load in addition to the weight W' at its extremity, we have (eqs. 8 and 12),

$$f = \frac{2W'l + wl^2}{2ad} \qquad (13)$$

If the semi-girder be of uniform strength, f will be constant and a will vary as $l\,(2W' + wl)$, and if the depth of the flange be uniform its breadth will vary in the same ratio. Consequently the plan of the flanges will if symmetrical be bounded by a pair of parabolas, differing however from Fig. 7 in the position of their vertices.

29. Elevation of semi-girder of uniform strength loaded uniformly and at the end, breadth constant.—If, however, one flange be horizontal and the other curved, f and a in eq. 13 apply to the horizontal flange only; hence, if its area be uniform, d will vary as $l\ (2W' + wl)$ and the elevation of the girder will be bounded by a parabola.

Ex. A semi girder, 44·7 feet long and 22·25 feet deep, supports a uniformly distributed load of 1·82 tons per foot and a weight of 161·6 tons in addition at the extremity. What is the inch-strain on the net section of the tension flange at the point of support, its gross area being 132·6 inches but reduced by rivet-holes to the extent of $\frac{2}{9}$ths?

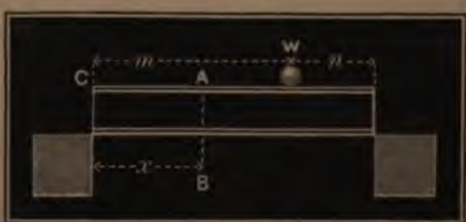
Here, $W' = 161{\cdot}6$ tons,
$l = 44{\cdot}7$ feet,
$d = 22{\cdot}25$ feet,
$w = 1{\cdot}82$ tons per foot,

$$a = \frac{7 \times 132{\cdot}6}{9} = 103{\cdot}13 \text{ square inches.}$$

Answer (eq. 13). $f = \dfrac{2 \times 161{\cdot}6 \times 44{\cdot}7 + 1{\cdot}82 \times 44{\cdot}7 \times 44{\cdot}7}{2 \times 103{\cdot}13 \times 22{\cdot}25} = 3{\cdot}94$ tons inch-strain.

CASE III.—FLANGED GIRDER SUPPORTED AT BOTH ENDS AND LOADED AT AN INTERMEDIATE POINT.

Fig. 11.



30. Flanges.—Let W = the weight dividing the girder into segments containing respectively m and n linear units,

$l = m + n$ = the length of the girder,

d = the depth at any given cross section A B,

x = the distance of this cross section from the end of the segment in which it occurs,

F = the horizontal strain exerted by either flange at A or B, that is, the horizontal component of the longitudinal strain if the flange be oblique.

On the principle of the lever (**10**), the reaction of the left abutment $= \frac{n}{l}$W, and A B C is held in equilibrium by the reaction of the left abutment, the horizontal flange-strains at A and B, the shearing-strain in the cross section A B, and the horizontal strains in the web when continuous. Neglecting these latter when the web is thin, and taking the moments of the other forces round A or B, we obtain the following relations:—

$$\frac{n}{l}\mathrm{W}x = \mathrm{F}d \tag{14}$$

whence

$$\mathrm{W} = \frac{\mathrm{F}dl}{nx} \tag{15}$$

and

$$\mathrm{F} = \frac{nx\mathrm{W}}{dl} \tag{16}$$

31. Maximum flange—strains occur at the weight.—If the cross section be taken at the weight, $x = m$, and eqs. 15 and 16 become

$$\mathrm{W} = \frac{\mathrm{F}dl}{mn} \tag{17}$$

$$\mathrm{F} = \frac{mn\mathrm{W}}{dl} \tag{18}$$

x attains its greatest value when it equals m; hence, comparing eqs. 16 and 18, we find that the horizontal strain at any point in either flange attains its greatest value when the weight rests there.

32. Single moving load, strains in flanges proportional to the rectangle under the segments.—If W is a moving load and the flanges are parallel the maximum strain at any point in either flange occurs when the load is passing that point and is proportional to mn, that is, to the rectangle under the segments.

33. Weight at centre.—This rectangle attains its greatest value when the weight is at the centre, in which case eqs. 17 and 18 become

$$W = \frac{4Fd}{l} \tag{19}$$

$$F = \frac{lW}{4d} \tag{20}$$

Ex. A cast-iron girder is 26 feet long and 27½ inches deep, and the area of the bottom flange = 16 × 3 = 48 inches. If the tearing inch-strain of cast-iron be 7 tons, what weight laid on the middle of the girder will break it across by tearing the bottom flange?

Here, l = 26 feet,
d = 27·5 inches,
f = 7 tons inch strain,
a = 48 inches,
F = 7 48 = 336 tons.

Answer (eq. 19). $W = \frac{4Fd}{l} = \frac{4 \times 336 \times 27{\cdot}5}{12 \times 26} = 118{\cdot}5$ tons nearly.

34. Web—Shearing-strain.—*The shearing-strain in each segment is uniform throughout that segment and equals the pressure which is transmitted through it to the abutment.* Thus, in Fig. 11, the shearing-strain at $AB = \frac{n}{l}W$ = the reaction of the left abutment. This shearing-strain is uniform throughout the left segment, while that in the right segment is also uniform and equals $\frac{m}{l}W$. When both flanges are horizontal the entire shearing-strain is transmitted through the web (**15**), and each segment should have its web of uniform area adequate to sustain a shearing-strain equal to the reaction of the adjacent abutment. Consequently, when a single weight rests on the centre of a girder with horizontal flanges, the section of the web should be uniform throughout its whole length, as it sustains a uniform shearing-strain $= \frac{W}{2}$.

35. Several weights resting on the girder.—When several weights rest upon the girder the simplest method of calculating the strain at any point in either flange is to determine the strain for each weight separately. The sum of the separate strains will be the total strain due to the collective weights.

36. Single fixed load—Plan of girder of uniform strength, depth constant.—When both flanges are horizontal, we have from eq. 16,

$$f = \frac{nxW}{adl} \qquad (21)$$

where f and a represent the unit-strain and area of either flange at a distance x measured from the abutment. When the girder is of uniform strength, f is constant throughout each flange and a will vary as x. Hence, if the depth of the flange be uniform, its width will vary as x, and the plan of the flange will be two triangles united at their bases as in Fig. 12.

Fig. 12.—Plan.

Ex. 1. A girder (see Fig. 11), 50 feet long and 4 feet deep, supports a load of 16 tons at 9 feet from one end; what should be the area in the middle of the top flange so that the inch-strain may not exceed 4 tons?

Here, $W = 16$ tons,
$l = 50$ feet,
$d = 4$ feet,
$f = 4$ tons inch-strain,
$n = 9$ feet,
$x = 25$ feet.

Answer (eq. 21). $a = \frac{nxW}{dfl} = \frac{9 \times 25 \times 16}{4 \times 4 \times 50} = 4\frac{1}{2}$ square inches.

Ex. 2. What is the strain in either flange at the load?

Here, $m = 41$ feet.

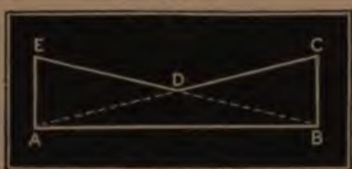
Answer (eq. 18). $F = \frac{mnW}{dl} = \frac{41 \times 9 \times 16}{4 \times 50} = 29{\cdot}5$ tons.

37. Single fixed load—Elevation of girder of uniform strength, breadth constant.—If, however, one flange be horizontal and the other sloped, f and a in eq. 21 apply to the horizontal flange only, and if its area be uniform d will vary as x, and the elevation of the girder will be a triangle whose apex is at the weight, Fig. 13. In this case the shearing-strain is transmitted through the oblique flange; the web may therefore be omitted and the girder becomes the simplest form of truss. The longitudinal strain in the oblique flange may be calculated according to the principle explained in **9**. When the weight rests upon the horizontal flange, a strut h is required of sufficient strength to support W and transmit its weight to the apex.

Fig. 13.—Elevation.

38. Single moving load—Shearing-strain.—If the load be a moving load the shearing-strain in either segment varies directly as the length of the other segment (**34**). Consequently it attains its greatest value at each point just as the weight passes, when it suddenly changes both in amount and in the direction in which it is transmitted, to the right or left abutment as the case may be. In this case the maximum shearing-strain at each section is proportional to its distance from the farther abutment and, if both flanges be horizontal, the area of the web should increase in the same ratio also—*i.e.*, as the ordinates of the figure A B C D E, Fig. 14, in which the horizontal line A B represents the length of the girder, and each of the vertical lines A E and B C represents the weight of the passing load.

Fig. 14.

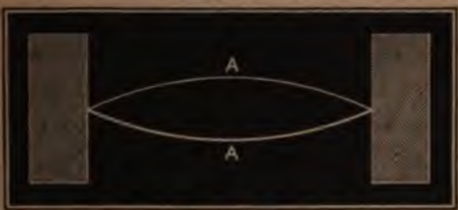


39. Single moving load—Plan of girder of uniform strength, depth constant.—In the case of a single load traversing a girder both of whose flanges are horizontal, we have at the place the weight is passing, from eq. 18,

$$f = \frac{mnW}{adl} \qquad (22)$$

where a and f represent the area and maximum unit-strain of either flange at the weight, and m and n represent the lengths of the two segments into which the weight divides the girder at the moment of passing. If the girder be of uniform strength, f will be constant throughout each flange and a will vary as the rectangle mn. Hence, if the depth of the flange be uniform, its breadth will vary as mn also, and the plan of the flange if symmetrical will be formed by the overlap of two parabolas whose vertices are at A A, Fig. 15.

Fig. 15.—Plan.



40. Single moving load—Elevation of girder of uniform strength, breadth constant.—If, however, one flange be horizontal and the other curved, f and a apply to the horizontal flange only, and, if its section be uniform, d will vary as mn. Hence the elevation of the curved flange will be a parabola whose axis is vertical and its vertex at A, Fig. 16.

Fig. 16.—Elevation.

41. Single moving load—Strain in curved flange.—The maximum longitudinal strain at any point in the curved flange of Fig. 16, *i.e.*, the strain when the weight rests over that point, may be thus obtained. Eq. 18 proves that the horizontal component of this longitudinal strain is equal to the strain in the horizontal flange at the same cross section; it is therefore a known quantity and the longitudinal strain may be found from it as follows:—Let the line A B, Fig. 17, represent F, *i.e.*, the horizontal component; draw A C parallel to the tangent of the curve at the given point and draw B C perpendicular to A B; then A C will represent the maximum longitudinal strain at the given point (9) and B C will represent its vertical component, or that portion of the shear-

Fig. 17.

ing-strain which is transmitted through the curved flange; the remainder of the shearing-strain passes through the web, which indeed prevents the girder from assuming a form similar to Fig. 13, a result that would occur were the curved flange flexible and the web absent.

42. Single moving load—Section of curved flange.—From what has just been stated it appears that the longitudinal strain in the curved flange $= \mathbf{F} \sec \theta$ where θ represents the inclination of the flange to a horizontal line. Its sectional area should increase therefore as it approaches the abutments in proportion to $\sec \theta$, since by hypothesis **F** is constant.

CASE IV.—FLANGED GIRDER SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

Fig. 18.

43. Flanges.—Let $l =$ the length of the girder,

$d =$ the depth of the girder at any given cross section **A B**,

$w =$ the load per unit of length,

$\mathbf{W} = wl =$ the whole load,

$\mathbf{F} =$ the horizontal strain exerted by either flange at **A** or **B**, that is, the horizontal component of the longitudinal strain if the flange be oblique,

m and $n =$ the segments into which the section **A B** divides the girder.

The forces which keep **A B C** in equilibrium are the reaction of the right abutment $\left(= \frac{wl}{2}\right)$, the weights uniformly distributed along **A C** $(= wn)$, the horizontal strains of compression and tension in the flanges at **A** and **B**, the shearing-strain in the plane of section **A B**, and the horizontal strains in the web when continuous. Neglecting these latter forces when the web is very thin and taking the moments of the remainder round either **A** or **B**, we have (**11**)—

$$\frac{wl}{2}n - wn\frac{n}{2} = \mathbf{F}d \qquad (23)$$

whence

$$\mathbf{F} = \frac{wmn}{2d} = \frac{mn\mathbf{W}}{2dl} \qquad (24)$$

and

$$\mathbf{W} = \frac{2\mathbf{F}dl}{mn} \qquad (25)$$

Ex. A girder, 50 feet long and 4 feet deep, supports a uniformly distributed load of 32 tons; what is the strain in either flange at 9 feet from one end?

Here, $\mathbf{W} = 32$ tons,
$l = 50$ feet,
$d = 4$ feet,
$m = 9$ feet,
$n = 50 - 9 = 41$ feet.

Answer (eq. 24). $\mathbf{F} = \dfrac{mn\mathbf{W}}{2dl} = \dfrac{9 \times 41 \times 32}{2 \times 4 \times 50} = 29{\cdot}5$ tons.

44. Strains at centre of girder.—At the centre of the girder $m = n = \frac{l}{2}$, and we have from eq. 24,

$$\mathbf{F} = \frac{\mathbf{W}l}{8d} = \frac{wl^2}{8d} \qquad (26)$$

and

$$\mathbf{W} = \frac{8\mathbf{F}d}{l} \qquad (27)$$

Ex. A girder, 101·2 feet long and 22·25 feet deep, supports a uniform load of 1·68 tons per running foot; what is the strain at the centre of either flange?

Here, $l = 101{\cdot}2$ feet,
$d = 22{\cdot}25$ feet,
$w = 1{\cdot}68$ tons per running foot.

Answer (eq. 26). $\mathbf{F} = \dfrac{wl^2}{8d} = \dfrac{1{\cdot}68 \times 101{\cdot}2 \times 101{\cdot}2}{8 \times 22{\cdot}25} = 96{\cdot}6$ tons.

45. Single weight equivalent to twice the weight uniformly distributed.—Comparing eqs. 18 and 24, we find that the horizontal strain at any point in either flange from a single weight resting there is double that which would be produced by the same load uniformly distributed.

46. Shearing-strain.—*The shearing-strain at the centre of the girder is cipher, and at any other cross section it equals the sum of the weights between it and the centre.* This will appear evident from the consideration that the shearing-strain at any section is the pressure which is transmitted to the abutment through that section. Hence, with a uniformly distributed load, the shearing-strain is proportional to the distance from the centre of the girder where it is cipher, and increases towards the ends, where it equals $\frac{W}{2}$, as the ordinates of a triangle. When both flanges are horizontal the sectional area of the web ought for economical reasons to vary in this ratio also. Any surplus material would be more valuable for sustaining horizontal strains if placed in the flanges, as its leverage would be thereby increased.

47. Plan of girder of uniform strength, depth constant.—From eq. 24 we have, when both flanges are horizontal,

$$f = \frac{wmn}{2ad} \tag{28}$$

a and f representing the area and unit-strain of either flange at any section which divides the girder into segments containing m and n linear units. If the girder be of uniform strength, f will be constant throughout each flange (**19**), and a will vary as mn. Hence, if the depth of the flange be uniform, its width will vary as mn, and the plan of the flange will if symmetrical be formed by the overlap of two parabolas whose vertices are at A A, Fig. 19.

Fig. 19.—Plan.

48. Elevation of girder of uniform strength, breadth constant.—If, however, the depth of the girder vary while the area of the horizonal flange remains uniform, d will vary as mn. Hence the elevation of the curved flange will be a parabola whose axis is vertical with its vertex at **A**, Fig. 20. In this case it may be shown that the whole shearing-strain passes through the curved flange and that therefore no web is required for diagonal strains.

Fig. 20.—Elevation.

When, however, the load rests upon the horizontal flange pillars, represented by vertical lines (or suspension rods, if Fig. 20 be inverted), are required to convey the vertical pressure of each weight to the curved flange.

The longitudinal strain in the curved flange increases towards the points of support and may be found by the method explained in **26**.

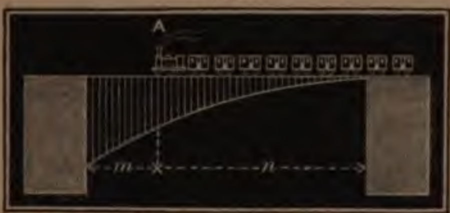
49. Suspension bridge—Curve of equilibrium.—The horizontal flange, Fig. 20, prevents the ends of the curved flange from approaching each other; the same effect may be produced by fastening the ends of the curved flange to the abutments, in which case, the load being suspended below the curved flange, we have the suspension bridge for a uniform horizontal load. The curve which an unloaded chain of uniform section assumes from its own weight is the catenary, which, however, differs but slightly from a parabola when the ratio of the deflection to the span does not exceed that commonly adopted for suspension bridges, viz., $\frac{1}{15}$.

If Fig. 20 be inverted and the horizontal flange replaced by solid abutments, to keep the arch from spreading, we have the arch of equilibrium for a uniform horizontal load, and when the arch has merely its own weight to support the inverted catenary becomes the arch of equilibrium.

Every change in the position of a load alters the form of the curve of equilibrium, *whose horizontal component is uniform throughout the whole curve;* for it is obvious that, if the horizontal strain at one point of a flexible chain exceed that at another point, the intermediate portion will move towards that side on which the stronger pull is exerted so as to conform to the position of equilibrium. A suspension bridge being flexible accommodates itself to each change of load, assuming at each moment the position of equilibrium for the particular load to which it is temporarily subjected; but neither the rigid flanges of a girder nor the voussoirs of a stone arch can thus suit themselves to the changing position of the load. The web of the former and the spandrils of the latter are therefore requisite to enable a rigid structure to sustain a variable load without fracture, which they do by converting what would otherwise be transverse strains in the arch or flanges into longitudinal ones.

50. Passing train of uniform density—Shearing-strain.—When a passing load, such as a railway train, Fig. 21, traverses a girder, the shearing-strain throughout the unloaded segment may be found as follows. Let the train be of uniform density per running foot, and its total length not less than that of the girder.

Fig. 21.



Let l = the length of the girder,

w = the weight of the train per unit of length,

m and n = the segments into which the front of the train divides the girder,

R = the reaction of the left or unloaded abutment = the shearing-strain in the segment m.

The girder is held in equilibrium by the upward reaction of each abutment and the downward pressure of the train. This latter = wn, which we may conceive collected at its centre of gravity whose distance from the right abutment $= \frac{n}{2}$ (**11**). Taking moments round this abutment we have $\text{R}l = wn\frac{n}{2}$. Hence

$$\text{R} = \frac{wn^2}{2l} \qquad (28^*)$$

This is the shearing-strain throughout the unloaded segment since it is transmitted through every section between the front of the train and the left abutment (14). As the train moves forward the shearing-strain in front increases as the square of the loaded segment, and varies therefore as the ordinates of a parabola, represented by the vertical lines of shading in the figure.

51. Maximum strains in web occur at the end of a passing train.—It can be easily proved that the shearing-strain at any point A is greater when the load covers the longer segment than when it covers the whole girder. In the latter case the load is uniformly distributed all over, and the shearing-strain at A $= \frac{w(n-m)}{2}$ (46), but when the load covers the greater segment only, the shearing-strain at A $= \frac{wn^2}{2(m+n)}$. Subtracting the former from the latter quantity we obtain the following result. The shearing-strain at the end of a passing train of uniform density covering the greater segment of a girder exceeds that produced by a load of equal density, but extending over the whole girder, by a quantity equal to $\frac{wm^2}{2l}$, where m represents the shorter segment. It will be observed that this excess is equal to the shearing-strain throughout the greater segment when the train covers the lesser segment only.

52. Uniform load and passing train—Web.—Let D E, Fig. 22, represent a railway girder, and let the ordinates D A and E C represent the shearing-strains at its extremities from a load uniformly distributed over its whole length, such as the permanent bridge-load. Draw A B and C B to the centre of D E and the ordinates of the figure DABCE will represent the shearing-strains at each point due to this uniformly distributed load. Again, let D E and E H represent the shearing-strains at the extremities from the greatest passing load of uniform

Fig. 22.

density (say engines) when covering the whole girder. Draw the parabolas **D G H** and **E G F**, and the ordinates of the figure **D F G H E** will represent the greatest shearing-strains due to this maximum passing load. The ordinates of the two figures combined, namely **A B C H G F** will represent the greatest possible shearing-strains to which the girder is liable whatever may be the position of the passing load.*

53. Maximum strain in flanges occur with load all over.— The horizontal strains in the flanges attain their greatest value when the load covers the whole girder, for the strain at each point equals the sum of those produced by each weight acting separately, and is consequently diminished by the removal of any one weight.

* Appendix to Paper on Lattice Beams. By W. B. Blood, Esq., *Proc. Inst. Civ. Eng.*, Vol. xi., p. 9.

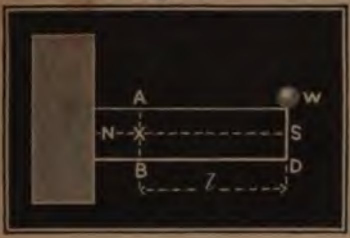
Page 7, Chap.

CHAPTER III.

TRANSVERSE STRAIN.

54. Transverse strain.—Let Fig. 23 represent a semi-girder of any form whatever of cross section loaded at the extremity with the weight **W**, and let $l =$ the distance of **W** from any plane of section **A B**. We know from experience that whenever a semi-girder such as that described is subject to transverse strain, deflection takes place, the upper edge being extended and the lower edge compressed. This longitudinal elongation and shortening are not confined to the outside fibres merely, but affect those in the interior of the girder, their change of length becoming less and less in direct proportion as their distance from the edge increases, as is proved by the lines **A B** and **W D** remaining straight after deflection.

Fig. 23.



55. Neutral surface.—The surface of unaltered length **N S** at or near the centre of the girder, where extension ceases and compression begins, is called the *Neutral surface*—a term calculated to produce a false impression that this part of a girder is free from all strain, whereas, as has been already stated (**14**), the weight, which is a vertical force, could not produce longitudinal strains in the fibres except through the medium of certain diagonal strains, which, as will be shown hereafter, act probably with their greatest intensity in the vicinity of the neutral surface. *The Neutral surface of any girder is therefore that surface along which the resultant of the horizontal components of all the diagonal forces equals cipher;* and according to this definition it may be said to exist in diagonally braced girders, in those at least in which the systems of triangulation are numerous. The reader will find his physical conceptions of these diagonal strains much clearer after he has studied the action of diagonal bracing in succeeding chapters.

56. Neutral axis—Centres of strain—Resultant of horizontal forces in any cross section equals cipher.—The line at **X** perpendicular to the plane of the figure and formed by the intersection of the neutral surface with any cross section of the girder is called the Neutral line, or more generally, the *Neutral axis* of that particular section. *The Neutral axis of any section is therefore the line of demarcation between the horizontal elastic forces of tension and compression exerted by the fibres in that particular section of the girder.* For these tensile and compressive forces we may substitute their resultants.

Let T = the resultant of the horizontal tensile forces above the neutral axis,

C = the resultant of the horizontal compressive forces below the neutral axis,

δ = the distance between the points of application of these resultants,

called the *Centres of strain*, or for distinction's sake, the *Centres of tension and compression.* The segment **A B W D** is held in equilibrium by the weight **W**, the horizontal resultants **T** and **C**, and the shearing-strain at the section **A B**. Taking moments round the centres of compression and tension successively, we have

$$\mathsf{W}l = \mathsf{T}\delta = \mathsf{C}\delta \tag{29}$$

whence

$$\mathsf{T} = \mathsf{C} \tag{30}$$

Thus in every girder of whatsoever form, *the resultant of all the horizontal forces in any cross section equals cipher*, or in other words, *the horizontal forces in any cross section balance each other.*

57. We may arrive at the same conclusion from the following consideration. Suppose a loaded girder to rest on rollers at both ends so as to be perfectly free to move in a horizontal direction. If we consider the forces acting at any cross section we find that they may be resolved into three series, the first of which is vertical, viz., the shearing-strain; the second is horizontal tending to thrust the segments apart, and the third is likewise horizontal tending to draw them together. These horizontal forces must balance, otherwise the girder would separate at the section, since by hypothesis the segments are free to move horizontally on the points of support.

58. Moment of rupture, M.—The sum of the moments of the horizontal forces in any transverse section round any point whatsoever is called the *Moment of forces tending to produce rupture*, or more briefly, the *Moment of rupture* of that particular section, because rupture generally ensues by the horizontal fibres giving way. Representing the moment of rupture by the symbol **M**, we have for a semi-girder loaded at the extremity,

$$Wl = M \tag{31}$$

where l = the distance of **W** from the transverse section.

It will be observed that the moment of rupture of any particular section is constant, no matter round what point the moments of the horizontal forces may be taken, since the sum of the tensile forces is equal to the sum of the compressive forces so that they form a couple.

The general case of a girder of any form of cross section is similar to that of a flanged girder whose flanges are at the centres of horizontal strain, and the formulæ in the several cases of flanged girders would be applicable to this general case, if we only knew the resultants of these horizontal tensile and compressive strains and also the distance between their points of application.

59. Semi-girder loaded at the extremity—Coefficient of rupture, S.—The following method is frequently adopted for calculating the breaking weight of solid rectangular or solid round girders, though applicable to other forms also, and possesses the advantage of being founded on general reasoning independently of any assumption relating to the laws of elastic reaction or of direct experiments on the tensile and compressive strength of materials, which generally require special apparatus and are therefore less easily made than experiments on transverse strength. We have just seen that the relation between the weight, length, horizontal elastic forces and distance between the centres of strain of a semi-girder fixed at one end and loaded at the other, is expressed by the equation

$$\mathsf{W} = \frac{\mathsf{F}\delta}{l}$$

in which F represents indifferently the sum of the horizontal elastic forces either above or below the neutral axis, and is therefore proportional in girders of similar section to the number of horizontal fibres in the girder, that is, to its sectional area; $\delta =$ the distance between the centres of strain and is evidently proportional to the depth, and $l =$ the length (**83**). Hence we obtain the following relations for semi-girders loaded at the extremity:—

$$\mathsf{W} = \frac{ad\mathsf{S}}{l} \tag{32}$$

$$\mathsf{S} = \frac{l\mathsf{W}}{ad} \tag{33}$$

in which $\mathsf{W} =$ the breaking weight,
$a =$ the sectional area,
$d =$ the depth,
$l =$ the length,

and S is a constant which must be determined for each material by finding experimentally the breaking weight of a girder of known dimensions and similar in section to that whose strength is required. The constant S is called the *Coefficient of rupture** of that particular material and section from which it is derived, and equals the breaking weight of any semi-girder of similar section in which the quantity $\frac{ad}{l} = 1$.

By reasoning similar to that adopted in the several cases of Chapter II. we have the following formulæ for girders supported and loaded in various ways:—

60. Semi-girder loaded uniformly.

$$\mathsf{W} = \frac{2ad\mathsf{S}}{l} \tag{34}$$

$$\mathsf{S} = \frac{l\mathsf{W}}{2ad} \tag{35}$$

* Sometimes called the *Modulus of rupture*.

61. Girder supported at both ends and loaded at an intermediate point, the segments containing m and n linear units and l representing the length $(= m + n)$.

$$\mathsf{W} = \frac{adl\mathsf{S}}{mn} \qquad (36)$$

$$\mathsf{S} = \frac{mn\mathsf{W}}{adl} \qquad (37)$$

62. Girder supported at both ends and loaded at the centre.

$$\mathsf{W} = \frac{4ad\mathsf{S}}{l} \qquad (38)$$

$$\mathsf{S} = \frac{l\,\mathsf{W}}{4ad} \qquad (39)$$

63. Girder supported at both ends and loaded uniformly.

$$\mathsf{W} = \frac{8ad\mathsf{S}}{l} \qquad (40)$$

$$\mathsf{S} = \frac{l\,\mathsf{W}}{8ad} \qquad (41)$$

64. Table of coefficients of rupture.—These formulæ, though generally restricted in practice to solid rectangular and solid round girders, are also applicable to girders of any form provided they are similar in section to the experimental girder from which the coefficient S for that form is derived. In each class we must obtain the coefficient of rupture for its particular section by experimentally breaking a model girder. This has been done for certain forms of section and the results are given in the following tables which contain the values of S, or the *coefficients of rupture*, which in the case of square or round sections are the breaking weights of solid semi-girders whose length, depth, and breadth are each one inch, fixed at one end and loaded at the other. Hence, when using these coefficients in the preceding equations, all the dimensions should be in inches. The reader may easily satisfy himself that the value of S is constant for all rectangular sections of the same depth, from the consideration that any number of rectangular girders of equal depth placed side by side have the same collective strength as the single girder which they would become if united laterally. Hence $\frac{\mathsf{W}}{a}$ has the same value for the multiple girder as for one of its component girders, and therefore, from eq. 33, **S** is the same in both.

CAST-IRON.	Value of S in Tons.
Small rectangular bars (not exceeding one inch in width),	3·40
Large rectangular bars (three inches wide),	2·25
Small round bars,	2·00
Circular tubes of uniform thickness,	2·85
Square tubes of uniform thickness,	3·42
WROUGHT-IRON.	
New rectangular bars whose deflection limits their utility,	3·82
Rectangular bars previously strained by bending them while hot and straightening them when cold (410),	5·58
New round bars,	2·25
Circular welded tubes of uniform thickness (boiler tubes),	5·23
Circular riveted tubes of plate iron with transverse joints double riveted,	3·26

N.B.—The preceding values of **S** are taken from *Clark on the Britannia and Conway Tubular Bridges*, pages 436, 743.

WOOD.

Solid Rectangular Girders and Semi-Girders.

DESCRIPTION OF WOOD.	Initials of Experimenters.	Specific Gravity.	Value of S in lbs.
Acacia,	B.	710	1,867
Ash, English,	B.	760	2,026
„ American,	D.N.	626	1,795
„ „ Swamp,	D.	925	1,165
„ „ Black,	D.	533	861
Beech, English,	B.	696	1,556
„ American White,	D.	711	1,380
„ „ Red,	D.N.	775	1,739
Birch, Common,	B.	711	1,928
„ American Black,	B.D.N.	670	2,061
„ „ Yellow,	D.	756	1,335
Box, Australian,	T.	1,280	2,445
Bullet Tree, Demerara,	B.Y.	1,052	2,692
Cabacally,	B.	900	2,518
Canada Balsam,	D.	548	1,123
Cedar, Bermuda,	N.Y.	748	1,443
„ Guadaloupe,	N.	756	2,044
„ American White,	D.	354	766
„ of Lebanon,	D.	330	1,493
Crab Wood, Demerara,	Y.	648	1,875
Deal, Christiana,	B.	689	1,562
Elm, English,	B.D.	579	782
„ Canada Rock,	D.N.	725	1,970

DESCRIPTION OF WOOD.	Initials of Experimenters.	Specific Gravity.	Value of S in lbs.
Fir, Mar Forest,	B.	698	1,232
„ Spruce,	M.	503	1,346
„ „ American Black,	D.	772	1,036
Greenheart, Demerara,	B.Y.	985	2,615
Hemlock,	D.	911	1,142
Hickory, American,	D.N.M.Y.	831	2,129
„ Bitter Nut,	D.	871	1,465
Iron Bark, Australia,	T.	1,211	2,288
Iron Wood, Canada,	D.	879	1,800
Kakarally, Demerara,	Y.	1,223	2,379
Larch,	B.D.M.	556	1,335
„ American or Tamarack,	D.	433	911
Lignum Vitæ,	N.	1,082	2,013
Locust, Demerara,	B.	954	3,430
Mahogany, Nassau,	M.N.Y.	668	1,719
Mangrove, Bermuda Black,	N.	1,188	1,699
„ „ White,	N.	951	1,985
Maple, Soft Canada,	D.	675	1,694
Norway Spar,	B.	577	1,474
Oak, Adriatic,	B.M.	855	1,471
„ African,	B.D.M.N.	988	2,523
„ American Live,	N.	1,160	1,862
„ „ Red,	D.N.	952	1,687
„ „ White,	B.D.M.N.	779	1,743
„ Dantzic,	B.M.	720	1,518
„ English,	B.D.M.N.	829	1,694
„ Italian,	M.	796	1,688
„ Lorraine,	M.	796	1,483
„ Memel,	M.	727	1,665
Pine, American Red,	B.D.M.N.Y.	576	1,527
„ „ Pitch,	B.D.	740	1,727
„ „ White,	B.N.Y.	432	1,229
„ „ Yellow,	B.D.M.	508	1,185
„ Archangel,	M.	551	1,370
„ Dantzic,	M.	649	1,426
„ Memel,	M.	601	1,348
„ Prussian,	M.	596	1,445
„ Riga,	B.M.	654	1,383
„ Virginian,	M.	590	1,456
Poon,	B.M.	673	1,954
Sneezewood, South Africa,	N.	1,066	3,305
Spotted Gum, Australia,	T.	1,035	2,006
Stringy Bark, Australia,	T.	937	1,818
Teak,	B.M.N.	729	2,108
Wallaba, Demerara,	Y.	1,147	1,643
Yellow Wood, West Indies,	N.	926	2,103

The coefficients for wood are chiefly taken from the *Professional Papers of the Corps of Royal Engineers*, Vol. v. The initial letters refer to the following experimenters :—B, Mr. Barlow ; D, Lt. Denison; M, Mr. Moore ; N, Lt. Nelson ; T, Mr. Tricket; Y, Captain Young; two or more letters signify that the tabulated number is the mean result of the experimenters whom they represent.

