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# UNITED STATES DEPARTMENT OF THE INTERIOR BUREAU OF MINES HELIUM ACTIVITY HELIUM RESEARCH CENTER

# INTERNAL REPORT

APPLICATION OF THE METHOD OF THE AUXILIARY MATRIX IN EVALUATING

VIRIAL COEFFICIENTS FROM PVT DATA

BY

B. J. Dalton

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Fundamental Research

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# APPLICATION OF THE METHOD OF THE AUXILIARY MATRIX IN EVALUATING VIRIAL COEFFICIENTS FROM PVT DATA

by

B. J. Dalton<sup>1</sup>/

# ABSTRACT

An important application of the method of the auxiliary matrix is in the least squares evaluation of virial coefficients from PVT data. This analytical method for solving constants from general equations containing these coefficients is much shorter than a solution by determinents and is most desirable for desk calculation work as the only writing involved is that of writing the auxiliary matrix and the final results.

This report gives a brief summary of the method of the auxiliary matrix, previously published by Crout, and its application to rational, integral functions of one to four degrees together with formulas for evaluating the standard error in a single measurement, the standard error in each coefficient, and the standard error of these functions.

<u>1</u>/ Chemist (Physical), Helium Research Center, Bureau of Mines, Amarillo, Texas.

Work on manuscript completed November 1964

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> [] Chemist (Physical), Helium Hasearch Center, Sureau of Mines, Amarillo, Texas.

> > Work on manuscript completed November 190

The formulas presented in this report for evaluating the standard errors mentioned above are expressed in terms of the elements of the auxiliary matrix. These formulas were derived by expressing the constants in terms of the original data and applying the law for the "Propagation of Errors", unit weight being assigned to each of the observed measurements.

# INTRODUCTION

In a previous publication  $(3)^{2/2}$ , formulas were presented for eval-

<u>2</u>/ Underlined numbers in parentheses refer to items in the list of references at the end of this report.

uating the constants of rational integral functions of one to four degrees together with formulas for calculating the standard error of a single measurement, the standard error in each coefficient, and the standard error of the resulting function. The derivations for calculating the errors mentioned above were based on the law for the "Propagation of Errors" (1, 4), unit weight being given to each of the observed measurements.

After publication, this author became aware of the Crout Method  $(\underline{2})$ , the method of the auxiliary matrix. This method is a modification of the elimination method and it is most desirable for desk calculation work. It is a fact that this analytical method for solving a set of normal equations is much shorter than a solution by determinents as The formulas presented in this report for evaluating the standard errors mantioned above are expressed in terms of the elements of the surfliary matrix. These formulas were derived by expressing the constants in terms of the original data and applying the law for the "Propagation of Errors", unit weight being sasigned to each of the observed measurements.

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The purpose of this report is to give, without proof, the Crout Method as applied to rational, integral functions of one to four degrees together with formulas which I have derived for evaluating the standard error in a single Y<sub>i</sub>, in each coefficient, and in the resulting function. The formulas presented for evaluating these errors are expressed in terms of the elements of the auxiliary matrix and they were derived on the basis of the law for the "Propagation of Errors", each observation having unit weight.

# CALCULATION OF THE STANDARD ERROR

An important application of the method of the auxiliary matrix is in the least squares evalation of the coefficients from a general equation containing these constants. For example, the PV product of a gas is represented as a function of either the pressure or the density by an equation such as

$$Y = A + Bx_1 + Cx_2 + Dx_3 + Ex_4 + \dots$$

in which Y is the PV product;  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ , ... are forms of either the pressure or the density variable; and A, B, C, D, E, ... represent so-called virial coefficients to be determined by least squares solution using the Crout method.

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# Y = A + Bxy + Cxg + Dxg + A = Y

in which Y is the FV product; x1, x2, x3, x, ... are force of either the pressure of the doualty variable; and A, S, C, D, E, ... represent so-called virial coefficients to be determined by least squares The form of the general equation given above is desirable when considered from the standpoint of the independent variable. That is, the PV-dependent variable may or may not be expressed in terms of a power series expansion

$$x = x_1; x^2 = x_2; x^3 = x_3; x^4 = x_4; \dots$$

of the independent variable.

The following symbolism has been used in the derivation of all equations for evaluating standard errors contained in this report:  $Y_i, x_{1_i}, x_{2_i}, \dots$  designate observed values; Y refers to the function;  $s_A^2, s_B^2, \dots$  are the variances in the coefficients;  $S_{Y_i}^2$  and  $S_{Y_i}$  are the variance and the standard error in a single  $Y_i$ , respectively;  $S_Y^2$  and  $S_Y$  are the variance and the standard error in the function, respectively; R and C represent the words row and column, respectively, and refer to those elements of the given coefficient matrix; and r and c represent the words row and column, respectively, and refer to those elements of the auxiliary matrix.

Now if one has a function, F, of a number of independently observed quantities,  $Y_1$ ,  $Y_2$ ,  $Y_3$ , ..., whose standard errors  $S_{Y_1}$ ,  $S_{Y_2}$ ,  $S_{Y_3}$ ,  $S_{Y_3}$ , ... are known, then the variance,  $S_F^2$ , in F is given by the formula (<u>1</u>, <u>4</u>) for the "Propagation of Errors":

$$s_{F}^{2} = \left(\frac{\partial F}{\partial Y_{1}} \cdot s_{Y_{1}}\right)^{2} + \left(\frac{\partial F}{\partial Y_{2}} \cdot s_{Y_{2}}\right)^{2} + \dots + \left(\frac{\partial F}{\partial Y_{n}} \cdot s_{Y_{n}}\right)^{2}$$
(1)

the summation being over i for 1 to n for the n observations. Taking the partial derivative of equation (5) with respect to first A and The form of the general equation given above is desirable when considered from the standpoint of the independent variable. That is, the PV-dependent variable may or may not be expressed in terms of a power series expansion

 $x = x_1 : x^2 = x_2 : x^3 = x_3 : x^4 = x_4 : \dots$ 

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The following symbolism has been used in the derivation of all equations for evaluating standard errors contained in this report:  $Y_1$ ,  $x_1$ ,  $x_2$ , ... designate observed values; Y refers to the function;  $S_1^2$ ,  $S_3^2$ , ... are the variances in the coefficients;  $S_{Y_1}^2$  and  $S_{Y_1}$  are the variance and the standard error in a single  $Y_{v}$ , respectively;  $S_Y^2$  and  $S_Y$  are the  $S_Y$  are the variance and the standard error in a single  $Y_v$ , respectively;  $S_Y^2$  and to the sectively the function, respectively the function, respectively the coefficient error in the function, respectively the function for the standard error in the function, respectively the function, respectively for the standard error in the function, respectively and refer to those elements of the given coefficient matrix; and r and r represent the words row and column, respectively, and refer to those elements of the given coefficient matrix; and r and r represent the words row and column, respectively, and refer to those elements of the given coefficient matrix; and refer to those elements of the given coefficient matrix; and refer to those elements of the given coefficient matrix; and refer to those elements of the given coefficient matrix; and refer to those elements of the surfice of

Now if one has a function,  $\mathbb{P}$ , of a number of independently observed quantities,  $\mathbb{Y}_1$ ,  $\mathbb{Y}_2$ ,  $\mathbb{Y}_3$ , ..., whose standard errors  $\mathbb{S}_2$ ,  $\mathbb{S}_2$ ,  $\mathbb{S}_3$ , ... are known, then the variance,  $\mathbb{S}_2^2$ , in  $\mathbb{P}$  is given by the formula  $(\underline{1}, \underline{4})$ for the "Propagation of Errors":

$$\frac{2}{F} = \left(\frac{3\Gamma}{3Y_1} \cdot S_{Y_1}\right)^2 + \left(\frac{3F}{3Y_2} \cdot S_{Y_2}\right)^2 + \dots + \left(\frac{3F}{3Y_n} \cdot S_{Y_n}\right)^2$$
(1)

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Extracting the square root of the variance, one obtains a value on the same scale as the original measurements; this value is called the standard error or standard deviation.

Suppose we have a function of the kind

$$Y = A + B_{X_1}$$
(2)

where A and B are to be evaluated by least squares solution (using the method of the auxiliary matrix) along with the following errors: the standard error of a single measurement; the standard error in each coefficient; and the standard error of the resulting function. The estimated variance,  $S_{Y_i}^2$ , of a single  $Y_i$  is given as

$$S_{Y_{i}}^{2} = \frac{\sum (i_{obs} - Y_{i}_{cal})^{2}}{n - 2}$$
(3)

Now in order to evaluate the standard error in both intercept and slope, we must express these quantities in terms of the original data. The residuals, W<sub>i</sub>, are

$$W_{i} = (Y_{i} - A - Bx_{1})$$
 (4)

and the sum of the squares of the residuals is

where A.C. - n.

$$\Sigma W_{i}^{2} = \Sigma (Y_{i} - A - Bx_{1_{i}})^{2}$$
(5)

the summation being over i for 1 to n for the n observations. Taking the partial derivative of equation (5) with respect to first A and Extracting the equare root of the variance, one obtains a value on the same scale as the original measurements; this value is called the standard error or simulari deviation.

