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On Transient Solutions of the "Baffled Piston" Problem

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The acoustic field produced by the movement of a piston in an infinite rigid wall for arbitrary time dependency of the motion is given.

1. Introduction

The case of the time harmonic movement of a piston membrane in an infinite rigid wall ("baffle") can easily be generalized to the case when the motion of the piston is not periodic but an arbitrary function of time. Such transient solutions have become of considerable interest in recent times (for a detailed treatment of the propagation of such sound pulses, see [5][2]). The procedure for the case treated here is the same as used elsewhere [7], i.e., the Green's function for the exponential decay case (modified wave equation $\Delta U-\gamma^2U=0$, $\gamma=ik$) is used to obtain the solution for the pulse problem. The accustic field (velocity potential) for the time harmonic movement of the piston include representations given by Bouwkamp [1], King [6], and Wells and Leitner [9]. The first of these contributions gives the solution in the form of a series expression while the second and third involve integral representations that are obtained using integral transform methods (Hankel transform [4, p. 73] and Lebedev transform [4, p. 75] respectively). These representations can be used to treat the general case of an arbitrary movement of the piston. In view of the method to be employed here, such representations should be used for which the inverse Laplace transform of the velocity potential with respect to the purely imaginary wave parameter $\gamma=ik$ can be given. Such an expression can be obtained in a direct way by regarding each point of the moving disk as an accustic point source and integrating over all points of the disk.

2. Exponential Decay and Time Harmonic Solution

The piston is represented by an infinitely thin circular disk of radius a located in the x,y-plane with its center at the origin of a three-dimensional Cartesian system of coordinates. The remaining part of the x,y-plane consists of a rigid wall. The movement of the disk is time harmonic of the form

$$v_z=e^{i\omega t} \tag{1}$$

where v_z is the velocity of the disk.

Each point Q of the disk can be considered as a point source with the velocity potential

$$\Phi=-\eta\,\frac{e^{-ikr}}{4\pi r}\,e^{i\omega t}\rho'd\varphi'd\rho', \quad v=PQ, \tag{2}$$

($\omega=kc$, c is the velocity of sound), which represents Green's function of the wave equation in free space (see fig. 1).

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[2] Figures in brackets indicate the literature references at the end of this paper.

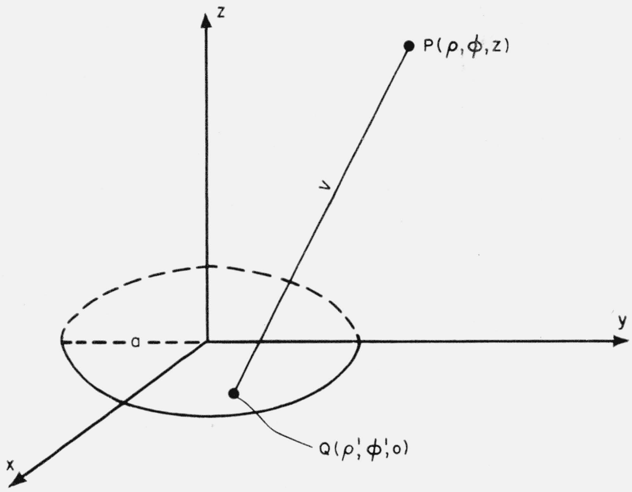

FIGURE 1.

Upon substituting

$$k = -i\gamma, \quad \omega = -i\gamma c, \tag{3}$$

one obtains, after isolation of the time-dependent factor $e^{\gamma ct}$ for the free-space Green's function of

$$\Delta u - \gamma^2 u = 0, \tag{4}$$

the expression

$$u = -\eta \frac{e^{-\gamma r}}{4\pi r} \rho' d\varphi' d\rho', \tag{5}$$

$$r = [\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + z^2]^{1/2}. \tag{6}$$

The transition from k to γ by (3) amounts to the transition from a wave problem to an exponential decay problem. At first the exponential decay problem for positive real γ is solved and the solution for the wave problem can be obtained by returning from γ to k. This, however, is not necessary here, since the exponential decay solution will be used in order to solve the problem of an arbitrary moving disk. If the expression in (5) is integrated over the whole disk, one obtains—observing (6):

$$U = -\frac{\eta}{4\pi} \int_{\varphi'=0}^{2\pi} \int_{\rho'=0}^{a} [\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + z^2]^{-1/2} \cdot \exp\{[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + z^2]^{1/2}\} \rho' d\rho' d\varphi'. \tag{7}$$