STONE.

DESCRIPTION OF STONE. (Generally used for Paving).	Value of S in lbs.	Authority.
Green Moor Yorkshire Blue Stone,	335	G. Rennie.
Ditto ditto White do.,	359	"
Caithness, Scotland,	857	"
Valentia, Ireland,	871	"
Welsh Slate,	1,961	"

The coefficients for stone are taken from *Barlow on the Strength of Materials*, p. 187.

Ex. 1. In an experiment made by the author, a wrought-iron bar, 4 inches deep and ¾ inch wide, had a weight of 1,568 lbs. hung from one end, the other end being rigidly fixed. It commenced bending at 2 ft. 8 in. from the load, at a part which had been previously softened in the fire and allowed to cool slowly. What is the value of S?

Here, $W = 1{,}568$ lbs.

$l = 32$ inches,

$d = 4$ inches,

$a = 3$ square inches.

Answer (eq. 33). $S = \frac{lW}{ad} = \frac{32 \times 1{,}568}{3 \times 4 \times 2{,}240} = 1{\cdot}86$ tons.

Comparing this with the tabular value of S for "new rectangular bars whose deflection limits their utility," it would appear that the useful strength of annealed bars is only one-half that of new bars fresh from the rolls. This result is confirmed by two of Mr. Hodgkinson's experiments on annealed wrought-iron bars heated to redness and allowed to cool slowly.—(Appendix to *Report of the Commissioners on the Application of Iron to Railway Structures*, pp. 45, 46.)

Ex. 2. The teeth of a cast-iron wheel are 3·5 inches long, 2·3 inches thick and 7 inches wide; what is the breaking weight of a tooth?

Here, $l = 3{\cdot}5$ inches,

$d = 2{\cdot}3$ inches,

$a = 16{\cdot}1$ square inches,

$S = 2{\cdot}25$ tons.

Answer (eq. 32). $W = \frac{adS}{l} = \frac{16{\cdot}1 \times 2{\cdot}3 \times 2{\cdot}25}{3{\cdot}5} = 23{\cdot}8$ tons.

Ex. 3. A round wrought-iron shaft, 5 feet long and supported at the extremities, sustains a transverse strain of 30 tons at 14 inches from one end; what should be its diameter when on the point of yielding?

Here, $W = 30$ tons,

$l = 5$ feet,

$m = 14$ inches,

$n = 46$ inches,

$S = 2{\cdot}25$ tons.

From eq. 37, $ad = \frac{mnW}{lS} = \frac{14 \times 46 \times 30}{60 \times 2{\cdot}25} = 143{\cdot}1$ inches; but $ad = \frac{\pi d^3}{4}$, whence

$$\textit{Answer.}\quad d = \sqrt[3]{\frac{4 \times 143{\cdot}1}{3{\cdot}1416}} = 5{\cdot}7 \text{ inches.}$$

Ex. 4. In an experiment made by William Anderson, Esq., a piece of memel timber, 2 inches deep and $1\frac{15}{16}$ inches wide, was securely fixed at one extremity, the projecting part being 2 feet long. It sustained a load of 504·5 lbs. at the end for twenty hours without breaking right across. This load, however, upset the timber on the lower or compression side next the fulcrum. What is the value of S derived from this experiment?

Here, $W = 504{\cdot}5$ lbs.
$l = 24$ inches,
$d = 2$ inches,
$b = 1{\cdot}94$ inches,

$$\textit{Answer (eq. 33).}\quad S = \frac{lW}{ad} = \frac{24 \times 504{\cdot}5}{1{\cdot}94 \times 2 \times 2} = 1{,}560 \text{ lbs.}$$

This value of S exceeds that given in the table, namely, 1,348 lbs. The piece of memel in this experiment was, however, remarkably straight-grained and well seasoned, and consequently above the average.

65. Strength of similar girders—Limit of size.—It appears from the foregoing investigation that the strength of similar girders varies as the square of their linear dimensions, for $\frac{l}{d}$ is constant in similar girders, and consequently the breaking weight W varies as a. The weight of the girder itself, however, varies as al, *i.e.*, as the cube of its linear dimensions. If this weight, which we shall call G, equals $\frac{1}{n}$th of the breaking weight, we have in the case of girders loaded uniformly (eqs. 34 and 40),

$$nG = \frac{KadS}{l}$$

in which $K = 2$ for a semi-girder and 8 for a girders, upported at both ends. The breaking weight W of a similar girder n times longer is as follows:—

$$W = \frac{nK^2adS}{l} = n^3G.$$

But n^3G is the weight of the second girder. Hence, *if the weight of any girder is* $\frac{1}{n}$*th of its breaking weight, a similar girder* *n times longer will break from its own weight.* This defines the theoretic limit of size of similar girders.

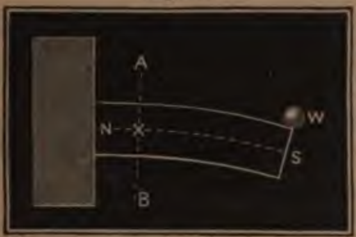
66. Neutral axis passes through centre of gravity.—If the law of uniform elastic reaction holds good in girders subject to transverse strains, the horizontal elastic reaction exerted by each fibre will be in proportion to the extension or compression of the fibre, that is, in direct proportion to its distance from the neutral axis (7). Its amount will also be proportional to the sectional area of the fibre, and if the variable distance from the neutral axis be called y and the sectional area $d\sigma$ (differential of σ), then the elastic force of the fibre may be represented by $yd\sigma$ multiplied by a constant, and F, or the sum of the horizontal elastic forces on either side of the neutral axis, equals $\int yd\sigma$ taken within proper limits and multiplied by the same constant. This integral for the horizontal elastic forces on the upper side of the neutral axis is equal to the similar expression for the horizontal elastic forces on the lower side (eq. 30). Now this equality is also the condition which determines the position of the centre of gravity of the section. Hence it follows, that within the limits of uniform elastic reaction, *the neutral axis of any cross section passes through its centre of gravity*, and we have the following practical rule for finding the position of the neutral axis where the section is unsymmetrical, as in T iron, or a girder with unequal flanges. Cut a model of the cross section of the girder out of card-board and find its centre of gravity by balancing it on a knife-edge. This will give the position of the neutral axis of the girder accurately enough for practical purposes.

CHAPTER IV.

GIRDERS OF VARIOUS SECTIONS.

67. Moment of rupture.—The following method of investigating the strength of girders of any form whatsoever of cross section is based on the assumption that the law of uniform elastic reaction is true, that is, that the fibres exert forces which are proportional to their change of length, and therefore directly proportional to their distance from the neutral axis (7). Suppose a girder composed of longitudinal fibres of infinitesimal thickness, and let us consider the horizontal elastic forces developed by the weight **W** in any cross section **AB**,

Fig. 24.



Let **M** = the moment of rupture of the section **AB** (58),

d = the depth of the girder,

y = the variable distance of any fibre in the section **AB** either above or below the neutral axis,

β = the breadth of the section at the distance y from the neutral axis, and consequently a variable, except in the case of rectangular sections,

f = the horizontal unit-strain exerted by fibres in the section **AB** at a given distance c from the neutral axis.

According to our assumption, the unit-strain in any other fibres at a distance y from the neutral axis will be $\frac{fy}{c}$. Let the depth of the latter fibres $= dy$ (differential of y); then the total horizontal force exerted by the fibres in the breadth β will $= \frac{f}{c}\beta y dy$. The moment of this force round the neutral axis $= \frac{f}{c}\beta y^2 dy$, and the integral of this quantity will be the sum of the moments of all the horizontal elastic forces in the section **AB** round its neutral axis, that is, the *moment of rupture* of the section in question (**58**). Representing this as before by the symbol **M**, we have

$$\mathbf{M} = \frac{f}{c}\int \beta y^2 dy \tag{42}$$

in which the integral must be taken within proper limits for each form of cross section and may be readily found for those sections which occur in practice in the following manner.*

68. Let h_1 = the distance of the top of the girder above the neutral axis,

h_2 = the distance of the bottom of the girder below the neutral axis.

The expression for the moment of rupture becomes

$$\mathbf{M} = \frac{f}{c}\int_0^{h_1} \beta y^2 dy + \frac{f}{c}\int_0^{h_2} \beta y^2 dy \tag{43}$$

in which β if variable must be expressed in terms of y.

69. M for sections symmetrically disposed around the centre of gravity.—When the material is symmetrically disposed above and below the centre of gravity the neutral axis bisects the depth (**66**), and if d = the depth, we have $h_1 = h_2 = \frac{d}{2}$, and

$$\mathbf{M} = \frac{2f}{c}\int_0^{\frac{d}{2}} \beta y^2 dy \tag{44}$$

The values of **M** for the usual forms of cross section are as follows, recollecting that f = the unit-strain in fibres whose distance from the neutral axis = c.

70. M for a solid rectangle.

Let b = the breadth,

d = the depth.

From eq. 44,

$$\mathbf{M} = \frac{2bf}{c}\int_0^{\frac{d}{2}} y^2 dy = \frac{bd^3f}{12c} \tag{45}$$

* The reader will recognize the integral $\int \beta y^2 dy$ as that which expresses the *Moment of Inertia* of the cross section round its neutral axis.

71. M for a solid square with one diagonal vertical.

Let a = the semi-diagonal,

b = the side of the square.

The variable breadth β expressed in terms of $y = 2\,(a - y)$; substituting this value in eq. 44, we have

$$\mathbf{M} = \frac{4f}{c}\int_0^a (a - y)\, y^2 dy$$

Integrating and reducing,

$$\mathbf{M} = \frac{a^4 f}{3c} = \frac{b^4 f}{12c} \qquad (46)$$

72. M for a circular disc.

Let r = the radius.

The variable breadth β expressed in terms of y becomes $2\sqrt{r^2 - y^2}$. Substituting this value in eq. 44, we have

$$\mathbf{M} = \frac{4f}{c}\int_0^r \sqrt{r^2 - y^2}.y^2 dy$$

Integrating and reducing,

$$\mathbf{M} = \frac{\pi f r^4}{4c} \qquad (47)$$

73. M for a circular ring of uniform thickness.

Let r = the external radius,

r_1 = the internal radius.

The moment of rupture of a ring is equal to that of the external circle minus that of the internal one, and we have from eq. 47,

$$\mathbf{M} = \frac{\pi f}{4c}(r^4 - r_1^4) \qquad (48)$$

If t = the thickness of the ring, $r_1 = r - t$; whence by substitution in eq. 48,

$$\mathbf{M} = \frac{\pi f}{4c}(4r^3 t - 6r^2 t^2 + 4rt^3 - t^4)$$

If the thickness be small compared with the radius the last three terms may be neglected, and we have

$$\mathbf{M} = \frac{\pi f r^3 t}{c} \qquad (49)$$

74. M for an elliptic disc with one axis horizontal.

Let b = the horizontal semi-axis,

d = the vertical semi-axis.

gravity of the two flanges (**66**), and we have $h_1 = \frac{a_2 d}{\mathsf{A}}$ and $h_2 = \frac{a_1 d}{\mathsf{A}}$. Hence, by substitution,

$$\mathsf{M} = \frac{a_1 a_2 d^2 f}{\mathsf{A}c} \qquad (54)$$

78. M for the section of a flanged girder or rectangular tube, including the web.—When, however, the horizontal strains in the web are too considerable to be safely neglected, the moment of rupture of the web, derived from eq. 43, must be added to that just obtained for the flanges (eq. 53), when we have

$$\mathsf{M} = \frac{f}{c}\left\{\left(a_1 + \frac{a_3}{3}\right)h_1^2 + \left(a_2 + \frac{a_4}{4}\right)h_2^2\right\} \qquad (55)$$

79. M for the section of a flanged girder or rectangular tube with equal flanges, including the web.—If the flanges have equal areas the neutral axis will be in the middle of the depth, in which case $h_1 = h_2 = \frac{d}{2}$, and eq. 55 becomes

$$\mathsf{M} = \frac{d^2 f}{12c}(6a + a') \qquad (56)$$

where a = the area of either flange,
a' = the area of the web.

The moment of rupture of a rectangular tube with flanges of equal area may also be obtained from eq. 45 by subtracting the moment of rupture of the inner from that of the outer rectangle as follows,

$$\mathsf{M} = \frac{f}{12c}(bd^3 - b_1 d_1^3) \qquad (57)$$

where b = the external breadth,
b_1 = the internal breadth,
d = the external depth,
d_1 = the internal depth.

80. M for the section of a square tube of uniform thickness, either with the sides or one diagonal vertical.—From eqs. 45 and 46,

$$\mathsf{M} = \frac{f(b^4 - b_1^4)}{12c} \qquad (58)$$

where b = the external breadth of the tube,
b_1 = the internal breadth of the tube.

If t = the thickness of the tube, $b_1 = b - 2t$; substituting this value in eq. 58, expanding, and neglecting the terms in which the higher powers of t occur, we have when the thickness is small compared with the breadth of the tube,

$$\mathbf{M} = \frac{2fb^3t}{3c} \tag{59}$$

When the value of **M** is known for any particular section of girder we can easily find the value of the weight **W** in terms of f, or *vice versâ*, as explained in the following cases:—

CASE I.—SEMI-GIRDERS LOADED AT THE EXTREMITY.

Fig. 25.

81. Let **W** = the weight at the extremity,

l = the distance of **W** from any cross section **A B**,

M = the moment of rupture of the section **A B**.

The forces which keep the segment **A B W** in equilibrium are the weight **W**, the shearing strain at **A B**, and the horizontal elastic forces developed in the same section. Taking the moments of all these forces round the neutral axis we have (eq. 31),

$$\mathbf{W}l = \mathbf{M} \tag{60}$$

82. Solid rectangular semi-girders.

Let b = the breadth,

d = the depth.

From eqs. 45 and 60,

$$\mathbf{W}l = \frac{fbd^3}{12c}$$

where f = the unit-strain in fibres whose distance from the neutral axis = c.

If, however, f = the unit-strain in the extreme fibres, $c = \frac{d}{2}$, and we have

$$W = \frac{fbd^2}{6l} \tag{61}$$

Ex. A piece of teak, 2 inches deep and $1\frac{15}{16}$ inches wide, is fixed at one extremity; what weight hung 2 feet from the point of attachment will break it across, the crushing inch-strain of teak being 12,000 lbs.?

Here, l = 2 feet,
b = 1·94 inches,
d = 2 inches,
f = 12,000 lbs.

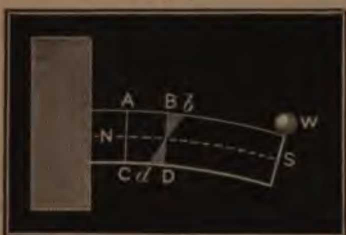
Answer. $W = \frac{fbd^2}{6l} = \frac{12,000 \times 1·94 \times 2 \times 2}{6 \times 24} = 647$ lbs.

The crushing strength of teak being considerably less than its tearing strength, rupture will ensue from the crushing of the fibres on the compressed side.

83. Geometrical proof.—Eq. 61 may be easily deduced from geometrical consideration as follows:—

Let the rectangle **ABCD**, Fig. 26, represent in an exaggerated degree the side view of any small transverse slice whose breadth before deflection = **AB**. Suppose the upper edge after deflection stretched out to the length **A***b*, and the lower edge compressed to **C***d*; then the lines of shading in the two triangles will represent the alteration of length of the intermediate fibres, **N S** being the neutral surface which divides the section into equal parts (**54**).

Fig. 26.



The sum of the horizontal forces exerted by the fibres in either the upper or the lower half of the section is equal to the product of the half section by the mean unit-strain of the fibres, and if f = the unit-strain in the extreme fibres, then $\frac{f}{2}$ is the mean unit-strain of all the fibres, for it equals the unit-strain exerted by the fibres lying mid-way between the neutral surface and either the upper or the lower edge (7). The total strain of tension in the upper half, and that of compression in the lower half, are therefore each equal to $\frac{f}{2} \times \frac{bd}{2}$, where b and d represent the breadth and depth of the section. Moreover, since the horizontal forces in the various fibres are proportional to the lines of shading in the two triangles (7), the centres of tension and compression (56) coincide with the centres of gravity of the two triangles, and their distance apart therefore $= \frac{2}{3}d$. Hence, taking moments round either centre of strain, we have as before,

$$Wl = \frac{fbd^2}{6}$$

84. Solid square semi-girders with diagonal vertical—Solid square girders with the sides vertical are 1·414 times stronger than with one diagonal vertical.—If one diagonal is vertical, we have from eqs. 46 and 60,

$$Wl = \frac{fb^4}{12c}$$

where $f =$ the unit-strain in fibres whose distance from the neutral axis $= c$.

If, however, $f =$ the unit-strain in the extreme fibres, $c = \frac{b}{\sqrt{2}}$, and we have

$$W = \frac{fb^3}{8{\cdot}5l} \qquad (62)$$

Comparing eqs. 61 and 62, we find that the transverse strength of a solid square girder with the sides vertical $= \frac{8{\cdot}5}{6} = 1{\cdot}414$ times the strength of the same girder with the diagonal vertical.

85.—The strength of square semi-girders in the direction of their diagonals may be investigated in a different manner as follows.

Fig. 27.

Let Fig. 27 represent a cross section of the girder, and let the line **AB** represent the shearing force acting downwards. We may conceive this replaced by its components **AC** and **AD** parallel to the sides of the girder. Since the section is square each component will equal $\frac{AB}{\sqrt{2}}$. Now the force **AC** will produce tension in the side parallel to **A D**, and the force **A D** will produce tension in the side parallel to **A C**; the corner will therefore sustain twice the strain that either component alone would produce, that is, it will sustain a strain which would be produced by a force equal to $\frac{2\,\mathsf{AB}}{\sqrt{2}}$ $(= 1{\cdot}414\,\mathsf{AB})$, acting in the direction of one side, which result agrees with that already obtained.

86. Rectangular girder of maximum strength cut out of a cylinder.—It is sometimes required to cut a rectangular girder of maximum strength out of round timber.

Fig. 28.

Let D = the diameter of the log,
b = the breadth of the girder of maximum strength,
d = its depth.

From eq. 61 the strength of a rectangular girder is maximum when bd^2 is maximum, or, since $d^2 = \mathsf{D}^2 - b^2$, when $b\mathsf{D}^2 - b^3$ is maximum. Equating the differential coefficient of this quantity to cipher, we have

$$b^2 = \frac{1}{3}\mathsf{D}^2$$

from which we derive the following rule. Erect a perpendicular p at one-third of the length of the diameter; from the point where this perpendicular intersects the circumference draw two lines b and d to the extremities of the diameter; then $b^2 = \frac{1}{3}\mathsf{D}^2$ *

87. Solid round semi-girders.

Let r = the radius.

From eqs. 47 and 60,

$$\mathsf{W}l = \frac{\pi f r^4}{4c}$$

where f = the unit-strain in fibres whose distance from the neutral axis = c.

If, however, f = the unit-strain in the extreme fibres, $c = r$, and

$$\mathsf{W} = \frac{\pi f r^3}{4l} \qquad (63)$$

* *Euclid*, Book VI.; Cor., prop. 8.

88. Solid square girders are 1·7 times as strong as the inscribed circle, and 0·6 times as strong as the circumscribed circle.—Comparing eqs. 61 and 63, we find that the strength of a solid square girder is 1·7 times that of the solid inscribed cylinder, whereas its strength is only $\frac{5·66}{9·42} = 0·6$ times that of the solid circumscribed cylinder.

89. Hollow round semi-girders of uniform thickness.

Let r = the external radius,

r_1 = the internal radius,

From eqs. 48 and 60,

$$Wl = \frac{\pi f}{4c}(r^4 - r_1^4)$$

where f = the unit-strain in fibres whose distance from the neutral axis = c.

If, however, f = the unit-strain in the extreme fibres, $c = r$, and

$$W = \frac{\pi f}{4lr}(r^4 - r_1^4) \qquad (64)$$

If, moreover, the thickness of the tube be small compared with the radius, we have from eqs. 49 and 60,

$$W = \frac{\pi f r^2 t}{l} \qquad (65)$$

where t represents the thickness of the tube.

Ex. A tubular crane post of plate iron is 11 feet high and 2·4 feet diameter at the base. The load on the crane produces a cross-strain of 20 tons acting at right angles to the top of the post; what should be the thickness of the plating at the base in order that the inch-strain may not exceed 3 tons?

Here, W = 20 tons,

l = 11 feet,

r = 1·2 feet,

f = 3 tons per square inch.

Anwer (eq. 65). $t = \frac{Wl}{\pi f r^2} = \frac{20 \times 11}{3·1416 \times 3 \times 12 \times (1·2)^2} = 1·35$ inches.

90. Centre of solid round girders nearly useless.—The centre or core of a cylindrical girder may be removed without sensibly diminishing its transverse strength; for it appears from eqs. 63 and 64 that the strengths of two cylinders of equal diameters, one solid and the other hollow, are as $\frac{r^4}{r^4 - r_1^4}$, in which r and r_1 are the external and internal radii; let $r = nr_1$, then the ratio becomes $\frac{n^4 - 1}{n^4}$; if, for example, $n = 2$, the loss of strength in the hollow cylinder amounts to only $\frac{1}{16}$th of that of the solid cylinder while the saving of material amounts to $\frac{1}{4}$th. For this, among other reasons, cylindrical castings, such as crane posts, should be made hollow.

91. Hollow and solid cylinders of equal weight.—It may also be shown that the transverse strength of a thin hollow cylinder is to that of a solid cylinder of equal weight as the diameter of the former is to the radius of the latter. By eqs. 65 and 63 the ratio of the strength of a hollow to that of a solid cylinder $= \frac{4r^2t}{r_1^3}$, in which r and t represent the radius and thickness of the hollow cylinder, and r_1 represents the radius of the solid cylinder; since by hypothesis the two cylinders are of equal weight, we have $2rt = r_1^2$; whence, by substitution, the ratio of strength becomes $\frac{2r}{r_1}$, that is, as the diameter of the hollow cylinder is to the radius of the solid cylinder.

92. Solid elliptic semi-girders.

Let b = the horizontal semi-axis,

d = the vertical semi-axis.

From eqs. 50 and 60 we have

$$Wl = \frac{\pi f b d^3}{4c}$$

where f = the unit-strain in fibres whose distance from the neutral axis = c.

If, however, f = the unit-strain in the extreme fibres, $c = d$, and

$$W = \frac{\pi f b d^2}{4l} \qquad (66)$$

93. Hollow elliptic semi-girders.

Let b = the external horizontal semi-axis,

b_1 = the internal horizontal semi-axis,

d = the external vertical semi-axis,

d_1 = the internal vertical semi-axis.

From eqs. 51 and 60 we have

$$Wl = \frac{\pi f}{4c}(bd^3 - b_1 d_1^3)$$

where f = the unit-strain at the distance c from the neutral axis.

If, however, f = the unit-strain in the extreme fibres, $c = d$, and

$$W = \frac{\pi f}{4dl}(bd^3 - b_1d_1^3) \tag{67}$$

If, moreover, the thickness of the tube is small compared with the shorter axis, we have from eqs. 52 and 60

$$W = \frac{\pi fdt}{4l}(3b + d) \tag{68}$$

where t = the thickness of the tube.

94. Flanged semi-girder or rectangular tube, taking the web into account.

Let a_1 = the area of the top flange,
a_2 = the area of the bottom flange,
a_3 = the area of the web above the neutral axis,
a_4 = the area of the web below the neutral axis,
h_1 = the distance of the top flange above the neutral axis,
h_2 = the distance of the bottom flange below the neutral axis,
f = the unit-strain in fibres whose distance from the neutral axis = c.

From eqs. (55) and (60), we have

$$W = \frac{f}{cl}\left\{\left(a_1 + \frac{a_3}{3}\right)h_1^2 + \left(a_2 + \frac{a_4}{3}\right)h_2^2\right\} \tag{69}$$

95. Flanged semi-girder or rectangular tube with equal flanges.—If the flanges are equal, we have from eqs. 56 and 60,

$$Wl = \frac{fd^2}{12c}(6a + a')$$

where d = the depth of web,
a = the area of either flange,
a' = the area of the web,
f = the unit-strain in fibres whose distance from the neutral axis = c.

If f = the unit-strain in either flange, $c = \frac{d}{2}$, and we have

$$W = \frac{fd}{l}\left(a + \frac{a'}{6}\right) \tag{70}$$

In the case of a rectangular tube with equal flanges the following equation, derived from eqs. 57 and 60, may be used instead of eq. 70,

$$W = \frac{f}{6dl}(bd^3 - b_1d_1^3) \qquad (71)$$

where b = the external breadth,
b_1 = the internal breadth,
d = the external depth,
d_1 = the internal depth,
f = the unit-strain in the extreme fibres, in which case $c = \frac{d}{2}$

96. Square tubes, with sides vertical.—If the tube is square with vertical sides of uniform thickness, we have from eq. 71,

$$W = \frac{f}{6bl}(b^4 - b_1^4) \qquad (72)$$

If, moreover, the thickness of the tube is small compared with its breadth, we have from eqs. 59 and 60,

$$W = \frac{4fb^2t}{3l} \qquad (73)$$

where t = the thickness of the side of the tube.

97. Square tubes with diagonal vertical—Square tubes of uniform thickness with the sides vertical are 1·414 times stronger than with one diagonal vertical.—If one diagonal of the square tube is vertical, the sides being of equal thickness, we have from eqs. 58 and 60,

$$Wl = \frac{f}{12c}(b^4 - b_1^4)$$

where f = the unit-strain at the distance c from the neutral axis. If f = the unit-strain in the extreme fibre, $c = \frac{b}{\sqrt{2}}$, and we have

$$W = \frac{f}{8 \cdot 5bl}(b^4 - b_1^4) \qquad (74)$$

If, moreover, the thickness of the tube is small compared with its breadth, we have from eqs. 59 and 80,

$$W = \frac{\cdot 94fb^2t}{l} \qquad (74^*)$$

where t = the thickness of the side of the tube.

Comparing eqs. 72 and 74, we find that the strength of a square tube of uniform thickness with the side vertical equals 1·414 times the strength of the same tube with the diagonal vertical (84).

98.—Square tubes of uniform thickness with the sides vertical are 1·7 times as strong as the inscribed tube of equal thickness, and 0·85 times as strong as the circumscribed tube of equal thickness—Square and round tubes of equal thickness and weight are of nearly equal strength.— Comparing eqs. 73 and 65, we find that the strength of a square tube with vertical sides is to that of a round tube of equal thickness and whose diameter equals the side of the square (inscribed circle) as $\frac{16}{9{\cdot}42} = 1{\cdot}7$; whereas the strength of the square tube with vertical sides is to that of a round tube of equal thickness but whose diameter equals the diagonal of the square (circumscribed circle) as $\frac{8}{9{\cdot}42} = 0{\cdot}85$. It also appears that the strength of the circumscribed circle is twice that of the inscribed circle of equal thickness. If square and round tubes are of equal thickness and weight their peripheries will be equal, that is, $4b = 2\pi r$, or $b = \frac{\pi}{2}r$, substituting this value for b in eq. 73, and comparing the result with eq. 65, we find that the relative strength of square tubes with vertical sides and round tubes of equal weight and thickness $= \frac{\pi}{3} = 1{\cdot}0472$, or very nearly a ratio of equality, the square tube being very slightly stronger than the other. When semi-girders are subject to transverse strain in various directions like crane posts, the round tube is generally preferable to a square tube of equal weight, as the latter is much weaker in the direction of the diagonals (**97**).

99. Value of web in aid of the flanges.—The strength of a girder with equal flanges and continuous web, in which full credit is given to the web for the horizontal strains which it sustains, is equal to the strength derived from the flanges alone plus that derived from the web acting as an independent rectangular girder (**76**). Eqs. 5 and 70 prove that *a continuous web, in a girder with flanges of equal area, does theoretically as much duty in aid of the flanges as if one-sixth of the web were added to each flange and the web were made of bracing.*

100. Plan of solid rectangular semi-girder of uniform strength, depth constant.—From eq. 61, the unit-strain in the

Fig. 29.—Plan.

extreme fibres of a solid rectangular semi-girder $f = \frac{6l\mathbf{W}}{bd^2}$. If the semi-girder be of uniform strength (**19**), f will be constant, and consequently the quantity $\frac{l}{bd^2}$, to which f is proportional, will also be constant. Hence, if the depth of the girder be uniform, b will vary as l, that is the plan of the girder will be triangular, Fig. 29.

101. Elevation of solid rectangular semi-girder of uniform strength, breadth constant.—If, however, the breadth be uniform, d^2 will vary as l, and if the top of the girder be horizontal the bottom will be bounded by a parabola whose vertex is at **W** and its axis horizontal, Fig. 30.

Fig. 30.—Elevation.

102. Solid round semi-girder of uniform strength.—From eq. 63, the unit-strain in the extreme fibres of a solid round semi-girder $f = \frac{4l\mathbf{W}}{\pi r^3}$. If its strength be uniform, r^3 will vary as l, and the semi-girder will be a solid formed by the revolution of a cubic parabola round a horizontal axis, Fig. 31. The beak of an anvil is a rude approximation to this form of semi-girder.

Fig. 31.

103. Hollow round semi-girder of uniform strength.—From eq. 65, the unit-strain in the extreme fibres of a thin round tube $f = \frac{l\mathbf{W}}{\pi r^2 t}$. If its strength be uniform, f will be constant and r^2t will vary as l. When the thickness is constant, r^2 will vary as l, and a hollow semi-girder, formed by the revolution of a parabola round a horizontal axis, will result. This, for instance, is the theoretic form for a hollow crane post of plate iron; the circumscribing cone, however, is preferable in practice, as it is more easily constructed.

CASE II.—SEMI-GIRDERS LOADED UNIFORMLY.

Fig. 32.

104. Let l = the distance of any cross section **AB** from the extremity of the girder,

w = the load per linear unit,

W $= wl$ = the sum of the weights resting on **AC**,

M = the moment of rupture of the section **AB**.

The forces which keep **ABC** in equilibrium are the weights uniformly distributed along **AC**, the shearing-strain at **AB**, and the horizontal elastic forces developed in the same section. Taking the moments of all these forces round the neutral axis of the section **AB**, and recollecting that the sum of the moments of the separate weights is equivalent to the moment of a single weight equal to their sum and placed at their centre of gravity (**11**), we have (**58**),

$$\mathbf{W}\frac{l}{2} = \frac{wl^2}{2} = \mathbf{M} \qquad (75)$$

105. Solid rectangular semi-girders.—From eqs. 45 and 75, we have

$$\mathbf{W} = \frac{fbd^2}{3l} \qquad (76)$$

in which b and d represent the breadth and depth of the girder, and f = the unit-strain in the outer fibres at top and bottom, in which case $c = \frac{d}{2}$.

106. Solid round semi-girders.—From eqs. 47 and 75,

$$\mathbf{W} = \frac{\pi f r^3}{2l} \qquad (77)$$

where $r =$ the radius and $f =$ the unit-strain in the extreme fibres at top and bottom, in which case $c = r$.

107. Hollow round semi-girders of uniform thickness.—From eqs. 48 and 75,

$$W = \frac{\pi f}{2lr}(r^4 - r_1^4) \tag{78}$$

in which r represents the external and r_1 the internal radius, and $f =$ the unit-strain in the extreme fibres at top and bottom. If, moreover, the thickness t is inconsiderable compared with the radius we have from eqs. 49 and 75,

$$W = \frac{2\pi f r^2 t}{l} \tag{79}$$

108. Flanged semi-girders on rectangular tubes, taking the web into account.—From eqs. 55 and 75,

$$W = \frac{2f}{cl}\left\{\left(a_1 + \frac{a_3}{3}\right)h_1^2 + \left(a_2 + \frac{a_4}{3}\right)h_2^2\right\} \tag{80}$$

where $a_1 =$ the area of the top flange,
$a_2 =$ the area of the bottom flange,
$a_3 =$ the area of the web above the neutral axis,
$a_4 =$ the area of the web below the neutral axis,
$h_1 =$ the distance of the top flange above the neutral axis,
$h_2 =$ the distance of the bottom flange below the neutral axis,
$f =$ the unit-strain in fibres whose distance from the neutral axis $= c$.

If the flanges are equal and if $f =$ the unit-strain in either flange, we have from eqs. 56 and 75,

$$W = \frac{2df}{l}\left(a + \frac{a'}{6}\right) \tag{81}$$

where a = the area of either flange,
a' = the area of the web,
d = the depth of the web.

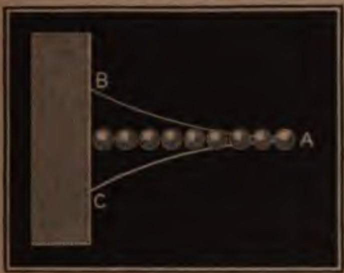
109. Plan of solid rectangular semi-girder of uniform strength, depth constant.—From eq. 76, the unit-strain in the outer fibres of a solid rectangular semi-girder loaded uniformly,

$$f = \frac{3wl^2}{bd^2}$$

in which w represents the load on the unit of length, $= \frac{W}{l}$.

When the strength of the girder is uniform throughout its length (**19**), the quantity $\frac{l^2}{bd^2}$, to which f is proportional, is constant, and if d be uniform, b will vary as l^2, and the plan of the girder will if symmetrical be bounded by two parabolas whose common vertex is at **A** and the axis vertical, Fig. 33.