Suppose we have a function of the kind

where A and Nore to be sealuated by later squares solution (using the method of the sumiliary matrix) along with the following errors: the standard error of a single measurement; the standard error in seccoefficient; and the standard error of the resulting (smatter, T); estimated variance, 5<sup>2</sup>, of a single T, is given as

$$\frac{1}{(1-\frac{1}{2})^2} = \frac{1}{(1-\frac{1}{2})^2} = \frac{1}{(1-\frac{1}{2})^2}$$

Now in order to evaluate the standard error in both intercept and slope, we must express these quantities in terms of the original date. The residuals, M., sto

$$(a) = (x_1 - b - bx_1)$$
 (a)

and the sum of the squares of the residuals is

$$(e) = E(Y_1 - h - 0x_1)^2$$
 (s)

the summation boing over 1 for 1 to n for the n observations. Taking the summation boing over 1 for 1 to not separate

then B and setting each derivative equal to zero gives the normal equations

$$(Y_1 - A - Bx_{1_1}) + \dots + (Y_n - A - Bx_{1_n}) = 0$$
 (6)

$$x_{1_1} (Y_1 - A - Bx_{1_1}) + \dots + x_{1_n} (Y_n - A - Bx_{1_n}) = 0$$
 (7)

or their equivalents

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wh

R<sub>2</sub>

$$An + B\Sigma x_{1_{i}} = \Sigma Y_{i}$$
(8)

$$A \Sigma x_{1_{i}} + B\Sigma x_{1_{i}}^{2} - \Sigma x_{1_{i}}^{Y}$$
(9)

Now equations (8) and (9) can be written as a general coefficient matrix as

n 
$$\Sigma x_{1_{i}}$$
  $\Sigma Y_{i}$  (Matrix 10)  
 $\Sigma x_{1_{i}}$   $\Sigma x_{1_{i}}^{2}$   $\Sigma x_{1_{i}}^{\gamma} Y_{i}$ 

or, abbreviating the above, using the letters R and C to represent the words row and column, respectively, as

$$\begin{vmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{vmatrix}$$
(Matrix 11)  
ere  $R_1C_1 = n$ ;  $R_1C_2 = \Sigma x_{1_i}$ ;  $R_1C_3 = \Sigma Y_i$ ;  $R_2C_1 = \Sigma x_{1_i}$ ;  $R_2C_2 = \Sigma x_{1_i}^2$ ; and  
 $C_3 = \Sigma x_{1_i} Y_i$ . The solution of the constants of equation (2) requires

then 5 and metting each derivative equal to arro given the normal

,

$$(x_1 - A - Bx_1) + \dots + (x_n - A - Bx_1) = 0$$
 (6)

$$i_{1} \left( i_{1} - A - B x_{1} \right) + \dots + x_{1} \left( i_{n} - A - B x_{1} \right) = 0 \quad (7)$$

or their equivalents

Now equations (8) and (9) can be written as a general committene antitu

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or, abbreviating the above, mater the lockers & and C to represent the words row and column, respectively, as

where  $R_1C_1 = n$ ;  $R_1C_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n}$ 

and our final results are

$$B = r_2 c_3 \tag{13}$$

$$A = r_{1}c_{3} - r_{1}c_{2} \cdot B$$
 (14)

The expressions derived by me for evaluating the variances in A and in Bare of the form

$$s_{A}^{2} = \left(\frac{r_{2}c_{2}}{R_{1}c_{1}}\right) \cdot s_{B}^{2} - (r_{1}c_{2}) \cdot s_{AB}^{2}$$
 (15)

$$s_{B}^{2} = S_{Y_{i}}^{2} \left(\frac{1}{r_{2}c_{2}}\right)$$
 (16)

where  $s_{AB}^2$  is

$$s_{AB}^2 = (-r_1 c_2) \cdot s_B^2$$
 (17)

From the variance in a single  $Y_i$ , in A, and in B, we can evaluate the variance in our function, equation (2). This is done by applying the general equation for the "Propagation of Errors", equation (1). To avoid repetition, familarity with the techniques given in reference 3 is assumed. The formula for evaluating the variance in our function, equation (2), can be determined from the equation (3): that we evaluate an autiliary matrix and a final regult. Now our . autiliary matrix is of the form

and our final results are

$$h = r_2 r_3$$
 (13)

The expression derived by as for evaluating the variance in A and

$$u_{A}^{2} = \left(\frac{\tau_{A} c_{A}}{x_{1} c_{1}}\right) = u_{B}^{2} - \left(\tau_{1} c_{2}\right) + u_{AB}^{2}$$
(15)

$$r_{\rm B}^{2} = s_{\rm T_1}^2 \left(\frac{1}{r_2 c_2}\right)$$
 (16)

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$$a_{AB}^{2} = (-r_{1}c_{2}) - a_{B}^{2}$$
 (17)

From the variance in a single Y<sub>1</sub>, is A, and in B, we can evaluate the variance in our function, equation (2). This (a done by applying the general equation for the "Fromgation of Errors", equation (1). To avoid resettion, familarity with the techniques given in reference is assumed. The formula for evaluating the variance in our function, equation (2), can be determined from the equation (1):

$$S_{Y}^{2} = s_{A}^{2} + 2x_{1}s_{AB}^{2} + x_{1}^{2}s_{B}^{2}$$
 (18)

# 1. <u>Rules for obtaining the auxiliary matrix from the given coefficient</u> <u>matrix</u>

The steps to follow in going from the given coefficient matrix, (Matrix 11), to the auxiliary matrix, (Matrix 12), are as follows:

(1) The first column of the auxiliary matrix is identical to the first column of the given coefficient matrix. Each element of the first row of the auxiliary matrix, except the first element, is obtained by dividing the corresponding element of the coefficient matrix by the first element of the coefficient matrix.

(2) "Each element on or below the principal diagonal (of the auxiliary matrix) is equal to the corresponding element of the given matrix (Matrix 11) minus the sum of those products of elements in its row and corresponding elements in its column (in the auxiliary matrix) which involve only previously computed elements" (2).

(3) "Each element to the right of the principal diagonal (of the auxiliary matrix) is given by a calculation which differs from rule 3 (rule 2 of this report) only in that there is a final division by its diagonal elment (in the auxiliary matrix)" (2).

Suppose we follow the above steps and proceed to go from our coefficient matrix to our auxiliary matrix. Now the coefficient matrix, in abbreviated form, is given as

# 

# 1. Rules for obtaining the Auxiliary matrix from the given coefficient

The steps to follow in going from the given coefficient estein.

(1) The first column of the suxfilery matrix is identical to the first column of the sives medificted matrix. Each element of the first row of the sumfliery matrix, except the first element, is obtained by dividing the corresponding element of the coefficient matrix by the

(2) "Each element on or below the principal disgonal (af the auxiliary matrix) is equal to the corresponding element of the given matrix (Matrix 11) whole the sum of those products of elements in its row and corresponding elements in its column (in the sumiliary matrix) which involve only previously computed elements" (2).

(3) "Each element to the right of the principal disgumal (of the sumfiliary matrix) is given by a calculation which differs from rule 3 (rule 2 of this report) only in that there is a final division by its disgonal element (is the auxiliary matrix)" (2).

Suppose we follow the shove stars and proceed to go from our coefficient matrix to our sumiliary matrix. Now the coefficient matrix, in abbreviated form, is given as

Now rule 1 says that the first column of the auxiliary matrix is identical to the first column of the coefficient matrix. Hence,  $R_1C_1$  and  $R_2C_1$  appear in the first column of the auxiliary matrix. Rule 1 also says that each element which lies in the first row of the auxiliary matrix, excluding  $R_1C_1$ , is equal to the corresponding element in the coefficient matrix divided by  $R_1C_1$ . Hence,  $R_1C_1$ ,  $R_1C_2/R_1C_1$ , and  $R_1C_3/R_1C_1$  appear in the first row of the auxiliary matrix.

Rule 2 states that any element in the auxiliary matrix which lies on or below the principal diagonal<sup> $\frac{3}{}$ </sup> of the auxiliary matrix

<u>3</u>/ The principal diagonal is composed of those elements which have the same row and column index—i.e.: R<sub>1</sub>C<sub>1</sub>, R<sub>2</sub>C<sub>2</sub>,...R<sub>n</sub>C<sub>n</sub>. The principal diagonal starts with that element in the upper left hand corner and slopes down to the right. In our coefficient matrix, the principal diagonal is made up of R<sub>1</sub>C<sub>1</sub> and R<sub>2</sub>C<sub>2</sub>.

is equal to the corresponding element in the given matrix,  $R_2C_2$ , minus the sum of those products of elements in its row and corresponding elements in its column,  $R_2C_1 \cdot r_1c_2$ . Therefore, from our auxiliary



(Hatrix 11)

Now rule 1 says that the first column of the samillary matrix ts identical to the first column of the coefficient matrix. Mence,  $R_1C_1$  and  $R_2C_1$  appear in the first column of the samiltary matrix. Rule 1 slass nave that each element which ites in the first row of the samiltary matrix, excluding  $R_1C_1$ , is equal to the corresponding element in the coefficient matrix divided by  $R_1C_1$  hence,  $R_1C_1$ ,  $R_1C_2/R_1C_1$ , and  $R_1C_3/R_1C_1$  appear in the first row of matrix.

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3/ The principal diagonal is composed of those elements which have the same row and column indus—i.e.: R<sub>1</sub>C<sub>1</sub>. R<sub>2</sub>C<sub>2</sub>...R<sub>n</sub>C<sub>n</sub>. The principal diagonal starts with that clammat is the upper left hand corner and slopes down to the tipe tight. In our coefficient matrix, the principal diagonal is made up of R<sub>1</sub>C<sub>1</sub> and R<sub>2</sub>C<sub>2</sub>.

is equal to the corresponding element in the given metrix,  $R_2^{C_2}$ , wince the sum of those products of elements in its row and corresponding elements in its column,  $R_2^{C_1, c_1 c_2}$ . Therefore, from our auxiliary

11.

matrix

R <sub>1</sub> C <sub>1</sub>	r <sub>1</sub> c <sub>2</sub>	<sup>r</sup> 1 <sup>c</sup> 3	
<sup>R</sup> 2 <sup>C</sup> 1	<sup>r</sup> 2 <sup>c</sup> 2	<sup>r</sup> 2 <sup>c</sup> 3	(Matrix 12)

we see from rule 2 that  $r_2c_2 = R_2c_2 - R_2c_1 \cdot r_1c_2$ .