Here η is the "density distribution" over the disk. Since the disk moves as a whole, this property is a constant; it is determined by the fact that according to (1)

$$v_z = -\left(\frac{\partial U}{\partial z}\right)_{z=0} = \begin{cases} 1, \text{ for } \rho < a, \\ 0, \text{ for } \rho > a. \end{cases} \tag{8}$$

This function U as defined in (7) satisfies

(a) $$\Delta U - \gamma^2 U = 0,$$

(b) $$\frac{\partial U}{\partial z} = -1 \text{ for } \rho < a, \quad \frac{\partial U}{\partial z} = 0 \text{ for } \rho > a \quad \text{at } z = 0,$$

(c) $$\lim U \to 0 \text{ when } \rho \text{ or } |z| \to \infty.$$

In order to perform the integration in (7), one replaces the integrand in (7) using the well-known formula [3, vol. 1, p. 191, 9],

$$\int_{\gamma}^{\infty} J_0[\lambda(t^2-\gamma^2)^{1/2}]e^{-zt}dt=(z^2+\lambda^2)^{-1/2}e^{-\gamma(z^2+\lambda^2)^{1/2}}, \quad z\geqq 0$$

or, upon substituting

$$t^2-\gamma^2=\tau^2, \; t^2=\tau+\gamma^2$$

$$(z^2+\lambda^2)^{-1/2}e^{-\gamma(z^2+\lambda^2)1/2}=\int_0^{\infty} J_0(\lambda\tau)\tau(\tau^2+\gamma^2)^{-1/2}e^{-z(\tau^2+\gamma^2)^{1/2}}d\tau \tag{9}$$

If this formula is used to replace the integrand in (7), one obtains upon interchanging the order of integration

$$U=-\frac{\eta}{4\pi}\int_0^{\infty}\tau(\tau^2+\gamma^2)^{-1/2}e^{-z(\tau^2+\gamma^2)^{1/2}}\times\left\{\int_0^{\alpha}\int_0^{2\pi}J_0(\tau[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')]^{1/2})\rho'd\rho'd\varphi'\right\}d\tau. \tag{10}$$

To evaluate the integral in (10), one uses the addition theorem for the Bessel functions [4, p. 101]

$$J_0\{\tau[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')]^{1/2}\}=\sum_{m=0}^{\infty}\epsilon_m J_m(\tau\rho)J_m(\tau\rho')\cos[m(\varphi-\varphi')],$$

$$\epsilon_0=1, \quad \epsilon_m=2, \quad m=1,2,3, \ldots$$

Integrating with respect to φ' and observing that only the term $m=0$ gives a contribution, yields

$$U=-\frac{1}{2}\eta\int_0^{\infty}\tau(\tau^2+\gamma^2)^{-1/2}e^{-z(\tau^2+\gamma^2)1/2}J_0(\tau\rho)\left\{\int_0^a\rho' J_0(\tau\rho')d\rho'\right\}d\tau.$$

Finally, using the result [4, p. 45, 1]

$$\int_0^a J_0(\tau\rho')\rho'd\rho'=\frac{a}{\tau}J_1(\tau a),$$

one obtains

$$U=-\frac{1}{2}a\eta\int_0^{\infty}(\tau^2+\gamma^2)^{-1/2}e^{-z(\tau^2+\gamma^2)^{1/2}}J_0(\tau\rho)J_1(\tau a)d\tau. \tag{11}$$

Now, according to (1)

$$-\left(\frac{\partial U}{\partial z}\right)_{z=0}=1, \quad \rho<a; \qquad -\left(\frac{\partial U}{\partial z}\right)_{z=0}=0, \quad \rho>a.$$

therefore by [3, vol. 2, p. 14],

$$-\left(\frac{\partial U}{\partial z}\right)_{z=0}=-\frac{1}{2}a\eta\begin{cases}a^{-1}, & \rho<a,\\ 0, & \rho>a.\end{cases}$$

Hence

$$\eta=-2.$$

The final representation becomes therefore

$$U=a\int_0^{\infty}(\tau^2+\gamma^2)^{-1/2}e^{-z(\tau^2+\gamma^2)^{1/2}}J_0(\tau\rho)J_1(\tau a)d\tau, \tag{12}$$

$$z\geqq 0.$$

Clearly U satisfies the conditions a, b, c, stated following (8). The restriction $z\geqq 0$ in (12) means no loss of generality, since it is sufficient to consider the upper half space. If one wishes to return to the wave problem one has to replace γ by ik according to (3) and one arrives in this case at King's solution [6].