Fig. 33.—Plan.



110. Elevation of solid rectangular semi-girder of uniform strength, breadth constant.—If, however, the breadth be uniform, d will vary as l, and the elevation of the girder will be triangular.

111. Solid round semi-girder of uniform strength.—From eq. 77 the unit-strain in the extreme fibres of a solid round semi-girder loaded uniformly,

$$f = \frac{2wl^2}{\pi r^3}$$

If the strength be uniform, r^3 will vary as l^2, and the semi-girder will be a solid formed by the revolution of a semi-cubic parabola round a horizontal axis.

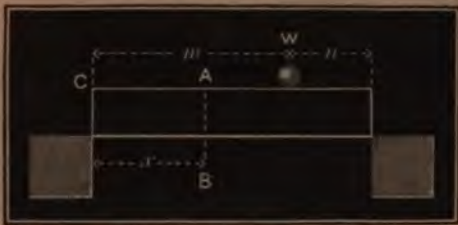
112. Hollow round semi-girder of uniform strength.—From eq. 79 the unit-strain in the extreme fibres of a thin round tube,

$$f = \frac{wl^2}{2\pi r^2 t}$$

If the strength be uniform, r^2t will vary as l^2. Hence, if t be constant, r will vary as l and the tube will be conical.

CASE III.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED AT AN INTERMEDIATE POINT.

Fig. 34.



113. Let W = the weight, dividing the girder into segments containing respectively m and n linear units,

$l = m + n$ = the length of the girder,

x = the distance of any cross section **AB** from that end of the girder which is remote from **W**,

M = the moment of rupture of the section **AB**.

On the principle of the lever the reaction of the left abutment = $\frac{n}{l}\mathsf{W}$, and **ABC** is held in equilibrium by this reaction, the shearing-strain at **AB**, and the horizontal elastic forces developed in the same section. Taking the moments of all these forces round the neutral axis of the section **AB**, we have (**58**),

$$\frac{n}{l}\mathsf{W}x = \mathsf{M} \tag{82}$$

When f = the unit-strain in the extreme fibres at top or bottom, c = the distance of the top or bottom from the neutral axis, and we have the following expressions for the strength of each class of girder.

114. Solid rectangular girders.—

Let b = the breadth,

d = the depth.

From eqs. 45 and 82,

$$\mathsf{W} = \frac{fbd^2l}{6nx} \tag{83}$$

If both the weight and cross section are at the centre of the girder, $x = n = \frac{l}{2}$, and

$$\mathsf{W} = \frac{2fbd^2}{3l} \tag{84}$$

115. Solid round girders.—From eqs. 47 and 82,

$$W = \frac{\pi f l r^3}{4nx} \tag{85}$$

in which r = the radius,

If both the weight and cross section are at the centre,

$$W = \frac{\pi f r^3}{l} \tag{86}$$

116. Hollow round girders of uniform thickness.—From eqs. 48 and 82,

$$W = \frac{\pi f l}{4nrx}(r^4 - r_1^4) \tag{87}$$

where r and r_1 represent the external and internal radii.

If both the weight and cross section are at the centre,

$$W = \frac{\pi f}{lr}(r^4 - r_1^4) \tag{88}$$

If the thickness t is inconsiderable compared with the radius, we have from eqs. 49 and 82,

$$W = \frac{\pi f l r^2 t}{nx} \tag{89}$$

If, moreover, the weight and cross section are at the centre,

$$W = \frac{4\pi f r^2 t}{l} \tag{90}$$

117. Flanged girders or rectangular tubes, taking the web into account.—From eqs. 55 and 82,

$$W = \frac{fl}{cnx}\left\{\left(a_1 + \frac{a_3}{3}\right)h_1^2 + \left(a_2 + \frac{a_4}{3}\right)h_2^2\right\} \tag{91}$$

where a_1 = the area of the upper flange,

a_2 = the area of the lower flange,

a_3 = the area of the web above the neutral axis,

a_4 = the area of the web below the neutral axis,

h_1 = the height of the web above the neutral axis,

h_2 = the height of the web below the neutral axis,

f = the unit-strain in fibres whose distance from the neutral axis = c.

If the flanges are equal and f = the unit-strain in either flange, we have from eqs. 56 and 82,

$$W = \frac{fdl}{nx}\left(a + \frac{a'}{6}\right) \tag{92}$$

in which a = the area of either flange,

a' = the area of the web,

d = the depth from centre to centre of flange.

If, moreover, the weight and cross section are at the centre,

$$\mathbf{W} = \frac{4fd}{l}\left(a + \frac{a'}{6}\right) \tag{93}$$

118. Plan of solid rectangular girder of uniform strength, depth constant.—From eq. 83 the unit-strain in the extreme fibres of a solid rectangular girder,

$$f = \frac{6nx\mathbf{W}}{bd^2l}$$

When the strength of the girder is uniform, the quantity $\frac{x}{bd^2}$, to which f is proportional, will be constant. Hence, if the depth d is uniform, b will vary as x and the plan of the girder will be two triangles joined at their bases, Fig. 35.

Fig. 35.—Plan.

119. Elevation of solid rectangular girder of uniform strength, breadth constant.—If, however, the breadth be uniform, d^2 will vary as x, and if the top of the girder is horizontal the bottom will be bounded by two parabolas which intersect underneath the weight, with horizontal axes and their vertices at the extremities of the girder, Fig. 36.

Fig. 36.—Elevation.

120. Solid round girder of uniform strength.—From eq. 85 the unit-strain in the extreme fibres of a solid round girder,

$$f = \frac{4nx\mathbf{W}}{\pi lr^3}$$

If the strength be uniform, r^3 will vary as x, and the girder will be formed by two spindles joined at their base, each spindle being produced by the revolution of a cubic parabola round its axis.

121. Hollow round girder of uniform strength.—From eq. 89 the unit-strain in the extreme fibres of a thin hollow cylinder,

$$f = \frac{nx\mathbf{W}}{\pi l r^2 t}.$$

In a girder of uniform strength the quantity $\frac{x}{r^2 t}$, to which f is proportional, will be constant; hence, if t be uniform, r^2 will vary as x, and the girder will be formed by two hollow spindles joined at their bases, each spindle being generated by the revolution of a parabola round its axis. This, for instance, is the form which the hollow axis of a transit instrument should theoretically have, though a double cone is preferred in practice from its greater facility of construction.

122. Single moving load—Plan of solid rectangular girder of uniform strength, depth constant—Elevation of same, breadth constant.—If **W** be a *single moving load*, the maximum strain at each point will occur as the load passes that point, for x attains its greatest value when it equals m; hence, eq. 83, the unit-strain in the extreme fibres of the section where the weight occurs,

$$f = \frac{6mn\mathbf{W}}{bd^2 l} \qquad (94)$$

If the strength of the girder be uniform, $\frac{mn}{bd^2}$ will be a constant quantity, and if d be uniform, b will vary as the rectangle under the segments; hence the plan of the girder if symmetrical will be bounded by two overlapping parabolas whose vertices are at **A A**, Fig. 37. If, however, the breadth be uniform, d^2 will vary as mn and the elevation of the girder will be a semi-ellipse, Fig. 38.

Fig. 37.—Plan.

Fig. 38.—Elevation.

CASE IV.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

Fig. 39.

123. Let l = the length of the girder,

w = the load per linear unit,

$\mathsf{W} = wl$ = the whole load,

m and n = the segments into which any given cross section **AB** divides the girder,

M = the moment of rupture of the section **AB** (**58**).

The forces which hold **ABC** in equilibrium are the reaction of the right abutment, $= \frac{wl}{2}$, the weights uniformly distributed along **AC** ($= wn$), the shearing-strain at **AB**, and the horizontal elastic forces in the same section. Taking the moments of all these forces round the neutral axis of **AB** we have (**58**),

$$\frac{mnw}{2} = \mathsf{M} \tag{95}$$

Multiplying the left side of the equation by $\frac{l}{l}$, we have

$$\frac{mn\mathsf{W}}{2l} = \mathsf{M} \tag{96}$$

When f = the unit-strain in the extreme fibres of the section, c = their distance from the neutral axis, and we have the following expressions for the strength of each class of girder.

124. Solid rectangular girders.

Let b = the breadth,

d = the depth.

From eqs. 45 and 96,

$$\mathsf{W} = \frac{fbd^2l}{3mn} \tag{97}$$

If the cross section is at the centre, $m = n = \frac{l}{2}$, and

$$W = \frac{4bd^2f}{3l} \qquad (98)$$

125. Solid round girders.—From eqs. 47 and 96,

$$W = \frac{\pi flr^3}{2mn} \qquad (99)$$

in which r = the radius.

If the section is at the centre, $m = n = \frac{l}{2}$, and

$$W = \frac{2\pi fr^3}{l} \qquad (100)$$

126. Hollow round girders of uniform thickness.

Let r = the external radius,

r_1 = the internal radius.

From eqs. 48 and 96,

$$W = \frac{\pi fl}{2mnr}(r^4 - r_1^4) \qquad (101)$$

At the centre of the girder $m = n = \frac{l}{2}$, and

$$W = \frac{2\pi f}{lr}(r^4 - r_1^4) \qquad (102)$$

If the thickness t is inconsiderable in comparison with the radius, we have from eqs. 49 and 96,

$$W = \frac{2\pi fr^2tl}{mn} \qquad (103)$$

If, moreover, the plane of section is at the centre,

$$W = \frac{8\pi fr^2t}{l} \qquad (104)$$

127. Flanged girders or rectangular tubes, taking the web into account.—From eqs. 55 and 96,

$$W = \frac{2fl}{cmn}\left\{\left(a_1 + \frac{a^3}{3}\right)h_1^2 + \left(a_2 + \frac{a^4}{3}\right)h_2^2\right\} \qquad (105)$$

Where a_1 = the area of the upper flange,

a_2 = the area of the lower flange,

a_3 = the area of the web above the neutral axis,

a_4 = the area of the web below the neutral axis,

h_1 = the height of the web above the neutral axis,

h_2 = the height of the web below the neutral axis,

f = the unit-strain in fibres whose distance from the neutral axis = c.

If the flanges are equal and if $f =$ the unit-strain in either flange, $c = \frac{d}{2}$, and we have from eqs. 56 and 96,

$$\mathbf{W} = \frac{2dfl}{mn}\left(a + \frac{a'}{6}\right) \tag{106}$$

in which a = the area of either flange,
a' = the area of web,
d = the depth of the web.

At the centre $m = n = \frac{l}{2}$, and eq. 106 becomes

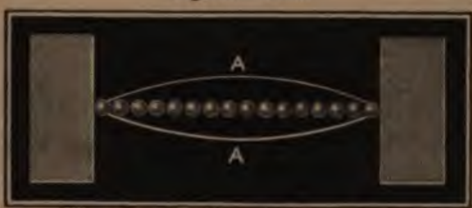
$$\mathbf{W} = \frac{8df}{l}\left(a + \frac{a'}{6}\right) \tag{107}$$

128. Plan of solid rectangular girder of uniform strength, depth constant.—From eq. 97 the unit-strain in the extreme fibres of a solid rectangular girder,

$$f = \frac{3mn\mathbf{W}}{bd^2l}$$

When the strength of the girder is uniform and the material consequently disposed in the most economical manner, the unit-strain f will be uniform (**19**), and the quantity $\frac{mn}{bd^2}$, to which it is proportional, will be constant. Hence, if the depth d be uniform, b will vary as mn, and the plan of the girder if symmetrical will be formed by the overlap of two parabolas whose vertices are at AA, Fig. 40.

Fig. 40.—Plan.



129. Elevation of solid rectangular girder of uniform strength, breadth constant.—If, however, the breadth be uniform, d^2 will vary as mn and the elevation of the girder will be a semi-ellipse, Fig. 41.

Fig. 41.—Elevation.

130. Discrepancy between experiments and theory—Shifting of neutral axis—Longitudinal strength of materials derived from transverse strains erroneous.—The student will naturally conclude that the formulæ investigated in the present and preceding chapters should give identical or nearly identical results when they are applied to the same girder; that, for instance, the breaking weight of a solid rectangular semi-girder, calculated by eq. 32, should closely agree with its breaking weight calculated by eq. 61; and, if our theory were complete, this would no doubt be the case. To test its accuracy let us compare these two equations, when we obtain this result,

$$\mathbf{S} = \frac{f}{6}$$

that is, the value of **S** for solid rectangular girders of any given material should equal one-sixth of the ultimate tearing or crushing strength of that material according as it yields by tearing or crushing. In many instances, however, this will be found to be far from the truth: for example, the value of **S** for small rectangular bars of cast-iron = 3·4 tons, and 6 times this (= 20·4 tons) far exceeds the tensile strength of cast-iron, which is about 7 tons per square inch. It must, indeed, be confessed that our theory is defective, and that the formulæ for *solid* girders investigated in the present chapter give the breaking weight much under what it really is for many materials, and this discrepancy will probably be found more marked in those whose ultimate tearing strain differs widely from their ultimate crushing strain. Confidence, however, may be placed in the formulæ relating to hollow and flanged girders.

Mr. Hodgkinson endeavours to explain this discrepancy by a change in the position of the neutral axis and gives some reasons for this hypothesis derived from his experiments on cast-iron, at page 384 of his *Experimental Researches on the Strength of Cast-iron*. This seems a plausible hypothesis, for if the neutral axis of a solid rectangular cast-iron girder approach its compressed edge as the weight increases, and after the limit of tensile elasticity has been passed by the fibres along the extended edge, we shall have a larger proportion than one-half the girder subject to tension, and consequently the total horizontal tensile strain may exceed that derived from our theory which assumes that the neutral axis always passes through the centre of gravity of the cross section (**66**). Mr. Hodgkinson concludes from his experiments that the neutral axis of a rectangular girder of cast-iron divides the depth in the proportion of $\frac{1}{5}$ or $\frac{1}{6}$ at the time of fracture, that is, the compressed section is to the extended section nearly in the inverse proportion of the compressive to the tensile strength of the material.

The experiments on timber by the elder Barlow, given at page 133 of his treatise *On the Strength of Materials*, also corroborate this view. Mr. W. H. Barlow, however, controverts Mr. Hodgkinson's conclusions in two papers which will be found at page 225 of the Philosophical Transactions for 1855, and at page 463 of the Transactions for 1857. In the former of these papers Mr. Barlow gives the results of micrometrical measurements on two cast-iron rectangular girders, each 7 feet long, 6 inches deep and 2 inches thick, which he subjected to transverse strain; his inference from these experiments is that the neutral axis does not shift its position, and this view is in accordance with experiments made long ago by Sir D. Brewster who transmitted polarized light through a little rectangular glass girder 6 inches long, 1·5 inch broad and 0·28 inch thick; when this was bent by transverse pressure the neutral surface remained in the centre and colours due to strain were developed above and below it in curved lines, which may at some future period aid the physicist in investigating the strains in continuous webs.* Unless, however, the tensile and compressive elasticities of glass are materially different, near the point of rupture, as they are in cast-iron when approaching its limit of tearing (**402**), this experiment does not throw much light on the subject. The whole question, it must be confessed, is one of great difficulty, and may require numerous experiments before it can be satisfactorily solved. One practical inference, however, is of great importance, namely, that *the tearing and crushing strengths of materials derived from experiments on the transverse strength of solid girders are often erroneous*, and have even led astray men of such capacity as Tredgold.

* See Encycl. Metrop. Art. Light, par. 1090.

CHAPTER V

SIMPLE FORMS OF BRACING.

131. Object of bracing.—The primary object of bracing is to convert transverse strains into others which act in the direction of the length of the material employed and tend either to shorten or extend it, according as the material performs the function of a strut or tie.

This object is attained by dividing the structure into one or more triangles; for since the triangle or some modification of it is the only geometric figure which possesses the property of preserving its form unaltered so long as the lengths of its sides remain constant, it is therefore that which is best adapted for structures in which rigidity is essential for stability. Hence three pieces at least are required to constitute a braced structure. Take, for instance, the common roof truss which is an example of one of the simplest forms of bracing (Fig. 42). The weight **W** is transmitted through a pair of struts **S** and **S′**, to the walls. As, however, the oblique thrust of the struts would overthrow the walls it is necessary to connect their feet by a tie-beam **T**. The strains in the different parts may be derived from the principle enunciated in (**9**).

Fig. 42.

132. Derrick crane.—The derrick crane, Fig. 43, consists of a revolving post **P**, a jib **J**, a chain or tie-bar **T**, and two back-sta[illegible] one of which is shown at **B**, the other, lying in a plane at right ang[illegible] to that of the figure, is not represented, being hidden by the p[illegible] The derrick crane is generally made of wood. It is simple in c[illegible] struction and easily erected. Hence it is well adapted for tempor[illegible] works, as also for quarries or other situations where the back stays do not interfere with the traffic. At the peak **A** three forces meet, viz., the downward pull of **W**, the tension of the tie-bar **T** and the oblique thrust of the jib **J**. Since these three forces are in equilibrium their relative amounts may be represented by the sides of the triangle **PTJ** (**9**).

Fig. 43.

Hence the tension of the tie-bar $= \frac{T}{P}W$ and the compression of the jib $= \frac{J}{P}W$.

If the chain pass along **T**, and so over a pulley at *b* down to the chain barrel bolted to the foot of the post, it relieves the tie-bar of an amount of tension equal to that in the chain, namely, **W** divided by the number of falls in the hanging part of the chain.*

If, however, the chain pass along the jib, the compression of the latter is increased by an amount equal to the tension of the chain. The tension in **T** being known, the strains in the post and back-stays, which are its components, are easily found. This operation is most conveniently performed by the aid of a skeleton diagram (Fig. 43) drawn accurately to scale. Let the jib and one back-stay lie in the same plane. Lay off *bc* by scale to represent the tension in $T\left(=\frac{T}{P}W\right)$ and draw *cd* parallel to **B**; then *cd*, measured by the same scale, will represent the tension in the back-stay, and *bd* the compression of the post. In this case the second back-stay is free from strain, but when the jib does not lie in the same plane with

* This is not accurately true, for the friction of the blocks, pulley, &c., increases or diminishes the tension of the chain according as the weight happens to be raised or lowered.

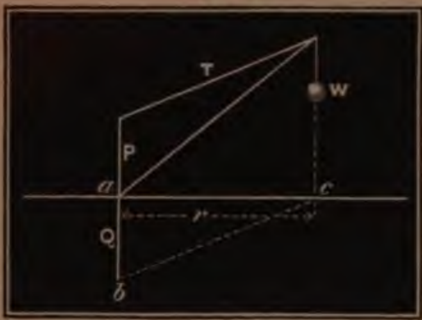
either back-stay, both back-stays are subject to strain; to a less degree, however, than in the case already considered as will appear from the following considerations. Let Fig. 44 represent a plan of the crane, *bh* and *bk* being the horizontal projections of the back-stays, and *b*A that of the tie-bar and jib; let *be* represent the horizontal component of the tension in the tie-bar (equal *ce* in Fig. 43), then *bf* and *bg* will represent the horizontal components of the strains in the back-stays, and hence the strains in the back-stays can be found. It is obvious, however, that either *bf* or *bg* will attain its maximum when the tie-bar lies in the same plane with one of the back-stays. Hence the former case, in which the jib and one back-stay lie in the same plane, is sufficient for us to consider when calculating the requisite strength of the stays.

Fig. 44.

The strain in the post attains its greatest value when the plane of the tie-bars and jib bisects the angle between the back-stays, for then the sum of *bf* and *bg* is maximum, and consequently the sum of the vertical components of the strains in the stays is maximum also. But the strain transmitted through the post is equal to the sum of these vertical components + or — the vertical component of the tension in the tie-bar, according as the latter slopes downwards or upwards from the head of the post. The back-stays act sometimes as struts, sometimes as ties, and when the jib is swung round so as to lie alongside one of the back-stays, the latter will sustain its maximum compression equal to the maximum tension produced when the jib and stay are in the same plane. The radius of the circle described by the jib, or the range of the derrick, is generally capable of adjustment by lengthening or shortening the tie-bar, which is then a chain attached to a small auxiliary crab-winch fastened to the post near the working barrel, in which case the working chain passes along the jib. This form of derrick is convenient for setting masonry as its range is equal to a circle described by the jib when nearly horizontal, in which position moreover the crane is most severely strained.

133. Wharf crane.—The wharf crane, unlike the derrick crane, has no back-stays. Consequently the post is subject to transverse strain from the oblique pull of the tie-bar; it is in fact a semi-girder fixed in the ground and loaded at the extremity. The strains in the tie-bar and jib are calculated in the same way as for the derrick crane. The moment of rupture (**58**) of the post attains its greatest value at its intersection with the ground and equals the horizontal component of the tension in **T** multiplied by the height of the post above ground. It may, however, be more conveniently found as follows:—

Fig. 45.



The whole crane above ac (the ground line) is a bent semi-girder held in equilibrium by the weight and the elastic forces at a (in this case vertical). Taking moments round either the centre of tension or the centre of compression at a (**56**), we have the moment of rupture $= \mathbf{W}r$, where $r =$ the radius of the circle described by the jib. From this it follows, that the transverse strain at a is not affected by increasing the height of the post, which, however, diminishes the strains in the jib and the tie-bar, and is so far attended with advantage; neither is it affected by raising or lowering the peak of the jib in the same vertical line. It also follows that the strain on the post is increased when the weight is farther out than the circle described by the jib, for the leverage of **W** is then increased and attains its greatest value when the chain is at right angles to the jib. If the post be fixed in the ground the frame, to which the jib, tie-bar and wheelwork are attached, is generally suspended by a cross head from the top of the post which forms a pivot round which the cross-head turns. In this form of crane the weight is transmitted from the pivot through the whole length of the post in addition to the longitudinal strains to which as a semi-girder it is liable, and the section of the post should theoretically be circular (**98**), since it may be equally strained in all directions.* When the post revolves on its axis, the jib and wheelwork are bolted to it and all move together on a pivot at the toe-plate *b*. In this case the post should be double-flanged. The underground portion is subject to a vertical compression equal to the weight (viz., the difference of the vertical components of the strains in the jib and tie-bar) in addition to the longitudinal strain derived from its acting as a semi-girder. When the post moves round its axis friction rollers may be advantageously placed between it and a curb plate which is secured to the masonry at *a*.

To find the amount and direction of the pressure at the toe, join *b* with a point *c* vertically beneath **W**. The whole structure is balanced by three forces, viz., the weight **W**, the horizontal pressure against the curb plate at *a*, and the pressure on the toe at *b*. The two former forces pass through *c*; consequently the latter intersects them at the same point (**9**). Hence the sides of the triangle *abc* represent the relative amounts of these forces, and we have the horizontal component of the oblique pressure at *b* equal $\frac{r}{\mathbf{Q}}\mathbf{W}$. The vertical components equals **W**, which is otherwise evident.†

134. The A roof.—In the common **A** roof, the span of which seldom exceeds 40 feet, each pair of rafters is kept from exerting a lateral thrust against the wall by a tie-beam which is often placed a few feet above the wall-plate for the sake of the head-room which this arrangement allows. Consequently each pair of rafters with

Fig. 46.

* Square tubular posts of boiler plate with angle iron at the corners form very efficient fixed posts for small cranes not exceeding four or five tons.

† The reader is referred for practical information on the subject of cranes to the rudimentary treatise on the *Art of Constructing Cranes* by J. Glynn, F.R.S., C.E.

their tie-beam constitutes a simple truss which supports so much of the roof as lies between two adjacent pairs of rafters.

Let W = the weight uniformly distributed over each pair of rafters,

l = the span of the roof,

l' = the length of each rafter,

d = the height of the ridge above the tie-beam, *i.e.*, the depth of the truss,

h = the height of the ridge above the wall-plates,

T = the tension in the tie-beam.

Each rafter is held in equilibrium by the uniformly distributed weight of the roof (equivalent to $\frac{\mathsf{W}}{2}$ acting downwards at the middle of the rafter), the upward reaction of the wall-plate $\left(=\frac{\mathsf{W}}{2}\right)$, the horizontal thrust of the opposite rafter at the ridge and the horizontal tension of the tie-beam. Taking the moments of these forces round the ridge we have

$$\frac{\mathsf{W}}{2} \times \frac{l}{2} - \frac{\mathsf{W}}{2} \times \frac{l}{4} = \mathsf{T}d$$

whence

$$\mathsf{T} = \frac{l\mathsf{W}}{8d}$$

By taking moments round the foot of the rafter it may be shown that the horizontal thrust of the rafters against each other at the ridge = T. This investigation of the horizontal strains in a simple trussed girder is, it will be perceived, merely a repetition of that given in **43** (eq. 26). Each rafter is subject to transverse strains as a girder and to longitudinal compression as a pillar. The transverse strains are produced by the components of W and of T at right angles to the rafter. The former $= \frac{l\mathsf{W}}{4l'}$ distributed uniformly. The latter $= \frac{h}{l'}\mathsf{T} = \frac{hl\mathsf{W}}{8dl'}$ applied at the intersection of the rafter and tie-beam. Hence the transverse strength of the rafter may be calculated by eqs. 83 and 98, or perhaps more conveniently by eqs. 36 and 40. The longitudinal component of W compresses the rafter like a pillar and accumulates gradually from the ridge, where it equals cipher, to the wall-plate, where it equals $\frac{h\mathsf{W}}{2l'}$. The longitudinal component of $\mathsf{T} = \frac{l\mathsf{T}}{2l'} = \frac{l^2\mathsf{W}}{16dl'}$. It compresses that part of the rafter which lies between the ridge and tie-beam and is balanced by the longitudinal component of the thrust of the opposite rafter at the ridge. When the tie-beam is placed high for the sake of room beneath, d is shortened and T increased in the same proportion. The transverse strain and deflection of the rafter is, however, increased in a higher ratio, for not only is the component of T at right angles to the rafter increased, but its moment of rupture also, in consequence of its acting nearer to the centre of the rafter and farther from the wall-plate which acts the part of an abutment. When rafters are in danger of sagging from their great length, a horizontal *collar-beam* is attached midway between the ridge and the tie-beam. This collar-beam resists the tendency of the rafters to approach each other and is subject to compression, in which case each rafter is a continuous girder supported at both ends and at the collar-beam, and subject to a transverse pressure from the roofing material equal to $\frac{l\mathsf{W}}{4l'}$ distributed uniformly. If the tie-beam connect the feet, and the collar-beam the centres of each pair of rafters, $\frac{5}{8}$ths of this pressure is sustained by the collar-beam, the remaining $\frac{3}{8}$ths being supported by the thrust of the opposite rafter and the reaction of the wall-plate (eq. 167). Hence $\frac{5l\mathsf{W}}{32l'}$ is the pressure (measured at right angles to the rafter) against the collar-beam; resolving this horizontally we have the longitudinal compression of the collar-beam $= \frac{5l\mathsf{W}}{32h}$. A collar-beam increases the tension of the tie-beam. This tension may be found when the strain in the collar-beam is known by taking moments round the ridge.

The foregoing investigation is only an approximation to the truth. The longitudinal strains produced in the rafter by the forces acting at its ends will modify the longitudinal strains due to the transverse forces, and an accurate investigation would be very complicated if not altogether impracticable, for we cannot say how much of these longitudinal strains pass through the tension fibres or lower side of the rafter, and how much pass through its compression fibres or upper side. If there be any tendency in the rafter to sag, the probability is that they will pass altogether through the compression fibres, and therefore the topside of the rafter should be strong enough to sustain the longitudinal strains produced by the end forces in addition to the longitudinal strain due to the transverse components of the load and tie-beam; but in general it is unnecessary to take these longitudinal compression strains into consideration, for when rafters fail they commonly give way on the under side which is in tension. Of course, if the sag be very considerable so that a line joining the ridge and wall plate passes above the rafter, the longitudinal compression will increase the strain in the tension flange in proportion to the versine of the deflection.

CHAPTER VI.

GIRDERS WITH PARALLEL FLANGES AND WEBS FORMED OF ISOSCELES BRACING.

135. Isosceles bracing.—The class of girders which I purpose investigating in this chapter is that in which the flanges are parallel and connected by diagonals which form one or more systems of *isosceles* triangles. This class of bracing includes girders whose web consists of a single system of triangles, such as "Warren's" girder, as well as girders whose web consists of two or more systems of equal-sided triangles, such as isosceles "lattice girders."

Definitions.

136. Brace.—The term *Brace* includes both struts and ties.

137. Apex.—The intersection of a brace with either flange is called an *Apex*.

138. Bay.—The portion of a flange between two adjacent apices is called a *Bay*.

139. Counterbraced brace.—A brace is said to be *counterbraced* when it is capable of acting either as a strut or as a tie.

140. Counterbraced girder.—A girder is said to be *counterbraced* when it is rendered capable of supporting a moving load. This may be effected either by counterbracing the existing braces or by adding others.

141. Symbols.—The symbol + placed before a number which represents a strain signifies that the strain is compressive; the symbol — signifies that the strain is tensile.

Axioms.

142. *The strain in each brace or bay is uniform throughout its length and acts in the direction of the length only.* This will be obvious if we consider a braced girder to be an assemblage or framework of straight bars connected with each other by pins passing through their extremities merely.

143. *A brace cannot undergo tension and compression simultaneously.*

144. *If several weights, acting one at a time, produce in any brace strains of the same kind, either all tensile or all compressive, the strain produced by all these weights acting together will equal in amount the sum of those produced by each weight acting separately.*

145. *If several weights, acting one at a time, produce in any brace strains of different kinds, some tensile, some compressive, the strain resulting from all these weights acting together will equal the algebraic sum of all the strains;* in other words, *their resultant will equal the difference between the sum of the tensile and the sum of the compressive strains.*

146. *A uniformly distributed load may without sensible error be assumed to be grouped into weights resting on the apices, each apex supporting a weight equal to the load resting on the adjoining half bays.* This view is evidently correct if each bay be connected with the adjoining bays and diagonals by a single pin at their intersection, as in "Warren's" girder. In this case each loaded bay is a short girder covered by a uniform load, the vertical pressure of which is transferred to the adjoining diagonals. In addition to the transverse strain each bay sustains a longitudinal strain which it transmits to the adjacent bays, from which, however, it derives no aid to its transverse strength on the principle of continuity. In practice, the cross girders on which the flooring rest generally occur at the apices, so that no bay is subject to transverse strain except from its own weight.

CASE I.—SEMI-GIRDERS LOADED AT THE EXTREMITY.

Fig. 47.

147. Web.—Let W = the load at the extremity of the girder,
Σ = the strain in any diagonal,
F = the strain in any given bay of either flange,
n = the number of diagonals between the centre of the given bay and the weight,
θ = the angle which the diagonals make with a vertical line.

The weight W is supported by the first diagonal and the upper flange, the former sustaining compression, the latter tension. At a three forces meet and balance, namely, the weight, the horizontal tension of the upper flange and the oblique thrust of the diagonal; their relative amounts may therefore be represented by the sides of the triangle abc (**9**). Hence the tension in the first bay of the upper flange is to W as ac is to cb, that is, $\mathsf{F} = \mathsf{W}tan\theta$, and the compression in the first diagonal is to W as ab is to cb, that is, $\Sigma = \mathsf{W}sec\theta$. The tension of ad is transmitted throughout the upper flange to its connexion with the abutment, but the compression in diagonal 1 is resolved at b into its components in the directions of diagonal 2 and the lower flange, producing tension in the former and compression in the latter. Thus there are three forces in equilibrium meeting at b, and their relative amounts may be represented to the same scale as before by the sides of the triangle edb; whence the tension in diagonal 2 equals the compression in diagonal 1, and the compression in the first bay of the lower flange equals twice the tension in the first bay of the upper flange $= 2\ \mathsf{W}tan\theta$.

In this way it may be shown that all the diagonals are strained equally, but by forces alternately tensile and compressive (**18**), while the flanges receive at each apex equal increments of strain each equal to $2\mathsf{W}tan\theta$. The general formula for the strain in any diagonal is therefore

$$\Sigma = \mathsf{W}sec\theta \qquad (108)$$

Ex. 1. If $\theta = 45°$, $sec\theta = 1{\cdot}414$, and we have $\Sigma = 1{\cdot}414\,\mathsf{W}$.*

148. Flanges.—Since the flanges receive at each apex successive increments of strain, each equal to $2\mathsf{W}tan\theta$, the resultant strains in the successive bays, being the sum of these successive increments, increase as they approach the abutment in an arithmetic progression whose difference $= 2\mathsf{W}tan\theta$; they are therefore for any given bay proportional to the number of diagonals between it and the load. Hence

$$\mathsf{F} = n\mathsf{W}tan\theta \qquad (109)$$

where n represents the number of diagonals between the centre of any given bay and the weight.

Ex. 2. In the last bay of the upper flange $n = 7$, and if $\theta = 45°$, $tan\theta = 1$, and we have $\mathsf{F} = 7\,\mathsf{W}$.

The tension in the last diagonal may be resolved at g into a vertical force pressing downwards through the abutment and a horizontal force tending to pull the abutment towards the weight. The relative amounts of these three forces may be represented by the sides of the triangle fgh; whence the vertical pressure $= \mathsf{W}$ and the horizontal force $= \mathsf{W}tan\theta$; the latter added to the tension in the last bay of the upper flange gives the total horizontal force exerted by the upper flange to pull the abutment towards W. It will be observed that the horizontal thrust of the lower flange against the abutment is equal and opposite to the pull of the upper flange, so that they form a couple whose tendency is to overturn the abutment on its lower edge next the weight.