Finally, rule 3 says that in evaluating an element which lies to the right of the principal diagonal of the auxiliary matrix, we follow step 2 and then divide by the diagonal element. That is,  $r_2c_3$ is equal to the corresponding element in our coefficient matrix,  $R_2C_3$ , minus the sum of those products of elements in its row and corresponding elements in its column of the auxiliary matrix,  $R_2C_1 \cdot r_1c_3$ , and then all of this is to be divided by the diagonal element of the auxiliary matrix which lies in the same row as  $r_2c_3$ . Therefore,

$$r_{2}c_{3} = \frac{R_{2}C_{3} - R_{2}C_{1}r_{1}c_{3}}{r_{2}c_{2}}$$

Now all of the elements which make up our auxiliary matrix have been defined and we proceed to outline the procedure for going from our auxiliary matrix to a set of final results.

# 2. Rules for obtaining the final results from the auxiliary matrix

In going from the auxiliary matrix, (Matrix 12), to the final result, equations (13) and (14), we proceed as follows:

(1) We evaluate our coefficients of equation (2) in reverse order; for example, we evaluate first B and then A.

(2) The last coefficient, B, is numerically equal to the corresponding element in the last column of our auxiliary matrix.

matrix

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we are from rule 2 that they = R262 - R261. F142.

Winally, rule 3 says that is evaluating an element which lies to the right of the principal diagonal of the sumiliary matrix, we follow atup 2 and then divide by the diagonal element. That is,  $r_2 c_3$ is equal to the corresponding element in our coefficient matrix,  $R_2 C_3$ , minus the sum of those products of elements in its row and corresponding elements in its column of the sumiliary matrix,  $h_2 C_1, r_1 c_3$ , and then all of this is to be divided by the diagonal element of the availiary matrix which lies in the same row as  $r_2 c_3$ . Therefore,

Now all of the elements which make up our sumiliary matrix have been defined and we proceed to outline the procedure for going from our sumiliary matrix to a set of final results.

2. Mules for obtaining the final results from the suchiary sating

In going from the auxiliary matrix, (Matrix 12), to the Heal result, equations (13) and (14), we proceed as follows:

(1) We avaluate our coefficients of equation (2) in reverse order; for example, we evaluate first 5 and then A.

(2) The last coefficient, B, is numerically equal to the corresponding element in the last column of our auxiliary matrix.

(3) Each of the other coefficients is "...equal to the corresponding element of the last column of the auxiliary matrix minus the sum of those products of elements in its row in the auxiliary matrix and corresponding elements in its column in the final matrix which involve only previously computed elements" (2).

Now suppose we follow the above steps in going from our auxiliary matrix to our final solution. Now the auxiliary matrix, in abbreviated form, is given as

Rule 1 says that the first constant we evaluate is B and the second constant we evaluate is A. Rule 2 says that B is numerically equal to the last element in the last column of the auxiliary matrix. Therefore,  $B = r_2c_3$ . Rule 3 then says that A is equal to the next to the last element in the last column of the auxiliary matrix  $(r_1c_3)$ minus the sum of those products of elements in its row in the auxiliary matrix and corresponding elements in the last column of our final results. Therefore,  $A = r_1c_3 - r_1c_2 \cdot B$ .

For curves of higher degree, the following formulas for evaluating coefficients are given along with formulas derived by me for evaluating the above-mentioned errors.

For a function of the kind

 $Y = A + Bx_1 + Cx_2$ 

(19)

(3) Each of the other coefficients is "...equal to the corrasponding element of the last column of the samilitary matrix minus the sum of those products of elements in its row in the sumlilary matrix and corresponding elements in its column in the final matrix which involve only previously computed elements" (2).

Now suppose we follow the above steps is going from our succinary matrix to our final solution. Now the auxiliary metrix, is abbreviated form, is given as

Bule I ways that the first constant we conjuste is A and the second constant we evaluate is A. Rule 2 mays that B is mamerically equal to the last element in the last column of the surfitary matrix. Therefore,  $B = r_2 r_3$ . Rule 3 then mays that A is equal to the mext to the last element in the last column of the auxiliary matrix ( $r_1 r_3$ ) minus the sum of those products of elements in its row in the sumlifiery matrix and corresponding elements in the last column of our final results Therefore,  $A = r_1 r_3 - r_1 r_3 - 8$ .

For curves of higher degree, the following formulas for evaluating coefficients are given along with formulas derived by me for evaluating

For a function of the kind

$$f = h + Bx, + Ex,$$

El

the method of the auxiliary matrix can be used for the solution of the set of normal equations. The normal equations are:

An + 
$$B\Sigma x_1 + C\Sigma x_2 = \Sigma Y_i$$
 (19a)

$$A\Sigma x_{1_{i}}^{2} + B\Sigma x_{1_{i}}^{2} + C\Sigma x_{1_{i}}^{2} z_{i}^{2} = \Sigma x_{1_{i}}^{2} y_{i}$$
(19b)

$$\Delta \Sigma x_{2i} + B \Sigma x_{1i} x_{2i} + C \Sigma x_{2i}^{2} = \Sigma x_{2i}^{Y}$$
(19c)

Therefore, our coefficient matrix is of the form

n 
$$\Sigma x_{1_{i}}$$
  $\Sigma x_{2_{i}}$   $\Sigma Y_{i}$   
 $\Sigma x_{1_{i}}$   $\Sigma x_{1_{i}}^{2}$   $\Sigma x_{1_{i}} x_{2_{i}}$   $\Sigma x_{1_{i}} Y_{i}$  (Matrix 19d)  
 $\Sigma x_{2_{i}}$   $\Sigma x_{1_{i}} x_{2_{i}}$   $\Sigma x_{2_{i}}^{2}$   $\Sigma x_{2_{i}} Y_{i}$ 

or, if we use the symbolism R and C to represent the words row and column, respectively, we can abbreviate (Matrix 19d) as

and from the rules cited for obtaining the auxiliary matrix from the given coefficient matrix, we have as our auxiliary matrix $\frac{4}{}$ 

4/ The relations for obtaining each element of the auxiliary matrix from the given coefficient matrix for third and fourth order matrices, respectively, are given in the appendix of this report. the mathed of the anguitary matrix can be used for the solution of the set of normal equations. The normal equations are;

Therefore, bur coefficient matrix is of the form

or, if we use the symbolian h and C to represent the words row and column, respectively, we can abbreviate Garris 19d) as

> <sup>2</sup>1<sup>6</sup>1 <sup>2</sup>1<sup>6</sup>2 <sup>2</sup>1<sup>6</sup>3 <sup>2</sup>1<sup>6</sup>2 <sup>2</sup>2<sup>6</sup>1 <sup>2</sup>2<sup>6</sup>2 <sup>2</sup>2<sup>6</sup>3 <sup>2</sup>2<sup>6</sup>3 <sup>2</sup>3<sup>6</sup>1 <sup>2</sup>3<sup>6</sup>2 <sup>2</sup>3<sup>6</sup>3

SALLYIN 19a

and from the rules cited for obtaining the auxiliary matrix

A) The relations for ontaining each element of the auxiliary carries, respectively, are given in the appendix of this report.

p.

and our final result is of the form

$$C = r_3 c_4 \tag{19g}$$

$$B = r_{2}c_{4} - r_{2}c_{3}C$$
(19h)

$$A = r_{1}c_{4} - r_{1}c_{3} \cdot C - r_{1}c_{2} \cdot B$$
(19i)

The variance in a single  $Y_i$  is given as

$$s_{Y_{i}}^{2} = \frac{\sum_{i} \frac{Y_{i} - Y_{i}}{obs} \frac{2}{cal}}{n - 3}$$
 (19j)

The error in our function, equation (19), can be evaluated from the expression (3)

$$s_{Y}^{2} = s_{A}^{2} + 2x_{1}s_{AB}^{2} + x_{1}^{2}s_{B}^{2} + 2x_{1}x_{2}s_{BC}^{2} + 2x_{2}s_{AC}^{2} + x_{2}^{2}s_{C}^{2}$$
 (19k)

where

$$s_{A}^{2} = (r_{3}c_{3}/R_{1}C_{1}) \cdot s_{C}^{2} - (r_{1}c_{3}) \cdot s_{AC}^{2} - (r_{1}c_{2}) \cdot s_{AB}^{2}$$
 (19m)

$$s_{AB}^2 = -(r_1c_3) \cdot s_{BC}^2 - (r_1c_2) \cdot s_B^2$$

$$s_{AC}^2 = -(r_1c_3) \cdot s_C^2 - (r_1c_2) \cdot s_{BC}^2$$
 (190)

$$s_{\rm B}^2 = (r_3 c_3 / r_2 c_2) \cdot s_{\rm C}^2 - (r_2 c_3) \cdot s_{\rm BC}^2$$
 (19p)

and our final result is of the form

The variance in a single Y, is given as

.

$$\frac{2}{2} = \frac{E(\frac{1}{2} \cos \theta - \frac{1}{2} \cos \theta)}{\alpha - 3}$$
(193)