Approximate evaluations of King's and related integrals for various cases have been carried out [2, 6, 8].

3. Transient Solutions

The expression (12) is now used to investigate the corresponding pulse problem. Let the movement of the piston be represented by a pulse starting at $t=0$; let the pulse function be $g(t)$, and $g(t)=0$ for $t<0$. Under the assumption that the pulse function $g(t)$ is such that it admits its representation by Laplace's integral formula

$$g(t)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\left[\int_0^\infty g(\tau)e^{-\gamma\tau}d\tau\right]e^{\gamma t}d\gamma, \tag{13}$$

one can represent the expression for the acoustic field generated by the pulse in the form

$$\Phi(t)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\left[\int_0^\infty g(\tau)e^{-\gamma\tau}d\tau\right]Ue^{\gamma t}d\gamma \tag{14}$$

where U is the representation (12) for the "exponential decay" field. (The velocity of sound has been chosen to be unity.)

If the order of integration can be interchanged one obtains

$$\Phi(t)=\int_0^\infty g(\tau)\left[(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} Ue^{\gamma(t-\tau)}d\gamma\right]d\tau. \tag{15}$$

The integral in the bracket is the inverse Laplace transform of the "exponential decay" solution with respect to the wave parameter γ. If this is known, then the velocity potential $\Phi(t)$ of the acoustic field for the case of an arbitrary movement of the piston can be obtained by a further integration involving the pulse function $g(t)$ according to (15). The inverse Laplace transform of the integrand in (12) with regard to γ is known [3, vol. 1, p. 248]:

$$(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}(\tau^2+\gamma^2)^{-\frac{1}{2}}e^{-z(\tau^2+\gamma^2)^{1/2}}e^{\gamma t}d\gamma=\begin{cases}J_0[\tau(t^2-z^2)^{\frac{1}{2}}]\\0\end{cases} \tag{16}$$

according as $t>z$ or $t<z$, respectively.

The acoustic velocity potential due to a pulse function $g(t)$ ("Dirac" pulse movement of the piston at $t=0$) is by (14)

$$\Phi_D(t)=2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}U_o{}^{\gamma t}d\gamma. \tag{17}$$

Hence, (15) can be written as

$$\Phi(t)=\int_0^\infty g(\tau)\Phi D(t-\tau)\,d\tau. \tag{18}$$

Therefore, by (12) and (16)

$$\Phi_D(t)=\begin{cases}0, & t<z\\ a\int_0^\infty J_1(\tau a)J_0(\tau\rho)J_0[\tau(t^2-z^2)^{\frac{1}{2}}]d\tau, & t>z.\end{cases} \tag{19}$$

The integral is known. One has [3, vol. 2, p. 21]:

$$\int_0^\infty J_0(bx)J_0(cx)J_1(xy)dx=\begin{cases}0, & 0<y<|b-c|\\ (\pi y)^{-1}\text{ arc cos}\left(\dfrac{b^2+c^2-y^2}{2bc}\right), & |b-c|<y<b+c\\ y^{-1}, & y>b+c\end{cases} \tag{20}$$

In order to evaluate (19) identify y with a, b with $(t^2-z^2)^{1/2}$ and c with ρ. Two different cases have to be distinguished:

$$\text{Case A.} \qquad a>\rho,$$

$$\text{Case B.} \qquad a<\rho,$$

i.e., the two possible different cases where the projection S of the point of observation P into the x,y-plane lies either inside or outside the disk are treated separately (fig. 2 and fig. 3 respectively).