149. Strains in braced webs may be deduced from shearing-strain.—When the flanges are parallel and the bracing consists of a single system of triangulation, the strain in any brace is equal to

* See the table in **276** for the numerical values of the tangents and secants of different angles.

the shearing-strain multiplied by *sec*θ. Hence the strains in the bracing might be deduced from the shearing-strain in the web calculated in the manner explained in Chap. II. (**14, 18**). The method of the resolution of forces just described is, however, better calculated to give a correct perception of the properties of diagonal bracing, and it has moreover the advantage of being applicable to lattice girders as well as those whose bracing consists of a single system of triangles.

CASE II.—SEMI-GIRDERS LOADED UNIFORMLY.

Fig. 48.

150. Web.—Let W = the weight of so much of the load as covers one bay, *i.e.*, the weight resting on each apex of the loaded flange (**146**),

n = the number of these weights between any given diagonal and the outer end of the girder,

Σ = the strain in the given diagonal,

F = the strain in any bay of either flange,

θ = the angle which the diagonals make with a vertical line.

The weight on the apex farthest from the abutment equals $\frac{W}{2}$, since it is assumed to support the load spread over the outer half bay, while the load spread over the half bay next the abutment is assumed to rest on the apex in contact with it and may therefore be neglected. If each weight be supposed acting alone, it would, as in Case I., produce strains of equal amount, but of opposite kinds, in each diagonal between its point of application and the abutment without affecting that part of the girder which lies outside it; consequently, when the whole load is applied, each diagonal sustains the sum of the strains produced by the several weights which occur between it and the outer end of the girder (**23, 144**) and we have

$$\Sigma = n\mathbf{W}sec\theta \qquad (110)$$

Ex. 1. The value of n for diagonal 5 is $2\frac{1}{2}$; if $\theta = 45^\circ$, $sec\,\theta = 1{\cdot}414$, and we have $\Sigma = 3{\cdot}535\,\mathbf{W}$.

151. Strains in intersecting diagonals.—When the apex of any pair of diagonals supports a weight **W** the strain in that diagonal which is nearer the abutment exceeds that in the more remote by $\mathbf{W}sec\theta$. But when an apex does not support a weight (those, for instance, in the lower flange of Fig. 48), the strains in the two diagonals are equal in amount but of opposite kinds.

152. Increments of strain in flanges.—In the case of semi-girders loaded uniformly the *increments* of strain at the apices increase as they approach the wall in an arithmetic ratio whose difference $= 2\mathbf{W}tan\theta$, and the resultant strains in each bay consequently increase in a much more rapid ratio, viz., as the square of their distance from the outer end of the girder (eq. 11).

153. Resultant strains in flanges.—The resultant strains in the bays may be represented by equations if desirable. For the loaded flange

$$\mathsf{F} = \left\{m(m-1) + \tfrac{1}{2}\right\}\mathbf{W}tan\theta \qquad (111)$$

For the unloaded flange

$$\mathsf{F} = m^2\,\mathbf{W}tan\theta \qquad (112)$$

where m represents the number of the bay measured along its own flange from the outer end of the girder. These equations are obtained by summation; their proof will afford instructive practice to the student.

154. General law of strains in horizontal flanges of braced girders.—The strains in the flanges may also be derived from the following law which is applicable to all braced girders or semi-girders with horizontal flanges, no matter how loaded, or whether the bracing be isosceles, or the triangulation be single or lattice. *The increment of strain developed in the flange at any apex is equal to the algebraic sum (i.e., the resultant) of the horizontal components of the strains in the intersecting diagonals.* Keeping this in our recollection we may readily exhibit on a rough diagram—first, the strains in the diagonals; secondly, their horizontal components at the apices; and lastly, the successive sums of these components, that is, the total strains in the several bays of each flange.

Ex. 2. Let Fig. 49 represent such a diagram, the load being on the upper flange.

$$\text{Let } W = 10 \text{ tons},$$
$$\theta = 30^\circ,$$
$$Sec\,\theta = 1{\cdot}154,$$
$$Tan\,\theta = 0{\cdot}577.$$

Fig. 49.

The horizontal numbers attached to the diagonals are the coefficients *n* in eq. 110; these multiplied by $W sec\theta$ give the strains in each diagonal (see the numbers written alongside). The horizontal numbers at each apex are obtained by adding the coefficients of the two intersecting diagonals, and when multiplied by $W tan\theta$ give the horizontal components of the strains in the diagonals, *i.e.*, the increments of flange-strain at each apex (see the vertical numbers at each apex). Finally, the successive additions of these increments give the resultant strains in each bay (see the vertical numbers at the centre of each bay). These may be checked by eqs. 111 and 112; thus in the 3rd bay of the upper flange $F = (3 \times 2 + \frac{1}{2}) \times 10 \times {\cdot}577 = 37{\cdot}5$ tons, which differs merely in the decimals from the number obtained by the diagram.

155. Lattice web—No theoretic advantage over single system—Practical advantage of lattice web—Long pillars.— If two or more systems of triangulation be substituted for the single system just described, we have a lattice girder; and here I may remark that lattice bracing has no theoretic advantage over a single system of triangulation; its advantages are entirely of a practical nature, consisting in the frequent support which the tension diagonals give to those in compression, and which both afford the flanges. Long pillars are serious practical difficulties owing to their tendency

Fig. 50.

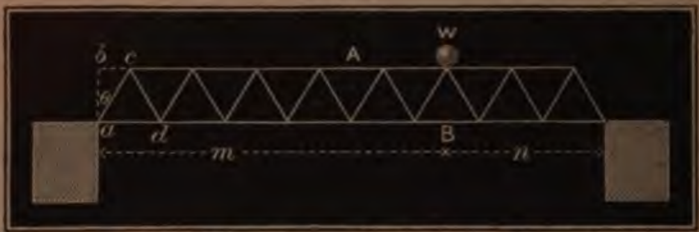
to flexure, and lattice tension bars subdivide the struts, which would otherwise be long unsupported pillars, into a series of shorter pillars and hold them in the direction of the line of thrust. That this does not injuriously affect the tension diagonals will be evident, when we reflect that the longitudinal strain produced in a tension diagonal by the deflection of a strut crossing it at right angles *in the plane of the girder* bears the same ratio to the weight on the strut, as twice the versine of the deflection curve bears to the length of the half strut—an amount quite inappreciable in practice. If, for instance, a strut *adc*, Fig. 50, be ten feet long, and if its central deflection under strain, *bd*, equal half-an-inch (an amount much greater than ever occurs in practice), the transverse force in the direction of *bd*, which will sustain the thrust due to deflection, is to the longitudinal pressure as $\frac{2bd}{dc}$, that is, it is only $\frac{1}{60}$th of the weight passing through the pillar; so that in most cases a stout wire in tension would be sufficiently strong to keep the pillar from deflecting in the plane of the girder. Again, if the force requisite to resist the tendency of a strut to deflect *at right angles to the plane of the girder* were supplied altogether by a tension brace the longitudinal strain in that brace would equal the weight on the strut, but it does not follow that this strain is developed in the tension brace. In fact, the force with which the ends of the tension brace are pulled asunder is practically independent of the strut, for the increase in the strain on the tension brace is only due to the difference between the lengths *bc* and *dc*. These considerations show that a moderate force will keep a pillar from bending, and the apprehension of long compression bars yielding by flexure in the plane of the girder, or producing undue strains in the tension bars, need not deter us from applying lattice bracing to girders exceeding in length any girder bridge hitherto constructed. They also explain the otherwise anomalous strength and rigidity of plate girders and lattice girders whose webs are formed merely of thin plates or thin bars. Such modes of construction are, however, more or less defective. The struts should be formed of angle, T, or channel iron, or the material should be thrown into some other form than that of a thin bar, which is quite unsuitable for resisting flexure at right angles to the plane of the web. A very effective method of stiffening thin compression bars has been applied to tubular lattice girders. It consists of a species of light internal cross-bracing between the opposite compression bars of the double web; this stiffens them at right angles to the plane of the web, while the tension braces keep them from deflecting in the plane of the web (**337**).

156. Multiple and single triangulation compared—Lattice semi-girders loaded uniformly.—The effect of latticing compared with a single system of triangulation is, as far as theory is concerned, merely to distribute the load over a greater number of apices and consequently to reduce the strain in each of the original diagonals in proportion to the increased number of systems; for, since the several systems are, as we have just seen, practically independent of each other, each diagonal sustains the strain due to those weights alone which are supported on the apices of the system to which it belongs. Eq. 110 will therefore give the strain in any brace of a lattice semi-girder loaded uniformly, observing that the coefficient n will now express the number of those weights alone which are supported by that system to which the brace in question belongs, and which occur between it and the outer end of the semi-girder. The strains in the flanges of a lattice semi-girder increase less abruptly than when one system of triangulation is adopted, and are most conveniently obtained by a diagram similar to Fig. 49.

157. Girder balanced on a pier.—The case of a girder balanced midway on a pier is obviously included in the preceding cases, since each segment is a semi-girder.

CASE III.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED AT AN INTERMEDIATE POINT.

Fig. 51.



158. Web.—Let $\mathbf{W}$ = the weight dividing the girder into segments containing respectively m and n bays,

$l = m + n$ = the number of bays in the span,

Σ = the strain in any diagonal,

F = the strain in any bay of either flange,

θ = the angle which the diagonals make with a vertical line,

x = the number of diagonals between any bay and either abutment, measured from the centre of the bay.

On the principle of the lever (**10**), the reaction of the right abutment $= \frac{m}{l}\mathbf{W}$; that of the left abutment $= \frac{n}{l}\mathbf{W}$. Since the flanges are capable of transmitting strains in the direction of their length only (**142**), they cannot transfer vertical pressures to the abutments; $\frac{m}{l}\mathbf{W}$ must therefore be transmitted through the diagonals on the right side of $\mathbf{W}$ to the right abutment, while $\frac{n}{l}\mathbf{W}$ pass through the diagonals on the left side of $\mathbf{W}$ to the left abutment. These quantities are in fact the shearing-strains described in **34**, that is, they are the vertical components of the strains in the diagonals of each segment. The actual strain in any diagonal is to its vertical component as the length of the diagonal is to the depth of the girder, or, calling the angle of inclination of a diagonal to a vertical line θ, we have the strain in each diagonal in the right segment

$$\Sigma = \frac{m}{l}\mathbf{W}sec\theta \qquad (113)$$

in the left segment

$$\Sigma = \frac{n}{l}\mathbf{W}sec\theta \qquad (114)$$

The diagonals which intersect at the weight are both subject to the same kind of strain, while the strains in the diagonals of each segment are alternately tensile and compressive. If the weight be at the centre of the girder all the diagonals will be equally strained.

159. Flanges.—The tensile strain in the second diagonal *cd* is resolved at *c* into its components in the directions of the top flange and the first diagonal. The former $= \frac{2n}{l}\mathbf{W}tan\theta$ and is transmitted throughout the flange as far as **W**, receiving at the intervening apices successive increments of strain each equal to $\frac{2n}{l}\mathbf{W}tan\theta$. At **W** these horizontal strains are met and balanced by a similar series of horizontal increments developed at each apex to the right of **W** and acting in the opposite direction to the first series. The strains in the lower flange may be found in a similar manner, for the thrust of the first diagonal *ac* is resolved at *a* into a vertical pressure on the abutment $\left(= \frac{n}{l}\mathbf{W}\right)$ and a horizontal tensile strain in the lower flange which acts as a tie. As these three forces which meet at *a* balance, their relative amounts may be represented by the sides of the dotted triangle *abc*; hence the horizontal strain in the first bay of the lower flange $= \frac{n}{l}\mathbf{W}tan\theta$ which is transmitted throughout the flange as far as the bay underneath **W**, receiving at each intervening apex successive increments each equal to $\frac{2n}{l}\mathbf{W}tan\theta$. Beneath **W** these strains are met and balanced by the reverse series generated at the several apices in the right segment.

The resultant strain in any bay of either flange equals the sum of the increments generated at the several apices between it and the abutment of the segment in which it occurs. If the bay be in the right segment and x be measured from the right abutment,

$$F = \frac{mx}{l} W tan\theta \qquad (115)$$

If the bay be in the left segment and x be measured from the left abutment,

$$F = \frac{nx}{l} W tan\theta \qquad (116)$$

The maximum strains in the flanges occur at **W** and are represented by the equation

$$F = \frac{2mn}{l} W tan\theta \qquad (117)$$

160. Ex.—In Fig. 51,

Let $\theta = 30^\circ$,
$l = 8$,
$m = 5{\cdot}5$,
$n = 2{\cdot}5$,
$Sec\theta = 1{\cdot}154$,
$Tan\theta = 0{\cdot}577$.

From eqs. 113 and 114 the strains in each diagonal of the right segment = 0·7934 W, and those in each diagonal of the left segment = 0·3606 W. From eq. 116 the compressive strain in bay A = 1·4425 W, and the tensile strain in bay B = 1·9834 W.

161. Single moving load.—If the load move, the strains in the diagonals will vary according to its position, changing from tension to compression and *vice versâ*, as it passes each apex (**38**). If the upper flange supports the load the maximum compression in any diagonal occurs when the weight is passing its upper extremity, and the maximum tension when passing the adjoining apex at that side to which the diagonal slopes downwards. If the lower flange supports the load, the maximum tensile strain in any diagonal occurs when the weight is passing its lower end, and the maximum compressive strain when passing the adjoining apex on that side to which the diagonal slopes upwards. The maximum strain in any bay of the unloaded flange occurs when the moving load is in the vertical line passing through that bay, as may be seen from eq. 115 or 116, for mx and nx are at their maximum when they become mn (**32**). The maximum strain in any bay of the loaded flange occurs when the passing load rests on the adjoining apex on the side next the centre, for the product mn in eq. 117 is greater for this apex than for the adjoining apex on the side remote from the centre.

162. Lattice girder traversed by a single load.—In this case the strains in the diagonals may be calculated by eqs. 113 and 114, for the reasoning by which these equations were deduced is equally applicable to lattice girders. It will also be observed that only one system of triangulation is strained at a time, *i.e.*, supposing the load to rest on a single apex, which however is seldom the case, as generally two or more adjacent apices are loaded together.

CASE IV.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

Fig. 52.

163. Web.—Let W = the weight of so much of the load as covers one bay, *i.e.*, the weight resting on each apex of the loaded flange,

l = the number of bays in the span,

n = the number of weights between any given diagonal and the centre of the girder,

Σ = the strain in the given diagonal,

F = the strain in any bay of either flange,

θ = the angle which the diagonals make with a vertical line.

If the load be uniformly distributed so that an equal weight rests upon each apex, the strains in the diagonals gradually increase from the centre toward the ends. Any two diagonals equally distant from the centre sustain all the intermediate load. If they are tension diagonals the weight is suspended as it were between them; if they are compression diagonals it is supported by them as oblique props. Each diagonal conveys therefore to the abutment the pressure of the weights between it and the centre, and the sum of these weights constitutes its vertical component or shearing-strain (**46**). Hence we have for a uniform load

$$\Sigma = n\text{W}\sec\theta \qquad (118)$$

164. Flanges.—The strains in the flanges may be derived from the law stated in **154** by the aid of a rough diagram, as explained in the following example:—

Ex. 1. Let Fig. 53 represent one-half of a girder 80 feet long, with the bracing formed of 8 equilateral triangles and supporting a uniform load of half a ton per running foot. From these data we have

$$\text{W} = 5 \text{ tons},$$
$$\theta = 30^\circ,$$
$$l = 8,$$
$$Tan\theta = 0{\cdot}577$$
$$Sec\theta = 1{\cdot}154,$$
$$\text{W}\,tan\theta = 2{\cdot}885 \text{ tons},$$
$$\text{W}\,sec\theta = 5{\cdot}770 \text{ tons}.$$

Fig. 53.

The horizontal numbers attached to the diagonals are the coefficients n in eq. 118; these multiplied by W$sec\theta$ give the strains in the several diagonals (see the numbers written alongside them). The horizontal numbers at each apex are the sums of the coefficients attached to the intersecting diagonals; these multiplied by W$tan\theta$ give the horizontal components of the strains in the diagonals, that is, the increments of flange-strain at each apex (see the numbers written in a vertical direction at each apex). Finally, the successive additions of these increments give the resultant strains in the flanges (see the numbers written in a vertical direction at the centre of each bay).

165. Ex 2. Let Fig. 52 represent a girder 80 feet long, with the bracing formed of 8 right-angled triangles, and supporting a uniform load of half a ton per running foot.

$$\text{Here,}\quad W = 5 \text{ tons},$$
$$\theta = 45^\circ,$$
$$l = 8,$$
$$Tan\theta = 1,$$
$$Sec\theta = 1{\cdot}414,$$
$$W tan\theta = 5 \text{ tons},$$
$$W sec\theta = 7{\cdot}07 \text{ tons}.$$

The strains in tons will be as follows :—

Diagonals, . . .	1	2	3	4	5	6	7	8
Strains in tons (eq. 118),	−24·7	+24·7	−17·7	+17·7	−10·6	+10·6	− 3·5	+ 3·5
Flanges, . . .	A	B	C	D	E	F	G	H
Strains in tons, . .	+17·5	+47·5	+67·5	+77·5	− 35	− 60	− 75	− 80

166. Web, second method.—The strains in the diagonals may also be obtained by forming a table of the strains which each weight would produce if acting separately, and then taking as the resultant strain from all acting together the sum or difference of the tabulated strains according as they are of the same or opposite kinds. Thus, diagonal 4, Fig. 52, is subject to compressive strains from all the weights except the first; the resultant strain is therefore found by subtracting the tensile strain produced by the first weight from the sum of the compressive strains produced by the remaining six weights (**145**). This method, as applied to the example in **164**, is exhibited in the annexed table, the numerals in the first column of which represents the diagonals, and the letters in the upper row the weights, in order of position. The numbers found at the intersection of a diagonal with a weight represent in tons the strain produced in that diagonal by the weight in question (see eq. 113). The last column contains the strains which the load produces when distributed uniformly all over. These are obtained by adding algebraically the several horizontal rows, and the strains thus obtained should agree with those derived from eq. 118.

Diagonals.	W_1	W_2	W_3	W_4	W_5	W_6	W_7	Strains due to a uniform load.
	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.
1	− 5·1	− 4·3	− 3·6	− 2·9	− 2·2	− 1·4	− ·72	− 20·2
2	+ 5·1	+ 4·3	+ 3·6	+ 2·9	+ 2·2	+ 1·4	+ ·72	+ 20·2
3	+ 0·7	− 4·3	− 3·6	− 2·9	− 2·2	− 1·4	− ·72	− 14·4
4	− 0·7	+ 4·3	+ 3·6	+ 2·9	+ 2·2	+ 1·4	+ ·72	+ 14·4
5	+ 0·7	+ 1·4	− 3·6	− 2·9	− 2·2	− 1·4	− ·72	− 8·7
6	− 0·7	− 1·4	+ 3·6	+ 2·9	+ 2·2	+ 1·4	+ ·72	+ 8·7
7	+ 0·7	+ 1·4	+ 2·2	− 2·9	− 2·2	− 1·4	− ·72	− 2·9
8	− 0·7	− 1·4	− 2·2	+ 2·9	+ 2·2	+ 1·4	+ ·72	+ 2·9

It will be observed that when once the strain produced by $\mathbf{W}_7$ in diagonal 1 is obtained, all the other strains may be derived from it by addition.

167. Increments of strain in flanges.—The flanges receive successive increments of strain at each apex as they approach the centre where the maximum strains occur. These increments *diminish* as they approach the centre in an arithmetic progression whose difference $= 2\mathbf{W}tan\theta$. Hence the strain in the bays might be expressed by an equation; they may, however, be more conveniently found by the aid of a rough diagram as already described in **164**.

168. Strains in flanges calculated by moments.—The strains in any given bay may also be obtained by taking moments round the apex immediately above or below it. To obtain the strain in bay **C**, Fig. 52, for example, take moments round the apex a. The left segment of the girder is held in equilibrium by the reaction of the left abutment (= 17·5 tons), the two first weights, $\mathbf{W}_1$ and $\mathbf{W}_2$ the horizontal tension in **C** and the strains at a. Taking moments round the latter point we have

$$\mathbf{F}d = 17{\cdot}5 \times 2{\cdot}5b - 5\ (1{\cdot}5 + 0{\cdot}5)b$$

where $\mathbf{F}$ = the strain in the flange at **C**,
b = the length of one bay,
d = the depth of the girder.

If $\theta = 45°, b = 2d$, and we have $\mathbf{F} = 67{\cdot}5$ tons as in ex. 2 (**165**).

This method is, it will be perceived, merely a modification of that described in Chap. II. (**43**). It is sometimes convenient for checking results obtained by the resolution of forces.

169. Girder loaded unsymmetrically.—If the load be distributed in an unsymmetrical manner, the strains produced by each weight acting separately should first be tabulated, and then the resultant strains may be obtained as indicated in **166** (**35**).

170. Girder loaded symmetrically.—If the central part of a symmetrically loaded girder be free from load, the central braces may be removed without affecting the strength of the structure. If, for example, the girder represented in Fig. 52 support only W_1, W_2, W_6, W_7, the braces in the interval, 5, 6, 7, 8, 8′, 7′, 6′, 5′, may be removed. If the central weight alone be wanting, then braces 7, 8, 8′, 7′, may be removed.

171. Strains in end diagonals and bays.—When the load is symmetrical, each of the end diagonals sustains a strain equal to one-half the load multiplied by $sec\theta$, and the extreme bays of the longer flange sustain a strain equal to one-half the load multiplied by $tan\theta$. Consequently, when $\theta = 45°$, the strain in each of these extreme bays equals half the load.

172. Strains in intersecting diagonals—General law of strains in intersecting diagonals of isosceles bracing with parallel flanges.—When two diagonals intersect at a *loaded* apex of a girder loaded uniformly, the strain in that diagonal which is more remote from the centre exceeds that from the other by $W sec\theta$. The following law is applicable to all girders with parallel flanges and isosceles bracing whether single or lattice; *when two diagonals intersect at an unloaded apex, no matter how the load may be distributed, the strains in the two diagonals are equal in amount, but of opposite kinds.*

CASE V.—GIRDERS SUPPORTED AT BOTH ENDS AND TRAVERSED BY A TRAIN OF UNIFORM DENSITY.

Fig. 54.

123. Web.—Let $\mathbf{W}$ = the weight of so much of the uniformly distributed load as covers one bay, *i.e.*, the permanent load resting on each apex,

$\mathbf{W}'$ = the weight of so much of the passing load as covers one bay, *i.e.*, the passing weight on each apex,

l = the number of bays in the span,

n = the number of apices loaded by the passing load between any given diagonal and either abutment,

Σ = the strain in the given diagonal due to the permanent load,

Σ' = the maximum strain in the given diagonal due to the passing load,

θ = the angle the diagonals make with a vertical line.

The strains in the diagonals vary according to the position of the passing train not only in amount, but also in kind. If, for instance, $\mathbf{W}_1$ alone rests upon the girder, diagonal 4 is subject to tension. If now $\mathbf{W}_2$ be added, its tendency will be to produce compression in diagonal 4, that is, a strain of an opposite kind to that produced by W_1, and the true strain which this diagonal sustains, when both weights rest upon the girder is equal to the difference of the strains produced by each weight acting separately (**145**) The third, fourth, fifth, sixth, and seventh weights tend to increase the compression in diagonal 4, while the first weight alone tends to produce tension. Consequently the maximum compression in this diagonal takes place when all the weights except the first rest upon the girder, and the maximum tension occurs when all the weights are removed except the first.

The same result may be arrived at in any particular case by means of a table of strains such as that in (**166**), where we find at the intersection of diagonal 4 and W_1, that this weight produces a tension of 0·7 tons in the diagonal, while each of the remaining weights produces compression. When all the weights rest upon the girder the first and last produce no effect on diagonal 4, since the strains due to these weights are equal and have opposite signs. In fact, these weights are supported exclusively by the flanges and the last pair of diagonals at each end, and as far as they alone are concerned all the intermediate diagonals might be omitted. If, however, W_1 be removed, the eighth part of W_7 is transmitted to the left abutment, and consequently increases the compression in diagonal 4 by the strain found in the table at the intersection of W_7 and 4. If on the other hand W_7 be removed, the eighth part of W_1 is transmitted to the right abutment, diminishing the compression in diagonal 4 by the strain found at the intersection of W_1 and 4. In a similar manner we find from the table that any other diagonal, 7 for instance, sustains the greatest amount of compression when the first, second, and third weights alone rest upon the girder, and the greatest tension when these are removed and the other weights remain.

174. Maximum strains in web—Strains in intersecting diagonals.—*The maximum strain in any diagonal occurs when the passing train covers only one segment* (**51**); and in general terms, *the maximum tensile strain in any diagonal occurs when the passing train covers the segment from which the diagonal slopes upwards, and the maximum compressive strain when it covers the segment towards which* *the diagonal slopes upwards.* When a pair of diagonals meet at the *unloaded* flange, the strains in the two diagonals are equal in amount but of opposite kinds, and the maximum tensile strain in one is equal to the maximum compressive strain in the other, and *vice versâ* (**172**).

175. Permanent load—Absolute maximum strains.—In all the foregoing investigations the weight of the girder and roadway has been left out of consideration, but in practice the permanent load materially modifies the strains, especially in bridges of large span where the ratio of the permanent to the passing load is considerable. If the supported load be uniformly distributed, its weight may be added to that of the structure, provided the latter be also uniform, and the calculations made for their combined weights as already explained for uniform loads. But when the load moves, the strains in the bracing produced by the weight of the permanent structure will be increased or diminished, or even a strain of an opposite kind produced, according to the position of the passing load. In order to obtain the absolute maximum strains to which the bracing is liable under these circumstances, we must calculate—first, the strains produced by the permanent structure alone, and afterwards the maximum strains, both tensile and compressive, due to the passing load alone. These latter, when added to, or subtracted from, the strains produced by the permanent load, according as they are of the same or opposite kinds, will give the absolute maximum strains to which each brace is liable in any position of the passing load.

176. Web, first method.—Perhaps the simplest method of obtaining the strains in the diagonals from a passing train is by forming a table of strains produced by each weight separately as in **166**. Then adding, first the tensile, and afterwards the compressive, strains in each horizontal row, we obtain the required maximum strains of each kind.

Ex. 1. The following example of a girder of eight bays will illustrate this method of calculating the absolute maximum strains when the bridge is traversed by a load of uniform density whose length is not less than the span. Let Fig. 54 represent a railway girder 80 feet long and 5 feet deep, the bracing of which is formed of 8 right-angled isosceles triangles, with the roadway attached to the upper flange. Let the permanent bridge-load equal half a ton per running foot, and the greatest passing train of uniform density equal one ton per foot; we then have

$$W = 5 \text{ tons from the permanent load,}$$
$$W' = 10 \text{ tons from the passing train,}$$
$$l = 8,$$
$$\theta = 45^\circ,$$
$$\tan\theta = 1,$$
$$\sec\theta = 1{\cdot}414,$$
$$W\sec\theta = 7{\cdot}07 \text{ tons,}$$
$$\frac{W'}{l}\sec\theta = 1{\cdot}77 \text{ tons,}$$
$$(W + W')\tan\theta = 15 \text{ tons.}$$

Diagonals.	W_1	W_2	W_3	W_4	W_5	W_6	W_7	C′	T′	Σ	C	T
	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.
1	−12·4	−10·6	−8·9	−7·1	−5·3	−3·5	−1·8	...	−49·6	−24·7	...	−74·3
2	+12·4	+10·6	+8·9	+7·1	+5·3	+3·5	+1·8	+49·6	...	+24·7	+74·3	...
3	+ 1·8	−10·6	−8·9	−7·1	−5·3	−3·5	−1·8	+ 1·8	−37·2	−17·7	...	−54·9
4	− 1·8	+10·6	+8·9	+7·1	+5·3	+3·5	+1·8	+37·2	− 1·8	+17·7	+54·9	...
5	+ 1·8	+ 3·5	−8·9	−7·1	−5·3	−3·5	−1·8	+ 5·3	−26·6	−10·6	...	−37·2
6	− 1·8	− 3·5	+8·9	+7·1	+5·3	+3·5	+1·8	+26·6	− 5·3	+10·6	+37·2	...
7	+ 1·8	+ 3·5	+5·3	−7·1	−5·3	−3·5	−1·8	+10·6	−17·7	− 3·5	+ 7·1	−21·2
8	− 1·8	− 3·5	−5·3	+7·1	+5·3	+3·5	+1·8	+17·7	−10·6	+ 3·5	+21·2	− 7·1

The numbers in the first column represent the diagonals, and the seven first letters in the upper row the passing weights, in order of position. The numbers found at the intersection of a diagonal with a weight represent in tons the strains produced in the diagonals by the passing load resting on each apex separately; these are derived from eqs. 113 and 114. The columns marked C′ and T′ contain the maximum strains of compression and tension which the passing load can produce; they are obtained by adding, first the compressive, and afterwards the tensile, strains in each row in the first part of the table. The column marked Σ contains the strains due to the uniform permanent load. These are derived from eq. 118. Finally, the two last columns (marked C and T) contain the absolute maximum strains which the combination of permanent and passing loads can produce; these are obtained by adding algebraically column Σ to column C′ and T′. If one ton per foot be the greatest passing load to which the girder is liable, the strains in the bracing can never exceed these absolute maximum strains.

177. Flanges.—The maximum strains in the flanges occur when the passing load covers the whole girder (**53**).

In our example this occurs when the girder supports a uniformly distributed load of 1·5 tons per running foot, equivalent to 15 tons at each apex. The strains in the several bays are given in the following table; they are obtained by the aid of a diagram as described in **164**.

Bays.	A	B	C	D	E	F	G	H
Strains in tons.	+ 52·5	+ 142·5	+ 202·5	+ 232·5	— 105	— 180	— 225	— 240

178. Counterbracing.—On examining the table in **176** it will be seen that diagonals 7 and 8 are the only braces which are liable to both tensile and compressive strains. Consequently the four central diagonals alone require to be counterbraced (**139**); whereas, if the permanent load had been left out of consideration, all the diagonals except the extreme pair at each end would require counterbracing; and if, on the other hand, the strains from the passing load had been calculated on the supposition of its being a uniformly distributed in place of a passing load, none of the diagonals would require counterbracing.

179. Permanent load diminishes counterbracing.—In bridges of large span the permanent load will materially diminish the amount of counterbracing that would be required if the passing load alone had to be provided for; and when the span is very large it will be more accurate to consider the permanent load as resting part on the upper, and part on the lower flange. In small spans this nicety of calculation may be neglected, since the cross road-girders and roadway with the flange to which they are attached form the greater portion of the permanent load.

180. Web, second method.—The maximum strains in the diagonals due to a passing train of uniform density may be expressed by equations similar to those given in the preceding cases, for which purpose it is necessary to divide girders into two classes.

Class A.

Girders in which the extreme apices of the loaded flanges are each distant one *whole bay* from the abutments, as in Fig. 55.

Fig. 55.

From eq. 113 the strain in any diagonal from the passing weight at—

$$\text{The 1st apex} = \frac{W'}{l} sec\theta,$$

$$\text{2nd apex} = 2\frac{W'}{l} sec\theta,$$

$$\text{3rd apex} = 3\frac{W'}{l} sec\theta,$$

* * * *

$$nth \text{ apex} = n\frac{W'}{l} sec\theta,$$

where n, as before, represents the number of loaded apices between the diagonal and one abutment. The maximum strain is equal to the sum of these separate strains; hence

$$\Sigma' = (1 + 2 + 3 + \ldots n)\frac{W'}{l} sec\theta,$$

or by summation,

$$\Sigma' = \frac{n(n+1)}{2} \cdot \frac{W'}{l} sec\theta. \tag{119}$$

Class B.

181. Girders in which the extreme apices of the loaded flange are each distant one *half-bay* from the abutment, as in Fig. 56.

Fig. 56.

The strain in any diagonal from the passing weight at—

$$\text{The 1st apex} = \frac{W'}{2l} sec\theta,$$

$$\text{2nd apex} = 3\frac{W'}{2l} sec\theta,$$

$$\text{3rd apex} = 5\frac{W'}{2l} sec\theta,$$

* * * *

$$nth \text{ apex} = (2n - 1)\frac{W'}{2l} sec\theta.$$

Adding these together we have the strain due to the passing load,

$$\Sigma' = (1 + 3 + 5 + \ldots \overline{2n - 1})\frac{W'}{2l}\sec\theta,$$

or by summation,

$$\Sigma' = \frac{n^2}{2} \cdot \frac{W'}{l}\sec\theta. \qquad (120)$$

Eq. 120 proves that the strains in the diagonals produced by a passing load are proportional to the square of the loaded segment (**50**).

Ex. 2. The following example of a girder of 8 bays with equilateral triangles, belonging to *Class A*, will illustrate this method of calculating the maximum strains produced by a passing train of uniform density sufficiently long to extend over the whole bridge.