The error in our function, squation (19), can be evaluated from the expression (3)

$$s_{x}^{2} = s_{x}^{2} + 2x_{1}s_{x}^{2} + x_{1}s_{y}^{2} + 2x_{1}x_{2}s_{x}^{2} + 2x_{2}s_{x}^{2} + 2x_{2}s_{x}^{2} + (196)$$

stade

$$a_{A}^{2} = (x_{3}c_{3}/\pi_{1}c_{1}) \cdots a_{C}^{2} - (x_{1}c_{3}) \cdots a_{AC}^{2} - (x_{1}c_{2}) \cdots a_{AC}^{2}$$
(19m)

$$a_{AB}^2 = -(r_1c_3) \cdot a_{BC}^2 - (r_1c_2) \cdot a_B^2$$

$$x_{AC}^{2} = (x_{1}c_{3}) + x_{C}^{2} - (x_{1}c_{2}) - x_{C}^{2}$$
(190)

$$a_{0}^{2} = (\tau_{3} \epsilon_{3} / \epsilon_{2} \epsilon_{2}) - a_{0}^{2} - (\tau_{3} \epsilon_{3}) - a_{2}^{2} - (\tau_{3} \epsilon_{3}) - a_{3}^{2}$$
(199)

$$s_{BC}^2 = -(r_2 c_3) \cdot s_C^2$$
 (19q)

$$s_{C}^{2} = S_{Y_{i}}^{2} (1/r_{3}c_{3})$$
 (19r)

For an equation of the form

$$Y = A + Bx_1 + Cx_2 + Dx_3$$
 (20)

our normal equations are

An 
$$+ B\Sigma x_{1_{i}} + C\Sigma x_{2_{i}} + D\Sigma x_{3_{i}} = \Sigma Y_{i}$$
 (20a)  
 $A\Sigma x_{1_{i}} + B\Sigma x_{1_{i}}^{2} + C\Sigma x_{1_{i}} x_{2_{i}} + D\Sigma x_{1_{i}} x_{3_{i}} = \Sigma x_{1_{i}} Y_{i}$  (20b)  
 $A\Sigma x_{2_{i}} + B\Sigma x_{1_{i}} x_{2_{i}} + C\Sigma x_{2_{i}}^{2} + D\Sigma x_{2_{i}} x_{3_{i}} = \Sigma x_{2_{i}} Y_{i}$  (20c)  
 $A\Sigma x_{3_{i}} + B\Sigma x_{1_{i}} x_{3_{i}} + C\Sigma x_{2_{i}} x_{3_{i}} + D\Sigma x_{3_{i}}^{2} = \Sigma x_{3_{i}} Y_{i}$  (20d)

Hence, our coefficient matrix is of the form

n  $\Sigma x_{1_i}$   $\Sigma x_{2_i}$   $\Sigma x_{3_i}$   $\Sigma Y_i$  $\begin{bmatrix} \Sigma \mathbf{x}_{1_{i}} & \Sigma \mathbf{x}_{1_{i}}^{2} & \Sigma \mathbf{x}_{1_{i}}^{\mathbf{x}_{2_{i}}} & \Sigma \mathbf{x}_{1_{i}}^{\mathbf{x}_{3_{i}}} & \Sigma \mathbf{x}_{1_{i}}^{\mathbf{x}_{1_{i}}} \\ \Sigma \mathbf{x}_{2_{i}} & \Sigma \mathbf{x}_{1_{i}}^{\mathbf{x}_{2_{i}}} & \Sigma \mathbf{x}_{2_{i}}^{2} & \Sigma \mathbf{x}_{2_{i}}^{\mathbf{x}_{3_{i}}} & \Sigma \mathbf{x}_{2_{i}}^{\mathbf{x}_{3_{i}}} & \Sigma \mathbf{x}_{2_{i}}^{\mathbf{x}_{3_{i}}} & \Sigma \mathbf{x}_{2_{i}}^{\mathbf{x}_{3_{i}}} \\ \Sigma \mathbf{x}_{3_{i}} & \Sigma \mathbf{x}_{1_{i}}^{\mathbf{x}_{3_{i}}} & \Sigma \mathbf{x}_{2_{i}}^{\mathbf{x}_{3_{i}}} & \Sigma \mathbf{x}_{3_{i}}^{2} & \Sigma \mathbf{x}_{3_{i}}^{\mathbf{x}_{3_{i}}} & \Sigma \mathbf{x}_{3_{i}}^{\mathbf{x}_{3_{i}}} \\ \end{bmatrix}$ 

(Matrix 20e)

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For an equation of the form

1

$$y = x + 8x_1 + 0x_2 + 0x_3$$
 (20)

our normal squattons are

$$ABx_{1} + Bx_{1}^{2} + Cx_{1}x_{2} + Bx_{1}x_{1}x_{3} - Ex_{1}x_{4}$$
(206)

$$\frac{1}{2}x_{2} + \frac{1}{2}x_{1}^{2}x_{2} + \frac{1}{2}x_{2}^{2}x_{3} + \frac{1}{2}x_{2}x_{3} + \frac{1}{2}x_{2}x_{3} + \frac{1}{2}x_{2}x_{3} + \frac{1}{2}x_{2}x_{3} + \frac{1}{2}x_{2}x_{3} + \frac{1}{2}x_{3} + \frac{1}{2$$

$$\Delta \Sigma x_{1} + B \Sigma x_{1}^{x} + C x_{2}^{x} x_{3}^{x} + D x_{3}^{2} = \Sigma x_{3}^{x} x_{3}^{x}$$
 (20d)

Hence, our coefficient untrix is of the form

(Materia 20s)

We can write (Matrix 20e) in abbreviated form as

Proceeding from (Matrix 20f), we can write our auxiliary matrix as

R <sub>1</sub> C <sub>1</sub>	r1 <sup>c</sup> 2	r1 <sup>c</sup> 3	r <sub>1</sub> c <sub>4</sub>	<sup>r</sup> 1 <sup>c</sup> 5	(r2 22) · s2 (
R <sub>2</sub> C <sub>1</sub>	r2 <sup>c</sup> 2	<sup>r</sup> 2 <sup>c</sup> 3	<sup>r</sup> 2 <sup>c</sup> 4	<b>r</b> 2 <sup>c</sup> 5	(Matrix 20a)
R <sub>3</sub> C <sub>1</sub>	r <sub>3</sub> c <sub>2</sub>	r3 <sup>c</sup> 3	r3c4	<sup>r</sup> 3 <sup>c</sup> 5	(Matrix 20g)
R <sub>4</sub> C <sub>1</sub>	r4 <sup>c</sup> 2	<sup>r</sup> 4 <sup>c</sup> 3	r <sub>4</sub> c <sub>4</sub>	<sup>r</sup> 4 <sup>c</sup> 5	- 2 (

and our final result is of the form

$$D = r_4 c_5$$
 (20h)

$$C = r_3 c_5 - (r_3 c_4) \cdot D$$
 (20i)

$$B = r_2 c_5 - (r_2 c_4) \cdot D - (r_2 c_3) \cdot C$$
 (20j)

$$A = r_{1}c_{5} - (r_{1}c_{4}) \cdot D - (r_{1}c_{3}) \cdot C - (r_{1}c_{2}) \cdot B \qquad (20k)$$

The variance in our function, equation (20), can be determined from the expression (3)

We can write (Matrix 20s) in abbreviated form as .

		P1 <sup>C</sup> 2	R1C1
			1262
		8462	

Proceeding from (Matrix 206), we can write our sumiliary matrix as

			1258

and our final result is of the form

$$D = \pi_{0}^{2} \pi_{0}^{2}$$
 (20b)

$$a = z_{2}c_{2} - (z_{3}a_{4}) - a$$
 (201)

$$B = r_2 c_5 - (r_2 c_6) - D - (r_2 c_3) - C \qquad (203)$$

$$h = e_1 e_2 - (e_1 e_4) - D - (e_1 e_3) - C - (e_1 e_2) - B$$
(204)

The variance in our function, squation (20), can be determined iron the expression (3)

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$$s_{A}^{2} + 2x_{1}s_{AB}^{2} + 2x_{2}s_{AC}^{2} + 2x_{3}s_{AD}^{2}$$

$$s_{Y}^{2} = + x_{1}^{2}s_{B}^{2} + 2x_{1}x_{2}s_{BC}^{2} + 2x_{1}x_{3}s_{BD}^{2}$$

$$+ x_{2}^{2}s_{C}^{2} + 2x_{2}x_{3}s_{CD}^{2} + x_{3}^{2}s_{D}^{2}$$
(20m)

0

where

.