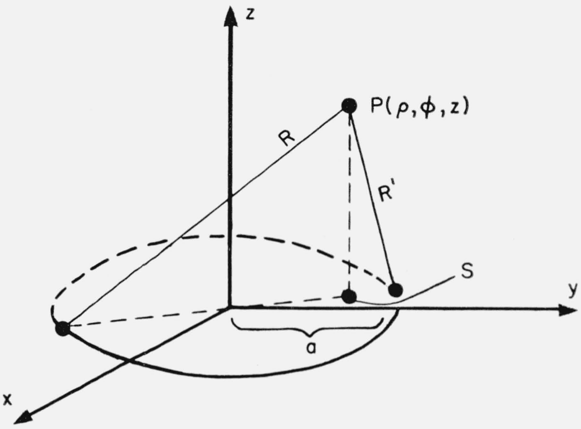

FIGURE 2.

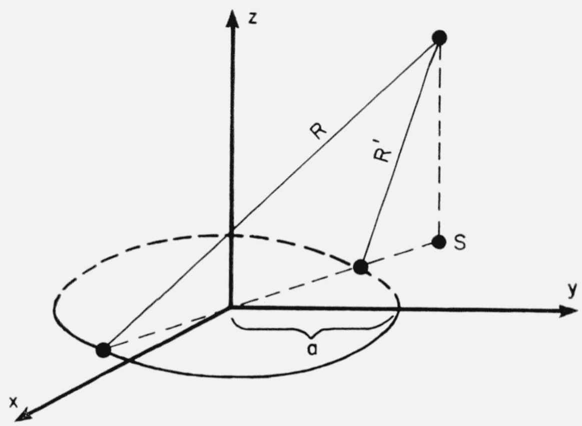

FIGURE 3.

The largest and the smallest distances of a point on the circumference of the disk from the point of observation P are denoted by R and R' respectively.

$$R=[z^2+(a+\rho)^2]^{1/2}, \qquad R'=[z^2+(a-\rho)^2]^{1/2}. \tag{21}$$

One obtains then for $\Phi_D(t)$ by (19) under consideration of (20) and (21):
For the case A, $a>\rho$:

$$\begin{aligned}
\Phi_D(t)&-0, && t<z \\
&=1, && z<t<R' \\
&=\pi^{-1}\arccos\left[\frac{t^2-z^2+\rho^2-a^2}{2\rho(t^2-z^2)^{1/2}}\right], && R'<t<R \\
&=0, && t>R.
\end{aligned} \tag{22}$$

For the case B, $a<\rho$:

$$\begin{aligned}
\Phi_D(t)&=0, && t<R' \\
&=\pi^{-1}\arccos\left[\frac{t^2-z^2+\rho^2-a^2}{2\rho(t^2-z^2)^{1/2}}\right], && R'<t<R \\
&=0, && t>R.
\end{aligned} \tag{23}$$

The physical interpretation of these results is obvious. The parts in (22) and (23) that are different from zero represent the transient acoustic field at a point P due to a "Dirac" pulse movement of the piston for a time which lies between the smallest and the largest distance of the point of observation P from the disk. Outside of this the field is zero. (Note that the velocity of sound was assumed to be unity.) The "Dirac" pulse solutions (22) and (23) can now be used to construct the acoustic field in case of an arbitrary time dependency $g(t)$ of the movement of the piston according to (18). One obtains immediately:

For the case A, $a>\rho$:

$$\Phi(t)=\pi^{-1}\int_{t-R}^{t-R'} g(\tau) \arccos\left\{\frac{(t-\tau)^2+\rho^2-a^2-z^2}{2\rho[(t-\tau)^2-z^2]^{1/2}}\right\}d\tau+\int_{t-R'}^{t-z} g(\tau)d\tau. \tag{24}$$

For the case B, $a<\rho$:

$$\Phi(t)=\pi^{-1}\int_{t-R}^{t-R'} g(\tau) \arccos\left\{\frac{(t-\tau)^2+\rho^2-a^2-z^2}{2\rho[(t-\tau)^2-z^2]^{1/2}}\right\} d\tau. \tag{25}$$

The case $\rho=0$ (point of observation on the axis of the disk) reduces to

$$\Phi(t)=\int_{t-(a^2+z^2)^{1/2}}^{t-z} g(\tau)d\tau. \tag{26}$$

It has to be remembered that $g(t)=0$ for $t<0$. Therefore no possibly negative part of the interval of integration in (24) to (26) gives a contribution. Furthermore, the time parameter t in the above formulas has to be replaced by ct, where c is the velocity of sound.

4. References

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