Let the girder be 80 feet long, the permanent load 0·5 ton per running foot, and the passing load of greatest density (say engines) one ton per foot; we then have, using the same notation as before,

$$\begin{aligned} W &= 5 \text{ tons from the permanent load,} \\ W' &= 10 \text{ tons from the passing train,} \\ l &= 8, \\ \theta &= 30^\circ, \\ \tan\theta &= 0{\cdot}5773, \\ \sec\theta &= 1{\cdot}154, \\ W\sec\theta &= 5{\cdot}77 \text{ tons,} \\ \frac{W'}{l}\sec\theta &= 1{\cdot}442 \text{ tons,} \\ (W + W')\tan\theta &= 8{\cdot}66 \text{ tons.} \end{aligned}$$

Diagonals	$\frac{n(n+1)}{2}$	Σ	C′	T′	C	T
		Tons.	Tons.	Tons.	Tons.	Tons.
1	− 28	− 20·2	...	− 40·4	...	− 60·6
2	− 0	+ 20·2	+ 40·4	...	+ 60 6	...
3	− 21	− 14·4	+ 1·4	− 30·3	...	− 44·7
4	− 1	+ 14·4	+ 30·3	+ 1·4	+ 44·7	...
5	− 15	− 8·7	+ 4·3	− 21·6	...	− 30·3
6	− 3	+ 8·7	+ 21·6	− 4·3	+ 30·3	...
7	− 10	− 2·9	+ 8·7	− 14·4	+ 5·8	− 17·3
8	− 6	+ 2·9	+ 14·4	− 8·7	+ 17·3	− 5·8

The numerals in the first column represent the diagonals (see Fig. 54). The second column contains the coefficients for each diagonal, $\frac{n(n+1)}{2}$ in eq. 119, n being measured alternately from the right and the left abutment. Column Σ contains the strains in tons produced by the permanent bridge-load; these are calculated by eq. 118. Columns C′ and T′ contain the maximum strains in tons produced by the passing load; these are calculated by the aid of the second column and eq. 119 (see **174**). Finally, the two last columns contain the absolute maximum strains of either kind in the bracing, taking both permanent and passing loads into consideration; these are obtained by adding columns C′ and T′ algebraically to column Σ. The strains in the flanges are as follows (**164**):—

Bays.	A	B	C	D	E	F	G	H
Strains in tons.	+30·3	+82·3	+117·0	+134·2	−60·6	−103·9	−129·9	−138·6

CASE VI.—LATTICE GIRDERS SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

182. Approximate rule for strains in lattice web.—It has been already shown (**156**) that the effect of increasing the number of diagonals, so as to form a lattice girder, is merely to distribute the load over a greater number of apices and thus diminish the strain in each diagonal in proportion to the increased number of systems. This suggests the following approximate rule for finding the strains in the bracing of lattice girders. *Calculate the strains on the supposition that there is only one system of triangles. These divided by the number of systems will give the strains in the corresponding lattice diagonals.* As, however, more exact methods of calculation are of easy application, they are preferable to a rule which is merely approximate.

183. Web—Flanges.—In the case of a uniform load the strains in the bracing may be calculated by eq. 118, observing that the

Fig. 57.

coefficient n will represent in a lattice girder the number of those weights which occur between any given diagonal and the centre of the girder, and which rest only on the apices belonging to its own system of triangulation. This assumes that the strains from weights belonging to different systems but at equal distances on opposite sides of the centre, such as W_5 and W_{11} in Fig. 57, do not pass through the intermediate diagonals, but merely through the flanges and those diagonals of their respective systems which occur between them and the abutments. This is the simplest way of calculating the strains due to a uniform load, but they may also be calculated for each system separately (**166**), in which case the strains in the diagonals will differ somewhat from those obtained by the first method (**186**). The strains in the flanges are most conveniently obtained by the aid of a diagram of strains (**164**).

Ex. The following example of a lattice girder, 80 feet long and 10 feet deep, with four systems of right-angled triangles, *i.e.* 16 bays, will illustrate the mode of calculation (see Fig. 57). If the uniform load equal half a ton per running foot we have

$W = 2{\cdot}5$ tons = the weight on each apex,

$\theta = 45^\circ$,

$W sec\theta = 3{\cdot}535$ tons,

$W tan\theta = 2{\cdot}5$ tons,

n = the number of weights belonging to its own system between any given diagonal and the centre of the girder.

Fig, 58.

The numbers attached to the diagonals in the above diagram of strains are the coefficients n, in eq. 118; these multiplied by $W sec\theta$ give the strains in the diagonals as in the following table, the upper row of which represents the diagonals in order of position (see Fig. 57), and the lower row the corresponding strains in tons :—

Diagonals.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Strains in tons.	+7·1	+7·1	+7·1	+5·3	+3·5	+3·5	+3·5	+1·8				—1·8	—3·5	—3·5	—3·5	—5·3	—7·1

The horizontal numbers at the apices are obtained by adding the coefficients of the intersecting diagonals. These numbers multiplied by $W tan\theta$ are the increments of strain in the flanges (**154**) (see the vertical figures at each apex). Finally, the successive additions of these increments give the resultant strains in the flanges in tons (see the vertical figure at the centre of each bay).

CASE VII.—LATTICE GIRDERS SUPPORTED AT BOTH ENDS AND TRAVERSED BY A TRAIN OF UNIFORM DENSITY.

184. Web, first method.—Perhaps the simplest method of obtaining the strains in the case of a passing train is to tabulate the strains produced by each weight separately, and thence infer what condition of the load will produce the maximum strains in each diagonal (**166**).

Ex. 1. The following example of a lattice girder, 80 feet long and 10 feet deep, with 4 systems of right angled triangles, will illustrate this method (see Fig. 59) :—

Fig. 59.

Let the permanent bridge-load equal half a ton per running foot and the passing train equal one ton per running foot. From these data we have

$$W = 2·5 \text{ tons at each apex from the permanent load,}$$
$$W' = 5·0 \text{ tons at each apex from the passing train,}$$
$$l = 16 = \text{the number of bays in the span,}$$
$$\theta = 45°,$$
$$W sec\theta = 3·535 \text{ tons,}$$
$$\frac{W'}{l} sec\theta = 0·442 \text{ tons,}$$
$$\frac{W + W'}{l} tan\theta = 0·47 \text{ tons.}$$

Diagonals	W'_1	W'_2	W'_3	W'_4	W'_5	W'_6	W'_7	W'_8	W'_9	W'_{10}	W'_{11}	W'_{12}	W'_{13}	W'_{14}	W'_{15}	C′	T′	Σ	C	T
																Tons.	Tons.	Tons.	Tons.	Tons.
1	+6·6	...	...	...	+4·9	...	...	...	+3·1	...	...	...	+1·3	...	...	+15·9	...	+7·1	+23·0	...
2	...	+6·2	...	...	...	+4·4	...	...	...	+2·7	...	...	...	+0·9	...	+14·2	...	+7·1	+21·3	...
3	...	...	+5·7	...	...	...	+4·0	...	...	...	+2·2	...	...	...	+0·4	+12·3	...	+7·1	+19·4	...
4	...	...	...	+5·3	...	...	...	+3·5	...	...	...	+1·8	...	...	...	+10·6	...	+5·3	+15·9	...
5	−0·4	...	...	...	+4·9	...	...	...	+3·1	...	...	...	+1·3	...	...	+ 9·3	− 0·4	+3·5	+12·8	...
6	...	−0·9	...	...	...	+4·4	...	...	...	+2·7	...	...	...	+0·9	...	+ 8·0	− 0·9	+3·5	+11·5	...
7	...	...	−1·8	...	...	...	+4·0	...	...	...	+2·2	...	...	...	+0·4	+ 6·6	− 1·8	+3·5	+10·1	...
8	...	...	...	−1·8	...	...	...	+3·5	...	...	...	+1·8	...	...	...	+ 5·3	− 1·8	+1·8	+ 7·1	...
9	−0·4	...	...	...	−2·2	...	...	...	+3·1	...	...	...	+1·8	...	...	+ 4·4	− 2·6	...	+ 4·4	− 2·6
10	...	−0·9	...	...	...	−2·7	...	...	...	+2·7	...	...	...	+0·9	...	+ 3·6	− 3·6	...	+ 3·6	− 3·6
11	...	...	−1·3	...	...	...	−3·1	...	...	...	+2·2	...	...	...	+0·4	+ 2·6	− 4·4	...	+ 2·6	− 4·4
12	...	...	...	−1·8	...	...	...	−3·5	...	...	...	+1·8	...	...	...	+ 1·8	− 5·3	−1·8	...	− 7·1
13	−0·4	...	...	...	−2·2	...	...	...	−4·0	...	...	...	+1·3	...	...	+ 1·3	− 6·6	−3·5	...	−10·1
14	...	−0·9	...	...	...	−2·7	...	...	...	−4·4	...	...	...	+0·9	...	+ 0·9	− 8·0	−3·5	...	−11·5
15	...	...	−1·3	...	...	...	−3·1	...	...	...	−4·9	...	...	...	+0·4	+ 0·4	− 9·3	−3·5	...	−12·8
16	...	...	...	−1·8	...	...	...	−3·5	...	...	...	−5·3	...	...	...	...	−10·6	−5·3	...	−15·9
17	−0·4	...	...	...	−2·2	...	...	...	−4·0	...	...	...	−5·7	...	...	...	−12·3	−7·1	...	−19·4

The upper row in the preceding table represents the passing weights, and the first column represents the diagonals. The next fifteen columns contain the strains produced in the diagonals by each weight acting separately; these are derived from eqs. 113 and 114. The next two columns marked C′ and T′ contain the maximum strains of compression and tension produced by the passing load; these are obtained by adding the strains of compression and tension in each row separately. The column headed Σ contains the strains produced by the permanent load; it is copied from the previous example in **183**. Finally, the last two columns marked C and T contain the absolute maximum strains which the combined passing and permanent loads can produce; these are obtained by adding column Σ to columns C′ and T′ successively. From this table it appears that diagonals 9, 10, and 11 are subject to both compression and tension; consequently the six central diagonals require counterbracing (**139**). The maximum strains in the flanges occur when the passing load extends uniformly over the whole girder (**53**); they may be obtained by means of a diagram of strains as explained in **183**. In this example the flange-strains are three times greater than in the example in **183**, for the passing load per running foot equals twice the permanent load.

185. End pillars.—The end pillars of lattice girders are sometimes subject to transverse strain from the horizontal components of the diagonals which intersect them midway between the flanges. This transverse strain is, however, of slight amount, as it is merely a differential quantity, being due to the excess of the strain in the tension diagonals over those in compression, or *vice versâ*. In Fig. 59, for example, the vertical component of the diagonals meeting at *c* is transmitted through the lower half of the pillar to the abutment in addition to any pressure which it may receive from the upper half. Their horizontal component, however, tends to deflect the pillar outwards or inwards, according as the strain in the compression or tension diagonal is in excess, and this transverse strain converts the pillar into a vertical girder whose abutments are the flanges. This excess does not attain its greatest value when the girder is uniformly loaded; for since the load is on the upper flange, the tension in diagonal 17′ equals the compression in diagonal 3, and on examining the preceding table we find that the greatest excess of strain in diagonal 1 over that in diagonal 3 occurs when all the apices of the system to which the former diagonal belongs are loaded while those of the latter are free from load. This of course is a condition of load which is very unlikely to occur in practice, but it is quite possible that passing weights may rest on two apices of the first system, say $\mathbf{W}_1$ and $\mathbf{W}_5$, while the apices belonging to the other system are free from load. This might occur, for instance, if a pair of engines or heavy wagons were to cross with a proper interval between them. If this were to occur in our example, the horizontal component of the strain in diagonal 1 would $= \frac{(15 + 11)\ \mathbf{W}'}{l} tan\theta = 8{\cdot}1$ tons. The pillars ought accordingly to be designed with adequate strength to meet such transverse strains as well as those of compression in the direction of their length.

186. Ambiguity respecting strains in lattice bracing.— When a lattice girder contains three or more systems of triangles, a slight ambiguity may occur respecting the strains if the load be disposed on both sides of the centre. Take for example $\mathbf{W}_7$ and $\mathbf{W}_9$, Fig. 59, which belong to different systems, but rest on apices equally distant from the centre; the whole of $\mathbf{W}_7$ may be transmitted to the left abutment through diagonals 7, 13′, 3 and 17′, and the whole of $\mathbf{W}_9$ to the right abutment through diagonals 7′, 13, 3′ and 17 without producing strains in the other diagonals of either system, which indeed might be safely removed as far as these weights are concerned. The method of calculation described in **183** assumes this to be the case. But again, $\frac{7}{16}$ths of $\mathbf{W}_7$ may be transmitted to the right abutment, and $\frac{9}{16}$ths to the left, through the diagonals of its own system, and similarly with $\mathbf{W}_9$ (**10**). This is assumed to be the case for the passing load in the example in **184**. Hence there is a slight ambiguity respecting the strains, as they may go in either way, or partly in one, partly in the other, just as it is impossible to say how much pressure is transmitted through any one leg of a four-legged table. If, however, the girder be strong enough to sustain the strain in whichever way it can be conveyed the safety of the structure is secured, and practically there is a very slight difference in the resulting strains whichever method of calculation is adopted.

187. Flange-strains calculated by moments.—When calculating the strain in any bay of a lattice girder by the method of moments (**168**), we must not neglect the moments of the strains in the diagonals. That part of the girder represented in Fig. 59, for instance, which is to the left of a line drawn through bays a and b, is held in equilibrium by the reaction of the left abutment, the weights $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{W}_3$, the horizontal forces at a and b, and the oblique forces in diagonals 4, 5, 13′ and 14′. The moments of the former pair of diagonals are opposed to those of the latter pair, but they seldom balance exactly. Hence the strains in two bays vertically over each other are rarely precisely the same in value, but differ by an amount equal to the horizontal component of the strains in the diagonals which are intersected by a line joining them; this, indeed, is true whether the bays lie vertically over each other or not, and is merely a modification of the law stated in **56** and **57**. Again, it would be erroneous to expect that the strains in the bays of braced girders when uniformly loaded must necessarily agree precisely with those obtained by eqs. 24 or 26. In some cases it happens that they do so agree, but in general they are only close approximations. This arises from our assuming that the load in braced girders is concentrated at the apices in place of being uniformly distributed. In Fig. 59, for instance, the load on the extreme half-bays is assumed to rest directly over the pillars, while that on the two central half-bays is assumed to rest exactly on the central apex; consequently these portions of the load are neglected in calculating the central strains in the flanges by the method of moments. If, however, the moments be calculated on the supposition that these loads act at their centres of gravity, *i.e.*, at a distance from the pillars equal to a quarter-bay, and at a distance from the centre also equal to a quarter-bay, the strain at the centre will agree with that obtained by eq. 26.

188. Web, second method.—The strains in the bracing of lattice girders subject to passing loads of uniform density may be expressed by an equation obtained in the following manner:—

Let $\mathbf{W}'$ = the passing weight on each apex,

l = the number of bays in the span (= 16 in Fig. 60),

k = the number of systems of triangles, *i.e.*, the number of bays in the base of one of the primary triangles (= 6 in Fig. 60).

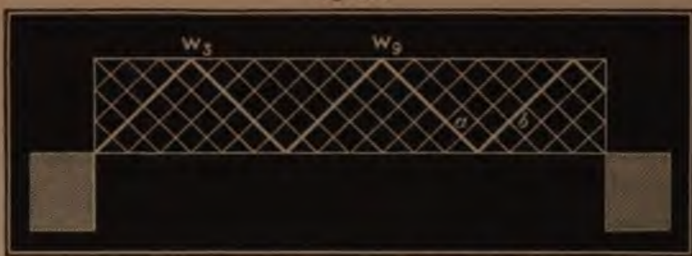
Σ' = the maximum strain which any given diagonal sustains from the passing load,

n = the number of bays between the given diagonal and one of the abutments, measured along the loaded flange,

p = the integral number of times that its own system occurs between the given diagonal and the same abutment, measured also along the loaded flange (= the integral part of $\frac{n}{k}$),

θ = the angle which the diagonals make with a vertical line.

Fig. 60.



Suppose the load traversing the upper flange of Fig. 60; diagonal a sustains the maximum compressive strain when $\mathbf{W}_3$ and $\mathbf{W}_9$ rest upon the girder, and in general, each brace will sustain the maximum strain when the passing load covers only one segment (which segment may be easily seen by inspection **124**), but the strain it sustains is due to those weights alone which rest on the apices of its own system. If, for example, there be n bays between the top of diagonal a and the left abutment, then, on the principle of the lever, the portion of $\mathbf{W}_9$ which is transmitted to the right abutment through $a = \frac{n}{l}\mathbf{W}'$; and of $\mathbf{W}_3 = \frac{n-k}{l}\mathbf{W}'$. The maximum compressive strain in diagonal a is equal to the sum of these quantities multiplied by $sec\theta$, and equals $(n + \overline{n-k})\frac{\mathbf{W}'}{l}sec\theta$; and in general the maximum strain in any given diagonal due to the passing load,

$$\Sigma' = (n + \overline{n-k} + \overline{n-2k} + \overline{n-3k} + \ldots \overline{n-pk})\frac{\mathbf{W}'}{l}sec\theta$$

or summing these up,

$$\Sigma' = \left(n - \frac{pk}{2}\right)(p+1)\frac{W'}{l}\sec\theta. \qquad (121)$$

The maximum tension in a = the maximum compression in b (**174**), and Σ' will represent compressive or tensile strains according as the load traverses the upper or lower flange.

189. Ex. 2. Let Fig. 61 represent a lattice girder 80 feet long and 5 feet deep, whose bracing consists of two systems of right-angled triangles with the load traversing the upper flange.

Fig. 61.

Let the permanent bridge-load equal half a ton per running foot and the heaviest passing train of uniform density equal one ton per foot. Then we have

$W = 2{\cdot}5$ tons at each apex from the permanent load,

$W' = 5$ tons at each apex from the passing train,

$\theta = 45°$,

$l = 16$,

$k = 2$,

$W\sec\theta = 3{\cdot}54$ tons,

$\frac{W'}{l}\sec\theta = 0{\cdot}44$ tons,

$(W + W')\tan\theta = 7{\cdot}5$ tons.

The strains in tons are given in the following table, the numbers in the first column of which represent the diagonals in Fig. 61. The 2nd, 3rd, and 4th columns are the coefficients in eq. 121, from which the maximum strains produced by the passing load (columns C′ and T′) are derived. The strains produced by the permanent bridge-load (column Σ) are obtained from eq. 118, observing that the coefficient n in that equation now represents the number of weights belonging to its own system which occur between any given diagonal and the centre of the girder (**183**). The last two columns (C and T) give the absolute maximum strains due to both permanent and passing loads; these are obtained by adding columns C′ and T′ successively to column Σ.

Diagonals.	n	p	$\left(n-\frac{pk}{2}\right)(p+1)$	C′	T′	Σ	C	T
				Tons.	Tons.	Tons.	Tons.	Tons.
1	15	7	64	+ 28·2	...	+ 14·2	+ 42·4	...
2	14	7	56	+ 24·6	...	+ 12·4	+ 37·0	...
3	13	6	49	+ 21·6	− 0·4	+ 10·6	+ 32·2	...
4	12	6	42	+ 18·5	− 0·9	+ 8·9	+ 27·4	...
5	11	5	36	+ 15·8	− 1·8	+ 7·1	+ 22·9	...
6	10	5	30	+ 13·2	− 2·6	+ 5·3	+ 18·5	...
7	9	4	25	+ 11·0	− 4·0	+ 3·5	+ 14·5	− 0·5
8	8	4	20	+ 8·8	− 5·3	+ 1·8	+ 10·6	− 3·5
9	7	3	16	+ 7·0	− 7·0	...	+ 7·0	− 7·0
10	6	3	12	+ 5·3	− 8·8	− 1·8	+ 3·5	− 10·6
11	5	2	9	+ 4·0	− 11·0	− 3·5	+ 0·5	− 14·5
12	4	2	6	+ 2·6	− 13·2	− 5·3	...	− 18·5
13	3	1	4	+ 1·8	− 15·8	− 7·1	...	− 22·9
14	2	1	2	+ 0·9	− 18·5	− 8·9	...	− 27·4
15	1	0	1	+ 0·4	− 21·6	− 10·6	...	− 32·2
16	0	0	0	...	− 24·6	− 12·4	...	− 37·0

The maximum strains in the flanges occur when the passing load covers the whole girder. They are most conveniently obtained by the aid of a diagram as described in **183**, and are given in the following table, the letters in the upper rows of which refer to the bays in Fig. 61. The figures in the lower row represent the strains in tons.

Bays, . .	A	B	C	D	E	F	G	H
Strains in tons,	+26·3	+78·8	+123·8	+161·3	+191·3	+213·8	+228·8	+236·3
Bays, . .	I	J	K	L	M	N	O	P
Strains in tons,	−30	−82·5	−127·5	−165	−195	−217·5	−232·5	−240

The compressive strain in each of the end pillars is equal to the vertical component (shearing-strain) of the end tension diagonal plus the load resting on the last half-bay; it reaches its maximum when the girder is loaded all over, in which case it equals 26·25 + 3·75 = 30 tons on each pillar.

CHAPTER VII.

GIRDERS WITH PARALLEL FLANGES CONNECTED BY VERTICAL AND DIAGONAL BRACING.

190. Introductory.—In the preceding chapter our attention was confined to that form of braced web which consists of isosceles triangles. There is, however, another class of bracing in common use which consists of right-angled triangles, the braces being alternately vertical and oblique. Besides its employment in the webs of girders, this species of bracing is extensively used in scaffolding and for stiffening the platforms of suspension bridges, but more especially for horizontal cross-bracing between the flanges of large girder bridges, so as to strengthen them against side pressure, whether arising from the wind or other sources. The ordinary form of plate girder is, as will be shown hereafter, a modification of this form of bracing (**430**).

191. Since the verticals may act as struts, and the diagonals as ties, or *vice versâ*, each of the following cases might be subdivided; this, however, is unnecessary, as in each case it will be evident on inspection whether any given brace is designed to act as a strut or a tie.

CASE I.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED AT AN INTERMEDIATE POINT.

Fig. 62.

192. Let W = the weight dividing the girder into segments containing respectively m and n bays,
$l = m + n$ = the number of bays in the span,
θ = the angle between the diagonal and vertical braces,
Σ = the strain in a diagonal brace,
Σ' = the strain in a vertical brace.

On the principle of the lever, $\frac{m}{l}$ W is transmitted to the right abutment through the bracing of the right segment (**10**). Hence the strain in each vertical of the right segment,

$$\Sigma' = \frac{m}{l} W \qquad (122)$$

Similarly in the left segment,

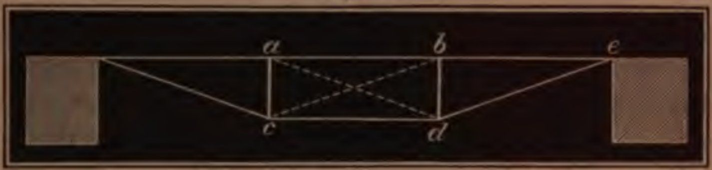
$$\Sigma' = \frac{n}{l} W \qquad (123)$$

These strains in the verticals are identical with the shearing-strain of **34**. The strains in the diagonals are the same as in Case III. of the preceding chapter, that is, they equal the foregoing strains in the verticals multiplied by $sec\theta$ (see eqs. 113 and 114). The strains in the flanges may be found by the aid of a rough diagram of coefficients in the diagonals (**154**), or more simply, by adding the successive increments at the apices, each of which is equal to $\frac{m}{l}$ W$tan\theta$ or $\frac{n}{l}$ W$tan\theta$, according as the apex lies to the right or left of W.

193. Single moving load.—If the load move, the girder must be counterbraced (**140**); this may be effected either by counterbracing the existing braces, or by adding a second series of diagonals. In the latter case there will always be certain braces not acting when the load is in any given position; thus, when the weight rests as represented in Fig. 62, and the verticals are in compression, the dotted diagonals are free from strain.

194. Trussed beam.—The trussed beam of the gantry or travelling crane, Fig. 63, is a familiar example of this kind of bracing. It is, however, seldom counterbraced by the dotted

Fig. 63.



diagonals; hence, when the weight rests on *a*, the tension rod *cde* tends to straighten itself and thrust *b* upwards. This is counteracted by the stiffness of the beam *abe* which is generally formed of a whole balk of timber. Fig. 63 when counterbraced is a simple form of girder for small bridges.*

CASE II.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

Fig. 64.

195. By reasoning similar to that used in Case IV. of the preceding chapter, it may be shown that each brace sustains a strain which is due to all the weights between it and the centre of the girder.

Let $\mathbf{W}$ = the weight resting on each apex,
n = the number of weights between any given brace and the centre of the girder,
θ = the angle between the diagonal and vertical braces,
Σ = the strain in a diagonal,
Σ' = the strain in a vertical.

The strain in each vertical equals the shearing-strain of **46**, that is,

* The railway bridge over the Wye, near Chepstow, erected by the late Mr. Brunel, is an example of this truss on a gigantic style. (See *Clark on the Tubular Bridges*, p. 101). The road, however, is attached to the lower flange, but in small bridges it is usual to place the truss upwards, like Fig. 63 inverted, for this arrangement leaves greater headway beneath, and as the truss forms part of the hand-rail it answers a double purpose.

$$\Sigma' = n\mathbf{W} \qquad (124)$$

The strain in each diagonal

$$\Sigma = n\mathbf{W}sec\theta$$

The increment of strain at each apex $= n\mathbf{W}tan\theta$, where $n =$ the number of weights between the diagonal which intersects that apex and the centre; the successive additions of these increments will give the resultant strains in the several bays.

CASE III.—GIRDERS SUPPORTED AT BOTH ENDS AND TRAVERSED BY A TRAIN OF UNIFORM DENSITY.

Fig. 65.

Fig. 66.

196. Web.—When the load traverses the upper flange, each vertical, if acting as a strut (Fig. 65), sustains the maximum strain when the passing load rests on its own apex and on those between it and the farther abutment: if acting as a tie (Fig. 66), when its own apex is free from load and those between it and the farther abutment are loaded.

When the load traverses the lower flange, each vertical, if acting as a strut (Fig. 65), sustains the maximum strain when its own apex is free from load and those between it and the farther abutment are loaded; if acting as a tie (Fig. 66), when its own apex and those between it and the farther abutment are loaded.

The maximum strain in any diagonal, if in tension (Fig. 65), occurs when the load rests on each apex between it and the abutment from which it slopes *upwards;* if in compression (Fig. 66), when the load rests on each apex between it and the abutment from which it slopes *downwards* (**174**).

Let W' = the passing weight on each apex,

n = the number of weights resting on the girder in the foregoing cases of maximum strain,

l = the number of bays in the span,

θ = the angle between the diagonal and vertical braces,

Σ = the maximum strain in a diagonal,

Σ' = the maximum strain in a vertical.

The maximum strain in any vertical is represented by the following arithmetical series:—

$$\Sigma' = (1 + 2 + 3 + 4 + \dots n)\frac{W'}{l}$$

$$\Sigma' = \frac{n(1+n)}{2} \cdot \frac{W'}{l} \qquad (126)$$

Similarly, the maximum strain in any diagonal,

$$\Sigma = \frac{n(1+n)}{2} \cdot \frac{W'}{l} \sec\theta \qquad (127)$$

The absolute maximum strains in girders subject to both fixed and passing loads are found by tabulating the strains produced by each class of load separately, and then adding or subtracting them according as they are of the same or of opposite kinds (**175**).

CASE IV.—LATTICE GIRDERS SUPPORTED AT BOTH ENDS AND TRAVERSED BY A TRAIN OF UNIFORM DENSITY.

Fig. 67.

197. Web.—In this form of latticing the verticals are generally constructed so as to act as struts and the diagonals as ties, in which case the dotted diagonals are theoretically unnecessary.

Let $\mathsf{W'}$ = the passing weight on each apex,

l = the number of bays in the span (= 10 in Fig. 67),

k = the number of systems of right-angled triangles, *i.e.*, the number of bays in the base of one of the primary right-angled triangles (= 2 in Fig. 67),

T = the maximum tensile strain which any given diagonal sustains from the passing load,

n = the number of bays between the foot of the given diagonal and that abutment from which it slopes upwards,

p = the integral number of times that its own (right-angled) system occurs between the foot of the diagonal and the same abutment $\left(= \text{the integral part of } \frac{n}{k}\right)$,

θ = the angle between the diagonal and vertical braces.

It may be shown by reasoning similar to that employed in **188** that the maximum tensile strain in any diagonal,

$$\mathsf{T} = \left(n - \frac{pk}{2}\right)(p+1)\frac{\mathsf{W'}}{l}\sec\theta \qquad (128)$$

The maximum compression in any vertical equals the maximum tension in one of the conterminous diagonals divided by $\sec\theta$. If the load traverse the upper flange, take the diagonal intersecting at bottom on the side remote from the centre. If the load traverse the lower flange, take the diagonal intersecting it at top on the side next the centre.

198. End pillars—Ambiguity respecting strains in faulty designs.—In this form of latticing the end pillars are subject to a severer transverse strain than in the isosceles latticing described in the preceding chapter (**185**). In the present case the end pillars must be made sufficiently strong to sustain the horizontal components of *all* the diagonals which intersect them between the flanges. This inconvenience may be remedied by introducing short diagonal struts such as *a*, *a*, Fig. 67, which will relieve the end pillars of a certain, though indefinite, amount of transverse strain and at the same time diminish the compression in the bay *c* and the vertical *d*. Both diagonals and verticals are occasionally constructed so as to act either as struts or ties; in such designs calculation is at fault, for the strains may pass through the isosceles system of triangles alone, or through the vertical and diagonal system alone, or partly through one and partly through the other In such designs there will generally be found a certain waste of material.

CHAPTER VIII.

BRACED GIRDERS WITH OBLIQUE OR CURVED FLANGES.

199. Introductory—Calculation by diagram.—The class of braced girders to which our attention has been directed in the two preceding chapters is characterized by the parallelism of the flanges. We have seen that the strains in each part vary according to the position of the load, and that they may be calculated by simple formulæ with a degree of accuracy which leaves nothing further to be desired. I now propose investigating braced girders, one or both of whose flanges are oblique or curved. The "bowstring" girder may be taken as the chief representative of this class, which also includes the arch with external bracing and roadway above, the bent crane, and the various kinds of arched girders now so common for large roofs. Formulæ for strains are unsuited to this species of bracing on account of the various inclinations of the several parts of the structure. Instead, we have recourse to carefully constructed diagrams in which strains are represented to scale, by the aid of which, however, a degree of accuracy is attainable which is practically nearly as perfect as that obtained by the application of formulæ to the girders described in previous chapters.*

CASE I.—BENT SEMI-GIRDERS LOADED AT THE EXTREMITY.

200. Bent crane.—This form of semi-girder has been adopted for wharf cranes where head-room is required close to the post. The flanges may be equi-distant as in Fig. 68, though a

* The curved flanges are assumed to be polygonal, *i.e.*, formed of straight lines joining the apices (**146**).

more pleasing form is produced by bringing them closer together as they approach the peak.*

The weight **W** is supported by diagonal 1 and the first bay in the lower flange **E**, producing tension in the former, compression in the latter. The tension of diagonal 1 is resolved at *d* into its components in the direction of **A** and diagonal 2. The resultant of the strains in diagonal 2 and **E**, found by a triangle of force, is resolved at *g* into its components in the directions of the third diagonal and **F**. In a similar manner the resultant of the strains in diagonal 3 and **A** is resolved into its components in diagonal 4 and **B**, and so on throughout the girder.

Fig. 68.

An example (see Fig. 68) will illustrate this fully, and the student is recommended to work it out for himself by the aid of a diagram drawn accurately to a scale of not less than five feet to one inch. The strains may be represented to a scale of ten tons to one inch, though in many cases a larger scale will be found preferable.† The

* Tubular cranes of this form were first made with plate webs by Mr. Fairbairn (*Proc. Inst. Mech. Eng.*, Part I., 1857), and the braced web was first adopted by William Anderson, Esq., in a six-ton crane erected for the Government at the Pigeon House Fort, near Dublin. Mr. Anderson also designed a very fine twenty-ton bent crane, with plate webs, for the Russian Government, 60 feet high, and 31′·6″ radius of peak (*Trans. Inst. C. E. of Ireland*, Vol. vi., and Vol. viii., p. 167).

† Rolling parallel rules, 15 or 18 inches in length, will be found useful for laying off parallel lines of strain.

flanges are equidistant, forming quadrants of two circles whose radii are respectively 20 and 24 feet. The inner flange is divided into four equal bays on which stand equal isosceles triangles, and a weight of 10 tons is suspended from the peak. Draw *ab* vertically and equal to 10 tons measured on the scale representing strains, and draw *bc* parallel to **E** so as to meet the diagonal 1 produced; *bc* and *ac* represent the strains in **E** and diagonal 1, and measure on the scale of strains + 10·8 tons and — 13·1 tons respectively (9). Next, take *de* equal 13·1 tons (= *ac*), and draw *ef* parallel to diagonal 2 so as to meet **A** produced; *ef* and *df* represent the strains in diagonal 2 and **A**, and measure + 18·8 tons and — 21·7 tons respectively. Next, produce diagonal 2 so that *gh* may equal 18·8 tons (= *ef*), and draw *hi* parallel to **E** and equal 10·8 tons (= *bc*); *ig* is the resultant of the strains in diagonal 2 and **E**, and is transmitted through **F** and diagonal 3. Draw *ik* parallel to **F**; *ik* and *kg* will represent the strains in **F** and diagonal 3, and measure + 30·5 tons and — 5·4 tons respectively. Proceeding in this manner we obtain the strains given in the following table:—

Diagonals, . . .	1	2	3	4	5	6	7	8
Strains in tons, . .	−13·1	+18·8	−5·4	+21·4	+3·2	+20·5	+11·2	+8·8
Flanges, . . .	A	B	C	D	E	F	G	H
Strains in tons, . .	−21·7	−39·7	−51·3	−49·0	+10·8	+30·5	+45·3	+52·8

201. Calculation by moments.—It is prudent to check the calculation by computing the strains in some of the bays by the method of moments. That portion of the crane which extends above **B***l*, for instance, is held in equilibrium by the tension in **B**, the weight **W**, and the forces which meet at *l*. Taking moments round the latter point we obtain the strain in **B**. In this example **B***l* measures 3·55 feet, and the horizontal distance of *l* from **W** measures 14·12 feet; hence we have

$$3·55 \times \text{strain in } \mathbf{B} = 14·12 \times 10 \text{ tons};$$

whence the strain in **B** = 39·8 tons, which agrees closely with the former result.