$$S_{Y_{i}}^{2} = \frac{\sum (i_{obs} - Y_{i}_{cal})}{n - 4}$$
 (20n)

$$\mathbf{s}_{A}^{2} = (\mathbf{r}_{4}\mathbf{c}_{4}/\mathbf{R}_{1}\mathbf{c}_{1}) \cdot \mathbf{s}_{D}^{2} - (\mathbf{r}_{1}\mathbf{c}_{4}) \cdot \mathbf{s}_{AD}^{2} - (\mathbf{r}_{1}\mathbf{c}_{3}) \cdot \mathbf{s}_{AC}^{2} - (\mathbf{r}_{1}\mathbf{c}_{2}) \cdot \mathbf{s}_{AB}^{2}$$
(200)

$$s_{AB}^2 = -(r_1c_4) \cdot s_{BD}^2 - (r_1c_3) \cdot s_{BC}^2 - (r_1c_2) \cdot s_B^2$$
 (20p)

$$s_{AC}^2 = -(r_1c_4) \cdot s_{CD}^2 - (r_1c_3) \cdot s_{C}^2 - (r_1c_2) \cdot s_{BC}^2$$
 (20q)

$$s_{AD}^2 = -(r_1c_4) \cdot s_D^2 - (r_1c_3) \cdot s_{CD}^2 - (r_1c_2) \cdot s_{BD}^2$$
 (20r)

$$s_{\rm B}^2 = (r_4 c_4 / r_2 c_2) \cdot s_{\rm D}^2 - (r_2 c_4) \cdot s_{\rm BD}^2 - (r_2 c_3) \cdot s_{\rm BC}^2$$
 (20s)

$$s_{BC}^2 = -(r_2c_4) \cdot s_{CD}^2 - (r_2c_3) \cdot s_C^2$$
 (20t)

$$s_{BD}^2 = -(r_2c_4) \cdot s_D^2 - (r_2c_3) \cdot s_{CD}^2$$
 (20u)

$$s_{\rm C}^2 = (r_4 c_4 / r_3 c_3) \cdot s_{\rm D}^2 - (r_3 c_4) \cdot s_{\rm CD}^2$$
 (20v)

$$s_{CD}^2 = -(r_3c_4) \cdot s_D^2$$
 (20w)

$$\begin{aligned}
\begin{aligned}
\begin{aligned}
\dot{a}^{2} + \dot{a}_{2} + \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{2} \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{2} \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{2} \dot{a}_{2} \dot{a}_{2}^{2} \\
\dot{a}^{2} \dot{a}_{1}^{2} - \dot{a}_{1}^{2} \dot{a}_{1}^{2} + \dot{a}_{1} \dot{a}_{1}^{2} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2}^{2} \dot{a}_{2}^{2} \\
& \dot{a}^{2} \dot{a}_{1}^{2} + \dot{a}_{1} \dot{a}_{1} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} \\
& \dot{a}^{2} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} \\
& \dot{a}^{2} \dot{a}_{1}^{2} + \dot{a}_{1} \dot{a}_{1} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} \\
& \dot{a}^{2} \dot{a}_{1}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{1}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} \\
& \dot{a}^{2} \dot{a}_{1}^{2} - \dot{a}_{1} \dot{a}_{2} \dot{a}_{1}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} + \dot{a}_{1} \dot{a}_{2} \dot{a}_{2}^{2} & \dot{a}_{1} \dot{a}_{2}^{2} & \dot{a}_{1} \dot{a}_{2}^{2} & \dot{a}_{2}^{2} & \dot{a}_{1} \dot{a}_{2} & \dot{a}_{1} \dot{a}_{1} \dot{a}_{2} & \dot{a}_{1} \dot{a}_{2} & \dot{a}_{1} \dot{a}_{2} & \dot{a}_{$$

$$s_D^2 = s_{Y_i}^2 (1/r_4 c_4)$$
 (20x)

Attention has been given to functions of the kind

$$Y = 1 + Bx_1 + Cx_2$$
 (21)

and of higher degree and to the standard error associated with each coefficient. The normal equations for equation (21) are

$$B\Sigma x_{1_{i}}^{2} + C\Sigma x_{1_{i}} x_{2_{i}} = (\Sigma x_{1_{i}} Y_{i} - \Sigma x_{1_{i}})$$
(21a)

$$B\Sigma x_{1_{i}}^{2} x_{2_{i}}^{2} + C\Sigma x_{2_{i}}^{2} = (\Sigma x_{2_{i}}^{2} y_{i} - \Sigma x_{2_{i}})$$
(21b)

and our coefficient matrix is of the form

$$\begin{array}{c} \Sigma \mathbf{x}_{1_{i}}^{2} & \Sigma \mathbf{x}_{1_{i}}^{2} \mathbf{x}_{2_{i}}^{2} & (\Sigma \mathbf{x}_{1_{i}}^{Y} \mathbf{1}_{i}^{-\Sigma \mathbf{x}_{1_{i}}}) \\ & & & \\ \Sigma \mathbf{x}_{1_{i}}^{X} \mathbf{x}_{2_{i}}^{2} & \Sigma \mathbf{x}_{2_{i}}^{2} & (\Sigma \mathbf{x}_{2_{i}}^{Y} \mathbf{1}_{i}^{-\Sigma \mathbf{x}_{2_{i}}}) \end{array}$$
(Matrix 21c)

Writing (Matrix 21c) in abbreviated form, we have

as our given coefficient matrix. The auxiliary matrix is given by

Attention has been given to functions of the kind

$$Y = I + Im_1 + Cm_2$$
 (21)

and of Manner degrees and to the standard error associated with sach coefficient. The normal equations for equation (21) are

$$m_{1}^{2} + m_{1}^{2} + m_{1$$

$$\max_{l_1} \pi_{l_1} + \max_{l_2} = (\max_{l_2} T_1 - \Sigma \pi_{l_1})$$
(21b)

and our coefficient matrix is of the form

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Writing ( Matrix 21c) in abbreviated form, we have

(Matrin 213)

as our given coefficient metrix. The sumiliary matrix is given by

(Mateix 21a)

Thus, our final result is

$$C = r_2 c_3 \tag{21f}$$

$$B = r_1 c_3 - (r_1 c_2) \cdot C$$
 (21g)

The variance in our function, equation (21), is given by (3)

$$S_{Y}^{2} = x_{1}^{2}s_{B}^{2} + 2x_{1}x_{2}s_{BC}^{2} + x_{2}^{2}s_{C}^{2}$$
 (21h)

where

$$s_{\rm B}^2 = (r_2 c_2 / R_1 c_1) \cdot s_{\rm C}^2 - (r_1 c_2) \cdot s_{\rm BC}^2$$
 (21i)

$$s_{BC}^2 = -(r_1c_2) \cdot s_C^2$$
 (21j)

$$s_{\rm C}^2 = S_{\rm Y_i}^2 (1/r_2c_2)$$
 (21k)

and

$$S_{Y_{i}}^{2} = \frac{\sum (i_{obs} - Y_{i}_{cal})^{2}}{n - 2}$$
 (21m)

For the function

$$Y = 1 + Bx_1 + Cx_2 + Dx_3$$
 (22)

our normal equations are

$$B\Sigma \mathbf{x}_{1_{i}}^{2} + C\Sigma \mathbf{x}_{1_{i}} \mathbf{x}_{2_{i}}^{2} + D\Sigma \mathbf{x}_{1_{i}} \mathbf{x}_{3_{i}}^{2} = (\Sigma \mathbf{x}_{1_{i}} \mathbf{y}_{i} - \Sigma \mathbf{x}_{1_{i}})$$
(22a)

$$B\Sigma x_{1_{i}} x_{2_{i}}^{2} + C\Sigma x_{2_{i}}^{2} + D\Sigma x_{2_{i}} x_{3_{i}}^{2} = (\Sigma x_{2_{i}} y_{i}^{2} - \Sigma x_{2_{i}})$$
(22b)

$$B\Sigma \mathbf{x}_{1_{i}} \mathbf{x}_{3_{i}} + C\Sigma \mathbf{x}_{2_{i}} \mathbf{x}_{3_{i}} + D\Sigma \mathbf{x}_{3_{i}}^{2} = (\Sigma \mathbf{x}_{3_{i}} \mathbf{y}_{i} - \Sigma \mathbf{x}_{3_{i}})$$
(22c)

Thus, our final result is

(211)

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The variance in our function, equation (21), is given by (3)

$$x_2^2 = x_1^2 x_2^2 + 2x_1 x_2^2 x_3^2 + x_2^2 x_3^2$$
 (21b)

DI BI

$$= (r_2 e_2/R_1 e_1) \cdot e_0^2 - (r_1 e_2) \cdot e_{10}^2$$
(211)

$$a_{BC}^2 = -ia_1 a_2 - a_C^2$$
 (21))

$$\frac{1}{2} = \frac{2}{2} (1/r_2 \epsilon_2)$$
 (21k)

bga

$$\frac{2}{2\chi} = \frac{\Sigma (\frac{L_{obs} - \chi_{call}}{m - 2})^2}{m - 2}$$
 (21m)

For the function

$$x = 1 + nx_1 + Gx_2 + Dx_3$$
 (22)

our normal aquations are

$$(22a)_{1} + (2x_{1}_{1}_{2}_{2}_{1} + 0x_{1}_{2}_{2}_{2}_{2} + 0x_{1}_{2}_{2}_{2}_{2}_{2} + (2x_{1}_{1}_{2}_{2}_{1} - 2x_{1}_{1})$$
(22a)

$$\sum_{i=1}^{N} \sum_{i=1}^{N} \sum_{i$$

$$8Ex_{1_{1}}x_{3_{1}} + Ex_{2_{1}}x_{3_{1}} + DEx_{3_{1}}x_{3_{1}} = (Ex_{3_{1}}x_{1} - Ex_{3_{1}})$$
(22c)

From equations (22a), (22b), and (22c), we can write our coefficient matrix as

In abbreviated form, (Matrix 22d) is given as

and from ( Matrix 22e), we can write our auxiliary matrix as

R <sub>1</sub> C <sub>1</sub>	<sup>r</sup> 1 <sup>c</sup> 2	r1 <sup>c</sup> 3	r <sub>1</sub> c <sub>4</sub>	A CP
R <sub>2</sub> C <sub>1</sub>	<sup>r</sup> 2 <sup>c</sup> 2	<sup>r</sup> 2 <sup>c</sup> 3	r2 <sup>c</sup> 4	(Matrix 22f)
R <sub>3</sub> C <sub>1</sub>	r3 <sup>c</sup> 2	r <sub>3</sub> c <sub>3</sub>	r <sub>3</sub> c <sub>4</sub>	

Hence, our final result is given to be

$$D = r_3 c_4 \tag{22g}$$

$$C = r_2 c_4 - (r_2 c_3) \cdot D$$
 (22h)

$$B = r_1 c_4 - (r_1 c_3) \cdot D - (r_1 c_2) \cdot C$$
 (22i)

From equations (22a), (22b), and (22c), we can write our coefficient matrix as

$$\sum_{x_{1_{1}}}^{x_{1_{1}}} \sum_{x_{1_{1}}}^{x_{1_{1}}} \sum_{x_{1}}^{x_{1}} \sum_{x_{1}}$$

In abbreviated form, ( Matrix 22d) is given as

and from ( Matrix 22e), we can write our auxiliary matrix an

Rence, our final result is given to be

$$D = r_3 c_4$$
 (22g)

$$c = r_2 c_4 - (r_2 c_3) \cdot D$$
 (22h)

$$B = r_1 c_4 - (r_1 c_3) \cdot D - (r_1 c_2) \cdot C$$
(221)

We can determine the error in our function, equation (22), from the expression (3)

$$S_{Y}^{2} = x_{1}^{2}s_{B}^{2} + 2x_{1}x_{2}s_{BC}^{2} + 2x_{1}x_{3}s_{BD}^{2} + x_{2}^{2}s_{C}^{2} + 2x_{2}x_{3}s_{CD}^{2} + x_{3}^{2}s_{D}^{2}$$

$$(22j)$$

$$+ 2x_{2}x_{3}s_{CD}^{2} + x_{3}^{2}s_{D}^{2}$$

where

$$s_{B}^{2} = (r_{3}c_{3}/R_{1}c_{1}) \cdot s_{D}^{2} - (r_{1}c_{3}) \cdot s_{BD}^{2} - (r_{1}c_{2}) \cdot s_{BC}^{2}$$
 (22k)

$$s_{BC}^{2} = -(r_{1}c_{3}) \cdot s_{CD}^{2} - (r_{1}c_{2}) \cdot s_{C}^{2}$$
 (22m)

$$s_{BD}^{2} = -(r_{1}c_{3}) \cdot s_{D}^{2} - (r_{1}c_{2}) \cdot s_{CD}^{2}$$
 (22n)

$$s_{C}^{2} = (r_{3}c_{3}/r_{2}c_{2}) \cdot s_{D}^{2} - (r_{2}c_{3}) \cdot s_{CD}^{2}$$
 (220)

$$s_{CD}^{2} = \succ (r_{2}c_{3}) \cdot s_{D}^{2}$$
(22p)

$$s_D^2 = S_{Y_i}^2 (1/r_3 c_3)$$
 (22q)

and

$$S_{Y_{i}}^{2} = \frac{\sum (i_{obs} - Y_{i_{cal}})^{2}}{n - 3}$$
 (22r)

Finally, for the function

$$Y = 1 + Bx_1 + Cx_2 + Dx_3 + Ex_4$$
(23)

we have the set of normal equations

$$B\Sigma x_{1_{i}}^{2} + C\Sigma x_{1_{i}} x_{2_{i}}^{2} + D\Sigma x_{1_{i}} x_{3_{i}}^{2} + E\Sigma x_{1_{i}} x_{4_{i}}^{2} = (\Sigma x_{1_{i}} y_{i}^{2} - \Sigma x_{1_{i}})$$
(23a)

$$s_{\chi}^{2} = x_{1}^{2}s_{B}^{2} + 2x_{1}x_{2}s_{BC}^{2} + 2x_{1}x_{3}s_{BD}^{2} + x_{2}^{2}s_{C}^{2}$$

(223)

where

$$\frac{2}{8} = (r_3 c_3' R_1 c_1) \cdot a_2^2 - (r_1 c_3) \cdot a_{BD}^2 - (r_1 c_2) \cdot a_{BC}^2 , \quad (22k)$$

$$a_{BC}^{2} = -(\pi_{1}c_{3}) \cdot a_{CD}^{2} - (\pi_{1}c_{2}) \cdot a_{C}^{2}$$
 (22m)

$$a_{BD}^2 = -(x_1c_3) \cdot a_D^2 - (x_1c_2) \cdot a_{CD}^2$$
 (22a)

$$a_{\rm C}^2 = (r_3 c_3 / r_2 c_2) \cdot a_{\rm D}^2 - (r_2 c_3) \cdot a_{\rm CD}^2$$
(220)

$$a_{CD}^2 = -(x_2 \sigma_3) \cdot a_D^2$$
 (22p)

$$s_{\rm D}^2 = s_{\rm Y_{\rm I}}^2 (1/r_3 c_3)$$
 (22q)

bus

$$s_{Y_{4}}^{2} = \frac{\Sigma (Y_{0}bs - Y_{call})^{2}}{b - 3}$$
 (22r)

Finally, for the function

$$Y = 1 + Bx_1 + Cx_2 + Dx_3 + Ex_4$$
(23)

we have the set of normal equations

$$3\Sigma x_{1_{4}}^{2} + CE x_{1_{4}} x_{2_{4}}^{2} + DE x_{1_{4}} x_{3_{4}}^{2} + EE x_{1_{4}} x_{4_{4}}^{2} = (Ex_{1_{4}} Y_{4} - Ex_{1_{4}})$$
(23a)

$$B\Sigma x_{1_{i}}^{2} x_{1_{i}}^{2} + C\Sigma x_{2_{i}}^{2} + D\Sigma x_{2_{i}}^{2} x_{3_{i}}^{2} + E\Sigma x_{2_{i}}^{2} x_{4_{i}}^{2} = (\Sigma x_{2_{i}}^{2} Y_{1_{i}} - \Sigma x_{2_{i}})$$
(23b)

$$B\Sigma x_{1_{i}} x_{3_{i}} + C\Sigma x_{2_{i}} x_{3_{i}} + D\Sigma x_{3_{i}}^{2} + E\Sigma x_{3_{i}} x_{4_{i}} = (\Sigma x_{3_{i}} x_{i} - \Sigma x_{3_{i}})$$
(23c)

$$B\Sigma x_{1_{i}}^{X_{i}} + C\Sigma x_{2_{i}}^{X_{i}} + D\Sigma x_{3_{i}}^{X_{i}} + E\Sigma x_{4_{i}}^{2} = (\Sigma x_{4_{i}}^{Y_{i}} - \Sigma x_{4_{i}})$$
(23d)

from which we can evaluate B, C, D, and E. Our given coefficient matrix, written in abbreviated form, is given as

Proceeding from (Matrix 23e), we can write our auxiliary matrix as

Hence, from our auxiliary matrix, (Matrix 23f), we have as our final result

$$E = r_{4}c_{5}$$
(23g)

$$BEx_{1_{4}}x_{2_{4}}^{2} + GEx_{2_{4}}^{2} + DEx_{2_{4}}x_{3_{4}}^{2} + BEx_{2_{4}}x_{4_{4}}^{2} = (Ex_{2_{4}}^{Y} - Ex_{2_{4}})$$
(23b)

$$BEx_{1_{4}}x_{3_{4}} + CEx_{2_{4}}x_{3_{4}} + BEx_{3_{4}} + EEx_{3_{4}}x_{4_{4}} = CEx_{3_{4}}x_{1} - Ex_{3_{4}}$$
(23c)

$$8\Sigma x_{1_{1}} x_{4_{1}}^{*} + 6\Sigma x_{2_{1}} x_{4_{1}}^{*} + 9\Sigma x_{3_{1}} x_{4_{1}}^{*} + 8\Sigma x_{4_{1}}^{2} = 6\Sigma x_{4_{1}} x_{1}^{*} - \Sigma x_{4_{1}}^{*}$$
(23d)

from which we can evaluate B, C, D, and E. Our given coefficient matrix, written in abbraviated form, is given as

Proceeding from (Matrix 23e), we can write our noxiliary matrix as

	<sup>2</sup> 1 <sup>c</sup> 2	

(Matrix 231)

Hence, from our auxiliary matrix, ( Matrix 231), we have as our final result

(232)

$$D = r_{3}c_{5} - (r_{3}c_{4}) \cdot E$$
 (23h)

$$C = r_2 c_5 - (r_2 c_4) \cdot E - (r_2 c_3) \cdot D$$
 (23i)

$$B = r_1 c_5 - (r_1 c_4) \cdot E - (r_1 c_3) \cdot D - (r_1 c_2) \cdot C$$
(23j)