When only one system of triangulation is adopted, the strains in the flanges may be obtained in this manner by moments, and those in the diagonals may afterwards be derived from the flanges. This method is perhaps more simple in practice than that first described, and has a farther advantage that errors do not accumulate.

202. Lattice webs not suited for powerful bent cranes.— The chief merit claimed for the bent crane is the large amount of head-room it allows underneath the jib, which enable boilers or other bulky articles to be brought close up to the peak. This merit, however, is balanced, and in many cases more than balanced, by the greater simplicity of the ordinary crane. The lattice web is not well suited for cranes exceeding 10 tons, as the diagonals become so wide and leave so little open space that plating may be advantageously substituted for bracing.

CASE II.—THE BRACED SEMI-ARCH.

Fig. 69.

203. Swing bridge.—This form of semi-girder is a modification of the previous case, in which the radius of the upper flange becomes infinite; it is suitable for swing bridges, in which case the end next the abutment is prolonged backwards with parallel flanges and loaded at the inner extremity with a counterpoise weight to balance the projecting part. This backward continuation resembles the semi-girder described in Case I., Chap. VI. In order to obtain the maximum strains when a single load or a passing train traverses the girder, we must first calculate the strains produced by the weight on each apex separately, and tabulating these we can find what position of the load, if it be a single one, or what weights, if there be several, will produce maximum strains in each part of the structure, and the methods of calculation described in the preceding case are applicable to this also.

204. Single triangulation.—When, however, there is but one system of triangles in the bracing, the following plan is more simple in practice, and as errors do not accumulate, it is less liable to inaccuracy. Suppose a weight resting on the extremity of the girder; on examining the forces which hold any portion $\mathbf{C}a\mathbf{W}_1$ in equilibrium, we find that two of them, viz., the weight and the horizontal tension in **C** pass through $\mathbf{W}_1$; consequently the third force, viz., the resultant of the strains in bay **G** and diagonal 6 also passes through $\mathbf{W}_1$ (**9**). In the same way it can be shown that the resultants at each of the other lower apices pass through $\mathbf{W}_1$. If the weight rest on any other apex, $\mathbf{W}_2$ for example, the resultant strains produced by it at each lower apex pass through $\mathbf{W}_2$; or, to express this more generally, *the resultant strain at each apex in the lower flange from a weight at any apex in either flange will pass through the intersection of the horizontal flange with a vertical line drawn through the weight, provided there be but one system of triangulation.*

Again, since the horizontal flange transmits no vertical strains, the weight must be conveyed to the wall through these resultant strains at each lower apex. Their vertical components are in fact the shearing-strain and equal to the weight; hence, knowing both their directions and their vertical components, we can find their amounts. Thus the resultant strain at a from $\mathbf{W}_1$ may be found as follows:—Draw a vertical line ab equal (by a scale of strains) to $\mathbf{W}_1$, and draw bc horizontally till it meet $\mathbf{W}_1a$ produced; ac is the required resultant, and may be resolved into its components in bay **G** and diagonal 6. The strain in the latter may next be resolved at $\mathbf{W}_4$ in the directions of bay **D** and diagonal 7. The former component is the *increment* of horizontal strain at the apex, and when added to the sum of the preceding increments gives the resultant strain in D. The strains in the other parts may be obtained in a similar manner.

205. Example.—The following example, Fig. 69, in which the strains have been worked out on a diagram drawn to a scale of 5 feet to one inch, will be found useful practice for the student. The projecting portion of the girder is 40 feet long, and 10 feet deep at the wall, with a circular lower flange which has a horizontal tangent two feet below the extremity of the girder. Consequently the versine of the arch is 8 feet, and its radius 104 feet. The load is uniform and equal to one ton per running foot, which for calculation is supposed collected into weights of 10 tons at each upper apex except the outer one, which has only 5 tons, or the load which rests on half a bay. The strains have been calculated for each weight separately.

		W_1	W_2	W_3	W_4	Uniform Load.	Max. Compn.	Max. Tension.
		Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.
BRACING.	1	+ 12·7	...	...	...	+ 12·7	+ 12·7	...
	2	− 8·0	...	...	...	− 8·0	...	− 8·0
	3	+ 5·9	+ 19·0	...	...	+ 24·9	+ 24·9	...
	4	− 0·3	− 9·8	...	...	− 10·1	...	− 10·1
	5	+ 0·2	+ 7·3	+ 14·0	...	+ 21·5	+ 21·5	...
	6	+ 2·7	− 1·1	− 7·5	...	− 5·9	+ 2·7	− 8·6
	7	− 2·3	+ 0·9	+ 6·3	+ 11·7	+ 16·6	+ 18·9	− 2·3
	8	+ 3·1	+ 1·7	− 2·8	− 7·2	− 5·2	+ 4·8	− 10·0
FLANGES.	A	− 11·7	...	...	...	− 11·7		
	B	− 24·1	− 16·1	...	...	− 40·2		
	C	− 24·7	− 29·7	− 9·9	...	− 64·3		
	D	− 21·6	− 30·8	− 18·5	− 6·2	− 77·1		
	E	+ 19·3	...	...	...	+ 19·3		
	F	+ 25·2	+ 25·2	...	...	+ 50·4		
	G	+ 23·9	+ 31·8	+ 15·9	...	+ 71·6		
	H	+ 21·4	+ 32·1	+ 21·4	+ 10·7	+ 85·6		

The reader will perceive that the strain produced in bay **H** by W_4 is half that produced by W_3, and one-third of that produced by W_2, and in general, the strains produced by the different weights in any given bay will be sub-multiples of the strain produced by the most remote weight, for they are proportional to the leverage of the weights round the apex above or below the given bay. This check on the accuracy of the work is, however, applicable only in the case of a single system of triangulation. The strains in girders of this form are not always such as might perhaps be expected at first sight; W_1, for instance, produces compression in both diagonals 6 and 8, and in bay **D** a strain of less amount than in bay **C**. These apparent anomalies occur when the resultant at the lower apex, *oc* for example, passes altogether *above* the lower flange.

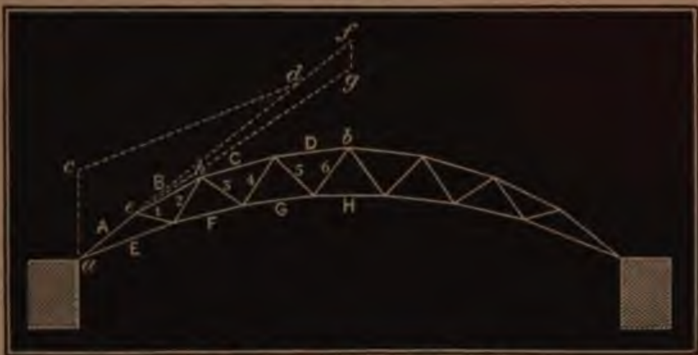
206. Lattice semi-arch—Triangular semi-girder.—When two or more systems of triangulation are introduced, the strains in one system produce strains in the others in consequence of the curvature of the arched flange, and this renders the calculations more tedious than would otherwise occur. This remark applies to all arched girders with lattice webs. In this particular case the calculations would be much simpler if the girder were triangular with a straight lower flange, since each bay would communicate its strain directly to the adjoining bay without affecting the diagonals at their junction, but this form of semi-girder has the disadvantage of being somewhat unsightly in appearance, which in some cases might prevent its adoption, whatever merits, and they are considerable, it may possess in other respects.*

207. Inverted semi-arch.—When head-room beneath is required, we may invert the girder represented in Fig. 69, so that it will resemble one-half of a suspension bridge. By so doing we change the strains in kind but not in amount.

* A large iron swing bridge, a drawing of which appeared in the *Illustrated London News* for October 12, 1861, has been constructed at Brest, in France; it is formed of two triangular semi-girders with vertical and diagonal bracing.

CASE III.—BENT GIRDERS SUPPORTED AT BOTH ENDS AND LOADED SYMMETRICALLY.

Fig. 70.



208. Suitable for roofs—Flanges.—Frequent modifications of this form of girder occur in the roofs of our railway stations and crystal palaces, to which its graceful outline and lightness of appearance impart an air of elegance which no other form possesses to the same degree. It may also be employed for bridges where greater headway is required beneath the centre than at the abutments. I shall, however, merely investigate the strains produced by a load symmetrically disposed on either side of the centre, such as a roof principal generally sustains. When the girder is subject to a partial or a passing load, the more general method of investigation treated of in the next case becomes necessary. The horizontal strains at the centre of the flanges are equal and of opposite kinds; their amount depends upon the central depth of the girder and may be found by the method of moments as follows:—

Let W = the load symmetrically distributed,

l = the span,

d = the central depth from flange to flange, *ah b*H,

l' = the distance of the centre of gravity of each half load from the centre of the girder,

T = the tension at the centre of the lower flange,

C = the compression at the centre of the upper flange.

The half girder ab**H** is held in equilibrium by the reaction of the left abutment $\left(=\frac{W}{2}\right)$, by the left half load (which we may conceive collected at its centre of gravity), and by the horizontal strains of compression and tension at b and **H**. Taking moments round each of these latter points successively, we have $\frac{W}{2}\left(\frac{l}{2}-l'\right)=Td=Cd$ whence,

$$T=C=\frac{W\,(l-2l')}{4d} \qquad (129)$$

This, which is merely a particular form of eq. 26, proves that the strains at the centre do not depend upon the height of the lower flange above the chord line, but upon the depth of the girder from flange to flange. The method of calculating the strains in other parts of the girder consists in working by the resolution of forces from either abutment, whose reaction is a known quantity, towards the centre. The following examples, which have been worked out on a diagram drawn to a scale of 5 feet to one inch and with strains represented by 4 tons to one inch, will explain this clearly.

209. Example 1.—The span of the girder, Fig. 70, is 80 feet, the versines of the flanges respectively 10 and 16 feet; both flanges are circular and each flange is divided into equal bays, with the exception of the extreme bays of the lower flange, which are each half as long again as the other bays. The load is supposed equal to 8 tons distributed, so that each apex sustains a weight of one ton; hence the reaction of each abutment equals 4 tons, of which, however, half a ton is at once balanced by the weight of the first half bay of the roof which rests directly on the wall-plate. Consequently the resultant of the forces in **A** and **E** = 3·5 tons pressing downwards on the wall. Draw ac = 3·5 tons, and draw cd parallel to **E** until it meets **A** produced. The lines ad and cd represent the strains in **A** and **E**, and measure by scale + 12·25 tons and — 10·43 tons respectively. Next, lay off $ef = ad$ and draw fg vertically equal to one ton, that is, equal to the weight at the first apex. The line eg is the resultant of the strain in **A** and the weight at *e*, and the strains in **B** and diagonal 1 are its components, and can therefore be found by resolving *eg* in their directions. Similarly, the resultant of **E** and diagonal 1 may be resolved in the directions of **F** and diagonal 2.

At *h* we must find the resultant of *three* forces, viz., the strain in **B**, the strain in diagonal 2, and the weight resting on the apex. From this resultant the strains in **C** and diagonal 3 are derived, and so on to the centre. The following table contains these strains:—

Bracing, . .	1	2	3	4	5	6		
Strains in tons, . .	−2·4	−1·05	−1·36	−0·91	−1·04	−1·0		
Flanges, . . .	A	B	C	D	E	F	G	H
Strains in tons, . .	+12·3	+13·5	+13·1	+12·9	−10·4	−11·7	−12·2	−12·2

The accuracy of the work may be checked by comparing the strain in **H** with the central strain in the flanges obtained by the method of moments. As the distance of the centre of gravity of the half load from the centre of the girder is unknown, the most convenient method for obtaining the leverage of the weights is by accurately measuring on the diagram the distance of each weight from the centre. Doing this, and taking moments round the centre of either flange, we have

$$6{\cdot}15\ \mathbf{F} = 40 \times 3{\cdot}5 \text{ tons} - (31{\cdot}4 + 21{\cdot}6 + 11{\cdot}1)$$

whence the strain at the centre of either flange,

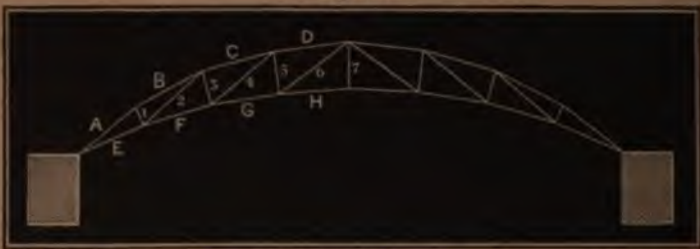
$$\mathbf{F} = 12{\cdot}34 \text{ tons}$$

in place of 12·2 tons, an amount of discrepancy which is immaterial.

The central depth by which **F** is multiplied has been obtained by measurement, and is, it will be observed, slightly in excess of 6 feet, arising from the central bay of the lower flange being a straight line and therefore slightly farther from the upper flange than the arc of which it is the chord.

210. Example 2—Ambiguity in the strains when a girder rests on more than two points.—The girder represented in Fig. 71 has the same span, depth and versine as the preceding example, but the mode of bracing is similar to that described in Chapter VII. Each flange is divided into eight equal bays and every alternate brace is nearly radial to the lower flange.

Fig. 71.



The strains due to a load of one ton at each apex of the upper flange are as follows:—

Bracing, . . .	1	2	3	4	5	6	7	
Strains in tons, . .	−1·75	+0·6	−1·65	+04·5	−1·7	+0·2	−1·4	
Flanges, . . .	A	B	C	D	E	F	G	H
Strains in tons, . .	+13·7	+12·7	+12·6	+12·6	−11·8	−12·3	−12·6	−12·7

The horizontal strain at the centre of either flange equals 12·68 tons. Checking this as before by the method of moments, we have

$$6\mathsf{F} = 40 \times 3{\cdot}5 \text{ tons} - (31{\cdot}4 + 21{\cdot}6 + 11{\cdot}1)$$

whence the strain at the centre of either flange

$$\mathsf{F} = 12{\cdot}65 \text{ tons.}$$

This class of girder is occasionally constructed with equidistant flanges, in which case it is essential for accurate calculation that the girder rest on two points only, either the extremities of the inner, or the extremities of the outer, flange; otherwise we cannot say how much pressure any one point sustains, just as the pressure on any one leg of a four-legged table is indefinite

CASE IV.—BOWSTRING GIRDER.

Fig. 72.

211. Single load.—Let a single weight W_3 rest upon one of the apices which divides the girder into segments containing respectively *m* and *n* segments. On the principle of the lever the pressure on the right abutment $= \frac{m}{m+n} W_3$, that on the left $= \frac{n}{m+n} W_3$. This latter quantity is the resultant of the strains in bays **A** and **F**, which can therefore be obtained from it by a diagram of strains. Again, the strains in **B** and diagonal 1 may be derived from that in **A**, and by resolving the strain in diagonal 1 in the directions of diagonal 2 and bay **G** we obtain the strain in the former and the horizontal increment of strain developed at the first apex of the lower flange. This increment added to the strain in **F** gives the total strain in **G**. The resultant of the strains in **B** and diagonal 2 is also the resultant of those in **C** and diagonal 3, which can therefore be derived from it, and so on.

212. Passing load—Little counterbracing required—Bowstring girder suited for large spans.—When the load is a single passing load or a train, we must tabulate the strains produced by the weight on each apex separately, and thence deduce what position of the load produces maximum strains. It will be found that the maximum strains in the flanges occur when the train covers the whole girder, and that they are of nearly uniform magnitude throughout each flange, while the maximum strains in the diagonals increase as they approach the centre, just the reverse of what occurs in the webs of girders with horizontal flanges. The following example, Fig. 72, will illustrate fully the mode of calculating the strains in this important form of girder. They have been worked out on a diagram drawn to a scale of 5 feet to one inch.

The span is 80 feet divided into 8 equal bays, and the bow is a circular arc whose versine equals 10 feet, but as there is no apex at the crown the central depth of the inscribed polygon, measured by scale, equals 9·85 feet in place of 10 feet. The load is supposed to traverse the lower flange and to be of uniform density equal to one ton per running foot, which is equivalent to 10 tons at each apex.

		W_1	W_2	W_3	W_4	W_5	W_6	W_7	Uniform Load.	Max. Comp^n	Max. Tens^n.
		Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.
Bracing.	1	−0·39	− 0·8	− 1·2	− 1·6	− 2·0	− 2·3	− 2·7	−11·0		−11·0
	2	+0·23	+ 0·5	+ 0·7	+ 0·9	+ 1·1	+ 1·4	−11·4	− 6·6	+4·8	−11·4
	3	−0·56	− 1·1	− 1·7	− 2·2	− 2·8	− 3·4	+ 4·8	− 7·0	+4·8	−11·8
	4	+0·51	+ 1·0	+ 1·5	+ 2·0	+ 2·6	− 8·6	− 4·3	− 5·3	+7·6	−12·9
	5	−0·90	− 1·8	− 2·7	− 3·6	− 4·5	+ 4·7	+ 2·4	− 6·4	+7·1	−13·5
	6	+0·88	+ 1·8	+ 2·6	+ 3·5	− 6·9	− 4·6	− 2·3	− 5·0	+8·8	−13·8
	7	−1·40	− 2·8	− 4·2	− 5·6	+ 4·2	+ 2·8	+ 1·4	− 5·6	+8·4	−14·0
Flanges.	A	+2·82	+ 5.6	+ 8·5	+11·3	+14·1	+16·9	+19·7	+78·9		
	B	+3·08	+ 6·2	+ 9·2	+12·3	+15·4	+18·5	+21·6	+86·3		
	C	+3·47	+ 6·9	+10·4	+13·9	+17·3	+20·8	+10·4	+83·2		
	D	+4·11	+ 8·2	+12·3	+16·4	+20·5	+13·7	+ 6·8	+82·0		
	E	+5·11	+10·2	+15·3	+20·4	+15·3	+10·2	+ 5·1	+81·6		
	F	−2·52	− 5·0	− 7·6	−10·1	−12·6	−15·1	−17·6	−70·5		
	G	−3·01	− 6·0	− 9·0	−12·0	−15·0	−18·1	−13·1	−76·2		
	H	−3·62	− 7·2	−10·9	−14·5	−18·1	−15·9	− 7·9	−78·1		
	I	−4·46	− 8·9	−13·4	−17·8	−17·1	−11·4	− 5·7	−78·8		

On examining the foregoing table we observe that, when the permanent (uniform) load is equal to or less than the passing load, a large number of the diagonals require counterbracing; in this example, for instance, diagonals 4, 5, 6, 7, and their counterparts at the other side of the centre, require counterbracing. If, however, the permanent load be much greater than the passing load, it may happen that the diagonals will always be in tension and thus relieve the engineer of one difficulty in large girders, namely, that of providing against flexure in long struts. Hence the bowstring girder seems well suited for large spans (**454**).

213. Calculation by moments.—The work may be checked by calculating the strains in some of the bays by the method of moments. Thus, in the central bay **E**, the strain

$$\mathsf{F} = \frac{\overset{\text{tons}}{35} \times 40 - \overset{\text{tons}}{10} \times 60}{9 \cdot 85} = 81 \cdot 2 \text{ tons compression,}$$

a close approximation to the amount in the table as the discrepancy is only 0·4 tons, or $\frac{1}{203}$rd of the whole. On examining the table we find that all the intermediate strains are multiples of those in the columns under either $\mathbf{W}_1$ or $\mathbf{W}_7$. They agree also in sign with their sub-multiples. This arises from the reaction of each abutment being directly proportional to the length of the remote segment and indicates a speedy method of filling up the table, viz., by calculating on a diagram the strains produced by the two extreme weights and thence deriving those due to all the intermediate weights.

214. Uniformly distributed load—Little bracing required. If a uniform horizontal load be suspended by vertical rods from a circular bow, the diagonal bracing will scarcely come into action and the tension throughout the string will be very nearly uniform, for a small arc of a circle differs but slightly from the parabola which a chain (inverted arch) assumes when loaded uniformly per horizontal foot (**48, 49**). In this case the horizontal component of strain is nearly uniform throughout the bow and equals the compression at the crown or the tension in the string. The vertical component at the springing is equal to the half load, and at any other point it equals the half load supported above the level of that point. The longitudinal compression at any point in the bow is the resultant of these horizontal and vertical components, and would be strictly tangential to the curve if it were a parabola, *i.e.*, the curve of equal horizontal thrust for a uniform horizontal load. The bow forms a considerable item of the total weight of a bridge of large span, and the annexed method of calculating the strains will be found more accurate than one which supposes the whole permanent load resting on the lower flange:—

1°. Calculate the maximum strains in both flanges and bracing produced by the passing load of greatest uniform density, as already explained.

2°. Calculate the strains produced by the permanent load which rests on the lower flange, including in this the string, roadway and bracing. These may be obtained by proportion from the strains produced by the passing load when the latter covers the whole bridge.

3°. Calculate the (nearly) uniform strain produced throughout the bow and string by the weight of the former (eq. 26). If greater accuracy is required the longitudinal strains in the bow may be obtained by the method explained in **26**.

Having these arranged in a tabular form we can easily find the maximum strains which each part sustains. The 2nd and 3rd of the foregoing calculations may be replaced by the method described in the preceding case for calculating the strains due to a permanent load, without however simplifying the operation in practice.

215. Inverted bowstring or fish-bellied girder—Bow and invert, or double bow.—The method of calculating the strains of the bowstring girder is also applicable to its inverse—the fish-bellied girder, *i.e.*, the arc in tension with a horizontal flange in compression, as well as the lenticular girder compounded of the two, *i.e.*, a bow and invert connected by bracing, such as the Royal Albert Bridge, Saltash. Examples of these forms are, however, comparatively rare, except in cast-iron girders and beams of steam engines, but the fish-bellied girder is sometimes used for cross road-girders.

216. Single triangulation.—When the bracing consists of a single system of triangulation, as in Fig. 72, the strains may be calculated by a method similar to that described in **204**. Suppose, for example, that W_3 alone rests upon the girder, dividing the lower flange into segments containing respectively m and n bays; the segment abc is held in equilibrium by three forces, viz., the reaction of the right abutment, the horizontal tension at c and the resultant of the strains in K and diagonal 10. The two former meet at a; consequently the third, the resultant at b, passes through the same point (**9**). Again, since the lower flange is horizontal, it cannot convey a vertical pressure to the abutment; hence $\frac{m}{m+n}\mathbf{W}_3$ (= the reaction of the abutment) must be conveyed through the bow and diagonals to the right abutment, forming the vertical component of the resultant at each upper apex. This suggests the following method of calculating the strains. Draw bd vertically equal to $\frac{m}{m+n}\mathbf{W}_3$, and draw de horizontally till it meets ba produced: be represents the resultant at b, and hence we can find its component in **K** and diagonal 10, or in **L** and diagonal 11. The same reasoning will apply if all the apices to the left of $\mathbf{W}_3$ are loaded, in which case diagonals 10 and 11 will sustain the maximum strains of tension and compression which a passing train can produce in them. At the several apices in the bow over the *unloaded* segment resultant strains will be developed, each of which will pass through a and have the same vertical component, viz., the reaction of the right abutment, provided there be but one system of triangles. In the case of the train, bd will represent $\frac{\mathbf{W}}{m+n}(1+2+3+4+5) = \frac{15}{8}\mathbf{W}$, since there are 5 loaded apices in the left segment and 8 bays in the span. This operation must be repeated at each apex of the bow.

The maximum strains in the diagonals of the example in **212** are calculated by this method and are given in the annexed table They agree closely with those previously obtained:—

Diagonals.	Maximum compression.	Maximum tension.
	Tons.	Tons.
1	...	− 11·0
2	+ 4·7	− 11·4
3	+ 4·8	− 11·8
4	+ 7·6	− 12·8
5	+ 7·1	− 13·6
6	+ 8·7	− 13·6
7	+ 8·4	− 14.0

CASE V.—THE BRACED ARCH.

Fig. 73.

217. Law of the lever applicable to the braced arch.—In the braced arch the upper flange is generally horizontal and supports the roadway. Both flanges are subject to compression and the lower one exerts an oblique pressure against the abutments. In this respect the braced arch resembles its prototype the stone arch, while it also resembles the girder in its capability of sustaining transverse strain. The horizontal components of the pressures against the abutments are equal and in opposite directions; equal—since, if the horizontal reaction of one abutment exceed that of the other, the arch will move towards that side which exerts the weaker thrust, a thing manifestly impossible. We may therefore conceive a horizontal tie substituted for the horizontal reaction of the abutments, and the arch will then follow the laws of girders, exerting a vertical pressure only on the points of support. The principle of the lever (**10**) is consequently applicable to this form of bracing, and hence we can find the direction and amount of the thrust against either abutment for each position of the load. The lower flange should not be continued across the crown of the arch, for if it were, the strains in every part would be uncertain, since the central bay of this flange would be subject to tensile strains of indefinite amount, varying with the load and temperature, and modifying therefore to an unknown extent the horizontal reaction of the abutments. To illustrate this, let us suppose for a moment that the reaction of the abutments is replaced by a tie-bar; we then have three unknown horizontal forces, viz., compression in the top flange, tension in the lower flange at the crown and tension in the tie-bar; also three known vertical forces, viz., the weight and the vertical reaction of each abutment. Now it is evident that we cannot determine the three unknown forces by the method of moments from these data, and we must therefore get rid of the difficulty by supposing the lower flange discontinued at the crown, which indeed is not far from the truth in practice, for the two flanges generally merge into one, and the less in depth is the line of junction of the two semi-arches, *i.e.*, the depth of the arch at the crown, the nearer will the following theory and practice agree.

Let us now consider the effect of a single weight W_6. The left semi-arch is subjected to two forces only, viz., the pressure of the other semi-arch at the crown and the reaction of the left abutment at *a*. Since equilibrium exists these forces are equal and opposite; consequently the reaction of the left abutment acts in the direction aW_4. Again, the whole arch is balanced by the weight W_6 and the reactions of the abutments. The weight and the reaction of the left abutment intersect at *b*, consequently that of the right abutment passes through the same point (**9**). Resolving W_6 in the directions *ba* and *bc* we obtain these reactions, and once they are known, we can work from the abutments towards the weight by the resolution of forces and thus find the strains produced by W_6 throughout the arch. Performing similar operations for each weight and tabulating the results, we can obtain the maximum strains of each kind produced in every part of the structure. Those produced in the arch represented in Fig. 73, by weights of 10 tons at each apex are given in the following table. The arch is 80 feet in span with a rise or versine of 8 feet, and the depth measured from the springing to the upper flange is 10 feet. The upper flange is divided into 8 equal bays, and the bracing consists of a series of isosceles triangles of which these bays form the bases.

		W_1	W_2	W_3	W_4	W_5	W_6	W_7	Uniform Load.	Max. Compn.	Max. Tensn.
		Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.	Tons.
BRACING.	1	+ 3·2	+ 6.4	+ 9·6	+ 12·7	− 9·6	− 6·4	− 3·2	+ 12·7	+ 31·9	− 19.2
	2	− 2·0	− 4·0	− 6·0	− 8·0	+ 6.0	+ 4·0	+ 2·0	− 8·0	+ 12·0	− 20·0
	3	+ 1·5	+ 3·0	+ 4·5	+ 5·9	+ 14·5	− 3·0	− 1·5	+ 24·9	+ 29·4	− 4·5
	4	− 0·07	− 0·1	− 0·2	− 0·3	− 9 7	+ 0·1	+ 0·07	− 10·2	+ 0·2	− 10·4
	5	+ 0·05	+ 0·1	+ 0·15	+ 0·2	+ 7·1	+ 13·9	− 0·05	+ 21·4	+ 21·5	− 0·1
	6	+ 0·7	+ 1·4	+ 2·1	+ 2·7	− 3·2	− 8·9	− 0·7	− 5·9	+ 6·9	− 12·8
	7	− 0·6	− 1·2	− 1·8	− 2·3	+ 2·7	+ 7·5	+ 12·3	+ 16·6	+ 22·5	− 5·9
	8	+ 0·8	+ 1·6	+ 2·4	+ 3·1	− 0·6	− 4·4	− 8·0	− 5·1	+ 7·9	− 13·0
	9	− 0·7	− 1·4	− 2·1	− 2·8	− 0·4	+ 3·6	+ 6·8	+ 3·0	+ 10·4	− 7·4
FLANGES.	A	+ 2·0	+ 4·0	+ 6·0	+ 8·1	+ 24·0	+ 16·0	+ 8·0	+ 68·1	+ 68·1	...
	B	− 1·1	− 2·2	− 3·3	− 4·4	+ 17·1	+ 22·4	+ 11·2	+ 39·7	+ 50·7	− 11·0
	C	− 1·2	− 2·4	− 3·6	− 4·7	+ 3·8	+ 12·6	+ 11·3	+ 15·8	+ 27·7	− 11·9
	D	− 0·4	− 0·8	− 1·2	− 1·7	+ 0·3	+ 2·3	+ 4·2	+ 2·7	+ 6·8	− 4·1
	E	+ 4·8	+ 9·7	+ 14·5	+ 19·3	− 14·5	− 9·7	− 4·8	+ 19·3	+ 48·3	− 29·0
	F	+ 6·3	+ 12·6	+ 18·9	+ 25·2	+ 6·3	− 12·6	− 6·3	+ 50·4	+ 69·3	− 18·9
	G	+ 6·0	+ 12·0	+ 18·0	+ 23·9	+ 13·9	+ 3·9	− 6·0	+ 71·7	+ 77·7	− 6·0
	H	+ 5·3	+ 10·7	+ 16·0	+ 21·4	+ 16·0	+ 10·7	+ 5·3	+ 85·4	+ 85·4	...

218. Strains in the braced arch loaded symmetrically resemble those in the semi-arch.—On examining this table it will be observed that the strains produced in the right semi-arch by W_1, W_2, and W_3 are sub-multiples of those produced by W_4; this arises from the circumstance, that the reactions of the right abutment from the weights on the left semi-arch act all in the same direction, viz., cW_4, and are proportional to the distance of each weight from the left abutment. Hence, having calculated the strains produced by W_4, we can deduce thence the strains produced by the three other weights. On comparing this table with that in **205**, we find that the strains produced by a symmetric load in the diagonals and lower flange of the braced arch and semi-arch are identical. If the weight of the structure be small compared with that of the moving load, some of the bays may sustain tensile strains. These are the end bays of the upper flange and the central bays of the lower flange.

219. Flat arch or arch with horizontal flanges.—If the radius of the lower flange be infinite both flanges will be horizontal, and this flat arch will resemble girders of the ordinary form, Fig. 56, but with their lower flanges severed at the centre so as to exert a lateral thrust against the abutments. When the load is uniform this thrust will equal the central compression in the upper flange. This modification of the braced arch possesses some qualities which merit our attentive consideration. In the first place the quantity of material required for its lower flange is less than in girders of the usual form, for the increments of strain increase as they approach the abutments, and it is therefore more economical to convey them *from* than *towards* the centre; and again, the heavier parts of the lower flange are near the abutments in place of the centre, which is a matter of some importance in very large girders whose own weight forms a large proportion of the total load.

220. Rigid suspension bridge.—When inverted, the braced arch becomes a rigid suspension bridge. Other modifications might be suggested, such as Fig. 70 inverted with a horizontal roadway suspended beneath. The railway bridge over the Donau Canal in Vienna, 83·44 mètres long, is constructed on this latter system There are two suspension chains on each side formed of flat links, and equi-distant one above the other with bracing between; a trussed platform for the rails is suspended beneath by vertical rods in the usual manner. The chains being equi-distant and therefore hung from four points, there must be an ambiguity in the strains as already explained in **210.**

221. Triangular Girder.—If the lower flange of the braced arch be formed of two straight bars meeting at the centre like the letter A, so that the arch becomes two braced triangles, the calculations as well as the construction will be much simplified, especially where multiple systems of bracing are employed. The chief objection to this arrangement is the inelegance of its outline, which, however, will be an immaterial objection in many situations.

222. Cast-iron arches.—The spandrils of cast-iron arches frequently consist of vertical or radial struts without any diagonal bracing whatever. This form of arch resembles the common suspension bridge inverted; and since the spandrils do not brace the flanges together so as to change their transverse into longitudinal strains, but resemble in their action the rungs of a ladder placed on its side, it is necessary to make the flanges sufficiently deep to act as girders and sustain the transverse strain when the moving load causes the line of thrust to pass outside the curved flange (**49**). Unless very massive, arches with verticle spandrils may be expected to be more subject to vibration and deflection than those with braced spandrils.