The variance in our function, equation (23), can be determined from the equation  $(\underline{3})$ 

$$x_{1}^{2}s_{B}^{2} + 2x_{1}x_{2}s_{BC}^{2} + 2x_{1}x_{3}s_{BD}^{2}$$

$$S_{Y}^{2} = + 2x_{1}x_{4}s_{BE}^{2} + x_{2}^{2}s_{C}^{2} + 2x_{2}x_{3}s_{CD}^{2}$$

$$+ 2x_{2}x_{4}s_{CE}^{2} + x_{3}^{2}s_{D}^{2} + 2x_{3}x_{4}s_{DE}^{2}$$

$$+ x_{4}^{2}s_{E}^{2}$$

$$(23k)$$

where

$$s_{B}^{2} = (r_{4}c_{4}/R_{1}c_{1}) \cdot s_{E}^{2} - (r_{1}c_{4}) \cdot s_{BE}^{2} - (r_{1}c_{3}) \cdot s_{BD}^{2} - (r_{1}c_{2}) \cdot s_{BC}^{2}$$
(23m)  
$$s_{BC}^{2} = -(r_{1}c_{4}) \cdot s_{CE}^{2} - (r_{1}c_{3}) \cdot s_{CD}^{2} - (r_{1}c_{2}) \cdot s_{C}^{2}$$
(23n)

$$s_{BD}^{2} = -(r_{1}c_{4}) \cdot s_{DE}^{2} - (r_{1}c_{3}) \cdot s_{D}^{2} - (r_{1}c_{2}) \cdot s_{CD}^{2}$$
 (230)

$$s_{BE}^{2} = -(r_{1}c_{4}) \cdot s_{E}^{2} - (r_{1}c_{3}) \cdot s_{DE}^{2} - (r_{1}c_{2}) \cdot s_{CE}^{2}$$
 (23p)

$$s_{C}^{2} = (r_{4}c_{4}/r_{2}c_{2}) \cdot s_{E}^{2} - (r_{2}c_{4}) \cdot s_{CE}^{2} - (r_{2}c_{3}) \cdot s_{CD}^{2}$$
 (23q)

$$s_{CD}^2 = -(r_2 c_4) \cdot s_{DE}^2 - (r_2 c_3) \cdot s_D^2$$
 (23r)

$$s_{CE}^2 = -(r_2c_4) \cdot s_E^2 - (r_2c_3) \cdot s_{DE}^2$$
 (23s)

$$E = E_3 E_5 - (E_3 E_4) - E$$
 (23b)

$$c = r_2 c_5 - (r_2 c_4) \cdot E - (r_2 c_3) \cdot D$$
(231)

$$x = x_1 c_5 - (c_1 c_4) \cdot z - (x_1 c_3) \cdot z - (x_1 c_2) \cdot c$$
 (23))

The variance in our function, equation (23), can be determined from the equation (3)

$$\begin{aligned} & \frac{2}{x_1^2 e_B^2} + \frac{2}{x_1 x_2 e_B^2 c} + \frac{2}{x_1 x_3 e_B^2 c} \\ & \frac{2}{y} = + \frac{2}{x_1 x_4 e_B c} + \frac{2}{x_2^2 e_C^2} + \frac{2}{x_2 x_3 e_C^2 c} \\ & + \frac{2}{x_2 x_4 e_C c} + \frac{2}{x_3^2 e_C^2} + \frac{2}{x_3 x_4 e_D c} \end{aligned}$$
(23k)  
$$& + \frac{2}{x_4^2 e_C^2} + \frac{2}{x_3^2 e_D^2} + \frac{2}{x_3 x_4 e_D c} \end{aligned}$$

where

$$= (r_{4}e_{4}/8_{1}e_{1}) \cdot s_{E}^{2} \cdot (r_{1}e_{4}) \cdot s_{EE}^{2} - (r_{1}e_{3}) \cdot s_{ED}^{2} - (r_{1}e_{2}) \cdot s_{EC}^{2} (23m)$$

$$s_{EC}^{2} = -(r_{1}e_{4}) \cdot s_{CE}^{2} - (r_{1}e_{3}) \cdot s_{CD}^{2} - (r_{1}e_{2}) \cdot s_{C}^{2} (23m)$$

$$s_{ED}^{2} = -(r_{1}e_{4}) \cdot s_{DE}^{2} - (r_{1}e_{3}) \cdot s_{CD}^{2} - (r_{1}e_{2}) \cdot s_{C}^{2} (23m)$$

$$(23m)$$

$$s_{ED}^{2} = -(r_{1}e_{4}) \cdot s_{DE}^{2} - (r_{1}e_{3}) \cdot s_{D}^{2} - (r_{1}e_{2}) \cdot s_{CD}^{2} (23m)$$

$$(23m)$$

$${}^{2}_{BE} = -(r_{1}c_{A}) \cdot s_{E}^{2} - (r_{1}c_{3}) \cdot s_{DE}^{2} - (r_{1}c_{2}) \cdot s_{CE}^{2}$$
(230)

$$s_{\rm C}^2 = (r_A c_A / r_2 c_2) \cdot s_{\rm E}^2 - (r_2 c_A) \cdot s_{\rm CE}^2 - (r_2 c_3) \cdot s_{\rm CD}^2$$
 (23q)

A

$$s_{\rm D}^2 = (r_4 c_4 / r_3 c_3) \cdot s_{\rm E}^2 - (r_3 c_4) \cdot s_{\rm DE}^2$$
 (23t)

$$s_{DE}^2 = -(r_3 c_4) \cdot s_D^2$$
 (23u)

$$s_{E}^{2} = S_{Y_{i}}^{2} (1/r_{4}c_{4})$$
 (23v)

$$S_{Y_{i}}^{2} = \frac{\sum_{i=1}^{Y_{i}} - Y_{i}^{2}}{n - 4}$$
(23w)

## CONCLUSIONS

and

The method of treating PVT data on gases consists of expressing the isothermal variation of the PV product of a gas in terms of a power series in either the pressure or the density and of evaluating so-called virial coefficients by least squares solution. This report gives a brief summary of the method of the auxiliary matrix, previously published by Crout, and its application to rational integral functions of one to four degrees, without going into any details of the mathematical background of statistics or into the various approaches to the problem of curve fitting.

Formulas for calculating the standard error in a single measurement, the standard error in each coefficient, and the standard error of the respective functions are given, without proof, and are presented on the basis of unit weight being given to each observed measurement.

(232)

$$2 = \frac{\Sigma(\frac{Y_{tobs} - Y_{t_{call}}}{n - 4})^2}{\pi - 4}$$

2 -- (r3°4)-=2

# -----

The method of treating PVT data on games consists of expressing the isothermal variation of the PV product of a gas in terms of a power series in either the pressure or the density and of evaluating so-called virial coefficients by least squares solution. This report gives a brief summary of the method of the auxiliary matrix, previously published by Grout, and its application to restonal integral functions of one to four degrees, without going into any details of the mathematical background of statistics or into the various approaches to the problem of curve fitting.

Formulas for calculating the standard error in a single measurement, the standard error in each coefficient, and the standard error of the respective functions are given, without proof, and are presented on the basis of unit weight being given to each observed measurement.

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# APPENDIX

In evaluating the elements of the auxiliary matrix, it is most convenient to determine these elements in the following order: First, we evaluate all elements which lie in the first column of our auxiliary matrix. We then evaluate all elements which lie in the first row of our auxiliary matrix. Next we evaluate all elements in the second column and then all elements in the second row and so on until all elements of our auxiliary matrix are defined.

Now suppose we have a third order coefficient matrix of the form

22) Are third order	R <sub>1</sub> C <sub>4</sub>	R <sub>1</sub> C <sub>3</sub>	<sup>R</sup> 1 <sup>C</sup> 2	R <sub>1</sub> C <sub>1</sub>
(Matrix 19d)	R <sub>2</sub> C <sub>4</sub>	R <sub>2</sub> C <sub>3</sub>	R2C2	R2C1
or of the form	R <sub>3</sub> C <sub>4</sub>	R <sub>3</sub> C <sub>3</sub>	R <sub>3</sub> C <sub>2</sub>	R <sub>3</sub> C <sub>1</sub>

and a third order auxiliary matrix of the form

From the rules given on page 10 of this report, the elements of our third order matrix, (Matrix 19e), can be determined from the following relations

$$r_1 c_2 = R_1 c_2 \div R_1 c_1$$

$$r_1 c_3 = R_1 c_3 \div R_1 c_1$$

 $r_3c_3 = R_3c_3 - R_3c_1 \cdot r_1c_3 - r_3c_2 \cdot r_2c_3$ 

That is, To

mined as outlined also

### APPENDIX .

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In evaluating the elements of the auxiliary matrix, it is most conveniant to determine these elements in the following order: First, we evaluate all elements which lie in the first column of our auxiliary matrix. We then evaluate all elements which lie in the first row of our auxiliary matrix. Next we evaluate all elements in the second column and then all elements in the second row and so on until all elements of our auxiliary matrix are defined.

Now suppose we have a third order coefficient matrix of the form

$$\begin{vmatrix} B_{1}c_{1} & B_{1}c_{2} & B_{1}c_{3} & B_{1}c_{4} \\ B_{2}c_{1} & B_{2}c_{2} & B_{2}c_{3} & B_{2}c_{4} \\ B_{3}c_{1} & B_{3}c_{2} & B_{3}c_{3} & B_{3}c_{4} \\ \end{vmatrix}$$
(Matrix 194)
(Matrix 194)
(Matrix 194)
$$\begin{vmatrix} B_{1}c_{1} & F_{1}c_{2} & F_{1}c_{3} & F_{1}c_{4} \\ B_{1}c_{1} & F_{1}c_{2} & F_{1}c_{3} & F_{1}c_{4} \\ P_{2}c_{1} & F_{2}c_{2} & F_{2}c_{3} & F_{2}c_{4} \\ \end{vmatrix}$$
(Matrix 19a)

From the rules given on page 10 of this report, the elements of our third order matrix, (Matrix 19s), can be determined from the following relations

$$r_1 c_2 = R_1 c_2 = R_1 c_1$$
  
 $r_1 c_3 = R_1 c_3 + R_1 c_1$   
 $s^2 = R_3 c_3 - R_3 c_1 \circ r_1 c_3 - r_3 c_2 \circ r_2 c_3$ 

$$r_{1}c_{4} = R_{1}C_{4} \div R_{1}C_{1}$$

$$r_{2}c_{2} = R_{2}C_{2} - R_{2}C_{1} \cdot r_{1}c_{2}$$

$$r_{3}c_{2} = R_{3}C_{2} - R_{3}C_{1} \cdot r_{1}c_{2}$$

$$r_{2}c_{3}^{5/} = [R_{2}C_{3} - R_{2}C_{1} \cdot r_{1}c_{3}] \div r_{2}c_{2}$$

$$r_{2}c_{4} = [R_{2}C_{4} - R_{2}C_{1} \cdot r_{1}c_{4}] \div r_{2}c_{2}$$

$$r_{3}c_{4} = [R_{3}C_{4} - R_{3}C_{1} \cdot r_{1}c_{4} - r_{3}c_{2} \cdot r_{2}c_{4}] \div r_{3}c_{3}$$
It is to be noted that (Matrix 19e) and (Matrix 22f) are third order

auxiliary matrices. Therefore, the elements of (Matrix 22f) are determined as outlined above.