CASE VI.—THE BRACED TRIANGLE.

Fig. 74.

223. Suitable for roofs and timber bridges.—Various modifications of this class of trussed girder are used for roof principals, for which it is well adapted when the roofing material is slate. The strains may be calculated on the principle already explained, viz., that of finding the reactions of the abutments and working thence towards the centre by the resolution of forces. The braced triangle, whether erect or inverted, is suitable for bridges, especially in colonies where timber abounds and labour is scarce.

CHAPTER IX.

DEFLECTION.

CLASS 1.—*Girders whose sections are proportioned so as to produce uniform strength.*

224. Girders of uniform strength—Deflection curve circular.—The equations generally used for calculating the deflections of loaded girders are based on the assumption that the section of the girder is uniform throughout its entire length, that is, that there is the same amount of material at the centre as at the ends. In scientifically constructed girders, however, this is not the case. Each part is duly proportioned to the maximum strain which can pass through it, so that no material is wasted; and when this occurs in a girder with horizontal flanges and a uniformly distributed load, that is, the load which produces the maximum strain in the flanges, these latter will, as has been already shown (**47**), taper from the centre, where their section is greatest, towards the ends as the ordinates of a parabola. The girder is then said to be of uniform strength, because the unit-strain in each flange is uniform throughout the whole length of the flange and no part has an excess of material or is unduly strained beyond the rest (**19**). Now, as the contraction and elongation are according to Hooke's law proportional to the unit-strain (**7**), the contraction per running foot of the upper flange will be uniform throughout its length, and the extension per running foot of the lower flange will likewise be uniform throughout its length; and this uniform contraction and elongation must produce a circular deflection, since the circle is the only curve that is due to a uniform cause.

At first sight it may be thought that the continuous web of the plate girder, or the braced web of the lattice girder, will seriously affect the amount of the deflection curve; but it can be readily shown by carefully constructed diagrams, in which the alterations of length due to the load are drawn to a highly exaggerated scale, that the construction of the web has scarcely any influence on the curvature so long as the unit-strains in the flanges are unaltered in amount by the method of construction, and it is only when this is the case that a fair comparison can be instituted between the rival girders.

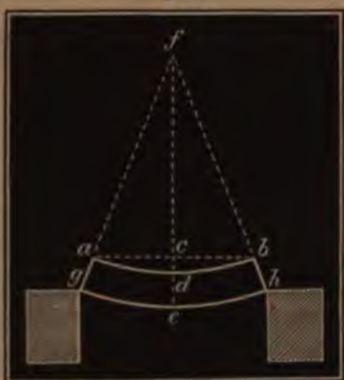
Fig. 1, Plate 1, represents one-half of a diagonally braced girder of the simplest form, namely, a girder with one system of triangles before the load rests upon it. Every part is then in its normal state, and the girder will be horizontal. Now suppose that a uniform load deflects it and shortens each bay of the top or compression flange by a certain quantity, while it lengthens each bay of the lower or tension flange to a similar extent; and further, let us suppose that the diagonals are alternately shortened and lengthened by equal amounts according as they are struts or ties. Fig. 2 now represents the girder; the deflection curve forms a segment of a circle whose centre is at **A**, a little to the left of the vertical line drawn through the middle of the girder. Next, suppose that the flanges are compressed and extended as in Fig. 2, but that the diagonals remain of their original length as in Fig. 1, that is, that their length is not affected by the load. Fig. 3 is the result, which, it will be perceived, is circular and differs but slightly from Fig. 2, having its centre, however, at **B** in the vertical line drawn through the middle of the girder.

It may at first seem strange that **A**, the centre of Fig. 2, is not in the vertical line passing through the middle of the girder. This is due to the circumstance that, with a uniform load, the two central diagonals d, d', are subject to the same strain, either both lengthened or both shortened, while all the other diagonals are alternately lengthened and shortened. Hence a very slight angle is produced at the centre, as shown in Fig. 4, where the flanges are unaltered as in Fig. 1, while the diagonals are alternately lengthened and shortened as in Fig. 2. Considering, however, the exaggerated scale of the diagrams, Fig. 4 is practically horizontal when compared with Figs. 2 or 3, and the chief effect of this common change in the length of the two central diagonals is to throw the centre of each half of the girder in Fig. 2 a little to the right or left of the middle line. These diagrams give very interesting results; they show that the curvature of flanged girders is practically independent of change of form in the web and almost entirely due to the shortening of the upper, and the elongation of the lower, flange; and a further inference may be derived from them, viz., that deflection is practically unaffected by the nature of the web, whether it be formed of plates or lattice bars, so long as the unit-strains in the flanges are not increased or diminished by a different formation of web. Consequently, if there be two girders of equal length and depth, one a lattice, the other a plate girder, having the same unit-strains transmitted throughout their respective flanges, they will both deflect to the same extent.

225. Formula for the deflection of circular curves.—The circumstance of the curve of a loaded girder of uniform strength being circular enables us to find a very simple equation for calculating its deflection.

Let *adbgeh*, Fig. 75, represent a girder supported at both ends and of uniform strength for the load, which generally occurs when the load is uniformly distributed.

Fig. 75.



Let $l = adb$ = the length of the girder,

$d = de$ = the depth,

$\mathsf{R} = af$ = the radius of curvature,

$\lambda = geh - adb$ = the difference in length of the flanges after deflection,

$\mathsf{D} = cd$ = the central deflection.

Since the deflection is very small compared with the radius of curvature, we may assume $cf = af = \mathsf{R}$, and $ab = adb = l$; then (*Euclid*, prop. 35, book iii.),

$$\mathsf{D} = \frac{l^2}{8\mathsf{R}}.$$

By similar triangles, $$\mathsf{R} = \frac{dl}{\lambda}$$

whence by substitution, $$\mathsf{D} = \frac{\lambda l}{8d} \qquad (130)$$

in which the value of λ is known, as it depends on the coefficients of elasticity of the flanges and the strains to which they are subject. This equation for the deflection curve confirms the previous investigation, for the depth d is the only quantity in the equation which can be affected by a change in the length of the diagonals, and it is obvious that a slight change in the value of d will not affect that of D to any appreciable extent.

CLASS 2.—*Girders whose section is uniform throughout their length.*

226. Let W = the weight,

M = the moment of rupture of any given cross section of the girder (**58**),

x = the horizontal distance of the section from the left abutment,

y = the vertical distance of any fibre in the section, either above or below the neutral axis,

β = the breadth of the section at the distance y from the neutral axis, and consequently a variable, except in the case of rectangular sections.

f = the horizontal unit-strain exerted by fibres in the given section at a distance c from its neutral axis,

I = the moment of inertia of any cross section round its neutral axis, and consequently a constant quantity throughout the whole length of the girder when the latter is of uniform section,

R = the radius of curvature.

E = the coefficient of elasticity.

It has already been shown (eq. 42) that **M**, the moment of the horizontal elastic forces of any cross section round its neutral axis, may be expressed by the equation

$$\mathbf{M} = \frac{f}{c}\int \beta y^2 dy.$$

When the girder is of uniform section throughout its length, the integral $\int \beta y^2 dy$, being a definite integral, will be a constant throughout the girder, and as it happens to express the moment of inertia of the cross section round its neutral axis (**67**), we may substitute for this integral the symbol **I**, when we have

$$\mathbf{M} = \frac{f}{c}\mathbf{I} \qquad (131)$$

In order to transform this equation into one involving the co-ordinates of the deflection curve we must substitute for the three variables, **M**, f and c, their values in terms of the coordinates x and y. Let us first deal with f and c.

Fig. 76.

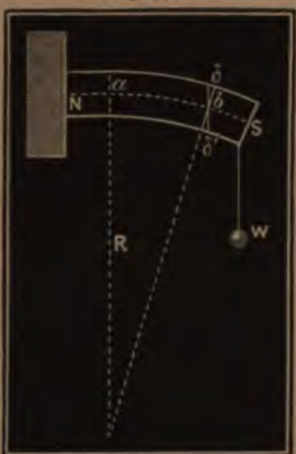


Fig. 76 represents a deflected semi-girder, whose neutral surface is **N S**.

Let ab = a unit of length,

δ and δ' = the increment and decrement in length of a linear unit of the extreme fibres after deflection.

If the law of elasticity (7) hold good, we have the following relation,

$$\frac{f}{c} = \frac{\mathbf{E}}{\mathbf{R}}$$

Substituting this in eq. 131, we have the moment of rupture

$$\mathbf{M} = \frac{\mathbf{E}}{\mathbf{R}}\mathbf{I} \qquad (132)$$

From the principles of the differential calculus we know that, where the deflection is small compared with the length of the curve,

$$\frac{1}{\mathbf{R}} = -\frac{d^2y}{dx^2} \text{ nearly,}$$

whence, by substitution in eq. 132, we have

$$\mathbf{M} = -\mathbf{E}\mathbf{I}\frac{d^2y}{dx^2} \qquad (133)$$

in which **M** is a positive or negative moment according as the upper flange is in compression or tension, y being measured downwards. This equation expresses the moment of the horizontal elastic forces, the moment of rupture (58), at any section of a girder in terms of the ordinates of the deflection curve, the coefficient of elasticity, and the moment of inertia of the cross section round its neutral axis. In order to solve eq. 133, there still remains before integration to substitute for the variable **M** its value in terms of the ordinates of the deflection curve, which may be derived from the leverage of the weight, observing that the moments of forces are to be taken as positive or negative according as they tend to compress or extend the upper flange. To effect this substitution we must consider each case separately, and after integration the value of **I**, which is a different constant for each form of section, may be obtained by multiplying the values of **M**, already determined in (70) and the succeeding articles, by $\frac{c}{f}$ (eq. 131).

CASE I.—SEMI-GIRDERS OF UNIFORM SECTION LOADED AT THE EXTREMITY.

227. Let $\mathbf{W}$ = the load at the extremity,

l = the length of the semi-girder,

x = the abscissa of the deflection curve measured from the fixed end,

y = the ordinate of the deflection curve measured downwards,

$\mathbf{D}$ = the deflection at the extremity,

$\mathbf{M}$ = the moment of the horizontal elastic forces at any given section, whose distance from the fixed ends = x (**58**),

$\mathbf{I}$ = the moment of inertia of any cross section,

$\mathbf{E}$ = the coefficient of elasticity.

Taking moments round the neutral axis of the given section, we have

$$\mathbf{M} = -\mathbf{W}\,(l - x)$$

Substituting this in eq. 133, we have

$$\mathbf{EI}\frac{d^2y}{dx^2} = \mathbf{W}\,(l - x)$$

Integrating,

$$\mathbf{EI}\frac{dy}{dx} = \mathbf{W}\left(lx - \frac{x^2}{2}\right) + \text{const.}$$

The constant = 0, for when $x = 0$, $\frac{dy}{dx}$ also = 0, since the tangent of the curve is horizontal at the fixed end. Integrating again, and determining that the new constant = 0 from the consideration that $y = 0$ when $x = 0$, we have

$$\mathbf{EI}y = \mathbf{W}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \qquad (134)$$

This is the equation of the deflection curve, y being the deflection at any point whose distance from the fixed end equals x.

At the extremity where $x = l$, $y = \mathbf{D}$, and we have

$$\mathbf{EID} = \mathbf{W}\frac{l^3}{3}$$

whence

$$\mathbf{D} = \frac{\mathbf{W}l^3}{3\mathbf{EI}} \qquad (135)$$

228. Solid rectangular semi-girders.—Let b = the breadth and d = the depth. From eqs. 45, 131, and 135,

$$D = \frac{4Wl^3}{Ebd^3} \qquad (136)$$

Comparing eqs. 45 and 46, we find that the deflection of square girders is the same whether the diagonal or one side be vertical. Their strength however is not the same (84).

Ex. The piece of Memel timber described in Ex. 4 (84), deflected 0·66 inch from a load of 336 lbs. hung at its extremity; what is the value of E?

Here, $W = 336$ lbs.
$l = 24$ inches,
$b = 1·94$ inches,
$d = 2$ inches,
$D = 0·66$ inch.

Answer (eq. 136). $E = \frac{4Wl^3}{Dbd^3} = 1,800,000$ lbs.

229. Solid round semi-girders.—Let r = the radius. From eqs. 47, 131, and 135,

$$D = \frac{4Wl^3}{3E\pi r^4} \qquad (137)$$

230. Hollow round semi-girders of uniform thickness.—Let t = the thickness of the tube, supposed small in proportion to its radius r. From eqs. 49, 131, and 135,

$$D = \frac{Wl^3}{3E\pi r^3 t} \qquad (138)$$

231. Semi-girders with parallel flanges.—Let $A = a_1 + a_2$ = the sum of the areas of the two flanges, and let d = the depth of the web. When the web is formed of bracing, or if continuous, is yet so thin that we may safely neglect the support it gives the flanges (76), we have from eqs. 54, 131, and 135,

$$D = \frac{WAl^3}{3Ea_1a_2d^2} \qquad (139)$$

When the web is taken into account and the flanges are of equal area,

let a = the area of either flange,
a' = the area of the web.

From eqs. 56, 131, and 135,

$$D = \frac{4Wl^3}{E(6a + a')d^2} \qquad (140)$$

232. Square tubes of uniform thickness, with the sides or one diagonal vertical.—From eqs. 58, 131, and 135,

$$D = \frac{4Wl^3}{E(b^4 - b_1^4)} \qquad (141)$$

where b and b_1 are the external and internal breadths.

If the thickness of the tube be small compared with the breadth, we have from eqs. 59, 131, and 135,

$$D = \frac{Wl^3}{2Eb^3t} \qquad (142)$$

in which t represents the thickness of one side.

CASE II.—SEMI-GIRDERS OF UNIFORM SECTION LOADED UNIFORMLY.

233. Let l = the length of the semi-girder,

x = the abscissa of the deflection curve measured from the fixed end,

y = the ordinate of the deflection curve measured downwards,

w = the load per unit of length,

$W = wl$ = the whole load,

D = the deflection at the extremity,

M = the moment of the horizontal elastic forces at any given section, whose distance from the fixed end $= x$ (**58**),

E = the coefficient of elasticity.

Taking moments round the neutral axis of the given section we have

$$M = -\frac{w}{2}(l-x)^2$$

Substituting this in eq. 133, we have,

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}(l-x)^2$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{w}{6}(l-x)^3 + \text{const.}$$

When $x = 0$, $\frac{dy}{dx} = 0$ also; hence the constant equals $\frac{wl^3}{6}$. Substituting this value and integrating again,

$$\mathsf{EI}y = \frac{w}{24}(l - x)^4 + \frac{wl^3x}{6} + \text{const.}$$

Determining the second constant by the consideration that $y = 0$ when $x = 0$, we have

$$\mathsf{EI}y = \frac{w}{24}(l - x)^4 + \frac{wl^3x}{6} - \frac{wl^4}{24}$$

At the extremity where $x = l$, $y = \mathsf{D}$, and we have

$$\mathsf{D} = \frac{wl^4}{8\mathsf{EI}} = \frac{\mathsf{W}l^3}{8\mathsf{EI}} \tag{143}$$

234. Deflection of a semi-girder loaded uniformly equals three-eighths of its deflection with the same load concentrated at its extremity.—Comparing eqs. 143 and 135, we see that the deflection of a semi-girder loaded uniformly is to its deflection with the same load concentrated at the extremity as $\frac{3}{8}$. Hence, to obtain the deflections of the various classes of semi-girders in the case of a uniform load, we have merely to multiply the formulæ in the preceding case by $\frac{3}{8}$, recollecting that **W** will now represent the uniform load.

CASE III.—GIRDERS OF UNIFORM SECTION SUPPORTED AT BOTH ENDS AND LOADED AT THE CENTRE.

235. Let l = the length of the girder,

x = the abscissa of the deflection curve measured from the left end of the girder,

y = the ordinate of the deflection curve measured downwards,

W = the load at the centre,

D = the deflection at the centre,

M = the moment of the horizontal elastic forces at any given section whose distance from the left end = x (**58**),

E = the coefficient of elasticity.

Taking moments round the neutral axis of the given section, we have

$$\mathsf{M} = \frac{\mathsf{W}x}{2}$$

[redacted]

Substituting this in eq. 133, we have

$$\mathsf{EI}\frac{d^2y}{dx^2} = -\frac{\mathsf{W}x}{2}$$

Integrating,

$$\mathsf{EI}\frac{dy}{dx} = -\frac{\mathsf{W}x^2}{4} + \text{const.}$$

To determine the constant we must recollect that the tangent of the curve is horizontal at the centre; hence $\frac{dy}{dx} = 0$ when $x = \frac{l}{2}$, and the constant $= \frac{\mathsf{W}l^2}{16}$; substituting this,

$$\mathsf{EI}\frac{dy}{dx} = \frac{\mathsf{W}}{4}\left(\frac{l^2}{4} - x^2\right)$$

Integrating again, and observing that the second constant $= 0$ from the consideration that $y = 0$ when $x = 0$, we have

$$\mathsf{EI}y = \frac{\mathsf{W}}{4}\left(\frac{l^2x}{4} - \frac{x^3}{3}\right)$$

which is the equation of the deflection curve.

At the centre where $x = \frac{l}{2}$, $y = \mathsf{D}$, and we have

$$\mathsf{D} = \frac{\mathsf{W}l^3}{48\mathsf{EI}} \qquad (144)$$

236. Solid rectangular girders.—From eqs. 45, 131, and 144,

$$\mathsf{D} = \frac{\mathsf{W}l^3}{4\mathsf{E}bd^3} \qquad (145)$$

in which b and d represent the breadth and depth of the girder.

Ex. From the mean of five experiments made by Mr. Hodgkinson on Blaenavon cast-iron, No. 2, (see *Clark on the Tubular Bridges*, p. 441,) it appears that the breaking weight and ultimate deflection of a bar 13 feet 6 inches between points of support, 3 inches wide and 1½ inch deep, are respectively 819 lbs. and 10·46 inches; what is the value of the coefficient of elasticity at the limit of rupture?

Here, $\mathsf{W} = 819$ lbs.
$l = 13{\cdot}5$ feet,
$b = 3$ inches,
$d = 1{\cdot}5$ inches,
$\mathsf{D} = 10{\cdot}46$ inches.

Ans. (eq. 145). $\mathsf{E} = \frac{\mathsf{W}l^3}{4\mathsf{D}bd^3} = \frac{819 \times (13{\cdot}5 \times 12)^3}{4 \times 10{\cdot}46 \times 3 \times (1{\cdot}5^3)} = 8{,}200{,}000$ lbs. per square inch.

237. Solid round girders.—From eqs. 47, 131, and 144,

$$D = \frac{Wl^3}{12E\pi r^4} \qquad (146)$$

in which r represents the radius.

238. Hollow round girders of uniform thickness.—From eqs. 49, 131, and 144,

$$D = \frac{Wl^3}{48E\pi r^3 t} \qquad (147)$$

in which t represents the thickness of the tube, supposed small in proportion to its radius r.

239. Girders with parallel flanges.—When the vertical web is formed of bracing, or if continuous, yet so thin that it affords but slight assistance to the flanges in sustaining horizontal strains, its stiffness as an independent girder may be neglected, and we have from eqs. 54, 131, and 144,

$$D = \frac{WAl^3}{48Ea_1a_2d^2} \qquad (148)$$

in which $A = a_1 + a_2 =$ the sum of the areas of the top and bottom flanges, and $d =$ the depth of the web.

When the web is taken into account, and the flanges are of equal area, from eqs. 56, 131, and 144,

$$D = \frac{Wl^3}{4E(6a + a')d^2} \qquad (149)$$

in which $a =$ the area of one flange and $a' =$ that of the web.

240. The deflections of girders of other forms of section may be obtained in a similar manner from eqs. 131 and 144 by substituting for **M** the corresponding values given in Chap. IV.

CASE IV.—GIRDERS OF UNIFORM SECTION SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

241. Let l = the length of the girder,
w = the load per linear unit,
$\mathbf{W} = wl$ = the whole load,
x = the abscissa of the deflection curve measured from the left end of the girder,
y = the ordinate of the deflection curve measured downwards,
$\mathbf{D}$ = the deflection at the centre,
$\mathbf{M}$ = the moment of the horizontal elastic forces at any given section whose distance from the left end $= x$ (**58**),
$\mathbf{E}$ = the coefficient of elasticity.

Taking moments round the neutral axis of the given section, we have

$$\mathbf{M} = \frac{w}{2}(lx - x^2)$$

Substituting this in eq. 133, we have

$$\mathbf{EI}\frac{d^2y}{dx^2} = -\frac{w}{2}(lx - x^2) \qquad (150)$$

Integrating,

$$\mathbf{EI}\frac{dy}{dx} = -\frac{w}{2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + \text{const.}$$

When $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$, and the constant becomes $\frac{wl^3}{24}$; substituting this,

$$\mathbf{EI}\frac{dy}{dx} = \frac{w}{2}\left(\frac{x^3}{3} - \frac{lx^2}{2} + \frac{l^3}{12}\right)$$

Integrating again, and observing that the second constant $= 0$ from the consideration that $y = 0$ when $x = 0$,

$$\mathbf{EI}y = \frac{w}{24}(x^4 - 2lx^3 + l^3x)$$

which is the equation of the deflection curve.

At the centre where $x = \frac{l}{2}$, $y = \mathbf{D}$, and we have

$$\mathbf{D} = \frac{5wl^4}{384\mathbf{EI}} = \frac{5\mathbf{W}l^3}{384\mathbf{EI}} \qquad (151)$$

242. Central deflection of a girder loaded uniformly equals five-eighths of its deflection with the same load concentrated at the centre.—Comparing eqs. 151 and 144 we find that the central deflection of a girder loaded uniformly is $\frac{5}{8}$ths of the deflection if the same load were concentrated at the centre.

243. Solid rectangular girders.—From eqs. 45, 131, and 151,

$$\mathsf{D} = \frac{5wl^4}{32\mathsf{E}bd^3} = \frac{5\mathsf{W}l^3}{32\mathsf{E}bd^3} \qquad (152)$$

where b and d represent the breadth and depth of the girder.

Comparing eqs. 45 and 46, we find that the deflection of solid square girders is the same, whether one side or the diagonal be vertical. The former, however, is 1·414 times stronger than the latter (84).

244. Solid round girders.—From eqs. 47, 131, and 151,

$$\mathsf{D} = \frac{5wl^4}{96\mathsf{E}\pi r^4} = \frac{5\mathsf{W}l^3}{96\mathsf{E}\pi r^4} \qquad (153)$$

where r represents the radius of the cylinder.

245. Hollow round girders of uniform thickness.—From eqs. 49, 131, and 151,

$$\mathsf{D} = \frac{5wl^4}{384\mathsf{E}\pi r^3 t} = \frac{5\mathsf{W}l^3}{384\mathsf{E}\pi r^3 t} \qquad (154)$$

where r = the radius, and t = the thickness of the tube, supposed small in comparison with the radius.

246. Girders with parallel flanges.—When the web is formed of bracing, or if continuous, yet so thin that its strength as an independent girder may be neglected (76), we have from eqs. 54, 131, and 151,

$$\mathsf{D} = \frac{5\mathsf{A}wl^4}{384\mathsf{E}a_1a_2d^2} = \frac{5\mathsf{A}\mathsf{W}l^3}{384\mathsf{E}a_1a_2d^2} \qquad (155)$$

where $\mathsf{A} = a_1 + a_2$ = the sum of the areas of top and bottom flanges, and d = the depth of the web.

If the web be taken into account and if the flanges have equal areas, from eqs. 56, 131, and 151,

$$\mathsf{D} = \frac{5wl^4}{32\mathsf{E}(6a + a')d^2} = \frac{5\mathsf{W}l^3}{32\mathsf{E}(6a + a')d^2} \qquad (156)$$

where a = the area of one flange, and a' = that of the web.

CHAPTER X.

CONTINUOUS GIRDERS.

247. Continuity—Contrary flexure—Points of inflexion.—A girder is said to be continuous when it overhangs its bearings, or is sub-divided into more than one span by one or more intermediate points of support.

When a loaded girder is balanced on a single pier at or near its centre, like the beam of a pair of scales, the upper flange is subject to tension, the lower one to compression, and the girder becomes curved with the convex flange uppermost. If, however, the same girder be supported at its extremities, the pier being removed, the strains in the flanges are reversed, the upper flange being now compressed and the lower one extended, and in this case the convex flange is underneath. If, while in this latter position, we replace the central pier so as to form two spans, the girder becomes continuous and partakes of the nature of both the independent girders. Each flange is in part extended, in part compressed, and the curve becomes a waved line. Let Fig. 77 represent a continuous girder of two spans uniformly loaded.

Fig. 77.

The central segment **BB′** resembles the independent girder in the first case, namely, when balanced over a pier; the extreme segments, **AB**, **B′A′**, resemble it in the second case, since one end of each rests upon an abutment and the other end is supported by the central segment, which thus sustains besides its own proper load an additional weight suspended from each extremity, equal to the half load on each of the end segments.

The points B, B′, where the curvature alters its direction, are called the *points of contrary flexure*, or more briefly, the *points of inflexion*. The curves of the end and central segments have common tangents at these points, and here the strains in the flanges change from tension to compression, and *vice versâ*. Exactly at these points the strains in the flanges are cipher; consequently the flanges might be severed there without altering the conditions of equilibrium in any respect. In fact, a continuous girder may be regarded as formed of independent girders connected merely by chains at the points of inflexion. In braced girders the bracing acts as the chain, in others the continuous web.

248. Passing load.—For the investigation of the strains in a continuous girder it is necessary—first, to find the points of inflexion, and afterwards to calculate the strains in the separate segments on the principles already laid down for independent girders. A passing load complicates the question, for its effect is to alter the position of the points of inflexion, and consequently the lengths of the component segments; if, for instance, a passing train covers the left span, its deflection will be increased and that of the right span diminished, or even altogether removed, if the passing load be sufficiently heavy to lift the right end off the abutment A′. The effect of this partial loading on the points of inflexion will be to bring B nearer to, and remove B′ farther from, the central pier, and this is that disposition of the load which gives the greatest length to the segment AB; it is necessary therefore in the case of a passing load to find this new position of the points of inflexion and calculate the strains in AB as an independent girder of this maximum length. Of course the same calculations will suit B′A′ when it is of maximum length, that is, when the right span only is loaded. The central segment, BB′, becomes of maximum length when the load is uniformly distributed over the whole girder, and the points of inflexion have to be determined under this condition of the load also. Having thus calculated the strength of each part when subject to the load which produces the maximum strain in the flanges of that part, we may assume that there is sufficient strength for any other disposition of the load, since the motion of the points of inflexion is restricted within these limits. The reaction of either abutment is equal to half the load on the adjacent segment; thus the reaction of the left abutment equals half the load resting upon **AB**. The reaction of the pier equals the load resting upon the central segment, **BB′**, plus the sum of the reactions of the two abutments.

249. Experimental method of finding the points of inflexion.—The following method of finding the points of inflexion depends partly on theory, partly on experiment, and is applicable to continuous girders containing any number of spans. Take a long rod of clean yellow pine or other suitable material to represent the continuous girder, and let it be supported at intervals corresponding to the spans of the real girder. Next, load this model uniformly all over, or each span separately, or in pairs, or make any other disposition of the load which can occur in practice. Now it is clear that, if the model and its load be a tolerably accurate representation of the girder and its load, the points of inflexion of the former will correspond with those of the latter; they might therefore be at once obtained by projecting the curves of the model on a vertical plane. It is difficult, however, to do this so as to determine the points of inflexion with the requisite accuracy, for the exact place where the curvature alters is never very precisely defined to the eye. The pressures on the points of support may, however, be measured with considerable accuracy, taking the precaution of keeping them all in the same horizontal line, as a slight error in their level would seriously affect the curvature and lengths of the component segments. We shall assume therefore that the reactions of the points of support have been thus found experimentally.*

Let Fig. 78 represent a continuous girder containing any number of spans, each loaded uniformly, and let *o*,*o*,*o*, &c., represent successive points of inflexion, the intervals between which are called segments.

* It is a safe precaution to measure the pressures on the points of support with the rod turned upside down as well as erect, and then take the mean measurement as the true result.

Fig. 78.

Let R_1, R_2, R_3, &c. = the reactions of the successive points of support as found by experiment,

l, l', &c. = the lengths of the successive spans,

w, w', &c. = the loads per lineal unit on each span,

a, b, c = the lengths of certain parts of the girder, as represented in the figure,

Q = the centre of the third segment.

R_1, the reaction of the left abutment, is equal to half the load on the first segment a, whence $R_1 = \frac{aw}{2}$, and

$$a = \frac{2R_1}{w} \qquad (157)$$

This equation gives the distance of the first point of inflexion from the left abutment, since R_1 is known from experiment.

R_2, the reaction of the first pier, is equal to the load resting on the girder as far as Q minus the reaction of the first abutment; that is, $R_2 = wl + w'b - R_1$, whence

$$b = \frac{R_1 + R_2 - wl}{w'} \qquad (158)$$

Again, taking moments round either flange at Q, which is now a known point, we have

$$Fd = R_1(l + b) + R_2 b - wl\left(\frac{l}{2} + b\right) - \frac{w'b^2}{2}$$

in which F = the strain in either flange at Q, and d = the depth of the girder; but from eq. 26 we have

$$Fd = \frac{w'c^2}{8}$$

c being the length of the third segment as marked in the figure; substituting this value of Fd and arranging, we have

$$c = \sqrt{\frac{8}{w'}\left\{R_1(l + b) + R_2 b - wl\left(\frac{l}{2} + b\right) - \frac{w'b^2}{2}\right\}} \qquad (159)$$

The distance of the second point of inflexion from the first pier $= b - \frac{c}{2}$, and so on. It will be observed that the depth of the girder does not enter into these equations and therefore does not affect the position of the points of inflexion.

250. Practical method of fixing the points of inflexion—Economical position of points of inflexion.—I shall here briefly describe a method by which the points of inflexion of braced girders may be fixed in any particular bay at will, so that there may be no uncertainty respecting their position, or so that they may, if desirable, be made to assume that position which is most advantageous for economy in the flanges.

Fig. 79.

Let Fig. 79 represent a continuous lattice girder capable of free horizontal motion on the points of support. Suppose that the point of inflexion as determined by theory is at *a*, but that it is desirable to fix it at *b*, that is, to make that part of the upper flange which lies between *a* and *b* subject to tension in place of compression. This may be effected by severing the flange at *b* and lowering the end of the girder on the left abutment slightly, so as just to separate the parts at *b*. The left segment *cb* will then assume the condition of an independent girder supported at one extremity by the abutment and at the other by the oblique forces in diagonals *d* and *e*. The upper flange from *c* to *b* will undergo compression, from *b* to some corresponding point in the second span, tension. Further, the operation of fixing the point of inflexion in the upper flange determines its position in the lower one also, for the only horizontal forces acting upon the segment *cbf* are the strains in the lower flange at *f* and the horizontal component of the strains in diagonals *d* and *e*. This component must therefore be exactly equal and opposite to the strain at *f*, otherwise the left segment *cbf* will move either to the right or left, since by hypothesis it is free to move horizontally on the abutment (57). Hence it is evident that the point of inflexion in the lower flange is not far from f, probably not farther than the adjoining bay. Its position is determined by the condition that *the horizontal component of the strains in the diagonals intersected by a line joining the points of inflexion in the two flanges is equal to cipher.*

Thus by leaving any particular bay in one of the flanges of a continuous girder of two spans permanently severed, we have the point of inflexion in that span fixed under all conditions of the load; and when this is determined we can find the strains in the flanges over the pier, and thence deduce the position of the point of inflexion in the second span. If the severed flange be united when any given load rests upon the girder, though the point of inflexion will move with every change of load, yet it will return to its original position whenever a similar load rests on the girder in the same position as when the flange was first severed.

If there be three spans the central span may have both points of inflexion fixed independently of each other, and these again will determine the corresponding points in the side spans. The operation is safe in practice, as was proved at the Boyne Viaduct, where the points of inflexion in the centre span were fixed by severance in those bays in which theory had previously indicated their probable existence. (See Appendix, Vol. II.) The most economical arrangement in theory for the flanges of a large girder of one span consists in forming points of inflexion at the quarter-spans. In this case the end segments of the upper flange must be held back by land chains as in suspension bridges, while those of the lower flange exert a horizontal thrust against the abutments like the flat arch (219). The two extreme segments of the girder thus form semi-girders, while the central segment is an independent girder suspended between them by the web.

The following theoretic investigations respecting continuous girders are based on the assumption that the material is perfectly elastic, and that the girder is of uniform section throughout its whole length.

CASE I.—CONTINUOUS GIRDERS OF TWO EQUAL SPANS, EACH LOADED UNIFORMLY THROUGHOUT ITS WHOLE LENGTH.*

Fig. 80.

251. Pressures on points of support—Points of inflexion—Deflection.—Let $l = \mathsf{AB} = \mathsf{BC}$ = the length of each span,

w = the load per lineal unit of **AB**,

w' = the load per lineal unit of **BC**,

R_1, R_2, R_3 = the reactions of the three points of support **A**, **B**, and **C**, respectively,

$x = \mathsf{A}h$ = the horizontal distance of any point **P** from the abutment **A**,

$y = h\mathsf{P}$ = the deflection at that point,

M = the moment of the horizontal elastic forces at **P** (**58**),

β = the inclination to the horizon of the tangent to the curve at **B**,

I = the moment of inertia of any cross section round its neutral axis, and consequently a constant quantity throughout the whole length of the girder when the section of the latter is uniform from end to end,

E = the coefficient of elasticity.