Now suppose we have a fourth order coefficient of the form

	R <sub>1</sub> C <sub>5</sub>	R <sub>1</sub> C <sub>4</sub>	R <sub>1</sub> C <sub>3</sub>	R <sub>1</sub> C <sub>2</sub>	R <sub>1</sub> C <sub>1</sub>
	R <sub>2</sub> C <sub>5</sub>	R <sub>2</sub> C <sub>4</sub>	R <sub>2</sub> C <sub>3</sub>	R <sub>2</sub> C <sub>2</sub>	R <sub>2</sub> C <sub>1</sub>
(Matrix 20f)	R <sub>3</sub> C <sub>5</sub>	R <sub>3</sub> C <sub>4</sub>	R <sub>3</sub> C <sub>3</sub>	R <sub>3</sub> C <sub>2</sub>	R <sub>3</sub> C <sub>1</sub>
	R <sub>4</sub> C <sub>5</sub>	R <sub>4</sub> C <sub>4</sub>	R <sub>4</sub> C <sub>3</sub>	R4C2	R <sub>4</sub> C <sub>1</sub>

and a fourth order auxiliary matrix of the form

5/ Since our auxiliary matrix is symmetrical about the principal diagonal, then this particular element can be evaluated by dividing its symmetrically opposite element below the principal diagonal,  $r_3c_2$ , by the diagonal element which lies in the same row as  $r_2c_3$ . That is,  $r_2c_3 = r_3c_2 \div r_2c_2$ . 2564 - [256 - 2561 - 2563 + 2563 259 - 2563 - 2561 - 265 259 - 2563 - 2561 - 265 259 - 2563 - 2561 - 265

It is to be noted that (Matrix 19s) and (Matrix 22f) are third order auxiliary matrices. Therefore, the elements of (Matrix 22f) are determined as outlined above.

Now suppose we have a fourth order coefficient of the form

	e <sup>D</sup> r <sup>g</sup> ·		

(Matrix 20f)

and a fourth order auxillary matrix of the form

 $\frac{5}{2}$  Since our surfiftery matrix is symmetrical about the principal diagonal, then this particular element can be evaluated by dividing its symmetrically opposite element below the principal diagonal,  $r_3c_2$ , by the disgonal element which lies in the same row as  $r_2c_3$ . That is,  $r_2c_3 = r_3c_2 + r_2c_3$ .

Now we determine the elements of our auxiliary matrix in the same order as cited above — i.e., first column and then first row ; second column and then second row, and so on until all of the elements are defined.

Now from the rules given on page 10 of this report, we see that the elements of (Matrix 20g) can be evaluated from the following relations

$r_1c_2 = R_1c_2 \div R_1c_1$
$r_1c_3 = R_1C_3 \div R_1C_1$
$r_1c_4 = R_1C_4 \div R_1C_1$
$r_{1}c_{5} = R_{1}C_{5} \div R_{1}C_{1}$
$r_2c_2 = R_2c_2 - R_2c_1 \cdot r_1c_2$
$r_{3}c_{2} = R_{3}C_{2} - R_{3}C_{1}r_{1}c_{2}$
$r_4c_2 = R_4C_2 - R_4C_1 \cdot r_1c_2$
$r_2 c_3 \frac{6}{3} = [R_2 C_3 - R_2 C_1 \cdot r_1 c_3] \div r_2 c_2$

<u>6</u>/ Since our auxiliary matrix is symmetrical about the principal diagonal, then this particular element can be evaluated by dividing its symmetrically opposite element below the principal diagonal,  $r_3c_2$ , by the diagonal element which lies in the same row as  $r_2c_3$ . That is,  $r_2c_3 = r_3c_2 \div r_2c_2$ .

Now we determine the elements of our suxiliary matrix in the same order as cited above - i.e., first column and then first row ; second column and then second row, and so on until all of the elements are defined.

Now from the rules given on page 10 of this report, we see that the elements of (Matrix 20g) can be evaluated from the following relations

$$\begin{array}{l} r_{1}c_{2} = R_{1}c_{2} + R_{1}c_{1} \\ r_{1}c_{3} = R_{1}c_{3} + R_{1}c_{1} \\ r_{1}c_{3} = R_{1}c_{3} + R_{1}c_{1} \\ r_{1}c_{4} = R_{1}c_{4} + R_{1}c_{1} \\ r_{1}c_{5} = R_{1}c_{5} + R_{1}c_{1} \\ r_{2}c_{2} = R_{2}c_{2} - R_{2}c_{1}r_{1}c_{2} \\ r_{3}c_{2} = R_{3}c_{2} - R_{2}c_{1}r_{1}c_{2} \\ r_{4}c_{2} = R_{5}c_{2} - R_{5}c_{1}r_{1}c_{2} \\ r_{4}c_{2} = R_{5}c_{5} - R_{5}c_{1}r_{1}c_{2} \\ \end{array}$$

 $\frac{5}{2}$  Since our auxfiltery matrix is symmetrical about the principal diagonal, then this particular element can be evaluated by dividing its symmetrically opposite element below the principal diagonal,  $r_3c_2$ , by the diagonal element which lies in the same row as  $r_2c_3$ . That is,

$$r_{2}c_{4}^{2'} = [R_{2}C_{4} - R_{2}C_{1} \cdot r_{1}c_{4}] \div r_{2}c_{2}$$

$$r_{2}c_{5} = [R_{2}C_{5} - R_{2}C_{1} \cdot r_{1}c_{5}] \div r_{2}c_{2}$$

$$r_{3}c_{3} = R_{3}C_{3} - R_{3}C_{1} \cdot r_{1}c_{3} - r_{3}c_{2} \cdot r_{2}c_{3}$$

$$r_{4}c_{3} = R_{4}C_{3} - R_{4}C_{1} \cdot r_{1}c_{3} - r_{4}c_{2} \cdot r_{2}c_{3}$$

$$r_{3}c_{4}^{8'} = [R_{3}C_{4} - R_{3}C_{1} \cdot r_{1}c_{4} - r_{3}c_{2} \cdot r_{2}c_{4}] \div r_{3}c_{3}$$

$$r_{3}c_{5} = [R_{3}C_{5} - R_{3}C_{1} \cdot r_{1}c_{5} - r_{3}c_{2} \cdot r_{2}c_{5}] \div r_{3}c_{3}$$

$$r_{4}c_{4} = R_{4}C_{4} - R_{4}C_{1} \cdot r_{1}c_{4} - r_{4}c_{2} \cdot r_{2}c_{4} - r_{4}c_{3} \cdot r_{3}c_{4}$$

$$[R_{A}C_{5} - R_{4}C_{1} \cdot r_{1}c_{5} - r_{4}c_{2} \cdot r_{2}c_{5} - r_{4}c_{3} \cdot r_{3}c_{5}] \div r_{4}c_{4}$$

It is to be noted that (Matrix 20g) and (Matrix 23f) are fourth order auxiliary matrices. Therefore, the elements of (Matrix 23f) are determined as outlined above.

 $r_4 c_5 =$ 

- <u>7</u>/ Since our auxiliary matrix is symmetrical about the principal diagonal, then this element can be determined by dividing its symmetrically opposite element below the principal diagonal,  $r_4c_2$ , by the diagonal element which lies in the same row as  $r_2c_4$ . That is,  $r_2c_4 = r_4c_2 \div r_2c_2$ .
- <u>8</u>/ Since our auxiliary matrix is symmetrical about the principal diagonal, then this element can be determined by dividing its symmetrically opposite element below the principal diagonal,  $r_4c_3$ , by the diagonal element which lies in the same row as  $r_3c_4$ . That is,  $r_3c_4 = r_4c_3 \div r_3c_3$ .

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1042 - [8402 - 8401 - 105 - 1402 - 1403 - 1403 - 2041 - 2042

It is to be noted that (Matrix 20g) and (Matrix 23f) are fourth order auxiliary matrices. Therefore, the elements of (Matrix 23f) are determined as outlined above.

2/ Since our auxiliary varies is symmetrical about the principal diagonal, then this alemant can be determined by dividing its symmetrically opposite element below the principal diagonal,  $r_4^c_2$ , by the diagonal element which lies in the same row as  $r_2^c_4$ . That is,  $r_2^c_4 = r_4^c_2 + r_2^c_2$ .

Solution this element can be determined by dividing its symmetrically than this element can be determined by dividing its symmetrically opposite element below the principal disconal,  $r_4 c_3$ , by the disconal element which lies in the same row as r.c.. That is, r.c. - r.c. - r.c.

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