The forces which hold the segment **AP** in equilibrium are the reaction of the left abutment R_1; the load wx uniformly distributed over **AP**; the vertical shearing-strain at **P**, and the horizontal elastic forces at the same place. Taking the moments of these forces round the neutral axis at **P**, we have

$$M = R_1 x - \frac{wx^2}{2} \qquad (160)$$

* See Mr. Pole's paper on the "Investigation of general formulæ applicable to the Torksey Bridge," *Proc. Inst. C. E.*, Vol. ix., p. 261.

M

Substituting for **M** its value in eq. 133,

$$\mathbf{EI}\frac{d^2y}{dx^2} = \frac{wx^2}{2} - \mathbf{R}_1 x.$$

Integrating this, and determining the constant by the consideration that $\frac{dy}{dx} = tan\beta$ when $x = l$, we have

$$\mathbf{EI}\left(\frac{dy}{dx} - tan\beta\right) = \frac{w}{6}(x^3 - l^3) - \frac{\mathbf{R}_1}{2}(x^2 - l^2)$$

Integrating again, and determining the second constant by the consideration that $y = 0$ when $x = 0$, we have

$$\mathbf{EI}(y - x\,tan\beta) = \frac{w}{6}\left(\frac{x^4}{4} - l^3x\right) - \frac{\mathbf{R}_1}{2}\left(\frac{x^3}{3} - l^2x\right) \qquad (161)$$

which is the equation of the deflection curve from **A** to **B**.

At the point **B**, $x = l$ and $y = 0$; substituting these values in eq. 161, we have

$$tan\beta = \frac{l^2}{24\mathbf{EI}}(3wl - 8\mathbf{R}_1) \qquad (162)$$

Applying a similar process to the second span, and remembering that the angle β must in this case have a contrary sign, we have

$$tan\beta = \frac{l^2}{24\mathbf{EI}}(8\mathbf{R}_3 - 3w'l) \qquad (163)$$

Again, taking moments round **B**, we have

$$\mathbf{R}_1 l - \frac{wl^2}{2} = \mathbf{R}_3 l - \frac{w'l^2}{2} \qquad (164)$$

also

$$\mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 = (w + w')l \qquad (165)$$

By solving these last four simultaneous equations we obtain the reactions of the points of support as follows:—

$$\mathbf{R}_1 = \frac{7w - w'}{16}l \qquad (166)$$

$$\mathbf{R}_2 = \frac{5}{8}(w + w')l \qquad (167)$$

$$\mathbf{R}_3 = \frac{7w' - w}{16}l \qquad (168)$$

At the points of contrary flexure the horizontal forces become cipher. Hence the distance of the point of inflexion in the left span from **A** may be obtained from eq. 160 by making **M** $= 0$ and substituting for $\mathbf{R}_1$ its value in eq. 166 as follows:—

$$x = \frac{2\mathbf{R}_1}{w} = \frac{7w - w'}{8w}\,l \qquad (169)$$

Similarly, the distance of the point of inflexion in the right span measured from **C**,

$$x' = \frac{2\mathbf{R}_3}{w'} = \frac{7w' - w}{8w}\,l \qquad (170)$$

The deflection y in the left span may be derived from eq. 161 by substituting for $tan\beta$ its value in eq. 162 as follows:—

$$y = \frac{wx}{24\mathbf{EI}}\left\{x_3 - l_3 + \frac{4\mathbf{R}_1}{w}(l^2 - x^2)\right\} \qquad (171)$$

The value of **I** for each form of cross section may be obtained from **70** and the succeeding articles by the aid of eq. 131.

The maximum strains in the flanges occur over the pier and half way between the abutments and the points of inflexion, and when the latter are known, may be easily determined on the principles laid down in the second and fourth chapters for calculating the strains in independent girders (see eqs. 13 and 24, or 69, 80 and 105.)

252. Both spans loaded uniformly.—If both spans have the same load per running foot, $w = w'$, and we have

$$\mathbf{R}_1 = \mathbf{R}_3 = \frac{3}{8}wl \qquad (172)$$

$$\mathbf{R}_2 = \frac{5}{4}wl \qquad (173)$$

The distance of each point of inflexion from the near abutment,

$$x = \frac{3}{4}l \qquad (174)$$

For an example of the application of these formulæ the reader is referred to the description of the Torksey Bridge in the Appendix to Vol. II.

CASE II.—CONTINUOUS GIRDERS OF THREE SYMMETRICAL SPANS LOADED SYMMETRICALLY.*

Fig. 81.

253. Pressure on points of support—Points of inflexion—Deflection.—Let **Q** be the centre of the centre span,

$AB = CD = l =$ the length of each side span,

$AQ = nl$,

$w =$ the load per lineal unit on each side span,

$w' =$ the load per lineal unit on the centre span,

$R_1 =$ the reaction of either abutment, **A** or **D**,

$R_2 =$ the reaction of either pier, **B** or **C**,

$x = Ah =$ the horizontal distance of any point **P** from the abutment **A**,

$y = hP =$ the deflection at that point,

$M =$ the moment of the horizontal elastic forces at **P** (**58**),

$\beta =$ the inclination to the horizon of a tangent to the curve at **B** or **C**,

$I =$ the moment of inertia of any cross section round its neutral axis, and consequently a constant quantity throughout the whole length of the girder when the section of the latter is uniform.

$E =$ the coefficient of elasticity.

It can be shown by the same process of reasoning as that adopted in **251** that the equation of equilibrium for any point **P** in the side span **AB** is

$$M = R_1 x - \frac{wx^2}{2} \tag{175}$$

whence as before,

$$tan\beta = \frac{l^2}{24EI}(3wl - 8R_1) \tag{176}$$

* For the elegant investigation in **253** and **254** the author is indebted to William B. Blood, Esq., sometime Professor of Civil Engineering in Queen's College, Galway.

The equation of equilibrium for any point in the centre span is

$$\mathbf{M} = \mathbf{R}_1 x + \mathbf{R}_2 (x - l) - wl\left(x - \frac{l}{2}\right) - \frac{w'}{2}(x - l)^2 \qquad (177)$$

Substituting for **M** its value in eq. 133,

$$\mathbf{EI}\frac{d^2y}{d^2x} = wl\left(x - \frac{l}{2}\right) + \frac{w'}{2}(x - l)^2 - \mathbf{R}_1 x - \mathbf{R}_2 (x - l)$$

Integrating and determining the constant by the consideration that $\frac{dy}{dx} = tan\beta$ when $x = l$, we have

$$\mathbf{EI}\frac{dy}{dx} = \mathbf{EI}\ tan\beta + \frac{w}{2}lx(x - l) + \frac{w'}{6}(x - l)^3 - \frac{\mathbf{R}_1 + \mathbf{R}_2}{2}(x^2 - l^2)$$
$$+ \mathbf{R}_2 l(x - l) \qquad (178)$$

which is the equation of the deflection curve from **B** to **C**.

Since $\frac{dy}{dx} = 0$ when $x = nl$, we have

$$tan\beta = \frac{l^2}{\mathbf{EI}}\left\{-\frac{l}{2}\left(n(n-1)w + \frac{(n-1)^3}{3}w'\right) + \frac{n^2 - 1}{2}(\mathbf{R}_1 + \mathbf{R}_2)\right.$$
$$\left. - (n - 1)\mathbf{R}_2\right\} \qquad (179)$$

also

$$\mathbf{R}_1 + \mathbf{R}_2 = l\left\{w + (n - 1)w'\right\} \qquad (180)$$

From eqs. 176, 179, and 180 we obtain the reactions of the points of support as follows:—

$$\mathbf{R}_1 = l\frac{(1{\cdot}5n - 1{\cdot}125)\,w - (n - 1)^3 w'}{3n - 2} \qquad (181)$$

$$\mathbf{R}_2 = l\frac{(1{\cdot}5n - 0{\cdot}875)\,w + (n^3 - 2n + 1)w'}{3n - 2} \qquad (182)$$

The distance of the point of inflexion in either side span from the abutment is obtained from eq. 175 by making $\mathbf{M} = 0$.

$$x = \frac{2\mathbf{R}_1}{w} \qquad (183)$$

The distances of the points of inflexion in the centre span from **A** are obtained from eq. 177 by making $\mathbf{M} = 0$, substituting for $\mathbf{R}_1$ its value in eq. 180, and solving the resulting quadratic, as follows:—

$$x = l\left\{n \pm \sqrt{n^2 - 1 + \frac{w}{w'} - \frac{2\mathbf{R}_2}{w'l}}\right\} \qquad (184)$$

The equation for the deflection of the side spans is the same as eq. 171.

That for the deflection at the centre of the centre span where $x = nl$, is obtained by integrating eq. 178 and determining the constant by the consideration that $y = 0$ when $x = l$ as follows:—

$$\mathsf{E}\,\mathsf{I}y = \frac{wl^4}{12}(2n^3-3n^2+1) + \frac{w'l^4}{24}(n-1)^4 - \frac{\mathsf{R}_1+\mathsf{R}_2}{6}l^3(n^3-3n+2)$$
$$+ \frac{\mathsf{R}_2 l^3}{2}(n-1)^2 + \mathsf{E}\,\mathsf{I}\,tan\beta\, l\,(n-1) \qquad (185)$$

The value of I for each form of cross section may be obtained from 70 and the following articles by the aid of eq. 131.

254. Three spans loaded uniformly.—If the girder be loaded uniformly throughout the three spans, $w = w'$, and the pressures on the points of support become

$$\mathsf{R}_1 = wl\left\{\frac{n^3-3n^2+\frac{3n}{2}+0{\cdot}125}{2-3n}\right\} \qquad (186)$$

$$\mathsf{R}_2 = wl\left\{\frac{n^3-\frac{n}{2}+0{\cdot}125}{3n-2}\right\} \qquad (187)$$

The distance of the point of inflexion in each side span from the abutment is as before,

$$n = \frac{2\mathsf{R}_1}{w} \qquad (188)$$

The distances of the points of inflexion in the centre span from A are

$$x = l\left\{n \pm \sqrt{n^2-\frac{2\mathsf{R}_2}{wl}}\right\} \qquad (189)$$

If the radicle in eqs. 184 or 189 vanish, there will be no strain at Q, and the centre span will be cambered throughout. If the value of R_1 in eqs. 181 or 186 be negative, the ends of the girder will be lifted off the abutments, owing to the excess of load on the centre span.

255. Maximum strains in flanges.—The maximum strains in the flanges occur as follows: in the side spans when the passing load covers both side spans, leaving the centre span free from load; in the centre span, when the passing load covers it alone, leaving both side spans free from load; and over either pier when the passing load covers the centre span and the adjacent side span, leaving the remote side span free from load. When the lengths of the component segments are determined, the strains in the flanges may be calculated by eqs. 13 and 24 if the girders are diagonally braced, or by eqs. 69, 80 and 105 if they are plate girders.

The hypothesis of the load being symmetrically disposed on either side of the centre prevents us from finding the points of inflexion when the segment over either pier is of maximum length; we have, however, a close approximation to its maximum length in the case of a passing load covering all three spans, and if desirable, a small extra allowance may be made for greater security. When the maximum length of the segment over either pier is thus determined, the calculation for the strains in its flanges are made as indicated in previous chapters, recollecting that each of these pier segments supports not only its own proper load, but also the weight of half the adjoining segments with their load suspended from its extremities by the vertical web.

256. Maximum strains in web—Ambiguity in calculation.—Though we obtain by these means the maximum strains of either kind to which the flanges are subject, it does not follow that we have also got the maximum strains in the web. Let *o*, for example, in Fig. 81, be the point of inflexion when the segment A*o* is of maximum length. Now this segment does not remain of this maximum length while a train is passing from A to B, that is, while the maximum strains are being produced in the web of A*o*; the point of inflexion is much closer to A when the train first comes upon the bridge (especially if the centre span happens to be traversed at the same time by another train), and gradually moves forward towards B as the train advances. It is incorrect therefore to calculate the maximum strains in the web on the hypothesis that A*o* is the length of the segment while the load advances. The maximum strain in a diagonal, at P for instance, takes place when the load covers AP, but the point of inflexion is then really nearer A than the point *o* is, and the maximum strain in the diagonal at P is therefore greater than if we assume the segment constant in length during the advance of the train. A similar or even greater uncertainty occurs in the centre span, for there neither end of the segment is fixed.

257. Permanent load, Shearing-strain.—When a continuous girder supports a fixed load, the strains in the web are not modified at the points of inflexion. The horizontal strains in the flanges change from tension to compression, or *vice versâ*, at these points, but the vertical or diagonal strains are transmitted through the web just as if no points of inflexion existed. The effect of contrary flexure is merely this: the horizontal increments of strain developed in the flanges *pull* from the piers in place of *thrusting* towards the centres of the component segments, and *vice versâ*. Hence, when a continuous girder of three, five, or any uneven number of spans is symmetrically loaded, the strains throughout the web of the centre span are the same as if the centre span were an independent girder supported at its extremities. This perhaps will be made clearer from the consideration that the shearing-strain at any section in the centre span, when the points of inflexion are symmetrical, is equal to the weight between the section and the centre of the span, and this is the case whether there be any point of inflexion or not. Thus the shearing-strain at any point f, Fig. 81, is equal to the load on fo' + that on $o'\mathbf{Q}$; but if the central span were an independent girder, resting on abutments at **B** and **C** and uniformly loaded, the shearing-strain at f would equal the load on $f\mathbf{Q}$, that is, it would be the same as before.

258. Advantages of continuity—Not desirable for small spans with passing loads, or where the foundations are insecure.—The advantage of continuity arises from two causes; first, from the smaller amount of material required in the flanges; secondly, from the removal of a certain portion of their weight from the central part of each span to a position nearer the piers. The latter is but a trifling advantage in continuous girders of moderate spans, say under 150 feet, which support passing loads, for the part so removed forms but a small proportion of the total weight. In the case of a fixed load, however, the saving from this cause is considerable; but when the load is passing the advantages of continuity are liable to be over-rated, especially in girders of small spans, for on a little reflection it will be evident that, when the points of inflexion move under the influence of the passing load, a greater amount of material is required than if their position remained stationary, and this moreover introduces the necessity of providing for both tension and compression in those parts of the flanges which lie within the range of the points of inflexion; this latter objection is perhaps of little consequence when wrought-iron is the material employed.

A subsidence of any of the points of support of a continuous girder will cause a change of strain whose amount it is quite impossible to foresee, and which may seriously injure the structure or perhaps render it dangerous. Hence continuous girders should be avoided where the foundations of the piers are insecure. In bridges of large span, where the permanent load constitutes a very large portion of the whole weight, the advantage of continuity is very considerable. The position of each point of inflexion alters but little with a passing load, and a considerable portion of the permanent weight, which would otherwise rest at or near the centre of each span, is brought close to the points of support.

CASE III.—GIRDERS OF UNIFORM SECTION FIXED AT BOTH ENDS AND LOADED UNIFORMLY.

Fig. 82.

259. Points of inflexion—Central flange-strain one-third of that in girders not fixed at the ends.—When both ends of a girder are built into a wall so as to be rigidly held there, the tangent to the girder at its intersection with the wall is horizontal, and the strains closely resemble those which occur in the centre span of a continuous girder of three spans when the load is so disposed that the tangents over the piers are horizontal.

Let l = the span from wall to wall,

w = the load per lineal unit,

M' = the moment of the horizontal elastic forces at the intersection of the girder with the wall (**58**),

M = the moment of the horizontal elastic forces at any cross section P,

x and y = the ordinates of P measured from a as origin,

I = the moment of inertia of any cross section round its neutral axis,

E = the coefficient of elasticity.

Taking moments round P (eq. 133),

$$\mathsf{M} = -\mathsf{EI}\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2} - \mathsf{M}' \qquad (190)$$

Integrating and determining that the constant = 0 from the consideration that $\frac{dy}{dx} = 0$ when $x = 0$,

$$\mathsf{EI}\frac{dy}{dx} = \frac{wx^3}{6} - \frac{wlx^2}{4} + \mathsf{M}'x$$

Making $x = l$ we have $\frac{dy}{dx} = 0$, and

$$\mathbf{M}' = \frac{wl^2}{12}$$

Substituting this value in eq. 190, we have

$$\mathbf{M} = -\frac{w}{2}\left(x^2 - lx + \frac{l^2}{6}\right)$$

At the points of inflexion $\mathbf{M} = 0$, and we have $x^2 - lx + \frac{l^2}{6} = 0$, whence

$$x = l\left(\frac{1}{2} \pm \frac{1}{\sqrt{12}}\right) = \cdot 211l \text{ or } \cdot 789l \qquad (191)$$

The length of the middle segment $= \cdot 578l$, and the central strain in either flange (eq. 26) $= \frac{(\cdot 578)^2 wl^2}{8d} = \frac{\cdot 334wl^2}{8d} = \frac{\cdot 0418wl^2}{d}$, in which $d =$ the depth of the girder. This central strain is just $\frac{1}{3}$rd of what it would be were the ends merely resting on the wall in place of being built therein. From eq. 13 we find that the strain in either flange at the wall $= \frac{\cdot 08324wl^2}{d}$, which is just double the strain at the centre of the flanges.

CHAPTER XI.

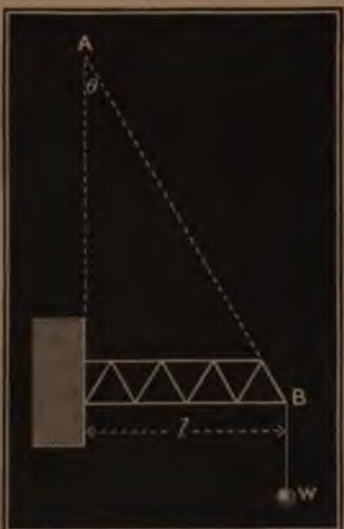
QUANTITY OF MATERIAL IN BRACED GIRDERS.

CASE I.—SEMI-GIRDERS LOADED AT THE EXTREMITY, ISOSCELES BRACING.

260. Web.

Let W = the weight at the extremity,
l = the length of the semi-girder,
d = its depth,
θ = the angle the diagonals make with a vertical line,
f = the unit-strain,
Q = the cubical quantity of material in the diagonals,
Q' = the cubical quantity of material in either flange.

Fig. 83.



The cubical quantity of material required for the diagonal bracing is equal to the sum of the products of the length and section of each brace. When the triangles are isosceles and the load is a single weight, the section, if proportional to the strain, is the same for all the diagonals, and the quantity of material is therefore equal to the product of their aggregate length by their common section. The line AB, Fig. 83, is equal in length to the sum of the several diagonals; expressing its length in terms of l and θ, we have

$$\mathsf{AB} = l\,\mathrm{cosec}\,\theta$$

The section of each brace is equal to the total strain passing through it divided by the unit-strain, $= \frac{\mathbf{W} sec\theta}{f}$ (eq. 108). Multiplying this by the foregoing value for the length we have

$$\mathbf{Q} = \frac{\mathbf{W}l}{f} sec\theta . cosec\theta \tag{192}$$

261. Flanges.—The quantity of material in the flanges is most conveniently deduced from the principles stated in Chapter II. as follows:—The sectional area of either flange at the wall $= \frac{\mathbf{W}l}{df}$ (eq. 8), and when the girder is of uniform strength gradually diminishes towards the extremity as the ordinates of a triangle (**20**). Hence the quantity of material in one flange equals its sectional area at the wall multiplied by $\frac{l}{2}$, and we have

$$\mathbf{Q}' = \frac{\mathbf{W}l^2}{2df} \tag{193}$$

CASE II.—SEMI-GIRDERS LOADED AT THE EXTREMITY, VERTICAL AND DIAGONAL BRACING.

Fig. 84.

262. Web.—When every alternate brace is vertical, as in Fig. 84, we must divide the material in the web into two parts, namely, that in the vertical, and that in the diagonal bracing.

Let $\mathbf{Q}$ = the quantity of material in the diagonals,

$\mathbf{Q}''$ = the quantity of material in the verticals, and the other symbols as before

The quantity of material required for the diagonal bracing is as before,

$$Q = \frac{Wl}{f} \sec\theta \cdot \operatorname{cosec}\theta \qquad (194)$$

The strain transmitted through each vertical $= W$; hence its sectional area $= \frac{W}{f}$. Multiplying this by the aggregate length of the verticals ($= l \cot\theta$), we have

$$Q'' = \frac{Wl}{f} \cot\theta. \qquad (195)$$

CASE III.—SEMI-GIRDERS LOADED UNIFORMLY, ISOSCELES BRACING.

268. Web, length containing a whole number of bays.—
Let W = the total weight resting on the girder,

n = the number of bays in the longest flange, supposed a whole number, and the other symbols as in case I.

When the bracing is formed of isosceles triangles the length of one bay equals $2d \cdot \tan\theta$, whence

$$l = 2nd \cdot \tan\theta. \qquad (196)$$

The quantity of material that the weight at any given apex would require in the bracing if it alone were supported by the girder may be obtained from eq. 192 by substituting for W and l the load resting on the apex $\left(= \frac{W}{n}\right)$ and the distance of the weight from the wall. The quantity required for the whole load is equal to the sum of the quantities required for the separate weights. Hence, recollecting that the weight on the last apex equals half that on each of the other apices (**146**), we have when there is no half bay in the length, that is, where n is a whole number,

$$Q = \frac{W}{fn} 2d.\tan\theta \left\{(1 + 2 + 3 + \ldots n) - \frac{n}{2}\right\} \sec\theta \cdot \operatorname{cosec}\theta$$

$$= \frac{W}{f} nd \cdot \tan\theta \cdot \sec\theta \cdot \operatorname{cosec}\theta.$$

Substituting for $nd \cdot \tan\theta$ its value in eq. 196 we have

$$Q = \frac{Wl}{2f} \sec\theta \cdot \operatorname{cosec}\theta \qquad (197)$$

264. Web, length containing a half-bay.—When the length contains a half-bay the quantity of material in the bracing, derived from eq. 192,

$$Q = \frac{Wl}{2f} sec\theta \,.\, cosec\theta + \frac{Wd^2}{2fl} sec^2\theta \,.\, tan\theta. \qquad (198)$$

265. Flanges.—From eq. 12 the area of either flange at the wall $= \frac{Wl}{2fd}$ and diminishes towards the extremity as the ordinates of a parabola (**24**), but from the well-known properties of the parabola the area of **ABC**, Fig. 7, equals one-third of the circumscribed rectangle. Hence the quantity of material in either flange equals its area at the wall multiplied by $\frac{l}{3}$, that is,

$$Q' = \frac{Wl^2}{6fd} \qquad (199)$$

CASE IV.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED AT AN INTERMEDIATE POINT, ISOSCELES BRACING.

266. Quantity of material in the web is the same for each segment.—Let W = the weight resting on the girder,

l = its length, and the other symbols as in Casė I.

Let the weight divide the girder into segments containing respectively m and n linear units, as in Fig. 51. The strains throughout the girder will in no respect be altered if we conceive it inverted, resting on a pier at **W**, and loaded with $\frac{m}{l}W$ at the right extremity and with $\frac{n}{l}W$ at the left. Each segment will then become a semi-girder loaded at its extremity. Hence the quantity of material in the bracing of each segment $= \frac{mnW}{fl} sec\theta \,.\, cosec\theta$ (eq. 192). The quantity of the material in the bracing of both segments together is equal to twice this, that is,

$$Q = \frac{2mnW}{fl} sec\theta \,.\, cosec\theta \qquad (200)$$

267. Flanges.—From eq. 21 the sectional area of either flange at the point where the weight rests $= \frac{nm\mathbf{W}}{fdl}$ and diminishes gradually towards each extremity as the ordinates of a triangle (**26**). Hence the quantity of material in one flange equals its area at the weight multiplied by $\frac{l}{2}$, and we have

$$\mathbf{Q}' = \frac{nm\mathbf{W}}{2fd} \qquad (203)$$

CASE V.—GIRDERS SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY, ISOSCELES BRACING.

268. Web, length containing an even number of bays.—
Let $\mathbf{W}$ = the total weight on the girder,

l = the length, and the other symbols as in Case I.

In order to avoid unnecessary minuteness in this case I shall first assume that the number of bays in the half-length is a whole number, in other words, that the length contains an even number of bays. Let us consider each half of the girder by itself; the vertical forces which act upon each half are the upward reaction of its abutment and the downward pressure of the weights between the abutment and the centre. The former pressure, if acting alone, would require a certain amount of material for the bracing, obtained by eq. 192, while the weights, leaving the reaction of the abutment out of consideration, would require an amount of material which may be obtained from eq. 197; since the latter forces tend to relieve the strain produced by the reaction of the abutment, the true quantity of material required is equal to the difference of the amounts which would be required were each set of forces to act independently of the other. Hence, subtracting eq. 197 from 192, and bearing in mind that $\mathbf{W}$ and l have twice the value they had in the semi-girder, we have the quantity of material in the web of the whole girder,

$$\mathbf{Q} = \frac{\mathbf{W}l}{4f}\, sec\theta \,.\, cosec\theta \qquad (202)$$

269. Web, the length containing an odd number of bays.—
If the half-length contain a half-bay the quantity of material in the bracing is obtained by subtracting eq. 198 from eq. 192, that is,

$$\mathbf{Q} = \frac{\mathbf{W}l}{4f}\, sec\theta \,.\, cosec\theta - \frac{\mathbf{W}d^2}{fl}\, sec^2\theta \,.\, tan\theta \qquad (203)$$

270. Flanges.—From eq. 26 the sectional area of either flange at the centre of the girder $= \frac{\mathbf{W}l}{8fd}$ and diminishes towards either end as the ordinates of a parabola (**47**). But the area of Fig. 19 equals ⅔rds of the circumscribed rectangle; hence the quantity of material required for either flange equals its central section multiplied by $\frac{2}{3}l$, and we have

$$\mathbf{Q}' = \frac{\mathbf{W}l^2}{12fd} \qquad (204)$$

CASE VI.—BOWSTRING GIRDERS UNIFORMLY LOADED.

271. Flanges.—When a bowstring girder is uniformly loaded the strains are nearly uniform and equal throughout both flanges (**214**); hence we can find a close approximation to the quantity of material by multiplying the length of each flange by its sectional area.

Let $\mathbf{W} =$ the total weight uniformly distributed over the girder,
$l =$ the length of the string,
$nl =$ the length of the bow,
$d =$ the depth of girder at centre,
$\mathbf{Q}' =$ the quantity of material in the string,
$\mathbf{Q}'' =$ the quantity of material in the bow,
$f =$ the unit-strain.

The strain at the centre of either flange $= \frac{\mathbf{W}l}{8d}$ (eq. 26): hence the sectional area of the flange $= \frac{\mathbf{W}l}{8df}$; multiplying this latter quantity by the respective lengths of the string and bow, we have

$$\mathbf{Q}' = \frac{\mathbf{W}l^2}{8df} \qquad (205)$$

$$\mathbf{Q}'' = \frac{n\mathbf{W}l^2}{8df} \qquad (206)$$

272. The following table contains the corresponding values of $\frac{d}{l}$ and n, the depth being expressed in fractional parts of the length

$\frac{d}{l}$.	n.
$\frac{1}{4}$	1·158
$\frac{1}{6}$	1·073
$\frac{1}{8}$	1·040
$\frac{1}{10}$	1·027
$\frac{1}{12}$	1·019
$\frac{1}{14}$	1·014
$\frac{1}{16}$	1·010

n, or the ratio of the bow to the string, is thus found.

Let $\lambda =$ the half span $= \frac{l}{2}$,

$r =$ the radius of the bow,

$\theta =$ the angle the bow subtends at the centre of the circle.

Fig. 85.

$$n = \frac{\text{length of bow}}{l} = \frac{r\theta}{2\lambda} \qquad (a)$$

also,
$$(2r - d)\,d = \lambda^2$$

whence,
$$r = \frac{\lambda^2 + d^2}{2d}$$

again,
$$\frac{d}{\lambda} = \tan\frac{\theta}{4}$$

whence,
$$\theta = 4\tan^{-1}\frac{d}{\lambda}$$

Substituting in eq. (a) these values for r and θ, we have

$$\frac{\lambda^2 + d^2}{d\lambda} \tan^{-1}\frac{d}{\lambda} = \left(\frac{\lambda}{d} + \frac{d}{\lambda}\right) \tan^{-1} \frac{d}{\lambda} \qquad (207)$$

whence we can obtain the values of n corresponding to different values of $\frac{d}{l}$.

CHAPTER XII.

ANGLE OF ECONOMY.

273. 45° is Angle of Economy for Isosceles bracing.—On examining those equations in the last chapter which express the quantity of material required for the vertical web of girders whose bracing consists of isosceles triangles, we find that they may all be expressed by one general equation,

$$Q = K sec\theta \,.\, cosec\theta,$$

in which K for each case is a constant quantity depending upon the length, weight, and unit-strain. Q is therefore proportional to the variable quantity $sec\theta \,.\, cosec\theta$, or to its equivalent $\frac{2}{sin 2\theta}$, which is a minimum when $\theta = 45°$. This proves that the angle of 45° is the most economical inclination for the diagonals of isosceles bracing, and it is to be observed that certain of the diagonals being in compression, and therefore practically requiring a greater amount of material to stiffen them than others, does not materially affect this conclusion (**240**); for, let the compression diagonals take m times the quantity of material they would require on the supposition that they were subject to tension in place of compression, then, since every alternate diagonal is in compression when the load is fixed (**172**), the foregoing expression becomes

$$Q = \frac{m+1}{2} K sec\theta \,.\, cosec\theta$$

but the variable part of this expression is $sec\theta \,.\, cosec\theta$ as before, and therefore the angle of economy is 45°.*

274. 55° is Angle of Economy for Vertical and diagonal bracing.—The angle of economy in girders with vertical and diagonal bracing differs from that in girders whose webs are formed of isosceles triangles. From eqs. 194 and 195 we find that the quantity of material in the bracing may be expressed as follows:—

$$Q + Q'' = K\,(sec\theta \,.\, cosec\theta + cot\theta).$$

* Mr. Bow first drew attention to the fact that 45° is the angle of economy for isosceles bracing; see his *Treatise on Bracing*. Edinburgh, 1851.

It is necessary to equate the differential coefficient of the bracketed part of this equation to cipher in order to find the value of θ which makes $Q + Q''$ a minimum. Doing so, we have

$$cosec\theta \,.\, sec\theta \,.\, tan\theta - sec\theta \,.\, cosec\theta \,.\, cot\theta - cosec^2\theta = 0,$$

dividing by $cosec\theta \,.\, sec\theta$ and transposing,

$$tan\theta = 2cot\theta$$

whence

$$tan\theta = \sqrt{2}, \text{ and } \theta = 54° \, 44' \, 8{\cdot}2'' = 55° \text{ nearly},$$

which therefore is the angle of economy for this form of bracing, and has moreover the merit of forming lozenge-shaped openings which have a more agreeable appearance than square ones.

275. Isosceles more economical than Vertical and diagonal bracing.—The superior economy of the isosceles over the vertical and diagonal system of bracing will be now apparent, for the quantity of material required in the latter exceeds that in the former by an amount never less than Q'', and exceeds Q'' when θ differs from 45°.

276. Trigonometrical functions of θ.—The following table contains the value of different trigonometrical functions of θ.

Angle of bracing, θ.	$Sec\theta$.	$Sec\theta \,.\, cosec\theta$.	$Cot\theta$.	$Sec\theta \,.\, cosec\theta + cot\theta$.	$Tan\theta$.
20°	1·064	3·11	2·747	5·857	·364
25°	1·103	2·61	2·144	4·754	·466
30°	1·154	2·31	1·732	4·041	·577
35°	1·221	2·13	1·428	3·557	·700
40°	1·305	2·03	1·192	3·222	·839
45°	1·414	2·00	1·000	3·000	1·000
50°	1·515	2·03	·839	2·869	1·192
55°	1·743	2·13	·700	2·829	1·428
60°	2·000	2·31	·577	2·886	1·732
65°	2·369	2·61	·466	3·076	2·144
70°	2·924	3·11	·364	3·474	2·747

277. Relative economy of different kinds of bracing.—By means of this table we can at once compare the relative economy of different descriptions of bracing as follows:—

Values of θ.	Value of Q.	Comparative quantities of material required in web.
Isosceles bracing, . . $\theta = 45°$	$Q = 2 \cdot 00\ K$	100
Ditto (Warren's girder), . $\theta = 30°$	$Q = 2 \cdot 31\ K$	115·5
Vertical and diagonal bracing, $\theta = 55°$	$Q \times Q'' = 2 \cdot 83\ K$	141·5

Hence equilateral bracing ("Warren's girder") requires 15½ per cent., and vertical and diagonal bracing of the best form requires 41½ per cent. more material in the web than isosceles bracing at an angle of 45°.

278. Quantity of material in the bracing independent of depth.—The reader will observe that the depth of the girder does not enter into those equations which express the quantity of material required in the bracing, whereas it enters into the denominator of those which express the quantity of material in the flanges. Hence we conclude that altering the depth of braced girders does not affect the amount of bracing (**18**); but the quantity of material in the flanges varies inversely as the depth, and consequently the deeper a girder is made the greater will be the economy, theoretically speaking. In practice, the additional material required to stiffen long struts generally defines the limit to which this increase of depth can be judiciously extended; but of this in succeeding chapters.